On The Palatini Variation and Connection Theories of Gravity

by

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Abstract

This thesis involves a rigorous treatment of the Palatini Variational Principle of gravitational actions in an attempt to fully understand the rôle of the connection in such theories. After a brief geometrical review of affine connections, we examine N-dimensional dilatonic theories via the Standard Palatini principle in order to highlight the potential differences arising in the dynamics of theories obtained by utilizing the Hilbert and Palatini formalisms. We then develop a more generalized N-dimensional, torsion-free, Einstein-Hilbert-type action which is shown to give rise to Einsteinian dynamics but can be made, for certain choices of the associated arbitrary parameters, to yield either weak constraints or *no* constraints on the connection, Γ. The latter case is referred to as a "maximally symmetric" action.

In the following Chapter this analysis is extended to the realm of a potentially non-vanishing torsion tensor, where it is seen that such actions do *not*, in general, lead to Einsteinian dynamics under a Palatini variation. Following another brief geometrical review, which highlights some elements of fibre bundle theory appropriate to our later analysis, we examine the so-called Palatini Tetrad formalism and show that it must be modified for a proper Palatini variation - i.e. to not assume anything *a priori* about the relevant connection. We then analyze this modified approach from a geometrical perspective and show that, for the torsion-free case at least, a proper treatment of the Palatini Tetrad procedure is equivalent to the "maximally symmetric" case alluded to earlier.

Furthermore, we recognize that the Palatini Tetrad approach should effectively be regarded as little more than a calculational technique resulting from analysis of the more generalized action $S = \int \mathrm{tr}\,[R \wedge \star(\beta \wedge \beta)] + S_m$, where R, β are local versions of the curvature 2-form and solder 1-form from the $GL(N, R)$ full bundle of

frames respectively. Our above-mentioned modification of this approach not only renders treatment of this action geometrically consistent (i.e. by considering all of the terms in the action as pertaining to the same principal bundle), but also enables one to clearly see the manifest *connection invariance* of the full theory (given some connection independent matter action, S_m). Hence a rigorous Palatini analysis of Einstein-Hilbert - like actions leads one to the rather unexpected conclusion that generalized Einsteinian actions of the type $S = \int \mathrm{tr}\,[R \wedge \star(\beta \wedge \beta)] + S_m$ are connection invariant and naturally give rise to Einsteinian dynamics if S_m is independent of the connection, w; and otherwise only give rise to Einsteinian dynamics by an *a posteriori* symmetry-breaking-type condition, $T = 0$ (for T the torsion tensor), identical to that of Einstein-Cartan theory.

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To my parents,

who set me on the path of learning

and supported me every step of the way.

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Glossary of Terms and Symbols

∇ - Affine Connection

$\Gamma_{\mu}{}^{\epsilon}{}_{\nu}$ - Connection Coefficients

$T_{\mu}{}^{\epsilon}{}_{\nu}$ - Torsion Tensor

$T_{\mu}{}^{\epsilon}{}_{\nu} \equiv \Gamma_{\mu}{}^{\epsilon}{}_{\nu} - \Gamma_{\nu}{}^{\epsilon}{}_{\mu}$ (in holonomic coordinates)

$R_{\alpha\beta}{}^{\lambda}{}_{\epsilon}$ - Riemannian Curvature Tensor

$R_{\alpha\beta}{}^{\lambda}{}_{\epsilon} = \Gamma_{\beta}{}^{\lambda}{}_{\epsilon,\alpha} - \Gamma_{\alpha}{}^{\lambda}{}_{\epsilon,\beta} + \Gamma_{\alpha}{}^{\lambda}{}_{\eta}\,\Gamma_{\beta}{}^{\eta}{}_{\epsilon} - \Gamma_{\beta}{}^{\lambda}{}_{\eta}\,\Gamma_{\alpha}{}^{\eta}{}_{\epsilon}$ (in holonomic coordinates)

$R_{\beta\epsilon} := R_{\sigma\beta}{}^{\sigma}{}_{\epsilon}$ - Ricci Tensor

$R := g^{\beta\epsilon} R_{\beta\epsilon}$ - Ricci Scalar

$\{_{\mu}{}^{\epsilon}{}_{\nu}\} := \frac{1}{2} g^{\kappa\epsilon}\left[g_{\mu\kappa,\nu} + g_{\nu\kappa,mu} - g_{\mu\nu,\kappa}\right]$ - Christoffel Symbol

$\mathcal{D}_{\mu}$ - Covariant Derivative with $\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\}$

$\mathcal{D}^2 := \mathcal{D}^{\lambda}\mathcal{D}_{\lambda}$

$V_{\lambda} := \frac{\nabla_{\lambda}\sqrt{-g}}{\sqrt{-g}}$

$Z^{\lambda} := \nabla_{\rho} g^{\lambda\rho}$

$\mathcal{G}$ - Lie Group

$\mathcal{L}(\mathcal{G})$ - Lie Algebra of $\mathcal{G}$

$\mathbf{D}$ - Exterior Covariant Derivative

A - Principal Bundle Connection 1-form

$\mathbf{G}$ - Principal Bundle Curvature 2-form

σ - Local Cross-Section of Principal Bundle

$\sigma^{*}A := w$ - Local Connection 1-form

$\sigma^{*}\mathbf{G} := R$ - Local Curvature 2-form

$\mathcal{B}(M)$ - Bundle of Frames

θ - Solder 1-form on $\mathcal{B}(M)$

$\sigma^{*}\theta := \beta$ - Local Solder Form

Chapter 1

Introduction

Among the various schisms throughout the theoretical physics community, perhaps none is more contentious than that which separates researchers into the camps of metric-oriented or connection-oriented perspectives of gravity.

Historically, of course, Einstein founded General Relativity as a dynamical theory of the metric tensor with the connection necessarily being that of the unique torsion-free metric-compatible Levi-Civita connection, $\{_\mu{}^\epsilon{}_\nu\}$. This was more than a choice of mathematical convenience or simplicity as there are several important *physically* motivated reasons to opt for this connection, many of which were integral to Einstein's deep physical intuitions about the nature of space-time.

Nonetheless, from the earliest days since the advent of General Relativity, attempts have been made to generalize it, sometimes explicitly with regards to the connection. One of the first such attempts made in this regard is the so-called Palatini variation [14, 15, 36][1] where one subjects the generalized Einstein-Hilbert action,

$$S_{EH} = \int d^4x \left[\sqrt{-g}\left(R(\Gamma) + 16\pi\mathcal{L}_m\right)\right] \tag{1.1}$$

with Γ no longer *a priori* regarded as any particular function of the metric, to a

[1]An interesting irony is that much of what is now referred to as a "Palatini Variation" was actually independently proposed by Einstein - see [15]

variation $\delta_\Gamma S_{EH} = 0$ in addition to the usual $\delta_g S_{EH} = 0$. As is well known, one finds that in addition to the usual field equation resulting from the metric variation, i.e.

$$G_{\mu\nu}(\Gamma) = 8\pi T_{\mu\nu}, \tag{1.2}$$

one obtains

$$\partial_\lambda g_{\mu\nu} - g_{\eta\nu}\Gamma_\lambda{}^\eta{}_\mu - g_{\eta\mu}\Gamma_\lambda{}^\eta{}_\nu = 0 \tag{1.3}$$

from the connection constraint, which is nothing more (for Γ torsion-free) than the familiar condition of metric compatibility, whose solution,

$$\Gamma_\mu{}^\epsilon{}_\nu = \{_\mu{}^\epsilon{}_\nu\}, \tag{1.4}$$

is the Levi-Civita Christoffel symbol.

Thus, as far as the Einstein-Hilbert action was concerned, attempts to regard the (torsion-free) connection as any potential generalization of the Christoffel symbol were relatively short-lived. The question of *why* in fact, a Palatini variation happened to lead to the relationship (1.4) was considered mildly peculiar [36, p.454], but little more than that.

The resurgence of interest in connection-based theories of gravity occurred due to two related latter-day phenomena: the growing awareness of structural similarities between the mathematical framework of General Relativity and that of Yang-Mills Gauge Theories [6, 12, 18, 35], and the development of the Ashtekar programme of Canonical Gravity [2, 3, 4, 5].

As the role of the gauge principle achieved preeminence among particle physicists, it became clear that by using more abstract geometrical techniques one could regard all gauge theories as fundamentally dependent upon connection 1-forms which took values in the Lie Algebra of some gauge group $\mathcal{G}$, and analyze their dynamics according to a Hamiltonian formulation involving the connection 1-form and its canonically conjugate momentum as the relevant configuration variables. Such a viewpoint has since been often referred to as "connection dynamics" [34], and its development led many to wonder how gravity, with its obvious geometrical structure involving a "ready-made" connection 1-form, Γ, could be similarly interpreted.

Meanwhile, with the development of the ADM formalism [1], General Relativity could be put into a consistent Hamiltonian framework involving the dynamical evolution of spacelike submanifolds Σ (with induced Riemannian metric q_{ij}), of space-time $\mathcal{M}$; and whose configuration variables involve functions of the induced metric q_{ij} and the extrinsic curvature, K_{lm}, of Σ with respect to $\mathcal{M}$ on the general configuration space $Met(\Sigma)$, the space of all Riemmanian (3-) metrics on Σ. The relevant Hamiltonian can be written as a sum of two fundamental constraints, the diffeomorphism constraint, $C(\vec{N})$, and the Hamiltonian constraint, $C(N)$, each (densitized) complex functions of the shift vector, $\vec{N}$, and lapse function, N, respectively. Hence one has

$$H = C(\vec{N}) + C(N) \tag{1.5}$$

where one finds that the allowed states of the configuration space are those for which the constraints (and hence the Hamiltonian) vanish identically.

The Poisson brackets of these constraints form the so-called Dirac algebra,

$$\{C(\vec{N}), C(\vec{M})\} = C([\vec{N}, \vec{M}]) \tag{1.6}$$

where

$$[\vec{N}, \vec{M}] := \mathcal{L}_{\vec{N}}\vec{M}, \tag{1.7}$$

$$\{C(\vec{N}, C(N)\} = \mathcal{L}_{\vec{N}}N \tag{1.8}$$

and

$$\{C(N), C(M)\} = C(\vec{K}) \tag{1.9}$$

where

$$K^i := q^{ij}(N\partial_j M - M\partial_j N) \tag{1.10}$$

We see that while the constraints are closed under Poisson brackets, owing to the last constraint they do not form a Lie Algebra.

Of course it should be emphasized here that nowhere in this analysis is there a connection variable - the definition of the extrinsic curvature K_{lm} above (on which the constraints implicitly depend) specifically utilizes D_ϵ, the covariant derivative with respect to the Christoffel symbol, $\{_\mu{}^\epsilon{}_\nu\}$. It is thus solely a metric-oriented perspective and one which in no way suggests, let alone necessitates, the treatment of Γ as a dynamical variable. Many people, after Wheeler, have referred to this treatment of gravity as "geometrodynamics".

Shortly after the development of the ADM analysis, those interested in the issue of quantum gravity with a bias towards the canonical perspective attempted to quantize this classical geometrodynamical picture by applying to it the well-known concepts of the Dirac quantization procedure [13, 23, 25] in the hopes of leading to the Wheeler-DeWitt equation,

$$\hat{H}\psi = 0 \tag{1.11}$$

for all choices of $\vec{N}$ and N of some quantized Hamiltonian, $\hat{H}$, and for $\psi \in \mathcal{H}$, with $\mathcal{H}$ some physically relevant Hilbert space[2].

This agenda encountered several insurmountable technical difficulties, one of which (though by no means the only one) being the fact that in attempting to quantize the Hamiltonian one must effectively quantize the constraints, since, as mentioned above, the Hamiltonian reduces to a sum of the two constraints.

[2]This Hilbert space itself is rather difficult to find, in fact. The usual choice of $L^2(Q)$, the set of square integrable functions on the configuration space, Q, leads us to consider $L^2(Met(\Sigma))$, which, owing to the infinite-dimensional nature of $Met(\Sigma)$ makes defining an inner product rather problematic.

The issue of the constraints turns out to be a substantial stumbling block, owing to the fact that elevating the two constraints to the operator level leads one to encounter operator ordering problems in the fundamental canonical variables p_{ij} and q_{ij}, combined with the fact that the constraints are non-polynomial in p_{ij} and q_{ij}.

Attempts to circumvent these difficulties with the constraints led those with a connection dynamics perspective to suggest re-interpreting gravity as a theory where the metric becomes relegated to the status of a derived variable, with a frame field e and (Lorentz) connection w the sole primary dynamical variables - the so-called Palatini Tetrad Formalism. If one recasts the Hamiltonian dynamics in this manner, one finds that the constraints do, indeed simplify and become polynomial, but are now no longer closed under Poisson brackets and hence the route to quantization is as mysterious as ever.

In the late 1980s, Ashtekar [2, 3] breathed new energy into the canonical quantum gravity programme by showing that one can modify the above Palatini Tetrad Formalism into one which extends the relevant geometrical spaces to their complexified counterparts, and, taking advantage of the uniqueness of $N = 4$, regarded the primary dynamical variables as a complex frame field and a *self-dual* (Lorentz) connection. Using this perspective, the constraints were closed and polynomial, but one now needed to impose reality conditions to force the metric (defined via the now-complex frame field e) to be real-valued.

Roughly simultaneous to this development, Witten [38] added a further boost to the spirits of those in the canonical quantum gravity camp by showing that the Palatini Tetrad treatment of 2+1 gravity was equivalent to Chern-Simons theory of the inhomogeneous Lie Group ISO(2,1) and thus could be explicitly canonically quantized.

Meanwhile, of course, those who were interested in different routes to the holy grail of quantum gravity [25] were avidly pursuing other avenues - most notably that of superstrings [19] and Euclidean quantum gravity [17], both of which regard the metric as the fundamental dynamical variable and have little interest in recasting General Relativity in a connection framework[3]. Most researchers in classical General Relativity, meanwhile, have understandably never regarded General Relativity as anything other than a primarily metric theory and tend to dismiss any attempts to reformulate it in connection language as mere mathematical chicanery.

We have thus reached the schism alluded to above, where strong views are prevalent and theorists tend to find themselves perched rather quickly on one or the other side of the connection-metric divide.

This thesis will begin, at least, in the connection camp and examine various different gravitational theories from a connection-oriented perspective, but the emphasis will lie with General Relativity. Climbing down from the rarefied heights of quantum gravity outlined in this introduction, we shall only examine classical dynamics and that only from a Lagrangian perspective. We shall attempt to relate the Palatini variation of page one to the Palatini Tetrad formalism alluded to above, en route clarifying certain key assumptions the latter subtlely makes.

This work was motivated by a general desire to visualize General Relativity in a rigorously geometrical connection-based framework like Yang-Mills Theory[4]. More specifically, a key concern was to clarify the mysterious origins of the Christoffel condition (1.4) in the standard Palatini treatment - a condition which seemed all the more mysterious owing to the fact that Ashtekar had already shown some time

[3]There are, of course, those who have feet in both camps (or neither) - Witten is perhaps the most illustrious, but by no means only, example.

[4]The gravitational action strictly analagous to Yang-Mills Theory is the Lovelock action[30], $S_L = \int R \wedge \star R$, which is completely well-defined geometrically, but yields non-Einsteinian dynamics.

ago that when one writes it in Hamiltonian form, 3+1 Palatini theory collapses back to the standard geometrodynamical description of general relativity and hence "3+1 Palatini theory does *not* (my emphasis) succeed in recasting general relativity as a connection dynamical theory" [34]. If such a statment is, indeed true, one would imagine that it should be possible to see its validity on the Lagrangian level alone - something which, according to (1.4) at least, appears *not* to be the case.

The thesis is divided into three basic sections. Chapters 2 and 5 review and clarify some necessary geometrical concepts involving the connection, while Chapters 3 and 4 (together with the Appendices) rigorously examine various gravitational actions under the Standard Palatini formalism, en route developing an Extended Action, which, for the torsion-free case at least, gives rise to the dynamics of General Relativity while leaving the connection completely undetermined. In Chapters 6 and 7 we turn to the Palatini Tetrad approach and develop an associated generalized prescription which again enables one to derive General Relativity (*in vacuuo*) while leaving the connection undetermined. Perhaps not surprisingly, this generalized prescription will be shown to be directly related to our Extended Action of Chapter 3 while simultaneously leading the way towards establishing a more consistent treatment of the relevant gravitational action $S_g = \int tr\,[R \wedge *(\beta \wedge \beta)]$ in terms of its basic constituent geometrical elements. Finally, we consider the explicit effect on the above analysis of adding a matter term to the action S_m.

Chapter 2

Affine Connections

2.1 Overview

Throughout Chapters 3 and 4, in what we call the Standard Palatini Variation, we deal exclusively with the usual affine connection of Riemannian geometry. As we shall see in Chapter 5, this connection has a natural geometric generalization to other (gauge) theories which is most vividly illustrated via the framework of fibre bundles, but for the purposes of the next three chapters this is both unnecessary and potentially obfuscatory.

Moreover, affine connections are fundamentally different from more generalized connections in that they admit two additional properties: metric-compatibility (or lack thereof) and torsion, both of which are undefined for more generalized connections owing to the fact that they are unique to the nature of the tangent bundle, TM (or, more precisely, to the bundle of frames, $B(M)$ and its related bundles).

Both metric compatibility and torsion are independent attributes of a general affine connection and thus serve to specify, at least to some extent, the nature of our connection. This chapter will briefly highlight and review the mathematical definitions and physical manifestations of each of these two attributes and their combination, while simultaneously specifying the relevant notation which will be used throughout much of the thesis.

2.2 Definition of an Affine Connection

Following mathematical convention [32, 10], we define an affine connection as ∇, a map from $Vec(M) \times Vec(M)$ to $Vec(M)$, where $Vec(M)$ represents the set of vector fields on the differentiable manifold, M; while the N^3 functions, $\Gamma_\mu{}^\epsilon{}_\nu(x)$, are technically referred to as "connection coefficients".

This definitional approach is somewhat at odds with the more generalized view from fibre bundle theory, where $\Gamma_\mu{}^\epsilon{}_\nu$ represents a $GL(N, R)$-valued *connection 1-form* and ∇ represents the covariant derivative. In keeping with these ideas, we shall often use the word "connection" to refer to the connection coefficients (1-form), Γ, rather than to its associated map (covariant derivative) ∇. This potential abuse of notation should not prove too confusing owing to the clear distinction in symbols used (i.e. Γ vs. ∇) combined with its prevalence throughout relativistic physics.

We thus have the following definition:

An *Affine Connection* is a map

$$\begin{aligned} \nabla \; := \; & Vec(M) \times Vec(M) \Rightarrow Vec(M) \\ & (X, Y) \rightarrow \nabla_X Y \end{aligned} \tag{2.1}$$

which satisfies the following conditions for f any smooth, real-valued function on the differentiable manifold M and $X, Y, Z \in Vec(M)$:

$$\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z \tag{2.2}$$

$$\nabla_{(X+Y)} Z = \nabla_X Z + \nabla_Y Z \tag{2.3}$$

$$\nabla_{fX} Y = f \nabla_X Y \tag{2.4}$$

$$\nabla_X (fY) = X(f) Y + f \nabla_X Y \tag{2.5}$$

For $\{\hat{e}_\alpha\}$ any (local) basis of the tangent space $T_p M$ at some point $p \in M$, we define the connection coefficients Γ by:

$$\nabla_{\hat{e}_\alpha} \hat{e}_\beta := \nabla_\alpha \hat{e}_\beta := \Gamma_\alpha{}^\gamma{}_\beta \hat{e}_\gamma \tag{2.6}$$

We write the indices of the connection (coefficients) $\Gamma_\mu{}^\epsilon{}_\nu$ in this somewhat unorthodox manner so as to simplify comparison with more generalized connections in later chapters.

Thus $\nabla_\alpha v$ for some vector field locally expanded as $v = v^\sigma \hat{e}_\sigma$ can be written in component form as:

$$\nabla_\alpha v = z \ \in \ T_p(M) \tag{2.7}$$

where

$$z = z^\lambda \ \hat{e}_\lambda; \ \ z^\lambda = (\nabla_\alpha v)^\lambda = \partial_\alpha (v^\lambda) + \Gamma_\alpha{}^\lambda{}_\beta \, v^\beta \tag{2.8}$$

As usual, if some vector field, X, satisfies the condition

$$\nabla_V X = 0 \tag{2.9}$$

it is said to be parallel transported along $c(t)$, for $c(t)$ the relevant integral curve of V; while if we have

$$\nabla_V V = 0 \tag{2.10}$$

then $c(t)$ is labelled a geodesic.

Using (2.2) - (2.5) together with the definitions:

$$\nabla_X f := X(f) \equiv \mathcal{L}_X f \tag{2.11}$$

and the Leibniz rule:

$$\nabla_X(\mathcal{T}_1 \otimes \mathcal{T}_2) := (\nabla_X \mathcal{T}_1) \otimes \mathcal{T}_2 + \mathcal{T}_1 \otimes (\nabla_X \mathcal{T}_2) \tag{2.12}$$

we can extend the action of ∇ to any generalized tensor $\mathcal{T}$ on M (this is another manifestation of the particular nature of the affine connection - in general one cannot do this.) which will thus enable us to define the concept of metricity.

Finally, we define two supplementary tensors, the *torsion* tensor, T, and the *curvature* tensor, R, as follows:

$$\begin{aligned} T &:= Vec(M) \times Vec(M) \times Vec^*(M) \to \Re \\ &\quad (X, Y, A) \Rightarrow \langle A, \nabla_X Y - \nabla_Y X - [X, Y] \rangle \end{aligned} \tag{2.13}$$

where $A \in Vec^*(M)$, the set of 1-forms on M and

$$[X, Y] \Leftrightarrow \mathcal{L}_X Y; \tag{2.14}$$

and

$$\begin{aligned} R &:= Vec(M) \times Vec(M) \times Vec(M) \times Vec^*(M) \Rightarrow \Re \\ &\quad (X, Y, Z, A) \to \langle A, \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \rangle, \end{aligned} \tag{2.15}$$

where we have used the usual notation, $\langle A, X \rangle$ to denote the element of $\Re$ obtained by the 1-form, A, acting on the vector field, X, i.e. $\langle A, X \rangle = A(X)$.
In holonomic coordinates (i.e. for $\{e_\alpha\} = \{\frac{\partial}{\partial x^\alpha}\}$) [1] we find:

$$T_\mu{}^\epsilon{}_\nu = \Gamma_\mu{}^\epsilon{}_\nu - \Gamma_\nu{}^\epsilon{}_\mu \tag{2.16}$$

and

$$R_{\alpha\beta}{}^\lambda{}_\epsilon = \Gamma_\beta{}^\lambda{}_{\epsilon,\alpha} - \Gamma_\alpha{}^\lambda{}_{\epsilon,\beta} + (\Gamma_\alpha{}^\lambda{}_\eta)(\Gamma_\beta{}^\eta{}_\epsilon) - (\Gamma_\beta{}^\lambda{}_\eta)(\Gamma_\alpha{}^\eta{}_\epsilon) \tag{2.17}$$

[1] We shall henceforth use such coordinates as a default and all tensor components will be expressed in terms of these coordinates unless otherwise specified.

The Ricci tensor,

$$R_{\beta\epsilon} := R_{\sigma\beta}{}^{\sigma}{}_{\epsilon} \tag{2.18}$$

and, in the presence of a metric, the Ricci scalar

$$R := g^{\beta\epsilon} R_{\beta\epsilon} \tag{2.19}$$

are defined in the usual way. We note that the curvature tensor, $R_{\alpha\beta}{}^{\lambda}{}_{\epsilon}$ exhibits the symmetry:

$$R_{\alpha\beta}{}^{\lambda}{}_{\epsilon} = -R_{\beta\alpha}{}^{\lambda}{}_{\epsilon} \tag{2.20}$$

by definition.

2.3 Metricity

Given the presence of a Riemannian metric, $g_{\alpha\beta}$, on M, one can mandate that the affine connection (extended to tensors as per (2.11) and (2.12)) is one which keeps the metric "covariantly constant" - i.e.

$$\nabla x(g) = 0 \Leftrightarrow \partial_\lambda g_{\mu\nu} - \Gamma_\lambda{}^\kappa{}_\mu g_{\kappa\nu} - \Gamma_\lambda{}^\kappa{}_\nu g_{\kappa\mu} = 0 \tag{2.21}$$

This condition, referred to as both "metricity" and "metric compatibility", assures that the inner product of any two vectors parallel transported along any curve remains constant and is a necessary consequence of the equivalence principle, but is by no means mathematically pre-ordained. It is worth noting that there have been those who have explicitly examined the dynamics arising from theories of gravity where the metricity condition does *not* hold [21], but for our purposes it is enough to recognize it as a "mathematical degree of freedom" of the affine connection.

2.4 Torsion

In general relativity, the torsion tensor, T, is also assumed to vanish, but of course mathematically this is also not generally the case. Owing to the fact that in holonomic components

$$T_{\mu}{}^{\epsilon}{}_{\nu} = \Gamma_{\mu}{}^{\epsilon}{}_{\nu} - \Gamma_{\nu}{}^{\epsilon}{}_{\mu} \tag{2.22}$$

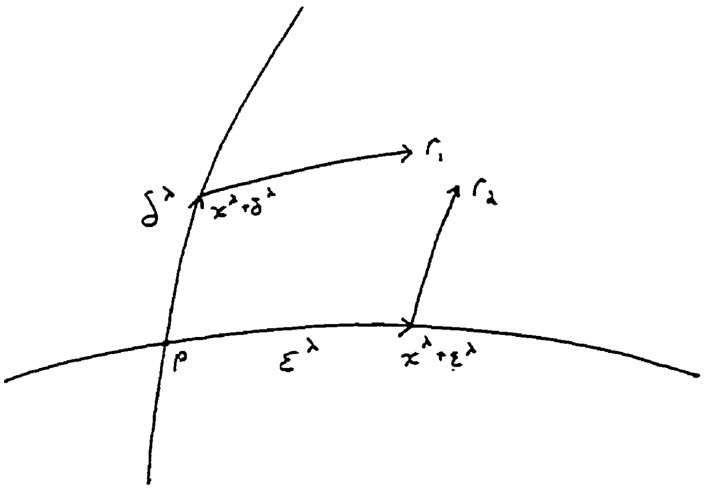
this (non) vanishing of the torsion term is also referred to as the (non) symmetry of the connection, but it is important to remember that this direct link only holds when working in a T_pM basis where the commutator of the basis vectors of T_pM, i.e. $[\hat{e}_\alpha, \hat{e}_\beta]$, is identically zero. It should also be noted here that in Chapter 4 we also encounter the contracted form of the torsion tensor $T_{\sigma}{}^{\sigma}{}_{\rho} = -T_{\rho}{}^{\sigma}{}_{\sigma}$, which we refer to as the torsion vector.

A large amount of work has been done involving the torsion tensor, both mathematically and physically [20, 22, 27, 31] but again we emphasize that for our purposes it is enough to recognize that it is merely a characteristic of a general affine connection independent of metricity.

One can get a basic geometrical picture of the torsion tensor by comparing the parallel transport of two infinitesimal vectors in T_pM along each other's flow lines. Consider the following diagram[2]:

[2]from [32], p.218

Figure 2.1: Torsion as Non-Closure of an Infinitesimal Parallelogram



If we define $X := \epsilon^\lambda \partial_\lambda$ and $Y := \delta^\lambda \partial_\lambda$ with both ϵ^λ and δ^λ infinitesimal[3], as elements of T_pM and we parallel transport X along Y and compare it to the parallel transport of Y along X, we find that the difference between the two resulting vectors, $r_2r_1 := pr_2 - pr_1$, can be expressed, to lowest order in ϵ and δ, directly in terms of the torsion tensor. That is,

$$r_2r_1 := pr_2 - pr_1 = (T_\mu{}^\lambda{}_\nu)\delta^\mu \, \epsilon^\nu \tag{2.23}$$

So we can view the vanishing/nonvanishing of the torsion tensor as a measure of the closure/non-closure of the infinitesimal parallelogram made up of infinitesimal tangent vectors at p and their respective parallel transports.

[3]Infinitesimals are used here in order to utilize distances via the metric and hence write r_2r_1 in terms of the coordinates at p via a Taylor series expansion.

2.5 Levi-Civita Connections

Of course, it is a well-established fact of Riemannian geometry that there is a unique affine connection which is *both* metric compatible *and* torsion-free. This connection is known as the Levi-Civita connection and in coordinate components it becomes the Christoffel symbol, i.e.

$$\Gamma_{\mu}{}^{\epsilon}{}_{\nu} \Rightarrow \{_{\mu}{}^{\epsilon}{}_{\nu}\} := \frac{1}{2} g^{\kappa\epsilon} \left[g_{\mu\kappa,\nu} + g_{\nu\kappa,\mu} - g_{\mu\nu,\kappa} \right] \tag{2.24}$$

This unique affine connection, solely dependent on the metric tensor, enables any other hitherto connection-dependent quantity to also be re-expressed merely in terms of the metric. Hence the curvature tensor can now be regarded solely as a function of the metric and its first and second derivatives, while it now displays the added symmetry:

$$R_{\alpha\beta}{}^{\lambda}{}_{\epsilon} = -R_{\alpha\beta\epsilon}{}^{\lambda} \tag{2.25}$$

in addition to (2.20).

The Christoffel symbol is assumed in General Relativity and its presence is visible in many related ways ranging from the ability to view the geodesic equation as a resultant Euler-Lagrange equation obtained by extremizing path length, to treating curvature as a manifestation of the geodesic deviation equation, to a mathematical consequence of the equivalence principle. Hence the physical ramifications of doing away with the assumption $\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\}$ are vast, indeed - yet that is hardly a reason to treat it as a mathematical necessity. In fact, quite the opposite is the case, for if we can determine under what class of mathematical scenarios this Levi-Civita constraint does or does not occur, we might well be closer to determining under what general mathematical circumstances one is irrevocably led to General Relativity. This knowledge should be just as valuable for those convinced of the validity of General

Relativity as for those who remain unconvinced of it.

Chapter 3

Standard Palatini Formalism (Torsion-Free)

3.1 Introduction

Early attempts to put General Relativity in a Lagrangian framework led Einstein and Hilbert to independently discover the action

$$S_{EH} = \int d^4x \left[\sqrt{-g}\,(R(\Gamma) + 16\pi\mathcal{L}_m)\right] \tag{3.1}$$

as one which yields the dynamics of General Relativity from a variational principle. That is, if one begins with the assumption (henceforth called the "Hilbert assumption")

$$\Gamma_\mu{}^\epsilon{}_\nu = \{_\mu{}^\epsilon{}_\nu\} \tag{3.2}$$

together with the identification

$$\sqrt{-g}T^{\alpha\beta} := 2\frac{\delta\left[\int d^4x\sqrt{-g}\mathcal{L}_m\right]}{\delta g_{\alpha\beta}}, \tag{3.3}$$

one obtains, by varying S_{EH} with respect to $g_{\alpha\beta}$ (henceforth called a Hilbert variation), the Einstein field equations:

$$G_{\mu\nu}(\{\}) = 8\pi T_{\mu\nu} \tag{3.4}$$

We note here that, in order to make the Hilbert variational principle well-defined, we must either specify a boundary term, S_B, to be added to the original action, (3.1), or *a priori* mandate that first derivatives of $g_{\alpha\beta}$ are fixed on this boundary. This extra boundary term is, however, non-dynamical, and hence may be incorporated into our action by a suitable redefinition of S_{EH}.

