
Single Killing Vector Gauss-Bonnet Boson Stars and Single Killing Vector Hairy Black Holes in $D \geq 5$ Odd Dimensions

by
Laura HENDERSON

A thesis
presented to the University of Waterloo
in fulfilment of the
thesis requirement for the degree of
Master of Science
in
physics

Waterloo, Ontario, Canada, 2014

© Laura Henderson 2014

Declaration of Authorship

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

I construct anti-de Sitter boson stars in Einstein-Gauss-Bonnet gravity coupled to a $\frac{D-1}{2}$ -tuple of complex massless scalar field both perturbatively and numerically in $D = 5, 7, 9, 11$ dimensions. Due to the choice of scalar fields, these solutions possess just a single helical Killing symmetry. For each choice of the Gauss-Bonnet parameter $\alpha \neq \alpha_{\text{cr}}$, the central energy density at the center of the boson star, q_0 completely characterizes the one parameter family of solutions. These solutions obey the first law of thermodynamics, in the case of the numerics, to within 1 part in 10^6 . I describe the dependence of the boson star mass, angular momentum and angular velocity on α and on the dimensionality. For $\alpha < \alpha_{\text{cr}}$ and $D > 5$, these quantities exhibit damped oscillations about finite central values and the central energy density tends to infinity. The Kretschmann invariant at the center of the boson star diverges in the limit of diverging central energy. This contrasts the $D = 5$ case, where the Kretschmann invariant diverges at a finite value of the central energy density. Solutions where $\alpha < \alpha_{\text{cr}}$, correspond to negative mass boson stars, and for all dimensions the boson star mass and angular momentum decrease exponentially as the central energy density tends toward infinity with the Kretschmann invariant diverging only when in the limit the central energy density diverges.

I also briefly discuss the difficulties of numerically obtaining single Killing vector hairy black hole solutions and present the explicit boundary conditions for both Einstein gravity and Einstein-Gauss-Bonnet gravity.

Acknowledgements

I would first like to thank my supervisor Robb Mann for providing me with this opportunity, for his support throughout this project and for his assistance proof-reading. Next, I thank Sean Stotyn for all of his input and assistance, especially with the coding. Finally, I thank my parents, who taught me that I should never give up; this wouldn't have been possible without them.

This work was supported in part by the Natural Sciences and Engineering.

Contents

Declaration of Authorship	ii
Abstract	iii
Acknowledgements	iv
Contents	v
List of Figures	vii
List of Tables	viii
Abbreviations	ix
1 Introduction	1
1.1 Single Killing Vector Solutions	1
1.2 Einstein-Gauss-Bonnet Gravity	3
1.3 Boson Stars	4
2 Setup	6
2.1 Metric and Scalar Field Ansatz	6
2.2 Boundary Conditions	8
2.2.1 Boundary Conditions at the Origin	8
2.2.2 Asymptotic Boundary Conditions	9
2.3 Physical Charges	11
3 Perturbative Solutions	12
3.1 Perturbative Boson Star	12
4 Numerical Construction	15
4.1 Chebyshev Approximation	15
4.2 Implementation	16
4.3 Error Checking	19

5	Numeric Boson Star Solutions	20
5.1	The $\alpha < \alpha_{\text{cr}}$ solutions	21
5.1.1	The $D > 5$ cases	21
5.1.2	The $D = 5$ case	22
5.2	Beyond α_{cr}	29
6	Preliminary Black Hole Numerics	41
6.1	Black Hole Boundary Conditions	41
6.1.1	Black Hole Asymptotics	41
6.1.2	Black Hole Horizon	41
6.2	Implementation	42
7	Conclusions and Future Work	46
A	Field Equations	48
B	Perturbative Fields	53
C	Conserved Charges and Thermodynamic Quantities	64
D	Kretschmann Scalar and Critical q'_h Values	66
	Bibliography	69

List of Figures

5.1	The boson star mass versus the perturbative parameter ϵ .	22
5.2	The boson star angular momentum versus the perturbative parameter ϵ .	23
5.3	Detail of the boson star mass and angular momentum versus the perturbative parameter q_0 for $D = 5$.	24
5.4	The angular frequency ω versus the perturbative parameter ϵ .	24
5.5	The boson star mass versus the parameter q_0 .	25
5.6	The boson star angular momentum versus the parameter q_0 .	26
5.7	Detail of the boson star mass and angular momentum versus the parameter q_0 for $D = 5$.	26
5.8	The angular frequency ω versus the parameter q_0 .	27
5.9	The Kretschmann scalar K versus the parameter q_0 .	28
5.10	Detail of the Kretschmann scalar K versus the parameter q_0 .	29
5.11	$q_{h'}^* - q_h'(0)$ versus the parameter q_0 .	30
5.12	The boson star mass versus the perturbative parameter ϵ for $\alpha > \alpha_{\text{cr}}$.	31
5.13	The boson star angular momentum versus the perturbative parameter ϵ for $\alpha > \alpha_{\text{cr}}$.	32
5.14	Boson star mass and angular momentum versus the perturbative parameter ϵ for $\tilde{\alpha} = 100$.	32
5.15	The angular frequency versus the perturbative parameter ϵ for $\alpha > \alpha_{\text{cr}}$.	33
5.16	The boson star mass versus the parameter q_0 for $\alpha > \alpha_{\text{cr}}$.	34
5.17	The boson star angular momentum versus the parameter q_0 for $\alpha > \alpha_{\text{cr}}$.	35
5.18	Boson star mass and angular momentum versus the parameter ϵ for $\tilde{\alpha} = 100$.	35
5.19	The natural log of the boson star mass versus the parameter q_0 for $\alpha > \alpha_{\text{cr}}$.	36
5.20	The natural log of the boson star angular momentum versus the parameter q_0 for $\alpha > \alpha_{\text{cr}}$.	37
5.21	The Kretschmann scalar K versus the parameter q_0 for $\alpha > \alpha_{\text{cr}}$.	38
5.22	$q_{h'}^* - q_h'(0)$ versus the parameter q_0 with $\tilde{\alpha} = 1.5$.	38
5.23	Boson star mass versus angular momentum for $\alpha > \alpha_{\text{cr}}$.	39
5.24	Boson star mass versus angular momentum.	40

List of Tables

5.1 Estimate of the critical value of q_0 in $D = 5$ 27

Abbreviations

SKV	Single K illing V ector
EGB	E insein G auss B onnet
AdS	A nti-de S itter
ODE	O rdinary D ifferential E quation

Chapter 1

Introduction

The goal of my thesis project is to find asymptotically anti-de Sitter (AdS) single Killing vector (SKV) boson star solutions to Einstein-Gauss-Bonnet gravity. These solutions are interesting for two primary reasons. The first is that since Gauss-Bonnet gravity is a higher curvature theory of gravity it can show up as curvature corrections in theories of quantum gravity. Therefore, it is always useful to have a stock pile of solutions, especially ones with relatively little symmetry, such as having only one Killing vector. Additionally, since these solutions are asymptotically AdS, they play a role in holographic gauge theories, such as the AdS/CFT correspondence and in particular boson stars describe finite energy excitations above the vacuum state. I will expand further on these concepts throughout my introduction.

1.1 Single Killing Vector Solutions

Most of the boson star and black hole solutions to Einstein's equations, and their higher curvature generalizations such as Lovelock gravity, have relied on assuming a relatively large set of symmetries about the space-time in order to be analytically or numerically tractable. These symmetries usually take the form of Killing vectors ξ , infinitesimal generators of the space-time's isometries which, in a region, obey

$$\mathcal{L}_\xi g = 0 \tag{1.1}$$

where $\mathcal{L}$ is the Lie derivative and g is the metric. Additionally, there are many theorems which show that under physically reasonable assumptions Killing symmetries are ubiquitous. For example, a rigidity theorem states that if a space-time is stationary, it must also possess an axis of symmetry; as a consequence the horizon of the multiply rotating black hole must be a Killing horizon with respect to the Killing field

$$\zeta = \frac{\partial}{\partial t} + \sum_I \Omega_I \frac{\partial}{\partial \phi_I}$$

where $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi_I}$ generate the respective stationary and rotational symmetries, with Ω_I the associated angular velocities. Therefore these solutions must have a minimum of two Killing symmetries [1–3].

However, if a generic matter field is introduced, then the gravitational field outside of the matter source may have reduced Killing symmetries or eliminate them entirely. In particular, black hole and especially boson star solutions possessing only a single Killing vector (SKV) have been recently explored in several settings. In 5 dimensions with anti-de Sitter (AdS) boundary conditions, there is a clever choice of metric and co-rotating scalar field ansatz [4] for which rotating boson star solutions exhibit a single helical Killing vector [5] even though the stress energy tensor has the same symmetries as the metric. The scalar field, which consists of a scalar doublet, has a harmonic time dependence which breaks the continuous rotational symmetry. This ansatz has a straightforward generalization to any odd dimension [6, 7] by using Hopf fibrations to parametrize the $D - 2$ -sphere. Physically, this configuration is based on superradiance. A mode of a scalar field given by $e^{-i\omega t + im\phi}$ will increase in amplitude by scattering off the horizon of a rotating black hole with angular velocity Ω_H provided $\omega < m\Omega_H$, which mines energy from the black hole and decreases Ω_H . They are then reflected back to the horizon by the AdS boundary conditions, where they can scatter again, further increasing their amplitude. This process continues until $\omega = m\Omega_H$, at which point the black hole is surrounded by “lumpy” co-rotating scalar hair.

1.2 Einstein-Gauss-Bonnet Gravity

Einstein-Gauss-Bonnet (EGB) gravity is the first higher curvature term of Lovelock gravity [8] which is a class of higher curvature theories of gravity whose resulting field equations consist of no higher than second order differential equations. The general Lovelock Lagrangian is given by

$$\mathcal{L} = \sqrt{-g} \sum_{n=0}^t \hat{\alpha}_n \left(\frac{1}{2^n} \delta^{\mu_1 \nu_1 \dots \mu_n \nu_n}_{\alpha_1 \beta_1 \dots \alpha_n \beta_n} \prod_{r=1}^n R^{\alpha_r \beta_r}_{\mu_r \nu_r} \right) \quad (1.2)$$

where $R^{\alpha_r \beta_r}_{\mu_r \nu_r}$ is the Riemann tensor and $\hat{\alpha}_n$ is an arbitrary coupling constant. Each term in the sum corresponds to the Euler density of a $2n$ dimensional manifold, and so it only contributes to the equations of motion when $D > 2n$. In the EGB case, $n = 2$ implying that it only contributes when $D \geq 5$.

These higher derivative theories are of interest in quantum gravity as the Einstein action is expected to be an effective gravitational action, valid for only small energies (or curvatures), and it will be modified by higher curvature terms. In fact, when the tension of a string is large compared to the energy scale of other variables, the Gauss-Bonnet term becomes the first curvature correction to general relativity [9].

The biggest advantage of using Lovelock gravity over other theories when looking at these higher curvature corrections is that although the Lagrangian (1.2) does contain higher derivatives of the Riemann tensor, the resulting field equations do not contain derivatives higher than second order. This means that the theory does not suffer from the Ostrogradski linear instability [10]. If higher derivatives were present in a (non-degenerate) theory, then it would have linear instability the resulting Hamiltonian would be unbounded from below. Attempts to quantize a theory with this instability would either result in negative energy states leading to unbounded pair production, or negative norm states (“ghosts”) which are undefined [11].

While the Gauss-Bonnet parameter α can take on any value, there is a critical value α_{cr} that exists in any dimension which corresponds to a class of quadratic curvature theories that have drastically different properties than standard EGB [12]. They admit BTZ-like solutions, where there is a mass gap between the zero mass black hole and the AdS space-time and do not obey the falloff boundary conditions required for $\alpha \neq \alpha_{\text{cr}}$ solutions.

It is worth noting that although EGB theories have two solution branches, since they are a quadratic in the Riemann tensor, I will only be considering the branch that reduces down to the Einstein case in the $\alpha \rightarrow 0$ limit.

1.3 Boson Stars

Boson stars are smooth horizonless geometries composed of self-gravitating and self-interacting bosonic matter [13, 14]. They are bundles of field energy that decay at large distances from their centers, meaning that unlike ordinary stars, they do not have a sharp edge. Although their physical existence is unknown, boson stars in asymptotically flat 4 dimensional space-time have been proposed as a dark matter halo candidate that may help explain galaxy rotation curves [15]. While excited states typically produce flatter and more realistic rotation curves than those in the ground state, they typically decay to the ground state unless in specific mixed states [16]. Additionally, when not coupled to a Maxwell field, boson stars provide a dark alternative to astrophysical black hole candidates, which could be discerned by gravitational wave astronomy [17, 18].

On the theoretical side, asymptotically AdS boson stars are thought to play important roles in holographic gauge theories through the AdS/CFT correspondence. Since all solutions of this are zero-temperature horizonless objects, they describe finite energy excitations above the vacuum state. The non-linear stability of these objects is of interest if AdS boundary conditions are present since it determines if the corresponding holographic dual CFT will thermalize and, if so, on what time-scale. Due to evidence of a gravitational turbulent instability in asymptotically AdS space-times [19–22], one might expect all AdS boson stars to be non-linearly unstable to black hole formation. However it has been shown that there is a wide range of initial data for which boson stars are nonlinearly stable and avoid this instability given sufficiently small perturbations [23], suggesting on the gauge theory side, that there is a family of strongly coupled CFT states that do not thermalize in finite time.

Due to this discovery, one of the immediate ongoing tasks is to map out the territory of boson star solutions. There is a wide range of boson star solutions for various forms of scalar matter. These include a complex doublet of massive [24, 25] and massless [5, 6] scalar fields,

self-interacting scalar fields [26, 27], scalars with gauge charges [28, 29], scalars in space times with de Sitter [30, 31], flat [26, 32], or anti-de Sitter [24, 33] boundary conditions, solutions in $(2 + 1)$ dimensions [7, 24, 34], and rotating doublets [5] and multiplets [6, 35] of scalars.

The latter solutions are especially interesting as they are single Killing vector (SKV) solutions, while most other have relatively high levels of symmetry in order to be either analytically or numerically solved. Recent work suggests that the turbulent instability of global AdS is due to a high level of symmetry [36]. That is, the normal mode frequencies are all integer multiples of the AdS frequency, which leads to a large number of resonances responsible for the nonlinear instability. As a result, solutions with less symmetry tend to be nonlinearly stable.

The hope is that a better understanding of the gravitational aspects of boson stars, particularly those with less symmetry, will lead to insights into physical realizable systems.

The rest of my thesis is structured as follows: in Chapter 2, I present the metric and scalar field ansatz, obtain the ordinary differential equations (ODEs) that must be solved and describe the boundary conditions and the basic physical properties of boson stars. I then, in Chapter 3, analytically construct perturbative solutions for rotating SKV Einstein-Gauss-Bonnet boson stars. I discuss the numerical methods to construct full non-perturbative solutions in Chapter 4 and present the results in Chapter 5 for $\alpha < \alpha_{\text{cr}}$ and $\alpha > \alpha_{\text{cr}}$. In Chapter 6, I present the explicit boundary conditions for rotating SKV hairy black holes in both Einstein and Einstein-Gauss-Bonnet gravity and briefly discuss the difficulties facing the numerics. Finally in Chapter 7, I make my concluding remarks.

Chapter 2

Setup

2.1 Metric and Scalar Field Ansatz

I begin with $D = n + 2$ dimensional Einstein-Gauss-Bonnet gravity with a negative cosmological constant minimally coupled to an $\frac{n+1}{2}$ -tuple of complex massless scalar fields $\vec{\Pi}$. The action is given by (Newton's constant, G , is set to 1)

$$S = \frac{1}{16\pi} \int d^D x \sqrt{-g} \left(R - 2\Lambda + \alpha L_{GB} - 2|\nabla \vec{\Pi}|^2 \right) \quad (2.1)$$

where

$$L_{GB} = R_{abcd}R^{abcd} - 4R_{ab}R^{ab} + R^2 \quad (2.2)$$

and Λ is the cosmological constant. The equations of motion resulting from this action are

$$G_{ab} + \Lambda g_{ab} - 2\alpha \left(-R_{acde}R_b{}^{cde} + 2R_{acbd}R^{cd} + 2R_{ac}R_b{}^c - RR_{ab} + \frac{1}{4}g_{ab}L_{GB} \right) = T_{ab} \quad (2.3)$$

$$\nabla^2 \vec{\Pi} = 0 \quad (2.4)$$

where

$$T_{ab} = \left(\partial_a \vec{\Pi}^* \partial_b \vec{\Pi} + \partial_a \vec{\Pi} \partial_b \vec{\Pi}^* \right) - g_{ab} \left(\partial_c \vec{\Pi} \partial^c \vec{\Pi}^* \right) \quad (2.5)$$

is the stress-energy tensor of the scalar fields. Since I am only interested in solutions that asymptote to AdS, I require global AdS to be a solution to equation (2.3) with $\vec{\Pi} = 0$.

Requiring this fixes the bare cosmological constant Λ to

$$\Lambda = \frac{n(n+1)((n-1)(n-2)\alpha - \ell^2)}{2\ell^4}, \quad (2.6)$$

where ℓ is the effective AdS length. Note that global AdS is still a solution regardless of the sign of Λ , provided that the appropriate choice of α is made. This is because in the absence of matter sources, the Gauss-Bonnet term acts like a negative cosmological constant.

I take the following metric and scalar field ansatz [6]

$$ds^2 = -f(r)g(r)dt^2 + \frac{dr^2}{f(r)} + r^2 \left(h(r)(d\chi + A_i dx^i - \Omega(r)dt)^2 + g_{ij}dx^i dx^j \right) \quad (2.7)$$

$$\Pi_i = \Pi(r)e^{-i\omega t}z_i, \quad i = 1 \dots \frac{n+1}{2} \quad (2.8)$$

where z_i are complex coordinates such that $\sum_i dz_i d\bar{z}_i$ is the metric of the unit n -sphere. A convenient choice for the z_i is

$$z_i = \begin{cases} e^{i(\chi + \phi_i)} \cos \theta_i \prod_{j < i} \sin \theta_j, & i = 1 \dots \frac{n-1}{2} \\ e^{i\chi} \prod_{j=1}^{\frac{n-1}{2}} \sin \theta_j, & i = \frac{n+1}{2} \end{cases} \quad (2.9)$$

in which case $\sum_i dz_i d\bar{z}_i = (d\chi + A_i dx^i)^2 + g_{ij}dx^i dx^j$ is the Hopf fibration of the unit n -sphere, where

$$A_i dx^i = \sum_{i=1}^{\frac{n-1}{2}} \cos^2 \theta_i \left[\prod_{j < i} \sin^2 \theta_j \right] d\phi_i \quad (2.10)$$

and g_{ij} is the metric on a unit complex projective space $\mathbb{CP}^{\frac{n-1}{2}}$. In these coordinates, χ and the ϕ_i all have a period of 2π while the θ_i take value in the range $[0, \pi]$. This construction only works in odd dimensions, and so I will restrict my analysis to $n = 3, 5, 7, 9$; however my results can be extended to any odd $n \geq 3$.

This construction was first used in the $n = 3$ form [4] to obtain boson star solutions in 5-dimensional Einstein gravity coupled to a doublet of self-interacting scalars and is crucial in obtaining rotating boson star solutions in any odd dimension. The scalar fields can be

viewed as coordinates on $\mathbb{C}^{\frac{n+1}{2}}$, with $\vec{\Pi}$ tracing out a round n -sphere with a time-varying (but otherwise constant) phase at each value of r . Constant r surfaces in the metric (2.7), on the other hand, correspond to squashed rotating n -spheres. Since the first term in the stress energy tensor T_{ab} is the pull-back of the round metric on the n -sphere, and the second term is proportional to g_{ab} , it has the same symmetries as the metric (2.7).

It is important to note that while the stress-energy tensor has the same symmetries as the metric, the scalars themselves do not. The scalar field (2.8) is only invariant under the combination

$$\kappa = \partial_t + \omega \partial_\chi. \quad (2.11)$$

whereas the metric is invariant under ∂_t , ∂_χ and the rotations of $\mathbb{CP}^{\frac{n-1}{2}}$. Any solution with non-trivial scalar field content will only be invariant under the single Killing vector field κ (2.11).

The equations of motion yield a system of five coupled second order ordinary differential equations which are listed in Appendix A as they are quite cumbersome to write down.

2.2 Boundary Conditions

In order to solve the EGB equations of motion, boundary conditions need to be imposed at the boson star origin as well as asymptotically. In this section I will employ the same boundary conditions as in the Einstein case [35].

2.2.1 Boundary Conditions at the Origin

Boson star geometry is smooth and horizonless, which means that all metric functions must be regular at the origin. Additionally, due to the slow physical rotation of points as $r \rightarrow 0$, surfaces of constant t are described by round n -spheres with r as the proper radial distance. Multiplying (A.5) by r^2 gives the boundary condition on Π as it must vanish at the origin to yield consistent equations of motion. Thus the boundary conditions at the center of the

boson star take the form

$$\begin{aligned} f|_{r \rightarrow 0} &= 1 + \mathcal{O}(r^2), & g|_{r \rightarrow 0} &= g(0) + \mathcal{O}(r), & h|_{r \rightarrow 0} &= 1 + \mathcal{O}(r^2), \\ \Omega|_{r \rightarrow 0} &= \Omega(0) + \mathcal{O}(r), & \Pi|_{r \rightarrow 0} &= q_0 \frac{r}{\ell} + \mathcal{O}(r^2), \end{aligned} \quad (2.12)$$

for all n , where q_0 is a dimensionless parameter such that the energy scalar field T^{00} at the origin (hereafter called the central energy density) is proportional to q_0^2 . It is formally defined as $q_0 \equiv \ell \Pi'(0)$. It is of particular importance to the numeric work presented below, as given a particular value of α , q_0 uniquely parametrizes the one-parameter family of boson-star solutions in each dimension.

2.2.2 Asymptotic Boundary Conditions

First I will make note of a residual gauge freedom in order to simplify the boundary conditions. The transformation

$$\chi = \tilde{\chi} + \lambda t, \quad \Omega(r) = \tilde{\Omega}(r) + \lambda, \quad \omega = \tilde{\omega} + \lambda \quad (2.13)$$

for some arbitrary constant λ leaves both the metric (2.7) and the scalar field (2.8) unchanged. It is convenient for the numerical analysis to set $\lambda = \omega$ so that I can set $\tilde{\omega} = 0$: in this frame, the coordinates are asymptotically rigidly rotating, so that $\tilde{\Omega}(r) \rightarrow -\omega$ as $r \rightarrow \infty$. In what follows, I use $\tilde{\chi}$, $\tilde{\Omega}$ and ω but I drop the tildes for notational convenience.

As $r \rightarrow \infty$, the boson stars will asymptote to AdS with corrections for mass and angular momentum. This determines the metric functions up to constants C_f , C_h and C_Ω . Requiring Π to be normalizable sets the boundary condition for the scalar field. Explicitly, the boundary conditions are given by

$$\begin{aligned} f|_{r \rightarrow \infty} &= \frac{r^2}{\ell^2} + 1 + \frac{C_f \ell^{n-1}}{r^{n-1}} + \mathcal{O}(r^{-n}), & g|_{r \rightarrow \infty} &= 1 - \frac{C_h \ell^{n+1}}{r^{n+1}} + \mathcal{O}(r^{-(n+2)}), \\ h|_{r \rightarrow \infty} &= 1 + \frac{C_h \ell^{n+1}}{r^{n+1}} + \mathcal{O}(r^{-(n+2)}), & \Omega|_{r \rightarrow \infty} &= -\omega + \frac{C_\Omega \ell^n}{r^{n+1}} + \mathcal{O}(r^{-(n+2)}), \\ \Pi|_{r \rightarrow \infty} &= \frac{\epsilon \ell^{n+1}}{r^{n+1}} + \mathcal{O}(r^{-(n+2)}), \end{aligned} \quad (2.14)$$

where ϵ is a dimensionless measure of the amplitude of the scalar field at infinity and the constants C_f , C_g and C_h can be found by solving the equations of motion. Note that the leading order corrections to $g(r)$ and $h(r)$ are not independent. When the equations are solved perturbatively in the next section, I will find (as in the Einstein case [6]) that ϵ uniquely parameterizes the boson star solutions. This is not the case in the non-perturbative regime, which will be explicitly demonstrated in Chapter 5.

At this point, it is important to note that I am using ℓ to define the asymptotic behavior of $f(r)$, and as long as $\ell^2 > 0$, the space time will be asymptotically AdS. By using the definition of the AdS length, the relation (2.6) for the cosmological constant follows from the field equations. This convention differs from another common convention [12, 37], which sets

$$\Lambda = \frac{n(n+1)}{2L^2} \quad (2.15)$$

(where L is the would-be AdS length in the absence of the Gauss-Bonnet term) and leads to an upper bound on α

$$\alpha < \frac{L^2}{4(n-1)(n-2)} \quad (2.16)$$

in order to satisfy AdS boundary conditions [37]. This inequality can be expressed in terms of the convention used here, as

$$(2(n-1)(n-2)\alpha - \ell^2)^2 > 0, \quad (2.17)$$

which immediately implies

$$\alpha \neq \frac{\ell^2}{2(n-1)(n-2)} \equiv \alpha_{\text{cr}}. \quad (2.18)$$

Note that this gives the critical value of α for the Gauss-Bonnet coupling in any dimension [12], which also appears in the perturbative boson star solutions as will be seen in Chapter 3. Additionally, since this convention does not put an upper bound on α I am able to explore boson star solutions for values of $\alpha > \alpha_{\text{cr}}$, which are significantly different from those for which $\alpha > \alpha_{\text{cr}}$.

2.3 Physical Charges

The SKV boson stars are invariant under the single Killing field (2.11) which is a linear combination of ∂_t and ∂_χ . However, since the scalar fields vanish with sufficient fall-off, both ∂_t and ∂_χ are each asymptotic Killing fields (the metric alone being invariant under them). Thus they define conserved charges; specifically, ∂_t is readily associated with a conserved energy and ∂_χ with angular momentum. I follow the definitions proposed by Ref. [38], which is an extension of the conserved charges for asymptotically AdS EGB solutions calculated in the same paper via the counterterm method.

$$\begin{aligned} M &= \frac{(n+1)\pi^{\frac{n-1}{2}}\ell^{n-1}}{16\left(\frac{n+1}{2}\right)!} \left(1 - \frac{\alpha}{\alpha_{\text{cr}}}\right) ((n+1)C_h - nC_f), \\ J &= \frac{(n+1)^2\pi^{\frac{n-1}{2}}\ell^n}{16\left(\frac{n+1}{2}\right)!} \left(1 - \frac{\alpha}{\alpha_{\text{cr}}}\right) C_\Omega \end{aligned} \quad (2.19)$$

where C_f , C_h and C_Ω are the constants appearing in the asymptotic boundary conditions (2.14).

The existence of these asymptotic Killing symmetries also guarantees that the boson stars satisfy the first law of thermodynamics. Since they have a vanishing temperature and entropy, the first law takes the form

$$dM = \omega dJ. \quad (2.20)$$

This relation provides a convenient and important numerical tool: by requiring the solutions respect the first law (2.20) to at least one part in 10^6 , I obtain a primary cross-check on the validity of the numerical methods used.

