

Nonlinear Fields in Gravitation: Investigations in Black Hole and Cosmological Spacetimes

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners. I understand that my thesis may be made electronically available to the public.

Statement of Contributions

This thesis is based upon the following published articles, and in-writing manuscripts. Any important distinctions for composition during writing is indicated below.

Chapter 4 is based on:

Brayden R. Hull and Fil Simovic. *Exotic black hole thermodynamics in third-order Lovelock gravity*, Classical and Quantum Gravity 40.14 (2023). DOI: 10.1088/1361-6382/acdb3d

Chapter 5 is based on:

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Chapter 6 is based on:

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★ Calculations of equation 6.61 in **Section 6.2.1** was conducted by J. Menšíková.

Chapter 7 is based on:

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Chapter 8 is based on:

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★ **Section 8.1** was jointly written by myself and C. Zanoletti. I performed all calculations in **Section 8.2**, as well as **Appendix A. Section 8.3 & 8.4** were primarily analyzed by C. Zanoletti and C. D. Leonard.

Chapter 9 is based on:

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★ I carried out initial analysis **Section 9.4.1** before it was completed by E. Moghtaderi along with **Section 9.4.2**. I was the primary author and completed analysis for **Section 9.5**. I made contributions in writing to **Section 9.1,9.2,9.6**.

Abstract

This thesis is devoted to the study of non-linear effects in gravity, with a focus on black hole and cosmological spacetimes. We wish to study the phenomenological consequences that non-linearity can play on these particular spacetimes. These non-linear effects are primarily focused through higher curvature corrections, non-linear electrodynamics, and scalar fields. As we focus on black hole and cosmological space time we split this thesis into three parts, with Part I serving as a review of classical gravity, and alternative theories studied within.

Part II is dedicated to the discussion and study of black holes within these theories. We study various regimes with both single and multiple non-linear effects present. We study the thermodynamics of black holes in higher curvature and higher dimensional black holes, where higher curvature corrections serve as the source of non-linear effects. We show novel phase behaviour for a certain black hole class we refer to as exotic black holes. We show a further result of these exotic black holes is that they can possess negative mass in spacetimes which are pseudo-de Sitter spaces – asymptotically possessing positive curvature, but not constant. We investigate lower dimensional black hole solutions in general relativity coupled to nonlinear electrodynamics, and show how they all follow similar solution structure. We also look at lower dimensional black holes non-minimally coupled to non-linear scalar fields with linear and non-linear electrodynamics. We show that all black hole solutions sourced via a radial electric field are characterized by the solution from the general relativity case.

Part III focuses on the study of cosmology. We look at cosmological constraints on the novel 4-dimensional Einstein-Gauss-Bonnet Gravity. This is done through construction of a complete set of equations of motion for 1st order scalar perturbations, and makes use of ACTpol cosmological data. We also look to determine whether this theory of gravity can resolve cosmological fine tuning problems such as the horizon and flatness problems. We then discuss constraints of quantum energy conditions on FLRW spacetimes in general relativity. How a variable equations of state parameter will be constrained, and its connection with the recent DESI results. We also look at how a bouncing universe through a non-minimally coupled non-linear massive scalar can also be constrained.

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Part I

Gravitation

Preface

Gravitation is unique amongst physical phenomena; all inhabitants of Earth cannot escape its consequences, even those orbiting on the International Space Station. Why this fundamental interaction is unique will be expanded upon in the forthcoming sections, but the underlying feature that makes it special is that it affects all bodies, celestial and earth-bound, in the same manner. As one of the four fundamental interactions in physics, we have a duty to understand gravity to the best of our abilities. Not only will the understanding of this phenomenon on an individual level help us unravel our understanding of the universe that we inhabit, it will also give glimpses into the field of physics as a whole, as we now understand that there are deep connections between gravitational physics and quantum physics.

Describing the history of gravitational physics is somewhat difficult. This may be in part due to the fact that the notion of discovery is rather strange. There is and was naturally an understanding that an object near Earth when dropped will fall; the understanding of this process is more difficult than the result. The history of gravity is also deeply tied into the understanding of physics as an entire science. Observations and experiments on gravitational phenomena were conducted by many prominent scientists, with the most influential being Aryabhata, Aristotle, Ptolemy, Copernicus, Kepler, and Galileo¹. The discoveries of these aforementioned scientists paved the way for our current understanding of gravity.

Our current paradigm of gravity, thanks to Einstein, could not be complete or appreci-

¹There are numerous other individuals who contributed to the understanding of gravity, who should not go unrecognized.

ated without Newtonian gravity. The Newtonian reign on gravitational physics was almost 230 years, and in that time period it provided the opportunity to resolve and describe many physical observations. From ecliptic orbits to perturbations of planets, Newtonian gravity was supreme in describing the celestial bodies that had fascinated observers for centuries. While flaws were eventually discovered in Newton's theory, which required a more correct theory of gravity, it still has many important and reasonable practical uses to this day. The Newtonian understanding of gravity, along with its consequences and axioms, along with the foundational work in differential geometry by Riemann, are what paved the way for Einstein to establish his general theory of relativity. Fundamentally changing our view of this everyday phenomena.

As this thesis touches on gravitational physics in a variety of manners, it seems rather appropriate to include an introduction to the subject in a manner that appropriately sets up the remaining parts of the thesis. We believe this to be necessary since portions of Part II and III will touch on modified (or alternative) theories of gravity. It would seem rather premature to have discussions on these theories without formally introducing the theory that we are "modifying". As such, Part I will be broken down into the following chapters.

In Chapter 1, we will discuss Newtonian gravity. We wish to introduce the mathematics of the theory in the form of the Newtonian field equation. We will discuss the consequences of Newtonian gravity as it applies to free-fall, and how this provided the essential framework for Einstein to develop his theory. We will also provide some examples of applications where it was used to great avail, and also provide discussion of the eventual incompleteness of the theory.

Chapter 2 is dedicated to general relativity. We will discuss how the theory is formulated on the grounds of the equivalence principle. We will discuss the underlying features of differential geometry that provide the notion that gravity is the manifestation of the curvature of spacetime. We reserve any concrete applications and consequences of general relativity to their respective parts further in the thesis.

Lastly, Chapter 3 discusses alternative physical theories theories. This comes in the form of modified gravity and non-linear electrodynamics, both which have important impacts in Parts II and III. We will introduce and provide a rationale for modified gravity

before considering some specific theories that have been studied and focused on in this thesis. The discussion of nonlinear electrodynamics will be brief but concise, and has applications in Part II of this thesis.

Each chapter is meant to serve as an accelerated recap of their respective materials. In doing this, we of course neglect some finer details – while trying to capture the important foundational aspects therein. As such, we refer the readers to materials upon which each chapter was constructed to find the finer details. The textbook by Will & Poisson [7] provides excellent discussion on Newtonian gravity and weak-field general relativity. Carroll [8] provides a nice introduction to the subject of general relativity at an advanced undergraduate/beginning graduate level, with a concise discussion of manifolds and differential geometry. Hartle [9] introduces the topic of general relativity from a results-first perspective and makes connections with experiments paramount throughout the reading. The classics of Wald [10] and MTW [11] need no introduction, they are dense in material, and still serve as useful references today. Lastly, some additional sources that are more specialized are: Poisson [12], which focuses on black holes in great detail, along with Frolov & Novikov [13]; Mukhanov [14] provides detailed discussion on cosmological problems and methods in both general relativity and Newtonian gravity.

Chapter 1

Newtonian Gravity

Tis unconceivable that inanimate brute matter should (without the mediation of something else which is not material) operate upon & affect other matter without mutual contact; as it must if gravitation in the sense of Epicurus be essential & inherent in it. And this is one reason why I desired you would not ascribe innate gravity to me. That gravity should be innate inherent & essential to matter so that one body may act upon another at a distance through a vacuum without the mediation of any thing else by & through which their action or force may be conveyed from one to another is to me so great an absurdity that I beleive no man who has in philosophical matters any competent faculty of thinking can ever fall into it. Gravity must be caused by an agent acting constantly according to certain laws, but whether this agent be material or immaterial is a question I have left to the consideration of my readers.

– Issac Newton, 25 February 1692/3

1.1 Gravitational Equations

Newton’s universal law of gravitation is presented in many forms. Introductory discussions usually rely on the famous inverse square law for the gravitational force between two

massive isolated objects separated by a distance r between their centres

$$F_G = -\frac{GM_1M_2}{r^2}\hat{\mathbf{r}}$$

where G is Newton's gravitational constant¹. In this form, the masses are treated as point particles, with all mass concentrated at their centres. The use of $\hat{\mathbf{r}}$ indicates that the force direction is along the line connecting the centre's of the two objects. For multi-object systems the universal law is simply extended to

$$F_{Gm} = -Gm \sum_{i=1}^N \frac{M_i}{r_i^2} \hat{\mathbf{r}}_i$$

where the net force on mass m is dictated by the other masses M_i . While this is elementary in its application and discussion, it still serves as a reasonable place to begin discussion of Newtonian gravity.

A natural progression from the point particle interpretation, is the use of continuous matter. In this formalism, Newton's law is written in differential form through Poisson's equation

$$\nabla^2\Phi = 4\pi G\rho. \tag{1.1}$$

Here ρ represents a certain matter distribution, and in general depends on time and spatial coordinates $\rho \equiv \rho(t, \mathbf{x})$; Φ represents the gravitational potential, which is also coordinate and time dependent, $\Phi = \Phi(t, \mathbf{x})$; ∇^2 represents the spatial Laplacian. Solving (1.1) for a given matter distribution will provide the gravitational potential Φ , and the gravitational field associated with that distribution is given by $\mathbf{g} = -\nabla\Phi$. In more direct terms: matter induces a gravitational field.

An important feature of (1.1) is that it is an elliptic partial differential equation – even with Φ being time dependent – meaning that we only need to solve it for a suitable set of boundary conditions. Since no initial conditions are required for finding Φ , this means that any change in ρ would propagate to Φ instantaneously. Additionally, ρ is a source term in the equation; thus we could choose a peculiar distribution and see what gravitational field

¹Current data suggests a numeric value of $6.67430 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}$ [15].

it would generate. This could lead to completely un-physical gravitational fields. To avoid this, the matter distribution is constructed to act in accordance with fluid dynamics.

Assuming an uncharged, non-relativistic, incompressible fluid with zero viscosity, the matter will adhere to the Euler equations:

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \\ \rho \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) &= \rho \nabla \Phi - \nabla p\end{aligned}\tag{1.2}$$

where $\nabla \Phi$ is to account for the effect that the induced gravitational field has on the matter itself, even as it sources (1.1). Also introduced is $p \equiv p(t, \mathbf{x})$ and $\mathbf{v} \equiv \mathbf{v}(t, \mathbf{x})$ the flow velocity. These two equations contain three unknown functions: $\rho, p, \mathbf{v}$; therefore the system is under-constrained. To negate this issue, one needs to introduce an equation of state relating pressure and density² $p \equiv p(\rho)$, allowing for a solution to more be readily accessible. The combination of (1.2) and (1.1) create the gravitational-matter system of equations, and must be solved simultaneously in all variables.

For a spherically symmetric, time-independent, uniform density object, (1.2) would no longer apply for the field outside of the object. Instead, one may directly use (1.1) and arrive at

$$\Phi = -\frac{GM}{r}.$$

for the potential outside the body. M represents the total mass of through $M = \frac{4\pi\rho_0 R^3}{3}$ with R being the radius of the object. Considering a particle outside the object with mass m insufficient in comparison to M , under only the force of gravity it would adhere to Newton's second law

$$m_I \frac{d^2 x^i}{dt^2} = m_g \left(-\frac{\partial \Phi}{\partial x^i} \right).\tag{1.3}$$

We have m_I representing the inertial mass, and m_g representing the gravitational mass. The weak equivalence principle, as it is called, is the concept that gravitational mass and inertial mass are equivalent: $m_I = m_g$. This principle – which has been shown to extremely

²Under the consideration of less simplistic fluids the equation of state can be a function of other intrinsic properties.

high accuracy [16] – then allows us to consider a simplified scenario for the particle, where it is only under going ”vertical” motion above the more massive object M . The second law for the height of the test object is given by

$$\frac{d^2h}{dt^2} = -\frac{GM}{(R+h)^2} \quad (1.4)$$

where the mass terms for the test particle are no longer present. This defines the universality of free-fall: all point-like test particles, independent of their composition and mass will fall at the same rate for a given gravitational field, and the motion only depends on initial position and velocity.

If we consider a more general scenario, where the gravitational field is given through $\mathbf{g} = -\nabla\Phi$, and are not restricting motion to be vertical, we can consider Newton’s second law for our particle as

$$\ddot{\mathbf{r}} = \mathbf{g}. \quad (1.5)$$

In this particle formalism we are taking the origin of our frame of reference to be the surface of the object inducing the gravitational field. A second frame of reference can be constructed with position $\mathbf{y}$ and has acceleration $\mathbf{a}$. The position of the particle in that frame is given by $\mathbf{r}' = \mathbf{h} - \mathbf{y}$, and the equation of motion in the new frame will be

$$\begin{aligned} \ddot{\mathbf{h}}' &= \ddot{\mathbf{h}} - \ddot{\mathbf{y}} \\ \ddot{\mathbf{h}}' &= \mathbf{g} - \mathbf{a} \\ \ddot{\mathbf{h}}' &= \mathbf{g}'. \end{aligned} \quad (1.6)$$

Thus, the motion as described in the new frame can still be modelled via a gravitational field, but with a new field strength g' . If the gravitational field of the earth was “turned off”, then $g = 0$, and the new frame simply says that $a = -g'$, meaning the new frame is mimicking a gravitational field simply by accelerating³. If the new frame had no gravitational field, $\mathbf{g}' = 0$, this would occur if $a = g$. The reference frame was under free-fall to the object’s surface. These results guided Einstein in the quest for the general theory of relativity, and his formulation of the equivalence principle which will be further discussed in Chapter 2.

³A secondary consequence of the principle of equivalence and universality of free-fall

1.2 Success and Downfalls

With the arrival of Newton’s law of gravity, there were semi-immediate implications for its use. One of the most important consequences of Newton’s law was the discovery of Neptune. The orbit of Uranus did not appear to follow motion in accordance with Newton’s law; instead, it showed deviations from its expected orbit. These observations led to the conclusion that there must be another planet in the solar system outside of Uranus that was responsible for these deviations. Through the use of Newton’s law and orbital data, astronomers were able to *predict* where the mysterious planet would be located at a given instant. Pointing their telescopes in the desired direction in the night sky, they were able to locate Neptune just as the theory had predicted.

The use of Newtonian gravity in the discovery of Neptune was a momentous occasion. The law had been used in far reaches of the solar system to make accurate and detectable predictions with the technology of the time. With this new tool at the disposal of astronomers, they could also try to rectify another anomaly visible in the solar system: Mercury’s perihelion advance. Like its cousin Uranus, astronomers set out to discover where a mystery planet could be located that would account for the advance being observed – there wasn’t one. Either they were not looking hard enough, or something else was missing. As we very well now know, the missing piece was general relativity. This was just one scenario upon which Newtonian theory was beginning to show its limits.

A second, more pertinent rationale for the incompleteness of Newtonian gravity is the lack of compatibility with special relativity. This can be seen in a few different manners. First, in (1.1) the functions are time-dependent; therefore any change in matter distribution would impose an instantaneous change in the gravitational field. Additionally, the presence of the Laplacian in (1.1) instead of the d’Alembertian operator makes the application of the Lorentz transformations invalid. This is no major surprise, Newtonian physics is based upon absolute space and absolute time, while special relativity deals with relative space and relative time. By construction, special relativity ignores gravity, leading to the understanding that for gravity and relativity to be safe, a general theory must be constructed.

Chapter 2

General Relativity

2.1 Einstein Equivalence Principle: Gravity as Curvature

The consequence of the weak equivalence principle, and universality of free-fall for that matter, is rather restrictive. This is due to the fact that they are constructed on the discussion of test bodies solely under the influence of gravity. Physics is much more than gravity, accordingly, any general theory of relativity must also consider how the non-gravitational laws of physics are influenced via gravitational fields. This notion is what leads to the Einstein equivalence principle: the weak equivalence principle holds, and locally in a gravitational field one may establish an inertial frame of reference where non-gravitational laws of physics adhere to special relativity in Minkowski space.

The requirement of local experiments is a necessity, not just for the Einstein principle, but the weak equivalence principle as well. If any experiment is conducted over a sufficiently large region of space, both inhomogeneous and/or tidal effects from the gravitational field will be detected and skew experimental results in comparison to a *purely* inertial reference frame. With this however, a new theory of gravity must hold in non-inertial frames. Why might this be? Well, since a new theory of gravity must be able to consider inhomogeneous fields, only local patches of spacetime are said to be inertial; the global structure is not

going to be inertial. As inertial frames have preferred coordinate systems, non-inertial frames do not. This leads to the principle of covariance, that any choice of coordinate system will be adequate in describing the theory and the consequences therein¹.

The weak equivalence principle in conjunction with Einstein’s equivalence principle lead to a profound understanding of the nature of gravity. Gravity does not act like a “traditional” force, such as electromagnetism. For a charged object, say in an electric field, the amount of charge and mass play a crucial role in the object’s motion; this is not the case with gravity. Gravitational trajectories are governed by the structure of spacetime. As such gravity is geometrical and should be described as such. The necessity of describing gravity as geometry means employing tools of differential geometry. Through differential geometry, one may give a definition of a spacetime: In general relativity spacetime is a four-dimensional Lorentzian manifold $\mathcal{M}$ with a metric tensor $g_{\mu\nu}$ which gives rise to curvature through the Riemann curvature tensor. The components of the metric are interpreted as the gravitational potentials, leading to derivatives of the metric capturing the essence of the gravitational field.

Just as the Minkowski metric $\eta_{\mu\nu}$ captures the essence of special relativity through the line element

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

the metric tensor conveys the same information

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

but now $g_{\mu\nu}$ is no longer taken to be globally flat. With curved spacetime, the notion of directional derivatives needs to be generalized into consideration of the curved space. This is due to the fact that basis vectors along the manifold $\mathcal{M}$ are not constant. This introduces the connection, and covariant derivative. The covariant derivative serves as the generalization of the directional derivative that can be extended beyond vectors to tensor fields as well. The connection provides a description for how basis vectors change throughout nearby points on the manifold. For a second rank tensor, the covariant derivative is

¹This provides another backbone for why a general theory of relativity needs to be constructed via a tensor formalism.

given by

$$\nabla_\beta T_{\mu\nu} = \partial_\beta T_{\mu\nu} - T_{\alpha\nu} \Gamma_{\mu\beta}^\alpha - T_{\mu\alpha} \Gamma_{\nu\beta}^\alpha \quad (2.1)$$

with the connection coefficients $\Gamma_{\alpha\beta}^\gamma$, defined through the directional derivative of the basis vectors $\nabla_\nu \mathbf{e}_\mu = \Gamma_{\mu\nu}^\alpha \mathbf{e}_\alpha$.

Assuming that the metric is compatible with covariant derivative, which is foundational feature of general relativity, introduces the *Levi-Civita connection*, or Christoffel symbols in terms of the metric tensor components

$$\Gamma_{\nu\rho}^\mu = \frac{1}{2} g^{\mu\sigma} (g_{\sigma\nu,\rho} + g_{\sigma\rho,\nu} - g_{\nu\rho,\sigma})$$

which will depend on the choice of coordinates. It is the Christoffel symbols which represent the gravitational field in general relativity. Test particles will follow paths that satisfy the geodesic equation

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\nu\rho}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0 \quad (2.2)$$

which is commonly viewed as a generalization of Newton's second law to curved spacetimes.

The curvature of spacetime, or any manifold for that matter, is captured through the Riemann curvature tensor. In a coordinate basis it is expressed through

$$R_{\nu\rho\sigma}^\mu = \partial_\rho \Gamma_{\nu\sigma}^\mu - \partial_\sigma \Gamma_{\nu\rho}^\mu + \Gamma_{\nu\sigma}^\tau \Gamma_{\tau\rho}^\mu - \Gamma_{\nu\rho}^\tau \Gamma_{\tau\sigma}^\mu. \quad (2.3)$$

If spacetime is flat, then the Riemann tensor will be identically zero. Contraction of the first and third indices give rise to the Ricci curvature tensor

$$R_{\mu\nu} = \partial_\mu \Gamma_{\nu\rho}^\rho - \partial_\nu \Gamma_{\mu\rho}^\rho + \Gamma_{\mu\rho}^\sigma \Gamma_{\sigma\nu}^\rho - \Gamma_{\mu\nu}^\sigma \Gamma_{\sigma\rho}^\rho \quad (2.4)$$

and a further contraction with the metric of the Ricci tensor gives rise to the Ricci scalar

$$R = g^{ij} R_{ij}. \quad (2.5)$$

In general relativity the metric is fundamental, and all associated gravitational phenomena are captured through the metric, or derivable therein (i.e., curvature, geodesics, tidal forces). As a result, general relativity is what is referred to as a metric theory of

gravity. The metric tensor encodes the geometry of spacetime. Up to this point, we have only discussed what the metric tensor is and how it relates to curvature. The purpose of a metric theory of gravity, and therefore general relativity, is to provide a framework for determining what the metric tensor is, that is where the field equations of general relativity come into the picture.

2.2 Einstein Field Equations

One of the postulates of general relativity is the field equations, also called the Einstein field equations, or simply Einstein's equations. While postulated by Einstein, the field equations may also be “derived” through a Lagrangian formalism, which plays an important role in discussing alternative theories of gravity. When presented, the field equations are commonly displayed as follows:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} \quad (2.6)$$

where κ is a coupling constant, and $T_{\mu\nu}$ is the stress energy tensor of any matter fields that may be present. The left-hand side is called the Einstein tensor

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$$

leaving

$$G_{\mu\nu} = \kappa T_{\mu\nu}.$$

The conservation of stress-energy is indicated through $\nabla_\mu T^{\mu\nu} = 0$, as such the Einstein tensor also adheres to this identity, $\nabla_\mu G^{\mu\nu} \equiv 0$. These features are true without any coordinate choice, a consequence of the diffeomorphism invariance of general relativity. In addition, by metric compatibility $\nabla_\rho g_{\mu\nu} = 0$. any additional terms containing the metric and a constant may be added to the left side of the field equations and retain that they are divergence free. This consequence persuaded Einstein to include the cosmological constant term in the field equations to obtain a static universe solution. Thus the field equations may also be presented as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \quad (2.7)$$

In order to match the Newtonian limit in the case of weak gravitational fields, the constant κ (in SI units) is set to $\kappa = \frac{8\pi G}{c^4}$.

To arrive at the field equations of general relativity one may also use an action principle

$$S = S_{\text{EH}} + S_{\text{M}} \quad (2.8)$$

where S_{EH} is the Einstein-Hilbert action, and S_{M} are any associated matter fields. Together they can be explicitly written as

$$S = \int_{\mathcal{M}} \left[\frac{c^4}{16\pi G} (R - 2\Lambda) + \mathcal{L}_M \right] \sqrt{-g} d^4x, \quad (2.9)$$

where g represents the metric determinant. A necessary inclusion to define an invariant volume form, otherwise covariance is not satisfied.

The gravitational Lagrangian is given by $\mathcal{L} = R - 2\Lambda$ where R is the Ricci scalar, $\mathcal{L}_M$ is the matter Lagrangian. To obtain the field equations, we take the variation of the action with respect to the inverse metric, and hold that it must be zero via the stationary-action principle, $\delta S = 0$. Since the Ricci scalar is $R \equiv R_{\mu\nu} g^{\mu\nu}$, the variation will be:

$$\delta S = \int_{\mathcal{M}} \left[\frac{c^4}{16\pi G} (\sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} + \sqrt{-g} R_{\mu\nu} \delta g^{\mu\nu} + R_{\mu\nu} g^{\mu\nu} \delta \sqrt{-g} - 2\Lambda \delta \sqrt{-g}) + \delta(\mathcal{L}_M \sqrt{-g}) \right] d^4x$$

The first term that can be evaluated is $\sqrt{-g}$ via Cramer's rule that states:

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\mu\nu} \delta g^{\mu\nu}. \quad (2.10)$$

The variation of the Ricci tensor yields

$$\delta R_{\mu\nu} = \nabla_\lambda (\delta \Gamma_{\mu\nu}^\lambda) - \nabla_\nu (\delta \Gamma_{\mu\lambda}^\lambda) \quad (2.11)$$

which under the product of the inverse metric will become

$$g^{\mu\nu} \delta R_{\mu\nu} = \nabla_\lambda V^\lambda.$$

Thus, the variation of the Ricci tensor can be viewed as the divergence of a vector field V . Putting this together will yield

$$\delta S = \int_{\mathcal{M}} \left[\frac{c^4}{16\pi G} (\nabla_\lambda V^\lambda + R_{\mu\nu} \delta g^{\mu\nu} - R_{\mu\nu} g^{\mu\nu} \frac{1}{2} g_{\mu\nu} \delta g^{\mu\nu} + \frac{\delta(\mathcal{L}_M \sqrt{-g})}{\sqrt{-g}}) \right] \sqrt{-g} d^4 x \quad (2.12)$$

The first term in the variation is a total derivative, hence it will not enter the field equations and does not contribute to the dynamics of the spacetime. This leaves us with

$$\delta S = \int_{\mathcal{M}} \left[\frac{c^4}{16\pi G} (R_{\mu\nu} - R_{\mu\nu} g^{\mu\nu} \frac{1}{2} g_{\mu\nu}) - \frac{1}{2} T_{\mu\nu} \right] \delta g^{\mu\nu} \sqrt{-g} d^4 x \quad (2.13)$$

Requiring $\delta S = 0$ implies the integrand must be zero, yielding the field equations. The action method for deriving, or constructing solutions, is an important feature when discussing alternative gravity theories.

Einstein equations in the form of (2.6), or (2.7) represent a set of 10 – in 4-dimensional spacetime – coupled, non-linear partial differential equations for the metric $g_{\mu\nu}$. The matter fields that may or may not be present are determined via their own field equations – that are coupled to the gravitational field equations – and the condition that stress-energy must be conserved: $\nabla_\mu T^{\mu\nu} = 0$. The underlying important aspect of the field equations in general relativity is that Einstein tensor is linear in curvature. Both the Ricci tensor and Ricci scalar are not coupled to any other curvature terms. This is also evident in the Einstein-Hilbert action as the gravitational Lagrangian is linear in the Riemann curvature tensor. In the following chapter we will discuss how one can modify (2.6) or (2.7) to achieve alternative gravity theories to general relativity, with a focus on non-linear effects.

Chapter 3

Nonlinear Fields: Modified Gravity and Electrodynamics

There is no debate that general relativity has been an extremely successful physical theory. Even before the field equations [17] were published, Einstein was able to use his theory of curved spacetime to account for the perihelion of mercury to near perfect accuracy. And in the over 100 years since then, it continues to pass all experimental tests [18] with intense rigour. This begs the question: why study theories beyond, or alternative to general relativity? There is not one uniform answer to this question. A shaky response to this might be that Newtonian gravity was right until it wasn't, and who is to say that the same might not happen to general relativity. A more robust answer lies in the fact that general relativity predicts its own demise through the creation of spacetime singularities. The singularities at the centre of black holes, or the origin of the universe indicate the breakdown of spacetime itself, so how will a theory that describes spacetime survive if the spacetime itself is non-existent?

A classical field theory, such as GR, at the level of the Planck scale is not believed to be a valid theory, and as such, the description of singularities should be viewed through a quantum theory of gravity. While there have been many attempts to construct a quantum theory of gravity [19], two approaches stand at the forefront of the problem: String Theory and Loop Quantum Gravity [20, 21]. While both of these methods have their elegance

and pitfalls, neither has come up with a full covariant formalism of quantum gravity; the main criticism being the lack of experimental verification and predictability, with varying degrees of harshness[22, 23].

It is generally assumed that GR is a low energy effective field theory of a more fundamental theory (quantum gravity). With this in mind, it suggests that at a higher energy level there will be modifications to the Einstein-Hilbert action (2.9). Modifications to the action can occur in many forms; one of the most popular is adding nonlinear curvature terms; however, this is not the only possibility. A motivation for modifying (2.9) can be argued from the perspective of quantum field theory in curved spacetime [24], along with the ability to re-normalize such theories when nonlinear curvature terms are present.

This raises the question as to why should we study different gravity theories other than GR. The easiest answer to this question is two-fold. Are the results and phenomena that occur in general relativity unique? If not, what other possible phenomena do these other theories predict, and could we propose a test and observe our predictions? These questions are subject to criticism and discussion with varying degrees of harshness. Primarily, from the opening statement that GR has withstood immense scrutiny, why bother looking elsewhere? If the response is along the lines of production of unavoidable space time singularities one can quote the Cosmic Censorship Conjecture [25], which says we are blinded from the singularity by an event horizon as the universe does not permit naked singularities. These conversations between theoretical physicists are common and they produce valid points of discussion. We shall properly define modified gravity below, and give a brief discussion of the field before focusing on a few particular classes of theories.

3.1 Modified Gravity Theories

To talk about modified gravity, we should be careful and more precise on what exactly we are modifying. In the field equations of general relativity the left hand side is commonly referred to as the “gravity” side, since this side contains the terms associated with curvature. This may be a naive comment to make, since the entirety of the field equations are necessary for gravity, but the fact remains the same: the left side contains curvature

and the right side matter. This is rather important, since it begs the question is general relativity unique? This question was addressed by Lovelock [26, 27] in a set of papers nearly 55 years ago.

Lovelock's theorem was an attempt to find the uniqueness in the field equations of general relativity, but more specifically on the Einstein tensor. The theorem assumes that gravitational action must solely be constructed through local invariants of the metric tensor

$$S \propto \int d^4x \mathcal{L}(g_{\mu\nu}) \quad (3.1)$$

and contains at most second derivatives, and the spacetime dimension is 4. Taking a variation of (3.1) with respect the metric, or conversely using the Euler-Langrange equations, the following can be stated [28]: *The only 4 dimensional expression with second order derivatives of the metric derived from a general scalar $\mathcal{L}(g_{ab})$ is given by*

$$E^{ab} = \alpha \sqrt{-g} \left[R^{ab} - \frac{1}{2} g^{ab} R \right] + \lambda \sqrt{-g} g^{ab} \quad (3.2)$$

where R is the Ricci scalar, and R_{ab} is the Ricci tensor. Therefore, the Einstein Field Equations are the unique 2nd order equations of motion for the gravitational field in 4 dimensions that come from an action principle. Again, it is more accurate to state that the Einstein tensor is the unique 4-dimensional result from a purely gravitational action without any additional fields.

The result of the theorem makes no comment on how the gravitational Lagrangian needs to look, or if there is a unique choice. The Einstein-Hilbert action is a particular choice that gives the field equations, but it is not the only choice. The most general scalar Lagrangian density from which the Einstein field equations can be derived from in 4 spacetime dimensions is given by [28]

$$\mathcal{L} = \alpha \sqrt{-g} R - 2\lambda \sqrt{-g} + \beta \epsilon^{abcd} R_{ab}^{ef} R_{efcd} + \gamma \sqrt{-g} (R^2 - 4R_a^b R_b^a + R_{cd}^{ab} R_{ab}^{cd}) \quad (3.3)$$

where $\alpha, \lambda, \beta, \gamma$ are all constants and $R_{\alpha\beta ab}$ is the Riemann curvature tensor. Neither the third term or the fourth term contribute to the field equations as they are total derivatives and hence do not contribute to gravitational field dynamics.

This leads us to Lovelock's theorem, which states that for a gravitational theory to differ from general relativity one or more of the following must occur [28]:

- Consider other fields, beyond (or rather than) the metric tensor.
- Accept higher than second derivatives of the metric in the field equations.
- Work in a space with a different dimensionality from four.
- Give up on either rank $(2, 0)$ tensor field equations, symmetry of the field equations under exchange of indices, or divergence-free field equations.
- Give up locality.

It is far too difficult to summarize all the branches of modified theories here; there are several elegant reviews on the subject [29, 28].

Alternative gravity theories are not a new idea. In fact, there were individuals such as Nordström prior to 1915 who were considering alternative gravity theories that differed from Newton. In particular, Nordström was trying to construct a scalar theory of gravity by invoking a Lorentz scalar as a replacement for the Newtonian potential. Although he was successful in “creating” a theory with this ideas, the theory fundamentally was of no importance as it could not account or explain observations. Since the publication of Einstein’s general relativity there have been numerous alternative theories presented, each with various level of deviation from general relativity. In the following sections we will focus discussion on alternative theories of gravity that will be pursued in Parts II and III of the thesis, in addition to a discussion of non-linear electrodynamics.

3.1.1 Lovelock Gravity: Higher Dimensions and Nonlinear Curvature

One class of modified theories that has attracted a lot of attention for well over 50 years is higher dimensional gravity, and one specific sub-class in particular to receive a bulk of this is Lovelock theory [27]. Lovelock gravity is most frequently stated as a generalization of Einstein gravity to higher dimensions that keeps the field equations at most second order in the metric. One of the key aspects about Lovelock theory, alongside the higher dimensions, is the inclusion of nonlinear curvature terms in the action, which do indeed

produce dynamics in the field equations. Just like Lovelock's theorem is needed to be treated with some care, Lovelock gravity must also be considered with care – which we will now discuss.

The action for a d-dimensional Lovelock theory is provided by

$$S = \int d^d x \sqrt{-g} \left(\frac{1}{16\pi G_N} \sum_{k=0}^K \alpha_k \mathcal{L}^{(k)} \right) \quad (3.4)$$

where K refers to the order of the theory, G_N is the d-dimensional Newton constant, and $\mathcal{L}^{(k)}$ are the Euler densities, computed through contractions of the Riemann tensor

$$\mathcal{L}^{(k)} = \frac{1}{2^k} \delta_{c_1 d_1 \dots c_k d_k}^{a_1 b_1 \dots a_k b_k} R_{a_1 b_1}^{c_1 d_1} \dots R_{a_k b_k}^{c_k d_k}. \quad (3.5)$$

The δ is the generalized Kronecker delta symbol:

$$\delta_{a_1 \dots a_p}^{b_1 \dots b_p} = \begin{cases} 1 & \text{if } a_1 \dots a_p \text{ are distinct integers and are an even permutation of } b_1 \dots b_p \\ -1 & \text{if } a_1 \dots a_p \text{ are distinct integers and are an odd permutation of } b_1 \dots b_p \\ 0 & \text{in all other cases.} \end{cases} \quad (3.6)$$

which forms unique contributions of the Euler characteristics. The first few Euler invariants are:

$$\mathcal{L}^{(0)} = 1 \quad \mathcal{L}^{(1)} = R \quad \mathcal{L}^{(2)} = R^2 - 4R_{ab}R^{ab} + R_{abcd}R^{abcd}. \quad (3.7)$$

The first term is nothing other than unity, the second term works out to the Ricci scalar – the Einstein-Hilbert term, and the third term is the Gauss-Bonnet invariant. The numerical coefficients, α_k are the unique portion that the Euler characteristic instills in the theory. The variation of (3.5) with respect to the metric yields the field equations

$$\sum_{k=0}^K \hat{\alpha}_{(k)} \mathcal{G}_b^{(k)a} = 0 \quad (3.8)$$

where $\mathcal{G}_b^{(k)a}$ are the Lovelock tensors, given through

$$\mathcal{G}_b^{(k)a} = -\frac{1}{2^{(k+1)}} \delta_{b e_1 f_1 \dots e_k f_k}^{a c_1 d_1 \dots c_k d_k} R_{c_1 d_1}^{e_1 f_1} \dots R_{c_k d_k}^{e_k f_k} \quad (3.9)$$

– another set of contractions of the Riemann tensor. Now, in (3.5) we neglected any other additional matter fields. If they are included in such a way that they have the form $\mathcal{L}_m \equiv \mathcal{L}(g_{\mu\nu}, \psi)$ where the “field” is represented by ψ and it is minimally coupled to the gravitational sector, the field equations will become

$$\sum_{k=0}^K \hat{\alpha}_{(k)} \mathcal{G}_b^{(k)a} = 8\pi G T_b^a \quad (3.10)$$

and T_b^a is the stress energy tensor associated with the matter field $\mathcal{L}_m$. The stress-energy tensor maintains the same definition as it does in the general relativity case, just modified to an arbitrary dimension.

Some bookkeeping needs to be established before moving forward. The α_k are referred to as the Lovelock coupling constants, and the $\hat{\alpha}_k$ are the rescaled constants given by

$$\alpha_0 = \frac{\hat{\alpha}_0}{(d-1)(d-2)}, \quad \alpha_1 = \hat{\alpha}_1, \quad \alpha_k = \hat{\alpha}_k \prod_{n=1}^{2k} (d-n) \quad \text{for } k \geq 2 \quad (3.11)$$

where we can identify the cosmological constant as $\Lambda = -\frac{\hat{\alpha}_0}{2}$. Lastly K is an integer that must satisfy the following inequality $K \leq \lfloor \frac{d-1}{2} \rfloor$, where d is the spacetime dimension. If a K -th order theory does not satisfy this inequality, any additional terms present with $K > \frac{d-1}{2}$ in the action will not contribute to the dynamics (field equations) as the terms are total derivatives. Thus, for the Gauss-Bonnet invariant, $\mathcal{L}^{(2)}$, to induce gravitational dynamics in the spacetime it must be $d \geq 5$.

Lovelock gravity is an extremely nice theory to work with that goes beyond general relativity. While introducing higher dimensions, and higher curvature, it keeps the field equations at most second order in the metric. This of course differs from general relativity in higher dimensions which would use the same field equations as (2.7) (with $G = G_N$) and the metric is of arbitrary dimension. A characteristic of Lovelock gravity is that under the limit of $\alpha_k \rightarrow 0 \forall k \geq 2$ it will approach d -dimensional general relativity, and $d \rightarrow 4$ will yield general relativity.

Although Lovelock gravity requires the usage of higher dimensions to produce any new dynamics, and thus is a higher dimensional theory, it is also part of a class of theories

that are referred to as purely geometric. This should not come as a surprise since no matter fields are coupled to the geometry sector, and additional curvature invariants are introduced. Like Lovelock gravity, purely geometric theories have been highly studied; but they suffer from a downfall of producing field equations with derivatives of the metric being higher than second order. A classic example of an alternative purely geometric theory of gravity is $F(R)$ gravity, where instead of a simple linear term in the Ricci scalar appear in the action, there is an arbitrary function

$$S_{F(R)} = \frac{1}{16\pi G} \int d^4x \sqrt{-g} f(R) \quad (3.12)$$

where matter fields are neglected, but could be present. The variation of this action with respect to the metric will yield field equations of fourth-order

$$f'(R)R_{\mu\nu} - \frac{f(R)}{2}g_{\mu\nu} = \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \square f'(R) \quad (3.13)$$

where $\square = \nabla_\mu \nabla^\mu = g^{\mu\nu} \nabla_\mu \nabla_\nu$. The Einstein tensor can be isolated in (3.13)

$$G_{\mu\nu} = \frac{1}{f'(R)} \left\{ \nabla_\mu \nabla_\nu f'(R) - g_{\mu\nu} \square f'(R) + g_{\mu\nu} \frac{[f(R) - f'(R)R]}{2} \right\} \quad (3.14)$$

and in this form, the equations look similar to general relativity but with a curvature induced stress-energy sector – the right hand side of the equations. Even with this presentation, the theory is of course not general relativity. The unique and consequential feature of $F(R)$ gravity is that it can be mapped into a different class of alternative theories called scalar-tensor theories, upon which a scalar field is non-minimally coupled to the gravitational sector, but the field equations remain second order in the metric.

3.1.2 Scalar-Tensor Theory

Both general relativity and Lovelock gravity are by construction tensor theories, built from the metric tensor. The theory of gravity attempted by Nordstöm was a pure scalar theory, which had suffered from the fact it did not satisfy the equivalence principle that was foundational in Einstein's quest. While scalar theories of gravity suffer from downfalls immediately in their construction, they can be combined with metric theories – or other

tensor theories in that matter – to create “scalar-tensor” theories of gravity. This does not mean that one simply adds a scalar field into the action as an additional matter field, but that it is coupled to the gravitational sector – the other matter fields are still minimally coupled.

The roots of scalar-tensor theory begin with Jordan in the late 1950’s while investigating manifold embedding. This embedding was implemented through a curved 4-dimensional manifold being imposed onto a 5-dimensional flat space. In Jordan’s work it was discovered that constraints must be imposed on the embedding, and these constraints can be associated with a scalar field coupled to the gravity sector; thus scalar-tensor theory originated.

In a scalar-tensor theory of gravity, the most general Lagrangian density is given by [28, 30, 31, 32]

$$\mathcal{L} = \frac{1}{16\pi G} \sqrt{-g} \left[\phi R - \frac{\omega(\phi)}{\phi} \nabla_a \phi \nabla^a \phi - 2\Lambda(\phi) \right] + \mathcal{L}_m(\Psi, g_{ab}) \sqrt{-g} \quad (3.15)$$

where $\omega(\phi)$ is an arbitrary function, and $\Lambda(\phi)$ is another arbitrary function that generalizes the concept of the cosmological constant and $\mathcal{L}_m$ is any associated Lagrangian for the other matter field(s) Ψ . A very well known example of scalar-tensor theory is the Brans-Dicke theory [33], which is a special case of (3.15) – Brans-Dicke theory is an extension of the work by Jordan. In Brans-Dicke theory, $\omega(\phi) \rightarrow \text{const.}$ and $\Lambda(\phi) \rightarrow 0$; one can recover General Relativity for $\omega \rightarrow \infty$, $\omega'/\omega^2 \rightarrow 0$ and $\Lambda \rightarrow \text{const.}$ Nevertheless, employing the action (3.15) and doing a variational principle with respect to the metric yields the field equations

$$\phi G_{ab} + \left[\square \phi + \frac{1}{2} \frac{\omega}{\phi} (\nabla \phi)^2 + \Lambda \right] g_{ab} - \nabla_a \nabla_b \phi - \frac{\omega}{\phi} \nabla_a \phi \nabla_b \phi = 8\pi G T_{ab} \quad (3.16)$$

where a variation with respect to the scalar field ϕ gives the scalar equation

$$(2\omega + 3) \square \phi + \omega' (\nabla \phi)^2 + 4\Lambda - 2\phi \Lambda' = 8\pi T \quad (3.17)$$

after substituting (3.16), with T being the trace of the stress-energy tensor. An interesting property of scalar-tensor theories is that they are *conformally* equivalent to General Relativity. It is important to distinguish that this does not mean that they are the

same theory. Simply put, you can do a conformal transformation on the metric and re-write the equations of motion as a portion containing the Einstein tensor with the scalar field ϕ acting like an additional matter field. To obtain the conformal equations of motion, one makes use of the following set of transformations: $g_{ab} = e^{2\Gamma(x)}\bar{g}_{ab}$, $e^{2\Gamma} = \phi^{-1}$, $g_{ab} = \bar{g}_{ab}/\phi$, $\sqrt{4\pi/(3+2\omega)} \equiv \partial\Gamma/\partial\psi$ and $8\pi V(\psi) \equiv e^{4\Gamma}\Lambda$ [28]

$$\begin{aligned}\bar{G}_{ab} &= 8\pi \left[\bar{T}_{ab} + \bar{\nabla}_a\psi\bar{\nabla}_b\psi - \left(\frac{1}{2}\bar{\nabla}_\alpha\psi\bar{\nabla}^\alpha\psi + V \right) \bar{g}_{ab} \right] \\ \tilde{\square}\psi - \frac{dV}{d\psi} &= -\sqrt{4\pi}\alpha\tilde{T}.\end{aligned}\tag{3.18}$$

The form of (3.15) and (3.16) are referred to as the Jordan frame, while (3.18) is referred to as the Einstein frame. While it is true that one can isolate the Einstein tensor in (3.16), there will be the $\frac{1}{\phi}$ term on the right side; the conformal transformations remove this feature.

It is also possible to map $F(R)$ theories into general scalar-tensor theories as stated below (3.18). This procedure is completed through a Legendre transformation on the action [34]. Ignoring the matter section, the $F(R)$ action (3.12) is identical to

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} [f(\varphi) + (R - \varphi)f'(\varphi)].\tag{3.19}$$

This is due to the later producing $(\varphi - R)f'' = 0$ if the variation with respect to the field ϕ is computed. Requiring that $f'' \neq 0$ would revert (3.19) back to (3.12). Thus, a more general action:

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} [F(\varphi)R - Z(\varphi)g^{\mu\nu}\nabla_\mu\varphi\nabla_\nu\varphi - 2V(\varphi)]\tag{3.20}$$

reduces to the integrand of (3.15) with

$$F(\phi) = \phi, \quad Z(\phi) = \frac{\omega}{\phi}, \quad 2V(\phi) = \varphi f'(\varphi) - f(\varphi).$$

The purpose of this discussion was to introduce the notion of conformal and Legendre transformations for gravitational theories – and showing that different alternative theories

can be mapped to each other ¹. The features of scalar-tensor and higher-curvature theories came back into the limelight a little more than 5 years ago when a pre-print [38] appeared to dispute Lovelock’s theorem. This paper brought rise to the newly discovered theory referred to as 4-dimensional-Einstein-Gauss-Bonnet Gravity which we will now discuss.

3.1.3 4 Dimensional Einstein-Gauss-Bonnet Gravity(4DEGB)

In 2020 Glavin and Lin produced the paper [38], where they proceeded to show that you could construct a new 4-dimensional gravity theory that included the Gauss-Bonnet invariant (second Euler density)

$$\mathcal{L}^{(2)} = R^2 - 4R^{\mu\nu}R_{\mu\nu} + R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \equiv \mathcal{G}. \quad (3.21)$$

Now, in general this would not cause any harm since the term can be coupled with a scalar field into a scalar-tensor theory of gravity. The problem came from the fact that they claimed it was a pure metric-only theory of gravity alternative to general relativity, thus conflicting with Lovelock’s theorem. The drawback of their theory was in the construction, they began with a theory of d -dimensions, and tried to map into a lower dimension through a naive re-scaling.

Their procedure begins with the following action (3.22)

$$S_{\text{GB}} = \int d^d x \sqrt{-g} \alpha \mathcal{G}, \quad (3.22)$$

which occurs in arbitrary dimensions d and contains the coupling constant α . The inclusion of G_N is not necessary for the details of the argument as it occurs merely on qualitative grounds. The variation of the action with respect to the metric yields the field equations

$$15\alpha\delta_{[\nu}^{\mu}R_{\rho\sigma}^{\rho\sigma}R_{\alpha\beta]}^{\alpha\beta} \equiv \alpha \left[-2R_{\rho\sigma}^{\mu\alpha}R_{\nu\alpha}^{\rho\sigma} + 4R_{\nu\beta}^{\mu\alpha}R_{\alpha}^{\beta} + 4R_{\alpha}^{\mu}R_{\nu}^{\alpha} - 2RR_{\nu}^{\mu} + \frac{1}{2}\mathcal{G}\delta_{\nu}^{\mu} \right] = 0 \quad (3.23)$$

in arbitrary dimensions. They are anti-symmetrized over 5 indices, thus if $D = 4$ then the resulting expression is zero identically – these field equation components would not

¹Additional consequences of these mappings and further discussion of $F(R)$ and scalar tensor theories are found in [34, 35, 36, 37]. Such discussions include geodesics, degrees of freedom, and energy conditions.

provide dynamics in 4-dimensions if added to the Einstein-Hilbert action. This is more readily deduced if one considers the trace of (3.23), which is

$$(d-4) \times \frac{\alpha}{2} \mathcal{G} \quad (3.24)$$

and for $d = 4$ is identically zero.

The fact that there was a $d - 4$ term present led the subsequent idea to re-scale α

$$\alpha \rightarrow \frac{\alpha}{(d-4)} \quad (3.25)$$

and consider the limit of $d \rightarrow 4$. Similar to idea of renormalization in quantum field theory, they wish to remove the zero that comes from the Gauss-Bonnet term and instead introduce a finite contribution. Unlike the quantum field theory procedure, this introduces a singularity. Several “new” (limiting) solutions were discovered in the context of this theory, varying from black holes to cosmological solutions [39, 40, 41, 42, 43, 44]. In this theory, the degrees of freedom are preserved and remain at 2 for the graviton in $d = 4$. Additionally, since this theory was constructed to attempt a circumvention of Lovelock’s theorem, it should only be considered for $d < 5$.

With this new novel theory, skepticism came, especially at the level of the action. The authors’ conclusion that there was a non-trivial contribution from quadratic curvature only came from an effective transformation of the field equations, not from the action. From Hamilton’s principle, the field equations cannot be derived when $d = 4$ [45]. It was eventually agreed upon [45, 46, 47] that there is no *pure* 4-dimensional Einstein-Gauss-Bonnet theory as presented in [38].

Shortly after [38], Lu & Pang showed [48] that a 4d Einstein-Gauss-Bonnet gravity (4DEGB) theory could be constructed through a compactification of a d -dimensional Einstein-Gauss-Bonnet theory onto a $(d - 4)$ maximally symmetric space. It was also shown that a 4DEGB theory could be constructed [49] similar to the method used for the $D \rightarrow 2$ limit of General Relativity [50]. In this method, one does not need any conditions or assumptions on the higher dimensional space as in [48]. Technicalities aside, the action

for [49] can be written as

$$S = \int d^d x \sqrt{-g} \left[R - 2\Lambda + \alpha \left(\phi \mathcal{G} + 4G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 4(\partial\phi)^2 \square\phi + 2((\partial\phi)^2)^2 \right) \right] \quad (3.26)$$

$$\mathcal{G} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4R_{\mu\nu} R^{\mu\nu} + R^2$$

where the variation of the scalar field produces the equation of motion

$$\begin{aligned} \mathcal{E}_\phi \equiv & -\mathcal{G} + 8G^{ab} \nabla_b \nabla_a \phi + 8R^{ab} \nabla_a \phi \nabla_b \phi - 8(\square\phi)^2 + 8(\nabla\phi)^2 \square\phi + 16\nabla^a \phi \nabla^b \phi \nabla_b \nabla_a \phi \\ & + 8\nabla_b \nabla_a \phi \nabla^b \nabla^a \phi = 0 \end{aligned} \quad (3.27)$$

and variation of the metric yields

$$\begin{aligned} \mathcal{E}_{\mu\nu} = & \Lambda g_{\mu\nu} + G_{\mu\nu} + \alpha \left[\phi H_{\mu\nu} - 2R [(\nabla_\mu \phi)(\nabla_\nu \phi) + \nabla_\nu \nabla_\mu \phi] + 8R_{(\mu}{}^\rho \nabla_{\nu)} \nabla_\rho \phi + 8R_{(\mu}{}^\rho (\nabla_\nu \phi)(\nabla_\rho \phi) \right. \\ & - 2G_{\mu\nu} [(\nabla\phi)^2 + 2\square\phi] - 4[(\nabla_\mu \phi)(\nabla_\nu \phi) + \nabla_\nu \nabla_\mu \phi] \square\phi - [g_{\mu\nu}(\nabla\phi)^2 - 4(\nabla_\mu \phi)(\nabla_\nu \phi)] (\nabla\phi)^2 \\ & + 8(\nabla_{(\mu} \phi)(\nabla_{\nu)} \nabla_\rho \phi) \nabla^\rho \phi - 4g_{\mu\nu} R^{\rho\sigma} [\nabla_\rho \nabla_\sigma \phi + (\nabla_\rho \phi)(\nabla_\sigma \phi)] + 2g_{\mu\nu} (\square\phi)^2 \\ & - 2g_{\mu\nu} (\nabla_\rho \nabla_\sigma \phi)(\nabla^\rho \nabla^\sigma \phi) - 4g_{\mu\nu} (\nabla^\rho \phi)(\nabla^\sigma \phi)(\nabla_\rho \nabla_\sigma \phi) + 4(\nabla_\rho \nabla_\nu \phi)(\nabla^\rho \nabla_\mu \phi) \\ & \left. + 4R_{\mu\rho\nu\sigma} [(\nabla^\rho \phi)(\nabla^\sigma \phi) + \nabla^\sigma \nabla^\rho \phi] \right] = 0 \end{aligned}$$

with

$$H_{\mu\nu} = 2 \left[RR_{\mu\nu} - 2R_{\mu\rho\sigma\nu} R^{\rho\sigma} + R_{\mu\rho\sigma\lambda} R_\nu{}^{\rho\sigma\lambda} - 2R_{\mu\rho} R_\nu{}^\rho - \frac{1}{4} \mathcal{G} g_{\mu\nu} \right] \quad (3.28)$$

being the variation of the Gauss-Bonnet invariant. A secondary feature of these two field equations is that they have a geometric connection, that being

$$0 = g^{\mu\nu} \mathcal{E}_{\mu\nu} + \frac{\alpha}{2} \mathcal{E}_\phi = 4\Lambda - R - \frac{\alpha}{2} \mathcal{G}. \quad (3.29)$$

Equation (3.29) provides a nice consistency check on the solutions obtained from the field equations; and in particular it shows that solutions from [38] are not in fact valid 4d solutions [41, 51]. An important note is that the field equations are valid for d -dimensions, and H vanishes for $d \leq 4$ and $\mathcal{G} = 0$ for $d < 4$. So while these equations can be used in 4-dimensions, they may almost be used in lower dimensions as well. Additionally, in $d = 4$, the graviton retains spin-2 degrees of freedom but there is now an additional degree of freedom due to the scalar field. This will be used in both Parts II and III, where the

later will focus on cosmological implications and the former discusses black hole solutions in 3-dimensions in general relativity and 3-dimensional-Einstein-Gauss-Bonnet (3DEGB) gravity.

3.2 Electrodynamics

3.2.1 Maxwell Electrodynamics

Having focused on modified gravity in the last two sections, we step away for the moment to address another form of nonlinear theories that are becoming more prominent in the literature. These are non-linear electrodynamic theories (NLED), and they have motivation from numerous angles, with modern arguments being found in string theory [52, 53]. NLED theories are also making their way into the field of black hole research, with interest in the resulting thermodynamics [54]. The construction of an NLED theory relies on properly understanding the basis upon which it should be made, that being Maxwell electrodynamics.

The start of any modern electrodynamics theory is the field strength tensor $F_{\mu\nu}$, which in an anti-symmetric 2nd rank tensor that encapsulates the electric and magnetic fields. As a matrix, it can be expressed as

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \quad (3.30)$$

where we assume a metric signature $-+++$ and coordinates t, x, y, z . The reason is is so fundamental is that it allows the Maxwell equations to be written in a covariant form. The

Maxwell equations

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{1}{\epsilon_0} \rho \\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} \\
\nabla \cdot \mathbf{B} &= 0 \\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\end{aligned} \tag{3.31}$$

can be recast with the field strength tensor, into two separate equations. The 2nd and 3rd equations of (3.31) are the homogeneous equations, and they can be combined into a tensorial equation with the field strength tensor as

$$\partial_\gamma F_{\alpha\beta} + \partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} = \partial_{[\gamma} F_{\alpha\beta]} = 0 \tag{3.32}$$

The 1st and 4th equations are non-homogeneous due to the sources ρ and $\mathbf{J}$, and can be written as

$$F^{\alpha\beta}{}_{,\alpha} = \mu_0 j^\beta \tag{3.33}$$

with the field strength tensor, and current j . Equation (3.32) can be written in differential form via $\mathbf{dF} = 0$, which requires that $\mathbf{F}$ is closed, with $\mathbf{F} = \frac{1}{2} F_{\mu\nu} \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu$ being the Faraday tensor. Equation (3.33) can also be written in differential form

$$\mathbf{d} \star \mathbf{F} = \mu_0 \star \mathbf{J} \tag{3.34}$$

with $\star$ being the Hodge star operator, and $\mathbf{J}$ the current 1-form.

To construct an action for the electromagnetic field

$$S = \int d^4x \mathcal{L} \tag{3.35}$$

we need to maintain that $\mathcal{L}$ is a scalar constructed from electromagnetic terms. With the Faraday tensor, there are three immediate scalars that can be formulated:

$$F^{\mu\nu} F_{\mu\nu}, \quad \frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F^{\alpha\beta} F^{\gamma\delta}, \quad \text{and} \quad \text{double dual contraction} \tag{3.36}$$

The third is redundant to the first through a minus sign, thus not of any use. The first expands out to

$$F_{\mu\nu}F^{\mu\nu} = 2(\mathbf{E}^2 - \mathbf{B}^2)$$

while the second is

$$\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\alpha\beta}F^{\gamma\delta} = 4\mathbf{B} \cdot \mathbf{E}.$$

The second invariant of (3.36) is parity-odd, therefore it is not a true Lorentz scalar. To include it in the Lagrangian of an electrodynamic theory as a true scalar one would need to consider $(\frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\alpha\beta}F^{\gamma\delta})^2$.

With the Faraday tensor being a two-form, we can construct it from a 1-form $\mathbf{A}$ we call the 4-potential, $\mathbf{F} = \mathbf{dA}$. In component notation we write $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Due to this feature, the requirement that $\mathbf{F}$ be closed will automatically be satisfied via $\mathbf{dF} \equiv \mathbf{d}(\mathbf{dA}) = 0$ as $\mathbf{F}$ is exact. The 4-potential provides our fundamental fields, but is not the *physical* field; the electric and magnetic field are produced via derivatives of the scalar and vector potentials contained in $\mathbf{A}$. Using the 1-form $\mathbf{A}$ and 1-form² $\mathbf{J}$ another scalar may be formed: $A_\mu J^\mu$.