Yet the Hilbert variational principle is not the only one open to us. As mentioned in Chapter 1, motivated by more general geometrical considerations, one can envision utilizing a variational principle where the connection is no longer *a priori* determined but is instead elevated to the status of an independent graviational field variable. Referred to here as the Standard Palatini variation, this approach begins with the same action as previously, i.e. that of equation (3.1), with the proviso that the Ricci tensor is now solely a function of a now-independent affine connection, only assumed to be torsion-free. That is, our gravitational action now becomes:

$$S_P[g,\Gamma] = \int d^4x\sqrt{-g}\,[g^{\mu\nu}R_{\mu\nu}(\Gamma) + 16\pi\mathcal{L}_m] \tag{3.5}$$

with

$$R_{\mu\nu} = \partial_\eta(\Gamma_\mu{}^\eta{}_\nu) - \partial_\mu(\Gamma_\epsilon{}^\epsilon{}_\nu) + (\Gamma_\epsilon{}^\epsilon{}_\tau)(\Gamma_\mu{}^\tau{}_\nu) - (\Gamma_\mu{}^\lambda{}_\rho)(\Gamma_\lambda{}^\rho{}_\nu) \tag{3.6}$$

and

$$\Gamma_\alpha{}^\eta{}_\beta = \Gamma_\beta{}^\eta{}_\alpha \tag{3.7}$$

Variation of S_P with respect to the metric results in the more general constraint:

$$G_{\mu\nu}(\Gamma) = 8\pi T_{\mu\nu}, \tag{3.8}$$

while variation with respect to the connection now gives the additional constraint [1] :

$$g_{\mu\nu,\lambda} - g_{\eta\nu}\Gamma_\lambda{}^\eta{}_\mu - g_{\mu\eta}\Gamma_\lambda{}^\eta{}_\nu = 0, \tag{3.9}$$

[1]A feature of the Palatini approach is that, unlike the Hilbert variation, there is now no need to include a boundary term since the action no longer countains any derivatives of the metric and all of the field variables are assumed to vanish on the boundary.

which is the familiar condition of metric compatibility, whose solution

$$\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\} \tag{3.10}$$

is the Christoffel symbol.

The combination of the two constraints induced by the Palatini variation, (3.8) and (3.10), once again leads to the (Einsteinian) dynamics deduced by the Hilbert variation, i.e. equation (3.4),

$$G_{\mu\nu}(\{\}) = 8\pi T_{\mu\nu}$$

Thus we see that in the case of the Einstein-Hilbert action, S_{EH}, the Hilbert and Palatini variations lead to the same results, both for the specific form of the connection as well as the final dynamics.

This equivalence of the two approaches, however, is by no means always the case in all theories of gravity. To illustrate this crucial point, we now turn our attention to a generalized N-dimensional dilaton gravitational action and examine how the two variational methods differ when applied to this action.

3.2 Hilbert and Palatini Dynamics of a Generalized N-Dimensional Dilaton Action

Dilaton theories of gravity are playing an increasing role in the study of gravitational physics. The prototype of this class of theories is the Brans-Dicke theory [7], whose original motivation stemmed from a desire to develop a theory which incorporated Mach's principle by relating the gravitational constant G to the mean value of a scalar field which was coupled to the mass density of the universe (see, for example, [37]). More recently, this motivation has been largely supplanted by superstring theories [19], which generally predict that the low-energy effective Lagrangian governing gravitational dynamics is that of a dilaton theory of gravity.

From the usual Hilbert perspective, the generic expression for such gravitational actions is of the following form:

$$S_{DH} = \int d^n x \sqrt{-g} \left[D(\Psi) R(g) + A(\Psi)(\partial\Psi)^2 + 16\pi \mathcal{L}_m(\Phi, \Psi) \right] + S_B \tag{3.11}$$

where Ψ is the dilaton field and Φ symbolically denotes the matter fields whose Lagrangian may or may not also have an explicit dependence on Ψ. The Hilbert assumption, $\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\}$, is here manifested by the notation $R = R(g)$ for the Ricci scalar.

Once again we note the appearance of a boundary term, S_B, owing to the fact that our Hilbert variation involves a curvature term of second order in metric derivatives. The inclusion of such boundary terms is necessary to correctly evaluate the thermodynamics of a system of matter fields coupled to dilaton gravity [11], but, as previously mentioned, is not directly directly relevant in ascertaining the basic Hilbert dynamics and will thus be henceforth ignored.

Regarding the issue of dilaton gravity from a Palatini perspective, however, forces us to generalize our action further. If we begin our programme constrained solely by the assumptions that our generalized action is first order in curvature terms, at most quadratic in derivatives of Ψ and with a matter action only dependent on the metric (and hence independent of both the (torsion-free) connection and the dilaton field), we find that our action is necessarily of the following form [9]:

$$\begin{aligned} S_{DP} \; = \; & \int d^N x \sqrt{-g} [D(\Psi) R(\Gamma) + A(\Psi)(\nabla\Psi)^2 + B(\Psi)(\nabla^{\nu}\Psi) g_{\alpha\beta}(\nabla_{\nu} g^{\alpha\beta}) \\ & + C(\Psi)(\nabla_{\nu}\Psi)(\nabla_{\mu} g^{\mu\nu}) + F(\Psi) \nabla^2 \Psi + 16\pi \mathcal{L}_m] \end{aligned} \tag{3.12}$$

and is clearly a function of three independent gravitational variables: the connection, the metric and the dilaton field (we also tacitly assume the necessity of the curvature term, $R(\Gamma)$ and hence demand that $D \neq 0$).

Note that although $\nabla_\mu \Psi = \partial_\mu \Psi$ because Ψ is a scalar, since metricity is not assumed, $\nabla^2 \Psi$ above is given explicitly by $\nabla^\mu \nabla_\mu \Psi$ or $g^{\mu\nu} \nabla_\mu \partial_\nu \Psi$. Clearly in an *a priori* metric theory, both the third and fourth terms above are identically zero, while the fifth merely adds a total divergence combined with a redefinition of the $A(\Psi)$ term. Hence (modulo the non-dynamical boundary term S_B) for the special case of $\Gamma_\mu{}^\epsilon{}_\nu = \{_\mu{}^\epsilon{}_\nu\}$ above, S_{DP} reduces to an action of the form S_{DH}, and thus can be seen to be its proper Palatini generalization.

A Hilbert variation of S_{DP} yields the following two dynamical equations obtained by varying with respect to the metric and the dilaton field respectively:

$$\begin{aligned} \delta_g S_{DP} = 0 \Rightarrow 8\pi T_{\mu\nu} &= DG_{\mu\nu}(\{\}) + \left[D'' + \frac{1}{2}(F' - A)\right] g_{\mu\nu}(\partial\Psi)^2 \\ &\quad - \left[D'' + (F' - A)\right](\partial_\mu\Psi)(\partial_\nu\Psi) \\ &\quad - D'\left[\mathcal{D}_\mu(\partial_\nu\Psi) - g_{\mu\nu}\mathcal{D}^2\Psi\right] \end{aligned} \tag{3.13}$$

and

$$\delta_\Psi S_{DP} = 0 \Rightarrow D'R(\{\}) + (F' - A)'(\partial\Psi)^2 + 2(F' - A)\mathcal{D}^2\Psi = 0 \tag{3.14}$$

where we have dropped the explicit Ψ dependence in $D(\Psi), A(\Psi)$, etc..., $\mathcal{D}_\lambda$ represents the covariant derivative with respect to the Christoffel symbol and F', say, represents $\frac{\partial F}{\partial \Psi}$.

Meanwhile, the corresponding Palatini variation of (3.12) gives the following connection constraint:

$$\begin{aligned} \frac{-1}{\sqrt{-g}} \nabla_\epsilon \left[D\sqrt{-g}g^{\mu\nu}\right] + \frac{1}{2\sqrt{-g}} \nabla_\rho \left[D\sqrt{-g}\left(\delta^\nu_\epsilon g^{\mu\rho} + \delta^\mu_\epsilon g^{\nu\rho}\right)\right] & \\ +(B + \frac{1}{2}C)\left[\delta^\mu_\epsilon(\partial^\nu\Psi) + \delta^\nu_\epsilon(\partial^\mu\Psi)\right] + (C - F)(\partial_\epsilon\Psi)g^{\mu\nu} &= 0 \end{aligned} \tag{3.15}$$

We can derive two supplementary equations by tracing (3.15) with $g_{\mu\nu}$ and by contracting (3.15) over λ and μ (or ν, since (3.15) is symmetric in μ and ν owing to the assumed torsion-free nature of the connection).

Contracting (3.15) over λ and μ yields:

$$\frac{\nabla_\rho}{\sqrt{-g}}\left[D\sqrt{-g}g^{\rho\nu}\right] = (\partial^\nu\Psi)\left[\frac{2}{1-N}\right]\left[(N+1)B + \left(\frac{N+3}{2}\right)C - F\right] \tag{3.16}$$

while tracing (3.15) gives

$$(1-N)\frac{\nabla_\lambda}{\sqrt{-g}}\left[D\sqrt{-g}\right] + D\left[g_{\epsilon\lambda}(\nabla_\rho g^{\epsilon\rho}) - g_{\mu\nu}\,\nabla_\lambda\, g^{\mu\nu}\right] = \left[2B + (N+1)C - NF\right](\partial_\lambda\Psi) \tag{3.17}$$

Combining (3.16) and (3.17), together with the realization that

$$\frac{\nabla_\lambda\sqrt{-g}}{\sqrt{-g}} = -\frac{1}{2}g_{\mu\nu}\,\nabla_\lambda\, g^{\mu\nu} \tag{3.18}$$

allows us to substitute for $\nabla_\rho g^{\lambda\rho}$ above and eventually find the constraint

$$\begin{aligned}(2-N)D(N-1)\left(\frac{\nabla_\lambda\sqrt{-g}}{\sqrt{-g}}\right) &= (\partial_\lambda\Psi)[N(N-1)D' + 4B + (4-N^2+N)C \\ &\quad +(N-2)(N+1)F]\end{aligned} \tag{3.19}$$

Upon substitution of this derived constraint back into (3.15), we find that, for $(N \neq 2)$[2]

$$\nabla_\lambda g^{\mu\nu} = X(\partial_\lambda\Psi)g^{\mu\nu} + Y\left[\delta^\nu_\lambda(\partial^\mu\Psi) + \delta^\mu_\lambda(\partial^\nu\Psi)\right] \tag{3.20}$$

where

$$X := \frac{2\left[(1-N)D' - 2B + (N-3)C + (2-N)F\right]}{D(N-2)(1-N)} \tag{3.21}$$

and

$$Y := \left[\frac{2(B+C) - F}{D(1-N)}\right] \tag{3.22}$$

[2]Clearly the case of $N = 2$ merely adds a constraint between our variables D', B, C, F and does not allow us to solve explicitly for the connection using this procedure. For details of how one handles this scenario, see Appendix A.

By permuting (3.20) one can find an explicit form for the connection (again for $N \neq 2$)

$$\Gamma_\mu{}^\epsilon{}_\nu = \{_\mu{}^\epsilon{}_\nu\} + \left(Y(\Psi) - \frac{1}{2}X(\Psi)\right)(\partial^\epsilon\Psi)g_{\mu\nu} + \frac{1}{2}X(\Psi)\left[\delta^\epsilon_\mu(\partial_\nu\Psi) + \delta^\epsilon_\nu(\partial_\mu\Psi)\right] \quad (3.23)$$

in terms of the metric and the dilaton. Combining (3.23) with the two equations obtained by varying (3.12) with respect to $g_{\alpha\beta}$ and Ψ leads to the following "Palatini dynamics"

$$\begin{aligned} 8\pi T_{\mu\nu} = \ & DG_{\mu\nu}(\{\}) + \left[D'' + \tfrac{1}{2}(F' - A + \tilde{Q})\right] g_{\mu\nu}(\partial\Psi)^2 \\ & - \left[D'' + (F' - A + \tilde{Q})\right](\partial_\mu\Psi)(\partial_\nu\Psi) \\ & - D'\left[\mathcal{D}_\mu(\partial_\nu\Psi) - g_{\mu\nu}\mathcal{D}^2\Psi\right] \end{aligned} \quad (3.24)$$

and

$$D'R(\{\}) + (F' - A + \tilde{Q})'(\partial\Psi)^2 + 2(F' - A + \tilde{Q})\mathcal{D}^2\Psi = 0 \quad (3.25)$$

where

$$\tilde{Q} := \frac{1}{2}(1 - N)D\left[\left(\frac{N-2}{2}\right)X^2 - 2Y^2 - (N-2)XY\right] \quad (3.26)$$

Comparing (3.13) to (3.24) and (3.14) to (3.25) yields some interesting conclusions. We see that for $\tilde{Q} \equiv 0$, the two dynamics are mathematically identical. Clearly for

$$X = Y = 0 \Rightarrow D' + 2B + C = 0 \quad (3.27)$$

this will always be the case, and here the Palatini dynamics reduce to the identical form of that of the Hilbert dynamics.

Moreoever, we see that for $\tilde{Q}$ directly proportional to $(F' - A)$, the dynamics are fundamentally equivalent physically and only result in a rescaling of some constant parameter.

An example may serve to clarify this point of the case of $\tilde{Q} \propto F' - A$.

Consider the above generalized dilaton action, (3.12), with the following parameters:

$$A = ae^{k\Psi};\ B = be^{k\Psi};\ C = ce^{k\Psi};\ D = de^{k\psi};\ F = fe^{k\Psi} \tag{3.28}$$

where a, b, c, d, f, k are some constants. Under this parametrization, we find that X and Y become:

$$X = \frac{2\left[(1-N)kd - 2b + (N-3)c + (2-N)f\right]}{d(N-2)(1-N)} \tag{3.29}$$

and

$$Y := \left[\frac{2(b+c) - f}{d(1-N)}\right] \tag{3.30}$$

that is, mere constants themselves. Hence under this parametrization, (3.28), $\tilde{Q}$ becomes:

$$\tilde{Q} = q\, e^{k\Psi} \tag{3.31}$$

where q is the constant

$$q = \frac{1}{2}(1-N)d\left[\left(\frac{N-2}{2}\right)X^2 - 2Y^2 - (N-2)XY\right] \tag{3.32}$$

where here X and Y are of course given by (3.29) and (3.30) above. It is clear that since we have:

$$F' - A = (kf - a)e^{k\Psi}, \tag{3.33}$$

we are in a domain where $\tilde{Q} \propto F' - A$.

A Hilbert variation of (3.12) complete with (3.28), leads to the following dynamics:

$$\begin{aligned} 8\pi T_{\mu\nu} \;=\;& d\, e^{k\Psi} G_{\mu\nu}(\{\}) + \left[k^2 d + \frac{1}{2}(\gamma)\right] e^{k\Psi}\, g_{\mu\nu}(\partial\Psi)^2 \\ & - \left[k^2 d + (\gamma)\right] e^{k\Psi}\, (\partial_\mu\Psi)(\partial_\nu\Psi) \\ & - kd\, e^{k\Psi}\left[\mathcal{D}_\mu(\partial_\nu\Psi) - \mathcal{D}^2\Psi\right] \end{aligned} \tag{3.34}$$

and

$$kd\, e^{k\Psi} R(\{\}) + k(\gamma) e^{k\Psi}\, (\partial\Psi)^2 + 2(\gamma)\, e^{k\Psi} \mathcal{D}^2\Psi = 0 \tag{3.35}$$

where the constant γ is defined as

$$\gamma := kf - a \tag{3.36}$$

Meanwhile, a Standard Palatini variation of (3.12) together with (3.28) gives:

$$\begin{aligned} 8\pi T_{\mu\nu} &= d\, e^{k\Psi} G_{\mu\nu}(\{\}) + \left[k^2 d + \frac{1}{2}(\hat{\gamma})\right] e^{k\Psi}\, g_{\mu\nu}(\partial\Psi)^2 \\ &\quad - \left[k^2 d + (\hat{\gamma})\right] e^{k\Psi}\, (\partial_\mu\Psi)(\partial_\nu\Psi) \\ &\quad - kd\, e^{k\Psi}\left[\mathcal{D}_\mu(\partial_\nu\Psi) - \mathcal{D}^2\Psi\right] \end{aligned} \tag{3.37}$$

and

$$kd\, e^{k\Psi} R(\{\}) + k(\hat{\gamma}) e^{k\Psi}\, (\partial\Psi)^2 + 2(\hat{\gamma})\, e^{k\Psi} \mathcal{D}^2\Psi = 0 \tag{3.38}$$

where the constant γ of (3.34) and (3.35) has merely been rescaled in the following manner

$$\gamma \Rightarrow \hat{\gamma} = \gamma + q \tag{3.39}$$

for q the constant given by (3.32) above. This rescaling is merely algebraic and has no physical manifestation[3]; thus one finds that the Hilbert and (Standard) Palatini variations result in the same dynamics for this particular case.

In general, however, we have a situation where *neither* $X, Y \neq 0$ *nor* $\tilde{Q}$ is proportional to $(F' - A)$. In this general case, then, we see that a Palatini variation does *not* yield identical dynamics, either mathematically or physically, from those of a Hilbert variation.

[3]Lindström [28, 29] has shown this for a smaller class of actions where $B = C = F = 0$ and $A(\Psi)$ is Ψ^a for $a \in \mathcal{I}$

3.3 Generalized N-Dim. Einstein-Hilbert Action

Convinced that a Palatini variation is thus no longer generally equivalent to that of a Hilbert variation, we turn our attention back upon the ordinary Einstein- Hilbert action of equation (3.1) and ask ourselves *why*, then a Palatini variation of this action gives in fact the Christoffel constraint

$$\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\}. \tag{3.40}$$

Again it is to be emphasized here that this question is not posed in support of some hidden agenda to necessarily pursue alternative theories of gravity where $\Gamma_{\mu}{}^{\epsilon}{}_{\nu} \neq \{_{\mu}{}^{\epsilon}{}_{\nu}\}$ - although, as previously mentioned in the previous chapter, there has been considerable effort devoted to this end, both with regards to non-metric connections and connections with torsion (see, for example, [21, 27]).

Regardless of whatever *physical* preferences one might have for the Christoffel symbol as the connection of choice, the *mathematical* singling out of this particular (metric compatible) connection from the space of all torsion-free connections via a general variational principle must strike one as curious, to say the least [36], while the sometimes associated view that the Palatini principle offers some sort of pseudo-teleological "proof" of the necessity of the connection "being" the Christoffel symbol, at least with regards to the Einstein- Hilbert action, is even more suspicious.

A natural question thus suggests itself: Is it possible to somehow modify the Einstein-Hilbert action so that a Palatini variation does *not* isolate the Levi-Civita connection but still gives rise to Einsteinian dynamics?

To this end, we move to N-dimensions and consider a generalized Einstein-Hilbert action which, for simplicity, includes all possible terms that are at most quadratic in derivatives and/or connection variables [8]. We also henceforth drop the explicit matter action term, $\mathcal{L}_m$, thus regarding all derived dynamics as *in vacuuo*, with the understanding that the relevant stress energy tensor can always be recovered via the identification as per equation (3.3) above. Throughout this chapter we are only concerned with torsion-free connections (generalizations to actions with torsion will occur in Chapter 4).

The most general action in N dimensions that one can construct subject to the above constraints is

$$\begin{aligned} S_{EHE} &= \int d^N x \sqrt{-g}[R + H(\nabla_\nu g^{\alpha\beta})(\nabla^\nu g_{\alpha\beta}) + IV^2 \\ &\qquad + J(\nabla_\epsilon g_{\mu\nu})(\nabla^\mu g^{\epsilon\nu}) + KV \cdot Z + LZ \cdot Z], \end{aligned} \tag{3.41}$$

where we have used the convenient definitions:

$$V_\rho := \frac{\nabla_\rho \sqrt{-g}}{\sqrt{-g}} \qquad Z^\lambda := \nabla_\eta g^{\eta\lambda} \tag{3.42}$$

and where the coefficients H, I,J,K and L are constants.

Other scalar quantities exist, but they can either be rewritten as linear combinations of the above terms up to total derivatives or they are higher order in derivatives and/or connection variables. Once again it should be noted that, just as in the usual Palatini analysis for the Einstein-Hilbert action, (3.1), since we assume $(\delta g_{\mu\nu})$ and $(\delta \Gamma_\mu{}^\epsilon{}_\nu)$ to vanish at the boundary, no additional boundary terms in (3.41) are required.

Since the connection is assumed, *a priori*, to be arbitrary, we can express it for this torsion-free case as

$$\Gamma_\mu{}^\epsilon{}_\nu = \{_\mu{}^\epsilon{}_\nu\} + Q_\mu{}^\epsilon{}_\nu \tag{3.43}$$

for $Q_\mu{}^\epsilon{}_\nu$ some (initially) completely undetermined tensor whose sole constraint is that it be symmetric in its first and third indices (i.e. the torsion-free condition.)

Looked at in this light, we see that we have the following relationships:

$$V_\lambda = -Q_\lambda{}^\epsilon{}_\epsilon \tag{3.44}$$

$$Z^\lambda = Q_\rho{}^{\rho\lambda} + Q_\rho{}^{\lambda\rho} = Q_\rho{}^{\lambda\rho} - V^\lambda, \tag{3.45}$$

where we see that V^λ and Z^λ are representative of the two independent quantities $Q_\rho{}^{\rho\lambda}$ and $Q_\rho{}^{\lambda\rho}$. The following analysis can thus be regarded as a determination of the constraints put on $Q_\mu{}^\epsilon{}_\nu$ due to the variational principle.

Variation of (3.41) with respect to the connection $\Gamma_\lambda{}^\rho{}_\sigma$ leads to the following constraint:

$$\begin{aligned}
&\frac{1}{\sqrt{-g}}\left(\nabla_\lambda\left[\sqrt{-g}g^{\rho\sigma}\right] - \frac{1}{2}\nabla_\epsilon\left[\sqrt{-g}g^{\rho\epsilon}\right]\delta^\sigma_\lambda - \frac{1}{2}\nabla_\epsilon\left[\sqrt{-g}g^{\sigma\epsilon}\right]\delta^\rho_\lambda\right) \\
&\quad + H\left[\left(\nabla^\rho g^{\sigma\gamma} + \nabla^\sigma g^{\rho\gamma}\right)g_{\gamma\lambda} - (\nabla^\rho g_{\lambda\gamma})g^{\sigma\gamma} - (\nabla^\sigma g_{\lambda\gamma})g^{\rho\gamma}\right] \\
&\quad + I\left[V^\rho\delta^\sigma_\lambda + V^\sigma\delta^\rho_\lambda\right] + J\left[2\nabla_\lambda g^{\rho\sigma} - \left(g^{\alpha\sigma}g^{\epsilon\rho} + g^{\rho\alpha}g^{\epsilon\sigma}\right)\nabla_\epsilon g_{\alpha\lambda}\right] \\
&+K\left[\frac{1}{2}\left(Z^\sigma\delta^\rho_\lambda + Z^\rho\delta^\sigma_\lambda\right) - \frac{1}{2}\left(V^\sigma\delta^\rho_\lambda + V^\rho\delta^\sigma_\lambda\right) - V_\lambda g^{\rho\sigma}\right] - L\left[Z^\rho\delta^\sigma_\lambda + Z^\sigma\delta^\rho_\lambda + 2Z_\lambda g^{\sigma\rho}\right] = 0
\end{aligned} \tag{3.46}$$

where we have explicitly incorporated the (torsion-free) symmetry of the connection (i.e. symmetry in ρ and σ). The solution of (3.46) determines the connection as a function of the metric in a manner which generalizes (3.9).

We thus seek to find the conditions under which (3.46) may be solved for Γ in terms of the metric. Tracing (3.46) on the (ρ, σ) indices yields

$$[(N-3) + 2I - 4J - (N+1)K]\,V_\lambda + [4H + 2J + K - 2L(N+1) - 1]\,Z_\lambda = 0, \tag{3.47}$$

henceforth written as

$$PV_\lambda + QZ_\lambda = 0 \tag{3.48}$$

whilst a $\rho - \lambda$ contraction of (3.46) gives

$$\begin{aligned} &[(N-1) + 8H - 2(N+1)I + 4J + (N+3)K]\, V_\lambda \\ &+ [(N-1) - 4H - 6J - (N+1)K + 2(N+3)L]\, Z_\lambda \;=\; 0. \end{aligned} \tag{3.49}$$

henceforth written as

$$RV_\lambda + SZ_\lambda = 0 \tag{3.50}$$

Equations (3.48) and (3.50) are two equations in the two unknown vector fields V_λ and Z_λ. We therefore examine the (2×2) matrix Ω, defined by

$$\Omega := \begin{vmatrix} P & Q \\ R & S \end{vmatrix} \tag{3.51}$$

There are clearly three distinct possibilities with regards to this matrix - namely that it is either rank 2, rank 1 or trivial. We shall examine each case in turn.

The first possibility leads to the necessary relationship:

$$V_\lambda = Z_\lambda = 0 \tag{3.52}$$

which constrains $Q_\mu{}^\epsilon{}_\nu$ above to be traceless over any two indices.

If Ω is rank 1, we are left with some relationship:

$$Z_\lambda = r\, V_\lambda \tag{3.53}$$

for some constant r = r(H,I,J,K,L,N) and thus leads to the following constraint on $Q_\rho{}^{\lambda\rho}$ and $Q_\rho{}^{\rho\lambda}$:

$$Q_\rho{}^{\lambda\rho} = -(1+r) Q_\rho{}^{\rho\lambda}, \tag{3.54}$$

while for Ω being trivial, we see that we have the necessary condition $P = Q = R = S = 0$ relating our coefficients H, I, J, K, L and the dimensionality, N, but can derive *no* information about $Q_\mu{}^\epsilon{}_\nu$ (or, more specifically, its associated traced and contracted vectors). For arbitrary N, one finds that the unique set of values of H, I, J, K, L satisfying $P = Q = R = S = 0$ is the following:

$$H = \frac{1}{4}; J = -\frac{1}{2}; I = K = 1; L = 0 \tag{3.55}$$

which we shall see again later in a related context (i.e. (3.72)).

We now return to our variational method and consider explicity the results for Ω rank 2. Insertion of (3.52) into (3.46) yields:

$$\nabla_\lambda g^{\rho\sigma}\,[1 + 2J] + (2H + J)\,[g_{\lambda\gamma}\,(\nabla^\rho g^{\sigma\gamma} + \nabla^\sigma g^{\rho\gamma})] = 0 \tag{3.56}$$

It is straightforward to show that

$$\Gamma_\mu{}^\epsilon{}_\nu = \{_\mu{}^\epsilon{}_\nu\} \Leftrightarrow Q_\mu{}^\epsilon{}_\nu = 0 \tag{3.57}$$

is the only solution to (3.56) provided that

$$4H + 4J \neq -1 \tag{3.58}$$

or

$$2H - J \neq 1, \tag{3.59}$$

where the constraints (3.58), (3.59) shall be referred to as "indeterminacy constraints" as their actualization prevents determination of an explicit form of the connection (i.e. $Q_\mu{}^\epsilon{}_\nu$) and merely limits $Q_\mu{}^\epsilon{}_\nu$ to a subset of possible values. We will see that such conditions recur throughout the analysis of the next two chapters.

Consequently we see that metric compatibility arises within the Palatini formalism under quite general conditions unless $4H + 4J = -1$, in which case, for $2H - J \neq 1$, it can be shown that $\Gamma_\mu{}^\epsilon{}_\nu$ is of the form:

$$\Gamma_\mu{}^\epsilon{}_\nu = \{_\mu{}^\epsilon{}_\nu\} - \frac{1}{2} g^{\lambda\epsilon}\,\nabla_\lambda\, g_{\mu\nu} \tag{3.60}$$

i.e.

$$Q_{\mu}{}^{\epsilon}{}_{\nu} = -\frac{1}{2}\left[Q^{\epsilon}{}_{\mu\nu} + Q^{\epsilon}{}_{\nu\mu}\right] \tag{3.61}$$

Similarly if $2H - J = 1$ and $4H + 4J \neq -1$, we see that $\Gamma_{\mu}{}^{\epsilon}{}_{\nu}$ is necessarily of the form:

$$\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\} + g^{\lambda\epsilon}\,\nabla_{\lambda}\,g_{\mu\nu} \tag{3.62}$$

i.e.

$$Q_{\mu}{}^{\epsilon}{}_{\nu} = \left[Q^{\epsilon}{}_{\mu\nu} + Q^{\epsilon}{}_{\nu\mu}\right] \tag{3.63}$$

We further note that the condition that trivializes (3.56), i.e. $J = -\frac{1}{2}$, $H = \frac{1}{4}$, is a simultaneous solution of both of the above special cases and thus leaves $\nabla_{\lambda}g^{\rho\sigma}$ completely undetermined modulo the conditions given in (3.52).

For Ω rank 1, the analysis is similar except that one finds that the right hand side of (3.56) is no longer zero, but instead some general function of the metric tensor and V_{λ} (or alternatively, one of $Z_{\lambda}, Q_{\lambda}{}^{\rho}{}_{\rho}$ or $Q_{\rho\lambda}{}^{\rho}$). Again one finds that the indeterminacy constraints must be satisfied for one to get an explicit expression for $\nabla_{\lambda}g_{\alpha\beta}$ (otherwise we are merely reduced to some weak constraint, as mentioned above, on $Q_{\mu}{}^{\epsilon}{}_{\nu}$). But now, even given that these constraints are satisfied, we no longer have

$$\nabla_{\lambda}g_{\alpha\beta} = 0 \Rightarrow Q_{\mu}{}^{\epsilon}{}_{\nu} = 0 \tag{3.64}$$

but instead

$$\nabla_{\lambda}g_{\alpha\beta} = E_{\alpha\beta\lambda}(g_{\alpha\beta}, Q_{\lambda}{}^{\rho}{}_{\rho}, H, I, J, K, L), \tag{3.65}$$

which merely entails some other weak constraint on $Q_{\mu}{}^{\epsilon}{}_{\nu}$. Finally, we note that the trivial case, $P = Q = R = S = 0$, amounts to changing the right hand side of (3.56) to include *both* traced Q tensors (i.e. V_{λ} *and* Z_{λ}), thereby eventually yielding an additional (weak) constraint on $Q_{\mu}{}^{\epsilon}{}_{\nu}$, different from that arising from the rank 1 case above.

Thus we have found that $Q_\mu{}^\epsilon{}_\nu$ can be weakly constrained or fully determined, depending on the rank of Ω and the indeterminacy conditions, (3.58), (3.59). If it is fully determined, then we see that it is forced to be identically zero and the connection consequently reduces to the Christoffel symbol. For the standard Einstein-Hilbert action, with $H = I = J = K = L = 0$, this is, indeed, the case.