Chapter 3

Perturbative Solutions

In this chapter, I will present solutions to the Einstein Gauss-Bonnet equations satisfying the boundary conditions (2.12) and (2.14) which are constructed as perturbations about AdS. I will use the scalar field condensate parameter ϵ as the perturbative parameter for the space-time dimensions $D = 5, 7, 9, 11$, since they are of interest to string theory. These solutions are generalizations of the perturbative solutions found in [6].

3.1 Perturbative Boson Star

The expansion of the fields is given by

$$F(r, \epsilon) = \sum_{i=0}^m \tilde{F}_{2i}(r) \epsilon^{2i} \quad \Pi(r, \epsilon) = \sum_{i=0}^m \tilde{\Pi}_{2i+1}(r) \epsilon^{2i+1} \quad \omega(\epsilon) = \sum_{i=0}^m \tilde{\omega}_{2i} \epsilon^{2i} \quad (3.1)$$

where $F \in \{f, g, h, \Omega\}$ is shorthand for each of the metric functions in (2.7). Expanding the metric functions in even powers, and the scalar fields in odd powers, allows the 0th order terms, the terms that remain when the scalar field vanishes, to be global AdS.

The next step is to solve the scalar equation (A.5) using this background to obtain $\tilde{\Pi}_1(r)$ and satisfy the equations of motion (A.1–A.5) up to order ϵ . The order ϵ^2 stress-energy for the scalars induces source corrections to the gravitational fields $\tilde{F}_2(r)$. These then back react on the scalar fields as described by (A.5), giving the correction $\tilde{\Pi}_3$ and satisfying equations of

motion to order ϵ^3 . The perturbative solutions can be, in principle, extended to any order in ϵ by continuing this bootstrapping procedure.

The angular frequency ω must also be expanded in ϵ to account for the non-trivial frame-dragging effects induced by the back reaction of the scalar field on the metric which affects the rotation of the scalar field. These corrections are found by imposing the boundary conditions.

Global AdS is given by

$$f_0 = 1 + \frac{r^2}{\ell^2}, \quad g_0 = 1, \quad h_0 = 1, \quad \Omega_0 = 0 \quad (3.2)$$

In this background, the most general massless scalar field solution to (A.5) that is consistent with the asymptotic boundary conditions (2.14) is

$$\Pi_1(r) = \frac{r\ell^{n+1}}{(r^2 + \ell^2)^{\frac{n+2}{2}}} {}_2F_1 \left[\frac{n+2-\omega\ell}{2}, \frac{n+2+\omega\ell}{2}; \frac{n+3}{2}; \frac{\ell^2}{r^2 + \ell^2} \right] \quad (3.3)$$

where ${}_2F_1$ is the hypergeometric function. In order to satisfy the boundary conditions (2.12), the following condition is imposed

$$\omega\ell = n+2+2k, \quad k = 0, 1, 2, \dots \quad (3.4)$$

where k describes the possible radial modes of the scalar field. In principle any radial profile can be build up as a linear combination of these radial modes; however, this introduces multiple frequency parameters, ω_k which is inconsistent with the existence of the Killing vector field (2.11). The ground state mode occurs when $k = 0$ and higher modes represent excited states. In what follows, I consider only the ground state, in which (3.3) simplifies to

$$\Pi_1(r) = \frac{r\ell^{n+1}}{(r^2 + \ell^2)^{\frac{n+2}{2}}}. \quad (3.5)$$

Next, proceeding with the bootstrapping method described above, I insert expansion (3.1) and (3.5) into the equations of motion (A.1–A.5), expand in order ϵ , and solve for order ϵ^2 using the boundary conditions (2.12) and (2.14) to fix the two constants of integration. The resulting corrections to the metric functions $\tilde{F}_2$ are inserted into the equation of motion of the scalar field (A.5), and the order ϵ^3 correction, $\tilde{\Pi}_3$, is obtained. This procedure is continued

up to, but not including, order ϵ^6 , and gives

$$f(r) = 1 + \frac{r^2}{\ell^2} - \frac{r^2 \ell^{n+1} f_{n;2}}{(r^2 + \ell^2)^{n+1} (\ell^2 - 2(n-1)(n-2)\alpha)} \epsilon^2 - \frac{r^2 \ell^{n+3} f_{n;4}}{(r^2 + \ell^2)^{2n+3} (\ell^2 - 2(n-1)(n-2)\alpha)^3} \epsilon^4 + \mathcal{O}(\epsilon^6) \quad (3.6)$$

$$g(r) = 1 - \frac{2\ell^{2n+4}((n+1)r^2 + \ell^2)}{n(r^2 + \ell^2)^{n+2} (\ell^2 - 2(n-1)(n-2)\alpha)} \epsilon^2 - \frac{\ell^{n+5} g_{n;4}}{(r^2 + \ell^2)^{2n+4} (\ell^2 - 2(n-1)(n-2)\alpha)^3} \epsilon^4 + \mathcal{O}(\epsilon^6) \quad (3.7)$$

$$h(r) = 1 + \frac{r^2 \ell^{n+5} h_{n;4}}{(r^2 + \ell^2)^{2n+3} (\ell^2 - 2(n-1)(n-2)\alpha)^3} \epsilon^4 + \mathcal{O}(\epsilon^6) \quad (3.8)$$

$$\Omega(r) = \frac{\ell^{n+2} \Omega_{n;2}}{(r^2 + \ell^2)^{n+1} (\ell^2 - 2(n-1)(n-2)\alpha)} \epsilon^2 + \frac{\ell^{n+4} \Omega_{n;4}}{(r^2 + \ell^2)^{2n+3} (\ell^2 - 2(n-1)(n-2)\alpha)^3} \epsilon^4 + \mathcal{O}(\epsilon^6) \quad (3.9)$$

$$\begin{aligned} \Pi(r) &= \frac{r \ell^{n+1}}{(r^2 + \ell^2)^{\frac{n+2}{2}}} \epsilon + \frac{r \ell^{n+5} \Pi_{n;3}}{(r^2 + \ell^2)^{\frac{3n+4}{2}} (\ell^2 - 2(n-1)(n-2)\alpha)} \epsilon^3 \\ &+ \frac{r \ell^{n+7} \Pi_{n;5}}{(r^2 + \ell^2)^{\frac{5(n+2)}{2}} (\ell^2 - 2(n-1)(n-2)\alpha)^3} \epsilon^5 + \mathcal{O}(\epsilon^7) \end{aligned} \quad (3.10)$$

where the fields $\{f_{n;s}, g_{n;s}, h_{n;s}, \Omega_{n;s}, \Pi_{n;s}\}$ are simple polynomials in r ; in this notation s labels the order in ϵ and $n = D - 2$ labels the space-time dimension¹. These fields are cataloged in Appendix B for $n = 3, 5, 7, 9$ up to order ϵ^6 with the associated energies and angular momenta in Appendix C. These obey the first law (2.20) to order ϵ^6 .

The Gauss-Bonnet parameter does not enter the perturbative solutions until the second order in ϵ , and in the $\alpha \rightarrow 0$ limit they reduce down to the Einstein case [6]. When $\alpha = \alpha_{\text{cr}}$, perturbative solutions which obey boundary conditions (2.14) do not exist in any dimension.

¹The notation is the same as that of ref [6], except that an additional “0” index (relevant for the perturbative black hole solutions) is dropped.

Chapter 4

Numerical Construction

Numerical boson star solutions were constructed using the same method as the Einstein case [34, 35], by approximating the metric functions and scalar field function $\{f(r), g(r), h(r), \Omega(r), \Pi(r)\}$ as Chebyshev polynomials and using a relaxation method on a Chebyshev grid. This grid is more dense at the end points and sparser in the middle, which allows us to get better resolution at the ends (particularly the origin where the curvature is high) where it is required, while still keeping the total number of grid points to a minimum saving on computational time.

4.1 Chebyshev Approximation

Chebyshev polynomials of the first kind are a set of orthogonal polynomials defined by

$$T_k(x) = \cos(k \cos^{-1}(x)), \quad x \in [-1, 1]. \quad (4.1)$$

They obey the discrete orthogonality relation

$$\sum_{i=0}^n \frac{1}{\bar{c}_i} T_k(x_i) T_l(x_i) = \frac{n}{2} \bar{c}_k \delta_{kl}, \quad (4.2)$$

where

$$\bar{c}_i = \begin{cases} 2, & i = 0 \\ 1, & 1 \leq i \leq (n-1) \\ 2, & i = n. \end{cases} \quad (4.3)$$

It is relatively straight forward to show that if $u(x)$ is a continuous function on $x \in [-1, 1]$, then it can be approximated by

$$u(x) \approx \sum_{i=0}^N \tilde{u}_i T_i(x) \quad (4.4)$$

and this sum is an equality at the $N+1$ extrema of the N^{th} Chebyshev polynomial $x_k = \cos\left(\frac{k\pi}{N}\right)$, $i = 0, 1, \dots, N$. The approximation of the derivative $u'(x)$ is also easy to define as

$$u'(x) \approx \sum_{i=0}^N \tilde{u}'_i T_i(x) \quad (4.5)$$

where the spectral coefficients of the derivative can be given by the recursion relation [39]

$$\begin{aligned} \tilde{u}'_i &= \tilde{u}'_{i+2} + 2(i+1)\tilde{u}'_{i+1} \quad i = 1, \dots, N-1, \\ \tilde{u}'_0 &= \frac{1}{2}\tilde{u}'_2 + \tilde{u}'_1, \end{aligned} \quad (4.6)$$

with $\tilde{u}_{N+1} = \tilde{u}_N = 0$. The coefficients of the second derivative $u''(x)$ can be found by applying (4.6) twice.

4.2 Implementation

In order to construct the Chebyshev grid, the infinite domain $r \in [0, \infty)$ needs to be transformed into the finite domain $y \in [0, 1]$ through the coordinate transformation

$$y = \frac{r^2}{r^2 + \ell^2}. \quad (4.7)$$

The grid is then constructed as the set of points $y_k = (\cos(k\pi/N) + 1)/2$, $k = 0, \dots, N$, that are the $N+1$ extrema of the N^{th} order Chebyshev polynomial, $T_N(2y-1)$.

Next, use the boundary conditions (2.12) and (2.14) to define auxiliary functions $\{q_f, q_g, q_h, q_\Omega, q_\Pi\}$, which are analytic over the entire domain $y \in [0, 1]$.

$$f(y) = \frac{y}{1-y} + 1 + (1-y)^{\frac{n-1}{2}} q_f(y), \quad (4.8)$$

$$g(y) = 1 + (1-y)^{\frac{n+1}{2}} q_g(y), \quad (4.9)$$

$$h(y) = 1 + (1-y)^{\frac{n+1}{2}} q_h(y), \quad (4.10)$$

$$\Omega(y) = \frac{q_\Omega(y)}{\ell}, \quad (4.11)$$

$$\Pi(y) = \sqrt{y}(1-y)^{\frac{n+1}{2}} q_\Pi(y). \quad (4.12)$$

Boundary conditions on the auxillary q-functions are found by first transforming the field equations (A.1–A.6) into functions of y , substituting in the definitions of the auxillary q-functions (4.8–4.12), Taylor expanding them around the two endpoints $y_0 = 0, 1$ and requiring that they vanish order by order in $(y - y_0)$. The lowest order terms give non-trivial relationships between the q-functions and their first derivatives, which are the boundary conditions. At $y_0 = 0$, we have

$$q_f(0) = 0, \quad q_h(0) = 0, \quad q_\Pi(0) - q_0 = 0,$$

$$\begin{aligned} q'_\Omega(0) + \frac{2\ell^2 q_0^2 q_\Omega(0)}{(n+3) \left(2\alpha(n-1) \left((n-2) \left(q'_f(0) + 1 \right) + 3q'_h(0) \right) - \ell^2 \right)} &= 0, \\ q'_g(0) + \frac{1}{2\alpha(n-2)(n-1) \left(n \left(q'_f(0) + 1 \right) + 3q'_h(0) \right) - \ell^2 n} \\ &\times \left[q'_f(0) \left(\alpha(n-2)(n-1) \left(3(n-1) (q_g(0) + 1) q'_h(0) + n(n+1) \right) - \frac{1}{2} \ell^2 n(n+1) (q_g(0) + 1) \right) \right. \\ &+ \frac{1}{2} \alpha(n-2)(n-1) n(n+1) (q_g(0) + 1) \left(q'_f(0) \right)^2 - \ell^2 (n-1) q_0^2 (q_g(0) + 1) \\ &+ (n-1) q'_h(0) \left(\alpha(n-2) (3(n-1) - 6q_g(0)) - \frac{3}{2} \ell^2 (q_g(0) + 1) \right) \\ &\left. + \frac{15}{2} \alpha(n-3)(n-1) (q_g(0) + 1) \left(q'_h(0) \right)^2 + n(n+1) q_g(0) \left(\frac{\ell^2}{2} - \alpha(n-2)(n-1) \right) \right] = 0, \end{aligned} \quad (4.13)$$

where $q'_f(0)$ is given by

$$\begin{aligned}
 q'_f(0) = & -\frac{1}{2\alpha(n-2)(n-1)n} \left[\left(-24\alpha^2(n-2)(n-1)^2(n+3)q'_h(0)^2 + \ell^4 n^2 \right. \right. \\
 & \left. \left. - 4\alpha\ell^2(n-2)(n-1)n^2 + 8\alpha\ell^2(n-2)(n-1)nq_0^2 + 4\alpha^2(n-2)^2(n-1)^2n^2 \right)^{1/2} \right. \\
 & \left. + 6\alpha(n-2)(n-1)q'_h(0) + \ell^2(-n) + 2\alpha(n-2)(n-1)n \right]. \quad (4.14)
 \end{aligned}$$

The $y_0 = 1$ boundary conditions are

$$\begin{aligned}
 q'_f(1) + \frac{n-1}{2}q_f(1) + q_h(1) &= 0, & q_g(1) + q_h(1) &= 0, & q'_\Omega(1) &= 0, \\
 q'_h(1) + \frac{n+1}{2}q_h(1) &= 0, & q'_\Pi(1) + \frac{1}{2(n+3)}((n+2)^2 - q_\Omega(1)^2)q_\Pi(1) &= 0. \quad (4.15)
 \end{aligned}$$

I generate solutions to the equations of motion by first using (4.4) to approximate each of the q-functions as an order N Chebyshev approximation. The expansions were then substituted into the equations of motion (A.1–A.6) at the $N - 1$ interior grid points and into the boundary conditions (4.13) and (4.15) at the two boundary points. This has reduced the numeric integration to a set of $5(N + 1)$ algebraic equations in the spectral coefficients of the Chebyshev expansions. The algebraic equations are then linearized with respect to the spectral coefficients and a standard Newton-Raphson method is used to solve the resulting system of linear equations. Convergence is defined as occurring when a change in the spectral coefficients between two iterations is less than 10^{-30} .

A one parameter family of solutions, parameterized by q_0 , is built up using a step size of $\Delta q_0 = 0.01$ (although it is sometimes necessary to reduce the step size); the previous numeric solution is used as seed for the next one along this chain of discrete solutions. The initial seed that is used to start the procedure (when looking for boson star solutions) is given by the perturbative solution (3.6)–(3.10), where the numeric parameter q_0 is related to the perturbative parameter ϵ through the equality

$$q_0 = \lim_{r \rightarrow 0} \frac{(r^2 + \ell^2)^{\frac{n+2}{2}}}{r\ell^{n+1}} \Pi(r),$$

and $\Pi(r)$ is taken to be the perturbative solution of equation (3.10).

4.3 Error Checking

In general, convergence alone does not guarantee that a sufficiently dense Chebyshev grid ensures a physical solution, and as a result I demand that the numeric solutions satisfy the first law of thermodynamics to one part in 10^6 . When this limit is no longer satisfied, I increase N to refine the grid and, in the case of $\alpha > \alpha_{\text{cr}}$ boson stars, decrease the step size between consecutive solutions. This is not the only cross check I use to ensure the validity of the solutions. Due to the exponential convergence of the Chebyshev approximation [39], the truncation error in the approximations of the q-functions can be estimated via

$$|\text{error}| \leq \sum_{j=N+1}^{\infty} C_j \sim \int_N^{\infty} C_0 e^{-kj} dj = \frac{C_0 e^{-kN}}{k} \quad (4.16)$$

for some real number k characteristic of the expansion. Here we have used the property that $-1 \leq T_j(x) \leq 1$ and that $C_j \approx C_0 e^{-kj}$. Using the % error $\approx \frac{C_N}{C_0}$ to quantify the truncation error, I find that when the constraint on the first law is satisfied, this error is less than 10^{-7} .

The numerical integration method I use requires that the equations of motion are solved here on the grid points, but not everywhere. In practice, I found that the largest error occurs midway between the gridpoints, *i.e.* the zeros of the N^{th} Chebyshev polynomial. As long as the grid is dense enough to satisfy the constraint on the first law, this error is on the order of 10^{-5} or less.

Chapter 5

Numeric Boson Star Solutions

Since the numeric integration finds the auxiliary q -functions, it is convenient to write the expressions for the physical quantities of the boson star in terms of their boundary values and derivatives. For example, the expressions for the thermodynamic quantities are [38]

$$M = \frac{\pi^{\frac{n-1}{2}} \ell^{n-1}}{8 \left(\frac{n-1}{2}\right)!} \left(1 - \frac{\alpha}{\alpha_{\text{cr}}}\right) ((n+1)q_h(1) - nq_f(1)), \quad (5.1)$$

$$J = \frac{(-1)^{\frac{n+1}{2}} \pi^{\frac{n-1}{2}} \ell^n}{4 \left(\left(\frac{n-1}{2}\right)!\right)^2} \left(1 - \frac{\alpha}{\alpha_{\text{cr}}}\right) q_{\Omega}^{(\frac{n+1}{2})}(1), \quad (5.2)$$

$$\omega = -q_{\Omega}(1) \quad (5.3)$$

which can be easily calculated using (2.19) and (4.8 – 4.12). The Kretschmann scalar $K = R_{abcd}R^{abcd}$ at the center of the boson star is given by

$$\begin{aligned} K_n = \frac{2(n+1)}{\ell^4} & \left(\frac{2q'_f(0) (2q'_g(0) + 3(q_h(0) + 1)q'_h(0) + q_h(0) + n + 2)}{q_h(0) + 1} + (n+2)q'_f(0)^2 \right. \\ & - \frac{2q'_g(0) ((n-1)q_h(0) - 2)}{(q_h(0) + 1)^2} + \frac{2q'_g(0)^2}{(q_h(0) + 1)^2} + 3(n+2)q'_h(0)^2 \\ & \left. + \frac{\left(\frac{1}{2}(n-1)(n+1) + 1\right) q_h(0)^2}{(q_h(0) + 1)^2} + \frac{n+2}{(q_h(0) + 1)^2} + 6q'_h(0) + \frac{2q_h(0)}{(q_h(0) + 1)^2} \right), \end{aligned} \quad (5.4)$$

which further simplifies using the boundary conditions (4.13), yielding an expression in terms of just $q'_h(0)$. The resulting expressions are rather cumbersome, and so they can be seen in

appendix D.

These quantities are used to study the behaviour of boson stars as a function of the Gauss-Bonnet parameter α for $D = 5, 7, 9, 11$ ($n = 3, 5, 7, 9$). In order to easily make comparisons between the different dimensions, I look at values of α that are fractions of the critical value: $\tilde{\alpha} \equiv \frac{\alpha}{\alpha_{\text{cr}}} = (0, \frac{1}{10}, \frac{3}{10}, \frac{1}{2}, \frac{7}{10}, \frac{9}{10}, \frac{12}{10}, \frac{15}{10}, \frac{18}{10}, 100)$. The $\alpha = 0$ matches the results of the Einstein case [35] and provides a useful cross-check on the analysis, since the form of the equations of motion I used (A.1–A.6) is significantly different than those used in [35].

The behaviour of boson stars differs drastically depending on whether $\alpha < \alpha_{\text{cr}}$ or $\alpha > \alpha_{\text{cr}}$ with the former having a surprising distinction depending on $D = 5$ or $D > 5$.

5.1 The $\alpha < \alpha_{\text{cr}}$ solutions

5.1.1 The $D > 5$ cases

When M and J are plotted against ϵ , as seen in figures 5.1 and 5.2, the familiar spiral-like behaviour that was seen in the Einstein case [35] is apparent. The spirals become tighter with increasing dimension, becoming barely visible when $D = 11$ ($n = 9$). In each dimension, greater than 5, the spirals become larger and begin forming at earlier with increasing $\tilde{\alpha}$. A similar spiral behaviour is seen in the for ω plotted against ϵ , as seen in figure 5.4, and, just as they did with M and J , the spirals for ω become larger with increasing $\tilde{\alpha}$ and tighten with increasing dimension.

The M and J versus q_0 curves reveal that the damped oscillations that were present in the Einstein case [35] are still seen when $\tilde{\alpha}$ is different from 0 as seen in figures 5.5 and 5.6. The maximum of these oscillations decrease non-linearly with increasing space-time dimension, and except in the $D = 7$ case, decrease with increasing values of the Gauss-Bonnet parameter. The location of the maximum on the mass curve may correspond to the maximum central energy density that the boson star can have before it becomes unstable, analogous to the behaviour seen in rotating neutrons stars [40].

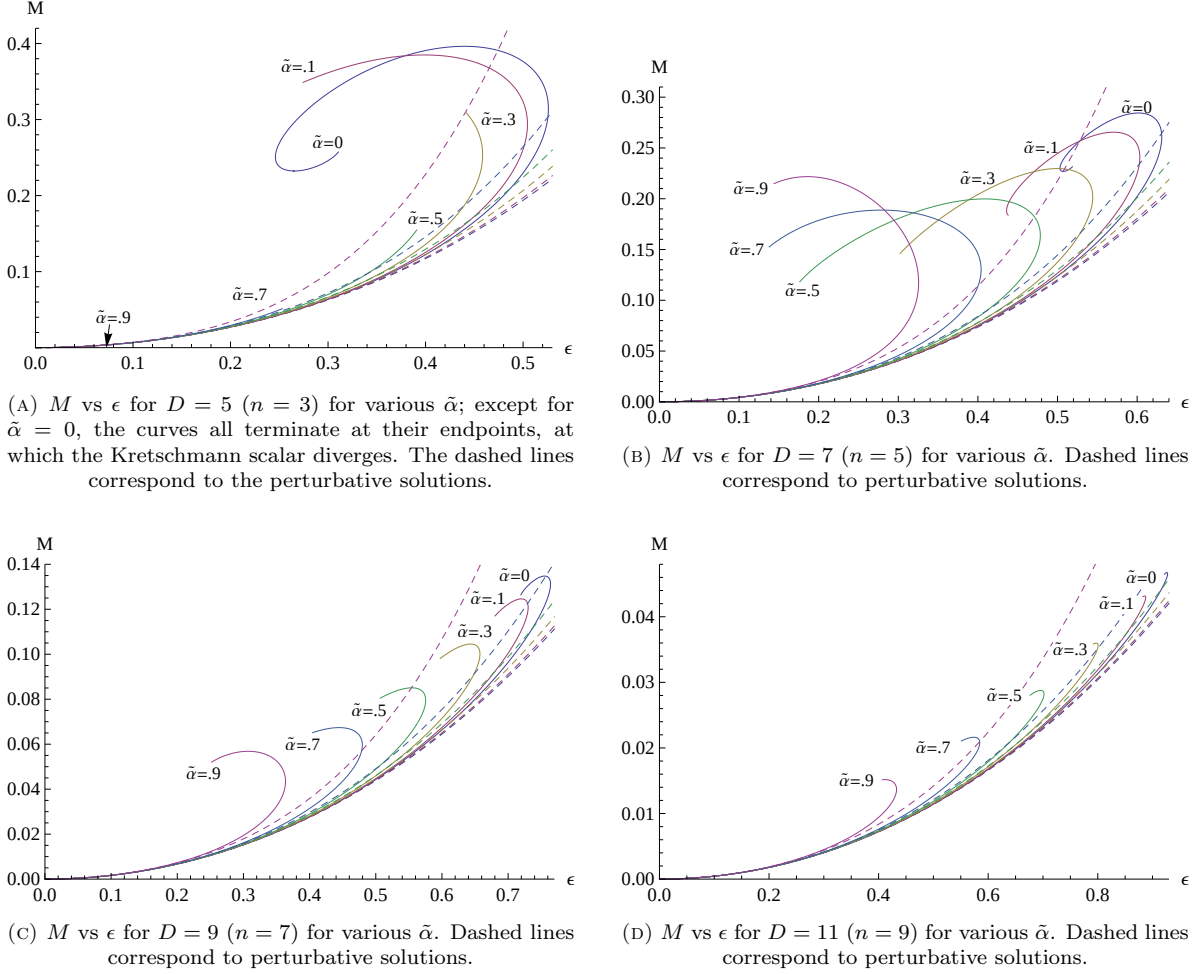


FIGURE 5.1: The boson star mass plotted against the perturbative parameter ϵ in (A) 5, (B) 7, (C) 9, and (D) 11 dimensions for various values of $\tilde{\alpha}$. In the non-perturbative regime, ϵ no longer uniquely parameterizes the boson star family. The spirals tighten with increasing dimension and decreasing α . The truncated behaviour for $D = 5$ is associated with diverging curvature at finite q_0 .

5.1.2 The $D = 5$ case

When $D = 5$, the spiral-like behaviour is no longer present when M (5.1) and J (5.2) are plotted against ϵ for values of $\tilde{\alpha}$ as low as 0.1 (further detail is provide in 5.3). Instead the curves terminate at some finite value of (M, J) , after which the code breaks down at some finite, relatively small, value of q_0 . The consequence of this critical q_0 value can be more easily seen in figures 5.5 and 5.6 with a close up of the behaviour of $\tilde{\alpha} = \frac{7}{10}$ and $\tilde{\alpha} = \frac{9}{10}$ curves provided in figure 5.7. For increasing values of $\tilde{\alpha}$ the curves terminate at a decreasing value of q_0 , some of which are small enough that no oscillatory behaviour is manifest.