For Maxwell electrodynamics the Lagrangian density is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A_\mu J^\mu \quad (3.37)$$

the Euler-Lagrange equations can be used with respect to the potential A_μ

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} \right) - \mu_0 \frac{\partial \mathcal{L}}{\partial A_\nu} = 0. \quad (3.38)$$

The first and second terms are given by

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu)} = 4F^{\mu\nu}, \quad \frac{\partial \mathcal{L}}{\partial A_\nu} = -J^\nu \quad (3.39)$$

yielding

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu \quad (3.40)$$

²The current form contains the charge density ρ and a vector current $\tilde{\mathbf{j}}$.

while the second set of equations still occurs solely from a geometric perspective. The pseudo-scalar term is not present in the Lagrangian for Maxwell, but it could be included since it can be written as a total derivative, thus not contributing to the equations of motion in anyway, even with the fact it is not a true Lorentz scalar. Note, the Maxwell equations were constructed in Minkowski space, hence the partial derivatives. To consider the Maxwell equations in general relativity, we simply employ the minimal coupling feature for derivatives: partial derivatives get replaced with covariant derivatives³.

$$\nabla_\mu F^{\mu\nu} = \mu_0 J^\mu, \quad \nabla_{[\gamma} F_{\alpha\beta]} = 0 \quad (3.41)$$

Having the machinery of Maxwell electrodynamics it is now appropriate to discuss generalized nonlinear theories.

3.3 Nonlinear Electrodynamics

Nonlinear theories of electrodynamics are built similar to nonlinear gravity theories. We have two invariants that can be created by the field strength tensor. First,

$$\mathcal{S} = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \quad (3.42)$$

which we call the Maxwell invariant, and then

$$\mathcal{P}^2 = \left(\frac{1}{2} \epsilon_{\alpha\beta\gamma\delta} F^{\alpha\beta} F^{\gamma\delta} \right)^2 \quad (3.43)$$

which we call the dual invariant. The quadratic feature in this invariant is due to the fact that $\mathcal{P}$ violates parity, and thus is a pseudo-Scalar. To make it a true Lorentz scalar, we need the square of $\mathcal{P}$. If an electrodynamics theory is to be constructed of pure Lorentz scalars, this requires our fundamental invariants to be $\mathcal{S}$ and $\mathcal{P}^2$. For a linear theory of electrodynamics, with respect to these invariants, the only possible Lagrangian is

$$\mathcal{L} \propto \alpha \mathcal{S} + \beta \mathcal{P}^2. \quad (3.44)$$

³This is not necessary in the definition of the field strength tensor due to the anti-symmetric nature of the indices.

For a nonlinear theory of electrodynamics, the Lagrangian density contains arbitrary functions of these two invariants⁴ in (3.42), (3.43)

$$\mathcal{L}_{\text{EM}} = \mathcal{L}(\mathcal{S}, \mathcal{P}^2) \quad (3.45)$$

and any source/current terms (if present) the same way they were introduced into the Maxwell Lagrangian. The field equations associated with (3.45) are

$$\nabla_{[\gamma} F_{\alpha\beta]} = 0, \quad \nabla_{\mu} (\mathcal{L}_{\mathcal{S}} F^{\mu\nu} + 2\mathcal{L}_{\mathcal{P}} \star F^{\mu\nu}) = \mu_0 J^{\nu} \quad (3.46)$$

with

$$\mathcal{L}_{\mathcal{S}} = \frac{\partial \mathcal{L}}{\partial \mathcal{S}}, \quad \mathcal{L}_{\mathcal{P}} = \frac{\partial \mathcal{L}}{\partial \mathcal{P}}$$

For NLED theories there are some properties that one wishes to have to consider the theory to be “nice”. The first is that under weak field approximation, $F_{\mu\nu} \rightarrow 0$ one should be able to recover the Maxwell Lagrangian (3.37); this property is referred to as the principle of correspondence. The electro-magnetic duality, interchanging electric and magnetic fields in the Maxwell equations in such a way the equations remain unchanged, is an additional feature that Maxwell dynamics possesses that may or may not be present in an NLED theory. Theories that avoid the use of $\mathcal{P}^2$ are referred to as restrictive, and the equations of motions are easily modified by setting $\beta = 0$.

Perhaps the most famous NLED theory is that of Born and Infeld, which was constructed to fix divergences for charged point particles⁵. The electric field is infinite at the position of the particle, which in turn makes the Lagrangian density unbounded. In order

⁴Some of the literature on this topic does not consider the square on (3.43), instead just uses $\mathcal{P}$ which is not a loretnz scalar [55, 56].

⁵A brief note aside: The motivation of [57] is to renormalize infinities, and electrodynamics is only one place where said infinities occur. Researchers have also used the Born-Infeld model in other theories, in particular for gravitational theories [58]. The model that was considered used a gravity action of the form

$$\mathcal{S} = \int d^4x \sqrt{-\det(ag_{ab} + bR_{ab})}. \quad (3.47)$$

This however creates instabilities due to non-linear curvature terms in the action and hence higher order equation of motion for the metric. A discussion about how to avoid this problem is provided in [58].

to resolve this, Born & Infeld [57] proposed a Lagrangian density different from Maxwell, that being

$$\mathcal{L} = b^2 \left(\sqrt{1 - \frac{F_{ab}F^{ab}}{2b^2}} - 1 \right). \quad (3.48)$$

where b is a constant (assumed to be large) that will put a bound on E [59]. This Lagrangian density is only valid for small fields, as [57] were trying to re-normalize the unbounded electric field for a point particle. It goes without saying that, in the limit of $b \rightarrow \infty$, the Born-Infeld Lagrangian approaches Maxwell.

There are many different non-linear models that have been studied not only in the context of flat-space, but also in the context of black holes. The motivation for specific theories shall be treated on a case-by-case basis, as some models are constructed to satisfy certain requirements, while others are inspired by string theory. Nevertheless, they do provide an interesting arena to study phenomenological consequences in black hole spacetimes. One such regime that has been of focus is the notion of regular black holes. Regular black holes possess an event horizon but no singularity at their centre. One such mode of constructing these solutions is through the use of nonlinear electrodynamics as a source term [60].

There are two routes usually taken to trying to find regular black holes: pose the metric solution and determine the 4-potential that the theory must have, along with its Lagrangian, or, start with a particular Lagrangian and see if it provides a black hole solution. A drawback of regular black holes is that almost always there is violation of classical energy conditions. Whether this is an issue is still up for debate as quantum energy conditions are becoming more and more prevalent in the discussion of physically viable spacetimes. In a rather surprising turn however, it was shown recently that regular black holes are possible in pure gravity theory if you consider an infinite tower of curvature invariants in the action [61].

Having laid the foundational discussion of alternative gravity theories and non-linear electrodynamics, we now begin our quest into the second and third portions of this thesis, and explore how these nonlinear contributions effect black hole and cosmological spacetimes.

Part II

Black Holes

Introduction

One of the the crown jewel's of general relativity is the prediction of black holes [62]. Once a mathematical curiosity, they have been studied by theorists for decades as a peculiar solution to Einstein's equations. With technological improvements in the field of astronomy, these mathematical objects have now been experimentally verified through gravitational waves and the Event Horizon Telescope. It is somewhat remarkable that not long after the publication of the field equations of general relativity we had the first black hole solution by Schwarzschild [63]. Finding exact solutions to the field equations are easier said than done; this might be why it took another 50 years until we had the Kerr-Newman solution to describe a rotating charged black hole⁶ [64]. The most general solution depends on 3 parameters: mass, spin, and charge.

Defining a black hole is not as straightforward as a one would think. Depending on the perspective that someone researching black holes has [65], they may define them differently than their colleague in an adjacent office. Very simply speaking, a black hole is a region of spacetime from which exterior observers cannot receive signals from interior observers. The point of no return that induces this phenomena is the event horizon, which is also referred to as the boundary of the black hole. The exterior observers are located outside the event horizon, while the interior observers are inside. The event horizon creates a one-way channel for causal contact. The interior observer can receive signals from outside the black hole, but the converse is not possible. At the core of *classical* black holes lies a spacetime singularity ⁷.

⁶Even then, the techniques used for finding those solutions relied more heavily on algebraic components.

⁷More to be discussed in later sections.

As mentioned in Part I, singularities indicate the breakdown of spacetime. So it is rather bittersweet that the objects that are a signature of general relativity ultimately cause its downfall. Now, since the singularity is always located behind the event horizon – shielded from observers outside the black hole – the universe is still safe for one to use general relativity as a theory of spacetime. Singularities that are not shielded via an event horizon are said to be naked. However the cosmic censorship hypothesis posed by Penrose [66] suggests that *all* singularities are shielded via a horizon.

Black hole formation is commonly associated with gravitational collapse of stars. These stars, much more massive than our Sun, when they get close to the end of their life will undergo collapse. The final remanent of the star is dictated by how massive the star is. While black holes are one possibility, collapse can also result in a white dwarf or a neutron star, which referred to are ultra-dense objects. The first proper calculation using general relativity to model stellar collapse was undertaken by Oppenheimer and Snyder [67]. Their calculations revealed that under continual gravitational collapse, there would be inevitably a region of spacetime created that could not communicate with its surroundings – a black hole is born⁸.

While black holes were classically thought of as being dull objects, which acted as nature’s large-scale vacuum cleaner, work in the early 70s greatly shifted this perspective. Black holes were discovered to be lively objects that exhibited behaviour analogous to systems on earth, in particular thermodynamic systems. To be more accurate, they are thermodynamic systems. The spark that started the resulting 50+ years of black hole research was thanks to Hawking and his area theorem [68]. The theorem simply states that black hole areas can never decrease, and that black hole can never bifurcate. The former statement spurred the idea that black hole must have an entropy, simply due to fact that the area theorem is analogous to the 2nd law of thermodynamics. This led Bekenstein to discover that black holes have entropy proportional to their horizon area [69, 70] $S \propto A_H$.

The fact that black holes were looking more and more like thermodynamic systems led to production of the “four laws of black hole mechanics” by Hawking, Carter, and Bardeen

⁸One interesting note, the term black hole was not used until the 1960’s by Wheeler

[71]. These four laws are:

1. The surface gravity κ is constant over the event horizon of a stationary black hole.
2. For a rotating charged black hole with a mass M , an angular momentum J , and a charge Q ,

$$\delta M = \frac{\kappa}{8\pi G} \delta A + \Omega \delta J + \Phi \delta Q$$

where κ is its surface gravity, Ω its angular velocity, and Φ its electric potential.

3. Hawking's area theorem: $\delta A \geq 0$, i.e., the area A of a black hole's event horizon can never decrease.
4. It is impossible to reduce the surface gravity κ to zero in a finite number of steps⁹.

A Killing vector field ξ , of a spacetime with metric g and connection ∇ , satisfies the Killing equation

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0. \quad (3.49)$$

A hypersurface ($\mathcal{H}$) in the spacetime manifold is said to be a Killing horizon if $\mathcal{H}$ is null, and ξ is a null vector normal to $\mathcal{H}$, $\xi|_{\mathcal{H}} \neq 0$. The surface gravity is defined by a Killing horizon through

$$\xi^a \nabla_a \xi^b = \kappa \xi^b \quad (3.50)$$

and the event horizon is taken to be a Killing horizon. In the 1st law surface gravity plays the role of temperature in comparison to the standard first law of thermodynamics. This would also indicate that mass plays the role of energy in the system. For black holes that are purely classical, with no quantum effects considered or present, the temperature will be zero.

If black holes are allowed to enter the quantum regime, through quantum field theory in curved spacetime, it was shown by Hawking that black holes do indeed have a temperature

⁹This was not formally proved until 1986 [72]; however, articulation of the third law is still openly discussed in the literature [73].

and radiate as black bodies [74] – no pun intended. This temperature is proportional to the surface gravity

$$k_B T = \frac{\hbar \kappa}{2\pi c} \Rightarrow T = \frac{\hbar \kappa}{2\pi c k_B} \quad (3.51)$$

with k_B being Boltzman’s constant, c the speed of light, and $\hbar$ Planck’s constant. The discovery of black hole temperature through black holes leads us directly into the centre of discussions about a quantum theory of gravity. Not only do they have singularities at their cores, but non-black hole thermodynamics is understood through statistical mechanics, an inherently quantum science. So, while black holes are created gravitationally, they are deeply connected to quantum features.

With the black hole temperature formally defined, the exact relation between temperature and area could be calculated:

$$S = \frac{Ac^3}{4\hbar G}. \quad (3.52)$$

This in turn rewrites the 1st law of black hole mechanics, into the first law of black hole *thermodynamics*

$$\delta M = T\delta S + \Omega\delta J + \Phi\delta Q. \quad (3.53)$$

Comparison with the first law of thermodynamics

$$\delta U = T\delta S - P\delta V + \sum_j \mu_j \delta N_j + \Phi\delta Q \quad (3.54)$$

shows the similarities to an uncanny level. Again, bearing further truth to the statement that black holes are not analogous to thermodynamic systems, but indeed are thermodynamic systems. There is one notable omission however, the lack of a pressure-volume term in (3.53). This feature at face value does not present any challenges for black hole mechanics, but it does draw some concerns with thermodynamics. Without the PV term, black hole thermodynamics cannot be truly mapped into everyday thermodynamics we experience and observe on earth. Is there a resolution? Yes, by considering another dynamical variable in the spacetime – apart from M, Q, J .

The black hole considered for (3.53) to be true is one which is considered to only be in the presence of a electromagnetic background. As mentioned previously however, it is totally reasonable to add the cosmological constant term into the field equations and

also obtain exact black hole solutions. Whether one considers the cosmological constant term on the left side of the equations (2.7), or incorporates it into the right hand side as a vacuum stress-energy is a matter of preference, but the result remains the same. The inclusion of the cosmological constant, treating it as a thermodynamic variable, allows (3.53) to be modified to include a PV term

$$\delta M = T\delta S + V\delta P + \Omega\delta J + \Phi\delta Q. \quad (3.55)$$

In this formalism, the cosmological constant acts as thermodynamic pressure if it is taken to be negative, and a tension if it is assumed to be positive. This notion of associating Λ with a thermodynamic variable occurred shortly after the publication of the first law of black hole mechanics. It was first considered as a dynamical variable in [75, 76], but the notion of being a thermodynamic variable followed gradually. Considering a thermodynamic variable is only one-half of the problem; understanding the conjugate variable in an additional task. Incorporating Λ into the first law was completed in [77], but the interpretation of the conjugate variable as thermodynamic volume was only assimilated 20 years ago [78]. With the introduction of pressure and volume into the first law the interpretation of mass changes; it is no longer associated with the energy of the system but instead as the *enthalpy* of the system

$$H = U + PV.$$

The pressure associated with the cosmological constant is given by the formula

$$P = -\frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi l^2} \quad (3.56)$$

and the *thermodynamic* volume is given by [79, 80]

$$V \equiv \left(\frac{\partial M}{\partial P} \right)_{S,Q,J}. \quad (3.57)$$

This realm is referred to as extended phase space thermodynamics, and has been shown to have immense phenomena for black holes. In the studying of black holes in this regime it has been shown that black holes can possess behaviour comparable to Van der Waals fluids, and exhibit phase transition behaviour. This phenomena is seen through the use of

the Gibbs free energy potential ¹⁰:

$$G = M - TS.$$

It is also straightforward to find an equation of state of the system, which for black holes takes the form

$$P \equiv P(T, r_+, Q, J)$$

where inflection points of P indicate critical points. Through identification of critical points – if present – it is again straightforward to compute phase diagrams along with $G - T$ diagrams.

Part II of this thesis is devoted to the discussion of black holes in the presence of non-linear fields as discussed in Chapter 3, whether that is: Lovelock gravity, nonlinear electrodynamics, d-dimensional EGB gravity, or combinations of said theories. We find new solutions, new thermodynamic behaviour, new parameter space constraints, and generalized features of nonlinear electrodynamic black holes.

In Chapter 4, we discuss the thermodynamics of exotic black holes in Lovelock gravity. We discuss the notion of exotic black holes and how they differ from “standard” black holes with constant-curvature event horizons. We investigate their thermodynamic behaviour in third order Lovelock gravity in 7 and 8 dimensions. We show that they can possess rich phase structure, and undergo the Hawking-Page transition, as well as display van der Waals behaviour. We also show that they have the same novel triple point shown in quadratic Lovelock gravity where three distinct phases of black hole size exist analagous to a ice-liquid-steam configuration of water.

In Chapter 5, we investigate negative mass black holes in quadratic Lovelock gravity. These black holes are possible via the exotic black hole construction. These black holes are shown to posses negative mass in spaces that are asymptotically comparable to de Sitter space, which we are calling pseudo-de Sitter space. We show that not only can black hole solutions exist with negative mass, but also zero mass black holes are possible.

In Chapter 6, we discuss lower dimensional black holes in general relativity. We re-examine the BTZ black hole and provide an alternative description of the solution. We

¹⁰ M replaces enthalpy H for black hole procedures.

investigate how general solutions may be constructed with nonlinear electrodynamics, and how static solutions can be transformed into rotating solutions through a coordinate boost.

In Chapter 7, we discuss lower dimensional black holes in 3DEGB gravity. We investigate black holes coupled to nonlinear electrodynamics with a pure radial electric field. We show that all solutions can be constructed once the corresponding general relativity solution is known. We also discuss rotating solutions obtainable via a modified coordinate boost on the static solutions.

Chapter 4

Thermodynamics of Exotic Black Holes in Lovelock Gravity

4.1 Exotic Black Holes

Exotic black holes are a class of black hole solutions that only exist in higher dimensions, in particular, Lovelock gravity¹. Their discovery [81] coincided with the revived discussion of Birkhoff's theorem [82] in alternative gravity theories. Birkhoff's theorem in general relativity is a classic result that states: Any C^2 solution of the vacuum Einstein equations, which is spherically symmetric in an open set U , is locally isometric to the maximally extended Schwarzschild solution in U .

In the coordinate chart (Schwarzschild coordinates) $\{t, r, \theta, \phi\}$ the line element famously reads

$$ds^2 = - \left(1 - \frac{2M}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (4.1)$$

with $G = c = 1$ unit convention, and M represents the mass of the body inducing the gravitational field. The important feature of this line element is that it does not dis-

¹The name exotic black holes has been used quite frequently in the literature when referring to solutions that authors claim to be *exotic*. We use this name as it corresponds to the original work that this chapter was based upon

criminate what object is inducing it. The object could be a black hole or a star; this spacetime exists in the exterior of its source. It can be seen from the metric form that it possess the $SO(3)$ isometry group indicative of spherical symmetry. In higher dimensional general relativity the theorem also holds, but the associated solution is referred to as the Schwarzschild-Tangherlini metric

$$ds^2 = - \left(1 - \frac{2M}{r^{d-3}}\right) dt^2 + \left(1 - \frac{2M}{r^{d-3}}\right)^{-1} dr^2 + r^2 d\Omega_{d-2}^2 \quad (4.2)$$

with $d\Omega_{d-2}^2$ being the metric of a $d - 2$ sphere. It is common place to refer to the metric $d\Omega_{d-2}^2$ as the base manifold, or angular portion. In 4-d general relativity the 2 sphere is the unique solution, but in higher dimensions it need not be that way.

With the introduction of higher dimensions, vacuum solutions no longer need to possess spherical symmetry. The only restriction that the base manifold must have is that it is an Einstein space, thus the metric tensor is proportional to the Ricci tensor. This feature allows a more generalized Birkhoff theorem; there are many other possible solutions for vacuum spacetimes in higher dimensions. The consequence of this is that the base manifold no longer needs to possess constant curvature. A classic example is the class of Bohm metrics, which are constructed through the product of n -spheres and possess topology $S^n \times S^p$ [83, 84].

A version of Birkhoff's theorem also holds in Lovelock gravity, wherein solutions with constant curvature base manifolds with spherical, plane, or hyperboic symmetry are locally isometric to the corresponding static Lovelock black hole solutions [85, 86]. However in Lovelock gravity, non-constant curvature base manifolds are also permitted [87, 88, 89, 90]. In this setting, the constants characterizing the Euler invariants and Lovelock tensors appear in the metric function, and determine the transverse part of the metric. Black hole solutions therefore exist in Lovelock gravity with 'exotic' base manifolds, with additional constraints on the Weyl tensor severely restricting the allowed geometries [91, 92, 93, 94, 95, 96]. Thermodynamic properties of black hole solutions in this new context have yet to receive significant attention, with the case of Gauss-Bonnet gravity being only very recently examined [97]. In this work we aim to extend this analysis to third-order Lovelock gravity, where the additional coupling introduces interesting new thermodynamic features.

4.2 Lovelock Black Holes and Thermodynamics

4.2.1 Black Hole Solutions

We will use the following ansatz² for the metric and gauge field:

$$ds^2 = g_{ij}dy^i dy^j + \gamma_{\alpha\beta}dx^\alpha dx^\beta = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Sigma_{d-2}^2, \quad F = \frac{Q}{r^{d-2}}dt \wedge dr. \quad (4.3)$$

Latin indices are reserved for coordinates in the $t - r$ plane while Greek indices refer to coordinates on the base manifold $d\Sigma_{d-2}^2$. The importance of this metric is how one defines $\gamma_{\alpha\beta}$, which determines the geometry of the transverse sections (including the horizon). In this work we take the base manifold with metric $\gamma_{\alpha\beta}$ to be a $(d - 2)$ compact space not necessarily of constant curvature. The constant curvature case (corresponding to ‘ordinary’ Lovelock black holes) has been studied extensively [98], while the non-constant curvature case has only been examined in the context of Gauss-Bonnet gravity [97]. Inserting the metric (4.3) into the field equations (3.8) produces two simplified expressions:

$$\begin{aligned} \mathcal{G}_j^i &\equiv \frac{-(d-2)\delta_j^i}{2r^{d-2}} \frac{d}{dr} \sum_{n=0}^K \left\{ b_n \left(r^{d-2n-1} A_n \left(\frac{-f(r)}{r^2} \right) \right) \right\} = 8\pi G_N T_j^i, \\ \mathcal{G}_\beta^\alpha &\equiv \frac{-\delta_\beta^\alpha}{2r^{d-3}} \frac{d^2}{dr^2} \sum_{n=0}^K \left\{ b_n \left(r^{d-2n-1} A_n \left(\frac{-f(r)}{r^2} \right) \right) \right\} = 8\pi G_N T_\beta^\alpha, \end{aligned} \quad (4.4)$$

with

$$T_{ab} = F_{ac}F_b^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}$$

being the stress-energy tensor associated with Maxwell electrodynamics. Additionally, we have

$$A_n \left(\frac{-f(r)}{r^2} \right) \equiv \sum_{k=n}^K \alpha_k \binom{k}{n} \left(\frac{-f(r)}{r^2} \right)^{k-n}, \quad (4.5)$$

where b_n are arbitrary constants which depend on the geometry of the base manifold $d\Sigma_{d-2}^2$. These field equations determine the metric function $f(r)$, which is obtained as a solution

²One could in principle consider a metric where $g_{tt} \neq g^{rr}$ but for simplicity we do not do this here.

to the following polynomial of degree K in $f(r)$:

$$\sum_{n=0}^K \frac{b_n}{r^{2n}} \left(\sum_{k=n}^K \alpha_k \binom{k}{n} \left(\frac{-f(r)}{r^2} \right)^{k-n} \right) = \frac{16\pi G_N M}{(d-2)\Sigma_{d-2} r^{d-1}} - \frac{8\pi G_N Q^2}{(d-2)(d-3)} \frac{1}{r^{2d-4}}. \quad (4.6)$$

The two field equations are redundant to each other, and the consistent solution is found via (4.6). In (4.6) M represents the ADM mass, and Q is the charge defined by

$$Q = \frac{1}{2\Sigma_{d-2}} \int \star F. \quad (4.7)$$

Finally, α_k are the re-scaled Lovelock coupling constants defined by

$$\alpha_0 = \frac{\hat{\alpha}_{(0)}}{(d-1)(d-2)} = \quad \alpha_1 = \hat{\alpha}_{(1)}, \quad \alpha_k = \hat{\alpha}_{(k)} \prod_{n=3}^{2k} (d-n) \quad \text{for } k \geq 2. \quad (4.8)$$

Note that constant curvature base manifolds have $b_n = \kappa^n$ with $\kappa \in \{-1, 0, +1\}$, corresponding to negative, flat and positive curvature respectively. This can be shown by direct comparison of (4.6) with the familiar form of the constant curvature $f(r)$ polynomial given by

$$\sum_{k=0}^K \alpha_k \left(\frac{\kappa - f}{r^2} \right)^k = \frac{16\pi G_N M}{(d-2)\Sigma_{d-2}^{(\kappa)} r^{d-1}} - \frac{8\pi G_N Q^2}{(d-2)(d-3)} \frac{1}{r^{2d-4}}. \quad (4.9)$$

where M , Q and the coupling constants are as defined above, and $\Sigma_{d-2}^{(\kappa)}$ is the volume of the constant curvature compact space.

In what follows we take $K = 3$ in the above expressions, corresponding to third-order Lovelock gravity, and set $\alpha_1 = 1$ in order to recover general relativity in the $\{\alpha_2, \alpha_3\} \rightarrow 0$ limit. Useful for analyzing the structure of the resulting solutions is the Kretschmann scalar, a curvature invariant which for our general metric (4.3) can be written as

$$\begin{aligned} \mathcal{K} &= R^{abcd} R_{abcd} \\ &= \left(\frac{d^2 f(r)}{dr^2} \right)^2 + \frac{2(d-2)}{r^2} \left(\frac{df(r)}{dr} \right)^2 + \frac{2(d-2)(d-3)f(r)^2}{r^4} - \frac{4R[\gamma]f(r)}{r^4} + \frac{\mathcal{K}[\gamma]}{r^4}. \end{aligned} \quad (4.10)$$

It is a function of both the metric function $f(r)$ and the Ricci and Kretschmann scalars of the base manifold ($R[\gamma]$ and $\mathcal{K}[\gamma]$, respectively), and provides a coordinate-free measure of curvature that can be used to locate singularities and other pathologies in the spacetime.

The primary motivation of this work is to examine the role that horizon geometry plays in determining the thermodynamic features of black hole spacetimes. Ordinarily, Birkhoff's theorem in general relativity strictly limits the types of allowed horizon geometries. In Lovelock gravity however, the base manifold is constrained through a number of relations among its Lovelock tensors, as discussed in [95]. The metric function $f(r)$ is a solution to the polynomial (4.6) with the additional requirement that

$$\hat{\mathcal{G}}_{\beta}^{(n)\alpha} = -\frac{(d-3)!b_n}{2(d-2n-3)!}\delta_{\beta}^{\alpha}, \quad (4.11)$$

where $\hat{\mathcal{G}}_{\beta}^{(n)\alpha}$ are simply the Lovelock tensors of the base manifold. The constants b_n appear in the polynomial (4.6) defining the metric function and are arbitrary, though they depend on the geometry of the base manifold through (4.11).

When $d = 2K + 1$ the tranverse space must be of constant curvature [87, 99], so the first dimension in which the K -th Lovelock term begins to contribute only admits spherical, flat, and hyperbolic transverse sections and has $b_n = \kappa^n$ with $\kappa \in \{-1, 0, +1\}$. For third-order Lovelock gravity, this means that only black hole solutions in $d \geq 8$ can possess exotic horizon geometries. Furthermore, we emphasize that significant restrictions on the allowed geometries arise from the field equations. For example, the Bohm metrics

$$ds^2 = d\rho^2 + a(\rho)^2 d\Omega_p^2 + b(\rho)^2 d\Omega_q^2 \quad (4.12)$$

represent an infinite family of local metrics that can be extended to manifolds of topology S^{p+q+1} . Black holes with Bohm horizons have non-constant curvature, yet of the infinitely many metrics given by different choices of $a(\rho)$ and $b(\rho)$ in (4.12), only one choice turns out to be admissible as a horizon due to the Gauss-Bonnet correction appearing when $K \geq 2$ [91]. Higher orders in Lovelock theory likely further constrain the allowed choices of horizon geometry, though this remains a currently unexplored feature of exotic black holes in higher-dimensional Lovelock gravity. In this chapter we focus on choices of b_n that produce everywhere real and analytic metric functions, ensuring that curvature scalars are finite and that the spacetime has the appropriate asymptotics, rather than attempting to derive general constraints on the b_n 's. We will see that many interesting features appear even in the limited parameter space we study herein.

4.2.2 Lovelock Thermodynamics

We operate in an *extended* thermodynamic phase space, which arises from considering variations of the cosmological constant Λ when constructing the first law for black hole systems. The inclusion of such a variation is motivated by the observation that the ‘ordinary’ first law of black hole mechanics contains no pressure-volume term, since there are no obvious notions of such quantities for a black hole. Since the cosmological constant is equivalent to a perfect fluid stress-energy tensor, a variable Λ provides a mechanism for would-be pressure variations in a gravitational system. Precisely, we identify the cosmological constant with a fluid pressure through the relation

$$P = -\frac{\Lambda}{8\pi} = \frac{(d-1)(d-2)}{16\pi l^2}, \quad (4.13)$$

where l is the AdS length scale. The consequences of such a variation were first explored by Kastor and collaborators [100, 101, 102], who showed through a Hamiltonian analysis that the resulting first law is supplemented by a term $\Theta\delta\Lambda$, where the potential Θ conjugate to Λ plays the role of the thermodynamic volume, and, at least for Schwarzschild-AdS black holes, is exactly the renormalized geometric volume of AdS minus the volume of the black hole. In this way, a pressure-volume term naturally enters into the first law. From this extended phase space emerged the area of research known as black hole chemistry, so named because the inclusion of pressure in the thermodynamic description gives rise to a number of phenomena in the gravitational sector that bear striking resemblance to thermodynamic phenomena observed in everyday systems such as water, helium, and dry ice [103, 104]. The first law in the extended phase space formally reads

$$\delta M = T\delta S + V\delta P, \quad (4.14)$$

where additional terms enter if angular momentum or charge are present. In Lovelock theory, the mass M , temperature T , entropy S , and charge Q of static, asymptotically

AdS black holes are given by [97]

$$M = \frac{\Sigma_{d-2}(d-2)}{16\pi G_N} \sum_{k=0}^K \alpha_k b_k r_+^{d-1-2k} + \frac{\Sigma_{d-2}}{2(d-3)} \frac{Q^2}{r_+^{d-3}} , \quad (4.15)$$

$$S = \frac{\Sigma_{d-2}(d-2)}{4G_N} \sum_{k=0}^K \frac{k b_{k-1} \alpha_k r_+^{d-2k}}{d-2k} , \quad (4.16)$$

$$T = \frac{1}{4\pi r_+ D(r_+)} \left[\sum_{k=0}^K b_k \alpha_k (d-2k-1) r_+^{-2(k-1)} - \frac{8\pi G_N Q^2}{(d-2) r_+^{2(d-3)}} \right] , \quad (4.17)$$

where r_+ is the largest positive real root of $f(r_+) = 0$ and we have defined

$$D(r_+) \equiv \sum_{k=1}^K k \alpha_k b_{k-1} r_+^{-2(k-1)} . \quad (4.18)$$

These quantities obey the following first law and Smarr relation:

$$\begin{aligned} \delta M &= T \delta S - \frac{1}{16\pi G_N} \sum_h \hat{\Psi}^{(k)} \delta \hat{\alpha}_{(k)} + \Phi \delta Q , \\ (d-3)M &= (d-2)TS + \sum_k 2(k-1) \frac{\hat{\Psi}^{(k)} \hat{\alpha}_{(k)}}{16\pi G_N} + (d-3)\Phi Q . \end{aligned} \quad (4.19)$$

Here, $\hat{\Psi}^{(k)}$ are potentials conjugate to the Lovelock coupling constants $\hat{\alpha}_{(k)}$ which are promoted to thermodynamic variables. The potentials are defined by

$$\hat{\Psi}^{(k)} = 4\pi T \mathcal{A}^{(k)} + \mathcal{B}^{(k)} + \Theta^{(k)} , \quad (4.20)$$

with

$$\begin{aligned} \mathcal{B}^{(k)} &= -\frac{16\pi k G_N M (d-1)!}{b(d-2k-1)!} \left(-\frac{1}{\ell^2} \right)^{k-1} , \quad b = \sum_k \frac{\hat{\alpha}_k k (d-1)!}{(d-2k-1)!} \left(-\frac{1}{\ell^2} \right)^{k-1} , \\ \Theta^{(k)} &= \int_{\Sigma} \sqrt{-g} \mathcal{L}^{(k)}[s] - \int_{\Sigma_{\text{AdS}}} \sqrt{-g_{\text{AdS}}} \mathcal{L}^{(k)}[s_{\text{AdS}}] . \end{aligned} \quad (4.21)$$

The thermodynamic entropy is also modified in a non-trivial way compared to the Einstein-AdS case, and is no longer one-quarter the area of the horizon. It is now given by

$$S = \frac{1}{4G_N} \sum_k \hat{\alpha}_k \mathcal{A}^{(k)} , \quad \mathcal{A}^{(k)} = k \int_{\mathcal{H}} \sqrt{\sigma} \mathcal{L}^{(k-1)} , \quad (4.22)$$

where σ is the induced metric on the black holes horizon and $\mathcal{L}^{(k-1)}$ are the Euler densities of the horizon.

4.3 Exotic Black Holes in Third-Order Lovelock Gravity

In this section, we present general solutions for charged black hole metrics in third-order Lovelock gravity with exotic horizon geometries. We solve the polynomial (4.6) to determine the metric function, and examine various constraints on the couplings $\{\alpha_i\}$ that arise from considerations of analyticity and asymptotic behaviour. We also determine the equation of state and free energy, and place bounds on the allowed pressure (cosmological constant) for these black holes.

4.3.1 Solutions

Setting $K = 3$ in (4.6) gives the following polynomial for the metric function f :

$$-\frac{\alpha_3 f^3}{r^6} + \left(\frac{\alpha_2}{r^4} + \frac{3b_1 \alpha_3}{r^6} \right) f^2 + \left(-\frac{1}{r^2} - \frac{2b_1 \alpha_2}{r^4} - \frac{3b_2 \alpha_3}{r^6} \right) f + \alpha_0 + \frac{b_1}{r^2} + \frac{b_2 \alpha_2}{r^4} + \frac{b_3 \alpha_3}{r^6} = A(r) , \quad A(r) \equiv \frac{16\pi M}{(d-2)\Sigma_{d-2} r^{d-1}} - \frac{8\pi Q^2}{(d-2)(d-3)r^{2d-4}} . \quad (4.23)$$

This is a cubic equation for f possessing three distinct solutions. The first is given by

$$f_E = \frac{\left(r^4 (3\alpha_3 + \alpha_2^2) - 9(b_1^2 - b_2) \alpha_3^2 \right) + (r^2 \alpha_2 + 3\alpha_3 b_1) \left(\frac{1}{2}(Y + \sqrt{X}) \right)^{1/3} - \left(\frac{1}{2}(Y + \sqrt{X}) \right)^{2/3}}{3\alpha_3 \left(\frac{1}{4}(Y + \sqrt{X}) \right)^{1/3}} \quad (4.24)$$

where we have defined

$$\begin{aligned} X &\equiv \left(27\alpha_3^3 (2b_1^3 - 3b_1 b_2 + b_3) - (27\alpha_3^2 (A(r) - \alpha_0) - 2\alpha_2^3 + 9\alpha_2 \alpha_3) r^6 \right)^2 \\ &\quad + 4 \left(9\alpha_3^2 (b_2 - b_1^2) - (\alpha_2^2 + 3\alpha_3) r^4 \right)^3 , \\ Y &\equiv (81b_1 b_2 - 54b_1^3 - 27b_3) \alpha_3^3 + \left(27\alpha_3^2 (A(r) - \alpha_0) + 9\alpha_2 \alpha_3 - 2\alpha_2^3 \right) r^6 . \end{aligned}$$

The subscript E indicates that this branch gives the appropriate Einstein limit when successively taking the limits $\alpha_3 \rightarrow 0$ and $\alpha_2 \rightarrow 0$. The other two (conjugate) branches are given by the solutions

$$f = \frac{(1 \mp i\sqrt{3}) \left(r^4 (3\alpha_3 + \alpha_2^2) - 9(b_1^2 - b_2) \alpha_3^2 \right) + (r^2 \alpha_2 + 3\alpha_3 b_1) \sqrt[3]{4(Y + \sqrt{X})} + (1 \pm i\sqrt{3}) \left(\frac{1}{2}(Y + \sqrt{X}) \right)^{2/3}}{3\alpha_3 \sqrt[3]{4(Y + \sqrt{X})}}.$$

These two solutions are distinguished by a lack of a convergent $\alpha_2 \rightarrow 0$ limit. We further differentiate between the two by considering their Gauss-Bonnet ($\alpha_3 \rightarrow 0$) limits. The solution f_{GB} , given by the upper sign, converges in this limit and yields the appropriate metric function for the Schwarzschild-AdS black hole (when $\{b_1, b_2, b_3\} = \{\kappa, \kappa^2, \kappa^3\}$) and is therefore referred to as the Gauss-Bonnet branch. The solution f_L , given by the lower sign, has a divergent $\alpha_3 \rightarrow 0$ limit and exists only in third-order Lovelock gravity. We refer to this as the Lovelock branch.

A number of considerations arise from the form of the solutions (4.24)–(4.25). As discussed in [98], a bound on the couplings $\{\alpha_2, \alpha_3\}$ can be obtained by requiring that all three solutions to (4.23) are asymptotically AdS. This is determined by positivity of the discriminant of the cubic equation for f at large- r , which is equivalent to the following constraint on the couplings:

$$27\alpha_0^2 \alpha_3^2 + 4\alpha_0 \alpha_2^3 - 18\alpha_0 \alpha_2 \alpha_3 - \alpha_2^2 + 4\alpha_3 \leq 0 \quad (4.25)$$

When (4.25) is satisfied, all solutions are asymptotically AdS and one can smoothly take the Einstein limit by appropriate choice of solution.

The inequality (4.25) puts an upper (and sometimes lower) bound on the allowed pressure for certain values of the couplings, given by

$$P_{\pm} = \frac{(d-1)(d-2)}{432\pi} \frac{9\alpha_2 \alpha_3 - 2\alpha_2^3 \pm 2(\alpha_2^2 - 3\alpha_3)^{\frac{3}{2}}}{\alpha_3^2}. \quad (4.26)$$

Now introducing the following dimensionless variables

$$\alpha = \frac{\alpha_2}{\sqrt{\alpha_3}}, \quad p = 4\sqrt{\alpha_3}P, \quad r_+ = v\alpha_3^{\frac{1}{4}}, \quad T = \frac{t\alpha_3^{-\frac{1}{4}}}{d-2}, \quad m = \frac{16\pi M}{(d-2)\Sigma_{d-2}^{(\kappa)}\alpha_3^{\frac{d-3}{4}}}, \quad Q = \frac{q}{\sqrt{2}}\alpha_3^{\frac{d-3}{4}} \quad (4.27)$$

the pressure can be expressed as

$$p_{\pm} = \frac{(d-1)(d-2)}{108\pi} \left(9\alpha - 2\alpha^3 \pm 2(\alpha^2 - 3)^{\frac{3}{2}} \right) . \quad (4.28)$$

This matches our expectations for the constant curvature case, since these bounds are sensitive only to the asymptotic structure of the solutions that are independent of the horizon geometry. We see that both branches terminate at the same values of $\alpha = \sqrt{3}$, below which the pressure becomes imaginary and the spacetime is no longer asymptotically anti-de Sitter. We therefore limit our analysis to couplings for which $\alpha \geq \sqrt{3}$. There is also a narrow region in the parameter space where a *minimum* pressure exists along with a maximum; in this case when $\sqrt{3} < \alpha < 2$. When $\alpha \geq 2$ there is no lower bound on the pressure.

4.3.2 Equation of State and Free Energy

For ordinary thermodynamic systems, the equation of state provides a closure relation between the thermodynamic degrees of freedom, and is usually phenomenological in nature. For black holes, the equation of state can be constructed directly from (4.13) and (4.17). For $K = 3$, we have that

$$\begin{aligned} P = & \frac{(d-2)T}{4r_+} - \frac{(d-2)(d-3)b_1}{16\pi r_+^2} + \frac{(d-2)\alpha_2 b_1 T}{2r_+^3} - \frac{(d-2)(d-5)\alpha_2 b_2}{16\pi r_+^4} \\ & + \frac{3(d-2)\alpha_3 b_2 T}{4r_+^5} - \frac{(d-2)(d-7)\alpha_3 b_3}{16\pi r_+^6} + \frac{Q^2}{2r_+^{2d-4}} , \end{aligned} \quad (4.29)$$

and in terms of the dimensionless variables defined above we have

$$\begin{aligned} p = & \frac{t}{v} - \frac{(d-2)(d-3)b_1}{4\pi v^2} + \frac{2\alpha b_1 t}{v^3} - \frac{(d-2)(d-5)\alpha b_2}{4\pi v^4} + \frac{3b_2 t}{v^5} \\ & - \frac{(d-2)(d-7)b_3}{4\pi v^6} + \frac{q^2}{v^{2d-4}} . \end{aligned} \quad (4.30)$$

From the equation of state, critical points are easily identified as locations where

$$\frac{\partial p}{\partial v} = 0 , \quad \frac{\partial^2 p}{\partial v^2} = 0 . \quad (4.31)$$

The first of these conditions yields the critical temperature t_c :

$$t_c = \frac{(d-2) \left(-4v_c^{10-2d} q^2 \pi + b_1(d-3)v_c^4 + 2\alpha b_2(d-5)v_c^2 + 3b_3(d-7) \right)}{2\pi (6\alpha b_1 v_c^2 + v_c^4 + 15b_2) v_c} \quad (4.32)$$

The second condition determines the critical volume v_c as a solution to the following polynomial:

$$\begin{aligned} 0 = & 4\pi v_c^{10-2d} \left[(2d-5)v_c^4 + 6\alpha b_1(2d-7)v_c^2 + 15b_2(2d-9) \right] (d-2)q^2 \\ & + (d-2) \left[6\alpha((d-3)b_1^2 - b_2(d-5))v_c^6 - b_1(d-3)v_c^8 - 45b_3b_2(d-7) \right] \\ & + (d-2) \left[6\alpha(5b_2^2(d-5) - 9(d-7)b_1b_3) v_c^2 - 3(b_2(4(d-5)\alpha^2 - 15d + 45) b_1 \right. \\ & \left. + 5b_3(d-7)) v_c^4 \right] \end{aligned} \quad (4.33)$$

Once the critical volume and temperature have been found, the equation of state (4.30) determines the critical pressure p_c . As (4.33) is generally a high-degree polynomial in v_c , multiple solutions may exist. We restrict our analysis to choices of parameters for which $\{v_c, t_c, p_c\}$ are all positive and real. Additional restrictions on the parameter space arise from entropic considerations, namely the positivity of (4.22) which implies that

$$(d-6)(d-4)v^4 + 2b_1\alpha(d-2)(d-6)v^2 + 3(d-2)(d-4)b_2 > 0. \quad (4.34)$$

Since $d \geq 7$ and $\alpha > 0$, this is automatically satisfied when $b_1 \geq 0$ and $b_2 \geq 0$.

The Gibbs free energy G and its dimensionless counterpart g can likewise be constructed. The equilibrium state of the system globally minimizes the Gibbs free energy, and coexistence phases are easily identified as locations where $r_+(G)$ is multivalued. They are defined, respectively, as

$$G = M - TS, \quad g = \frac{1}{\Sigma_{d-2}} \alpha_3^{\frac{3-d}{4}} G, \quad (4.35)$$

giving

$$\begin{aligned}
g = & \frac{q^2 \left((d-4) \left((v^4 + 3b_2) d^2 - \left(\frac{17v^4}{2} + \frac{39b_2}{2} \right) d + 15v^4 + 27b_2 \right) + 2b_1 \left(d - \frac{7}{2} \right) (d-2) \alpha v^2 (d-6) \right)}{2(d-3)(d-2)(d-4) (2\alpha v^2 b_1 + v^4 + 3b_2) (d-6) v^{d-3}} \\
& - \frac{p \left(15b_2(d-2)(d-4)v^{d-1} + (d-6) \left(6b_1\alpha(d-2)v^{d+1} + v^{d+3}(d-4) \right) \right)}{4(d-6)(d-2)(d-4) (2\alpha v^2 b_1 + v^4 + 3b_2) (d-1)} \\
& + \frac{3b_2b_3(d-2)(d-4)v^{d-6} + 6 \left(b_3(d-6)b_1 - \frac{b_2^2(d-4)}{2} \right) \alpha (d-2)v^{d-4}}{16\pi(d-6)v(d-4) (2\alpha v^2 b_1 + v^4 + 3b_2)} \\
& + \frac{((d-2)b_2(36-9d)b_1 + 5b_3(d-4)(d-6) + 2(d-2)b_2(\alpha^2 d - 6\alpha^2)b_1)v^{d-2}}{16\pi(d-6)v(d-4) (2\alpha v^2 b_1 + v^4 + 3b_2)} \\
& - \frac{\alpha (2(d-2)b_1^2 - 3b_2(d-4))v^d - b_1(d-4)v^{d+2}}{16\pi v(d-4) (2\alpha v^2 b_1 + v^4 + 3b_2)} . \tag{4.36}
\end{aligned}$$

This can be plotted parametrically against the equilibrium temperature of the system, given by

$$t = \frac{-4v^{10-2d}q^2\pi + 4v^6p\pi + b_1(d-2)(d-3)v^4 + \alpha b_2(d-2)(d-5)v^2 + b_3(d-2)(d-7)}{4\pi (2\alpha v^2 b_1 + v^4 + 3b_2) v} \tag{4.37}$$

and using the horizon radius r_+ as the parameter. The black hole size thus serves as the order parameter characterizing distinct equilibrium states and/or phases of the system. One can verify that these quantities agree with known results for spherical and hyperbolic black holes with constant curvature horizons [98]. Our analysis will be predicated on the free energy and equation of state, which provide complementary information about the location and nature of critical points of the black hole system. We will see how moving past the restriction of constant curvature transverse spaces markedly alters the nature of the transitions present in these systems.

4.4 Thermodynamics of $d = 7$ Exotic Black Holes

In the following sections we examine thermodynamic aspects of the general solutions presented above, focusing on seven and eight-dimensional charged and uncharged examples. We study the equation of state and thermodynamic free energy in each case, and locate critical points that separate different phases of the system. We work in the canonical ensemble where the physical charge is fixed. Seven-dimensional black holes in third-order Lovelock gravity do not admit horizons of non-constant curvature, so this case corresponds to ‘ordinary’ black holes that have been studied previously [98]. Nonetheless, we reexamine this case and discover new features previously unseen for this class of black holes.

In seven dimensions, we are restricted to constant curvature horizon cross sections. This arises from a dimensional analysis of the field equations and the requirement that the base manifold be of constant curvature [99]. This restricts our topological terms to be $b_1 = b_3 = \pm 1$ and $b_2 = 1$ (we may also have $b_n = 0 \ \forall n$ but we will not be examining this case). We consider the charged and uncharged cases separately.

4.4.1 Uncharged Solutions

We begin with the uncharged solutions in seven dimensions. With $q = 0$ the equation of state becomes

$$p = \frac{t}{v} - \frac{5b_1}{\pi v^2} + \frac{2\alpha b_1 t}{v^3} - \frac{5\alpha}{2\pi v^4} + \frac{3t}{v^5} , \quad (4.38)$$

and critical points as determined by (4.31) occur when

$$t_c = \frac{10(b_1 v_c^2 + \alpha) v_c}{\pi(6\alpha b_1 v_c^2 + v_c^4 + 15)} , \quad 10b_1 v_c^6 + 60\alpha^2 b_1 v_c^2 - 30\alpha v_c^4 - 450b_1 v_c^2 - 150\alpha = 0 . \quad (4.39)$$

We first examine the case when $b_1 = b_2 = b_3 = 1$, where the above reduces to

$$t_c = \frac{10(v_c^2 + \alpha) v_c}{\pi(v_c^4 + 6\alpha v_c^2 + 15)} , \quad 10v_c^6 - 30\alpha v_c^4 - (450 - 60\alpha^2) v_c^2 - 150\alpha = 0 . \quad (4.40)$$

The critical volume in seven dimensions is therefore given by the solution to a cubic function in v_c^2 . Only one solution is real and positive for $\alpha \geq \sqrt{3}$. Positivity of the entropy (4.22)

will be automatically satisfied for black holes of any size since all coefficients are positive. In what follows, we fix the value of α to limit the size of the parameter space, noting where variations of α lead to important qualitative changes in the phase structure. Otherwise, changing α simply changes the numerical values of the various critical points and bounds we find herein. We set $\alpha = 3$ where all solutions for the metric function f are real and positive, and a smooth Einstein limit exists. We are bounded only by a maximum pressure p_+ , while the minimum pressure is null:

$$p_+ = \frac{5(4\sqrt{6} - 9)}{6\pi} \approx 0.211, \quad p_- = 0. \quad (4.41)$$

In Figure 4.1 we show the equation of state $p(v)$, the Gibbs free energy $g(t)$, and the phase diagram for these black holes. In Figure 4.1a, the equation of state is shown for temperatures above, below, and equal to the critical temperature t_c . Below t_c , we observe Van der Waals oscillations characteristic of a small-large black hole phase transition. States in the negative pressure region are not only thermodynamically unstable (since $\partial p / \partial v > 0$), but also correspond to asymptotically *de Sitter* configurations (since $\Lambda > 0$). The physical state of the system instead follows the coexistence line as determined by Maxwell's equal area law, and is always positive in pressure due to the long tail of the $p(v)$ curve.

In Figure 4.1b, we plot the Gibbs free energy as a function of equilibrium temperature at fixed pressure above, below, and at the critical pressure p_c , using r_+ as a parameter. The radiation phase (sometimes referred to as *thermal anti-de Sitter* or *empty anti-de Sitter*) corresponds to the $g = 0$ line. Along each curve, the black hole size increases as the curve is followed from left to right. As the temperature increases, the physical state of the system corresponds to whichever branch (including $g = 0$) globally minimizes the free energy. Above the critical pressure (as in the red curve) we observe a Hawking-Page transition between the ‘empty’ radiation phase and a large black hole where the red line crosses $g = 0$. As the pressure decreases, a kink eventually forms at the critical pressure where a second-order phase transition between a small and large black hole occurs. Below p_c , a crossing forms where the small and large black hole branches are separated by an unstable intermediate branch, characterized by a negative specific heat. As the temperature increases, the system follows the lower curve and a first-order small-large transition occurs

since the horizon size is discontinuous at the crossing.

A new feature appearing in our analysis is the presence of a *triple point* at sufficiently low pressure. As the pressure decreases towards zero the second-order phase transition occurring at the crossing within the swallowtail eventually intersects the radiation phase at $g = 0$, as seen in Figure 4.2. At this novel triple point, the large and small black hole phases coexist with the radiation phase, with all three globally minimizing g . This is in contrast to an ordinary triple point which consists of three phases that all correspond to black hole states. The vertical line in Figure 4.2 extends upwards and terminates at the maximum pressure p_+ (not shown). This novel triple point was also observed in Gauss-Bonnet black holes with non-constant curvature horizons [97], and has not previously been noted in studies of the phase structure of anti-de Sitter black holes in third-order Lovelock gravity. Despite receiving significant attention in the hyperbolic ($\kappa = -1$) and flat ($\kappa = 0$) cases, notably in [99], the spherical $\kappa = 1$ case has not been analyzed as thoroughly.

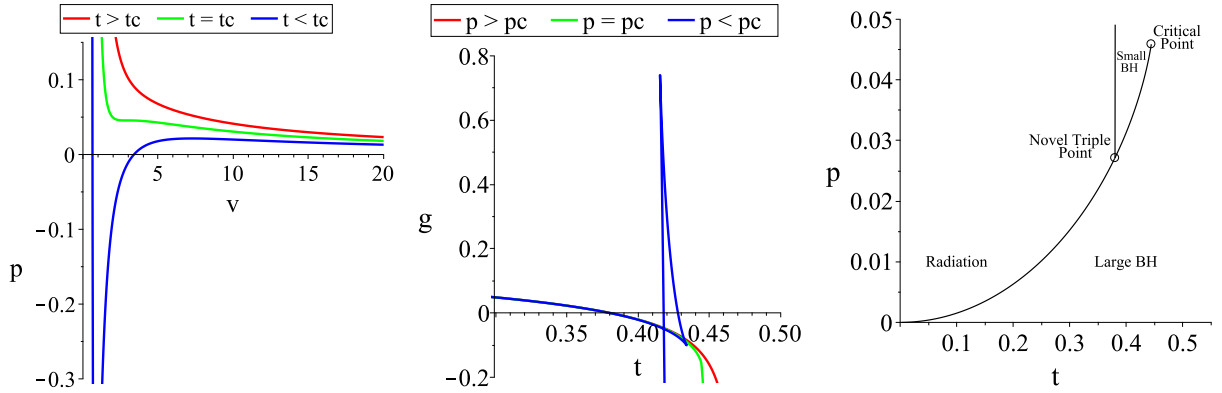


Figure 4.1: Seven-dimensional uncharged black holes with $b_1 = b_2 = b_3 = 1$, $\alpha = 3$ and $q = 0$. **Left:** p - v isotherms with Van der Waals oscillations occurring for temperatures below t_c . **Middle:** g - t isobars demonstrating the emergence of a swallowtail for pressures below p_c . **Right:** Phase diagram, with coexistence lines (black) separating regions where the small, large, and radiation phases occur.

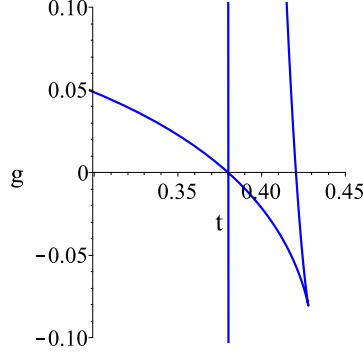


Figure 4.2: Free energy of the seven-dimensional uncharged black hole with $b_1 = b_2 = b_3 = 1$, $\alpha = 3$ and $q = 0$. With $p \approx 0.027$, the small-large transition lies on the $g = 0$ line, and a triple point occurs where the small and large black hole phases coexist with the radiation phase.

Finally, we observe that the phase structure is highly sensitive to variations of α . As α increases, the critical pressure decreases, until $\alpha \approx 3.85$ where a turning point occurs and p_c begins to increase monotonically with α . When $\alpha \approx 4.56$ the critical pressure coincides with the maximum pressure p_+ , while above this value, no critical points occur. Despite this, one can still examine phases with $\alpha \gtrsim 4.56$, however in the uncharged case the phenomenology is limited to Hawking-Page phase transitions between radiation and a large black hole. When $\alpha \gtrsim 3.87$ the novel triple point no longer occurs at any allowed pressure. In this case the free energy at the small-large transition always lies above $g = 0$, and only a Hawking-Page transition occurs (for any value of p). Finally as the value of α increases, the triple point moves to the right, eventually merging with the critical point. Above this value of α only a Hawking-Page transition occurs. Henceforth we fix $\alpha = 3$, as variations in α simply correspond to a rescaling of the b_n terms when $d \geq 8$.

4.4.2 Charged Solutions

We now briefly examine charged solutions with positive-curvature transverse sections. When $q \neq 0$ and $b_1 = b_2 = b_3 = 1$, the critical temperature and volume are given by

$$t_c = \frac{10v_c^8 + 10\alpha v_c^6 - 10\pi q^2}{\pi v_c^5 (v_c^4 + 6\alpha v_c^2 + 15)} , \quad (4.42)$$

$$v_c^{12} - 3\alpha v_c^{10} + (6\alpha^2 - 45) v_c^8 - 15\alpha v_c^6 - (9\pi v_c^4 + 42\pi\alpha v_c^2 + 75\pi) q^2 = 0 . \quad (4.43)$$

The critical pressure is given by

$$p_c = \frac{10v_c^{12} - 5\alpha v_c^{10} + (10\alpha^2 - 90) v_c^8 - 15\alpha v_c^6 - 18\pi q^2 v_c^4 - 28\pi\alpha q^2 v_c^2 - 30\pi q^2}{2\pi v_c^{10} (v_c^4 + 6\alpha v_c^2 + 15)} \quad (4.44)$$

where v_c will be the solution to the critical volume expression while the temperature has been replaced by the critical temperature. The critical volume equation can be re-expressed as the condition that $w_1 = w_2$ where we define

$$w_1 = (9\pi v_c^4 + 42\pi\alpha v_c^2 + 75\pi) q^2, \quad w_2 = v_c^{12} - 3\alpha v_c^{10} + (6\alpha^2 - 45) v_c^8 - 15\alpha v_c^6. \quad (4.45)$$

In this form the number of possible critical points can be easily determined through Descartes' rule of signs (graphically, we are concerned with the number of intersections between w_1 and w_2). w_1 is a quadratic in v_c^2 and no sign changes occur between terms, therefore no positive roots exist (q acts simply as a rescaling factor). For w_2 there are three sign changes, so a maximum of three positive roots exist and a minimum of one. For the case considered here where $\alpha = 3$ there is only one positive root, therefore for any value of q there exists only one value of v_c for which $w_1 = w_2$, and only one critical point exists. In what follows the charge is fixed to $q = 1$ unless charge variations are being considered.