But is it possible for our analysis to leave $Q_\mu{}^\epsilon{}_\nu$ completely *undetermined*? We have seen that Ω trivial leaves $Q_\rho{}^\rho{}_\lambda$ and $Q^\epsilon{}_{\lambda\epsilon}$ (i.e. V_λ and Z_λ) completely undetermined, while the combination of the indeterminacy constraints, (3.58) and (3.59), i.e.

$$J = -\frac{1}{2}; H = \frac{1}{4} \tag{3.66}$$

necessitates that the left hand side of (3.56) vanish identically. Thus in this special case (3.56) becomes:

$$0 = \delta^\rho_\lambda \left[\left(I - \frac{1}{2}K \right) V^\sigma + \left(\frac{1}{2}K - L - \frac{1}{2} \right) Z^\sigma \right] + \rho \Leftrightarrow \sigma + g^{\rho\sigma} \left[(1-K) V_\lambda + 2LZ_\lambda \right] \tag{3.67}$$

We can see that in general, (3.67) again leads to some weak constraint relating V_λ to Z_λ, but the special case of

$$L = 0, I = K = 1 \tag{3.68}$$

merely trivializes (3.67)

Thus for the special case of $H = \frac{1}{4}, J = -\frac{1}{2}; I = K = 1; L = 0$, (3.46) gives a simple triviality, tells us *nothing* about the connection, and is thus completely redundant. We expect that this redundancy is manifested by a general invariance of the connection To this end, consider the following general transformation of the connection:

$$\Gamma_\mu{}^\epsilon{}_\nu \Rightarrow \hat{\Gamma}_\mu{}^\epsilon{}_\nu + Q_\mu{}^\epsilon{}_\nu \tag{3.69}$$

for $Q_\mu{}^\epsilon{}_\nu$, as before, an arbitrary tensor field with the sole restriction that it is symmetric in its first and third indices. This type of transformation is sometimes called a deformation transformation [20]. Under the above transformation we find that the action (3.41) is correspondingly transformed

$$S_{EHE} \Rightarrow \hat{S}_{EHE} = S_{EHE} + \delta S, \tag{3.70}$$

where

$$\begin{aligned}
\delta S \;=\;& [1+2J](\nabla^\epsilon g^{\mu\nu})Q_{\mu\epsilon\nu} + [2H+J](\nabla^\epsilon g^{\mu\nu})(Q_{\epsilon\mu\nu} + Q_{\epsilon\nu\mu}) \\
& -[1+2H+3J]Q^{\epsilon\mu\nu}Q_{\epsilon\nu\mu} - [2H+J]Q^{\epsilon\mu\nu}Q_{\epsilon\mu\nu} \\
& +[I-K+L]Q_\lambda{}^{\lambda\rho}Q^\sigma{}_{\sigma\rho} + [1-K+2L]Q^\lambda{}_{\lambda\rho}Q_\sigma{}^{\rho\sigma} \\
& +LQ_{\sigma\rho}{}^\sigma Q_{\lambda\rho}{}^\lambda + [1-2I+K]V_\lambda Q_\rho{}^{\rho\lambda} + [K-1]V_\lambda Q_\eta{}^{\lambda\eta} \\
& +2LZ^\lambda Q_{\eta\lambda}{}^\eta + [1+2L-K]Z^\lambda Q^\eta{}_{\eta\lambda}
\end{aligned} \tag{3.71}$$

For $\Gamma_\mu{}^\epsilon{}_\nu$ to be completely unconstrained, we must have $\delta S = 0$ regardless of the choice of $Q_\mu{}^\epsilon{}_\nu$. As expected, we see that this can only happen if

$$H = \frac{1}{4}; J = -\frac{1}{2}; I = K = 1; L = 0, \tag{3.72}$$

which was the same condition we found that led to our redundancy.

Conversely, consider subsitution of (3.69) for $\Gamma_\mu{}^\epsilon{}_\nu$ into the general action (3.41), and then varying the (transformed) action with respect to $Q_\mu{}^\epsilon{}_\nu$. This yields a set of complicated algebraic equations for $Q_\mu{}^\epsilon{}_\nu$. Insertion into (3.41) of their solution for $Q_\mu{}^\epsilon{}_\nu$ in terms of $\Gamma_\mu{}^\epsilon{}_\nu$ and $g_{\mu\nu}$ leads directly to a modified action of the form given in (3.41) whose specific values for H, I, J, K, L are given by (3.72) above.[4]

In other words, (3.72) is clearly the unique set of values such that our action is invariant under the transformation (3.69) with $Q_\mu{}^\epsilon{}_\nu$ completely unconstrained other

[4]See Appendix B for more explicit details

than being symmetric in its first and third indices. Accordingly, the values (3.72) will henceforth be called the "maximally symmetric" values.[5]

From this perspective one can say that the compatibility condition, (3.57), obtained by applying the Palatini variational principle to the Einstein-Hilbert action, is an example of a constraint induced by a broken symmetry. That is, the EH action is a special case of our general action (3.41) above, with the particular requirement that $H = I = J = K = L = 0$. That these values of H, I, J, K, L break the general symmetry is obvious from the above analysis, and it is this breaking of this "connection symmetry" which singles out the Christoffel symbol.

3.4 Extended Action Dynamics

Momentarily putting aside our consideration of the "connection-dynamics" of our extended action and calculating the ordinary "metric-dynamics", we find

$$\begin{aligned} 0 \;=\; & G_{(\mu\nu)}(\Gamma) + (I-K)V_\mu V_\nu - \frac{1}{2}K\left[\nabla_\mu V_\nu + \nabla_\nu V_\mu\right] \\ & -2\left(\nabla_\lambda + V_\lambda\right) g^{\lambda\epsilon}\left[H\,\nabla_\epsilon\, g_{\mu\nu} + \frac{1}{2}J\left(\nabla_\mu g_{\nu\epsilon} + \nabla_\nu g_{\mu\epsilon}\right)\right] \\ & -L\left[V_\mu Z_\nu + V_\nu Z_\mu + \nabla_\mu V_\nu + \nabla_\nu V_\mu + Z_\mu Z_\nu\right] \\ & +H\left[(\nabla_\mu g^{\alpha\beta})(\nabla_\nu g_{\alpha\beta}) + 2g^{\alpha\beta}(\nabla^\lambda g_{\alpha\mu})(\nabla_\lambda g_{\beta\nu})\right] \\ & +\frac{1}{2}J\left[(\nabla^\eta g_{\alpha\mu})(\nabla^\alpha g_{\nu\eta}) + (\nabla^\eta g_{\alpha\nu})(\nabla^\alpha g_{\mu\eta})\right] \\ & +g_{\mu\nu}\{-\frac{1}{2}H(\nabla_\rho g^{\alpha\beta})(\nabla^\rho g_{\alpha\beta}) + \frac{1}{2}IV^2 - \frac{1}{2}J(\nabla_\epsilon g_{\alpha\beta})(\nabla^\alpha g^{\epsilon\beta}) \\ & \qquad -\frac{1}{2}LZ^2 + \nabla_\epsilon\left(IV^\epsilon + \frac{1}{2}KZ^\epsilon\right)\} \end{aligned} \tag{3.73}$$

[5] "maximally" symmetric to distinguish them from other partial symmetries which occur when one assumes some particular tensorial structure of $Q_\mu{}^\epsilon{}_\nu$ derived from one of the above weaker constraints.

upon variation of (3.41) with respect to the metric. Provided the constants H, I, J, K, L are chosen so that (3.44) and (3.45) are satisfied (i.e. our coefficients are chosen so that $4H + 4J \neq -1$ and $2H - J \neq 1$), then all terms on the right hand side of (3.73) vanish except for the first one, which becomes the usual expression for the Einstein tensor in terms of the metric.

Consider next the condition of maximal symmetry. Insertion of our maximally symmetric values, (3.72), into the above dynamical equation yields

$$\begin{aligned} 0 \;=\; & G_{(\mu\nu)}(\Gamma) + \frac{1}{4}\left[\nabla_\mu P_{\nu\eta}{}^{\eta} + \nabla_\nu P_{\mu\eta}{}^{\eta}\right] + \left[\nabla_\lambda - \frac{1}{2}(P_{\lambda\eta}{}^{\eta})\right](E^{\lambda}{}_{\mu\nu}) \\ & + \frac{1}{4}\left[2(P^{\lambda\beta}{}_{\mu})(P_{\lambda\beta\nu}) - (P_{\mu}{}^{\lambda\eta})(P_{\nu\lambda\eta}) - 2(P^{\lambda}{}_{\eta\mu})(P^{\eta}{}_{\lambda\nu})\right] \\ & + \frac{1}{8}g_{\mu\nu}\left[2(P_{\lambda\eta}{}^{\eta})(P^{\lambda\rho}{}_{\rho}) - 2(P_{\epsilon\lambda\eta})(P^{\lambda\epsilon\eta}) + (P_{\lambda\eta\epsilon})(P^{\lambda\eta\epsilon}) + 4\,\nabla_\epsilon\left(P_{\lambda}{}^{\lambda\epsilon} - P^{\epsilon\eta}{}_{\eta}\right)\right] \end{aligned} \tag{3.74}$$

where

$$P_{\eta}{}^{\mu\nu} := \nabla_\eta g^{\mu\nu} \tag{3.75}$$

and

$$E^{\lambda}{}_{\mu\nu} := \frac{1}{2}\left[P^{\lambda}{}_{\mu\nu} - P_{\mu\nu}{}^{\lambda} - P_{\nu\mu}{}^{\lambda}\right] = \left[\{{}_{\mu}{}^{\epsilon}{}_{\nu}\} - \Gamma_{\mu}{}^{\epsilon}{}_{\nu}\right], \tag{3.76}$$

thus enabling us to put some quantities directly in terms of the Christoffel symbol.

Hence the field equations in the case of maximal symmetry consist of (3.74) alone – there is no equation which determines the connection in terms of the metric. In this sense the maximally symmetric action is a theory of gravity determined in terms of metric dynamics alone, with the connection freely specifiable.

Since the connection may be freely specified, one choice is to make it compatible with the metric, *i.e.* to demand that (3.57) hold. In this case all $P_{\eta}{}^{\mu\nu} = 0$, and (3.74) reduces to

$$G_{\mu\nu}(\{\}) = 0 \tag{3.77}$$

which are the field equations for general relativity. Alternatively, suppose we choose $\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = 0$. In this case (3.74) becomes

$$
\begin{aligned}
0 \;=\;& G_{(\mu\nu)}(\Gamma) + \frac{1}{4}\left[\nabla_{\mu}\hat{P}_{\nu\eta}{}^{\eta} + \nabla_{\nu}\hat{P}_{\mu\eta}{}^{\eta}\right] + \left[\nabla_{\epsilon} - \frac{1}{2}\hat{P}_{\epsilon\eta}{}^{\eta}\right]\{_{\mu}{}^{\epsilon}{}_{\nu}\} \\
&+\frac{1}{4}\left[2(\hat{P}^{\lambda\beta}{}_{\mu})(\hat{P}_{\lambda\beta\nu}) - (\hat{P}_{\mu}{}^{\lambda\eta})(\hat{P}_{\nu\lambda\eta}) - 2(\hat{P}^{\lambda}{}_{\eta\mu})(\hat{P}^{\eta}{}_{\lambda\nu})\right] \qquad (3.78) \\
&+\frac{1}{8}g_{\mu\nu}\left[2(\hat{P}_{\lambda\eta}{}^{\eta})(\hat{P}^{\lambda\rho}{}_{\rho}) - 2(\hat{P}_{\epsilon\lambda\eta})(\hat{P}^{\lambda\epsilon\eta}) + (\hat{P}_{\lambda\eta\epsilon})(\hat{P}^{\lambda\eta\epsilon}) + 4\,\nabla_{\epsilon}\left(\hat{P}_{\lambda}{}^{\lambda\epsilon} - \hat{P}^{\epsilon\eta}{}_{\eta}\right)\right]
\end{aligned}
$$

where $\hat{P}_{\eta}{}^{\mu\nu} := \partial_{\eta}g^{\mu\nu}$. Further simplification of the right-hand side of (3.78) yields

$$G_{(\gamma\sigma)}(g) = 0 \qquad (3.79)$$

where $G_{(\gamma\sigma)}(g)$ is the Einstein tensor expressed as a functional of the metric, *i.e.* $G_{(\gamma\sigma)}(g) = G_{(\gamma\sigma)}(\{\})$. Hence (3.79) also yield the equations of general relativity. The above case of examining $\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = 0$ raises an interesting curiosity. Clearly, as the maximally symmetric case only restricts the connection to be torsion-free, $\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = 0$ is an available option. But the fact that we are able to choose such a connection *globally* enables us to say something additional about the geometry of our manifold - namely that it is flat; or rather, that it can be made flat with no physical sacrifice. Hence for the maximally symmetric theory, one can always model the dynamics equivalently in flat space.

The preceding situation is also a generalization of a result obtained by Gegenberg *et. al.* for $(1+1)$ gravity [16]. Consider the action (3.41) for $N = 2$ with each of H, I, J, K, L set to zero. In this case the determinant of coefficients in eqs. (3.48) and (3.50) vanishes, and the general solution to (3.46) is given by [16]

$$\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\} + \left(\delta^{\epsilon}_{\mu}B_{\nu} + \delta^{\epsilon}_{\nu}B_{\mu} - g_{\mu\nu}B^{\epsilon}\right) \qquad (3.80)$$

where B_{μ} is an arbitrary vector field. The Einstein tensor is given by

$$G_{(\sigma\gamma)}(\bar{G}) = G_{(\sigma\gamma)}(\{\}) = 0 \qquad (3.81)$$

and so renders the $(1+1)$ dimensional field equations trivial, as in the usual Hilbert case. We see from the preceding analysis of (3.74) that an analogous situation holds in higher dimensions for the maximally symmetric action: although the field equations do not determine the connection in terms of the metric, one can choose the connection to be compatible with the metric by appropriately choosing $Q_{\mu}{}^{\epsilon}{}_{\nu}$ in (3.69) and recover the metric field equations of general relativity.

More generally, the choice of connection is completely irrelevant to the theory in the maximally symmetric case. One has only equation (3.74), which determines the evolution of the metric and is equivalent to the Einstein Field Equations.

Thus by beginning with a generalized Einstein-Hilbert action given by (3.41), we have obtained a result whereby, for all of the coefficients taking the specific "maximally symmetric" values of (3.72), a Palatini variation of this action yields Einsteinian dynamics together with a completely undetermined connection. Looked at from this perspective, the standard Einstein-Hilbert action is merely a particular case (i.e. that for which $H = I = J = K = L = 0$) of our generalized action, (3.41).

3.5 Summary

In attempting to better understand the origin of the Christoffel constraint, (3.10), arising from the Standard Palatini variation of the Einstein-Hilbert action, (3.5) we first examined a generalized dilaton theory of gravity, (3.12), to see if, under a Palatini variation, it too will yield the Christoffel constraint and consequent identical dynamics to that of a Hilbert variation. We find in general that the Christoffel constraint does *not* in general occur, *nor* are the two dynamics generally equivalent, although there are situations where the dynamics are equivalent without the connection necessarily satsifying the Christoffel constraint. We return to the Einstin-Hilbert action and succeed in finding a generalized version of this action which results in the final dynamics of General Relativity while simultaneously leaving the connection completely *indeterminate*. We denote such an action as the "maximally symmetric" action and note that it is invariant under a deformation transformation, $\Gamma_{\mu}{}^{\epsilon}{}_{\nu} \Rightarrow \Gamma_{\mu}{}^{\epsilon}{}_{\nu} + Q_{\mu}{}^{\epsilon}{}_{\nu}$, for $Q_{\mu}{}^{\epsilon}{}_{\nu}$ some arbitrary tensor symmetric in its first and third indices.

It is worth emphasizing here that, viewed from the perspective of our generalized action with arbitrary H, I, J, K, L, a Palatini variation of this action invariably leads to the Christoffel constraint *unless* H, I, J, K, L satisfy the particular values determined by the indeterminacy constraints (3.58), (3.59). Meanwhile, complete freedom for the connection only occurs when H, I, J, K, L satisfy the *unique* maximally symmetric values - which can be regarded as a particular point in H, I, J, K, L parameter space whose specific relevance lies in its relationship to deformation invariance of the connection.

Chapter 4

Palatini Variation of Actions with Torsion

4.1 Overview

We now consider further generalization of our actions where the connection is no longer necessarily torsion-free. There are two principal motivations for pursuing this particular avenue, the first of which being mathematical completeness. Since torsion is one of the two fundamental "degrees of freedom" of our affine connection (along with metric compatibility), it seems unwise not to at least investigate modifying our methods based upon its potential presence.

Secondly, one is spurred on to consider the question of torsion due to the nature of the so-called Palatini Tetrad formalism, which we shall see explicitly in Chapter 6. In this treatment, one proceeds in the opposite way to the technique of the previous chapter by implicity assuming a metric compatible connection and going on to derive the no-torsion constraint from a variational principle. This suggests that a variational approach which assumes *neither* metric compatibility *nor* zero torsion - i.e. a "true" Palatini variation, if you will - would be worth examining.

4.2 Einstein-Hilbert Action with Torsion

We begin then, by returning to our standard N-dimensional Einstein-Hilbert action,

$$S_{EH} = \int d^N x \sqrt{-g} R(\Gamma) \tag{4.1}$$

where we generally no longer have a symmetric affine connection and thus define the torsion tensor by

$$T_{\mu}{}^{\epsilon}{}_{\nu} := \Gamma_{\mu}{}^{\epsilon}{}_{\nu} - \Gamma_{\nu}{}^{\epsilon}{}_{\mu} \tag{4.2}$$

Owing to the generally non-symmetric nature of the connection, we now note that total derivative terms of the form

$$\int d^N x \, \nabla_\lambda \left(\sqrt{-g} X^\lambda \right) \tag{4.3}$$

for some vector X^λ, can no longer be ignored, but rather instead result in a net contribution (modulo the Gaussian term) of:

$$\int d^N x \sqrt{-g} X^\lambda (T_{\rho}{}^{\rho}{}_{\lambda}) \tag{4.4}$$

Variation of (4.1) with respect to Γ results in

$$T_{\lambda}{}^{\mu\nu} - \nabla_\lambda g^{\mu\nu} + g^{\mu\nu} \left[T_{\sigma}{}^{\sigma}{}_{\lambda} - V_\lambda \right] + \delta^{\mu}_{\lambda} \left[V^\nu + Z^\nu - T_{\sigma}{}^{\sigma\nu} \right] = 0 \tag{4.5}$$

Tracing and contracting (4.5) yields the constraints:

$$V^\lambda = \left(\frac{N}{N-1} \right) T_{\rho}{}^{\rho\lambda} \tag{4.6}$$

and

$$Z^\lambda = \left(\frac{2}{1-N} \right) T_{\rho}{}^{\rho\lambda} \tag{4.7}$$

Substitution of (4.6) and (4.7) into (4.5) eventually yields the following explicit expression for the connection:

$$\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\} + \left(\frac{1}{2(1-N)} \right) \left[\delta^{\epsilon}_{\nu} (T_{\rho}{}^{\rho}{}_{\mu}) + \delta^{\epsilon}_{\mu} (T_{\rho}{}^{\rho}{}_{\nu}) \right] + \frac{1}{2} T_{\mu}{}^{\epsilon}{}_{\nu} \tag{4.8}$$

The metric variation is, of course, unchanged, giving as before:

$$G_{(\mu\nu)}(\Gamma) = 0 \tag{4.9}$$

Combining (4.9) with (4.8) we get a final expression for the dynamics of the Einstein-Hilbert action for non-zero torsion:

$$\begin{aligned} G_{\mu\nu}(\{\}) &= \left[\frac{1}{4(N-1)}\right]\left(T_\sigma{}^\sigma{}_\mu T_\rho{}^\rho{}_\nu - \frac{1}{2}g_{\mu\nu}T_\rho{}^{\rho\epsilon}T_\rho{}^\rho{}_\epsilon\right) \\ &+\frac{1}{4}\left[T_\mu{}^\lambda{}_\eta T_\lambda{}^\eta{}_\nu - \frac{1}{2}g_{\mu\nu}T_\epsilon{}^\lambda{}_\eta T_\lambda{}^{\eta\epsilon}\right] \end{aligned} \tag{4.10}$$

A few points are worth emphasizing here. First, the Palatini variation (here manifested explicitly by equation (4.8)) now gives a connection where the torsion terms explicitly contribute to the *symmetric* part of the connection, via the term $\left(\frac{1}{2(1-N)}\right)\left[\delta^\epsilon_\mu(T_\sigma{}^\sigma{}_\nu) + \delta^\epsilon_\nu(T_\rho{}^\rho{}_\mu)\right]$. That is, a Palatini variation of this action does not allow one to express a torsion-laden connection in the usual way as:

$$\Gamma_\mu{}^\epsilon{}_\nu = \{_\mu{}^\epsilon{}_\nu\} + f(T)^\epsilon_{[\mu\nu]} \tag{4.11}$$

where $f(T)^\epsilon_{[\mu\nu]}$ represents the antisymmetric part of the connection.

In addition, we have seen in equations (4.6) and (4.7) that a $T \neq 0$ Palatini variation of the Einstein-Hilbert action allows us to explicitly express both non-metricity factors V_λ and Z_λ in terms of the Torsion vector, $T_\rho{}^{\rho\lambda}$, and thus each in terms of the other (i.e. $V_\lambda = -\left(\frac{N}{2}\right)Z_\lambda$).

4.3 Extended Einstein-Hilbert Action with Torsion

We now regard our Extended Action of the previous chapter and apply to it a Palatini variation whilst dropping the no-torsion requirement.

We thus begin again with the action:

$$S_{EHE} = \int d^N x\sqrt{-g}[R(\Gamma) + H(\nabla_\rho g_{\alpha\beta})(\nabla^\rho g^{\alpha\beta}) + IV^2 + \\ +J(\nabla_\rho g_{\alpha\beta})(\nabla^\alpha g^{\rho\beta}) + KV\cdot Z + LZ^2] \quad (4.12)$$

In an analogous fashion to (3.43), we can now assume a connection of the form:

$$\Gamma_\mu{}^\epsilon{}_\nu = \{_\mu{}^\epsilon{}_\nu\} + S_\mu{}^\epsilon{}_\nu + A_\mu{}^\epsilon{}_\nu \quad (4.13)$$

for $S_\mu{}^\epsilon{}_\nu$, $A_\mu{}^\epsilon{}_\nu$ respectively symmetric and anti-symmetric tensors in μ and ν.

Therefore we now have the relationships:

$$Z^\lambda = S_\sigma{}^{\sigma\lambda} + S_\sigma{}^{\lambda\sigma} + A_\sigma{}^{\sigma\lambda} \quad (4.14)$$

and

$$V_\lambda = A^\epsilon{}_{\epsilon\lambda} - S^\epsilon{}_{\epsilon\lambda} \quad (4.15)$$

Hence unlike the torsion-free case, V_λ and Z_λ now reflect *three* independent quantities: $S_\sigma{}^\sigma{}_\lambda, S_{\sigma\lambda}{}^\sigma, A_{\lambda\sigma}{}^\sigma$, but now we supplement (4.14) and (4.15) by:

$$T_{\lambda\sigma}{}^\sigma = 2A_{\lambda\rho}{}^\rho \quad (4.16)$$

. Therefore $S_\sigma{}^\sigma{}_\lambda, S_{\sigma\lambda}{}^\sigma, A_{\lambda\sigma}{}^\sigma$ can be represented by V_λ, Z_λ and $T_{\lambda\rho}{}^\rho$.

A Palatini variation thus yields, for $T \neq 0$:

$$T_\lambda{}^{\mu\nu} - \nabla_\lambda g^{\mu\nu} + g^{\mu\nu}[T_\sigma{}^\sigma{}_\lambda - V_\lambda] + \delta^\mu_\lambda[V^\nu + Z^\nu - T_\sigma{}^{\sigma\nu}] \\ +4H[g^{\nu\sigma}\nabla^\mu g_{\lambda\sigma}] - 2I(\delta^\nu_\lambda V^\mu) + 2J[g^{\mu\sigma}g^{\epsilon\nu}\nabla_\epsilon g_{\sigma\lambda} - \nabla_\lambda g^{\mu\nu}] \\ +K[V_\lambda g^{\mu\nu} + \delta^\mu_\lambda V^\nu - \delta^\nu_\lambda Z^\mu] + 2L[\delta^\mu_\lambda Z^\nu + Z_\lambda g^{\mu\nu}] = 0 \quad (4.17)$$

Owing to the fact that we are now dealing with a generally non-symmetric connection, tracing and contracting (4.17) yields *three* independent equations:

$$(2-N)T_\rho{}^{\rho\lambda} + [(N-1) - 2I + 4J + (N+1)K]\,V^\lambda \\ +[(N-1) - 4H - 2J - K + 2(N+1)L]\,Z^\lambda = 0 \quad (4.18)$$

$$[8H - 2NI + 2K]\, V^{\lambda} + [4L - 4J - NK]\, Z^{\lambda} = 0 \tag{4.19}$$

and

$$(N-2)T_{\rho}{}^{\rho\lambda} + PV^{\lambda} + QZ^{\lambda} = 0 \tag{4.20}$$

where P, Q are the same combination of functions of H, I, J, K, L as (3.47) of the previous chapter. We note that (4.18) can be combined with (4.19) to give:

$$(2-N)T_{\rho}{}^{\rho\lambda} + RV^{\lambda} + SZ^{\lambda} = 0 \tag{4.21}$$

We see clearly here that now the non-degeneracy of Ω, $PS - QR \neq 0$ no longer implies that $V_{\lambda} = Z_{\lambda} = 0$, but rather one has to examine the determinant of the 3×3 matrix Λ, defined by:

$$\Lambda := \begin{vmatrix} (2-N) & \kappa & \iota \\ 0 & \vartheta & \varrho \\ (2-N) & P & Q \end{vmatrix} \tag{4.22}$$

where

$$\kappa := (N-1) - 2I + 4J + (N+1)K \tag{4.23}$$

$$\iota := (N-1) - 4H - 2J - K + 2(N+1)L \tag{4.24}$$

$$\vartheta := 8H - 2NI + 2K \tag{4.25}$$

$$\varrho := 4L - 4J - NK \tag{4.26}$$

and where, as previously mentioned, we have the relationships

$$\kappa + \vartheta = R \tag{4.27}$$

$$\iota + \varrho = S \tag{4.28}$$

Now, in contrast to our earlier case of Ω, we have *four* possibilities to consider for Λ: rank 3, rank 2, rank 1 and trivial.

A quick inspection of Λ yields that it is necessarily non-trivial for $N \neq 2$, and hence there is no analogous "maximally symmetric" N-dimensional set of values for H, I, J, K, L when the torsion tensor is generally non-zero. Meanwhile, for the cases of Λ of rank 2 and 1, we can see that, analogously to the torsion-free case, the redundancy manifested by the non-invertibility of Λ will generate various weak constraints on $S_\mu{}^\epsilon{}_\nu$ and $A_\mu{}^\epsilon{}_\nu$ in terms of their various contracted and traced quantities. Meanwhile, if Λ is invertible (rank 3), we find the usual restriction:

$$T_\rho{}^{\rho\lambda} = V^\lambda = Z^\lambda = 0 \tag{4.29}$$

To get an idea of the structure of the results, we carry out a general calculation for Λ of rank 2 (i.e. the same rank as one finds from the Einstein-Hilbert action). In this case we can express both V_λ and Z_λ in terms of the torsion vector, $T_\lambda{}^\sigma{}_\sigma$ (i.e. re-express $S_\sigma{}^\sigma{}_\lambda$ and $S_{\sigma\lambda}{}^\sigma$ in terms of $A_{\lambda\sigma}{}^\sigma$).

We proceed, then, and simplify (4.17) to give something of the form:

$$A \, \nabla_\epsilon \, g_{\alpha\beta} + B \, \nabla_\alpha \, g_{\beta\epsilon} + C \, \nabla_\beta \, g_{\epsilon\alpha} = \chi_{\epsilon\alpha\beta} \tag{4.30}$$

where

$$A := 1 + 2J \tag{4.31}$$

$$B = 4H \tag{4.32}$$

$$C = 2J \tag{4.33}$$

and $\chi_{\epsilon\alpha\beta}$ is some complicated function of the metric, torsion vector $T_\sigma{}^\sigma{}_\lambda$ and torsion tensor (see Appendix C) Of course, in the rank 3 case, $\chi_{\epsilon\alpha\beta}$ is solely a function of the metric tensor and the torsion tensor, while in the rank 2 case, $\chi_{\epsilon\alpha\beta}$ becomes a function of V_λ (or Z_λ) as well.

At this point we break up the analysis into two distinct sections, depending on whether or not C (i.e. J) is non-zero.

If C (i.e. J) $\neq 0$, then, permuting (4.30) yields an undetermined connection iff:

$$(AC - B^2)(C^2 - AB + AC - B^2) = (BC - A^2)(C^2 - AB + BC - A^2) \qquad (4.34)$$

which is the $T \neq 0$ generalization of the $T = 0$ indeterminacy constraints (3.58) and (3.59) of Chapter 3.

Just as we found in the torsion-free case, with constraints (3.61) and (3.63) following from actualization of the torsion-free indeterminacy constraints, (3.58), (3.59), we note that actualization of our more generalized indeterminacy constraint will similarly lead to various constraints among $S_\mu{}^\epsilon{}_\nu, A_\mu{}^\epsilon{}_\nu$ and $S_\sigma{}^\sigma{}_\lambda, S_{\rho\lambda}{}^\rho$ and $A_\sigma{}^\sigma{}_\lambda$ in terms of H, I, J, K, L and N, the details of which will in turn depend on the particular rank of Ω.

Meanwhile, if equation (4.34) does *not* hold, we can therefore solve explicitly for the connection, as before. We find, after much manipulation[1], that our connection is of the form:

$$\begin{aligned} \Gamma_\mu{}^\epsilon{}_\nu &= \{_\mu{}^\epsilon{}_\nu\} + \tilde{F}\left[g_{\mu\nu}(\tilde{d} - \tilde{e} - \tilde{f})T_\rho^{\ \rho\epsilon} - \tilde{d}\left(\delta^\epsilon_\mu\, T_\rho{}^\rho{}_\nu + \delta^\epsilon_\nu\, T_\sigma{}^\sigma{}_\mu\right)\right] \\ &\quad + \left[\tilde{F}\tilde{s} - \frac{1}{2}\right]\left[T_{\mu\nu}^{\ \ \epsilon} + T_{\nu\mu}^{\ \ \epsilon}\right] + \frac{1}{2}T_\mu{}^\epsilon{}_\nu \end{aligned} \qquad (4.35)$$

where $\tilde{F}, \tilde{d}, \tilde{e}, \tilde{f}, \tilde{s}$ are all in turn complicated functions of H, I, J, K, L.