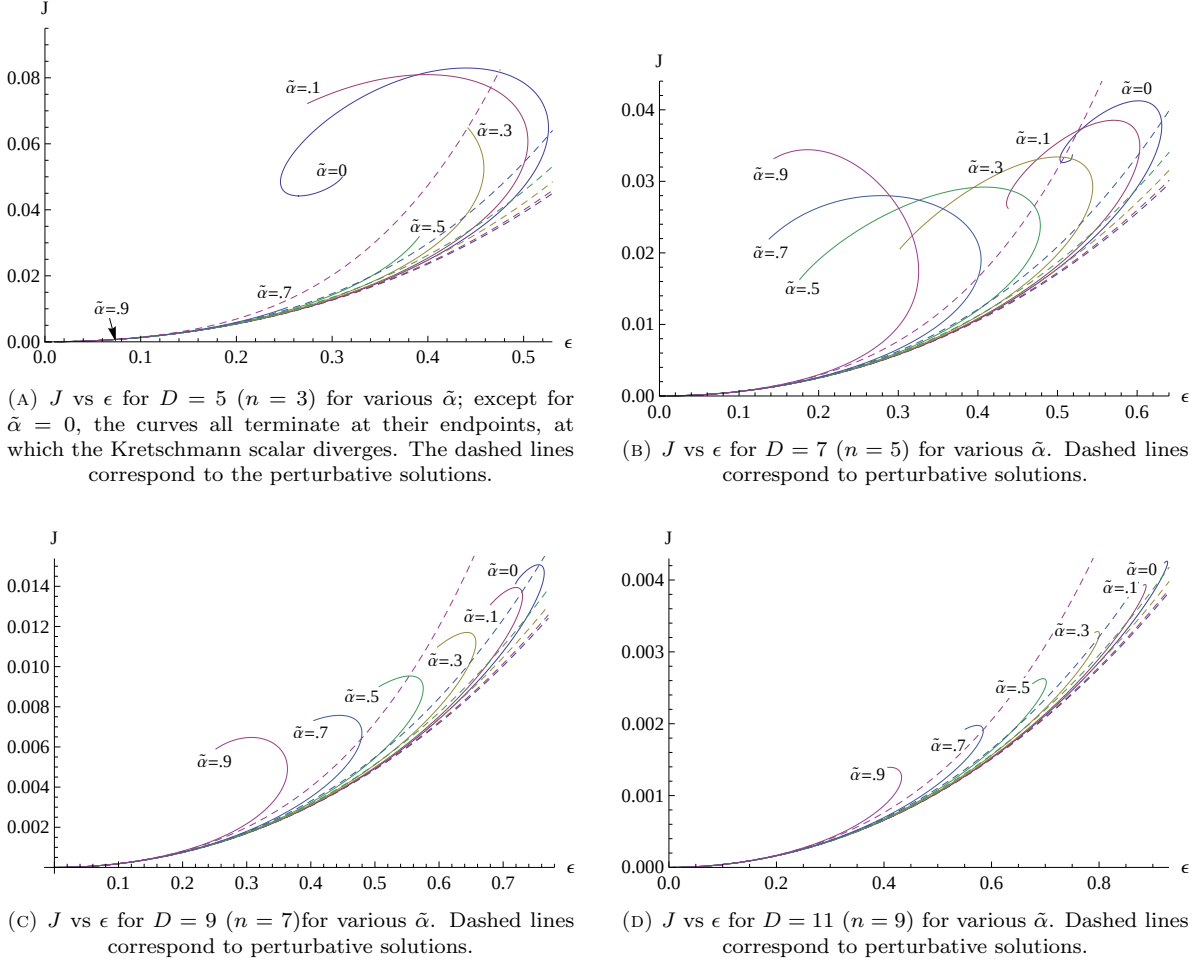


FIGURE 5.2: The boson star angular momentum plotted against the perturbative parameter ϵ in (A) 5, (B) 7, (C) 9, and (D) 11 dimensions for various values of $\tilde{\alpha}$. In the non-perturbative regime, ϵ no longer uniquely parameterizes the boson star family. As with the mass, the spirals tighten with increasing dimension and decreasing α . The truncated behaviour for $D = 5$ is associated with diverging curvature at finite q_0 .

The underlying reason behind this behaviour can be traced to the Kretschmann scalar, as seen in figure 5.9. For all values of n , the numerics show that the Kretschmann scalar decreases to a minimum, after which it strictly increases as a function of q_0 (this can be seen more easily in figure 5.10. While it remains finite for finite values of q_0 , when $D = 7, 9, 11$ ($n = 5, 7, 9$) it has a vertical asymptote at a finite q_0 in $D = 5$ ($n = 3$). Further investigation of the expression for the Kretschmann scalar (see Appendix D) shows that it diverges at a critical value of $q_{h'}^*$ of $q_h'(0)$. In $D = 5$ this value is

$$q_{h'}^* = - \frac{\sqrt{144\alpha^2 + 9\ell^4 - 72\alpha\ell^2 + 48\alpha\ell^2 q_0^2}}{24\alpha}. \quad (5.5)$$

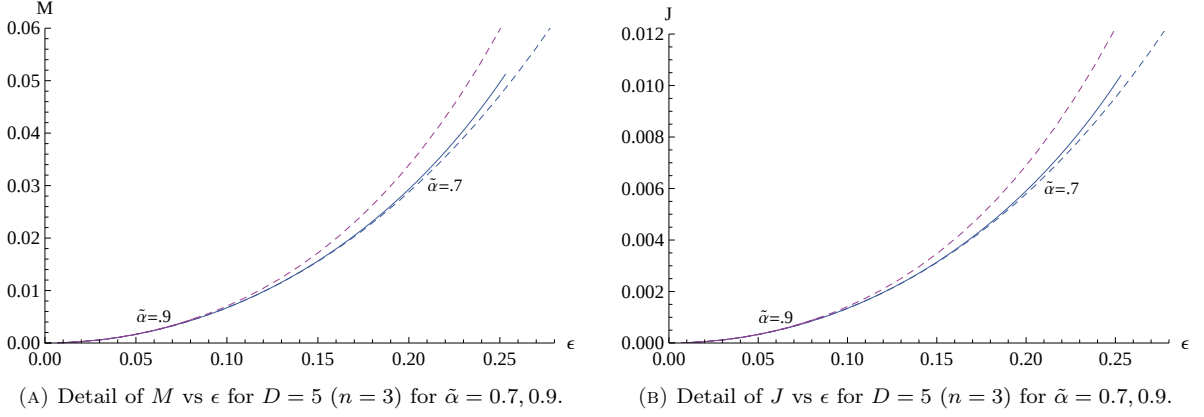


FIGURE 5.3: Detail plots of mass (A) and angular momentum (B) against the perturbative parameter ϵ in 5 dimensions for $\tilde{\alpha} = \frac{7}{10}, \frac{9}{10}$. The two curves almost lie on top of each other. Perturbative results are plotted as dashed lines.

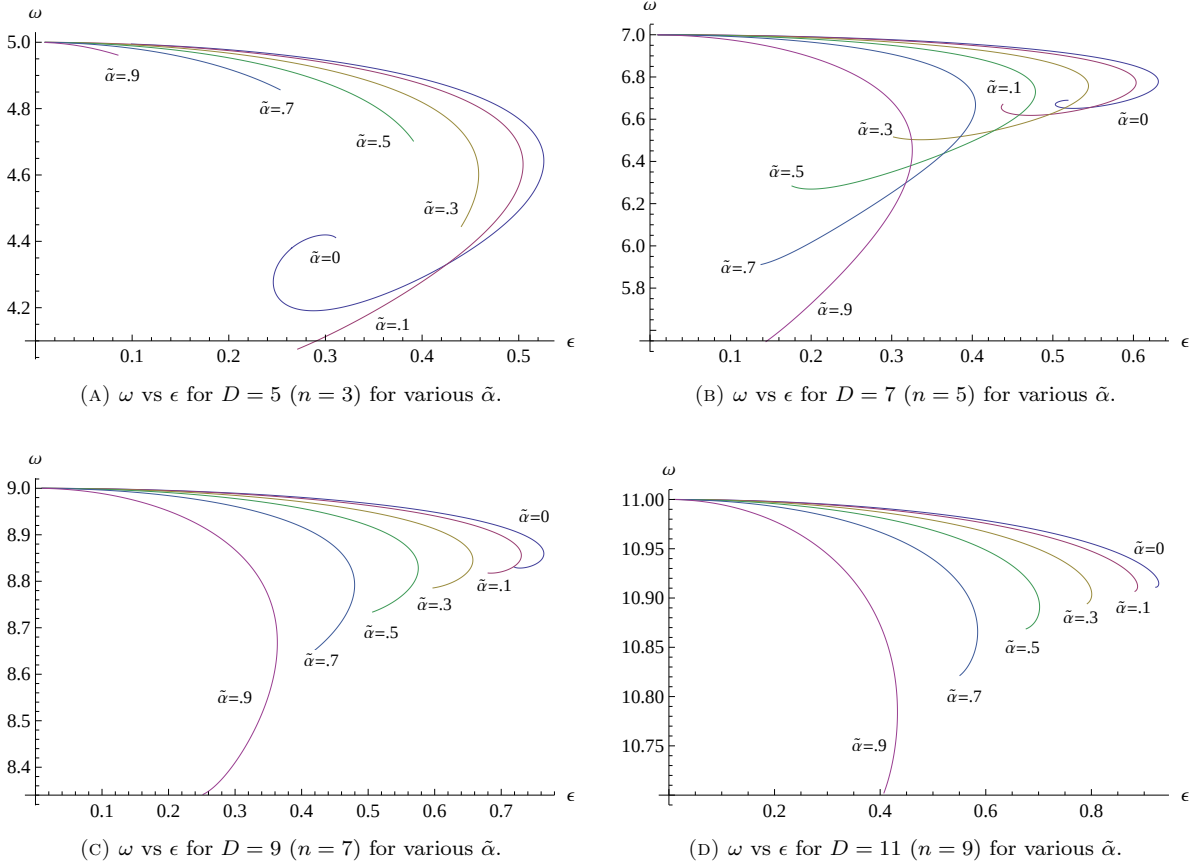
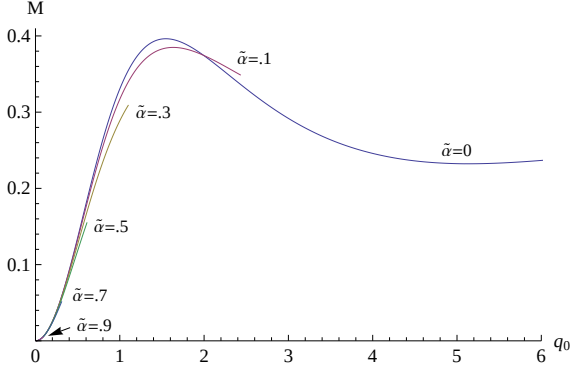
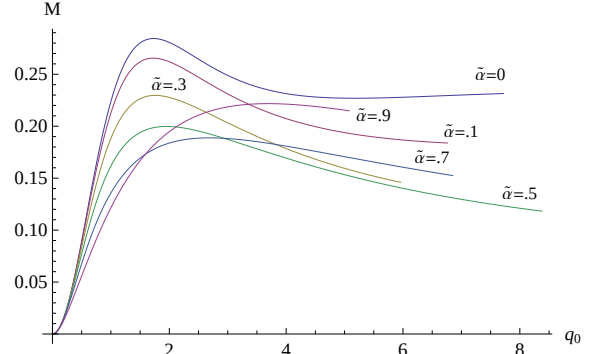


FIGURE 5.4: The angular frequency ω plotted against the perturbative parameter ϵ in (A) 5, (B) 7, (C) 9, and (D) 11 dimensions for various values of $\tilde{\alpha}$.

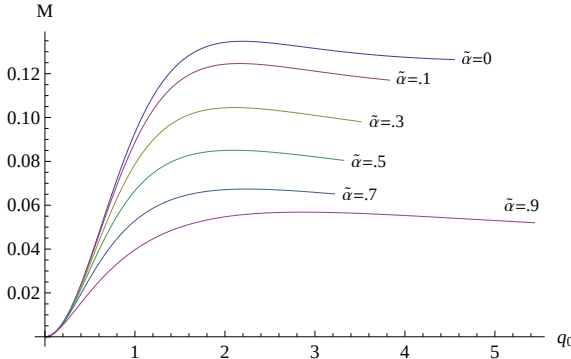
Although there is a critical value $q_h'^*$ in any dimension, which I have listed in Appendix D, I find that numerically that as q_0 increases $q_h'(0)$ departs further from this critical value when



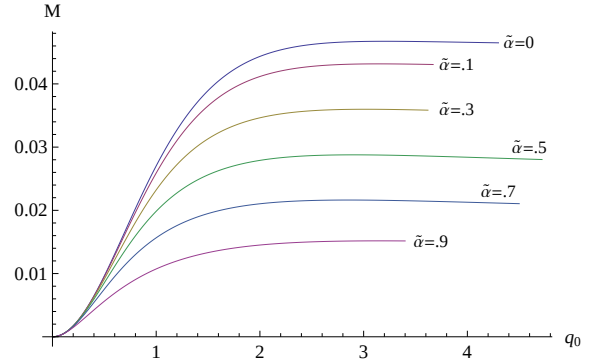
(A) M vs q_0 for $D = 5$ ($n = 3$), with $\tilde{\alpha}$ increasing from bottom to top; except for $\tilde{\alpha} = 0$, the curves all terminate at their endpoints, at which the Kretschmann scalar diverges.



(B) M vs q_0 for $D = 7$ ($n = 5$), with $\tilde{\alpha}$ increasing from bottom to top.



(C) M vs q_0 for $D = 9$ ($n = 7$), with $\tilde{\alpha}$ increasing from bottom to top.

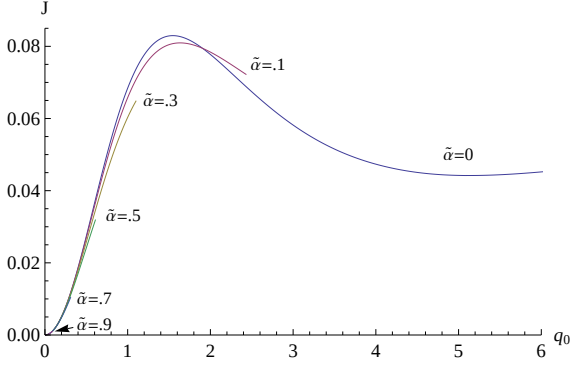


(D) M vs q_0 for $D = 11$ ($n = 9$), with $\tilde{\alpha}$ increasing from bottom to top.

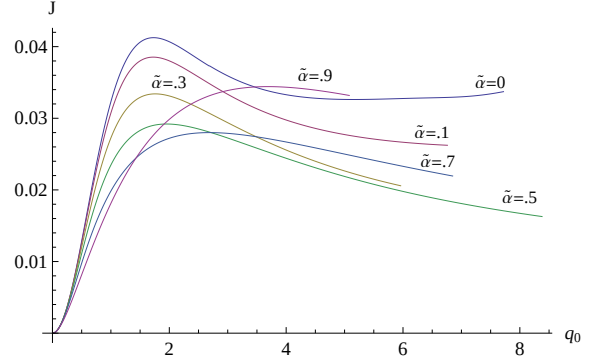
FIGURE 5.5: The boson star mass plotted against the parameter q_0 in (A) 5, (B) 7, (C) 9, and (D) 11 dimensions for various values of $\tilde{\alpha}$. The maximum mass decreases nonlinearly with the space-time dimension.

$D = 7, 9, 11$, implying that the Kretschmann scalar remains finite for all finite q_0 (see figure 5.11). However, when $D = 5$, the numerics show that $q'_h(0) \rightarrow q_h^*$ at finite q_0 , indicating a divergence in the Kretschmann scalar. To estimate the critical value of q_0 , I plot $q'_h(0) - q_h^*$ and employ a 9th order polynomial fit to interpolate the curve. These plots are shown for $\tilde{\alpha} = 0.1, 0.9$ in figure 5.11.

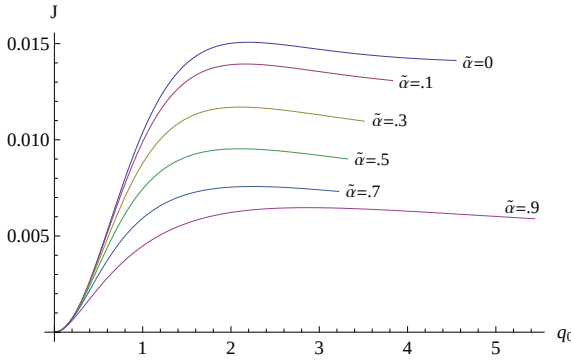
This behaviour was not present in the Einstein case nor was it seen in the perturbative solutions. It suggests that the behaviour of a $D = 5$ boson star must change drastically at or before some critical central energy density. This is reminiscent of the numerical $D = 3$ boson star solutions obtained in [34] as the mass, angular momentum and angular velocity all approach a finite terminal values at a critical value of the central energy density, whereas



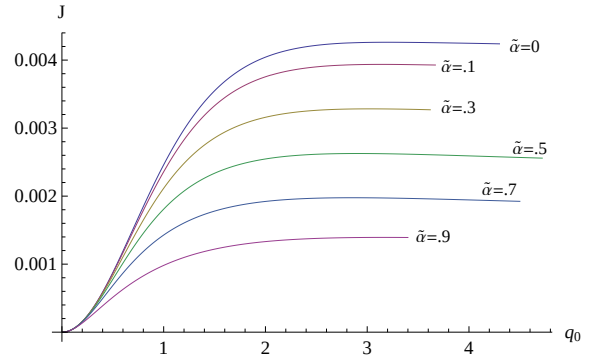
(A) J vs q_0 for $D = 5$ ($n = 3$), with $\tilde{\alpha}$ increasing from bottom to top; except for $\tilde{\alpha} = 0$, the curves all terminate at their endpoints, at which the Kretschmann scalar diverges.



(B) J vs q_0 for $D = 7$ ($n = 5$), with $\tilde{\alpha}$ increasing from bottom to top.

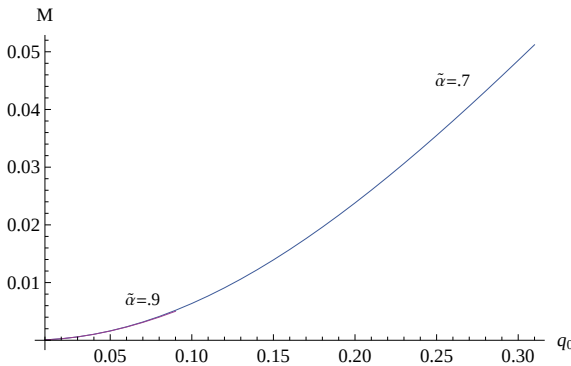


(C) J vs q_0 for $D = 9$ ($n = 7$), with $\tilde{\alpha}$ increasing from bottom to top.

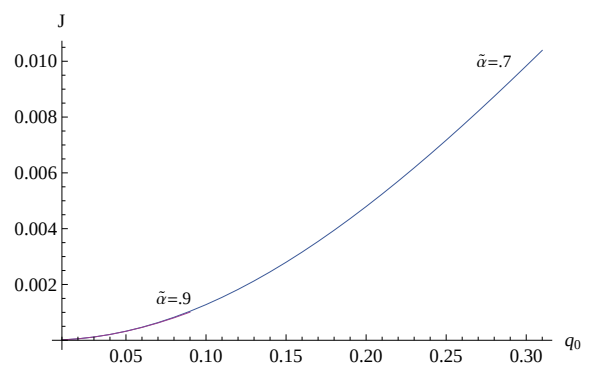


(D) J vs q_0 for $D = 11$ ($n = 9$), with $\tilde{\alpha}$ increasing from bottom to top.

FIGURE 5.6: The boson star angular momentum plotted against the parameter q_0 in (A) 5, (B) 7, (C) 9, and (D) 11 dimensions for various values of $\tilde{\alpha}$. As with the mass, the maximum angular momentum decreases nonlinearly with the space-time dimension.



(A) Detail of M vs q_0 for $D = 5$ ($n = 3$) for $\tilde{\alpha} = 0.7, 0.9$.



(B) Detail of J vs q_0 for $D = 5$ ($n = 3$) for $\tilde{\alpha} = 0.7, 0.9$.

FIGURE 5.7: Detail plots of mass (A) and angular momentum (B) against the parameter q_0 in 5 dimensions for $\tilde{\alpha} = \frac{7}{10}, \frac{9}{10}$. The two curves almost lie on top of each other.

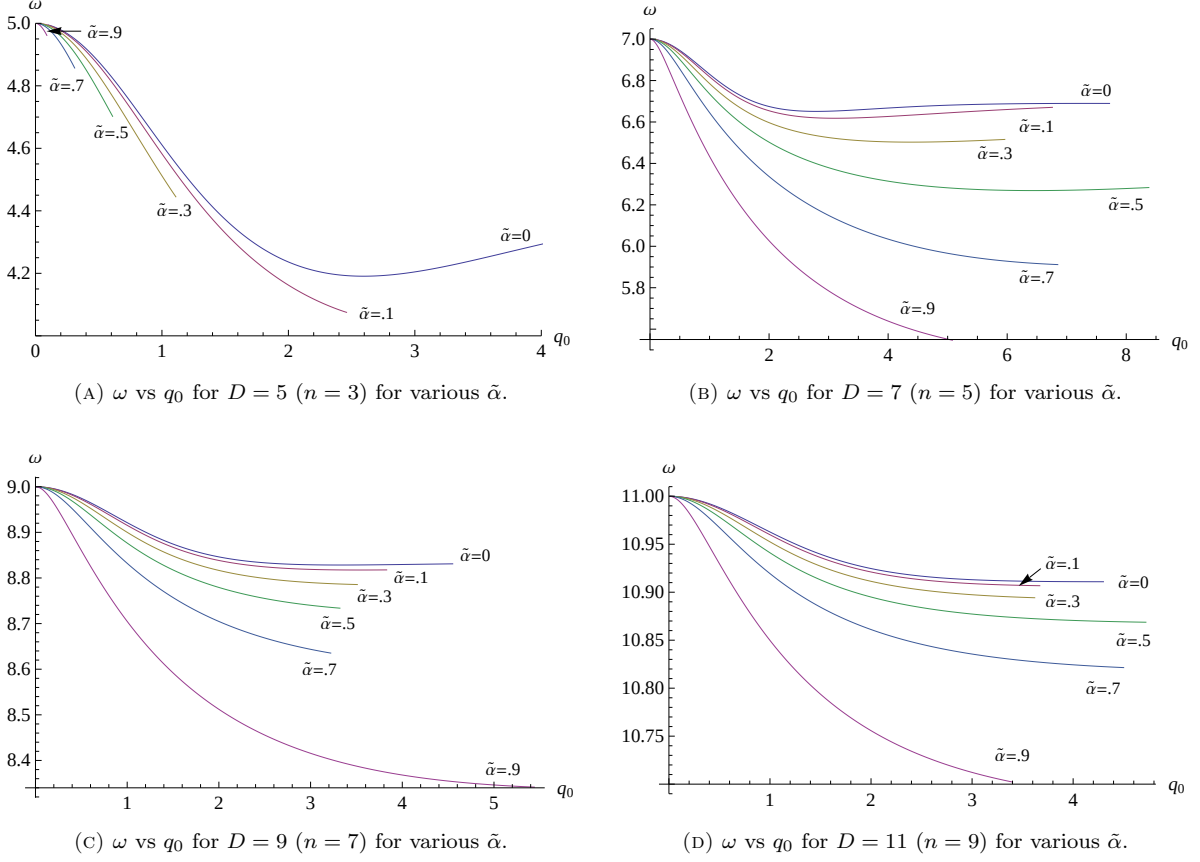


FIGURE 5.8: The angular velocity ω plotted against the parameter q_0 in (A) 5, (B) 7, (C) 9, and (D) 11 dimensions for various values of $\tilde{\alpha}$. The termination of the curves in $D = 5$ is due to the divergence of the Kretschmann scalar at the end points, except for $\tilde{\alpha} = 0$ which is immune to this divergence.

TABLE 5.1: Estimate of the critical value of q_0 at which the Kretschmann scalar diverges in $D = 5$ for various values of $\tilde{\alpha}$ using the best fit polynomials in figure 5.11.

$\tilde{\alpha}$	$q_{0\text{critical}}$
0.1	2.54
0.3	1.15
0.5	0.648
0.7	0.329
0.9	0.0950

in higher dimensions the central energy density is unbounded and the corresponding physical quantities all exhibit damped harmonic oscillations about finite limiting values. However, the numerical evidence for the $D = 3$ case indicates that an extremal BTZ black hole forms at the critical value, and the scalar field vanishes in the exterior. In the $D = 5$ EGB boson star, the scalar field does not vanish, and there is no indication of the formation of a horizon. Even so,

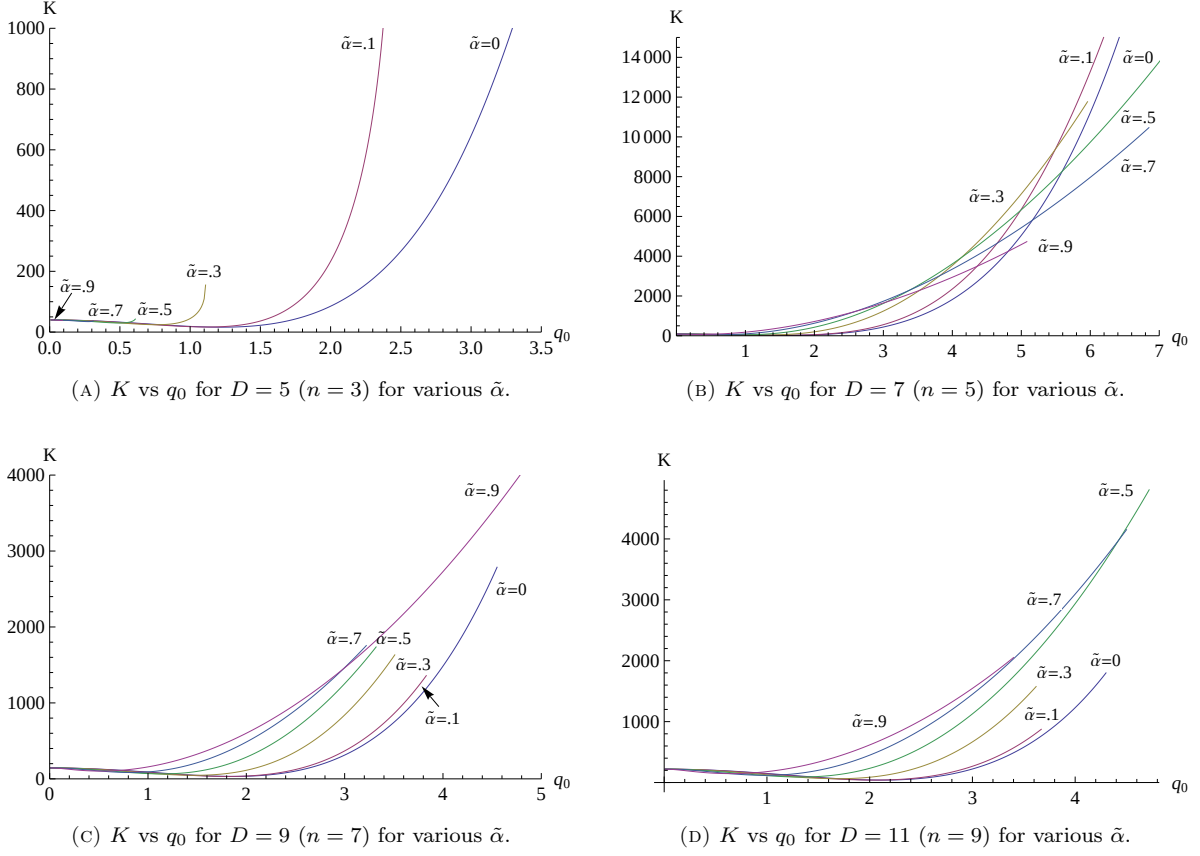


FIGURE 5.9: The Kretschmann scalar $K = R^{abcd}R_{abcd}$ plotted against the parameter q_0 in (A) 5, (B) 7, (C) 9, and (D) 11 dimensions for various values of $\tilde{\alpha}$. The divergent behaviour of K in $D = 5$ is drastically different than in higher dimensions for reasons explained in the text.

it is expected that these solutions are dynamically unstable to the formation of a hairy black hole for some $q_0 \geq q_{0\text{critical}}$.