As with the uncharged solutions, we examine the thermodynamics behaviour through the equation of state and free energy, as shown in Figure 4.3. On the right, the same oscillatory behaviour seen in the uncharged case is present, though the transition to the oscillatory region is now sensitive to the charge. As the value of charge is increased, the critical temperature decreases. In Figure 4.3b, the behaviour of the Gibbs free energy is qualitatively similar to the uncharged case, but with a different interpretation. In the

canonical ensemble the physical charge is fixed, which places a lower bound on the allowed black hole size. In the case of $d = 4$ Einstein gravity for example, Reissner-Nördstrom black holes require that $m^2 > q^2$ otherwise a naked singularity forms. This constraint generalizes to higher-dimensional Lovelock black holes, placing a lower bound on the black hole size which depends on $\{\alpha, q, \Lambda\}$. Thus, Hawking-Page transitions do not occur since the radiation phase corresponds to an $m = 0$ black hole which cannot exist when $q \neq 0$. Instead, we observe a first-order phase transition between a small and large black hole when $p < p_c$, a second-order phase transition when $p = p_c$, and no transitions when $p > p_c$.

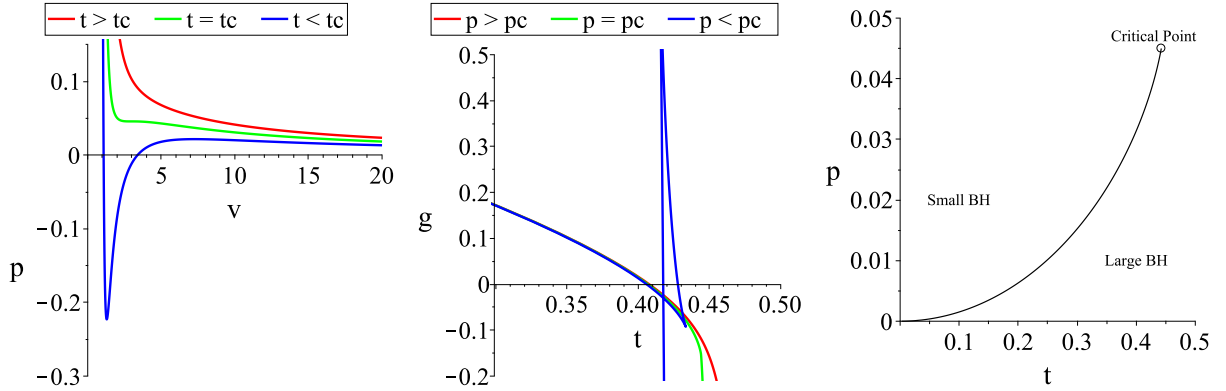


Figure 4.3: Seven-dimensional charged black holes with $b_1 = b_2 = b_3 = 1$, $\alpha = 3$, and $q = 1$. **Left:** p - v isotherms with Van der Waals oscillations occurring for temperatures below t_c . **Middle:** g - t isobars demonstrating the emergence of a swallowtail for pressures below p_c . **Right:** Phase diagram, with coexistence lines (black) separating regions where the small and large black hole phases occur.

Before moving on to eight dimensions, we make a brief comment about the hyperbolic case. When $b_1 = -1, b_2 = 1, b_3 = -1$, there are no critical points for $\alpha = 3$. Critical points are possible for values of $\alpha < 3$ as seen in [98], however these values of α appear to correspond to spacetimes where there is no smooth limit between the three branches of solutions to (4.23), hence their properties and structures are not entirely understood. Furthermore, hyperbolic geometries contain discontinuities in both the Gibbs free energy and temperature making the thermodynamic analysis more difficult. We therefore limit our analysis to $b_1 > 0$ where such issues are not present.

4.5 Thermodynamics of $d = 8$ Exotic Black Holes

In eight dimensions, we no longer require that $b_1 = \kappa$, $b_2 = \kappa^2 = 1$ and $b_3 = \kappa^3$. However some restrictions can be placed on the b_n by considering the character of the resulting metric functions. In the exotic black hole solutions given by (4.23), b_1 can in principle be chosen arbitrarily. Having $0 < |b_1| < 1$ or $|b_1| > 1$ is possible, but such a choice introduces a conical deficit (or excess) in the spacetime. The metric is no longer Ricci flat, instead resembling metrics that arise in the presence of a global monopole charge [105]. In the simplest case, such metrics are sourced by $\mathcal{O}(3)$ scalar fields with Higgs potentials whose vacuum expectation value does not vanish. Being interested in vacuum solutions with no external fields, we wish to avoid these cases. The case where $b_1 = -1$ is also possible, however a preliminary analysis reveals issues with discontinuities in the Gibbs free energy and a lack of critical points. As a result, we will restrict our attention to $b_1 = 1$ throughout.

Further restrictions can be placed on b_2 and b_3 by considering the behaviour of the Kretschmann invariant $\mathcal{K} = R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$. In d -dimensions, its functional form is given by

$$\mathcal{K} = \left(\frac{d^2 f(r)}{dr^2} \right)^2 + \frac{2(d-2)}{r^2} \left(\frac{df(r)}{dr} \right)^2 + \frac{2(d-2)(d-3)f(r)^2}{r^4} - \frac{4R[\gamma]f(r)}{r^4} + \frac{\mathcal{K}[\gamma]}{r^4}, \quad (4.46)$$

whereas before $f(r)$ corresponds to the chosen branch of (4.23), $R[\gamma]$ is the scalar curvature of the base manifold, and $\mathcal{K}[\gamma]$ is the Kretschmann scalar of the base manifold. $\mathcal{K}$ will diverge when $f(r)$ or its derivatives diverge, which occurs when the following quantity vanishes:

$$\begin{aligned} X(r) \equiv & 81 b_2 - 27 b_3 - 54 + (9\alpha - 2\alpha^3 - 27\alpha_0) r^6 \\ & + \sqrt{4(9 b_2 - 9 - (\alpha^2 - 3) r^4)^3 - (81 b_2 - 27 b_3 - 54 + (9\alpha - 2\alpha^3 - 27\alpha_0) r^6)^2}, \end{aligned} \quad (4.47)$$

4.5.1 Uncharged Solutions

In $d = 8$ with $q = 0$ and $\alpha = 3$, the maximum and minimum pressures are

$$p_+ = \frac{7(-27 + 12\sqrt{6})}{18\pi} \approx 0.296, \quad p_- = 0, \quad (4.48)$$

and the equation of state becomes

$$p = \frac{t}{v} - \frac{15}{2\pi v^2} + \frac{6t}{v^3} - \frac{27b_2}{2\pi v^4} + \frac{3b_2 t}{v^5} - \frac{3b_3}{2\pi v^6} . \quad (4.49)$$

The resulting critical volume and temperature are then given by

$$t_c = \frac{3(5v_c^4 + 18b_2 v_c^2 + 3b_3)}{\pi v_c (v_c^4 + 18v_c^2 + 15b_2)} , \quad (4.50)$$

$$15v_c^8 + (162b_2 - 270)v_c^6 + (297b_2 + 45b_3)v_c^4 - (810b_2^2 - 486b_3)v_c^2 + 135b_2 b_3 = 0 . \quad (4.51)$$

While the topological parameters are independent of each other, thermodynamic considerations lead to further constraints on their values. Re-arranging the equation of state (4.30) as

$$t \rightarrow t(p, v, b_2, b_3) = \frac{2\pi p v^6 + 15v^4 + 27v^2 b_2 + 3b_3}{2\pi v (v^4 + 6v^2 + 3b_2)} \quad (4.52)$$

reveals that negative values of b_2 lead to a divergent temperature due to the $(v^4 + 6v^2 + 3b_2)$ term in the denominator. We therefore restrict our analysis to the case where $b_2 \geq 0$. This $(v^4 + 6v^2 + 3b_2)$ term is also present in the free energy g , which then diverges along with the temperature. From equation (4.51) we can determine the possible number of positive real v_c solutions through Descartes' rule of signs, which are summarized in Table 4.1 below.

	$b_2 = 0$	$b_2 > 0$
$b_3 = 0$	1	3,1
$b_3 < 0$	1	3,1
$b_3 > 0$	2,0	4,2,0

Table 4.1: Possible number of positive real roots to 4.51 from Descartes' rule of signs applied to the topological parameters b_2 and b_3 .

We begin with the case where $b_2 = 0$ and $b_3 = -1$. This choice results in thermodynamic behaviour similar to the seven-dimensional case, with small-large transitions appearing generically below the critical temperature and pressure, as shown in Figure 4.4. Note that there is no Hawking-Page transition at low temperatures, and that $G \rightarrow 0$ as

$T \rightarrow 0$.

Criticality is also present when $b_2 = 0$ and $b_3 = 0$. The free energy in this case is qualitatively similar to that of Figure 4.4b, and the $p-v$ curves likewise follow Figure 4.4a. The critical pressure and temperature both increase as b_3 increases.

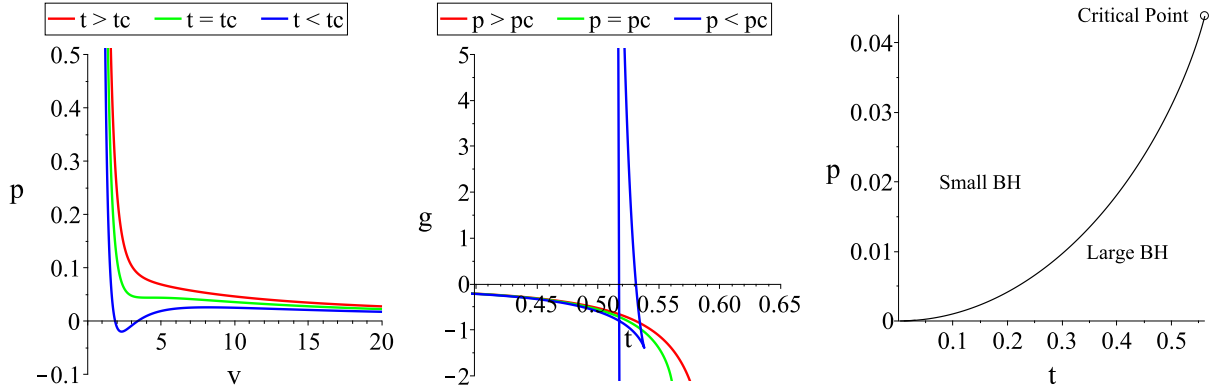


Figure 4.4: Eight-dimensional uncharged black holes with $b_1 = 1$, $b_2 = 0$, $b_3 = -1$, and $\alpha = 3$. **Left:** $p-v$ isotherms with Van der Waals oscillations occurring for temperatures below t_c . **Center:** $g-t$ isobars demonstrating the emergence of a swallowtail for pressures below p_c . **Right:** Phase diagram, with coexistence lines (black) separating regions where the small and large black hole phases occur.

When $b_2 = 0$ and $b_3 > 0$, two real and positive solutions for v_c will exist for $b_3 < \frac{324}{25}$. When $b_3 \geq 1$, the novel triple point emerges (seen also in the uncharged seven-dimensional case above) with notable qualitative differences. Increasing b_3 likewise increases the temperature and pressure at which this triple point occurs. Eventually, when $b_3 \approx 6$, the triple point is no longer present as the novel triple point merges with the smaller of the two critical points and only the Hawking-Page transition is seen.

Next we consider the case where $b_2 \neq 0$. When $b_2 = 1$ and $b_3 = 0$, Van der Waals behaviour is observed corresponding to a small-large black hole transition, though no triple point occurs at any temperature/pressure. When $b_3 > 0$ instead, we observe the same qualitative behaviour noted above for $b_2 = 0$ and $b_3 > 0$: the existence of a triple point

below a certain value of the parameter (in this case $0 < b_3 < 0.8$) , and Hawking-Page transitions when $b_3 > 0.8$.

4.5.2 Charged Solutions

When $d = 8$ and $q \neq 0$ the equation of state (4.30) becomes

$$p = \frac{t}{v} - \frac{15}{2\pi v^2} + \frac{6t}{v^3} - \frac{27b_2}{2\pi v^4} + \frac{3b_2 t}{v^5} - \frac{3b_3}{2\pi v^6} + \frac{q^2}{v^{12}} , \quad (4.53)$$

with the critical temperature and volume given by

$$t_c = -\frac{3(-5v_c^{10} - 18b_2v_c^8 - 3b_3v_c^6 + 4\pi q^2)}{\pi v_c^7(v_c^4 + 18v_c^2 + 15b_2)} , \quad (4.54)$$

$$15v_c^{14} + (162b_2 - 270)v_c^{12} + (297b_2 + 45b_3)v_c^{10} - (810b_2^2 - 486b_3)v_c^8 + 135b_2b_3v_c^6 - (132\pi v_c^4 + 1944\pi v_c^2 + 1260\pi b_2)q^2 = 0 \quad (4.55)$$

In principle, a systematic determination of the root structure of the polynomial above can be done using Descartes' rule of signs; however, the presence of b_3 and q -dependent terms means such an analysis quickly becomes cumbersome. Instead we proceed on a case-by-case basis, as only a few examples are required to demonstrate the salient features of charged $d = 8$ black holes.

As with the uncharged case, we begin with the *spherical* base manifold for which $b_n = 1$. In this case, we observe only a small-large transition terminating at a critical point $\{t_c, p_c\}$ for all values of the charge q as shown in Figure 4.6. This behaviour is identical to what is shown in Figure 4.3 for the $d = 7$ case. As q increases, t_c and p_c decrease, until they eventually achieve the $q = 0$ values shown in Figure 4.4. That there can *only* be a small-large transition for 8-dimensional charged black holes is easily seen by separating the charge-dependent and charge-independent portions of (4.55), and studying the number of intersections of the resulting functions. Let

$$F_1 = F_2 \quad (4.56)$$

with

$$\begin{aligned} F_1 &\equiv (132\pi v_c^4 + 1944\pi v_c^2 + 1260\pi) q^2 \\ F_2 &\equiv 15v_c^{14} - 108v_c^{12} + 342v_c^{10} - 324v_c^8 + 135v_c^6. \end{aligned} \quad (4.57)$$

Plotting F_1 and F_2 for varying values of q reveals that for any finite value of the charge there is a unique value of v_c which satisfies (4.56), as displayed in Figure 4.5.

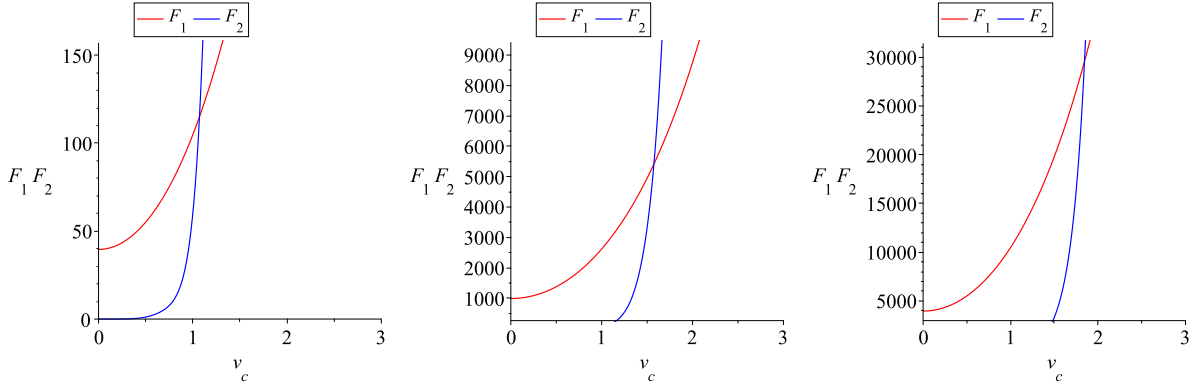


Figure 4.5: Plots for F_1 and F_2 as a function of v_c for varying values of q . **Left:** $q = 0.1$, **Center:** $q = 0.5$, **Right:** $q = 1$.

As in the uncharged case, we plot the equation of state $p(v)$ and free energy $g(t)$, searching for Van der Waals oscillations in the $p-v$ plane and the formation of a swallowtail in the $g-t$ plane. Note that even though critical points may occur above $g = 0$, the radiation phase is inaccessible to the system since the charge is fixed in the canonical ensemble, so no Hawking-Page transitions will occur in any of the cases we examine below.

Beginning with $b_2 = 0$, when $b_3 \leq 0$ we observe a single critical point and the corresponding Van der Waals-like transition between a large and small black hole, for all values of q . Only quantitative changes in the critical temperature and volume occur when q is varied. When $b_3 > 0$, a triple point may occur for very specific values of q and b_3 . For example, when $b_3 = 0.01$, the standard small-intermediate-large black hole triple point occurs when $0 < q \lesssim 0.0069$. When $q \gtrsim 0.0069$ the triple point vanishes and only a small-large transition occurs. On the other hand, when the charge is fixed the existence of the triple

point is lower bounded by the parameter b_3 . For example, when $q = 0.05$ we must have $b_3 > 0.4$ for a triple point to occur. This is shown in Figure 4.7.

Next we consider the case where $b_2 > 0$. For any choice of $b_2 > 0$ there is no choice of q and $b_3 < 0$ that yields a triple point. The only phenomena observed are small-large phase transitions. This is also the case when $b_3 = 0$. When $b_3 > 0$, we have the possibility of a small-intermediate-large triple point. As above, the presence of this triple point is highly sensitive to the values of q and b_2 . If $b_2 = 0.1$ and $b_3 = 0.1$, the triple point only occurs for $0 < q < 0.006$. If b_3 is increased while b_2 is kept fixed, the triple point will occur at a larger value of q . Correspondingly, the location of the triple point occurs at a larger temperature t_c and pressure p_c . Eventually, if both b_3 and q are sufficiently large, the triple point moves above the maximum allowed pressure p_{max} , and only a small-large transition is physically allowed. Therefore as long as the maximum pressure bound is not violated, a triple point can occur for any choice of $b_2 > 0$ and $b_3 > 0$ provided the black hole has the correct charge.

The behaviour for fixed b_3 and varying b_2 and q is qualitatively different. As a concrete example, take $b_3 = 0.1$ and increase b_2 and q from 0.1 and 0 respectively. A triple point occurs as long as $0.1 < b_2 < 0.2$ and $0 < q \lesssim 0.003$. If *both* $b_2 \geq 0.2$ and $q \gtrsim 0.003$ only a small-large transition occurs. Generically, when the charge *decreases* the temperature and pressure at which the triple point occurs will *increase*. In all cases, analytic solutions to the locations of the triple/critical points are not available, so the accuracy in the bounds presented are constrained by the choice of numerical solver and precision used.

The equation of state, free energy, and phase diagram in the presence of a triple point are shown in Figure 4.7. The triple point occurs at the intersection of the small, intermediate, and large black hole branches of the $g - t$ diagram (indicated also by a red dot in the phase diagram). At this point, three distinct black hole states with different sizes all globally minimize the free energy at the same temperature. Figure 4.7c contains the phase diagram, which displays new features compared to the uncharged case. The right-most branch distinguishes the large and intermediate black hole phases and terminates at a critical point, beyond which the system is in a ‘superfluid’ phase where the two states are no longer distinguishable. The nearly vertical branch separates the intermediate and small black hole phases, and extends upward until it terminates at the maximum allowed pres-

sure p_{max} . This is conjectured to always occur before any critical point forms regardless of parameter choice, though a lack of analytic solutions for the location of the critical points prevents us from excluding the possibility of a second critical point occurring below p_{max} . Above the critical temperature/pressure which terminates the intermediate-large transition, the two phases are no longer distinguishable (there is no discontinuity in the horizon size). However, we still observe a phase transition between this ‘merged’ intermediate-large black hole branch and the small black hole branch.

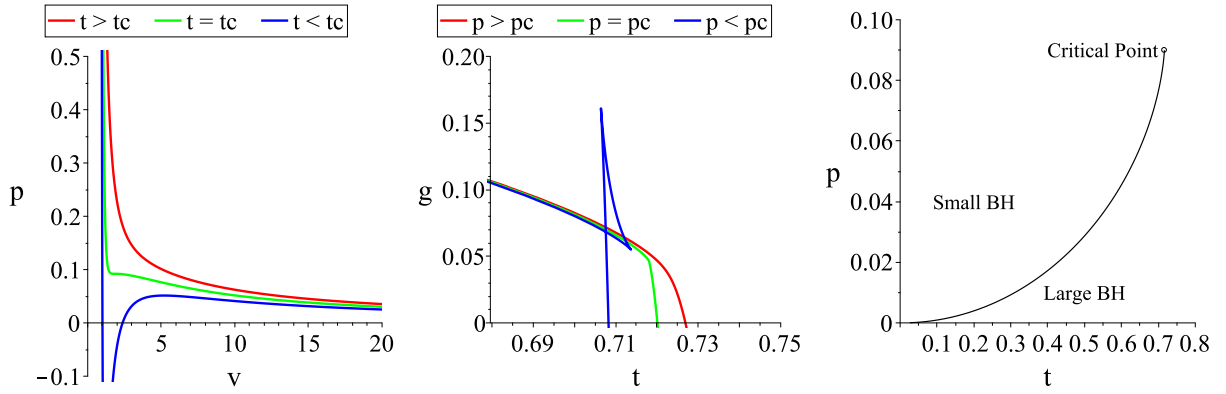


Figure 4.6: Eight-dimensional charged black holes with $b_1 = b_2 = b_3 = q = 1$. **Left:** $p-v$ diagram showing oscillation. **Center:** $g-t$ diagram with standard swallow tail structure. **Right:** Phase diagram with coexistence line showing the separation of large and small black holes which terminates at the critical point.

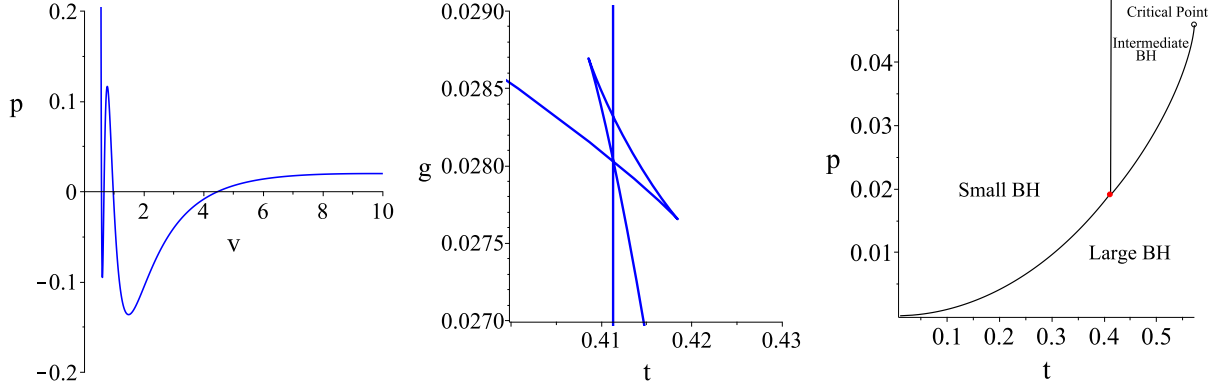


Figure 4.7: Eight-dimensional charged black hole with $b_1 = 1$, $b_2 = 0.1$, $b_3 = 0.5$, $q = 0.05$. **Left:** $p-v$ diagram for $t \approx 0.415$ showing two oscillations. **Center:** $g-t$ diagram for $p \approx 0.19$ showing the intersection of the two swallow tails. **Right:** Phase diagram showing coexistence lines that all join together at the red triple point. The intermediate-large line will terminate at the critical point. The intermediate/small coexistence line will terminate at the maximum pressure of the spacetime.

4.6 Discussion

Higher-curvature generalizations of Einstein gravity admit black hole solutions with ‘exotic’ horizon geometries. In particular, Lovelock gravity allows for black holes that are distinguished by non-constant curvature horizon cross-sections. This is in stark contrast to vacuum solutions in $d = 4$ Einstein gravity, which are constrained by Birkhoff’s theorem to have transverse metrics isometric to $\mathcal{S}^2$ (in the case of spherical symmetry). In this work, we have examined the phase structure of such black hole solutions in third-order Lovelock gravity, focusing on both $U(1)$ charged and uncharged asymptotically AdS solutions in seven and eight dimensions and in an extended phase space where the cosmological constant acts as a thermodynamic pressure. Our work can be seen as a natural extension of some preliminary efforts at understanding the thermodynamic properties of these exotic black hole solutions as seen in [99, 97].

The generalization to third-order Lovelock gravity (and by necessity $d \geq 7$) retains

much of the qualitative features of exotic black holes in second-order Gauss-bonnet gravity in $d = 5$. Nevertheless we observed and accounted for a wide variety of thermodynamic phenomena which are generic to asymptotically AdS black holes, and found that novel triple point phenomena as seen in [97] persist in higher-order Lovelock gravity. We also clarified the interpretation of earlier results in the case of $d = 5$ exotic black holes and $d = 7$ constant curvature black holes, and analyzed the space of physically reasonable exotic black hole solutions.

General ‘exotic’ black hole solutions in third-order Lovelock gravity depend on a set of arbitrary constants $\{b_1, b_2, b_3\}$ characterizing the base manifold, as well as the Lovelock coupling constants $\{\alpha_0, \alpha_2, \alpha_3\}$. We obtained a discriminant condition for the metric function $f(r)$ that establishes a maximum and minimum allowed pressure, which all branches of solution are constrained by. The analysis focused on the Einstein branch of solutions, which has a smooth limit to Einstein gravity when $\alpha_2, \alpha_3 \rightarrow 0$. The topological parameters $\{b_n\}$ were treated as constants, which can in principle take any values in $\mathbb{R}$. To our knowledge there is currently no known prescription for systematically determining all allowed combinations of topological parameters in third-order Lovelock gravity. Nevertheless, we were able to rule out a large portion of the parameter space by studying how the formation of naked singularities is sensitive to the choice of transverse geometry. We found that most choices of transverse space are pathological, containing a divergence in the Kretschmann scalar at a finite radius outside of the event horizon. This is not surprising as the allowed exotic solutions are already severely constrained in Gauss-Bonnet gravity [106]. In principle, additional constraints on the Lovelock couplings could be obtained through considerations of classical energy conditions and observational constraints on deviations from general relativity, though such investigations lie beyond the scope of this work.

In seven dimensions, we found that uncharged black holes with a *spherical* base manifold contain a novel triple point akin to the one found in [97], which was not previously seen in the analyses of [98] and [99]. This triple point is found to be sensitive to the rescaled Gauss-Bonnet coupling α , terminating at the critical point once α reaches a certain value (after which only a Hawking-Page transition occurs). Furthermore, small-large black hole phase transitions are forbidden as they require $\alpha < 3$ which precludes the existence of a smooth Einstein limit. When charge is present, the triple point vanishes and only small-

large transitions occur with a single critical point. We clarify that although Hawking-Page transitions were reported previously for this case [99], the canonical (fixed-charge) ensemble does not allow for such transitions, even if they appear energetically favourable.

In eight dimensions, black holes without charge exhibit three distinct phase transition behaviours: small-large transitions, Hawking-Page transitions, or triple point phenomena, depending on the choice of topological parameters. For charged black holes, the Hawking-Page transition no longer occurs and we only observe a small-large transition or a triple point. Surprisingly, new types of phase transitions are not seen even with the inclusion of a new topological parameter b_3 . One might expect a new branch of phase transitions (or even a quadruple point with two intermediate branches) to emerge in $d = 8$, because the equation of state contains new terms in higher inverse powers of r . It turns out, however, that the constraints present do not allow for choices of parameters that would produce the necessary oscillations/branch structure for such phenomena to occur.

Along the way, we have found solutions for black holes with vanishing mass parameter that nevertheless possess horizons and non-trivial spacetime singularities. These can be found directly from (4.24) by an appropriate choice of parameters. Investigating the thermodynamic properties and geometric structure of these solutions would be an interesting future endeavour. Another ambitious task would be to examine exotic black holes in asymptotically de-Sitter spaces, where the notion of thermodynamic equilibrium is more subtle. This is an area which has been gaining attention in recent years [107, 108, 109, 110, 111] with many new thermodynamic features being uncovered for a wide class of black holes. Finally, understanding fully how to constrain the topological parameters would be a significant step towards determining exactly which types of exotic black holes deserve further investigation.

Chapter 5

Negative Mass Black Holes in pseudo-de-Sitter Space

5.1 Introduction

In 4 dimensions the assumption that the metric has radial symmetry restricts black hole solutions to have horizons of constant curvature in Einstein gravity. These can be spherical, planar, or hyperbolic if a negative cosmological constant $\Lambda < 0$ is introduced [112, 113, 114]. If the geometry is hyperbolic then these black holes can have negative mass [115] and can form from gravitational collapse [116]. However if $\Lambda \geq 0$ then the situation is notably different. While it has recently been shown that de Sitter Schwarzschild spacetimes with negative mass in [117, 118] can exist in Einstein gravity, these solutions consist of spacetime bubbles. No negative mass black hole vacuum solutions have been obtained.

We show that asymptotically pseudo-de Sitter black hole vacuum solutions of negative mass exist in Lovelock gravity. This feature is due to the exotic nature of the base manifolds discussed in the previous chapter. We explicitly illustrate this for Gauss-Bonnet gravity, but it is clear that this will occur in higher-order Lovelock gravity and we expect in more general higher-curvature theories as well. Interest in de Sitter black holes is motivated primarily by cosmological considerations [119], though some motivation comes from

considerations of holographic duality [120]. Now, we refer to these black holes as pseudo-de Sitter because of the nature of the base manifold being not of constant curvature. While the scalar curvature is still positive it no longer represent a constant curvature space such as 4-dimensional de Sitter space, or d-dimensional de Sitter space.

We also note that the concept of mass in de Sitter space is not straightforward. This is directly due to the fact that the presence of a positive cosmological constant introduces a cosmological horizon, and thus spatial infinity is not accessible. The inaccessibility of spatial infinity does not allow the ADM formulation of mass to be used explicitly. When the cosmological horizon is much larger than the event horizon, an approximate definition of mass was constructed in [121]. The procedure in [122] is also expected to hold in the scenario where the cosmological constant is positive but small. For regimes where the cosmological constant is significant, the Wald or Dolan procedure [123, 124, 125, 126] provides a more accurate understanding of black hole mass with a positive cosmological constant. However, we make note that for this chapter the importance is not directly in the formal definition of black hole mass, but the notion of a constant of integration in the metric solution is associated with black hole mass. This can be seen explicitly in the behaviour of our solutions in the Einstein limit, and the form in powers of r .

5.2 Gauss-Bonnet Solutions

Setting $K = 2$ in (4.6) yields

$$\frac{\alpha_2 f^2}{r^4} + \left(-\frac{1}{r^2} - \frac{2b_1\alpha_2}{r^4} \right) f + \alpha_0 + \frac{b_1}{r^2} + \frac{b_2\alpha_2}{r^4} = \frac{16\pi M}{(d-2)\Sigma_{d-2}r^{d-1}} \quad (5.1)$$

with the solutions

$$f = f_{\pm}(\text{m}) \equiv \frac{r^2 + 2b_1\alpha_2 \pm \sqrt{(b_1^2 - b_2)4\alpha_2^2 + r^4(1 - 4\alpha_2\alpha_0) + \frac{8\text{m}\alpha_2}{r^{d-5}}}}{2\alpha_2} \quad (5.2)$$

where we have written

$$\text{m} \equiv \frac{8\pi M}{(d-2)\Sigma_{d-2}}. \quad (5.3)$$

The two solutions are distinguished by their behaviour in the $\alpha_2 \rightarrow 0$ limit. The f_- branch is referred to as the Einstein branch as it has a smooth $\alpha_2 \rightarrow 0$ limit, giving

$$\lim_{\alpha_2 \rightarrow 0} f_-(m) = \alpha_0 r^2 + b_1 - \frac{2m}{r^{d-3}} \quad (5.4)$$

whereas the f_+ solution does not have a smooth limit as $\alpha_2 \rightarrow 0$; it is referred to as the Gauss-Bonnet branch.

For the remainder of this paper we will only consider the Einstein branch for analysis; we will also choose $b_1 = 1$, so that as $\alpha_2 \rightarrow 0$ we recover the standard Schwarzschild (Anti) de Sitter solution. We will only examine solutions with $d \geq 6$, since the $d = 5$ case only admits constant curvature solutions in the transverse space. The cosmological constant is

$$\Lambda = -\frac{\hat{\alpha}_0}{2} = -\frac{\alpha_0(d-1)(d-2)}{2} \quad (5.5)$$

and we shall set $\alpha_0 < 0$ since we are interested in asymptotically de Sitter solutions. This allows us to write

$$f_- = \frac{r^2 + 2\alpha_2 - \sqrt{(1-b_2)4\alpha_2^2 + r^4 \left(1 + 4\alpha_2 \frac{2\Lambda}{(d-1)(d-2)}\right) + \frac{8m\alpha_2}{r^{d-5}}}}{2\alpha_2} \quad (5.6)$$

where we note $\Lambda > 0$. In the large r limit, the function (5.6) has the same features as the metric function of an asymptotically de Sitter spacetime.

It is convenient at this point to introduce a set of dimensionless variables

$$r = x\sqrt{\alpha_2}, \quad \Lambda = \frac{(d-1)(d-2)z}{2\alpha_2}, \quad m = m\alpha_2^{\frac{d-3}{2}}, \quad z > 0 \quad (5.7)$$

so that

$$f = 1 + \frac{x^2}{2} - \frac{\sqrt{(1+4z)x^4 + 4(1-b_2) + \frac{8m}{x^{d-5}}}}{2}. \quad (5.8)$$

The horizon(s) are located at the roots of f , which can be found from solving the polynomial:

$$x^{d-1}z - x^{d-3} - b_2x^{d-5} + 2m = 0 \quad \text{with} \quad d \geq 6, \quad z > 0 \quad (5.9)$$

For a black hole solution we must have $f(x) > 0$ for $x > x_+$ for some range of x , where x_+ locates the outermost event horizon of the black hole.

Superficially it appears that there will be 2 or 0 positive roots to this polynomial from Descartes' rule of signs. However if we allow both b_2 and m to be either positive or negative the number of possible positive roots are then the following:

$$\begin{aligned}
&\text{If } m > 0 && \text{then 2 roots } \rightarrow (x_c, x_+) \text{ or 0 roots for any } b_2 \\
&\text{If } m < 0 \text{ and } b_2 > 0, && \text{then 1 root } \rightarrow x_c \\
&\text{If } m < 0 \text{ and } b_2 < 0, && \text{then 3 roots } \rightarrow (x_-, x_+, x_c) \text{ or 1 root } \rightarrow x_c.
\end{aligned} \tag{5.10}$$

where x_c is the cosmological horizon, x_- is the inner horizon, and x_+ is the outer horizon of the black hole.

5.2.1 Positive Mass

Consider first positive mass. We plot the behaviour of f in Figure 5.1 for sample values $b_2 = -2.5$, $z = 0.05$, and $d = 6$, varying m . For small nonzero m we have an asymptotically de Sitter black hole (left diagram in Figure 5.1) where as for sufficiently large m there is no static black hole solution and we have an expanding spacetime with r and t interchanging roles (middle diagram). As m increases from small to large values, there is one value of m where f has a double root and the horizons merge, illustrated in the right diagram in Figure 5.1. This is a Gauss-Bonnet generalization of the Nariai solution. This structure is the same in all dimensions $d \geq 6$. Only quantitative differences are seen as we increase the dimension: the spread between x_+ and x_c increases and the maximum value of f slightly increases. Consequently we will illustrate our results in 6 dimensions unless otherwise stated.

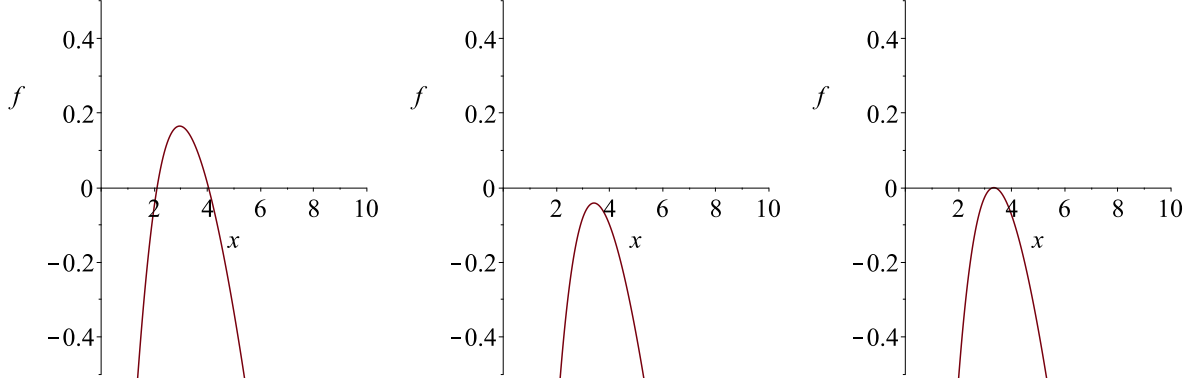


Figure 5.1: Positive Mass Black Hole $b_2 = -2.5, z = 0.5, d = 6$. **Left:** Black hole with mass $m = 1$ showing two distinct real roots. **Center:** Mass $m = 5$, showing no roots, and a naked singularity. **Right:** Mass $m = 4.0638$, showing a double root.

We may also examine what happens when we hold all the parameters fixed except for b_2 , which is seen in Figure 5.2. For a fixed mass m and cosmological parameter z increasing the value of b_2 will shift f upwards. In this manner the geometry of the horizon plays an important role regarding the existence of a black hole solution, namely that a minimum allowed value of b_2 is required. This minimum value occurs when f has a double root, corresponding to a Nariai type of black hole solution. In de Sitter space this is referred to as the maximum allowed mass a black hole can have, other parameters being fixed. To determine the value of $x_+ = x_c$ for the Nariai-type solution, the equations

$$f = 0, \quad \frac{\partial f}{\partial x} = 0 \quad (5.11)$$

must be simultaneously solved. In 6 dimensions there is no closed form solution since the latter equation is quintic polynomial in x . For the parameters in Figure 2 we obtain numerically $b_{2_{min}} \approx -1.62$. For b_2 less than this the spacetime will have a naked singularity at the origin.

We can also make another observation, shown in Figure 5.3. That is, for a certain set of parameters f can become discontinuous for some range of x . This discontinuity is due to the argument inside the square root becoming negative. Whenever this occurs

the Kretschmann scalar (4.10) diverges, yielding a spacetime singularity. Solutions to the following polynomial

$$(4 - 4b_2)x^{d-5} + (4z + 1)x^{d-1} + 8m = 0 \quad (5.12)$$

yield the singularity location(s). From Decartses' rule of signs this will only have a root when $b_2 > 1$. This is a necessary but not sufficient condition; as illustrated in the right image of Figure 5.3, increasing the value of z allows the two discontinuous branches to join, yielding a smooth continuous metric function f . If we hold all parameters fixed, increasing z shifts f down vertically and reduces the spread between the black hole and cosmological horizons, allowing the possibility of a smooth metric function between them.

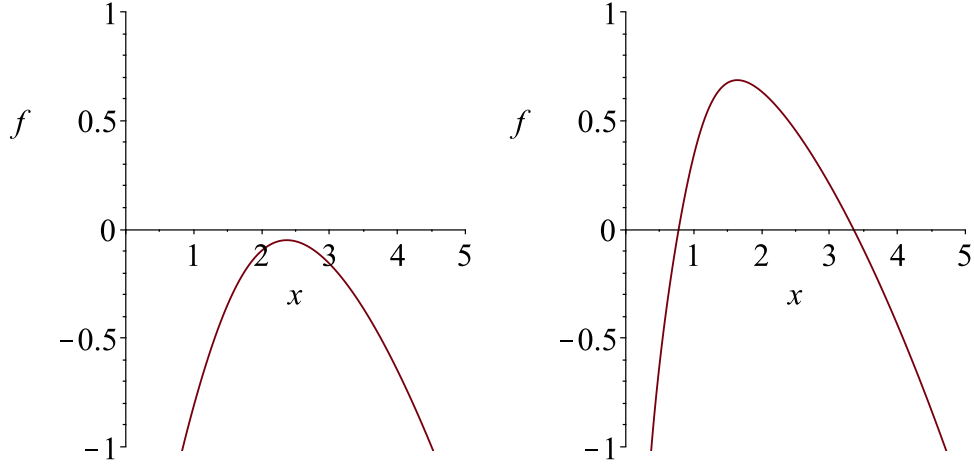


Figure 5.2: Positive Mass $m = 1, z = 0.1, d = 6$. Left: Naked singularity with $b_2 = -2$ showing no real roots. **Right:** de Sitter black hole with $b_2 = 2$ show a cosmological horizon and event horizon $x_c > x_+$

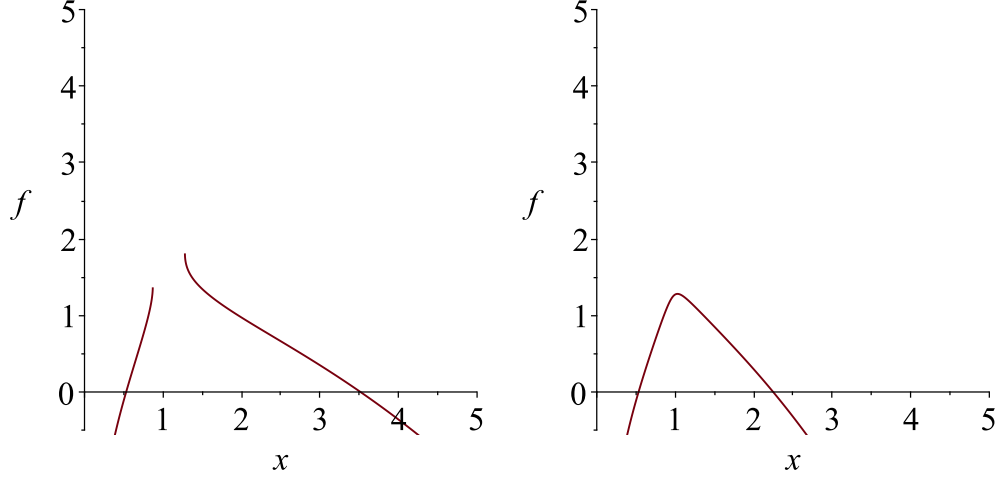


Figure 5.3: Positive mass $m = 1, d = 6, b_2 = 3.5$. **Left:** $z = 0.1$ Two discontinuous branches with two spacetime singularities. **Right:** $z = 0.3$ de Sitter black hole.

5.2.2 Negative Mass

Negative mass solutions have more interesting behaviour, since if $b_2 < 0$ there can now be three roots to (5.9). We illustrate this in Figure 5.4. The left image depicts three horizons, while the center and right image show the two possible cases where horizons can merge. In the center image $x_c = x_+$ (a Nariai-type solution) and in the right $x_- = x_+$ (an extremal black hole with a cosmological horizon). These latter two extremes constrain the range of allowed values of $b_2 < 0$ that yield black hole solutions of negative mass. For the case illustrated in Figure 5.4, we approximately have $-1.3 \geq b_2 > -3$.

We also find that as $x \rightarrow 0$, f does not have a smooth limit, and instead becomes complex at some finite positive value of x . For $m < 0$ and $b_2 < 0$ there is *only one* sign change in the coefficients of the terms in (5.12) and hence this equation has only one positive root, at which point the Kretschmann scalar (4.10) diverges. For the admissible range of b_2 , this singularity is always behind an horizon as can be seen in Figure 5.4. For the more generic 3-horizon case shown in the left diagram in Figure 5.4, the solution to (5.12) is $x = 0.1999637662$.

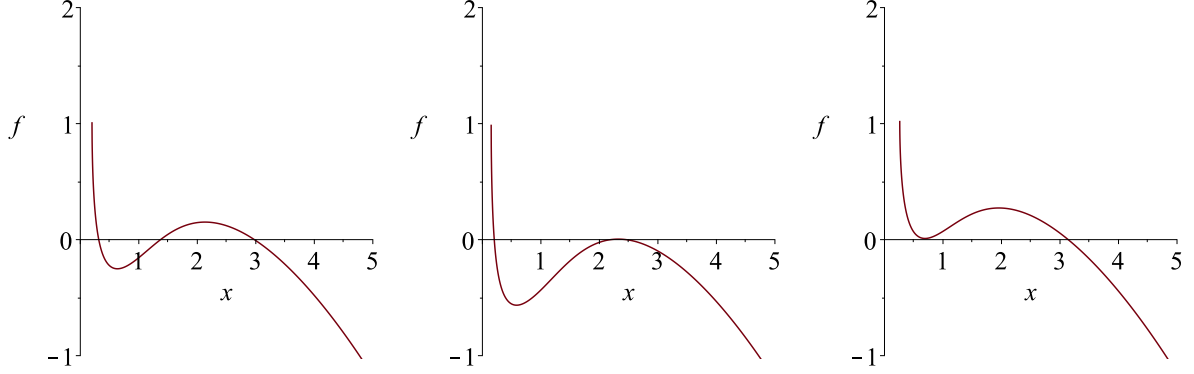


Figure 5.4: Negative Mass: $m = -0.3$, $z = 0.09$, $d = 6$ **Left:** $b_2 = -2$, showing three distinct horizons. **Center:** $b_2 = -3$, showing one double root depicting a Nariai like solution **Right:** $b_2 = -1.3$, showing an extremal black hole with a distinct cosmological horizon.

5.2.2.1 Allowed values of b_2 for valid negative mass black hole solutions

To determine the the range of b_2 that allows for a valid black hole solution of negative mass in any dimension, we must solve the following set of equations

$$f = 0, \quad \frac{\partial f}{\partial x} = 0 \text{ for } b_2, x \quad \text{with} \quad \{x, b_2\} \in \mathbb{R} > 0\}. \quad (5.13)$$

The first equation is given by (5.9) and yields

$$b_2 = x_+^4 z - x_+^2 + 2x_+^{5-d} m \quad (5.14)$$

where x_+ solves

$$2x_+^{d-1} z - x_+^{d-3} - m(d-5) = 0. \quad (5.15)$$

from the latter equation. There is no general solution to this set of equations except in certain dimensions.

For $d = 6$ equation (5.15) is a quintic, and there is no analytic solution for arbitrary m, z . For the parameters used in Figure 5.4, we obtain numerically

$$\{x_{1+} = 0.6897087623, b_{2_{max}} = -1.325264590\}, \{x_{2+} = 2.328863088, b_{2_{min}} = -3.033847194\}. \quad (5.16)$$

refining the values given above.

In 7 dimensions however, we may find a closed form expression. The equations become

$$b_2 = \frac{x_+^6 z - x_+^4 + 2m}{x_+^2} \quad (5.17)$$

$$2x_+^6 z - x_+^4 - 2m = 0. \quad (5.18)$$

which admits analytic solutions for arbitrary m, z . The latter equation is a cubic in x_+^2 ; writing $m = -|\mathbf{m}|$ we get

$$2y^3 z - y^2 + 2|\mathbf{m}| = 0 \quad \text{with} \quad y = x_+^2, \quad z > 0 \quad (5.19)$$

Now through Decartes' rule of signs, the possible number of distinct positive roots is either 2 or zero, while the number of negative roots will *always* be one; this latter case is inadmissible since we must have $y > 0$.

To have two *distinct* positive real roots for y (and thus the same number of distinct positive values for x_+), we require the discriminant

$$\Delta_3 = 8|\mathbf{m}| (1 - 54z^2|\mathbf{m}|) > 0 \quad (5.20)$$

for (5.19), implying

$$-\frac{1}{54z^2} < m < 0. \quad (5.21)$$

This condition must be satisfied to have three distinct roots. Writing (5.19) as a depressed cubic, the solutions (for x) are easily written as

$$x_k = \sqrt{\frac{1}{3z} \cos \left(\frac{1}{3} \arccos (1 - 108|\mathbf{m}|z^2) - \frac{2\pi k}{3} \right) - \frac{1}{6z}} \quad k = 0, 1, 2 \quad (5.22)$$

For all values of m satisfying (5.20), the only $k = 0, 1$ yield real solutions, where $x_0 > x_1$. The value x_0 corresponds to the location of the location of the Nariai-like black hole solution whereas x_1 locates the extremal black hole of negative mass. From (5.17), the respective

minimal and maximal values of b_2 are

$$\begin{aligned}
b_{2_{min}} &= \frac{8 \cos(A)^3 - 432|\mathbf{m}|z^2 - 36 \cos(A)^2 + 30 \cos(A) - 7}{72z^2 \cos(A) - 36z} \\
A &= \frac{1}{3} \arccos(1 - 108|\mathbf{m}|z^2) \\
b_{2_{max}} &= \frac{8 \cos(B)^3 - 432|\mathbf{m}|z^2 - 36 \cos(B)^2 + 30 \cos(B) - 7}{72z^2 \cos(B) - 36z} \\
B &= \frac{1}{3} \arccos(1 - 108|\mathbf{m}|z^2) - \frac{2\pi}{3}
\end{aligned} \tag{5.23}$$

and provided $b_{2_{max}} > b_2 > b_{2_{min}}$, there will be a valid negative mass black hole with two horizons and a de Sitter cosmological horizon.

We can make a further interesting observation about (5.20). If we saturate the inequality (5.21) from above, setting set $m = 0$, a range of interesting phenomena can occur. It is possible to have a black hole, a Nariai like black hole, pure de Sitter space or a naked singularity. These scenarios are analyzed in the next section.

5.3 Massless Black Holes

The saturation of (5.21) from above allows for the possibility of zero mass exotic black holes. A intriguing feature of these possible black holes is that they are independent of the dimension of the spacetime. Taking (5.8) and setting the mass to zero gives us

$$f = 1 + \frac{x^2}{2} - \frac{\sqrt{(1+4z)x^4 + 4 - 4b_2}}{2}. \tag{5.24}$$

The root equation (5.15) is now

$$x^4 z - x^2 - b_2 = 0 \tag{5.25}$$

which is a biquadratic equation in x^2 so it can be written as

$$y^2 z - y - b_2 = 0. \tag{5.26}$$

We may again analyze the roots of this polynomial by Decartses' rule of signs. Assuming $z > 0$, then

$$\begin{aligned} \text{for } b_2 < 0 & \quad 2 \text{ or } 0 \text{ roots} \\ \text{for } b_2 \geq 0 & \quad 1 \text{ root} \end{aligned} \tag{5.27}$$

As a quadratic the discriminant is immediately written down as: $\Delta_2 \equiv 4zb_2 + 1$. For a valid black hole solution, we must have $\Delta_2 \geq 0$. The saturation of this inequality suggests the Nariai solution. The unsaturated inequality gives us, similar to the negative mass case, a condition on b_2 as a function of z that must be satisfied for a black hole to exist: $-\frac{1}{4z} < b_2 < 0$. If b_2 is not within these bounds there will be a naked singularity located at the origin.

For $b_2 > 0$ the space is pure de Sitter, provided (5.12) has no solutions so as to avoid singularities. These equations are easily solved for any d , yielding

$$x = \frac{\sqrt{2}(b_2 - 1)^{1/4}}{(1 + 4z)^{1/4}}. \tag{5.28}$$

and so we conclude that we must have $0 \leq b_2 < 1$ to obtain a pure de Sitter space without any naked singularities. The naked singularity that would occur for $b_2 \geq 1$ is at a finite value of x given by (5.28). The different scenarios are outlined in Figure 5.5. The 3 leftmost images display negative values of b_2 respectively showing a naked singularity, Nariai black hole and de Sitter black hole. The two rightmost images are for zero and positive values of b_2 respectively, with the former being pure de Sitter space and the latter a naked singularity outside of the origin. For increasing values of $b_2 > 0$ the termination point of f at $x = 0$ moves up the vertical axes until it reaches a maximum point at $f(x, b_2, z) \rightarrow f(0, 1, 0.1) = 1$.

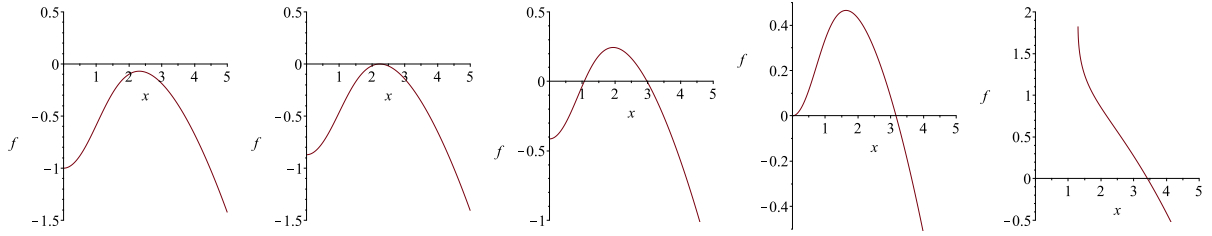


Figure 5.5: Massless Solutions $z = 0.1$ From left to right the b_2 values are: -3, -2.5, -1, 0, 2. It depicts a naked singularity at the origin, a Nariai like black hole, a de Sitter black hole, pure de Sitter space, and then a non-trivial naked singularity outside of the origin, respectively.

5.4 Discussion

We have shown that metrics of the form (4.3), with $\gamma_{\alpha\beta}$ being the metric of an arbitrary Einstein space, can possess both negative mass and massless asymptotically de Sitter black hole solutions in Gauss-Bonnet gravity. These $M \leq 0$ black holes are all exotic black holes, and their existence depends on the relative values of the parameters b_2 , z and m . For the massless case, for there is a direct relationship between b_2 and z with b_2 being bounded from below by z . For $M < 0$ black holes there are more restrictions on the parameters. The range of m is bounded by z , as given by (5.21). There is also a minimum and maximum allowed value of b_2 admitting such solutions.

We close by noting that the existence of such horizon geometries is still an open question. This is due to the fact that field equations for the transverse space are an over constrained PDE system, where certain topological parameters b_n may not provide a solution. However we are not aware of any theorem forbidding the existence of non-trivial solutions to (4.11).

With the discovery of negative mass black holes in de Sitter space interesting questions arise. These include the behaviour of geodesics, formation from gravitational collapse, and possible pair production in the early universe of such objects. Their thermodynamic behaviour is also likely to yield interesting surprises in comparison to their Anti de Sitter counterparts [97, 127].

Chapter 6

3-Dimensional Black Holes in General Relativity

Having focused on black hole solutions in higher dimensions, it is now time to turn our interest into black hole solutions in lower dimensions, in particular 3-dimensional or 2+1 dimensional black holes. General relativity in 3-dimensional spacetime is much simpler than its 4-dimensional counterpart [128]. A principal consequence of 3-dimensional spacetime is that the Weyl tensor is zero. Consequently, the Riemann tensor is entirely characterized by the Ricci tensor. While this feature allows us to globally classify vacuum solutions to the Einstein equations as flat, it also establishes that no propagating degrees of freedom exist in 3 dimensions. On mathematical grounds, the loss of one spacetime dimension - in general - leads to differential equations that are more amenable to yielding solutions. One such solution, the Banados-Teitelboim-Zanelli (BTZ) solution [129], describing a 3-dimensional black hole, has been of considerable interest to theorists for over three decades. For a review on the 3-dimensional black hole physics see, e.g., [130].

While the discussion of vacuum and Maxwell solutions are quite prevalent in the literature, nonlinear electrodynamic theories coupled to lower dimensional gravity are less understood. This is in particular true for modified or alternative lower dimensional gravity. The most heavily studied solutions are static solutions – even with 3-dimensional gravity producing differential equations that under axial symmetry are ordinary. The difficulty

in the ordinary differential equations is the fact that even though they only depend on one variable, they are heavily coupled and non-linear. This in turn leads to the use of alternative methods to find rotating solutions, such as coordinate boosts.

The rotating solutions are only difficult to find when the solutions are no longer vacuum. The rotating vacuum BTZ black hole is easily derivable from the field equations with elementary methods on the differential equations. As soon as matter is introduced, the problem becomes exponentially more difficult. While the solution for a rotating Maxwell charged 3-dimensional black hole was originally published along with the vacuum solution, it was quickly shown that it was indeed an incorrect solution [131, 132].

This chapter is dedicated to the study of lower dimensional black holes coupled to nonlinear electrodynamics. However, we wish to also conduct a study of the original BTZ solution and provide an alternate description of the solution. We will discuss black holes in the 3-dimensional Einstein-Gauss-Bonnet gravity and how generalized solutions can be produced for the static case, and how all solutions are able to be constructed through the general relativity solution.

