Excluded Structures for Pinch-graphic Matroids and Tournaments

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

All research detailed in this thesis was completed during my time in the Master of Mathematics program at the University of Waterloo.

- Results detailed in Chapter 2 were completed in collaboration with Bertrand Guenin.
- Results detailed in Chapter 3 were completed in collaboration with Sophie Spirkl, David Evangelista and Xinyue Fan.

Abstract

In this thesis, we study several problems related to excluded structures in matroids and tournaments. For matroids, we focus on finding the excluded minors for pinch-graphic matroids. For tournaments, we bound the size of excluded induced subgraphs of k-degenerate tournaments and we bound the number of backedge graphs of a tournament that are P_3-free or K_t-free. Furthermore, we introduce a new way to color the arcs of a tournament and we propose a conjecture that connects this coloring to tournaments that do not contain Paley tournaments as an induced subgraph.

For pinch-graphic matroids, we prove that if M is minimally non-pinch-graphic and M is not 3-connected, then $|M| \leq 20$. Furthermore, we characterize all minimally non-pinch-graphic matroids that are not 3-connected. For the 3-connected case, we prove that in a special case, a 3-connected but not internally 4-connected minimally non-pinch-graphic matroid has at most 21 elements. We also improve a characterization of 3-separations in pinch-graphic matroids by Guenin and Heo [9].

For tournaments, we show that an excluded induced subgraph for 1-in(out)-degenerate tournaments have at most $\frac{k^2+5k+6}{2}$ vertices, and an excluded induced subgraph for 1-degenerate tournaments have at most k^2+5k+6 vertices. We also prove that a tournament with n vertices can have at most exponentially many P_3-free or K_t-free backedge graphs, despite there being $n!$ backedge graphs for a tournament with n vertices. We then introduce a new way to arc color a tournament T with the corresponding chromatic number denoted as $\chi_\Delta(T)$. The rule is to arc color a tournament such that every directed cycle of length 3 gets 3 distinct colors. We prove that for tournaments T with n vertices, the growth rate of $\chi_\Delta(T)$ is $\Theta(\log n)$.

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Chapter 1

Introduction

This thesis studies combinatorial problems related to excluded structures in different combinatorial objects. The study of excluded structures plays a central role in both graph theory and matroid theory. Many important classes of graphs and matroids — including planar graphs, perfect graphs, and graphical matroids — have a characterization in terms of excluded structures. For graphs, there are two major types of studies on excluded structures: excluded minors and excluded induced subgraphs. For matroids, most studies are related to excluded minors. Motivated by these studies, this thesis mainly focuses on two topics: excluded minors of pinch-graphic matroids and excluded induced subgraphs of special classes of tournament graphs. We discuss some results in an attempt to bound the size of excluded minors of pinch-graphic matroids. We also include results on the characterization of 1-degenerate tournaments, enumeration of tournaments with excluded induced subgraphs, and a new rule to color the arcs of tournaments.

1.1 Literature Review

The study of excluded structures is a major area of graph theory and has a long history. Dating back to the 1930s, Kuratowski proved that a graph is planar if and only if it has no K_5 or $K_{3,3}$ subdivision [12]. This shows that excluded structures can be a useful way to describe global properties of a class of graphs. The most significant result in this area is the Graph Minor Theorem by Robertson and Seymour, which states that every minor-closed class of graphs has a finite set of minimal excluded minors [21].

The study of excluded structures is a useful tool to connect different areas of graph theory. For example, the results on perfect graphs use excluded structures to eventually characterize the class of perfect graphs and leading to an efficient algorithm to recognize perfect graphs. A perfect graph is a graph whose chromatic number equals the size of a maximum clique in every induced subgraph. This definition is clear but seemingly difficult to work with. Fortunately, the Strong Perfect Graph Theorem states that perfect graphs are exactly the graphs with no holes (odd induced cycles with at least 5 edges) or antiholes (complement of holes) as induced subgraphs [4]. In 2004, Chudnovsky, Cornuéjols, Liu, Seymour and Vušković found a polynomial time algorithm to detect perfect graphs by using the Strong Perfect Graph Theorem [3]. In this example, excluded structures serve as a tool that connects global properties of graphs to recognition algorithms.

In addition, excluded structures can be generalized to other combinatorial objects, such as matroids. In 1958, Tutte proved that binary matroids are exactly the matroids without a $U_{2,4}$-minor [25]. In 1959, Tutte proved that graphic matroids have five forbidden minors [25]. There is also an analogue of the Graph Minor Theorem in matroids. In 2014, Geelen, Gerards, and Whittle announced that every minor-closed class of representable matroids has finitely many excluded minors.[1] These results give us useful insight into the structural properties of these classes of matroids, and these properties are helpful in other areas of matroid theory, such as algorithmic or extremal matroid theory.

There are two main topics in this thesis: excluded minors for pinch-graphic matroids and excluded induced subgraphs for tournaments. For pinch-graphic matroids, we aim to bound the number of elements in excluded minors. For excluded induced subgraphs in tournaments, there are three different topics in this thesis. We first discuss the exact excluded induced subgraphs of 1-degenerate tournaments, and then we work on enumeration of tournaments whose backedge graphs have an excluded induced subgraph. Additionally, we introduce a new way to arc-color a tournament and some interesting results related to this coloring. The details of these results are given in the next section.

1.2 A Quick Glimpse of This Thesis

This section serves as a quick summary of the results of the thesis. We first introduce the questions of interest, and then we present the results. Basic definitions and common terminology in graph and matroid theory are postponed to Chapter 2 and Chapter 3. There are no detailed proofs in this section.

[1] The theorem is unpublished but generally accepted as true by the community.

1.2.1 Pinch-graphic matroids

We start with the definitions of graphic and pinch-graphic matroids. A *graphic matroid* is a matroid whose circuits are exactly polygons[2] of an undirected graph. A *signed graph* is a pair (G, Σ) where G is a graph and Σ is a subset of $E(G)$, and a cycle in a signed graph (G, Σ) is a Eulerian subgraph of (G, Σ). A cycle C of (G, Σ) is *even* (respectively odd) if $|C \cap \Sigma|$ is even (respectively odd). M is an *even-cycle matroid* if the circuits of M are exactly even cycles of a signed graph (G, Σ). (G, Σ) is called a *signed-graph representation* of M. Note that a matroid M may have more than one signed-graph representation.

Definition 1.1. A *blocking pair* of a signed graph (G, Σ) is a pair of vertices $\{u, v\}$ such that $(G, \Sigma)\backslash\{u, v\}$ has no odd cycles. A matroid is *pinch-graphic* if it is an even-cycle matroid that has a signed graph representation with a blocking pair.

For example, the Fano plane is a pinch-graphic matroid by the following representation:

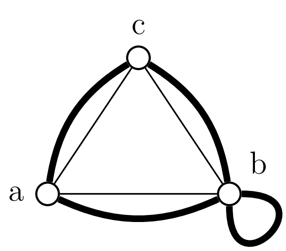

Figure 1: A signed graph representation of F_7 (Bold edges are in Σ.)

Here, $\{a, b\}$ is a blocking pair, since every edges in Σ are incident with at least one of a, b. $\{b, c\}$ can also serve as a blocking pair.

The class of pinch-graphic matroids is minor-closed and representable. Thus, by the Matroid Minor Theorem, there is a finite list of excluded minors for pinch-graphic matroids. Our ultimate question is:

Problem 1. What are the excluded minors for pinch-graphic matroids?

Problem 1 seems out of reach so far. Hence, we attempt to partially solve the problem. First of all, we want an explicit upper bound of the size of a minimally non-pinch-graphic matroid.

[2]Here, a polygon in a graph G is a connected 2-regular subgraph of G.

Problem 2. Can we bound the size of the matroids that are excluded minors for pinch-graphic matroids?

One of the main results of this thesis is:

Theorem 1.2. *If M is a matroid that is minimally non-pinch-graphic and not 3-connected, then $|M| \leq 20$.*

In fact we characterize all excluded minors for pinch-graphic matroids that are not 3-connected:

Theorem 1.3. *If $M = M_1 \oplus_1 M_2$ is minimally non-pinch-graphic, then we have $M_1, M_2 \in \{F_7, F_7^*, M^*(K_5), M^*(K_{3,3})\}$.*

Theorem 1.4. *If $M = M_1 \oplus_2 M_2$ is 2-connected and minimally non-pinch-graphic, then we have $M_1, M_2 \in \{F_7, F_7^*, M^*(K_5), M^*(K_{3,3}), M^*(K'_{3,3})\}$, where $K'_{3,3}$ is the simple graph obtained by adding an edge to $K_{3,3}$.*

A natural follow-up question is: can we bound the size of a minimally non-pinch-graphic matroid M with higher connectivity? One main obstacle here is that an even-cycle matroid can have exponentially many inequivalent[3] representations (with respect to number of elements in the matroid). Guenin and Heo showed that internally 4-connected pinch-graphic matroids only have polynomial many equivalent representations that has a blocking pair [10]. Thus, our goal is to restrict ourself to the internally 4-connected case. But before that, we need to address the 3-connected case.

Problem 3. Can we bound the size of a minimally non-pinch-graphic matroid that is not internally 4-connected?

To solve this problem, we first want to understand 3-separations in M. In a binary 3-connected matroid M, a 3-separation can be expressed as a 3-sum $M = M_1 \oplus_3 M_2$. Guenin and Heo managed to classify 3-separations in pinch-graphic matroids up to "homologous 3-separations" [9]. We improve this result by partially removing the "homologous" condition. These results are discussed in Section 2.7.

1.2.2 Tournaments

A *tournament* is a directed graph with exactly one arc between each pair of vertices. A subtournament is an induced subgraph of a tournament. In this thesis, various aspects of

[3]Here, two representations are equivalent if they are related by Whitney flips and resigning.

tournaments are discussed. Our first topic is k-degenerate tournaments, which are defined as follows:

Definition 1.5. A tournament T is *k-in-degenerate* (respectively *k-out-degenerate*) if, for every induced subgraph of T, there exists a vertex of indegree (respectively outdegree) at most k. A tournament is *k-degenerate* if, for every induced subgraph of T, there exists a vertex of indegree or outdegree at most k. A tournament T is *minimally not k-degenerate* if T is not k-degenerate but every proper induced subgraph of T is k-degenerate.

In graphs, degeneracy is very useful for coloring. A k-degenerate graph is a graph in which every subgraph has at least one vertex of degree at most k. We know that 1-degenerate graphs are exactly forests, and it is not hard to show that k-degenerate graphs have chromatic number at most $k+1$. Degeneracy can also be useful in digraphs. In [7], Golowich introduces the m-degenerate chromatic number $\chi_m(D)$, where $\chi_m(D)$ is the minimum number of sets into which the vertices in D can be partitioned into sets that each induces a m-degenerate digraph. This is useful to improve certain bounds on dichromatic number $\vec{\chi}(D)$ of a digraph D, as the definition of $\chi_0(D)$ is identical to the definition of $\vec{\chi}(D)$. It now seems reasonable to introduce degeneracy of tournaments, since tournaments are a special class of digraphs.

Our main results on k-degenerate tournaments are:

Theorem 1.6. *If a tournament T is minimally not k-in(out)-degenerate, then it has at most $\frac{k^2+5k+6}{2}$ vertices.*

Theorem 1.7. *If a tournament T is minimally not k-degenerate, then it has at most k^2+5k+6 vertices.*

With the results above, we are able to find the complete list of excluded subtournaments for 1-in(out)-degenerate tournaments and 1-degenerate tournaments. The complete list can be found in Appendix A.

Our next topic is related to backedge graphs of tournaments. Let $(v_1, ..., v_n)$ be a (linear) ordering of vertices of a tournament T. If v_i is an in-neighbor of v_j with $j < i$, we call (v_i, v_j) a backedge. Let B be the undirected graph with vertices $v_1, ..., v_n$ such that $(v_i, v_j) \in E(B)$ if and only if v_iv_j is a backedge. We call B the backedge graph of T under the enumeration $(v_1, ..., v_n)$. Note that for a tournament T with n vertices, there are $n!$ ways to enumerate the vertices. Hence, there are at most $n!$ distinct backedge graphs of T. We can view each enumeration as a permutation, so for $\sigma \in S_n$, we can denote the backedge graph of T under the permutation σ as $B_\sigma(T)$.

Let C_H be the class of undirected graphs that have no induced subgraph that is isomorphic to H. Let $f(H,n) = \max_{|T|=n} |\{\sigma \in S_n : B_\sigma(T) \in C_H\}|$, where T is a tournament. We are interested in the following question: can we bound the function $f(H,n)$ for some particular graph H? In other words, we want to understand for a fixed tournament with n vertices, what is the largest possible number of backedge graphs that do not have H as an induced subgraph. We study the case where $H = P_3$, the 3-vertex path and the case where $H = K_t$, the complete graph on t vertices.

Here are the main results:

Theorem 1.8. *For all H with at least 2 edges, $f(H,n) \geq 2^{\lfloor n/2 \rfloor}$.*

Theorem 1.9. *There exists a constant c such that for sufficiently large n, $f(P_3, n) \leq c^n$.*

Theorem 1.10. *For every positive integer t, there exists a constant c_t such that for sufficiently large n, $f(K_t, n) \leq c_t^n$.*

We now proceed to our third topic, rainbow coloring.

A cyclic triangle is a directed cycle with 3 vertices. A rainbow coloring of a tournament T is a coloring of all arcs in T such that every cyclic triangle is colored by 3 distinct colors. It is clear that a rainbow coloring exists for every tournament, since we can color every edge with a distinct color. For example, here is a rainbow coloring of a 5-vertex tournament:

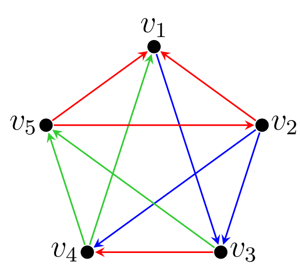

Figure 2: A rainbow coloring of a 5-vertex tournament

The rainbow chromatic number of T, denoted as $\chi_\Delta(T)$, is the minimum number of colors required to rainbow color T. We find that in general, $\chi_\Delta(T)$ is much smaller than the number of edges, but it is impossible to use a constant number of colors to rainbow color all tournaments.

Theorem 1.11. *For all $k > 0$, there exists a tournament T such that $\chi_\Delta(T) > k$.*

In fact, with a few modifications to the proof of Theorem 1.11, we can get a better lower bound:

Theorem 1.12. *For all integer $n > 0$, there exists a tournament T with n vertices such that $\chi_\Delta(T) > \log\log\sqrt{n/2}$.*

We improve the bounds further to have the exact growth rate of $\chi_\Delta(T)$:

Theorem 1.13. *Every tournament T with n vertices has $\chi_\Delta(T) \leq 3\lceil \log_3 n \rceil$.*

Theorem 1.14. *For all $\varepsilon > 0$, there exists $n > 0$ such that there is a tournament T with n vertices with $\chi_\Delta(T) > (1-\varepsilon)\log n$.*

1.3 Outline

The organization of this thesis is as follows. Chapter 2 focuses on problems related to pinch-graphic matroids, and Chapter 3 addresses problems related to tournament graphs. Chapter 4 concludes the thesis and presents several open problems that arise from our study.

In Chapter 2, we first introduce the motivation for studying pinch-graphic matroids, as well as some previous results. We then provide proofs for our results on excluded minors for pinch-graphic matroids, as well as the improvements on classification of 3-separations in pinch-graphic matroids.

In Chapter 3, we study three subtopics related to excluded structures in tournaments. The first topic addresses the forbidden induced subgraphs of k-degenerate tournaments. The second topic concerns the enumeration of tournaments whose backedge graphs are K_t-free or P_3-free. The third topic investigates the rainbow coloring of tournaments and provides an exact bound for the rainbow chromatic number.