The parallel between the $D = 3$ Einstein boson star and the $D = 5$ EBG boson star leads to an interesting hypothesis. Given that in $D = 3$ the Ricci scalar is the highest-curvature term that can appear in that dimension, and the Gauss-Bonnet term is the highest that can appear in $D = 5$, it is possible that analogous critical central energies appear in all odd dimensions when the gravitational theory includes the highest possible curvature terms. It would be difficult, but not impossible, to numerically check 3rd order Lovelock in $D = 7$ and even 4th order in $D = 9$.

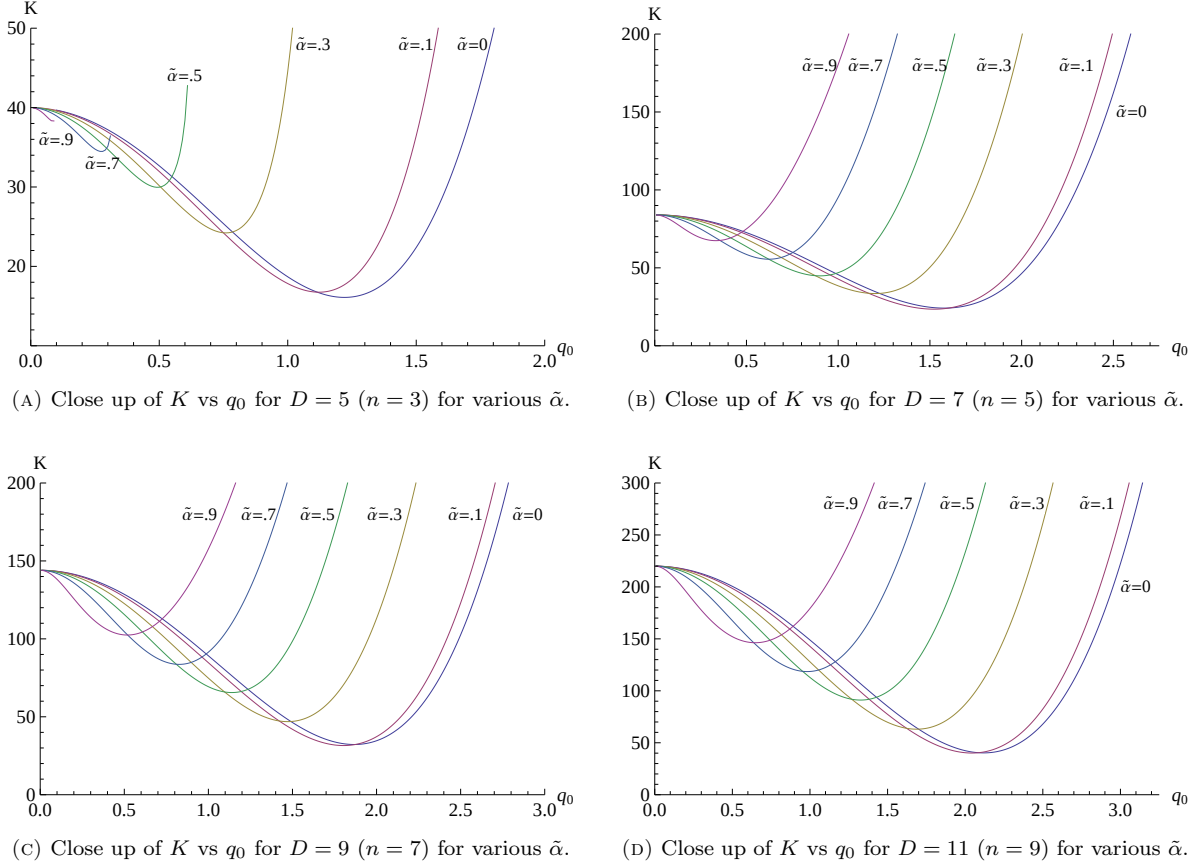


FIGURE 5.10: Detail of the Kretschmann scalar K plotted against the parameter q_0 in (A) 5, (B) 7, (C) 9, and (D) 11 dimensions for various values of $\tilde{\alpha}$. Except for the scale on the K axis, dimensions $D = 7, 9, 11$ behave qualitatively the same as $\tilde{\alpha}$ increases, whereas $D = 5$ exhibits distinct behaviour.

5.2 Beyond α_{cr}

Due to the choice of parameterization of the cosmological constant (2.6), it is possible to find boson star solutions for values of α greater than α_{cr} , including values for which cosmological constant is positive (*i.e.* $\alpha > 2\alpha_{\text{cr}}$). One surprising result is that unlike for $\alpha < \alpha_{\text{cr}}$, spirals do not appear when M (5.12, 5.14a) and J (5.13, 5.14a) are plotted against ϵ in any dimension, but rather M and J both increase monotonically with ϵ (the $\tilde{\alpha} = 100$ are plotted separately due to scale). The numeric curves do match the perturbative results for sufficiently small values of ϵ . The spiral behaviour is also absent when ω is plotted against ϵ (5.15); again it increases monotonically.

Similarly, when M (5.16, 5.18a) and J (5.17, 5.18b) are plotted as functions of q_0 they also

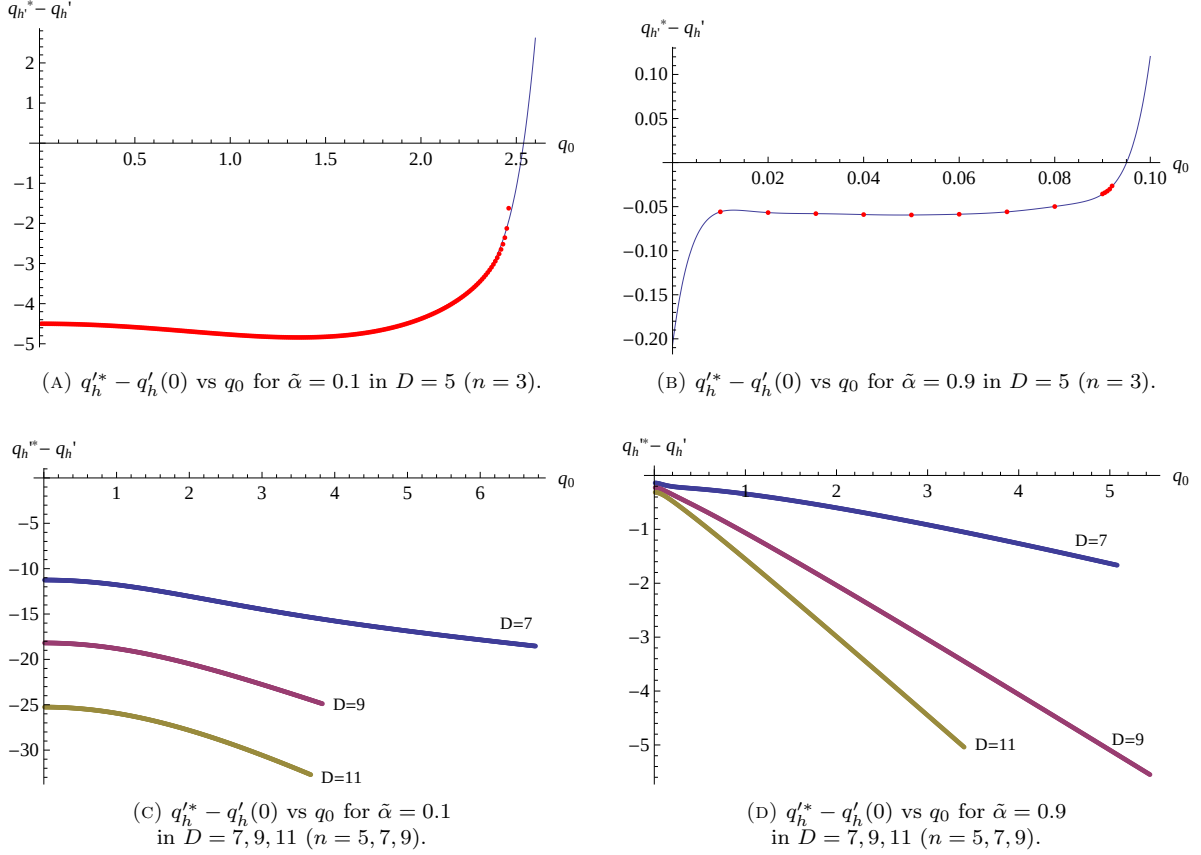


FIGURE 5.11: Plots of $q_h^{**} - q_h^{*'}(0)$ against the parameter q_0 in 5, 7, 9, and 11 dimensions for various values of $\tilde{\alpha}$. The solid curves in sub-figures (A), (B) are the best-fit 9th-order polynomials to the points used to obtain $q_{0\text{critical}}$.

increase monotonically; the damped oscillations that appear for $\alpha < \alpha_{\text{cr}}$ are no longer present (again, the $\tilde{\alpha} = 100$ curves are plotted separately due to scale). Not only do they increase monotonically, but plotting $\ln(M)$ (5.19) and $\ln(J)$ (5.20) as functions of q_0 reveal that after a certain value of q_0 they increase exponentially with the value of this point of inflection increasing both with increasing dimension and increasing α .

There is no significant difference between the boson star behaviour in $D = 5$ and the $D > 5$; the Kretschmann scalar remains finite for all finite q_0 (see figure (5.21) and $q_h^{**} - q_h^{*'}(0)$ is nowhere vanishing and monotonically decreases with increasing q_0 (5.22).

In contrast to the $\alpha < \alpha_{\text{cr}}$ case, the numerics break down because of the exponential decrease in M and J between consecutive values of q_0 rather than the curvature becoming large. This means that rather than increasing the resolution of the Chebyshev grid, the step size, Δq_0 ,

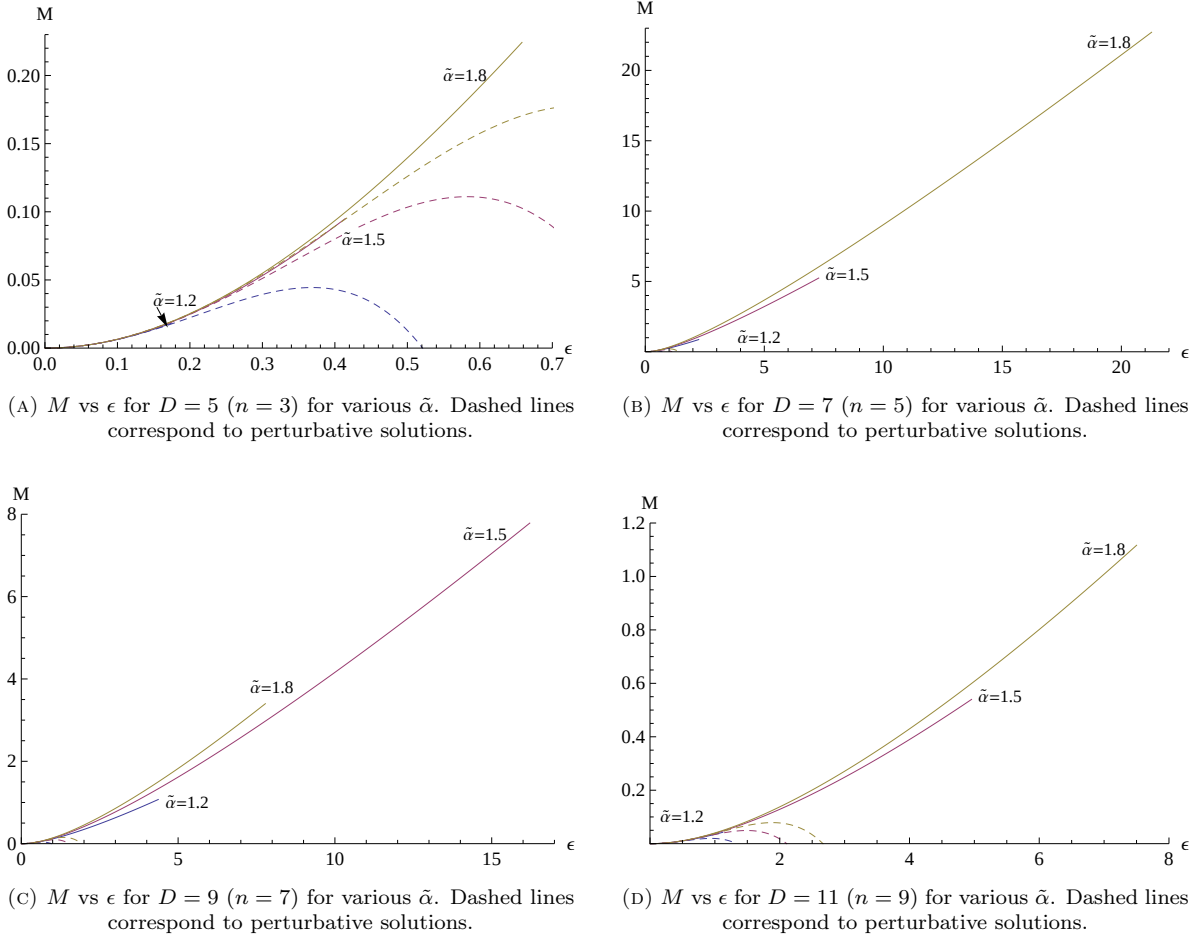


FIGURE 5.12: The boson star mass plotted against the perturbative parameter ϵ in (A) 5, (B) 7, (C) 9, and (D) 11 dimensions for various values of $\tilde{\alpha}$. The boson star mass increases monotonically with ϵ .

between solutions needed to be decreased by a factor of 2 so that the previous solution remains a good seed for the subsequent one.

Finally, I plot M versus J for each dimension and the values of $\tilde{\alpha}$ indicated (5.23, 5.24). For each dimension and for small values of (M, J) , which correspond to small values of ϵ , all of the solutions lie on nearly the line, regardless of the value of $\tilde{\alpha}$. This is due to the first law (2.20) which states that the slope of the (M, J) curve is given by ω . When ϵ is small, the value of ω is fairly constant for each dimension, regardless of the value of α , as seen in figure (5.4) and 5.15) as well as in the perturbative expressions in Appendix C. When $\alpha < \alpha_{\text{cr}}$ these curves “turn back”, making a zig-zag pattern familiar from the Einstein case [35], albeit much tighter. As α increases, while remaining less than α_{cr} , the turnaround point for the zig-zag

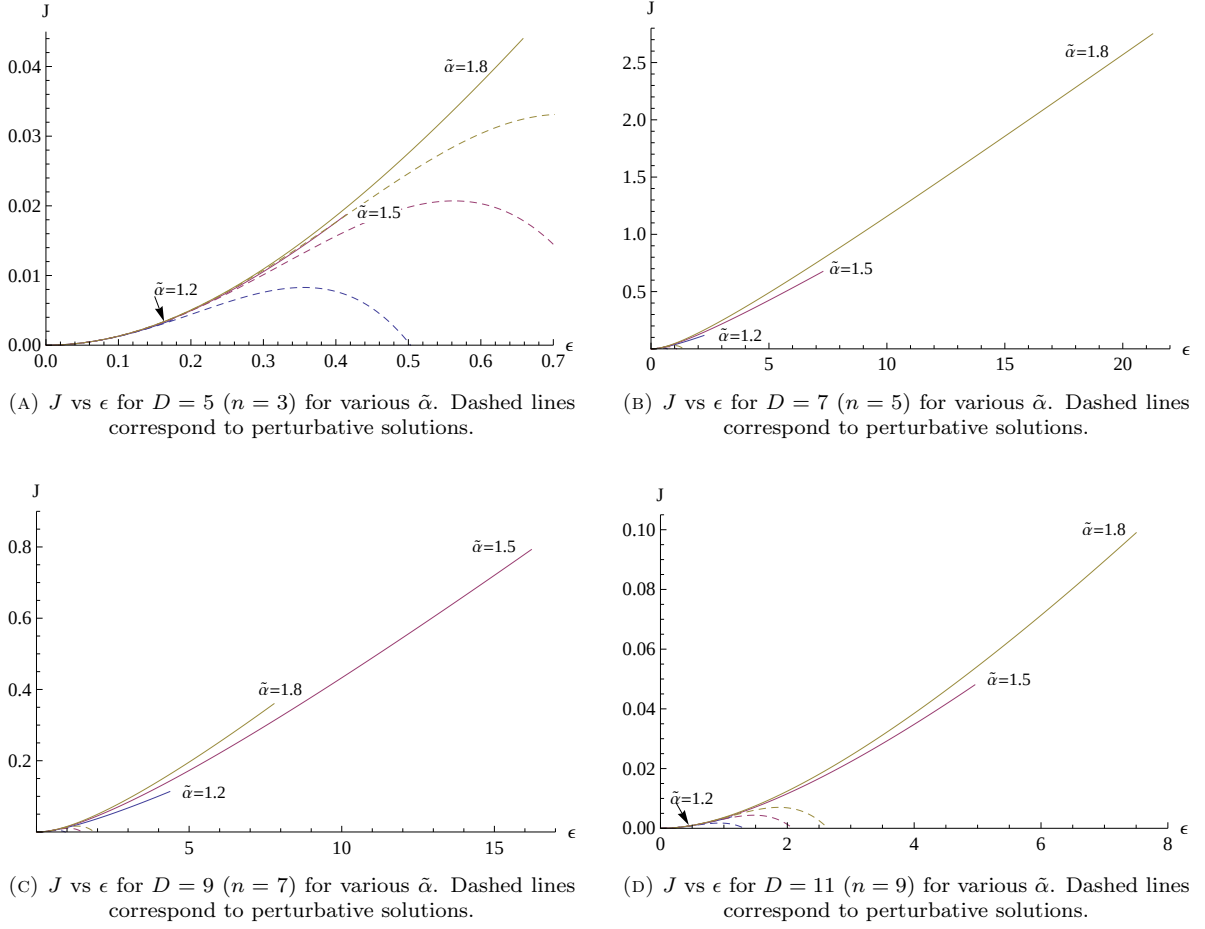


FIGURE 5.13: The boson star angular momentum plotted against the perturbative parameter ϵ in (A) 5, (B) 7, (C) 9, and (D) 11 dimensions for various values of $\tilde{\alpha}$. The angular momentum increases monotonically with ϵ .

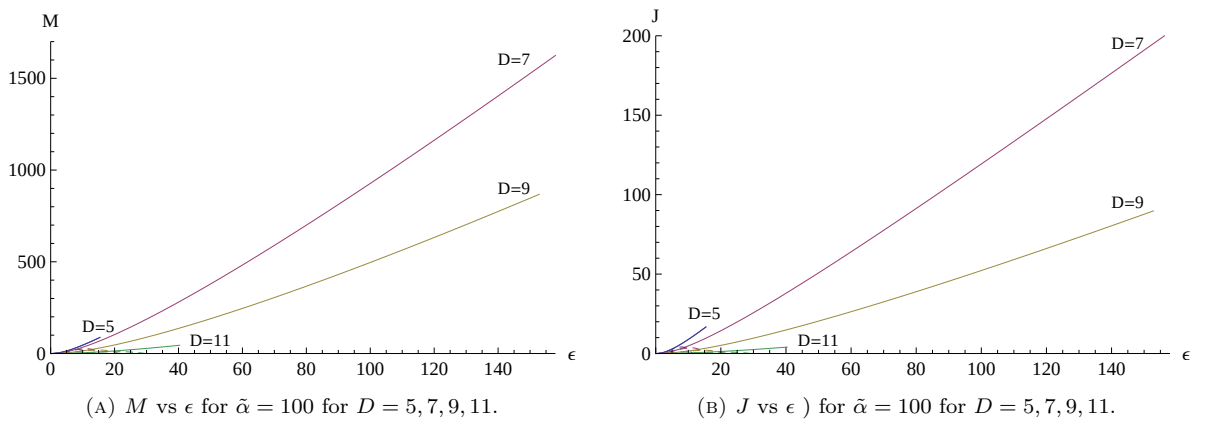


FIGURE 5.14: The boson star mass (A) and angular momentum (B) plotted against the perturbative parameter q_0 in 5, 7, 9 and 11 dimensions for $\tilde{\alpha} = 100$. Both the mass and the angular momentum increase monotonically with ϵ .

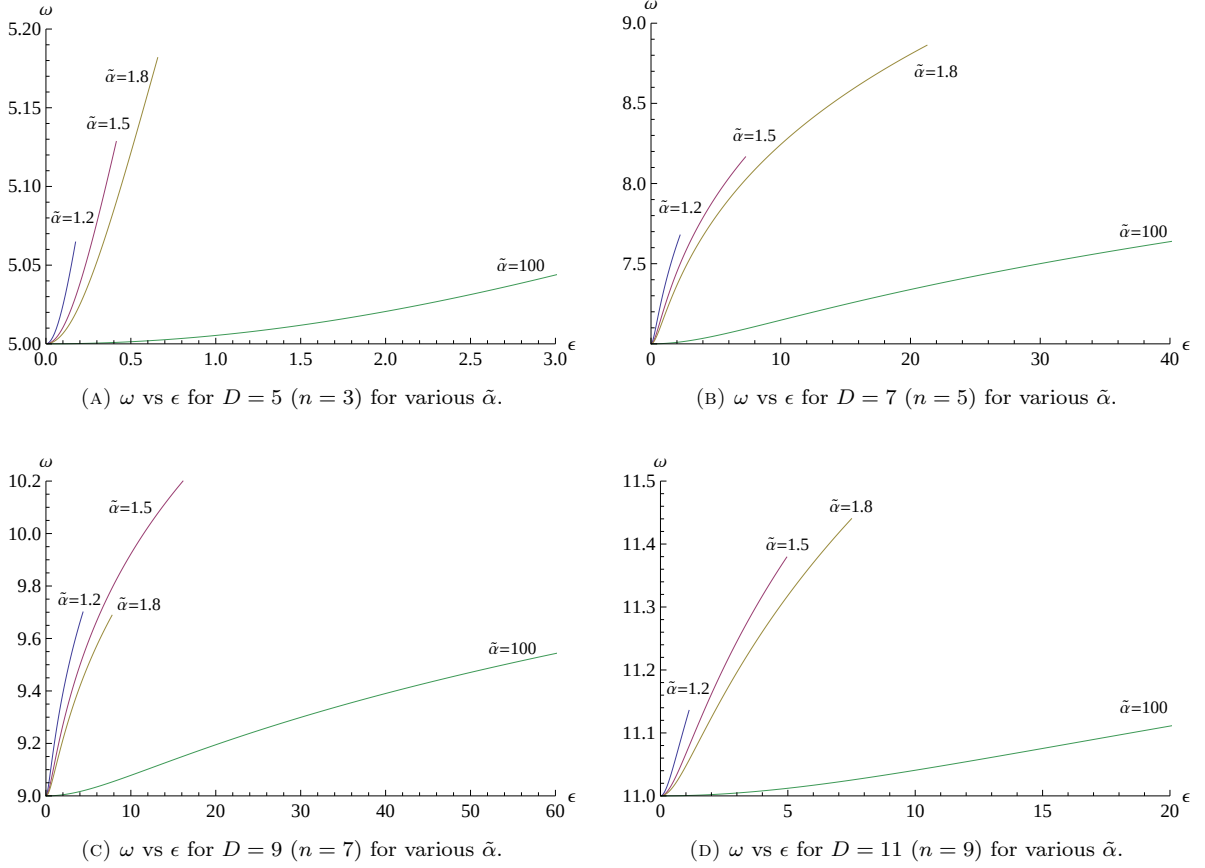


FIGURE 5.15: The angular frequency plotted against the perturbative parameter ϵ in (A) 5, (B) 7, (C) 9, and (D) 11 dimensions for various values of $\tilde{\alpha}$. The angular frequency increases monotonically with increasing ϵ .

is farther along the curve (at larger values of (M, J)). The effect of the zig-zag is suppressed with increasing dimensionality, which is most apparent in $D = 5$ solutions when $\alpha \leq 0.1\alpha_{\text{Max}}$ (5.24). After this point there is no numerical evidence for the pattern. When $\alpha > \alpha_{\text{cr}}$ the lines do not zig-zag and “turn back” at any value of q_0 due to the lack of damped oscillations in M and J .

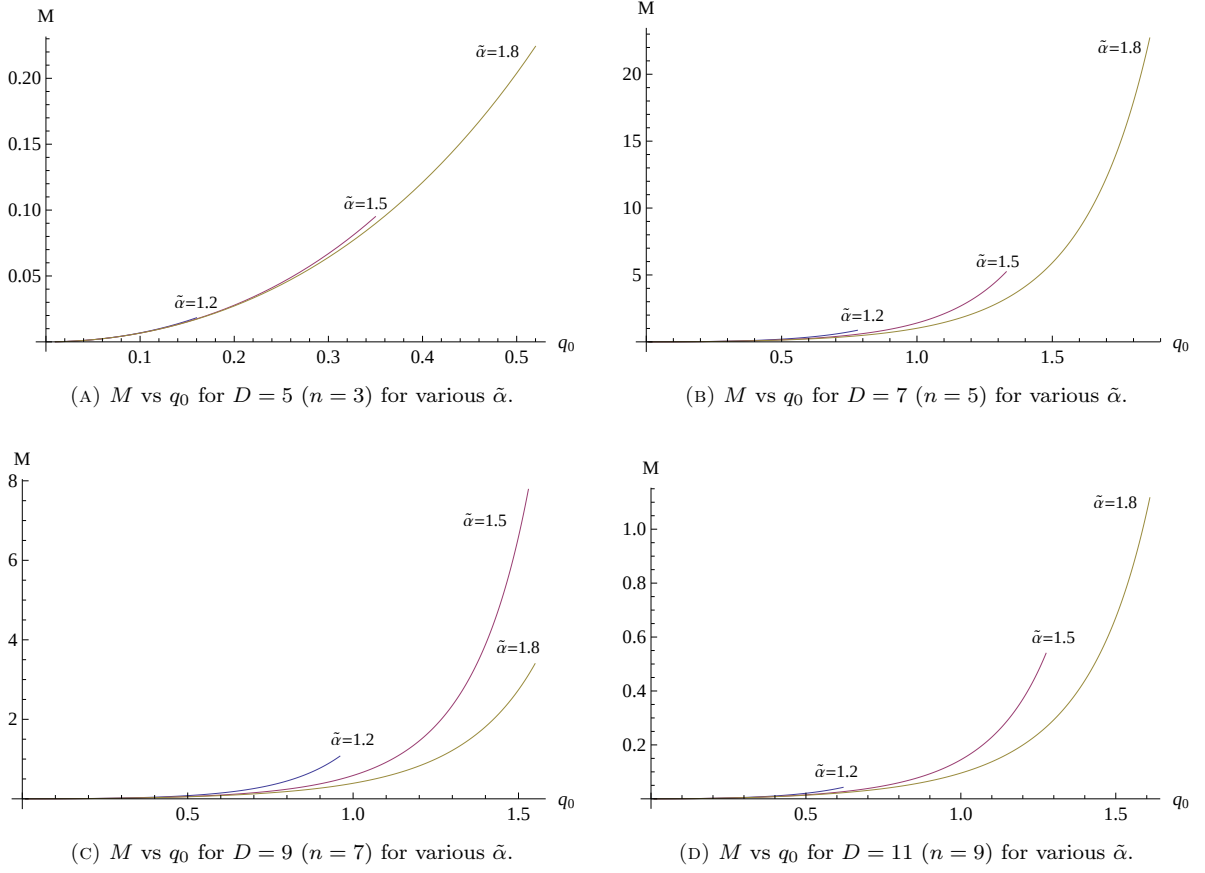


FIGURE 5.16: The boson star mass plotted against the parameter q_0 in (A) 5, (B) 7, (C) 9, and (D) 11 dimensions for various values of $\tilde{\alpha}$. The mass increases monotonically with q_0 .