6.1 2+1 Dimensional Black holes in General Relativity

6.1.1 Static Solutions

Any static spherically symmetric (SSS) line element can be written as

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2d\varphi^2 \quad (6.1)$$

with φ being identified with periodicity 2π . The general form of $f(r)$ is given by

$$\frac{r^2}{l^2} - m - h(r) \quad (6.2)$$

where $h(r)$ is a function of only the radial coordinate and is dictated by the matter Lagrangian – in our case associated with a $U(1)$ electromagnetic charge. All of the work we

do is going to be based on the following action

$$S = \int d^3x \sqrt{-g} \left(\frac{1}{16\pi} (R - 2\Lambda) + \mathcal{L} \right) \Rightarrow R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi T_{\mu\nu} \quad (6.3)$$

where $\mathcal{L}$ is the matter Lagrangian density. We will write the Maxwell Lagrangian as

$$\mathcal{L}_M = -\frac{1}{4} F \quad (6.4)$$

with $F = F_{\mu\nu} F^{\mu\nu}$, leading to the Born-Infeld Lagrangian having the form

$$\mathcal{L}_{BI} = -b^2 \left(\sqrt{1 + \frac{F}{2b^2}} - 1 \right) \quad (6.5)$$

which approaches $\mathcal{L}_m$ when $b \rightarrow \infty$. The generalized Maxwell equations that come from the action are

$$\nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial F} F^{\mu\nu} \right) = 0. \quad (6.6)$$

We are solely interested in radial electric charged solutions. Therefore the 3-potential we will consider only has 1 component and is given via

$$A = \phi(r) dt \quad (6.7)$$

where we will indicate $\mathcal{L}_F \equiv \frac{\partial \mathcal{L}}{\partial F}$. Therefore, any theory in this class will only have 1 non-zero component of the Maxwell equations, that being the t -component. This equation is explicitly

$$\phi'' \mathcal{L}_F r + \phi' (r \mathcal{L}_F' + \mathcal{L}_F) = 0 \quad (6.8)$$

and the solution for ϕ can be written as an integral solution

$$\phi(r) = A + B \int \frac{1}{\mathcal{L}_F r} dr. \quad (6.9)$$

It is much easier to view this as a differential equation in ϕ'

$$\phi' \mathcal{L}_F r = B \quad (6.10)$$

where we have the integration constant B . Thus given a theory with Lagrangian $\mathcal{L}$, calculating $\mathcal{L}_F$ and inserting the result into (6.10) will yield the differential equation necessary

for ϕ . For the unknown function $h(r)$ in the metric we turn to the field equations for that solution. For a general matter Lagrangian $\mathcal{L}$, the stress energy tensor can be defined as

$$T_{\mu\nu} = 4\mathcal{L}_F F_{\beta\mu} F_{\nu}^{\beta} + \mathcal{L}g_{\mu\nu} \quad (6.11)$$

and this can be used to solve for $h(r)$. Using either tt or rr from the field equations, which are first order – $\varphi\varphi$ is redundant – we have

$$h' = -16\pi \left(4\mathcal{L}_F \phi'^2 + \mathcal{L} \right) r. \quad (6.12)$$

Now again the solution could be written down as an integral; but instead if we use (6.10) this can be re-written as

$$h' = -16\pi \left(\frac{4B^2}{L_F r^2} + L \right) r \quad (6.13)$$

where again if we have the Lagrangian and calculate $\mathcal{L}_F$ we can solve for h assuming that ϕ is of course known. Let us look at some simple examples.

6.1.1.1 Maxwell

With $\mathcal{L} = -\frac{1}{4}F$, this gives us $\mathcal{L}_F = -\frac{1}{4}$ and so we have for ϕ

$$\begin{aligned} \phi &= A + B \int \frac{1}{\mathcal{L}_F r} dr \\ \phi &= A - 4B \ln(r) \end{aligned} \quad (6.14)$$

using the integral solution since $\mathcal{L}_F$ is a constant. For the metric function h we have

$$\begin{aligned} h' &= -16\pi \left(\frac{4B^2}{L_F r^2} + L \right) r \\ h' &= -16\pi \left(-\frac{16B^2}{r^2} + \frac{\phi'^2}{2} \right) r \end{aligned} \quad (6.15)$$

Now let us call $4B = Q$, such that we now have

$$\begin{aligned} \phi &= -Q \ln(r/r_0) \\ h &= 8\pi Q^2 \ln(r/r_0) \end{aligned} \quad (6.16)$$

We have absorbed the constants into the $\ln$ functions to create a dimensionless argument, hence fixing the gauge.

6.1.1.2 Born-Infeld

Consider the Born-Infeld Lagrangian above

$$\mathcal{L}_{BI} = -b^2 \left(\sqrt{1 + \frac{F}{2b^2}} - 1 \right) \quad (6.17)$$

this gives us

$$\begin{aligned} \mathcal{L}_F &= -\frac{1}{2\sqrt{4 + \frac{2F}{b^2}}} \\ \mathcal{L}_F &= -\frac{b}{4\sqrt{-\phi_r'^2 + b^2}}. \end{aligned} \quad (6.18)$$

To solve for ϕ we use (6.10)

$$\begin{aligned} \phi' &= -\frac{4\sqrt{-\phi'^2 + b^2}B}{br} \\ \phi'^2 b^2 r^2 + 16B^2 \phi'^2 &= 16B^2 b^2 \\ \phi &= -Q \ln \left(\frac{r + \sqrt{r^2 + Q^2/b^2}}{r_0 + \sqrt{r_0^2 + Q^2/b^2}} \right). \end{aligned} \quad (6.19)$$

Similar to the Maxwell case the chosen constant fixes the gauge. Now for h we have

$$\begin{aligned} h' &= -16\pi \left(\frac{4B^2}{L_F r^2} + L \right) r \\ h' &= \frac{16\pi \left(b^2 r^2 \sqrt{\frac{-\phi'^2 + b^2}{b^2}} - b^2 r^2 + 16\sqrt{\frac{-\phi'^2 + b^2}{b^2}} B^2 \right)}{r} \\ h' &= \frac{16\pi b (b^2 r^2 - \sqrt{b^2 r^2 + 16B^2} br + 16B^2)}{\sqrt{b^2 r^2 + 16B^2}} \\ h(r) &= -8\pi b^2 r^2 + 8\pi b \sqrt{b^2 r^2 + 16B^2} r + 128\pi B^2 \ln \left(br + \sqrt{b^2 r^2 + 16B^2} \right) + C \\ h(r) &= -8\pi b^2 r^2 + 8\pi b \sqrt{b^2 r^2 + Q^2} r + 8\pi Q^2 \ln \left(br + \sqrt{b^2 r^2 + Q^2} \right) + C \\ h(r) &= 8\pi b^2 r \left(\sqrt{r^2 + Q^2/b^2} - r \right) + 8\pi Q^2 \ln \left(\frac{r + \sqrt{r^2 + Q^2/b^2}}{r_0 + \sqrt{r_0^2 + Q^2/b^2}} \right) - 8\pi b^2 r_0 \left(\sqrt{r_0^2 + Q^2/b^2} - r_0 \right) \end{aligned} \quad (6.20)$$

this combination of constants will allow or h to be zero when $Q = 0$ or $r = r_0$ and in the large r limit the solution approaches the Maxwell case.

6.1.2 Rotating Solutions

By requiring that solutions have $\partial/\partial t$ and $\partial/\partial\phi$ as Killing vectors, stationary black holes in $(2 + 1)$ dimensions are described by metrics that contain only functions of the radial variable r . This feature, in turn, will produce equations of motion that are solely ordinary differential equations; a property that their $(3 + 1)$ counterparts will not have, apart from slowly rotating solutions. While ordinary differential equations are forgiving in comparison to partial differential equations, the complexity of the system of equations grows significantly once one moves beyond the vacuum case. This is true even for the Einstein-Maxwell case.

We begin this section with a brief review and re-derivation of the Einstein case for vacuum via the field equations and then implement coordinate boosts. We will then discuss the Maxwell case and why it is so formidable before moving onto discussion of currents.

For a rotating charged solution one must consider the general metric ansatz

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2(d\phi + L(r)dt)^2 \quad (6.21)$$

if one wishes to solve the field equations. Anything less and there are not enough degrees of freedom in the system. Here f, g, L are all functions of coordinate r . Under this ansatz the angular coordinate can be identified with period 2π . As well one must make the ansatz

$$A = \phi dt + \omega d\varphi \quad (6.22)$$

for the 3-potential, and only considering homogeneous fields. With the metric and 3-potential ansatz, the gravitational and electrodynamic field equations result in 5 non-linear coupled ordinary differential equations. The highly non-linear nature of these equations result in no closed-form solution for even the case of Maxwell dynamics. In order to find rotating solutions, another method must be employed, that being the boost method. The boost method is similar to the Newman-Janis algorithm in 4-dimensions, but in 3-dimensions it

simply employs linear coordinate boosts analogous to the Lorentz transformations for the time and angular coordinate¹.

6.1.2.1 Boosted Solutions

To obtain the rotating solutions we will use the following coordinate transformations

$$\phi \rightarrow \frac{at}{l^2} - \Xi\varphi, \quad t \rightarrow \Xi t - a\varphi \quad \text{with} \quad \Xi^2 = 1 + a^2/l^2. \quad (6.23)$$

For the general metric (6.1) with (6.23) we get the following 'boosted' line element

$$ds^2 = - \left(\frac{r^2}{l^2} - \frac{a^2 + l^2}{l^2} (m + h) \right) dt^2 + \frac{dr^2}{f} + ((h + m)a^2 + r^2) d\varphi^2 - 2\Xi(g + m)d\phi dt \quad (6.24)$$

and the potential (6.7) becomes

$$A = \phi(\Xi dt - a d\varphi). \quad (6.25)$$

This is a completely viable solution in Kerr-like coordinates, as φ cannot be identified with period 2π since the metric coefficient is not r^2 . While it may be valid that asymptotically the coefficient is r^2 , since it is not true for all r the identification cannot be made. This metric can be transformed into the ADM form

$$ds^2 = -N^2 dt^2 + \frac{d\hat{r}^2}{K^2} + \hat{r}^2 (d\varphi^2 + N^\varphi dt) \quad (6.26)$$

where now φ is identified with period 2π through an appropriate radial coordinate transformation. We can see that in (6.24) the transformation takes the following form

$$\hat{r}^2 = (h + m)a^2 + r^2 \quad (6.27)$$

thus all surfaces of constant t, r will have circumference $2\pi r$. We will now look at the Maxwell dynamics case and provide the proper ADM form of the metric, and associated 3-potential for a rotating black hole with an electric and magnetic field.

¹Numerical methods may also be used but we did not investigate these cases and the differential equations are boundary value problems and not initial values problems, leading to further difficulty in trying to find solutions.

For Maxwell electrodynamics, h is given by (6.16) and the radial transformation equation is given by (6.27) as

$$\hat{r}^2 = \left(8\pi Q^2 \ln \left(\frac{r}{r_0} \right) + m \right) a^2 + r^2. \quad (6.28)$$

This allows $r(\hat{r})$ to be written as

$$r = 2Q\sqrt{\pi} \sqrt{W \left(\frac{r_0^2 e^{\frac{-a^2 m + \hat{r}^2}{4\pi Q^2}}}{4\pi Q^2} \right)} \quad (6.29)$$

where W represents the Lambert function $W(x) \exp(W(x)) = x$. Now since we have a new radial coordinate we must also have a new radial scale $\hat{r}_0$. Evaluating at $r = r_0$ yields

$$\hat{r}_0^2 = a^2 m + r_0^2 \quad (6.30)$$

and as such we have

$$r_0 = \sqrt{\hat{r}_0^2 - a^2 m} \quad (6.31)$$

Now using this transformation we have the metric in the following form

$$\begin{aligned} ds^2 = & \hat{r}^2 d\varphi^2 + \frac{8 \left(\pi a^2 Q^2 W(x) - \frac{\hat{r}^2}{4} \right) dt d\varphi \sqrt{a^2 + l^2}}{al} \\ & + \frac{l^2 \hat{r}^2 W(x) d\hat{r}^2}{16 \left(a^2 Q^2 \pi (a^2 + l^2) W(x) - \frac{\hat{r}^2 l^2}{4} \right) \pi (1 + W(x))^2 Q^2} - \frac{8 \left(Q^2 a^2 \left(a^2 + \frac{l^2}{2} \right) \pi W(x) - \frac{\hat{r}^2 (a^2 + l^2)}{8} \right) dt^2}{a^2 l^2} \end{aligned} \quad (6.32)$$

where

$$x = \frac{(-a^2 m + \hat{r}_0^2) e^{\frac{-a^2 m + \hat{r}^2}{4\pi a^2 Q^2}}}{4\pi a^2 Q^2}. \quad (6.33)$$

We can put this into full ADM form after a little simplification

$$\begin{aligned} ds^2 = & - \left[\frac{4\pi Q^2 (2a^2 + l^2) W(x)}{l^2} - \frac{r^2 (a^2 + l^2)}{a^2 l^2} \right] dt^2 + \frac{dr^2}{4\pi Q^2 \left[\frac{4a^2 \pi Q^2 (a^2 + l^2)}{r^2 l^2} - \frac{1}{W(x)} \right] [(1 + W(x))^2]} \\ & + r^2 d\varphi^2 + \frac{2\sqrt{a^2 + l^2} (4\pi a^2 Q^2 W(x) - \hat{r}^2)}{al} dt d\varphi \end{aligned} \quad (6.34)$$

$$ds^2 = - \left[\frac{\hat{r}^2 (a^2 + l^2) (\hat{r}^2 - 1)}{l^2 a^2} + \frac{4\pi Q^2 W(x) (4\pi W(x) a^4 Q^2 + 4\pi W(x) a^2 Q^2 l^2 - 2\hat{r}^2 a^2 - 2\hat{r}^2 l^2 + 2a^2 + l^2)}{l^2} \right] dt^2$$

$$+ \frac{d\hat{r}^2}{4\pi Q^2 \left[\frac{4a^2 \pi Q^2 (a^2 + l^2)}{r^2 l^2} - \frac{1}{W(x)} \right] [(1 + W(x))^2]} + r^2 \left(d\varphi + \frac{\sqrt{a^2 + l^2} (4\pi a^2 Q^2 W(x) - \hat{r}^2)}{al} dt \right)^2$$

For the potential, the boosted solution is

$$A = -Q \ln(r/r_0) (\Xi dt - a d\varphi) \quad (6.35)$$

and the change of radial coordinate yields

$$A = -Q \ln \left(\frac{2Qa\sqrt{\pi} \sqrt{W \left(\frac{(-a^2 m + \hat{r}_0^2) e^{-\frac{a^2 m + \hat{r}^2}{4\pi Q^2 a^2}}}{4\pi Q^2 a^2} \right)}}{\sqrt{-a^2 m + \hat{r}_0^2}} \right) (\Xi dt - a d\varphi) \quad (6.36)$$

or

$$A = \frac{4W \left(\frac{(-a^2 m + \hat{r}_0^2) e^{-\frac{a^2 m + \hat{r}^2}{4\pi Q^2 a^2}}}{4\pi Q^2 a^2} \right) \pi Q^2 a^2 + a^2 m - \hat{r}^2}{8Q\pi a^2} (\Xi dt - a d\varphi). \quad (6.37)$$

This solution is not what was originally posed by [133] in the original work on 2+1-dimensional black hole solutions, of course this can be immediately deduced as they used a static 3-potential. It also does not match the solution presented by [134] which used a boost technique, but kept the solution in Kerr-like coordinates without proper angular identification. We believe this is the first closed-form presentation of the charged ADM form black hole solution coupled with Maxwell electrodynamics with a magnetic field in the 3-potential.

6.1.2.2 Solutions with Current

Consider the rotating vacuum solution

$$ds^2 = -\hat{f} dt^2 + \hat{r}^2 \left(d\varphi - \frac{J}{2\hat{r}^2} dt \right)^2 + \frac{d\hat{r}^2}{\hat{f}}, \quad \hat{f} = \frac{\hat{r}^2}{\ell^2} - \hat{m} + \frac{J^2}{4\hat{r}^2} \quad (6.38)$$

where $T_{\mu\nu} = 0$. Assuming for a moment that we can add the static charged metric contribution into the metric, what does this do for the Maxwell equations?

$$\begin{aligned} ds^2 = & - \left(-m - r^2\Lambda + \frac{J^2}{4r^2} - 8\pi Q^2 \ln \left(\frac{r}{r_0} \right) \right) dt^2 + \frac{dr^2}{-m - r^2\Lambda + \frac{J^2}{4r^2} - 8\pi Q^2 \ln \left(\frac{r}{r_0} \right)} \\ & + r^2 \left(d\varphi - \frac{Jdt}{2r^2} \right)^2 \\ A = & \phi dt. \end{aligned} \tag{6.39}$$

There are two Maxwell equations now

$$\begin{aligned} \frac{\phi''r + \phi'}{4r} &= 0 \\ \frac{J(\phi''r - \phi')}{8r^3} &= 0 \end{aligned} \tag{6.40}$$

Now when $J=0$ we must surely recover the static solution and as such let us solve the first equation, yielding

$$\phi = -Q \ln(r/r_0). \tag{6.41}$$

unsurprisingly. This turns the second equation into

$$\frac{JQ}{4r^4} = 0. \tag{6.42}$$

This product of QJ is discussed heavily in [132] as the rationale for why attempts to derive a fully rotating solution cannot use a metric of the form (6.38) but must instead be (6.21). What does this do to the field equations? This solution of ϕ solves the field equations and also satisfies the conservation of stress energy. So what we have is a rotating solution which possess a current $j^\mu = \frac{JQ}{r^4} d\varphi$. Current solutions have been known since [131] and [135] but were not found in [136].

What if we treat it more generally and consider nonlinear electrodynamics? Suppose the metric is

$$ds^2 = - \left(-m - r^2\Lambda + \frac{J^2}{4r^2} - h(r) \right) dt^2 + \frac{dr^2}{-m - r^2\Lambda + \frac{J^2}{4r^2} - h(r)} + r^2 \left(d\varphi - \frac{Jdt}{2r^2} \right)^2 \tag{6.43}$$

and the potential $A = \phi dt$. The t -component of the NLED equations (6.6) can be cast as

$$\phi' = \frac{c_2}{L_F r} \quad (6.44)$$

while the φ -component is

$$\frac{J c_2}{r^4} = 0 \quad (6.45)$$

where we have made use of ϕ' from (6.44). So, similar to the Maxwell case, regardless of what NLED theory we use, we still get the same current form. The conservation of stress energy $\nabla_\mu T^{\mu\nu} = 0$ only posses' one component, that being $\nabla_\mu T^{\mu r} = 0$ and it reads

$$\frac{4(L'_F \phi' r + L_F \phi'' r + L_F \phi') \phi'}{r} = 0. \quad (6.46)$$

Inserting (6.44) will solve this equation identically, irregardless of the NLED theory. Lastly, in the field equations (6.3), there are 3 differential equations for h . Focusing on only the first-order equation, which comes from the rr -component, it reads

$$-\frac{2r(64\phi'^2 L_F \pi r + 16L(-2\phi'^2)\pi r + h')}{-4r^4 \Lambda - 4hr^2 - 4mr^2 + J^2} = 0 \quad (6.47)$$

which can readily integrate into

$$h = \int -16\pi r \left(4L_F \phi'^2 + L(-2\phi'^2) \right) dr + c_1. \quad (6.48)$$

Placing this expression in the $\varphi\varphi$ and $t\varphi$ components of the field equations will solve these remaining components. What this shows is that any NLED theory will always posses a metric of the form (6.43), where h solely relies on the solution of (6.44). The difficulty and problems that arise from this discussion is, where does the current come from? This leads into the discussion of new interpretation of the BTZ black hole, and how the current can be understood with the introduction of new fields.

6.2 New Interpretation of BTZ Black Hole

As introduced in the previous section, the rotating BTZ black hole with charge is not a viable solution to the field equations. While it does solve the gravitational field equations,

it does not properly solve the associated Maxwell equations. A correct solution for the charged rotating BTZ black hole was subsequently found in 1995 [134] (see also [137]) by applying a boost in the azimuthal direction to the charged static solution as also shown in the previous section.

If we want the metric and 3-potential to indeed be proper solutions to our field equations, we need to modify the Maxwell equations to be non-homogeneous – if the 3-potential will only posses an electric field. Now, one intermediate way to do this is to simply introduce the current and 3-potential invariant into the action

$$S = \int d^3x \sqrt{-g} \left(\frac{1}{16\pi} (R - 2\Lambda) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - A_\mu J^\mu \right) \quad (6.49)$$

and the variation with respect to the A_μ will produce the corrected Maxwell equations

$$\nabla_\mu F^{\mu\nu} = J^\nu.$$

The issue with this correction is that $A_\mu J^\mu \equiv 0$ for the solution presented

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \left(d\varphi - \frac{Jdt}{2r^2} \right)^2 \\ f(r) &= -m - r^2\Lambda + \frac{J^2}{4r^2} - 8\pi Q^2 \ln \left(\frac{r}{r_0} \right) \\ \phi &= -Q \ln \left(\frac{r}{r_0} \right) \end{aligned} \quad (6.50)$$

and thus trying to verify the solutions from a Lagrangian perspective will fail as the "source" of the current is not well defined. The gravitational field equations will still remain unchanged

$$\begin{aligned} G_{\mu\nu} + \Lambda g_{\mu\nu} &= 8\pi T_{\mu\nu} \\ T_{\mu\nu} &= \frac{1}{4\pi} \left(F_{\mu\alpha} F_\nu^\alpha - \frac{1}{4} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) \end{aligned}$$

as does the Maxwell stress-energy tensor.

In addition to the field equations being satisfied, the conservation of stress-energy is also satisfied

$$\begin{aligned}
4\pi\nabla_\mu T^{\mu\nu} &= 0 \\
\frac{(-\nabla^\nu (F_{\alpha,\beta}) - 2\nabla_\alpha (F_\beta{}^\nu)) F^{\alpha\beta}}{2} + \nabla_\beta (F_\alpha{}^\beta) F^{\alpha\nu} &= 0 \\
\nabla_\beta (F_\alpha{}^\beta) F^{\alpha\nu} &= 0
\end{aligned} \tag{6.51}$$

where in the second equation of (6.51), the first term is simply $d\mathbf{F} = 0$ in index form. Through the construction of $F_{\mu\nu}$ with a static 3-potential, and the metric tensor in ADM form, the only component of this differential equation that is non-zero is $\nu = r$. And with this the equation that returns is simply just the differential equation on ϕ that produces

$$\phi = -Q \ln \left(\frac{r}{r_0} \right).$$

Understanding that the Maxwell equations must be corrected to $\nabla_\mu F^{\mu\nu} = J^\nu$, we can combine this with (6.51) and it gives $F_{\mu\nu} J^\nu = 0$. This formula is a feature of what is referred to as force-free electrodynamics [138]. It makes an appearance in the Blandford-Znajek process of electromagnetic energy extraction from rotating black holes [139]. With the gravitational field equations and stress-energy conservation satisfied, the BTZ solution can be interpreted as a back-reacting spacetime in force-free electrodynamics; the spacetime supports itself without a properly defined current.

6.2.1 Maxwell-Current as Topological Contribution

It is well-known that it is possible to add topological matter terms to the action that leave the gravitational field equations invariant, but alter the matter field equations. One of the most common and well studied of these theories are of the Chern-Simons form [140]. The Chern-Simons contribution to the Lagrangian can be written as $L = \lambda \varepsilon^{\nu\alpha\beta} F_{\alpha\beta} A_\nu$, with ε representing the Levi-Civita symbol, in index form, or $L = \lambda A \wedge F$ in differential form language. There has been additional work done on considering a generalized Chern-Simons term in odd dimensions through

$$I_{\text{CS}}^{(2n+1)} = \frac{\lambda}{4\pi} \int A \wedge \underbrace{F \wedge \cdots \wedge F}_{n\text{-times}}, \tag{6.52}$$

in $d = (2n + 1)$ dimensions. It has been shown however that in $d = 3$ dimensions [141] there are no analytic black hole solutions with this term.

In 5 dimensions, it was shown that in gauged supergravity, the introduction of (6.52) resulted in a doubly spinning analytic black hole solution [142]. This result in turn led to the discovery of a generalization of this solution to all higher odd dimensions [143] where a modified Chern-Simons term is used. This was conceived by Deshpande and Lunin, where the Lagrangian used is

$$L_{DL} = \frac{\lambda}{4\pi} A \wedge H \wedge K \quad (6.53)$$

where $H = dB$, $K = dC$, with B and C are two new non-dynamical $(n - 1)$ -form fields, and λ is an arbitrary dimensionless coupling constant; this Lagrangian can be applied in any $d = 2n + 1$ dimension. Coupling this to the Maxwell Lagrangian will produce the following field equations

$$\nabla_\mu F^{\mu\nu} = \mathcal{J}^\nu, \quad (6.54)$$

where the current is given by

$$\mathcal{J} = \lambda * (H \wedge K), \quad (6.55)$$

in terms of the (auxiliary) field strengths H and K . These quantities satisfy the two algebraic constraints

$$F \wedge H = 0, \quad F \wedge K = 0, \quad (6.56)$$

in any odd dimension [143].

If we focus on the BTZ metric, in coordination with the static 3-potential A , we can find the associated fields that will satisfy (6.55), (6.56) and provide the appropriate current to satisfy (6.54). In general, the algebraic equations (6.56) are a set of PDEs that need to be solved, and they need to provide the proper current in (6.54). One such set that we found to satisfy (6.56) is given by

$$B = \frac{t}{\lambda r}, \quad C = -\frac{2jQ}{r}. \quad (6.57)$$

In turn, these two fields will provide the appropriate current to satisfy the Maxwell equations. This current is given by

$$H = dB = \frac{dt}{\lambda r} - \frac{t dr}{\lambda r^2}, \quad K = dC = \frac{2jQ}{r^2} dr, \quad (6.58)$$

and thence

$$\begin{aligned}\mathcal{J} &= \lambda * (H \wedge K) = \frac{2jQ}{r^3} * (dt \wedge dr) \\ &= -\frac{2j^2Q}{r^4} dt - \frac{2jQ}{r^2} d\varphi,\end{aligned}\tag{6.59}$$

which, upon raising the index, yields the current

$$j^\mu = \frac{JQ}{r^4} d\varphi\tag{6.60}$$

Therefore, the BTZ black hole (6.50) may also be regarded as a solution to the DL theory, where the current is provided by the newly introduced fields B, C .

The thermodynamic quantities that are associated with the BTZ black hole are now given by

$$\begin{aligned}M &= \frac{m}{8}, \quad T = \frac{f'(r_+)}{4\pi} = \frac{r_+^4 - j^2\ell^2 - Q^2\ell^2r_+^2}{2\pi r_+^3\ell^2}, \\ S &= \frac{\pi r_+}{2}, \quad \Omega = \frac{j}{r_+^2}, \quad J = \frac{j}{4}, \\ \phi &= -\frac{Q}{2} \log(r/r_0), \\ V &= \pi r_+^2, \quad P = \frac{1}{8\pi\ell^2}, \quad \Pi_{r_0} = \frac{Q^2}{4r_0},\end{aligned}\tag{6.61}$$

and show that these obey the extended first law

$$\delta M = T\delta S + \phi\delta Q + \Omega\delta J + V\delta P + \Pi_{r_0}\delta r_0,\tag{6.62}$$

together with the standard Smarr relation

$$0 = TS - 2PV + \Omega J + \Pi_{r_0}r_0.\tag{6.63}$$

Interestingly, even though new fields are introduced into the action, they provide no modifications to the thermodynamic variables. The variables are all still characterized by only the Maxwell contributions.

6.2.2 NLED Generalizations

We are able to extend the DL theory into non-linear electrodynamics, by considering an action of the form

$$I_{\text{DL-NLE}} = \frac{1}{16\pi} \int d^d x \sqrt{-g} (R - 2\Lambda + 4\mathcal{L}) + \frac{\lambda}{4\pi} \int A \wedge H \wedge K, \quad (6.64)$$

where we have replaced the Maxwell Lagrangian, $\mathcal{L} = \mathcal{L}_{\text{M}} = -\frac{1}{2}\mathcal{S}$, with an ‘arbitrary’ function

$$\mathcal{L} = \mathcal{L}(\mathcal{S}) \quad (6.65)$$

of the electromagnetic invariant $\mathcal{S}$. This yields the Einstein equations with the modified electromagnetic energy-momentum tensor:

$$T^{\mu\nu} = -\frac{1}{4\pi} (2F^\mu{}_\alpha F^{\nu\alpha} \mathcal{L}_{\mathcal{S}} - \mathcal{L} g^{\mu\nu}), \quad (6.66)$$

together with the NLED field equations:

$$\nabla_\mu D^{\mu\nu} = \mathcal{J}^\nu, \quad \mathcal{J} = \lambda * (H \wedge K), \quad (6.67)$$

where we have denoted

$$D_{\mu\nu} \equiv -2\mathcal{L}_{\mathcal{S}} F_{\mu\nu}, \quad \mathcal{L}_{\mathcal{S}} \equiv \frac{\partial \mathcal{L}}{\partial \mathcal{S}}, \quad (6.68)$$

along the Deshpande–Lunin algebraic equations

$$F \wedge H = 0, \quad F \wedge K = 0, \quad (6.69)$$

which remain unchanged. As shown in the previous section, all NLED theories that are characterized by an electrostatic 3-potential possess the same current, therefore the fields B, C that solve the Maxwell case will also solve any generic NLED theory.

6.3 Discussion

This chapter was dedicated to the study of black holes solutions in general relativity with non-linear electrodynamics. We began with showing that there is a very generic solution

structure for static black holes coupled to restrictive NLED theories. The electromagnetic portion of the metric is entirely captured via the electromagnetic Lagrangian. This is of no surprise, but it does allow for the possibility of reverse engineering Lagrangians solutions that allows h (electromagnetic contribution to the metric) to follow certain criteria. For example, we could impose a solution of h to approach zero as $r \rightarrow \infty$, which would have the form of an inverse power law in r , and we could find the resultant Lagrangian.

We then explicitly derived the metric in ADM form the Maxwell sourced rotating 2+1-dimensional black hole. This was achieved through a coordinate boost and radial coordinate redefinition. This was significant in two ways. First, it appears to be the first explicit ADM form of the solution to appear in any literature or text. Second, the appearance of the Lambert function deduces a further rationale for why the differential equations arising from the field equations seem so formidable. However, it is also worth noting there has been 4-dimensional Einstein-Maxwell solutions derived from the field equations with a plane-symmetric space [144].

Lastly, we showed that there can be a new interpretation and understanding of the original rotating BTZ solution [129]. We show that the solution can be regarded as a self-reacting force-free solution, and that it is also a solution to the Deshpande–Lunin theory [143]. The latter provides a formal understanding of the current present in the BTZ solution, where a static Maxwell field sources any electromagnetic contributions in the spacetime. We then show that any NLED theory will possess the same current as the Maxwell case, as such, the additional fields introduced in Maxwell case to provide this current also provide the current for any NLED theory.

There are many naturally progressing questions that have emerged during the course of this work. Some elementary questions, that may serve as more pedagogical in nature, would be to investigate charged particle behaviour around these black holes which are sourced through non-linear electrodynamics, and Deshpande–Lunin fields. A further more fundamental question would be to calculate the thermodynamic variables using (6.34) and (6.37) directly.

In the next chapter we will investigate an extension of the aforementioned work by investigating 3DEGB black holes, showing that they may always be constructed once their

Einsteinian counterparts are known.

Chapter 7

3-Dimensional Einstein-Gauss-Bonnet Black Holes

Since we are considering $D = 3$ the electromagnetic invariants are solely constructed from the invariant $\mathcal{S} = F_{\mu\nu}F^{\mu\nu}$ where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is our field strength tensor with vector potential A_μ . This is due to the fact that the dual of F is a 1-form since our spacetime dimension is 3. We thus consider the Lagrangian density

$$\mathcal{L} = -\beta L(\mathcal{S}) \quad (7.1)$$

for nonlinear electrodynamics, where β is a coupling constant and $L(\mathcal{S})$ is an arbitrary function of the invariant $\mathcal{S}$. The generalized (source free) Maxwell equations will now read

$$\begin{aligned} \partial_{[\alpha} F_{\beta\gamma]} &= 0 \\ \nabla_\mu (L_S F^{\mu\nu}) &\equiv \frac{1}{\sqrt{-g}} \partial_\alpha (\sqrt{-g} L_S F^{\alpha\beta}) = 0 \end{aligned} \quad (7.2)$$

where $L_S = \partial L(\mathcal{S})/\partial \mathcal{S}$. The first equation is unchanged from general relativity since it relies on the anti-symmetric nature of F , whereas the second equation picks up the details of the nonlinear portion. With this form of our NLED Lagrangian we may easily write down the corresponding stress-energy tensor

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_M)}{\delta g^{\mu\nu}} \quad (7.3)$$

which will give us

$$T_{\mu\nu} = -4\beta L_{\mathcal{S}} F_{\delta\mu} F_{\nu}^{\delta} - \beta L(\mathcal{S}) g_{\mu\nu}. \quad (7.4)$$

7.0.1 The Generic 3-potential

We shall consider both static and stationary black hole solutions. It is possible to construct the 3-potential that gives rise to the most general field strength for such solutions regardless of the nonlinear electrodynamic theory in question. The most general stationary axially symmetric metric is of the form

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2(d\varphi + \omega(r)dt)^2 \quad (7.5)$$

and the most general 3-potential that can give all possible static electric/magnetic fields supported by this spacetime is

$$A = \Phi(r)dt + H(r)d\varphi + ct \, d\varphi \quad (7.6)$$

whose associated field strength is

$$F_{\mu\nu} = \begin{bmatrix} 0 & -\Phi' & c \\ \Phi' & 0 & H' \\ -c & -H' & 0 \end{bmatrix}. \quad (7.7)$$

The field strength contains two electric fields and one magnetic field, although not all will be present in a given solution. These fields will be restricted by the field equations and if the spacetime is static ($\omega(r) = 0$) we will be even further restricted to only one possible field. These restrictions are referred to as the Ricci circularity conditions [145, 146, 147]; we will elaborate on them in the following sections.

7.1 Static Black Holes

Before we consider rotating solutions we carefully review the static case. We will begin with the most general such static metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2 d\varphi^2 \quad (7.8)$$

which contains two arbitrary metric functions, with coordinate identifications $(t, \varphi) \rightarrow (t, \varphi + 2\pi)$.

We first look at the scalar field equation (3.27), which simplifies to

$$\begin{aligned} & -4g\phi'f(\phi'r-1)f'' - 6\phi'f(\phi'r-1)(-2\phi'f+f')g' + 2g\phi'(\phi'r-1)f'^2 \\ & + 4g(f(6\phi'^2fr - 2\phi'f'r - 4\phi'f + f')\phi'' + f\phi'^2(\phi'r-2)f' + 2f^2\phi'^3) = 0. \end{aligned} \quad (7.9)$$

assuming the scalar field is only dependent on the radial coordinate r . What is immediately apparent is that the first 3 terms are zero if $\phi = \ln(r/r_0)$, where r_0 is an arbitrary length scale. What is also interesting - although not immediately apparent - is that the 4th term is also identically zero with this choice of ϕ . We shall use this form for ϕ , noting that an alternate (and much more complicated) solution for ϕ in terms of $f(r)$ and $g(r)$ exists.

For the 3-potential displayed in the previous section

$$A = \Phi(r)dt + H(r)d\varphi + ct \, d\varphi$$

the field strength tensor that follows is given by

$$F_{\mu\nu} = \begin{bmatrix} 0 & -\Phi' & c \\ \Phi' & 0 & H' \\ -c & -H' & 0 \end{bmatrix} \quad F^{\mu\nu} = \begin{bmatrix} 0 & \frac{g\Phi'}{f} & -\frac{c}{fr^2} \\ -\frac{g\Phi'}{f} & 0 & \frac{gH'}{r^2} \\ \frac{c}{fr^2} & -\frac{gH'}{r^2} & 0 \end{bmatrix} \quad (7.10)$$

yielding

$$\mathcal{S} = -\frac{2\Phi'^2g}{f} - \frac{2c^2}{fr^2} + \frac{2H'^2g}{r^2}. \quad (7.11)$$

for the invariant $\mathcal{S}$. For the generalized Maxwell equations (7.2) there are two nonzero components: one for t , and one for φ . The t component is

$$\frac{\Phi''}{\Phi'} = -\frac{\partial_r(L_S)}{L_S} - \frac{g'}{2g} + \frac{f'}{2f} - \frac{1}{r} \quad (7.12)$$

and a simple integration yields the following condition on Φ'

$$\Phi'^2 = \frac{Q^2f}{gr^2L_S^2} \quad (7.13)$$

where Q is an arbitrary constant of integration. In the same manner we can retrieve the φ component of (7.2), obtaining

$$\frac{H''}{H'} = -\frac{f'}{2f} - \frac{g'}{2g} - \frac{\partial_r(L_S)}{L_S} + \frac{1}{r} \Rightarrow H'^2 = \frac{r^2 c_2^2}{fgL_S} \quad (7.14)$$

where c_2 is a constant of integration. With these functional forms of Φ and H we can recast the field strength tensor as

$$F_{\mu\nu} = \begin{bmatrix} 0 & -\frac{\sqrt{f}Q}{\sqrt{g}rL_S} & c \\ \frac{\sqrt{f}Q}{\sqrt{g}rL_S} & 0 & \frac{rc_2}{\sqrt{f}\sqrt{g}L_S} \\ -c & -\frac{rc_2}{\sqrt{f}\sqrt{g}L_S} & 0 \end{bmatrix} \quad F^{\mu\nu} = \begin{bmatrix} 0 & \frac{\sqrt{g}Q}{\sqrt{f}rL_S} & -\frac{c}{r^2f} \\ -\frac{\sqrt{g}Q}{\sqrt{f}rL_S} & 0 & \frac{\sqrt{g}c_2}{r\sqrt{f}L_S} \\ \frac{c}{r^2f} & -\frac{\sqrt{g}c_2}{r\sqrt{f}L_S} & 0 \end{bmatrix} \quad (7.15)$$

which is agreement with [147, 145, 146]

The diagonal components of the gravitational field equations become

$$\begin{aligned} \frac{(r^3 + 2r\alpha g)g' + 2\Lambda r^4 - 2g^2\alpha}{2r^4} &= -\frac{\beta(Lfr^2L_S + 4c^2L_S^2 + 4Q^2f)}{L_Sr^2f} \\ \frac{2\Lambda fr^4 + 2g^2f'\alpha r + gf'r^3 - 2g^2f\alpha}{2fr^4} &= -\frac{\beta(Lfr^2L_S - 4c_2^2r^2L_S + 4Q^2f)}{r^2L_Sf} \\ \frac{2r^2fg(r^2 + 2\alpha g)f'' + (-r^4g - 2r^2\alpha g^2)f'^2 + ((r^3 + 6r\alpha g)g' - 8g^2\alpha)rf f' + 4f^2(\Lambda r^4 - 2gg'\alpha r + 3g^2\alpha)}{4f^2r^4} &= -\frac{\beta(Lfr^2L_S - 4c_2^2r^2L_S + 4Q^2f)}{r^2L_Sf} \end{aligned} \quad (7.16)$$

while the off-diagonal components are

$$\begin{aligned} 0 &= 4\beta cc_2 \\ 0 &= 4\beta Qc_2 \\ 0 &= 4\beta Qc. \end{aligned} \quad (7.17)$$

Satisfying these requirements are what are referred to as the Ricci circularity conditions [147, 146, 145]. Since $\beta = 0$ is the vacuum case, we disregard this possibility. In that case these conditions can be satisfied by setting any two of $\{Q, c, c_2\}$ to zero.

If $c = 0$ then either Q or c_2 must be set to zero. Setting $c_2 = 0$ corresponds to a purely radial electric field solution, while $Q = 0$ will purely be a magnetic solution. For the case

of $Q = c_2 = 0$ this will again be an electric field solution, however along the angular coordinate φ . We will restrict our focus to the purely radial electric field solutions for this in what follows. We first revisit the vacuum case.

7.1.1 Vacuum Solution

The vacuum solution is characterized by setting $\beta = 0$, which results in the following differential equations on f and g

$$\frac{(r^3 + 2r\alpha g) g' + 2\Lambda r^4 - 2g^2\alpha}{2r^4} = 0 \quad (7.18)$$

$$\frac{g^2(-f + f'r)\alpha}{fr^4} + \Lambda + \frac{f'g}{2rf} = 0 \quad (7.19)$$

$$\begin{aligned} & \frac{2r^2fg(r^2 + 2\alpha g)f'' + (-r^4g - 2r^2\alpha g^2)f'^2}{4f^2r^4} \\ & + \frac{((r^3 + 6r\alpha g)g' - 8g^2\alpha)rf f' + 4f^2(\Lambda r^4 - 2gg'\alpha r + 3g^2\alpha)}{4f^2r^4} = 0 \end{aligned} \quad (7.20)$$

If we subtract the first two equations we get

$$\frac{(g'f - gf')(r^2 + 2\alpha g)}{2r^3f} = 0 \quad (7.21)$$

and so

$$g'f - gf' = 0$$

which states that $g \propto f$ up to a constant. Now this constant can always appear in f and can thus be absorbed into the definition of the t coordinate; hence without loss of generality $f \equiv g$. In doing this the first two field equations become identical and read

$$(r^3 + 2r\alpha g) g' + 2\Lambda r^4 - 2g^2\alpha = 0 \quad (7.22)$$

while the angular component is now

$$(r^4 + 2r^2\alpha g) g'' + 2\Lambda r^4 + 2g'^2\alpha r^2 - 8gg'\alpha r + 6g^2\alpha = 0. \quad (7.23)$$

These two equations are redundant, differentiation of (7.22) will match (7.23), and as such we only focus on the former. The solution to (7.22) is given by

$$g = -\frac{r^2}{2\alpha} \pm \frac{r\sqrt{-4\Lambda\alpha r^2 - 4M\alpha + r^2}}{2\alpha} \quad (7.24)$$

where the positive root has the following small α limit

$$\lim_{\alpha \rightarrow 0} g_+ \approx -\Lambda r^2 - M + O(\alpha) \quad (7.25)$$

where M is a constant of integration interpreted as the black hole's mass. This small alpha limit of g_+ corresponds to the Einstein solution, whereas g_- has a divergent small α limit. This allows us to finalize the solution into the form

$$g = -\frac{r^2}{2\alpha} + \frac{r^2}{2\alpha} \sqrt{1 + \frac{4\alpha}{r^2} F_E} \quad (7.26)$$

$$F_E = -\Lambda r^2 - M.$$

This solution is known [148, 149]; its importance lies in the function form of g , namely it is characterized by the Einstein solution F_E .

This is a guide for obtaining additional solutions in non-linear electrodynamics, as we shall see.

7.1.2 Radial Electric Solutions

We now wish to discuss the solutions that correspond to purely radial electric fields; in the context of (7.16),(7.17) this means $c = c_2 = 0$ as previously mentioned. Using (7.8) the first two diagonal field equations read

$$\begin{aligned} \frac{(r^3 + 2r\alpha g)g' + 2\Lambda r^4 - 2g^2\alpha}{2r^4} &= -\frac{\beta(Lr^2L_S + 4Q^2)}{L_S r^2} \\ \frac{2\Lambda f r^4 + 2f'g^2\alpha r + g f' r^3 - 2fg^2\alpha}{2f r^4} &= -\frac{\beta(Lr^2L_S + 4Q^2)}{L_S r^2}. \end{aligned} \quad (7.27)$$

Subtracting these two equations again implies (by the same reasoning as the vacuum case) that $f \equiv g$, regardless of the nonlinear electrodynamic theory. To solve for the metric

function g we exploit the form

$$g = -\frac{r^2}{2\alpha} + \frac{r^2}{2\alpha} \sqrt{1 + \frac{4\alpha}{r^2} F_E} \quad (7.28)$$

that the solution had in the vacuum case, where F_E is now an unknown function of r . We have chosen the positive branch for the appropriate small alpha limit. If we introduce this form of g into the field equations we arrive at two differential equations: one from either tt or rr (as they are now equivalent) and $\varphi\varphi$. These are

$$\begin{aligned} \frac{2\Lambda r + F'_E}{2r} &= -\frac{\beta (Lr^2 L_S + 4c_l^2)}{L_S r^2} \\ \Lambda + \frac{F''_E}{2} &= -L\beta. \end{aligned} \quad (7.29)$$

and are *identically* the equations of the $\alpha = 0$ Einstein case. In other words, any solution in the Einstein case yields a 3DEGB solution via (7.28) as long as the metric is of the form (7.8). We consider a few examples before discussing the rotating solutions.

7.1.2.1 Maxwell Electrodynamics

For the Maxwell case we set $L(\mathcal{S}) = \mathcal{S}$ (and so $L_S = 1$). The differential equation for Φ is simply

$$\Phi' = \frac{Q}{r} \quad (7.30)$$

and so

$$\Phi = -Q \ln(r/r_1)$$

where r_1 is some constant of integration; note that it is not related to the constant of integration for ϕ . From this we get

$$\mathcal{S} = -\frac{2Q^2}{r^2}. \quad (7.31)$$

in turn giving the final form

$$\begin{aligned} F'_E &= -\frac{4Q^2\beta}{r} - 2\Lambda r \\ F''_E &= \frac{4Q^2\beta}{r^2} - 2\Lambda \end{aligned} \quad (7.32)$$

for the field equations. These are redundant, leaving us with the final solution

$$F_E = -\Lambda r^2 - 4\beta Q^2 \ln(r) + c_3. \quad (7.33)$$

The constant of integration c_3 must be broken up into two components: one associated with the mass term $-M$, the other to ensure there is a dimensionless argument inside of the logarithm, so that

$$F_E = -\Lambda r^2 - 4\beta Q^2 \ln(r/r_2) - M$$

where r_2 is another constant of integration that not the same as r_1 . This ‘split’ of the constant has been used to argue that a mass term for the charged 2+1 black hole can be eliminated [134]. The problem with this assertion is that if $\beta = 0$ the black hole has no mass.

7.1.2.2 Born-Infeld

For Born-Infeld electrodynamics

$$L(\mathcal{S}) = b^2 \left(\sqrt{1 + \frac{2\mathcal{S}}{b^2}} - 1 \right) \quad (7.34)$$

where for large $b = 1/z$

$$L(\mathcal{S}) = \mathcal{S} - \frac{1}{2}\mathcal{S}^2 z^2 + \mathcal{O}(z^4) \quad (7.35)$$

indicating that in the limit that $b = \infty$ we recover Maxwell electrodynamics. Returning to the t component of the generalized Maxwell equation (7.2)

$$\Phi' = \frac{Q}{rL_{\mathcal{S}}} \quad (7.36)$$

and nothing that

$$L_{\mathcal{S}} = \frac{b}{\sqrt{b^2 + 2\mathcal{S}}} \quad (7.37)$$

for the Born-Infeld theory, where

$$\mathcal{S} = -2\Phi'^2$$

we find

$$\frac{d}{dr}\Phi(r) = \frac{Q\sqrt{b^2 - 4\left(\frac{d}{dr}\Phi(r)\right)^2}}{rb} \quad (7.38)$$

as the final differential equation for Φ . We can simplify this to be

$$\left(\frac{d}{dr}\Phi(r)\right)^2 r^2 b^2 + 4\left(\frac{d}{dr}\Phi(r)\right)^2 Q^2 = b^2 Q^2 \quad (7.39)$$

and thus

$$\Phi' = \pm \frac{Qb}{\sqrt{r^2 b^2 + 4Q^2}} \quad (7.40)$$

with

$$\Phi' = \pm \frac{Q}{r} \quad (7.41)$$

in the large b limit, yielding the Maxwell case. The solution is

$$\Phi = -Q \ln \left(r + \sqrt{r^2 + \frac{4Q^2}{b^2}} \right) + C = -Q \ln \left(\frac{r + \sqrt{r^2 + \frac{4Q^2}{b^2}}}{r_1 + \sqrt{r_1^2 + \frac{4Q^2}{b^2}}} \right) \quad (7.42)$$

choosing the minus sign without loss of generality. The constant of integration C has been chosen so that $\Phi = 0$ at $r = r_1$.

As with the Maxwell case, the remaining equations are redundant. We find

$$F_E' = -\frac{2b^2 r^2 \beta}{\sqrt{r^2 + \frac{4Q^2}{b^2}}} + 2b^2 \beta r - \frac{8Q^2 \beta}{\sqrt{r^2 + \frac{4Q^2}{b^2}}} - 2\Lambda r \quad (7.43)$$

which simply integrated yields

$$F_E = \left(\beta r^2 - \beta r \sqrt{r^2 + \frac{4Q^2}{b^2}} \right) b^2 - 4Q^2 \beta \ln \left(r + \sqrt{r^2 + \frac{4Q^2}{b^2}} \right) - \Lambda r^2 + \mathbf{C} . \quad (7.44)$$

We can automatically deduce that $\mathbf{C}$ must have a term of the form $+4Q^2 \beta \ln \left(r_2 + \sqrt{r_2^2 + \frac{4Q^2}{b^2}} \right)$ where r_2 is an arbitrary constant. Hence we have

$$F_E = \left(r - \sqrt{r^2 + \frac{4Q^2}{b^2}} \right) \beta r b^2 - 4Q^2 \beta \ln \left(\frac{r + \sqrt{r^2 + \frac{4Q^2}{b^2}}}{r_2 + \sqrt{r_2^2 + \frac{4Q^2}{b^2}}} \right) - \Lambda r^2 + \tilde{C} \quad (7.45)$$

and we choose $\tilde{C}$ so that

$$F_E = \left(r - \sqrt{r^2 + \frac{4Q^2}{b^2}} \right) \beta r b^2 - 4Q^2 \beta \ln \left(\frac{r + \sqrt{r^2 + \frac{4Q^2}{b^2}}}{r_2 + \sqrt{r_2^2 + \frac{4Q^2}{b^2}}} \right) - \Lambda r^2 - M - \left(r_2 - \sqrt{r_2^2 + \frac{4Q^2}{b^2}} \right) \beta r_2 b^2 \quad (7.46)$$

which has no electromagnetic contribution at $r = r_2$.

7.1.2.3 Conformal Electrodynamics

For conformal electrodynamics

$$L(\mathcal{S}) = -\frac{2}{3}(-\mathcal{S})^{3/4} \quad (7.47)$$

implying $L_{\mathcal{S}} = \frac{1}{2(-\mathcal{S})^{1/4}}$. The differential equation for Φ is

$$\Phi'^2 = \frac{Q\Phi'}{r^2} \quad (7.48)$$

where Q is a constant of integration and the solution is given by

$$\Phi(r) = -\frac{Q}{r} + c_4 \quad (7.49)$$

where we may set $c_4 = 0$ without loss of generality. The differential equations appearing in the field equations are again redundant for this NLED and are

$$F_E' = -\frac{2\beta 2^{3/4} Q^{3/2}}{3r^2} - 2\Lambda r \quad (7.50)$$

whose solution through a simple integration is

$$F_E = -\Lambda r^2 + \frac{2\beta 2^{3/4} Q^{3/2}}{3r} - M \quad (7.51)$$

where we have introduced M as a constant of integration. The unique fact about this NLED is that there is no need to break up the constant of integration in F_E so as to create a dimensionless argument for the function coupled to Q .

7.1.2.4 Generalized Power Maxwell

Generalized power-Maxwell is characterized by the following Lagrangian functional

$$L(S) = \sum_{n=1}^N a_n (S)^n$$

which will give us

$$L_S = \sum_{n=1}^N n S^{n-1} a_n.$$

For the generalized Maxwell equation

$$\Phi'^2 = \frac{Q^2}{r^2 L_S^2} \quad (7.52)$$

it is much easier for the purposes of this functional to write it as

$$L_S \Phi' = -\frac{Q}{r}. \quad (7.53)$$

and noting that $S = -2\Phi'^2$ this becomes

$$\left(\sum_{n=1}^N (-1)^n n 2^{n-1} a_n \Phi'^{-1+2n} \right) = \frac{Q}{r}. \quad (7.54)$$

Now since the physical electric field is given by $E(r) \propto \Phi'$ we wish to find solutions that are divergence free in the presence of this functional. One simple way to do this is by considering the following ansatz

$$\Phi' = \sum_{i=1}^{\infty} \frac{b_i}{r^i} \quad (7.55)$$

and as such Φ is found through an easy integration. Setting $a_1 = 1$ without any loss of generality, the first few constants b_n are

$$\begin{aligned}
b_1 &= -Q \\
b_3 &= -4Q^3 a_2 \\
b_5 &= -12Q^5 (4a_2^2 - a_3) \\
b_7 &= -32Q^7 (24a_2^3 - 12a_2 a_3 + a_4) \\
b_9 &= -80Q^9 (176a_2^4 - 132a_2^2 a_3 + 16a_2 a_4 + 9a_3^2 - a_5) \\
&\vdots \\
b_{2n} &= 0 \quad \forall \quad n
\end{aligned} \tag{7.56}$$

To determine the metric function F_E we write the field equation as

$$\frac{4\beta Q^2}{r} = \frac{\left(2r\beta \left(\sum_{n=1}^N a_n \left(-2 \left(\frac{d}{dr}\Phi(r)\right)^2\right)^n\right) + 2r\Lambda + \frac{d}{dr}f(r)\right) \left(\sum_{n=1}^N n \left(-2 \left(\frac{d}{dr}\Phi(r)\right)^2\right)^n a_n\right)}{4 \left(\frac{d}{dr}\Phi(r)\right)^2} \tag{7.57}$$

and using

$$\Phi' = \sum_{i=1}^{\infty} \frac{b_i}{r^i}$$

in conjunction with

$$f(r) = -M - \Lambda r^2 - c_1 \ln(r) - \left(\sum_{n=1}^{\infty} \frac{c_{n+1}}{r^n}\right) \tag{7.58}$$

we obtain the following coefficients for c_n :

$$\begin{aligned}
c_1 &= 4\beta Q^2 \\
c_2 &= 0 \\
c_3 &= -4Q^4 \beta a_2 \\
c_4 &= 0 \\
c_5 &= -112Q^6 \beta a_2^2 + 28Q^6 \beta a_3 - 2\beta Q (-48Q^5 a_2^2 + 12Q^5 a_3) \\
&\vdots \\
c_{2n} &= 0 \quad \forall \quad n
\end{aligned} \tag{7.59}$$

which can be further simplified into

$$c_i = \beta Q b_i \quad i \geq 3. \quad (7.60)$$

7.2 Stationary Black Holes

7.2.1 Einstein Gravity

We will quickly review the procedure for obtaining the rotating BTZ solution [129] in Einstein gravity as it is rather elegant. To do this we begin with following line element ansatz

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{g(r)} + r^2(d\varphi + \omega(r)dt)^2 \quad (7.61)$$

where we have 3 unknown metric functions, now with coordinate identifications $(t, \varphi) \rightarrow (t, \varphi + 2\pi)$. For the field equations the $t\varphi$ component can be written as

$$\frac{\omega''}{\omega'} = \frac{f'}{2f} - \frac{g'}{2g} - \frac{3}{r} \quad (7.62)$$

which has a differential solution of the form

$$\omega' = \frac{\sqrt{f}c_2}{r^3\sqrt{g}}. \quad (7.63)$$

Inserting this back into the field equations will yield the simplified tt component and solution:

$$\frac{4\Lambda r^4 + 2g'r^3 + c_2^2}{4r^4} = 0 \quad (7.64)$$

$$g = -r^2\Lambda + \frac{c_2^2}{4r^2} + c_3. \quad (7.65)$$

The remaining field equation components can be summed up by the following differential equation

$$\frac{f'}{f} = \frac{8\Lambda r^4 + 2c_2^2}{4\Lambda r^5 - 4c_3r^3 - c_2^2r} \quad (7.66)$$

with straight forward solution

$$f = c_4 \left(4r^2 \Lambda - 4c_3 - \frac{c_2^2}{r^2} \right). \quad (7.67)$$

Since the constant of integration c_4 can be absorbed into the time coordinate, without loss of generality we may choose it to be $c_4 = -1/4$, leaving us with

$$f = -\Lambda r^2 + c_3 + \frac{c_2^2}{4r^2}. \quad (7.68)$$

Therefore in this case $f \equiv g$ and we can consider the constants of integration in the following way

$$g \equiv f = -\Lambda r^2 - M + \frac{J^2}{4r^2} \quad (7.69)$$

for mass and angular momentum, respectively. This leaves us a simple solution for ω

$$\omega = -\frac{J}{2r^2} + c_3 \quad (7.70)$$

where we can set the constant $c_3 = 0$ without so as to obtain zero angular momentum at infinity.

7.2.1.1 Boosted Vacuum solution

The second way to derive the rotating solution instead of the “brute force” approach of the previous subsection is to apply coordinate boosts on the static metric

$$ds^2 = - \left(-M + \frac{r^2}{l^2} \right) dt^2 + \frac{dr^2}{-M + \frac{r^2}{l^2}} + r^2 d\tilde{\varphi}^2 \quad (7.71)$$

with identifications $(\tilde{t}, \tilde{\varphi}) \rightarrow (\tilde{t}, \tilde{\varphi} + 2\pi)$. to induce a differential rotation [150, 145, 147, 134].