Meanwhile, for $J = 0$ the analysis simplifies somewhat. Here the indeterminacy constraint becomes simply $H \neq \pm\frac{1}{4}$ (i.e. $\Gamma_\mu{}^\epsilon{}_\nu$ is explicitly soluble *unless* $H = \pm\frac{1}{4}$), where in this case $\Gamma_\mu{}^\epsilon{}_\nu$ takes the explicit form:

$$\begin{aligned} \Gamma_\mu{}^\epsilon{}_\nu &= \xi\left[g_{\mu\nu}T_\sigma^{\ \sigma\epsilon}\left([\tilde{b} - \tilde{c}](1 + 4H) + \tilde{a}(4H - 1)\right) + (4H\tilde{c} - \tilde{b})\left(\delta^\epsilon_\mu\, T_\sigma{}^\sigma{}_\nu + \delta^\epsilon\nu\, T_\rho{}^\rho{}_\mu\right)\right] \\ &\quad \{_\mu{}^\epsilon{}_\nu\} + (\xi - \frac{1}{2})\left[T_{\mu\nu}^{\ \ \epsilon} + T_{\nu\mu}^{\ \ \epsilon}\right] + \frac{1}{2}T_\mu{}^\epsilon{}_\nu \end{aligned} \qquad (4.36)$$

[1]See Appendix C

where

$$\xi := \left[\frac{1}{2(1+4H)(1-4H)}\right] \tag{4.37}$$

and $\tilde{a}, \tilde{b}, \tilde{c}$ are (still) more complicated functions of H, I, J, K, L as shown in Appendix C.

Owing to the complicated nature of $\tilde{a}, \tilde{b}, ..., \tilde{f}, \tilde{F}$ etc., it is difficult to get an intuitive feel for the final results of the connection in terms of our original H, I, J, K, L parameters, but the general structure is readily apparent. As in the case for the Einstein-Hilbert action, we once again find that, given that the indeterminacy constraint is satisfied, we end up with a connection of the form:

$$\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\} + [f(g,T)]^{\epsilon}{}_{(\mu\nu)} + \frac{1}{2}T_{[\mu}{}^{\epsilon}{}_{\nu]} \tag{4.38}$$

where the $[f(g,T)]^{\epsilon}{}_{(\mu\nu)}$ term is a combination of metric, torsion tensor and torsion vector terms together with various complicated functions of H, I, J, K, L such that the entire term is symmetric in its μ and ν indices.

4.3.1 Final Dynamics

The procedure for calculating the final dynamics of our potentially torsion-laden system is fairly straightforward: we substitute the derived explicit form for the connection, i.e. (4.35) or (4.36), into the action (4.12) and vary the resultant expression with respect to the metric tensor. In reality, however, we find that such a programme becomes highly complicated not only because of the complicated nature of the various parameters $\tilde{a}...\tilde{F}$ in (4.35) and (4.36), but more significantly because these expressions themselves implicitly or explicitly involve the metric tensor and one must be extremely careful to keep all the metric terms separate so as to successfully vary with respect to them when the time comes.

At the end of an admittedly very long day, however, one comes to a final dynamical expression of the following form:

$$G_{\mu\nu}(\{\}) + S_{(\mu\nu)} + W^{\rho\ \epsilon}_{\ \rho\ \epsilon(\mu\nu)} + W^{\rho}_{\ \rho(\mu\nu)\ \epsilon}^{\ \ \ \ \ \ \ \epsilon} - W_{(\mu\nu)\ \rho\ \epsilon}^{\ \ \ \ \ \rho\ \epsilon} - \frac{1}{2}g_{\mu\nu}(S+W) = 0 \tag{4.39}$$

where $S_{\alpha\beta}$ and $W^{\rho\gamma}_{\ \ \eta\lambda\alpha\beta}$ are some complicated tensors comprised solely of torsion tensors and their various associated contractions (See Appendix D).

4.4 Extra Torsion Terms to the Action

Finally, we note that dropping the $T = 0$ requirement from our original assumptions which eventually led to the extended action, S_{EHE} of Chapter 2 now allows us to include more terms to such an action consistent with the previous constraints of being only up to second order in derivatives and/or field variables.

That is, we now find a possible six additional independent terms involving the torsion tensor. We examine the dynamics of this system, where for simplicity we regard it independently from our previous S_{EHE} action. We consider the combined dynamics in the next section.

The relevant action for this section, then, is of the form

$$\begin{aligned} S_T &= \int d^N x\sqrt{-g}[R(\Gamma) + a(\nabla^{\mu}g^{\epsilon\nu})T_{\mu\epsilon\nu} + bT_{\sigma}{}^{\sigma}{}_{\lambda}Z^{\lambda} + cT_{\rho}{}^{\rho}{}_{\lambda}V^{\lambda} \\ &\qquad + dT_{\epsilon\sigma}^{\ \ \sigma}\,T^{\epsilon\rho}_{\ \ \rho} + eT_{\lambda\epsilon\rho}\,T^{\lambda\rho\epsilon} + fT_{\lambda\epsilon\rho}\,T^{\lambda\epsilon\rho}] \end{aligned} \tag{4.40}$$

We see that the d, e, f terms arise solely from the various scalar contractions of the torsion tensor, $T_{\mu}{}^{\epsilon}{}_{\nu}$ combined with the metric tensor, while the a, b, c terms occur through potential torsion-non-metricity contributions.

Variation of S_T with respect to the connection leads to the following expression:

$$\begin{aligned}
\delta^\mu_\epsilon \left[(1+b)V^\nu + (1+c)Z^\nu + (c+2d-1)T_\rho{}^{\rho\nu}\right] - \delta^\nu_\epsilon \left[bV^\mu + cZ^\mu + (b+2d)T_\rho{}^{\rho\mu}\right] & \\
+ g^{\mu\nu}\left[(1+c)T_\sigma{}^\sigma{}_\epsilon - V_\epsilon\right] + (1+2e)\,T_\epsilon{}^{\mu\nu} + (a+4f)\,T^\mu{}_\epsilon{}^\nu + (a+2e)\,T^{\mu\nu}{}_\epsilon &= 0 \\
- \nabla_\epsilon\, g^{\mu\nu} + a\left[g_{\rho\epsilon}\left(\nabla^\mu g^{\rho\nu} - \nabla^\nu g^{\rho\mu}\right)\right] &
\end{aligned} \tag{4.41}$$

We again trace and contract (4.41) to obtain relations for V^λ and Z^λ in terms of $T_\rho{}^{\rho\lambda}$, thereby finding the three relationships:

$$\begin{aligned}
\left[(N-1)(2d-1) + (N+1)c + (1+b+2e+a+4f)\right] T_\rho{}^{\rho\lambda} & \\
+ \left[(N-1)(1+b) + 2a\right] V^\lambda + \left[(N-1)(1+c) + a\right] Z^\lambda &= 0
\end{aligned} \tag{4.42}$$

$$\begin{aligned}
\left[2d(1-N) + 2c - Nb - (2e+4f+2a)\right] T_\rho{}^{\rho\lambda} & \\
+ \left[b(1-N) - 2a\right] V^\lambda + \left[c(1-N) - a\right] Z^\lambda &= 0
\end{aligned} \tag{4.43}$$

$$\left[(N-2) + c(N+2) + (a-b)\right] T_\rho{}^{\rho\lambda} + (3-N)V^\lambda + Z^\lambda = 0 \tag{4.44}$$

which may be rewritten as:

$$\zeta\, T_\rho{}^{\rho\lambda} + \alpha\, V^\lambda + \beta\, Z^\lambda = 0 \tag{4.45}$$

$$\bar\zeta\, T_\rho{}^{\rho\lambda} + \bar\alpha\, V^\lambda + \bar\beta\, Z^\lambda = 0 \tag{4.46}$$

$$\hat\zeta\, T_\rho{}^{\rho\lambda} + \hat\alpha\, V^\lambda + \hat\beta\, Z^\lambda = 0 \tag{4.47}$$

If we define

$$\Delta := \begin{vmatrix} \zeta & \alpha & \beta \\ \bar\zeta & \bar\alpha & \bar\beta \\ \hat\zeta & \hat\alpha & \hat\beta \end{vmatrix} \tag{4.48}$$

and proceed as in the previous section, we again consider the relevant possibilities of Δ - i.e. rank 3, rank 2, rank 1 and trivial.

The $\hat{\beta}$ term ensures that Δ cannot be trivial; whereas, as before, the case of Δ of rank 1 amounts to a weak constraint, allowing one to express one of $V_\lambda, Z_\lambda, T_\sigma{}^\sigma{}_\lambda$ in terms of the other 2; Δ of rank 2 amounts to a stronger constraint, allowing two of $V_\lambda, Z_\lambda, T_\sigma{}^\sigma{}_\lambda$ to be written in terms of the remaining one, and the non-degeneracy of Δ ensures that:

$$V_\lambda = Z_\lambda = T_\sigma{}^\sigma{}_\lambda = 0 \tag{4.49}$$

We see that for the special case of $a = b = c = d = e = f = 0$ as seen in Section 4.2 above, the matrix Δ clearly *is* of rank 2 and (4.42)-(4.44) reduce to the previously found relationships (4.6), (4.7):

$$V^\lambda = \left[\frac{N}{N-1}\right] T_\rho{}^{\rho\lambda} \tag{4.50}$$

and

$$Z^\lambda = \left[\frac{2}{1-N}\right] T_\rho{}^{\rho\lambda} \tag{4.51}$$

Once again we push forth the analysis for the case of rank 2, thereby enabling us to eventually express the connection explicitly solely in terms of the torsion tensor and the torsion vector.

Thus we say that (4.42)-(4.44) reduce to expressions of the form:

$$V^\lambda := \Theta\, T_\rho{}^{\rho\lambda} \tag{4.52}$$

$$Z^\lambda := \Upsilon\, T_\rho{}^{\rho\lambda} \tag{4.53}$$

where Θ and Υ are functions of a, b, c, d, e, f and the dimensionality N.

This transforms the general expression for the connection variation (4.41) to:

$$\nabla_\epsilon g^{\mu\nu} + a\left[g_{\rho\epsilon}\left(\nabla^\nu g^{\rho\mu} - \nabla^\mu g^{\rho\nu}\right)\right] = \delta^\mu_\epsilon\ A(T_\rho{}^{\rho\nu}) - \delta^\nu_\epsilon\ B(T_\rho{}^{\rho\mu}) + g^{\mu\nu}\ C(T_\sigma{}^\sigma{}_\epsilon) + (1+2e)T_\epsilon{}^{\mu\nu} + (a+4f)T^\mu_\epsilon{}^\nu + (a+2e)T^{\mu\nu}_\epsilon \tag{4.54}$$

where we define

$$A := (1+b)\Theta + (1+c)\Upsilon + (c+2d-1) \tag{4.55}$$

$$B := b\Theta + c\Upsilon + (b+2d) \tag{4.56}$$

and

$$C := (1+c) - \Theta \tag{4.57}$$

As before, for the rank 1 case, the left hand side of (4.54) is unaltered, while the right hand side generalizes to include terms of the extra degree of freedom (i.e. V_λ or Z_λ; or $S_\sigma{}^\sigma{}_\lambda$ or $S_{\sigma\lambda}{}^\sigma$). Of course in the rank 3 case the right hand side of (4.54) is solely a function of the torsion tensor and the metric tensor. Equation (4.54) is thus the pure torsion term analogy to (4.30).

Once again, we find that analysis of (4.54) gives rise to an indeterminacy condition, this time dependent solely on a, as one might expect.

First we consider the case of $a = 0$:

For $a = 0$ we find that manipulation of equation (4.54) eventually yields an explicit form of the connection as follows:

$$\Gamma_\mu{}^\epsilon{}_\nu = \{_\mu{}^\epsilon{}_\nu\} + \frac{1}{2}(A+B-C)g_{\mu\nu}T_\rho{}^{\rho\epsilon} + \frac{1}{2}C\left[\delta^\epsilon_\mu T_\sigma{}^\sigma{}_\nu + \delta^\epsilon_\nu T_\rho{}^\rho{}_\mu\right] + \frac{1}{2}T_\mu{}^\epsilon{}_\nu \tag{4.58}$$

For $a \neq 0$, however, life is somewhat trickier. We define

$$\Phi_{\epsilon\alpha\beta} := Ag_{\epsilon\alpha}T_\beta{}^\sigma{}_\sigma + Bg_{\epsilon\beta}T_\alpha{}^\rho{}_\rho + Cg_{\alpha\beta}T_\epsilon{}^\lambda{}_\lambda - (1+2e)T_{\epsilon\alpha\beta} + (a+4f)T_{\beta\epsilon\alpha} + (a+2e)T_{\epsilon\beta\alpha} \tag{4.59}$$

and

$$\Xi_{\alpha\beta\epsilon} := \left[\frac{1}{(1-2a)(1+a)}\right]\left[\left(\frac{1-a}{a}\right)\Phi_{\alpha\beta\epsilon} + \Phi_{\epsilon\alpha\beta} + \Phi_{\beta\epsilon\alpha}\right] \tag{4.60}$$

and note that we have an indeterminacy condition at $a = \frac{1}{2}, a = -1$. For $a \neq \frac{1}{2}$ or $a \neq -1$, we eventually find that our connection has the following form

$$\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\} + \frac{1}{2}g^{\lambda\epsilon}\left[(\Xi_{\lambda\mu\nu} + T_{\lambda\mu\nu}) - (\Xi_{\mu\nu\lambda} + T_{\mu\nu\lambda}) - (\Xi_{\nu\lambda\mu} + T_{\nu\lambda\mu})\right], \tag{4.61}$$

which we can rewrite as:

$$\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\} + Y_{\mu}{}^{\epsilon}{}_{\nu} \tag{4.62}$$

where $Y_{\mu}{}^{\epsilon}{}_{\nu}$ is defined by

$$Y_{\mu}{}^{\epsilon}{}_{\nu} = \hat{p}\delta^{\epsilon}_{\mu}T_{\sigma}{}^{\sigma}{}_{\nu} + \hat{q}\delta^{\epsilon}_{\nu}T_{\rho}{}^{\rho}{}_{\mu} + \hat{r}g_{\mu\nu}T_{\rho}{}^{\rho\epsilon} + \hat{s}T^{\epsilon}{}_{\mu\nu} + \hat{t}T^{\epsilon}{}_{\nu\mu} + \hat{u}T_{\mu}{}^{\epsilon}{}_{\nu} \tag{4.63}$$

for $\hat{p}, \hat{q}, \hat{r}, \hat{s}, \hat{t}, \hat{u}$ some complicated functions of a, b, c, d, e, f, explicitly found by the recursive utilization of (4.52),(4.53),(4.55)-(4.57),(4.59), (4.60) and (4.61).

Of course, if $a = \frac{1}{2}, -1$ we cannot solve explicitly for the connection and we instead are reduced to finding some constraint relating $S_{\mu}{}^{\epsilon}{}_{\nu}, A_{\mu}{}^{\epsilon}{}_{\nu}$ and (since Δ is assumed to be singly degenerate here) $A_{\sigma}{}^{\sigma}{}_{\lambda}$, say.

In general, though, for $a \neq \frac{1}{2}, -1$ one can say that in the singly degenerate (or non-degenerate) case, we can express our connection, Γ, explicitly in tersm of a, b, c, d, e, f, the metric tensor, $g_{\mu\nu}$ and the torsion tensor, $T_{\mu}{}^{\epsilon}{}_{\nu}$. Substition of this connection, (4.61) or (4.58) into the action (4.40), and varying with with respect to the metric, $g_{\alpha\beta}$, eventually allows us to obtain a final dynamical result of the form

$$G_{\mu\nu}(\{\}) + \hat{\mathcal{S}}_{\mu\nu} + \hat{\mathcal{W}}^{\rho}{}_{\rho}{}^{\epsilon}{}_{\epsilon(\mu\nu)} + \hat{\mathcal{W}}^{\rho}{}_{\rho(\mu\nu)}{}^{\epsilon}{}_{\epsilon} - \hat{\mathcal{W}}_{(\mu\nu)}{}^{\rho\epsilon}{}_{\rho\epsilon} - \frac{1}{2}g_{\mu\nu}(\hat{\mathcal{S}} + \hat{\mathcal{W}}) = 0 \tag{4.64}$$

where $\hat{\mathcal{S}}_{\alpha\beta}$ and $\hat{\mathcal{W}}^{\rho\gamma}{}_{\eta\lambda\alpha\beta}$ are again complicated tensors solely comprised of $T_{\mu}{}^{\epsilon}{}_{\nu}$ and its contractions, but different from $\mathcal{S}_{\alpha\beta}$ and $\mathcal{W}^{\rho\gamma}{}_{\eta\lambda\alpha\beta}$, previously defined for the S_{EHE}

dynamics of Section(4.3) above and solved for explicitly (if recursively) in Appendix D. Of course as mentioned above, if Δ is doubly degenerate, an explicit solution of Γ in terms of the torsion tensor and torsion vector is no longer possible, as Ξ above will now explicitly depend on V_λ or Z_λ (i.e. $S_\sigma{}^\sigma{}_\lambda$ or $S_{\sigma\lambda}{}^\sigma$).

4.5 Summary

We now briefly summarize this chapter's results while simultaneously sketching out an argument for the form of the results of a Palatini variation of the generalized torsion-laden action which arises from the *combined* effects of Section 4.3 and 4.4, i.e. the generalized action

$$S_C = \int d^N x \sqrt{-g}\ [R(\Gamma) + H(\nabla_\rho g_{\alpha\beta})(\nabla^\rho g^{\alpha\beta}) + IV^2 \tag{4.65}$$

$$+J(\nabla_\rho g_{\alpha\beta})(\nabla^\alpha g^{\rho\beta}) + KV \cdot Z + LZ^2$$

$$+a(\nabla^\mu g^{\epsilon\nu})T_{\mu\epsilon\nu} + bT_\sigma{}^\sigma{}_\lambda V^\lambda + cT_\rho{}^\rho{}_\lambda Z^\lambda$$

$$+dT_{\epsilon\sigma}{}^\sigma T^{\epsilon\rho}{}_\rho + eT_{\lambda\epsilon\rho}T^{\lambda\rho\epsilon} + fT_{\lambda\epsilon\rho}T^{\lambda\epsilon\rho}]$$

Variation with respect to Γ, leads to three independent relations between $T_\rho{}^{\rho\lambda}, V^\lambda$ and Z^λ (or, likewise, three independent relations between $S_\sigma{}^\sigma{}_\lambda, S_{\sigma\lambda}{}^\sigma$ and $A_\sigma{}^\sigma{}_\lambda$):

$$\tau\, T_\rho{}^{\rho\lambda} + \epsilon\, V^\lambda + \gamma\, Z^\lambda = 0 \tag{4.66}$$

$$\bar{\tau}\, T_\rho{}^{\rho\lambda} + \bar{\epsilon}\, V^\lambda + \bar{\gamma}\, Z^\lambda = 0 \tag{4.67}$$

$$\hat{\tau}\, T_\rho{}^{\rho\lambda} + \hat{\epsilon}\, V^\lambda + \hat{\gamma}\, Z^\lambda = 0 \tag{4.68}$$

which can be represented by the matrix:

$$\Sigma = \begin{vmatrix} \tau & \epsilon & \gamma \\ \bar{\tau} & \bar{\epsilon} & \bar{\gamma} \\ \hat{\tau} & \hat{\epsilon} & \hat{\gamma} \end{vmatrix} \tag{4.69}$$

where

$$\Sigma := \Lambda + \Delta \tag{4.70}$$

Σ can either be trivial, rank 1, rank 2 or rank 3, leading one to derive various constraints between V_λ, Z_λ and $T_\sigma{}^\sigma{}_\lambda$ depending on the rank of Σ.

Substitution of the relevant rank of Σ into the connection variation equation gives the following general scenario:

$$\hat{A} \bigtriangledown_\epsilon g_{\alpha\beta} + \hat{B} \bigtriangledown_\alpha g_{\beta\epsilon} + \hat{C} \bigtriangledown_\beta g_{\epsilon\alpha} = \hat{\chi}_{\epsilon\alpha\beta} \tag{4.71}$$

where

$$\hat{A} := 1 + 2J \tag{4.72}$$

$$\hat{B} := 4H - a \tag{4.73}$$

$$\hat{C} := 2J + a \tag{4.74}$$

and $\hat{\chi}_{\epsilon\alpha\beta}$ is some complicated function of $g_{\mu\nu}, T_\mu{}^\epsilon{}_\nu$ and either none, one, two or all three of $V_\lambda, Z_\lambda, T_\sigma{}^\sigma{}_\lambda$ (or, alternatively, $S_\sigma{}^\sigma{}_\lambda, S_{\sigma\lambda}{}^\sigma, A_\sigma{}^\sigma{}_\lambda$) depending on whether Σ is rank 3, rank 2, rank 1 or trivial, respectively.

For $\hat{C} \neq 0$ we find that (4.71) yields an undetermined connection iff:

$$(\hat{A}\hat{C} - \hat{B}^2)(\hat{C}^2 - \hat{A}\hat{B} + \hat{A}\hat{C} - \hat{B}^2) = (\hat{B}\hat{C} - \hat{A}2)(\hat{C}^2 - \hat{A}\hat{B} + \hat{B}\hat{C} - \hat{A}^2) \tag{4.75}$$

while for $\hat{C} = 0$, we have the following indeterminacy condition:

$$\hat{A}^2 = \hat{B}^2 \tag{4.76}$$

If (4.76) or (4.75) do *not* apply, we can solve for the connection explicitly in terms of the metric, torsion tensor and remaining variables unconstrained by the above degeneracy of Σ, otherwise we are left merely with weak constraints on relations between $S_\mu{}^\epsilon{}_\nu, A_\mu{}^\epsilon{}_\nu$ and $S_\sigma{}^\sigma{}_\lambda, S_{\sigma\lambda}{}^\sigma$ and $A_\sigma{}^\sigma{}_\lambda$.

Furthermore, the indeterminacy constraints do not apply, and if Σ is rank 3 or rank 2 then we can express our connection explicitly in terms of only the metric tensor, torsion tensor and its contraction (i.e. the torsion vector). We see that in that case we find an explicit solution for the connection of the general form:

$$\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\} + \bar{p}\,\delta^{\epsilon}_{\mu}(T_{\sigma}{}^{\sigma}{}_{\nu}) + \bar{q}\,\delta^{\epsilon}_{\nu}(T_{\rho}{}^{\rho}{}_{\mu}) + \bar{r}\,g_{\mu\nu}(T_{\rho}{}^{\rho\epsilon}) + \bar{s}\,T^{\epsilon}{}_{\mu\nu} + \bar{t}\,T^{\epsilon}{}_{\nu\mu} + \frac{1}{2}\,T_{\mu}{}^{\epsilon}{}_{\nu} \quad (4.77)$$

where $\bar{q}, \bar{r}, \bar{s}, \bar{t}, \bar{u}$ represent various complicated functions of $H, I, J, K, L, a, b, c, d, e, f$ obtained in the usual recursive manner.

Finally, substitution of (4.77) into (4.65) followed by varying with respect to the remaining dynamical variable, $g_{\alpha\beta}$, yields a final dynamical expression of the form:

$$G_{\mu\nu}(\{\}) + \bar{\mathcal{S}}_{(\mu\nu)} + \bar{\mathcal{W}}^{\rho}{}_{\rho}{}^{\epsilon}{}_{\epsilon(\mu\nu)} + \bar{\mathcal{W}}^{\rho}{}_{\rho(\mu\nu)}{}^{\epsilon}{}_{\epsilon} - \bar{\mathcal{W}}_{(\mu\nu)}{}^{\rho}{}_{\rho}{}^{\epsilon}{}_{\epsilon} - \frac{1}{2}g_{\mu\nu}(\bar{\mathcal{S}} + \bar{\mathcal{W}}) = 0 \qquad (4.78)$$

where $\bar{\mathcal{S}}_{\mu\nu}$ and $\bar{\mathcal{W}}^{\rho\gamma}{}_{\eta\lambda\alpha\beta}$ are again some complicated tensors comprised solely of torsion tensors and their various associated contractions (i.e. independent of the metric)[2].

We note that the final dynamics arising from this combined action modify the Einsteinian dynamics only by factors of T^2, that is, by factors second order in the torsion tensor.

Among those researchers concerned with the effects of torsion, there have been some who have sought a more dynamical role for the torsion tensor by examining a dynamical scenario containing covariant derivatives of the torsion tensor in addition to the Einstein tensor - i.e. by including terms of the form $\nabla_{\mu}(T_{\sigma}{}^{\sigma}{}_{\nu})$, say, on the left hand side of (4.78). It is clear from the above analysis that this does not occur in our case, where we have excluded any terms from the original action which are

[2]For the sake of simplicity, we have maintained our assumption that Σ is non-degenerate or singly degenerate here. Otherwise $\bar{\mathcal{S}}$ and $\bar{\mathcal{W}}$ would also be functions of V_{λ} and/or Z_{λ}.

greater than second order in the connection. However, if one removes this constraint and considers higher order connection terms, it begins to look like such "dynamical torsion" terms can arise from a Palatini analysis. It turns out that it is impossible to construct scalar quantities from $\nabla\lambda, T_{\mu}{}^{\epsilon}{}_{\nu}, g_{\alpha\beta}$ and $T_{\sigma}{}^{\sigma}{}_{\rho}$ to only third order in Γ. In order to pursue this line of inquiry, then, we are forced to go to fourth order in Γ. We thus find that a Palatini variation of the term

$$S_{4th} := \int d^N x \sqrt{-g} (\nabla_\sigma T_{\mu}{}^{\epsilon}{}_{\nu}) g^{\sigma\alpha} g^{\nu\beta} (T_{\alpha}{}^{\mu}{}_{\beta})(T_{\epsilon}{}^{\rho}{}_{\rho}), \tag{4.79}$$

say, leads to the following constraint:

$$\begin{aligned} &(T_{\alpha}{}^{\mu}{}_{\beta})(T_{\epsilon}{}^{\rho}{}_{\rho}) \left[g^{\sigma\alpha} g^{\nu\beta} T_{\lambda}{}^{\lambda}{}_{\sigma} - (V^\alpha + Z^\alpha) g^{\nu\beta} - \nabla^\alpha g^{\nu\beta} \right] \\ &\quad + g^{\mu\beta} \, \nabla_\sigma \left[(T_{\alpha}{}^{\nu}{}_{\beta})(T_{\epsilon}{}^{\rho}{}_{\rho}) \right] + g^{\mu\sigma} g^{\nu\rho} (T_{\eta}{}^{\xi}{}_{\xi})(\nabla_\sigma T_{\epsilon}{}^{\eta}{}_{\rho}) \\ &\quad\quad + \delta^\nu_\epsilon (\nabla_\sigma T_{\alpha}{}^{\mu}{}_{\beta})(T_{\rho}{}^{\alpha}{}_{\eta}) g^{\sigma\rho} g^{\beta\eta} - (\mu \Longleftrightarrow \nu) \quad = \quad 0 \end{aligned} \tag{4.80}$$

Although a rigorous analysis of such higher order terms has not, in fact, been done, the above constraint is highly suggestive of an eventual expression for the connection involving covariant derivatives of the torsion tensor and hence the eventual appearance of such terms in the final dynamical expression. For those whose interests lie in developing such a theory, it is conceivable that a fourth order generalized Palatini action might well be worth considering further.

Chapter 5

Geometrical Divertimento

5.1 Fibre Bundle Review

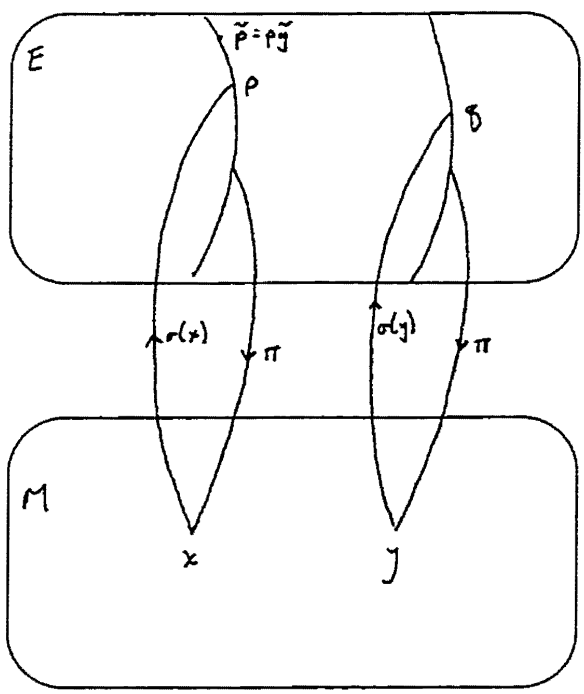
At this point we pause in our analysis of the Palatini procedure to briefly review[1] and clarify some methods of abstract geometry which will prove necessary to our cause in later chapters - specifically key elements of fibre bundle theory.

Fibre bundles have become fairly popular with physicists over the last several decades largely because of their utility in visualizing gauge theories. We begin with principal fibre bundles. Briefly put, a principal fibre bundle allows one to simultaneously view the physical space, M, referred to as the base space, and the bundle space E, where the bundle space generally reflects the symmetry group of the theory by associating with each point $x \in M$ a fibre in E diffeomorphic to some Lie Group $\mathcal{G}$, referred to as the gauge group or structure group.

More technically, for E some topological space which is equipped with some free right $\mathcal{G}$-action (where "free" here necessitates that all fibres are diffeomorphic to one another), π some smooth map from E to some other topological space M, we see that E can be grouped as sets of fibres, $\pi^{-1}(x)$, over each $x \in M$, where each fibre somehow represents the gauge freedom of our theory with any point on the fibre

[1]This brief review relies heavily upon [26]

Figure 5.1: Principal Fibre Bundle



over x, p, being related to another, q, via the action of the gauge group $\mathcal{G}$ - i.e. $\forall p, q \in \pi^{-1}(x) \, \exists \, g \in \mathcal{G} \ni p = q\,g$.

Meanwhile a cross-section, σ, a smooth map $\sigma : M \to E$ which associates with each $x \in M$ some unique $p \in \pi^{-1}(x)$ in the fibre over x amounts to, in this picture, a gauge choice.