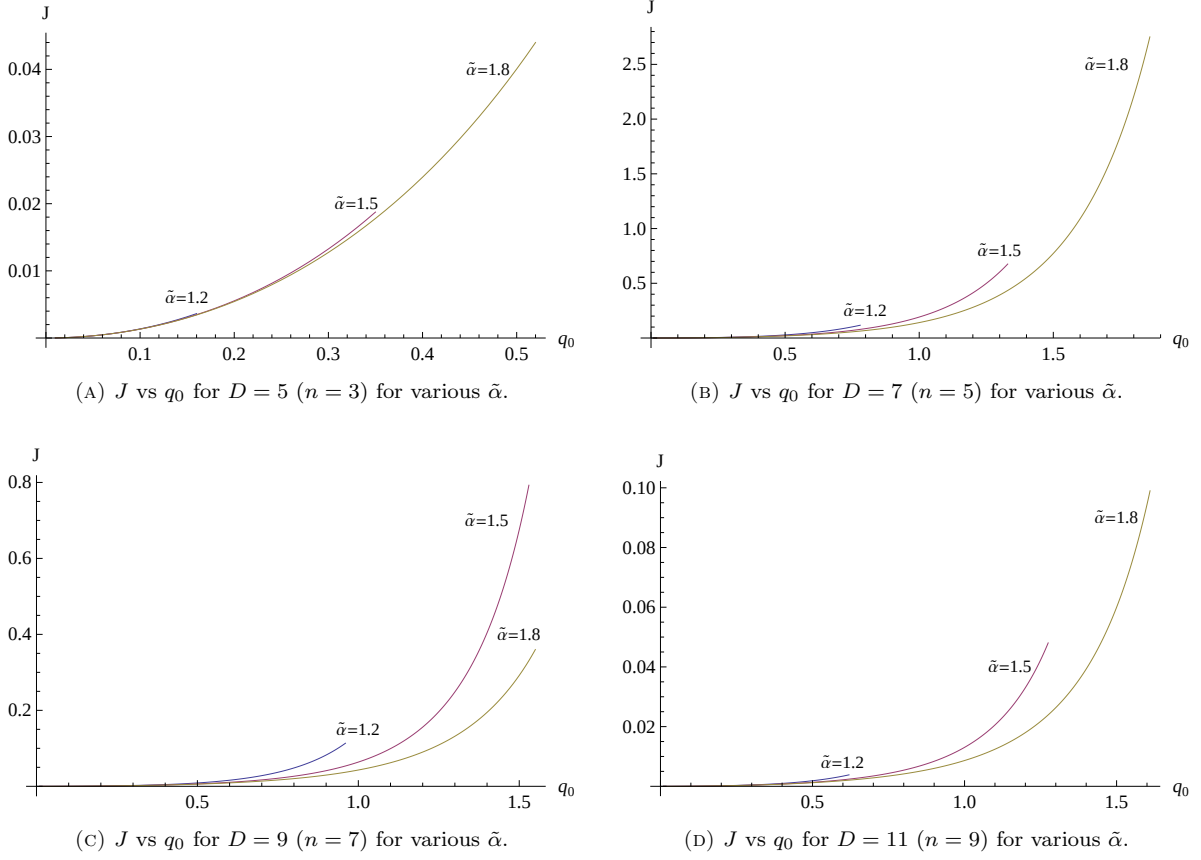


FIGURE 5.17: The boson star angular momentum plotted against the parameter q_0 in (A) 5, (B) 7, (C) 9, and (D) 11 dimensions for various values of $\tilde{\alpha}$. The angular momentum increases monotonically with q_0 .

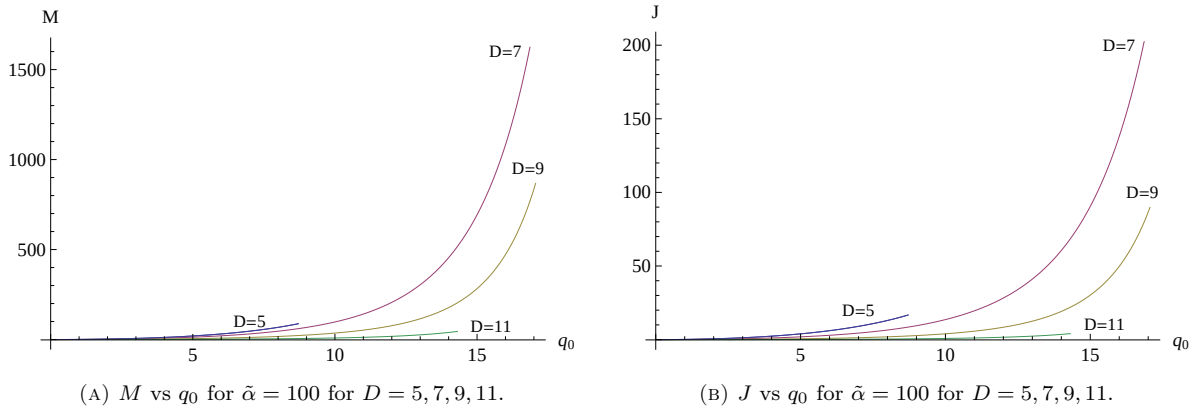


FIGURE 5.18: The boson star mass (A) and angular momentum (B) plotted against the perturbative parameter q_0 in 5, 7, 9 and 11 dimensions for $\tilde{\alpha} = 100$. Both the mass and the angular momentum increase monotonically with q_0 .

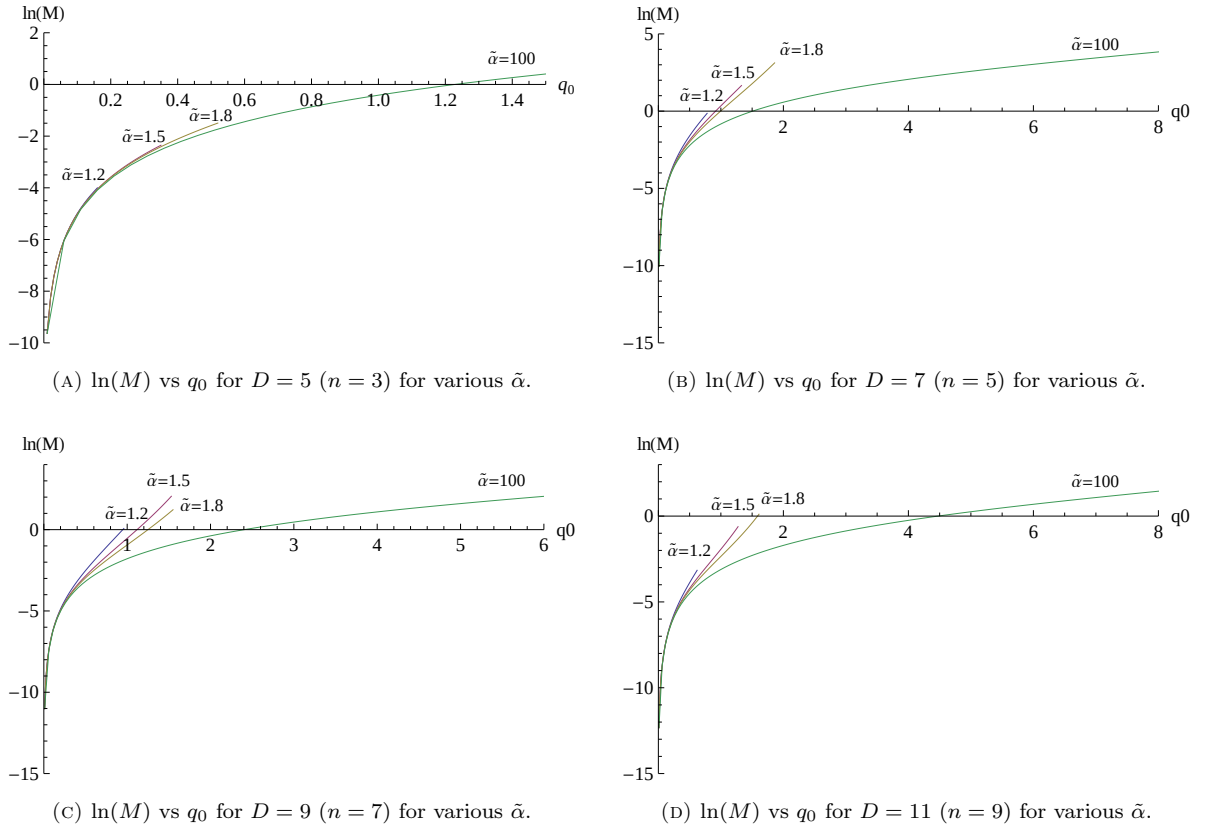


FIGURE 5.19: The natural log of the boson star mass plotted against the parameter q_0 in (A) 5, (B) 7, (C) 9, and (D) 11 dimensions for various values of $\tilde{\alpha}$. The mass approaches exponential growth with increasing q_0 .

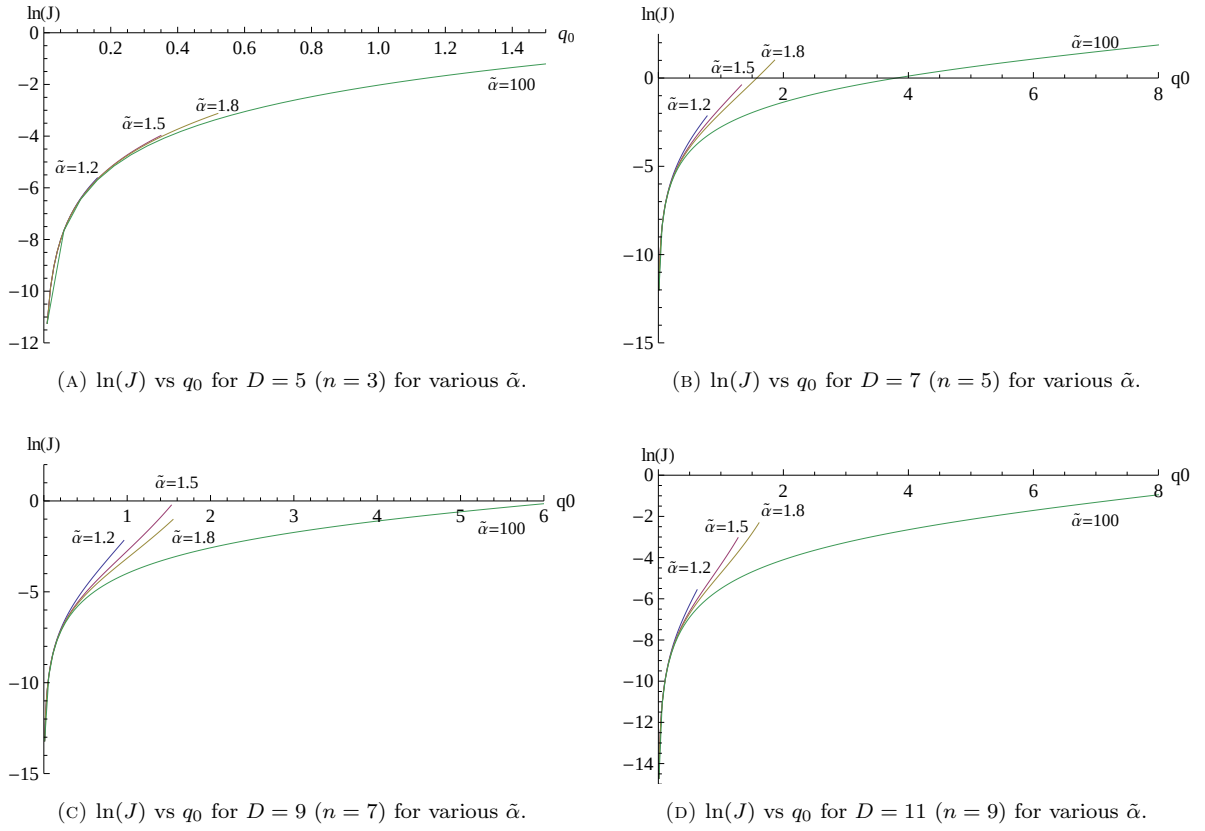


FIGURE 5.20: The natural log of the boson star angular momentum plotted against the parameter q_0 in (A) 5, (B) 7, (C) 9, and (D) 11 dimensions for various values of $\tilde{\alpha}$. The angular momentum approaches exponential growth with increasing q_0 .

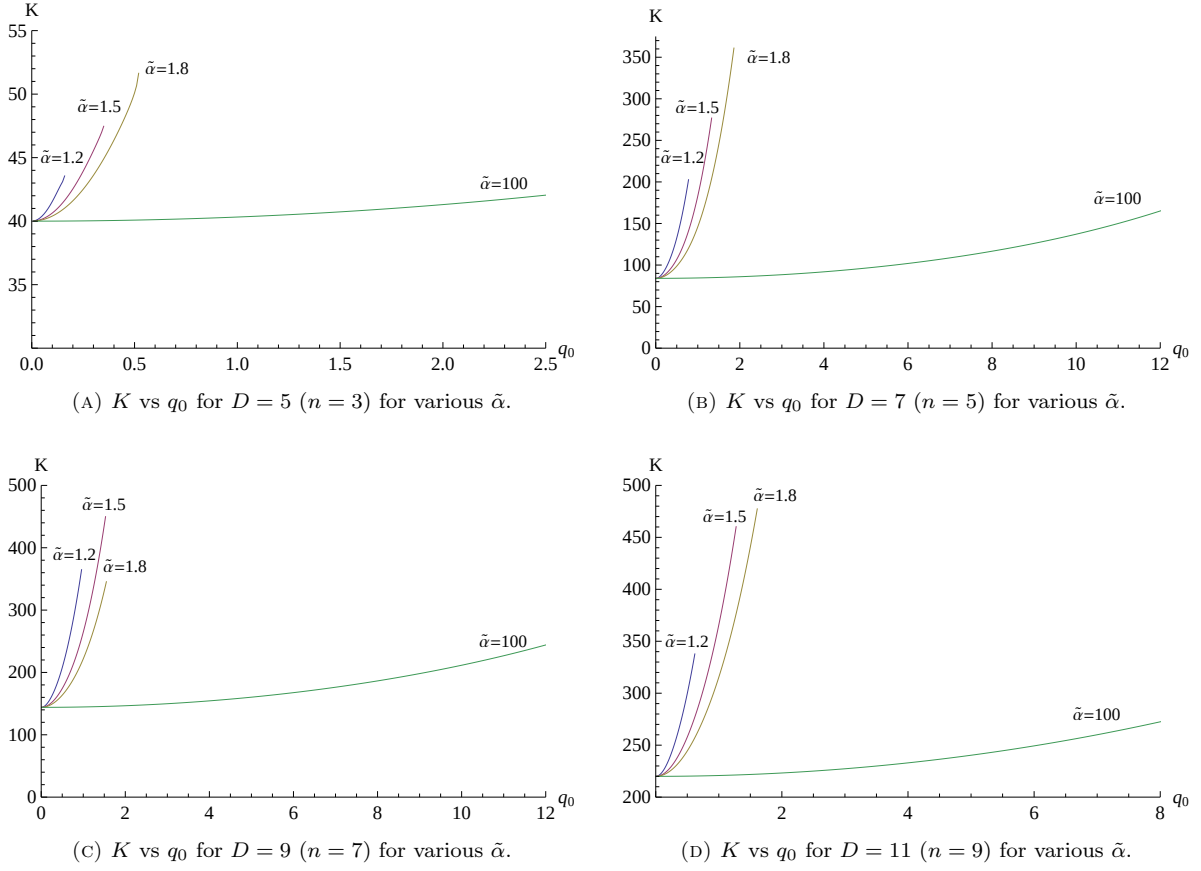


FIGURE 5.21: The Kretschmann scalar $K = R^{abcd}R_{abcd}$ plotted against the parameter q_0 in (A) 5, (B) 7, (C) 9, and (D) 11 dimensions for various values of $\tilde{\alpha}$. The Kretschmann scalar does not diverge for any value of D .

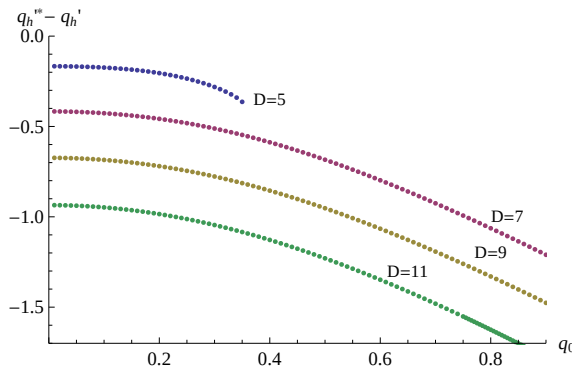


FIGURE 5.22: Plots of $q_h^{**} - q_h'(0)$ against the parameter q_0 in 5, 7, 9, and 11 dimensions for $\tilde{\alpha} = 1.5$. It is nowhere vanishing.

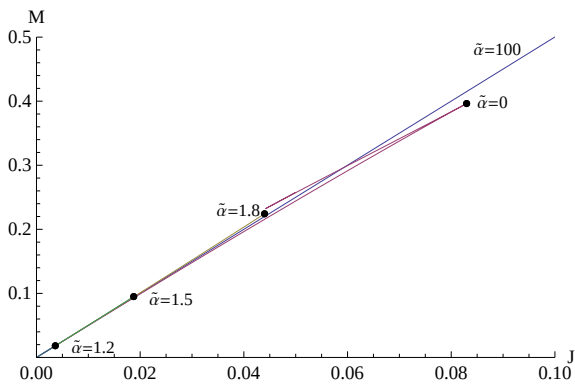
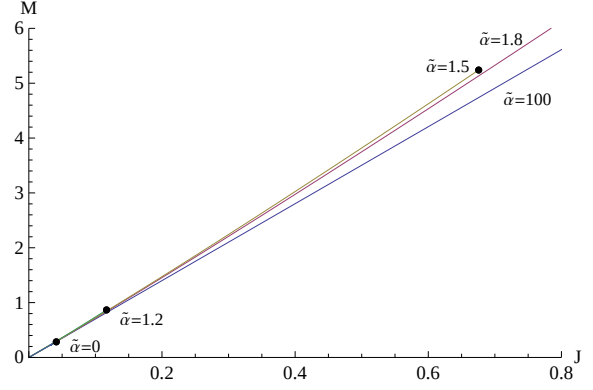
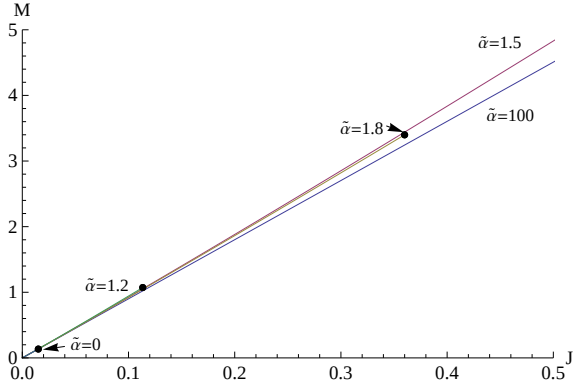
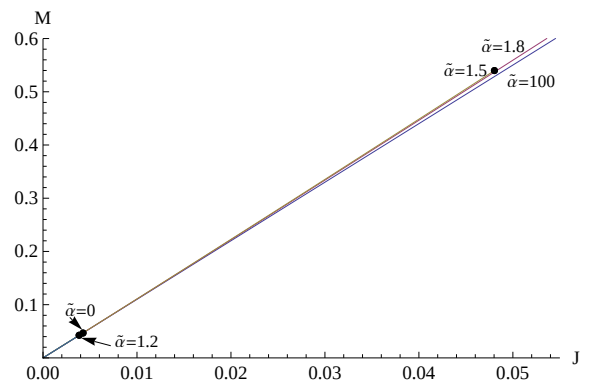

 (A) M vs J for $D = 5$ ($n = 3$) for various $\tilde{\alpha}$.

 (B) M vs J for $D = 7$ ($n = 5$) for various $\tilde{\alpha}$.

 (C) M vs J for $D = 9$ ($n = 7$) for various $\tilde{\alpha}$.

 (D) M vs J for $D = 11$ ($n = 9$) for various $\tilde{\alpha}$.

FIGURE 5.23: Boson star mass plotted against angular momentum in (A) 5, (B) 7, (C) 9, and (D) 11 dimensions for various values of $\tilde{\alpha}$. For small values of M and J , these curves almost lie on top of each other regardless of the value of $\tilde{\alpha}$. The black dots correspond to the maximum (M, J) reached on each curve.

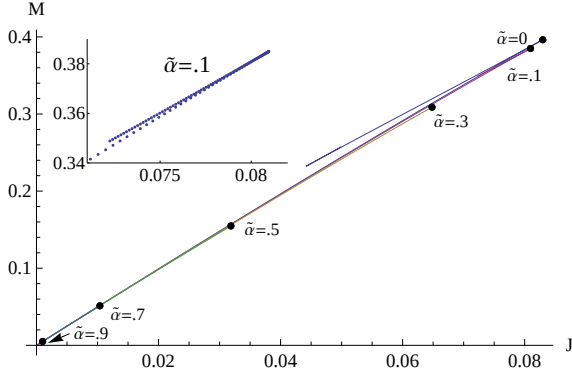
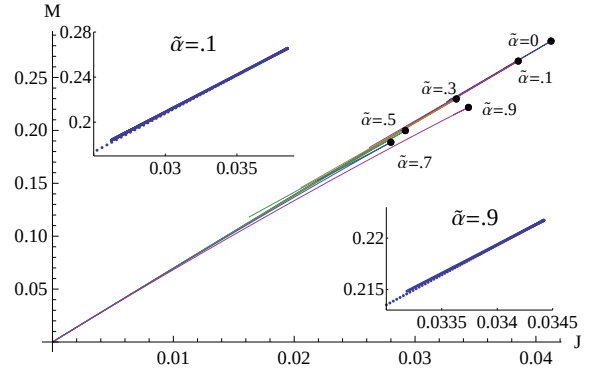
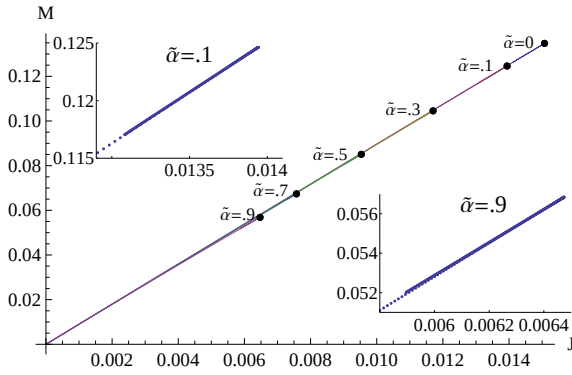
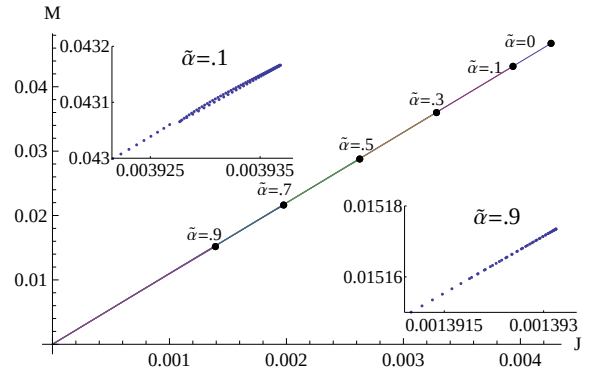

 (A) M vs J for $D = 5$ ($n = 3$) for various $\tilde{\alpha} < 1$.

 (B) M vs J for $D = 7$ ($n = 5$) for various $\tilde{\alpha} < 1$.

 (C) M vs J for $D = 9$ ($n = 7$) for various $\tilde{\alpha} < 1$.

 (D) M vs J for $D = 11$ ($n = 9$) for various $\tilde{\alpha} < 1$.

FIGURE 5.24: Boson star mass plotted against angular momentum in (A) 5, (B) 7, (C) 9, and (D) 11 dimensions for various values of $\tilde{\alpha}$. The black dots correspond to the point at which these curves turn back for a particular value of $\tilde{\alpha}$. The insets are close ups of the zig-zag patterns for $\tilde{\alpha} = 0.1$ (top left corner) and $\tilde{\alpha} = 0.9$ (bottom right corner) for each dimension.

Chapter 6

Preliminary Black Hole Numerics

The methods outlined in Chapter 4 can be used to find asymptotically AdS SKV hairy black hole solutions in addition to boson star solutions.

6.1 Black Hole Boundary Conditions

6.1.1 Black Hole Asymptotics

The asymptotic boundary conditions of the black hole will be identical to those of the boson star (2.14) since they are both asymptote to AdS with corrections for mass and angular momentum as $r \rightarrow \infty$. Note that I am still free to use the gauge freedom (2.13) and again, for numeric convenience, take $\lambda = -\omega$ which sets $\tilde{\omega} = 0$.

6.1.2 Black Hole Horizon

I am looking for non-extremal black hole solutions with scalar hair, and so I define $r = r_+$ to be the location of the non-degenerate outermost horizon. This means that $f(r)$ must have a simple zero at r_+ and the other functions must remain regular here. It was shown in (ref) that $\Pi(r)$ does not need to vanish at the horizon if the black hole and scalar field are co-rotating, and indeed, requiring that $\Pi(r_+)$ is finite and non-zero and then multiplying equation (A.5)

by f^2 , shows that the equations of motion remain consistent across the horizon as long as the frequency of the scalar field $\tilde{\omega}$ equals the angular velocity of the black hole Ω_H . That is

$$\tilde{\omega} = \Omega(r_+). \quad (6.1)$$

Once I set the gauge $\lambda = -\omega$ stated in the above subsection, the boundary conditions at the horizon are

$$\begin{aligned} f|_{r \rightarrow r_+} &= \mathcal{O}(r - r_+), & g|_{r \rightarrow r_+} &= g(r_+) + \mathcal{O}(r - r_+), & h|_{r \rightarrow r_+} &= h(r_+) + \mathcal{O}(r - r_+), \\ \Omega|_{r \rightarrow r_+} &= \mathcal{O}(r - r_+), & \Pi|_{r \rightarrow r_+} &= \tilde{q}_0 + \mathcal{O}(r - r_+), \end{aligned} \quad (6.2)$$

for all n . In this case I define $\tilde{q}_0 \equiv \Pi(r_+)$ to uniquely parameterize the family of black hole solutions.