Chapter 2

Excluded Structures in Pinch-graphic Matroids

2.1 Matroids and Connectivity

We start with the standard definition of matroids. There are multiple equivalent sets of axioms to define matroids. Here, we choose the ones based on circuits. All of the following definitions can be found in [17] by Oxley:

A *matroid* is a pair $M = (E, \mathcal{C})$ where E is a finite ground set, and $\mathcal{C}$ is a family of subsets of E such that:

(C1) $\varnothing \notin \mathcal{C}$

(C2) If $C_1, C_2 \in \mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$

(C3) If $C_1, C_2 \in \mathcal{C}$, $C_1 \neq C_2$, and there is an element $e \in C_1 \cap C_2$, then there exists $C_3 \in \mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) \setminus e$.

Members of $\mathcal{C}$ are called *circuits*. A set $X \subseteq E$ is *independent* if no circuit is a subset of X. The *rank* of a set $X \subseteq E$ is defined as $r(X) := \max_{I \subseteq X}\{|I| : I \text{ is independent}\}$. The *rank of a matroid* M is $r(E)$, and r is the rank function of the matroid M. A maximal independent set is called a *basis*. The *closure* of a set $X \subseteq E$ is the set $\mathrm{cl}(X) = \{x \in E : r(X \cup \{x\}) = r(X)\}$. For $X, Y \subseteq E$, we say X *spans* Y if $Y \subseteq \mathrm{cl}(X)$. We say $e \in E$ is in the span of X if $e \in \mathrm{cl}(X)$. A set $F \subseteq E$ is a *flat* if $\mathrm{cl}(X) = X$. A *hyperplane* is a flat of rank $r(M)-1$. The dual matroid of M, denoted as M^*, is the matroid with the same ground set E such that X is a circuit in M^* if and only if $E\backslash X$ is a hyperplane in M. The circuits of M^* are called *cocircuits* of M.

A circuit of size one is called a *loop*, a circuit of size two is called a *parallel pair*, and a circuit of size three is called a *triangle*. A cocircuit of size one is called a *coloop*, a cocircuit of size two is called a *series pair*, and a cocircuit of size three is called a *triad*. A matroid is *simple* if it does not have loops and parallel pairs. A matroid is *cosimple* if its dual matroid is simple. The simplification of M, denoted as $\mathrm{si}(M)$, is the matroid obtained from M by deleting all loops and keeping only one element in each parallel pair. The cosimplification of M, denoted as $\mathrm{cosi}(M)$, is the dual of the simplification of M^*.

Given $e \in E$, the *deletion* $M\backslash e$ is the matroid $(E-e, \{C \in \mathcal{C} : e \notin \mathcal{C}\})$. The *contraction* M/e is the matroid $(E-e, \mathcal{C}')$ where $\mathcal{C}'$ consists of the minimal members of the set $\{C-e : C \in \mathcal{C}\}$ unless e is a loop, in which case, $M/e = M\backslash e$. We say N is a *restriction* of M if N can be obtained from M by a series of deletions. In this case, let A be the ground set of N, we often denote N as $M|A$. We say N is a *minor* of M if N can be obtained from M by a series of deletions and contractions. A *minor-closed* class of matroids is a collection of matroids that are closed under taking minors.

For $X \subseteq E$, the connectivity of the set X is defined by $\lambda_M(X) = r(X)+r(E\backslash X)-r(M)$. A set X is k-separating if $\lambda_M(X) \leq k-1$. $(X, E\backslash X)$ is called a k-separation if X is k-separating and $|X|, |E\backslash X| \geq k$. A k-separation $(X, E\backslash X)$ is *proper* if $|X|, |E\backslash X| \geq k+1$. A matroid is k-connected if it has no t-separations for all $t < k$. An exact k-separation $(X, E\backslash X)$ is a k-separation such that $\lambda_M(X) = k-1$.

Matroids can be constructed via matrices. If A is a matrix over a field $\mathbb{F}$, we can construct a matroid $M = (E, \mathcal{C})$ where E is the collection of columns of A and $\mathcal{C}$ is the collection of nonempty sets of columns that are minimally linear dependent. We denote the matroid M as $M[A]$, and we say M is representable over a field $\mathbb{F}$ if $M = M[A]$ for some matrix A over $\mathbb{F}$ (with the linear dependency over $\mathbb{F}$ as well). A matroid is *binary* if it is representable over the field $\mathbb{F}_2$.

2.2 Matroid Sums

Let $M_1 = (E_1, \mathcal{C}_1), M_2 = (E_2, \mathcal{C}_2)$ be two matroids. If $E_1 \cap E_2 = \varnothing$, we define the *1-sum* of M_1, M_2 , denoted by $M_1 \oplus_1 M_2$, as the matroid $M = (E_1 \cup E_2, \mathcal{C})$, where $C \in \mathcal{C}$ if and only if $C \in \mathcal{C}_1$ or $C \in \mathcal{C}_2$. If $E_1 \cap E_2 = \{e\}$ and e is not a loop or coloop, we define the *2-sum* of M_1, M_2 , denoted by $M_1 \oplus_2 M_2$, as the matroid $M = (E_1 \triangle E_2, \mathcal{C})$, where $C \in \mathcal{C}$

if and only if $C \in \mathcal{C}_1$ or $\mathcal{C}_2$ with $e \notin C$, or $C = C_1 \triangle C_2$ where $C_i \in \mathcal{C}_i$ for $i \in \{1, 2\}$ and $e \in C_1, C_2$. If M_1, M_2 are binary with $|M_1|, |M_2| \geq 7$, and $|E_1 \cap E_2| = 3$ forms a triangle T in both M_1, M_2 and T contains no cocircuits of M_1, M_2, we define the *3-sum* of M_1, M_2 , denoted by $M_1 \oplus_3 M_2$, as the matroid $M = (E_1 \triangle E_2, \mathcal{C})$, where $C \in \mathcal{C}$ if and only if $C \in \mathcal{C}_1$ or $C \in \mathcal{C}_2$ with $T \cap C = \varnothing$, or $C = C_1 \triangle C_2$ where $C_i \cap T = \{e\}$ for some $e \in T$ and $C_i \in \mathcal{C}_i$ for $i \in \{1, 2\}$. The following propositions of 1,2 and 3-sums of matroids can be found in [9]:

Proposition 1 ([9])**.** Suppose that M is a k-connected matroid such that $M = M_1 \oplus_k M_2$ for $k \in \{1, 2, 3\}$, and suppose M is binary when $k = 3$, then M_1, M_2 are minors of M.

Before we proceed with more definitions, we require the following observation:

Observation 1 ([9])**.** Let M be a matroid with matrix representation A and let $X \subseteq E(M)$. Let $\langle X \rangle$ be the vector space spanned by $A|X$. Then $\lambda_M(X) = \dim [\langle X \rangle \cap \langle E \backslash X \rangle]$.

Let M be a binary matroid with matrix representation A in $\mathbb{F}_2$. Let $(X, E\backslash X)$ be an exact 2-separation of M. Then, by 1, there exists a unique nonzero binary vector v that belongs to $\langle X \rangle \cap \langle E\backslash X \rangle$. Let A^+ be the matrix obtained from A by adding column v. Then $M[A^+]$ is the *completion of M with respect to the 2-separation* $(X, E\backslash X)$. Now we let $(X, E\backslash X)$ be an exact 3-separation in M. Then $\dim [\langle X \rangle \cap \langle E\backslash X \rangle] = 2$, so there are three unique nonzero binary vectors in $\langle X \rangle \cap \langle E\backslash X \rangle$. Let these vectors be $p, q, p + q$ and let A^+ be the matrix obtained from A by adding columns $p, q, p + q$. Then $M[A^+]$ is the *completion of M with respect to the 3-separation* $(X, E\backslash X)$.

The completion allows us to connect k-separations to k-sums for $k \in \{2, 3\}$:

Proposition 2 ([9])**.** Let M be a 2-connected binary matroid with an exact 2-separation $(X, E\backslash X)$ and let N be the completion of M with respect to $(X, E\backslash X)$. Then we have $M = (N\backslash X) \oplus_2 (N\backslash(E\backslash X))$.

Proposition 3 ([9])**.** Let M be a 3-connected binary matroid with a proper 3-separation $(X, E\backslash X)$ and let N be the completion of M with respect to $(X, E\backslash X)$. Then we have $M = (N\backslash X) \oplus_3 (N\backslash(E\backslash X))$.

Proposition 4 ([9])**.** For $k \in \{1, 2, 3\}$ k-connected binary matroids has a k-separation if and only if it can be expressed as a k-sum $M_1 \oplus_k M_2$.

2.3 Pinch-graphic Matroids

A matroid is *graphic* if its circuits are precisely the polygons of a graph. A *signed graph* is a pair (G, Σ) where G is a graph and Σ is a subset of $E(G)$. A cycle C of (G, Σ) is even (respectively odd) if $|C \cap \Sigma|$ is even (respectively odd). M is an *even-cycle matroid* if the circuits of M are precisely the minimal even cycles of a signed graph (G, Σ). (G, Σ) is called a *signed-graph representation* of M. We say e is odd if $e \in \Sigma$ and e is even otherwise.

A signed graph representation for an even-cycle matroid is clearly not unique. In fact, if δ is a cut in G and we replace Σ by $\Sigma \triangle \delta$, we end up with the same matroid. We call this process *resigning*, and we say Σ is equivalent to Σ' if Σ can be obtained by a sequence of resigning from Σ'. Note that, an even-cycle matroid can still have more than one signed-graph representation up to equivalence. An example is the dual of the fano plane F_7^*, which has two non-equivalent signed-graph representations. The following figure can be found in Appendix B of [18]:

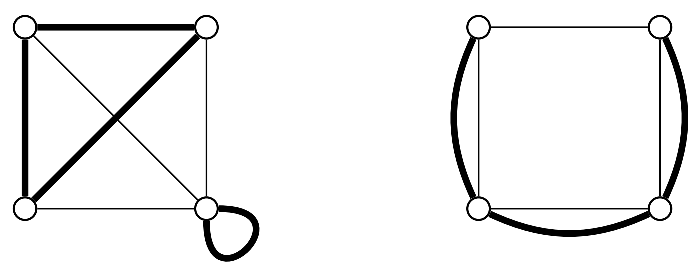

Figure 3: Two signed graph representations of F_7^* (Bold edges are odd.)

A *blocking vertex* of a signed-graph representation (G, Σ) is a vertex that intersects every odd cycle of (G, Σ). A *blocking pair* of a signed graph representation (G, Σ) is a pair of vertices such that every odd cycle intersects at least one of the vertices in the pair. Note that graphic matroids and even-cycle matroids are all binary. Now we are ready to define the pinch-graphic matroids:

Definition 2.1. A matroid M is *pinch-graphic* if it is an even-cycle matroid and it has a signed graph representation with a blocking pair.

As the name suggests, pinch-graphic matroids are very close to graphic matroids. At the moment, it is not very clear why and how they are related. The following well-known results would help:

Proposition 5. Let M be a 3-connected even-cycle matroid with a signed-graph representation (G, Σ). If (G, Σ) has a blocking vertex, then M is graphic.

Proof. Let v be a blocking vertex of (G, Σ). Then, $(G \backslash v, \Sigma)$ has no odd cycle. Hence, Σ is a cut in $G \backslash v$, which can be resigned to an empty set [11]. Thus, up to resigning we may assume all odd edges of (G, Σ) are incident on v. We can split v into two vertices v_{odd} and v_{even} where v_{odd} is adjacent to vertices that are connected to v via odd edges in (G, Σ), v_{even} is adjacent to vertices that are connected to v via even edges in (G, Σ). One can check that cycles in the resulting graph are exactly even cycles in (G, Σ). Thus, the even-cycle matroid with representation (G, Σ) is graphic. ■

Now it is straightforward to see that, pinch-graphic matroids are just "lifts" of graphic matroids. Instead of a blocking vertex, the signed-graph representation of a pinch-graphic matroid has a pair of vertices that is "blocking".

2.4 Motivations

The world of graphic matroids and binary matroids is well-studied in the past century. For excluded structures in graphic matroids, excluded minor is the most well-known field in the area. In 1959, Tutte proved the excluded minor characterization of graphic matroids:

Theorem 2.2 (Tutte, [25])**.** *A matroid is graphic if and only if it has no minor isomorphic to $U_{2,4}, F_7, F_7^*, M^*(K_5)$ or $M^*(K_{3,3})$.*

We expect that there is a similar result for pinch-graphic matroids. The main concern of this section is to find an analogous result of Theorem 2.2 for pinch-graphic matroids. Note that we assume all matroids discussed in this chapter are binary. Here is a Venn diagram of all the classes of matroids discussed in this thesis:

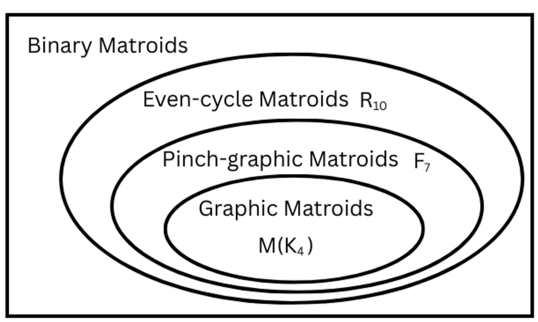

Figure 4: The world of matroids discussed in this thesis

Of course, our ultimate goal is the following question:

Problem 4. What is the exact list of excluded minors for pinch-graphic matroids?

From this stage, it seems too ambitious to hope for an exact list of excluded minors. Therefore, at this stage, we only want an explicit upper bound on the size of excluded minors for pinch-graphic matroids.

Problem 5. Can we bound the size of the matroids that are excluded minors for pinch-graphic matroids?

To partially answer this question, we need to use some previous results due to Guenin and Heo in [9]. Before we state their results, we need some more propositions and definitions:

Proposition 6 (Guenin and Heo, [9])**.** Let $M = M_1 \oplus_k M_2$ for $k \in \{1, 2, 3\}$ where M_1 is graphic. Assume that M is k-connected and binary. Then M is pinch-graphic if and only if M_2 is pinch-graphic.

This proposition is very useful in a sense that, if the matroid does not admit high connectivity, then the converse is also true:

Definition 2.3. For $k \in \{1, 2, 3\}$, consider a k-connected binary matroid M where $M = M_1 \oplus_k M_2$. Let $X = E(M_1)\backslash E(M_2)$. If M_1 or M_2 is graphic, we say the k-separation $(X, E\backslash X)$ is *reducible.*

Theorem 2.4 (Guenin and Heo, [9])**.** *Every 1-separation and 2-separation of a pinch-graphic matroid is reducible.*

2.5 Results on 1,2-separations in Excluded Minors

To bound the size of excluded minors for pinch-graphic matroid, we first want to classify these excluded minors. Similar to Tutte's proof in [25] of excluded minors for graphic matroids, we want to classify these minors by their connectivity. To warm up, we start with a theorem for matroids that has a 1-separation, i.e. matroids that are not connected.

Theorem 2.5. *If $M = M_1 \oplus_1 M_2$ is minimally non-pinch-graphic, then we have $M_1, M_2 \in \{F_7, F_7^*, M^*(K_5), M^*(K_{3,3})\}$.*

Proof. Let M be a minimally non-pinch-graphic matroid with a 1-separation. Then by Proposition 4, we can write M as $M = M_1 \oplus_1 M_2$. Suppose for contradiction that M_1' is a non-graphic proper minor of M_1. Since $M_1' \oplus M_2$ is a minor of M, $M_1' \oplus M_2$ is pinch-graphic. Then by Theorem 2.4, M_2 is graphic. But by Proposition 6, this suggests that $M = M_1 \oplus_1 M_2$ is pinch-graphic. This contradicts our assumption that M is minimally non-pinch-graphic. Thus, all proper minors of M_1 are graphic, but M_1 is non-graphic, Thus, M_1 is an excluded minor for graphic matroids. Similarly, M_2 is also an excluded minor for graphic matroids. The list follows from Theorem 2.2, except for $U_{2,4}$. Since $U_{2,4}$ is not binary, $M' = U_{2,4} \oplus M_2'$ where M_2' is a proper minor of M_2 is not binary, which means M' is not pinch-graphic. Thus $M_1, M_2 \neq U_{2,4}$. ■

Corollary 2.6. If $M = M_1 \oplus_1 M_2$ is minimally non-pinch-graphic, then $|M| \leq 20$.