Furthermore, we note that the so-called "Vertical Vector Fields" on E, $V(E)$, which arise from the action of $\mathcal{G}$ on E are directed "along the fibre"[2] and faithfully

[2]Owing to the above one-dimensional pictorial representation of the fibres, it becomes difficult to visualize more than one such direction for this vector field. A better model is one of an onion,

reflect the Lie Algebra, $\mathcal{L}(\mathcal{G})$, of $\mathcal{G}$ by the isomorphism:

$$
\begin{aligned}
\imath \ : \ & \mathcal{L}(\mathcal{G}) \rightarrow V(E) \\
& L \rightarrow X^L,
\end{aligned}
\tag{5.1}
$$

where X^L represents the vector field on E induced by acting on every element of p by the one-parameter subgroup of $\mathcal{G}$ which corresponds to the Lie Algebra element L. We note that vertical vector fields on E can be defined by the relation:

$$V_pE := \{\tau \in T_pE \ni \pi_\star \tau = 0\} \tag{5.2}$$

We can then define a *connection* on a principal bundle as a smooth assignment of a subspace (H_pE) $\forall p \in E$ such that

$$T_pE = V_pE \oplus H_pE \tag{5.3}$$

where H_pE is compatible with the action of $\mathcal{G}$ on E, i.e.

$$\delta_{g\star}(H_pE) = (H_{pg}E) \ \forall g \in \mathcal{G}, p \in E. \tag{5.4}$$

This above definition can be shown to be equivalent to defining an $\mathcal{L}(\mathcal{G})$- valued 1-form, A, where we have

$$A_p(\tau) := \imath^{-1}(ver(\tau))\forall \tau \in T_pE \tag{5.5}$$

where $\imath$ corresponds to the isomorphism defined above by equation (5.1)

Armed with a connection we can thus compare points in neighbouring fibres by the notion of parallel transport. Since $\pi_\star(V_pE) \equiv 0$, once we have defined a connection on our bundle, we not that we have an isomorphism given by

$$\pi_\star : H_pE \rightarrow T_{\pi(p)}M, \tag{5.6}$$

where each layer (shell) represents a fibre and where one can imagine many different such (vertical) vector fields.

which enables us to uniquely write the "horizontal lift" of any vector field (or thus any curve) in the base space M - where by "horizontal lift" we simply mean a unique vector field which is necessarily strictly horizontal (i.e. $\tau^{\dagger} \in H_pE \; \forall p$ over which τ is defined). We find that for any curve $\alpha : [a,b] \to M$ in M there is a unique horizontal lift $\alpha^{\dagger}$ of α in E for each "starting point" - i.e. $\forall\, p \in \pi^{-1}(\alpha(a))$. We then regard the *parallel transport* along α as the map between the two fibres $\pi^{-1}(\alpha(a))$ and $\pi^{-1}(\alpha(b)$ defined by:

$$\begin{aligned} p_{II} \;:=\; & \pi^{-1}(\alpha(a)) \to \pi^{-1}(\alpha(b)) \\ & p \to \alpha^{\dagger}(b) \end{aligned} \tag{5.7}$$

We note that the exterior covariant derivative, $\mathbf{D}$, of any k-form, ξ, on E is defined by

$$\mathbf{D}\xi\ [X_1, X_2, ..., X_{k+1}] := d\xi\ [hor(X_1), hor(X_2), ..., hor(X_k)] \tag{5.8}$$

For $\mathbf{G} := \mathbf{D}\ w$ the exterior covariant derivative of the connection 1-form, w, we have, for any pair of vector fields X, Y on E, the Cartan Structure Equation:

$$\mathbf{G}(X,Y) = dA(X,Y) + [A(X), A(Y)] \tag{5.9}$$

and the Bianchi identity:

$$\mathbf{D}\ \mathbf{G} \equiv 0 \tag{5.10}$$

It should be emphasized that these quantities are written here in their most general principal bundle form. In order to obtain the usual connection 1-forms and curvature 2-forms which take arguments in the base space M, we must pull back the respective principal bundle quantities using some local cross-section. Hence we define:

$$R := \sigma^{\star}\ \mathbf{G} \tag{5.11}$$

and

$$w := \sigma^{\star}\ A \tag{5.12}$$

and thus can express (5.9) and (5.10) in their more familiar local form

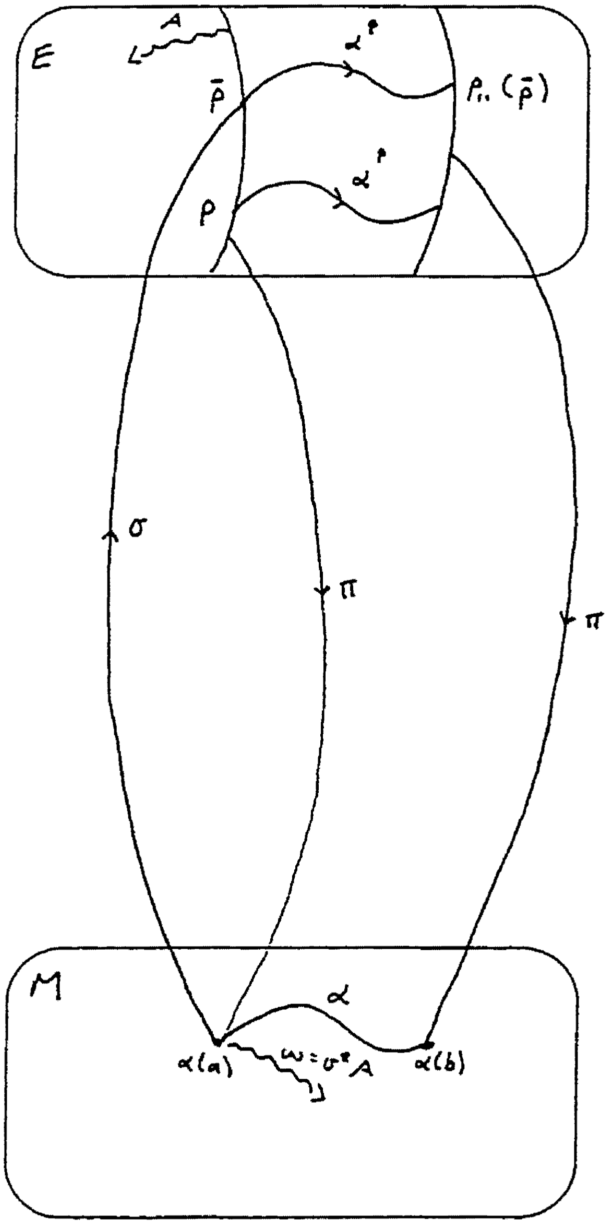
$$R = dw + w \wedge w \tag{5.13}$$

and

$$\mathbf{D}R = 0 \tag{5.14}$$

For reasons which we shall shortly see, we find that covariant derivatives are only defined for associated vector bundles, which necessitates a short digression in this direction.

Figure 5.2: Parallel Transport in Principal Fibre Bundles



We begin with (E, π, M), a principal $\mathcal{G}$ bundle and V some generalized vector space which has a *left* $\mathcal{G}$-action. Then we define the associated vector bundle (E_V, π_V, M) where E_V is composed of the set of equivalence classes of points $[p, v]$ with the equivalence class defined by:

$$(p, v) \equiv (\tilde{p}, \tilde{v}) \text{ iff } \exists\, g \in \mathcal{G} \ni \tilde{p} = p\, g;\ \tilde{v} = g^{-1}\, v \tag{5.15}$$

and π_V is defined by

$$\pi_V([p, v]) := \pi(p) \tag{5.16}$$

It is important to realize that this associated vector bundle has a fibre which is diffeomorphic to the vector space, V. Now we can extend our previous relations for parallel transport to these associated vector bundles by defining the horizontal lift of α in the associated vector bundle (E_V, π_V, M) passing through the point $[p, v]$ in the fibre $\pi_V^{-1}(\alpha(a))$ as:

$$\alpha_V^{\uparrow}(b) := \left[\alpha^{\uparrow}(b), v\right] \tag{5.17}$$

with the associated parallel transport defined accordingly (c.f. (5.7)).

Finally, then, we can define a *covariant derivative* in the following way. Given some associated vector bundle (E_V, π_V, M) with some smooth cross-section, $\psi : M \to E_V$, and α some curve in M such that $\alpha(0) = x_0$, then the covariant derivative of ψ at x_0 with respect to the vector $[\alpha]$ is:

$$\nabla_{[\alpha]}\psi := \lim_{t \to 0} \frac{1}{t} \left[p_{II}(\psi(\alpha(t))) - \psi(x_0)\right] \tag{5.18}$$

and we can see why covariant derivates are only defined on *vector* bundles, as it is only on a *vector* bundle that the above *difference* of the two points on the fibre, $\pi_V^{-1}(x_0)$, is well-defined.

In local holonomic coordinates the above definition becomes the familiar

$$(\nabla_\mu \psi)(x) = \partial_\mu \psi(x) + w_\mu(x)\psi(x) \tag{5.19}$$

where the index μ in the covariant derivative refers to ∂_μ as usual and the last term reflects matrix multiplication between the $(N \times N)$ matrix-valued element of the Lie Algebra, $w_\mu(x)$, and the (N-dim) vector $\psi(x)$.

Of course all of this analysis is purely kinematical - that is, the *dynamics* for any theory, regardless of whatever relevant bundles one is using, must be specified by some other means, usually an action principle. We will discuss this point further in Chapter 7. For now it is enough to be aware of the fact that the principal bundles of relevance will be $\mathcal{B}(M)$ and its reduced bundle, $\mathcal{SO}(3,1)(M)$, while the vector bundles of relevance will be their respective associated bundles on R^N, i.e. $(\mathcal{B}(M)_{\Re^N}, \pi_{\Re^N}, M)$ and $(\mathcal{SO}(3,1)(M)_{\Re^N}, \pi_{\Re^N}, M)$, *both* of which are isomorphic to the tangent bundle, TM (see Section 5.3).

5.2 Bundle of Frames

The key principal fibre bundle in General Relativity is the Bundle of Frames, $\mathcal{B}(M)$. It consists of a bundle space, E, containing all the possible ordered sets of basis vectors $(b_1, b_2, ..., b_N)$ of T_xM for all the points $x \in M$. Clearly this is a principal bundle with structure group $GL(N, R)$ and the bundle space, E, can be easily shown to be a differential manifold of dimension $N + N^2$.

As alluded to above, the tangent bundle, TM, can be regarded as an associated vector bundle to $\mathcal{B}(M)$ in the following way. From $\mathcal{B}(M)$, we first form the associated vector bundle $(\mathcal{B}(M)_{\Re^N}, \pi_{\Re^N}, M)$ with associated covariant derivative ∇. But one can then associate this vector bundle, in turn, with the tangent bundle, TM, by the prescription:

$$[\tilde{b}, \mathbf{r}] \rightarrow \tilde{b}_i \, r^i \in T_xM \tag{5.20}$$

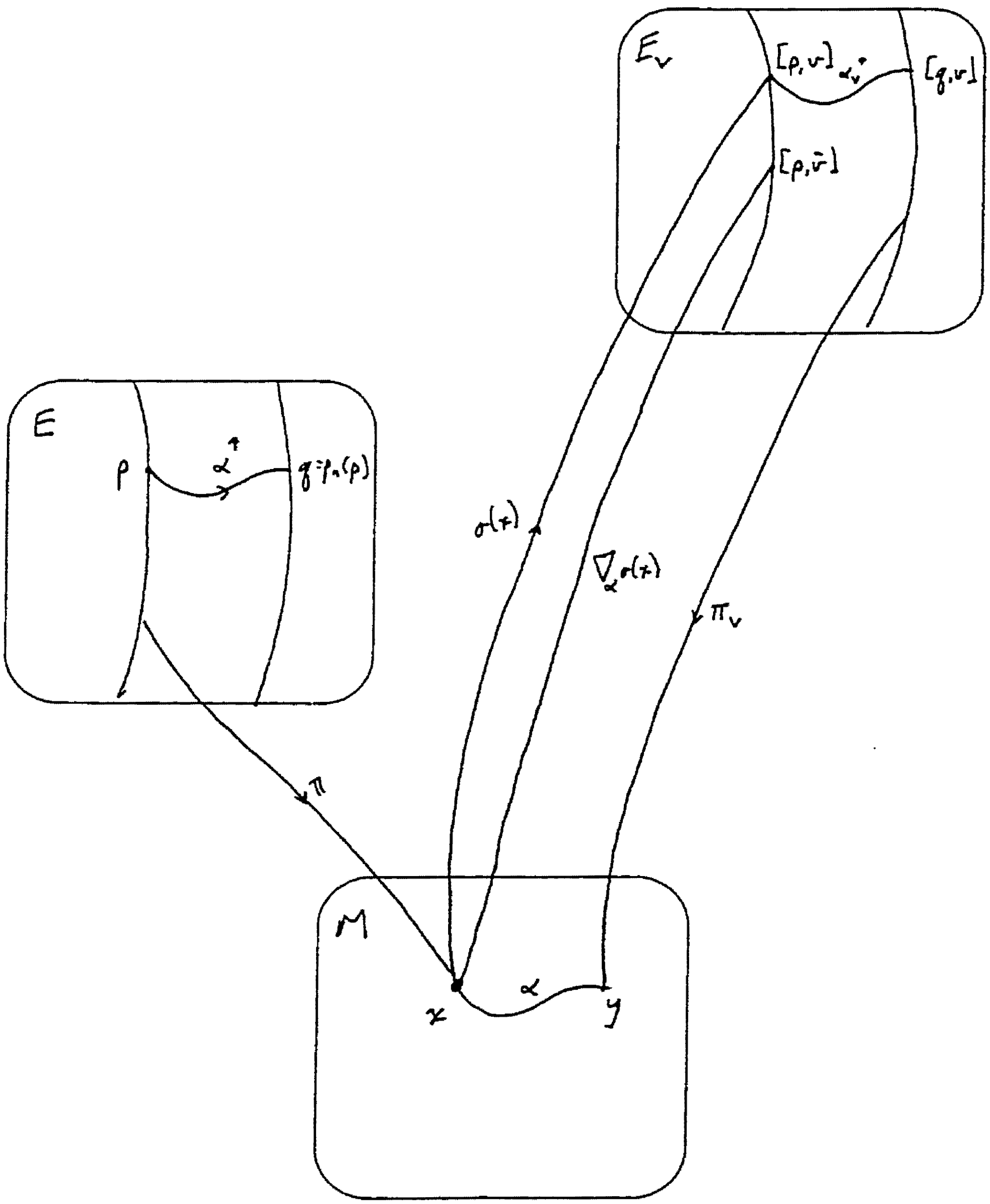
for $\tilde{b}$, some particular frame at $x \in M$.

Recall that in Chapter 2 we went to the trouble of specifying something called an "affine connection", which started out being a map from the space of $Vec(M) \times Vec(M)$ to $Vec(M)$ and became further generalized to a map involving vector fields and arbitrary rank tensors, which thereby enabled us to define torsion tensors and discuss issues of metric-compatibility. How does this relate to our more general fibre bundle picture?

There are many ways to visualize this, but the important fact to remember is that $\mathcal{B}(M)$ has cross-sections, $\psi(x)$, which are themselves directly related to elements of T_xM. If one moves to the associated tangent bundle, TM, this fact remains unchanged and we are now in the unique situation where for

$$\nabla_{[\alpha]}(\psi(x)) = \psi'(x), \tag{5.21}$$

Figure 5.3: Parallel Transport in Associated Vector Bundles



we can effectively regard $\bigtriangledown$ as a map from $Vec(M) \times Vec(M)$ (i.e. $[\alpha], \psi(x)$) to $Vec(M)$ (i.e. $\psi'(x)$). It is this unique attribute of these particular bundles (i.e. the explicit relationship between the argument of the covariant derivative, $[\alpha]$, and the sections of the relevant bundle itself, $\psi(x)$) which enables us to define torsion tensors and generalize $\bigtriangledown$ to acting on arbitrary tensors *in* M (from whence the concept of metricity arises).

There is, however, another way to look at this unique peculiarity of $\mathcal{B}(M)$ which directly leads to the concept of torsion and is vital to formulating gravitational theories in an action framework.

We define a *solder form* as a $\Re^N$-valued 1-form, θ, on $\mathcal{B}(M)$ by

$$\langle \theta, v \rangle_p := (\pi_\star v)^\alpha \, \forall \, p \in B(M) \tag{5.22}$$

where the $\Re^N$-valued set of components of $\pi_\star v$, i.e. $(\pi_\star v)^\alpha$ are necessarily with respect to the local frame $(b_1, b_2, ..., b_N)$ defined at $p \in B(M)$

That is, the solder form takes any vector $v \in T_p(B(M))$ and gives its components with respect to the particular basis at $p \in \pi^{-1}(x)$. Since we know that $\pi_\star v = 0$ defines any vertical vector v_p at p we see that the solder form only gives non-trivial results for horizontal vectors, $v \in H_p(B(M))$. Projecting down to the base space, M, given some cross-section, σ, it is clear that the local representation of θ in M, $\sigma^\star\theta := \beta$, is merely given by:

$$\beta(v) = \left[\beta^1(v), \beta^2(v), ..., \beta^N(v)\right] \tag{5.23}$$

for $\{\beta^\lambda\}$ the set of co-vectors associated with the particular frame chosen by the cross-section σ. Hence the solder form θ or β is intimately related to the cross-sections of $\mathcal{B}(M)$, σ, which serve to select particular frames over some $x \in M$ and are hence often referred to as *frame fields*. In a holonomic coordinate system, the standard frame to which all others are compared is the frame $(\partial_1, \partial_2, ..., \partial_N)$ and hence the frame field, $e \equiv \sigma$ can be written as the $GL(N, R)$ function e^μ_ν with the understanding that

$$e(x) = (b_1, b_2, ..., b_N) \tag{5.24}$$

and

$$b_\lambda = e^\rho_\lambda \, \partial_\rho \tag{5.25}$$

where we see the equivalence between our frame field and the usual N-beins or tetrads explicitly. Since in this coordinate system the solder form acting on any $v = v^\eta \partial_\eta \in M$ merely gives $v^\eta e^\lambda_\eta$, it is often said that in these coordinates the frame field *is* the solder form, where the technical difference between the frame field as a cross-section and the N-bein as a $GL(N, R)$ function is glossed over.

As before, we see that the solder form is unique to the frame bundle and its related bundles as, for a general principal bundle, the definition above (i.e. (5.22)) is nonsensical owing to the fact that in general $\pi_* v$ produces a vector with basis vectors (and potentially dimension) completely unrelated to $p \in \pi^{-1}(x)$.

This existence of θ enables us to define further geometrical entities on $B(M)$ and M, namely the exterior covariant derivative of θ, the 2-form

$$\mathcal{T} := \mathbf{D}\theta, \tag{5.26}$$

known as the *torsion* two-form, and its associated local representative,

$$T := \sigma^* \mathcal{T}, \tag{5.27}$$

the torsion 2-form on M.

In this way we can find the analogous principal bundle equations to the structure equation and the Bianchi identity, i.e.

$$\mathcal{T} = d\theta + A \wedge \theta \tag{5.28}$$

and

$$\mathbf{D}\mathcal{T} = \mathbf{G} \wedge \theta \tag{5.29}$$

and their more familiar local representations:

$$T = d\beta + w \wedge \beta \tag{5.30}$$

and

$$dT + w \wedge w = R \wedge w \tag{5.31}$$

respectively.

5.3 Metrics and The Bundle $SO(N-1,1)(M)$

It is significant to note that, until now, we have not mentioned the presence of a metric at all. We have merely demonstrated that, for a principal bundle with connection A and some local cross-section, σ, we can establish $\mathcal{L}(\mathcal{G})$-valued forms on M, R and w, obeying (5.13) and (5.14); while if there also exists some solder form, θ, on the principal bundle, we can additionally obtain additional $\Re^N$-valued forms on M, T and β, such that (5.30) and (5.31) hold.

We now turn our attention to the issue of a metric, i.e. some symmetric, non-degenerate (0,2) tensor defined on M. If we *define* a metric, g, in terms of the local cross-sections of $\mathcal{B}(M)$, i.e. the frame field, e, so that it is necessarily orthogonal, then we have:

$$g_{\alpha\beta} = e^I_\alpha \, e^J_\beta \, \eta_{IJ} \tag{5.32}$$

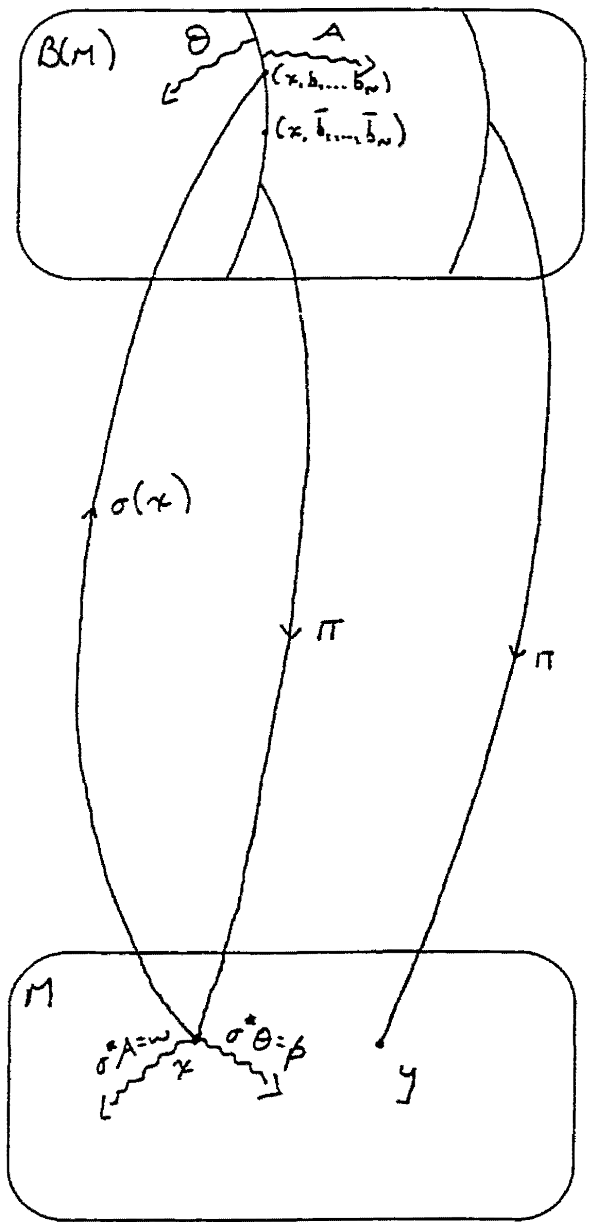
for η_{IJ} the Minkowski metric

$$\eta_{IJ} = diag\{-1, 1, 1, ..., 1\} \tag{5.33}$$

Since $g_{\alpha\beta}$ has $\frac{N(N+1)}{2}$ degrees of freedom, while the N-bein, e^I_α has N^2 degrees of freedom, one would expect that there are many different N-beins which can be combined via (5.32) above to give the same metric, $g_{\alpha\beta}$. This redundancy is manifested by the Lorentz group, i.e. the group $SO(N-1,1)$ of dimension $\frac{N(N-1)}{2}$ (where, of course, $\frac{N(N-1)}{2} = N^2 - \frac{N(N+1)}{2}$), by the fact that the metric $g_{\alpha\beta}$ defined above by (5.32) remains invariant under a general $SO(N-1,1)$ - i.e. Lorentz, transformation.

In bundle language, it is said that, $SO(N-1,1)(M)$, the principal bundle over M with structure group $SO(N-1,1)$ instead of the "full" $GL(N,R)$ of $\mathcal{B}(M)$, is a reduction of $\mathcal{B}(M)$, where this reduction process is intimately tied to the existence of

Figure 5.4: Bundle of Frames with Solder Forms



a metric on M[3]

From our perspective, however, this is unimportant. It is enough to recognize that $\mathcal{SO}(N-1,1)(M)$ is a sort of "limited" bundle of frames, and hence similarly also contains a solder form, $\hat{\theta}$, and can give rise to its own affine connection, $\hat{\nabla}$ on TM, complete with torsion tensor and potential metricity. In fact, we shall explicitly see in the next Chapter that the fact that $\mathcal{SO}(N-1,1)(M)$ is intimately related to the existence of a Lorentzian metric on M is manifested by the fact that $\hat{\nabla}(g) \equiv 0$, which was one of the principal motivations for working with it in the first place.

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[3]In general, for Lorentzian metrics, there may exist topological obstructions which can limit this "reduction" process - such obstructions lead the way towards the concept of "metric kinks"

Chapter 6

Palatini Tetrad Formalism

6.1 Overview

There is a certain amount of ambiguity in the literature concerning the notion of a Palatini variation. This results from the fact that, while some regard a Palatini variation as what we have called the "Standard Palatini Formalism" in Chapter 3 and 4 - that is, the variation of some action $S(\Gamma, g)$ with respect to an independent Γ as well as g - there are others who refer to a Palatini variation as the variation of some frame-space action with respect to the relevant frame-field variable, e (through which a metric is defined) and generalized connection w. This latter approach, which we will call the "Palatini Tetrad Formalism", has steadily increased in popularity owing to its direct relevance for those interested in a connection dynamics perspective of gravity.

In the Palatini Tetrad approach, one no longer works directly with an affine connection (or connection coefficient, Γ) derived from $\mathcal{B}(M)$, but rather with the aforementioned $SO(N-1,1)$ connection associated with the bundle $\mathcal{SO}(N-1,1)(M)$;

Nevertheless, as previously mentioned in the last chapter, just as TM can be regarded as an associated vector bundle of the "full" $GL(N, R)$ bundle of frames,

$\mathcal{B}(M)$, with the help of the frame field, e, TM can also be eventually viewed as an associated vector bundle of the reduced bundle $\mathcal{SO}(N-1,1)(M)$, and hence one can consistently talk of a covariant derivative, $\hat{\nabla}$ induced on the tangent bundle TM due to w of $\mathcal{SO}(N-1,1)(M)$ in addition to the usual covariant derivative, ∇, associated with the "full" connection Γ on $\mathcal{B}(M)$. Both $\hat{\nabla}$ and ∇, owing to their applicability to TM, are affine connections as defined in Chapter 2.

In what follows we limit ourselves for simplicity to the case $N = 4$, but we note that the forthcoming analysis extends to arbitrary dimension.

6.2 3+1 Palatini Tetrad Formalism

The following is a brief synopsis of the 3+1 Palatini Tetrad approach [33]. As alluded to above, here one begins with an action solely dependent on a $\mathcal{SO}(3,1)(M)$ connection and a frame field e^{α}_{I}, through which one defines a metric. Variation of this action with respect to the two dynamical variables, w and e, eventually results in Einstein's equations.

We begin, then, with the following action:

$$S_{PT} = \int d^4x \; e \; e^{\alpha}_{I} \; e^{\beta}_{J} \; R_{\alpha\beta}{}^{IJ}(w) \tag{6.1}$$

where Cartan's equation defines:

$$R_{\alpha\beta}{}^{I}{}_{K} := 2\partial_{[\alpha} w_{\beta]}{}^{I}{}_{K} + w_{\alpha}{}^{I}{}_{L}\, w_{\beta}{}^{L}{}_{K} - w_{\beta}{}^{I}{}_{J}\, w_{\alpha}{}^{J}{}_{K} \tag{6.2}$$

and we use the convention

$$R_{\alpha\beta}{}^{IJ} := \eta^{KJ} R_{\alpha\beta}{}^{I}{}_{K} \tag{6.3}$$

for η^{IJ} defined as the (raised) Minkowski (4×4) metric,

$$\eta_{IJ} := diag\{-1, 1, 1, 1\} \tag{6.4}$$

We follow the standard procedure of defining the metric $g_{\alpha\beta}$ in terms of the frame fields, e such that it is necessarily orthonormal, that is:

$$g_{\alpha\beta} := e^I_\alpha e^J_\beta \eta_{IJ} \tag{6.5}$$

where e^I_α, often called the coframe field, is defined such that

$$e^I_\alpha e^\beta_I = \delta^\beta_\alpha \tag{6.6}$$

and

$$e^I_\alpha e^\alpha_J = \delta^I_J \tag{6.7}$$

We see that the coframe field represents the (4×4 here) matrix inverse of the frame field, which is always well defined because we know from Chapter 5 that $e^\alpha_I \in GL(4, R)$ and hence has a unique inverse.

Hence (6.5) enables us to express $\sqrt{-g}$ in terms of $det(e^I_\alpha)$ (or, more rigorously, the invariant volume element $\sqrt{-g}dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$ in terms of the coframe field).

Furthermore, we note that, since ${w_\alpha}^I{}_K$ takes values in the Lie Algebra of the group $SO(3,1)$, we necessarily have the relationship

$${w_\alpha}^I{}_K = -{w_{\alpha K}}^I \tag{6.8}$$

that is, w is anti-symmetric in its Lie Algebra indices. Using the convention of (6.3) above, this becomes:

$${w_\alpha}^{IJ} = -{w_\alpha}^{JI} \tag{6.9}$$

Owing to the difference between the Lie Algebra indices, $(I, J, K...)$ and TM indices $(\alpha, \beta\gamma...)$, the induced connection $\hat{\nabla}$ on TM owing to the principal bundle $SO(3,1)(M)$ must be defined in a slightly roundabout manner which invokes the frame field, e, rigorously a cross section of $B(M)$. Hence in order to firmly establish the covariant derivative on TM, $\hat{\nabla}$ owing to the reduced principal bundle $\mathcal{SO}(3,1)(M)$, we need to first find local cross-sections of the *full* principal bundle, $\mathcal{B}(M)$. The fact that the treatment to defining $\hat{\nabla}$ is not self-contained within $\mathcal{SO}(3,1)(M)$ is an important subtlety which will be pivotal to our future analysis (see Appendix E and Chapter 7)

The natural vector bundle associated with $\mathcal{SO}(3,1)(M)$ is $(\mathcal{SO}(3,1)(M)_{\Re^N}, \pi_{\Re^N}, M)$, with basis vectors ξ_I where I now corresponds directly to the Lie Algebra index. Each element of each fibre of the vector bundle $(\mathcal{SO}(3,1)(M)_{\Re^N}, \pi_{\Re^N}, M)$ can be identified with a corresponding point on the related fibre of TM and hence we have an isomorphism between $(\mathcal{SO}(3,1)(M)_{\Re^N}, \pi_{\Re^N}, M)$ and TM provided by the frame field, e. This isomorphism is illustrated by the accompanying figure (Figure 6.1).

