Mathematical Aspects of Higgs & Coulomb Branches

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Aiden Suter was the sole author of Chapters 1 and 2. The research conducted in Chapter 2 will be published in a joint work with Andrea Ferrari.

Chapter 3 was co-authored with Ben Webster and will be published as a joint work.

Abstract

This thesis contains results pertaining to different aspects of the $3d$ mirror symmetry between Higgs and Coulomb branches.

In Chapter 2 we verify the 3d A-model Higgs branch conjecture formulated in [BF23] for SQED with $n > 3$ hypermultiplets. The conjecture claims that the associated variety of the boundary VOA for the 3d A-model is isomorphic to the Higgs branch of the physical theory. We demonstrate that the boundary VOA is $L_1(\mathfrak{psl}_{n|n})$ and show that its associated variety is the closure of the minimal nilpotent orbit, verifying the conjecture.

In Chapter 3 we build on the work of [Web19a; Web22] by explicitly constructing a tilting generator for the derived category of coherent sheaves on $T^*\mathrm{Gr}(2,4)$. This variety is the Coulomb branch for a quiver gauge theory and has functions described by a KRLW algebra. We achieve this result by constructing generators for modules over this diagrammatic algebra and identifying the coherent sheaves these correspond to.

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Dedication

Dedicated to the memory of Dale Suter.

Table of Contents

Examining Committee Membership ii

Author's Declaration iii

Statement of Contributions iv

Abstract v

Acknowledgements vi

Dedication vii

1 Introduction 1

1.1 $3d$ mirror symmetry 1

1.2 $3d$ A-model Higgs branch conjecture 2

1.3 Coulomb branch tilting generator 5

2 Associated variety of $L_1(\mathfrak{psl}_{n|n})$ 8

2.1 Introduction 8

2.2 Preliminaries 9

2.2.1 Notation & conventions 9

2.2.2 Vertex algebras 11

2.2.3 Sheets & nilpotent orbits of type A 19
2.3 BRST cohomology & 3d A-model boundary VOA 22
2.4 Singular vectors of $V^1(\mathfrak{psl}_{n|n})$ 25
2.5 Associated variety of $L_1(\mathfrak{psl}_{n|n})$ 29

3 Tilting generator for $T^*\mathrm{Gr}(2,4)$ **32**
3.1 Introduction 32
3.2 Preliminaries 35
3.2.1 $T^*\mathrm{Gr}(n-k,n)$ and its vector bundles 35
3.2.2 Quiver gauge theories and cylindrical KLRW algebras 40
3.2.3 Grassmannians from quiver gauge theories 45
3.3 Vector bundles on $T^*\mathrm{Gr}(2,4)$ 54
3.3.1 General procedure 54
3.3.2 Rank 1 cases 59
3.3.3 Rank 2 cases 60
3.3.4 Summary 68
3.4 Comparison to other constructions 68

References **70**

Chapter 1

Introduction

1.1 $3d$ mirror symmetry

The two chapters comprising the bulk of this thesis both regard mathematical results occurring in the study of $3d$ *mirror symmetry.* Physically, $3d$ mirror symmetry arises as a duality of topological twists of 3d quantum field theories (QFTs) with $\mathcal{N} = 4$ supersymmetry. Such a theory $\mathcal{T}$ admits two distinguished topological twists known as the A-twist, $\mathcal{T}_A$, and the B-twist $\mathcal{T}_B$. These are sometimes also referred to as the $3d$ A-model and $3d$ B-model respectively. The $3d$ mirror symmetry conjecture posits the following:

Conjecture 1.1.1. *Given a 3d $\mathcal{N} = 4$ theory $\mathcal{T}$, there exists a mirror theory $\mathcal{T}^\vee$ such that as fully extended topological quantum field theories (TQFTs) we have*

$$\mathcal{T}_A \cong \mathcal{T}_B^\vee, \qquad\qquad \mathcal{T}_B \cong \mathcal{T}_A^\vee.$$

Such a result is interesting for a number of reasons. For example, this duality can interchange perturbative and non-perturbative data and hence provide new methods for rigorously studying non-perturbative phenomena in QFT.

The manifestation of $3d$ mirror symmetry that has been most intensely studied however is the duality between the objects the functors defining these TQFTs assigns to a 2-sphere. Mathematically this manifests in a duality between symplectic varieties known as the *Higgs branch* and *Coulomb branch*. These varieties are in-fact expected to be Hyperkähler, as has been proved in the case of theories with cotangent matter in [BF24], however we will note consider this additional structure here. Physically, these spaces arise as branches of

the moduli space of vacua for $\mathcal{T}$ and contain information about the algebra of operators for these theories.

Mathematically, these varieties are typically singular and are equipped with resolutions whose theory generalizes that of the Springer resolution $T^*(G/B) \to \mathcal{N}$. It is for this reason that Okounkov said "*Symplectic resolutions are the Lie algebras of the 21st Century*" [Kam22]. To get a feeling for what symplectic duality does, we provide some examples of dual pairs in Figure 1.1.

Symplectic resolution	**Symplectic dual**
$T^*\mathbb{P}^n$	$\mathbb{C}^2/\mathbb{Z}_{n+1}$
$T^*(G/B)$	$T^*(G^\vee/B^\vee)$
hypertoric varieties	Gale dual hypertoric varieties
ADE quiver varieties	affine Grassmanian slices

Figure 1.1: Some examples of symplectic dual pairs. G is an algebraic group and $G^\vee$ denotes the Langlands dual group of G.

The symplectic dual to a variety is conjectured to interchange different algebraic and geometric data; for example one can associate a category $\mathscr{O}$ to a symplectic singularity and symplectic duality interchanges this category with its Koszul dual category $\mathscr{O}^!$ [Web19b]. For a more comprehensive overview of the subject including connections to elliptic stable envelopes and the geometric Langlands correspondence, the interested reader is encouraged to read the survey papers [Kam22; WY23].

1.2 $3d$ A-model Higgs branch conjecture

One promising way to explore $3d$ mirror symmetry is through a conjectured relationship between vertex operator algebras (VOAs) constructed from the boundary data of twisted 3d $\mathcal{N} = 4$ theories and their Higgs and Coulomb branches [CCG18; CG19]. Physically, the boundary VOA arises by fixing holomorphic boundary conditions and restricting the bulk algebra of local operators to the boundary. For the A-twist of a gauge theory with gauge group G, the boundary VOA is given by a gauged $\beta\gamma$ VOA. More precisely, this is the BRST reduction, or semi-infinite cohomology, of the VOA $\beta\gamma^{\otimes n} \otimes bc^{\otimes n}$ where n is the number of symplectic bosons as well as the number of free fermions used to cancel gauge anomalies.

One should think of the $\beta\gamma$ VOA as being the chiral[1] analogue of the Weyl algebra of differential operators and the bc VOA as being the chiral analogue of the exterior algebra of differential forms. Hence, when a sheaf of $\beta\gamma$ VOAs exists on a space X, it is known as the *chiral differential operators* (CDOs) of X. Similarly, the sheaf of $\beta\gamma bc$ VOAs on X is known as the the *chiral de Rham complex* (CDRs) of X.

The A-twist boundary VOA can be obtained as the global sections of the CDOs or CDR complex for a quotient stack N/G, where here we take N to be a G-representation. From this point of view the BRST reduction of the $\beta\gamma^n \otimes bc^n$ VOA corresponds to the symplectic reduction of T^*N by G.

It was posited in [CCG18, §6] that a certain category of "*spectral flow modules*" for the boundary VOA corresponds to D-modules on the loop space $\mathcal{L}(X/G)$ which physically describe line defects of the A-twist theory. Local operators at a point can be recovered by considering endomorphisms of the trivial line defect which gives the BFN Coulomb branch defined in [BFN18]. This reasoning leads to the following conjecture:

Conjecture 1.2.1 ([CCG18])**.** *Let V^A be the boundary VOA for the A-twist of a 3d $\mathcal{N}=4$ gauge theory and let $\mathcal{C}$ denote the subcategory of V^A-mod containing spectral flow modules. Then the algebra of functions on the Coulomb branch is isomorphic to* $\mathrm{Ext}_{\mathcal{C}}(V^A, V^A)$.

A mirror conjecture also states that for appropriate theories, the A-twist boundary VOA also produces the Higgs branch. Given a VOA V, one may construct its *associated variety* X_V by passing to a quotient of its mode algebra [Ara10].

Conjecture 1.2.2 ([BF23])**.** *Let V^A be the boundary VOA for the A-twist of a gauge theory with purely free fermionic boundary degrees of freedom. Then the Higgs branch is isomorphic to X_{V^A}.*

Proving these conjectures is an attractive prospect as it provides a systematic method for constructing both sides of the duality from a single algebraic object. In recent work [Bal+23] Conjecture 1.2.1 was proved in the case that G is abelian and the mirror symmetry was verified at the level of module categories. However many technical difficulties remain in proving this statement for the non-abelian case.

In Chapter 2, we will prove Conjecture 1.2.2 in the case of super quantum electrodynamics (SQED) with $n > 3$ pairs of symplectic bosons and free fermions. Interestingly, the resulting associated variety appears to depend on the choice of anomaly cancellation used; for example see Remark 2.3.1. The proposed boundary VOA for this theory, $L_1(\mathfrak{psl}_{n|n})$,

[1] We use adjective "*chiral*" here to mean the VOA analogue of that object.

is the simple quotient of the universal affine VOA $V^1(\mathfrak{psl}_{n|n})$. In order to mathematically verify this statement, we must check that the associated variety of $L_1(\mathfrak{psl}_{n|n})$ is indeed the correct Higgs branch which was found by physicists to be the type A minimal nilpotent orbit closure $\overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}$.

This claim was proved for the case $n = 2$ in [BF23, §2.2], however their argument does not work for higher rank cases. To study the higher rank cases, we need to know the structure of the maximal submodule of $V^1(\mathfrak{psl}_{n|n})$. This is because the relations imposed by quotienting by the maximal submodule will determine the defining ideal of the associated variety. Understanding the submodule structure of affine Lie superalgebras is a difficult and interesting problem in general, however a number of known facts give us enough to work with to make the problem tractable here.

The first thing to note is that $V_1(\mathfrak{psl}_{n|n})$ contains both $V_1(\mathfrak{sl}_n)$ and $V_{-1}(\mathfrak{sl}_n)$ as subalgebras. It follows that their associated varieties satisfy

$$X_{V_1(\mathfrak{psl}_{n|n})} \subseteq X_{V_1(\mathfrak{sl}_n)} \times X_{V_{-1}(\mathfrak{sl}_n)}.$$

It is known that $X_{V_1(\mathfrak{sl}_n)} = \{\text{pt}\}$ and in [AM19] it was shown for $n > 3$ that $X_{V_{-1}(\mathfrak{sl}_n)}$ is the closure of the sheet closure $\overline{\mathbb{S}_{\min}(\mathfrak{sl}_n)}$ containing the nilpotent orbit closure $\overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}$. It follows that $X_{V_1(\mathfrak{psl}_{n|n})} \subseteq \overline{\mathbb{S}_{\min}(\mathfrak{sl}_n)}$.

The difference between $\overline{\mathbb{S}_{\min}(\mathfrak{sl}_n)}$ and $\overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}$ is the presence of semisimple elements in former. Hence the maximal submodule of $V^1(\mathfrak{psl}_{n|n})$ must contain singular or subsingular vectors which under the image of the quotient map give polynomials that correspond to the vanishing of these semisimple elements. Using this information, we prove the following theorem:

Theorem 1.2.1. *For $n > 3$ the associated variety of the boundary VOA for the A-twist of 3d SQED[n] is the Higgs branch:*

$$X_{V_1(\mathfrak{psl}_{n|n})} \cong \overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}$$

This result additionally proves that $L_1(\mathfrak{psl}_{n|n})$ satisfies what is known as the *quasi-lisse* condition in VOA literature, something that was not previously known. Furthermore this work has revealed new information about the submodule structure of $V^1(\mathfrak{psl}_{n|n})$. In particular we will describe a conjectural description of the its maximal submodule. We leave proof of this conjecture to later work, possibly employing the techniques used in [Feh23].

1.3 Coulomb branch tilting generator

As suggested earlier, the (derived) category of line defects for $\mathcal{T}_A$ is described by the (derived) category of coherent sheaves on the Coulomb branch. In Chapter 3, we study this category for the quiver gauge theory associated to the quiver seen in Figure 1.2 which has Coulomb branch $T^*\mathrm{Gr}(2,4)$. The goal of this is to provide a concrete algebraic description of the derived category of coherent sheaves $D^b(\mathsf{Coh}(T^*\mathrm{Gr}(2,4)))$.

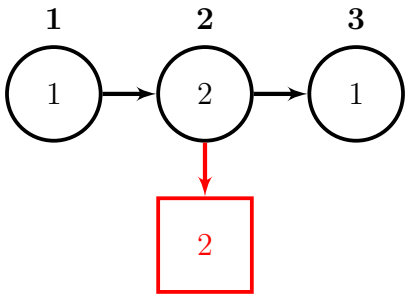


Figure 1.2: Quiver diagram for the quiver describing the theory with Coulomb branch $T^*\mathrm{Gr}(2,4)$.

To achieve this goal we employ techniques for studying non-commutative resolutions of symplectic varieties over fields of characteristic $p > 0$ developed in [BK04]. Given a non-commutative resolution R of a symplectic variety X, it was shown in [Kal08] that there exists a coherent sheaf $\mathcal{E}$ called a *tilting generator* such that $R = \mathrm{End}(\mathcal{E})$ and provides an equivalence of derived categories

$$\begin{aligned} D^b(\mathsf{Coh}(X)) &\to D^b(R\text{−mod}^{\mathrm{fg}}) \\ \mathcal{F} &\mapsto \mathbb{R}\mathrm{Hom}(\mathcal{E}, \mathcal{F}). \end{aligned}$$

In [Web19a] these methods were used to show that any BFN Coulomb branch $\mathfrak{M}$ has a non-commutative resolution A with tilting generator $\mathcal{Z}$ such that for any BFN resolution $\tilde{\mathfrak{M}} \to \mathfrak{M}$ there is a derived equivalence[2]

$$D^b(\mathsf{Coh}(\tilde{\mathfrak{M}})) \cong D^b(A\text{−mod}^{\mathrm{fg}}). \tag{3.1}$$

[2]While the proof of this statement makes use of quantization in characteristic p, the result holds in characteristic 0.

In the sequel [Web22] it was shown that if the Coulomb branch is that of a quiver gauge theory, then A is a *cylindrical KLRW algebra*, a type of diagrammatic algebra generated by string diagrams on a cylinder.

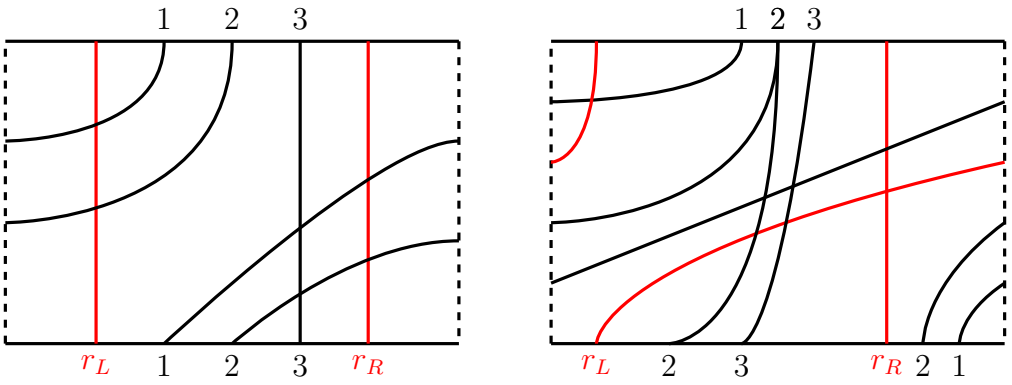


Figure 1.3: Examples of cylindrical KLRW diagrams. The left diagram is an element of the algebra A while the right diagram is an element of an A-module labelled by the idempotent 223221. The colours and labels indicate the quiver data that a strand corresponds to and action of the algebra is given by stacking diagrams from the top.

The aforementioned constructions however are not explicit and so a major motivation of this work was to provide a concrete construction of the tilting generator $\mathcal{Z}$ for $D^b_{\mathrm{coh}}(T^*\mathrm{Gr}(2,4))$. This was achieved by exploiting the geometry of $\mathrm{Gr}(2,4)$, and constructing six modules for the KLRW algebra A corresponding to idempotents labelled by strings of quiver vertices. These modules were identified with coherent sheaves on $T^*\mathrm{Gr}(2,4)$ as summarised in Table 1.4.

KLRW idempotent	$T^*\mathrm{Gr}(2,4)$ **sheaf**
$212^{(2)}32$	$\mathcal{O}$
$2212^{(2)}3$	$\mathcal{L}^{-1}$
221322	$\mathcal{H}\otimes\mathcal{L}$
222312	$\mathcal{H}$
223221	$\mathcal{T}^{\perp}$
221223	$\mathcal{T}$

Figure 1.4: Table showing coherent sheaves on $T^*\mathrm{Gr}(2,4)$ and their corresponding KRLW algebra modules. The idempotent labels indicate the order of the strands appearing in the KLRW diagrams for that module. $\mathcal{T}$ and $\mathcal{L}$ denote the pullbacks of the tautological bundle and ample line bundle on $\mathrm{Gr}(2,4)$ respectively and $\mathcal{H}$ is an indecomposable extension of line bundles on $T^*\mathrm{Gr}(2,4)$.

It is worth noting that this collection of coherent sheaves is different to those constructed in [Kap88] by applying Schur functors to $\mathcal{T}$. Utilising the characteristic p quantization methods developed in [BK04; Kal08] and applied to the Coulomb branch setting in [Web19a], we argue the following result:

Theorem 1.3.1. *The coherent sheaf on $T^*\mathrm{Gr}(2,4)$ given by*

$$\begin{aligned}\mathcal{Z} &= \mathcal{Z}^{221322}\oplus\mathcal{Z}^{223221}\oplus\mathcal{Z}^{221223}\oplus\mathcal{Z}^{223221}\oplus\mathcal{Z}^{221223}\oplus\mathcal{Z}^{212322}\\ &\cong \mathcal{O}\oplus\mathcal{L}^{-1}\oplus\mathcal{H}\oplus\mathcal{H}\otimes\mathcal{L}\oplus\mathcal{T}\oplus\mathcal{T}^{\perp}\end{aligned}$$

is a tilting generator that realises the derived equivalence (3.1).

It is also worth noting that this tilting generator likely differs to that given in [TU10], although we postpone a proof of this fact to later work. This result acts as a proof of concept and we believe studying further examples would similarly provide novel examples of tilting generators on interesting symplectic varieties.

Chapter 2

Associated variety of $L_1(\mathfrak{psl}_{n|n})$

2.1 Introduction

In [CG19], Costello and Gaiotto constructed VOAs on the boundaries of the 3d A-model and 3d B-model, which we denote V^A and V^B respectively. It was conjectured that the derived endomorphism algebra of the A-side boundary VOA is isomorphic to the algebra of functions for the Coulomb branches. It was further conjectured that certain modules for the boundary VOA would correspond to bulk line operators of the theory. The structure of this category of modules and many aspects of this conjecture were proved for abelian gauge theories in [Bal+23].

In work of Beem and Ferrari, a mirror conjecture was formulated:

Conjecture 2.1.1 ([BF23])**.** *For theories with purely fermionic boundary degrees of freedom, the associated variety X_{V^A} is isomorphic to the Higgs branch of that theory.*

One class of examples investigated in [BF23] was that of SQED[n]. This is an abelian gauge theory with gauge group $U(1)$ acting with matter valued in a quarternionic representation T^*N with $\dim N = n$. The Higgs branches for these theories are known to be isomorphic to the minimal nilpotent orbit closure of type A, which we denote $\overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}$. It was shown explicitly in the $n = 2$ case that:

Theorem 2.1.1 ([**beem˙free˙2023**])**.**

$$X_{L_1(\mathfrak{psl}_{2|2})} \cong \overline{\mathbb{O}_{\min}(\mathfrak{sl}_2)}.$$

Unfortunately, the technique to prove Theorem 2.1.1 does not readily generalise for $n > 2$. To prove Conjecture 2.1.1 for SQED[n] with $n > 2$ two key statements need to be proved:

1. $V^A \cong L_1(\mathfrak{psl}_{n|n})$
2. $X_{L_1(\mathfrak{psl}_{n|n})} \cong \overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}$.

The first statement was conjectured in [BF23] and involves the identification of a certain relative BRST cohomology. A general expression for the BRST cohomology defining V^A for abelian theories in terms of simple current extensions is given in [Bal+23] while free field realisations for the BRST cohomology for certain classes of abelian theories are given in [BF23]. Combining this with results expressing $L_1(\mathfrak{psl}_{n|n})$ as a simple current extension of $L_{-1}(\mathfrak{sl}_n) \otimes L_1(\mathfrak{sl}_n)$ [Ada+19; CY21] makes proving the first statement relatively straight forward.

The second step is at first glance quite challenging, as the submodule structure of $V^1(\mathfrak{psl}_{n|n})$, and hence the structure of $L_1(\mathfrak{psl}_{n|n})$, is not well-understood. To ameliorate this, one can exploit the inclusion of a $L_{-1}(\mathfrak{sl}_n) \otimes L_1(\mathfrak{sl}_n)$ subalgebra and use results due to Arakawa and Moreau proving that the associated variety for this subalgebra is the unique minimal sheet containing the minimal nilpotent orbit [AM19]. The sheet is a deformation of the nilpotent orbit as a symplectic singularity and contains elements that are semisimple elements of $\mathfrak{sl}_n^*$. To prove the isomorphism, it suffices to find elements in the maximal submodule of $V^1(\mathfrak{psl}_{n|n})$ that are not contained in the maximal submodule of $V^{-1}(\mathfrak{sl}_n) \otimes V^1(\mathfrak{sl}_n)$.

We succeed in finding a new singular vector for $V^1(\mathfrak{psl}_{n|n})$ whose image in the Zhu C_2-algebra generates the 2×2-minors of $\mathfrak{sl}_n$ not present in the defining ideal of the sheet. We conjecture that this singular vector may generate the maximal submodule of $V^1(\mathfrak{psl}_{n|n})$, however leave this to future work.