A similar result holds for 2-connected matroids. However, the proof is not as straightforward as the case of disconnected matroids.

Theorem 2.7. *If $M = M_1 \oplus_2 M_2$ is 2-connected and minimally non-pinch-graphic, then we have $M_1, M_2 \in \{F_7, F_7^*, M^*(K_5), M^*(K_{3,3}), M^*(K_{3,3}')\}$, where $K_{3,3}'$ is the simple graph obtained by adding an edge to $K_{3,3}$.*

In order to prove Theorem 2.7, we need to use the concept of *1-roundedness*. The following definitions can be found in [17] by Oxley:

Definition 2.8. A class $\mathcal{N}$ of matroid is 1-*rounded* if every member of $\mathcal{N}$ is 2-connected and for all 2-connected matroid M having a $\mathcal{N}$-minor, each element of M is contained in an $\mathcal{N}$-minor.

The following proposition is due to Seymour in 1977:

Proposition 7 (Seymour, [22])**.** The class $\mathcal{N} = \{U_{2,4}, F_7^*, F_7, M^*(K_5), M^*(K_{3,3}), M^*(K_{3,3}')\}$ is 1-rounded, where $K_{3,3}'$ is the simple graph obtained by adding an edge to $K_{3,3}$.

We will also use the following elementary properties of pinch-graphic matroids:

Proposition 8. A matroid M is pinch-graphic if and only if the simplification of M is pinch-graphic.

Proof. Suppose M is pinch-graphic, then since $\mathrm{si}(M)$ is a minor of M, it is pinch-graphic. On the other hand, suppose $\mathrm{si}(M)$ is pinch-graphic with representation (G, Σ). It is clear that if we add loops to (G, Σ), it is still pinch-graphic, as loops are in no odd cycles. To add an element parallel to an exists element e in $\mathrm{si}(M)$, we simply copy the edge represented by e. It is easy to verify that the resulting signed graph still has a blocking pair. Hence, after adding back all loops and parallel pairs, we can still get a signed graph representation of M with a blocking pair, which means M is pinch-graphic. ■

Proposition 9. A matroid M is pinch-graphic if and only if the cosimplification of M is pinch-graphic.

Proof. Similar as the proof of Proposition 8, we can assume $\text{cosi}(M)$ is pinch-graphic with a signed-graph representation (G, Σ), and this time we consider adding back coloops and series pairs. To add a coloop e, we can create a new vertex and connect that vertex to any vertex of (G, Σ), and we label this new edge e. To add an element e' in series pair with e where $e \in (G, \Sigma)$, we can subdivide the edge into a path of 2 edges, e and e' and we force $e' \notin \Sigma$. One can check that after adding back coloops and series pairs, we still have a blocking pair, which means M is also pinch-graphic. ■

We are now ready to prove our characterization of minimally non-pinch-graphic matroids with 2-separations.

Proof of Theorem 2.7:
Let

$$\mathcal{N} = \{U_{2,4}, F_7^*, F_7, M^*(K_5), M^*(K_{3,3}), M^*(K'_{3,3})\}.$$

Let $M = M_1 \oplus_2 M_2$ be a 2-connected minimally non-pinch-graphic matroid that is not 3-connected, and let $e = E(M_1) \cap E(M_2)$.

Claim 1: M_1, M_2 are both simple and cosimple.

Suppose M_1 is not simple, then $\text{si}(M_1)$ is a 2-connected proper minor of M_1. We may assume $e \in E(\text{si}(M_1))$ since e is not a loop and for parallel elements we can choose which element to keep.

Since simplification does not decrease connectivity, we have $\text{si}(M_1) \oplus_2 M_2$ is well-defined and is a proper minor of $M_1 \oplus_2 M_2$, and thus $\text{si}(M_1) \oplus_2 M_2$ is pinch-graphic. Then, by Theorem 2.4, either $\text{si}(M_1)$ or M_2 is graphic. If $\text{si}(M_1)$ is graphic, then M_1 is graphic and M_2 is pinch-graphic. By Proposition 6, $M_1 \oplus_2 M_2$ is pinch-graphic, a contradiction. On the other hand, if M_2 is graphic and $\text{si}(M_1)$ is pinch-graphic, by Proposition 8 we have M_1 is pinch-graphic and thus M is pinch-graphic. Thus, $\text{si}(M) = M$. Similarly by Proposition 9 we can show that $\text{cosi}(M) = M$. □

Claim 1 ensures that if we delete or contract an element f of $E(M_1)\backslash\{e\}$, the common element e will not become a loop or coloop. Hence, if M_1/f or $M_1\backslash f$ is still 2-connected, $(M_1/f) \oplus_2 M_2$ amd $(M_1\backslash f) \oplus_2 M_2$ are well-defined and are proper minors of $M_1 \oplus_2 M_2$.

Now let $f \in E(M_1)\backslash\{e\}$.

Claim 2: $(E(M_1)\backslash\{e, f\}, E(M_2)\backslash\{e\})$ is a 2-separation in $M\backslash f$ and M/f.

Let $E(M_1)\backslash\{e,f\} = A, E(M_2)\backslash\{e\} = B$. Consider the matroid M/f and suppose for contradiction that (A,B) is a 1-separation in M/f. Since

$$\lambda_{M/f}(A) = r_{M/f}(A) + r_{M/f}(B) - r(M/f) = 0$$

$$\lambda_M(A \cup \{f\}) = r_M(A) + r_M(B) - r(M) = 1$$

Since M_1 is simple, f is not a loop, thus $r(M/f) = r(M) - 1$. Therefore we have $r_{M/f}(A) = r_M(A) - 1, r_{M/f}(B) = r_M(B) - 1$. This means f is in span of both A, B, which means f is parallel to e. This contradicts that M_1 is simple. Hence, (A,B) is not a 1-separation in M/f. The same argument works for $M\backslash f$ since M_1 is cosimple. □

Claim 3: For every 1-separation (A,B) of M_1/f and $M_1\backslash f$, if $e \notin A$, $(M/f)|A$ and $(M\backslash f)|A$ is graphic.

Suppose $M_1\backslash f$ has a 1-separation (A,B) where $e \in B$. Then, $(A, E(M)\backslash A)$ is also a 1-separation in $M\backslash f$, which means either $M|A$ is graphic or $M|(E(M)\backslash A)$ is graphic. If $M|(E(M)\backslash A)$ is graphic, M_2 is graphic, by Proposition 6 and Theorem 2.4, $M_1 \oplus_2 M_2$ is not pinch-graphic if and only if M_1 is not pinch-graphic, which contradicts the minimality. Thus, $M|A$ is graphic. The same proof works for M_1/f. □

Claim 4: $M_1/f, M_1\backslash f$ are graphic.

Consider M_1/f. If it has a 1-separation (A,B) with $e \in B$, then $M|A$ is graphic, so M_1/f is non-graphic if and only if $(M_1/f)|B$ is non-graphic. Note that we may assume $(M_1/f)|B$ is 2-connected. Thus, $((M_1/f)|B) \oplus_2 M_2$ is well-defined and is a proper minor of $M_1 \oplus_2 M_2$. If $(M_1/f)|B$ is non-graphic, since $((M_1/f)|B) \oplus_2 M_2$ is pinch-graphic, M_2 is graphic. This is a contradiction because then $M_1 \oplus_2 M_2$ is pinch-graphic if and only if M_1 is pinch-graphic, which contradicts that $M = M_1 \oplus_2 M_2$ is minimally non-pinch-graphic. The same proof works for $M_1\backslash f$. □

Claim 5: $M_1, M_2 \in \mathcal{N}$.

We first prove that M_1 is not graphic. If M_1 is graphic, then by Proposition 6 and Theorem 2.4, M is pinch-graphic if and only if M_2 is pinch-graphic. Since M is not pinch-graphic, M_2 is also not pinch-graphic, but this contradicts minimality of M. Thus, M_1 is not graphic.

Since $\mathcal{N}$ contains all excluded minors for graphic matroids, M_1 has a $\mathcal{N}$-minor. Furthermore, consider the element e in $M_1 \cap M_2$. Since $\mathcal{N}$ is 1-rounded (by Proposition 7), e is contained in a $\mathcal{N}$-minor of M_1. But by Claim 4, every proper minor of M_1 containing e is graphic. Hence, this $\mathcal{N}$-minor is not proper, which means $M_1 \in \mathcal{N}$. By symmetry, this also proves that $M_2 \in \mathcal{N}$. □

Note that $U_{2,4}$ is not binary, thus if $M_1 = U_{2,4}$, M_1 is already not pinch-graphic, which contradicts the minimality of M. Hence $M_1, M_2 \in \{F_7, F_7^*, M^*(K_5), M^*(K_{3,3}), M^*(K'_{3,3})\}$, which completes the proof. ■

This characterizes all 2-connected but not 3-connected excluded minors for pinch-graphic matroids. As a corollary, we can bound the size of these excluded minors:

Corollary 2.9. If M is minimally non-pinch-graphic, connected but not 3-connected, then $|M| \leq 19$.

2.6 Results on 3-separations in Excluded Minors

A natural question to ask is whether the Theorem 2.7 can be generalized to 3-separations in pinch-graphic matroids. we will see later, as Theorem 2.13 suggests, 3-separations in 3-connected pinch-graphic matroids are more complicated, since we do not always have reducible 3-separations. However, for a 3-connected pinch-graphic matroid where all 3-separations are reducible, we still yield a similar result. This time, we have to use a theorem in [20] by Reid:

Theorem 2.10 (Reid, [20])**.** *Let $T = \{e, f, g\}$ be a triangle of a 3-connected binary matroid M, N being a 3-connected minor M with $e \in E(N)$. Then there exists a 3-connected minor M' of M containing T such that M' has a minor isomorphic to N and $|M'| \leq |N| + 2$.*

This is a nice tool as it is an analog to the definition of 1-roundedness but substitute an element with a triangle.

Theorem 2.11. *Let $M = M_1 \oplus_3 M_2$ be a 3-connected matroid that is minimally not pinch-graphic. Let $M_1 \cap M_2 = T = \{p, q, r\}$. If for every $e \in M_1 \backslash M_2$, $(E(M_1)\backslash T, E(M_2)\backslash T)$ is a reducible 3-separation for M/e, and $(E(M_1)\backslash T, E(M_2)\backslash T)$ is a reducible 3-separation for $M\backslash e$, then $|M_1|, |M_2| \leq 12$.*

Proof. Let

$$\mathcal{N} = \{U_{2,4}, F_7^*, F_7, M(K_5)^*, M(K_{3,3})^*, M(K'_{3,3})^*\}.$$

If M_1 is graphic, then M_2 is not pinch-graphic, otherwise M is pinch-graphic. Thus, similar as in the proof of Theorem 2.7, we have $M = M_2$ and $M_1 = T$. If M_1 is not graphic, it has a $\mathcal{N}$-minor N. Since $\mathcal{N}$ is 1-rounded, there is a $\mathcal{N}$-minor using the element $p \in T$. Let us call this minor N. Therefore, by Theorem 2.10, there is a 3-connected minor M'_1 of M_1 such that all of the following holds:

1. $|M_1'| \leq |N| + 2$

2. $T \subseteq E(M_1')$

3. M_1' has a minor isomorphic to N

Assume M_1' is a proper minor of M_1. Since $T \subseteq E(M_1')$ and T is a triangle, $M' = M_1' \oplus_3 M_2$ is well-defined. Since M_1' is 3-connected, then M' is a proper minor of M. By minimality of M, M' is pinch-graphic. Then by Proposition 6, M_1' or M_2 is graphic. It is clear that M_2 is not graphic, otherwise M is pinch-graphic. Thus, M' is graphic, but M_1' has a minor isomorphic to N. Note that every matroid in $\mathcal{N}$ is non-graphic, hence N is non-graphic. This leads to a contradiction as M_1' is graphic but it has a non-graphic minor. Therefore, our assumption fails, which means $M_1' = M_1$. Then we have $|M_1| = |M_1'| \leq |N| + 2$. Since $|N| \leq 10$, we have $|M_1'| \leq 12$. ■

This also gives us a nice bound on the size of the matroid:

Corollary 2.12. Let $M = M_1 \oplus_3 M_2$ be a 3-connected matroid that is minimally not pinch-graphic. Let $M_1 \cap M_2 = T = \{p, q, r\}$. If for every $e \in M_1 \backslash M_2$, $(E(M_1) - e, M_2)$ is a reducible 3-separation for M/e, and $(E(M_1) - e, M_2)$ is a reducible 3-separation for $M \backslash e$, then $|M| \leq 21$.

2.7 Classification of 3-separations in Pinch-graphic Matroids

To study 3-separations, we are going to use previous results from Guenin and Heo in [9]. The following definitions can also be found in [9].

Given a graph G with $X \subseteq E(G)$, we denote $G|X$ as the graph with edge set X and vertex set $\{v \in V(G) : v \text{ is an end of an edge in } X\}$. Let $Y = E(G) \backslash X$. We denote $\partial_G(X)$ as the set of vertices common to $G|X$ and $G|Y$. For a signed graph (G, Σ), we denote by $(G, \Sigma)|X$ the signed graph induced by the edges in X, namely $(G, \Sigma)|X = (G|X, \Sigma \cap X)$. The triple (G, Σ, X) is of *Type I or II* if $|X|, |Y| \geq 4$, $G|X, G|Y$ are connected and $|\partial_G(X)| = 2$. For any signed graph (G, Σ), we say it is bipartite if the signed graph has no odd cycle. We define $p[(G, \Sigma)] = 0$ if (G, Σ) is bipartite and $p[(G, \Sigma)] = 1$ otherwise. For a graph H, we define the number of components in G by $\kappa(G)$.

For Type I in addition, there exists a blocking pair u, v where $u \in V(G|X) \backslash V(G|Y)$. For Type II, $\partial_G(X)$ is a blocking pair. Suppose M is a matroid with $X \subseteq E(M)$ and let $Y = E(M)\backslash X$. Further suppose $|X| \geq 5$ and $(X, E(M) - X)$ is a 3-separation of the matroid M, and there exists $e \in X$ such that $e \in \mathrm{cl}^*(Y) \cap \mathrm{cl}^*(X - e)$, then it is easy to show that $(X - e, Y + e)$ is also a 3-separation. We say $(X - e, Y + e)$ is *homologous* to (X, Y) and so is any set obtained by repeating the procedure aforementioned.

Now, consider a pinch-graphic matroid M with a proper 3-separation $(X, E(M) - X)$. We say $(X, E(M) - X)$ is *compliant* if there is a representation (G, Σ) of M such that (G, Σ, X) is of Type I. We say $(X, E(M) - X)$ is *recalcitrant* if there is a representation (G, Σ) of M such that (G, Σ, X) is of Type II. The following figure can be found in [9]:

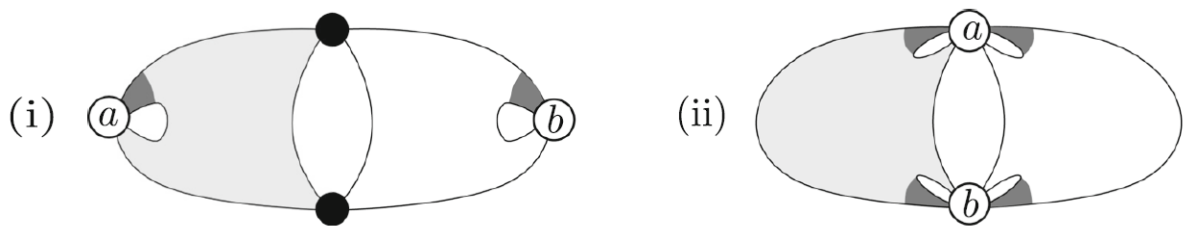

Figure 5: Compliant and recalcitrant 3-separations

In the figure above, a, b serves as a blocking pair, and the dark gray part represents for edges that are in the signature Σ. Now we are ready to state a full characterization when we have 3-separations in our pinch-graphic matroids:

Theorem 2.13 (Guenin and Heo, [9])**.** *If (A, B) is a 3-separation of a pinch-graphic matroid S, then (A, B) is homologous to a 3-separation that is either reducible, compliant or recalcitrant.*

We also improve Theorem 2.13 by trying to remove the homologous condition. Note that, for pinch-graphic matroids, if (A, B) is a 3-separation, then by Observation 1, there are at most three elements in $\mathrm{cl}(A) \cap \mathrm{cl}(B)$ or $\mathrm{cl}^*(A) \cap \mathrm{cl}^*(B)$. We have a theorem that characterize the 3-separations that are "1-element away" from recalcitrant or compliant.