The coordinate boosts that can be used in the Einstein case are:

$$\tilde{t} \rightarrow \Xi t - a\varphi, \quad \tilde{\varphi} \rightarrow \frac{at}{l^2} - \Xi\varphi, \quad \Xi^2 = 1 + \frac{a^2}{l^2} \quad (7.72)$$

where l represents the AdS length through $-\Lambda = 1/l^2$. Applying (7.72) yields after some simplification

$$ds^2 = \left(\frac{r^2}{l^2} + \frac{(a^2 + l^2) M}{l^2} \right) dt^2 + \frac{dr^2}{-M + \frac{r^2}{l^2}} + (Ma^2 + r^2) d\varphi^2 - 2a\Xi M dt d\varphi. \quad (7.73)$$

One difficulty is how to interpret the line element (7.73). To interpret this metric as a rotating black hole, we now impose the identifications $(t, \varphi) \rightarrow (t, \varphi + 2\pi)$. In the large r limit we have

$$ds^2 \approx \left(\frac{r^2}{l^2}\right) dt^2 + \frac{dr^2}{\frac{r^2}{l^2}} + (r^2) d\varphi^2 \quad (7.74)$$

which suggests that the limit is Anti-de Sitter space, but also for a circle at large r the circumference is given by $2\pi r$. However at finite r we have

$$ds^2 = (Ma^2 + r^2) d\varphi^2$$

for fixed r and t , which does not yield $2\pi r$ for the circumference of a circle. By writing

$$\hat{r}^2 = (Ma^2 + r^2) \quad (7.75)$$

which is simply inverted to give

$$r(\hat{r}) = \sqrt{\hat{r}^2 - Ma^2}$$

we then obtain

$$ds^2 = -\left(\frac{\hat{r}^2}{l^2} - \frac{(2a^2 + l^2)M}{l^2}\right) dt^2 + \frac{d\hat{r}^2}{\frac{\hat{r}^2}{l^2} - \frac{(2a^2 + l^2)M}{l^2} + \frac{a^2 \Xi^2 M^2}{\hat{r}^2}} + \hat{r}^2 d\varphi^2 - 2a\Xi M dt d\varphi \quad (7.76)$$

$$= -\left(\frac{\hat{r}^2}{l^2} - m + \frac{J^2}{4\hat{r}^2}\right) dt^2 + \frac{d\hat{r}^2}{\frac{\hat{r}^2}{l^2} - m + \frac{J^2}{4\hat{r}^2}} + \hat{r}^2 \left(d\varphi - \frac{J}{2\hat{r}^2} dt\right)^2 \quad (7.77)$$

with

$$m = \frac{(2a^2 + l^2)M}{l^2} \quad J = 2a\Xi M.$$

So with the boost method it is rather straightforward to put the metric into the ADM form and as such in this coordinate chart we have that for any fixed value of r and t the circumference of a circle is 2π .

7.2.2 Gauss-Bonnet

Searching for rotating solutions from first principles appears to be a hopeless task. We can no longer use the solution $\phi \propto \ln(r)$ as the scalar field equation is not satisfied with the use

of (7.5), leaving us with an over-constrained system for $f(r), g(r), \omega(r)$. Therefore, without an a priori solution for ϕ we are left with solving 4 coupled non-linear ordinary differential equations for $\phi(r), f(r), g(r), \omega(r)$ which has no apparent closed form solutions in vacuum or *any* NLED scenario. We can however resort to “boosting” the static solutions to obtain the rotating cases, identical to the Einstein scenario. For the Gauss-Bonnet case these are

$$\begin{aligned} t &\rightarrow \Xi_{\text{eff}} t - a\varphi, \quad \varphi \rightarrow \frac{at}{\ell^2} - \Xi_{\text{eff}}\varphi, \quad \Xi_{\text{eff}}^2 = 1 + \frac{a^2}{\ell^2} \\ \ell &= \sqrt{\frac{2\alpha}{\sqrt{X_\alpha} - 1}}, \quad X_\alpha = 1 + \frac{4\alpha}{l^2}. \end{aligned} \quad (7.78)$$

The reasoning for the modified boosts is due to the fact that the nonlinear curvature terms alter the AdS length scale of the theory.

7.2.2.1 Generic Case

For the fully generic case we consider a metric of the form

$$\begin{aligned} ds^2 &= -f_{GB}(r)dt^2 + \frac{dr^2}{f_{GB}(r)} + r^2 d\varphi^2 \\ f_{GB}(r) &= -\frac{r^2}{2\alpha} + \frac{r^2}{2\alpha} \sqrt{1 + \frac{4\alpha}{r^2} F_E} \end{aligned} \quad (7.79)$$

and apply the boost to obtain

$$\begin{aligned} ds^2 &= \frac{f_{GB}(r)r^2(-\ell\Xi_{\text{eff}} + a)^2(\ell\Xi_{\text{eff}} + a)^2 dt^2}{\ell^4(-\Xi_{\text{eff}}^2 r^2 + a^2 f_{GB}(r))} + \frac{dr^2}{f_{GB}(r)} \\ &\quad + (-a^2 f_{GB}(r) + \Xi_{\text{eff}}^2 r^2) \left(d\varphi - \frac{a\Xi_{\text{eff}}(-f_{GB}(r)\ell^2 + r^2) dt}{\ell^2(-a^2 f_{GB}(r) + \Xi_{\text{eff}}^2 r^2)} \right)^2 \end{aligned} \quad (7.80)$$

The radial coordinate transformation must take the form

$$-a^2 f_{GB}(r) + \Xi_{\text{eff}}^2 r^2 = R^2$$

and thus

$$-a^2 f_{GB}(G(R)) + \Xi_{\text{eff}}^2 G(R)^2 \equiv R^2$$

so we can rewrite the metric as

$$\begin{aligned} ds^2 = & \frac{f_{GB}(G)G^2(-\ell\Xi_{eff}+a)^2(\ell\Xi_{eff}+a)^2 d(T)^2}{-\ell^4 R^2} + \frac{G'^2 d(R)^2}{f_{GB}(G)} \\ & + R^2 \left(d(\Phi) + \frac{a\Xi_{eff}(f_{GB}(G)\ell^2 - G^2) d(T)}{\ell^2 R^2} \right)^2. \end{aligned} \quad (7.81)$$

To find $G(R)$ we must solve

$$\begin{aligned} & \frac{(a^2 - \alpha) \left(a^2 \sqrt{\frac{l^2 + 4\alpha}{l^2}} - a^2 + \alpha \right) r^4}{\alpha^2} \\ & + \frac{r^2 \left(\sqrt{\frac{l^2 + 4\alpha}{l^2}} R^2 a^2 + a^4 E(r) - a^4 m - 2R^2 a^2 + 2R^2 \alpha \right)}{\alpha} - R^4 = 0 \end{aligned} \quad (7.82)$$

for $r(R)$. Here we have explicitly used the form of f_{GB} and the Einstein general solution for a radial electric field.

We can now consider a few different cases, beginning with the vacuum case. In this case we have $E(r) = 0$ and thus the simple biquadratic (7.82) yields

$$r^2 = \frac{-a^4 m l + \sqrt{l^2 + 4\alpha} R^2 a^2 + 2R^2 \alpha l \pm a^2 \sqrt{-2l\sqrt{l^2 + 4\alpha} R^2 a^2 m + (m^2 a^4 + R^4 - 4R^2 \alpha m) l^2 + 4R^4 \alpha}}{2\sqrt{l^2 + 4\alpha} a^2 + 2\alpha l} \quad (7.83)$$

whose small alpha limits are

$$\begin{aligned} \lim_{\alpha \rightarrow 0} r^2_+ & \approx -a^2 m + R^2 + O(\alpha) \\ \lim_{\alpha \rightarrow 0} r^2_- & \approx \frac{R^4}{a^2 (-a^2 m + R^2)} \alpha + O(\alpha^2) \end{aligned} \quad (7.84)$$

indicating that the positive branch gives the appropriate limit to Einstein gravity. Hence

$$r^2 = \frac{-a^4 m l + \sqrt{l^2 + 4\alpha} R^2 a^2 + 2R^2 \alpha l + a^2 \sqrt{-2l\sqrt{l^2 + 4\alpha} R^2 a^2 m + (m^2 a^4 + R^4 - 4R^2 \alpha m) l^2 + 4R^4 \alpha}}{2\sqrt{l^2 + 4\alpha} a^2 + 2\alpha l} \quad (7.85)$$

as our change of variables.

For conformal electrodynamics we have

$$\left(a^2\sqrt{\frac{l^2+4\alpha}{l^2}}+\alpha\right)r^4+\left(-\sqrt{\frac{l^2+4\alpha}{l^2}}R^2a^2+a^4m-2R^2\alpha\right)r^2-\frac{2\beta Q^{3/2}2^{3/4}ra^4}{3}+R^4\alpha \quad (7.86)$$

which is a depressed quartic. So in principle a solution may be found, which is algebraically quite complicated. This generic procedure will work for *any* NLED once the Einstein case is known, although an explicit solution in the ADM form may not be practical for phenomenological purposes.

7.3 Discussion

This chapter was meant to serve as a continuation of Chapter 6. Initially, we show that the 3DEGB theory introduces non-linear differential equations in the metric functions for vacuum spacetime. While not a major advancement on its own, it does show that even for the simplest case (static spherically symmetric vacuum) the equations of motion are drastically more difficult than their general relativity counterparts. We further show that even with this feature all static, purely radial, NLED solutions in 3DEGB can be solely written in terms of their general relativity counterparts. This was something that was conjectured in [151] but not explicitly proven.

We further derive a new charged black hole solution that is sourced via an infinite tower of Maxwell invariants referred to as power-Maxwell. This solution was derived in the context of the general relativity equations of motion, hence it serves as a solution in both theories from the proceeding paragraph. We lastly discuss the generation of rotation solutions through a coordinate boost. Like the situation discussed in the general relativity section, the key feature lies in identifying and solving a coordinate transformation in the radial variable. In the 3DEGB case, however, the non-linear features of the static solution propagate extensively into the coordinate transformation – most noticeably with the square root terms. This feature makes the conversion into the ADM metric algebraically tedious, and the solutions would be of best, and easiest, use in the Kerr-like coordinates.

The thermodynamic properties of 3DEGB black holes is an almost entirely uncharted

territory, especially with rotating solutions. While vacuum solutions have been investigated [151], the charged cases have never been fully investigated. This would serve as the most pertinent work to be considered, particularly in the cases with NLED coupling. The coupling constant α in the 3DEGB theory serves as an additional variable that may be altered when investigating phase behaviour. It is not known whether this parameter will indeed induce novel phase behaviour, but it serves as a promising avenue to consider.

Part III

Cosmology

Unlike the study of black holes, cosmology did not need a gravitational theory to stir the interest of individuals. Even before modern or formal science was constructed, humanity would glance into the night sky and ponder where we are, and how did we get here. Trying to understand these questions about the universe at its most fundamental level encapsulates the field of cosmology. With the broad spectrum of curiosity that encapsulates humanity's quest for understanding, the origin debate about cosmology came from a personal and somewhat teleological level. The advent of modern science and observational information allowed the nature of personal thoughts to take a backseat and propel cosmology into a truly formal scientific field with empirical facts and consequences.

Like the study of black holes, it took general relativity to provide us with a framework to begin properly understanding the universe. The understanding of modern cosmology as we currently know it would not be complete without the advent of general relativity. With the tools of general relativity, it was now possible to construct and interpret cosmological models/solutions to the field equations of general relativity; it also allowed the movement from a static universe into a dynamical time dependent one. Some of the pioneering work on the subject came from de Sitter, Friedmann, and Lemaître [152, 153, 154, 155]. While the discussion regarding a static vs. dynamic universe was hinged on theoretical grounds, the discovery by Hubble [156], through techniques by Leavitt [157], that galaxies were receding quicker from earth the further they were away indicated an expanding universe.

The junction of theory and observation allowed the field of physical cosmology to be born, conceiving physical models of the universe to predict/explain cosmological phenomena. The first and most well-known models studied are homogeneous and isotropic models, which claim no favourable direction or points in the universe: on large scales the universe appears the same to every observer in every direction. Accomplished by Walker and Robertson [158, 159], the most general metric that can admit these two properties is given by the line element

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right) \quad (\text{III.1})$$

where $a(t)$ is referred to as the scale factor, $k = -1, 0, 1$ and parametrizes the spatial curvature as either negative, flat or positive restively. The name given to the spacetime (III.1) varies depending on who you are speaking to, but we will refer to it as the FLRW

metric – or line element. The importance of (III.1) is that it is determined entirely on geometric grounds. The only use of general relativity is in the determination of a solution for the scale factor which comes from solving the field equations.

Currently, the big bang model of cosmology is based upon: the notion of an approximately homogeneous and isotropic universe, general relativity is the theory of gravity, and the matter content of the universe can be modelled by a perfect fluid. These assumptions undergird the Λ –CDM model of the “standard model of cosmology”. This model assumes: a cosmological constant in the field equations that is associated with dark energy, providing the currently observed accelerated expansion of the universe; cold dark matter, which is associated with structure formation and observations of galaxy rotation curves; ordinary matter. Additionally, this model assumes that there was a period of cosmic inflation [160] in the very early universe. The introduction of inflation provides a framework for rectifying two rather fundamental problems in big bang cosmology: the horizon and flatness problems. Additionally, inflation converts microscopic quantum fluctuations into macroscopic density perturbations, providing the initial conditions for structure formation in the universe.

The horizon problem is most evident in the fact that the temperature of the CMB (cosmic microwave background) appears very isotropic. This on its own is not problematic. The problem arises with the fact that assuming ordinary matter, both relativistic and non-relativistic, regions of the CMB are out of causal contact, and never were. How could the apparent thermodynamic equilibrium be communicated across patches (or regions) of the CMB which were not in causal contact? This is the essence of the horizon problem. The flatness problem, is related to the spatial curvature of the universe. Assuming a perfect fluid stress-energy $T^\mu{}_\nu = \text{diag.}[-\rho, p, p, p]$, using (III.1) and the Einstein equations (2.7) the *first* Friedmann equation is given by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \quad (\text{III.2})$$

with $\dot{}$ the derivative with respect to t . Using $H \equiv \frac{\dot{a}}{a}$, and setting $\Lambda = 8\pi G\rho_\Lambda$, this equation becomes

$$H^2 = \frac{8\pi G}{3}\rho_T - \frac{k}{a^2} \quad (\text{III.3})$$

with $\rho_T \equiv \rho + \rho_\Lambda$. Now, if the density of the universe, which is time dependent, is

$$\rho_T = \frac{3H^2}{8\pi G}$$

this would indicate that $k \equiv 0$; this density is referred to as the critical density. Rewriting the Friedman equation as

$$1 = \frac{8\pi G \rho_T}{3H^2} - \frac{k}{H^2 a^2} \quad (\text{III.4})$$

we have that the first term on the right can be written as

$$1 = \Omega - \frac{k}{H^2 a^2} \quad (\text{III.5})$$

with $\Omega = \frac{\rho_T}{\rho_c}$. Defining $\Omega_k = -\frac{k}{H^2 a^2}$ the Friedman equation is simply written as

$$1 = \Omega + \Omega_k. \quad (\text{III.6})$$

Current observations place the density of the universe very close to the critical density, therefore $\Omega \approx 1$, which yields that $\Omega_k \approx 0$ and the universe is very close to being flat. Since Ω is time dependent, and depends on the matter content of the universe and scale, early cosmic times would have Ω even closer to unity, thus the universe was even flatter than it is now. We do not wish to go into great detail on the mechanics of inflation, as there are plenty of resources on this matter, such as [161, 162]. Nevertheless, it would be inappropriate to continue any discussion of cosmology without a brief introduction: after all, we are going to be discussing theories that challenge inflation.

Inflation is characterized by as a phase during in the very early universe where the expansion of the universe was accelerating near exponentially, and gravity acts as a repulsive force, this corresponds to the scale factor condition $\ddot{a} > 0$. Observing the so-called second Friedmann equation, which comes from the combination of the tt and any space-space component of the field equations, it reads

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \quad (\text{III.7})$$

For $\ddot{a} > 0$ we require $\rho + 3p < 0$, which would be in violation of the strong energy condition. It is known that a universe of pure cosmological constant will undergo an

exponential acceleration, and therefore violate the strong energy condition, but this model is not sufficient for an inflationary period. A pure cosmological constant universe would never exit this stage. A time dependent Hubble parameter is necessary for an exit of inflation.

The classic method of introducing inflation is through a scalar field φ , with Lagrangian density $\mathcal{L} = \frac{1}{2}\nabla\varphi^2 + V(\varphi)$. Coupled to the FLRW metric (III.1), the equation of motion at zeroth order for the scalar field is given by

$$\ddot{\varphi} + 3H\dot{\varphi} + V_{,\varphi} = 0. \quad (\text{III.8})$$

The scalar field can mimic a perfect fluid with density and pressure through

$$\begin{aligned} \rho &= \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \\ p &= \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \end{aligned} \quad (\text{III.9})$$

and thus can be inserted in the Friedmann equation for the Hubble parameter. Under the direction of inflation and the slow-roll approximation, so long as $V(\varphi)$ is large but roughly flat, and $\dot{\varphi}^2 \ll V(\varphi)$, near exponential growth of the scale factor will occur. For the potential itself, the conditions necessary to induce a slow-roll of the scalar field are given by

$$\left(\frac{V_{,\varphi}}{V}\right)^2 \ll 1, \quad \left|\frac{V_{,\varphi\varphi}}{V}\right| \ll 1. \quad (\text{III.10})$$

The previous paragraph solely considers general relativity, but it introduces an important point: Where does this scalar field (the inflaton) come from? Now this choice of words is rather off-putting, as we do not really mean who delivered it. More technically, what physical model or mechanism provides the most probable prospect? As discussed in Part I, higher-curvature theories of gravity can be mapped to scalar-tensor theories of gravity, thus suggesting that the higher-curvature terms in the action can source this field. The literature of modified gravity and cosmology is immense [28]. The details of all the theories is beyond the scope of this writing, but the overlapping themes are these. Can the said theory provide an alternative to inflation? Can it solve any cosmological fine-tuning problems (i.e. horizon, flatness problems, generate the right initial conditions for large-scale structure)? This part of the thesis is dedicate to that study in 2 chapters.

In Chapter 8, we will investigate the cosmological implications of 4DEGB in a completely untested regime. This regime is due to the fact that a free parameter arises from the scalar field solution in the background spacetime, and has been set to zero in all previous literature. We wish to determine whether 4DEGB could provide a framework for resolving flatness and horizon problems. In addition to the background investigations, we also discuss a brief analysis on the 1st order perturbative effects and the observational constraints they impose.

In Chapter 9, we investigate quantum energy conditions in cosmological spacetimes. We discuss the smeared null energy condition and its implications on FLRW spacetime, and how it could be used to constrain semi-classical gravity. We do this through the use of a variable equation of state parameter, and compare to recently announced observational data. We also probe the idea of a bouncing cosmology as an alternative to inflation, and we impose constraints on the duration of the bounce phase and energy scale.

Chapter 8

Observational Constraints on 4DEGB

Ever since the discovery of 4DEGB gravity [163, 164], and discussed in Section 3.1.3, there have been investigations into how it effects gravitational phenomena on all scales [165, 166, 167, 168, 169, 170, 171, 172, 173, 174, 175, 176, 177, 178]. Not only are investigations undertaken on phenomenological grounds, but also into what constraints can be placed on the theory based on current data. Working with the action

$$S = \int d^4x \sqrt{-g} \frac{1}{16\pi G} \left[R - 2\Lambda + \alpha \left(\phi \mathcal{G} + 4G^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi - 4(\partial\phi)^2 \square \phi + 2((\partial\phi)^2)^2 \right) \right] + S_m \quad (8.1)$$

weak-field solutions have provided a constraint on α to be

$$\alpha < 10^{10} \text{ m}^2 \implies \tilde{\alpha} \lesssim 10^{-43} \quad (8.2)$$

which is the tightest constraint thus far. This result was constructed via the LAGEOS observations, and $\tilde{\alpha} = \alpha H_0^2$, with H_0 being the Hubble parameter. Additional constraints have been found for gravitational wave data [2], and for requiring that horizon does not shield nuclei [174].

Here we investigate some of the implications of 4DEGB gravity for cosmological observations, contributing to the already extensive efforts [2] at constraining this theory. Specifically, we consider a largely untested sector of 4DEGB parameterized by C , a free parameter that comes from solutions to the scalar field equations of motion derived from

the action (8.1), in addition to the coupling constant α . Since the Glavin-Lin formulation [179] is obtained by considering the $D \rightarrow 4$ limit of various D -dimensional solutions, it has no counterpart for the parameter C .

8.1 Theoretical Considerations

8.1.1 FLRW Solutions

In this section we will discuss the familiar FLRW metric in the context of 4DEGB gravity. We will first discuss the background spacetime and the associated field equations with a perfect fluid stress energy tensor. We will then move onto the perturbed spacetime, looking at scalar perturbations only. We will treat the perturbations of the stress energy tensor to be those of an anisotropic fluid, again with only scalar perturbations.

The cosmological background and scalar perturbation equations in 4DEGB gravity have been previously studied [165, 172]; in this section, we re-visit their construction afresh. We present our derivation in detail, which leads in the case of scalar perturbations to a more complete set of equations than previously presented in the literature. Furthermore, we expand upon the existing literature by explicitly including the possibility of a non-zero spatial curvature in the FLRW metric, with the resulting equations presented in Appendix A.

8.1.1.1 Background spacetime

We begin with the familiar FLRW line element in co-moving coordinates with arbitrary spatial curvature

$$ds^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin(\theta)^2 d\varphi^2) \right) \quad (8.3)$$

where we parameterize to $k = +1, -1, 0$ respectively corresponding to positive, negative, or flat spatial curvature, with $a(t)$ the scale factor. Isotropy and homogeneity assumptions

imply $\phi = \bar{\phi}(t)$ from which we obtain

$$(a\ddot{\phi} + \dot{a}\dot{\phi} - \ddot{a}) \left(a^2\dot{\phi}^2 - 2a\dot{a}\dot{\phi} + \dot{a}^2 + k \right) = 0 \quad (8.4)$$

from (3.27) using (8.3), and the dot represents a derivative with respect to the coordinate time t , the overbars denoting background quantities. We will also make use of the standard Hubble parameter

$$H \equiv \frac{\dot{a}}{a}.$$

The solutions to (8.4) are

$$\dot{\phi} = H \pm \frac{\sqrt{-k}}{a} \quad (8.5a)$$

$$\dot{\phi} = H + \frac{A}{a} \quad (8.5b)$$

where the former comes from the second term in (8.4); clearly $k \leq 0$ for a real solution. The latter solution (8.5b) comes from the first term in (8.4), with A an arbitrary constant. This is the most general solution, and we will henceforth take it to be the solution for the background scalar field.

For the field equations (4.4) we will use the perfect fluid stress energy with components

$$T^\mu_\nu = \text{diag}(-\bar{\rho}, \bar{p}, \bar{p}, \bar{p})$$

where both $\bar{\rho}$ and $\bar{p}$ are understood to be functions of coordinate time t . Using this, the tt component of (4.4) yields

$$\frac{\left(-\dot{\phi}^4 a^3 + 4\dot{\phi}^3 \dot{a} a^2 - 2\dot{\phi}^2 (3\dot{a}^2 + k) a + 4\dot{a}\dot{\phi} (\dot{a}^2 + k) \right) \alpha}{a^3} + \frac{\dot{a}^2 + k}{a^2} = \frac{8\pi G \bar{\rho}}{3} \quad (8.6)$$

which through the use of (8.5b) and the Hubble parameter becomes

$$\alpha \left(H^2 - \frac{A^2}{a^2} \right) \left(H^2 + \frac{A^2 + 2k}{a^2} \right) + H^2 + \frac{k}{a^2} = \frac{8\pi G \bar{\rho}}{3}. \quad (8.7)$$

Without loss of generality, we can set $A^2 = -k + C$, obtaining

$$\left(H^2 + \frac{k}{a^2} \right)^2 \alpha + H^2 + \frac{k}{a^2} = \frac{8\pi G \bar{\rho}}{3} + \frac{\alpha C^2}{a^4} \quad (8.8)$$

where the arbitrary constant C now acts as a geometrical “dark radiation” parameter [165]. If $C = 0$ we recover the generalization of the Friedmann equations obtained in [38].

The other generalized Friedmann equation can be obtained from any one of the diagonal terms of (4.4), yielding

$$\frac{\left(4(H^2 a^4 + k a^2) \dot{H} + 3H^4 a^4 + 2kH^2 a^2 + A^2(A^2 + 2k)\right) \alpha}{a^2} + 3H^2 a^2 + 2\dot{H} a^2 + k = -8\pi G a^2 \bar{p} \quad (8.9)$$

or, setting $A^2 = -k + C$,

$$\frac{(H^2 a^2 + k) \left(3H^2 a^2 + 4\dot{H} a^2 - k\right) \alpha}{a^2} + 3H^2 a^2 + 2\dot{H} a^2 + k = -8\pi G a^2 \bar{p} - \frac{\alpha C^2}{a^2}. \quad (8.10)$$

The conservation of stress energy yields

$$\begin{aligned} \nabla_\mu T^{\mu\nu} = 0 \quad \rightarrow \quad \dot{\rho} a + 3\dot{a} \bar{\rho} + 3\dot{a} \bar{p} &= 0 \\ \dot{\rho} &= \frac{-3(\bar{\rho} + \bar{p})\dot{a}}{a} \end{aligned} \quad (8.11)$$

and the system (8.8), (8.10), and (8.11) comprises the background equations for 4DEGB. Note that (8.10) is consistent with (8.8) and (8.11), as in general relativity.

Before we consider perturbations about the background solution, it will be useful to consider these equations in conformal time, where the line element (8.3) is now written as

$$ds^2 = a(\eta)^2 \left(-d\eta^2 + \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin(\theta)^2 d\phi^2) \right) \quad (8.12)$$

with

$$d\eta = \frac{dt}{a(t)}. \quad (8.13)$$

Letting prime denote a derivative with respect to η , the Hubble parameter becomes

$$\mathcal{H}(\eta) \equiv \frac{a'(\eta)}{a(\eta)} \quad \Rightarrow \quad \mathcal{H}(\eta) = a(t)H(t) \quad (8.14)$$

with $\bar{\rho}$ and $\bar{p}$ now functions of η , along with the scalar field $\bar{\phi}$, with solution

$$\bar{\phi}' = \mathcal{H} + A \Rightarrow \bar{\phi} = \ln(a) + A\eta + B. \quad (8.15)$$

In conformal time the generalized Friedmann equations are now written as

$$\begin{aligned} \frac{(\mathcal{H}^2 + k)^2 \alpha}{a^2} + \mathcal{H}^2 + k &= \frac{8\pi G a^2 \bar{\rho}}{3} + \frac{\alpha C^2}{a^2} \\ \frac{(\mathcal{H}^2 + k)(4\mathcal{H}' - \mathcal{H}^2 - k) \alpha}{a^2} + \mathcal{H}^2 + 2\mathcal{H}' + k &= -8\pi G a^2 \bar{p} - \frac{\alpha C^2}{a^2} \\ \bar{\rho}' &= -\frac{3a'(\bar{p} + \bar{\rho})}{a}. \end{aligned} \quad (8.16)$$

The use of a conformal or co-moving metric is a matter of taste. We shall employ the former (unless otherwise noted) since $\bar{\phi}'$ depends on the scale factor only when in terms of the Hubble parameter, as we see in (8.15).

8.1.1.2 Perturbed Spacetime

We will work in the context of the Conformal Newtonian gauge, and only consider scalar perturbations. All leading order equations are *linear* order in the perturbative quantities, which depend on all spacetime coordinates. The perturbed version of the line element (8.12) is

$$ds^2 = a^2 \left(-(1 + 2\Phi) d\eta^2 + (1 - 2\Psi) \left(\frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin(\theta)^2 d\phi^2) \right) \right) \quad (8.17)$$

where Φ and Ψ are the metric scalar perturbations in the longitudinal gauge. We follow the convention of [2] in the labelling of Φ and Ψ . Perturbations of the stress energy [165, 180] are given by $\delta\rho$ and δp for the density and pressure respectively, as well as the velocity perturbation v and the anisotropic fluid perturbation π_{ij} through the following [165, 180]

$$\delta T_0^0 = -\delta\rho, \quad \delta T_i^0 = (1 + w)\bar{\rho}\partial_i v, \quad \delta T_j^i = \delta_j^i \delta p + \pi_j^i \quad (8.18)$$

where π is defined via

$$\pi_{ij} = \left(\nabla_i \partial_j - \frac{1}{3} \gamma_{ij} \nabla^2 \right) \Pi \quad (8.19)$$

and where Π is a scalar function representing the anisotropy, ∇_i represents a covariant derivative¹, and ∇^2 is the Laplacian with respect to the spatial metric γ_{ij} .

¹In the absence of spatial curvature one may replace ∇_i with ∂_i in π_{ij} , consistent with [165, 180].

The relative energy density perturbation and speed of sound are respectively defined as

$$\delta = \frac{\delta\rho}{\bar{\rho}} \quad c_s^2 = \frac{\delta p}{\delta\rho} \quad (8.20)$$

and

$$\phi \rightarrow \bar{\phi} + \delta\phi \quad (8.21)$$

defines the perturbation of the scalar field.

We now consider the perturbed equations of motion, starting with the scalar field equation (3.27). Its leading order correction is

$$\alpha (A^2 + k) (\nabla^2(\delta\phi - \Phi) + 3A(\Psi' + \Phi') - 3(\Psi'' + \delta\phi'')) = 0 \quad (8.22)$$

upon using (8.15) and (8.17). Note that it has no dependence on the scale factor a , which would not be the case - as stated above - if we were working in co-moving time. Setting $A^2 = -k + C$, this becomes

$$\alpha C (\nabla^2(\delta\phi - \Phi) + 3\sqrt{-k + C}(\Psi' + \Phi') - 3(\Psi'' + \delta\phi'')) = 0. \quad (8.23)$$

Similar to the discussion above, when $C = 0$ the equation is trivially satisfied and no information about the perturbation $\delta\phi$ is given.

Henceforth we focus on the situation in which $C \neq 0$ and $k = 0$ (with the $k \neq 0$ case presented in Appendix A). Turning now to the gravitational field equations (4.4), the $\mathcal{E}_{\eta\eta} = 8\pi G T_{\eta\eta}$ component is

$$2\alpha A^2 (3\Phi A^2 - 6A(\Psi' + \delta\phi') - 2\nabla^2(\Psi + \delta\phi)) + \mathcal{A}(2\nabla^2\Psi - 6\mathcal{H}\Psi') - 6\alpha\Phi\mathcal{H}^4 = 8\pi G a^2 (a^2\delta\rho + 2a^2\Phi\rho) \quad (8.24)$$

at leading order, where $\delta T_{\eta\eta} = a^2\delta\rho + 2a^2\Phi\bar{\rho}$ and we follow [165] in defining

$$\mathcal{A} \equiv 2\alpha\mathcal{H}^2 + a^2 \quad \mathcal{B} \equiv 2\alpha\mathcal{H}^2 - a^2 \quad (8.25)$$

for convenience. From the zeroth order equation we identify

$$\bar{\rho} = \frac{-3A^4\alpha + 3\alpha\mathcal{H}^4 + 3a^2\mathcal{H}^2}{8a^4\pi G}$$

and inserting this into (8.24) we find

$$4\alpha A^2 (3A^2\Phi - 3A(\Psi' + \delta\phi') - \nabla^2(\Psi + \delta\phi)) + 2\mathcal{A} (\nabla^2\Psi - 3\mathcal{H}\Psi' - 3\Phi\mathcal{H}^2) = 8\pi G a^4 \delta\rho. \quad (8.26)$$

For any time-spatial component $\mathcal{E}_{\eta i} = 8\pi GT_{\eta i}$ we obtain

$$(4\alpha A^2(A(\Phi - \delta\rho) - \Psi' - \delta\rho') + 2\mathcal{A}\Psi' + 2\mathcal{H}\mathcal{A}\Phi)_{,i} = -8\pi Ga^4(1+w)\rho\partial_i v$$

to leading order, which readily integrates to

$$4\alpha A^2(A(\Phi - \delta\rho) - \Psi' - \delta\rho') + 2\mathcal{A}\Psi' + 2\mathcal{H}\mathcal{A}\Phi = -8\pi Ga^4(1+w)\rho v. \quad (8.27)$$

For the spatial-spatial components of $\mathcal{E}_{ij} = 8\pi GT_{ij}$ where $i \neq j$ we have

$$(2\alpha(A^2(\Psi + \Phi) + 2\mathcal{H}'\Psi) - \mathcal{B}\Psi - \mathcal{A}\Phi)_{,ij} = 8\pi Ga^4\Pi_{,ij}$$

to leading order, which again easily integrates into

$$2\alpha(A^2(\Psi + \Phi) + 2\mathcal{H}'\Psi) - \mathcal{B}\Psi - \mathcal{A}\Phi = 8\pi Ga^4\Pi \quad (8.28)$$

where we have set any integration functions of η to zero. This can be done without any loss of generality as the constraints must hold for all time; as such we can tune them to any value we wish. We note that for time-spatial and mixed spatial components there are no zeroth order counterparts.

Lastly we consider the diagonal components, $\mathcal{E}_{ii} = 8\pi GT_{ii}$. Since we lose homogeneity and isotropy at leading order, the equations are no longer equivalent for $i = x, y, z$, unlike the zeroth order case. Due to this feature we take the trace, $\mathcal{E}_i^i = 8\pi GT_i^i$ which to leading order can be written as

$$\delta G_{xx} + \delta G_{yy} + \delta G_{zz} = 8\pi G(3a^2(\delta p - 2\Psi\bar{p})). \quad (8.29)$$

Now isolating

$$\bar{p} = \frac{-A^4\alpha + \alpha H^4 - 4\dot{H}H^2\alpha - H^2a^2 - 2\dot{H}a^2}{8a^4\pi G} \quad (8.30)$$

from any zeroth order diagonal equation and inserting into $\delta G_{xx} + \delta G_{yy} + \delta G_{zz} = 8\pi G(3a^2(\delta p - 2\Psi\bar{p}))$ we get the final expression

$$\begin{aligned} & 2\alpha(3A^2\Phi - 3A(\delta\phi' - \Phi') - \nabla^2(\Phi + \Psi) - 3(\delta\phi'' + \Psi'')) + \mathcal{A}(3\mathcal{H}\Phi' + \nabla^2\Phi + 3\Psi'') + \mathcal{B}\nabla^2\Psi \\ & + 6\mathcal{H}'\mathcal{C}\Phi + 6\mathcal{H}\mathcal{D}\Psi' - 3\mathcal{H}^2\mathcal{B}\Phi - 4\mathcal{H}'\alpha\nabla^2\Psi = 12\pi Ga^4\delta p \end{aligned} \quad (8.31)$$

where $\mathcal{C} = 4\alpha\mathcal{H}^2 + a^2$ and $\mathcal{D} = 2\mathcal{H}'\alpha + a^2$. We note here a subtle point regarding the sum of the diagonal components, which yields three equations that are not identical at leading order. However making use of (8.28) renders these equations equivalent and hence the sum may be taken without any loss of generality. This subtle feature sometimes gets taken for granted, even when discussing General Relativity.

At this point we have 7 unknown scalar functions – $\Phi, \Psi, \delta\phi, v, \delta\rho, \delta p$ and Π – and 5 equations: (8.23) from the scalar field equation, and (8.26), (8.27), (8.28), and (8.31) from the gravitational field equations (4.4). The remaining two equations come from the conservation of stress-energy $\nabla_\mu T^{\mu\nu} = 0$. At zeroth order we retrieve only one equation, the cosmological fluid equation

$$\bar{\rho}' = -3\mathcal{H}(\bar{p} + \bar{\rho}) \quad (8.32)$$

obtained above. The first order $\nabla_\mu T^{\mu\eta}$ equation is

$$\bar{\rho} [(1+w)\nabla^2 v - 3(\Psi' + 2\mathcal{H}\Phi)] - 3\Psi'\bar{p} + \delta\rho' - 2(3\mathcal{H}\bar{p} + \bar{\rho}')\Phi + 3\mathcal{H}(\delta p + \delta\rho) = 0 \quad (8.33)$$

and after inserting the zeroth order solution (8.32) this becomes

$$\delta\rho' = (1+w)(3\Psi' - \nabla^2 v)\bar{\rho} - 3\mathcal{H}(\delta\rho + \delta p). \quad (8.34)$$

This can be rewritten as

$$\delta' = 3\mathcal{H}\delta(w - c_s^2) + (1+w)(3\Psi' - \nabla^2 v) \quad (8.35)$$

defining $\delta\bar{\rho} = \delta\rho \Rightarrow \delta\rho' = \delta'\bar{\rho} + \delta\bar{\rho}'$. Lastly, any $\nabla_\mu T^{\mu i} = 0$ component - they are all equivalent - in conjunction with (8.32) yields

$$\partial_i(2\nabla^2\Pi + 3\delta p + 3(1+w)\bar{\rho}[(1-3w)v + v' + \Phi]) = 0 \quad (8.36)$$

whose divergence is

$$2\nabla^4\Pi + 3\nabla^2\delta p + 3(1+w)\bar{\rho}[(1-3w)\nabla^2 v + \nabla^2 v' + \nabla^2\Phi] = 0, \quad (8.37)$$

a form that will prove useful later on.

Upon setting $A^2 = C$, the full set of equations for $k = 0$ is

$$\begin{aligned}
& \alpha C \left(\nabla^2(\delta\phi - \Phi) + 3\sqrt{C}(\Psi' + \Phi') - 3(\Psi'' + \delta\phi'') \right) = 0 \\
& 4\alpha C \left(3C\Phi - 3\sqrt{C}(\Psi' + \delta\phi') - \nabla^2(\Psi + \delta\phi) \right) + 2\mathcal{A}(\nabla^2\Psi - 3\mathcal{H}\Psi' - 3\Phi\mathcal{H}^2) = 8\pi G a^4 \delta\rho \\
& 4\alpha C \left(\sqrt{C}(\Phi - \delta\rho) - \Psi' - \delta\rho' \right) + 2\mathcal{A}\Psi' + 2\mathcal{H}\mathcal{A}\Phi = -8\pi G a^4(1+w)\bar{\rho}v \\
& 2\alpha(C(\Psi + \Phi) + 2\mathcal{H}'\Psi) - \mathcal{B}\Psi - \mathcal{A}\Phi = 8\pi G a^4\Pi \\
& 2\alpha \left(3C\Phi - 3\sqrt{C}(\delta\phi' - \Phi') - \nabla^2(\Phi + \Psi) - 3(\delta\phi'' + \Psi'') \right) + \mathcal{A}(3\mathcal{H}\Phi' + \nabla^2\Phi + 3\Psi'') \\
& + \mathcal{B}\nabla^2\Psi + 6\mathcal{H}'\mathcal{C}\Phi + 6\mathcal{H}\mathcal{D}\Psi' - 3\mathcal{H}^2\mathcal{B}\Phi - 4\mathcal{H}'\alpha\nabla^2\Psi = 12\pi G a^4\delta p \\
& \delta' = 3\mathcal{H}\delta \left(w - \frac{\delta p}{\delta\rho} \right) + (1+w)(3\Psi' - \nabla^2 v) \\
& 2\nabla^4\Pi + 3\nabla^2\delta p + 3(1+w)\bar{\rho}[(1-3w)\nabla^2 v + \nabla^2 v' + \nabla^2\Phi] = 0.
\end{aligned} \tag{8.38}$$

The full set of equations with $k \neq 0$ are given in Appendix A.

In what follows, we will consider these equations from two perspectives. First, we will look at the full solutions at the background level outlined by equations (8.16), which describe phenomena in the very early universe. We shall then investigate an unexplored region of parameter space in which α is small but C is unconstrained. Mathematically this is most easily studied by taking $\alpha \rightarrow 0$ but keeping αC^2 finite. This will have interesting implications for cosmological observables at CMB redshifts and later times.

8.2 The very early universe

When a is small or H is large the αH^4 term in (8.8) will be non-negligible (even for small α). Therefore, in the very early universe the parameter α will have an effect on background dynamics. We will primarily explore the effects of α on the sound and particle horizons in the early universe and on the horizon problem.

8.2.1 Diverging horizons close to the Big Bang

One striking feature of 4DEGB is its effect on the comoving sound horizon $r_s(z)$, where $a = \frac{1}{1+z}$. This is defined as the comoving distance a sound wave could travel from the beginning of the

universe in some time $t(z)$. The sound horizon at recombination is a fundamental scale shaped by early universe physics that leaves an imprint on the clustering of matter in the universe [181]. The equation describing $r_s(z)$ is

$$r_s(z) = \int_z^\infty \frac{c_s(z')}{H(z')} dz' = c \int_0^a \frac{1}{H(a')a'^2 \sqrt{3(1+R)}} da' \quad (8.39)$$

where $c_s(z) = \frac{c}{\sqrt{3(1+R)}}$ is the sound speed and $R = \frac{3\rho_b}{4\rho_r} = 3a/4$ (when assuming collisionless cold dark matter) where ρ_b and ρ_r are the energy densities of baryons and radiation respectively [182].

Let us see what effect $\tilde{\alpha}$ has on this equation. We notice that (8.8) gives us an expression for H :

$$H(a) = \frac{H_0}{\sqrt{2\tilde{\alpha}}} \sqrt{\frac{2\alpha k}{a^2} - 1 + \sqrt{1 + 4\tilde{\alpha} \left(\frac{\Omega_{m0}}{a^3} + \frac{\Omega_{r0}}{a^4} + \Omega_{\Lambda 0} + \frac{\tilde{\alpha}\tilde{C}^2}{a^4} \right)}}. \quad (8.40)$$

As a first step, this then allows us to find an analytic solution to (8.39) close to the Big Bang (in the limit $a \rightarrow 0$). Within GR we have the standard result

$$\left. \begin{array}{l} H \approx H_0 \sqrt{\Omega_{r0}} a^{-2} \\ R \rightarrow 0 \end{array} \right\} \implies r_s \approx \frac{c}{H_0} \int_0^a \frac{1}{\sqrt{3\Omega_{r0}}} da' \approx \frac{1}{H_0} \frac{1}{\sqrt{3\Omega_{r0}}} a \rightarrow 0$$

whereas for a 4DEGB universe

$$\left. \begin{array}{l} H \approx \mathcal{K} H_0 a^{-1} \\ R \rightarrow 0 \end{array} \right\} \implies r_s \approx \frac{1}{\sqrt{3}H_0\mathcal{K}} \int_0^a \frac{1}{a'} da' \approx \frac{1}{\sqrt{3}H_0\mathcal{K}} [\ln a]_0^a$$

where $\mathcal{K} = \sqrt[4]{\frac{\Omega_{r0} + \tilde{\alpha}\tilde{C}^2}{\tilde{\alpha}}}$ is a constant. This simplified version of Equation (8.39) diverges close to the Big Bang for a 4DEGB universe. In general, in a 4DEGB universe r_s diverges logarithmically as $a \rightarrow 0$ for any $\alpha \neq 0$ if we use Equation (8.39) to define the sound horizon. This seems to suggest 4DEGB predicts an infinite comoving sound horizon as long as $\alpha \neq 0$.

This effect is perhaps not as severe as it may seem at first glance. First, there will in general be a cutoff at small a at the Planck time $t_{\text{Planck}} = 5.39 \times 10^{-44}$ s, since quantum gravitational effects will become important. More generally, recall that what is physically relevant is an effective sound horizon, defined as the distance traveled by a sound wave that has started propagating after the end of inflation. Waves that left their sources earlier than the end of inflation underwent incredibly large Doppler shifts as the universe expanded, so it is reasonable to consider them to

be completely unobservable in practice. If this is the case, the lower bound of the integral in (8.39) is some scale factor at the end of inflation rather than at the beginning of the universe. This then gives us a new equation for our sound horizon:

$$r_s(a) = \int_{a_{\min}}^a \frac{c}{H(a')a'^2 \sqrt{3(1+R)}} da' \quad (8.41)$$

where $a_{\min}$ is close to the end of inflation, $a_{\min} \approx 10^{-28}$. We now encounter no divergences when calculating r_s . In GR+ Λ CDM, the contribution to the integral from early times is negligible and the exact value of the lower bound isn't important. In 4DEGB, the contributions are large at early times and an exact value of this lower bound is needed to find the sound horizon.

We can solve this apparent issue by assuming that Big Bang Nucleosynthesis (BBN) has to occur at the same time (and be described by similar physics) in 4DEGB as in GR because of primordial abundance constraints. This requires $\alpha \lesssim 10^{17} \implies \tilde{\alpha} \lesssim 10^{-36}$ [2]. We can then show that for this small value of $\tilde{\alpha}$, if the end of inflation occurs around $a = a_{\min} \approx 10^{-28}$, the exact lower bound of (8.41) isn't important. This small (but non-zero) α now causes a minimal modification to the sound horizon, which will be virtually the same in 4DEGB and GR as illustrated in Figure 8.1. Because the divergence of the integral in (8.41) is logarithmic, in general the lower cutoff does not meaningfully change the result. Even taking the cutoff $a_{\min}$ to be at the Planck time (before which any classical theory of gravity is believed to be invalid), the difference between the Λ CDM and 4DEGB solutions would still be negligible for $\tilde{\alpha} \lesssim 10^{-36}$.

In spite of this argument, having to impose a cutoff scale in the equation for the sound horizon is a theoretically unsatisfying characteristic of this theory. We have no firm knowledge of the correct $a_{\min}$, and the fact that we have to pick one is a complication of the theory.

8.2.2 The Horizon Problem

Another interesting feature of 4DEGB in the early Universe is that in the limit of early time, the first equation in (8.16) reduces to:

$$\alpha \mathcal{H}^4 = \Omega_{r0} H_0^2 + \alpha C^2 \quad (8.42)$$

and thus it follows that

$$a(\eta) = a_{\min} \exp \left[H_0 \sqrt[4]{\frac{\Omega_{r0} + \tilde{\alpha} \tilde{C}^2}{\tilde{\alpha}}} (\eta - \eta_{\min}) \right] = a_{\min} \exp [\mathcal{K} H_0 (\eta - \eta_{\min})] \quad (8.43)$$

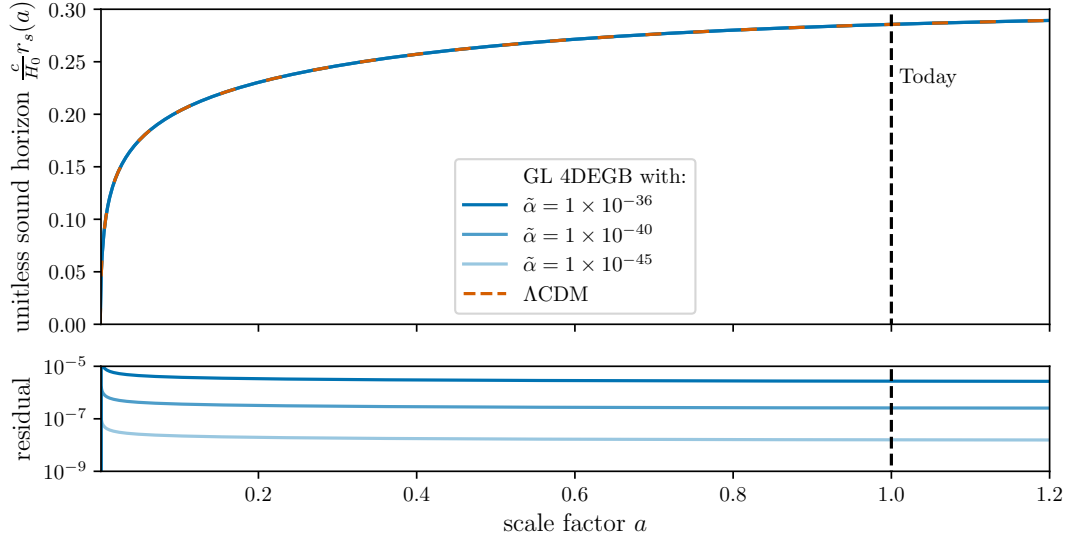


Figure 8.1: The dimensionless sound horizon as a function of redshift z in GL 4DEGB compared to Λ CDM, with its residual. We assume fiducial cosmological parameters from [1] and $a_{\min} \approx 10^{-28}$. We have set $C = 0$ for convenience, but non-zero C will not affect the sound horizon significantly. We display the sound horizon for values of $\alpha \lesssim 10^{17} \Rightarrow \tilde{\alpha} \lesssim 10^{-36}$ [2].

where $\eta_{\min}$ is the conformal time at which the cutoff scale factor $a_{\min}$ occurs. In terms of comoving time t , (8.43) means the universe isn't accelerating or decelerating ($\ddot{a} = 0$), just expanding at a constant velocity. This implies that the comoving Hubble radius $(aH)^{-1}$ is a constant, as shown in Figure 8.2.

Let us now compare the comoving Hubble radius $(aH)^{-1}$ in 4DEGB to that in Λ CDM, and see the effect this has on the horizon problem. Λ CDM without inflation predicts that patches on the CMB are causally disconnected for $\theta \gtrsim 1^\circ$, contrary to the observed uniformity of the CMB temperature map (which is 1 part in 10^5). This is called the horizon problem. If the universe had an early epoch during which the comoving Hubble radius decreased or stayed constant, then there would be more conformal time between the beginning of the universe and the CMB, so the past light cones of two points on the CMB would have had more time to be in causal contact. In the standard cosmological model, this is achieved by introducing an inflationary era at very

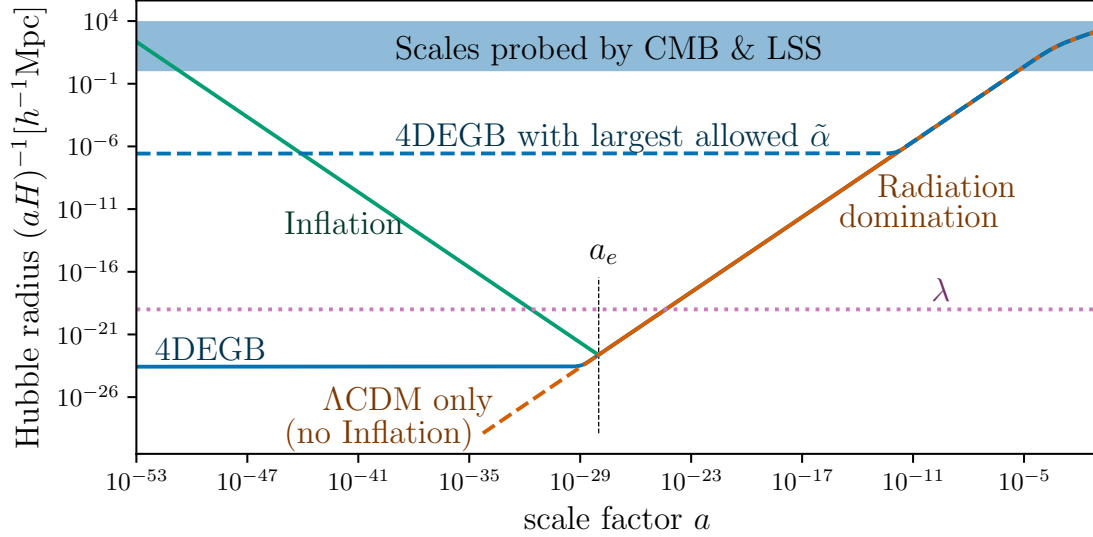


Figure 8.2: The comoving Hubble radius as a function of scale factor. Its logarithmic integral is the comoving particle horizon η . In general $\mathcal{H}^{-1} = (aH)^{-1} \propto a^n$, where for inflationary models $n = -1$, for radiation domination $n = 1$ and in 4DEGB $n = 0$. a_e is the transition scale factor between inflation domination and radiation domination. Here λ is an example of a primordial perturbation exiting the horizon during inflation and re-entering during radiation domination. We can see the same effect doesn't occur in 4DEGB. We assume fiducial cosmological parameters from [1] with $C = 0$ and with $\tilde{\alpha} = 10^{-112}$ and $\tilde{\alpha} = 10^{-44}$ for the blue and blue-dashed lines respectively. The cutoff for the integration in (8.44) occurs at scale factors below the lowest limit of the x-axis for all investigated values of α .

early times.

Alternatively, as mentioned above, 4DEGB also has this effect at early times (again, see Figure 8.2). As we see from Equation (8.43), 4DEGB is characterized by an early epoch where the Hubble radius $(aH)^{-1} = \dot{a}^{-1}$ is a constant. We then have a transition from an α -dominated to a radiation-dominated universe. In this scenario, points in the CMB have overlapping past light cones and therefore originated from a causally connected region of space (see Figure 8.3).

Another way of thinking about this is by looking at the comoving particle horizon $\eta(z)$, defined

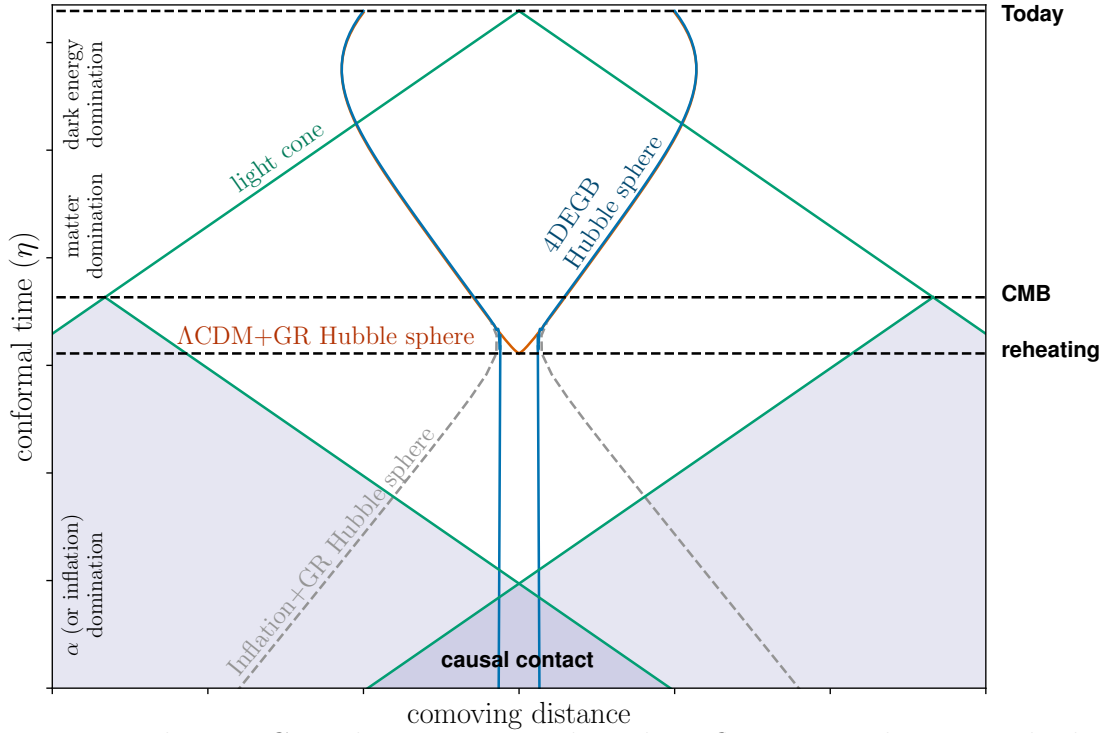


Figure 8.3: The 4DEGB solution compared to the inflationary solution to the horizon problem, visualized by looking at distances of interest as a function of conformal time (η). The plot assumes the exaggerated value of $\tilde{\alpha} = 10^{-7}$ to show the qualitative behaviour of the theory (and to aid the visual distinction between CMB and reheating space-like surfaces). We assume fiducial cosmological parameters from [1]. The Λ CDM+GR solution is plotted so that it matches 4DEGB at late conformal times. The reheating space-like surface is defined here as the conformal time at which the scale factor for the Λ CDM+GR solution goes to zero.

as:

$$\eta(z) = \int_z^{z_{\max}} \frac{c}{H(z')} dz' = \int_{a_{\min}}^a \frac{c}{H(a')a'^2} da' = \int_{a_{\min}}^a \frac{c}{H(a')a'} d \ln a'. \quad (8.44)$$

This is the logarithmic integral of the inverse Hubble radius; it diverges logarithmically as $a_{\min} \rightarrow 0$. A diverging comoving particle horizon (a diverging integral for (8.44)) means any two particles will have been in causal contact in some point in the past. However, as established above in Section 8.2.1, $a_{\min}$ must be finite and therefore Equation 8.44 does not diverge. As a result,

4DEGB won't necessarily provide a solution to the horizon problem. Whether it does depends on how large the value of $\tilde{\alpha}$ is. If we enforce $a_{\min} = a(t_{\text{planck}})$, 4DEGB provides an independent solution to the horizon problem when $\tilde{\alpha} \gtrsim 5 \times 10^{-11}$. Thus, while 4DEGB solves the horizon problem in principle, in practice this is a much larger $\tilde{\alpha}$ than current constraints [174] allow.

4DEGB doesn't provide an early period of acceleration and $\Omega_k(z)$ is a constant in the early 4DEGB universe. Therefore, while 4DEGB can in principle be sufficient to solve the horizon problem, it can never solve the flatness problem. 4DEGB without inflation is also not sufficient to have primordial fluctuations exit and then re-enter the horizon as in $\Lambda\text{CDM} + \text{GR}$ with inflation (again, see Figure 8.2).

8.3 Constraints from cosmological probes

8.3.1 A new formalism for cosmological parameter constraints

Although the cosmological impact of the parameter α alone is relatively weak other than in the very early universe (see Section 8.2), several studies have nevertheless placed constraints on α using late-time cosmological probes [173, 183]. While these do provide valuable independent constraints on α , they are uninformative compared to bounds due to other effects. More importantly, we have seen in Section 8.2.1 that to use observational information that relates to the sound horizon we must impose a strong prior from BBN of $|\alpha| \lesssim 10^{17} \text{m}^2$ [2]. This is the only meaningful way of using this information without assuming an inflationary model and doing a more detailed analysis of the sound horizon in 4DEGB. If we take this as a prior for a cosmological analysis using e.g. CMB and BAO data, the resulting constraints on α will then be completely prior-dominated.

Given the stringent constraints on α from [174], the GL formulation is very highly constrained. However, the full 4DEGB theory, with an additional free parameter C , has not been confronted with data (for one exception, see [171], which discusses how one might constrain C from theoretical considerations and existing observational bounds, but does not perform a full cosmological parameter inference analysis). Motivated by the lack of conclusive constraints on the parameter C in the literature, and by the stringent observational constraints on α as described above, we now introduce a version of the 4DEGB equations in the limit of small α .