If w induces the covariant derivative relationship $\hat{\nabla} b := c$ for $b = b^I \xi_I, c = c^J \xi_J$ on some fibre of $(\mathcal{SO}(3,1)(M)_{\Re^N}, \pi_{\Re^N}, M)$, we find that, given the isomorphism:

$$e(\xi_I) = e_I^\gamma \partial_\gamma \tag{6.10}$$

consistency requirements[1] mandate that the connection functions $\hat{\Gamma}$ on TM must be written in terms of w in the following way.

$$\hat{\Gamma}_\alpha{}^\gamma{}_\lambda = e_\lambda^J \, e_I^\gamma \, w_\alpha{}^I{}_J + \partial_\alpha(e_\lambda^J) e_J^\gamma \tag{6.11}$$

or

$$w_\alpha{}^K{}_J = \partial_\alpha(e_J^\gamma) \, e_\gamma^K + e_\gamma^K \, e_J^\epsilon \, \hat{\Gamma}_\alpha{}^\gamma{}_\epsilon \tag{6.12}$$

We note that (6.2) and (6.11) together give a general relationship linking the curvature tensors (2-forms) $R_{\alpha\beta}{}^I{}_K$ and $\hat{R}_{\alpha\beta}{}^\lambda{}_\epsilon$.

$$R_{\alpha\beta}{}^I{}_K = e_\epsilon^J \, e_I^\lambda \, \hat{R}_{\alpha\beta}{}^\lambda{}_\epsilon \tag{6.13}$$

Having thus derived an affine connection, $\hat{\nabla}$, based upon our $SO(3,1)$ connection, w, we can then apply it to various tensors on M. If we examine the application of $\hat{\nabla}$ to $g_{\alpha\beta}$ as defined by (6.5) above, we find that:

$$\hat{\nabla}_\rho g_{\alpha\beta} := g_{\alpha\beta,\rho} - \hat{\Gamma}_\rho{}^\kappa{}_\alpha g_{\beta\kappa} - \hat{\Gamma}_\rho{}^\tau{}_\beta g_{\alpha\tau} \tag{6.14}$$

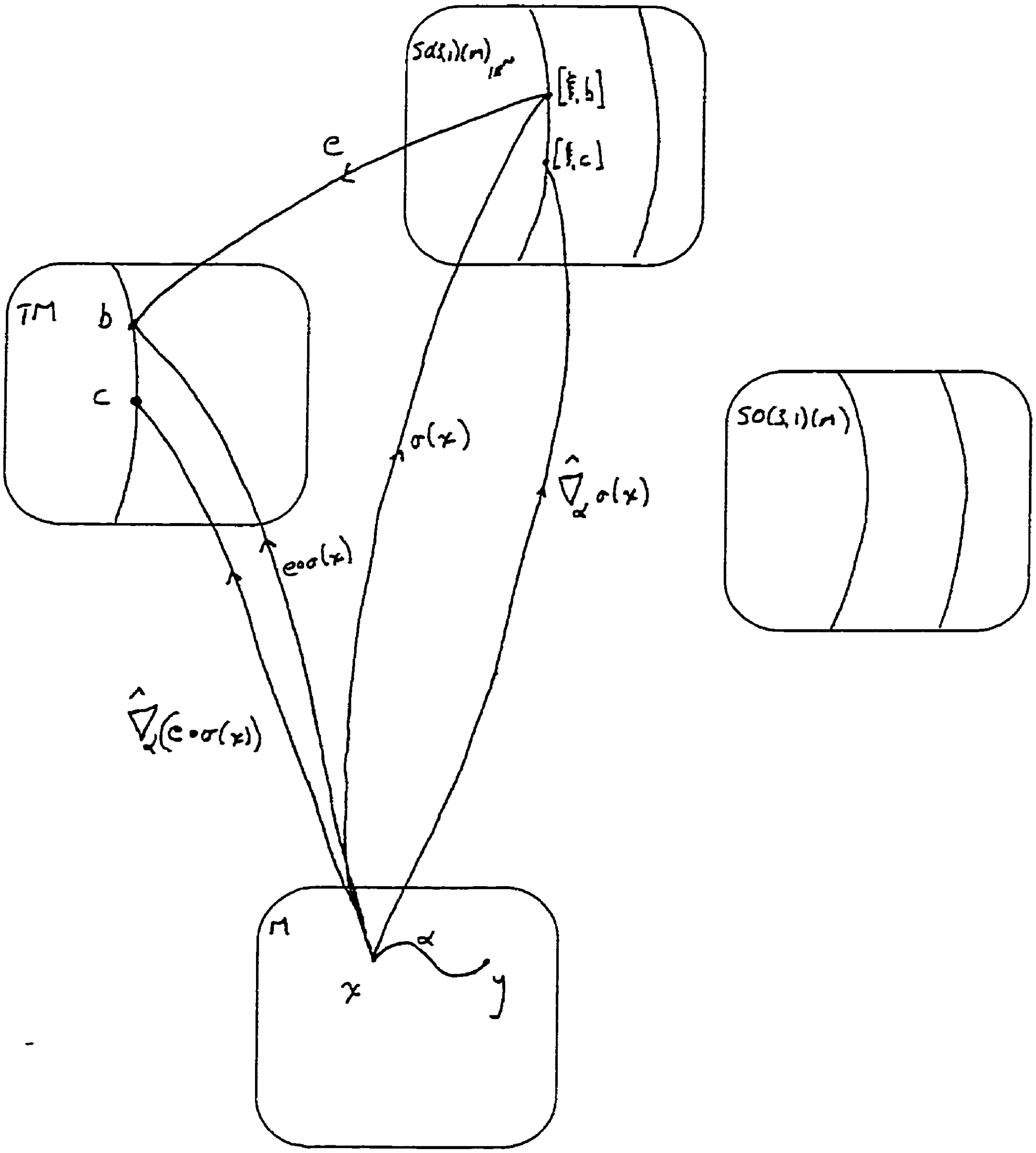
Invocation of (6.11) and (6.8) leads to the expression

$$\hat{\nabla}_\rho \, g_{\alpha\beta} = 0 \tag{6.15}$$

for all choices of ρ - i.e. under this prescription, $\hat{\nabla}$ is necessarily a metric connection. As mentioned earlier, this is not an entirely surprising development as it was the original motivation from moving away from $\mathcal{B}(M)$ to its reduction $\mathcal{SO}(M)$ to begin

[1]See Appendix E

Figure 6.1: $\mathcal{SO}(3,1)(M)$ - Induced Covariant Derivative on TM



with - that is, to cast General Relativity in a geometrical framework while somehow containing the metricity assumption vital to its dynamics. Yet from our perspective it is somewhat question-begging as once more the geometry of the problem has been set up so that one is not truly varying with respect to all possible connections, but merely a subset of them. This issue shall be addressed in the next section.

Let us now return to the dynamics induced by (6.1). Variation with respect to e and w leads to the following constraints:

$$\delta_e S_{PT} = 0 \Rightarrow \left[e^{\alpha}_{K}\, e^{\gamma}_{I}\, e^{\beta}_{J} - \frac{1}{2}\, e^{\alpha}_{I}\, e^{\gamma}_{K}\, e^{\beta}_{J}\right] R_{\gamma\beta}{}^{KJ} = 0 \tag{6.16}$$

and

$$\delta_w S_{PT} = 0 \Rightarrow D_{\alpha}\left[e \cdot e^{[\alpha}_{I}\, e^{\beta]}_{J}\right] = 0 \tag{6.17}$$

We find that, for $N = 4$, (6.17) can be reduced to[2]

$$D_{[\alpha}\, e^{K}_{\beta]} = 0 \tag{6.18}$$

Utilizing (6.11), one can easily see that (6.18) can be rewritten as

$$\hat{T}_{\mu}{}^{\epsilon}{}_{\nu} := \hat{\Gamma}_{\mu}{}^{\epsilon}{}_{\nu} - \hat{\Gamma}_{\nu}{}^{\epsilon}{}_{\mu} = 0 \tag{6.19}$$

In short, by geometrical definition we had an affine connection which was metric-compatible (6.15), whereas we see by the variational principle applied to the action S_{PT}, $\hat{\Gamma}$ is also torsion-free. Hence it must be the case that

$$\hat{\Gamma}_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\} \tag{6.20}$$

One must be rather careful here, though. It is worth noting that our "torsion-free" condition, (6.19) depends implicitly on our expression of $\hat{\Gamma}$ in terms of w, i.e. (6.11),

[2]for $N \neq 4$ one correspondingly adjusts one's action to enable a similar manipulation to occur.

which in turn implicitly depends on the fact that $\hat{\Gamma}$ is, in fact, metric-compatible. We shall have more to say about this potential question-begging in the next section.

For now, however, we note that we have achieved, through whatever means, the conclusion that our connection is once again the Christoffel symbol. Expressing (6.16) in terms of $\hat{R}_{\alpha\beta}{}^{\lambda}{}_{\epsilon}$ via (6.13), together with the fact that (6.20) above necessarily implies

$$\hat{R}_{\alpha\beta}{}^{\lambda}{}_{\epsilon} = \hat{R}_{\alpha\beta}{}^{\lambda}{}_{\epsilon}(\{\}) = -\hat{R}_{\alpha\beta\epsilon}{}^{\lambda} \tag{6.21}$$

enables us to finally express (6.16) in the final form[3]

$$G_{\alpha\beta}(\{\}) = 0 \tag{6.22}$$

i.e. the Einstein Field Equations.

6.3 Generalized 3+1 Palatini Tetrad Formalism

As mentioned in the previous section, the usual 3+1 Palatini Tetrad Formalism is devised so that one implicitly assumes the metricity of the relevant affine connection (6.15), while the no-torsion constraint arising from the variational principle of the action S_{PT} is itself dependent on the assumed form of the affine connection (6.11) and hence implicitly upon its metricity.

In an effort to once again take the spirit of the Palatini variation seriously -i.e. attempting to isolate the potential connection dependence of the dynamics by varying the relevant action in the space of *all* possible affine connections - we find ourselves once again modifying the procedure.

[3]It is worth noting here that, unlike in the Standard Palatini Formalism of Chapter 3-4, the Einstein Field Equations can only be derived by using the above antisymmetry condition of the curvature tensor when $\Gamma = \{\}$. Otherwise, unlike in the Standard Formalism, one doesn't find the result $G_{\mu\nu}(\Gamma) = 0$

In the Standard Palatini Formalism of Chapters 3-4, we found that two pivotal assumptions were made to the usual Einstein-Hilbert action which limited the ability to be as general as possible in one's connection and thus proved to go against the spirit of a "true" Palatini variation.

i) any potential explicitly non-metric terms were *a priori* not included in the action

ii) any potential torsion effects (both with the curvature term and any potential addition explicitly non-zero torsion terms) were also *a priori* neglected from the action

In the Palatini Tetrad formalism, meanwhile, one begins with a particular reduced bundle, $\mathcal{SO}(3,1)(M)$, of $\mathcal{B}(M)$ which is specifically engineered to give a resulting affine connection which is necessarily metric- preserving. In what follows, we shall counterbalance this result by redefining our relationship between w and $\hat{\Gamma}$ to include potential non-metricity and see how this affects the relevant dynamics induced by varying the action S_{PT}.

We thus return to (6.11) and now define:

$$\tilde{\Gamma}_{\alpha}{}^{\gamma}{}_{\lambda} := e^{J}_{\lambda}\, e^{\gamma}_{I}\, w_{\alpha}{}^{I}{}_{J} + \partial_{\alpha}(e^{J}_{\lambda})e^{\gamma}_{J} + K_{\alpha}{}^{\gamma}{}_{\lambda} \tag{6.23}$$

where $K_{\alpha}{}^{\gamma}{}_{\lambda}$ is *any* arbitrary tensor on M. Clearly, we can then define $\tilde{\nabla}$ as the affine connection on TM related to the connection, $\tilde{\Gamma}$.

Under (6.23), we find the following alterations to (6.15), and (6.13):

$$\tilde{\nabla}_{\lambda} g_{\mu\nu} = -\left(K_{\lambda\mu\nu} + K_{\lambda\nu\mu}\right) \tag{6.24}$$

and the associated

$$\tilde{\nabla}_{\lambda} g^{\mu\nu} = K_{\lambda}{}^{\mu\nu} + K_{\lambda}{}^{\nu\mu} \tag{6.25}$$

while we can relate the "new" curvature tensor, $\tilde{R}_{\alpha\beta}{}^{\lambda}{}_{\epsilon}$, to the $\mathcal{SO}(3,1)(M)$ curvature,

$R_{\alpha\beta}{}^{I}{}_{J}$, by

$$\tilde{R}_{\alpha\beta}{}^{\lambda}{}_{\epsilon} = e^{J}_{\epsilon}\, e^{\lambda}_{I}\, R_{\alpha\beta}{}^{I}{}_{J} + \tilde{\nabla}_{\alpha}\, K_{\beta}{}^{\lambda}{}_{\epsilon} - \tilde{\nabla}_{\beta}\, K_{\alpha}{}^{\lambda}{}_{\epsilon} + \tilde{T}_{\alpha}{}^{\eta}{}_{\beta}\, K_{\eta}{}^{\lambda}{}_{\epsilon} - K_{\beta}{}^{\rho}{}_{\epsilon}\, K_{\alpha}{}^{\lambda}{}_{\rho} + K_{\alpha}{}^{\rho}{}_{\epsilon}\, K_{\beta}{}^{\lambda}{}_{\rho} \tag{6.26}$$

We note we can express this as:

$$e^{J}_{\epsilon}\, e^{\lambda}_{I}\, R_{\alpha\beta}{}^{I}{}_{J} = \tilde{R}_{\alpha\beta}{}^{\lambda}{}_{\epsilon} - [f(K)]_{\alpha\beta}{}^{\lambda}{}_{\epsilon} \tag{6.27}$$

with

$$[f(K)]_{\alpha\beta}{}^{\lambda}{}_{\epsilon} := \tilde{\nabla}_{\alpha}\, K_{\beta}{}^{\lambda}{}_{\epsilon} - \tilde{\nabla}_{\beta}\, K_{\alpha}{}^{\lambda}{}_{\epsilon} + \tilde{T}_{\alpha}{}^{\eta}{}_{\beta}\, K_{\eta}{}^{\lambda}{}_{\epsilon} - K_{\beta}{}^{\rho}{}_{\epsilon}\, K_{\alpha}{}^{\lambda}{}_{\rho} + K_{\alpha}{}^{\rho}{}_{\epsilon}\, K_{\beta}{}^{\lambda}{}_{\rho} \tag{6.28}$$

. Furthermore, we note that in general for any tensor K,

$$R_{\alpha\beta}{}^{\lambda}{}_{\epsilon}(\Gamma - K) = R_{\alpha\beta}{}^{\lambda}{}_{\epsilon}(\Gamma) - [f(K)]_{\alpha\beta}{}^{\lambda}{}_{\epsilon} \tag{6.29}$$

Therefore (6.26) can be rewritten as:

$$e^{J}_{\epsilon}\, e^{\lambda}_{I}\, R_{\alpha\beta}{}^{I}{}_{J} = R_{\alpha\beta}{}^{\lambda}{}_{\epsilon}(\tilde{\Gamma} - K) \tag{6.30}$$

If we turn our attention now to the equations derived from varying S_{PT} with respect to e and w, we see that both (6.16) and (6.17) are *prima facie* unchanged under (6.23), i.e. we still have

$$\delta_e S_{PT} = 0 \Rightarrow \left[e^{\alpha}_{K}\, e^{\gamma}_{I}\, e^{\beta}_{J} - \frac{1}{2}\, e^{\alpha}_{I}\, e^{\gamma}_{K}\, e^{\beta}_{J}\right] R_{\gamma\beta}{}^{KJ} = 0$$

and

$$\delta_w S_{PT} = 0 \Rightarrow D_{\alpha}\left[e \cdot e^{[\alpha}_{I}\, e^{\beta]}_{J}\right] = 0 \Rightarrow (N=4) D_{[\alpha}\, e^{L}_{\beta]} = 0$$

But now, rather than yield the no-torsion condition (6.19),

$$D_{[\alpha}\, e^{L}_{\beta]} = 0 \Rightarrow K_{[\mu}{}^{\epsilon}{}_{\nu]} = \frac{1}{2}\, T_{\mu}{}^{\epsilon}{}_{\nu} \tag{6.31}$$

Permuting (6.24) gives:

$$\bar{\nabla}_\lambda g_{\mu\nu} - \bar{\nabla}_\nu g_{\mu\lambda} - \bar{\nabla}_\mu g_{\nu\lambda} = 2g_{\mu\epsilon}\, K_{[\nu}{}^{\epsilon}{}_{\lambda]} + 2g_{\nu\epsilon}\, K_{[\mu}{}^{\epsilon}{}_{\lambda]} + 2g_{\lambda\epsilon}\, K_{[\nu}{}^{\epsilon}{}_{\mu]} + 2K_{\mu\lambda\nu} \quad (6.32)$$

Combining (6.31) and (6.32) finally give:

$$\tilde{\Gamma}_\mu{}^\epsilon{}_\nu - K_\mu{}^\epsilon{}_\nu = \{_\mu{}^\epsilon{}_\nu\} \quad (6.33)$$

which is the generalized Palatini Tetrad analogue to (6.20). Substitution of (6.33) into (6.30) gives:

$$e^J_\epsilon\, e^\lambda_I\, R_{\alpha\beta}{}^I{}_J = \tilde{R}_{\alpha\beta}{}^\lambda{}_\epsilon(\{\}), \quad (6.34)$$

whereas substitution of (6.34) into (6.16) once again gives the requisite Einsteinian dynamics, (6.22),

$$G_{\mu\nu}(\{\}) = 0$$

This is a somewhat unexpected development - our generalized Palatini Tetrad formalism, obtained by using the prescription (6.23) for our affine connection in terms of the $SO(3,1)(M)$ connection $w_\alpha{}^I{}_J$ instead of the usual (6.11) leads to the same (Einsteinian) dynamics as before, (6.22), but now gives a fundamentally *indeterminate* result for the connection, $\tilde{\Gamma}$! The only thing we can say explicitly about our new connection is that it is related to the Christoffel symbol via: $\tilde{\Gamma}_\mu{}^\epsilon{}_\nu - K_\mu{}^\epsilon{}_\nu = \{_\mu{}^\epsilon{}_\nu\}$, which, since $K_\mu{}^\epsilon{}_\nu$ is arbitrary, does not tell us really anything at all about $\tilde{\Gamma}$. Hence our generalized prescription, together with the action S_{PT} (6.1), leads to a truly connection *independent* expression of General Relativistic (metric) dynamics, if not General Relativity proper.

This is strongly reminiscent of our Extended Action results of Chapter 3, except that now we are not necessarily mandating that $\tilde{\Gamma}$ be torsion-free and hence $K_\mu{}^\epsilon{}_\nu$ ($Q_\mu{}^\epsilon{}_\nu$, in our previous notation), be symmetric in any indices. Nonetheless, both our

Extended Action of Chapter 3 and the above Generalized Palatini Tetrad Formalism amount to connection independent means of generating General Relativistic dynamics, and hence we would expect them to be somehow related. This relationship shall be explicitly demonstrated in the next chapter.

Chapter 7

Analysis

7.1 Geometrical Picture of Generalized 3+1 Palatini Tetrad Formalism

In the last chapter we recognized that generalizing the usual prescription linking the $\mathcal{SO}(3,1)(M)$ connection to its affine connection

$$\hat{\Gamma}_\alpha{}^\gamma{}_\lambda := e^J_\lambda \, e^\gamma_I \, w_\alpha{}^I{}_J + \partial_\alpha(e^J_\lambda) \, e^\gamma_J \tag{7.1}$$

to

$$\tilde{\Gamma}_\alpha{}^\gamma{}_\lambda := e^J_\lambda \, e^\gamma_I \, w_\alpha{}^I{}_J + \partial_\alpha(e^J_\lambda) \, e^\gamma_J + K_\alpha{}^\gamma{}_\lambda, \tag{7.2}$$

i.e.

$$\tilde{\Gamma}_\mu{}^\epsilon{}_\nu = \hat{\Gamma}_\mu{}^\epsilon{}_\nu + K_\mu{}^\epsilon{}_\nu \tag{7.3}$$

for an arbitrary tensor $K_\alpha{}^\gamma{}_\lambda$, enabled us to replace the *a priori* metricity relationship

$$\hat{\nabla}_\lambda \, g_{\mu\nu} = 0 \tag{7.4}$$

with

$$\tilde{\nabla}_\lambda \, g_{\mu\nu} = -\,(K_{\lambda\mu\nu} + K_{\lambda\nu\mu})\,; \tag{7.5}$$

while the constraint

$$D_{[\lambda} e^L_{\beta]} = 0 \tag{7.6}$$

obtained from the action

$$S_{PT} = \int d^4x \; e \; e^\alpha_I \; e^\beta_J \; R_{\alpha\beta}{}^{IJ}(w) \tag{7.7}$$

by varying with respect to w changes from reducing to

$$\hat{T}_\mu{}^\epsilon{}_\nu = 0 \tag{7.8}$$

under (7.1) to

$$K_{[\mu}{}^\epsilon{}_{\nu]} = \frac{1}{2} T_\mu{}^\epsilon{}_\nu \tag{7.9}$$

under (7.3). Thus this relation combined with (7.5) gives

$$\tilde{\Gamma}_\mu{}^\epsilon{}_\nu - K_\mu{}^\epsilon{}_\nu = \{_\mu{}^\epsilon{}_\nu\}, \tag{7.10}$$

which winds up transforming the second constraint obtained from S_{PT} with respect to e, i.e.

$$\left[e^\alpha_K \; e^\gamma_I \; e^\beta_J - \frac{1}{2} \; e^\alpha_I \; e^\gamma_K \; e^\beta_J\right] R_{\gamma\beta}{}^{KJ} = 0 \tag{7.11}$$

into nothing less than the vacuum Einstein Field Equations,

$$G_{\mu\nu}(\{\}) = 0. \tag{7.12}$$

So our new prescription (7.3) gives us the same dynamics as before (7.12), but now, instead of the old connection constraint

$$\hat{\Gamma}_\mu{}^\epsilon{}_\nu = \{_\mu{}^\epsilon{}_\nu\} \tag{7.13}$$

we have effectively an indeterminacy relation given by (7.10) above.

All of these results have been generated in the previous chapter, but what we now need is some comprehensive geometrical framework by which we can understand the meaning of the difference between the two prescriptions (7.3) and (7.1). Accordingly, we now examine the following diagram, which may be viewed as an extension of the previous one of Section 6.2, where we now extend matters to explicitly include $\mathcal{B}(M)$.

This diagram enables us to visualize several things. In the first place we see that the generalized prescription, (7.3), $\tilde{\Gamma}_\mu{}^\epsilon{}_\nu = \hat{\Gamma}_\mu{}^\epsilon{}_\nu + K_\mu{}^\epsilon{}_\nu$, is manifested in terms of covariant derivatives acting on elements of TM as:

$$(\tilde{\nabla}_\alpha\, v)^\gamma = (\hat{\nabla}_\alpha\, v)^\gamma + K_\alpha{}^\gamma{}_\eta\, v^\eta \tag{7.14}$$

From the diagram, we see that the tensor $K_\alpha{}^\gamma{}_\eta\, v^\eta$ represents a fibre-preserving transformation from $v \in \pi_{TM}^{-1}(x)$ to some other element $z \in \pi_{TM}^{-1}(x)$. If we have

$$\hat{\nabla}_\alpha\, v \equiv w \tag{7.15}$$

and

$$[K_\alpha][v] \equiv z \tag{7.16}$$

where the above matrix notation is an abbreviation for

$$K_\alpha{}^\gamma{}_\eta\, v^\eta = z^\gamma, \tag{7.17}$$

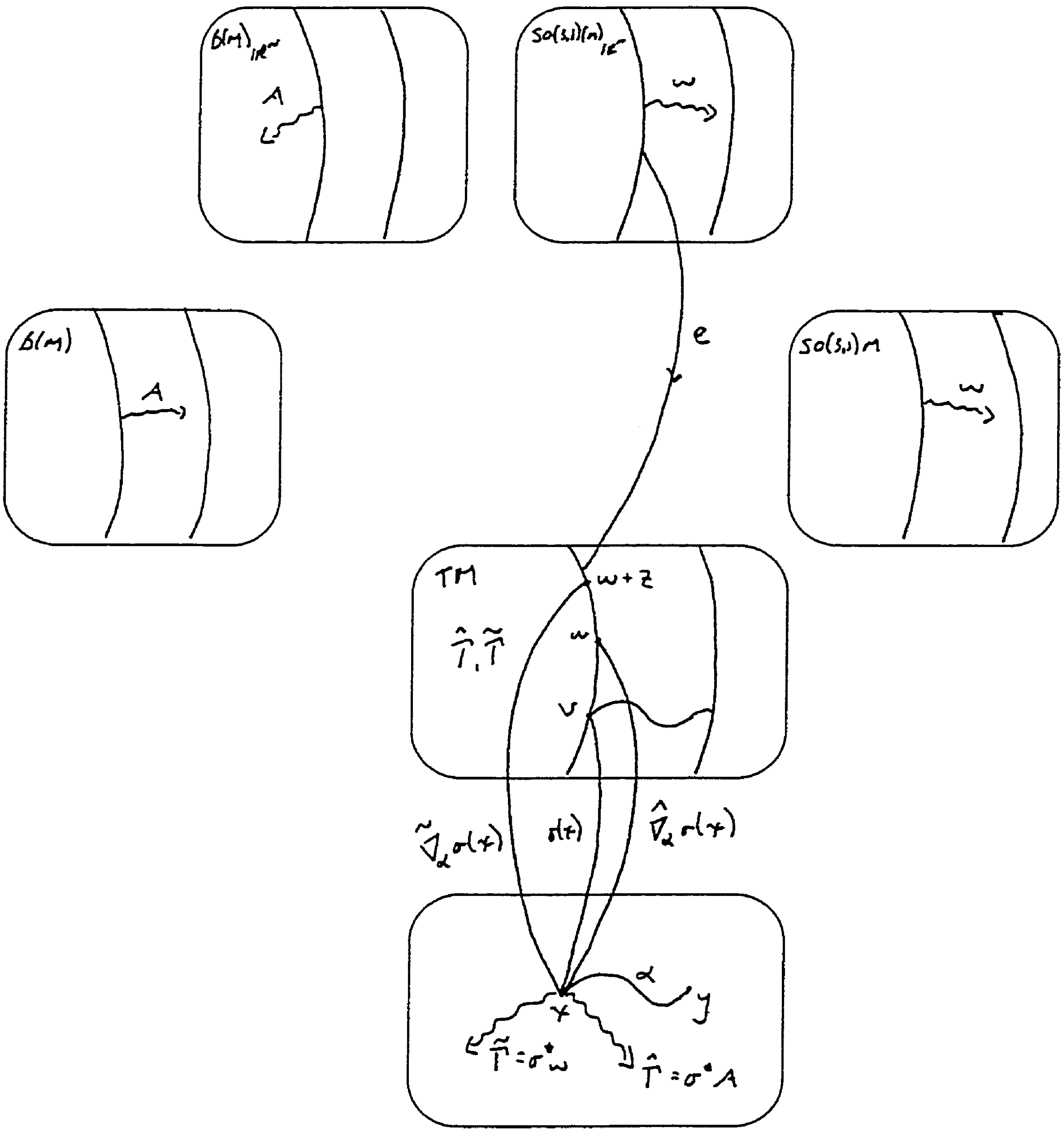
we thus have

$$\tilde{\nabla}_\alpha\, v = w + z \tag{7.18}$$

Now if we regard $\tilde{\nabla}$ as the general affine connection arising from the general (unconstrained) $\mathcal{B}(M)$ $GL(4, R)$-valued connection 1-form $\Gamma_\mu{}^\epsilon{}_\nu$ with $\hat{\nabla}$ the affine connection arising from $\mathcal{SO}(3,1)$ as before, we see that the arbitrary tensor, $K_\mu{}^\epsilon{}_\nu$, links the two by "undoing" the uniqueness invoked by moving from $\mathcal{B}(M)$ to the reduced bundle $\mathcal{SO}(3,1)(M)$. In other words, the resultant affine connection, $\tilde{\nabla}$, with associated connection coefficient, $\tilde{\Gamma}$, can once more be viewed as a general unconstrained affine connection where metric compatibility and torsion are completely undetermined.

This is a helpful picture, but an obvious question suggests itself: If the object of our generalized formalism was to return to a completely general affine connection

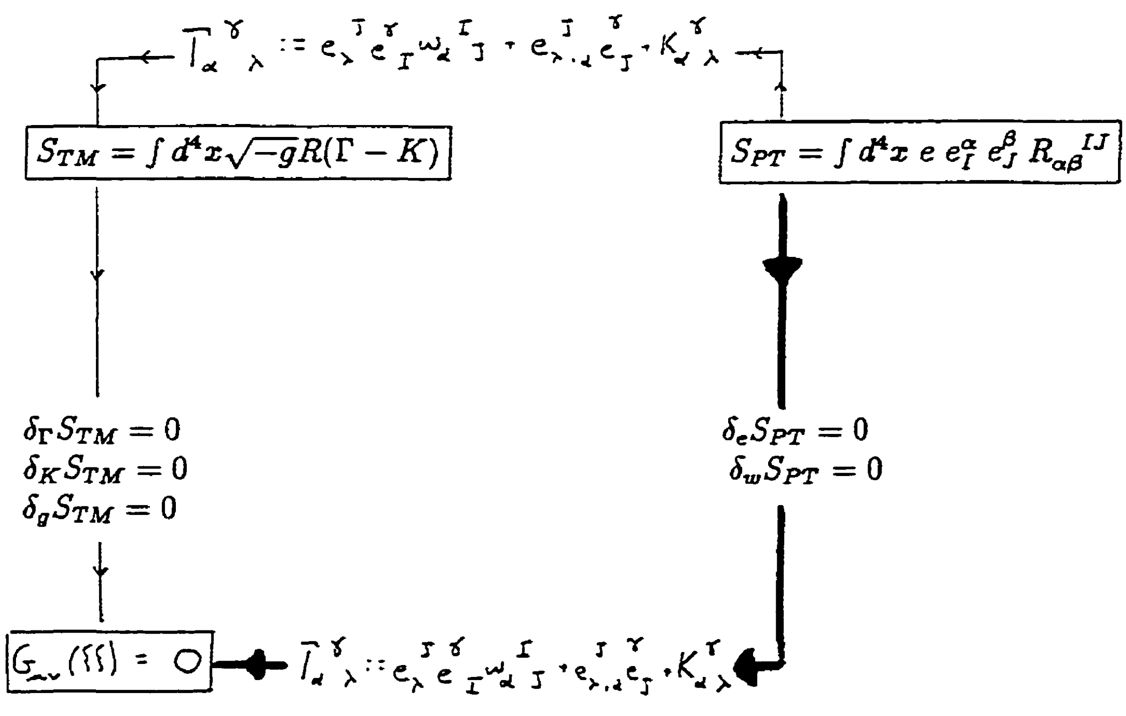
Figure 7.1: $\mathcal{B}(M)$ vs $\mathcal{SO}(3,1)(M)$ Covariant Derivatives on TM



derived from the full bundle of frames $\mathcal{B}(M)$, why didn't we simply do that to begin with? In other words, what is the purpose in moving over to $\mathcal{SO}(3,1)(M)$ and back to TM before then "undoing" matters?

The answer to this question lies in the matter of calculational ease. A full understanding of this will be readily apparent in the section dealing with actions later on in this chapter, but for now it is enough to point out that we can see from this diagram that the frame field, e plays a dual role in our generalized picture. The frame function, e^I_α, is a local cross section of the principal bundle $\mathcal{B}(M)$ and gives, for every $x \in M$ a corresponding element of $GL(4,R)$. As such it "belongs" to $\mathcal{B}(M)$ (the relevant analogous section of $\mathcal{SO}(3,1)(M)$ would be, of course, an element of the group SO(3,1) and would reflect a Lorentz transformation rather than a general coordinate transformation) and *not* to $\mathcal{SO}(3,1)(M)$. Yet, as we have seen in Appendix E, this same frame field is used to "push over" the covariant derivative, D, from $(\mathcal{SO}(3,1)(M)_{\Re^N}, \pi_{\Re^N}, M)$ over to TM. Therefore, although the frame field is necessarily associated with $\mathcal{B}(M)$, we also use it in conjunction with the connection on the reduced bundle $\mathcal{SO}(3,1)(M)$ and our generalized prescription (7.3) to *simulate* a full affine connection. And so our original choice of opting to study the connections on our reduced bundle, $\mathcal{SO}(3,1)(M)$ appears to be motivated more by calculational ease - like a clever choice of coordinates - than by anything geometrically intrinsic to the problem at hand. But the preceeding analysis also indicates that the often expressed idea that we are really limiting ourselves here to the reduced bundle $\mathcal{SO}(3,1)(M)$, is not an entirely honest one, for we indirectly use cross-sections of the *full* bundle of frames, $\mathcal{B}(M)$ to suitably define our affine connection, $\hat{\nabla}$, associated with w of $\mathcal{SO}(3,1)(M)$.

Figure 7.2: Comparison of Palatini Tetrad and Standard Palatini



7.2 Relating Palatini-Tetrad to Standard Palatini

In light of this new generalized prescription (7.3) we are finally able to relate "the two Palatinis" as promised earlier. Consider the following schematic diagram: where we have written both the Palatini-Tetrad and Standard Palatini formalisms explicitly.