Definition 2.14. We say a 3-separation (A, B) in a pinch-graphic matroid is a *teapot* if there exists a representation (G, Σ) such that
1. $(G, \Sigma)|A$ has 1 component,$(G, \Sigma)|B$ has 2 components, and $|\partial_G(A)| = 4$.
2. $(G, \Sigma)|A$ is bipartite,$(G, \Sigma)|B$ is non-bipartite.
3. One component of $(G, \Sigma)|B$ consists of a single edge.

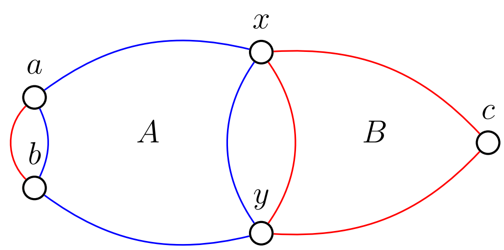

Figure 6: Example of a teapot

Theorem 2.15. Let $(A+e, B)$ be a 3-separation of a pinch-graphic matroid M and let $e \in \mathrm{cl}^*(A) \cap \mathrm{cl}^*(B)$. If $(A+e, B)$ is compliant but not reducible, then $(A, B+e)$ is reducible, compliant or teapot.

Before we prove the statement, we first introduce two useful lemmas in [8] and [9]:

Lemma 2.16 (Guenin and Heo, [8]). Let M be an even-cycle matroid with bipartition (X, Y) and a signed graph representation (G, Σ). Then

$$\lambda_M(X) = |\partial_G(X)| - \kappa(G|X) - \kappa(G|Y) + p[(G, \Sigma)|X] + \kappa(G|Y) + p[(G, \Sigma)|Y].$$

Lemma 2.17 (Guenin and Heo, [9]). Let M be an even-cycle matroid with representation (G, Σ) and let $A \subseteq E(M), e \in E(M) \backslash A$. Then the following are equivalent:
(1) $e \in \mathrm{cl}^*(A)$.
(2) there exists a signature D of (G, Σ) or a cut D of G where $D \backslash A = \{e\}$.

Now we are ready to proof Theorem 2.15.

Proof of Theorem 2.15: Let M be a pinch-graphic matroid with a Type I representation (G, Σ). Let $(A, B+e)$ be the corresponding compliant 3-separation. Let C_A be the component which consists of edges in $A+e$ and let C_B be the component which consists of edges in B. Let $\{x, y\}$ be the two vertices in $C_A \cap C_B$. Let $\{a, b\}$ be the blocking pair where $a \in C_A \backslash \{x, y\}, b \in C_B \backslash \{x, y\}$. In all figures below, blue edges are edges in A, red edges are edges in B and the green edge is e.

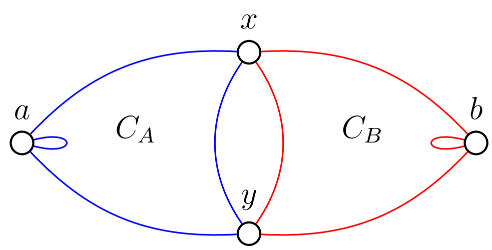

Figure 7: A compliant 3-separation

Since $e \in \mathrm{cl}^*(A) \cap \mathrm{cl}^*(B)$, by Lemma 2.17, there exists signatures or cuts D_1, D_2 where $D_1 \backslash A = \{e\}$ and $D_2 \backslash B = \{e\}$. This gives us four cases:

1. If D_1, D_2 are cuts of G:
   We claim that if e has both ends in $C_A \backslash \{x, y\}$, then $(A, B + e)$ is also compliant. If there is a circuit K in C_A that contains e, then any cut D_2 must intersect K even number of times, but $D_2 \backslash B = \{e\}$, a contradiction. Thus, e is a bridge of C_A. Note that, since M is 3-connected, G is at least 2-connected. Thus, x, y is not in the same component of $C_A \backslash e$.

   We claim that we can find a new representation of M where e is not a bridge of C_A. We first contract the edge e. Now, let $N_i(x)$ be the neighborhood of x in C_i for $i \in \{A, B\}$. We delete the vertex x and create two new vertices x_A, x_B where x_i is exactly adjacent to $N_i(x)$ for $i \in \{A, B\}$. Then, let $e = x_A x_B$. The resulting new signed graph has exactly the same even cycles as in our original graph. One can check in this case, $(A, B + e)$ is compliant with the blocking pair (a, b).

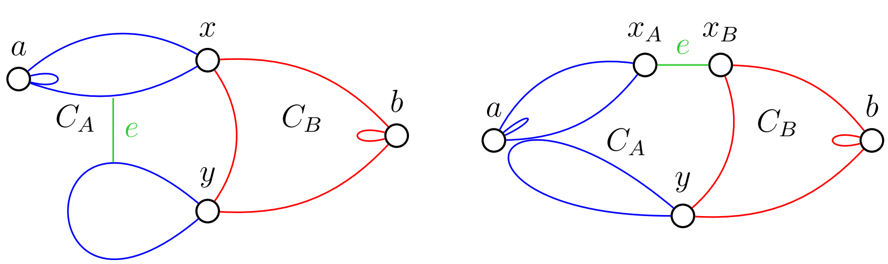

Figure 8: Equivalent representations when e is a bridge

Now, if e has at least one end in $\{x, y\}$. Without loss of generality, assume that one end of e is x. Since there is a cut D_2 with $D_2 \backslash B = \{e\}$, we claim that we may assume x is adjacent to only one vertex in C_A, and the corresponding edge is e. Suppose otherwise, then let $e = xm, e' = xn$ be two edges in A. Since D_2 contains e, we may assume D_2 is the cut induced by a set of vertices X where $x \in X$. Since e is the only edge in C_A that is in the cut, e' is not in the cut, and thus $n \in X$. Since $m \notin X$, if there is a path in $C_A \backslash e$ from m to n, $D_2 \not\subseteq B + e$, which is a contradiction, hence m, n are disconnected in C_A after deleting e. This means e is a bridge in C_A, and we can perform the same trick as above, resulting in an equivalent representation. In this representation, x_B is adjacent to only one vertex in C_A.

Now, if x is only adjacent to one vertex in C_A, we can check that $(A, B + e)$ is still compliant, except for the case that x is only adjacent to a. However, if $e = xa$, we can look at the completion of the matroid $M|A$. We can add an even edge and an odd edge between a, y, and then an odd loop on a. This give us a completion of $M|A$, which has a blocking vertex a. Thus by Theorem 5, $(A, B + e)$ is reducible.

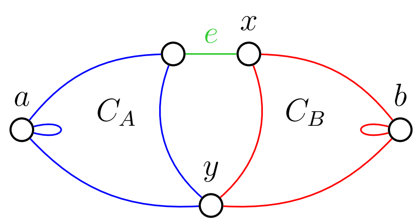

Figure 9: $(A, B + e)$ is compliant

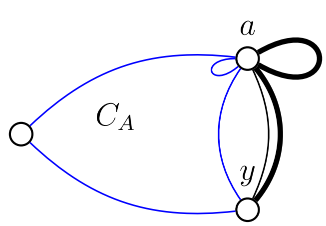

Figure 10: Completion of A has a blocking vertex a (Bold edges are odd.)

2. If D_1 is a cut and D_2 is a signature:
   e is in the component C_A since $(A + e, B)$ is compliant. If $e = xy$, then consider the

3-separation $(A, B+e)$ in (G, Σ). Since D_2 is a signature, the component $C_A - e$ is bipartite. The number of common vertices is still 2, and we have $\kappa(G|A) \geq 1, \kappa(G|B+e) = 1$. Thus, by Lemma 2.16 we have

$$\lambda_M(B+e) = |\partial_G(B+e)| - \kappa(G|A) - \kappa(G|(B+e)) + p[(G,\Sigma)|A] + p[(G,\Sigma)|(B+e)] < 2.$$

This is a contradiction, hence $e \neq xy$.
If e has exactly one end in $\{x, y\}$, without loss of generality say e has one end x and the other end in $C_A \backslash \{y\}$. Then, with the signature D_2, C_A only has one odd edge, which is e. We claim that the completion of $M|(A+e)$ is graphic: take the signed graph $(G, \Sigma)|(A+e)$, we can add two edges between x, y with one odd and one even. We then add an odd loop attached to x. This is the completion of $M|(A+e)$, and x is now a blocking vertex. Thus, $(A+e, B)$ is reducible, a contradiction.

Thus, e has both ends in $C_A \backslash \{x, y\}$. In this case, $(A, B+e)$ is a teapot.

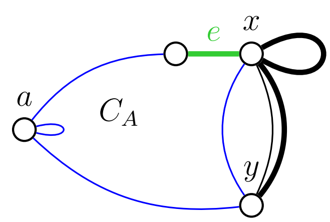

Figure 11: Completion of $A+e$ has a blocking vertex x (Bold edges are odd.)

3. If D_1 is a signature and D_2 is a cut:
Then the signature D_1 has all edges in $A+e$, which means the component C_B is bipartite. Note that by Lemma 2.16

$$\lambda_M(B) = |\partial_G(B)| - \kappa(G|(A+e)) - \kappa(G|B) + p[(G,\Sigma)|(A+e)] + p[(G,\Sigma)|B]$$

Since $(A+e, B)$ is compliant with (G, Σ), we have $|\partial_G(B)| = 2$, $\kappa(G|(A+e)) = \kappa(G|B) = 1$, hence $\lambda_M(B) < 2$, which is a contradiction.

4. If D_1,D_2 are both signatures:
If $e = xy$, the same reasoning in Case 2 works, hence we may assume $e \neq xy$. But then, e has one end in $C_A \backslash \{x, y\}$. Let v be a vertex in $C_A \backslash \{x, y\}$ such that v is one end of e. Then the cut induced by v is a cut with edges in $A+e$. Hence, this case

can be reduced to Case 2. ■

There is a similar theorem for the recalcitrant case, but the result is in fact different:

Theorem 2.18. Let $(A+e, B)$ be a 3-separation of a pinch-graphic matroid M and let $e \in \text{cl}^*(A) \cap \text{cl}^*(B)$. If $(A+e, B)$ is compliant but not reducible, then $(A, B+e)$ is reducible, recalcitrant or teapot.

Proof. Let M be a pinch-graphic matroid with a Type II representation (G, Σ). Let $(A, B+e)$ be the corresponding recalcitrant 3-separation. Let C_A be the component which consists of edges in $A+e$ and let C_B be the component which consists of edges in B. Let $\{x, y\}$ be the two vertices in $C_A \cap C_B$.

Similarly, since $e \in \text{cl}^*(A) \cap \text{cl}^*(B)$, by Lemma 2.17, there exists signatures or cuts D_1, D_2 where $D_1 \backslash A = \{e\}$ and $D_2 \backslash B = \{e\}$. Again, we have four cases:

1. If D_1, D_2 are cuts:
   If e has both ends in $C_A \backslash \{x, y\}$, then it is a bridge of C_A. Again we can switch to the equivalent representation as mentioned in the proof of Theorem 2.15. Now we can assume e has at least one end in $\{x, y\}$. Without loss of generality, we may assume $e = xx'$ where $x' \in C_A$. If $x' \neq y$, the completion of A can be constructed by adding an odd edge and an even edge between x', y together with an odd loop on y. Now, the completion of $A - e$ has a blocking vertex y, which means $(A, B+e)$ is reducible. If $x' = y$, $(A, B+e)$ is clearly recalcitrant.

2. If D_1 is a cut and D_2 is a signature:
   We can check the connectivity still fail if $e = xy$. If e has exactly one end in $\{x, y\}$, say $e = xx'$ where $x' \in C_A \backslash y$. Then, C_A only has e as an odd edge, Now, the completion of C_A has a blocking vertex x, which means $(A+e, B)$ is reducible, a contradiction. Hence we may assume e has both ends in $C_A \backslash \{x, y\}$, which is a teapot.

3. If D_1 is a signature and D_2 is a cut:
   Similar as Case 3 in the proof of Theorem 2.15, the connectivity function fails.

4. If D_1, D_2 are cuts:
   Similar as the proof of Theorem 2.15, this can be reduced to Case 2 if $e \neq xy$, and if $e = xy$ the same reasoning in Case 2 works. ■

Chapter 3

Excluded Structures in Tournaments

3.1 Tournaments and Backedge Graphs

A *tournament* is a directed graph where for each pair of distinct vertices (u, v), there is exactly one arc with ends $\{u, v\}$. If u, v are vertices in a tournament and the arc between them starts at u and ends at v, we denote this arc as (u, v) or uv. In this thesis, all tournaments are finite.

Here are some definitions used in this paper. All of the following definitions can be found in [24].

Let G be a tournament. A *subtournament* of G is an induced subgraph of G. A subtournament of G is *proper* if its vertex set is a proper subset of $V(G)$. Let $u, v \in V(G)$ and $X, Y \subseteq V(G)$. We say u is an in-neighbor of v if there is an arc in $E(G)$ that starts from u and ends at v, and we say v is an out-neighbor of u. We use $u \to v$ to denote that v is an out-neighbour of u, and $X \Rightarrow Y$ if each vertex in Y is an out-neighbour of all vertices in X. We use $N^-(v)$ to denote the set of in-neighbors of v and $N^+(v)$ to denote the set of out-neighbors of v. The *indegree* of a vertex v is defined as $d^-(v) = |N^-(v)|$ and the *outdegree* is defined as $d^+(v) = |N^+(v)|$. A tournament is *transitive* if for all $u, v, w \in V(G)$ such that $u \to v$, $v \to w$, we have $u \to w$. A *cyclic triangle* is the 3-vertex tournament that is not transitive.

For a directed graph D, the dichromatic number, $\vec{\chi}(D)$, is the minimum number to color vertices of D such that each color class induces an acyclic digraph [16]. Motivated by this definition by Neumann-Lara, we call a set $X \subseteq V(G)$ an *independent set* of G if the induced subgraph $G[X]$ is transitive. We define the *complement* $\bar{G}$ of a tournament G

as the directed graph obtained by reversing every arc of G. We also define the *complement* of a vertex set $S \subseteq V(G)$ as $V(G) \setminus S$.

Based on [1], we define the backedge graph of a tournament as follows. Let G be a tournament on n vertices, and let $(v_1, \ldots, v_n)$ be a linear ordering of the vertex set $V(G)$. An edge $v_iv_j \in E(G)$ is called a *backedge* if $i > j$. The *backedge graph* B of G is the undirected graph with vertex set $\{v_1, \ldots, v_n\}$, where an edge $v_iv_j \in E(B)$ exists if and only if v_iv_j is a backedge in G. We have the following well-known property for all tournaments:

Proposition 10. Let T be a tournament. The following are equivalent:
(a) T is transitive.
(b) T contains no cyclic triangle.
(c) T is acylic.
(d) T admits a topological ordering on its vertices $V(T)$.
(e) There exists an ordering of $V(T)$ that gives a backedge graph with no edges.