2.2 Preliminaries

2.2.1 Notation & conventions

Throughout this chapter all vector spaces are over $\mathbb{C}$ and we use the following convention

$$\mathbb{N} = \{0, 1, 2, \dots\}, \qquad \mathbb{Z}^{\pm} = \{\pm 1, \pm 2 \pm 3, \dots\}.$$

We denote by $\mathfrak{g}$ a finite-dimensional Lie superalgebra with superspace decomposition

$$\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$$

and a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-.$$

We will denote by G its associated Lie supergroup with $\mathrm{Lie}(G) = \mathfrak{g}$ and by $\widehat{\mathfrak{g}}$ the its affine Lie algebra. As a vector space this decomposes as

$$\widehat{\mathfrak{g}} = \mathfrak{g}((t)) \oplus \mathbb{C}K \oplus D\mathbb{C}$$

where K is central and D acts as a derivation with respect to t. We will denote the triangular decomposition of $\widehat{\mathfrak{g}}$ induced from the above triangular decomposition of $\mathfrak{g}$ by

$$\widehat{\mathfrak{g}} = \widehat{\mathfrak{n}}_+ \oplus \widehat{\mathfrak{h}} \oplus \widehat{\mathfrak{n}}_-$$

where

$$\widehat{\mathfrak{n}}_\pm = \mathfrak{n}_\pm \oplus (t^{\pm 1}\mathfrak{g}[[t^{\pm 1}]]), \qquad \widehat{\mathfrak{h}} = \mathfrak{h} \oplus K\mathbb{C} \oplus D\mathbb{C}.$$

Let $E^{i,j} \in \mathrm{Mat}_{n\times n}(\mathbb{C})$ denote the elementary matrix with a 1 in the $(i,j)^{\text{th}}$ position and zeroes elsewhere. We will make use of the following notation whereby

$$\begin{aligned} \varepsilon_i : \mathfrak{h} &\to \mathbb{C} \\ E^{i,j} &\mapsto \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases} \end{aligned}$$

$$\varepsilon_{i,j} = \varepsilon_i - \varepsilon_j, \qquad \alpha_i = \varepsilon_{i,i+1}.$$

We then fix the following notation for the roots, positive roots and simple roots of $\mathfrak{sl}_n$ respectively:

$$\begin{aligned} \Delta &= \{\varepsilon_{i,j} \mid 1 \leq i \neq j \leq n\} \\ \Delta_+ &= \{\varepsilon_{i,j} \mid 1 \leq i < j \leq n\} \\ \Delta_+^s &= \{\alpha_i \mid 1 \leq i \leq n-1\}. \end{aligned}$$

We will denote by Q_n the root lattice of $\mathfrak{sl}_n$ and the set of fundamental weights corresponding to our choice of simple roots by

$$\Omega = \{\omega_i \mid 1 \leq i \leq n-1\} \subset \mathfrak{h}^*.$$

2.2.2 Vertex algebras

We refer the reader to [FB04] for an introduction to the theory of vertex algebras. Given a vertex algebra V, we will denote by $|0\rangle$ its vacuum vector and by T its translation operator. For $a \in V$, we use the following notation for the modes of the corresponding field $a(z)$:

$$a(z) = \sum_{j \in \mathbb{Z}} a_{(j)} z^{-j-1}.$$

We also follow the convention where a vertex *operator* algebra specifically means a vertex algebra with a conformal structure.

For homogeneous elements

$$v = v^{j_1}_{(-k_1)} \dots v^{j_m}_{(-k_m)} |0\rangle \in V$$

we will refer to $\sum_{i=1}^{m} k_i$ as the *degree* of v.

Symplectic boson & free fermion VOAs

The symplectic boson vertex algebra is a chiral analogue of the Weyl algebra of differential operators and is an important building block for many chiral constructions.

Definition 2.2.1. *The **symplectic boson** vertex algebra, denoted $\beta\gamma^{\otimes n}$, is strongly generated by fields $\beta^i(z)$ and $\gamma^i(z)$ for $1 \leq i \leq n$ and satisfy the following operator product expansion (OPE):*

$$\beta^i(z)\gamma^j(w) \sim \frac{\delta_{i,j}}{z-w}.$$

Remark 2.2.1. *The symplectic boson vertex algebra is sometimes also called the $\beta\gamma$ system or chiral differential operators (CDOs).*

An odd analogue of the symplectic boson VOA is given by the free fermion VOA.

Definition 2.2.2. *The **free fermion vertex** superalgebra, denoted $bc^{\otimes n}$, is strongly generated by odd fields $b^i(z)$ and $c^i(z)$ for $1 \leq i \leq n$ satisfying the following OPE:*

$$b^i(z)c^j(w) \sim \frac{\delta_{i,j}}{z-w}.$$

Affine VOAs

Given a finite-dimensional Lie superalgebra $\mathfrak{g}$ we consider the following induced representation of its affinization $\widehat{g}_k$

$$V^k(\mathfrak{g}) = \mathrm{Ind}_{\mathfrak{g}[[t]]\oplus\mathbb{C}K}^{\widehat{g}}\mathbb{C}_k$$

where $\mathbb{C}_k$ is the one-dimensional representation of $\mathfrak{g}[[t]] \oplus \mathbb{C}K$ on which K acts as $k \in \mathbb{C}$.

Definition 2.2.3. *Let $\{J^a\}_{i=1}^{\dim\mathfrak{g}}$ be a basis for $\mathfrak{g}$. Then the **universal affine vertex algebra** is the vertex algebra structure on $V^k(\mathfrak{g})$ strongly generated by fields $J^i(z)$ satisfying the OPE*

$$J^i(z)J^j(w) \sim \frac{[J^i, J^j](w)}{z-w} + \frac{k(J^i, J^j)}{(z-w)^2}$$

where $(\cdot,\cdot)$ is a non-degenerate form on $\mathfrak{g}$.

If the level k is not critical (i.e. not the dual coxeter number of $\mathfrak{g}$), then $V^k(\mathfrak{g})$ is a VOA. Furthermore, as a $\widehat{\mathfrak{g}}$-representation, $V^k(\mathfrak{g})$ has a unique simple quotient, denoted $L_k(\mathfrak{g})$, which also has the structure of a VOA.

Definition 2.2.4. *The **simple affine VOA**, denoted $L_k(\mathfrak{g})$, is the vertex algebra structure inherited by the simple quotient of $V^k(\mathfrak{g})$*

We can also consider more general highest weight modules and their simple quotients. Given a $\mathfrak{g}$-weight λ, we denote the Verma module for $\widehat{\mathfrak{g}}$ of highest weight λ by $M(\mathfrak{g},\lambda)$ and we will denote its simple quotient by $L(\mathfrak{g},\lambda)$. The following definition is crucial to understanding the submodule structures of these modules:

Definition 2.2.5. *A vector $v \in V^k(\mathfrak{g})$ is called **singular** if for all $x \in \widehat{\mathfrak{n}}_+$ and $j \in \mathbb{N}$*

$$x_{(j)}v = 0.$$

The $V^1(\mathfrak{psl}_{n|n})$ VOA

In this subsection we review background relating to the Lie superalgebra $\mathfrak{psl}_{n|n}$ and its VOA which is the primary object of study for this paper.

Definition 2.2.6. *For $n \geq 2$, the projective special linear algebra $\mathfrak{psl}_{n|n}$ is the simple Lie superalgebra of type $A(n-1, n-1)$. This algebra can be described by the set of matrices $X \in \mathrm{End}(\mathbb{C}^{n|n})$ such that*

$$\mathrm{str} X = 0$$

and quotienting by the ideal generated by the identity matrix.

At times we will make use of the notation employed in [BF23] which highlights the subalgebra structure of $\mathfrak{psl}_{n|n}$. Given $X \in \mathfrak{psl}_{n|n}$ we write

$$X = \left(\begin{array}{ccc|ccc} L^{1,1} & \dots & L^{1,n} & \Theta^{1,1} & \dots & \Theta^{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ L^{n,1} & \dots & L^{n,n} & \Theta^{n,1} & \dots & \Theta^{n,n} \\ \hline \widetilde{\Theta}^{1,1} & \dots & \widetilde{\Theta}^{1,n} & J^{1,1} & \dots & J^{1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{\Theta}^{n,1} & \dots & \widetilde{\Theta}^{n,n} & J^{n,1} & \dots & J^{n,n} \end{array}\right).$$

The two diagonal blocks form commuting $\mathfrak{sl}_n$ subalgebras. We will also make use of the projections and embeddings to and from these subalgebras

$$\pi_{\pm} : \mathfrak{psl}_{n|n} \twoheadrightarrow \mathfrak{sl}_n$$
$$\iota_{\pm} : \mathfrak{sl}_n \hookrightarrow \mathfrak{psl}_{n|n}$$

where $+$ indicates the bottom right copy of $\mathfrak{sl}_n$ and $-$ indicates the upper left copy of $\mathfrak{sl}_n$. We note that the $\pi_{\pm}$ is not a Lie superalgebra morphism while $\iota_{\pm}$ is. We make use this notation as the upper-left block forms a $V^{-1}(\mathfrak{sl}_n)$-subalgebra, while the lower right block forms a $V^1(\mathfrak{sl}_n)$-subalgebra.

Following the conventions set in [FSS96], we choose simple roots analogously to those chosen for $\mathfrak{sl}_n$ in Section 2.2.1:

$$\Delta_s^+ = \{\alpha_i \mid 1 \leq i \leq 2n-1\}.$$

Note that each of the α_i are even except for α_n which is odd.

Let $E^{i,j}$ denote the elementary supermatrix acting on $\mathbb{C}^{n|n}$ with a 1 in the $(i,j)^{\text{th}}$ entry and zeroes elsewhere. The Chevalley generators corresponding to the root data are given by

$$e_{i,j} = E^{i,j}, \qquad f_{i,j} = E^{j,i}$$

for $1 \leq i < j \leq 2n$ and

$$h_i = E^{i,i} - E^{i+1,i+1}$$

for $1 \leq i \leq n-1$ and $n+1 \leq i \leq 2n-1$. These generators span $\mathfrak{psl}_{n|n}$, although they do not form a basis since the h_i are not linearly independent. We will generically denote elements of this set of generators by E^α, where α represents the appropriate single or double index. The strong generators of $V^1(\mathfrak{psl}_{n|n})$ corresponding to these Chevalley generators $E^\alpha(z)$ satisfy the following OPE:

$$E^\alpha(z)E^\beta(w) \sim \frac{[E^\alpha, E^\beta](w)}{z-w} + \frac{\mathrm{str}(E^\alpha E^\beta)\mathbf{1}}{(z-w)^2}.$$

Heisenberg & lattice vertex algebras

Let $\mathfrak{H}$ denote the level 1 Heisenberg Lie algebra with basis $\{\phi_{(j)}\}_{j\in\mathbb{Z}}$ and let $\mathfrak{H}_+ \subseteq \mathfrak{H}$ denote the subalgebra generated by $\{\phi_{(j)}\}_{j\in\mathbb{N}}$. $\mathfrak{H}$ has a vacuum representation

$$H^\phi = \mathrm{Ind}_{\mathfrak{H}_+}^{\mathfrak{H}} \mathbb{C}$$

where $\mathbb{C}$ indicates the 1-dimensional representation of $\mathfrak{H}_+$.

Definition 2.2.7. *The **Heisenberg** VOA is the VOA structure on H^ϕ strongly generated by a field $\phi(z)$ and satisfying the OPE*

$$\phi(z)\phi(w) \sim \frac{1}{(z-w)^2}.$$

All irreducible representations of H are in the form of the following class of modules.

Definition 2.2.8. *Given $\mu \in \mathbb{C}$, the **Fock module** of highest weight $\mu \in \mathbb{C}$ for H^ϕ and denoted F_μ^ϕ is*

$$F_\mu^\phi = \mathrm{Ind}_{H_+^\phi}^{H^\phi} \mathbb{C}_\mu$$

where $\mathbb{C}_\mu$ is the 1-dimensional representation of H_+^ϕ whereby $\phi_{(0)}$ acts as multiplication by μ and $\phi_{(j)}$ act as zero for all $j > 0$.

We will also require generalizations of the Heisenberg VOA and its Fock modules defined using the data of a lattice. Let L be a lattice of rank r with basis

$$L = \mathbb{Z}\phi^1 \oplus \cdots \oplus \mathbb{Z}\phi^r$$

and equipped with a symmetric bilinear form

$$B_L : L \times L \to \mathbb{C}.$$

From this data we define an algebra

$$\mathfrak{h}^L = \mathbb{C} \otimes_{\mathbb{Z}} L$$

and consider its central extension of with respect to B_L:

$$0 \to \mathbb{C} \to \mathfrak{H}^L \to \mathfrak{h}^L((t)) \to 0.$$

This extended algebra has a vacuum representation

$$V^L = \mathrm{Ind}_{\mathfrak{H}_+^L}^{\mathfrak{H}^L} \mathbb{C}$$

where as before $\mathfrak{H}_+^L \subset \mathfrak{H}^L$ is the subalgebra of non-negative modes and $\mathbb{C}$ denotes the 1-dimensional representation of $\mathfrak{H}_+^L$.

Definition 2.2.9. *The **lattice vertex algebra** associated to an integral lattice L is the vertex algebra structure on V^L strongly generated by fields $\phi^i(z)$ and $\exp(\phi^i)(z)$ for $1 \leq i \leq r$ which satisfy*

$$\phi^i(z)\phi^j(w) \sim \frac{B(v^i, v^j)}{(z-w)^2}, \qquad \exp\big(\phi^i\big)(z) \exp\big(\phi^j\big)(w) \sim \frac{:\exp(\phi^i + \phi^j):(w)}{(z-w)^{B_L(\phi^i,\phi^j)}}$$

and

$$T\exp\big(\phi^i\big)(z) =: \phi^i \exp\big(\phi^i\big) : (z).$$

Definition 2.2.10. *Given $\mu \in \mathbb{C}^r$, the Fock module of highest weight μ for the lattice vertex algebra V^L is given by*

$$F_\mu^\phi = \mathrm{Ind}_{V_+^L}^{V^L} \mathbb{C}_\mu^r$$

where $V_+^L \subset V^L$ is the subalgebra generated by non-negative modes and $\mathbb{C}_\mu^r$ is the r-dimensional representation generated by a heighest weight vector $|\mu\rangle$ such that

$$\phi_{(0)}^i |\mu\rangle = B_L(\mu, \phi^i) |\mu\rangle, \qquad \phi_{(n)}^i |\mu\rangle = 0$$

for all $n > 0$.

If the basis $\{\phi_1, \ldots, \phi_r\}$ of L is orthogonal, one can produce a decomposition into Heisenberg algebras and modules:

$$V^L = \bigotimes_{i=1}^{r} H^{\phi^i}, \qquad F^{\phi}_{\mu} = \bigotimes_{i=1}^{r} F^{\phi^i}_{\mu_i}.$$

The following result is one of two we will employ to identify the BRST cohomology used to define the 3d A-model boundary VOA:

Theorem 2.2.1 ([FK80; Seg81])**.** *If $\mathfrak{g}$ is a simply laced Lie algebra then there are VOA isomorphisms*

$$L_1(\mathfrak{g}) \cong V^{Q_n} \cong \bigoplus_{\mu \in Q_n} F^{Q_n}_{\mu},$$

where we recall that Q_n is the root lattice for $\mathfrak{sl}_n$.

Singlet VOA

One final vertex algebra we will need is the $p = 2$ **singlet algebra**, which we denote by W and is also known as the W_3-algebra with central charge $c = -2$. We will not explicitly define it here, however the interested reader can refer to Section 2.3 of [Bal+23].

Wang classified all irreducible modules for the singlet algebra and showed that given $\mu \in \mathbb{C}$ there exists an irreducible highest weight W-module which we will denote M_μ [Wan98]. Let

$$P_m = \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \sum_{i=1}^{n} \lambda_i = m\}$$

and define a lattice

$$L_\varphi = \mathbb{Z}\varphi^1 \oplus \cdots \oplus \mathbb{Z}\varphi^{n-1}, \qquad B_\varphi(\varphi^i, \varphi^j) = -\delta_{i,j}.$$

Theorem 2.2.2 ([AM18])**.** *There is an isomorphism of VOAs*

$$L_{-1}(\mathfrak{sl}_n) \cong \bigoplus_{\lambda \in P_0} M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \otimes F^{\varphi}_{\lambda}$$

where

$$\tilde{\lambda} = (\lambda_1 - \frac{\lambda_n}{n}, \dots, \lambda_{n-1} - \frac{\lambda_n}{n}).$$

Similarly we have

$$L(\mathfrak{sl}_n, \lambda(m)) \cong \bigoplus_{\lambda \in P_m} M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \otimes F^{\varphi}_{\tilde{\lambda}}$$

where

$$\lambda(m) = \begin{cases} m\omega_1, & j \geq 0 \\ m\omega_{n-1}, & j \leq 0 \end{cases}$$

and all irreducible ordinary $L_{-1}(\mathfrak{sl}_n)$ modules are of the form $L(\mathfrak{sl}_n, \lambda(m))$ for some $m \in \mathbb{Z}$.

Associated varieties

The associated variety of a vertex algebra V was first defined in [Ara10] and provides information about the structure and representation theory of the vertex algebra. Furthermore, the associated varieties of VOAs appearing in certain 3d and 4d QFTs are conjectured to be, and in some cases proven to be [Ara18], the Higgs branches of those theories. The motivation for this paper is to prove such a conjecture for the 3d A-model of SQED[n].

Definition 2.2.11. *Let V be a strongly finitely generated vertex algebra with strong generators $\{a^i, \dots, a^r\}$ and consider the subspace*

$$C_2(V) = \mathrm{span}_{\mathbb{C}}\{a^i_{(-n)}v \mid 1 \leq i \leq r,\ n \geq 2,\ v \in V\}.$$

*The **Zhu C_2-algebra** of V is defined to be*

$$R_V = V/C_2(V)$$

and has a commutative product given by

$$\overline{a} \cdot \overline{b} = \overline{a_{(-1)}b}, \qquad a, b \in V.$$

*The **associated variety** of V is defined as*

$$X_V = \mathrm{specm} R_V.$$

The Zhu C_2-algebra R_V is finitely generated if and only if V is strongly finitely generated and has the structure of a Poisson algebra whose bracket is given by

$$\begin{aligned}\{\cdot,\cdot\} : R_V \times R_V &\to R_V \\ (\overline{x},\overline{y}) &\mapsto \overline{x_{(0)}y}.\end{aligned}$$

It follows that the associated variety of a strongly finitely generated vertex algebra is a Poisson variety of finite type.

This construction also gives a good way of characterising how "large" a VOA is:

Definition 2.2.12 ([Zhu96],[AK17]). *A vertex algebra V is called C_2-**cofinite** or **lisse** if $C_2(V) < \infty$, or equivalently if $\dim X_V = 0$. We say V is **quasi-lisse** if X_V has finitely many symplectic leaves.*

The lisse condition is widely used throughout the study of VOAs and has many important consequences. For instance, lisse VOAs have finitely many irreducible representations and their characters satisfying a modular invariance property [Zhu96]. The quasi-lisse condition is a weaker version of this property and is of interest since VOAs constructed from supersymmetric QFTs, such as those appearing in this chapter, are typically expected to satisfy this condition. Quasi-lisse VOAs also enjoy certain properties, for example their characters are known to satisfy modular linear differential equations [AK17].

Example 1. Consider the associated universal affine Kac-Moody VOA $V^k(\mathfrak{g})$. Following [AM19], the Zhu C_2-algebra of this VOA is given by

$$\begin{aligned} & C_2(V^k(\mathfrak{g})) = t^{-2}\mathfrak{g}[t^{-1}]V \\ \implies \quad & R_{V^k(\mathfrak{g})} = \mathrm{Sym}(\mathfrak{g}).\end{aligned}$$

Since expressions involving odd elements are nilpotent, they do not contribute to associated variety. It follows that

$$X_{V^k(\mathfrak{g})} = \mathfrak{g}_{\bar{0}}^*.$$

If we let V be a quotient of $V^k(\mathfrak{g})$ we then have a surjective algebra morphism

$$\begin{aligned}\pi_V : \mathrm{Sym}(\mathfrak{g}) &\twoheadrightarrow V/(t^{-2}\mathfrak{g}[t^{-1}]V) \\ x &\mapsto \overline{x_{(-1)}} + t^{-2}\mathfrak{g}[t^{-1}]V.\end{aligned}$$

The associated variety of V is the zero locus of $\ker \pi_V$, which in turn implies that X_V is a closed G-invariant subvariety of $\mathfrak{g}_{\bar{0}}^*$. In the following two sections, we will recall results on varieties that appear as the associated varieties for $L_k(\mathfrak{sl}_n)$ at certain levels.

2.2.3 Sheets & nilpotent orbits of type A

Nilpotent orbits of type A

For a general introduction to the theory of nilpotent orbits we suggest consulting [Hen15]. Briefly put, given a simple Lie algebra $\mathfrak{g}$, one can define its nilpotent cone $\mathcal{N}(\mathfrak{g})$ which is a conic algebraic variety under the usual scalar multiplication. The variety $\mathcal{N}(\mathfrak{g})$ is the disjoint union of finitely many G-orbits which are irreducible if G is connected. The closure of a nilpotent orbit is a union of orbits, and thus, this defines a closure partial ordering which agrees with dominance order on partitions. For the purposes of this paper, we are only interested in minimal non-zero nilpotent orbits of type A and their closures which we define explicitly now.

Definition 2.2.13. *The* ***nilpotent cone*** *of type A is*

$$\mathcal{N}(\mathfrak{sl}_n) = \{X \in \mathfrak{sl}_n \mid \textit{all eigenvalues of } X \textit{ are } 0\}.$$

The ***minimal nilpotent orbit*** *of type A is*

$$\mathbb{O}_{\min}(\mathfrak{sl}_n) = \{X \in \mathfrak{sl}_n \mid \operatorname{rank}(X) = 1\}.$$

The closure of $\mathbb{O}_{\min}(\mathfrak{sl}_n)$ is found by taking its union with the trivial orbit:

$$\overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)} = \{X \in \mathfrak{sl}_n \mid \operatorname{rank}(X) \leq 1\}.$$

This condition is equivalent to requiring that the 2×2 minors of any matrix in $\overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}$ vanish. This allows us to express the coordinate ring of $\overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}$ as follows:

$$\begin{aligned} & I = \{X_{ij}X_{kl} - X_{il}X_{kj} \mid X \in \mathfrak{sl}_n,\ i < k,\ j < l\} \subset \mathbb{C}[X_{ij}] = \operatorname{Sym}(\mathfrak{sl}_n) \\ \implies \quad & \overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)} = \operatorname{Spec}(\mathbb{C}[X_{ij}]/I). \end{aligned}$$

Sheets of type A

In the same way nilpotent orbits are irreducible subvarieties of the nilpotent cone, a sheet of a Lie algebra is an irreducible subvariety of the subset

$$\mathfrak{g}^{(m)} = \{x \in \mathfrak{g} \mid \dim \mathfrak{g}^x = m\}, \qquad m \in \mathbb{N},$$

where $\mathfrak{g}^x$ denotes the centralizer of x in $\mathfrak{g}$. A key difference between sheets and nilpotent orbits is that sheets need not be contained in $\mathcal{N}(\mathfrak{g})$. For an introduction to the subject, we suggest consulting [TY05].

Sheets are closed, irreducible and G-invariant varieties and every sheet contains a unique nilpotent orbit. Since for our purposes we are only concerned with the closure of sheets of type A containing the minimal nilpotent orbit, we make use of the following result:

Lemma 2.2.1 ([AM19]). *Let $\mathfrak{l} \subseteq \mathfrak{sl}_n$ be a Levi subalgebra such that its centralizer is one-dimensional:*

$$\mathfrak{z}(\mathfrak{l}) = \mathbb{C}\xi, \qquad \xi \in \mathfrak{h}\backslash\{0\}.$$

Then the closure of the sheet associated to the Jordan class of $\mathfrak{l}$ is given by

$$\overline{\mathbb{S}_{\mathfrak{l}}(\mathfrak{sl}_n)} = SL_n \cdot (\mathbb{C}\xi + \mathfrak{p}_u)$$

where

$$\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{p}_u$$

is a parabolic subalgebra and $\mathfrak{p}_u$ is its nilradical.