We develop the equations of 4DEGB in the limit where terms in α , αA , αA^2 or αA^3 are negligible compared to terms in $\alpha A^4 = \alpha C^2$. In this small- α limit and with $k = 0$, Equation (8.40) takes the form:

$$H(z) = H_0 \sqrt{\Omega_{m0}(1+z)^3 + (\Omega_{r0} + \tilde{\alpha}_C)(1+z)^4 + \Omega_{\Lambda 0}} \quad (8.45)$$

where we define a new dimensionless parameter

$$\tilde{\alpha}_C \equiv \tilde{\alpha} \tilde{C}^2 \quad (8.46)$$

where $\tilde{C} \equiv C H_0^{-2}$, and $\alpha_C \equiv \alpha C^2$ such that $\tilde{\alpha}_C = \alpha_C H_0^{-2}$.

In the small- α limit, this will be the only parameter to enter both background and perturbation equations. We assume $\tilde{\alpha}_C$ is finite and seek to empirically constrain it. At times substantially past nucleosynthesis, the background is then equivalent to that of a Λ CDM universe with a “dark radiation” term that depends on $\tilde{\alpha}_C$.

We can now look at the set of equations (8.38) in the same limit. These reduce to:

$$\begin{aligned} k^2 \delta\varphi + 3\delta\varphi'' &= 3(\Phi' + \Psi') \\ 12\alpha_C \Phi - 12\alpha_C \delta\varphi' - 2a^2(k^2\Psi + 3\mathcal{H}\Psi' + 3\mathcal{H}^2\Phi) &= 8\pi G a^4 \delta\rho \\ \Psi' + \mathcal{H}\Phi &= -4\pi G a^2(1+\omega)\rho v \\ \Phi &= \Psi + 8\pi G a^2 \Pi \\ 3\Psi'' + 3\mathcal{H}(\Phi' + 2\Psi') + k^2(\Psi - \Phi) + (6\mathcal{H}' - 3\mathcal{H}^2)\Phi &= 12\pi G a^2 \delta\rho \\ \delta' &= 3\mathcal{H}\delta \left(w - \frac{\delta p}{\delta\rho} \right) + (1+w)(3\Psi' - \nabla^2 v) \\ 2\nabla^4 \Pi + 3\nabla^2 \delta p + 3(1+w)\bar{\rho} \left[(1-3w)\nabla^2 v + \nabla^2 v' + \nabla^2 \Phi \right] &= 0 \end{aligned} \quad (8.47)$$

where $\delta\varphi = \frac{\delta\phi}{\sqrt{C}}$ is the same order of magnitude as Φ with respect to the constant C . For a derivation of the equations above, we refer the reader to Appendix B.

This limit of 4DEGB at the background level behaves like a Λ CDM universe with an additional contribution to the dark relativistic degrees of freedom. Interestingly, the parameter $\tilde{\alpha}_C$ couples this background dark radiation term (which has a purely geometric origin) to the modified equations for structure growth. This offers the prospect of differentiating 4DEGB from modifications to the standard model of particle physics that cause dark radiation, since the equations for growth will be different.

We can also look at equations (146c) and (146d) from [2] in the same limit. These reduce to:

$$\begin{aligned} a^2 \partial^2 \dot{F}_i + 16\pi G(1 + \omega)a^4 \bar{\rho} v_i &= 0 \\ a^2 \ddot{\gamma}_{ij} + 2\mathcal{H}a^2 \dot{\gamma}_{ij} - a^2 \partial^2 \gamma_{ij} &= 8\pi G a^4 \Pi_{ij}. \end{aligned} \quad (8.48)$$

These tell us that the tensor and vector perturbations reduce to those from Λ CDM in the small- α limit, even though the scalar perturbations are modified. Then any data that depends only on the vector and tensor perturbations (such as data from gravitational waves) won't enforce additional constraints on our parameter $\tilde{\alpha}_C$.

In the following subsections we use cosmological probes of background equations to constrain our parameter $\tilde{\alpha}_C$. In Section 8.3.3 we also introduce some qualitative effects of the modifications to the scalar perturbations on the power spectrum.

8.3.2 Background constraints on $\tilde{\alpha}_C$ as a non-interacting dark radiation with a geometrical origin

Tests of dark radiation have been carried out extensively in the literature (see, e.g., [184, 185, 186, 122, 1]). We can take advantage of existing constraints and map them onto the question of constraining the $\tilde{\alpha}_C$ parameter of 4DEGB in the small- α limit. We first place background constraints by using datasets that don't require the assumption of a GR+ Λ CDM structure growth history.

We will find it productive here and in the following section to examine how analyses that consider additional beyond-standard model neutrinos can equally constrain a cosmological model with standard model neutrinos and 4DEGB gravity, parameterized by $\tilde{\alpha}_C$ in the small- α limit. Flexibility in the number of such early-Universe relativistic degrees of freedom is often parameterized by:

$$\Omega_{\nu 0} = N_{\text{eff}} \frac{7}{8} \left(\frac{4}{11} \right)^{7/8} \Omega_{\gamma 0} \quad (8.49)$$

where N_{eff} is the effective number of relativistic neutrino species (equal to 3.044 for standard model neutrinos only) and $\Omega_{\gamma 0}$ is the dimensionless photon density parameter. Constraints on N_{eff} from this Λ CDM+ N_{eff} model exist in the literature from a number of probes of the cosmological background dynamics (see, e.g., [184, 185]), and we will take advantage of this here.

At the level of the background only, we can treat $\tilde{\alpha}_C$ as if it were a contribution from an additional massless particle that neither interacts nor decays. Doing so, we can write down an equivalence between $\tilde{\alpha}_C$ in 4DEGB and N_{eff} in GR:

$$\tilde{\alpha}_C = (N_{\text{eff}} - 3.044) \frac{7}{8} \left(\frac{4}{11} \right)^{7/8} \Omega_{\gamma 0} = \Delta N_{\text{eff}} \frac{7}{8} \left(\frac{4}{11} \right)^{7/8} \Omega_{\gamma 0} \quad (8.50)$$

It is important to emphasize that $\tilde{\alpha}_C$ has a geometrical origin — we are not proposing to test a model with a varying number of neutrinos. We are testing a 4DEGB universe with standard model neutrinos (with $N_{\text{eff, SM}} = 3.044$). This means we will allow, for example, negative values of $\tilde{\alpha}_C$ which correspond to $N_{\text{eff}} < 3.044$, but the number of physical neutrino species $N_{\text{eff, SM}}$ will always be the standard 3.044. We emphasize also that this equivalence should only be seen as a valid description of *background* cosmological observables; those that are sensitive to the growth of structure require consideration of modifications to the perturbation equations.

With this in mind, we can initially consider constraints from ^4He abundances (which give a conservative bound of $|N_{\text{eff}}| \leq 3.5$ [184]). This maps to an upper bound on $\tilde{\alpha}_C$ of:

$$|\tilde{\alpha}_C| \lesssim 10^{-5}.$$

8.3.2.1 Constraints on $\tilde{\alpha}_C$ from the CMB

Next, we will look at data from the Atacama Cosmology Telescope (ACT), specifically that taken with its second-generation polarization-sensitive receiver ACTPol [187]. This dataset provides information about the CMB anisotropies, which in general depend on the perturbative behavior of the theory as well as the background behaviour. In practice, the effect of small modifications to the perturbation equations (which will change the Sachs-Wolfe and Integrated Sachs-Wolfe effects) is apparent only at angular multipoles $\ell \lesssim 200$ [188]. The ACTPol analysis that we build on here takes a minimum multipole of 600 in the CMB temperature auto-spectrum, and 350 in the E-mode polarization auto-spectrum as well as the cross-spectrum between E-mode polarization and temperature, which means we can treat it effectively as a background-only probe. We use posterior chains of cosmological parameters obtained with the `actpollite` likelihood for a $\Lambda\text{CDM}+N_{\text{eff}}$ model, which results in $N_{\text{eff}} = 2.42 \pm 0.43$ (using the maximum marginalized likelihood at a 68% confidence level) [189].

This translates to a constraint for 4DEGB (with standard neutrinos) of:

$$\tilde{\alpha}_C = (-3.5 \pm 2.4) \times 10^{-6} h^2 = (-9 \pm 6) \times 10^{-6} \quad (8.51)$$

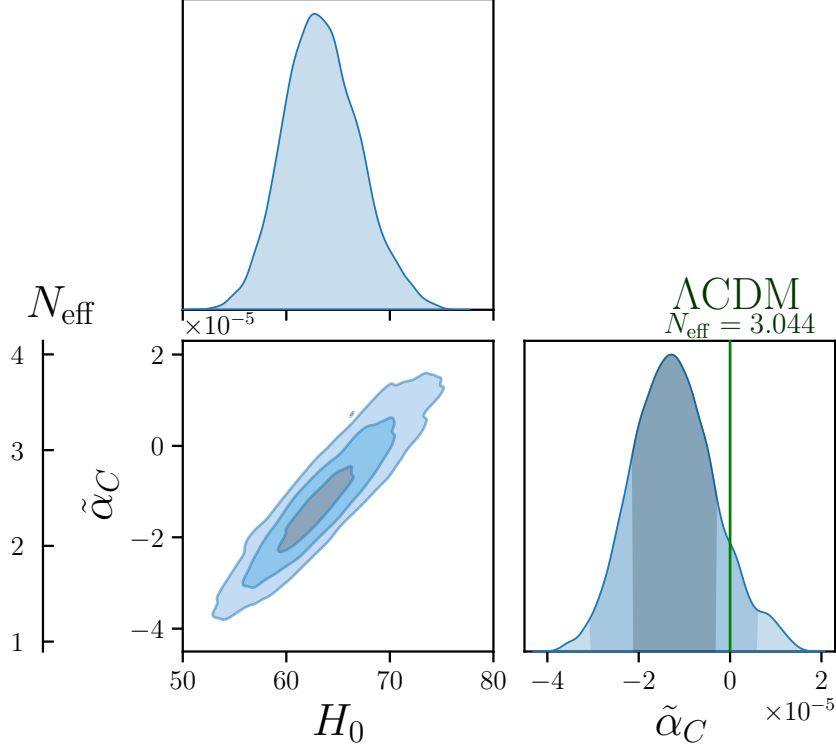


Figure 8.4: ACTPol posterior results for a Λ CDM universe with varying N_{eff} . We map this analysis onto our scenario of interest, with N_{eff} playing an equivalent role to $\tilde{\alpha}_C$ in 4DEGB for background-only probes. $\tilde{\alpha}_C$ is consistent with zero within 1.34σ . We are displaying the 1σ , 2σ and 3σ confidence regions in the contour plot, and the 1σ and 2σ regions in the 1D posterior probability distribution for $\tilde{\alpha}_C$.

at 68% confidence level, which is consistent with zero at the 1.34σ level. h is the dimensionless Hubble constant defined by $H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$ and we have used the maximum posterior value for h from the ACTPol analysis in question in our calculations.

As we can see in Figure 8.4, the analysis of [189] prefers a lower N_{eff} than the standard model (or in 4DEGB a negative $\tilde{\alpha}_C$). An interesting feature of 4DEGB is that it provides a modified gravity framework that mimics a continuous variation of N_{eff} from its Λ CDM value of 3.044, including values $N_{\text{eff}} < 3.044$ [1], so 4DEGB provides an interesting new precedent, which should be borne in mind should future

constraints also favour a negative value for ΔN_{eff} . While we cannot draw conclusions from all-sky CMB Planck data without accounting for a modification to the scalar perturbation equations, we note here for completeness that this preference for a lower value of N_{eff} does not seem to persist in Planck data [1] for a $\Lambda\text{CDM}+\text{GR}$ model.

8.3.3 Effect of modified perturbations in 4DEGB on the matter power spectrum

Having considered the background-only effect of $\tilde{\alpha}_C$, we now examine the growth of perturbations in this small α limit of 4DEGB. We incorporate the 4DEGB background and perturbation equations within a bespoke cosmological Boltzmann code to study the qualitative behaviour of small- α 4DEGB in a perturbed universe. We use a modified version of equations (8.45) and (8.47) (see Appendix B.1) to recover the matter power spectrum $P(k)$. The Boltzmann code we use is constructed following the methodology of [190]. In taking this approach, we prioritize ease of implementation rather than the computational efficiency which would be gained by modifying the highly optimised public packages **CAMB** [191] or **CLASS** [192]. We validate our Boltzmann code against these packages in the GR case and find excellent agreement.

The resulting power spectra can be seen in Figure 8.5. We see that the difference in $P(k)$ between 4DEGB and ΛCDM with the equivalent N_{eff} value is non-negligible for $\tilde{\alpha}_C \approx \pm 8 \times 10^{-6}$. The difference in the paired spectra is larger at large scales (low k); to understand this, we turn to the second equation in (8.47), which suggests that the $\tilde{\alpha}_C$ term is more dominant when k is smaller. Beyond this large-scale effect, there is also a scale-dependent deviation between the two theories as we move to smaller scales (larger k). Combined, these effects suggest that there is potential for the degeneracy between this sector of 4DEGB and $\Lambda\text{CDM}+N_{\text{eff}}$ to be broken by observational probes sensitive to changes in the power spectrum (such as weak gravitational lensing, galaxy clustering, CMB spectra at lower multipoles which probe perturbation effects, or CMB lensing). Although the fractional deviations between the two models in Figure 8.5 are small, upcoming surveys such as *Euclid* [193] and the Rubin Observatory Legacy Survey of Space and Time (LSST, [194]) may be capable of observationally distinguishing these percent-level effects. Such an analysis is beyond the scope of this paper but is of great interest for future work.

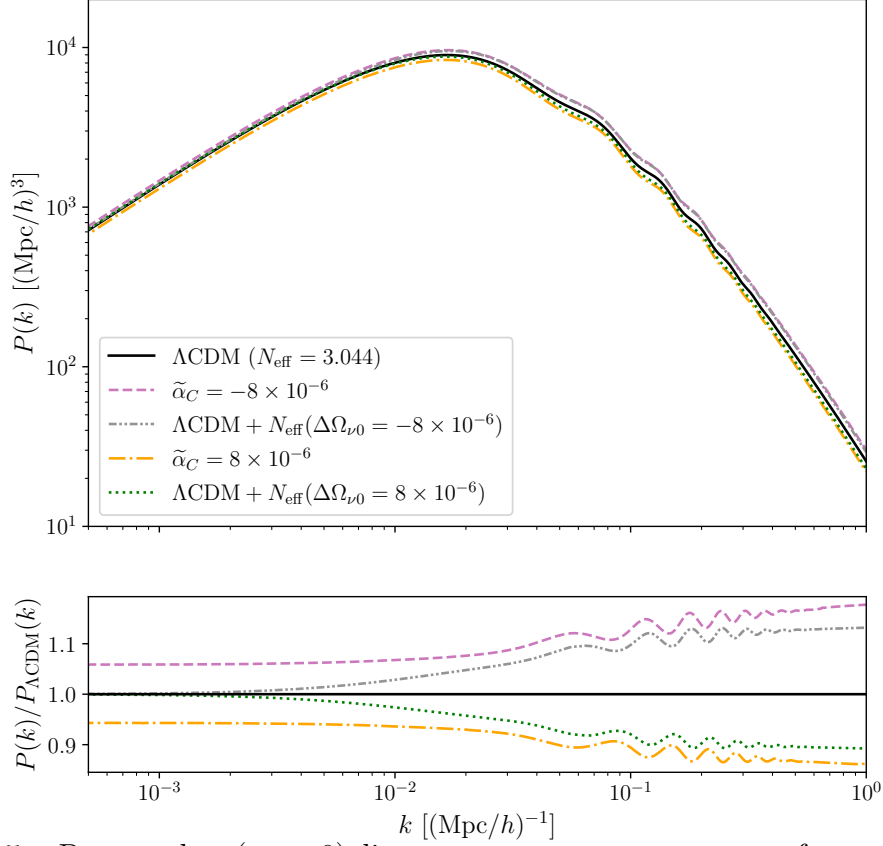


Figure 8.5: Present-day ($z = 0$) linear matter power spectrum for small- α 4DEGB. The middle (solid black) curve is the power spectrum for a Λ CDM model. The values of $\tilde{\alpha}_C$ displayed here are chosen to be of the same order of magnitude as background constraints of Section 8.3.2. The dotted (green) and dash-double dotted (gray) curves show the Λ CDM+ N_{eff} model with extra neutrino densities equivalent to the plotted values of $\tilde{\alpha}_C$ ($\Delta\Omega_{\nu 0} = \tilde{\alpha}_C = 8 \times 10^{-6}$ and -8×10^{-6} respectively), which correspond to $N_{\text{eff}} = 5.46$ and 0.62 respectively. When varying N_{eff} , we fix $\{Y_p, \Omega_M^0, \Omega_B^0, \Omega_R^0, h, A_s, n_s\}$ to be the fiducial cosmological parameters from [1]. We approximate all neutrinos as massless.

8.4 Discussion

4DEGB gravity has broad theoretical interest as a competitor to general relativity, primarily because it is a 4-dimensional higher-curvature theory of gravity that enjoys all the basic features of higher-dimensional Lovelock gravity. It has also provoked substantial observational interest as a theory with few extra parameters compared to Λ CDM+GR, which has resulted in the Glavin-Lin version being extensively investigated, and its single parameter α being well-constrained. Here we have for the first time studied the implications of scalar-tensor 4DEGB, providing the first observational constraints on a geometric term parametrized by a constant C .

Considering a background spacetime that is homogeneous and isotropic and a perturbed spacetime that only depends on scalar functions, we find that the stress-energy perturbations are that of a perfect fluid with anisotropic stress. This in turn dictates that there are 7 unknown functions correcting the cosmic background. We have provided the first derivation of the complete set of equations necessary for computing these 7 unknown functions for both flat and curved spatial hyper surfaces. The conservation of stress-energy provides two equations of motion (that do not differ from GR), while the gravitational and scalar field equations provide the other five. The complete set forms a nonlinear coupled PDE system that does not have a closed form solution. These equations become the first order perturbation equations of general relativity in the appropriate limit.

We have explored some consequences of α in the very early universe, where the small, allowable values of this parameter can still have important cosmological effects. We find that the sound horizon in 4DEGB depends on the behaviour of the universe at early times, meaning that the value of the parameter α and the exact redshift at which sound propagation begins to matter both affect the measured solution for the sound horizon. We show that this issue can be alleviated somewhat, though this requires imposing a minimum scale factor after which 4DEGB becomes relevant. Furthermore, we have shown that 4DEGB can resolve the horizon problem: the particle horizon diverges at early times, meaning that all particles will have been in causal contact at some very early time. This solution fails for very low values of α if we enforce a cutoff to this classical theory at the Planck time. This is because the aforementioned divergence is logarithmic and therefore is very slow to diverge even for a very small minimum time. This theory also fails to act as a replacement for an inflation-type model in the early Universe because it does not resolve the flatness problem.

Unlike the Glavin-Lin formulation, full 4DEGB contains a dark radiation-like term at the

background level, whose associated parameter C has remained largely unconstrained in the literature before now. Introducing a new parameter $\tilde{\alpha}_C = \frac{\alpha_C}{H_0^2} = \frac{\alpha_C C^2}{H_0^2}$ we have shown that, under the assumption that α is small (justified by prior constraints on this parameter [2]), the equations describing the theory at times substantially past inflation depend only on $\tilde{\alpha}_C$. The late-time behaviour of the theory can then be constrained as a combination of background “dark radiation” effects and a modification to structure growth. We have shown that the 4DEGB parameter is constrained using CMB data from ACTPol to

$$\tilde{\alpha}_C = \tilde{\alpha}\tilde{C}^2 = (-9 \pm 6) \times 10^{-6} \implies \alpha_C = (-10 \pm 7) \times 10^{-58} h^2 \text{ m}^{-2} \approx (-4 \pm 3) \times 10^{-58} \text{ m}^{-2} \quad (8.52)$$

where we have used the conversion factor $H_0^2 = (1.17 \times 10^{-52}) h^2 \text{ m}^{-2}$. This implies α_C is consistent with zero within 1.34σ . Interestingly, ACTPol data seems to prefer a lower value of N_{eff} than that suggested from the standard model (or in 4DEGB a negative $\tilde{\alpha}_C$). 4DEGB enjoys the interesting feature of being a geometrical alternative to continuously varying N_{eff} , including values of $N_{\text{eff}} < 3.044$ that are much less theoretically motivated in other settings. This may prove interesting in the future, should further cosmological observables favour such values of N_{eff} in a $\Lambda\text{CDM}+N_{\text{eff}}$ model.

While we completed a qualitative analysis of the power spectrum in this paper, modifying an efficient public Boltzmann solver to compute this quantity would enable further constraints on α_C using observables sensitive to cosmic structure growth. This would enable us to break the degeneracy between this branch of 4DEGB and $\text{GR}+\Lambda\text{CDM}$ with a varying N_{eff} . Figure 8.5 shows that the power spectrum (and therefore σ_8 and S_8) will be modified in 4DEGB compared to ΛCDM (or $\Lambda\text{CDM}+N_{\text{eff}}$). It would therefore be interesting to investigate in future work what impact this sector of 4DEGB might have on the S_8 tension.

Finally, we note that there are other potential avenues of exploration of the cosmological consequences of 4DEGB in the very early universe. We have not here discussed the implications of a 4DEGB universe in terms of the behaviour and evolution of super-horizon perturbations, or the implications regarding early-universe singularities or lack thereof in these theories. A universe dominated by 4DEGB at early times might have an effect on the scale dependence or non-gaussianity of the primordial power spectrum which would be interesting to investigate. As yet no such calculation has been attempted. The work presented here on the sound horizon and its divergent behaviour as $a \rightarrow 0$ suggests that these may be fruitful avenues for further work.

Chapter 9

Quantum Energy Conditions on Cosmological Spacetimes

9.1 Introduction

The role of energy conditions in classical general relativity (GR) is to determine what kind of matter fields one may ‘reasonably’ include in the theory and, correspondingly, what kind of spacetime geometry one may physically expect. Historically, these conditions were formulated to represent the non-negativity of the energy densities or fluxes measured locally by observers, and they played three major roles in GR. To start with, under the assumption that some energy conditions hold, there exist rigorous singularity theorems (e.g., [62, 195]) that point to the breakdown of classical GR. Second, they have been used to forbid traversable wormhole solutions (e.g., [196]) and third to prove positive asymptotic mass theorems for isolated objects (e.g., [197]). Notably, the strong energy condition was initially regarded as one of the ‘reasonable’ classical energy conditions. However, it is now generally accepted that it has been violated in the recent history of the cosmos by the current accelerated expansion of the universe (e.g., [198, 199, 200, 201, 202, 203]) and plausibly previously in the early universe, assuming an inflationary phase.

In the context of semiclassical gravity, determining what constitutes ‘reasonable’ energy conditions becomes even more subtle, as the right-hand side of the Einstein equations now represents the quantum expectation value of the energy-momentum tensor. In fact, in regimes where singularities are predicted in classical GR, one expects quantum effects to change the game. For one,

the hope is that a full theory of quantum gravity should be free of physically pathological singularities altogether. However, a more immediate impact of coupling quantum fields to gravity is that in quantum field theory (QFT) the energy densities at a point can be undoubtedly negative, and one can find solutions such that any of the local classical energy conditions are violated. A good example comes from the null energy condition (NEC), the weakest of these conditions, which plays a crucial role in GR theorems (see, e.g., [204]): when considering quantum fields in flat or curved spacetime, the NEC can notoriously be violated (see, e.g., [205, 206, 207, 208, 209]). In fact, the violation of the NEC has also been shown to be very generic in the context of Hawking evaporation (e.g., [210, 211, 212, 213, 214, 215, 216]), which is one of the most robust (theoretical) results in semiclassical gravity. Therefore, the relevant question is not whether the NEC holds at the fundamental level¹ locally, but rather if there is any bound on its violation. More specifically, physically speaking we do not expect the NEC to be violated systematically all the time and everywhere, arbitrarily. There should still exist some quantum or semiclassical energy conditions that tell us to what extent the NEC can be broken.

The averaged null energy condition (ANEC; e.g., [218, 219, 220, 221, 222, 223]) is one of the first proposals for a quantum-inspired, semiclassical version of the NEC. The ANEC stipulates that the quantum expectation value of the null energy, when averaged over an entire null (achronal) geodesic, is non-negative. It has been shown to hold for a wide class of QFTs in curved spacetimes and finds motivation from various approaches to quantum gravity (see, e.g., [224, 225, 226, 227, 228, 229, 230, 231, 232]). By definition, the ANEC is a global concept, though. Thus, it is difficult to apply the ANEC without proper knowledge of the entire spacetime, and correspondingly, it cannot always be used as a way of constraining local (or semilocal) violations of the NEC.

A more recent quantum-motivated semilocal null energy condition that has been conjectured to hold at the level of semiclassical gravity (or even further into a quantum gravity perturbative regime) is known as the smeared null energy condition (SNEC; [233]). The SNEC is similar to the ANEC, except that the standard integral average is replaced by a weighted average through a smearing function. The goal of the smearing is to act in a similar fashion to a window function to regulate the energy condition in a particular regime, specifically a regime where the NEC would normally be violated. The SNEC then puts a bound on how much (semilocal) NEC breaking

¹In particular, NEC-violating quantum fields do not have to be ‘exotic’ — this can already happen within the standard model of particle physics and, for instance, could lead to nonpathological traversable wormholes [217].

may be allowed within semiclassical gravity (i.e., within the regime of validity of the effective field theory). The SNEC can be heuristically understood as follows: quantum mechanically, we have Heisenberg’s uncertainty principle, which can be stated as $\Delta E \Delta t \geq \hbar/2$, i.e., if we perform an observation with a time spread Δt , then we expect fluctuations in the energy of the measurement as $\Delta E \geq \hbar/(2\Delta t)$. Therefore, if one (a classical ‘observer’) makes a measurement of the energy over a short smearing time, one may have large energy fluctuations that lead to negative energy, and the SNEC is a conjectured bound on how much negative energy one could actually observe given the coupling to gravity.² In other words, it is a form of ‘energy-spacetime uncertainty relation.’

The SNEC has passed a number of tests at the semiclassical level, has been generalized in various ways, and has led to novel singularity theorems with implications for black hole and cosmological spacetimes (see [234, 235, 236, 237, 238, 239]). For example, evaporating black holes satisfy the SNEC [234, 240]. Yet, there are many contexts in which the implications of the SNEC have not been explored, both in cosmological and black hole spacetimes. This work solely focuses on the former. Specifically, there are two main regimes in cosmology where the classical NEC might appear to be broken, but where a quantum modified version, like the SNEC, should hold: at late times and low energies; and at early times and high energies. The first regime is where we expect some form of dark energy (DE) to dominate the universe (e.g., [198, 199, 200, 201, 202, 203]). If this DE is of ‘phantom type’ (meaning it has an equation-of-state [EoS] parameter smaller than -1 ; see, e.g., [241, 242, 243, 244, 245, 246, 247, 248, 249]), then the classical NEC would be violated. The second regime pertains to the evolution of the universe prior to the start of standard big bang cosmology (and potentially before inflation if there was a phase of inflationary expansion). Indeed, in that regime, if a big bang singularity were avoided, then it might be that the universe underwent a period of NEC violation, resulting in, e.g., a big bounce (see, e.g., [250, 251, 252, 253]).

Since phantom DE and a bouncing universe are both speculations — though both lead to observable predictions that are testable — we take the stance in this work that the SNEC might hold fundamentally and apply it to these NEC-violating spacetimes as a way of theoretically constraining them. Naturally, the SNEC remains a conjecture (at the same level as, e.g., the swampland conjectures in string theory), but it is nevertheless informative to explore its implications should

²As we will see, the smearing is technically done along null geodesics (since this pertains to the *null* energy condition). Still, we will also see in what sense this can be related in cosmology to a smearing with respect to time or the scale factor.

the conjecture hold. In this work, we put aside theoretical issues that specific models may have related to having stable and consistent NEC violation in cosmology (such issues already put constraints on NEC-violating models; see, e.g., [254, 255, 256, 257, 258, 259, 260, 261, 262, 263, 264, 265, 266, 267, 268, 269, 270, 271, 272, 273, 274, 275, 276, 277, 278, 279, 280, 281, 282, 283]). We simply assume that the NEC can be violated and see what constraints the SNEC would put on such cosmologies.

Outline We start in Sec. 9.2 with a more thorough review of classical and semiclassical energy conditions, putting emphasis on the SNEC. Section 9.3 is devoted to the formulation of the SNEC in a cosmological background. This is applied to DE in Sec. 9.4, covering the cases of a constant and time-dependent EoS. In particular, we compared our constraints with current observational bounds from the Dark Energy Spectroscopic Instrument (DESI; [3, 4, 5, 6]). Section 9.5 is devoted to exploring the constraints that the SNEC puts on nonsingular bouncing cosmologies. We end with a discussion in Sec. 9.6.

Conventions Spacetime indices are denoted by Greek letters and are contracted with the spacetime metric tensor $g_{\mu\nu}$. We use the mostly positive metric signature and work with units where the speed of light c and the reduced Planck constant $\hbar$ are set to unity.

9.2 Review of some energy conditions

9.2.1 Classical versions

General relativity is a classical theory of gravity. It relates the spacetime geometry (described by the metric tensor $g_{\mu\nu}$) to the energy content in the universe (described by the energy-momentum tensor $T_{\mu\nu}$). From the metric, one has knowledge of the curvature through the Riemann tensor $R^\alpha_{\mu\beta\nu}$, the Ricci tensor $R_{\mu\nu} \equiv R^\alpha_{\mu\alpha\nu}$, the Ricci scalar $R \equiv R^\mu_{\mu}$, and the Einstein tensor $G_{\mu\nu} \equiv R_{\mu\nu} - (R/2)g_{\mu\nu}$. The relation between $g_{\mu\nu}$ and $T_{\mu\nu}$ is known as the Einstein field equations,

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \quad (9.1)$$

where G_N is Newton's gravitational constant, or equivalently, taking the trace of (9.1) to replace R by $T \equiv T^\alpha_\alpha$,

$$R_{\mu\nu} = 8\pi G_N \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right). \quad (9.2)$$

In principle, one can get any desired spacetime geometry $g_{\mu\nu}$ by simply computing $G_{\mu\nu}$ from $g_{\mu\nu}$ and postulating an energy-momentum tensor $T_{\mu\nu} = G_{\mu\nu}/(8\pi G_N)$. Although this is mathematically legitimate, the resulting spacetime and its energy content may not be physically reasonable. On the other hand, if the equation of motion for a particular type of matter or the action leading to it is known, one can aim at solving it (given reasonable initial/boundary conditions) along with the Einstein equations to obtain both the spacetime geometry and the evolution of the matter degrees of freedom. However, ideally, we would like to obtain some generic features about possible physical spacetime geometries independent of a particular type of matter.

For these reasons, one usually adds energy conditions to the theory, i.e., reasonable conditions on the non-negativity of local energy densities or fluxes, formulated as linear combinations of components of $T_{\mu\nu}$, which in turn restrict the kind of geometries that are allowed. We call these ‘standard’ energy conditions (null, strong, weak, dominant, etc.) the *classical energy conditions* (also known as pointwise energy conditions in the literature). For instance, the strong energy condition (SEC) states that, for any timelike vector U^μ (so $U^\mu U_\mu < 0$),

$$\left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right) U^\mu U^\nu \geq 0, \quad (9.3)$$

while the null energy condition (NEC) states that

$$T_{\mu\nu} K^\mu K^\nu \geq 0 \quad (9.4)$$

for any null vector K^μ (so $K^\mu K_\mu = 0$).

One of the main motivations for the classical energy conditions comes from their utility in proving the singularity theorems (of Penrose and Hawking for instance). Such geometrical theorems always have some kind of ‘convergence condition’ in their hypothesis, often conditions on $R_{\mu\nu}$ (akin to curvature bounds in mathematics); see, e.g., [284, 285]. If one assumes GR, there is often a straightforward one-to-one relation between classical energy conditions and various convergence conditions (null, timelike, etc.); see, e.g., [204, 286]. For instance, one can prove the Penrose black hole singularity theorem [62] assuming the null convergence condition (NCC), which says that $\forall K^\mu$ null,

$$R_{\mu\nu} K^\mu K^\nu \geq 0. \quad (9.5)$$

Similarly, the Hawking cosmological singularity theorem (e.g., [195]) needs the timelike convergence condition (TCC), which says that $\forall U^\mu$ timelike, $R_{\mu\nu}U^\mu U^\nu \geq 0$. It is then straightforward to check that, whenever the Einstein field equations hold, $\text{NCC} \Leftrightarrow \text{NEC}$, while $\text{TCC} \Leftrightarrow \text{SEC}$. For the former equivalence, one uses (9.1), while for the latter, one uses (9.2).

If the gravitational theory is not GR, then things are a little more complicated. Nevertheless, modified theories of gravity are often (though not always) extensions of GR, e.g., $f(R)$ gravity, quadratic gravity, scalar-tensor gravity (such as k -essence, Galileon, Horndeski, etc.), and more. In that case (that is, when an Einstein frame exists), one can define an ‘effective’ energy-momentum tensor that encapsulates the modifications to GR, such that the field equations read

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}^{(\text{eff})} \equiv 8\pi G_N \left(T_{\mu\nu}^{(\text{matter})} + T_{\mu\nu}^{(\text{ext})} \right). \quad (9.6)$$

Whether the extension of gravity is due to a scalar field, higher-curvature terms, or something else, one writes it ‘on the right-hand side’ as $T_{\mu\nu}^{(\text{ext})}$. Then, the key question becomes whether the standard classical convergence conditions (NCC, TCC, ...), which are useful for singularity theorems, can be translated as before into ‘reasonable’ energy conditions on the full $T_{\mu\nu}^{(\text{eff})}$, which includes $T_{\mu\nu}^{(\text{ext})}$. However, in these cases, determining what qualifies as ‘reasonable’ energy conditions, even excluding quantum effects, is yet more ambiguous.

A straightforward way to see this is that if one has a modified theory that is written as (9.6), i.e., if an appropriate Einstein frame exists (this is not always the case; see, e.g., [287, 288, 289, 290]), energy conditions may proceed in the ‘standard way’ in that frame. However, the difficulty and challenges arise in the interpretation of the energy conditions across different frames (e.g., Einstein and Jordan frames) and if the energy conditions in the Einstein frame are physical. It is easy to have models where in one frame (Einstein to begin) there is satisfaction or violation of, e.g., the NEC, but where this result does not carry over into the Jordan frame identically (see [291, 292] for an explicit example).

In general, to switch between the two frames one must apply the appropriate conformal (or even disformal) transformation on the metric and the additional fields present. In doing this, one obtains an ‘energy condition’ in the Jordan frame, which possesses more terms than the Einstein-frame condition (see, e.g., [293, 294]). For example, the NCC in the Einstein frame will read $R_{\mu\nu}^E K^\mu K^\nu \geq 0$, where $R_{\mu\nu}^E$ is the Einstein-frame Ricci tensor, i.e., the Ricci tensor computed from the Einstein-frame metric $g_{\mu\nu}^E$. This condition does not imply $R_{\mu\nu}^J K^\mu K^\nu \geq 0$, where the superscript refers to the Jordan frame this time and where the metrics across the two frames are related through a conformal transformation of the form $g_{\mu\nu}^E = \Omega^2 g_{\mu\nu}^J$, for some

spacetime-dependent function Ω^2 . There are also numerous subclasses of theories that have been investigated individually in regard to their energy conditions (e.g., [295, 296, 237]), though details can be theory dependent.

Despite the potential subtleties about energy conditions in modified gravity, some singularity theorems can nevertheless be rigorously proven in many modified gravity situations (e.g., [297, 298, 299, 300, 301, 302]) with certain adjustments to the convergence conditions to accommodate the modifications to gravity. Furthermore, there are also some singularity theorems in GR when energy conditions are weakened [303, 304, 305, 306, 307, 308] or when working in a nonsmooth setting [309, 310, 311, 312, 313, 314, 315, 316, 317, 318].

9.2.2 Semiclassical versions

As already mentioned, classical GR breaks down in the approach to high-curvature regimes, in particular near classical singularities (as predicted under the aforementioned singularity theorems, which need to assume classical energy/convergence conditions). Therefore, we expect some form of quantum gravity to address that. However, even in low-curvature regimes far below Planck scales, the quantum effects of matter/tensor perturbations can be significant/observable (e.g., generating the seeds of large-scale structures, Hawking radiation, etc.), while the background spacetime is still well approximated by classical geometry. In the most conservative approach, we could apply semiclassical gravity to account for such effects by maintaining a ‘classical’ spacetime geometry while quantizing the matter sector. In this case, assuming that the quantum fluctuations of the stress tensor are much smaller than its expectation value, the field equations would read³

$$G_{\mu\nu} = 8\pi G_N \langle \hat{T}_{\mu\nu} \rangle, \quad (9.7)$$

where $\langle \hat{T}_{\mu\nu} \rangle \equiv \langle \Psi | \hat{T}_{\mu\nu} | \Psi \rangle$ indicates the regulated quantum expectation value of the energy-momentum tensor quantum operator given some quantum state $|\Psi\rangle$. (See, e.g., [319] for examples of how this may be computed.) One may then ask if there are reasonable energy conditions that $\langle \hat{T}_{\mu\nu} \rangle$ satisfies (in addition to those for remaining in the semiclassical approximation). Those would be semiclassical energy conditions. (There is vast literature on the subject, but see, e.g.,

³In cosmological perturbation theory, this picture becomes slightly more intricate since not only are the gravitational waves quantized, but the distinction between scalar metric perturbations and quantum modes corresponding to matter fields is not as clear-cut.

[320, 321, 322, 323] for reviews.) An example is the ANEC, which stipulates that

$$\int_{\gamma} d\lambda \langle \Psi | \hat{T}_{\mu\nu} K^{\mu} K^{\nu} | \Psi \rangle \geq 0 \quad (9.8)$$

for any quantum state $|\Psi\rangle$ and for any achronal null geodesic γ . Note that K^{μ} in that case is the null tangent vector to γ , and λ is an affine parameter along γ . While the ANEC has been shown to hold for many physically motivated QFTs, its applicability is limited in the sense that verifying the ANEC requires knowledge of the full spacetime (and every null geodesic in that spacetime). The ANEC is a very nonlocal concept in that sense.

For this reason, we will be more interested in this work in semilocal energy conditions and specifically in a more recent proposal that is also applicable along null geodesics, namely the SNEC [233, 234]. The SNEC conjecture expands on the concept of quantum energy inequalities in (1+1)-dimensional (along timelike and null geodesics) and (3+1)-dimensional (along timelike geodesics) Minkowski spacetimes derived in [220] and further refined by [324]. It suggests introducing a real positive smearing function, $f_{\tau}^2(\lambda)$, where λ is an affine parameter for a null geodesic γ and τ denotes the smearing scale of the smearing function. The smearing function must be square integrable and normalized such that

$$\int_{\gamma} d\lambda f_{\tau}^2(\lambda) = 1. \quad (9.9)$$

Then, the smearing of a test function $F(\lambda)$ along the null geodesic is done through

$$\mathbb{E}_{\tau}[F] \equiv \int_{\gamma} d\lambda f_{\tau}^2(\lambda) F(\lambda), \quad (9.10)$$

and the corresponding characteristic smearing affine ‘length/time’ scale τ is given according to

$$\frac{1}{\tau^2} \equiv 4 \int_{\gamma} d\lambda \left(\frac{df_{\tau}}{d\lambda} \right)^2. \quad (9.11)$$

As such, $f_{\tau}(\lambda) = \sqrt{f_{\tau}^2(\lambda)}$ needs to be continuously differentiable, in principle. The SNEC then claims that

$$\mathbb{E}_{\tau} \left[\langle \Psi | \hat{T}_{\mu\nu} K^{\mu} K^{\nu} | \Psi \rangle \right] \geq -\frac{B}{G_N \tau^2} \quad (9.12)$$

for any achronal null geodesic with null tangent vector K^{μ} , any quantum state $|\Psi\rangle$ within the regime of perturbative quantum gravity, and any reasonable smearing function f_{τ}^2 (more about

this below). On the right-hand side, B is a real positive constant, whose precise value is unknown (again, more about this later).

The original goal of the SNEC [214, 233] was to find a general, (semi)local constraint on NEC violation in any dimensions that does not require additional quantities in QFT (such as entanglement entropy), with the motivation to restrict models (such as traversable wormholes or models with a large number of species that can lead to significant negative densities) that were not excluded by the ANEC or quantum inequalities in lower dimensions; see, e.g., [220, 324, 325, 326]. As discussed in [233], there are similarities between the SNEC and other quantum inequalities (e.g., in 1+1 dimensions [324, 325] and for classes of interacting conformal field theories [327, 328]). However, those results do not seem to extend to 3 + 1 dimensions.

The initial SNEC formulation [233] as described above, inspired by semilocal quantum inequalities [220], uses smeared stress tensors to address large negative fluctuations of the energy-momentum tensor. To ensure this, and to mitigate the potential accumulated effect of Hawking radiation for null geodesics in the proximity of a black hole horizon, the regime of validity of the bound was assumed to be such that the smearing length scale τ is small compared to the background spacetime curvature scale. However, in the follow-up work [234], it was claimed that the bound may also hold for smearing length scales τ comparable to or larger than the spacetime curvature radius $\mathcal{R} \sim |R|^{-1/2}$ (or $\mathcal{R} \sim |R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}|^{-1/4}$ if, e.g., the spacetime is Ricci flat), and this was shown for some explicit examples, such as evaporating black holes. As we will see in this work, the SNEC yields its most interesting constraint in cosmology when the smearing scale is of the order of the spacetime curvature.

An important property of the SNEC is that it reduces to the ANEC in the limit of infinite smearing scale. Indeed, taking $\tau \rightarrow \infty$, it is not too hard to see that (9.12) reduces to (9.8). This was formally showed by [233] with a rescaling method first used in [220, 329, 301, 306]: as τ goes to infinity, the smearing function converges to a constant distribution and the SNEC reduces to the ANEC. Another nice aspect of SNEC is that it is invariant under rescaling of the affine parameter. This addresses one of the arguments against the existence of null worldline inequalities in 3 + 1 dimensions [330]. Furthermore, [234] imposes a Lorentz-invariant ultraviolet cutoff to show that the counterexample provided in [330] also satisfies this bound.

The constant B that appears on the right-hand side of (9.12) is unknown. Its value would depend on the relation between the ultraviolet cutoff and Newton's constant. In the case of induced gravity on a brane in the anti-de Sitter/conformal field theory correspondence [331], it

was proven that $B = 1/(32\pi)$. In what follows, we often use this value as a benchmark, but we must keep in mind that B could potentially be as large as 1, though more realistically it could also be much smaller. In fact, [234] points to the fact that B is typically expected to be much smaller than unity when semiclassical gravity is well under control.

Given the SNEC (9.12) as a conjecture and assuming semiclassical gravity, namely (9.7), one can say that classical geometries should satisfy a kind of ‘smeared null convergence condition’,

$$\mathbb{E}_\tau [R_{\mu\nu} K^\mu K^\nu] \geq -\frac{8\pi B}{\tau^2}. \quad (9.13)$$

This is similar to the correspondence between the classical NEC and the NCC. In what follows, we wish to assume as little as possible about the nature of the quantum fields (or their specific quantum states) that would lead to violating the NEC. We simply assume they exist and drive the background cosmology so to yield, e.g., a phase of phantom DE acceleration or a bouncing cosmology. This allows us to do a purely ‘geometrical’ analysis and ask: given a background cosmology with metric $g_{\mu\nu}$ and Ricci tensor $R_{\mu\nu}$, is the SNEC — expressed as (9.13) as a curvature bound — satisfied or not? While geometrical, the bound (9.13) nevertheless takes its root as a physical, semiclassical quantum inequality. Therefore, if a certain spacetime geometry cannot respect the SNEC (9.13), then we may say that the NEC-violating quantum fields (or states) that would presumably yield such a cosmological background are still nonphysical, as they violate a potentially fundamental energy condition, even though they may appear to be within the semiclassical approximation.

9.3 The SNEC in cosmology

9.3.1 The full SNEC

We want to explore the consequences of the SNEC in semiclassical GR, which following (9.13) we express as

$$\int_\gamma d\lambda f_\tau^2(\lambda) R_{\mu\nu} \frac{dX^\mu}{d\lambda} \frac{dX^\nu}{d\lambda} \geq -\frac{8\pi B}{\tau^2}, \quad (9.14)$$

where τ satisfies (9.11), and where γ denotes a null geodesic parametrized by the curve $X^\mu(\lambda)$, so the tangent vector along the curve is $K^\mu \equiv dX^\mu/d\lambda$ such that $K^\mu K_\mu = 0$. Note that the

smearing function $f_\tau^2(\lambda)$ should have dimensions of mass, so that (9.9) is respected. An example may be a Gaussian,

$$f_\tau^2(\lambda) = \frac{1}{\sqrt{2\pi} \tau} \exp\left(-\frac{(\lambda - \bar{\lambda})^2}{2\tau^2}\right), \quad (9.15)$$

or a Lorentzian,

$$f_\tau^2(\lambda) = \frac{\tau}{\pi\sqrt{2} \left((\lambda - \bar{\lambda})^2 + \tau^2/2\right)}, \quad (9.16)$$

for some real constants $\bar{\lambda}$ (controlling the center of the smearing) and $\tau > 0$ (setting the smearing length scale). Both (9.9) and (9.11) hold in those cases over $\lambda \in (-\infty, \infty)$. Naturally for the Gaussian, the smearing scale τ also has the interpretation of the standard deviation of the distribution $f_\tau^2(\lambda)$ about its mean $\bar{\lambda}$.

Let us consider a flat Friedmann-Lemaître-Robertson-Walker (FLRW) cosmology, described by the metric $g_{\mu\nu}dx^\mu dx^\nu = a(\eta)^2(-d\eta^2 + dr^2 + r^2 d\Omega_{(2)}^2)$, where a is the scale factor, η is the conformal time related to the physical time through $d\eta \equiv a^{-1}dt$, and $d\Omega_{(2)}^2 = d\theta^2 + \sin^2(\theta)d\phi^2$ is the line element of the unit 2-sphere. An affine parameter λ for a null geodesic $X^\mu(\lambda)$ can be taken such that $d\lambda \equiv a^2 d\eta = a dt$ without loss of generality (see, e.g., [332]). Then, we consider $K^\mu(\eta, r, \theta, \phi) = a^{-2}(1, 1, 0, 0)$ as the generic (outgoing and nonrotating) null solution to the geodesic equation, $K^\nu \nabla_\nu K^\mu = dK^\mu/d\lambda + \Gamma_{\rho\sigma}^\mu K^\rho K^\sigma = 0$. A simple calculation then shows

$$R_{\mu\nu} K^\mu K^\nu = -2 \frac{\dot{H}}{a^2}, \quad (9.17)$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and a dot denotes a derivative with respect to physical time, t . Thus, the SNEC (9.14) becomes

$$\int_{\lambda_i}^{\lambda_f} d\lambda f_\tau^2(\lambda) \frac{\dot{H}}{a^2} \leq \frac{4\pi B}{\tau^2} \quad (9.18a)$$

$$\Leftrightarrow \int_{\lambda_i}^{\lambda_f} d\lambda \left(16\pi B \left(\frac{df_\tau}{d\lambda} \right)^2 - f_\tau^2(\lambda) \frac{\dot{H}}{a^2} \right) \geq 0, \quad (9.18b)$$

where one should view $\dot{H}/a^2$ as a function of λ . Note that we do not necessarily assume geodesic completeness here: if the null geodesic γ is past (future) incomplete, then $\lambda_i > -\infty$ ($\lambda_f < \infty$); $\lambda_i = -\infty$ ($\lambda_f = \infty$) otherwise.

Given a solution for the scale factor, $a(t)$, one can express the affine length of null geodesics as

$$\lambda(t) = \lambda_0 + \int_{t_0}^t d\tilde{t} a(\tilde{t}), \quad (9.19)$$

where one chooses the arbitrary reference scale $\lambda(t_0) = \lambda_0$. From this, one can reexpress the integral in the SNEC as a time integral,

$$\int_{t_i}^{t_f} dt f_\tau^2(\lambda(t)) \frac{\dot{H}}{a} \leq \frac{4\pi B}{\tau^2}, \quad (9.20)$$

where it is understood that $\lambda(t_i) = \lambda_i$ and $\lambda(t_f) = \lambda_f$. Alternatively, it is sometimes easier to find a solution for the Hubble parameter as a function of the scale factor, $H(a)$. In such a case, $\dot{H} = aHdH/da$ and $d\lambda = da/H$, so the SNEC becomes

$$\int_{a_i}^{a_f} \frac{da}{a} f_\tau^2(\lambda(a)) \frac{dH}{da} \leq \frac{4\pi B}{\tau^2}, \quad (9.21)$$

where

$$\lambda(a) = \lambda_0 + \int_{a_0}^a \frac{d\tilde{a}}{H(\tilde{a})}, \quad (9.22)$$

so $\lambda(a_i) = \lambda_i$ and $\lambda(a_f) = \lambda_f$. While (9.20) and (9.21) show that we can perform changes of variable and express the SNEC integral as a smearing over time or scale factor, we should keep in mind that, physically, the smearing is really done along a null geodesic (as opposed to a timelike geodesic), as it is defined and normalized in terms of the null affine parameter λ .

Clearly, if the classical NEC — or equivalently the NCC (9.5) — is satisfied in flat FLRW, then we know from (9.17) that $\dot{H}(t) \leq 0$, hence the SNEC (9.18a) is always satisfied. (This is always true, not just in FLRW, i.e., $\text{NEC} \Rightarrow \text{SNEC}$, but $\text{SNEC} \not\Rightarrow \text{NEC}$.) It thus becomes interesting to use the SNEC to constrain cosmologies that predict $\dot{H} > 0$, i.e., violation of the classical NEC/NCC.

9.3.2 The SNEC applied to individual matter components and the corresponding geometrical picture

We have so far considered the SNEC for the total geometry of a FLRW universe, but the matter content of the universe consists of several fields, not all of which may violate the NEC in itself. Let us consider the case where the matter sector consists of several noninteracting components,

$$T_{\mu\nu} = \sum_I T_{\mu\nu}^I, \quad (9.23)$$

where the index I labels the different components (radiation, pressureless matter, DE, etc.). Then, we can define a ‘contribution to the curvature’ for each matter component as

$$G_{\mu\nu}^I \equiv 8\pi G_N T_{\mu\nu}^I, \quad (9.24)$$

such that the Einstein equations (9.1) with (9.23) and (9.24) give $G_{\mu\nu} = \sum_I G_{\mu\nu}^I$. Equivalently, we could define $R_{\mu\nu}^I \equiv 8\pi G_N (T_{\mu\nu}^I - (T^I/2)g_{\mu\nu})$, $R^I \equiv g^{\mu\nu} R_{\mu\nu}^I$, with $T^I \equiv g^{\alpha\beta} T_{\alpha\beta}^I$, such that $G_{\mu\nu}^I = R_{\mu\nu}^I - (R^I/2)g_{\mu\nu}$, $R_{\mu\nu} = \sum_I R_{\mu\nu}^I$, and $R = \sum_I R^I$. It is important to emphasize that we are not decomposing the Einstein equations into separate, independent equations here. Indeed, it would be inappropriate to separate the metric tensor into the sum of individual ‘metric components’ because of the nonlinearities involved, hence $g_{\mu\nu}$ always appears as the ‘total’ metric in the above.

The above definitions suggest we can view the source contributions to curvature as being the sum of several ‘components’. With this definition, the NEC for a given component, $T_{\mu\nu}^I K^\mu K^\nu \geq 0$, can now be expressed in terms of the corresponding ‘curvature contribution’, $R_{\mu\nu}^I K^\mu K^\nu \geq 0$. If those hold for all components I , then naturally the full NEC and NCC, (9.4) and (9.5), are implied. However, the converse is not necessarily true, i.e., the validity of the full NEC does not mean that each component must individually respect the NEC. In other words, it is certainly possible to have some component violate the NEC, $T_{\mu\nu}^I K^\mu K^\nu \not\geq 0$, such that the full NEC (9.4) still holds.

Let us consider FLRW with a perfect fluid in (t, x, y, z) Cartesian coordinates, so $g_{\mu\nu} = \text{diag}(-1, a^2, a^2, a^2)$, $T^\mu{}_\nu = \text{diag}(-\rho, p, p, p)$, and $T_{\mu\nu} = \text{diag}(\rho, a^2 p, a^2 p, a^2 p)$. With

$$K^\mu = (a^{-1}, a^{-2}, 0, 0)$$

in these coordinates, we get

$$T_{\mu\nu} K^\mu K^\nu = \frac{1}{a^2}(\rho + p). \quad (9.25)$$

This is in agreement with (9.17) and the equation of motion one can derive from (9.1) in FLRW, namely $\dot{H} = -4\pi G_N(\rho + p)$. If the matter sector consists of several noninteracting components, then we may write $\rho = \sum_I \rho_I$ and $p = \sum_I p_I$, hence $\dot{H} = -4\pi G_N \sum_I (\rho_I + p_I)$ and

$$T_{\mu\nu} K^\mu K^\nu = \sum_I T_{\mu\nu}^I K^\mu K^\nu = \frac{1}{a^2} \sum_I (\rho_I + p_I). \quad (9.26)$$

Upon defining

$$\dot{H}_I \equiv -4\pi G_N (\rho_I + p_I), \quad (9.27)$$

we get $\dot{H} = \sum_I \dot{H}_I$ and

$$R_{\mu\nu} K^\mu K^\nu = \sum_I R_{\mu\nu}^I K^\mu K^\nu = -\frac{2}{a^2} \sum_I \dot{H}_I, \quad (9.28)$$

so $\dot{H}$ separates into several $\dot{H}_I$ ‘curvature contributions’. As noted previously, the metric does not separate into ‘contributions’, though, so in the above a is really the ‘total’ scale factor.

If we wish to impose the classical NEC on the $T_{\mu\nu}^I$ s individually, then this is equivalent to imposing $\rho_I + p_I \geq 0$ for all I s. However, if we want to impose the SNEC on individual components, then things are a little different. Indeed, let us write the SNEC (9.12) on individual components as

$$\int_{\lambda_i}^{\lambda_f} d\lambda f_\tau^2(\lambda) T_{\mu\nu}^I K^\mu K^\nu \geq -\frac{B}{G_N \tau^2}, \quad (9.29)$$

where we view $T_{\mu\nu}^I = \langle \hat{T}_{\mu\nu}^I \rangle$. Then in FLRW this means

$$\int_{\lambda_i}^{\lambda_f} d\lambda f_\tau^2(\lambda) \frac{\rho_I + p_I}{a^2} \geq -\frac{B}{G_N \tau^2} \quad (9.30a)$$

$$\Leftrightarrow \int_{\lambda_i}^{\lambda_f} d\lambda f_\tau^2(\lambda) \frac{\dot{H}_I}{a^2} \leq \frac{4\pi B}{\tau^2}. \quad (9.30b)$$

Following from our assumption that there are no ‘energy transfers’ (interactions) between components, we have $\nabla^\mu T_{\mu\nu}^I = 0$ individually for any I s. This means $\dot{\rho}_I + 3H(\rho_I + p_I) = 0$, and for example if we consider their respective equations of state as $p_I = w_I \rho_I$ such that the w_I s remain constant with negligible fluctuations, then we get

$$\rho_I = \rho_{I,0} a^{-3(1+w_I)}, \quad (9.31)$$

hence from (9.27),

$$\dot{H}_I = -4\pi G_N (1 + w_I) \rho_{I,0} a^{-3(1+w_I)}, \quad (9.32)$$

where the constants $\rho_{I,0}$ indicate the value of the ρ_I s when $a = a_0 = 1$ (normalized to be the value of the scale factor in the universe today; henceforth the subscript 0 indicates that a quantity is evaluated today). Recall H , the ‘total’ Hubble parameter, is given by Friedmann’s equation

$$\frac{3}{8\pi G_N} H^2 = \rho = \sum_I \rho_I = \sum_I \frac{\rho_{I,0}}{a^{3(1+w_I)}}. \quad (9.33)$$

Furthermore, since $d\lambda = da/H$, the SNEC (9.30b) becomes

$$L_I \equiv \int_{a_i}^{a_f} \frac{da}{H} f_\tau^2(\lambda(a)) \frac{\dot{H}_I}{a^2} \leq \frac{4\pi B}{\tau^2}. \quad (9.34)$$

Therefore, the integral on the left-hand side of the inequality, denoted as L_I , may be expressed as

$$L_I = -\sqrt{6\pi G_N} (1 + w_I) \rho_{I,0} \times \int_{a_i}^{a_f} da \frac{f_\tau^2(\lambda(a))}{a^{5+3w_I}} \left(\sum_J \frac{\rho_{J,0}}{a^{3(1+w_J)}} \right)^{-1/2}, \quad (9.35)$$

when combining (9.32) and (9.33), assuming an expanding universe with $H > 0$.

Note that, while convenient, the whole ‘curvature contribution’ terminology was not necessary here, i.e., one could simply express the SNEC in terms of the energy components, (9.30a), and use (9.31) and (9.33) to arrive at (9.35). Nevertheless, what it shows is that when applying the SNEC to a single component I , one still has to take into account all other components [the sum over J in (9.35)]. This is because evaluating the SNEC requires knowledge of the whole cosmology. The above shall be very useful in cosmological contexts where several fields ‘compete’ in the energy budget, as in the late era (see the next section). This will be less important for a bouncing universe since then a single field usually dominates the dynamics.