6.2 Implementation

As with the boson star case, the infinite domain $r \in [r_+, \infty)$ needs to be transformed into $y_{BH} \in [0, 1]$, which is finite by using the definition

$$y_{bh} = \frac{r^2 - r_+^2}{r^2 + \ell^2} \quad (6.3)$$

so that a Chebyshev grid can be defined. Note that when $r_+ \rightarrow 0$ the boson star coordinate transformation (4.7) is recovered. Auxillary functions, which are analytic in the y_{bh} domain, must also be defined, this time taking into account the black hole asymptotic behavior (ref).

$$f(y_{bh}) = \left(\frac{r_+^2 + \ell^2}{\ell^2(1 - y_{bh})} \right) \left(\frac{y_{bh}(r_+^2 + \ell^2)}{y_{bh}\ell^2 + r_+^2} \right) \tilde{q}_f(y_{bh}), \quad (6.4)$$

$$g(y_{bh}) = 1 + (1 - y_{bh})^{\frac{n+1}{2}} \tilde{q}_g(y_{bh}), \quad (6.5)$$

$$h(y_{bh}) = 1 + (1 - y_{bh})^{\frac{n+1}{2}} \tilde{q}_h(y_{bh}), \quad (6.6)$$

$$\Omega(y_{bh}) = \frac{1}{\ell} \left(\frac{y_{bh}(r_+^2 + \ell^2)}{y_{bh}\ell^2 + r_+^2} \right) \tilde{q}_\Omega(y_{bh}), \quad (6.7)$$

$$\Pi(y_{bh}) = \sqrt{\frac{y_{bh}\ell^2 + r_+^2}{r_+^2 + \ell^2}} \left(\frac{\ell^2(1 - y_{bh})}{r_+^2 + \ell^2} \right)^{\frac{n+1}{2}} \tilde{q}_\Pi(y_{bh}). \quad (6.8)$$

The non-trivial boundary conditions on the auxiliary q-functions are found for both Einstein and Einstein-Gauss-Bonnet black holes from the lowest order term that is obtained when the definitions of the q-functions are substituted into the equations of motion (A.1–A.6), which are then Taylor expanded about $y_0 = 0, 1$. At $y_0 = 1$ I obtain

$$\begin{aligned} \tilde{q}_f(1) - 1 &= 0, & \tilde{q}_g(1) + \tilde{q}_h(1) &= 0, & \tilde{q}'_\Omega(1) &= \frac{r_+^2 \tilde{q}_\Omega(1)}{r_+^2 + \ell^2}, \\ \tilde{q}'_h(1) + \frac{n+1}{2} \frac{\tilde{q}_h(1)}{r_+^2 + \ell^2} &= 0, & \tilde{q}'_\Pi(1) + \frac{\ell^2 \tilde{q}_\Pi(1) \left((n+2)^2 - \tilde{q}_\Omega(1)^2 \right)}{2(n+3)(r_+^2 + \ell^2)} &= 0. \end{aligned} \quad (6.9)$$

for both the Einstein and the Einstein-Gauss-Bonnet case. This is not surprising, since these boundary conditions require that the space-time is asymptotically AdS.

However, at $y_0 = 0$ the space-time does depend on the Gauss-Bonnet parameter, and I obtain different boundary conditions. In Einstein gravity, they are

$$\begin{aligned} \tilde{q}_f(0) + \frac{2\ell^2 \tilde{q}_h(0) + 2q_0^2 \ell^2 - (n-1)\ell^2 - (n+1)r_+^2}{2(r_+^2 + \ell^2)} &= 0, & \tilde{q}_\Pi(0) - \frac{\tilde{q}_0(r_+^2 + \ell^2)^{\frac{n+2}{2}}}{\ell^{n+1}r_+} &= 0, \\ \tilde{q}'_g(0) - \frac{2}{(\tilde{q}_h(0) + 1)r_+^2 (\ell^2(-n + 2\tilde{q}_h(0) + 2\tilde{q}_0^2 + 1) + (-n-1)r_+^2)^2} & \\ \times \left[\ell^6(\tilde{q}_g(0) + 1) \left((n+1)\tilde{q}_h(0)(\tilde{q}_h(0) + 1)^2 + \tilde{q}_0^2 \left(n + 2\tilde{q}_h(0) \left(\frac{n+1}{2} + \tilde{q}_h(0) \right) \right) \right) \right. & \\ + \ell^4 r_+^2 \left((\tilde{q}_h(0) + 1) \left(\frac{1}{4}(n+1)(n-1)^2 \tilde{q}_g(0) + (n+1)\tilde{q}_h(0)^2 + 2(n+1)\tilde{q}_g(0)\tilde{q}_h(0)^2 + (n-3)\tilde{q}_\Omega(0)^2 \right. \right. & \\ + (n-7)\tilde{q}_h(0)\tilde{q}_\Omega(0)^2 + (n+1)\tilde{q}_h(0) - (n-2)(n+1)\tilde{q}_g(0)\tilde{q}_h(0) - 4\tilde{q}_h(0)^2\tilde{q}_\Omega(0)^2 \Big) & \\ + (n+1)\tilde{q}_g(0)(\tilde{q}_h(0) + 1)\tilde{q}_0^4 + \tilde{q}_0^2 \left(n + \tilde{q}_g(0) (-n^2 + n + 2(n+2)\tilde{q}_h(0)^2 - (n-4)(n+1)\tilde{q}_h(0) + 1) \right. & \\ + \tilde{q}_h(0)(n + 2\tilde{q}_h(0) + 1) - 2(\tilde{q}_h(0) + 1)^2\tilde{q}_\Omega(0)^2 \Big) & \\ \left. - 2\ell^2(\tilde{q}_h(0) + 1)r_+^4 \left(\frac{1}{2}(n+1)^2\tilde{q}_g(0) \left(\frac{1-n}{2} + \tilde{q}_h(0) + \tilde{q}_0^2 \right) + (\tilde{q}_h(0) + 1)\tilde{q}_\Omega(0)^2 (-n + 2\tilde{q}_h(0) + \tilde{q}_0^2 + 1) \right) \right. & \\ \left. + (n+1)(\tilde{q}_h(0) + 1)r_+^6 \left(\frac{1}{4}(n+1)^2\tilde{q}_g(0) + (\tilde{q}_h(0) + 1)\tilde{q}_\Omega(0)^2 \right) \right] &= 0 \end{aligned}$$

$$\begin{aligned}
 & \tilde{q}'_h(0) - \frac{2}{(\tilde{q}_g(0) + 1)r_+^2 ((n+1)r_+^2 - \ell^2 (-n + 2\tilde{q}_h(0) + 2\tilde{q}_0^2 + 1))} \\
 & \times \left[(\tilde{q}_g(0) + 1)\tilde{q}_h(0) \left(\ell^4 \left(\frac{n+1}{2} + \frac{1}{2}(n+1)\tilde{q}_h(0) + \tilde{q}_0^2 \right) + \ell^2 r_+^2 \left(\frac{1}{4}(n+1)^2 - \frac{1}{2}(n-1)\tilde{q}_0^2 \right) + \frac{1}{4}(n+1)^2 r_+^4 \right) \right. \\
 & \left. - (\tilde{q}_h(0) + 1)^2 \tilde{q}_\Omega(0)^2 r_+^2 (\ell^2 + r_+^2) \right] = 0 \\
 \\
 & \tilde{q}'_\Omega(0) - \frac{\tilde{q}_\Omega(0) (\ell^2 + r_+^2)}{2(\tilde{q}_g(0) + 1)(\tilde{q}_h(0) + 1)r_+^2 (\ell^2 (-n + 2\tilde{q}_h(0) + 2\tilde{q}_0^2 + 1) + (-n-1)r_+^2)^2} \\
 & \times \left[\ell^4 (\tilde{q}_g(0) + 1) \left((\tilde{q}_h(0) + 1) \left(-\frac{1}{2}(n-1)^3 + 2(n+3)\tilde{q}_h(0)^2 + \frac{1}{2}(n-3)^2 \tilde{q}_h(0) \right) \right. \right. \\
 & \left. \left. - 2\tilde{q}_0^4 (n + (n-4)\tilde{q}_h(0) + 1) + \tilde{q}_0^2 (n(2n-1) + \tilde{q}_h(0) (2n^2 - 7n + (15-n)\tilde{q}_h(0) + 9)) \right) \right. \\
 & \left. + 2\ell^2 r_+^2 \left(\frac{1}{2}(n+1)(\tilde{q}_g(0) + 1) \left((-\tilde{q}_h(0) - 1) \left((n-1)^2 - \frac{1}{2}(n-7)\tilde{q}_h(0) \right) + \tilde{q}_0^2 (2n + (2n-5)\tilde{q}_h(0)) \right) \right. \right. \\
 & \left. \left. - (\tilde{q}_h(0) + 1)^2 \tilde{q}_\Omega(0)^2 (-2n + 5\tilde{q}_h(0) + 4\tilde{q}_0^2 + 3) \right) \right. \\
 & \left. + 4(n+1)(\tilde{q}_h(0) + 1)r_+^4 \left((\tilde{q}_h(0) + 1)\tilde{q}_\Omega(0)^2 - \frac{1}{8}(n-1)(n+1)(\tilde{q}_g(0) + 1) \right) \right] = 0 \quad (6.10)
 \end{aligned}$$

The Gauss-Bonnet boundary conditions are significantly more unwieldy, and as a result have not been written down here in the interest of space. They do reduce down to (6.10) in the $\alpha \rightarrow 0$ limit.

The largest obstacle to obtaining numeric black hole solutions is finding an appropriate seed. Unlike in the boson star case, the perturbative solutions, which are double expansions in ϵ and $\frac{r_+}{\ell}$ [6], cannot be used for the seed. This is because the solutions of small horizon, small ϵ black holes are very close to low ϵ boson stars, and are, in practice, numerically difficult to resolve from each other.

Instead, the black hole seeds are constructed as follows: take a numerical boson star solution found in Chapter 5 and put a small black hole inside of it by artificially inserting a non-zero horizon and demanding the metric functions have the correct boundary conditions there (6.2). This is done by noting that in the $r_+ \rightarrow 0$ limit the black hole metric and scalar functions should be equal to the boson star functions, which is used to set up the following relationship

between the black hole and boson star q-functions

$$\begin{aligned}\tilde{q}_f(r) &\approx 1 + \frac{q_f(r)}{\ell^2 + r^2}, & \tilde{q}_g(r) &\approx q_g(r), & \tilde{q}_h(r) &\approx q_h(r), \\ \tilde{q}_\Omega(r) &\approx q_\Omega(r), & \tilde{q}_\Pi(r) &\approx q_\Pi(r).\end{aligned}\tag{6.11}$$

Once the boson star solution is known in terms of the black hole q-functions the coordinates are transformed from r to y_{bh} (6.3) and the black hole seed knows about the horizon. While this seems like a relatively straight forward process, in practice it is quite difficult. Both the initial boson star and the horizon size need to be selected so that the resulting seed is far enough away from the perturbative regime that the numerics can resolve black hole solutions from boson star solutions, but not so far away that the Newton-Raphson method cannot converge. This leaves a very small range in the (ϵ, r_+) parameter space that will result in a good seed.

Chapter 7

Conclusions and Future Work

I have obtained asymptotically AdS rotating SKV boson star solutions in Einstein-Gauss-Bonnet gravity coupled to a $\frac{n+1}{2}$ -tuple of massless scalar fields on all odd space-time dimensions of interest to string theory, *i.e.* $D = 5, 7, 9, 11$ both perturbatively in powers of the scalar field and numerically. The same relaxation method on Chebyshev grid used in the Einstein case [34, 35] was used to obtain the numeric results and in all space-time dimensions studied the perturbative solutions match the numerical solutions at sufficiently small value of q_0 , as expected. Additionally, I have found that each dimension exhibits a critical value of the Gauss-Bonnet coupling parameter α_{cr} where the solutions no longer have the asymptotic behavior in (2.14). This critical value separates the solutions into two distinct groups, $\alpha < \alpha_{\text{cr}}$ and $\alpha > \alpha_{\text{cr}}$.

When $\alpha < \alpha_{\text{cr}}$ there is a striking distinction between the $D = 5$ case and the higher dimensional cases. When $D > 5$, the physical quantities (M, J, ω) as functions of q_0 , undergo damped oscillations about finite limiting values. In the $D = 5$ case when $\alpha > 0$, M , J and ω terminate at some finite value critical of $q_{0\text{critical}}$ at which point the Kretschmann scalar diverges. Even when $\alpha = 0.1\alpha_{\text{rc}}$ the oscillations are eliminated. This is reminiscent of the $D = 3$ AdS boson star solutions obtained by [34] where the boson star mass, angular momentum and angular velocity increase monotonically with the central energy density up to a critical value at which point the boson star solution connects smoothly with an extremal BTZ black hole. However, in the EGB $D = 5$ boson star studied here, there is no evidence

that a black hole forms at the critical value of q_0 , although some singularity seems to form as indicated by the divergent Kretschmann scalar. This boson star branch is likely dynamically unstable to forming a black hole at some value $q_0 < q_{0\text{critical}}$.

It seems likely that critical values of q_0 will appear in all odd dimensions in which the gravitational theory includes the highest possible curvature term that maintains 2nd order field equations provided that the mass is positive. It would be very interesting to numerically check this for the $D = 7$ 3rd-order Lovelock case and possibly even for $D = 9$ 4th-order Lovelock.

When $\alpha > \alpha_{\text{cr}}$ there is no obvious distinction between the $D = 5$ and $D > 5$ cases. In all dimensions, the physical quantities, mass and angular momentum increase monotonically as q_0 increases and there is no sign of the oscillations that appear when $\alpha < \alpha_{\text{cr}}$. In fact, after some value of q_0 , these physical quantities begin to increase exponentially with increasing q_0 .

There is a distinct possibility that space-times for which $\alpha > \alpha_{\text{cr}}$ contains gravitons with negative kinetic energy or ghosts [41]. An analysis similar to the one used in for the spherically symmetric case [42] would need to be employed in order to test for this.

Additionally, it would be worth numerically studying the $\alpha = \alpha_{\text{cr}}$ case, which requires a separate analysis due to the different asymptotic boundary conditions that must be employed.

A stability analysis is also something to look at in the future. Although I expect that the boson star solutions in this paper are non-pertubatively stable, since it has been shown that the AdS instability is due to the high level of symmetry present in AdS [36], it is something that should be explicitly verified.

I have also found the explicit boundary conditions for SKV hairy black holes in Einstein gravity for odd dimensions and in Einstein-Gauss-Bonnet gravity for $D = 5, 7, 9, 11$. Of course in this case, the next step is to numerically solving for these hairy black holes in both Einstein and EGB. The first step to accomplishing this, is to nail down where in the parameter space of (ϵ, r_+) that one needs to look to find an appropriate seed solution.

Appendix A

Field Equations

This appendix lists the five coupled ODEs for the five metric and scalar functions employed for both the perturbative and numeric analyses. While it is possible to rewrite these equations so that the second-derivative term is isolated, doing so produces significantly more cumbersome expressions than the ones listed here, so I did not carry out this step.

$$\begin{aligned}
& r^2 \left[2g \left(f^2 g h (n-2)(n-1) - 2h \Pi^2 r^2 (\omega - \Omega)^2 \right) \right. \\
& + f h r \left(g r \left(2g f'' + 2f g'' + 4f g (\Pi')^2 - 3h r^2 (\Omega')^2 \right) + g f' (3r g' + 4g(n-1)) + 2f g(n-1)g' - f r (g')^2 \right) \\
& + 2f g^2 \left(h (3h(n-1) - n^2 + 2\Lambda r^2 + 1) + 2\Pi^2 (h(n-1) - 1) \right) \Big] \\
& + 2\alpha f h (1-n) \left[g^2 (n-3) \left(f^2 (n-4)(n-2) + 2f(n-4)(3h-n-1) + 3h(5h-2(n+1)) + n^2 - 1 \right) \right. \\
& + r \left(g r \left(2(g f'' + f g'') (f(n-2) + 3h-n-1) - 3h r^2 (\Omega')^2 (f(n-2) + 5h-n-1) \right) \right. \\
& + 2g^2 (n-2)r (f')^2 + g f' \left(r g' (5f(n-2) + 9h-3(n+1)) + 4g(n-3)(f(n-2) + 3h-n-1) \right) \\
& \left. \left. + f r (g')^2 (-f(n-2) - 3h+n+1) + 2f g(n-3)g'(f(n-2) + 3h-n-1) \right) \right] = 0 \quad (\text{A.1})
\end{aligned}$$

$$\begin{aligned}
 & f^2 h r^4 \Omega (g')^2 + f g^2 r^2 \left(r \left(2 h r (f \Omega'' - \Omega f'') + f \Omega' (3 r h' + 2 h (n + 2)) \right) - 4 h (n - 1) r \Omega f' \right. \\
 & \quad \left. - 2 \Omega \left(h (n - 1) (f (n - 2) + 3 h - n - 1) + 2 \Pi^2 (h (n - 1) + 1) + 2 h \Lambda r^2 \right) - 4 f h r^2 \Omega (\Pi')^2 + 8 \Pi^2 \omega \right) \\
 & \quad + g h r^3 \left(r \Omega \left(-2 f^2 g'' + 3 f h r^2 (\Omega')^2 + 4 \Pi^2 (\omega - \Omega)^2 \right) - f g' (3 r \Omega f' + 2 f (n - 1) \Omega + f r \Omega') \right) \\
 & \quad + 2 \alpha f (n - 1) \left\{ g^2 h (n - 3) \Omega \left(f^2 (n - 4) (n - 2) + 2 f (n - 4) (3 h - n - 1) + 15 h^2 - 6 h (n + 1) + n^2 - 1 \right) \right. \\
 & \quad + r \left[g \left(2 h r ((f - 1) n - 2 f + 3 h - 1) (g \Omega f'' + f \Omega g'' - f g \Omega'') \right. \right. \\
 & \quad \left. \left. + (f(-n) + 2 f - 5 h + n + 1) \left(3 f g r h' \Omega' + 3 h^2 r^3 \Omega (\Omega')^2 \right) \right) \right. \\
 & \quad \left. + 2 g^2 h (n - 2) r \Omega (f')^2 + g h f' \left(r \Omega g' (5 f (n - 2) + 9 h - 3 n - 3) + 4 g (n - 3) \Omega ((f - 1) n - 2 f + 3 h - 1) \right. \right. \\
 & \quad \left. \left. - 2 f g (n - 2) r \Omega' \right) + 2 f g^2 h n \Omega' (f(-n) + 2 f - 3 h + n + 1) \right. \\
 & \quad \left. \left. + f g h g' (f (n - 2) + 3 h - n - 1) (2 (n - 3) \Omega + r \Omega') + f h r \Omega (g')^2 (-f (n - 2) - 3 h + n + 1) \right] \right\} = 0
 \end{aligned}
 \tag{A.2}$$

$$\begin{aligned}
 & r^2 \left\{ 2gh \left(-2h^2 \Pi^2 (\omega - \Omega)^2 \Omega^2 r^4 + fgh \left(2(-\omega^2 - 2\Omega\omega + (h(n-1) + 2)\Omega^2) \Pi^2 \right. \right. \right. \\
 & + h(-n^2 + 3h(n-1) + 2r^2\Lambda + 1)\Omega^2 \Big) r^2 - f^3 g^2 h(n-1)n \\
 & + f^2 g \left(-2g(h(n-1) + 1)\Pi^2 + h^2(n-2)(n-1)r^2\Omega^2 + gh(n^2 - h(n-1) - 2r^2\Lambda - 1) \right) \Big) \\
 & - fr \left[fr^3 \Omega^2 (g')^2 h^3 - 2fgr^2 \Omega g' ((n-1)\Omega + r\Omega') h^3 \right. \\
 & + gf' \left(g(2h(fgn - 2h(n-1)r^2\Omega^2) + fgrh') - 3h^2 r^3 \Omega^2 g' \right) h \\
 & + g \left(-f^2 g^2 r (h')^2 + 2fgh(3h\Omega\Omega' r^3 + fg(n+1)) h' + hr \left(2f^2 h'' g^2 + 4fh(fg - hr^2\Omega^2) (\Pi')^2 g \right. \right. \\
 & + 4fh^2(n+2)r\Omega\Omega' g + h^2 r^2 (3hr^2\Omega^2 + fg) (\Omega')^2 - 2h^2 r^2 \Omega (g\Omega f'' + f\Omega g'' - 2fg\Omega'') \Big) \Big) \Big] \Big\} \\
 & + 2f(n-1)\alpha \left\{ -2g^2 h^3 (n-2)\Omega^2 (f')^2 r^4 + fh^3(3h + f(n-2) - n-1)\Omega^2 (g')^2 r^4 \right. \\
 & - 2fgh^3(3h + f(n-2) - n-1)\Omega g' ((n-3)\Omega + r\Omega') r^3 \\
 & + ghf' \left[h^2(-9h - 5f(n-2) + 3n+3)\Omega^2 g' r^3 + g \left(fg(3h + 3f(n-2) - n-1)rh' \right. \right. \\
 & + 2h(2fh(n-2)\Omega\Omega' r^3 - 2h(n-3)(-2f + 3h + (f-1)n-1)\Omega^2 r^2 + fg(n-2)(h + (f-1)n-1)) \Big) \Big] r \\
 & + g \left[g(n-3) \left(fg((f-1)^2 n^2 + 2(f-1)(h-f)n + (h-1)(-4f + 3h + 1)) \right. \right. \\
 & - h((n-4)(n-2)f^2 + 2(3h-n-1)(n-4)f + 15h^2 + n^2 - 6h(n+1) - 1)r^2\Omega^2 \Big) h^2 \\
 & + r \left(f^2 g^2 (3h - f(n-2) + n+1)r (h')^2 \right. \\
 & + 2fgh(3h(-2f + 5h + (f-1)n-1)\Omega\Omega' r^3 + fg(n-1)(-2f + 3h + (f-1)n-1)) h' \\
 & + hr \left(r^2 (3h(5h + f(n-2) - n-1)r^2\Omega^2 + fg(3h + f(n-2) - n-1)) (\Omega')^2 h^2 \right. \\
 & + 4fg(3h + f(n-2) - n-1)nr\Omega\Omega' h^2 \\
 & + 2(3h + f(n-2) - n-1) \left(f(fh''g^2 + h^2 r^2 \Omega (2g\Omega'' - \Omega g'')) - gh^2 r^2 \Omega^2 f'' \right) \Big) \Big] \Big\} = 0
 \end{aligned}
 \tag{A.3}$$

$$\begin{aligned}
 & r^2 \left\{ 2fh \left(2\Pi^2 + h(2\Lambda r^2 + 2(n-3)\Pi^2 + h(n-5) + (3-n)(n+1)) \right) g^2 \right. \\
 & + 2h \left(f^2 gh(n-2)(n-1) - 2hr^2 \Pi^2 (\omega - \Omega)^2 \right) g \\
 & + fr \left[-fh^2 r (g')^2 + fgh(2h(n-1) + rh') g' + ghf' (4gh(n-1) + 3hrg' + 2grh') \right. \\
 & + g \left(-fgr(h')^2 + 2fghnh' + hr \left(h(4fg(\Pi')^2 - hr^2(\Omega')^2 + 2gf'' + 2fg'') \right) + 2fgh'' \right) \left. \right] \left. \right\} \\
 & - 2f\alpha \left\{ g^2 \left((n-4)(n-2)(n-1)(f-1)^2 + 2(h-1)(n-5)(n-4)(f-1) + 3(h-1)^2(n-9) \right) (n-3)h^2 \right. \\
 & + r \left[2g^2 h(n-2)r(h(n-1) + rh') (f')^2 + g \left(4g(n-3)(h(n-5) + f(n-2)(n-1) + (3-n)(n+1))h^2 \right. \right. \\
 & + r \left(-fg(n-2)r(h')^2 + 2gh(3h(n-3) + (-n-1)(n-3) + f(n-2)(3n-7))h' + hg' \right. \\
 & + 2(n-2) \left(h^3(\Omega')^2 r^3 + fghh''r \right) \left. \right) \left. \right] f' \\
 & - fhr(g')^2 \left(h(h(n-5) + f(n-2)(n-1) + (3-n)(n+1)) + f(n-2)rh' \right) \\
 & + fg' \left(2g(h(n-5) + (-n-1)(n-3) + f(n-2)(n-1))(n-3)h^2 \right. \\
 & + r \left(-h^3(n-2)(\Omega')^2 r^3 - fg(n-2)(h')^2 r + 2fgh(n-2)h''r \right. \\
 & + gh(3h(n-3) + (3-n)(n+1) + f(n-2)(3n-5))h' \left. \right) \left. \right) \\
 & + g \left(fg(n-3)(3h - f(n-2) + n+1)r(h')^2 \right. \\
 & + fh(n-2) \left(3h(\Omega')^2 r^4 + 2(gf'' + fg'')r^2 + 2g(3h + f(n-2) - n-1)(n-3) \right) h' \\
 & + hr \left(4fh^2(n-2)\Omega'\Omega''r^3 + h^2(-3h(n-5) + (n-3)(n+1) + f(n-2)(n+3))(\Omega')^2 r^2 \right. \\
 & + 2h(h(n-5) + f(n-2)(n-1) + (3-n)(n+1))(gf'' + fg'') \\
 & + fg(n-3)(6h + 2f(n-2) - 2(n+1))h'' \left. \right) \left. \right] \left. \right\} = 0 \tag{A.4}
 \end{aligned}$$

$$\Pi'' + \frac{\Pi'}{2fr} (2nfg h + 2rghf' + rf(gh)') + \frac{\Pi(\omega - \Omega)^2}{f^2 g} - \frac{(1 + (n-1)h)\Pi}{fhr^2} = 0 \tag{A.5}$$

where a $'$ denotes differentiation with respect to r . In addition to these second order ODEs, there is a first order ODE in the form of a constraint equation, given by:

$$\begin{aligned}
 & r^2 \left[4 \left(\frac{1}{2} f^2 g h (n-1) n + f g \left(h \left(\frac{1}{2} (n-1) (h-n+2\Pi^2-1) + \Lambda r^2 \right) + \Pi^2 \right) - h \Pi^2 r^2 (\omega - \Omega)^2 \right) \right. \\
 & \quad \left. + f r \left((fg)' (rh' + 2hn) + 4fg \left(\frac{1}{2} (n-1) h' - h r (\Pi')^2 \right) \right) + f h^2 r^4 (\Omega')^2 \right] \\
 & \quad - 2\alpha f(n-1) \left[r (fg)' \left(r h' (3f(n-2) + 3h - n - 1) + 2h(n-2)(fn + h - n - 1) \right) \right. \\
 & \quad \left. + g(n-3) \left(2f r h' (f(n-2) + 3h - n - 1) + h(2(f-1)(h-1)(n-2) + (f-1)^2(n-2)n + 3(h-1)^2) \right) \right. \\
 & \quad \left. + h^2 r^4 (\Omega')^2 (3f(n-2) + 3h - n - 1) \right] = 0 \tag{A.6}
 \end{aligned}$$