We are interested in the case where $\overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)} \subset \overline{\mathbb{S}_{\mathfrak{l}}(\mathfrak{sl}_n)}$. Since the minimal nilpotent orbit of type A corresponds to the partition $\beta = (2, 1, \ldots, 1)$, then the Levi subalgebra is determined by the conjugate partition $\overline{\beta} = (n-1, 1)$. Let

$$h_k = \operatorname{diag}\big(\underbrace{0, \ldots, 0}_{k-1}, 1, -1, \underbrace{0, \ldots, 0}_{n-k-1}\big)$$

be the standard basis for $\mathfrak{h} \subset \mathfrak{sl}_n$. The centre of the Levi subalgebra corresponding to the partition $\overline{\beta}$ can be expressed explicitly as

$$\begin{aligned}\mathfrak{z}(\mathfrak{l}) &= \left\{ Y_1 \sum_{i=1}^{N} (n-i) h_{n-i} \mid Y_1 \in \mathbb{C} \right\} \\ &= \left\{ \begin{pmatrix} Y_1(n-1) & 0 & \ldots & 0 \\ 0 & -Y_1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \ldots & \ldots & -Y_1 \end{pmatrix} \middle| Y_1 \in \mathbb{C} \right\}.\end{aligned}$$

The nilradical for the parabolic containing $\mathfrak{l}$ is given explicitly by

$$\mathfrak{p}_u = \bigoplus_{i=1}^{n-1} \mathfrak{g}^{\sum_{j=1}^{i} \alpha_j} = \left\{ \begin{pmatrix} 0 & Y_2 & \dots & Y_n \\ \vdots & \ddots & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} \middle| Y_i \in \mathbb{C} \right\}$$

Applying Lemma 2.2.1, we arrive at the following description for the sheet of type A containing $\overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}$.

$$\overline{\mathbb{S}_{min}(\mathfrak{sl}_n)} = \left\{ X \begin{pmatrix} Y_1(N-1) & Y_2 & \dots & Y_N \\ 0 & -Y_1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & -Y_1 \end{pmatrix} X^{-1} \middle| Y_i \in \mathbb{C},\ X \in \mathrm{SL}_n \right\}$$

To further elucidate the difference between $\overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}$ and $\overline{\mathbb{S}_{min}(\mathfrak{sl}_n)}$, we will determine some results regarding the functions on $\mathfrak{sl}_n^*$ that vanish on $\overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}$ but not $\overline{\mathbb{S}_{min}(\mathfrak{sl}_n)}$. As mentioned previously, $\overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}$ is the zero-locus of the functions in $\mathrm{Sym}(\mathfrak{sl}_n)$ given by the 2×2 matrix minors. In [Wey89, §4], it was shown that the space of 2×2 minors can be expressed as

$$\wedge^2 E^* \otimes \wedge^2 E \cong V_{1,2} \oplus U_{2,2} \tag{2.1}$$

where $E = \mathbb{C}^n$ and $U_{2,2}$ and $V_{1,2}$ are complementary SL_n subrepresentations. $U_{2,2}$ is an irreducible representation generated by the minor $X_{1,n}X_{2,n-1} - X_{1,n-1}X_{2,n}$ and $V_{1,2}$ is shown in [Wey89, pg. 237] to be isomorphic to $E^* \otimes E$. We will later use this presentation to identify functions that vanish on $\overline{\mathbb{O}_{\min}(\mathfrak{sl}_N)}$ but not $\overline{\mathbb{S}_{min}(\mathfrak{sl}_N)}$. In the following proposition we show that the elements of $U_{2,2}$ vanish on both varieties.

Lemma 2.2.2. *For all $n > 3$ and $f \in U_{2,2}$ and $X \in \overline{\mathbb{S}_{min}(\mathfrak{sl}_n)}$ we have $f(X) = 0$.*

Proof. Every element $X \in \overline{\mathbb{S}_{min}(\mathfrak{sl}_n)}$ may be expressed as

$$X = a\mathrm{Id} + uv^T$$

for some $a \in \mathbb{C}$ and vectors $u, v \in E$ satisfying $u \cdot v = -na$. Let $g = X_{1,n}X_{2,n-1} - X_{1,n-1}X_{2,n}$ and note that since uv^T is at most a rank 1 matrix we have $g(X) = 0$. Note that since $\overline{\mathbb{S}_{min}(\mathfrak{sl}_n)}$ is SL_n-invariant, the 2×2 minors that vanish on $\overline{\mathbb{S}_{min}(\mathfrak{sl}_n)}$ form a subrepresentation of $\wedge^2 E^* \otimes \wedge^2 E$. It follows that the elements of the subrepresentation generated by g vanish on $\overline{\mathbb{S}_{min}(\mathfrak{sl}_n)}$. Since g generates the irreducible SL_n representation $U_{2,2}$ and so for all $f \in U_{2,2}$ we have $f(X) = 0$. □

2.3 BRST cohomology & 3d A-model boundary VOA

In this section, we prove the statement made in [BF23] that the boundary VOA for the 3d A-model for SQED with $n > 3$ hypermultiplets is indeed $L_1(\mathfrak{psl}_{n|n})$. The proof of this fact follows rather directly from the results of [Bal+23; CY21; Ada+19]. We will briefly outline the data used to define the 3d A-model and boundary VOA in the case of interest, but the interested reader may refer to [**costello˙higgs˙2018**] for more details on the construction and the role it plays in the study of 3d mirror symmetry.

Let $G = U(1)^r$ be the gauge group and $V = T^*N$ be a quarternionic representation of G with $\dim N = n$. Let $\mathfrak{g} = \mathrm{Lie}(G)$ and $\widetilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}]$.

Definition 2.3.1. *The **boundary VOA** for the 3d A-model of an abelian gauge theory is defined by the following relative BRST cohomology.*

$$V_\rho^A = H^{\frac{\infty}{2}+\bullet}(\mathfrak{gl}_1^r, \widetilde{\mathfrak{gl}_1^r}, \beta\gamma^{\otimes n} \otimes bc^{\otimes n})$$

where ρ is a unimodular matrix encoding the action of G on V and determines the BRST differential.

Remark 2.3.1. *It is worth mentioning that physically the free fermions are included to cancel a gauge anomaly. Mathematically, we can think of this VOA as the global sections of a sheaf of (twisted) CDOs on the symplectic reduction of V with respect to G. It was shown in [GMS99] that there is often a topological obstruction to the construction of a sheaf of CDOs corresponding to this gauge anomaly.*

Other anomaly cancellations may be used, for example a different anomaly cancellation is used in [Kuw17] which results in a different associated variety. Here we use the anomaly cancellation data preferred by physicists as this makes some 3d mirror symmetric connections, for example the outer-automorphisms studied in [BF23], more apparent.

It was shown in [BF23; Bal+23] that for abelian gauge theories of this type, the cohomology defining V_ρ^A is concentrated in cohomological degree 0. In the same paper, they provide a decomposition for V_ρ^A in terms of singlet algebra modules and Fock modules resulting from a lattice decomposition determined by ρ. Here we will look at this decomposition for the case of interest in this paper.

Proposition 2.3.1. *For $r = 1$ and*

$$\rho = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

we have an isomorphism of vertex operator superalgebras

$$V_\rho^A \cong \bigoplus_{\lambda\in P_m,\, \nu\in Q} M_{\lambda_1}\otimes\cdots\otimes M_{\lambda_n}\otimes F_{\tilde{\lambda}}^{\varphi}\otimes F_\nu^Q.$$

Proof. To prove this, we use [Bal+23, Proposition 3.2]. The matrix $\rho^\vee$ is an $n\times(n-1)$ unimodular matrix associated to the mirror theory and in our case is given by

$$\rho^\vee = \begin{pmatrix} -1 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & -1 \end{pmatrix}.$$

These matrices provide an orthogonal decomposition

$$\mathbb{C}^n = \rho(\mathbb{C})\oplus\rho^\vee(\mathbb{C}^{n-1}).$$

The associated orthogonal projections are given as follows:

$$\begin{aligned} p &: \mathbb{C}^n \twoheadrightarrow \rho(\mathbb{C}) \\ \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} &\mapsto \frac{1}{n}\begin{pmatrix} v_1+\cdots+v_n \\ \vdots \\ v_1+\cdots+v_n \end{pmatrix} \\ p^\perp &: \mathbb{C}^n \twoheadrightarrow \rho^\vee(\mathbb{C}^{n-1}) \\ \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} &\mapsto \frac{1}{n}\begin{pmatrix} (n-1)v_1 - v_2 - \cdots - v_n \\ -v_1 + (n-1)v_2 - v_3 - \cdots - v_n \\ \vdots \\ -v_1 - \cdots - v_{n-1} + (n-1)v_n \end{pmatrix}. \end{aligned}$$

Using this data, we can compute the abelian group H appearing in the decomposition in [Bal+23, Proposition 3.2]:

$$H = N/\mathbb{Z} = N^\vee/\mathbb{Z}^{n-1} = \mathbb{Z}_n$$

where

$$N = \rho^{-1}(p(\mathbb{Z}^n)) = \frac{1}{n}\mathbb{Z}, \qquad N^\vee = (\rho^\vee)^{-1}(p^\perp(\mathbb{Z}^n)) = \frac{1}{n}\mathbb{Z}^{n-1}.$$

We then define three lattices

$$\begin{aligned} L_{\theta^\perp} &= \mathbb{Z}\theta_1^\perp \oplus \cdots \oplus \mathbb{Z}\theta_{n-1}^\perp \\ L_{\eta^\perp} &= \mathbb{Z}\eta_1^\perp \oplus \cdots \oplus \mathbb{Z}\eta_{n-1}^\perp \\ L_\phi &= \mathbb{Z}\phi_1 \oplus \cdots \oplus \mathbb{Z}\phi_n \end{aligned}$$

with bilinear forms

$$B_{\theta^\perp}(\theta_i^\perp, \theta_j^\perp) = -B_{\eta^\perp}(\eta_i^\perp, \eta_j^\perp) = B_\phi(\phi_i, \phi_j) = \delta_{i,j}.$$

Using this data, [Bal+23, Proposition 3.2] tells us that

$$V_\rho^A \cong \bigoplus_{\substack{\mu \in N,\ \mu^\vee, \nu^\vee \in N^\vee \\ [\mu]=[\mu^\vee]=[\nu^\vee]}} M^\phi_{(\rho(\mu)+\rho^\vee(\mu^\vee))\cdot\phi} \otimes F^{\eta^\perp}_{\mu^\vee \cdot \eta^\perp} \otimes F^{\theta^\perp}_{\nu^\vee \cdot \theta^\perp}$$

where $[\mu]$ denotes the equivalence class of μ in $\mathbb{Z}_n$. Note that the expression $\rho(\mu)+\rho^\vee(\mu^\vee)$ with $[\mu] = [\mu^\vee]$ is an orthogonal decomposition of a weight $\lambda \in P_m$ where m corresponds to the class $[\mu] = [\mu^\vee] \in \mathbb{Z}_n$. Also note note that $L_{\theta^\perp} = Q_n$ meaning we can rewrite the decomposition as

$$V_\rho^A \cong \bigoplus_{\substack{m \in \mathbb{Z},\ \lambda \in P_m \\ \nu \in Q_n}} M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \otimes F^\varphi_{\tilde{\lambda}} \otimes F^{Q_n}_\nu.$$

□

Theorem 2.3.1. *For all $n \geq 3$, the 3d A-model boundary VOA for SQED[n] is*

$$V_\rho^A \cong L_1(\mathfrak{psl}_{n|n}).$$

Proof. It was shown in [Ada+19; CY21] that $L_1(\mathfrak{psl}_{n|n})$ is a simple current extension of $L_{-1}(\mathfrak{sl}_n) \otimes L_1(\mathfrak{sl}_n)$:

$$L_1(\mathfrak{psl}_{n|n}) \cong \bigoplus_{i=1}^{n-1} \bigoplus_{m \in \mathbb{Z}} L_{-1}(\mathfrak{sl}_n, \lambda(m)) \otimes L_1(\mathfrak{sl}_n, \omega_i)$$

Applying Theorems 2.2.1 and 2.2.2 gives

$$L_1(\mathfrak{psl}_{n|n}) \cong \bigoplus_{\substack{m \in \mathbb{Z},\ \lambda \in P_m \\ \nu \in Q_n}} M_{\lambda_1} \otimes \ldots M_{\lambda_n} \otimes F^\varphi_{\tilde{\lambda}} \otimes F^{Q_n}_\nu$$

which is the expression for V_ρ^A given in Propsition 2.3.1. □

2.4 Singular vectors of $V^1(\mathfrak{psl}_{n|n})$

Remark 2.4.1. *From now on, we will assume $n > 3$. This is because the singular vectors and maximal submodules of $V^{-1}(\mathfrak{sl}_n)$, and hence the calculation of the associated varieties, for $L_{-1}(\mathfrak{sl}_n)$ differ for the cases where $n = 2$ and $n = 3$.*

As mentioned in Section 2.2.2 the vertex algebra $V^1(\mathfrak{psl}_{n|n})$ contains two even subalgebras $V^{\pm 1}(\mathfrak{sl}_n)$ with commuting actions. For $V^1(\mathfrak{sl}_n)$, the image of this singular vector in $V^1(\mathfrak{psl}_{n|n})$ is

$$\chi_+ = (E^{n+1,2n}_{(-1)})^2 \left|0\right\rangle$$

while the image of the singular vector for the $V^{-1}(\mathfrak{sl}_n)$ subalgebra is

$$\chi_- = (E^{1,n}_{(-1)}E^{2,n-1}_{(-1)} - E^{1,n-1}_{(-1)}E^{2,n}_{(-1)}) \left|0\right\rangle .$$

It can then be shown that $\chi_\pm$ are also singular within the full algebra $V^1(\mathfrak{psl}_{n|n})$.

Proposition 2.4.1. *The vectors $\chi_\pm$ are singular in $V^1(\mathfrak{psl}_{n|n})$.*

Proof. It is already known that $\chi_\pm$ are singular with respect to the $V^{\pm 1}(\mathfrak{sl}_n)$ subalgebras and since the actions of those subalgebras commute with one-another, the vectors are also singular with respect to the other subalgebra. It remains to show that $\chi_\pm$ are singular with respect to the odd generators.

To show this, it suffices to check that these vectors are annihilated by the generator corresponding to the negative of the odd simple root. Let $m \in \mathbb{N}$ and observe via direct computation that

$$\begin{aligned} E^{n+1,n}_{(m)}\chi_- &= E^{n+1,n}_{(m)}(E^{1,n}_{(-1)}E^{2,n-1}_{(-1)} - E^{1,n-1}_{(-1)}E^{2,n}_{(-1)}) \left|0\right\rangle = 0 \\ E^{n+1,n}_{(m)}\chi_+ &= E^{n+1,n}_{(m)}E^{n+1,2n}_{(-1)}E^{n+1,2n}_{(-1)} \left|0\right\rangle = 0. \end{aligned}$$

□

The associated varieties of both the $L_{\pm 1}(\mathfrak{sl}_n)$ subalegbras are known.

Theorem 2.4.1 ([AM19])**.** *For $n > 3$*

$$X_{L_{-1}(\mathfrak{sl}_n)} = \overline{\mathbb{S}_{min}(\mathfrak{sl}_n)}.$$

Theorem 2.4.2 ([Ara10])**.**

$$X_{L_1(\mathfrak{sl}_n)} = pt.$$

Using these results we can demonstrate the following:

Proposition 2.4.2. $X_{L_1(\mathfrak{psl}_{n|n})} \subseteq \overline{\mathbb{S}_{min}(\mathfrak{sl}_n)}$

Proof. First, note that there is a surjective map

$$\pi_{L_1(\mathfrak{psl}_{n|n})} : \mathbb{C}[\mathfrak{psl}_{n|n}] \twoheadrightarrow L_1(\mathfrak{psl}_{n|n})/(t^{-2}\mathfrak{psl}_{n|n}[t^{-1}]L_1(\mathfrak{psl}_{n|n}))$$

such that

$$X_{L_1(\mathfrak{psl}_{n|n})} \cong \operatorname{specm} \ker \pi_{L_1(\mathfrak{psl}_{n|n})}.$$

We also have surjective maps for each subalgebra

$$\pi_{L_{\pm 1}(\mathfrak{sl}_n)} : \mathbb{C}[\mathfrak{sl}_n] \twoheadrightarrow L_{\pm 1}(\mathfrak{sl}_n)/(t^{-2}\mathfrak{sl}_n[t^{-1}]L_{\pm 1}(\mathfrak{sl}_n))$$

such that

$$X_{L_{\pm 1}(\mathfrak{sl}_n)} \cong \operatorname{specm} \ker \pi_{L_{\pm 1}(\mathfrak{sl}_n)}.$$

Since the maximal submodules of the $V^{\pm 1}(\mathfrak{sl}_n)$ subalgebras do not generate $V^1(\mathfrak{psl}_{n|n})$ (in particular they do not produce the vacuum vector) it follows that the maximal submodule for $V^1(\mathfrak{psl}_{n|n})$ must contain the maximal submodules of the $V^{\pm 1}(\mathfrak{sl}_n)$ subalgebras. Hence

$$\begin{aligned} & \ker \pi_{L_{-1}(\mathfrak{sl}_n)} \times \ker \pi_{L_1(\mathfrak{sl}_n)} \subseteq \ker \pi_{L_1(\mathfrak{psl}_{n|n})} \\ \implies \qquad & X_{L_1(\mathfrak{psl}_{n|n})} \subseteq X_{L_{-1}(\mathfrak{sl}_n)} \times X_{L_1(\mathfrak{sl}_n)} \cong \overline{\mathbb{S}_{min}(\mathfrak{sl}_n)} \end{aligned}$$

□

Physical expectations from the 3d A-model for SQED$[n]$ tell us that we should have

$$X_{L_1(\mathfrak{psl}_{n|n})} = \overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}.$$

We note that $\overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)} \subset \overline{\mathbb{S}_{min}(\mathfrak{sl}_n)}$ and the difference between the nilpotent orbit and the sheet containing it is the presence of semisimple elements in the sheet. Correspondingly, there should be singular, or subsingular, vectors in $V^1(\mathfrak{psl}_{n|n})$ generating a maximal submodule containing the elements whose image in $R_{V^1(\mathfrak{psl}_{n|n})}$ correspond to the semisimple elements of $\mathfrak{sl}_n^*$.

In many cases, including $\mathfrak{g} = \mathfrak{sl}_n$, it is known that if $k \in \mathbb{N}$, then the maximal submodule of $V^k(\mathfrak{g})$ is generated by the singular vector

$$(E^{\theta}_{(-1)})^{k+1} |0\rangle \tag{4.1}$$

where θ denotes the longest root of the associated finite-dimensional Lie algebra and E^{θ} is the VOA generator associated to that root [GS18]. We use this as an ansatz to find a singular vector for $V^1(\mathfrak{psl}_{n|n})$ that is not contained in the maximal submodules of the even subalgebras. In this case our candidate singular vector is

$$\chi = E^{1,2n}_{(-1)} E^{1,2n}_{(-1)} |0\rangle .$$

Proposition 2.4.3. *The vector $\chi \in V^1(\mathfrak{psl}_{n|n})$ is singular.*

Proof. To show that χ, we must demonstrate that it is annihilated by $\widehat{\mathfrak{n}}_+$. For degree zero elements $E^{i,j}_{(0)} \in \mathfrak{n}_+ \subset \widehat{\mathfrak{n}}_+$ we have the following commutation relation:

$$[E^{i,j}_{(0)}, E^{1,2n}_{(-1)}] = [E^{i,j}, E^{1,2n}]_{(-1)}$$

which is only non-zero when either i, j is 1 or $2n$. It is also sufficient to show that χ is annihilated by the generators $E^{\alpha}_{(0)} \in \mathfrak{n}_+$ associated to simple roots $\alpha \in \Delta^s_+$. The only elements associated to simple roots with indices containing a 1 or $2n$ are $E^{1,2}_{(0)}$ and $E^{2n-1,2n}_{(0)}$. One can then verify that indeed

$$[E^{1,2}, E^{1,2n}] = [E^{2n-1,2N}, E^{1,2n}] = 0$$

and so $\mathfrak{n}_+$ annihilates χ. Next we want to check that the positive degree elements $E^{\alpha}_{(m)} \in t\mathfrak{g}[[t]] \subset \widehat{\mathfrak{n}}_+$ also annihilate χ. It is easy to verify that elements of the form $E^{\alpha}_{(m)}$ for $m > 1$ annihilate χ since

$$[E^{\alpha}_{(m)}, E^{1,2n}_{(-1)}] = [E^{\alpha}, E^{1,2n}]_{(m-1)}$$

and $x_{(m)} |0\rangle = 0$ for any $x \in V$ and $m \geq 0$. For degree 1 elements of $\widehat{\mathfrak{n}}_+$ we have

$$[E^{\alpha}_{(1)}, E^{1,2n}_{(-1)}] = [E^{\alpha}, E^{1,2n}]_{(0)} + \mathrm{str}(E^{\alpha} E^{1,2n}).$$

We see that this super-commutator is only non-zero when $\alpha = (m, 1)$ or $\alpha = (1, m)$ for some $1 \leq m \leq 2n$, or when E^{α} is a diagonal matrix. In the latter case it is easy to see that both terms vanish. For the first case suppose $m \neq 2n$. Then we have

$$[E^{m,1}_{(1)}, E^{1,2n}_{(-1)}] = E^{m,2n}_{(0)}.$$

Applying this to χ gives

$$\begin{aligned} E^{m,1}_{(1)}\chi &= (E^{m,2n}_{(0)}E^{1,2n}_{(-1)} \pm E^{1,2n}_{(-1)}E^{m,1}_{(1)}E^{1,2n}_{(-1)})\,|0\rangle \\ &= (\pm E^{1,2n}_{(-1)}E^{m,2n}_{(0)} \pm E^{1,2n}_{(-1)}E^{m,2n}_{(0)})\,|0\rangle \\ &= 0 \end{aligned}$$

where the signs depend on the value of m. Next we consider when $m = 2n$:

$$[E^{2n,1}_{(1)}, E^{1,2n}_{(-1)}] = D_{(0)} - \mathrm{Id}.$$

where

$$D_{(0)} = \sum_{k=1}^{n-1} \frac{1}{n}\left((n-k)E^{k}_{(0)} - kE^{n+k}_{(0)}\right).$$

Applying this to χ gives

$$\begin{aligned} E^{2n,1}_{(1)}\chi &= (D_{(0)}E^{1,2n}_{(-1)} - E^{1,2n}_{(-1)} - E^{1,2n}_{(-1)}E^{2n,1}_{(1)}E^{1,2n}_{(-1)})\,|0\rangle \\ &= (-E^{1,2n}_{(-1)} + E^{1,2n}_{(-1)})\,|0\rangle \\ &= 0. \end{aligned}$$

The remaining cases where $\alpha = (1, m)$ follow similarly. □

While the following result is not necessary to prove our primary theorem, it illuminates the submodule structure of $V^1(\mathfrak{psl}_{n|n})$ which is an interesting topic itself.

Proposition 2.4.4. *As $V^1(\mathfrak{psl}_{n|n})$ modules*

$$\langle \chi_-, \chi_+ \rangle \subset \langle \chi \rangle.$$

Proof. To see that $\chi_+ \in \langle \chi \rangle$ observe that

$$\chi_+ = \frac{1}{2}E^{n+1,1}_{(1)}TE^{n+1,1}_{(1)}T\chi.$$

Similarly to see that $\chi_- \in \langle \chi \rangle$ observe that

$$\chi_- = \frac{1}{2}E^{2n,n}_{(1)}TE^{2n,n-1}_{(1)}TE^{2,1}_{(1)}T\chi.$$

□

Henceforth we will denote the submodule of $V^1(\mathfrak{psl}_{n|n})$ generated by this new singular vector by

$$U = \langle \chi \rangle.$$

We end this section with a conjecture:

Conjecture 2.4.1. *The maximal proper submodule of $V^1(\mathfrak{psl}_{n|n})$ is U. Equivalently*

$$L_1(\mathfrak{psl}_{n|n}) = V^1(\mathfrak{psl}_{n|n})/U.$$

If this conjecture is true, it is likely that the following more general conjecture holds. The reason being that this result holds for a number of Lie algebras as indicated by the singular vector formula given in (4.1).