9.4 Dark energy

The onset of the DE era, which marks the current acceleration of cosmic expansion, is a relatively recent event in the history of the universe. The primary candidate for explaining this acceleration is a cosmological constant Λ , characterized by $w = -1$, with the time variation of the DE EoS considered negligible on cosmological scales. Interestingly, recent observations (DESI; [3, 4, 333, 334, 5, 6]) seem to support the assumption that the DE EoS may be time dependent. Consequently, current fitting models often use a constant w and, to next order, a parametrized time-dependent EoS, assuming that its variation is small, which we will discuss in subsequent sections. Theoretically, beyond the cosmological constant, these phenomenological models may serve as reasonable approximations during brief cosmological phases, but they are less applicable for extended periods of evolution. Additionally, this paper focuses on exploring the implications

of the SNEC on NEC-violating (phantom) DE driven by quantum effects, which are challenging to model and might reveal a more intricate time dependence in the EoS. Nevertheless, as a first step, our goal is not to construct such top-down models but rather to adopt an intermediate approach, where we assume these phenomenological models act as proxies for NEC violation (meaning the EoS variation is assumed to be minor, monotonic, and to occur over long durations) and explore the implications of the SNEC for such cosmologies. However, it would be intriguing to further investigate time-dependent equations of state and model NEC violation through top-down QFT constructions leading to phantom-like models.

9.4.1 Constant equation of state

For simplicity at first, and bearing in mind the caveats stated above, we consider a universe that has a phantom DE fluid with constant EoS parameter $w \equiv p_{\text{DE}}/\rho_{\text{DE}} < -1$ with the addition of pressureless matter (cold dark matter and baryons). As is evident from the sign of $-3(1+w_I)$ in (9.31) and as expected in an expanding universe, pressureless matter dominates at early times, whereas the phantom DE becomes dominant at later times. Denoting their respective fractional energy densities today by $\Omega_{\text{DE}} \equiv 8\pi G_{\text{N}}\rho_{\text{DE},0}/(3H_0^2)$ and $\Omega_{\text{m}} \equiv 8\pi G_{\text{N}}\rho_{\text{m},0}/(3H_0^2)$, we have $\Omega_{\text{m}} = 1 - \Omega_{\text{DE}}$. Then, the SNEC integral (9.35) applied to the DE component becomes

$$L_{\text{DE}} = -\frac{3}{2}H_0(1+w)\Omega_{\text{DE}}^{1/2} \times \int_{a_i}^{a_f} da \frac{f_{\tau}^2(\lambda(a))}{a^{(7+3w)/2}} \left(1 + \frac{1 - \Omega_{\text{DE}}}{\Omega_{\text{DE}}} a^{3w}\right)^{-1/2}, \quad (9.36)$$

and the affine length (9.22) satisfies

$$\begin{aligned} \lambda_0 - \lambda(a) &= \frac{1}{H_0\Omega_{\text{DE}}^{1/2}} \int_a^{a_0} d\tilde{a} \frac{a^{3(1+w)/2}}{\sqrt{1 + \frac{1-\Omega_{\text{DE}}}{\Omega_{\text{DE}}} a^{3w}}} \\ &= \frac{2\tilde{a}^{(5+3w)/2}}{(5+3w)H_0\Omega_{\text{DE}}^{1/2}} \\ &\quad \times {}_2F_1\left(\frac{1}{2}, \frac{5+3w}{6w}, \frac{5+9w}{6w}, -\frac{1-\Omega_{\text{DE}}}{\Omega_{\text{DE}}} \tilde{a}^{3w}\right) \Big|_{\tilde{a}=a}^{a_0}, \end{aligned} \quad (9.37)$$

where ${}_2F_1$ is the Gaussian hypergeometric function. One can check that in the limit $a \rightarrow \infty$, $\lambda(a)$ reaches a finite constant if $w < -5/3$, indicating future null geodesic incompleteness (i.e., a big rip; e.g., [243, 335, 336]), while null geodesics are complete if $w \geq -5/3$. As for the past,

since pressureless matter becomes dominant one naturally reaches a big bang singularity taking the scale factor to zero, but the model can only be applied until matter-radiation equality when $a_{\text{eq}} \approx 1/3400$. (Improving the model with the addition of a radiation component would not affect the subsequent results significantly, since what should matter most is the DE era at late times and where the NEC for DE is most motivated to be violated. In the same vein, what happens before standard big bang cosmology is irrelevant for this part.)

Let us evaluate (9.36), first making some rough approximations to gain some intuition. Specifically, let us take a continuous uniform distribution for our smearing function as

$$f_\tau^2(\lambda) = \begin{cases} 1/(\lambda_2 - \lambda_1) & \lambda \in [\lambda_1, \lambda_2] \\ 0 & \lambda \notin [\lambda_1, \lambda_2]. \end{cases} \quad (9.38)$$

Note that such a smearing function is not quite appropriate since one cannot use (9.11) to define the smearing scale τ (i.e., f_τ is not continuously differentiable).⁴ However, it does respect (9.9) as long as γ covers any range $[\lambda_i, \lambda_f]$ with $\lambda_i \leq \lambda_1$ and $\lambda_2 \leq \lambda_f$. Then, since the above smearing function is symmetric about its mean, $\mathbb{E}_\tau[\lambda] = (\lambda_1 + \lambda_2)/2$, we can take τ such that it corresponds to the standard deviation of the ‘probability’ distribution itself, $\tau = \sqrt{\mathbb{E}_\tau[\lambda^2] - \mathbb{E}_\tau[\lambda]^2} = \Delta\lambda / (2\sqrt{3})$, where $\Delta\lambda \equiv \lambda_2 - \lambda_1$. Correspondingly, the SNEC (9.34) using (9.36) becomes

$$\int_{a_1}^{a_2} \frac{da}{a^{(7+3w)/2} \sqrt{1 + \frac{1-\Omega_{\text{DE}}}{\Omega_{\text{DE}}} a^{3w}}} \leq \frac{1}{H_0 |1+w| \Omega_{\text{DE}}^{1/2} \Delta\lambda}, \quad (9.39)$$

where $\lambda(a_1) = \lambda_1$, $\lambda(a_2) = \lambda_2$, and for concreteness we take $B = 1/(32\pi)$. The integral on the left-hand side is another hypergeometric function, so once more for the purpose of a rough approximation, let us perform an expansion about $\Omega_{\text{DE}} \approx 1$, which is to say that DE should dominate over pressureless matter in the regime of interest. Then, to leading order one finds

$$\frac{16|1+w|}{(5+3w)^2} \sinh^2 \left(\frac{5+3w}{4} \ln \left(\frac{a_2}{a_1} \right) \right) \leq 1. \quad (9.40)$$

What this equation and (9.39) show is that the bound on the EoS w depends mainly on the chosen smearing scale $\tau \propto \Delta\lambda$, which controls a_2/a_1 in the above. For instance, if we set the

⁴One could instead work with a smooth version of the above (e.g., [337]), but for the purpose of the approximation, (9.38) is fine, and we will resort to other smooth smearing functions later.

final integration time to today, $a_2 = a_0 = 1$, and the lower integration limit to be matter-radiation equality, $a_1 = a_{\text{eq}}$, then the SNEC is satisfied as long as $w \gtrsim -1.003$, while the bound is much weaker if the integration only begins when DE starts dominating, e.g., $w \gtrsim -3.58$ when $a_1 = 4/7$ (corresponding to a redshift of 0.75). This is intuitively expected when assuming $w < -1$ persisted longer, because then NEC violation accumulates more and the bound should get tighter. Similarly, at fixed a_1 , we always find that the SNEC can only hold if $w \geq -1$ if we send $a_2 \rightarrow \infty$. Indeed, in such a limit ($\tau \rightarrow \infty$), we should recover the ANEC, which forbids the dominance of a NEC-violating phantom field forever in the future.

The above highlights an important fact about our implementation of the SNEC, which is that what matters is not only the effect of DE in the late-time universe when it initiates a period of accelerated expansion, but rather the presence of DE as a phantom field in the whole history of the observable universe. In our simple model with only matter [$\rho_{\text{m}}(a) = \rho_{\text{m},0}a^{-3}$] and DE [$\rho_{\text{DE}}(a) = \rho_{\text{DE},0}a^{-3(1+w)}$] such that $\Omega_{\text{m}} = 1 - \Omega_{\text{DE}}$, the transition between matter domination and DE domination in terms of the DE EoS occurs at the scale factor value [the solution to $\rho_{\text{m}}(a_{\star}) = \rho_{\text{DE}}(a_{\star})$]

$$a_{\star} = \left(\frac{\Omega_{\text{DE}}}{1 - \Omega_{\text{DE}}} \right)^{1/(3w)}. \quad (9.41)$$

The corresponding redshift is $z_{\star} = 1/a_{\star} - 1$. As long as $\Omega_{\text{DE}} > 1/2$, then $a_{\star} < 1$ for any $w < -1$, meaning that DE starts dominating at some point in the past. Note that the more negative w is, the closer to today DE starts dominating, so the shorter the DE-dominated phase. If we evaluate (9.39) from the start of DE domination ($a_1 = a_{\star}$) to today ($a_2 = 1$), then we find that the SNEC is always satisfied, for any EoS, no matter how negative (assuming the benchmark value of $B = 1/(32\pi)$; we discuss the value of B further below). The SNEC in this case could imply a constraint on w if we integrated to the future ($a_2 > 1$; in fact, integrating all the way to $a_2 \rightarrow \infty$ we would recover $w \not\geq -1$). However, we do not find it theoretically or observationally motivated since we cannot ‘observe’ the future. In particular, it could be that DE decays to the future (see, e.g., [338, 339, 340]). Therefore, it is more reasonable to consider $a_1 < a_{\star}$. Prior to DE domination, our ability to observationally probe the DE EoS is much lower, as its effect on cosmology is much weaker. Nevertheless, if we assume that DE, as a constant EoS phantom field, was present for essentially as long as we observe the universe, then the SNEC provides interesting constraints, qualitatively in agreement with our analytical estimate obtained in (9.39), to which we turn to next.

Approaching the problem numerically this time, let us take a smooth smearing function,

specifically the Gaussian (9.15), though we found the results to be quantitatively similar for different choices, such as the Lorentzian smearing function (9.16). We shall fix the lower and upper integration limits: we take $a_f = a_0 = 1$ so that there is no contribution to the SNEC coming from the future as explained above; and the lower integration limit is taken as small as reasonable, so $a_i = a_{\text{eq}}$ for concreteness. We then properly normalize the Gaussian (9.15) such that its integral from $\lambda_i = \lambda(a_i)$ to $\lambda_f = \lambda(a_f)$ is unity. The center of the Gaussian is parametrized according to $\bar{\lambda} = \lambda(\bar{a})$, and likewise, the smearing scale is parametrized according to $\tau = (\lambda(a_+) - \lambda(a_-))/2$, with $a_{\pm} = \bar{a} \pm \Delta a/2$ as long as $\bar{a} + \Delta a/2 \leq a_0$ and $\bar{a} - \Delta a/2 \geq a_{\text{eq}}$. If $\bar{a} + \Delta a/2 > a_0$, we instead set $a_+ = a_0$ and $a_- = a_0 - \Delta a$. Likewise if $\bar{a} - \Delta a/2 < a_{\text{eq}}$, we instead set $a_- = a_{\text{eq}}$ and $a_+ = a_{\text{eq}} + \Delta a$. In that sense, $\bar{a}$ controls the position of the smearing in time, and Δa (which is associated with the smearing scale) controls the total width. Recall that $\lambda(a)$ is given by (9.37), where the value of λ_0 is arbitrary. Likewise, the numerical value of H_0 cancels out, so it is irrelevant.

We can then numerically perform the integral of (9.36) and determine when the SNEC (9.34) is satisfied or not. This depends on the value of w , Ω_{DE} , $\bar{a}$, Δa , and B . For certain choices of parameters, we determine the critical value of the DE EoS w , beneath which the SNEC is violated. A first example is shown in Fig. 9.1, where the SNEC constraint on w is plotted as a function of Δa , with different curves showing different values of $\bar{a}$. The other parameters are fixed as $\Omega_{\text{DE}} = 0.7$ and $B = 1/(96\pi)$. Similar plots are shown in Fig. 9.2, where Ω_{DE} and B are varied.

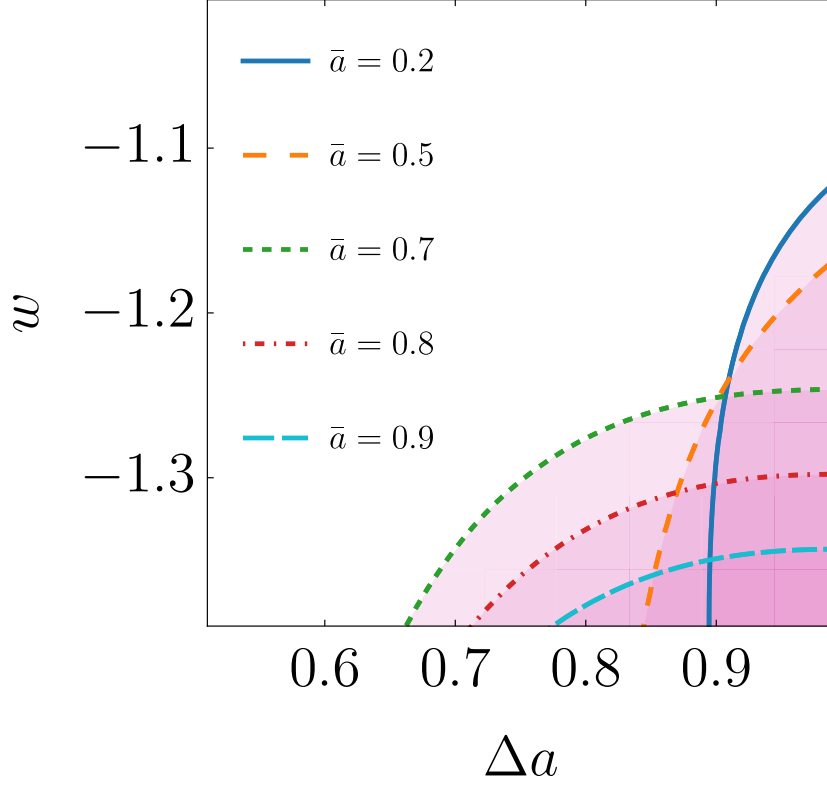


Figure 9.1: SNEC constraints on constant-EoS phantom DE. The constraint on the EoS w is shown as a function of Δa (related to the smearing width) for different values of $\bar{a}$ (related to the center of the smearing). The other parameters are fixed to $\Omega_{\text{DE}} = 0.7$ and $B = 1/(96\pi)$. The pink regions are where the SNEC is violated.

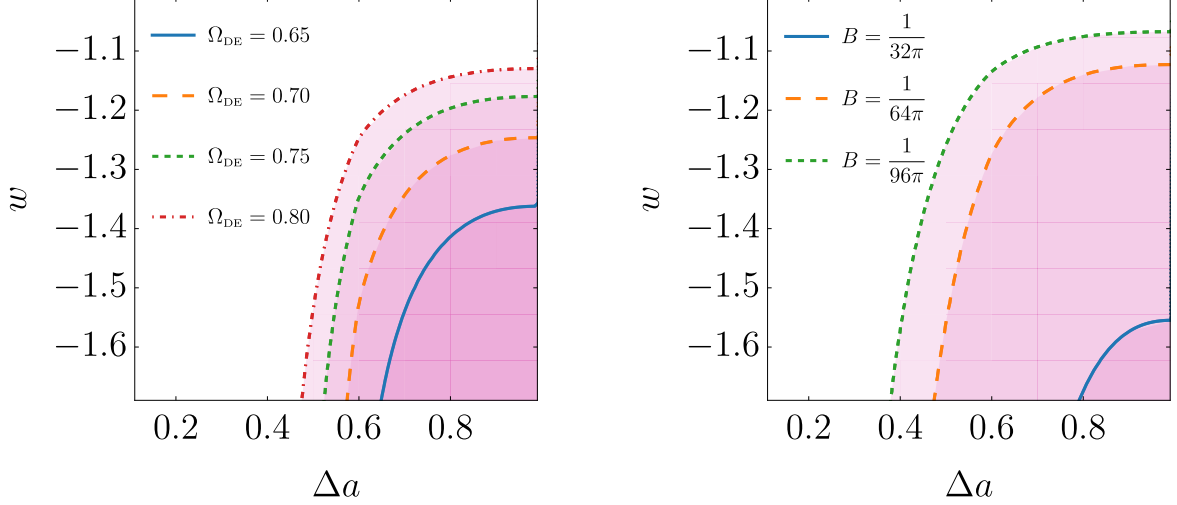


Figure 9.2: Same plots as Fig. 9.1, except now in the left plot we fix $\bar{a} = 0.7$, $B = 1/(96\pi)$, and show different values of Ω_{DE} ; and in the right plot we fix $\bar{a} = 0.7$, $\Omega_{\text{DE}} = 0.9$, and show different values of B .

A first thing we observe is that the constraint on w is always monotonically increasing in Δa . This was to be expected: the smearing scale enters both the ‘left- and right-hand sides’ of the SNEC [expressed as, e.g., (9.29)]. On the ‘left-hand side’, the smearing scale controls the normalization of the smearing and how long NEC violation can accumulate around the peak of the smearing, but its main effect is on the ‘right-hand side’, which scales as $1/\tau^2$. If Δa is small, the smearing scale τ is correspondingly small, and as such, the SNEC permits large amounts of negative energy, hence w can be very negative. Conversely, if Δa is large, the smearing scale τ is correspondingly large, and as such, the SNEC allows only small amounts of negative energy; hence the constraint on w is much closer to the NEC divide ($w = -1$). Consequently, the largest constraint (at fixed $\bar{a}$, B , and Ω_{DE}) always comes from the longest smearing scale allowed, which in our case corresponds to having $\Delta a = a_0 - a_{\text{eq}} \approx 0.999706$. This essentially amounts to the whole ‘observable’ history of the universe, hence one can say that the smearing length scale is of the order of the ‘curvature radius’ of the universe ($\mathcal{R} \sim |R|^{-1/2} \sim |H_0|^{-1}$). We recall that the original SNEC formulation [233] assumed that the smearing length scale τ had to be small compared to the background spacetime curvature scale $\mathcal{R}$, but subsequent work [234] found the SNEC could hold when τ is of the order of $\mathcal{R}$ (or even larger, as in the ANEC limit $\tau \rightarrow \infty$).

In cosmology, assuming B is not too small, the SNEC really appears to be most relevant as a constraint when τ is of the order of $\mathcal{R}$.

Under the premise that we want to smear only over ‘past observable scales’, the maximal value of Δa is naturally bounded by a_0 since we cannot observe the future. There is also a bound to the past, here conservatively taken to be a_{eq} , though in principle we are able to observationally infer the cosmology with confidence all the way to much smaller a values (e.g., down to big bang nucleosynthesis). At the same time, our inference of the DE EoS from observations is mostly dominated by late-time cosmology (see, e.g., [341, 342, 333, 334, 343, 344, 345, 346, 347, 348, 349, 5, 6, 350] for recent observational constraints on DE as a function of redshift). In that sense, we cannot say that we observationally ‘measure’ w in a robust manner over a large smearing scale $\Delta a \sim 1$. Yet in principle when a constant EoS is assumed for DE for a long cosmological period, it still implies that DE is present at all times (even though its contribution to the background evolution is potentially negligible at early times), and it has a significant effect on determining the SNEC bound.

The other parameter that controls the smearing is $\bar{a}$; it dictates where $f_\tau^2(\lambda)$ is centered and thus where it peaks. Different values of $\bar{a}$ are presented in Fig. 9.1, showing how it affects that SNEC constraint. The ‘amount of negative energy’ entering the SNEC has a nontrivial a dependence [cf. (9.36)]; in fact, for some range $-2 \lesssim w < -1$ it tends to pick up significant contributions at smaller a values (earlier times), hence the SNEC is more constraining the lower the center of the smearing function as it picks up more of that negative energy. At the same time, it might not be too realistic to set $\bar{a}$ to a very small value considering we observe DE best in the late universe ($z \lesssim 1$, so $a \gtrsim 0.5$). This is the motivation for fixing $\bar{a} = 0.7$ in Fig. 9.2, though we have to keep in mind that this is somewhat of an arbitrary choice simply to isolate the dependence on the other parameters ($\bar{a} \sim 0.7$ finds motivation in the case of a time-dependent EoS, which we discuss in the next subsection). Moreover, the SNEC — as a conjecture — states that within its regime of validity it should hold for all possible smearing functions, so in particular for any values of $\bar{a}$ and Δa . Under this interpretation, the most stringent constraint should be the one to hold. According to Fig. 9.1, this would imply $w \gtrsim -1.12$ [for $\Omega_{\text{DE}} = 0.7$ and $B = 1/(96\pi)$].

In the left plot of Fig. 9.2, we see how the constraint varies depending on the relative energy density of DE today, Ω_{DE} . The trend is intuitively understood: the larger Ω_{DE} is, the more DE there is in the universe over time, the stronger the constraint on w upon applying the SNEC. The dependence on B , depicted in the right plot of Fig. 9.2, is also straightforward to understand from (9.29): more ‘negative energy’ is allowed the larger B is, and this is what the right plot of

Fig. 9.2 shows. Note that we choose an artificially large amount of DE in that plot, $\Omega_{\text{DE}} = 0.9$, in order to depict a sizable constraint from the case $B = 1/(32\pi)$. Recall that this value of B comes from some examples of quantum inequalities similar to the SNEC, which have been derived in different contexts (lower dimensions, conformal field theories, etc.). In the present situation, the constraints on phantom DE with $B = 1/(32\pi)$ are rather weak (for $\Omega_{\text{DE}} \sim 0.7$, which is closer to what we observe). However, B remains an unknown constant, fundamentally, and as we pointed out earlier, it is postulated in [234] that it may be much lower for semiclassical gravity, hence we took the liberty of setting it to a smaller value in Fig. 9.1 and in the left plot of Fig. 9.2 as to present more stringent constraints on w .

Lastly, we point out that, although throughout this section we assumed a constant DE EoS down to a_{eq} , many results should be applicable if the EoS transitions to a phantom constant EoS at later times (e.g., after recombination or during the dark ages). This is due to the fact that dark energy does not significantly influence the background evolution at earlier times. Consequently, setting $\bar{a}$ after the transition period and considering the upper bound on Δa to be approximately from the transition to today should offer a reasonable estimate for these scenarios as well.

9.4.2 Varying equation of state

In the previous subsection, we assumed a constant EoS for DE. While a constant EoS parameter is a good approximation for ‘perfect fluids’ such as radiation ($w = 1/3$), dust ($w = 0$), or a cosmological constant ($w = -1$), it is harder to achieve this theoretically for phantom DE (besides perhaps with a scalar field that has an exponential potential; see, e.g., [351]). In fact, scalar field models of DE typically have a time-dependent EoS. As we mentioned before, it is even less motivated if we consider that the NEC violation is sourced by quantum fluctuations. Therefore, a more reasonable approach is to consider DE with a time-dependent EoS that can transition to the phantom regime over time. With the goal of having a model-independent approach, various time-dependent DE parametrizations have been developed. One of the most common is the Chevallier-Polarski-Linder (CPL) parametrization [352, 353] (see, e.g., [354, 355, 356] for alternatives; this is certainly not an exhaustive list), which makes the hypothesis that the variation of the EoS parameter linearly depends on the scale factor to leading order as

$$w(a) = w_0 + w_a(1 - a), \quad (9.42)$$

where w_0 and w_a are two real parameters. Here $a_0 = 1$ is already assumed, so $w_0 = w(a_0)$ and $w_a = -dw/da$ have the interpretation of today's value of the EoS and the negative of the EoS's derivative with respect to the scale factor, respectively.

As in the previous subsection, we make the simplifying assumption that the EoS, now (9.42), applies to DE for the whole past history of the universe. Thus, we take $p_{\text{DE}}(a) = w(a)\rho_{\text{DE}}(a)$, and the DE's conservation equation, $\dot{\rho}_{\text{DE}} + 3H(\rho_{\text{DE}} + p_{\text{DE}}) = 0$, can be rewritten as $d \ln \rho_{\text{DE}} / d \ln a = -3(1 + w(a))$, hence after integration one finds

$$\rho_{\text{DE}}(a) = \frac{3H_0^2 \Omega_{\text{DE}}}{8\pi G_{\text{N}}} a^{-3(1+w_0+w_a)} e^{-3w_a(1-a)}. \quad (9.43)$$

The relevant quantity to integrate in the SNEC (9.30a), which we recall can be expressed as

$$\int_{a_i}^{a_f} \frac{da}{H(a)} f_{\tau}^2(\lambda(a)) \frac{\rho_{\text{DE}}(a) + p_{\text{DE}}(a)}{a^2} \geq -\frac{B}{G_{\text{N}} \tau^2}, \quad (9.44)$$

is thus straightforward to compute from $\rho_{\text{DE}}(a) + p_{\text{DE}}(a) = \rho_{\text{DE}}(a)(1 + w(a))$ using (9.42) and (9.43). The last ingredient needed is $H(a)$, which also allows one to compute $\lambda(a)$ from (9.22), and it follows from Friedmann's constraint equation for a mix of dust and DE,

$$\frac{H(a)^2}{H_0^2} = \frac{1 - \Omega_{\text{DE}}}{a^3} + \frac{\Omega_{\text{DE}}}{a^{3(1+w_0+w_a)} e^{3w_a(1-a)}}. \quad (9.45)$$

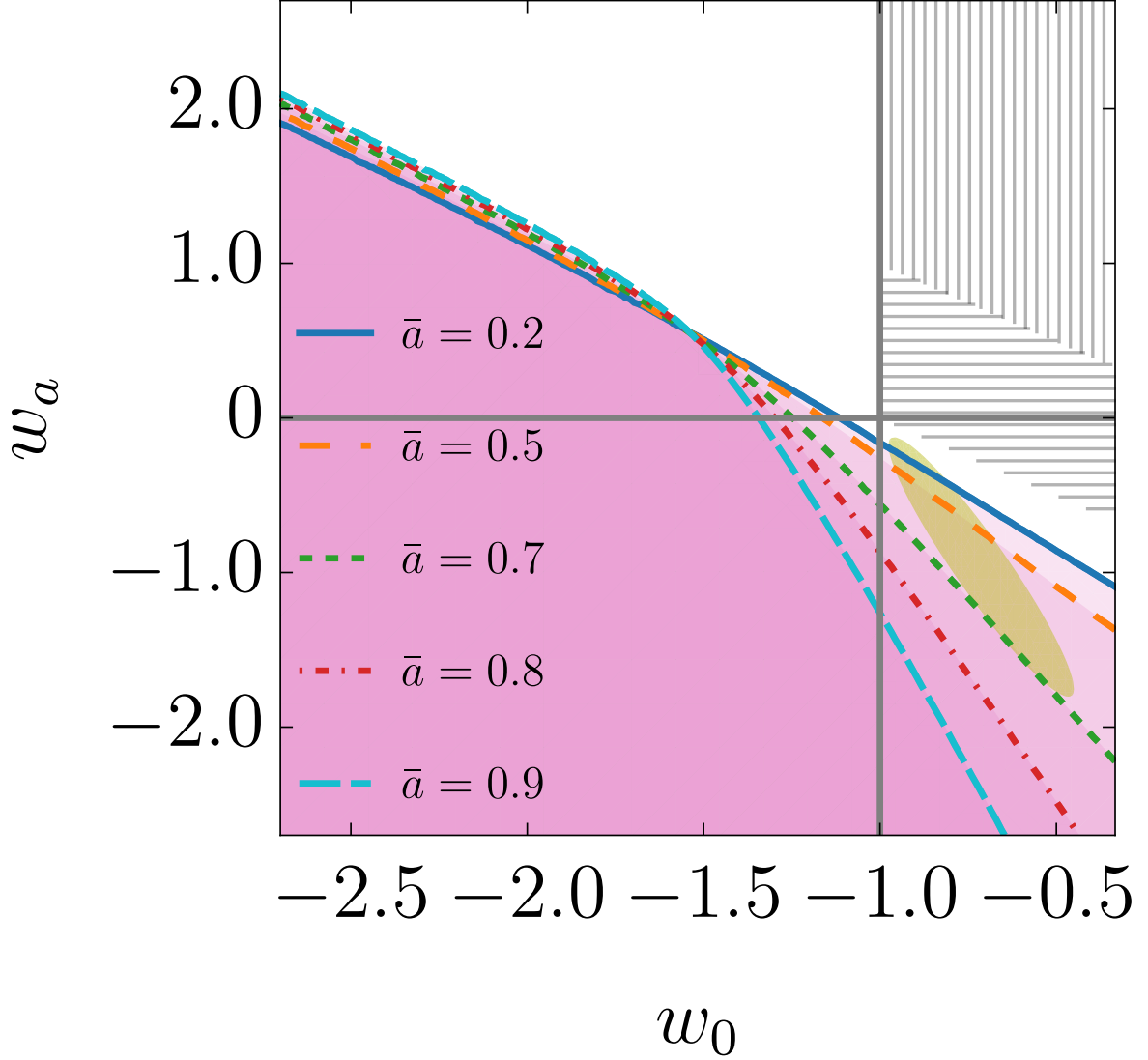


Figure 9.3: SNEC constraints on varying-EoS phantom (or quintom) DE. The constraint on the CPL EoS parameters w_0, w_a is shown for different values of $\bar{a}$ (related to the center of the smearing). The other parameters are fixed to $\Delta a = a_0 - a_{\text{eq}}$, $\Omega_{\text{DE}} = 0.7$, and $B = 1/(96\pi)$. The pink regions are where the SNEC is violated. The vertically dashed region shows where in $\{w_0, w_a\}$ -space DE never gets to dominate the universe, and the horizontally dashed region represents the $\{w_0, w_a\}$ -space where the classical NEC is always satisfied (and thus where the SNEC is also necessarily always satisfied). The olive ellipse gives a rough representation of current constraints from DESI [3, 4, 5, 6] when combined with CMB and supernova surveys. See text for more details.

The strategy to check whether the SNEC is satisfied is the same as in the previous subsection: we fix $a_i = a_{\text{eq}}$, $a_f = a_0$, and use a Gaussian smearing function, whose center $\bar{\lambda}$ and width τ are controlled by $\bar{a}$ and Δa , respectively, as explained in the previous subsection. We then numerically evaluate the integrals involved in the inequality (9.44) at fixed parameters Ω_{DE} and B . The results are first shown in Fig. 9.3, where $\Omega_{\text{DE}} = 0.7$, $B = 1/(96\pi)$, and where furthermore we fix Δa to its ‘largest allowed value’, $\Delta a = a_0 - a_{\text{eq}}$, since this yields the tightest, most conservative constraint (recall the discussion in the previous subsection). The curves of different styles in Fig. 9.3 depict the boundary of the SNEC for different choices of $\bar{a}$ as a function of the CPL EoS parameters w_0 and w_a ; on the pink side of the boundary, the SNEC is violated, while on the white side it is satisfied. One should ‘ignore’ the dashed regions that depict where the classical NEC is always satisfied, hence where the SNEC is trivially satisfied. In fact, in the vertically dashed region (where $w_0 + w_a > 0$ and $w_0 > -1$), DE never really dominates the universe, and in the horizontally dashed region (where $-1 < w_0 + w_a < 0$ and $w_0 > -1$), DE eventually dominates the universe, but it is never of phantom type. (See, e.g., [357, 358] for more refined presentations of the various subregions in the $\{w_0, w_a\}$ space.)

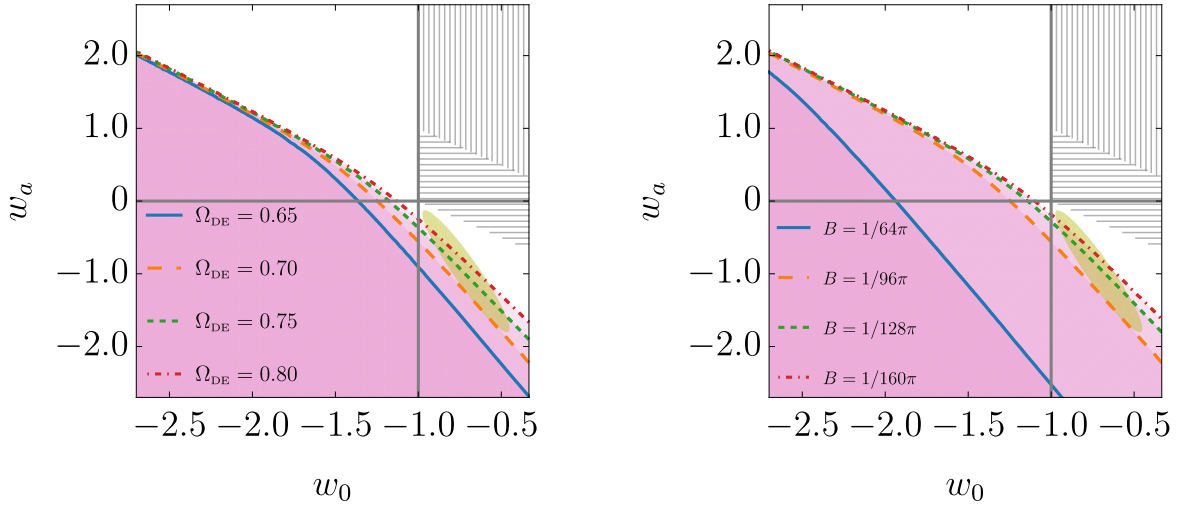


Figure 9.4: Same plots as Fig. 9.3, except now in the left plot we fix $\bar{a} = 0.7$, $B = 1/(96\pi)$, and show different values of Ω_{DE} ; and in the right plot we fix $\bar{a} = 0.7$, $\Omega_{\text{DE}} = 0.7$, and show different values of B .

To further navigate the plot, note that $w\text{CDM}$ (the constant EoS case analyzed in the previous

subsection) is recovered along the $w_a = 0$ horizontal solid gray line, and likewise, Λ CDM is recovered where this intersects the $w_0 = -1$ vertical solid gray line. To give an idea of what values of w_0 and w_a data currently favors, we added an olive ellipse, which is meant to roughly correspond to the current observational constraints at $\gtrsim 95\%$ confidence level from DESI’s baryonic acoustic oscillations (BAO) when combined with type-Ia supernova (SNIa) data and cosmic microwave background (CMB) data (see [3, 4, 5, 6]).⁵ While such a constraint comes with various caveats (see, e.g., [359, 360, 361, 362, 363]), there nevertheless is growing evidence for evolving DE, moreover potentially crossing the phantom divide in the past (further see, e.g., [364, 365, 366, 367, 368, 369, 370, 371, 372]), hence we use DESI’s results as a guiding reference for where observational constraints might lie.

Figure 9.3 shows how the center of the smearing function affects the SNEC constraint on w_0 and w_a . In the far top-left quadrant ($w_0 < -1$, $w_a > 0$), the resulting constraint depends marginally on $\bar{a}$. In that region, DE becomes phantom only close to today, so a larger $\bar{a}$ yields a tighter constraint (and correspondingly a smaller $\bar{a}$ yields a weaker constraint), but the difference is small since the smearing width is fixed at its largest with $\Delta a \sim 1$. The effect of changing $\bar{a}$ is more pronounced in the far bottom-right quadrant ($w_0 > -1$, $w_a < 0$), where DE is phantom in the far past before eventually satisfying the NEC today (a DE EoS that crosses the phantom divide is sometimes said to be of quintom type, as originally coined in [373]). There, the SNEC constraint is considerably stronger the farther back the smearing is centered (assuming the EoS is applicable that far in the past). Interestingly, this is the region where the current observational constraints from DESI lie.

Once again if the SNEC, as explained earlier, should hold for any choice of smearing, then one should take the union of all constraints, i.e., the whole pink region would thus violate the SNEC. According to this interpretation, most of the olive contour would fall in the SNEC-violating region, at least for $\Omega_{\text{DE}} = 0.7$, $B = 1/(96\pi)$, and assuming Δa can be as large as unity [the effect of these parameters is discussed below, and in fact, the DESI constraints are fully consistent with the SNEC if, e.g., $B = 1/(32\pi)$]. Another interpretation is that one should choose the smearing so that it roughly represents our ability to observe the DE’s ‘negative energy’ over the given window. Also, given that the linear approximation is likely to fail at early times, a more reasonable fundamental model for the time variation of the DE EoS might have less ‘negative

⁵Our olive ellipse does not correspond to any of the actual posteriors from DESI’s analysis. Rather, we are showing an approximate contour that encompasses the three different combinations of BAO+CMB+SNIa data that DESI reports in [5].

energy’ in past. Taking DESI as an example, if we use the galaxy density distribution as a proxy for the ‘window’ over which DE is observed, then we know that it peaks at around $z \lesssim 0.8$ [374, 5], which corresponds to $\bar{a} \gtrsim 0.56$.⁶ For $\bar{a}$ in that ballpark, only a fraction of the olive contour violates the SNEC when $B = 1/(96\pi)$. It certainly is interesting that the observational and theoretical constraints fall more or less in the same region. What this may imply will be further discussed in Sec. 9.6, together with the accompanying caveats.

As in the previous subsection, let us isolate the effect of changing the DE fractional density today, Ω_{DE} , and the SNEC constant B in Fig. 9.4. For concreteness, we fix $\bar{a} = 0.7$. In the left plot, we see that, following intuition, the more DE one has in the universe today, the more SNEC violation one has, the stronger the constraints on w_0 and w_a . Likewise, we see in the right plot that a smaller value of B always leads to a stronger SNEC constraint. Note that when we compute the SNEC for $B = 1/(32\pi)$ with $\Omega_{\text{DE}} = 0.7$, we find very weak constraints on w_0 and w_a , i.e., the constraint lies outside the range that is shown for w_0 and w_a . For $B = 1/(96\pi)$, though, the constraint gets very close to current observational bounds by DESI, and as the value of B is continuously lowered, a greater portion of the DESI contours falls into the regime of SNEC violation. As observations keep improving in the future, it is interesting to note that one could envision to observationally probe what the smallest allowed value of B is such that bestfit models are in agreement with the SNEC. This is another aspect of our results that will be further discussed in our conclusions.

9.5 Bouncing cosmology

Now that we have discussed NEC violation in the late universe with phantom DE, we move on to possible NEC violation in the very early universe. Inflationary cosmology is the paradigm of primordial cosmology, yet as it stands it cannot provide a complete description of the very early universe. Inflation resolves some problems of standard big bang cosmology (e.g., [375]), but it does so without resolving the initial cosmological singularity (see, e.g., [376, 332, 377]). As such, nonsingular cosmological models can serve as preludes or alternatives to inflation in the very early universe, and bouncing cosmologies are a subclass of such singularity resolving models (see, e.g., [250, 252, 378, 253] for reviews).

⁶In fact, DESI [6] finds that the energy density of DE peaks at around $z \approx 0.45$ (correspondingly $a \approx 0.69$) and that the matter-DE equality redshift is $z \approx 0.34$ (correspondingly $a \approx 0.75$).

In this section, we once more focus on flat FLRW cosmology since obtaining a bouncing universe with a flat background can only be achieved with some kind of NEC-violating field(s) or states. (This is not the case with positive spatial curvature; see, e.g., [379, 380, 381, 382, 340].) Indeed, we may express the flat FLRW equations with an ‘effective fluid’ satisfying $\dot{\rho}_{\text{eff}} + 3H(\rho_{\text{eff}} + p_{\text{eff}}) = 0$ as

$$\begin{aligned} 3H^2 &= 8\pi G_{\text{N}}\rho_{\text{eff}} , \\ \dot{H} &= -4\pi G_{\text{N}}(\rho_{\text{eff}} + p_{\text{eff}}) , \end{aligned} \tag{9.46}$$

so a smooth transition from contraction ($H < 0$) to expansion ($H > 0$) must pass through a ‘bounce point’ where $H = \rho_{\text{eff}} = 0$ and where the effective NEC is violated ($\dot{H} > 0$ and $\rho_{\text{eff}} + p_{\text{eff}} < 0$).⁷ A flat nonsingular bouncing cosmology can thus be described in terms of three phases: the prebounce contraction where $H < 0$ and $\dot{H} \leq 0$; the bounce phase where $\dot{H} > 0$ and where H crosses zero; and the postbounce expansion where $H > 0$ and $\dot{H} \leq 0$. See Fig. 9.5 for a depiction of the Hubble parameter and its derivative, assuming a symmetric bouncing phase centered at $t = 0$ (we are free to choose this origin) and where $2t_{\text{b}}$ represents the whole duration of the bounce phase. (More generally, it should be noted that the bounce need not be symmetrical about the bounce point, but we shall often assume so for simplicity.)

As stressed in the introduction, we put aside the challenges that one faces in trying to construct a viable NEC-violating bouncing cosmology (e.g., in terms of stability) since the focus is on exploring the implications of the SNEC if the NEC were violated (no matter how). For this reason, we simply work with a parametrization in the next subsection (as we did for DE), which is meant to describe the effective NEC-violating background evolution. However, we will look at a specific bouncing model in the subsequent subsection, the so-called Cuscuton bounce, since it is one of the most robust NEC-violating bouncing models in the literature so far.

⁷Yet, the ANEC may still be satisfied — see [383] — hence our main concern shall be about the SNEC.

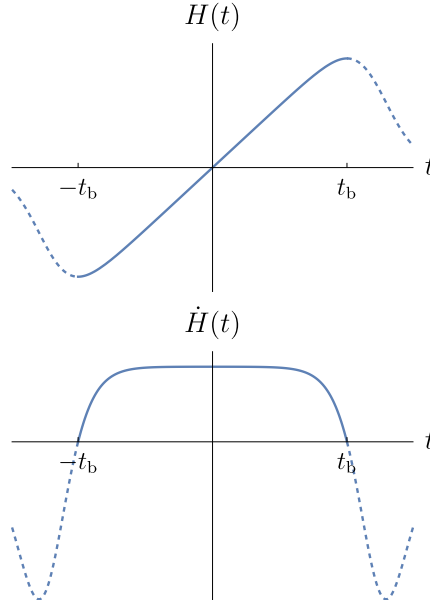


Figure 9.5: Sketches of the Hubble parameter and its time derivative through a prototypical nonsingular bounce. The solid curve is the bounce phase during which $\dot{H} > 0$, which is happening over the interval $t \in (-t_b, t_b)$. The dashed curve would correspond to standard NEC-satisfying cosmological evolution.

9.5.1 Parametrization

Let us consider symmetric nonsingular bouncing models. Without loss of generality, let us center the bounce about $t = 0$. We then parametrize the Hubble parameter through the effective NEC-violating bounce phase as

$$H(t) = \sum_{n \in \mathbb{N}_{\text{odd}}} \alpha_n t^n, \quad (9.47)$$

for some real parameters α_n , and where $\mathbb{N}_{\text{odd}} = \{1, 3, 5, 7, \dots\}$. The series expansion only contains odd powers to ensure $H(t)$ is an odd function by symmetry. Similar parametrizations have been used in the literature before (e.g., [384]), especially the case when $\alpha_n = 0$ for all $n > 1$ (i.e., keeping only a linear dependence) since it is a rather good approximation in many cases (e.g., [385, 386, 387, 388]). One can straightforwardly take respectively a time integral and derivative

of (9.47) to find the scale factor

$$a(t) = \exp\left(\int_0^t d\tilde{t} H(\tilde{t})\right) = \exp\left(\frac{\alpha_1 t^2}{2} + \frac{\alpha_3 t^4}{4} + \cdots\right), \quad (9.48)$$

and $\dot{H}(t) = \alpha_1 + 3\alpha_3 t^2 + \cdots$. Again without loss of generality, we parametrize the scale factor to unity at the bounce point when $t = 0$. The null affine parameter $\lambda(t)$ can subsequently be computed following (9.19), namely $\lambda(t) = \int_0^t d\tilde{t} a(\tilde{t})$ taking $\lambda(t = 0) = 0$ without loss of generality. In this case, however, we cannot write down a general, closed-form analytic expression for $\lambda(t)$. Still, we have all the ingredients needed to check the SNEC expressed as (9.20). Note that expressing the SNEC as such in terms of background evolution implies checking whether or not the *total* SNEC is satisfied, as opposed to isolating what specific NEC-violating ‘quantum field’ drives the bounce and checking the SNEC on that field only, as expressed in (9.30). This is fine assuming the NEC-violating effects dominate over any NEC-satisfying matter fields present in the bounce phase and assuming they decay very rapidly outside of that phase.

Let us see why this is a good approximation. We write $\rho_{\text{eff}} = \rho_s + \rho_e$ and $p_{\text{eff}} = p_s + p_e$, where the labels ‘s’ and ‘e’ denote the ‘standard’ (NEC-satisfying, $\rho_s + p_s \geq 0$) and ‘exotic’ (NEC-violating, $\rho_e + p_e < 0$) fields, respectively. The Friedmann equations (9.46) then tell us that at the start and end of the bouncing phase, when $\dot{H} = 0$, we must have $\rho_s + p_s = |\rho_e + p_e|$, indicating that the standard and exotic fields contribute equally to the total NEC at that point. Then, by the point the bounce occurs ($H = 0$, $\dot{H} > 0$), it must be that $\rho_e = -\rho_s$ and that $\rho_s + p_s < |\rho_e + p_e|$. While this is a minimal requirement, it is often the case that $\rho_s + p_s \ll |\rho_e + p_e|$ in order for $\dot{H}(t = 0)$ to be sufficiently large (for the bounce phase to be sufficiently short), which motivates our assumption that the NEC-violating quantum fields dominate the bounce. Naturally, outside the bounce phase one must have the opposite limit ($\rho_s + p_s \gg |\rho_e + p_e|$) in order to approach GR with standard classical matter sufficiently quickly, hence it is reasonable to assume that the SNEC is satisfied as the NEC-violating quantum fields quickly ‘decay’ outside the bounce phase. Altogether, this justifies the approximate equality through the bounce phase between the total SNEC (9.18a) and the SNEC (9.30b) applied to the exotic field(s) only.⁸

The parametrization (9.47) for the Hubble parameter, or (9.48) for the scale factor, should really be thought of as a series expansion. In that sense, the Hubble parameter depends linearly on time to leading order through the bounce, so $H(t) \simeq \alpha_1 t$, and $\dot{H} \simeq \alpha_1$ is approximately

⁸Technically, though, this means the constraints we will find could be even stronger if the SNEC were applied to the NEC-violating content only, as we did for DE.

constant. From here on, let us set $\alpha_1 \equiv \alpha$ to lighten the notation. Naturally, $\alpha > 0$ so that $\dot{H} > 0$. Recall that we can view the SNEC — in a similar fashion to Heisenberg’s uncertainty relation — as an ‘energy-spacetime uncertainty relation’, which puts an upper bound on the amount of negative-energy fluctuations of the form $\Delta E \Delta t \lesssim \#$, for some positive number $\#$. Through an NEC-violating bounce, we have negative-energy fluctuations $\Delta E \sim (\Delta H/\Delta t) \times \Delta t$, where $\Delta H/\Delta t \sim \dot{H} \sim \alpha$ and $\Delta t \sim t_b$, hence we expect the SNEC to provide a bound of the form $\Delta E \Delta t \sim (\Delta H/\Delta t) \times (\Delta t)^2 \sim \alpha t_b^2 \lesssim \#$. This makes sense if α and t_b are the only two dimensionful parameters in the problem: the SNEC should provide a bound on the dimensionless combination αt_b^2 . Since this quantity is used repeatedly below, let us label it as $x_b \equiv \alpha t_b^2/2$ (the factor of $1/2$ simplifies the presentation below).

If we include higher-order terms in the parametrization, this introduces additional dimensionful parameters, e.g., α_3 to next order. However, provided the parametrization is properly perturbative, the SNEC dependence on α_3 should be small, and one should still find a bound of the form $x_b \lesssim \#$. Let us see this more explicitly. We start from the SNEC as expressed in (9.18) and consider a Lorentzian smearing function (9.16) for concreteness, although this can be generalized to other smearing functions. Since we center the bounce at $\lambda(t=0)=0$, we center the smearing at the same point, so $\bar{\lambda}=0$. The oddness of (9.47) implies the scale factor $a(t)$ and the time derivative of the Hubble parameter $\dot{H}(t)$ are even functions of time [and thus of λ because $a(t)$ being an even function implies $\lambda(t) = \int dt a(t)$ is odd, hence $a(\lambda)$ is even; likewise for $\dot{H}(\lambda)$]. Therefore, by symmetry, we can write the SNEC as

$$\int_0^\infty d\lambda \frac{\dot{H}/a^2}{\lambda^2 + \tau^2/2} \leq \frac{2\sqrt{2}\pi^2 B}{\tau^3}. \quad (9.49)$$

If we consider (9.47) truncated to cubic order, $H(t) = \alpha t + \alpha_3 t^3$, we have $a(x) = e^{x(1+\beta x)}$ and $\dot{H}(x) = \alpha(1+6\beta x)$, where we defined the dimensionless parameter $\beta \equiv \alpha_3/\alpha^2$, and at fixed parameter α , we defined the rescaled dimensionless time variable $x \equiv \alpha t^2/2$. Then, from $\lambda(t) = \int_0^t d\tilde{t} a(\tilde{t})$, we find $\lambda(x) = \mathcal{F}(x)/\sqrt{2\alpha}$, where

$$\mathcal{F}(x) \equiv \int_0^x d\tilde{x} \frac{e^{\tilde{x}(1+\beta\tilde{x})}}{\sqrt{\tilde{x}}}. \quad (9.50)$$

From this, note $\tau = \lambda(t_b) = \mathcal{F}(x_b)/\sqrt{2\alpha}$. Further note that $a(x) = 1 + x + (\beta + 1/2)x^2 + \dots$ is a valid series expansion provided $x < 1$ and $-1/x < |\beta + 1/2| < 1/x$. In particular, we are considering the range $0 \leq x \leq x_b$, so we should ask for $-1/x_b < |\beta + 1/2| < 1/x_b$ and $x_b < 1$. In

fact, for $x \leq x_b \ll 1$ and $|\beta| \lesssim \mathcal{O}(1)$, we find $\mathcal{F}(x) \simeq 2\sqrt{x}(1 + x/3)$, with the dependence on β appearing only at $\mathcal{O}(x^{5/2})$. Now, the SNEC (9.49) can be reexpressed as⁹

$$\int_0^\infty dx \frac{(1 + 6\beta x)e^{-x(1+\beta x)}}{\sqrt{x}(\mathcal{F}(x)^2 + \mathcal{F}(x_b)^2/2)} \leq \frac{4\sqrt{2}\pi^2 B}{\mathcal{F}(x_b)^3}. \quad (9.51)$$

For completeness, the expression one finds when using a Gaussian smearing function is

$$\begin{aligned} \int_0^\infty \frac{dx}{\sqrt{x}} (1 + 6\beta x) \exp\left(-x(1 + \beta x) - \frac{\mathcal{F}(x)^2}{2\mathcal{F}(x_b)^2}\right) \\ \leq \frac{4\sqrt{2}\pi^{3/2}B}{\mathcal{F}(x_b)}. \end{aligned} \quad (9.52)$$

In either case, the result is an inequality to be solved for x_b , whose solution will depend on B (and marginally on β due to the previous perturbative argument). Importantly, any separate dependencies on α or t_b alone (i.e., not as the product αt_b^2) have cancelled out. Hence, in this context, the SNEC always yields a bound of the form $x_b \leq \#$, where $\#$ is a function of B and β . (This somewhat similar to DE, where the SNEC implied, e.g., a lower bound on the dimensionless EoS parameter w , except in the case of DE there was nontrivial dependence on the smearing scale due to the more intricate functionality of the negative energy and the limits of the integration.) Based on what we will show next, we can conjecture that the resulting bound has to be of the form

$$x_b(1 + \mathcal{O}(\beta x_b, x_b^2, \beta x_b^2)) \leq 2\pi B, \quad (9.53)$$

when assuming $x_b \ll 1$ and $|\beta| \lesssim \mathcal{O}(1)$.

Further insight can be gained from analyzing the linear case $H(t) = \alpha t$, which in terms of (9.47) means that we set $\alpha_1 = \alpha$ and all other $\alpha_n = 0$, $n > 1$. In this case, $\dot{H} = \alpha$ is just a constant. Integrating the Hubble function returns $a(t) = \exp(\alpha t^2/2)$, and the integrand of the SNEC (9.20) involves $\dot{H}/a = \alpha \exp(-\alpha t^2/2)$. To gain some analytical intuition about the SNEC in a bouncing universe, let us take the following rough smearing function,

$$f_\tau^2(\lambda(t)) = \begin{cases} 1/(2\tau) & |t| \leq t_b \\ 0 & |t| > t_b, \end{cases} \quad (9.54)$$

⁹As just argued, the parametrizaion is perturbative so long as $x \leq x_b < 1$, but then we wish to integrate it for $x \in (0, \infty)$. This remains consistent provided the smearing function brings the integrand to zero sufficiently quickly for $x > x_b$.

i.e., we take a continuous uniform distribution similar to (9.38), which is a positive constant during the bounce phase and zero otherwise. Such a smearing function satisfies the required normalization,

$$\begin{aligned} \int_{-\infty}^{\infty} d\lambda f_{\tau}^2(\lambda) &= \int_{-\infty}^{\infty} dt a(t) f_{\tau}^2(\lambda(t)) \\ &= \frac{1}{2\tau} \int_{-t_b}^{t_b} dt a(t) = 1, \end{aligned} \quad (9.55)$$

where the last equality follows from taking

$$\begin{aligned} \tau &= \frac{\lambda(t_b) - \lambda(-t_b)}{2} = \frac{1}{2} \int_{-t_b}^{t_b} dt a(t) = \frac{1}{2} \int_{-t_b}^{t_b} dt e^{\alpha t^2/2} \\ &= \sqrt{\frac{\pi}{2\alpha}} \operatorname{erfi} \left(t_b \sqrt{\frac{\alpha}{2}} \right) \simeq t_b \left(1 + \frac{\alpha}{6} t_b^2 + \dots \right), \end{aligned} \quad (9.56)$$

which involves the imaginary error function erfi , whose approximation holds when the parametrization is perturbative, i.e., when $\sqrt{x_b} = t_b \sqrt{\alpha/2} \ll 1$. Then, the SNEC integral becomes

$$\begin{aligned} \int_{-\infty}^{\infty} dt f_{\tau}^2(\lambda(t)) \frac{\dot{H}}{a} &= \frac{\sqrt{\alpha/(2\pi)}}{\operatorname{erfi} \left(t_b \sqrt{\alpha/2} \right)} \int_{-t_b}^{t_b} dt \alpha e^{-\alpha t^2/2} \\ &= \alpha \frac{\operatorname{erf} \left(t_b \sqrt{\alpha/2} \right)}{\operatorname{erfi} \left(t_b \sqrt{\alpha/2} \right)} \simeq \alpha \left(1 - \frac{\alpha}{2} t_b^2 + \dots \right). \end{aligned} \quad (9.57)$$

Expressed in terms of $x_b \equiv \alpha t_b^2/2$, the SNEC (9.20) thus reduces to

$$x_b (1 + \mathcal{O}(x_b^2)) \lesssim 2\pi B. \quad (9.58)$$

The calculation can be generalized upon reintroducing β , resulting in the similar looking perturbative bound (9.53).

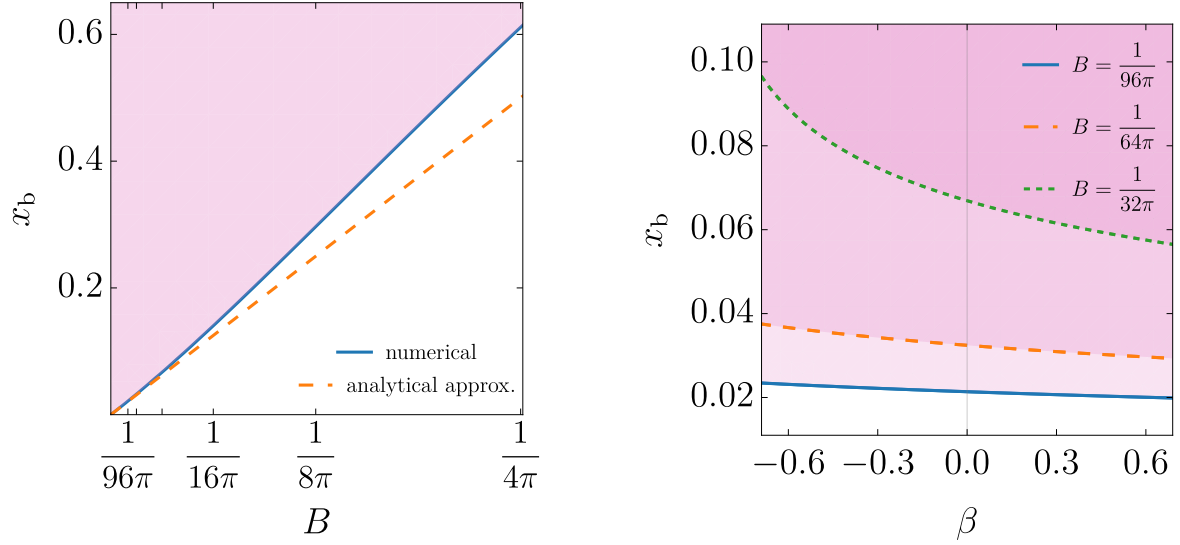


Figure 9.6: *Left:* SNEC constraint on the dimensionless bounce-phase parameter $x_b \equiv \alpha t_b^2/2$ as a function of the SNEC parameter B . Ticks on the horizontal axis are depicted for $1/(2^n\pi)$, $n \in \{2, 3, 4, 5, 6\}$, and for $1/(3 \cdot 32\pi)$, as representative values. The pink region with blue boundary shows where the SNEC is violated, using a Gaussian smearing function and a full numerical computation. In comparison, the dashed orange line shows the analytical approximation when (9.58) is saturated to leading order. *Right:* SNEC constraint on x_b as a function of the higher-order bouncing parameter $\beta \equiv \alpha_3/\alpha^2$. Different curves depict different values of the SNEC parameter B .