Proof. $(a) \to (b)$: If T is transitive, by definition it contains no cyclic triangle. $(b) \to (c)$: If T has no cyclic triangle, suppose for contradiction that G is not acyclic. Let C be a smallest directed cycle in T. If C has at least 4 vertices, consider 3 consecutive vertices $a, b, c \in C$ with $a \to b \to c$. Since C is shortest, $a \to c$. But then $C \backslash \{b\}$ induces a smaller directed cycle, which is a contradiction. $(c) \to (d)$: If G is acyclic, it is obvious that G has a topological ordering on its vertices. $(d) \to (e)$: The backedge graph on a topological ordering has no edges. $(e) \to (a)$: If T has a backedge graph with no edges, then by definition it is transitive. ■

3.2 Forbidden Induced Subgraphs for k-degenerate Tournaments

In this section, we consider the set of excluded induced subgraphs for k-degenerate tournaments. Note that, a tournament T is an excluded induced subgraph of a class $\mathcal{C}$ if and only if all proper subtournaments of T are in $\mathcal{C}$ but $T \notin \mathcal{C}$. Hence, a tournament T is an excluded induced subgraph if it is minimally (with respect to taking induced subgraphs) not in $\mathcal{C}$. We will first prove theorems for k-out-degenerate tournaments and then generalize the result to the k-degenerate case. We start with a refresher on the definitions:

Definition 3.1. A tournament T is *k-in-degenerate* (respectively *k-out-degenerate*) if every induced subgraph of T has a vertex of indegree (respectively outdegree) at most k.

A tournament is k-*degenerate* if every induced subgraph of T has a vertex of indegree or outdegree at most k.

Here is a useful observation regarding the k-out-degenerate tournaments:

Lemma 3.2. Let T be a tournament that is a minimally not k-out-degenerate tournament. Then every vertex of T has at least 1 in-neighbor with outdegree exactly $k+1$.

Proof. Suppose v is a vertex in T and v has no in-neighbor with outdegree $k+1$. Since T is not k-degenerate, every in-neighbor of v has outdegree at least $k+2$. Then the tournament $T\backslash\{v\}$ has no vertex of outdegree k, which contradicts minimality of T. ■

With Lemma 3.2, we are ready to bound the size of tournaments that are minimally non-k-degenerate.

Theorem 3.3. *Let T be a tournament that is a minimally not k-out-degenerate tournament. Then $|V(T)| \leq \frac{k^2+5k+6}{2}$.*

Proof. Let T be a minimally not k-out-degenerate tournament. It is obvious that there are no vertices with outdegree less than or equal to k. Let $X = \{v \in V(T) : d^+(v) = k+1\}$. By Lemma 3.2, X is not empty. It is easy to show that if a tournament G has n vertices, then the average outdegree of vertices in G is $\frac{n-1}{2}$. Thus, if $|X| \geq 2k+4$, then the subtournament $G[X]$ contains a vertex with outdegree at least $\frac{2k+3}{2} > k+1$, which contradicts the definition of X. Hence we can conclude that $1 \leq |X| \leq 2k+3$. Let $\delta^+(X) = \{ab : a \in X, b \notin X, ab \in E(T)\}$. The total outdegree of vertices in $|X|$ are $(k+1)|X|$, but $\binom{|X|}{2}$ of these arcs have both ends in $|X|$. Hence, $|\delta^+(X)| = (k+1)|X| - \binom{|X|}{2}$. By Lemma 3.2, every vertex of T has a in-neighbor in X, so the number of vertices not in X is at most $|\delta^+(X)|$. This gives $|V(T)\backslash X| \leq \frac{|X|(2k+3-|X|)}{2}$, which means $|V(T)| \leq \frac{|X|(2k+5-|X|)}{2}$. The right hand side is maximized when $|X| = k+2$ or $|X| = k+3$, with $|V(T)| \leq \frac{k^2+5k+6}{2}$. ■

By taking the complement of the tournament that is minimally not k-out-degenerate, we get a tournament that is minimally not k-in-degenerate, and vice versa. Hence we have the following corollary:

Corollary 3.4. Let T be a tournament that is a minimally not k-in-degenerate tournament. Then $|V(T)| \leq \frac{k^2+5k+6}{2}$.

If $k = 1$, this implies $|V(T)| \leq 6$, so it is possible to list all excluded induced subgraphs of 1-out-degenerate tournaments. The complete list can be found in Appendix A.

We now generalize Theorem 3.3 to k-degenerate tournaments:

Theorem 3.5. *Let T be a tournament that is a minimally not k-degenerate tournament. Then $|V(T)| \leq k^2 + 5k + 6$.*

Proof. Let T be a minimally not k-degenerate tournament. Similar as the proof in Theorem 3.3, we let $X = \{v \in V(T) : d^+(v) = k+1\}$ and $Y = \{v \in V(T) : d^-(v) = k+1\}$. By the same argument on average in(out)-degree, we have $0 \leq |X|, |Y| \leq 2k + 3$. Let $Z = V(G) \backslash (X \cup Y)$. By Lemma 3.2, every vertex in Z either has an in-neighbor in X or an out-neighbor in Y. Let Z_X be the vertices in Z with an in-neighbor in X and let Z_Y be vertices with an out-neighbor in Y. As in Theorem 3.3, we have $|Z_X \cup X| \leq \frac{k^2+5k+6}{2}$ and likewise for $Z_Y \cup Y$. Thus, $|V(T)| \leq k^2 + 5k + 6$.

Another question of interest is whether we can check if a tournament is k-degenerate in polynomial time.

Theorem 3.6. *There is an algorithm that checks whether a tournament with n vertices is k-degenerate in $O(n^3)$.*

Proof. (Sketch) Consider the following algorithm: we start with a tournament T and we search for a vertex v in T with either indegree at most k or outdegree at most k. If there is no such vertex, then the tournament is not k-degenerate, and we terminate the algorithm. Otherwise, we delete v and repeat the process until we have no vertex. It is clear that if T is k-degenerate, the algorithm will not terminate until we have no vertex. If k is not k-degenerate, it contains a subtournament T' with all vertices having indegree and outdegree at least $k + 1$. No vertices of this subtournament will be deleted, hence eventually the algorithm reach this subtournament. Thus, the algorithm correctly determine if a tournament is k-degenerate. If T has n vertices, we repeat the process n times and each time we iterate over all vertices. It also takes n steps to compute the indegree and outdegree of a vertex if we are given an adjacency matrix. Hence the running time is $O(n^3)$. ■

We proceed with an application of the theorem above. For tournaments G, H, we say G is *H-free* if G has no subtournament isomorphic to H. Recall that, a set S of vertices is independent if $G[S]$ is acylic.

We are interested in the following problems:

Problem 6. Given a tournament G, find the maximum size of an independent set in G.

This problem is proven to be NP-complete by Speckenmeyer in [23]. However, if we only look at a class of tournaments, things might be different.

Problem 7. For a fixed graph H, given an H-free tournament G, find the maximum size of an independent set in G.

Previous research showed:

Theorem 3.7 (Spirkl, Xing, [24])**.** *If Problem 7 can be solved in polynomial time, then H is 1-degenerate.*

By Theorem 3.6, we can determine whether a given tournament H is 1-degenerate in time $O(n^3)$. This allows us to efficiently verify the necessary condition in Theorem 3.7 for polynomial-time solvability of Problem 7.

3.3 Enumeration of Backedge Graphs with Forbidden Induced Subgraphs

In this section, we are interested in enumeration of backedge graphs. Recall that, a backedge graph of a tournament is defined as follows: given a tournament T with an ordering $(v_1, ...v_n)$, the backedge graph B of T is an undirected graph with $V(B) = V(T)$ and v_iv_j is an edge in B if and only if $i > j$ and $v_i \to v_j$ in T. By definition, a tournament T with n vertices can have at most $n!$ non-isomorphic backedge graphs. We can also view an ordering on $\{v_1, ...v_n\}$ as a permutation $\sigma \in Sym(n)$. The ordering corresponding to σ is $(v_{\sigma^{-1}(1)}, ..., v_{\sigma^{-1}(n)})$. Thus, for every $\sigma \in Sym(n)$, there is an associated backedge graph of T, and we denote this backedge graph as $B_\sigma(T)$. We are interested in the following problems:

Definition 3.8. Let $\mathcal{C}_H$ be the class of graphs with no induced subgraphs isomorphic to H. We say the graphs in $\mathcal{C}_H$ are *H-free*.
We define

$$f(H, n) = \max_{|T|=n} |\{\sigma : B_\sigma(T) \in \mathcal{C}_H\}|.$$

In other words, $f(H, n)$ is the maximum number of backedge graphs that are in H of a tournament T with n vertices.

Problem 8. For a fixed graph H, is it true that $f(H, n) \geq c^n$ for some constant $c > 1$?

Problem 9. For a fixed graph H, is it true that $f(H, n) \leq c^n$ for some constant $c > 1$?

If H is a simple graph with no isolated vertices, the answer to Problem 8 is simple. If $H = K_2$, i.e. H is just an edge, then $B_\sigma(T) \in \mathcal{C}_H$ if and only if it is a collection of isolated vertices. This happens only when T is transitive, and one can check that the only ordering of T such that $B_\sigma(T)$ has no edges is the topological ordering mentioned in Proposition 10. If H has at least two edges, we have the following theorem:

Theorem 3.9. *Let H be a simple graph with at least 2 edges. Then $f(H, n) \geq 2^{\lfloor n/2 \rfloor}$.*

Proof. Let $\bar{H}$ be the complement graph of H. Note that, if we reverse the ordering of vertices on a backedge graph $B_\sigma(T)$, we get the complement of $B_\sigma(T)$. If $B_\sigma(T)$ is H-free, then complement of $B_\sigma(T)$ is $\bar{H}$-free. Thus we can conclude that $f(H, n) = f(\bar{H}, n)$. Since at least one of H,$\bar{H}$ is connected, it suffices to consider connected H.

Let T be a transitive tournament with n vertices. Let $v_1, ...v_n$ be a topological ordering of the vertices in T such that $v_i \to v_j$ if and only if $i < j$. Suppose that n is even. Then we can group the vertices as follows:$\{v_1, v_2\}, \{v_3, v_4\}, ...\{v_{n-1}, v_n\}$. There are $n/2$ groups, and for each group, we have two choices: we either swap the positions of the two vertices or we make no changes. The components in resulting backedge graphs have at most 1 edge, hence it does not contain H as an induced subgraph. There are $2^{n/2}$ such backedge graphs, since we can decide whether to swap or not for each of the $n/2$ vertices. If n is odd, we fix the position of the last vertex, we still get $\lfloor n/2 \rfloor$ groups, hence $f(H, n) \geq 2^{\lfloor n/2 \rfloor}$. ■

Note that, the lower bound in Theorem 3.9 is far from optimal. The key takeaway here is that this lower bound is exponential, which is enough for our purpose. We will look into in two cases: $H = K_t$ for some constant t, the complete graph on t vertices, and $H = P_3$, the three vertex path, and we will show that in these two cases, $f(H, n) \leq c^n$ for some constant c for sufficiently large n.

Theorem 3.10. *For every positive integer t, there exists a constant c_t such that for sufficiently large n, $f(K_t, n) \leq c_t^n$.*

To prove Theorem 3.10, we will use the Marcus-Tardos Theorem (formerly the Stanley-Wilf conjecture). For two permutations $\alpha \in Sym(n), \beta \in Sym(k)$, we say α contains β if there exists integers $1 \leq x_1 < x_2 < \ldots < x_k \leq n$ such that for $1 \leq i, j \leq k$, $\alpha(x_i) < \alpha(x_j)$ if and only if $\beta(i) < \beta(j)$. Otherwise, we say α avoids β [15].

The theorem is stated as follows:

Theorem 3.11 (Marcus and Tardos, [15])**.** *For every permutation σ, there exists a constant C such that the number of permutations in $Sym(n)$ which avoid σ is at most C^n.*

This theorem suggests that though the total number of permutations of n elements are factorial, the number of permutations of n elements avoiding a "permutation pattern" is exponential. Recall that, for a tournament with n vertices, the number of backedge graphs is also factorial, and we aim to show that the number of backedge graphs avoiding some induced subgraph is exponential. Therefore, the theorem is very useful if we can relate backedge graphs to permutations.

Before we proceed to the proof of Theorem 3.10, we need some lemmas and definitions.

Definition 3.12 ([19])**.** We denote by $R(s,t)$ the minimum number of vertices a complete graph G needs to have so that in every red-blue edge coloring of G, there is either a red K_s or a blue K_t.

Lemma 3.13. Let T be a tournament with n vertices. Let G be a K_t-free backedge graph of T with the ordering $(v_1, ...v_n)$, and let H be a K_t-free backedge graph of T with the ordering $(v_{\sigma^{-1}(1)}, ...v_{\sigma^{-1}(n)})$. Let TT_n be the transitive tournament on vertices $\{v_1, ...v_n\}$ with the topological ordering $(v_1, ...v_n)$. Then $\omega(B_\sigma(TT_n)) < R(t,t)$.

Proof. Suppose for contradiction that K is a clique in $B_\sigma(TT_n)$ of size $R(t,t)$. Let $e = v_iv_j \in K$ and assume that $i < j$. Since $e \in B_\sigma(TT_n)$, $\sigma(i) > \sigma(j)$. We mark e as *red* if $v_iv_j \in E(G)$. Otherwise, if $v_iv_j \notin E(G)$, then $v_iv_j \in E(H)$ since $\sigma(i) > \sigma(j)$. In this case, we mark this edge *blue*. We mark all edges in K, and since $|K| = R(t,t)$, there is a clique of size t whose edges are all blue or all red. This contradicts that G, H are both K_t-free backedge graphs of T. ■

Definition 3.14. A *permutation graph* is a graph whose vertices represent the elements of a permutation, and whose edges represent pairs of elements that are reversed by the permutation.

For example, the following graph represents the permutation $(4,3,5,1,2)$ (in one-line notation).

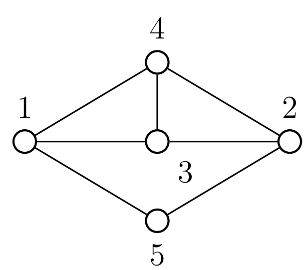

Figure 12: Permutation graph of the permutation $(4,3,5,1,2)$

Observation 2. Let TT_n be the transitive tournament with topological ordering $(v_1, ...v_n)$. Then, $B_\sigma(TT_n)$ is exactly the permutation graph of σ. Moreover, $\beta \in Sym(k)$ is a permutation pattern in σ if and only if $B_\beta(TT_k)$ is isomorphic to an induced subgraph of $B_\sigma(TT_n)$ such that corresponding vertices in $B_\sigma(TT_n)$ are in the same order as in $B_\beta(TT_k)$.

Now we can proceed to the proof of Theorem 3.10.

Proof of Theorem 3.10. Let TT_n be the transitive tournament with topological ordering $(v_1, ...v_n)$. Now, by Theorem 3.11 and Observation 2, if H is an ordered permutation graph, then there exists a constant c such that the number of H-free backedge graphs of TT_n is at most c^n. Note that $K_{R(t,t)}$ is an ordered permutation graph on $R(t,t)$ vertices. Hence, there is a constant c such that TT_n has at most c^n backedge graphs which have no ordered induced subgraph isomorphic to $K_{R(t,t)}$. By Lemma 3.13, for a tournament T, if $G = B_{id}(T), H = B_\sigma(T)$ are both K_t-free backedge graphs of T, then $\omega(B_\sigma(TT_n)) < R(t,t)$, and thus $B_\sigma(TT_n)$ cannot have an ordered induced subgraph isomorphic to $K_{R(t,t)}$. As a result, the number of candidates of σ in Lemma 3.13 is at most c^n. Hence, the number of backedge graphs of T that are K_t-free is at most c^n. ■

Note that, Observation 2 together with Theorem 3.11 imply the following result:

Theorem 3.15. *If H is a permutation graph, then there exists a constant C such that TT_n has at most C^n H-free backedge graphs.*

Finally, let us proceed to the case where $H = P_3$. We will use the following folklore structure theorem:

Theorem 3.16. *A graph G is P_3-free if and only if every component of G is a clique.*

Proof. Suppose $u, v \in V(G)$, u, v in the same component, and u is not adjacent to v. Let P be the shortest path from u to v. This path is an induced path of length at least 3, which is a contradiction. ■

Our goal will be proving the following.