Conjecture 2.4.2. *Let*

$$\chi_k = (E^{1,2n}_{(-1)})^{k+1} \left|0\right\rangle.$$

Then maximal submodule of $V^k(\mathfrak{psl}_{n|n})$ for $k \in \mathbb{N}$ is $\langle \chi_k \rangle$, or equivalently:

$$L_k(\mathfrak{psl}_{n|n}) = V^k(\mathfrak{psl}_{n|n})/\langle \chi_k \rangle.$$

One possible way to probe whether this conjecture is true and prove it is to employ the inverse quantum Hamiltonian reduction methods used, for example, in [Feh23].

2.5 Associated variety of $L_1(\mathfrak{psl}_{n|n})$

The strategy to proving the main theorem is to use our knowledge of the associated varieties of the $L^{\pm 1}(\mathfrak{sl}_n)$ subalgebras and incorporate the fact that the submodule U must be contained in the maximal submodule of $V^1(\mathfrak{psl}_{n|n})$.

Lemma 2.5.1. $X_{V^1(\mathfrak{psl}_{n|n})/U} \subseteq \overline{\mathbb{S}_{min}(\mathfrak{sl}_n)}$.

Proof. This follows from the same argument used to prove Proposition 2.4.2. □

Lemma 2.5.2. $X_{V^1(\mathfrak{psl}_{n|n})/U} \subseteq \overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}$.

Proof. Consider the map

$$\Psi : V^1(\mathfrak{psl}_{n|n}) \to R_{V^1(\mathfrak{psl}_{n|n})} = \mathbb{C}[\mathfrak{psl}^*_{n|n}]$$
$$v \mapsto \begin{cases} X_{\alpha_1} \dots X_{\alpha_k}, & v = E^{\alpha_1}_{(-1)} \dots E^{\alpha_k}_{(-1)} |0\rangle \notin C_2(V^1(\mathfrak{psl}_{n|n})) \\ 0, & v \in C_2(V^1(\mathfrak{psl}_{n|n})) \end{cases}$$

where the X_α correspond to the coordinates of elementary matrices in $\mathrm{End}(\mathbb{C}^{n|n})$ with indices matching those of α. For $1 \leq i < j \leq n$ consider the vector

$$\begin{aligned} u_{i,j} &= \frac{1}{2} E^{2n,j}_{(1)} T E^{j,1}_{(1)} T E^{2n,i}_{(1)} T E^{i,1}_{(1)} T \chi \\ &= (D^i_{(-1)} D^j_{(-1)} - E^{i,j}_{(-1)} E^{j,i}_{(-1)} + E^{i,2n}_{(-1)} E^{2n,i}_{(-1)} - E^{2n,j}_{(-1)} E^{j,2n}_{(-1)} - 5D^j_{(-2)} - H^{i,j}_{(-2)}) |0\rangle \end{aligned}$$

where

$$H^{i,j} = E^{i,i} - E^{j,j}$$

and

$$D^i = [E^{2n,i}, E^{i,2n}] = \frac{1}{n} \sum_{\substack{k=1 \\ k \neq i}}^{N} H^{i,k} - \frac{1}{n} \sum_{k=n+1}^{2n-1} H^{k,2n}.$$

By construction $u_{i,j} \in U$ and so $\Psi(u_{i,j})$ is zero in $\mathbb{C}[X_{V^1(\mathfrak{psl}_{n|n})/U}]$. Under the image of Ψ this vector becomes

$$\Psi(u_{i,j}) = X_{i,i}X_{j,j} - X_{i,j}X_{j,i} + X_{i,2n}X_{2n,i} - X_{2n,j}X_{j,2n}$$

where we have also employed the trace relations of $\mathbb{C}[\mathfrak{psl}^*_{n|n}]$. If we restrict under the pullback of the embedding $\iota_- : \mathfrak{sl}^*_n \hookrightarrow \mathfrak{psl}^*_{n|n}$ we get

$$\iota^*_-(\Psi(u_{i,j})) = X_{i,i}X_{j,j} - X_{i,j}X_{j,i}.$$

Recalling the decomposition of the 2×2 minors (2.1) given in [Wey89] and Proposition 2.2.2, any 2×2 minors that vanish on $\overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}$ but not $\mathbb{S}_{min}(\mathfrak{sl}_n)$ are not contained in the SL_n representation $U_{2,2}$. We can deduce that $V_{1,2} \cong E^* \otimes E$ is generated by a diagonal element $e_i \otimes e^*_i$ for some i, where $\{e_i\}$ and $\{e^*_i\}$ are dual bases of E and E^* respectively. Using the map given in [Wey89, pg. 237], this embeds into the space of 2×2 minors $\wedge^2 E^* \otimes \wedge^2 E$ as

$$e_i \otimes e^*_i \mapsto \sum_{j \neq i} X_{i,i}X_{j,j} - X_{i,j}X_{j,i}.$$

We have shown that this is contained in $\iota_-^*(\Psi(U))$ and hence vanishes on $X_{V^1(\mathfrak{psl}_{n|n})/U}$. We also know from Lemmas 2.2.2 and 2.5.1 that the 2×2 minor that generates $U_{2,2}$ vanishes on $X_{V^1(\mathfrak{psl}_{n|n})/U}$. Since these generate the entire space of 2×2 minors it follows that $X_{V^1(\mathfrak{psl}_{n|n})/U} \subseteq \overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}$. □

We are now in a position to prove the main theorem:

Theorem 2.5.1. *For $n > 3$*

$$X_{L_1(\mathfrak{psl}_{n|n})} = \overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}.$$

Proof. It is known that $L_1(\mathfrak{psl}_{n|n})$ is not C_2-cofinite [GK06, Section 0.4]. This implies $\dim X_{L_1(\mathfrak{psl}_{n|n})} \neq 0$ so we can conclude $X_{L_1(\mathfrak{psl}_{n|n})} \neq \{0\}$. Furthermore, we know $X_{L_1(\mathfrak{psl}_{n|n})}$ must be $PSL_{n|n}$-invariant. From Proposition 2.4.2 we have

$$X_{L_1(\mathfrak{psl}_{n|n})} \subseteq \overline{\mathbb{S}_{min}(\mathfrak{sl}_n)} \times pt.$$

and so $X_{L_1(\mathfrak{psl}_{n|n})}$ must be $SL_n \times SL_n \subset PSL_{n|n}$ invariant, with the second copy acting trivially. Hence $X_{L_1(\mathfrak{psl}_{n|n})}$ is SL_n-invariant. Since

$$\ker \pi_{V^1(\mathfrak{psl}_{n|n})/U} \subseteq \ker \pi_{L_1(\mathfrak{psl}_{n|n})}$$

Lemma 2.5.2 implies

$$X_{L_1(\mathfrak{psl}_{n|n})} \subseteq X_{V^1(\mathfrak{psl}_{n|n})/U} \subseteq \overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}.$$

The only non-trivial SL_n-invariant subvarieties of $\overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}$ are itself and $\mathbb{O}_{\min}(\mathfrak{sl}_n)$. However $X_{L_1(\mathfrak{psl}_{n|n})}$ must be closed, hence the result. □

Corollary 2.5.1. *For all $n > 3$ the VOA $L_1(\mathfrak{psl}_{n|n})$ is quasi-lisse.*

Proof. This follows since $\overline{\mathbb{O}_{\min}(\mathfrak{sl}_n)}$ is known to have finitely many symplectic leaves. □

Chapter 3

Tilting generator for $T^*\mathrm{Gr}(2,4)$

3.1 Introduction

One of the cornerstone ideas of geometric representation theory is the construction of equivalences between algebraic and geometric categories. Perhaps the most famous example of such an equivalence is the Beilinson-Bernstein theorem, relating modules over a semisimple Lie algebra and D-modules on the corresponding flag variety. The functor relating these categories is simply taking sections, and so is represented by the algebra of twisted differential operators itself, thought of as a twisted D-module.

If, instead of D-modules, we consider the coherent sheaves on a variety which is not affine, then the structure of this category conspires against the existence of a similar theorem: in particular, this category will effectively never have projective objects, so there are no representable exact functors. On the other hand, we can fix this defect by passing to derived categories and functors. For example, by [Bez+08, Th. 5.3.1], if we consider a base field $\Bbbk$ of large positive characteristic, the category of coherent sheaves on T^*G/B set-theoretically supported on G/B is equivalent to the principal block of the category of finite-dimensional representations of the Lie algebra $U(\mathfrak{g})$ over $\Bbbk$.

As with any equivalence to the category of modules over a ring, the key to the equivalence is the image of the ring as a module over itself. This is a little more subtle, due to the finiteness conditions, but this functor is of the form $\mathbb{R}\operatorname{Hom}(T,-)$ for T a vector bundle which is a **tilting generator**.

More generally, let X be a variety over a field $\Bbbk$ which is projective over an affine variety (that is, a closed subvariety of $\mathbb{A}^k_\Bbbk \times \mathbb{P}^n_\Bbbk$). For a vector bundle T, let $A = \operatorname{End}(T)^{\mathrm{op}}$ and A-mod be the abelian category of finitely presented A-modules.

Definition 3.1.1. *A vector bundle $\mathcal{Z}$ on a quasi-projective variety X is called a* **tilting generator** *if the functor*

$$\mathbb{R}\operatorname{Hom}(\mathcal{Z}, -)\colon D^b(\mathsf{Coh}(X)) \to D^b(A\text{-mod})$$

is an equivalence of categories.

Note that this functor sends $\mathcal{Z}$ to A as a left module over itself, and thus implies that:

1. For all $i > 0$, we have $\operatorname{Ext}^i(\mathcal{Z}, \mathcal{Z}) = 0$.
2. The object $\mathcal{Z}$ generates $D^b(\mathsf{Coh}(X))$, that is, $\mathbb{R}\operatorname{Hom}(\mathcal{Z}, M) = 0$ for a coherent sheaf M if and only if $M = 0$. In fact, by adjunction, we have:

$$M \cong \mathbb{R}\operatorname{Hom}(\mathcal{Z}, M) \overset{\mathbb{L}}{\otimes}_A \mathcal{Z}.$$

In more concrete terms, under this equivalence, projective A-modules correspond to summands of $\mathcal{Z}^{\oplus q}$. The module $\mathbb{R}\operatorname{Hom}(\mathcal{Z}, M)$ has a finite projective resolution, and we can reconstruct M by turning each term of the resolution into the corresponding projective.

Note, there are several natural operations which preserve the set of tilting generators:

i. The dual of a tilting generator is a tilting generator.

ii. The tensor product of a tilting generator with any line bundle is a tilting generator.

iii. Any sheaf $\mathcal{Z}'$ which is **equiconstituted** to a tilting generator $\mathcal{Z}$ (i.e. a module is a summand of $(\mathcal{Z}')^{\oplus r}$ for some r if and only if it is a summand of $\mathcal{Z}^{\oplus s}$ for some s) is a tilting generator.

In general, there is no guarantee that a given variety carries a tilting generator, but a remarkable theorem of Kaledin [Kal08] guarantees that a conical symplectic resolution of singularities must carry such a tilting generator. In fact, the same algebra A appears as the endomorphisms of a tilting generator on any such resolution of a given conical symplectic singularity, giving a beautiful proof of Kawamata's conjecture that K-equivalence implies D-equivalence in this case. The algebra A is a **noncommutative crepant resolution of singularities**, so we can think of this result as proving the D-equivalence of two usual resolutions of the singularities factoring through a noncommutative resolution.

Unfortunately, Kaledin's prescription is quite abstract and passes through quantization in characteristic p. In particular, our description of coherent sheaves on T^*G/B in terms of

$U(\mathfrak{gl}_n)$ has obvious defects—it only covers the coherent sheaves with set-theoretic support on the zero-section and only works in characteristic p.

In order to fix this defect, we must replace $U(\mathfrak{gl}_n)$ by the algebra $A = \mathrm{End}(T)^{\mathrm{op}}$ for T a corresponding tilting bundle, but while [Kal08] guarantees the existence of such a bundle, it does not provide much guidance on how to compute with it.

In some special cases, we can compute this tilting bundle and its endomorphism algebra more explicitly. For example, for the cotangent bundle $T^*\mathbb{CP}^{n-1}$, we can show on abstract grounds that this tilting generator is a sum of line bundles. An easy calculation confirms that the only sum of line bundles which gives a tilting generator is the pullback of the famous tilting generator of Beilinson $\mathcal{O}(a) \oplus \cdots \oplus \mathcal{O}(a-n)$ for some $a \in \mathbb{Z}$ [Bei78]. However, it is rare that an algebraic variety has a tilting generator which is a sum of line bundles, so we look for richer special classes we can study.

The one which will be relevant for us is when the symplectic singularity is a Coulomb branch in the sense of Braverman, Finkelberg, and Nakajima [BFN18]. This case is studied in the work of Webster [Web19a], especially in the case of **bow varieties**, which are realized as Coulomb branches by work of Nakajima and Takayama [NT17]. In particular, the algebra A in this case can be realised as a cylindrical version of a KLR algebra.

This description has its own defects, in that the resulting coherent sheaves, described in terms of modules over the projective coordinate ring of a resolution, are hard to compare with other descriptions of coherent sheaves on these varieties.

Our aim in this paper is to give a description of the resulting coherent sheaves in one of the most familiar of these singularities. The sheaves we will need below are: the structure sheaf $\mathcal{O}$, the unique ample line bundle $\mathcal{L}$ which generates $\mathrm{Pic}(T^*X) \cong \mathbb{Z}$, the tautological bundle $\mathcal{T} \subset \mathcal{O}^{\oplus 4}$, and $\mathcal{H}$, the unique extension $\mathcal{L}^{-2} \to \mathcal{H} \to \mathcal{O}$ such that $H^1(T^*X; \mathcal{H}) = 0$.

Theorem 3.1.1. *Under the natural operations mentioned above, any of Kaledin's tilting bundles will be equivalent to the tilting generator* $\mathcal{Z} \cong \mathcal{O} \oplus \mathcal{L}^{-1} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{L}^{-1}) \oplus \mathcal{T} \oplus \mathcal{T}^{\perp}$.

A natural question is how this construction compares with other known constructions of tilting bundles on $X = \mathrm{Gr}(2,4)$ and T^*X. Kapranov has famously constructed a tilting generator on the Grassmannian X [Kap88]; taking the dual of the version of this generator given in [DS14, §2.1.2], one of Kapranov's generators on $\mathrm{Gr}(2,4)$ is given by

$$\mathcal{O} \oplus \mathcal{L}^{-1} \oplus \mathcal{L}^{-2} \oplus \mathcal{T} \oplus \mathcal{T} \otimes \mathcal{L}^{-1} \oplus \mathrm{Sym}^2(\mathcal{T}).$$

As noted in [Kaw06, Rem. 3.6], the pullback of this vector bundle to T^*X is no longer tilting: $\mathrm{Ext}^1(\mathcal{O}, \mathcal{L}^{-2}) \neq 0$. Since $\mathcal{H}$ is essentially constructed to extend $\mathcal{L}^{-2}$ so as to kill these

higher extensions, there are some hints that our tilting generator might be a correction of Kapranov's, though there is another important difference—Kapranov's generator is always a sum of Schur functors applied to the tautological bundle and its dual, but $\mathcal{T}^{\perp}$ is not of this form.

A more implicit construction of a tilting generator on this space is given by Toda–Uehara [TU10]; note that this construction is not canonical, so we cannot speak of *the* Toda-Uehara tilting generator, only the class of generators they construct, which are unique up to equiconstitution. That is, the indecomposable summands that appear are well defined, but different choices can lead to different multiplicities in the final tilting generator. At the moment, due to the difficulty of making their construction explicit, we have not been able to prove or disprove the equivalence of our construction. However, we can show that this tilting generators has a large summand in common (Proposition 3.4.1), leading us to conjecture that:

Conjecture 3.1.1. *The tilting generator $\mathcal{Z}$ is equiconstituted with any Toda-Uehara tilting generator.*

3.2 Preliminaries

3.2.1 $T^*\mathrm{Gr}(n-k,n)$ and its vector bundles

Let $X = \mathrm{Gr}(n-k,n)$ be the Grassmannian of codimension k subspaces (or equivalently, dimension k quotients) in $\mathbb{C}^n$ and consider the Plücker embedding:

$$\begin{aligned} \iota : X &\hookrightarrow \mathbb{P}(\textstyle\bigwedge^{n-k}\mathbb{C}^n) \cong \mathbb{P}^{\binom{n}{k}-1} \\ V &\mapsto \textstyle\bigwedge^{n-k} V. \end{aligned}$$

We let $e_1, \dots, e_n$ be the usual coordinates of $\mathbb{C}^n$. We will use the induced Plücker coordinates, which satisfy the Plücker relations. The case of primary interest to us is $k = 2$ and $n = 4$, where Plücker coordinates correspond to ordered pairs $\{e_{ij} := e_i \wedge e_j\}$ and there is a single Plücker relation:

$$e_{12}e_{34} - e_{13}e_{24} + e_{14}e_{23} = 0. \tag{2.1}$$

We let $\mathcal{L}$ denote the pullback of $\mathcal{O}(1)$ under this map; this is the unique ample generator of the Picard group of X, and its pullback to T^*X is the unique ample generator of the Picard group there. We have an isomorphism $\Gamma(X;\mathcal{L}) \cong \bigwedge^k \mathbb{C}^n$ as representations of GL_n

by Borel-Weil; more generally $\Gamma(X;\mathcal{L}^{\otimes p})$ is an irreducible representation of GL_n whose highest weight is $p\omega_k$, where ω_k is the fundamental highest weight of $\bigwedge^k\mathbb{C}^n$.

We'll also want to consider the trivial bundle $\mathcal{E}=\mathcal{O}_X^{\oplus n}$ of rank n. This contains the rank $n-k$ tautological bundle $\mathcal{T}$ whose fiber over a subspace $U\subset\mathbb{C}^n$ is the subspace itself; let $\mathcal{Q}=\mathcal{E}/\mathcal{T}$ be the tautological quotient bundle of rank k. Note that we can think of e_1,e_2,e_3,e_4 as sections of $\mathcal{Q}$, and thus of e_{ij} as sections of $\bigwedge^k\mathcal{Q}\cong(\mathcal{E}\wedge\mathcal{E})/(\mathcal{E}\wedge\mathcal{T})\cong\mathcal{L}$; this map induces the isomorphism $\Gamma(X;\mathcal{L})\cong\bigwedge^k\mathbb{C}^n$. Dually, we have $\bigwedge^{n-k}\mathcal{T}\cong\mathcal{L}^{-1}$.

The bundle $\mathcal{E}=\mathcal{O}_X^{\oplus n}$ is self-dual via the usual dot product, so we have the dual filtration of $\mathcal{E}$ by the subbundle $\mathcal{T}^\perp\cong\mathcal{Q}^*$. We have an isomorphism $\mathrm{Gr}(n-k,n)\cong\mathrm{Gr}(k,n)$ which swaps the role of the tautological bundles and their perpendiculars. This map is not GL_n-equivariant; rather, it intertwines the action of $g\in GL_n$ with $(g^T)^{-1}$, its inverse-transpose. Depending on which of these GL_n actions we use, we can identify the sections of $\mathcal{L}$ with $\bigwedge^k\mathbb{C}^n$ or $\bigwedge^{n-k}\mathbb{C}^n$.

For the rest of the section below, we specialise to $k=2,n=4$ for simplicity of notation, though most of the results below extend to the general case. In this case, both $\mathcal{T}$ and $\mathcal{Q}$ have rank 2. Since any rank 2 vector bundle satisfies $V\cong V^*\otimes\bigwedge^2V$, we have $\mathcal{T}^\perp\cong\mathcal{Q}\otimes\mathcal{L}^{-1}$.

For any very ample line bundle on a projective variety, we can obtain an open cover by taking the non-vanishing sets of a basis of the space of sections. Applying this to $\mathcal{L}$, we find that X has an open cover $\mathcal{U}=\{U_{ij}\}$ where

$$U_{ij}=X\backslash\{e_{ij}=0\}.$$

We now consider the cotangent bundle $\pi:T^*X\to X$. This has an open cover $\mathcal{U}'=\{\pi^{-1}(U_{ij})\}$.

Given a vector bundle $\mathcal{Y}$ on T^*X, the restriction of $\mathcal{Y}$ to U'_{ij} must be trivial. Thus, if we choose trivialisations $\tau_{ij}\colon\mathcal{O}^{\mathrm{rk}\,\mathcal{Y}}\to\mathcal{Y}$ on U'_{ij}, we obtain the usual cocycle description of this bundle in terms of the transition functions $\gamma_{rs}^{ij}=\tau_{ij}^{-1}\circ\tau_{rs}$. We want to trivialise the bundles $\mathcal{L},\mathcal{T},\mathcal{T}^\perp$ on each of these open sets and compute the transition functions.

Of course, for a line bundle, this is simple: the section e_{ij} defines the desired trivialisation, and $\gamma_{rs}^{ij}=e_{rs}/e_{ij}$. More generally, for a power $\mathcal{L}^m$, we have transition functions $\gamma_{rs}^{ij}=(e_{rs}/e_{ij})^m$.

For the tautological bundles, it's useful to think about the following approach: given $(x_1,\dots,x_4)\in\mathcal{E}_V$, we can test whether $(x_1,\dots,x_4)\in V$ by whether the wedge product of $(x_1,\dots,x_4)$ with a basis of V is 0. The coordinate of $e_i\wedge e_j\wedge e_r$ in this triple wedge product is $x_ie_{jr}+x_je_{ri}+x_re_{ij}=0$. That is, $(x_1,\dots,x_4)\in V$ if it is perpendicular to the

sections of $\mathcal{E}(1)$ defined by:

$$A_1 = \begin{pmatrix} 0 \\ e_{34} \\ -e_{24} \\ e_{23} \end{pmatrix}, \quad A_2 = \begin{pmatrix} -e_{34} \\ 0 \\ e_{14} \\ -e_{13} \end{pmatrix}, \quad A_3 = \begin{pmatrix} e_{24} \\ -e_{14} \\ 0 \\ e_{12} \end{pmatrix}, \quad A_4 = \begin{pmatrix} -e_{23} \\ e_{13} \\ -e_{12} \\ 0 \end{pmatrix}.$$

That is, the image of these sections span $\mathcal{T}^\perp$. One can easily work out from the Plücker relations (2.1) that we have a similar set of maps $\mathcal{L}^{-1} \to \mathcal{T}$, defined by the vectors $B_i = (B_{1i}, \ldots, B_{4i})$ where:

$$B_{ij} = \begin{cases} e_{ij} & \text{if } i \neq j \\ 0 & \text{if } i = j. \end{cases}$$

On U'_{ij}, we have that e_{ij} is invertible, and we can always trivialise $\mathcal{T}$ by using the vectors $B_i/e_{ij}, B_j/e_{ij}$, and similarly for $\mathcal{T}^\perp$ using the vectors $A_k/e_{ij}, A_\ell/e_{ij}$ where $\{r,s\} = \{i,j\}^c$. We can always compute the transition matrices using the fact that any 3 A_i's or B_i's are linearly dependent. Note that while there are many pairs of $\{i,j\}$ in $\{1,2,3,4\}$, we only need the transition function for a single pair of open sets: the union $U_{ij} \cup U_{rs}$ has complement of codimension greater than or equal to 2, so a rational map between vector bundles defined on this open subset must extend to an isomorphism on T^*X.