We *begin* our treatment with the Palatini-Tetrad action in the top right hand corner. This starting point is not arbitrary, as we later find that, as alluded to earlier, the action S_{PT} together with the generalized prescription (7.3) is merely a rewritten form of a more general action.

We have already discovered that if we vary the action S_{PT} with respect to the relevant dynamical variables e and w and utilize the generalized prescription (7.3) (i.e. follow the flow of the thick arrows), we end up with the Einstein Field Equations in vaccuo,

$$G_{\mu\nu}(\{\}) = 0 \tag{7.19}$$

We now examine what occurs should we proceed in the other direction (i.e. that of the thinner arrows) by *first* transforming the Palatini Tetrad action S_{PT} via (7.3) and *then* varying with respect to the "new" relevant dynamical quantities.

From Chapter 5 we know that

$$e^J_\epsilon \, e^\lambda_I \, R_{\alpha\beta}{}^I{}_J = R_{\alpha\beta}{}^\lambda{}_\epsilon(\tilde{\Gamma} - K) \tag{7.20}$$

under (7.3). Therefore we see that the action S_{PT} smoothly transforms to

$$S_{TM} = \int d^4x \, \sqrt{-g} \, R(\Gamma - K)$$

where we have dropped the tildes. We note in general, for $T_\mu{}^\epsilon{}_\nu \neq 0$, we have:

$$\begin{aligned}\int d^Nx \, \sqrt{-g} \, R(\Gamma - K) = \int d^Nx\sqrt{-g} \, \Big[& R(\Gamma) + K_\lambda{}^\lambda{}_\gamma \, K_\sigma{}^{\gamma\sigma} - K_\epsilon{}^\lambda{}_\gamma \, K_\lambda{}^{\gamma\epsilon} \\ & \tfrac{1}{2} \left(K_\beta{}^\lambda{}_\epsilon + K_\epsilon{}^\lambda{}_\beta\right) \left[V_\lambda g^{\beta\epsilon} + \nabla_\lambda g^{\beta\epsilon}\right] \\ & + T_\rho{}^\epsilon{}_\epsilon \, (K_\sigma{}^{\rho\sigma} - K_\sigma{}^{\sigma\rho}) - (V^\epsilon + Z^\epsilon) \, K_\lambda{}^\lambda{}_\epsilon \\ & + T_\epsilon{}^\gamma{}_\lambda \, K_\gamma{}^{\lambda\epsilon} \Big]\end{aligned} \tag{7.21}$$

If we break up our arbitrary tensor $K_\mu{}^\epsilon{}_\nu$ into symmetric and anti-symmetric parts,

$$K_\mu{}^\epsilon{}_\nu := S_\mu{}^\epsilon{}_\nu + A_\mu{}^\epsilon{}_\nu \tag{7.22}$$

and substitute (7.22) into (7.21) we find, upon varying with respect to the antisymmetric tensor $A_\mu{}^\epsilon{}_\nu$:

$$\frac{1}{2} \, \delta^\mu_\epsilon \left[S_\sigma{}^{\nu\sigma} - (V^\nu + Z^\nu) + T_\rho{}^{\rho\nu}\right] + A_\epsilon{}^{\mu\nu} - \frac{1}{2} \, T_\epsilon{}^{\mu\nu} - (\mu \Rightarrow \nu) = 0 \tag{7.23}$$

Contraction ($\mu - \epsilon$, say) of (7.23) gives:

$$\frac{1}{2}(N-1)\left[S_{\sigma}{}^{\nu\sigma} - (V^{\nu} + Z^{\nu}) + T_{\rho}{}^{\rho\nu}\right] + A_{\sigma}{}^{\sigma\nu} - \frac{1}{2}T_{\sigma}{}^{\sigma\nu} = 0 \tag{7.24}$$

but from

$$\nabla_{\lambda}\, g^{\mu\nu} = K_{\lambda}{}^{\mu\nu} + K_{\lambda}{}^{\nu\mu} \tag{7.25}$$

we find

$$V^{\nu} + Z^{\nu} = 2\, A_{\sigma}{}^{\sigma\nu} + S_{\sigma}{}^{\nu\sigma} \tag{7.26}$$

Combining (7.24) with (7.26) leads to the relation

$$A_{\sigma}{}^{\sigma\nu} = \frac{1}{2}\, T_{\rho}{}^{\rho\nu} \tag{7.27}$$

which in turn leads to the following simplified version of (7.23)

$$U_{\epsilon}{}^{\mu\nu} - U_{\epsilon}{}^{\nu\mu} = 0 \tag{7.28}$$

where

$$U_{\epsilon}{}^{\alpha\beta} := A_{\epsilon}{}^{\alpha\beta} - \frac{1}{2}\, T_{\epsilon}{}^{\alpha\beta} \tag{7.29}$$

Finally we note that (7.28) leads to

$$U_{\epsilon\alpha\beta} = U_{\epsilon\beta\alpha}, \tag{7.30}$$

which, together with

$$U_{\epsilon\alpha\beta} = -U_{\beta\alpha\epsilon} \tag{7.31}$$

by definition, gives:

$$U_{\epsilon\alpha\beta} = 0 \Rightarrow A_{\mu}{}^{\epsilon}{}_{\nu} = \frac{1}{2}\, T_{\mu}{}^{\epsilon}{}_{\nu}, \tag{7.32}$$

that is,

$$2\, K_{[\mu}{}^{\epsilon}{}_{\nu]} = T_{\mu}{}^{\epsilon}{}_{\nu} \tag{7.33}$$

We have already seen that the constraint (7.33) in combination with the (non)metricity relationship,

$$\nabla_\lambda g_{\mu\nu} = -\left(K_{\lambda\mu\nu} + K_{\lambda\nu\mu}\right) \tag{7.34}$$

gives rise to the expression

$$\Gamma_\mu{}^\epsilon{}_\nu = \{_\mu{}^\epsilon{}_\nu\} + K_\mu{}^\epsilon{}_\nu \tag{7.35}$$

Insertion of (7.35) into our action S_{TM} clearly reduces it to one of *no* connection dependence, i.e. (7.35) transforms S_{TM} to:

$$S_{TM} = \int d^4x \sqrt{-g}\, R(\{\}) \tag{7.36}$$

If we now turn our attention to the case of a necessarily torsion-free connection, we see that we have the relationship (c.f.(7.35))

$$\Gamma_\mu{}^\epsilon{}_\nu = \{_\mu{}^\epsilon{}_\nu\} + S_\mu{}^\epsilon{}_\nu \tag{7.37}$$

which is nothing more than the full indeterminacy relationship for our torsion-free connection (3.32), whereas from Appendix A we know that under a general deformation transformation

$$\Gamma_\mu{}^\epsilon{}_\nu \Rightarrow \Gamma_\mu{}^\epsilon{}_\nu - S_\mu{}^\epsilon{}_\nu \tag{7.38}$$

our action is equivalent to our "maximally symmetric" action of Chapter 3. Hence in general, for $T = 0$, our corresponding action S_{TM} *is* our maximally symmetric action, and *not* the Einstein-Hilbert action. S_{TM} only becomes equivalent to the Einstein-Hilbert action if *both* $T = 0$ *and* $S = 0$, in which case, by the analysis above, we know that

$$\Gamma_\mu{}^\epsilon{}_\nu \equiv \{_\mu{}^\epsilon{}_\nu\} \tag{7.39}$$

So looked at in this light, we can finally answer our question of why the Christoffel relation,

$$\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\} \tag{7.40}$$

mysteriously pops out of the Einstein-Hilbert action under a Palatini variation. Since in writing down this action we had *a priori* set both the torsion tensor and the transformation tensor, S, to zero, we had determined ahead of time what the connection must be. That is, from a Palatini perspective, there is no real "connection variation" going on at all; $\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\}$ is a necessary consequence of writing down the Einstein-Hilbert in the first place.

7.3 Gravitational Actions

In relating the Standard Palatini to the Palatini Tetrad formalism we specified that one must "start" the analysis with the S_{PT} action,

$$S_{PT} = \int d^4x \; e \; e_I^{\alpha} \; e_J^{\beta} \; R_{\alpha\beta}^{IJ} \tag{7.41}$$

as opposed to

$$S_{TM} = \int d^4x \; \sqrt{-g} \; R(\Gamma - K) \tag{7.42}$$

This is because the generalized Einsteinian gravitational action [35, 18] is

$$S_E = \int tr \left[R \wedge \star(\beta \wedge \beta) \right] \tag{7.43}$$

where R is the local curvature 2-form from $\mathcal{B}(M)$ and β is the local solder form from $\mathcal{B}(M)$ (5.22). In keeping with our more general geometrical approach along the lines of Yang-Mills theory, we would like to define an action for *any* physical theory as that consisting of some scalar invariant composed of various quantities of the relevant principal bundle associated with the theory (together with, for those theories dependent on the existence of a metric, the hodge dual, $\star$).

For Yang-Mills theory, as we have mentioned, the relevant action is

$$S_{YM} = \int tr\,(F \;\wedge \star F) \tag{7.44}$$

for F the (local) curvature 2-form of some general principal bundle, and the relevant dynamics obtained by varying with respect to the connection 1-form, w - the only explicit variable one has to work with here - is:

$$\mathbf{D} \star F = 0 \tag{7.45}$$

With theories involving $\mathcal{B}(M)$, on the other hand, such as those pertaining to gravity, one now finds oneself with an extra mathematical entity, β with which one can build such an action. If we choose the particular action (7.43), one is left to assess the relevant dynamics by varying (7.43) with respect to **two** variables - w and β. The actual variation is somewhat tricky and one is forced, owing to the presence of β in the action, to pick some convenient coordinate frame in which to calculate it. We can simplify this calculation by choosing coordinates where the metric is defined to be orthonormal via the $GL(N, R)$ frames as per (5.32) and by using, for calculational ease, the connection from the reduced bundle $SO(N-1,1)(M)$, provided that we correct for this unwarranted specification of w at some later time in the calculation - i.e. by using our "generalized prescription" of Section 6.3.

If this generalization is *not* done - i.e. if one merely states that the relevant connection for our action *is* the $SO(N-1,1)$ connection - then one still gets the relevant final Einsteinian dynamics together with some definite restriction on the connection (the Christoffel constraint, again), just as one finds for the Standard Palatini approach to the Einstein-Hilbert action. And, just as one found for the latter case, this constraint is completely fictitious and arises from the fact that we have arbitrarily chosen some inappropriate subclass of our connection to start with. Since we know, with the benefit of hindsight, that our final dynamics resulting from (7.43) are connection *independent* anyway, it is hardly surprising that some *a priori* fixing of the connection does not manifest itself dynamically at the end of the day. But it certainly makes us discount any "information" we might find about the connection itself by such a procedure.

7.4 Matter actions

Virtually all of the previous analysis has been applied to Lagrangians without independent matter fields, and it is worth considering what would happen to our notions of a connection-independent action if we were to add an additional matter action term to the Lagrangian.

The usual prescription for such terms is to define the stress-energy tensor, $T^{\alpha\beta}$ according to (3.3), that is (for $N = 4$):

$$\sqrt{-g}\, T^{\alpha\beta} := 2\left[\frac{\delta\left(\int d^4x\sqrt{-g}\mathcal{L}_m\right)}{\delta g_{\alpha\beta}}\right] \tag{7.46}$$

where $\mathcal{L}_m$ represents the matter Lagrangian.

From either the Palatini Tetrad perspective or the Standard Palatini perspective, related via Section 7.2, addition of such matter terms change *none* of the preceeding analysis *if* the matter action is assumed to be *independent* of the generalized connection. The only thing affected in both cases is that, as one might expect, the consequent final dynamics is affected and thus moves from the vacuum Einstein relation,

$$G_{\mu\nu}(\{\}) = 0 \tag{7.47}$$

to the consequent full matter dynamics,

$$G_{\mu\nu}(\{\}) = 8\pi\, T_{\mu\nu} \tag{7.48}$$

We have added matter, but, owing to the *a priori* condition that such matter is independent of the generalized connection of the frame bundle, $\mathcal{B}(M)$, we naturally find that our connection invariance is preserved.

On the other hand, if we now allow our matter action to be dependent on the connection, we naturally find that the connection invariance of the action:

$$S_{tot} = \int tr\,[R \wedge \star(\beta \wedge \beta)] + S_m \tag{7.49}$$

no longer occurs, and we effectively move to the realm of Einstein-Cartan theory, where one must recover Einsteinian dynamics by breaking the symmetry of the connection and imposing extraneous conditions upon the connection (such as the "no torsion" condition).

Looked at from this perspective, we can regard the Einstein Field Equations as the necessary final dynamics of a sub-class of theories generated by the action:

$$S_{GR} = \int tr\,[R \wedge \star(\beta \wedge \beta)] + S_m \tag{7.50}$$

for S_m *independent* of w, which can be extended, by a sort of symmetry-breaking, to the more general class

$$S_{tot} = \int tr\,[R \wedge \star(\beta \wedge \beta)] + S_m \tag{7.51}$$

for general S_m, by imposing the additional external constraint, $T = 0$, where the torsion tensor is defined in terms of the solder form as per Chapter 5.

Chapter 8

Conclusions and Discussion

Largely motivated by the desire to understand the origins of the Christoffel constraint,

$$\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\} \tag{8.1}$$

from a (Standard) Palatini variation of the basic Einstein-Hilbert action,

$$S_{EH} = \int d^N x \left[\sqrt{-g}\, R(\Gamma)\right], \tag{8.2}$$

we began our investigation of the fundamental nature of connection dependence of the dynamics of General Relativity as formulated through an action principle. By utilizing dilaton theories of gravity (Section 3.2), we first noted that not *all* gravitational actions necessarily gave the Christoffel constraint, before proceeding to generalize the Einstein-Hilbert action into the so-called Extended Action, which yielded, as a special case, the "maximally symmetric" action

$$\begin{aligned} S_{MS} &= \int d^N x \sqrt{-g}[R(\Gamma) + \frac{1}{4}(\nabla_\nu g^{\alpha\beta})(\nabla^\nu g_{\alpha\beta}) + V^2 \\ &\qquad -\frac{1}{2}(\nabla_\epsilon g_{\mu\nu})(\nabla^\mu g^{\epsilon\nu}) + V \cdot Z, \end{aligned} \tag{8.3}$$

which was found to necessarily arise from a deformation transformation,

$$\Gamma_{\mu}{}^{\epsilon}{}_{\nu} \Rightarrow \hat{\Gamma}_{\mu}{}^{\epsilon}{}_{\nu} = \Gamma_{\mu}{}^{\epsilon}{}_{\nu} + Q_{\mu}{}^{\epsilon}{}_{\nu} \tag{8.4}$$

on any general Extended Action.

Under a Palatini variation, this "maximally symmetric" action gave (vacuum) Einsteinian dynamics but *didn't* give the aforementioned Christoffel constraint and instead left the connection indeterminate. We then examined the possibility of extending such actions into a domain where the torsion tensor did not necessarily vanish and examined the consequent non-Einsteinian dynamics together with the general form of the connection, $\Gamma_{\mu}{}^{\epsilon}{}_{\nu}$, before returning to the Einstein-Hilbert action in a somewhat more geometrical guise via the Palatini Tetrad formalism. We found that the conventional $3+1$ Palatini Tetrad approach contained some hidden assumptions which explicitly break the inherent *connection invariance* of the theory as manifested by the action,

$$S_E = \int tr\,[R \wedge \star(\beta \wedge \beta)]\,, \tag{8.5}$$

where the (local) curvature 2-form, R, and the solder form, β, are those corresponding to the full $GL(N, R)$ bundle of frames, $\mathcal{B}(M)$.

The above action, (8.5), when analysed properly, was shown to be equivalent to the action

$$S_{TM} = \int d^4x\,\sqrt{-g}\,R(\Gamma - K) \tag{8.6}$$

with

$$\nabla_{\lambda} g^{\mu\nu} = K_{\lambda}{}^{\mu\nu} + K_{\lambda}{}^{\nu\mu} \tag{8.7}$$

which in turn was shown to be equivalent to the maximally symmetric action above for the special case of $T = 0$.

Finally, we saw that, although (vacuum) Einsteinian dynamics necessarily arose from (8.5) above, we could generalize our action further to include matter terms, thereby producing:

$$S_{tot} = \int tr\,[R \wedge \star(\beta \wedge \beta)] + S_m \tag{8.8}$$

where the extra matter term, S_m plays the role of a (connection) symmetry breaker, thereby enabling us to regard (8.8) as the connection *invariant* action which necessarily gives rise to full Einsteinian dynamics,

$$G_{\mu\nu}(\{\}) = 8\pi\, T_{\mu\nu} \tag{8.9}$$

if S_m is independent of w, and otherwise can be made to yield (8.9) if one supplements (8.8) with the external constraint,

$$T = 0 \tag{8.10}$$

for the torsion tensor defined in the usual way via the solder form (5.27).

It is worth noting here that (8.9) does not generally lead to covariant conservation of stress-energy, $T_{\mu\nu}$, for arbitrary connection, Γ, as the usual form of the Bianchi identity,

$$\nabla^{\mu} G_{\mu\nu} = 0 \tag{8.11}$$

implicity assumes the Christoffel constraint, (8.1) in its derivation from the more general kinematical identity, (5.13),

$$R = dw + w \wedge w \tag{8.12}$$

of Chapter 5. The fact that most physical stress energy tensors *do* satisfy the covariant conservation

$$\mathcal{D}^{\mu} T_{\mu\nu} = 0 \tag{8.13}$$

for $\mathcal{D}$ the covariant derivative associated with the Christoffel symbol, clearly reduces the left hand side of (5.13) above to the usual (Christoffel assuming) Bianchi identity,

$$\mathcal{D}^{\mu} G_{\mu\nu}(\{\}) = 0 \tag{8.14}$$

but in no way negates the validity of the generalized connection invariance derived above.

The significance of this generalized connection invariance of (8.5) is not entirely clear. It is evident that, from a pragmatic perspective, this invariance enables one to work with principal bundles where the structure group is a *sub-group* of the "full" $GL(N, R)$ of the bundle of frames, $\mathcal{B}(M)$, rather than $GL(N, R)$ itself - an attribute which was exploited in the Palatini Tetrad Formalism and in Ashtekar's use of a complexified space. If all one demands from an action principle is that, at the end of the day, the required dynamics are produced, then (modulo such issues as reality conditions and the like) any particular connection from any generalized sub-group will likely do the job[1], and one might as well pick, *a priori*, a convenient space - i.e. a convenient connection. Ashtekar's complexified space was chosen, of course, because of its ability to simplify the associated Hamiltonian constraints and hence lead one further along the path towards quantization. But by arbitrarily selecting some other connection of some reduced bundle of $\mathcal{B}(M)$, one breaks the general connection invariance of the theory as manifested by (8.5). Does this matter? The answer to that question likely depends on the meaning of the connection invariance in the first place.

One would ideally like to see this invariance manifested in some way as some sort of "conserved quantity" or deep structural geometrical feature which separates General Relativity from other theories, or at least places it in some (potentially non-unique) distinguishing sub-class of connection invariant theories. One is tempted to conclude that the invariance is somehow intimately related to the diffeomorphism invariance of General Relativity, for example, but it is not immediately clear how one could formulate that link. Moreover, it is possible to imagine other gravitational theories that could satisfy diffeomorphism invariance which do not follow from a variation

[1]Barring potential topological obstructions.

of (8.5). With regards to the "full" action, (8.8), it is also unclear what it means, geometrically, to ascribe some sort of "symmetry breaking" property to matter actions that are connection *dependent*.

We have seen how the connection invariance is possible in (8.5) owing to the existence of the solder form, β, as another dynamical variable of the theory. Can one somehow generalize the notion of a solder form to include other spaces which are potentially unrelated to $\mathcal{B}(M)$ in order to construct a more general class of actions which would be connection-invariant?

Lastly, while the above arguments concerning the physical preference of the Christoffel connection due to stress-energy conservation might well serve as some sort of physical motivation for classical General Relativity (i.e. (5.13) *and* (8.1)), it is conceivable that it might well be important to consider the full connection invariance of the theory when considering the quantum regime.

A more careful examination of at least one of these issues would likely shed a great deal more light on the property of generalized connection invariance and its physical and mathematical ramifications. The investigation of such questions thereby represents a possible future avenue of research.

Appendix A

$N = 2$ Palatini Dynamics From a Generalized Dilaton Action

We can see from the form of equation (3.19) that for $N = 2$ the approach given above will break down: we will no longer be able to find an explicit expression for $\left(\frac{\nabla_\lambda \sqrt{-g}}{\sqrt{-g}}\right)$, and hence eventually $\nabla_\lambda g^{\mu\nu}$ in terms of functions of the dilaton field and its derivative. Instead, for $N = 2$, we are merely left with an added constraint:

$$D' + 2B + C = 0. \tag{A.1}$$

Note that if (A.1) does not hold then from (3.19) the dilaton must be constant $\Psi = \Psi_0$. The field equations (3.15), (3.24) then reduce to

$$\frac{-1}{\sqrt{-g}} \nabla_\lambda \left[D_0 \sqrt{-g} g^{\mu\nu}\right] = 0 \tag{A.2}$$

and

$$8\pi T_{\mu\nu} = D_0 G_{(\mu\nu)}(\Gamma) \tag{A.3}$$

where $D_0 = D(\Psi_0)$ is constant. This situation was previously investigated in [16]. Although it appears to yield non-trivial dynamics, this does not occur because eq. (A.2) is invariant under the transformation

$$\Gamma_\mu{}^\epsilon{}_\nu \Rightarrow \hat{\Gamma}_\mu{}^\epsilon{}_\nu = \Gamma_\mu{}^\epsilon{}_\nu + \delta^\epsilon_\nu A_\mu + \delta^\epsilon_\mu A_\nu - g_{\mu\nu} A^\epsilon, \tag{A.4}$$

where A_λ is an arbitrary vector field. From this it may be shown [16] that the general solution to (A.2) is

$$\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\} + \delta^{\epsilon}_{\nu} A_{\mu} + \delta^{\eta}_{\mu} A_{\nu} - g_{\mu\nu} A^{\epsilon} \tag{A.5}$$

where A_μ is undetermined. Insertion of this into the right hand side of (A.3) yields $G_{(\mu\nu)}(\Gamma) = 0$. Hence the theory is either inconsistent (if $T_{\mu\nu} \neq 0$) or trivial (if $T_{\mu\nu} = 0$).

For Ψ not constant we can understand the constraint (A.1) in the following way. For $N = 2$ the associated action (3.12) is invariant under the transformation (A.3) provided the constraint (A.1) is valid. Since A_λ is arbitrary, we can choose it in such a way as to achieve explicit dynamical equations for $N = 2$. Since under (A.4)

$$\frac{\widehat{\nabla}_\lambda \sqrt{-g}}{\sqrt{-g}} = \frac{\nabla_\lambda \sqrt{-g}}{\sqrt{-g}} - 2A_\lambda. \tag{A.6}$$

we chose

$$A_\lambda = \frac{1}{2}\left(\frac{\nabla_\lambda \sqrt{-g}}{\sqrt{-g}}\right) \tag{A.7}$$

so that

$$\nabla_\lambda g^{\mu\nu} = Y\left[(\partial^\mu \Psi)\delta^\nu_\lambda + (\partial^\nu \Psi)\delta^\mu_\lambda - (\partial_\lambda \Psi) g^{\mu\nu}\right] \tag{A.8}$$

and

$$\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\} - \frac{1}{2}Y\left[(\partial_\mu \Psi)\delta^\epsilon_\nu + (\partial_\nu \Psi)\delta^\epsilon_\mu - 3g_{\mu\nu}(\partial^\epsilon \Psi)\right] \tag{A.9}$$

where the hat notation has been dropped and $B(\Psi)$ has been eliminated using (A.1). If one combines (A.9) with the equations obtained by varying (3.12) with respect to $g_{\alpha\beta}$ and Ψ, one finds:

$$\begin{aligned} 8\pi T_{\mu\nu} \;=\;& \left[\frac{1}{2}D(Y' + Y^2) + \frac{1}{2}D'Y(4 - 3N)\frac{1}{2}A - B' - Y(C + 2B)\right](\partial\Psi)^2 g_{\mu\nu} \quad \text{(A.10)} \\ & + [A - C' - D(Y' + Y^2) - D'Y](\partial_\mu\Psi)(\partial_\nu\Psi) - D'[\mathcal{D}_\mu(\partial_\nu\Psi) - (\mathcal{D}^2\Psi)g_{\mu\nu}], \end{aligned}$$

and

$$\begin{aligned}&\{F'' - A' + 2Y'(F - C) + Y[(3N - 6)A + (2 - 3N)F']\\&+Y^2[7D' + 6(F - C) - 3N(F - C + D')]\}(\partial\Psi)^2\\&+D'R(\{\}) + 2\,[F' + Y(F - C + D') - A]\,(\mathcal{D}^2\Psi) = 0,\end{aligned} \tag{A.11}$$

That is,

$$D'R(\{\}) + \hat{\Pi}(\partial\Psi)^2 + \hat{\Lambda}(\mathcal{D}^2\Psi) = 0, \tag{A.12}$$

with the obvious definitions for $\hat{\Pi}$ and $\hat{\Lambda}$ in accordance with (A.11) above.

Appendix B

Variation of a (Torsion-Free) Extended Action Under a Deformation Transformation

The following is a proof of the claim made towards the end of Section 3.3 - namely that for any general action of the form (3.41) to be invariant under a deformation transformation, one finds that a necessary constraint is that the action must be maximally symmetric - i.e. H, I, J, K, L must satisfy the particular values of (3.72)

If one begins with our usual generalized action, with H,I,J,K,L arbitrary, that is:

$$\begin{aligned} S_{EHE} &= \int d^N x\sqrt{-g}[R + H(\nabla_\nu g^{\alpha\beta})(\nabla^\nu g_{\alpha\beta}) + IV^2 \\ &\qquad + J(\nabla_\epsilon g_{\mu\nu})(\nabla^\mu g^{\epsilon\nu}) + KV\cdot Z + LZ\cdot Z], \end{aligned} \tag{B.1}$$

and apply to it the variation:

$$\Gamma_\mu{}^\epsilon{}_\nu \Rightarrow \hat{\Gamma}_\mu{}^\epsilon{}_\nu = \Gamma_\mu{}^\epsilon{}_\nu + Q_\mu{}^\epsilon{}_\nu \tag{B.2}$$

for $Q_\mu{}^\epsilon{}_\nu$ any arbitrary tensor symmetric in first and third indices, we find that S consequently transforms to:

$$S_{EHE} \Rightarrow \hat{S}_{EHE} = S_{EHE} + \delta S, \tag{B.3}$$

where

$$
\begin{aligned}
\delta S &= [1+2J](\nabla^{\epsilon} g^{\mu\nu})Q_{\mu\epsilon\nu} + [2H+J](\nabla^{\epsilon} g^{\mu\nu})(Q_{\epsilon\mu\nu} + Q_{\epsilon\nu\mu}) \\
&\quad -[1+2H+3J]Q^{\epsilon\mu\nu}Q_{\epsilon\nu\mu} - [2H+J]Q^{\epsilon\mu\nu}Q_{\epsilon\mu\nu} \\
&\quad +[I-K+L]Q_{\lambda}{}^{\lambda\rho}Q^{\sigma}{}_{\sigma\rho} + [1-K+2L]Q^{\lambda}{}_{\lambda\rho}Q_{\sigma}{}^{\rho\sigma} \\
&\quad +LQ_{\sigma\rho}{}^{\sigma}Q_{\lambda\rho}{}^{\lambda} + [1-2I+K]V_{\lambda}Q_{\rho}{}^{\rho\lambda} + [K-1]V_{\lambda}Q_{\eta}{}^{\lambda\eta} \\
&\quad +2LZ^{\lambda}Q_{\eta\lambda}{}^{\eta} + [1+2L-K]Z^{\lambda}Q^{\eta}{}_{\eta\lambda} \qquad \text{(B.4)}
\end{aligned}
$$

If we subject this new action, $\hat{S}_{EHE}$, to a variation with respect to $Q_{\mu}{}^{\epsilon}{}_{\nu}$, we clearly have:

$$\delta_{Q_{\mu}{}^{\epsilon}{}_{\nu}}(\hat{S}_{EHE}) = \delta_{Q_{\mu}{}^{\epsilon}{}_{\nu}}(\delta S),$$

since $\delta_{Q_{\mu}{}^{\epsilon}{}_{\nu}}(S) = 0$. Now, from above we see that $\delta_{Q_{\mu}{}^{\epsilon}{}_{\nu}}(\hat{S}_{EHE}) = 0 = \delta_{Q_{\mu}{}^{\epsilon}{}_{\nu}}(\delta S)$ can be expressed as:

$$
\begin{aligned}
0 = \int d^N x\sqrt{-g}(\delta Q_{\alpha}{}^{\lambda}{}_{\beta})[&(1+2J)\nabla_{\lambda} g^{\alpha\beta} + (2H+J)g_{\lambda\mu}[\nabla^{\alpha} g^{\mu\beta} + \nabla^{\beta} g^{\mu\alpha}] \\
&-[1+2H+3J](Q^{\alpha\beta}{}_{\lambda} + Q^{\beta\alpha}{}_{\lambda}) - 2(2H+J)Q^{\alpha}{}_{\lambda}{}^{\beta} \\
&[I-K+L]\left[(Q_{\epsilon}{}^{\epsilon\beta})\delta^{\alpha}_{\lambda} + (Q_{\epsilon}{}^{\epsilon\alpha})\delta^{\beta}_{\lambda}\right] \\
&+[1-K+2L]\left(\frac{1}{2}\left[(Q^{\epsilon\beta}{}_{\epsilon})\delta^{\alpha}_{\lambda} + (Q^{\epsilon\alpha}{}_{\epsilon})\delta^{\beta}_{\lambda}\right] + Q^{\epsilon}{}_{\epsilon\lambda}g^{\alpha\beta}\right) \\
&+g^{\alpha\beta}\left[2L\left(Q_{\epsilon\lambda}{}^{\epsilon} + Z_{\lambda}\right) + (K-1)V_{\lambda}\right] \\
&+\frac{1}{2}\left[(1-2I+K)V^{\beta} + (1+2L-K)Z^{\beta}\right]\delta^{\alpha}_{\lambda} \\
&+\frac{1}{2}[(1-2I+K)V^{\alpha} + (1+2L-K)Z^{\alpha}]\,\delta^{\beta}_{\lambda}] \qquad \text{(B.5)}
\end{aligned}
$$

Clearly for arbitrary $\delta Q^{\lambda}{}_{\alpha\beta}$, we have the constraint:

$$[\quad] = 0 \qquad \text{(B.6)}$$

Taking the $g_{\alpha\beta}$ trace of B.6 yields:

$$AQ^{\epsilon}{}_{\epsilon\lambda} + B\left[Q_{\lambda\epsilon}{}^{\epsilon} + Z_{\lambda}\right] + CV_{\lambda} = 0 \qquad \text{(B.7)}$$

While contracting over, say, λ and α yields:

$$DQ^{\epsilon}{}_{\epsilon\lambda} + E\left[Q_{\lambda\epsilon}{}^{\epsilon} + Z_{\lambda}\right] + FV_{\lambda} = 0 \tag{B.8}$$

where

$$A = [(N-2) - 4H + 2I - 6J - K(N+2) + 2L(N+1)] \tag{B.9}$$

$$B = [1 - 4H - 2J - K + (1+N)L] \tag{B.10}$$

$$C = [(3-N) - 2I + 4J + (1+N)K] \tag{B.11}$$

$$D = [-6H + (N+1)I - 5J - (N+2)K + (N+3)L] \tag{B.12}$$

$$E = \left[\frac{1}{2}(N-1) - 2H - 3J - \frac{1}{2}(N+1)K + (N+3)L\right] \tag{B.13}$$

$$F = \left[\frac{1}{2}(N-1) + 4H - (N+1)I + 2J + \frac{1}{2}(N+3)K\right] \tag{B.14}$$

We note the following relationships:

$$BD - AE = CE - BF \tag{B.15}$$

and

$$F + D = E \tag{B.16}$$

Meanwhile, together (B.7) and (B.8) imply the following:

$$[BD - AE]Q^{\epsilon}{}_{\epsilon\lambda} + [BF - CE]V_{\lambda} = 0 \tag{B.17}$$

Therefore, (B.15),(B.16) and (B.17) in turn imply:

$$Q^{\epsilon}{}_{\epsilon\lambda} = V_{\lambda} \tag{B.18}$$

and

$$Q_{\epsilon\lambda}{}^{\epsilon} = -\left(V_{\lambda} + Z_{\lambda}\right) \tag{B.19}$$

Inserting (B.18) and (B.19) into (B.6), yields, after a bit of symmetrization and manipulation:

$$Q_{\alpha\lambda\beta} = \frac{1}{2}\left[P_{\lambda\alpha\beta} - P_{\alpha\beta\lambda} - P_{\beta\lambda\alpha}\right] \tag{B.20}$$

where $P_{\mu}{}^{\nu\rho} = \nabla_{\mu} g^{\nu\rho}$ and $P_{\mu\nu\lambda} = -\nabla_{\mu}\, g_{\nu\lambda}$. Inserting (B.18),(B.19) and (B.20) into (B.4) gives:

$$\delta S = -S_{EHE} + \int d^N x \sqrt{-g}\left[R + \frac{1}{4}(\nabla_{\alpha} g^{\mu\nu})(\nabla^{\alpha} g_{\mu\nu}) - \frac{1}{2}(\nabla_{\epsilon} g_{\mu\nu})(\nabla^{\mu} g^{\epsilon\nu}) + V^2 + V \cdot Z\right], \tag{B.21}$$

in other words, our maximally symmetric values.