If we use $B = 1/(32\pi)$ as a representative example, (9.58) gives us the approximate bound $x_b \lesssim 1/16$, or equivalently, $\alpha t_b^2 \lesssim 1/8$. The interpretation is once again quite straightforward: the SNEC puts an upper bound on the duration of the NEC-violating phase, t_b , and on the rate of NEC-violation (or the EoS), through $\alpha \approx \dot{H} \propto -(\rho_{\text{eff}} + p_{\text{eff}})$. At fixed duration, there is a lower bound on how negative $\rho_{\text{eff}} + p_{\text{eff}}$ can get, or vice versa at fixed α , there is an upper bound on the time interval of NEC violation. One could thus have, in principle, just as well a short, highly energetic bounce or a low-energy, elongated bouncing phase. To give an idea of the scales, if we take the bounce phase to be as short as a Planck time, then the SNEC implies the bound $\alpha \lesssim M_{\text{Pl}}^2/8$, hence the Hubble scale at the start and end of the bounce phase could be very high, with $|H(t_b)| \approx \alpha t_b \lesssim M_{\text{Pl}}/8 \approx 3 \times 10^{17} \text{ GeV}$. Conversely, if we say that the Hubble scale is

very low, e.g., $|H(t_b)| \approx \alpha t_b \approx 100 \text{ TeV}$, then the SNEC implies the bound $t_b \lesssim 3 \times 10^{12} M_{\text{Pl}}^{-1}$, so the bound phase could be very long (relative to a quantum gravity time scale).

The bound $x_b \lesssim 2\pi B$ comes from assuming a continuous uniform distribution for the smearing function, which allowed us to perform an analytical calculation and gain intuition. To properly apply the SNEC, the smearing function must not be nondifferentiable, though, so one should instead resort to a smooth Lorentzian or Gaussian smearing function for instance, in which case one has to solve (9.51) or (9.52) for x_b numerically given values for B and β . Results are shown in the left plot of Fig. 9.6 in the case $\beta = 0$ and applying a Gaussian smearing function (very similar results are found with a Lorentzian function). Given some values of B , the pink region shows for what values of x_b the SNEC is violated, and the boundary of that region (where the SNEC is saturated) is the solid blue line. In comparison, the line $x_b = 2\pi B$ is shown in dashed orange. The approximation is good for small values of B (since it corresponds to $x_b \ll 1$), but it starts to deviate from the full numerical solution, and the larger B and x_b are. Similar to phantom DE, the general dependence on B is as expected: the smaller the value of B , the tighter the bound, and in fact, as $B \rightarrow 0^+$, no bounce can satisfy the SNEC as the bound on x_b approaches zero (in that limit, the SNEC essentially reduces to the NEC). In contrast, a larger value of B implies a weaker bound.

The case $\beta = 0$ (meaning $H = \alpha t$) provides a good approximation in many bouncing instances and offers a straightforward analysis of the SNEC. However, it necessarily is an incomplete parametrization, in the sense that the cosmology can only smoothly evolve outside of the bounce phase to a NEC-satisfying regime if $\dot{H}$ crosses zero, as depicted earlier in Fig. 9.5. This provides motivation for exploring more general parametrizations, e.g., when at least one α_n is nonzero for $n > 1$, but for simplicity we only explore the case $\alpha_3 \neq 0$ (and we often rescale this parameter as $\beta \equiv \alpha_3/\alpha^2$). Then, we can write $H(t) = \alpha t + \alpha_3 t^3$ as before, so $\dot{H}(t) = \alpha + 3\alpha_3 t^2 = 0$ when $t = \pm 1/\sqrt{-3\beta\alpha}$ (assuming $\beta < 0$ here). Thus, the parametrization is valid so long as $t_b \leq 1/\sqrt{-3\beta\alpha}$ when $\beta < 0$. If $\beta \geq 0$, the bounce phase ‘never ends’ [i.e., there is no point where $\dot{H}(t) = 0$], in which case it is the smearing function that ensures no significant negative-energy contribution is picked up for $|t| \geq t_b$. If β is very large, though, $\dot{H}$ might be more important at the beginning and end of the bouncing phase, so one might have to refine the analysis further by exploring different smearing functions that better capture when NEC violation is most important.

To understand the effect of β , let us numerically solve the full expression (9.52) for x_b . We present the resulting constraint on x_b in the right plot of Fig. 9.6 as a function of β . The pink regions are where the SNEC is violated, and the boundaries of those regions for different choices

of B are represented by the difference lines. We see that when B is very small (so when the SNEC is very constraining), the bound on x_b has little dependence on β . Nevertheless, there is a dependence [it is most obvious in the case $B = 1/(32\pi)$ in the figure]: the more negative β is, the larger x_b can be, and vice versa (within the regime of validity of the approximation, which holds for the entire range shown in the right plot of Fig. 9.6). This can be understood as follows: if β is negative, it means that there is less integrated negative energy (compared to the $\beta = 0$ constant case $\dot{H} = \alpha$), while if β is positive, there is more integrated negative energy.

Let us end this subsection by mentioning that one could seek a ‘nonperturbative’ parametrization of a bouncing cosmology such as $H(t) = qt/(t^2 + t_b^2)$ with $q > 0$. This parametrization implies $a(t) = (1 + t^2/t_b^2)^{q/2}$, so near the bounce ($t \approx 0$), one recovers the linear approximation $H(t) \simeq \alpha t$ with $\alpha = q/(2t_b^2)$, hence $x_b = q/2$. Far away from the bounce, this reduces to $a(t) \sim |t|^q$, which corresponds to the expected evolution in classical GR with some matter content having an EoS $w = -1 + 2/(3q)$. Checking the SNEC on this parametrization, one finds the bound $q \lesssim 0.687$ [when $B = 1/(32\pi)$] with practically no other separate dependence on t_b . In essence, one recovers that the SNEC mainly puts a constraint on the quantity x_b , with an upper bound here of about 0.34. One should avoid attempting to compare this bound with, e.g., the right plot of Fig. 9.6, which assumed a perturbative parametrization. Indeed, here if we expanded $H(t)$ we would find $\beta = -2/q = -x_b$, so one cannot treat x_b and β as two independent parameters as in the right plot of Fig. 9.6.

Next, we consider the case of a proper bouncing cosmology model (beyond a mere parametrization) and observe the constraints that the SNEC places on the parameters of the given model.

9.5.2 Cuscuton bounce

The bouncing model we consider in this section is a product of the Cuscuton [389, 390], which is a modified gravity theory with some interesting features. Most importantly, the Cuscuton does not introduce any additional dynamical degrees of freedom on a cosmological background. While this theory in its own right and in relation to other theories has garnered quite a lot of discussion (e.g., [391, 392, 393, 394, 395, 396, 397, 398, 399, 400, 401, 289, 290, 402]), here we restrict ourselves to the Cuscuton as providing a full example of bouncing background and explore its parameter space in light of the SNEC.

The nondynamical nature of the Cuscuton has allowed it to produce stable and robust non-singular cosmological models (see [403, 404, 387, 388, 405, 406, 407]), something that is typically

very difficult to obtain in alternative modified gravity theories (which introduce new degrees of freedom). The readers interested in the details of this assertion are invited to explore the references above. Below we only provide a lightning review of the Cuscuton and of the corresponding bouncing model.

Before doing so, we should point out that we will apply the SNEC as expressed in (9.13) to the Cuscuton bounce model. We recall that the SNEC expressed as such, despite being a geometrical condition, is motivated by a bound on NEC-violating matter in semiclassical GR. However, the Cuscuton does not straightforwardly qualify as a matter field in semiclassical GR. The situation is blurry, similar to the discussion below (9.6). Indeed, the Cuscuton can be viewed purely as a geometrical modification to GR (still with a quantum origin; see, e.g., [389, 390, 391, 394, 402]). Therefore, SNEC constraints on the Cuscuton bounce must be taken with appropriate nuances, but nevertheless, the model provides a simple, complete, and robust nonsingular background, hence it remains an obvious choice to test the applicability of the SNEC in bouncing cosmology.

The theory consists of GR, standard matter fields, and the Cuscuton modification manifested as a scalar φ . For the matter content, one can assume the presence of a massless canonical scalar field ψ (i.e., ψ has no potential energy, and its kinetic term is $-\partial_\nu\psi\partial^\nu\psi/2$), which would eventually decay when the universe reheats. The Cuscuton's action is given by

$$S_\varphi = \int d^4x \sqrt{-g} \left(-\mu^2 \sqrt{-\partial_\nu\varphi\partial^\nu\varphi} - V(\varphi) \right), \quad (9.59)$$

where g denotes the determinant of the metric, μ is a new parameter, and $V(\varphi)$ is the Cuscuton's potential. Note that we set $8\pi G_N = 1$ for this whole subsection. One notices that the kinetic term is noncanonical due to the presence of the square root, and it is this feature that makes the Cuscuton nondynamical when $\partial_\nu\varphi\partial^\nu\varphi < 0$. (We should point out that one gets a different theory if the field's gradient is spacelike; see [389, 395, 397].) The Cuscuton's action could be more general than (9.59) by having an arbitrary function of φ in front of the square root (see, e.g., [389, 402]), but the form of (9.59) is ideally suited to find nonsingular cosmologies.

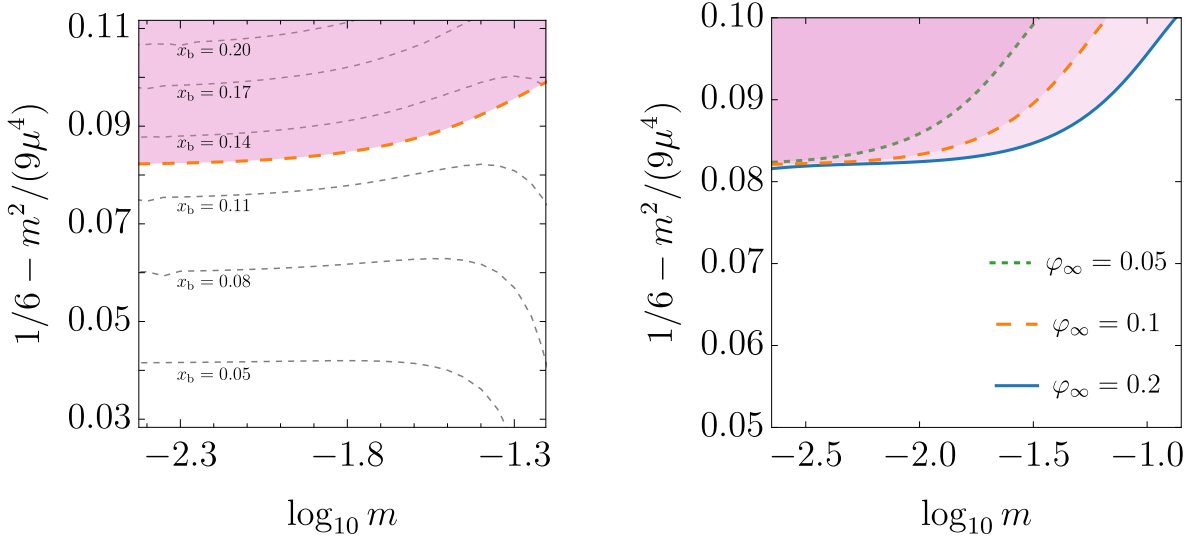


Figure 9.7: SNEC constraints on the Cuscuton bounce model. As before, the pink region indicates where the SNEC is violated, and the colored curves correspond to the pink regions' boundaries. The constraints on the Cuscuton bounce parameters $\{m, \mu\}$ are plotted in terms $\log_{10} m$ on the horizontal axis and $1/6 - m^2/(9\mu^4)$ on the vertical axis (recall $8\pi G_N = 1$ here). In the left plot, we fix $\varphi_\infty = 0.1$, and in the right plot, we show the constraints for three different values of φ_∞ . Also, we assume $B = 1/(32\pi)$.

Specializing to a flat FLRW background, the equations of motion are given by (we choose $\dot{\varphi} > 0$ without loss of generality)

$$3H^2 = \frac{\dot{\psi}^2}{2} + V(\varphi), \quad (9.60a)$$

$$2\dot{H} = -\dot{\psi}^2 + \mu^2\dot{\varphi}, \quad (9.60b)$$

$$3\mu^2 H = \frac{dV}{d\varphi}, \quad (9.60c)$$

in addition to the matter field's conservation equation, $\ddot{\psi} + 3H\dot{\psi} = 0$, which is readily solved by $\dot{\psi} \propto 1/a^3$. Note that (9.60c) represents the Cuscuton's 'evolution equation', though it truly is a constraint equation, as is evident by having no time derivatives of φ , indicating it is a nondynamical field. In fact, given a potential $V(\varphi)$ one could solve this algebraic constraint. Then, one could, in principle, reexpress all the dependence on φ in terms of H and $\dot{H}$ and simply

obtain (complicated) modified Friedmann equations. (This shows in what sense Cuscuton gravity has more to do with a nontrivial modified gravity theory than a theory of semiclassical GR.) In practice, though, it is often simpler to solve all the equations as a differential-algebraic system.

The equations make it clear how a bounce phase can occur: when $\mu^2\dot{\varphi}$ is sufficiently large, it can dominate over $-\dot{\psi}^2$ in (9.60b) and drive a phase of effective NEC violation with $\dot{H} > 0$. This can allow H to go through zero at the point where $V(\varphi) = -\dot{\psi}^2/2$. One potential that can achieve this is (see [387, 407] for details)

$$V(\varphi) = \frac{1}{2}m^2(\varphi^2 - \varphi_\infty^2) - \frac{m^4}{4} \left(e^{2(\varphi^2 - \varphi_\infty^2)/m^2} - 1 \right), \quad (9.61)$$

where m and φ_∞ are two positive constants. Provided $0 < m \leq \sqrt{3/2}\mu^2 < \varphi_\infty$ (see again [387, 388, 407]), a successful bounce occurs.

Our strategy is then the following: given the Cuscuton potential (9.61) and some values for the parameters (μ, m, φ_∞) , we numerically solve the system (9.60) for $a(t)$, from which we can compute $\dot{H}(t)$ and $\lambda(t)$ and numerically check whether the SNEC (9.20) is respected. As in the previous subsection when checking the SNEC numerically, we take a Gaussian smearing function centered at the bounce point and set the smearing scale to $\tau = \lambda(t_b)$. The bouncing time t_b is itself found numerically from the roots of $\dot{H}(t)$, from which we can further find the numerical value for $x_b = \alpha t_b^2/2$ by taking $\alpha = \dot{H}(t = 0)$.

We show our results in Fig. 9.7 assuming $B = 1/(32\pi)$. To facilitate reading the constraints, we use the quantity $1/6 - m^2/(9\mu^4)$ on the vertical axis. The reason is as follows: given the potential (9.61), analytical approximations from [407] suggest $H(t_b) \approx m^2\varphi_\infty/(3\mu^2)$ and $\dot{H}(0) \approx m^4\varphi_\infty^2/(3\mu^4 - 2m^2)$, hence we can estimate $x_b \approx H(t_b)^2/(2\dot{H}(0)) \approx 1/6 - m^2/(9\mu^4)$. Then, since we expect the SNEC to mainly constrain the quantity x_b , we plot constraints in terms of $1/6 - m^2/(9\mu^4)$ as functions of m (one could have equivalently chosen μ on the horizontal axis to span the $\{m, \mu\}$ space). Note, however, that $1/6 - m^2/(9\mu^4)$ is not necessarily the same as the actual value of x_b that we can compute numerically from the solutions. To see this, in the left plot of Fig. 9.7, we overlay level curves of constant x_b values in dashed black.

Our results confirm that the SNEC essentially provides an upper bound on the quantity x_b , roughly $x_b \lesssim 0.13$ when $B = 1/(32\pi)$, and that this bound has marginal dependence on the various parameters. As m — or equivalently μ — is varied by more than one order of magnitude, the bound on the ratio m^2/μ^4 only changes by a small fraction. Likewise, looking at the right plot of Fig. 9.7, we see that the bound appears to asymptote $1/6 - m^2/(9\mu^4) \lesssim 0.082$

as m (equivalently μ) is lowered to smaller and smaller values, hence the dependence on φ_∞ is marginal in that regime.

It is interesting to compare the bound that the SNEC puts on the Cuscuton bounce parameters with other (theoretical) constraints on the same model. For instance, it was found in [388, 407] that stability and strong coupling constraints forbid the ratio m^2/μ^4 from being too small or too close to $3/2$. In terms of $x_b \approx 1/6 - m^2/(9\mu^4)$, this suggests x_b should not be too close to zero, or too close to $1/6 \approx 0.167$. Here, we find that $x_b \approx 1/6$ would certainly violate the SNEC, but having a small value of x_b is consistent with the SNEC. In fact, it is interesting to note that the ‘sweet spot’ according to [388, 407] where $m \approx \mu^2$ (equivalently $x_b \approx 1/18 \approx 0.056$) also passes the test of the SNEC. Note that the SNEC can be even more constraining here if one takes smaller values of B , as we did for DE. We should end by stressing, though, that the applicability of the SNEC to the Cuscuton bounce remains a proof of principle and that any specific quantitative constraint should be taken with a grain of salt considering the fact the Cuscuton does not quite qualify as quantum matter in semiclassical GR.

9.6 Discussion

In this work, we applied the SNEC to cosmologies that would exhibit effective NEC violation. The SNEC is a proposed semilocal, semiclassical energy condition that would set a bound on how much NEC violation is physically admissible. In cosmology, there are two interesting eras that could feature matter violating the NEC: in today’s DE era and in the very early universe.¹⁰ In particular, phantom or quintom DE — DE with an EoS parameter $w \not\geq -1$ at all times (i.e., it violates the NEC at some point in the history) — has regained interest lately due to tantalizing observational evidence suggesting DE may have a time-dependent EoS, potentially crossing the phantom divide in the past (e.g., [6]).

¹⁰In primordial cosmology, NEC violation could occur in various scenarios: bouncing cosmology, emergent cosmology, inflationary cosmology, etc. We presented an analysis on bounces, but this could certainly be extended in the future. In particular, let us mention the case of stochastic fluctuations (e.g., during inflation) that can lead to NEC violation (e.g., a scalar field ‘jumping’ up its potential); see, e.g., [408, 409, 410]. It would thus be interesting to see how the SNEC could be applied in such a context. In particular, in the case of large fluctuations extending beyond the semiclassical regime, it might be interesting to explore whether one may define appropriate smeared spacetime energy conditions that apply to stochastic cosmological perturbation theory or full stochastic gravity (e.g., [411]).

If the SNEC were shown to hold fundamentally, then our analysis in this paper shows that it would have very interesting and important consequences for phantom or quintom DE, as well as bouncing scenarios in the early universe. That is, if the SNEC were rigorously ‘proven’ from some fundamental principles, one could apply the SNEC to constrain or rule out cosmological models.¹¹ Moreover, it could be used as a way to set more well-motivated priors on the ‘allowed’ parameter space when doing Bayesian inference (e.g., what prior to choose for w or w_0, w_a). The idea of using fundamental concepts (e.g., unitarity, causality, etc.) to constrain theoretical models and to set informed priors has been suggested in, e.g., [416, 417, 418, 419, 420, 421, 422, 423].

Likewise, observations can provide insights into the theory by first helping us determine whether a fundamental energy condition like the SNEC is valid, and second, if it is, offering information about its properties. For example, our preliminary results indicate that B values as low as $1/(96\pi)$ can be in agreement with the DESI results, and the theoretically motivated value $B = 1/(32\pi)$ is certainly consistent. Thus, by examining observational data through the lens of specific models where quantum effects lead to phantom or quintom DE, we can envision applying these models to the SNEC to determine, for instance, the minimum value of B data can support.

Let us highlight that when fitting models such as w CDM or w_0w_a CDM to data in observational analysis, one typically allows the DE EoS to be of phantom type in the prior (and it is not uncommon to find a bestfit of the EoS that is phantom; see, e.g., [424, 425]), but theoretically, it is a challenge to construct viable and consistent phantom models. Here, we did not explore building a top-down model either. Rather, we took a middle ground approach asking the question of whether phantom DE could be consistent with the SNEC. The short answer is yes when taking the w CDM or w_0w_a CDM phenomenological models as proxies for NEC violation. Our work also shows when this may no longer be true. For instance, for $\Omega_{\text{DE}} = 0.7$ and $B = 1/(96\pi)$, the SNEC is satisfied only if $w \gtrsim -1.25$ assuming the EoS is approximately constant for most of the past cosmic history (cf. Fig. 9.2, where the smearing function is centered at $\bar{a} = 0.7$). More interesting are the results for the case of a time-dependent EoS and modeling NEC violation in terms of the CPL parametrization. For example, when restricted to the region where $w_0 > -1$ and $w_a < 0$ and again assuming $\Omega_{\text{DE}} = 0.7$ and $B = 1/(96\pi)$, the constraint can be expressed as $w_a \gtrsim -2.49(w_0 + 1) - 0.57$ (fitting a straight line to the dashed orange curve in the bottom-right quadrant of the left plot of Fig. 9.4). This would imply that most of the $\{w_0, w_a\}$ points that fall into DESI’s 2σ -contours (when combined with CMB and SNIa data) may marginally satisfy the

¹¹A similar rationale is found in the swampland program. See, e.g., [338, 412, 413, 414, 415] for constraints on DE in that context.

SNEC. We should stress again that this constraint is obtained with a relatively small value of the SNEC parameter B . If we take the larger value $B = 1/(32\pi)$ suggested by some of the explicit QFT examples, the constraint is significantly reduced, while if $B \ll 1/(96\pi)$ the bound can rule out the $\{w_0, w_a\}$ values supported by DESI results.

One weakness of our analysis as already pointed out is that we only considered DE parametrizations, as opposed to specific QFT constructions leading to phantom or quintom models. While fits to data mostly focus on phenomenological parametrizations, it is hard to pin down the microphysics purely from parametrizations — see, e.g., [426, 427] — and one can certainly learn a great deal by constructing actual DE models that can fit the data (see, e.g., [428, 429, 430, 431, 432, 433, 434, 435, 436, 437, 438, 439, 440]). Naturally, constructing real phantom or quintom models on cosmological scales, where NEC violation arises from quantum fluctuations, may be challenging. However, as a follow-up work it would be very interesting to consider time-varying DE equations of state (such as oscillatory ones) that are better motivated from such effects and constrain those models (similar to what we did for the bounce, with parametrizations versus an actual model, specifically the Cuscuton bounce; we further discuss this below). In other words, DE parametrizations are best suited to model DE in the late universe (since this is where DE’s effects are most important). However, when applying the SNEC to a DE component, we assumed the parametrization was valid essentially at all times, i.e., we extrapolated it all the way to matter-radiation equality. Of course, in the region of interest the contribution of DE to the background evolution is not significant at early times, but it determines where it becomes dominant. At the same time, the smearing function in the SNEC acts as a window function that can constrain the accumulation of NEC violation at any time, hence it makes sense to apply it to the era where NEC violation is most expected. Consequently, we argued that one should center the smearing function around the times when we best measure DE (e.g., $\bar{a} \sim 0.7$). Yet, we also saw that the SNEC constraints were most stringent for large smearing scales (of the order of the whole universe), hence our smearing function necessarily picked up accumulation of NEC violation from the early universe. These caveats could be bypassed entirely with a better-motivated DE model since one would (in principle) trust its EoS throughout time, and hence one could robustly apply the SNEC to it and have more control on the smearing parameters. Additionally, although smearing scales might apply up to the scales of the radius of curvature [234], one could argue that taking this limit is quite conservative, and so, the constraints may be weaker.

In this work, we also explored possible NEC violation in the context of nonsingular bouncing universes. We found that the SNEC translates into an upper bound on the quantity $x_b \equiv \alpha t_b^2/2$,

where $2t_b$ represents the duration of the NEC-violating bounce phase and where α amounts to the value of $\dot{H}$ at the bounce point (when H crosses zero). At fixed B , we found the bound on x_b has little dependence on other model parameters. We confirmed this in the case of perturbative parametrizations of the bounce phase, as well as for a specific bouncing model, namely the Cuscuton bounce. With a simple parametrization and under some approximations, the bound can be analytically derived as $x_b \lesssim 2\pi B$, and with $B = 1/(32\pi)$, this yields $x_b \lesssim 0.0625$. In comparison, the bound is $x_b \lesssim 0.13$ in the case of a Cuscuton bounce. In the latter case, the bound on x_b mainly translates into constraints on the model parameters m and μ , which are consistent with other theoretical constraints on the model coming from stability and strong coupling considerations.

There is an interesting parallel to make in this case with [441], which demanded a certain quantum null energy condition (specifically the quantum focusing conjecture) to fundamentally hold over a certain smeared length scale. From this, they derived the cutoff scale of a semiclassical effective theory of gravity (specifically higher-dimensional quadratic gravity). We can share this interpretation, as the bound we found on m and μ from the SNEC could be translated into a strong coupling scale in light of [407]. However, it is also important to recognize that the Cuscuton is a ‘nonlocal’ modification of GR that does not introduce any new local dynamical degrees of freedom. Therefore, the extent to which the SNEC applies to the Cuscuton bounce specifically, compared to other bouncing theories derived from local perturbative quantum gravity theories, remains uncertain. We leave these interesting questions for future work.

Nonsingular bouncing cosmologies also share features with wormholes and nonsingular black holes. Restrictions on wormholes from quantum-inspired energy conditions have already been the subject of a lot of work (let us mention [442] as only one example), but for regular black holes it deserves closer scrutiny. Exploring the consequences of the SNEC on various nonsingular black hole spacetimes (e.g., [443, 444, 445, 446, 447, 448, 449, 450, 451, 452, 453, 454, 455, 456, 457, 458, 459, 460, 461, 462]) would constitute another interesting follow-up analysis.

We would like to end by mentioning that the SNEC is not the only proposal in terms of quantum-motivated energy conditions. As one additional example, we could mention the quantum null energy condition of [463], which has been proven in various contexts [464, 465, 466, 467, 468, 469, 470, 471, 472, 473, 474], and it would certainly be interesting to further explore its consequences on the cosmologies of this work. Nevertheless, it still seems very plausible that something akin to the SNEC might hold fundamentally (like Heisenberg’s uncertainty principle), so while it might not be exactly the SNEC as considered in this paper (in fact, the SNEC itself

has refined formulations [236, 237, 239] that could be further explored), it might still have the correct flavor. Exoticism in the universe may have its limit after all. Still, this is not incompatible with the universe having endured some NEC fracture on cosmological scales and if anything the recent observations of [3, 4, 333, 334, 5, 6] may already be hinting that to us. Time will certainly tell whether there really was a fracture, and if so, how swiftly the universe will recover from it.

Part IV

Epilogue

What have we accomplished, and where do we go?

Each individual chapter was self-contained in its conclusion and future outlook, so we won't dive into duplicating too many details here. The overall goal of this thesis was seeing the effects that non-linear fields had on gravitational phenomena. This was investigated through non-linear curvature invariants in the action, non-linear and (non)-minimally coupled scalar fields, and nonlinear electrodynamics.

We showed in Chapter 4, that in Lovelock gravity, the non-linear curvature invariants allowed exotic black holes to possess a rich phase structure. We focused strictly on the case of third order Lovelock gravity, and found the associated phenomena was similar to the Gauss-Bonnet (quadratic) Lovelock gravity case [97]. This gave us further insight into these exotic black holes, by determining that including more non-linear invariants was not a recipe for creating, or discovering, more novel phase behaviour. The identification in Chapter 5 of negative mass black holes in a pseudo-de Sitter space was an interesting indirect result of looking at the thermodynamics of exotic black holes in de Sitter space instead of Anti-de Sitter space.

While previously conjectured, we were able to show that any *static* black hole solution coupled to NLED which possesses a radial electric field in 3DEGB gravity can be entirely characterized by its general relativity counterpart, as seen in Chapter 7. We are particularly interested in the thermodynamic consequences of the power-Maxwell solution as a future field of study. As shown in [54], 4-dimensional black holes in anti-de Sitter space – in general relativity – are able to possess multi-criticality due to the power-Maxwell source. We wish to investigate if this is present for 3-dimensional black holes. In chapter 6 we properly constructed the ADM metric of a rotating Maxwell charged black hole with radial electric field and angular magnetic field potential. We also derived the additional field necessary to account for the current that is present in the original rotating BTZ solution with Maxwell electric charge, and extended it to generic NLED theories.

In Chapter 8 and 9 we studied the cosmological consequences of non-linear fields. In Chapter 8 we looked at scalar perturbed FLRW spacetime in 4DEGB gravity, and derived a full set of equations of motion for the most general set of perturbations. These equations were used to show that 4DEGB could solve the horizon problem, but never the flatness problem. The most striking feature was that 4DEGB can mimic general relativity and a modified Λ CDM model with a variable number of neutrinos. Although this appeared to be worrisome when first conceived, this apparent degeneracy can be broken through a power spectrum analysis on the CMB with a modified Boltzmann solver for 4DEGB – we are currently investigating this problem. Chapter 9 is the last

study, which is an in depth look at quantum energy conditions applied to the universe. Similar to the work done in Chapter 8, the purpose of the study was to constrain certain cosmological models. We particular focused on the work for bouncing cosmologies, which the SNEC constrains how long a cosmological bounce may occur, and how energetic it may be. We were also able to show how the SNEC can constrain an equation of state parameter $\omega < -1$ as a function of the smearing scale. While this work focused on the universe in a semi-classical level when using quantum energy conditions, there are many other regimes which it can be considered, such as black hole spacetimes. These are avenues which we have been considering for future studies.

Conclusion

As we approach the end of this thesis, we wish to close with a few additional comments. The field of gravitational physics research is vast, and by far from complete – if it ever can be. While we hope this thesis, and the work that embodies it, furthers our understanding of black holes, cosmology and theoretical physics as a whole, even in the slightest. We hope most of all, that it stimulates readers and upcoming eager physicists to continue to push the boundaries of theoretical physics and further our understanding of the universe and all it has to offer.

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APPENDICES

Appendix A

Equations of Motion for $k \neq 0$ spatial sections

Here we will present the gravitational and scalar field equations for non-flat ($k \neq 0$) spatial curvature where we compute in conformal time with the line element outlined by (8.12). While some of the presentations will be redundant, we nonetheless will proceed with clarity in our calculations. The background calculations are carried out in Section 8.1.1.1 so we will not repeat them here, and instead just note that the solution to (3.27) is given as

$$\bar{\phi}' = \mathcal{H} + A \Rightarrow \bar{\phi} = \ln(a) + A\eta + B \quad (\text{A.1})$$

which used in conjunction with (4.4) yields the finalized gravitational field equations

$$\begin{aligned} \frac{(\mathcal{H}^2 + k)^2 \alpha}{a^2} + \mathcal{H}^2 + k &= \frac{8\pi G a^2 \bar{\rho}}{3} + \frac{\alpha C^2}{a^2} \\ \frac{(\mathcal{H}^2 + k)(4\mathcal{H}' - \mathcal{H}^2 - k)\alpha}{a^2} + \mathcal{H}^2 + 2\mathcal{H}' + k &= -8\pi G a^2 \bar{p} - \frac{\alpha C^2}{a^2} \end{aligned} \quad (\text{A.2})$$

where we have made the substitution $A^2 = -k + C$. The conservation of stress-energy $\nabla_\mu T^{\mu\nu} = 0$ yields only one equation for the background, that being

$$\bar{\rho}' = -3\mathcal{H}(\bar{p} + \bar{\rho}) \quad (\text{A.3})$$

For the first order equations we will present first with the arbitrary constant A and then make the substitution $A^2 = -k + C$, placing a box around the finalized equation. Beginning with the

scalar field equation, which is found in Section 8.1.1.2

$$\alpha (A^2 + k) (\nabla^2(\delta\phi - \Phi) + 3A (\Psi' + \Phi') - 3 (\Psi'' + \delta\phi'')) = 0 \quad (\text{A.4})$$

$$\boxed{\alpha C (\nabla^2(\delta\phi - \Phi) + 3\sqrt{-k + C} (\Psi' + \Phi') - 3 (\Psi'' + \delta\phi'')) = 0.} \quad (\text{A.5})$$

For the gravitational field equations, we will consider the $\eta\eta$ component first. The stress energy tensor remains the same $\delta T_{\eta\eta} = a^2\delta\rho + 2a^2\Phi\bar{\rho}$. Computing $\delta\mathcal{E}_{\eta\eta} = 8\pi G(a^2\delta\rho + 2a^2\Phi\bar{\rho})$ and using the first equation of (A.2) we arrive at the simplified expression

$$4\alpha A^2 [3A^2\Phi - 3A(\delta\phi' + \Psi') - \nabla^2(\delta\phi + \Psi) - 3k(\Psi - \Phi)] - 4\alpha k [3A(\delta\phi' + \Psi') + \nabla^2\delta\phi] + 2\mathcal{A}\nabla^2\Psi - 6\mathcal{H}(\mathcal{A} + 2\alpha k)\Psi' - 12\alpha\mathcal{H}^2((k + \mathcal{H}^2)\Phi - k\Psi) + 6a^2(k\Psi - \mathcal{H}^2\Phi) = 8\pi G a^4\delta\rho. \quad (\text{A.6})$$

$$\boxed{2(\mathcal{A} + 2k\alpha)(\nabla^2\Psi - 3\mathcal{H}\Psi' + 3(k\Psi - \Phi\mathcal{H}^2)) + 4\alpha C(3\Phi C - \nabla^2(\Psi + \delta\phi) - 3\sqrt{-k + C}(\delta\phi' + \Psi')) = 8\pi a^4 G\delta\rho.}$$

Any one of the time-spatial equations $\delta\mathcal{E}_{0i} = 8\pi G\delta T_{0i}$ has no corresponding zeroth order equation so calculating this directly yields

$$4\alpha A^2 [A(\Phi_{,i} - \delta\phi_{,i}) - (\delta\phi'_{,i} + \psi'_{,i})] + 4\alpha k A(\Phi_{,i} - \delta\phi_{,i}) + 2\mathcal{H}(\mathcal{A} + 2\alpha k)\Phi_{,i} + 2\mathcal{A}\Psi'_{,i} - 4\alpha k\delta\phi'_{,i} = -8\pi G a^4(1 + w)\bar{\rho}v_{,i}$$

$$\boxed{2\mathcal{H}(\mathcal{A} + 2\alpha k)(\Phi + \Psi') + 4\alpha C(\sqrt{-k + C}(\Phi - \delta\phi) - \delta\phi' - \Psi') = -8\pi G(1 + w)a^4\bar{\rho}v.} \quad (\text{A.7})$$

For a mixed spatial-spatial component $\delta\mathcal{E}_{ij} = 8\pi G\delta T_{ij}$ for $i \neq j$ we have

$$\nabla_i\partial_j(2\alpha A^2(\Phi + \Psi) + 4\alpha\mathcal{H}'\Psi - \mathcal{A}\Phi - \mathcal{B}\Psi) = 8\pi G a^4\nabla_i(\partial_j\Pi) \quad (\text{A.8})$$

$$\boxed{2\alpha(-k + C)(\Phi + \Psi) + 4\alpha\mathcal{H}'\Psi - \mathcal{A}\Phi - \mathcal{B}\Psi = 8\pi G a^4\Pi.} \quad (\text{A.9})$$

For the diagonal spatial components we employ the same procedure as with the flat case. We consider all the components individually and compute the trace, $\delta\mathcal{E}_{rr} + \delta\mathcal{E}_{\theta\theta} + \delta\mathcal{E}_{\phi\phi} = 8\pi G(\delta T_{rr} + \delta T_{\theta\theta} + \delta T_{\phi\phi})$; using the 2nd equation of (A.2), this gives

$$4\alpha A^2 (3A^2\Phi + 3A(\Phi' - \delta\phi') - \nabla^2(\Phi + \Psi) - 3(\Psi'' + \delta\phi'') + 3k(\Phi - \Psi)) + 12k\alpha A(\Phi' - \delta\phi') + 4\mathcal{H}' (6\mathcal{H}\alpha\Psi' + 3(4\mathcal{H}^2 + a^2 + 2k\alpha)\Phi - 6k\alpha\Psi - 2\nabla^2\Psi\alpha) + 2\mathcal{A}\nabla^2\Phi + 6\mathcal{H}(\mathcal{A} + 2k\alpha)\Phi' + 2\mathcal{B}\nabla^2\Psi - 12\alpha k\delta\phi'' + 6\mathcal{A}\Psi'' + 12\mathcal{H}a^2\Psi' - 12\mathcal{H}^2(\Phi\mathcal{H}^2 + k(\Phi - \Psi)) + 6a^2(\mathcal{H}^2\Phi - k\Psi) = 24\pi G a^4\delta p \quad (\text{A.10})$$

$$\boxed{\begin{aligned} &(\mathcal{B} + 2k\alpha)\nabla^2\Psi + (\mathcal{A} + 2k\alpha)(\nabla^2\Phi + 3\Psi'') + 2\alpha C(-\nabla^2\Phi - 3(\delta\phi'' + \Psi'') - 3k(\Phi + \Psi) - 3\sqrt{-k + C}\delta\phi') \\ &2\mathcal{H}'(-2\alpha\nabla^2\Psi + 12\alpha\mathcal{H}\Phi + 3a^2\Phi + 6\alpha k\Phi - 6\alpha k\Psi + 6\alpha\mathcal{H}\Psi') + 6\alpha C^2\Phi + 6\mathcal{H}a^2\Psi' + 3a^2(\mathcal{H}^2 - k\Psi) \\ &+ 3(2\alpha\mathcal{H}^3 + 2\alpha(-k + C)^{3/2} + \mathcal{H}a^2 + 2\mathcal{H}\alpha k + 2k\alpha\sqrt{-k + C})\Phi' - 6\alpha(\mathcal{H}^2 + k)(\mathcal{H}^2\Phi - \Psi k) = 12\pi G a^4\delta p. \end{aligned}}$$

We now need to consider the conservation of stress energy, $\nabla_\mu T^{\mu\nu} = 0$. Considering the time component first $\nabla_\mu T^{\mu\eta} = 0$, we obtain the same zeroth and first order equations as in the flat case; hence

$$\boxed{\delta\rho' = (1+w) (3\Psi' - \nabla^2 v) \bar{\rho} - 3\mathcal{H}(\delta\rho + \delta p)} . \quad (\text{A.11})$$

For the spatial components there are some slight modifications. For any spatial coordinate i , the equation $\nabla_\mu T^{\mu i} = 0$ becomes

$$\partial_i (\nabla^2 \Pi + (3w+3)\bar{\rho}v' - 3(1+w)(3w-1)\mathcal{H}\bar{\rho}v + 3(1+w)\bar{\rho}\Phi + 3\delta p + 6k\Pi) = 0. \quad (\text{A.12})$$

Taking the spatial divergence of this yields

$$\nabla^i (\partial_i (\nabla^2 \Pi + (3w+3)\bar{\rho}v' - 3(1+w)(3w-1)\mathcal{H}\bar{\rho}v + 3(1+w)\bar{\rho}\Phi + 3\delta p + 6k\Pi)) = 0 \quad (\text{A.13})$$

$$\nabla^2 (\nabla^2 \Pi + (3w+3)\bar{\rho}v' - 3(1+w)(3w-1)\mathcal{H}\bar{\rho}v + 3(1+w)\bar{\rho}\Phi + 3\delta p + 6k\Pi) = 0 \quad (\text{A.14})$$

$$\boxed{\nabla^4 \Pi + 3(1+w)\bar{\rho}\nabla^2 v' - 3(1+w)(3w-1)\mathcal{H}\bar{\rho}\nabla^2 v + 3(1+w)\bar{\rho}\nabla^2 \Phi + 3\nabla^2 \delta p + 6k\nabla^2 \Pi = 0} \quad (\text{A.15})$$

Appendix B

Scalar Perturbations in the small- α limit

We start with equations (8.38). We derive an approximation for these scalar perturbations in the following limits:

1. α is small, meaning $\alpha/H_0^2 \ll 1$. This is a reasonable approximation given [165] gives us priors $-10^{-83} \lesssim \tilde{\alpha} \lesssim 10^{-43}$. We can then write $\mathcal{A} = \mathcal{C} = \mathcal{D} = a^2$ and $\mathcal{B} = -a^2$.
2. C is large. Specifically, we assume large enough that $\alpha_C = \alpha C^2$ is finite and affects late universe dynamics, but small enough that $\alpha C^{1/2}$, αC and $\alpha C^{3/2}$ terms can be neglected.
3. α is small enough that the background equation reduces to $H^2 = \frac{8\pi G\rho}{3} + \frac{\alpha_C}{a^4}$.

Using these approximations we can rewrite the first equation in (8.38) as:

$$k^2\delta\phi + 3\delta\phi'' = 3\sqrt{C}(\Phi' + \Psi') \quad (\text{B.1})$$

We can do this because we know $\nabla^2\Phi \ll \sqrt{C}\Phi'$ and $\Psi'' \ll \sqrt{C}\Psi'$. This suggests that $\delta\phi = \frac{\delta\phi}{\sqrt{C}}$ is the same order of magnitude as the metric perturbations Φ and Ψ , so that we can re-write our equation as:

$$k^2\delta\varphi + 3\delta\varphi'' = 3(\Phi' + \Psi') \quad (\text{B.2})$$

Then using the same approximations we can rewrite the second equation in (8.38) as:

$$12\alpha_C\Phi - 12\alpha_C\delta\varphi' - 2a^2(k^2\Psi + 3\mathcal{H}\Psi' + 3\mathcal{H}^2\Phi) = 8\pi Ga^4\delta\rho \quad (\text{B.3})$$

The remaining equations reduce to their GR equivalent:

$$\begin{aligned} \Psi' + \mathcal{H}\Phi &= -4\pi Ga^2(1 + \omega)\rho v \\ \Phi &= \Psi + 8\pi Ga^2\Pi \\ 3\Psi'' + 3\mathcal{H}(\Phi' + 2\Psi') + k^2(\Psi - \Phi) + (6\mathcal{H}' - 3\mathcal{H}^2)\Phi &= 12\pi Ga^2\delta\rho \\ \delta' &= 3\mathcal{H}\delta\left(w - \frac{\delta p}{\delta\rho}\right) + (1 + w)(3\Psi' - \nabla^2 v) \\ 2\nabla^4\Pi + 3\nabla^2\delta p + 3(1 + w)\bar{\rho}[(1 - 3w)\nabla^2 v + \nabla^2 v' + \nabla^2\Phi] &= 0 \end{aligned} \quad (\text{B.4})$$

B.1 Coupled Differential Equations for the Boltzmann Solver

The following derivations follow the procedures outlined in [190, 475].

The Boltzmann equations for photons, photon polarization, neutrinos, dark matter and baryons are independent of the cosmological model (they describe the equations in any given expanding spacetime with perturbations). They will therefore be the same in Λ CDM and 4DEGB. These can be written as (see equation (22) in [190]):

$$\begin{aligned} \Theta' + ik\mu\Theta &= \Psi' - ik\Phi - \tau' \left[\Theta_0 - \Theta + ik\mu v_b - \frac{1}{2}\mathcal{P}_2\Pi \right] \\ \Theta'_P + ik\mu\Theta_P &= -\tau' \left[\Theta_P + \frac{1}{2}(1 - \mathcal{P}_2)\Pi \right] \\ \delta' - k^2 v &= 3\Psi' \\ v' + \mathcal{H}v &= -\Phi \\ \delta'_b - k^2 v_b &= 3\Psi' \\ v'_b + \mathcal{H}v_b &= -\Phi + \frac{\tau'R}{k}(v_b + 3\Theta_1) \\ \mathcal{N}' + ik\mu\mathcal{N} &= \Psi' - ik\mu\Phi \end{aligned} \quad (\text{B.5})$$

where all quantities are defined as in [190], except for $\Phi_{[190]} = -\Psi$, $\Psi_{[190]} = \Phi$ and $v_{[190]} = kv$. Care should also be taken when comparing derivatives (the prime symbol in [190] indicates a

derivative with respect to $x = \ln a$). To explicitly solve the coupled ODEs we need to use the multipole expansion of the first two equations in (B.5).

Next, by looking at (B.3) and the second equation in (B.4) (when including photon and neutrino perturbations) we find that the relevant Einstein equations take the form:

$$\begin{aligned} k^2 \Psi + 3\mathcal{H}(\Psi' + \mathcal{H}\Phi) &= \frac{6\alpha_C(\Phi - \delta\varphi')}{a^2} + 4\pi G a^2 [\rho\delta + \rho_b\delta_b + 4\rho_\gamma\Theta_0 + 4\rho_\nu\mathcal{N}_0] \\ \Phi &= \Psi - \frac{12H_0^2}{k^2 a^2} [\Omega_{\gamma 0}\Theta_2 + \Omega_{\nu 0}\mathcal{N}_2] \end{aligned} \quad (\text{B.6})$$

We also need an additional differential equation for $\delta\varphi$ which is given by (B.2):

$$k^2 \delta\varphi + 3\varphi'' = 3(\Phi' + \Psi') \quad (\text{B.7})$$

B.1.1 Initial conditions for the coupled ODEs

In the early universe for small k , the first equation in (B.6) reduces to:

$$\begin{aligned} \frac{\Psi'}{\eta} + \frac{\Phi}{\eta^2} - \frac{2\tilde{\alpha}_C}{(\Omega_{r0} + \tilde{\alpha}_C)\eta^2}(\Phi - \delta\varphi') &= -2\frac{8\pi G a^2 \rho}{3} \left(\frac{\rho_{\gamma 0}}{\rho}\Theta_0 + \frac{\rho_{\nu 0}}{\rho}\mathcal{N}_0 \right) \implies \\ \frac{\Psi'}{\eta} + \frac{\Phi}{\eta^2} - \frac{2\tilde{\alpha}_C}{(\Omega_{r0} + \tilde{\alpha}_C)\eta^2}(\Phi - \delta\varphi') &= -\frac{2}{\eta^2} \left(\frac{\Omega_{r0}}{\Omega_{r0} + \tilde{\alpha}_C} \right) ((1 - f_\nu)\Theta_0 + f_\nu\mathcal{N}_0) \end{aligned} \quad (\text{B.8})$$

where we have used the background equation $\frac{8\pi G a^2 \rho}{3} = \mathcal{H}^2 - \frac{H_0^2 \tilde{\alpha}_C}{a^2}$, so that $\mathcal{H} = \frac{1}{\eta}$ and $a = H_0 \sqrt{\Omega_{r0} + \tilde{\alpha}_C} \eta$, and where $f_\nu = \frac{\rho_\nu}{\rho_r + \rho_\nu} = \frac{1}{\frac{8}{7N_{\text{eff}}}(\frac{11}{4})^{4/3+1}}$ when all neutrinos are massless. Now we use the fact that for small k , $\Psi' = \Theta'_0 = \mathcal{N}'_0$ together with the derivative of (B.8) $\times \eta^2$ to get:

$$\eta\Psi'' + \Psi' + \left(1 - \frac{2\tilde{\alpha}_C}{\tilde{\alpha}_C + \Omega_{r0}}\right)\Phi' + \frac{2\tilde{\alpha}_C}{\Omega_{r0} + \tilde{\alpha}_C}\delta\varphi'' = -\frac{2\Omega_{r0}}{\Omega_{r0} + \tilde{\alpha}_C}\Psi' \quad (\text{B.9})$$

Next we employ the ansatz $\Psi = (1 + \epsilon)\Phi$ in the early universe, where ϵ is a constant to be determined. Then, using (B.6) for small k (i.e. $\delta\varphi'' = (2 + \epsilon)\Phi'$), we get:

$$\begin{aligned} (1 + \epsilon)\eta\Psi'' + \left(2 + \epsilon - \frac{2\tilde{\alpha}_C}{\Omega_{r0} + \tilde{\alpha}_C} + \frac{2\Omega_{r0}}{\Omega_{r0} + \tilde{\alpha}_C}(1 + \epsilon) + \frac{2\tilde{\alpha}_C}{\Omega_{r0} + \tilde{\alpha}_C}(2 + \epsilon)\right)\Phi' &= 0 \implies \\ (1 + \epsilon)\eta\Psi'' + (4 + 3\epsilon)\Phi' &= 0 \end{aligned} \quad (\text{B.10})$$

This equation is solved by $\Phi = \eta^P$ for $P = 0, 1 - \frac{4+3\epsilon}{1+\epsilon}$, which are respectively a growing and a decaying mode for $\epsilon > -1$. This means the early-universe growing mode has solution $\Phi = \Phi_i$ (defined at some early time η_i) which is a constant. This implies $\delta\varphi' = \delta\varphi'_i$ is also a constant in the early universe (at some early time η_i).

We can now find the initial conditions for other quantities as a function of Φ_i and $\delta\varphi'_i$ both. We start with the bottom line of equation (B.8) with $\Psi' = 0$ and $\Theta_{0,i} = \mathcal{N}_{0,i}$ to get:

$$\begin{aligned} (1 - \frac{2\tilde{\alpha}_C}{\Omega_{r0} + \tilde{\alpha}_C})\Phi_i + \frac{2\tilde{\alpha}_C}{\Omega_{r0} + \tilde{\alpha}_C}\delta\varphi'_i &= -\frac{2\Omega_{r0}}{\Omega_{r,0} + \tilde{\alpha}_C}\Theta_{0,i} \implies \\ \Theta_{0,i} &= -\frac{1}{2}\left\{\frac{\Omega_{r0} - \tilde{\alpha}_C}{\Omega_{r0}}\Phi_i + \frac{2\tilde{\alpha}_C}{\Omega_{r0}}\delta\varphi'_i\right\} \end{aligned} \quad (\text{B.11})$$

Next, we use the third equation in (B.5) as well as the multipole expansion of the first equation in (B.5) (see the first equation in (22) from [190]) with adiabatic initial conditions ($\frac{\delta}{\Theta_0}$ is constant) to get:

$$\delta_i = 3\Theta_{0,i} \quad (\text{B.12})$$

We also use the first equation in (22) from [190] with $\Phi' = 0, 3\Theta_1 + v_b \approx 0$ and $\Theta_2 \ll 1$ to get:

$$\Theta_{1,i} = \frac{k\eta}{3}(\Phi_i + \Theta_{0,i}) \implies \Theta_{1,i} = \frac{k\Phi_i}{6\mathcal{H}}\left\{\frac{\Omega_{r0} + \tilde{\alpha}_C}{\Omega_{r0}} - \frac{2\tilde{\alpha}_C}{\Omega_{r0}}\frac{\delta\varphi'_i}{\Phi_i}\right\} \quad (\text{B.13})$$

The third equation in (B.5) as well as the first equation in (22) from [190] (with adiabatic initial conditions) also give:

$$-k^2v_i = (3\Phi' - \delta')_i = (3\Theta'_0 + 3k\Theta_1 - \delta')_i \approx 3k\Theta_{1,i} \implies \Theta_{1,i} = -\frac{kv_i}{3} \quad (\text{B.14})$$

To find the value of ϵ , we first note that the seventh and eighth equations in (22) from [190] in the early universe imply:

$$\mathcal{N}_2'' \approx \frac{2k}{5}\mathcal{N}_1' = \frac{2k^2}{15}(\mathcal{N}_0 - 2\mathcal{N}_2 + \Phi) \quad (\text{B.15})$$

and the second equation in (B.6) with $\Theta_2 \ll \mathcal{N}_2$ and $a = H_0\sqrt{\Omega_{r0} + \tilde{\alpha}_C}\eta$ implies:

$$\mathcal{N}_2 = -\frac{\Omega_{r0} + \tilde{\alpha}_C}{\Omega_{r0}}\frac{(k\eta)^2(\Phi - \Psi)}{12f_\nu} \quad (\text{B.16})$$

Differentiating (B.16) and equating to (B.15) (with $\mathcal{N}_2 \ll |\Phi + \mathcal{N}_0|$) we get:

$$\mathcal{N}_0 = \frac{5}{4f_\nu} \left(\frac{\Omega_{r0} + \tilde{\alpha}_C}{\Omega_{r0}} \right) \Psi - \left(\frac{5}{4f_\nu} \left(\frac{\Omega_{r0} + \tilde{\alpha}_C}{\Omega_{r0}} \right) + 1 \right) \Phi \quad (\text{B.17})$$

which, using $\mathcal{N}_{0,i} = \Theta_{0,i} = -\frac{1}{2} \left\{ \frac{\Omega_{r0} - \tilde{\alpha}_C}{\Omega_{r0}} \Phi_i + \frac{2\tilde{\alpha}_C}{\Omega_{r0}} \delta\varphi'_i \right\}$ implies:

$$\begin{aligned} \Psi_i &= \frac{4f_\nu}{5} \left(\frac{\Omega_{r0}}{\Omega_{r0} + \tilde{\alpha}_C} \right) \left\{ \left(1 - \frac{\Omega_{r0} - \tilde{\alpha}_C}{2\Omega_{r0}} + \frac{5}{4f_\nu} \left(\frac{\Omega_{r0} + \tilde{\alpha}_C}{\Omega_{r0}} \right) \right) \Phi_i - \frac{\tilde{\alpha}_C}{\Omega_{r0}} \delta\varphi'_i \right\} \Rightarrow \\ \boxed{\Psi_i &= \left(\frac{2f_\nu}{5} + 1 \right) \Phi_i - \frac{4f_\nu \tilde{\alpha}_C}{5(\Omega_{r0} + \tilde{\alpha}_C)} \delta\varphi'_i} \end{aligned} \quad (\text{B.18})$$

This satisfies the requirements of the ansatz for sufficiently small $\tilde{\alpha}_C$, f_ν and $\delta\varphi'_i$. It also allows us to find:

$$\boxed{\mathcal{N}_{2,i} = \frac{k^2 a^2}{12H_0^2 \Omega_{\nu 0} \left(\frac{5}{2f_\nu} + 1 \right)} \left(\Psi_i + \frac{2\tilde{\alpha}_C}{\Omega_{r0} + \tilde{\alpha}_C} \delta\varphi'_i \right)}, \quad \boxed{\mathcal{N}_\ell = \frac{k}{(2\ell + 1)\mathcal{H}} \mathcal{N}_{\ell-1}, \ell > 2} \quad (\text{B.19})$$

The derivation for the moments of the photon polarization perturbations can be found in [190]. Turning now to the scalar field, we notice that in the early universe (B.7) is the equation for a simple harmonic oscillator with frequency $\omega = \frac{k}{\sqrt{3}}$. This will have a general homogeneous solution:

$$\delta\varphi = \delta\varphi_i \cos\left(\frac{k}{\sqrt{3}}\eta\right) + \frac{\sqrt{3}\delta\varphi'_i}{k} \sin\left(\frac{k}{\sqrt{3}}\eta\right) \quad (\text{B.20})$$

where $\delta\varphi_i$ is the initial value of the scalar field at some early time η_i and $\delta\varphi'_i$ is the initial value of its derivative. When $\Phi' \neq 0$, there will also be a particular integral solution to equation (B.7). In the early universe, if $\Phi' \lesssim \mathcal{O}(\eta)$, this particular solution will be $\delta\varphi_i \lesssim \mathcal{O}(\eta^3) \Rightarrow \left(\frac{d\delta\varphi}{d(\ln a)} \right)_i = \mathcal{O}(\eta_i^3) \approx 0$.

In the interest of simplicity we look at the special initial condition in which the scalar field is initially at rest at a minimum of its effective potential, with $\delta\varphi_i = \delta\varphi'_i = 0$. However, we acknowledge that in general this theory is not necessarily constructed to behave like GR in the early universe. Therefore an interesting avenue for future research could be the exploration of the effect of different initial conditions on perturbations in 4DEGB.

We can then write down the relevant initial conditions as:

$$\begin{aligned}
\Theta_{0,i} &= -\frac{1}{2} \left(\frac{\Omega_{r0} - \tilde{\alpha}_C}{\Omega_{r0}} \Phi_i \right) \\
\delta_{m,i} &= 3\Theta_{0,i} \\
\Theta_{1,i} &= \frac{k\Phi_i}{6\mathcal{H}(\eta_i)} \left(\frac{\Omega_{r0} + \tilde{\alpha}_C}{\Omega_{r0}} \Phi_i \right) \\
v_{m,i} &= -\frac{3}{k} \Theta_{1,i} \\
\mathcal{N}_{0,i} &= \Theta_{0,i} \\
\left(\frac{d\delta\varphi}{d(\ln a)} \right)_i &= \mathcal{O}(\eta_i^3) \approx 0 \\
\delta\varphi_i &= \frac{1}{3} \left(\frac{d\delta\varphi}{d(\ln a)} \right)_i \approx 0
\end{aligned} \tag{B.21}$$

and all other initial conditions have the same form as their GR equivalent as described in [\[190\]](#).