To give an example we will use later, let us consider $i = 1, j = 3, r = 2, s = 4$. The transition functions in this case are defined on $U'_{13} \cap U'_{24}$. On this intersection, the Plücker

relation (2.1) tells us

$$e_{24}A_2 = \begin{pmatrix} e_{24}e_{34} \\ 0 \\ -e_{24}e_{14} \\ e_{12}e_{34} + e_{14}e_{23} \end{pmatrix} = e_{14}A_1 + e_{34}A_3$$

$$e_{24}A_4 = \begin{pmatrix} -e_{24}e_{23} \\ e_{14}e_{23} + e_{12}e_{34} \\ -e_{24}e_{12} \\ 0 \end{pmatrix} = e_{12}A_1 - e_{23}A_3$$

$$e_{24}B_1 = \begin{pmatrix} 0 \\ e_{24}e_{12} \\ e_{12}e_{34} + e_{14}e_{23} \\ e_{24}e_{14} \end{pmatrix} = e_{14}B_2 + e_{12}B_4$$

$$e_{24}B_3 = \begin{pmatrix} -e_{12}e_{34} - e_{14}e_{23} \\ -e_{24}e_{23} \\ 0 \\ e_{24}e_{34} \end{pmatrix} = e_{34}B_2 - e_{23}B_4.$$

Dividing by $e_{13}e_{24}$, we find that:

Lemma 3.2.1. *The transition functions γ_{24}^{13} are given by:*

1. *For $\mathcal{L}^k$, we have $\gamma_{13}^{24} = e_{13}^k/e_{24}^k, \gamma_{24}^{13} = e_{24}^k/e_{13}^k$.*
2. *For $\mathcal{T}$, we have $\gamma_{13}^{24} = \begin{bmatrix} e_{14}/e_{13} & e_{34}/e_{13} \\ e_{12}/e_{13} & -e_{23}/e_{13} \end{bmatrix}, \gamma_{24}^{13} = \begin{bmatrix} e_{23}/e_{24} & e_{12}/e_{24} \\ e_{34}/e_{24} & -e_{14}/e_{24} \end{bmatrix}$.*
3. *For $\mathcal{T}^\perp$, we have $\gamma_{13}^{24} = \begin{bmatrix} e_{14}/e_{13} & e_{12}/e_{13} \\ e_{34}/e_{13} & -e_{23}/e_{13} \end{bmatrix}, \gamma_{24}^{13} = \begin{bmatrix} e_{23}/e_{24} & e_{34}/e_{24} \\ e_{12}/e_{24} & -e_{14}/e_{24} \end{bmatrix}$.*

In [TU10, §7.2], it's shown that

$$\mathrm{Ext}^1_{T^*X}(\mathcal{O}, \mathcal{L}^{-2}) \cong \bigoplus_{j=1}^{\infty} \Gamma(X; \mathrm{Sym}^{j-1}(\mathcal{T})) \cong \Gamma(T^*X; \mathcal{O}) \tag{2.2}$$

where this decomposition gives the weight decomposition under the scaling $\mathbb{C}^*$-action. The summand for j has weight $2j$; in particular, $j = 1$ gives a unique line of nontrivial extensions of minimal weight which freely generate $\mathrm{Ext}^1_{T^*X}(\mathcal{O}, \mathcal{L}^{-2})$ as a module over $\Gamma(T^*X; \mathcal{O})$. We'll use later that:

Lemma 3.2.2. *There is a unique indecomposable extension $0 \to \mathcal{L}^{-2} \to \mathcal{H} \to \mathcal{O} \to 0$ which is equivariant for the squared scaling action, where $\mathcal{O}$ has the usual equivariant structure (where its lowest degree sections have weight 0) and $\mathcal{L}^{-2}$ is tensored with a space of weight 2 under the scaling action (where its lowest degree sections have weight 0). That is, this extension has transition matrix*

$$\gamma_{24}^{13} = \begin{bmatrix} e_{24}^2/e_{13}^2 & f \\ 0 & 1 \end{bmatrix} \tag{2.3}$$

where f is a function of weight 2 under the squared scaling action. We'll also use the tensor product $\mathcal{H} \otimes \mathcal{L}$, which has transition functions

$$\gamma_{24}^{13} = \begin{bmatrix} e_{24}/e_{13} & g \\ 0 & e_{13}/e_{24} \end{bmatrix}. \tag{2.4}$$

The extension $\mathcal{H}$ is uniquely characterised by the property $H^1(T^*X; \mathcal{H}) = 0$.

Proof. The uniqueness is determined by the fact that the $j = 1$ component of $\mathrm{Ext}^1_{T^*X}(\mathcal{O}, \mathcal{L}^{-2})$ is $\Gamma(X; \mathcal{O})$ which is 1-dimensional. The description of transition functions follows by the standard form for an extension of vector bundles.

For a given $\gamma \in \mathrm{Ext}^1_{T^*X}(\mathcal{O}, \mathcal{L}^{-2})$, let $\mathcal{H}_\gamma$ be the corresponding extension. Then, by the usual long exact sequence, $H^1(T^*X; \mathcal{H}_\gamma)$ is the cokernel of the boundary map $\Gamma(T^*X; \mathcal{O}) \to H^1(T^*X; \mathcal{L}^{-2})$. That is, it is trivial if and only if γ generates $H^1(T^*X; \mathcal{L}^{-2})$ as a $\Gamma(T^*X; \mathcal{O})$-module. Since $\Gamma(T^*X; \mathcal{O})$ has no units, this module has a unique line of such generators, which exactly give us $\mathcal{H}_\gamma \cong \mathcal{H}$. □

We will determine the functions f, g later, but by the uniqueness of this non-trivial extension, we do not need to determine them now.

We can also express X as the partial flag variety G/P for $G = SL(4, \mathbb{C})$ and

$$P = \left\{ \begin{pmatrix} A & 0 \\ C & D \end{pmatrix} \mid A, D \in SL(2, \mathbb{C}),\ C \in \mathfrak{gl}(2, \mathbb{C}) \right\}.$$

Since $T^*\mathrm{Gr}(k, n)$ is a GL_n-equivariant vector bundle, we can write it as the associated bundle for a representation of P, which is most naturally written as the perpendicular to $\mathfrak{p}$ in the dual $\mathfrak{g}^*$. Using the trace pairing to identify $\mathfrak{g}^* \cong \mathfrak{g}$, this perpendicular is identified with the unipotent radical of $\mathfrak{p}$, the strictly upper triangular block matrices. We thus have a Springer map $\sigma \colon T^*\mathrm{Gr}(k, n) \to \mathfrak{g}^* \cong \mathfrak{g}$, which we can also think of as the moment map

for the usual G-action. The image of this map is the closure $\overline{\mathcal{N}(\lambda)}$ of the orbit of nilpotent matrices whose Jordan type is $\lambda = (2^k, 1^{n-2k})$, and this map is a symplectic resolution of singularities. The cotangent bundle $T^*\mathrm{Gr}(k,n)$ is the only variety that appears as a symplectic resolution of $\overline{\mathcal{N}(\mu)}$, though there are two non-isomorphic symplectic resolution maps if $n > 2$—the map σ and its composition with the transposition map on $\mathfrak{g}$.

3.2.2 Quiver gauge theories and cylindrical KLRW algebras

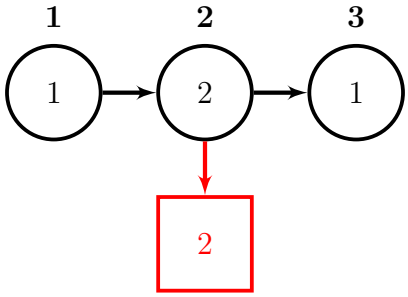
Let Γ be a finite quiver with vertices $\mathsf{V}(\Gamma)$, edges $\mathsf{E}(\Gamma)$ and dimension vectors $\mathbf{v}$,$\mathbf{w} : \mathsf{V} \to \mathbb{N}$ with the $\mathbf{w}$ corresponding to the framing and denote the number of vertices by $n = |\mathsf{V}(\Gamma)|$. The quiver gauge theory associated to Γ has gauge group

$$G(\Gamma) = \prod \mathrm{GL}(\mathbb{C}^{v_i})$$

and matter representation

$$V(\Gamma) = \left(\bigoplus_{i \to j} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}) \right) \oplus \left(\bigoplus_{i \in \mathsf{V}} \mathrm{Hom}(\mathbb{C}^{w_i}, \mathbb{C}^{v_i}) \right).$$

The theory of interest in this paper is determined by the data of the following framed quiver diagram:



It was shown in [Web22, Th. E] that the Coulomb branch of the corresponding $3d$ $\mathcal{N} = 4$ gauge theory is given by a cylindrical KLRW algebra defined by the quiver data. These diagrammatic algebras are generated by string diagrams on the cylinder $\mathbb{R}/\mathbb{Z} \times [0,1]$ with product given by vertical composition. We can define these diagrams from the quiver data more precisely as follows.

Throughout the rest of this paper, a **strand** will mean a curve in $\mathbb{R}/\mathbb{Z} \times [0,1]$ of the form $\{(\bar{\pi}(t), t) \mid t \in [0,1]\}$ for some path $\bar{\pi} \colon [0,1] \to \mathbb{R}/\mathbb{Z}$.

Definition 3.2.1. *An **(unflavoured) cylindrical KLRW diagram** associated to the framed quiver Γ with dimension vectors $\mathbf{v}$ and $\mathbf{w}$ is a diagram on the cylinder $\mathbb{R}/\mathbb{Z} \times [0,1]$ consisting of strands beginning at the bottom edge $\mathbb{R}/\mathbb{Z} \times 0$ and terminating at the top edge $\mathbb{R}/\mathbb{Z} \times 1$. For each i, there will be an unordered v_i-tuple of strands, which we call* **black** *and draw in that colour, and an unordered w_i-tuple of strands, which we call* **red** *and draw with that colour. We can also add dots at arbitrary positions on black strands (but not on red) that avoid crossings and the lines $y = 0$ and $y = 1$. In a cylindrical diagram, we require the red lines to be vertical (the corresponding $\bar{\pi}(t)$ is constant)*[1].

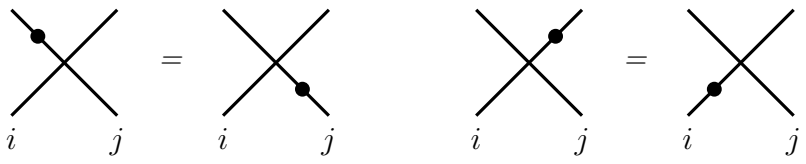
These must satisfy the usual genericity property of avoiding tangencies and triple points between any set of strands, as well as strands meeting at $y = 0$ or $y = 1$. We identify any diagrams that differ by isotopies that preserve these genericity conditions. For simplicity of drawing, we'll also assume genericity with respect to the line $x = 0$.

Let us now describe our conventions for drawing cylindrical diagrams. We'll draw these on the page in the rectangle $[0,1] \times [0,1]$ with seams on the left and right sides of the diagram where we should glue to obtain the cylindrical diagram.

Note that the order in which strands meet the circles $y = 0$ and and $y = 1$ induce two distinct cyclic orders on the set of strands, which we'll call the bottom and top orders. By reading the labels of the strands, red or black, in a positive direction starting at $x = 0$ (left to right in the convention discussed above), we obtain two words whose entries are v_i instances of i which are black and w_i which are red, ranging over $i \in \mathsf{V}(\Gamma)$. We'll call these words the **bottom** and **top** of our diagram.

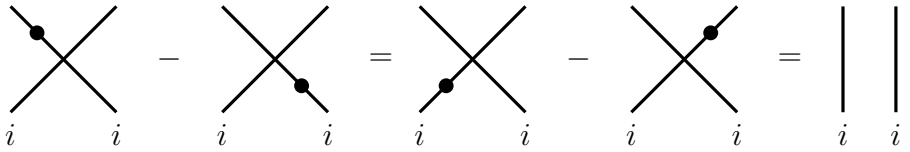
Note that the top and bottom of a KLRW diagram will have the same order on red strands, so we will typically take this order as fixed, and only consider diagrams with this order.

Definition 3.2.2. *The* **cylindrical (unflavored) KLRW algebra** $\mathring{R}$ *attached to the data $\Gamma, \mathbf{v}, \mathbf{w}$ and an order on the red strands is the quotient of the formal span over a commutative ring $\Bbbk$ of cylindrical KLRW diagrams for these data by the local relations below:*

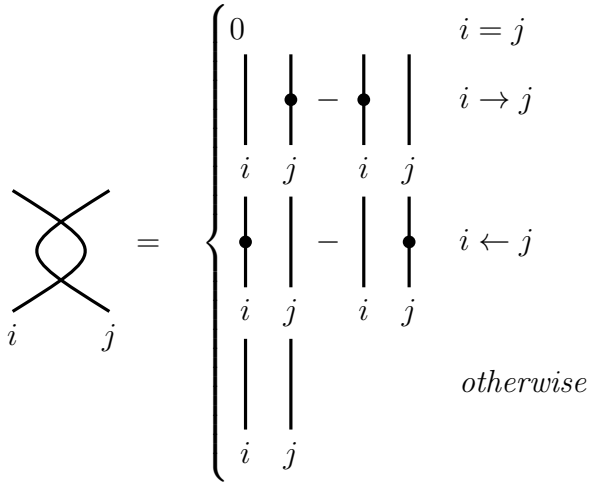


unless $i = j$ (2.5a)

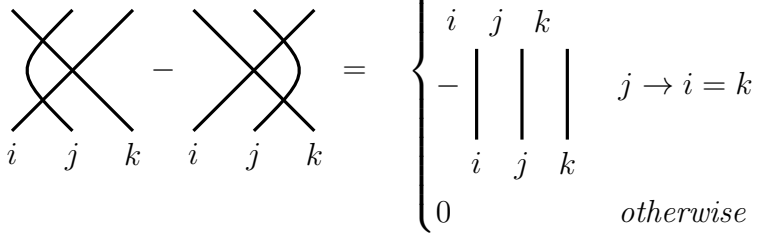
[1] We'll consider diagrams where these red strands cross later.



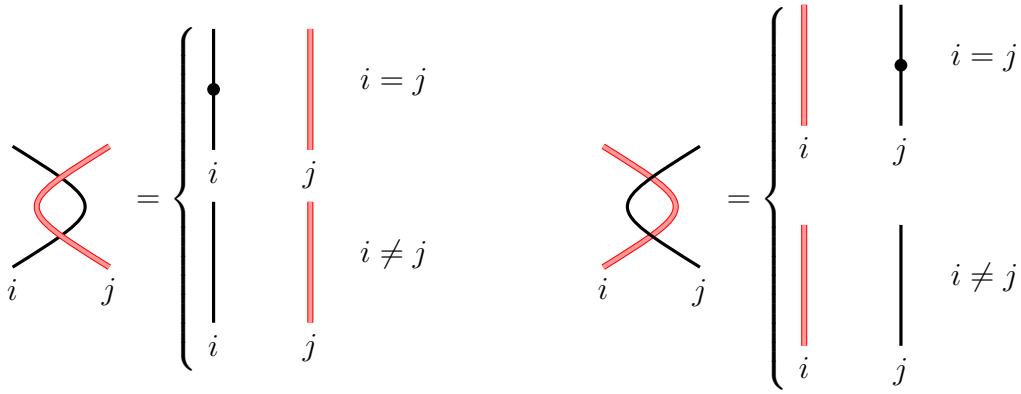
(2.5b)



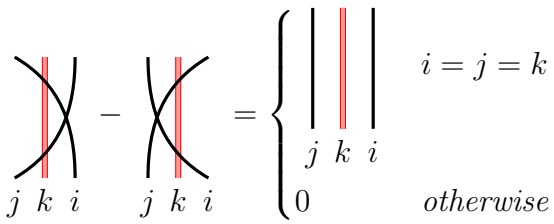
(2.5c)



(2.5d)



(2.5e)

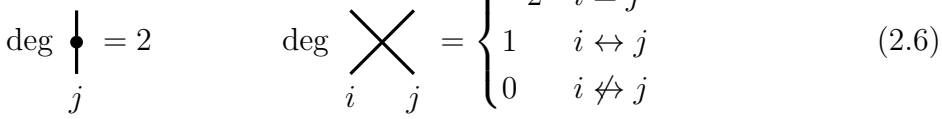


(2.5f)

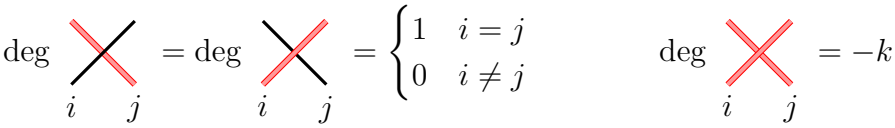
For all other triple points, we set the two sides of the isotopy through it equal.

Given a word in red and black copies of the symbols $\mathsf{V}(\Gamma)$, we can define a corresponding idempotent by having straight vertical strands on the cylinder, with no crossings or dots, where the strands we encounter are labelled by the vertices in the word, with the corresponding colour.

Note that the relations above are homogeneous under the **scaling grading** defined by:



(2.6)



(2.7)

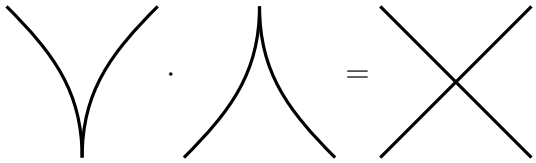
We also have a **winding grading** by the group $\mathbb{Z}^\Gamma$ whose i-component counts with sign the number of times that all strands with label i cross the vertical line $x = 0$; this is left unchanged by the relations eqs. (2.5a) to (2.5f) (though it can change which strand a crossing sits on) and isotopy can only cancel crossings with opposite sign.

When there are k consecutive strands with the same label, then crossings between these strands and dots on them form a copy of the nilHecke algebra of rank k. It's well-known that this algebra is isomorphic to a $k!$ matrix algebra over symmetric polynomials in the dots (see, for example, [Lau10, Prop. 3.5]) and so any object in an abelian (or more generally Karoubian) category with an action of this algebra is the direct sum of $k!$ isomorphic objects. We can make this object easier to access by replacing the cylindrical KLRW algebra with the Morita equivalent algebra where we add this divided power formally.

A graphical rule for doing this is already given in [Kho+12]. We now allow strands with the same label to meet at the top and bottom of diagrams. These should be generic in the sense that at these meeting points, all the strands which meet should have different

slopes, so there is a clear ordering on the strands as they leave the meeting point. When we compose diagrams with such meetings, the composition will be taken to be zero unless the labels and groupings of strands match—if k strands meet at the bottom of one diagram, k strands with the same label must meet at the top of the other.

We need a rule for interpreting a composition where we join the bottom of one diagram and the top of another where k points with the same label meet. This is done using the convention of [Kho+12, (2.69)]: if we join k strands and split them, then we can replace this join and split with the diagram D_k where we perform a half-twist on the strands which joined. The only case that we will use in this paper is that joining and then splitting two strands can be interpreted as a crossing of the strands.



We also add in idempotents with vertical strands, where we allow each strand to have a thickness > 1; describing these as a word, we write $i^{(k)}$ for a strand with label i and thickness k. As in [Kho+12, (2.63)], we can put symmetric functions in an alphabet of k dots on strands of thickness k; these can be constructed from dots on thickness 1 strands by [Kho+12, (2.72)].

Now, let e be any idempotent where for every label i, all v_i black strands with this label are joined. For example, we can take $e(\mathbf{i})$ for the word $\mathbf{i} = 1^{w_1} \cdots n^{w_n} 1^{(v_1)} 2^{(v_2)} 3^{(v_3)} \cdots (n-1)^{(v_{n-1})}$. We let $\bullet_i \in e\mathring{R}e$ denote the idempotent e with the elementary symmetric function of degree 1 applied to the single thick strand with label i (and no other dots added).

Theorem 3.2.1 ([Web22, Th. 6.15])**.** *The subalgebra $e\mathring{R}e$ is isomorphic to the Coulomb branch algebra for the quiver gauge theory defined by $\Gamma, \mathbf{v}, \mathbf{w}$. In particular, this subring is abelian and $\mathfrak{M} = \mathrm{Spec}(e\mathring{R}e)$ is the BFN Coulomb branch of this theory.*

In this paper, we'll only ever consider the vectors $\mathbf{w}$ of "level two" where there are two red strands total. Thus, we can distinguish these as the "left" and "right" red strands. Let us specialise to this case from now on, to simplify our discussion. In order to discuss line bundles on the Grassmannian, we'll need to consider twisted KLRW diagrams as well.

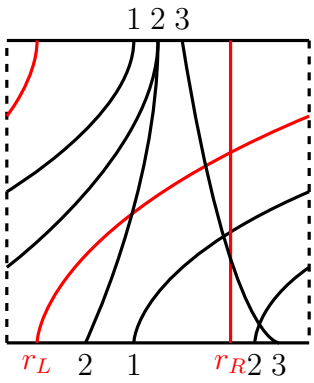
Definition 3.2.3. *A k-twisted (cylindrical) KLRW diagram for $k \in \mathbb{Z}$ is a diagram that satisfies all the same local rules as a KLRW diagram, except that instead of being vertical,*

the left red strand wraps around the cylinder k times (that is, if $k < 0$, it wraps $|k|$ times in the negative direction), while the right strand remains vertical. We assume that the red strands never make a bigon, that is, there are $|k|$ intersection points of the red strands, all with the same sign, and that the top and bottom of the left red strand is the same point of S^1.

The quotient of the formal span of the k-twisted diagrams by the relations (2.5a–2.5f) gives a bimodule $\mathring{T}^k$ over $\mathring{R}$, and the sum $\mathbf{R} = \oplus_{k\geq 0}\mathring{T}^k$ forms a graded ring with composition given by stacking.

To avoid confusion with other gradings, we say that elements of $\mathring{T}^k$ have **twist degree** k. Of course, since $\mathring{R}$ is the twist degree 0 subalgebra of $\mathbf{R}$, this means that we can think of e as an element of $\mathbf{R}$.

The following is an example of a diagram in $e\mathring{T}^1$:



Theorem 3.2.2 ([Web22, Th. 6.19])**.** *The ring $\mathbf{A} = e\mathbf{R}e$ is commutative and $\tilde{\mathfrak{M}} = \mathrm{Proj}(\mathbf{A})$ is a symplectic resolution of $\mathrm{Spec}(e\mathring{R}e)$.*

3.2.3 Grassmannians from quiver gauge theories

Fix integers n and $0 < k \leq n/2$, as before. Let $\Gamma = A_{n-1}$, with $\mathsf{V} = \{1, \ldots, n-1\}$ in the usual identification; since the orientation will occasionally be important, we choose the arrows $i \to i+1$. Consider the dimension vectors $\mathbf{w} = \mathbf{e}_k + \mathbf{e}_{n-k}$, and $\mathbf{v} = \sum_{i=1}^{n-1} \max(i, n-i-1, k)$. For example, if $n = 4, k = 2$, then $\mathbf{w} = (0, 2, 0)$ and $\mathbf{v} = (1, 2, 1)$. On the other hand, if $n = 7, k = 2$, then $\mathbf{w} = (0, 1, 0, 0, 1, 0)$ and $\mathbf{v} = (1, 2, 2, 2, 2, 1)$. If we consider a $k \times (n-k)$ grid of boxes indexed by $\{1, \ldots, k\} \times \{1, \ldots, n-k\}$, then v_m is the number of

boxes (i,j) that satisfy $m = i+j-1$. Let e be the idempotent $(n-k)1^{(v_1)}2^{(v_2)}3^{(v_3)}\cdots(n-1)^{(v_{n-1})}k$. In the case of $n=4$ and $k=2$, this idempotent is given by $212^{(2)}32$.