Theorem 3.17. *There exists a constant c such that for sufficiently large n, $f(P_3, n) \leq c^n$.*

Before we prove this theorem, we first define some terminology that is broadly used in our proof. In a backedge graph $B_\sigma(T)$ of a tournament T, we say a vertex u is *before* v if $\sigma(u) < \sigma(v)$. In other words, if $u \to v$ we say u is before v in a backedge graph G of T if and only if $uv \notin E(G)$, and vice versa. If u is not before v we say u is *after* v. A vertex is *between* u, v if it is before v and after u. Let R_5 be the 5-vertex tournament with the following backedge graph:

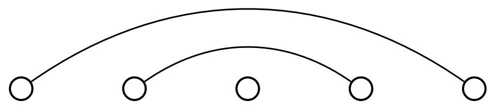

Figure 13: The 5-vertex tournament R_5

Now we are ready to prove a few useful lemmas:

Lemma 3.18. Let T be a tournament and G be a P_3-free backedge graph of T. If there exists a subtournament T' in T such that $u, v \in V(T'), T' \cong R_5$, and u is the unique vertex of indegree one in T', v is the unique vertex of outdegree one in T', then u is before v in any P_3-free backedge graph of T.

Proof. Suppose there exists such a subtournament $T' \cong R_5$. Let a, b, c be the three vertices other than u, v in T'. Then, the backedge graph of the vertices $\{u, v, a, b, c\}$ looks like the following picture:

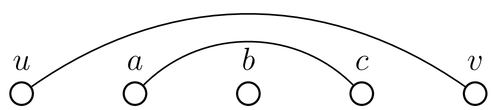

Figure 14: The subtournament $T' \cong R_5$

Now, consider any other P_3-free backedge graphs such that v is before u. If any of a, b, c are still in between v and u we have a P_3 :

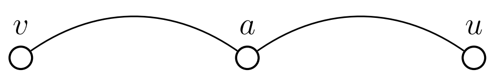

Figure 15: a is between u and v

If any two of a, b, c are before v, without loss of generality we may assume a, b are before v. For simplicity, we write $a - b$ to denote that a is before b. Note that this relationship is transitive: if $a - b$,$b - c$ then $a - c$, and in this case we write $a - b - c$ for simplicity. Since a, b are before v, we now have two possible orderings: either $a - b - v - u$ or $b - a - v - u$.

Figure 16: a, b are before v

The first case has a P_3. For the second case, consider the vertex c. If c is before u, we may assume it is not between v and u. But then, all of a, b, c are before v. Since a, b, c forms a cyclic triangle, no ordering of a, b, c gives a backedge graph that is a clique. Thus, we can always pick two of a, b, c such that there are no backedges in between the two vertices. This gives us the $a - b - v - u$ case, which is already solved. Hence, we may assume c comes after u in this ordering. In this case, $b - u - c$ forms a P_3.

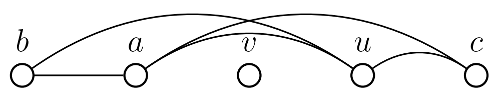

Figure 17: c is after u

On the other hand, if two of a, b, c are after u, we have either $v - u - a - b$ or $v - u - b - a$:

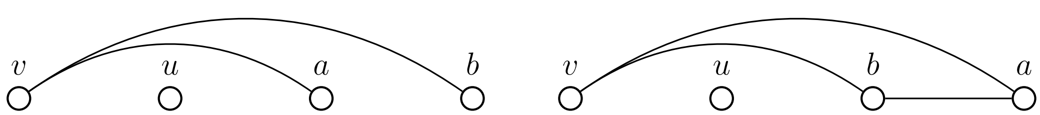

Figure 18: a, b are after u

Similarly, the first case has a P_3 and in the second case c must come before v. This means:

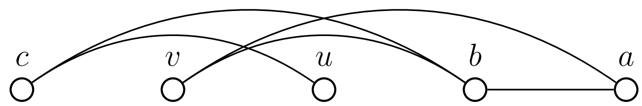

Figure 19: c is before v and a, b are after u

Now, $u - c - b$ forms a P_3. ∎

Definition 3.19. Let G be a P_3-free backedge graph of a tournament T and let uv be an backedge with u before v. We say uv is *good* if there is a subtournament $T' \cong R_5$ such that u is the unique vertex of indegree one in T' and v is the unique vertex of outdegree one in T'.

Lemma 3.20. Let G be a P_3-free backedge graph of a tournament T. If K is a clique in G and $e \in K$ is a good edge, then the vertices in K form as a clique in every P_3-free backedge graph of T. Moreover, the vertices in K appear in the exact same relative ordering as in G.

Proof. Let $a, b, c, u, v \in V(T)$ such that $T[u, v, a, b, c]$ is a copy of R_5 and $e = uv$ is a good edge. Since each of a, b, c forms a cyclic triangle with $u, v,$, it follows that $a, b, c \notin K$, and so a, b, c are anticomplete to K in G. Let K be the clique in G such that $e \in K$. By definition, if $u', v' \in K$ with u' before u and v' after v, then $u'v'$ is also a good edge. Hence, we may assume there exists no u', v' between u, v such that $u'v'$ is a good edge.

Let L, C, R be a partition of K such that L contains u and vertices in K before u in G; C contains vertices between u and v and R contains v and vertices in K after v in G. Then, all edges between L and R are good edges, and therefore, every P_3-free backedge graph G' of T contains a component D which contains both L and R. Since D is a clique, it follows that $L \cup R$ is a clique in G'.

For vertices of K between u and v, let x be a vertex after u before v. Let G' be another P_3-free backedge graph of T. One can check that x is adjacent to at least one of u, v in G', since u is always before v. Then x is in the same clique with u, v. This holds for every vertex between u and v, and hence vertices in K stay a clique in every P_3-free backedge graph.

Now let u, v be two vertices in K. If we swap their relative order, there is no backedge between u, v, contradicting that u, v are in a clique. Hence, the relative order of all vertices in K is fixed in any P_3-free backedge graphs of T, ∎

Definition 3.21. A clique K in a backedge graph is *good* if it contains a good edge.

Lemma 3.22. Let G be a P_3-free backedge graphs of a tournament T and let K, K' be two good cliques in G such that $K \cup K'$ is not transitive. If $u \in V(K), v \in V(K')$ and u is before v, then in all P_3-free backedge graphs of T, u is before v.

Proof. Let G, G' be a P_3-free backedge graphs of a tournament T. Let K, K' be two good cliques of G. By Lemma 3.20, both K and K' are cliques in G'. Since $K \cup K'$ is not transitive, $K \cup K'$ is not a subset of a large clique in G and G'. Since both G, G' are P_3-free, every component of G, G' is a clique, which means K, K' are in different components of both G, G'. Now, for any $u \in K, v \in K'$ such that u is before v, if in some P_3-free backedge graph G' v is before u, then uv becomes a backedge in G', so u, v are in the same component, which is a contradiction. ■

Lemma 3.23. Let G be a P_3-free backedge graphs of a tournament T and let K, K' be two distinct good cliques in G such that $K \cup K'$ is transitive. If $u \in V(K), v \in V(K')$ and u is before v, then in all P_3-free backedge graphs of T, u is before v.

Pf: Let G, G' be two P_3-free backedge graphs of T. By Lemma 3.20 both of K, K' are cliques in G and G'. Suppose we have $u \in K, v \in K'$ such that u is before v in G and u is after v in a P_3-free backedge graph G' of T. If $uv \in E(G)$ then $uv \notin E(G')$ and vice versa. Up to symmetry, we may assume $uv \notin E(G), uv \in E(G')$. In other words, we assume that $K \cup K'$ forms two components in G but one component in G'.

For simplicity, we use $x <_G y$ to represent that x is before y in G. If there exists $x, y \in K$, $z \in K'$ in G such that $x <_G z <_G y$, then $x \to z \to y \to x$ is a cyclic triangle of $K \cup K'$ in G, which contradicts that $K \cup K'$ is transitive. Hence in G, all vertices in K are before or after all vertices in K'. By symmetry, we assume that in G, we have all vertices in K are before all vertices in K'.

Then in G', we have that all vertices in K' are before all vertices in K, since these vertices form a single clique in G'. Now consider the backedge graph G. In G, K contains a good edge, so there exists a vertex $v \notin K$ between the first and last vertex in K. Thus in G, v is not incident with the first and the last vertex in K. These three vertices form a cyclic triangle, so $K + v$ is not transitive. Thus, in G', v is not in the same component with K. We also notice that v is not in the same component with K' in G, as we assume that all vertices in K' are after K. This means if $x \in K$ and x is before(after) v in G, then x is before(after) v in G'. Hence, v is still between the first and last vertex of K in G'.

Now in G', all vertices in K' are before all vertices in K. In G, all vertices in K' are after v and $v \notin K'$, so v is anticomplete to all vertices in K' in G. Since now all vertices in K' are before v, v is now complete to all vertices in K' in G', which means $K' + v$ is a clique in G'. Since $K' \cup K$ is a clique in G', $K' \cup K + v$ is a clique in G'. This contradicts that $K + v$ is not transitive. This is a contradiction, hence we cannot change the relative order for any pair of vertices in K and K'. ■

Now, we are ready to prove Theorem 3.17.

Proof of Theorem 3.17: Let T be a tournament with n vertices and let G be a P_3-free backedge graph of T. Let V_1 be the set of vertices in T that are in a good clique of G and let $V_2 = V(T) \backslash V_1$. By Lemma 3.22 and 3.23, the set of vertices in V_1 has a fixed ordering in all P_3-free backedge graphs.

Now, consider the induced subgraph $G[V_2]$. This graph has no R_5.

A tournament H is a hero if every H-free tournament has dichromatic number at most c for some constant c. In [1], it is showed that R_5 is a hero. Thus we have $\vec{\chi}(T[V_2]) \leq c_1$ for some c_1. Hence, V_2 can be partitioned into at most c_1 subsets such that every subset is transitive. For a transitive subset with k vertices, by Theorem 3.11, we know there are at most c_2^k P_3-free backedge graphs for some constant c_2.

Now we partition V_2 into a collection of at most c_1 transitive subsets $S_1, ..., S_{c_1}$. By Theorem 3.11 and Observation 2, there exists a constant c_2 such that a transitive subset of k vertices have at most c_2^k backedge graphs that are P_3-free. Now consider a P_3-free ordering on $G[V_2]$. Every vertex in V_2 belongs to one of $S_1, ..., S_{c_1}$, and for any transitive subset $S_i (i \in \{1, ..., c_1\})$, the induced subgraph $G[S_i]$ has at most $c_2^{|S_i|}$ P_3-free orderings. Then, there are at most $(c_1 c_2)^n$ P_3-free backedge graphs on $G[V_2]$.

If $x \in V_1, y \in V_2$, then they belong to different cliques in G. Hence, their relative order is fixed in all P_3-free backedge graphs. Thus, there are at most $(c_1 c_2)^n$ total possible orderings which give a P_3-free backedge graph. ■

3.4 Rainbow Coloring

In this section, three different types of chromatic number are used. The chromatic number of an undirected graph G, denoted as $\chi(G)$, is the smallest number to color $V(G)$ such that no two vertices in the same color are adjacent. The dichromatic number of a directed graph D, denote as $\vec{\chi}(D)$, is the smallest number to color $V(D)$ such that the induced subgraph of each color is acyclic. We then define a new type of coloring on arcs of a tournament T called "rainbow coloring". The corresponding notation for the rainbow chromatic number is $\chi_\Delta(T)$.

Definition 3.24. A rainbow coloring on a tournament T is a map from arcs to the color set $f : E(T) \to \{1, ..., k\}$ such that for every $a \to b \to c \to a$ in T, we have $f((a, b)) \neq f((b, c)), f((a, b)) \neq f((c, a)), f((b, c)) \neq f((c, a))$. In other words, for each cyclic triangle in T, each arc in the cyclic triangle has a distinct color. We define $\chi_\Delta(T) = k$ if k is the smallest numbers of colors required to rainbow color $E(T)$.

The first natural question to ask is whether this chromatic number is bounded by a constant, i.e. does there exist a constant k such that every tournament T has $\chi_\Delta(T) \leq k$? It turns out that the answer is false, and we do have an explicit counter example.

Theorem 3.25. *For all integers $k > 0$, there exists a tournament T such that $\chi_\Delta(T) > k$.*

Proof. Consider a tournament T whose vertices are distinct closed intervals on the number line, with both endpoints in $\{1, ..., n\}$. We define the arcs of T via a backedge graph. First, we order the vertices such that $[a, b] < [c, d]$ if $a < c$ or $a = c, b < d$. Now, two intervals $[a, b], [c, d]$ with $[a, b] < [c, d]$ are adjacent in the backedge graph if and only if they intersect. This implies that the three intervals $[a, b], [b, c], [c, d]$ form a cyclic triangle in T. Suppose $\chi_\Delta(T) = k$, i.e. we can color the arcs with k colors such that every cyclic triangle gets three colors. Let $f : E(T) \to \{1, ..., k\}$ be a k-rainbow-coloring of T.

Consider the graph G with $V(G) = \binom{[n]}{3}$. We denote the vertex $\{x, y, z\}$ as (x, y, z) and we assume that $1 \leq x < y < z \leq n$. For two ordered triples (x, y, z) and (p, q, r), (x, y, z) is adjacent to (p, q, r) if $y = p, z = q$ or $x = q, y = r$. Note this is exactly the shift graph $G(n, 3)$ as introduced in [5] by Erdős and Hajnal.

We can color the vertex (a, b, c) of G with the color $f([a, b][b, c])$. Now for any vertex in the form of (b, c, d), it is colored by $f([b, c][c, d])$. Since f is a rainbow coloring and $[a, b][b, c][c, d]$ is a cyclic triangle in T, $f([a, b][b, c]) \neq f([b, c][c, d])$. Thus this gives a proper k-coloring of the graph G. In [6] Erdős and Hajnal showed that G have asymptotic chromatic number $\chi(G(n, 3)) = (1 + o(1)) \log \log n$. If $n > 2^{2^k}$, $\chi(G(n, 3)) > k$, we get a contradiction since we proper color vertices of $G(n, 3)$ with k colors. ■

If we are careful enough in the numbers of the proof above, we get a lower bound on $\chi_\Delta(T)$ for T with n vertices when n is large enough.

Theorem 3.26. *For all integers $n > 0$, there exists a tournament T with n vertices such that $\chi_\Delta(T) > \log \log \sqrt{n/2}$.*

Proof. Suppose that $n = t^2$ for some integer t. Consider the tournament T in the proof of Theorem 3.25 with both endpoints in $\{1, ...t\}$. It has fewer than t^2 vertices. For the sake of contradiction, let us assume $\chi_\Delta(T) \leq \log \log \sqrt{t^2} < \log \log t$. Then the shift graph $G(t,3)$ can be properly colored with $\log\log t$ colors. This contradicts that $\chi(G(t,3)) = (1+o(1))\log\log t$. Hence, if $n = t^2$, the threshold is $\chi_\Delta(T) > \log\log\sqrt{n}$.