Appendix B

Perturbative Fields

In this Appendix I catalog all of the perturbative gravitational and scalar field solutions to the field equations on space-time dimension $D = n + 2$ for $n = 3, 5, 7, 9$. The fields are labeled as $F_{n,p}$, where p denotes the order in ϵ .

n=3 results

$$\begin{aligned} f_{3;2} &= \frac{6\ell^4 + 20\ell^2 r^2 + 5r^4}{9} \\ \Omega_{3;2} &= \frac{6\ell^4 + 4\ell^2 r^2 + r^4}{12} \\ \Pi_{3;3} &= \frac{1540\ell^6 + 5548\ell^4 r^2 + 3935\ell^2 r^4 + 900r^6}{2016} \\ f_{3;4} &= \left[\frac{\ell^2}{1270080} \left(1315860\ell^{14} + 11249595\ell^{12}r^2 + 36154839\ell^{10}r^4 + 51954798\ell^8r^6 + 40913902\ell^6r^8 \right. \right. \\ &\quad \left. \left. + 18512283\ell^4r^{10} + 4631027\ell^2r^{12} + 514952r^{14} \right) \right. \\ &\quad \left. - \frac{\alpha}{4762800} \left(38080980\ell^{14} + 165606315\ell^{12}r^2 + 564058887\ell^{10}r^4 + 809407494\ell^8r^6 \right. \right. \\ &\quad \left. \left. + 616096806\ell^6r^8 + 272183829\ell^4r^{10} + 67620861\ell^2r^{12} + 7535096r^{14} \right) \right] \end{aligned}$$

$$\begin{aligned}
 g_{3;4} &= \left[\frac{\ell^2}{1270080} \left(872290\ell^{16} + 7971520\ell^{14}r^2 + 25939155\ell^{12}r^4 + 34403186\ell^{10}r^6 + 16099475\ell^8r^8 \right. \right. \\
 &\quad \left. \left. + 3052440\ell^6r^{10} + 149055\ell^4r^{12} + 35410\ell^2r^{14} + 3541r^{16} \right) \right. \\
 &\quad \left. - \frac{\alpha}{1587600} \left(5891650\ell^{16} + 47634040\ell^{14}r^2 + 173336925\ell^{12}r^4 + 217174806\ell^{10}r^6 \right. \right. \\
 &\quad \left. \left. + 98887425\ell^8r^8 + 21549240\ell^6r^{10} + 2947095\ell^4r^{12} + 650010\ell^2r^{14} + 65001r^{16} \right) \right] \\
 h_{3;4} &= \left[\frac{\ell^2}{1270080} \left(22260\ell^{12} + 183645\ell^{10}r^2 + 311661\ell^8r^4 + 260694\ell^6r^6 + 123066\ell^4r^8 + 31869\ell^2r^{10} + 3541r^{12} \right) \right. \\
 &\quad \left. - \frac{\alpha}{529200} \left(1604820\ell^{12} + 2949765\ell^{10}r^2 + 2921877\ell^8r^4 + 1877358\ell^6r^6 + 787362\ell^4r^8 \right. \right. \\
 &\quad \left. \left. + 195003\ell^2r^{10} + 21667r^{12} \right) \right] \\
 \Omega_{3;4} &= \left[\frac{\ell^2}{2540160} \left(1598455\ell^{14} + 8795055\ell^{12}r^2 + 19004760\ell^{10}r^4 + 20982192\ell^8r^6 + 14139048\ell^6r^8 \right. \right. \\
 &\quad \left. \left. + 6020712\ell^4r^{10} + 1505178\ell^2r^{12} + 167242r^{14} \right) \right. \\
 &\quad \left. - \frac{\alpha}{635040} \left(2196535\ell^{14} + 11355375\ell^{12}r^2 + 22425240\ell^{10}r^4 + 22910832\ell^8r^6 + 14668248\ell^6r^8 \right. \right. \\
 &\quad \left. \left. + 6079512\ell^4r^{10} + 1505178\ell^2r^{12} + 167242r^{14} \right) \right] \\
 \Pi_{3;5} &= \left[\frac{\ell^2}{853493760} \left(1128452101\ell^{18} + 10678880150\ell^{16}r^2 + 41756607180\ell^{14}r^4 + 88872056182\ell^{12}r^6 \right. \right. \\
 &\quad \left. \left. + 115794392980\ell^{10}r^8 + 98320298706\ell^8r^{10} + 55586393870\ell^6r^{12} + 20395866890\ell^4r^{14} \right. \right. \\
 &\quad \left. \left. + 4416801537\ell^2r^{16} + 428716940r^{18} \right) \right. \\
 &\quad \left. - \frac{\alpha}{152562009600} \left(928722217423\ell^{18} + 8819401225090\ell^{16}r^2 + 33780454205940\ell^{14}r^4 \right. \right. \\
 &\quad \left. \left. + 71249955470946\ell^{12}r^6 + 92323311236940\ell^{10}r^8 + 78281035435638\ell^8r^{10} + 44381025035130\ell^6r^{12} \right. \right. \\
 &\quad \left. \left. + 16375305364670\ell^4r^{14} + 3570276679411\ell^2r^{16} + 349126822500r^{18} \right) \right]
 \end{aligned}$$

n=5 results

$$f_{5;2} = \frac{20\ell^6 + 105\ell^4r^2 + 42\ell^2r^4 + 7r^6}{50}$$

$$\begin{aligned}
 \Omega_{5;2} &= \frac{20\ell^6 + 15\ell^4 r^2 + 6\ell^2 r^4 + r^6}{60} \\
 \Pi_{5;3} &= \frac{152581\ell^{10} + 767826\ell^8 r^2 + 1035891\ell^6 r^4 + 773598\ell^4 r^6 + 313026\ell^2 r^8 + 53970r^{10}}{257400} \\
 f_{5;4} &= \left[\frac{\ell^2}{618377760000} \left(295557662400\ell^{20} + 3682611572640\ell^{18}r^2 + 17103932986140\ell^{16}r^4 \right. \right. \\
 &\quad + 36498131558460\ell^{14}r^6 + 47555951613885\ell^{12}r^8 + 41236810544215\ell^{10}r^{10} + 24529389790620\ell^8 r^{12} \\
 &\quad + 10021950443882\ell^6 r^{14} + 2733953554501\ell^4 r^{16} + 455760809811\ell^2 r^{18} + 35065241462r^{20} \Big) \\
 &\quad - \frac{\alpha}{360720360000} \left(5730125681280\ell^{20} + 54710235259680\ell^{18}r^2 + 261420130519548\ell^{16}r^4 \right. \\
 &\quad + 542618701892124\ell^{14}r^6 + 683370729631161\ell^{12}r^8 + 581979609281499\ell^{10}r^{10} + 343653106261036\ell^8 r^{12} \\
 &\quad + 140092289643826\ell^6 r^{14} + 38200555296657\ell^4 r^{16} + 6369004169047\ell^2 r^{18} + 490071416494r^{20} \Big) \Big] \\
 g_{5;4} &= \left[\frac{\ell^2}{618377760000} \left(198795488864\ell^{22} + 2586519623296\ell^{20}r^2 + 11564746522784\ell^{18}r^4 \right. \right. \\
 &\quad + 22932404913236\ell^{16}r^6 + 21539953150144\ell^{14}r^8 + 12564927910535\ell^{12}r^{10} + 4244040931130\ell^{10}r^{12} \\
 &\quad + 677658846865\ell^8 r^{14} + 31598878220\ell^6 r^{16} + 7946956745\ell^4 r^{18} + 1222608730\ell^2 r^{20} + 87329195r^{22} \Big) \\
 &\quad - \frac{\alpha}{360720360000} \left(3517759244608\ell^{22} + 44211086834432\ell^{20}r^2 + 200305583802688\ell^{18}r^4 \right. \\
 &\quad + 363929580852292\ell^{16}r^6 + 319063756957088\ell^{14}r^8 + 180755354654875\ell^{12}r^{10} \\
 &\quad + 61031317677330\ell^{10}r^{12} + 10223405437245\ell^8 r^{14} + 700305076380\ell^6 r^{16} + 175106329125\ell^4 r^{18} \\
 &\quad + 26939435250\ell^2 r^{20} + 1924245375r^{22} \Big) \Big] \\
 h_{5;4} &= \left[\frac{\ell^2}{123675552000} \left(769728960\ell^{18} + 9390693312\ell^{16}r^2 + 20983798836\ell^{14}r^4 + 25938182556\ell^{12}r^6 \right. \right. \\
 &\quad + 21295887327\ell^{10}r^8 + 12277654675\ell^8 r^{10} + 4978052794\ell^6 r^{12} + 1362335442\ell^4 r^{14} \\
 &\quad + 227055907\ell^2 r^{16} + 17465839r^{18} \Big) \\
 &\quad - \frac{\alpha}{369969600} \left(1593271680\ell^{18} + 3122255136\ell^{16}r^2 + 3848592748\ell^{14}r^4 + 3585574564\ell^{12}r^6 \right. \\
 &\quad + 2592129969\ell^{10}r^8 + 1419459613\ell^8 r^{10} + 565061926\ell^6 r^{12} + 153939630\ell^4 r^{14} \\
 &\quad + 25656605\ell^2 r^{16} + 1973585r^{18} \Big) \Big]
 \end{aligned}$$

$$\begin{aligned}
 \Omega_{5;4} = & \left[\frac{\ell^2}{46378332000} \left(15058082990\ell^{20} + 106176159440\ell^{18}r^2 + 311179210446\ell^{16}r^4 \right. \right. \\
 & + 505325794688\ell^{14}r^6 + 547149789758\ell^{12}r^8 + 420702504651\ell^{10}r^{10} + 234379619045\ell^8r^{12} \\
 & + 93593083298\ell^6r^{14} + 25525386354\ell^4r^{16} + 4254231059\ell^2r^{18} + 327248543r^{20} \Big) \\
 & - \frac{\alpha}{1932430500} \left(19521646094\ell^{20} + 131737647392\ell^{18}r^2 + 353356087188\ell^{16}r^4 \right. \\
 & + 537690140702\ell^{14}r^6 + 562006315442\ell^{12}r^8 + 424922932863\ell^{10}r^{10} + 235083023747\ell^8r^{12} \\
 & + 93647191352\ell^6r^{14} + 25525386354\ell^4r^{16} + 4254231059\ell^2r^{18} + 327248543r^{20} \Big) \Big] \\
 \Pi_{5;5} = & \left[\frac{\ell^2}{453635740958400000} \left(337633104499816268\ell^{26} + 4271077136958547132\ell^{24}r^2 \right. \right. \\
 & + 23112600885726752792\ell^{22}r^4 + 72350175319616674844\ell^{20}r^6 + 151159034180233581556\ell^{18}r^8 \\
 & + 228077768805842399885\ell^{16}r^{10} + 258620332265962810395\ell^{14}r^{12} + 223871564194330948248\ell^{12}r^{14} \\
 & + 147920037831416512914\ell^{10}r^{16} + 73557029936994344181\ell^8r^{18} + 26698941169899723207\ell^6r^{20} \\
 & + 6684077146010747418\ell^4r^{22} + 1032668691070620996\ell^2r^{24} + 74253000574956420r^{26} \Big) \\
 & - \frac{\alpha}{4498554431170800000} \left(88843628024978738912\ell^{26} + 1122507865343630559388\ell^{24}r^2 \right. \\
 & + 6023372115583107270188\ell^{22}r^4 + 18720948444225766469276\ell^{20}r^6 \\
 & + 38914019934303990489424\ell^{18}r^8 + 58601438194884845712305\ell^{16}r^{10} \\
 & + 66469446985987754507715\ell^{14}r^{12} + 57615205604166626638152\ell^{12}r^{14} \\
 & + 38129098662142936918386\ell^{10}r^{16} + 18989139612729591541089\ell^8r^{18} \\
 & + 6901386814262466690003\ell^6r^{20} + 1729600335134757290322\ell^4r^{22} \\
 & + 267442276275622329084\ell^2r^{24} + 19242295861536333180r^{26} \Big) \Big]
 \end{aligned}$$

n=7 results

$$\begin{aligned}
 f_{7;2} &= \frac{70\ell^8 + 504\ell^6r^2 + 252\ell^4r^4 + 72\ell^2r^6 + 9r^8}{245} \\
 \Omega_{7;2} &= \frac{70\ell^8 + 56\ell^6r^2 + 28\ell^4r^4 + 8\ell^2r^6 + r^8}{280}
 \end{aligned}$$

$$\begin{aligned}
 \Pi_{7;3} &= \frac{1}{62375040} \left(28301260\ell^{14} + 181648328\ell^{12}r^2 + 342224260\ell^{10}r^4 + 415360056\ell^8r^6 + 331552839\ell^6r^8 \right. \\
 &\quad \left. + 168013320\ell^4r^{10} + 48974940\ell^2r^{12} + 6252120r^{14} \right) \\
 f_{7;4} &= \left[\frac{\ell^2}{371502619488000} \left(96780316472840\ell^{26} + 1589105610832740\ell^{24}r^2 \right. \right. \\
 &\quad + 9521410725967140\ell^{22}r^4 + 25965374820716045\ell^{20}r^6 + 45556296988624565\ell^{18}r^8 \\
 &\quad + 56539135502871164\ell^{16}r^{10} + 51618427799679364\ell^{14}r^{12} + 35302266172890626\ell^{12}r^{14} \\
 &\quad + 18211924445078266\ell^{10}r^{16} + 7062641146585640\ell^8r^{18} + 2018090265782824\ell^6r^{20} \\
 &\quad \left. + 403651927726529\ell^4r^{22} + 50460227131621\ell^2r^{24} + 2968444009876r^{26} \right) \\
 &\quad - \frac{\alpha}{835880893848000} \left(16866537231824280\ell^{26} + 232049118692571180\ell^{24}r^2 \right. \\
 &\quad + 1414110709187636460\ell^{22}r^4 + 3716883650942351175\ell^{20}r^6 + 6304111318629442175\ell^{18}r^8 \\
 &\quad + 7700650359344929844\ell^{16}r^{10} + 6986591021140767244\ell^{14}r^{12} + 4767825442882086806\ell^{12}r^{14} \\
 &\quad + 2458034340313267006\ell^{10}r^{16} + 953081702075970200\ell^8r^{18} + 272332684743145624\ell^6r^{20} \\
 &\quad \left. + 54472333166578679\ell^4r^{22} + 6809680934566771\ell^2r^{24} + 400602893601676r^{26} \right) \Big] \\
 g_{7;4} &= \left[\frac{\ell^2}{371502619488000} \left(63735166423850\ell^{28} + 1080779423604460\ell^{26}r^2 \right. \right. \\
 &\quad + 6104455301649630\ell^{24}r^4 + 15807036948862760\ell^{22}r^6 + 21693866896535335\ell^{20}r^8 \\
 &\quad + 20860826548147390\ell^{18}r^{10} + 13769626720091151\ell^{16}r^{12} + 5953776409345128\ell^{14}r^{14} \\
 &\quad + 1535551141626138\ell^{12}r^{16} + 193867029381396\ell^{10}r^{18} + 10157293967310\ell^8r^{20} \\
 &\quad \left. + 2709965597136\ell^6r^{22} + 508118549463\ell^4r^{24} + 59778652878\ell^2r^{26} + 3321036271r^{28} \right) \\
 &\quad - \frac{\alpha}{835880893848000} \left(10750577761041750\ell^{28} + 178690917717167220\ell^{26}r^2 \right. \\
 &\quad + 1006526355431407410\ell^{24}r^4 + 2361158122174494840\ell^{22}r^6 + 3036925736949961845\ell^{20}r^8 \\
 &\quad + 2850729294425536090\ell^{18}r^{10} + 1868492713518324621\ell^{16}r^{12} + 807995702469461688\ell^{14}r^{14} \\
 &\quad + 209605974589864398\ell^{12}r^{16} + 27209163001515516\ell^{10}r^{18} + 1738728305273610\ell^8r^{20} \\
 &\quad \left. + 463697435964336\ell^6r^{22} + 86943269243313\ell^4r^{24} + 10228619910978\ell^2r^{26} + 568256661721r^{28} \right) \Big]
 \end{aligned}$$

$$\begin{aligned}
 h_{7;4} = & \left[\frac{\ell^2}{53071802784000} \left(153269396280\ell^{24} + 2477855239860\ell^{22}r^2 \right. \right. \\
 & + 6500468085540\ell^{20}r^4 + 9976953706005\ell^{18}r^6 + 10884488013685\ell^{16}r^8 + 9029812070708\ell^{14}r^{10} \\
 & + 5829937982596\ell^{12}r^{12} + 2930312879858\ell^{10}r^{14} + 1128813864010\ell^8r^{16} + 322614952040\ell^6r^{18} \\
 & \left. \left. + 64522990408\ell^4r^{20} + 8065373801\ell^2r^{22} + 474433753r^{24} \right) \right. \\
 & - \frac{\alpha}{17058793752000} \left(86486350630680\ell^{24} + 182245183220100\ell^{22}r^2 + 260482463266740\ell^{20}r^4 \right. \\
 & + 301178906390505\ell^{18}r^6 + 289138698210505\ell^{16}r^8 + 227450874603524\ell^{14}r^{10} \\
 & + 143878222471828\ell^{12}r^{12} + 71803208222954\ell^{10}r^{14} + 27603213721570\ell^8r^{16} + 7886010815720\ell^6r^{18} \\
 & \left. \left. + 1577202163144\ell^4r^{20} + 197150270393\ell^2r^{22} + 11597074729r^{24} \right) \right] \\
 \Omega_{7;4} = & \left[\frac{\ell^2}{25474465336320} \left(4684576834175\ell^{26} + 39868495607983\ell^{24}r^2 + 144325351642808\ell^{22}r^4 \right. \right. \\
 & + 301265955958360\ell^{20}r^6 + 438282129396128\ell^{18}r^8 + 473620370087280\ell^{16}r^{10} \\
 & + 391916979706512\ell^{14}r^{12} + 252029223792528\ell^{12}r^{14} + 126161196878808\ell^{10}r^{16} \\
 & + 48499288297880\ell^8r^{18} + 13856939513680\ell^6r^{20} + 2771387902736\ell^4r^{22} \\
 & \left. \left. + 346423487842\ell^2r^{24} + 20377852226r^{26} \right) \right. \\
 & - \frac{\alpha}{2122872111360} \left(30060965769595\ell^{26} + 246684097210475\ell^{24}r^2 + 809950806744472\ell^{22}r^4 \right. \\
 & + 1586524546896824\ell^{20}r^6 + 2238845007906592\ell^{18}r^8 + 2387640938645040\ell^{16}r^{10} \\
 & + 1965266521267920\ell^{14}r^{12} + 1261282443509712\ell^{12}r^{14} + 630948024962424\ell^{10}r^{16} \\
 & + 242504796816952\ell^8r^{18} + 69284697568400\ell^6r^{20} + 13856939513680\ell^4r^{22} + 1732117439210\ell^2r^{24} \\
 & \left. \left. + 101889261130r^{26} \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 \Pi_{7;5} = & \left[\frac{\ell^2}{21848348422059134976000} \left(8982162145045598581865\ell^{34} + 141066092907373532097690\ell^{32}r^2 \right. \right. \\
 & + 957479816306705256586845\ell^{30}r^4 + 3856281625419549220592040\ell^{28}r^6 \\
 & + 10789673943068638679272455\ell^{26}r^8 + 22792336247876448363452550\ell^{24}r^{10} \\
 & + 37918308373546561272000663\ell^{22}r^{12} + 50757619733081920144022472\ell^{20}r^{14} \\
 & + 55227488952051138771625005\ell^{18}r^{16} + 48974717480356234743070466\ell^{16}r^{18} \\
 & + 35284523399336392108933005\ell^{14}r^{20} + 20477493617041202568028488\ell^{12}r^{22} \\
 & + 9433307528652764598206937\ell^{10}r^{24} + 3370660892553798052350570\ell^8r^{26} \\
 & + 900812314937847677993001\ell^6r^{28} + 169475537803320197519400\ell^4r^{30} \\
 & \left. + 20022251781128501589300\ell^2r^{32} + 1117614384817280519400r^{34} \right) \\
 & - \frac{\alpha}{795644021703320165376000} \left(21464560511021140995490385\ell^{34} \right. \\
 & + 336419204700137912471998890\ell^{32}r^2 + 2265714121070327795119738725\ell^{30}r^4 \\
 & + 9045794851073754077922660840\ell^{28}r^6 + 25157542057967397600077180895\ell^{26}r^8 \\
 & + 53007966747953063314567065590\ell^{24}r^{10} + 88148563702087347039590470383\ell^{22}r^{12} \\
 & + 118053515649034646376315606792\ell^{20}r^{14} + 128548831954201729835945339445\ell^{18}r^{16} \\
 & + 114084293382704825962729579346\ell^{16}r^{18} + 82251607219034803579127543445\ell^{14}r^{20} \\
 & + 47764004511303199947728571528\ell^{12}r^{22} + 22014741352249962854054900097\ell^{10}r^{24} \\
 & + 7869719713738135878103963770\ell^8r^{26} + 2104015408041561787435884881\ell^6r^{28} \\
 & + 395980508360006822009399400\ell^4r^{30} + 46797072097258916197449300\ell^2r^{32} \\
 & \left. + 2612928661974905984879400r^{34} \right) \Big]
 \end{aligned}$$

n=9 results

$$\begin{aligned}
 f_{9;2} &= \frac{252\ell^{10} + 2310\ell^8r^2 + 1320\ell^6r^4 + 495\ell^4r^6 + 110\ell^2r^8 + 11r^{10}}{1134} \\
 \Omega_{9;2} &= \frac{252\ell^{10} + 210\ell^8r^2 + 120\ell^6r^4 + 45\ell^4r^6 + 10\ell^2r^8 + r^{10}}{1260}
 \end{aligned}$$

$$\begin{aligned}
 \Pi_{9;3} &= \frac{1}{1025589600} \left(356786123\ell^{18} + 2750576510\ell^{16}r^2 + 6313029495\ell^{14}r^4 + 10221019320\ell^{12}r^6 \right. \\
 &\quad + 11878432160\ell^{10}r^8 + 9822117536\ell^8r^{10} + 5636851220\ell^6r^{12} + 2135373240\ell^4r^{14} + 480720240\ell^2r^{16} \\
 &\quad \left. + 48768720r^{18} \right) \\
 f_{9;4} &= \left[\frac{\ell^2}{859498662576153600} \left(133340623834666248\ell^{32} + 2720901281792621928\ell^{30}r^2 \right. \right. \\
 &\quad + 19770363913882326030\ell^{28}r^4 + 63513374918908684770\ell^{26}r^6 + 135934002405968654888\ell^{24}r^8 \\
 &\quad + 213952704807502670100\ell^{22}r^{10} + 257756682509308369842\ell^{20}r^{12} + 242609187466772524790\ell^{18}r^{14} \\
 &\quad + 180579919814518993740\ell^{16}r^{16} + 107062480455455025288\ell^{14}r^{18} + 50684524900929500343\ell^{12}r^{20} \\
 &\quad + 19068904767424154265\ell^{10}r^{22} + 5608737609387866100\ell^8r^{24} + 1246433118183419250\ell^6r^{26} \\
 &\quad + 196811905684882635\ell^4r^{28} + 19681794632595465\ell^2r^{30} + 937254465799230r^{32} \left. \right) \\
 &\quad - \frac{\alpha}{2363621322084422400} \left(51858708461340364584\ell^{32} + 919343053698617972424\ell^{30}r^2 \right. \\
 &\quad + 6755425175241524371590\ell^{28}r^4 + 20763976427014460249610\ell^{26}r^6 \\
 &\quad + 42888828387704309354344\ell^{24}r^8 + 66469172496452595919140\ell^{22}r^{10} \\
 &\quad + 79612560563631174610986\ell^{20}r^{12} + 74782307435370609291070\ell^{18}r^{14} \\
 &\quad + 55625640941623976375420\ell^{16}r^{16} + 32972881325499331952424\ell^{14}r^{18} \\
 &\quad + 15608925102093402787239\ell^{12}r^{20} + 5872467180388490650745\ell^{10}r^{22} \\
 &\quad + 1727276938801702287300\ell^8r^{24} + 383856087397493107250\ell^6r^{26} \\
 &\quad \left. + 60611238678141544155\ell^4r^{28} + 6061339452062773545\ell^2r^{30} + 288644544654444590r^{32} \right) \left. \right]
 \end{aligned}$$