The choice of $\mathbf{w}, \mathbf{v}$ above might seem slightly random, but it is related to $T^*\mathrm{Gr}(k,n)$ by a result of Nakajima and Takayama, building on work of Maffei, Mirković-Vilonen and earlier work of Nakajima. In the notation of [NT17, §7.4], we see that $T^*\mathrm{Gr}(k,n)$ is the unique symplectic resolution of $\overline{\mathcal{N}(\lambda)} = \overline{\mathcal{N}(\lambda)} \cap \mathcal{S}(\mu)$ where

$$\lambda = (\underbrace{2,\cdots,2}_{k \text{ times}}, \underbrace{1,\ldots,1}_{n-2k \text{ times}}, 0, \ldots, 0) \qquad \mu = (1,\ldots,1).$$

Our choice of $\mathbf{w}$ and $\mathbf{v}$ arises from this choice of weights exactly as in [NT17, (7.7)].

Thus, we have that:

Theorem 3.2.3 ([NT17, Th. 7.11])**.** *The Coulomb branch $\mathfrak{M}$ is isomorphic as a Poisson variety to $\overline{\mathcal{N}(\lambda)}$ and its symplectic resolution $\tilde{\mathfrak{M}}$ is isomorphic to $T^*\mathrm{Gr}(k,n)$ as a symplectic variety.*

This isomorphism intertwines the weight 2 scaling $\mathbb{C}^*$-action on cotangent fibers with the scaling grading on KLRW algebras given in eq. (2.6) and the action of diagonal matrices with the winding grading by $\mathbb{Z}^{n-1}$, identifying $(a_1,\ldots,a_{n-1})$ with $\sum a_i\alpha_i$ in the root lattice of $\mathfrak{sl}_n$.

This isomorphism depends on a long chain of so-called "Hanany-Witten moves" and thus one does not immediately see explicit formulas for it. However, this isomorphism is essentially uniquely fixed by compatibility with scaling and winding gradings.

In particular, the embedding $\overline{\mathcal{N}(\lambda)} \subset \mathfrak{sl}_n^*$ induces a surjective map $\mathrm{Sym}(\mathfrak{sl}_n) \to \mathbb{C}[\overline{\mathcal{N}(\lambda)}]$, which is an isomorphism in degree 2. That is, the elements of $\mathbb{C}[\overline{\mathcal{N}(\lambda)}]$ of scaling degree 2 form a copy of $\mathfrak{sl}_n$ as a Lie algebra under Poisson bracket.

Thus, we can identify each of the root spaces of $\mathfrak{sl}_n$ by their scaling degree, and the fact that the root space for $\pm(\alpha_i + \cdots + \alpha_{j-1})$ for $i < j$ must have winding grading $\pm(0,\ldots,0,1,\ldots,1,0,\ldots,0)$.

For computing degrees, it is helpful to represent monopole operators with tableaux on $k \times (n-k)$ rectangles, like discussed above. For each m, we have an unordered v_m-tuple of winding numbers that describe how many times the strands with label m wrap around the cylinder. Let $a_1^{(m)} \le a_2^{(m)} \le \cdots \le a_{v_m}^{(m)}$ be these winding numbers in increasing order. Then we can represent these as a tableau (with no assumption on any inequalities) where we fill the boxes (i,j) satisfying $m = i+j-1$ (which we refer to as a **diagonal**) with the

integers $a^{(m)}_{j-s}$ where $s = \min(0, m-k)$ is the minimal column number of a box satisfying this equality $m = i + j - 1$ minus 1; that is, we fill these boxes with the entries $a^{(m)}_*$ increasing from left to right. The degree of the monopole operator for such a tableau is given by the sum of the contributions:

1. $-2|a^{(m)}_p - a^{(m)}_q|$ for $p < q$.
2. $|a^{(m)}_p - a^{(m\pm1)}_q|$ for all p, q.
3. $|a^{(k)}_p| + |a^{(n-k)}_p|$ for all p.

Lemma 3.2.3. *The only such tableau with degree 0 is the constant 0 tableau.*

Proof. Let r be the largest value appearing in the tableau. Assume $r > 0$. There is at least one box (i, j) with entry r such that neither $(i, j+1)$ nor $(i-1, j)$ has entry r. Let a and b be the number of other boxes in the same diagonal with entries $< r$ or $= r$ respectively, and $a^{\pm}, b^{\pm}$ be the number of boxes in adjacent diagonals with entries $< r$ or $= r$, respectively. In this case, changing this entry from r to $r-1$ changes the contribution from the same diagonal by $2a-2b$ and from adjacent diagonals by $b^+ + b^- - a^+ - a^- - \delta_{i+j-1,k} - \delta_{i+j-1,n-k}$. This is 0 if every box below or right of (i, j) in adjacent diagonals has label $\geq r$, and negative otherwise. Thus, there is a chain of moves which weakly decrease degree that we can use to turn every positive label into a 0. Note that if there is only one entry with label r, then decreasing it decreases the degree, so this chain will have strictly decreased degree if there were any positive entries.

A symmetric argument allows us to get rid of all negative entries and arrive at the tableau with all zeros. This has degree 0, so any tableau with non-zero entries has positive degree. □

If instead we consider diagrams with twisting degree ℓ, then the rules for calculating degrees are almost the same, except that the contribution from the red strands is $|a^{(k)}_p| + |a^{(n-k)}_p - \ell|$ for all p minus a global factor of $k\ell$. The same argument above shows that:

Lemma 3.2.4. *The diagrams of scaling degree 0 and twisting degree ℓ correspond to tableaux with entries in $[0, \ell]$ which satisfy $a(i-1, j) \leq a(i, j) \leq a(i, j-1)$.*

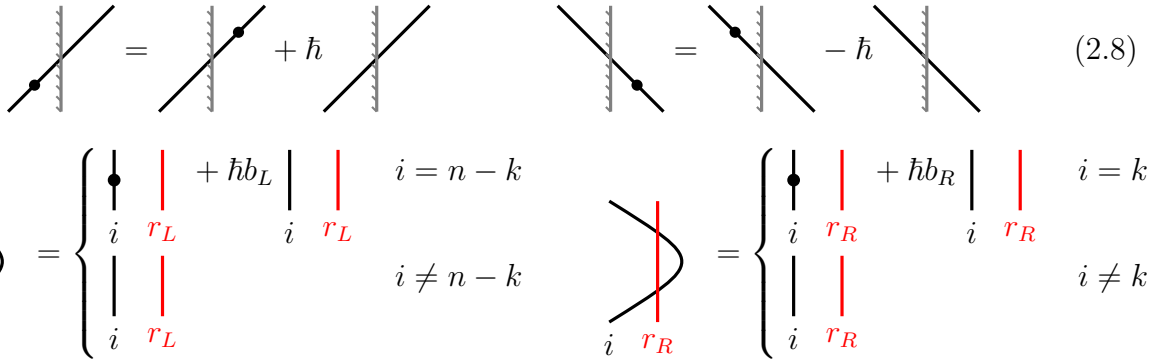
Note that such tableau are in bijection with basis vectors of the representation with highest weight $\ell\omega_k$; for example, if we extend our diagram by adding all boxes (i, j) satisfying $i + j \leq n + 1$ and fill all entries with $j > k$ with 0 and with $i > n - k$ with ℓ,

then reading along the diagonals gives a Gelfand-Tsetlin tableau for the corresponding representation. The usual Gelfand-Tsetlin formulas thus even provide explicit matrices for this representation.

Quantisation

In order to understand this isomorphism better, it is helpful to note that there is a quantum version of it, a special case of the "quantum Mirković-Vybornov isomorphism" of [WWY20]. In particular, this is the cleanest way of seeing that we have an isomorphism not just of varieties, but of Poisson varieties.

We first consider a noncommutative deformation $A_\hbar^{b_L,b_R}$ over $\mathbb{C}[\hbar]$ of $\mathbb{C}[\mathfrak{M}]$, where one deforms the relations so that sliding a dot past the "seam" $x = 0$ on the cylinder shifts the value of the dot by $\pm\hbar$, and the relations involving red strands are shifted by scalars b_L, b_R corresponding to the left and right strands:



(2.8)

(2.8a)

Note that this is still a graded algebra with $\deg \hbar = 2$ As usual, we can define the Poisson bracket on $\mathbb{C}[\mathfrak{M}] = A_\hbar/\hbar A_\hbar$ by the formula $\{a, b\} = \frac{1}{\hbar}[a, b] \pmod{\hbar}$. In this noncommutative algebra, the degree 2 elements form a Lie algebra under the bracket $\frac{1}{\hbar}[a, b]$ which is isomorphic to $\mathfrak{pgl}_n \oplus \mathbb{C}$.

Note that $A_\hbar^{b_L,b_R}$ only depends as an algebra on the difference $b_L - b_R$, since we can shift these parameters simultaneously by sending $\bullet \mapsto \bullet + a$ for any $a \in \mathbb{C}$.

By [Web22, Lem. 8.8], the algebra $A_\hbar^{b_L,b_R}$ is the quantum Coulomb branch as constructed by Braverman–Finkelberg–Nakajima [BFN18]. The parameter $b_L - b_R$ is a "flavour parameter" in this context. It will be easier to compare this with other papers where these flavour parameters appear by comparing with the "KLR Yangian" algebra Я defined in

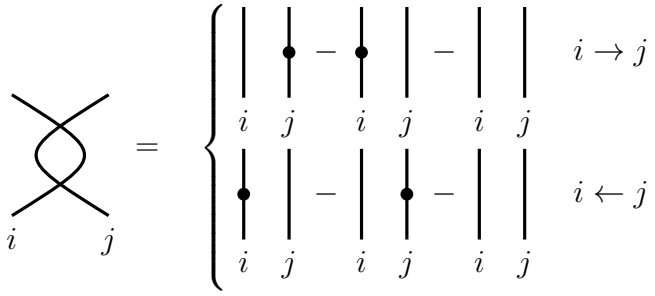
[Kam+19]. This algebra is very similar to the deformed KLR algebra discussed above, but with some key differences. Let

$$p_j(u) = \begin{cases} 1 & j \notin \{k, n-k\} \\ u + k - 2(b_R + 1) & j = k \neq n - k \\ u + (n-k) - 2b_L & j = n - k \neq k \\ (u + k - 2(b_R + 1))(u + k - 2b_L) & j = k = n - k. \end{cases} \tag{2.9}$$

Lemma 3.2.5. *We can define a homomorphism of the KLR Yangian $\mathcal{H}$ with the polynomials p_j as above into $A_\hbar^{b_L, b_R}$ specialized at $\hbar = -2$ by the rule:*

1. *a dotless diagram is sent to the same diagram with the red strand r_L placed as $x = \epsilon$, and r_R at $x = -\epsilon$ for $\epsilon > 0$ very small.*
2. *a dot $\bullet$ on a strand with the label i is mapped to $\bullet + i$.*

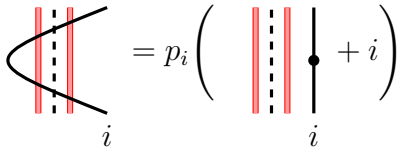
Proof. The KLR relations [Kam+19, (3.1-3.3)] are almost identical to eqs. (2.5a) to (2.5d); the only difference is that we use the polynomial $\bar{Q}_{ij}(u, v)$ which means that in $\mathcal{H}$ we have the relation



(2.9a)

Since our quiver is oriented with $i \mapsto i + 1$, the image of the RHS of this equation will agree with eq. (2.5c); all other KLR relations are the same.

The relation [Kam+19, (4.7a)] agrees with eq. (2.8) after the specialization $\hbar = -2$. In order for the relations [Kam+19, (4.7b–d)] to follow from (2.5f) and (2.8–2.8a) we must have



(2.10)

□

The appearance of $\hbar = -2$ might seem strange, but this is an artefact of notational choices in the literature on Yangians; all specialisations of $\hbar$ to non-zero scalars are isomorphic, via their natural isomorphism to the degree 0 subspace of $A_\hbar^{b_L,b_R}[\hbar^{-1}]$.

Thus, in the notation of [Kam+19; WWY20], we should take $\mathbf{R}_k = \{-k + 2(b_R + 1)\}$ and $\mathbf{R}_{n-k} = \{(k - n + 2b_L\}$ if $n > 2k$ and $\mathbf{R}_k = \{-k + 2b_L, -k + 2(b_R + 1)\}$ if $n = 2k$. By [WWY20, Th. 2.13], this means that $A_\hbar^{b_L,b_R}$ is a quotient of $U(\mathfrak{sl}_n)$ by the maximal ideal in its centre that corresponds to the set

$$\tilde{\mathbf{R}} = \{-n + 3 + 2b_R, -n + 5 + 2b_R, \dots, n - 2k + 1 + 2b_R, \\ - n + 1 + 2b_L, -n + 3 + 2b_L, \dots, -n - 1 + 2k + 2b_L\}. \quad (2.11)$$

Explicitly, this is the maximal ideal of the centre that vanishes on any Verma module for a weight λ of $\mathfrak{gl}_n$ such that the entries of $\lambda + \rho$ in the coordinates $\epsilon_i(\lambda + \rho)$ as an unordered multiset agree with $\tilde{\mathbf{R}}$; this maximal ideal is unchanged under the dot action of S_n and thus only depends on these coordinates as an unordered set. Furthermore, we are only interested in the intersection of this ideal with $U(\mathfrak{sl}_n)$, which is unchanged by simultaneous translation of the set $\tilde{\mathbf{R}}$. Examples of weights of this form which will be dominant for some values of $b_L - b_R$ are $(-n + k - 1 + b_L - b_R)\omega_k$ and $(-k + 1 + b_R - b_L)\omega_{n-k}$.

By [WWY20, Th. 4.3], the algebra $A_\hbar$ is also the algebra of sections of a sheaf $\mathscr{D}_{b_L,b_R}$ of twisted differential operators on X, where the twist depends on $b_L - b_R$ as determined in [WWY20, §3.3.2]. It's easy to clarify this dependence by noting that the parabolic Verma module with highest weight $\ell\omega_k$ is the space of sections of $\mathcal{L}^\ell$ on the open P orbit in X (for $\ell \in \mathbb{Z}$), and thus carries an action of twisted differential operators for this twist. On the other hand, if we twist the GL_n-action on X by the automorphism of taking inverse-transpose, then this space of sections will be the parabolic Verma module with highest weight $\ell\omega_{n-k}$ This shows that:

Lemma 3.2.6. *The algebra $A_\hbar^{b_L,b_R}$ is isomorphic to differential operators twisted by $\mathcal{L}^\ell$ when $b_L - b_R = \ell + n - k + 1$ or, using the inverse-transpose automorphism, when $b_L - b_R = -\ell - k + 1$.*

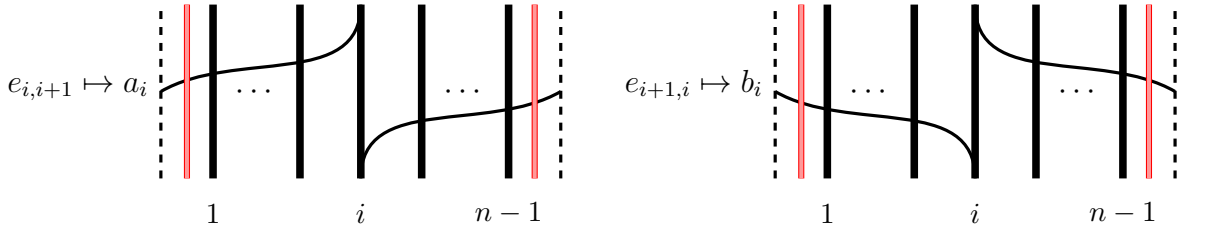
This quantisation will make it easier to see the connection to line bundles on the resolution.

The isomorphism

Armed with the Poisson structure induced by this quantisation, we can relatively easily give a more explicit description of the isomorphism between $e\mathring{R}e$ and $\mathbb{C}[T^*X]$. As discussed

above, the algebra $\mathbb{C}[T^*X]$ is generated by the matrix entries $e_{i,j}$, which span a copy of $\mathfrak{pgl}_n$ as a Lie algebra under Poisson bracket; these are precisely the elements of scaling degree 2. Any isomorphism of Lie algebras to the scaling degree 2 elements of $e\mathring{R}e$ (also considered under Poisson bracket) will induce an isomorphism $\mathbb{C}[T^*X] \to e\mathring{R}e$.

Since the space of elements of the correct winding and scaling degree is 1-dimensional, we must have:



(2.12)

These elements automatically satisfy the Serre relations, since the space of diagrams of the correct winding degree has no elements of scaling degree 2. Thus, in order to show that we have homomorphism of Lie algebras, we only need to choose the factors a_i, b_i so that these elements satisfy the relations of $\mathfrak{sl}_2$, which we know on abstract grounds to be possible.

In the case of primary interest to us, $k = 2, n = 4$, calculations too long to include here confirm that:

Theorem 3.2.4. *When $k = 2, n = 4$ and $a_1 = a_2 = a_3 = -1$, there is a unique isomorphism of Poisson algebras $\mathbb{C}[T^*X] \to e\mathring{R}e$ satisfying (2.12).*

This seems to suggest we can take $a_i = -1$ in all cases, but the confirmation of this sign is a very delicate computation, so we will leave more general confirmation of this pattern to another time.

Resolutions

Since thinking about resolutions is also important, we wish to more explicitly identify the resolution $\tilde{\mathfrak{M}}$. We know that it must be $T^*\mathrm{Gr}(n-k, n)$, since that is the only symplectic resolution of $\overline{\mathcal{N}}(\lambda)$. However, it is not immediately obvious, for example, which ample line bundle we recover. Let $\mathcal{O}(1)$ denote the canonical ample line bundle on $\tilde{\mathfrak{M}}$ induced by its construction as a Proj.

We can do this by defining a deformed version $e\mathring{T}^1_\hbar e$ of the module $e\mathring{T}^1 e$, which is a bimodule over $A_\hbar^{b_L+1,b_R}$ - $A_\hbar^{b_L,b_R}$. We can define this by applying the local relations (2.5a–2.5f,2.8), with some extra care to the red/black bigon relations: instead of (2.5e), we apply

(2.8a) with the values b_L and b_R below the y-value where the red strand r_L crosses the seam at $x = 0$, and $b_L + 1$ and b_R above this point. We can tensor together m of these bimodules for suitable choices of parameters to obtain a quantisation $e\mathring{T}^m_\hbar e$ of the module $e\mathring{T}^m e$.

By [Web22, Prop. 8.12], this is the natural deformation of $\Gamma(T^*X; \mathcal{O}(m))$ to a bimodule $\Gamma(X; \mathcal{O}(m) \otimes_{\mathcal{O}} \mathscr{D}_{b_L,b_R})$ over the corresponding rings of twisted differential operators. Using Lemma 3.2.6, we can see that if $\mathscr{D}_{b_L,b_R}$ is the ring of differential operators twisted by $\mathcal{L}^\ell$, then $\mathscr{D}_{b_L+m,b_R}$ is twisted by $\mathcal{L}^{\ell+m}$. This is only possible if:

Lemma 3.2.7. *There is an isomorphism $\tilde{\mathfrak{M}} \cong T^*X$ which induces an isomorphism $\mathcal{O}(1) \cong \mathcal{L}$.*

Thus, we have obtained the usual Plücker embedding of T^*X. We can recover the $\mathfrak{sl}_n$-equivariant structure on $\mathcal{L}$ by generalising the adjoint action of $\mathfrak{sl}_n$ on $U(\mathfrak{sl}_n)$. This defines a $\mathfrak{sl}_n$ action on $e\mathring{T}^1_\hbar e$ by the action $X \cdot t = \frac{1}{\hbar}(Xt - tX)$. This action preserves the scaling degree and thus preserves the subspace V of elements of scaling degree 0 in $e\mathring{T}^1_\hbar e$.

Lemma 3.2.8. *As a $\mathfrak{sl}_n$-module, we have an isomorphism $V \cong \bigwedge^k \mathbb{C}^n \cong \Gamma(X; \mathcal{L})$.*

Proof. The diagram where all strands have winding number 1 around the cylinder (in the case of $k = 2, n = 4$, this is e_{12}) is a highest weight vector since there is no element of V where a strand with winding number greater than 1 appears. Let λ be its weight. There is a non-zero element of the weight $\lambda - \alpha_i$ only when $i = k$, given by the diagram where a single strand with label k has winding number 0 and all others are 1 (in the case of $k = 2, n = 4$, this is e_{13}).

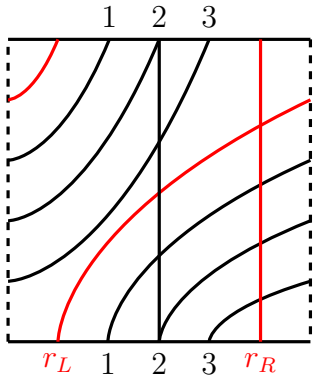
Furthermore, there is also no non-zero element of weight $\lambda - 2\alpha_i$ (no diagram in V where a strand with label k has winding number -1). This shows that the weight $\lambda = \omega_k$ and we have an equivariant map $\bigwedge^k \mathbb{C}^n \to V$ which must be injective by the simplicity of $\bigwedge^k \mathbb{C}^n$. On the other hand, by Lemma 3.2.4, we have that $\dim V = \binom{n}{k}$, so this map is an isomorphism. □

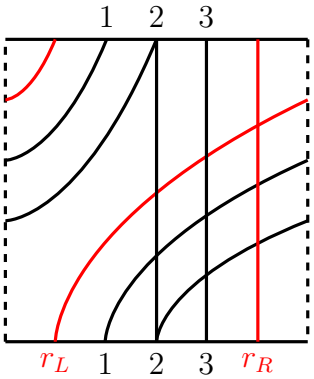
Lemma 3.2.9. *Under any $\mathbb{C}^* \times SL_n$ equivariant isomorphism $\tilde{\mathfrak{M}}$, the subspace V maps isomorphically to $\Gamma(X; \mathcal{L})$. In particular, the cover of T^*X and X obtained by inverting non-zero weight vectors in V is the usual affine cover of discussed in Section 3.2.1.*

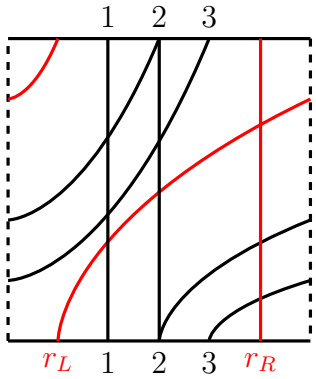
Proof. We have a map $\mathrm{Sym}^r(V) \to \Gamma(T^*X; \mathcal{L}^r)$, which for all r is surjective onto the elements of scaling degree 0. In particular, the rth power of an element of V never vanishes. This shows that the elements of V define non-zero sections of $\Gamma(T^*X; \mathcal{L})$. Since they have

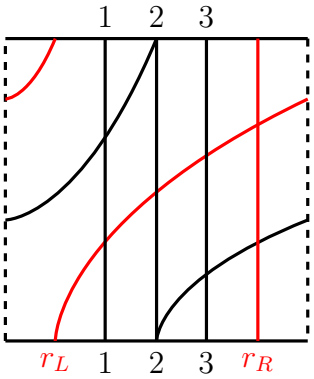
scaling degree 0, they are constant on the cotangent fibres, and thus are non-zero after restriction to X. The induced map to $\Gamma(X;\mathcal{L})$ is SL_n equivariant and not zero, so it must be injective. We already know that these vector spaces have the same dimension, so the induced map must be an isomorphism. $\square$

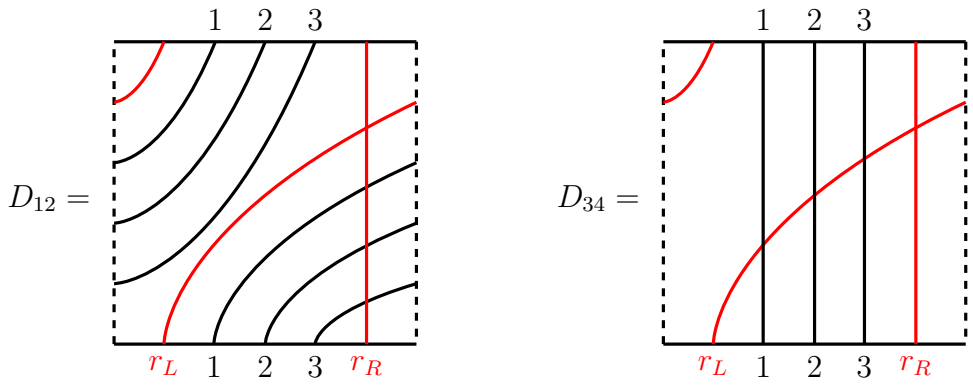
In particular, in the case $k = 2$ and $n = 4$, we can explicitly find diagrams D_{ij} which must agree up to scalar with the wedges $e_{ij} = e_i \wedge e_j$, since they are the only elements in V with the correct winding grading. These diagrams are:

$D_{13} =$ 

$D_{14} =$ 

$D_{23} =$ 

$D_{24} =$ 



3.3 Vector bundles on $T^*\mathrm{Gr}(2,4)$

3.3.1 General procedure

We can use the isomorphism above to construct a very interesting collection of vector bundles on T^*X via transport of structure from $\tilde{\mathfrak{M}}$. For any graded module M over the projective coordinate ring $\mathbf{A}$, we can define its localisation $\mathrm{Loc}(M)$. We can construct this explicitly as follows.