The theorem trivially holds for $n \leq 3$. For integers $n \geq 4$, there exists an integer t such that $n/2 \leq t^2 \leq n$. Let $m = t^2$ such that $n/2 \leq m \leq n$. Then we can construct a tournament M with m vertices such that $\chi_\Delta(M) > \log\log\sqrt{m}$. Let T be a tournament with n vertices such that M is a subtournament of T, then we have $\chi_\Delta(T) > \log\log\sqrt{m} > \log\log\sqrt{n/2}$. ■

The next natural question to ask is whether there is a function $f(n)$ such that $f(n) \geq \chi_\Delta(T)$ for every tournament T with n vertices. Of course, we can simply give each arc a distinct color, which uses $\binom{n}{2}$ colors hence $f(n) \sim O(n^2)$ works. However, this seems very unlikely to be optimal, as we are just coloring all arcs with different colors and disregard the structure of given tournament T. Thus, the question becomes: can we determine the smallest possible growth rate of $f(n)$ if $f(n) \geq \chi_\Delta(T)$ for every tournament T with n vertices? We have some nice results regarding this question.

Theorem 3.27. *Every tournament T with n vertices has $\chi_\Delta(T) \leq 3\lceil \log_3 n \rceil$.*

Proof. Let T be a tournament with n vertices. We add vertices to T until $|V(T)|$ is a power of 3 since this does not increase the value of $\lceil \log_3 n \rceil$. Inductively suppose the theorem holds for all tournaments with k vertices such that $k < n$. We can partition $V(T)$ into 3 sets A, B, C such that $|A| = |B| = |C| = \frac{n}{3}$. Let E_A be the arcs with both ends in A and let $T[A]$ be the tournament induced by A. Let $E' = E_A \cup E_B \cup E_C$.

Now, cyclic triangles in T are of two different flavors: they either have all 3 arcs in E_X for some $X \in \{A, B, C\}$, or they have at least two arcs in $E(T) \backslash E'$. We say a cyclic triangle is of Type 1 if they have all 3 arcs in E_X for some $X \in \{A, B, C\}$, and otherwise it is of Type 2. It is clear that every cyclic triangle in T is either of Type 1 or Type 2.

We first want to color the arcs in E_A. Inductively, we can rainbow color $T[A]$ with at most $3\lceil \log_3 n \rceil - 3$ colors. The same holds for $T[B], T[C]$. Thus, we need at most $3\lceil \log_3 n \rceil - 3$ colors to color all arcs in E'. In this way, we managed to rainbow color all Type 1 triangles.

Now, for arcs that start in A and end in B or C, we give them a new color x_A. Similarly, for arcs that start in B and end in A or C, we give them a new color x_B. For arcs that start in C and end in A or B we give them a new color x_c. Note that x_A, x_B, x_C are distinct from all colors used in the coloring of E', and $x_A \neq x_B, x_A \neq x_C, x_B \neq x_C$.

Now consider a cyclic triangle P of Type 2, up to symmetry we have 2 cases: either two vertices of P are in A and one vertex is in B, or P has one vertex from each of A, B, C. For the first case, the arc with both ends in A is colored inductively when we color all Type 1 triangles, and the other two arcs are given the color x_A, x_B, respectively, hence the triangle gets three colors. For the second case, the three arcs are colored in x_A, x_B, x_C. Thus, all triangles of Type 2 gets three distinct colors.

Now, we managed to color $E(T)$ with at most $3\lceil \log_3 n \rceil - 3 + 3 = 3\lceil \log_3 n \rceil$ colors. Hence $\chi_\Delta(T) \leq 3\lceil \log_3 n \rceil$. ■

The next theorem suggests that this is almost the best upper bound we can reach:

Theorem 3.28. *For all $\varepsilon > 0$, there exists $n > 0$ such that there is a tournament T with n vertices with $\chi_\Delta(T) > (1-\varepsilon)\log n$.*

To prove Theorem 3.28, we first need to prove some lemmas.

Lemma 3.29. For n large enough, almost all tournaments T with n vertices have no $A, B \subseteq V(T)$, $A \cap B = \varnothing$ such that $A \Rightarrow B$ and $|A|, |B| \geq \lfloor 10 \log n \rfloor$.

Proof. Let T_n be the tournament of n vertices such that each arc is decided by flipping a coin. Namely, for the arc between i and j, we have a $1/2$ chance of having $i \to j$. For simplicity, we say (A, B) is *homogeneous* if $A \cap B = \varnothing$, $A \Rightarrow B$ and $|A|, |B| \geq \lfloor 10 \log n \rfloor$. If $A \Rightarrow B$, then for all $A' \subseteq A, B' \subseteq B$ we have $A' \Rightarrow B'$. This implies homogeneous (A, B) exists if and only if homogeneous (A, B) exists for $|A|, |B| = \lfloor 10 \log n \rfloor$. Thus, we may assume $|A|, |B| = \lfloor 10 \log n \rfloor$.

Since each arc is decided by flipping a coin, for each particular choice of (A, B), the probability that $A \Rightarrow B$ is $2^{-|A||B|} = 2^{-\lfloor 10 \log n \rfloor^2}$. The number of choice of the ordered pair (A, B) is at most $\binom{n}{\lfloor 10 \log n \rfloor}^2$. Using the union bound, we have

$$P(\exists (A, B) \text{ regular in } T_n) \leq (\text{Number of choices for (A,B)}) \cdot P((A, B) \text{ is regular})$$

Thus we have

$$P(\exists (A, B) \text{ regular in } T_n) \leq \binom{n}{\lfloor 10 \log n \rfloor}^2 2^{-\lfloor 10 \log n \rfloor^2}$$

We can check that

$$\begin{aligned}
P(\exists(A,B) \text{ regular in } T_n) &\le \binom{n}{\lfloor 10\log n\rfloor}^2 2^{-\lfloor 10\log n\rfloor^2} \\
&\le \left(n^{\lfloor 10\log n\rfloor}\right)^2 2^{-\lfloor 10\log n\rfloor^2} \\
&\le n^{20\log n}\cdot 2^{-(10\log n-1)^2} \\
&= n^{20\log n}\cdot 2^{-(100\log^2 n-20\log n+1)} \\
&\le n^{20\log n}\cdot n^{20-100\log n}\cdot 2^{-1} \\
&= 2^{-1}n^{20-80\log n}
\end{aligned}$$

As $n\to\infty$, $2^{-1}n^{20-80\log n}\to 0$, thus as $n\to\infty$, we have $P(\exists(A,B) \text{ regular in } T_n)\to 0$. Therefore, almost all tournaments have no $A,B\subseteq V(T)$, $A\cap B=\varnothing$ such that $A\Rightarrow B$ and $|A|,|B|\ge\lfloor 10\log n\rfloor$. ■

Now we are ready to prove Theorem 3.28

Proof of Theorem 3.28:
Let T be a tournament with n vertices for some fixed n large enough. We fix $\varepsilon>0$ to be a constant. Let v be a vertex of T. Let f be a rainbow coloring of T with colors $\{1,...,c\}$. For $i\in\{1,...,c\}$, let

$$\begin{aligned}
E_i &= \{e\in E(T): e \text{ has an end } v, f(e)=i\},\\
A_i &= \{x\in N^-(v):(x,v)\in E_i\},\\
B_i &= \{x\in N^+(v):(v,x)\in E_i\}.
\end{aligned}$$

Now we have $A_i\Rightarrow v\Rightarrow B_i$. Suppose we have $a\in A_i, b\in B_i$ with $b\to a$, then since all $A-v$ arcs and $v-B$ arcs are in the same color i, the cyclic triangle $a\to v\to b\to a$ is not colored properly by f, which is a contradiction. Thus we have $A\Rightarrow B$. By Lemma 3.29, we have either $|A|<\lfloor 10\log n\rfloor$ or $|B|<\lfloor 10\log n\rfloor$. Thus, there exists a set $N_i\subseteq A_i\cup B_i$, $|N_i|\le\lfloor 10\log n\rfloor-1$ such that all monochromatic dipaths of length 2 in $A_i\Rightarrow v\Rightarrow B_i$ contain a vertex in N_i. Now, we can pick a N_i for each of the colors $i\in\{1,...,c\}$ and we let $N_v=\bigcup_{i=1}^c N_i$. Note that $|N_v|\le 10c\log n-1$. In this way, we can assign every vertex v of T with a vertex set N_v such that $|N_v|\le 10c\log n-1$.

Define a digraph D with $V(D)=V(T)$, and $v\to v'$ if and only if $v'\in N_v$. D has bounded outdegree: for all $v\in V(D)$, $d_D^+(v)\le 10c\log n-1$. Let G be the underlying undirected graph of D. Then the average vertex degree of D is at most $20c\log n$. Using the Caro-Wei bound [2] [26], G has a co-clique S such that

$$|S|\ge\frac{n}{20c\log n}.$$

Now consider the tournament $T[S]$ colored by f. Let $G[S]$ be the underlying undirected graph of $T[S]$ and let $T_i[S]$ be the digraph defined by

$$V(T_i[S]) = S, E(T_i[S]) = \{e \in T[S] : f(e) = i\}.$$

Let $G_i[S]$ be the underlying undirected graph of $T_i[S]$. We claim that $G_i[S]$ is bipartite: suppose there is an odd cycle in $G_i[S]$, then no matter how we orient the edges, there is a dipath of length 2. By definition of $T_i[S]$, this corresponds to a monochromatic dipath P in $T[S]$. Let $P = u \to v \to w$. Then either u or w is in N_v, which contradicts that S is a stable set in G. This means $\chi(G_i[S]) = 2$. Thus, $G[S]$ can be partitioned into c bipartite graphs, so we have

$$\chi(G[S]) \leq 2^c$$

On the other hand, $G[S]$ is the underlying graph of $T[S]$ and $T[S]$ is a tournament. Thus, $G[S]$ is complete, so $\chi(G[S]) = |S|$. Thus we have

$$\frac{n}{20c\log n} \leq |S| = \chi(G[S]) \leq 2^c$$

This means

$$n \leq 2^c \cdot 20c\log n$$

The right hand side is an increasing function over c. Suppose that $c \leq (1-\varepsilon)\log n$, we have

$$\begin{aligned} 2^c(20c\log n + 1) &\leq 2^{(1-\varepsilon)\log n}(20(1-\varepsilon)\log n) \\ &\leq n^{(1-\varepsilon)}(20\log n) \end{aligned}$$

Then for n large enough we have

$$\begin{aligned} \frac{n}{2^c(20c\log n)} &\geq \frac{n}{20\log n \cdot n^{(1-\varepsilon)}} \\ &= \frac{n^\varepsilon}{20\log n} \\ &> 1 \qquad\qquad \text{for } n \text{ large enough} \end{aligned}$$

This contradicts that

$$n \leq 2^c \cdot 20c\log n$$

Hence, for n large enough we must have $c > (1-\varepsilon)\log n$. ■

The result is satisfying in the sense that, we have precisely bounded the growth rate of $\chi_\Delta(T)$ with $|T| = n$ as a function of n. For the upper bound, as stated in Theorem 3.27, we have $O(\log n)$, and for the lower bound, as stated in Theorem 3.28, we have $\Omega(\log n)$. Hence, the growth rate of $\chi_\Delta(T)$ is $\Theta(\log n)$.

There are also some other interesting results about the rainbow coloring. It is not surprising that, the rainbow chromatic number χ_Δ of a tournament is related to the dichromatic number of the tournament. Here are some results:

Theorem 3.30. *For any tournament T, $\chi_\Delta(T) \leq \vec{\chi}(T) + 1$.*

Proof. Let T be a tournament with $\vec{\chi}(T) = k$. Then, we can partition T into k transitive subtournmanet $T_1, ..., T_k$. For all arcs in $T_1, .., T_k$, we color it with 0. Now for the arcs going out of T_i, we color it with i. This colors all arcs with $k+1$ colors. It is easy to verify that every cyclic triangle get three colors via this coloring. ■

Theorem 3.31. *There does not exist a function f such that for all tournament T, we have $\vec{\chi}(T) \leq f(\chi_\Delta(T))$.*

Proof. The following construction can be found in [1]. If A, B, C are tournament, we define $\Delta(A, B, C)$ be the tournament T on $V(A) \cup V(B) \cup V(C)$ such that $T[V(A)], T[V(B)], T[V(C)]$ are isomorphic to A, B, C respectively, and $A \Rightarrow B \Rightarrow C \Rightarrow A$. Let $S_1 = \{v\}$ and $S_i = \Delta(S_{i-1}, S_{i-1}, S_1)$. In [1], Berger et al. showed that $\vec{\chi}(S_i) \geq i$.

We now claim that $\chi_\Delta(S_i) \leq 3$ for all i: inductively we may assume $\chi(S_{i-1}) \leq 3$. We let $S_i = \Delta(A, B, C)$ such that $A, B \cong S_{i-1}$ and C is a single vertex. Observe that all cyclic triangles are of three different flavors: it either has all three vertices in A, or all three vertices in B, or it contains a vertex from each of A, B, C. Inductively, arcs within A, B, C can be colored with 3 colors, and this properly colors all cyclic triangles except for those who have one vertex from each of A, B, C. Since $A \Rightarrow B \Rightarrow C \Rightarrow A$, we can color all arcs going out of A by color 1, all arcs going out of B by color 2 and all arcs going out of C by color 3. This properly colors all cyclic triangles that have one vertex from each of A, B, C. Hence, $\chi_\Delta(S_i) \leq 3$.

Therefore, there is no function f such that for all tournaments T, $\vec{\chi}(T) \leq f(\chi_\Delta(T))$. ■

Another interesting result is about the prime tournaments.

Definition 3.32. A *homogeneous set* $X \subseteq V(G)$ is a set of vertices such that for each vertex $v \in V(G) \backslash X$, either $v \Rightarrow X$ or $X \Rightarrow v$. A homogeneous set X is *trivial* if $|X| \leq 1$ or $|X| = |V(G)|$, otherwise it is *nontrivial*. A tournament is *prime* if it has no nontrivial homogeneous set.

For a homogeneous set X in a tournament T, we denote by $N^-(X)$ the set of in-neighbors of X. We define a new tournament T/X as follows: the set of vertices is $V(T)\backslash X \cup \{x\}$, the arcs with both ends in $V(T)\backslash X$ remain same as in T and for the arcs incident on x we have $N^-(X) \Rightarrow x \Rightarrow N^+(X)$. In other words, we "merge" the homogeneous set X into a single vertex x and keep the in-neighborhood and out-neighborhood unchanged.

The following lemma is not hard to show but useful:

Lemma 3.33. If X is a homogeneous set in T, then $\chi_\Delta(T) = \max\{\chi_\Delta(T[X]), \chi_\Delta(T/X)\}$.

Proof. Since both $T[X]$ and T/X are induced subgraphs of T, it is clear that $\chi_\Delta(T) \geq \max\{\chi_\Delta(T[X]), \chi_\Delta(T/X)\}$. Now let $\chi_\Delta(T[X]) = a, \chi_\Delta(T/X) = b$. Let $c = \max\{a, b\}$. Then, we can rainbow color $T[X]$ and T/X with c colors. Let the two coloring be f_1, f_2, respectively. Now we can define a rainbow coloring g on T: if e has both ends in X, we have $g(e) = f_1(e)$; if e has both ends in $T\backslash X$, we have $g(e) = f_2(e)$; if e is in the cut induced by X, then $e = (u, v)$ or $e = (v, w)$ for $v \in X, u, w \notin X$, so we can have $g((u, v)) = f_2((u, x))$ and $g((v, w)) = f_2((x, w))$. It remains to show that g is a rainbow coloring. Let P be a cyclic triangle of T. If P has no arcs crossing the cut induced by X, then since each of f_1, f_2 is a rainbow coloring, P gets three distinct colors. If P has an arc crossing the cut induced by X, it has exactly two arcs crossing the cut, so we may assume P is in the form of $a \to b \to c \to a$ where $a \in N^-(X), b \in X, c \in N^+(X)$. The arcs in P get the same color as the arcs in the cyclic triangle $a \to x \to c \to a$ in T/X, and since f_2 is a rainbow coloring, P gets three distinct colors. Thus, g is a rainbow coloring of T. Thus, $\chi_\Delta(T) \leq \max\{\chi_\Delta(T[X]), \chi_\Delta(T/X)\}$. ■

Corollary 3.34. Let T be a tournament. Then

$$\chi_\Delta(T) = \max\{\chi_\Delta(T') : T' \text{ is a prime subtournament of } T\}.$$

Proof. It is obvious that $\chi_\Delta(T) \geq \max\{\chi_\Delta(T') : T' \text{ is a prime subtournament of } T\}$. On the other hand, we define a sequence $T_0, ..., T_k$ of tournaments as follows: $T_0 = T$, if T_i has been defined and has no non-trivial homogeneous set, we stop and let $k = i$. Otherwise, let X_i be a non-trivial homogeneous set in T_i, and we define $T_{i+1} = T_i/X$.