$$\begin{aligned}
 g_{9;4} = & \left[\frac{\ell^2}{859498662576153600} \left(84495843563523392\ell^{34} + 1768012312961670776\ell^{32}r^2 \right. \right. \\
 & + 11936492827687331904\ell^{30}r^4 + 37242538451355126890\ell^{28}r^6 + 65138644777217991980\ell^{26}r^8 \\
 & + 85422451772928300030\ell^{24}r^{10} + 82904154261107854584\ell^{22}r^{12} + 58562593998521246634\ell^{20}r^{14} \\
 & + 29286912971461691700\ell^{18}r^{16} + 9866747321987487030\ell^{16}r^{18} + 2041707149885182032\ell^{14}r^{20} \\
 & + 219306855955192929\ell^{12}r^{22} + 14460723267227622\ell^{10}r^{24} + 4017026312889975\ell^8r^{26} \\
 & + 845689750082100\ell^6r^{28} + 126853462512315\ell^4r^{30} + 12081282144030\ell^2r^{32} + 549149188365r^{34} \Big) \\
 & - \frac{\alpha}{787873774028140800} \left(10942174841871425152\ell^{34} + 225620421791864115016\ell^{32}r^2 \right. \\
 & + 1510935255902133245184\ell^{30}r^4 + 4208312886338787616390\ell^{28}r^6 \\
 & + 6894619427482858451380\ell^{26}r^8 + 8848332307639603148370\ell^{24}r^{10} \\
 & + 8534331180859836820104\ell^{22}r^{12} + 6020530420354129493814\ell^{20}r^{14} \\
 & + 3011536544232528981900\ell^{18}r^{16} + 1015910734943704195530\ell^{16}r^{18} \\
 & + 211151074883282761392\ell^{14}r^{20} + 23184156465480324699\ell^{12}r^{22} \\
 & + 1720346092586126082\ell^{10}r^{24} + 477878417772403725\ell^8r^{26} + 100605982688927100\ell^6r^{28} \\
 & \left. \left. + 15090897403339065\ell^4r^{30} + 1437228324127530\ell^2r^{32} + 65328560187615r^{34} \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 h_{9;4} = & \left[\frac{\ell^2}{95499851397350400} \left(149725307211480\ell^{30} + 3015895473831240\ell^{28}r^2 + 8797952237444370\ell^{26}r^4 \right. \right. \\
 & + 15541844723695290\ell^{24}r^6 + 20300574058072308\ell^{22}r^8 + 21034536810890148\ell^{20}r^{10} \\
 & + 17783794333160310\ell^{18}r^{12} + 12381092092392030\ell^{16}r^{14} + 7089196246488360\ell^{14}r^{16} \\
 & + 3310396957607208\ell^{12}r^{18} + 1241596238094513\ell^{10}r^{20} + 365184210262725\ell^8r^{22} \\
 & + 81152046725050\ell^6r^{24} + 12813481061850\ell^4r^{26} + 1281348106185\ell^2r^{28} + 61016576485r^{30} \Big) \\
 & - \frac{\alpha}{29180510149190400} \left(162655860396589560\ell^{30} + 362008206104224680\ell^{28}r^2 \right. \\
 & + 572645003750910090\ell^{26}r^4 + 763224818178352530\ell^{24}r^6 + 877801432409857956\ell^{22}r^8 \\
 & + 862035819332396436\ell^{20}r^{10} + 713514589774183470\ell^{18}r^{12} + 492832261815045110\ell^{16}r^{14} \\
 & + 281416001757949320\ell^{14}r^{16} + 131301969045434376\ell^{12}r^{18} + 49236215518455161\ell^{10}r^{20} \\
 & + 14481164174921325\ell^8r^{22} + 3218036483315850\ell^6r^{24} + 508111023681450\ell^4r^{26} \\
 & \left. \left. + 50811102368145\ell^2r^{28} + 2419576303245r^{30} \right) \right] \\
 \Omega_{9;4} = & \left[\frac{\ell^2}{59687407123344000} \left(6598585567818972\ell^{32} + 64940197030107672\ell^{30}r^2 + 273777806813998500\ell^{28}r^4 \right. \right. \\
 & + 681865813132481250\ell^{26}r^6 + 1221884704722504550\ell^{24}r^8 + 1674173405090497584\ell^{22}r^{10} \\
 & + 1807245829726785084\ell^{20}r^{12} + 1565834578187766250\ell^{18}r^{14} + 1101647489722639650\ell^{16}r^{16} \\
 & + 632417867298394200\ell^{14}r^{18} + 295268817053306040\ell^{12}r^{20} + 110708213943786165\ell^{10}r^{22} \\
 & + 32561239395231225\ell^8r^{24} + 7235830976718050\ell^6r^{26} + 1142499627902850\ell^4r^{28} \\
 & \left. \left. + 114249962790285\ell^2r^{30} + 5440474418585r^{32} \right) \right. \\
 & - \frac{\alpha}{532923277887000} \left(8546242063804172\ell^{32} + 81523891654804872\ell^{30}r^2 + 307318194150454500\ell^{28}r^4 \right. \\
 & + 715769933670769250\ell^{26}r^6 + 1245320391503010550\ell^{24}r^8 + 1686136326474110784\ell^{22}r^{10} \\
 & + 1811885990514247284\ell^{20}r^{12} + 1567209651830709250\ell^{18}r^{14} + 1101953061643293650\ell^{16}r^{16} \\
 & + 632466115496392200\ell^{14}r^{18} + 295273641873105840\ell^{12}r^{20} + 110708443697109965\ell^{10}r^{22} \\
 & + 32561239395231225\ell^8r^{24} + 7235830976718050\ell^6r^{26} + 1142499627902850\ell^4r^{28} \\
 & \left. \left. + 114249962790285\ell^2r^{30} + 5440474418585r^{32} \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 \Pi_{9;5} = & \left[\frac{\ell^2}{12928610477426464084962297600000} \left(2963660262582405195600812804780\ell^{42} \right. \right. \\
 & + 54960718928326817581432660068800\ell^{40}r^2 + 438680016278348277072828543787740\ell^{38}r^4 \\
 & + 2095459575089784573411052833767630\ell^{36}r^6 + 7126896792786249923435515161591030\ell^{34}r^8 \\
 & + 18828704309439914264745324330571272\ell^{32}r^{10} + 40318687255770901091487447290055224\ell^{30}r^{12} \\
 & + 71564603098088232189439094493589402\ell^{28}r^{14} + 106606113471306396324075091246911420\ell^{26}r^{16} \\
 & + 134129087251912145628690383503063720\ell^{24}r^{18} + 142873004760249240821838450887004074\ell^{22}r^{20} \\
 & + 128775176906725704442407617007965163\ell^{20}r^{22} + 97921202768932011707091446069275799\ell^{18}r^{24} \\
 & + 62477592200815789501550742105430020\ell^{16}r^{26} + 33171546764185616749586054262623220\ell^{14}r^{28} \\
 & + 14480110437447756571401543030317327\ell^{12}r^{30} + 5107957999464578930096857586903749\ell^{10}r^{32} \\
 & + 1419986713892478010760754706555122\ell^8r^{34} + 299476749663332693852942194632980\ell^6r^{36} \\
 & + 45035990011802132234460421933080\ell^4r^{38} + 4302621620873952400266415533360\ell^2r^{40} \\
 & \left. + 196278755076119819922240261840r^{42} \right) \\
 & - \frac{\alpha}{6691827369268356151636800000} \left(188034260736405759920667016480\ell^{42} \right. \\
 & + 3478119834944534785677233867800\ell^{40}r^2 + 27521476972944344576431164066840\ell^{38}r^4 \\
 & + 129974087795678824227161887469830\ell^{36}r^6 + 438663601013588577076579436614230\ell^{34}r^8 \\
 & + 1154908376153092917794909859416952\ell^{32}r^{10} + 2470488821740365277752497363025784\ell^{30}r^{12} \\
 & + 4385163888580609561595948244197282\ell^{28}r^{14} + 6535069763202647770491146331262470\ell^{26}r^{16} \\
 & + 8226462646319309998709010636408020\ell^{24}r^{18} + 8767061720853343920255745069666384\ell^{22}r^{20} \\
 & + 7905447916801781766415703529158883\ell^{20}r^{22} + 6013636958064400713013562652664609\ell^{18}r^{24} \\
 & + 3838221370012324936903467456571320\ell^{16}r^{26} + 2038449297489327713347987874728770\ell^{14}r^{28} \\
 & + 890063930280989733044566047178207\ell^{12}r^{30} + 314051732713817341652961541727309\ell^{10}r^{32} \\
 & + 87324111529482417321012534860802\ell^8r^{34} + 18420525186923716655392197784180\ell^6r^{36} \\
 & + 2770650184390283183984243528280\ell^4r^{38} + 264748247119254298796127491760\ell^2r^{40} \\
 & \left. + 12079435074425924874820591440r^{42} \right) \Big]
 \end{aligned}$$

Appendix C

Conserved Charges and Thermodynamic Quantities

In this Appendix, I catalog the perturbative thermodynamic charges and potentials entering the first law of thermodynamics in space-time dimension $D = n + 2$ for $n = 3, 5, 7, 9$. These were calculated from equations (2.19). Since boson stars are horizonless, they have zero temperature and the first law reads $dM = \omega dJ$.

n=3

$$M_3 = \frac{5\pi\ell^2\epsilon^2}{24} + \frac{77951\pi\ell^4\epsilon^4}{508032(\ell^2 - 4\alpha)} + \mathcal{O}(\epsilon^6)$$
$$J_3 = \frac{\pi\ell^3\epsilon^2}{24} + \frac{83621\pi\ell^5\epsilon^4}{2540160(\ell^2 - 4\alpha)} + \mathcal{O}(\epsilon^6)$$

$$\ell\omega_3 = 5 - \frac{15\ell^2\epsilon^2}{28(\ell^2 - 4\alpha)} - \frac{\ell^4(3211271921\ell^2 - 14548856100\alpha)\epsilon^4}{5085400320(\ell^2 - 4\alpha)^3} + \mathcal{O}(\epsilon^6)$$

n=5

$$M_5 = \frac{7\pi^2\ell^4\epsilon^2}{160} + \frac{314018183\pi^2\ell^6\epsilon^4}{17667936000(\ell^2 - 24\alpha)} + \mathcal{O}(\epsilon^6)$$

$$J_5 = \frac{\pi^2\ell^5\epsilon^2}{160} + \frac{327248543\pi^2\ell^7\epsilon^4}{12367552000(\ell^2 - 24\alpha)} + \mathcal{O}(\epsilon^6)$$

$$\ell\omega_5 = 7 - \frac{514\ell^2\epsilon^2}{2145(\ell^2 - 24\alpha)} - \frac{\ell^4(1023791506512739\ell^2 - 26707161768137988\alpha)\epsilon^4}{5355421941870000(\ell^2 - 24\alpha)^3} + \mathcal{O}(\epsilon^6)$$

n=7

$$M_7 = \frac{3\pi^3\ell^6\epsilon^2}{560} + \frac{1100829437\pi^3\ell^8\epsilon^4}{943498716160(\ell^2 - 60\alpha)} + \mathcal{O}(\epsilon^6)$$

$$J_7 = \frac{\pi^3\ell^7\epsilon^2}{1680} + \frac{10188926113\pi^3\ell^9\epsilon^4}{76423396008960(\ell^2 - 60\alpha)} + \mathcal{O}(\epsilon^6)$$

$$\ell\omega_7 = 9 - \frac{4135\ell^2\epsilon^2}{37128(\ell^2 - 60\alpha)} - \frac{\ell^4(1188837899283033191\ell^2 - 76263642821315268516\alpha)\epsilon^4}{20666078485800523776(\ell^2 - 60\alpha)^3} + \mathcal{O}(\epsilon^6)$$

n=9

$$M_9 = \frac{11\pi^4\ell^8\epsilon^2}{24192} + \frac{1065755263141\pi^4\ell^{10}\epsilon^4}{20836331213967360(\ell^2 - 112\alpha)} + \mathcal{O}(\epsilon^6)$$

$$J_9 = \frac{\pi^4\ell^9\epsilon^2}{24192} + \frac{1088094883717\pi^4\ell^{11}\epsilon^4}{229199643353640960(\ell^2 - 112\alpha)} + \mathcal{O}(\epsilon^6)$$

$$\ell\omega_9 = 11 - \frac{754\ell^2\epsilon^2}{14535(\ell^2 - 112\alpha)} - \frac{\ell^4(971718029741243000453\ell^2 - 115487747450683234769436\alpha)\epsilon^4}{58241769402253879260000(\ell^2 - 112\alpha)^3} + \mathcal{O}(\epsilon^6)$$

Appendix D

Kretschmann Scalar and Critical q'_h Values

In this Appendix, I catalog the “simplified” Kretschmann scalar in terms of $q'_h(0)$ and the corresponding critical $q_h'^*$ for each $D = n + 2$ dimension for $n = 3, 5, 7, 9$

n=3

$$\begin{aligned} K_3 = & \frac{1}{\alpha^2 \ell^4 (144\alpha^2 - 576\alpha^2 q'_h(0)^2 + 9\ell^4 - 72\alpha\ell^2 + 48\alpha\ell^2 q_0^2)} \\ & \times \left[\ell^2 \left(- (144\alpha^2 - 576\alpha^2 q'_h(0)^2 + 9\ell^4 - 72\alpha\ell^2 + 48\alpha\ell^2 q_0^2)^{3/2} \right) \right. \\ & - 2\ell^2 (48\alpha^2 + 3\ell^4 - 24\alpha\ell^2 + 8\alpha\ell^2 q_0^2) \sqrt{144\alpha^2 - 576\alpha^2 q'_h(0)^2 + 9\ell^4 - 72\alpha\ell^2 + 48\alpha\ell^2 q_0^2} \\ & + (24\alpha^2 + 4\ell^4 - 12\alpha\ell^2 + 8\alpha\ell^2 q_0^2) (144\alpha^2 - 576\alpha^2 q'_h(0)^2 + 9\ell^4 - 72\alpha\ell^2 + 48\alpha\ell^2 q_0^2) \\ & \left. + (48\alpha^2 + 3\ell^4 - 24\alpha\ell^2 + 8\alpha\ell^2 q_0^2)^2 \right] \end{aligned} \quad (D.1)$$

$$q_h'^* = - \frac{\sqrt{144\alpha^2 + 9\ell^4 - 72\alpha\ell^2 + 48\alpha\ell^2 q_0^2}}{24\alpha}$$

n=5

$$\begin{aligned}
 K_5 = & \frac{1}{48\alpha^2\ell^4 (14400\alpha^2 - 9216\alpha^2 q'_h(0)^2 + 25\ell^4 - 1200\alpha\ell^2 + 480\alpha\ell^2 q_0^2)} \\
 & \times \left[-2\ell^2 (14400\alpha^2 - 9216\alpha^2 q'_h(0)^2 + 25\ell^4 - 1200\alpha\ell^2 + 480\alpha\ell^2 q_0^2)^{3/2} \right. \\
 & - 4\ell^2 (2880\alpha^2 + 5\ell^4 - 240\alpha\ell^2 + 48\alpha\ell^2 q_0^2) \sqrt{14400\alpha^2 - 9216\alpha^2 q'_h(0)^2 + 25\ell^4 - 1200\alpha\ell^2 + 480\alpha\ell^2 q_0^2} \\
 & - (14400\alpha^2 - 9216\alpha^2 q'_h(0)^2 + 25\ell^4 - 1200\alpha\ell^2 + 480\alpha\ell^2 q_0^2)^2 \\
 & + (17280\alpha^2 + 37\ell^4 - 1440\alpha\ell^2 + 576\alpha\ell^2 q_0^2) (14400\alpha^2 - 9216\alpha^2 q'_h(0)^2 + 25\ell^4 - 1200\alpha\ell^2 + 480\alpha\ell^2 q_0^2) \\
 & \left. + 2 (2880\alpha^2 + 5\ell^4 - 240\alpha\ell^2 + 48\alpha\ell^2 q_0^2)^2 \right] \tag{D.2}
 \end{aligned}$$

$$q_h^* = - \frac{\sqrt{14400\alpha^2 + 25\ell^4 - 1200\alpha\ell^2 + 480\alpha\ell^2 q_0^2}}{96\alpha}$$

n=7

$$\begin{aligned}
 K_7 = & \frac{1}{225\alpha^2\ell^4 (176400\alpha^2 - 43200\alpha^2 q'_h(0)^2 + 49\ell^4 - 5880\alpha\ell^2 + 1680\alpha\ell^2 q_0^2)} \\
 & \times \left[-2\ell^2 (176400\alpha^2 - 43200\alpha^2 q'_h(0)^2 + 49\ell^4 - 5880\alpha\ell^2 + 1680\alpha\ell^2 q_0^2)^{3/2} \right. \\
 & - \left(4\ell^2 (25200\alpha^2 + 7\ell^4 - 840\alpha\ell^2 + 120\alpha\ell^2 q_0^2) \right. \\
 & \times \sqrt{176400\alpha^2 - 43200\alpha^2 q'_h(0)^2 + 49\ell^4 - 5880\alpha\ell^2 + 1680\alpha\ell^2 q_0^2} \Big) \\
 & - 2 (176400\alpha^2 - 43200\alpha^2 q'_h(0)^2 + 49\ell^4 - 5880\alpha\ell^2 + 1680\alpha\ell^2 q_0^2)^2 \\
 & + \left((378000\alpha^2 + 114\ell^4 - 12600\alpha\ell^2 + 3600\alpha\ell^2 q_0^2) \right. \\
 & \times (176400\alpha^2 - 43200\alpha^2 q'_h(0)^2 + 49\ell^4 - 5880\alpha\ell^2 + 1680\alpha\ell^2 q_0^2) \Big) \\
 & \left. + 2 (25200\alpha^2 + 7\ell^4 - 840\alpha\ell^2 + 120\alpha\ell^2 q_0^2)^2 \right] \tag{D.3}
 \end{aligned}$$

$$q_h'^* = - \frac{\sqrt{529200\alpha^2 + 147\ell^4 - 17640\alpha\ell^2 + 5040\alpha\ell^2 q_0^2}}{360\alpha} \quad (\text{D.4})$$

n=9

$$\begin{aligned} K_9 = & \frac{1}{28224\alpha^2\ell^4 (112896\alpha^2 - 14336\alpha^2 q_h'(0)^2 + 9\ell^4 - 2016\alpha\ell^2 + 448\alpha\ell^2 q_0^2)} \\ & \times \left[-270\ell^2 (112896\alpha^2 - 14336\alpha^2 q_h'(0)^2 + 9\ell^4 - 2016\alpha\ell^2 + 448\alpha\ell^2 q_0^2)^{3/2} \right. \\ & - \left(60\ell^2 (112896\alpha^2 + 9\ell^4 - 2016\alpha\ell^2 + 224\alpha\ell^2 q_0^2) \right. \\ & \times \sqrt{112896\alpha^2 - 14336\alpha^2 q_h'(0)^2 + 9\ell^4 - 2016\alpha\ell^2 + 448\alpha\ell^2 q_0^2} \Big) \\ & - 1215 (112896\alpha^2 - 14336\alpha^2 q_h'(0)^2 + 9\ell^4 - 2016\alpha\ell^2 + 448\alpha\ell^2 q_0^2)^2 \\ & + \left((142248960\alpha^2 + 11835\ell^4 - 2540160\alpha\ell^2 + 564480\alpha\ell^2 q_0^2) \right. \\ & \times (112896\alpha^2 - 14336\alpha^2 q_h'(0)^2 + 9\ell^4 - 2016\alpha\ell^2 + 448\alpha\ell^2 q_0^2) \Big) \\ & \left. + 10 (112896\alpha^2 + 9\ell^4 - 2016\alpha\ell^2 + 224\alpha\ell^2 q_0^2)^2 \right] \quad (\text{D.5}) \end{aligned}$$

$$q_h'^* = - \frac{\sqrt{1580544\alpha^2 + 126\ell^4 - 28224\alpha\ell^2 + 6272\alpha\ell^2 q_0^2}}{448\alpha}$$

Bibliography

- [1] Stephen W. Hawking. Black holes in general relativity. *Commun.Math.Phys.*, 25:152–166, 1972. doi: 10.1007/BF01877517.
- [2] Stefan Hollands, Akihiro Ishibashi, and Robert M. Wald. A Higher dimensional stationary rotating black hole must be axisymmetric. *Commun.Math.Phys.*, 271:699–722, 2007. doi: 10.1007/s00220-007-0216-4.
- [3] Vincent Moncrief and James Isenberg. Symmetries of Higher Dimensional Black Holes. *Class.Quant.Grav.*, 25:195015, 2008. doi: 10.1088/0264-9381/25/19/195015.
- [4] Betti Hartmann, Burkhard Kleihaus, Jutta Kunz, and Meike List. Rotating Boson Stars in 5 Dimensions. *Phys.Rev.*, D82:084022, 2010. doi: 10.1103/PhysRevD.82.084022.
- [5] Oscar J.C. Dias, Gary T. Horowitz, and Jorge E. Santos. Black holes with only one Killing field. *JHEP*, 1107:115, 2011. doi: 10.1007/JHEP07(2011)115.
- [6] Sean Stotyn, Miok Park, Paul McGrath, and Robert B. Mann. Black Holes and Boson Stars with One Killing Field in Arbitrary Odd Dimensions. *Phys.Rev.*, D85:044036, 2012. doi: 10.1103/PhysRevD.85.044036.
- [7] Sean Stotyn and Robert B. Mann. Another Mass Gap in the BTZ Geometry? *J.Phys.*, A45:374025, 2012. doi: 10.1088/1751-8113/45/37/374025.
- [8] D. Lovelock. The Einstein tensor and its generalizations. *J.Math.Phys.*, 12:498–501, 1971. doi: 10.1063/1.1665613.
- [9] David J. Gross and Edward Witten. Superstring modifications of einstein’s equations. *Nucl. Phys. B*, 277:1, 1986.

- [10] Mikhail Ostrogradski. Memoires sur les equations differentielles relatives au probleme des isoperimetres. *Mem. Ac. St. Petersbourg*, VI(385), 1850.
- [11] Tai-jun Chen, Matteo Fasiello, Eugene A. Lim, and Andrew J. Tolley. Higher derivative theories with constraints : exorcising Ostrogradski's ghost. *arXiv:1209.0583*.
- [12] Juan Crisostomo, Ricardo Troncoso, and Jorge Zanelli. Black hole scan. *Phys.Rev.*, D62:084013, 2000. doi: 10.1103/PhysRevD.62.084013.
- [13] David J. Kaup. Klein-Gordon Geon. *Phys.Rev.*, 172:1331–1342, 1968. doi: 10.1103/PhysRev.172.1331.
- [14] Remo Ruffini and Silvano Bonazzola. Systems of selfgravitating particles in general relativity and the concept of an equation of state. *Phys.Rev.*, 187:1767–1783, 1969. doi: 10.1103/PhysRev.187.1767.
- [15] Philippe Jetzer. Boson stars. *Phys.Rept.*, 220:163–227, 1992. doi: 10.1016/0370-1573(92)90123-H.
- [16] Argelia Bernal, Juan Barranco, Daniela Alic, and Carlos Palenzuela. Multi-state Boson Stars. *Phys.Rev.*, D81:044031, 2010. doi: 10.1103/PhysRevD.81.044031.
- [17] Emanuele Berti and Vitor Cardoso. Supermassive black holes or boson stars? Hair counting with gravitational wave detectors. *Int.J.Mod.Phys.*, D15:2209–2216, 2006. doi: 10.1142/S0218271806009637.
- [18] Michael Kesden, Jonathan Gair, and Marc Kamionkowski. Gravitational-wave signature of an inspiral into a supermassive horizonless object. *Phys.Rev.*, D71:044015, 2005. doi: 10.1103/PhysRevD.71.044015.
- [19] Oscar J.C. Dias, Gary T. Horowitz, and Jorge E. Santos. Gravitational Turbulent Instability of Anti-de Sitter Space. *Class.Quant.Grav.*, 29:194002, 2012. doi: 10.1088/0264-9381/29/19/194002.
- [20] Piotr Bizon and Andrzej Rostworowski. On weakly turbulent instability of anti-de Sitter space. *Phys.Rev.Lett.*, 107:031102, 2011. doi: 10.1103/PhysRevLett.107.031102.

- [21] Joanna Jalmuzna, Andrzej Rostworowski, and Piotr Bizon. A Comment on AdS collapse of a scalar field in higher dimensions. *Phys.Rev.*, D84:085021, 2011. doi: 10.1103/PhysRevD.84.085021.
- [22] Maciej Maliborski and Andrzej Rostworowski. Time-periodic solutions in Einstein AdS - massless scalar field system. *Phys.Rev.Lett.*, 111:051102, 2013. doi: 10.1103/PhysRevLett.111.051102.
- [23] Alex Buchel, Steven L. Liebling, and Luis Lehner. Boson Stars in AdS. *Phys.Rev.*, D87:123006, 2013. doi: 10.1103/PhysRevD.87.123006.
- [24] Dumitru Astefanesei and Eugen Radu. Boson stars with negative cosmological constant. *Nucl.Phys.*, B665:594–622, 2003. doi: 10.1016/S0550-3213(03)00482-6.
- [25] Yves Brihaye, Betti Hartmann, and Sardor Tojiev. Stability of charged solitons and formation of boson stars in 5-dimensional Anti-de Sitter space-time. *Class.Quant.Grav.*, 30:115009, 2013. doi: 10.1088/0264-9381/30/11/115009.
- [26] M. Colpi, S.L. Shapiro, and I. Wasserman. Boson Stars: Gravitational Equilibria of Self-interacting Scalar Fields. *Phys.Rev.Lett.*, 57:2485–2488, 1986. doi: 10.1103/PhysRevLett.57.2485.
- [27] Jeong-won Ho, F.C. Khanna, and Chul H. Lee. Boson stars with selfinteracting quantum scalar fields. *arXiv:gr-qc/0207073*.
- [28] Yves Brihaye, Betti Hartmann, and Eugen Radu. Boson stars in SU(2) Yang-Mills-scalar field theories. *Phys.Lett.*, B607:17–26, 2005. doi: 10.1016/j.physletb.2004.12.020.
- [29] Oscar J.C. Dias, Pau Figueras, Shiraz Minwalla, Prahar Mitra, Ricardo Monteiro, et al. Hairy black holes and solitons in global AdS_5 . *JHEP*, 1208:117, 2012. doi: 10.1007/JHEP08(2012)117.
- [30] Gyula Fodor, Peter Forgacs, and Mark Mezei. Boson stars and oscillatons in an inflationary universe. *Phys.Rev.*, D82:044043, 2010. doi: 10.1103/PhysRevD.82.044043.
- [31] Rong-Gen Cai, Jeong-Young Ji, and Kwang-Sup Soh. Hairs on the cosmological horizons. *Phys.Rev.*, D58:024002, 1998. doi: 10.1103/PhysRevD.58.024002.

- [32] Betti Hartmann and Juergen Riedel. Supersymmetric Q-balls and boson stars in $(d+1)$ dimensions. *Phys.Rev.*, D87:044003, 2013. doi: 10.1103/PhysRevD.87.044003.
- [33] Jefferson Bjoraker and Yutaka Hosotani. Stable monopole and dyon solutions in the Einstein-Yang-Mills theory in asymptotically Anti-de Sitter space. *Phys.Rev.Lett.*, 84: 1853, 2000. doi: 10.1103/PhysRevLett.84.1853.
- [34] Sean Stotyn, Melanie Chanona, and Robert B. Mann. Numerical Boson Stars with a Single Killing Vector II: the $D=3$ Case. 2013.
- [35] Sean Stotyn, C. Danielle Leonard, Marius Oltean, Laura J. Henderson, and Robert B. Mann. Numerical Boson Stars with a Single Killing Vector I: the $D \geq 5$ Case. 2013.
- [36] Oscar J.C. Dias, Gary T. Horowitz, Don Marolf, and Jorge E. Santos. On the Nonlinear Stability of Asymptotically Anti-de Sitter Solutions. *Class.Quant.Grav.*, 29:235019, 2012. doi: 10.1088/0264-9381/29/23/235019.
- [37] Yves Brihaye. Charged, rotating black holes in Einstein-Gauss-Bonnet gravity. *arXiv:1108.2779*.
- [38] Yves Brihaye and Eugen Radu. Black objects in the Einstein-Gauss-Bonnet theory with negative cosmological constant and the boundary counterterm method. *JHEP*, 0009:006, 2008.
- [39] David Gottlieb and Steven A. Orszag. *Numerical Analysis of Spectral Methods: Theory and Applications*. Society of Industrial and Applied Mathematics, 1977.
- [40] John L. Friedman, James R. Ipser, and Rafael D. Sorkin. Turning-point method for axisymmetric stability of rotating relativistic stars. *APJ*, 325:722–724, February 1988. doi: 10.1086/166043.
- [41] David G. Boulware and S. Deser. String-generated gravity models. *Phys. Rev. Lett.*, 55: 2656–2660, Dec 1985. doi: 10.1103/PhysRevLett.55.2656.
- [42] Robert C. Myers and Brandon Robinson. Black holes in quasi-topological gravity. *Journal of High Energy Physics*, 2010(8), 2010. doi: 10.1007/JHEP08(2010)067.