Given $f \in \mathbf{A}^m = e\mathring{T}^m e$, the scheme $\mathrm{Proj}(\mathbf{A})$ has an open subset $U_f = \mathrm{Spec}(\mathbf{A}_f^0)$ given by the spectrum of the degree 0 part of the localisation (in the sense of commutative algebra); by definition, $\mathrm{Proj}(\mathbf{A})$ is the colimit of these open subschemes over the natural inclusions $U_{fg} \subset U_f$. Given a module M, we can define $\mathrm{Loc}(M)$ by describing its pullback to U_f for each f, together with pullback maps under the inclusions above. We do this by localising the $\mathbf{A}_f^0$-module $\mathbf{M}_f^0$. Note that this means that we have a natural map $M^m \to \Gamma(\tilde{\mathfrak{M}}; \mathrm{Loc}(M) \otimes \mathcal{L}^m)$; if M is finitely generated, these will be an isomorphism for $m \gg 0$.

The most natural choice of $\mathbf{A}$-module (beyond $\mathbf{A}$ itself) is the module $e\mathbf{R} \cong \bigoplus_{m=0}^{\infty} e\mathring{T}^m$.

Theorem 3.3.1 ([Web22, Thm. 8.14])**.** *The vector bundle $\mathcal{Z} = \mathrm{Loc}(e\mathbf{R})$ is a tilting generator and agrees with the tilting generator constructed in [Kal08].*

Note that Kaledin only canonically constructs the Azumaya algebra $\mathcal{Z} \otimes \mathcal{Z}^\vee$; for general symplectic resolutions, different choices of quantisation parameters in characteristic p will give different Azumaya algebras, but there is only one such algebra in the case of T^*X.

Choosing idempotents other than e in $\mathbf{R}$ can give other tilting generators, as does tensoring with line bundles.

We can write $1 \in \mathbf{R}$ as a sum of idempotents corresponding to words, so $\mathcal{Z}$ decomposes into the sum of the images of these idempotents, which are $\mathcal{Z}^{\mathbf{i}} = \mathrm{Loc}(e\mathbf{R}e(\mathbf{i}))$. By [Web19a, Lem. 2.9], we have:

Lemma 3.3.1. *The sheaf $\mathcal{Z}^{\mathbf{i}}$ for $i_1^{(a_1)} \cdot i_r^{(a_r)}$ is a vector bundle of rank $\frac{v_1! \cdots v_{n-1}!}{a_1! \cdots a_r!}$. In particular, if $n = 4$ and $k = 2$, then this rank is 2 if all a_i's are 1, and 1 otherwise.*

Remark 3.3.1. *As mentioned in the introduction, in the special case $k = 1$ and general n, we have that $v_i = 1$ for all i, so we obtain a sum of line bundles, which must correspond to Beilinson's generator [Bei78].*

Our main purpose in this paper is to compute which representations appear as $\mathcal{Z}^{\mathbf{i}}$ in the case of $n = 4, k = 2$. Let us state one useful preparatory results. Due to topological constraints, we have written our words linearly; when we discuss "reordering" a word, that implicitly includes moving the last entry to be the first and vice versa. For example, we can obtain 213222 from 232221 by "reordering" the 1 and the first 2. Standard results in KLR algebras show that:

Lemma 3.3.2.

1. *If we reorder two letters both of which are not black 2's, then $\mathcal{Z}^{\mathbf{i}}$ is unchanged.*
2. $\mathcal{Z}^{\cdots 22 \cdots} \cong (\mathcal{Z}^{\cdots 2^{(2)} \cdots})^{\oplus 2}$
3. *If we have a subword $2i2$ for $i \in \{1, 3, 2\}$, then*
$$\mathcal{Z}^{\cdots 2i2 \cdots} \cong \mathcal{Z}^{\cdots i2^{(2)} \cdots} \oplus \mathcal{Z}^{\cdots 2^{(2)} i \cdots}$$

Thus, we will obtain a tilting generator equiconstituted to $\mathcal{Z}$ if we take the sum of $\mathcal{Z}^{\mathbf{i}}$ where $\mathbf{i}$ ranges over those where we have a $2^{(2)}$ (in which case we have a line bundle) or when the pair of black 2's is separated on both sides of the circle by two elements of the set $\{1, 3, 2\}$. Up to reordering using Lemma 3.3.2.1, there are only two cases where we have $2^{(2)}$, determined by which of the gaps between red 2's the single appearance of $2^{(2)}$ lies in:

$$212^{(2)}32 \qquad 2212^{(2)}3. \tag{3.1}$$

There are four possibilities where the pair of black 2's are separated:

$$221322 \qquad 222312 \qquad 223221 \qquad 221223 \tag{3.2}$$

From this point on, we need only consider these cases.

Lemma 3.3.3. *In the cases of (3.1–3.2), the corresponding vector bundle $\mathcal{Z}^{\mathbf{i}}$ is indecomposable.*

Proof. We can compute $\mathrm{End}(\mathcal{Z}^{\mathbf{i}}) = e(\mathbf{i})Re(\mathbf{i})$, so if $\mathcal{Z}^{\mathbf{i}}$ is decomposable, there is an endomorphism of scaling degree 0 given by projection to one of the summands. But one can readily check that the identity spans degree 0 diagrams in these cases. □

To identify which vector bundles these $\mathcal{A}$-modules correspond to, we will consider the localisation of these modules with respect to the open subsets where $D_{13} \neq 0$ and where $D_{24} \neq 0$. We do this for two reasons: the first is that it will assist us in constructing free generators for the modules and the second is that it will allow us to compute transition matrices of the corresponding locally free coherent sheaf which we can then compare with the results of Section 3.2.1.

We will outline here the argument used to find generators for these **A**-modules and this argument will essentially provide the proof that the pairs of diagrams given below in Sections 3.3.2 and 3.3.3 for each idempotent and each patch do indeed generate the corresponding module.

By definition, for a given word **i** consisting of one black 1, two black 2's, one black 3 and two red 2's, we have an idempotent for **A**. We are interested in the **A**-module $M^{\mathbf{i}} = e\mathbf{R}e(\mathbf{i})$ whose elements are KLRW diagrams with the word **i** labelling the bottom of the diagram and the word for the Coulomb branch idempotent **j** =21232 labelling the top.

This is not a free **A**-module, but we know by Lemma 3.3.1 that the module localised to the patch where $D_{ij} \neq 0$, which we denote $M^{\mathbf{i}}_{ij}$, is free and of rank 2 or 1 for any i, j. To calculate the transition functions, we find generators of $M^{\mathbf{i}}_{ij}$ for the patches with $(i, j) = (1, 3), (2, 4)$.

By applying Nakayama's lemma to the localisations at all different prime ideals, a surjective map between rank k free modules over **A** is an isomorphism, so a generating set whose size matches the rank of the vector bundle must necessarily be a free generating set.

To construct these generating sets, we will extensively use the following fact:

Lemma 3.3.4. *Given a diagram in $M^{\mathbf{i}}_{ij}$, let $(a_0, 0)$, $(b_0, 0)$ be a pair of points at the bottom of the diagram, $(a_1, 1), (b_1, 1)$ a pair of points at the top of the diagram, and α be a continuous path from $(a_0, 0)$ to $(a_1, 1)$. Then there exists a unique path up to isotopy β from $(b_0, 0)$ to $(b_1, 1)$ such that α and β do not intersect.*

We enumerate the red strands and black strands with label 2 "first" and "second" as we read from the left.

Algorithm 3.3.1. *Assume that $k \geq 2$. The generating diagrams of $M_{13}^{\mathbf{i}}$ with winding grading $k \in \mathbb{Z}$, which we denote $Z_{13,s}^{(k)}$ for $s = 1, 2$, can be constructed as follows:*

1. *Begin by wrapping the first red strand around the cylinder k times in the positive direction. The second red strand is vertical.*
2. *(a) If $s = 1$, then we connect the first black terminal with label 2 to the unique such terminal at the top of the diagram, avoiding intersecting the first red strand, and similarly with the second black terminal, avoiding intersection with the second red strand and minimising the number of crossing with other strands (i.e. not creating any bigons with other strands).*

   (b) If $s = 2$, we do the same but with the first black strand avoiding the second red and vice versa. For simplicity of the discussion below, we say that the black and red strand that avoid intersection above are **tied.**
3. *The unique terminals for 1 and 3 on the top and bottom are connected so that they avoid crossing the black strand tied to the first red strand. That is, if $s = 1$ (resp. $s = 2$), they avoid crossing the first (resp., second) black strand.*

If $k < 2$, we let $Z_{13,s}^{(k)} = D_{13}^{k-2} Z_{13,s}^{(2)}$.

It might be hard to see from this description how $(i, j) = (1, 3)$ is relevant; it is perhaps easiest to see when we consider why:

Lemma 3.3.5. *For all k, k', we have $Z_{13,s}^{(k)} = D_{13}^{k-k'} Z_{13,s}^{(k')}$.*

Proof. It's enough to show that if $k \geq 2$, then $Z_{13,s}^{(k+1)} = D_{13} Z_{13,s}^{(k)}$. Since $k \geq 2$, the black strand R tied to the first red will meet the other black strand R' one place from the left—the strand R must have winding number $k + 1$, while the other R' must have winding number ≤ 1 (depending on which side of the second red strand it lies on).

Thus, in $D_{13} Z_{13,s}^{(k)}$, the strand R connects to the strand which winds around in D_{13}, and R' to the one which remains vertical. Thus, in D_{13}, the first red strand and strands with labels 1 and 3 don't intersect the R, while the second red strand doesn't intersect R'. Thus, if the product diagram $D_{13} Z_{13,s}^{(k)}$ has no bigons, then we have $Z_{13,s}^{(k+1)}$. Such a bigon is impossible for $k \geq 2$, since it would have to involve the strand R, and for $k \geq 2$, all the other black strands have winding number ≥ 1 while R' must have winding number ≤ 1. □

Similar arguments apply when we consider $(i,j) = (1,4), (2,3), (2,4)$, suitably modifying step (3) so that strands that wrap in D_{ij} avoid crossing the black strand tied to the first red and those which are vertical in D_{ij}. In particular, for $(i,j) = (2,4)$, we have:

3'. Connect the unique terminals for 1 and 3 on the top and bottom so they avoid intersections with the black strand tied to the second red strand.

Remark 3.3.2. *This approach doesn't work for D_{12}, D_{34}; it's an interesting question how we modify this approach to work in these cases.*

Theorem 3.3.2. *For $k \gg 0$, the diagrams $Z^{(k)}_{13,1}, Z^{(k)}_{13,2}$ are a free basis for $M^{\mathbf{i}}_{13}$.*

Proof. As discussed above, it is enough to show that these are generators, and since we have inverted D_{13}, by Lemma 3.3.5, proving this for one large k establishes it for all k greater than that.

Let $m_1, \ldots, m_p$ be a set of generators of this module. By multiplying by powers of D_{13}, we can assume that these all have the same twisting degree k, and for some fixed $a \geq 0$, we have the winding number of the strands with labels $1, 3$ and the sum of the winding numbers for the black strands with label 2 are in the interval $[k-a, k+a]$. Note that multiplying by D_{13} increases all these quantities by 1, so this condition is unchanged by increasing k; in particular, we can assume that $k \geq 3a$.

We can also assume without loss of generality that our diagram is "pulled taut," i.e. no pair of strands make a bigon. In this case, there must be a first point where the black strands meet. Let h be the height of this intersection. Since the sum of the winding numbers of the two black strands is $\leq k + a$, at least one of these strands must have winding number $\leq \frac{k+a}{2} \leq k - a$. Thus, this intersection must come at the top of a triangle made by the two black strands, and by the tautness, the right side of the triangle is the black strand R of label 2 with lower winding number and the left side is the black strand L of label 2 with greater winding number.

If any red strand or strand with label 1 or 3 crosses both sides of the triangle, then we can shrink the triangle to remove this crossing, so each other strand crosses at most one side of this triangle. Since the first red strand and both strands with label 1 and 3 have greater winding number than R, they must cross it at least once, and thus, after isotopy, we can assume that these strands only intersect a black with label 2 below $y = h$ by crossing R at most once and never crossing L.

On the other hand, the second red strand has winding number 0, and thus we can arrange so that its only crossing with a black strand with label 2 below $y = h$ is crossing

L at most once, and never crossing R. Thus, by isotoping the strands with labels other than 2, we can arrange that the cross section at height $y = h$ is e and the portion of the diagram below $y = h$ is $Z^{(k)}_{13,1}$ or $Z^{(k)}_{13,2}$, depending on whether L is the first black strand or the second one. In fact, one can see from this argument that $k = 2$ suffices, since if $k > 2$, there are multiple intersection points between the black strands in $Z^{(k)}_{13,1}$ and $Z^{(k)}_{13,2}$. □

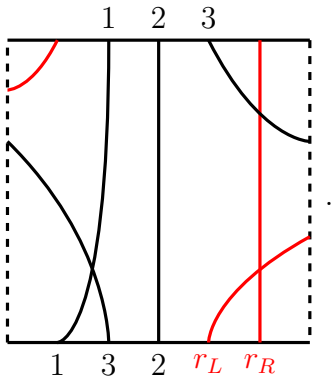
Given this basis, we now turn to computing the transition matrices with respect to it.

3.3.2 Rank 1 cases

In this section, we identify the line bundles corresponding to the words in (3.1):

$$\mathcal{Z}^{2\mathbf{1}2^{(2)}3\mathbf{2}}, \qquad \mathcal{Z}^{22\mathbf{1}2^{(2)}3}$$

For the former, simply note that $e(2\mathbf{1}2^{(2)}3\mathbf{2}) = e$, we have $\mathcal{Z}^{2\mathbf{1}2^{(2)}3\mathbf{2}} \cong \mathrm{Loc}(eRe) \cong \mathcal{O}$. For the latter, we note that $\mathcal{Z}^{22\mathbf{1}2^{(2)}3} \otimes \mathcal{L}$ is generated by



This diagram contains no relevant intersections and hence trivializes $\mathcal{Z}^{22\mathbf{1}2^{(2)}3} \otimes \mathcal{L}$ on all coordinate patches U_{ij}. This implies

$$\mathcal{Z}^{22\mathbf{1}2^{(2)}3} \otimes \mathcal{L} \cong \mathcal{O}$$
$$\implies \qquad \mathcal{Z}^{22\mathbf{1}2^{(2)}3} \cong \mathcal{L}^{-1}.$$

To summarize:

Lemma 3.3.6. *We have isomorphisms* $\mathcal{Z}^{2\mathbf{1}2^{(2)}3\mathbf{2}} \cong \mathcal{O}$ *and* $\mathcal{Z}^{22\mathbf{1}2^{(2)}3} \cong \mathcal{L}^{-1}$.

3.3.3 Rank 2 cases

In the four following subsections, we apply the algorithm 3.3.1 and its modification to the U_{24} patch to each of the remaining idempotents identified in (3.2) to find explicit diagrams for each generator $Z^{(k)}_{ij,s}$. The choice of generators produced is not unique.

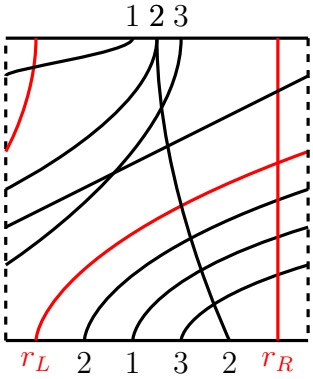
Once sets of generators for $M^{\mathbf{i}}_{13}$ and $M^{\mathbf{i}}_{24}$ have been found, we use the KLRW relations to find the transition matrices $\gamma^{13}_{24}[\mathbf{i}]$ for the vector bundles $\mathcal{Z}^{\mathbf{i}}$. As explained in section 2, this is sufficient to identify each vector bundle up to isomorphism.

To do this, we formulate an ansatz for the equations determining the transition functions. This is achieved by considering the winding degrees of the possible diagrams allowed in each equations. Since the winding degree of each strand must agree on each side of the equation, one can deduce which which coordinate diagrams may appear as coefficients to the $Z^{(k)}_{24,s}$ diagrams by considering the winding degrees appearing in a given $Z^{(k)}_{13,s}$ diagram. Since D_{13} and D_{24} are invertible on $U_{13} \cap U_{24}$, we may freely multiply $Z^{(k)}_{13,s}$ on the LHS by powers of these coordinate diagrams in order to increase or decrease the winding degree as needed.

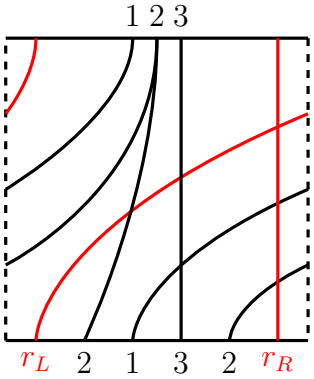
Sheaf with idempotent 221322

Applying algorithm 3.3.1 and its variant to the case where $\mathbf{i}$=221322 produces the following generating diagrams:

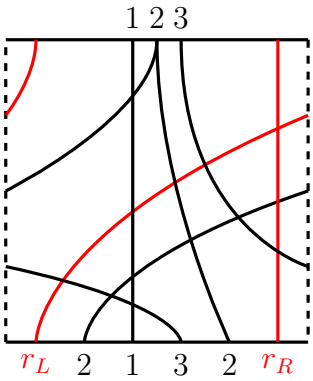
$Z^{(1)}_{13,1} =$



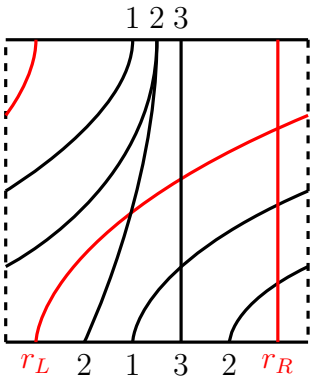
$Z^{(1)}_{13,2} =$



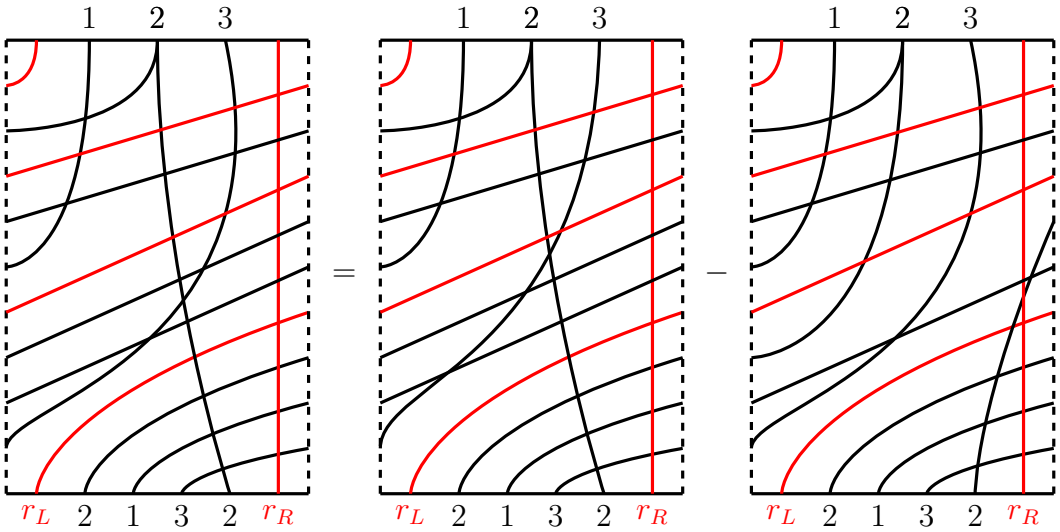
$Z^{(1)}_{24,1} =$



$Z^{(1)}_{24,2} =$



$D^2_{24} Z^{(1)}_{13,1} =$



$$= D^2_{13} Z^{(1)}_{24,1} + (\bullet_1 - \bullet_2 + \bullet_3) D_{24} D_{13} Z^{(1)}_{24,2}.$$

Several important identities were used in the above calculation. Most importantly was applying relation (2.5d) for $ijk = 232$. Secondly the bigon relations (2.5c) and (2.5e) were applied to arrive at the final result.

Upon inspection we immediately see that $Z^{(1)}_{13,2} = Z^{(1)}_{24,2}$ which implies the vector bundle contains a line subbundle. Expressing these equations using generators with winding degree zero yields

$$D_{24} Z^{(0)}_{13,1} = D_{13} Z^{(0)}_{24,1} + (\bullet_1 - \bullet_2 + \bullet_3) D_{24} Z^{(0)}_{24,2}, \qquad D_{13} Z^{(0)}_{13,2} = D_{24} Z^{(0)}_{24,2}.$$

From these we can deduce the transition matrix is

$$\gamma_{13}^{24}[221322] = \begin{pmatrix} D_{13}/D_{24} & \bullet_1 - \bullet_2 + \bullet_3 \\ 0 & D_{24}/D_{13} \end{pmatrix}$$

$$\Longrightarrow \qquad \gamma_{24}^{13}[221322] = \begin{pmatrix} D_{24}/D_{13} & -\bullet_1 + \bullet_2 - \bullet_3 \\ 0 & D_{13}/D_{24} \end{pmatrix}.$$

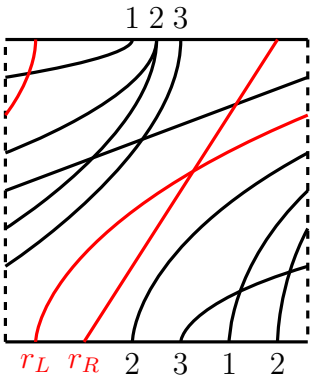
Lemma 3.2.2 then tells us that

Lemma 3.3.7. $\mathcal{Z}^{221322} \cong \mathcal{H} \otimes \mathcal{L}$.