By applying Lemma 3.33 repeatedly, we have

$$\chi_\Delta(T) = \max\{\chi_\Delta(S) : S \in \{T[X_1], ..., T[X_k], T_k\}$$

If $\chi_\Delta(T) = \chi_\Delta(T_k)$ then we are done. Otherwise, without loss of generality, we may assume $\chi_\Delta(T) = \chi_\Delta(T[X_1])$. Inductively, we have

$$\chi_\Delta(T[X_1]) = \max\{\chi_\Delta(T') : T' \text{ is a prime subtournament of } T[X_1]\}.$$

Since a prime subtournament of $T[X_1]$ is also a prime subtournament of T, we complete the proof. ■

This enables us to compute $\chi_\Delta(T)$ for T in some special classes of tournaments. Here are two five-vertex tournament, U_5 and W_5:

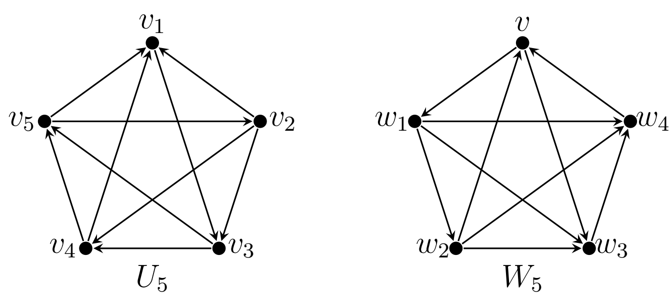

Figure 20: U_5 and W_5

Recall that, we say a tournament G is H-free if G has no subtournament isomorphic to H. In [13], Latka showed that if a prime tournament T is W_5-free, then either $T = Q_7$, the Paley tournament on 7 vertices, or $\vec{\chi}(T) \leq 2$. In [14], Liu proved that if a prime tournament T is U_5-free, then $\vec{\chi}(T) \leq 2$.

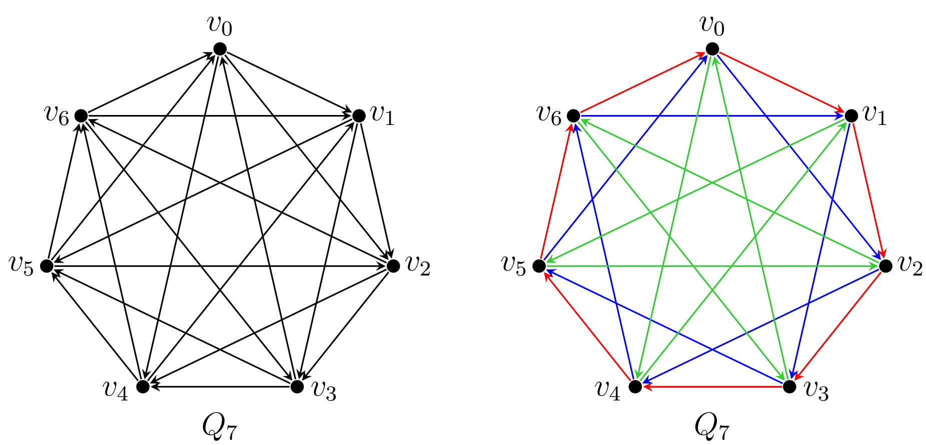

Figure 21: Q_7 and its rainbow coloring

Then, we have the following theorem:

Theorem 3.35. *If T is a U_5-free tournament, then $\chi_\triangle(T) \leq 3$.*

Proof. If T is prime, then since $\vec{\chi}(T) \leq 2$, by Theorem 3.30, $\chi_\Delta(T) \leq 3$. Otherwise, by Corollary 3.34, $\chi_\Delta(T) = \chi_\Delta(T')$ for some prime subtournament T' of T. Since T' is prime, $\chi_\Delta(T') \leq 3$. ∎

Theorem 3.36. *If T is a W_5-free tournament, then $\chi_\triangle(T) \leq 3$.*

Proof. If T is not Q_7, the proof is similar to the proof of Theorem 3.35. Hence, it suffices to show that $\chi_\Delta(Q_7) \leq 3$. Note that in Q_7, $v_i \to v_j$ if and only if $j - i \in \{1, 2, 4\}$ modular $\mathbb{Z}_7$. We can color arcs in the form of (v_i, v_j) with color 1 if $j - i = 1$, color 2 if $j - i = 2$ and color 3 if $j - i = 4$. One can check that in that way, all cyclic triangles of Q_7 get three colors. ∎

Chapter 4

Open Problems

In this section, we list some open problems related to the topic we have discussed in Chapter 2 and Chapter 3. These open problems do not lead to any significant results, but they arise naturally when we are studying our original question.

4.1 Open Problems: Pinch-graphic Matroids

Recall that our final goal is to bound the size of excluded minors for pinch-graphic matroids. A general approach is to divide into cases by connectivity. In Chapter 2, we bound the size of the excluded minors that are not 3-connected. We are on the way to solve the cases where the excluded minors are exactly 3-connected. To progress to our final goal, we also need to consider the cases where the excluded minors are internally 4-connected.

Problem 10. Can we bound the size of a minimally non-pinch-graphic matroid that is not internally 4-connected?

We are on the way to resolve Problem 10. In fact, our current goal is to bound a specific type of elements in a 3-connected minimally non-pinch-graphic matroid:

Problem 11. Let M be a 3-connected minimally non-pinch-graphic matroid. We say an element e is *dangerous* if M/e or $M\backslash e$ is not 3-connected. Can we bound the number of dangerous elements?

After solving Problem 10, we are approaching the final question, which is the internally 4-connected case:

Problem 12. Can we characterize all minimally non-pinch-graphic matroids that are internally 4-connected?

To generalize the problem to a higher level, recall that pinch-graphic matroids are special cases of even-cycle matroids. Again, even-cycle matroids are minor-closed, and we have the following question:

Problem 13. Can we bound the size of a matroid that is minimally not even-cycle?

4.2 Open Problems: Tournaments

4.2.1 Enumeration of Backedge Graphs with Forbidden Induced Subgraphs

Note that, we widely use the Marcos-Tardos Theorem to bound the number of backedge graphs. We notice that, if we let $H = K_{1,t}$ or even $K_{t,t}$, a similar argument may still work. The key argument is to find a construction such that for any tournament T and any ordering σ on the vertices $\{1, ..., n\}$, either $B_\sigma(TT_n)$ or $B_\sigma(T)$ is not H-free. Hence we have the two following conjectures:

Conjecture 1. If n is large enough, $f(K_{1,t}, n) \leq c^n$ for some constant c depending on t.

Conjecture 2. If n is large enough, $f(K_{t,t}, n) \leq c^n$ for some constant c depending on t.

4.2.2 Rainbow Coloring

A question of interest with regard to the rainbow coloring is whether we can characterize all tournaments T with $\chi_\Delta(T) \leq 3$. By Theorem 3.30, one may conjecture that all tournaments with $\chi_\Delta(T) \leq 3$ have dichromatic number at most 2. The conjecture is false because of Theorem 3.31, and also the 7-vertex Paley tournament Q_7 has $\vec{\chi}(Q_7) = \chi_\Delta(Q_7) = 3$. However, through our trials on several examples, we believe that the Paley tournaments are playing a crucial role here. Thus, we conjecture that if $\vec{\chi}(T) \geq \chi_\Delta(T) = 3$, then there is a Paley tournament hidden in T.

Conjecture 3. If T is a tournament with $\chi_\Delta(T) \leq 3$, then either $\vec{\chi}(T) \leq 2$ or T has an induced subgraph isomorphic to a Paley tournament.

References

[1] Eli Berger, Krzysztof Choromanski, Maria Chudnovsky, Jacob Fox, Martin Loebl, Alex Scott, Paul Seymour, and Stéphan Thomassé. Tournaments and colouring. *Journal of Combinatorial Theory, Series B*, 103(1):1–20, 2013.

[2] Y. Caro. New results on the independence number. Technical report, Tel Aviv University, 1979.

[3] Maria Chudnovsky, Gérard Cornuéjols, Xinming Liu, Paul Seymour, and Kristina Vuskovic. Recognizing berge graphs. *Combinatorica*, 25:143–186, 03 2005.

[4] Maria Chudnovsky, Neil Robertson, Paul Seymour, and Robin Thomas. The strong perfect graph theorem, 2002.

[5] Paul Erdős and András Hajnal. On chromatic number of graphs and set-systems. *Acta Mathematica Academiae Scientiarum Hungarica*, 17(1):61–99, Mar 1966.

[6] Paul Erdős and András Hajnal. On the chromatic number of infinite graphs. In *Theory of Graphs: Proceedings of the Colloquium at Tihany, 1976*, pages 83–98. Academic Press, 1976.

[7] Noah Golowich. The m-degenerate chromatic number of a digraph, 2018.

[8] Bertrand Guenin and Cheolwon Heo. Recognizing even-cycle and even-cut matroids. *Mathematical Programming*, 202:515–542, 2023.

[9] Bertrand Guenin and Cheolwon Heo. Small separations in pinch-graphic matroids. *Mathematical Programming*, 204(1–2):81–111, April 2023.

[10] Bertrand Guenin and Cheolwon Heo. Recognizing pinch-graphic matroids. *Mathematical Programming*, 204(1–2):113–134, 2024.

[11] Frank Harary. On the notion of balance of a signed graph. *Michigan Mathematical Journal*, 2(2):143–146, 1953.

[12] Kazimierz Kuratowski. Sur le problème des courbes gauches en topologie. *Fundamenta Mathematicae*, 15:217–283, 1930.

[13] Brenda Latka. Structure theorem for tournaments omitting N_5. *Journal of Graph Theory*, 42:165 – 192, 03 2003.

[14] Gaku Liu. Structure theorem for U_5-free tournaments. *Journal of Graph Theory*, 78(1):28–42, March 2014.

[15] Adam Marcus and Gábor Tardos. Excluded permutation matrices and the Stanley–Wilf conjecture. *Journal of Combinatorial Theory, Series A*, 107(1):153–160, 2004.

[16] V. Neumann-Lara. The dichromatic number of a digraph. *Journal of Combinatorial Theory, Series B*, 33(3):265–270, 1982.

[17] James Oxley. *Matroid Theory*, volume 21 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2nd edition, 2011.

[18] Irene Pivotto. *Even Cycle and Even Cut Matroids*. PhD thesis, University of Waterloo, 2011.

[19] Frank P. Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society*, s2-30(1):264–286, 1930.

[20] Talmage James Reid. Triangles in 3-connected matroids. *Discrete Mathematics*, 90(3):281–296, 1991.

[21] Neil Robertson and Paul Seymour. Graph minors. xx. wagner's conjecture. *Journal of Combinatorial Theory, Series B*, 92(2):325–357, 2004. Special Issue Dedicated to Professor W.T. Tutte.

[22] Paul Seymour. A note on the production of matroid minors. *Journal of Combinatorial Theory, Series B*, 22(3):289–295, 1977.

[23] Ewald Speckenmeyer. On feedback problems in digraphs. In *Graph-Theoretic Concepts in Computer Science*, pages 218–231, 1990.

[24] Sophie Spirkl and Yun Xing. Complexity of the feedback vertex set problem in tournaments with forbidden subtournaments, 2026.

[25] William T. Tutte. A homotopy theorem for matroids, i. *Transactions of the American Mathematical Society*, 88(1):144–160, 1958.

[26] Victor K. Wei. A lower bound on the stability number of a simple graph. Technical report, Bell Laboratories, 1981.

APPENDICES

Appendix A

Additional Figures

Here is the list of excluded induced subgraphs for 1-out-degenerate tournaments:

By Theorem 3.3, an excluded induced subgraph for 1-out-degenerate tournaments has at most 6 vertices. There is only one excluded induced subgraph with 5 vertices:

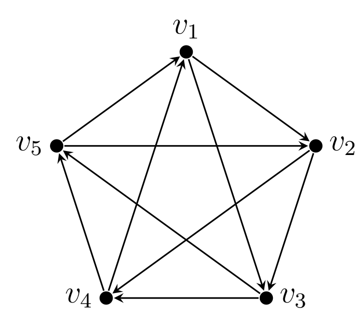

Figure 22: Excluded induced subgraph for 1-out-degenerate tournaments with 5 vertices

There are five excluded induced subgraph with 6 vertices. (See next page.)

The complete list of excluded induced subgraphs of 1-in-degenerate tournaments are exactly the complements of the tournaments listed.

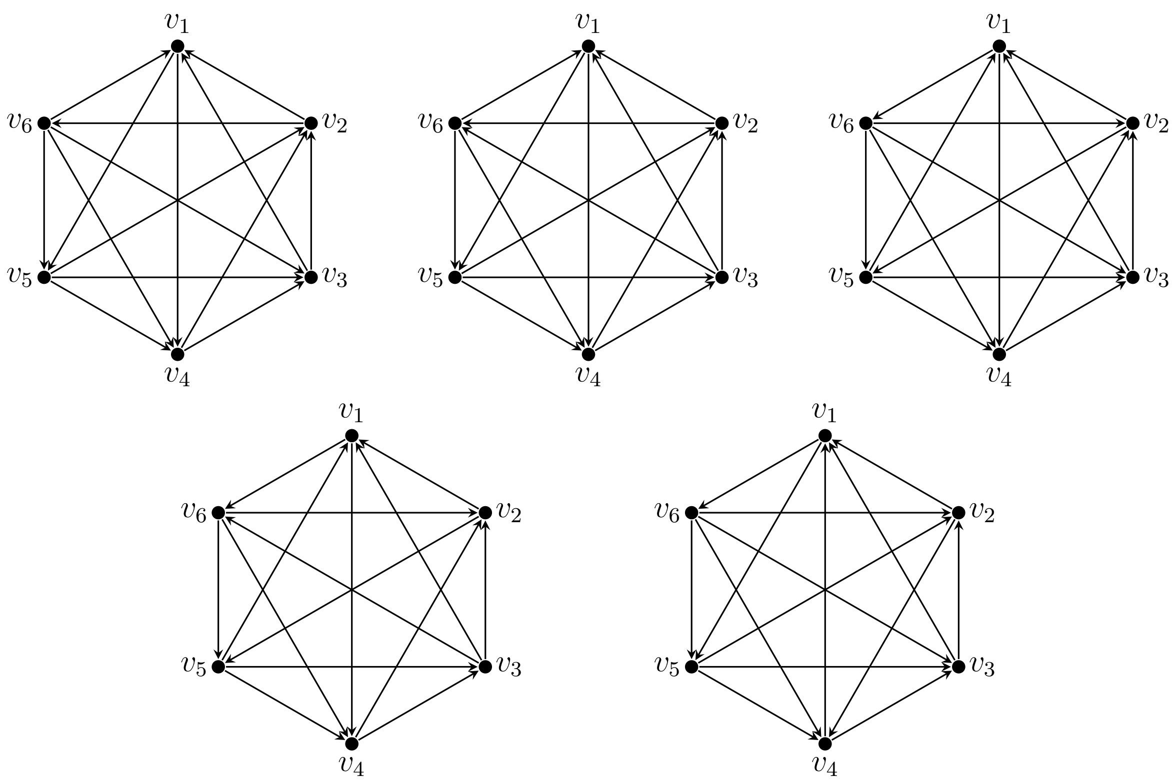

Figure 23: Excluded induced subgraphs for 1-out-degenerate tournaments with 6 vertices