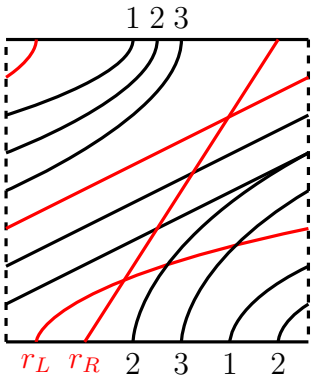
Sheaf with idempotent 222312

Applying algorithm 3.3.1 and its variant to $\mathbf{i} =$222312 yields the following set of generators:

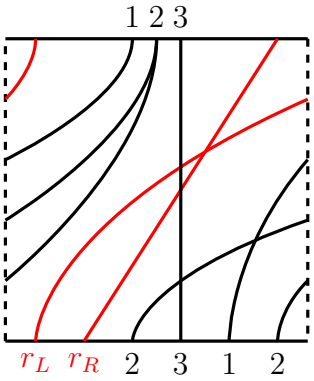
$Z_{13,1}^{(1)} =$



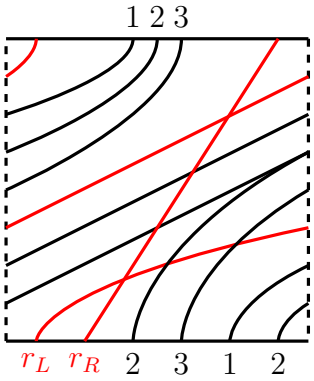
$Z_{13,2}^{(2)} =$



$Z_{24,1}^{(1)} =$

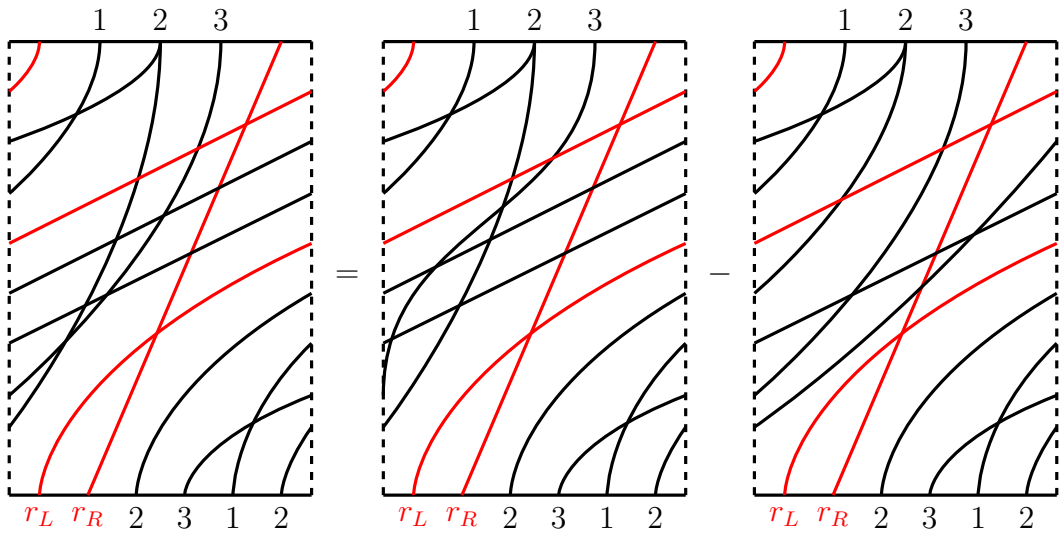


$Z_{24,2}^{(2)} =$



To compute the transition matrix, we proceed similarly to before:

$D_{24}Z^{(1)}_{13,1} =$



$$= D_{13}Z^{(1)}_{24,1} + (\bullet_1 - \bullet_2 + \bullet_3)Z^{(2)}_{24,2}.$$

Similarly to the previous case, in the above we have applied relation (2.5d) with $ijk = 232$ as well as the bigon relations (2.5c) and (2.5e).

Also similar to the previous case, we note that $Z^{(2)}_{13,2} = Z^{(2)}_{24,2}$ and so we again have a line subbundle. Rewriting these equations using a basis comprised of winding degree 0 generators yields:

$$D_{13}Z^{(0)}_{13,1} = D_{13}Z^{(0)}_{24,1} + (\bullet_1 - \bullet_2 + \bullet_3)D_{24}Z^{(0)}_{24,2}, \qquad D^2_{13}Z^{(0)}_{13,2} = D^2_{24}Z^{(0)}_{24,2}.$$

The transition matrix of this vector bundle on these patches is hence given by

$$\gamma^{24}_{13}[222312] = \begin{pmatrix} 1 & (\bullet_1 - \bullet_2 + \bullet_3)D_{24}/D_{13} \\ 0 & D^2_{24}/D^2_{13} \end{pmatrix}$$

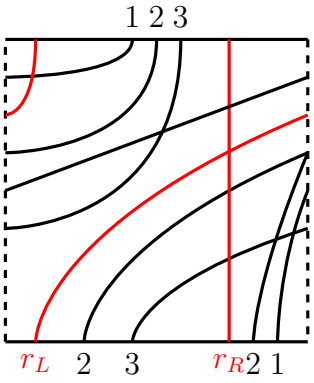
$$\implies \qquad \gamma^{13}_{24}[222312] = \begin{pmatrix} 1 & -(\bullet_1 - \bullet_2 + \bullet_3)D_{13}D_{24} \\ 0 & D^2_{13}/D^2_{24} \end{pmatrix}.$$

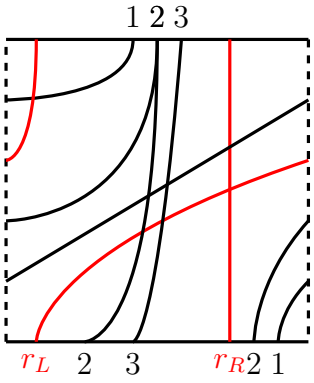
Lemma 3.2.2 then tells us that

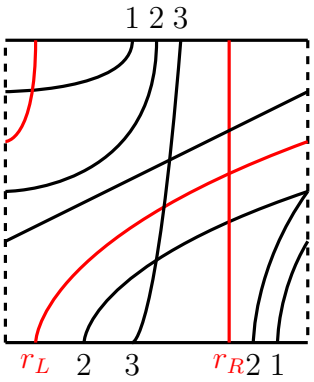
Lemma 3.3.8. $\mathcal{Z}^{222312} \cong \mathcal{H}$

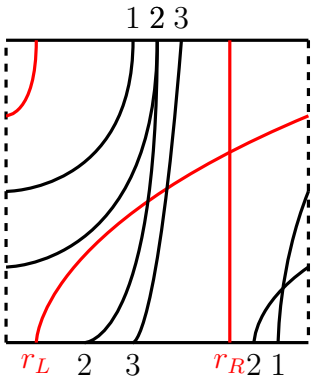
Sheaf with idempotent 223221

Applying algorithm 3.3.1 and its variant to $\mathbf{i}$ =223221 yields the following set of generators:

$Z^{(1)}_{13,1} =$ 

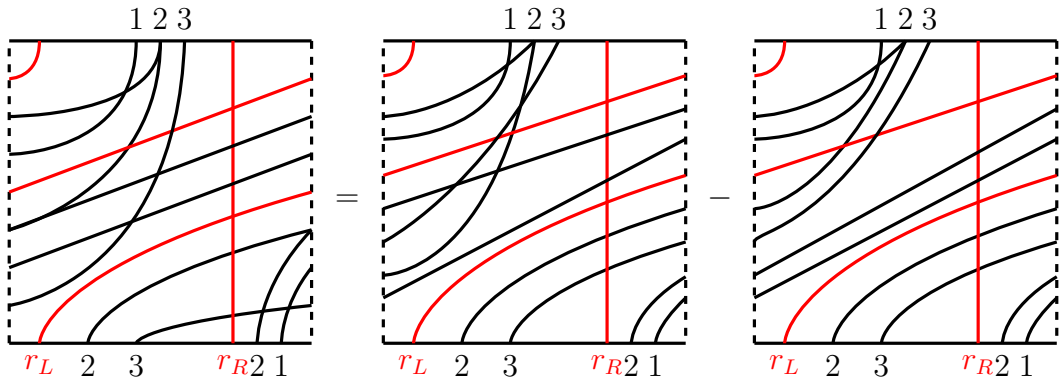
$Z^{(1)}_{13,2} =$ 

$Z^{(1)}_{24,1} =$ 

$Z^{(1)}_{24,2} =$ 

To compute the transition matrix $\gamma_{24}^{13}[223221]$ we perform the following manipulations:

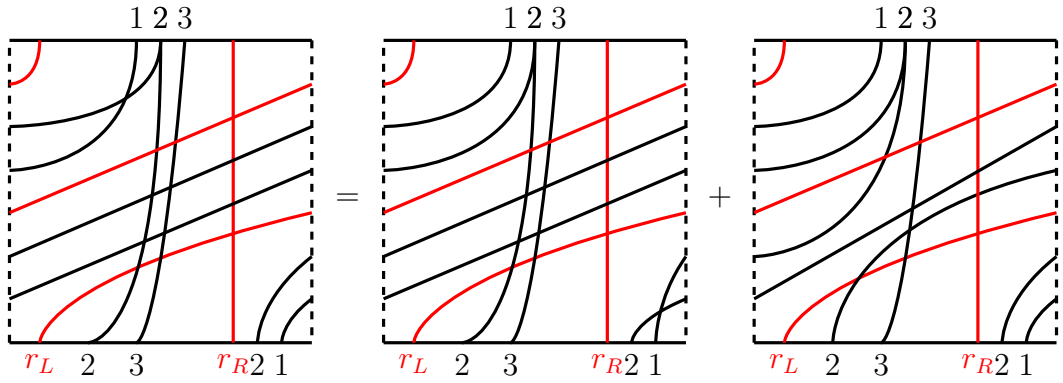
$D_{24}Z_{13,1}^{(1)} =$



$= D_{23}Z_{24,1}^{(1)} - D_{12}Z_{24,2}^{(1)}.$

The most important identity used in this calculation was the application of relation (2.5d) with $ijk = 232$. Similarly we have

$D_{24}Z_{13,2}^{(1)} =$



$= D_{34}Z_{24,1}^{(1)} + D_{14}Z_{24,2}^{(1)}.$

where the most notable step was applying relation (2.5d) with $ijk = 212$. Combined these two equations produce the transition matrix

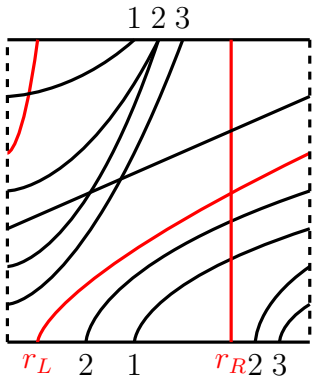
$$\gamma_{13}^{24}[223221] = \begin{pmatrix} D_{23}/D_{13} & -D_{12}/D_{13} \\ D_{34}/D_{13} & D_{14}/D_{13} \end{pmatrix}.$$

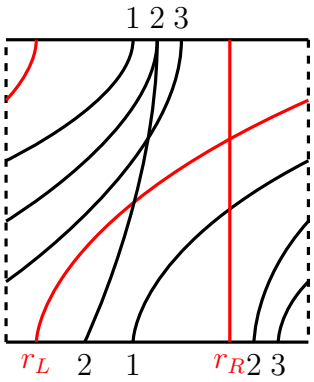
Lemma 3.2.1 then tells us that

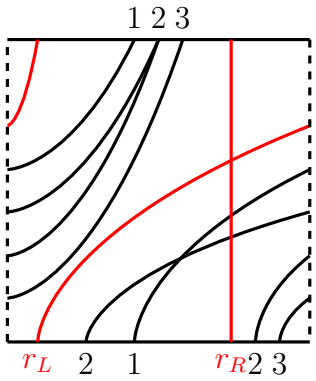
Lemma 3.3.9. $\mathcal{Z}^{223221} \cong \mathcal{T}^{\perp}$.

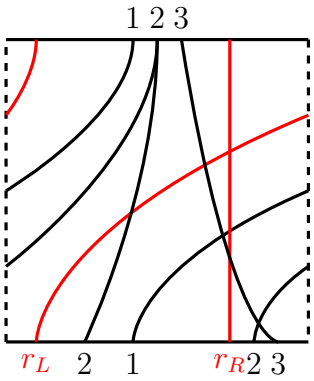
Sheaf with idempotent 221223

Applying the algorithms to the case where **i**=221223 produces the following set of generators:

$Z^{(1)}_{13,1} =$ 

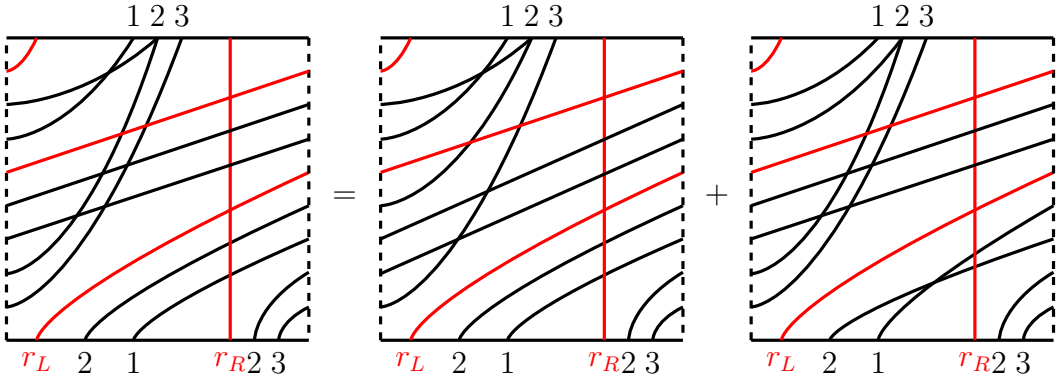
$Z^{(1)}_{13,2} =$ 

$Z^{(1)}_{24,1} =$ 

$Z^{(1)}_{24,2} =$ 

To compute the transition matrix $\gamma_{24}^{13}[221223]$ we perform the following manipulations:

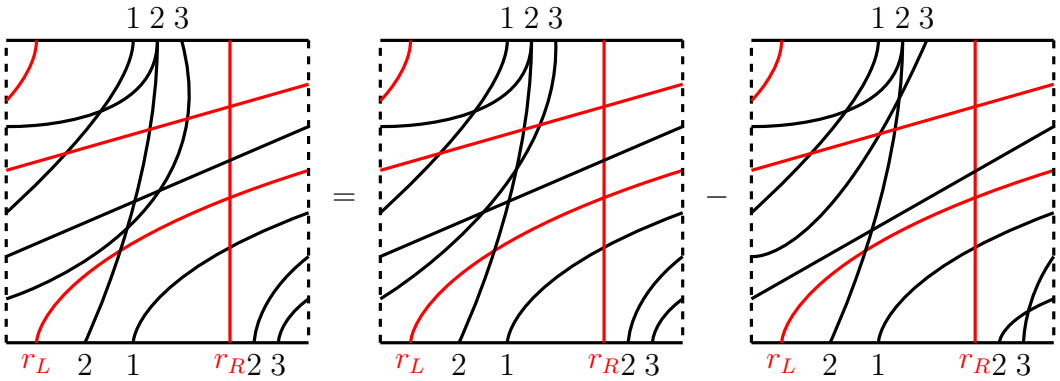
$D_{24}Z_{13,1}^{(1)} =$



$$= D_{14}Z_{24,1}^{(1)} + D_{12}Z_{24,2}^{(1)}.$$

The most notable relation used here was applying (2.5d) with $ijk = 212$. Similarly, we have

$D_{24}Z_{13,2}^{(1)} =$



$$= -D_{34}Z_{24,1}^{(1)} + D_{23}Z_{24,2}^{(1)}.$$

Here the most notable relation used was (2.5d) with $ijk = 232$. These equations then give us the following transition matrix:

$$\gamma_{13}^{24}[221223] = \begin{pmatrix} D_{14}/D_{13} & D_{12}/D_{13} \\ -D_{34}/D_{13} & D_{23}/D_{13} \end{pmatrix}.$$

Lemma 3.2.1 then tells us that:

Lemma 3.3.10. $\mathcal{Z}^{221223} \cong \mathcal{T}^{\perp}$

3.3.4 Summary

Combining Lemmata 3.3.6, 3.3.7, 3.3.8, 3.3.9, and 3.3.10, we summarise the results of the previous two sections in the following table:

Idempotent	Vector bundle
$212^{(2)}32$	$\mathcal{O}$
$2212^{(2)}3$	$\mathcal{L}^{-1}$
221322	$\mathcal{H} \otimes \mathcal{L}$
222312	$\mathcal{H}$
223221	$\mathcal{T}^{\perp}$
221223	$\mathcal{T}$

With these identifications established, we are now in a position to conclude our main theorem:

Theorem 3.3.3. *The vector bundle*

$$\begin{aligned}\mathcal{Z} &= \mathcal{Z}^{212^{(2)}32} \oplus \mathcal{Z}^{2212^{(2)}3} \oplus \mathcal{Z}^{221322} \oplus \mathcal{Z}^{222312} \oplus \mathcal{Z}^{223221} \oplus \mathcal{Z}^{221223} \\ &\cong \mathcal{O} \oplus \mathcal{L}^{-1} \oplus \mathcal{H} \oplus \mathcal{H} \otimes \mathcal{L} \oplus \mathcal{T} \oplus \mathcal{T}^{\perp}\end{aligned}$$

is isomorphic to the tilting generator of Theorem 3.3.1 on $D^b(\mathsf{Coh}(T^*X))$.

3.4 Comparison to other constructions

Let us discuss in a bit more detail the comparison of our construction with that of [TU10]. They construct a tilting generator by an inductive process. Define a series of vector bundles $\mathcal{E}_k$ by the construction:

Algorithm 3.4.1.

1. *Let* $\mathcal{E}_0 = \mathcal{O}_X$.
2. *Given* $\mathcal{E}_{k-1}$ *for* $0 < k < 4$, *consider* $\mathrm{Ext}^1(\mathcal{E}_{k-1}, \mathcal{L}^{-k})$; *this is the only non-zero Ext group between this pair, and represents the only obstruction to* $\mathcal{E}_{k-1} \oplus \mathcal{L}^{-k}$ *being a tilting bundle.*

As a module over $\mathrm{End}(\mathcal{E}_{k-1})$*, the group* $\mathrm{Ext}^1(\mathcal{E}_{k-1}, \mathcal{L}^{-k})$ *is finitely generated and any set of* r *generators gives an extension*

$$\mathcal{L}^{-k} \to \mathcal{N}_k \to \mathcal{E}_{k-1}^{\oplus r}$$

such that $\mathrm{Ext}^1(\mathcal{E}_{k-1}, \mathcal{N}_k) = 0$*.*

Note that we can often find a smaller summand $\mathcal{N}_k'$ *of* $\mathcal{N}_k$ *containing* $\mathcal{L}^{-k}$*; the complementary summand is already a summand of* $\mathcal{E}_{k-1}^{\oplus r}$*, so* $\mathcal{E}_{k-1} \oplus \mathcal{N}_k$ *and* $\mathcal{E}_{k-1} \oplus \mathcal{N}_k'$ *are equiconstituted. In particular, this shows that if we have two generating sets, the resulting bundles* $\mathcal{E}_{k-1} \oplus \mathcal{N}_k$ *are equiconsituted.*

Let $\mathcal{E}_k = \mathcal{E}_{k-1} \oplus \mathcal{N}_k'$*, which is unique up to equiconsitution.*

3. *If* $k = 4$*, we perform a more complicated process to cancel all* $\mathrm{Ext}^i(\mathcal{E}_3, \mathcal{L}^{-4})$ *with* $i > 0$*; this is more complicated because this can be non-zero for* $i \in [1, 4]$*.*

Proposition 3.4.1. *We can make choices so that* $\mathcal{E}_3 \cong \mathcal{O}_X \oplus \mathcal{L}^{-1} \oplus \mathcal{H} \oplus \mathcal{H} \otimes \mathcal{L}^{-1}$*. That is,* $\mathcal{E}_3$ *is a summand of* $\mathcal{Z}$*.*

Proof. By definition, we have $\mathcal{E}_0 = \mathcal{O}_X$. Furthermore, $\mathcal{E}_1 = \mathcal{O}_X \oplus \mathcal{L}^{-1}$ since $\mathrm{Ext}^1(\mathcal{E}_0, (\mathcal{L}^{-1}) = 0$.

On the other hand, by (2.2), we have $\mathrm{Ext}^1(\mathcal{E}_1, \mathcal{L}^{-2}) = H^0(T^*X; \mathcal{O})$, and so $\mathcal{N}_2$ comes from the unique lowest degree class, which is a generator of this module. That is,

$$\mathcal{N}_2 \cong \mathcal{H} \oplus \mathcal{L}^{-1}.$$

This is the first example where it is more convenient to take $\mathcal{N}_2' \cong \mathcal{H}$, so that we have

$$\mathcal{E}_2 \cong \mathcal{O}_X \oplus \mathcal{L}^{-1} \oplus \mathcal{H}.$$

Taking $\mathcal{N}_2$ instead would have just given us a second copy of $\mathcal{L}^{-1}$.

To construct $\mathcal{E}_3$, consider

$$\begin{aligned}\mathrm{Ext}^1(\mathcal{E}_2, \mathcal{L}^{-3}) &\cong \mathrm{Ext}^1(\mathcal{O}, \mathcal{L}^{-3}) \oplus \mathrm{Ext}^1(\mathcal{L}^{-1}, \mathcal{L}^{-3}) \oplus \mathrm{Ext}^1(\mathcal{H}, \mathcal{L}^{-3}) \\ &\cong H^0(T^*X; \mathcal{L}) \oplus H^0(T^*X; \mathcal{O}) \oplus H^0(T^*X; \mathcal{L})\end{aligned}$$

We have a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}^1(\mathcal{O}, \mathcal{L}^{-3}) \longrightarrow \mathrm{Ext}^1(\mathcal{H}, \mathcal{L}^{-3}) \longrightarrow \mathrm{Ext}^1(\mathcal{L}^{-2}, \mathcal{L}^{-3}) = 0.$$

In particular, the module $\mathrm{Ext}^1(\mathcal{E}_3, \mathcal{L}^{-3})$ is generated by the class in $\mathrm{Ext}^1(\mathcal{L}^{-1}, \mathcal{L}^{-3})$ corresponding to $\mathcal{H} \otimes \mathcal{L}^{-1}$. All other classes in other summands come from this one using elements of $\mathrm{Hom}(\mathcal{L}^{-1}, \mathcal{O}) \cong \mathrm{Hom}(\mathcal{L}^{-1}, \mathcal{H}) \cong H^0(T^*X; \mathcal{L})$. □

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