Therefore, the unique extended action unaffected by a deformation transformation, (B.3), is that for which H, I, J, K, L take on the particular maximally symmetric values.

Moreover, we find that *any* general action of the form (B.1), subjected to a deformation transformation, (B.2), is necessarily equivalent to a maximally symmetric action (see Section 7.2).

Appendix C

Explicit Calculation of The Connection for S_{EHE} With Torsion For Λ of Rank 2

We recall that a Palatini variation of S_{EHE} with torsion leads to (4.17), i.e:

$$\begin{aligned} T_\lambda{}^{\mu\nu} - \nabla_\lambda g^{\mu\nu} + g^{\mu\nu}[T_\sigma{}^\sigma{}_\lambda - V_\lambda] + \delta^\mu_\lambda[V^\nu + Z^\nu - T_\sigma{}^{\sigma\nu}] \\ +4H[g^{\nu\sigma}\nabla^\mu g_{\lambda\sigma}] - 2I(\delta^\nu_\lambda V^\mu) + 2J[g^{\mu\sigma}g^{\epsilon\nu}\nabla_\epsilon g_{\sigma\lambda} - \nabla_\lambda g^{\mu\nu}] \\ +K[V_\lambda g^{\mu\nu} + \delta^\mu_\lambda V^\nu - \delta^\nu_\lambda Z^\mu] + 2L[\delta^\mu_\lambda Z^\nu + Z_\lambda g^{\mu\nu}] &= 0 \end{aligned} \qquad \text{(C.1)}$$

while tracing and contracting the above yields the following (3×3) matrix for 3 relations involving V^λ, Z^λ and $T_\rho{}^{\rho\lambda}$:

$$\Lambda := \begin{vmatrix} (2-N) & \kappa & \iota \\ 0 & \vartheta & \varrho \\ (2-N) & P & Q \end{vmatrix} \qquad \text{(C.2)}$$

If Λ is singly degenerate, then we can express both V^λ and Z^λ in terms of $T_\rho{}^{\rho\lambda}$. That is, for some $\bar{\Theta}$, $\bar{\Upsilon}$, we have:

$$V^\lambda = \bar{\Theta} T_\rho{}^{\rho\lambda} \qquad \text{(C.3)}$$

$$Z^\lambda = \bar{\Upsilon} T_\rho{}^{\rho\lambda} \qquad \text{(C.4)}$$

Meanwhile we can rearrange (C.1) so as to present it in the following simplified form:

$$A \nabla_\epsilon g_{\alpha\beta} + B \nabla_\alpha g_{\beta\epsilon} + C \nabla_\beta g_{\epsilon\alpha} = \chi_{\epsilon\alpha\beta} \tag{C.5}$$

where

$$A := 1 + 2J \tag{C.6}$$

$$B = 4H \tag{C.7}$$

$$C = 2J \tag{C.8}$$

and

$$\begin{aligned} \chi_{\epsilon\alpha\beta} &= g_{\beta\epsilon}[2IV_\alpha + KZ_\alpha] - g_{\epsilon\alpha}\left[(1+K)V_\beta + (1+2L)Z_\beta - T_\sigma{}^\sigma{}_\beta\right] \\ &\quad - g_{\alpha\beta}\left[(K-1)V_\epsilon + 2LZ_\epsilon + T_\sigma{}^\sigma{}_\epsilon\right] - T_{\epsilon\alpha\beta} \end{aligned} \tag{C.9}$$

Using (C.3) and (C.4) we can re-express (C.9) in the following manner:

$$\chi_{\epsilon\alpha\beta} = \tilde{a} g_{\epsilon\alpha} T_\sigma{}^\sigma{}_\beta + \tilde{b} g_{\alpha\beta} T_\rho{}^\rho{}_\epsilon + \tilde{c} g_{\epsilon\beta} T_\lambda{}^\lambda{}_\alpha - T_{\epsilon\alpha\beta} \tag{C.10}$$

where

$$\tilde{a} := \left[1 - (1+K)\bar{\Theta} - (1+2L)\bar{\Upsilon}\right] \tag{C.11}$$

$$\tilde{b} := \left[(1-K)\bar{\Theta} - 2L\bar{\Upsilon} - 1\right] \tag{C.12}$$

$$\tilde{c} := \left[2I\bar{\Theta} + K\bar{\Upsilon}\right] \tag{C.13}$$

If $C \neq 0$, permuting (C.5) gives

$$D \nabla_\alpha g_{\beta\epsilon} + E \nabla_\beta g_{\epsilon\alpha} = \Pi_{\alpha\beta\epsilon} \tag{C.14}$$

where

$$\Pi_{\alpha\beta\epsilon} := \chi_{\beta\epsilon\alpha} + \chi_{\epsilon\alpha\beta} - \left(\frac{(A+B)}{C}\right) \chi_{\alpha\beta\epsilon} \tag{C.15}$$

and

$$D := \frac{(C^2 - AB + BC - A^2)}{C} \tag{C.16}$$

$$E := \frac{(C^2 - AB + AC - B^2)}{C} \tag{C.17}$$

Equation (C.14) is explicitly soluble so long as we don't have $D^2 = E^2$, which is exactly the constraint of (4.34) We therefore define the parameter $\tilde{F}$ to represent the determinacy of these equations, i.e. we define

$$\tilde{F} := 2\left(\frac{E}{E^2 - D^2}\right) \tag{C.18}$$

and note that it is undefined when the connection is indeterminate according to (4.34).

Therefore, for $\tilde{F}$ well-defined, permuting (C.14) yields

$$\nabla_\beta g_{\epsilon\alpha} = \tilde{F}\left[\Pi_{\alpha\beta\epsilon} - \frac{D}{E}\Pi_{\beta\alpha\epsilon}\right] \tag{C.19}$$

which can be eventually written as

$$\nabla_\beta g_{\epsilon\alpha} = \tilde{F}\left[2\tilde{d}g_{\epsilon\alpha}T_\sigma{}^\sigma{}_\beta + (\tilde{e}+\tilde{f})[g_{\beta\epsilon}T_\rho{}^\rho{}_\alpha + g_{\alpha\beta}T_\lambda{}^\lambda{}_\epsilon] + \tilde{s}[T_{\alpha\epsilon\beta} + T_{\beta\alpha\epsilon}\right] \tag{C.20}$$

where

$$\tilde{d} := \tilde{a}(1 - \frac{D}{E}) + \tilde{b}(1 - \frac{DX}{E}) + \tilde{c}(X - \frac{D}{E}) \tag{C.21}$$

$$\tilde{e} := \tilde{a}(1 - \frac{D}{E}) + \tilde{b}(X - \frac{D}{E}) + \tilde{c}(1 - \frac{DX}{E}) \tag{C.22}$$

$$\tilde{f} := \tilde{a}(X - \frac{DX}{E}) + \tilde{b}(1 - \frac{D}{E}) + \tilde{c}(1 - \frac{D}{E}) \tag{C.23}$$

$$\tilde{s} := \frac{D(1 - X)}{E} \tag{C.24}$$

with

$$X := -\left[\frac{A+B}{C}\right] \tag{C.25}$$

A final permutation of (C.20) thus yields our explicit form for the connection - i.e

$$\Gamma_\mu{}^\epsilon{}_\nu = \{_\mu{}^\epsilon{}_\nu\} + \tilde{F}\left[g_{\mu\nu}(\tilde{d}-\tilde{e}-\tilde{f})T_\rho{}^{\rho\epsilon} - \tilde{d}\left(\delta^\epsilon_\mu\, T_\rho{}^\rho{}_\nu + \delta^\epsilon_\nu\, T_\sigma{}^\sigma{}_\mu\right)\right]$$

$$+\left[\tilde{F}\left(\tilde{s}-\frac{1}{2}\right)\right]\left[T_{\mu\nu}{}^\epsilon + T_{\nu\mu}{}^\epsilon\right] + \frac{1}{2}T_\mu{}^\epsilon{}_\nu \tag{C.26}$$

For $C = 0$ (i.e. $J = 0$), on the other hand, (C.5) simplifies considerably, giving

$$\nabla_\alpha g_{\epsilon\beta} + 4H\,\nabla_\epsilon\, g_{\beta\alpha} = \chi_{\alpha\epsilon\beta} \tag{C.27}$$

Permuting this result gives

$$\nabla_\epsilon g_{\alpha\beta}[1-16H^2] = \chi_{\epsilon\alpha\beta} - 4H\chi_{\alpha\epsilon\beta} \tag{C.28}$$

and thus we see that our indeterminacy condition is here

$$H \neq \pm\frac{1}{4} \tag{C.29}$$

For $H \neq \pm\frac{1}{4}$, the same procedure for the above case for $J \neq 0$ leads eventually to the following explicit form for the connection:

$$\Gamma_\mu{}^\epsilon{}_\nu = \{_\mu{}^\epsilon{}_\nu\} + (\xi - \frac{1}{2})\left[T_{\mu\nu}{}^\epsilon + T_{\nu\mu}{}^\epsilon\right] + \frac{1}{2}T_\mu{}^\epsilon{}_\nu$$
$$+\xi\left[g_{\mu\nu}T_\sigma{}^{\sigma\epsilon}\left([\tilde{b}-\tilde{c}](1+4H) + \tilde{a}(4H-1)\right) + (4H\tilde{c}-\tilde{b})\left(\delta^\epsilon_\mu\, T_\sigma{}^\sigma{}_\nu + \delta^\epsilon{}_\nu\, T_\rho{}^\rho{}_\mu\right)\right] \tag{C.30}$$

where

$$\xi := \left[\frac{1}{2(1+4H)(1-4H)}\right] \tag{C.31}$$

Of course if the indeterminacy constraints *do* hold then $\tilde{F}, \tilde{s}$ and ξ become undefined - that is, we can not explicitly solve for Γ in terms of the remaining variables and our analysis reduces, as mentioned in Chapter 4, to a weak constraint on $S_\mu{}^\epsilon{}_\nu, A_\mu{}^\epsilon{}_\nu, S_\sigma{}^\sigma{}_\lambda, S_{\rho\lambda}{}^\rho$ and $A_\mu{}^\epsilon{}_\nu$, reflecting some partial invariance of Γ under $\Gamma_\mu{}^\epsilon{}_\nu \Rightarrow \Gamma_\mu{}^\epsilon{}_\nu + K_\mu{}^\epsilon{}_\nu$, for some limited (i.e. constrained) choice of $K_\mu{}^\epsilon{}_\nu$.

We can check the above analysis for the Einstein-Hilbert action of Section 4.2, which we know to be singly degenerate.

Here, since $H = I = J = K = L = 0$, we work in the domain of C (i.e. $J) = 0$. Thus we have a connection of the form (C.30), where we find, for this case (c.f. (4.6), (4.7):

$$\tilde{\Theta} = \left(\frac{N}{N-1}\right) \tag{C.32}$$

$$\tilde{\Upsilon} = \left(\frac{2}{1-N}\right) \tag{C.33}$$

and

$$\tilde{a} = \tilde{b} = \left[\frac{1}{N-1}\right] \tag{C.34}$$

$$\tilde{c} = 0 \tag{C.35}$$

$$\xi = \frac{1}{2} \tag{C.36}$$

Therefore we find that $\Gamma_\mu{}^\epsilon{}_\nu$ has the explicit form:

$$\Gamma_\mu{}^\epsilon{}_\nu = \{_\mu{}^\epsilon{}_\nu\} + \frac{1}{2}\, T_\mu{}^\epsilon{}_\nu + \left[\frac{1}{2(1-N)}\right] \left(\delta^\epsilon_\mu\, T_\sigma{}^\sigma{}_\nu + \delta^\epsilon_\nu\, T_\rho{}^\rho{}_\mu\right) \tag{C.37}$$

as expected from (4.8) above.

Appendix D

Explicit Calculation of the Final Dynamics for S_{EHE} (Λ of Rank 2)

It was thought to be appropriate to explicitly calculate the terms in the final dynamics for at least *one* of the three cases (i.e. (4.39),(4.64),(4.78))treated in Chapter 4, if only to show that it could be done in a reasonable amount of time. In this Appendix, we choose the case of the singly degenerate Extended Action (with Torsion) with $J \neq 0$, where we assume the indeterminacy condition (4.34) does *not* hold and hence the connection (and consequent dynamics) can be solved exactly in terms of $g_{\alpha\beta}, T_{\mu}{}^{\epsilon}{}_{\nu}$ and $T_{\sigma}{}^{\sigma}{}_{\lambda}$. Clearly the methods delineated in this Appendix can be extended to (4.64) and (4.78).

Starting, then, with the action

$$\begin{aligned} S_{EHE} &= \int d^N x \sqrt{-g}[R(\Gamma) + H(\nabla_\rho g_{\alpha\beta})(\nabla^\rho g^{\alpha\beta}) + IV^2 + \\ &\quad + J(\nabla_\rho g_{\alpha\beta})(\nabla^\alpha g^{\rho\beta}) + KV.Z + LZ^2] \end{aligned} \tag{D.1}$$

and a connection of the general form

$$\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\} + Y_{\mu}{}^{\epsilon}{}_{\nu} \tag{D.2}$$

or, more explicitly,

$$\Gamma_{\mu}{}^{\epsilon}{}_{\nu} = \{_{\mu}{}^{\epsilon}{}_{\nu}\} + X_{\mu}{}^{\epsilon}{}_{\nu} + \lambda g_{\mu\nu} g^{\epsilon\rho} T_{\sigma}{}^{\sigma}{}_{\rho} + \gamma g^{\tau\epsilon} \left[g_{\nu\rho} T_{\mu}{}^{\rho}{}_{\tau} + g_{\mu\rho} T_{\nu}{}^{\rho}{}_{\tau} \right] \tag{D.3}$$

with

$$X_{\mu}{}^{\epsilon}{}_{\nu} := \eta \left[\delta^{\epsilon}_{\mu} T_{\sigma}{}^{\sigma}{}_{\nu} + \delta^{\epsilon}_{\nu} T_{\rho}{}^{\rho}{}_{\mu} \right] + \frac{1}{2} T_{\mu}{}^{\epsilon}{}_{\nu} \tag{D.4}$$

where we have broken up $Y_{\mu}{}^{\epsilon}{}_{\nu}$ up explicitly into the non-metric term $X_{\mu}{}^{\epsilon}{}_{\nu}$ and the various metric factors represented by the λ and γ terms.

For our general case of Λ singly degenerate, $J \neq 0$, we found from Appendix C above:

$$\lambda := \tilde{F}(\tilde{d} - \tilde{e} - \tilde{f}) \tag{D.5}$$

$$\gamma := \tilde{F}\tilde{s} - \frac{1}{2} \tag{D.6}$$

$$\eta := -\tilde{F}\tilde{d} \tag{D.7}$$

where $\tilde{F}, \tilde{d}, \tilde{e}, \tilde{f}, \tilde{s}$ are defined as per Appendix C above, and for Λ non-degenerate, we have the simplification $\eta = \lambda = 0$.

We note that in general, for (D.2), the following relationships hold:

$$R_{\mu\nu} = R_{\mu\nu}(\{\}) + \mathcal{D}_{\alpha} Y_{\mu}{}^{\alpha}{}_{\nu} - \mathcal{D}_{\mu} Y_{\sigma}{}^{\sigma}{}_{\nu} + Y_{\sigma}{}^{\sigma}{}_{\rho} Y_{\mu}{}^{\rho}{}_{\nu} - Y_{\mu}{}^{\sigma}{}_{\beta} Y_{\sigma}{}^{\beta}{}_{\nu} \tag{D.8}$$

where $\mathcal{D}_{\alpha}$ represents the covariant derivative with respect to the Christoffel symbol.

$$(\nabla_{\nu} g_{\alpha\beta})(\nabla^{\nu} g^{\alpha\beta}) = -2 g^{\nu\sigma} \left[Y_{\nu}{}^{\rho}{}_{\alpha} \left(Y_{\rho}{}^{\alpha}{}_{\sigma} + T_{\sigma}{}^{\alpha}{}_{\rho} \right) + g_{\beta\epsilon} g^{\alpha\rho} Y_{\nu}{}^{\epsilon}{}_{\alpha} Y_{\sigma}{}^{\beta}{}_{\rho} \right] \tag{D.9}$$

$$(\nabla_{\epsilon} g_{\mu\nu})(\nabla^{\mu} g^{\epsilon\nu}) = -g^{\nu\sigma} \left[Y_{\alpha}{}^{\epsilon}{}_{\nu} (3 Y_{\sigma}{}^{\alpha}{}_{\epsilon} + T_{\epsilon}{}^{\alpha}{}_{\sigma}) + g^{\epsilon\alpha} g_{\beta\rho} Y_{\nu}{}^{\rho}{}_{\alpha} Y_{\epsilon}{}^{\beta}{}_{\sigma} \right] \tag{D.10}$$

$$V_{\epsilon} = \bar{\Theta}\, T_{\sigma}{}^{\sigma}{}_{\epsilon} \tag{D.11}$$

$$Z^{\epsilon} = \bar{\Upsilon}\, g^{\epsilon\rho}\, T_{\sigma}{}^{\sigma}{}_{\rho} \tag{D.12}$$

where

$$\bar{\Theta} := \left[\eta(1+N) + \lambda - \frac{1}{2} - \gamma\right] \tag{D.13}$$

$$\bar{\Upsilon} := \left[(N+3)\eta + \frac{1}{2} + (N+1)\lambda + \gamma\right] \tag{D.14}$$

We can thus now express (D.1) in terms of $T_{\mu}{}^{\epsilon}{}_{\nu}, T_{\sigma}{}^{\sigma}{}_{\rho}, g_{\alpha\beta}$ and $R_{\mu\nu}(\{\})$ by utilizing equations (D.8)-(D.12) above. We therefore find that our action S_{EHE} takes the form:

$$\begin{aligned} S_{EHE} = \int d^N x\sqrt{-g}g^{\mu\nu}\ [R_{\mu\nu}(\{\}) + Y_{\sigma}{}^{\sigma}{}_{\rho}Y_{\mu}{}^{\rho}{}_{\nu} - (1+2H+3J)Y_{\mu}{}^{\rho}{}_{\alpha}Y_{\rho}{}^{\alpha}{}_{\nu} \\ (J-2H)Y_{\nu}{}^{\rho}{}_{\alpha}T_{\mu}{}^{\alpha}{}_{\rho} - (2H+J)g_{\beta\epsilon}g^{\alpha\rho}Y_{\nu}{}^{\epsilon}{}_{\alpha}Y_{\mu}{}^{\beta}{}_{\rho} \\ +J\left(T_{\mu}{}^{\alpha}{}_{\epsilon}T_{\alpha}{}^{\epsilon}{}_{\nu} - g^{\epsilon\rho}g_{\alpha\beta}Y_{\nu}{}^{\alpha}{}_{\rho}T_{\epsilon}{}^{\beta}{}_{\mu}\right) \\ \left(I\bar{\Theta}^2 + K\bar{\Theta}\bar{\Upsilon} + L\bar{\Upsilon}^2\right)T_{\sigma}{}^{\sigma}{}_{\mu}T_{\rho}{}^{\rho}{}_{\nu}] \end{aligned} \tag{D.15}$$

Now we are *almost* ready to vary our transformed action with respect to the metric, $g_{\alpha\beta}$ to obtain our final dynamics. The difficulty here is, however, that $Y_{\mu}{}^{\epsilon}{}_{\nu}$ as defined in (D.2) and (D.3) above contains explicit factors of $g_{\alpha\beta}$. Separating out the relevant factors of $g_{\alpha\beta}$ from the above expression eventually yields the following expression for S_{EHE}:

$$S_{EHE} = \int d^N x\sqrt{-g}g^{\mu\nu}\left[R_{\mu\nu}(\{\}) + \mathcal{S}_{(\mu\nu)} + g^{\eta\lambda}g_{\rho\alpha}\mathcal{W}^{(\rho\alpha)}{}_{(\eta\lambda)(\mu\nu)}\right] \tag{D.16}$$

where

$$\begin{aligned} \mathcal{W}^{\rho\alpha}{}_{\eta\lambda\mu\nu} &:= \gamma(J-2H)T_{\mu}{}^{\alpha}{}_{\eta}T_{\nu}{}^{\rho}{}_{\lambda} + JT_{\mu}{}^{\rho}{}_{\lambda}X_{\nu}{}^{\alpha}{}_{\eta} \\ &\quad -(2H+J)\mathcal{M}^{\rho\alpha}{}_{\eta\lambda\mu\nu} - (1+2H+3J)\left[\mathcal{H}^{\rho\alpha}{}_{\eta\lambda\mu\nu} + \mathcal{I}^{\rho\alpha}{}_{\eta\lambda\mu\nu}\right] \end{aligned} \tag{D.17}$$

$$\mathcal{H}^{\rho\alpha}{}_{\eta\lambda\mu\nu} := \gamma\left[X_{\mu}{}^{\rho}{}_{\eta}T_{\nu}{}^{\alpha}{}_{\lambda} + T_{\mu}{}^{\alpha}{}_{\lambda}X_{\eta}{}^{\rho}{}_{\nu}\right] \tag{D.18}$$

$$\mathcal{I}^{\rho\alpha}{}_{\eta\lambda\mu\nu} := \gamma^2\left[T_{\mu}{}^{\rho}{}_{\lambda}T_{\eta}{}^{\alpha}{}_{\nu}\right] \tag{D.19}$$

$$\mathcal{M}^{\rho\alpha}{}_{\eta\lambda\mu\nu} := X_\nu{}^\rho{}_\eta X_\mu{}^\alpha{}_\lambda + 2\gamma^2\, T_\nu{}^\rho{}_\eta T_\mu{}^\alpha{}_\lambda \tag{D.20}$$

while

$$\begin{aligned} \mathcal{S}_{\mu\nu} &:= \mathcal{C}_{\mu\nu} + \aleph\left[T_\sigma{}^\sigma{}_\mu T_\rho{}^\rho{}_\nu\right] - (1+2H+3J)\mathcal{D}_{\mu\nu} \\ &\quad -(2H+J)\mathcal{N}_{\mu\nu} + (J-2H)\mathcal{Q}_{\mu\nu} + JT_\mu{}^\alpha{}_\epsilon T_\alpha{}^\epsilon{}_\nu \end{aligned} \tag{D.21}$$

where

$$\mathcal{C}_{\mu\nu} := \left[\eta\;(N+\frac{3}{2}) + \lambda - \gamma\right] X_\mu{}^\rho{}_\nu T_\sigma{}^\sigma{}_\rho \tag{D.22}$$

$$\aleph := \left[2\left(\eta\;[N+\frac{3}{2}] + \lambda - \gamma\right) + N\lambda + I\bar{\Theta}^2 + K\bar{\Theta}\bar{\Upsilon} + L\bar{\Upsilon}^2\right] \tag{D.23}$$

$$\begin{aligned} \mathcal{D}_{\mu\nu} &:= X_\mu{}^\rho{}_\sigma X_\rho{}^\sigma{}_\nu + (\lambda[\lambda-2\gamma])T_\sigma{}^\sigma{}_\nu T_\rho{}^\rho{}_\mu + 2\lambda X_\mu{}^\epsilon{}_\epsilon T_\rho{}^\rho{}_\nu \\ &\quad +2\gamma X_\mu{}^\sigma{}_\epsilon T_\sigma{}^\epsilon{}_\nu - \gamma^2 T_\mu{}^\sigma{}_\epsilon T_\nu{}^\epsilon{}_\sigma \end{aligned} \tag{D.24}$$

$$\begin{aligned} \mathcal{N}_{\mu\nu} &:= 2\lambda T_\sigma{}^\sigma{}_\rho X_\mu{}^\rho{}_\nu + 2\gamma\left[T_\mu{}^\rho{}_\sigma\left(X_\rho{}^\sigma{}_\nu + X_\nu{}^\sigma{}_\rho\right)\right] \\ &\quad +\lambda(N\lambda+4\gamma)T_\rho{}^\rho{}_\mu T_\sigma{}^\sigma{}_\nu + 2\gamma^2 T_\kappa{}^\tau{}_\mu T_\tau{}^\kappa{}_\nu \end{aligned} \tag{D.25}$$

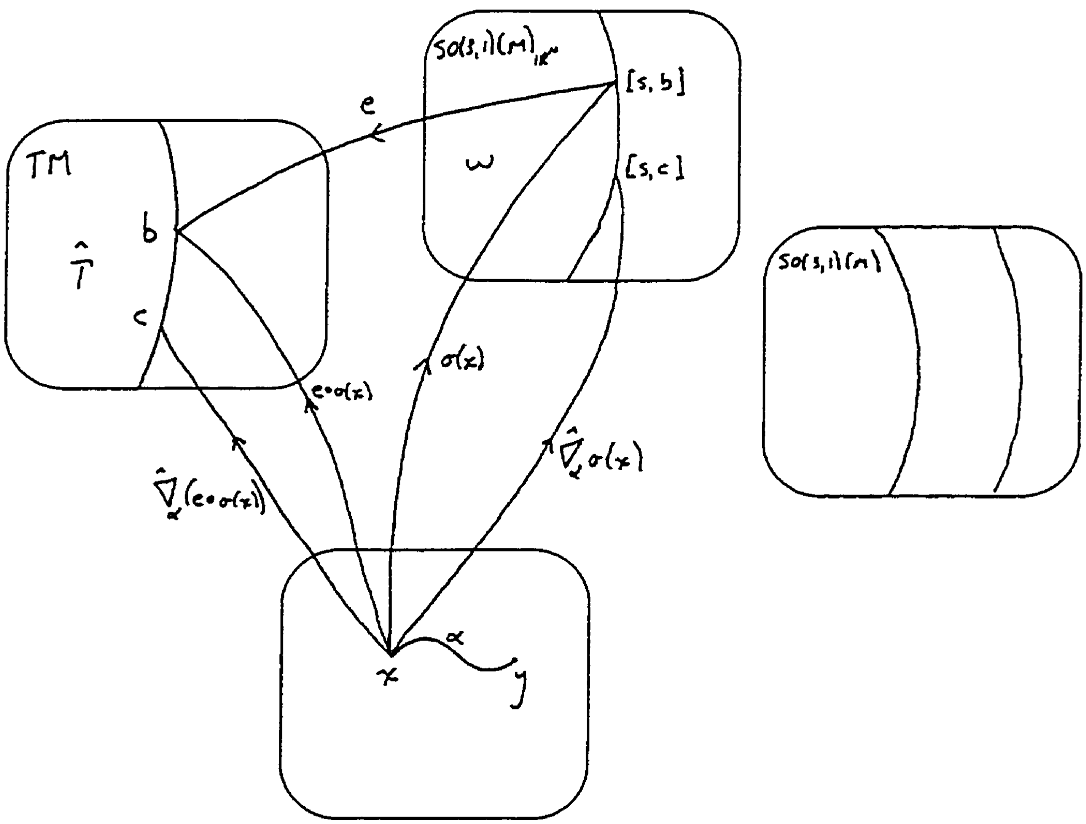
$$\mathcal{Q}_{\mu\nu} := X_\nu{}^\sigma{}_\epsilon T_\mu{}^\epsilon{}_\sigma + \lambda T_\sigma{}^\sigma{}_\mu T_\rho{}^\rho{}_\nu + \gamma T_\rho{}^\epsilon{}_\mu T_\epsilon{}^\rho{}_\nu \tag{D.26}$$

Therefore, assured that we have finally separated metric from non-metric terms in the action, we can finally vary (D.16) with respect to $g_{\alpha\beta}$, thereby producing our final dynamical relationship:

$$G_{\mu\nu}(\{\}) + S_{(\mu\nu)} + W^{\rho\epsilon}{}_{\rho\,\epsilon(\mu\nu)} + W^{\rho}{}_{\rho(\mu\nu)}{}^{\epsilon}{}_{\epsilon} - W_{(\mu\nu)}{}^{\rho\epsilon}{}_{\rho\epsilon} - \frac{1}{2}g_{\mu\nu}(S+W) = 0 \tag{D.27}$$

Appendix E

Moving $\hat{\nabla}$ from $(\mathcal{SO}(3,1)(M)_{\Re^N}, \pi_{\Re^N}, M)$ to TM



We first recognize that the equivalence class $[s, a]$ of points of $(SO(3,1)(M)_{\Re^N}, \pi_{\Re^N}, M)$ is only defined for the group $SO(3,1)$. That is, for the point $[s, a]$, one has

$$(s, a) \equiv (sg, g^{-1}a) \; \forall \, g \; \in \; SO(3,1) \tag{E.1}$$

We now that the associated bundle $(SO(3,1)(M)_{\Re^N}, \pi_{\Re^N}, M)$ has dimensionality $2N$ just as does TM, where each fibre of each has dimensionality N. We would like to link up the two so as to "push forward" the covariant derivative defined on the former by the connection in its principal bundle w. Thus suppose we decide to describe each vector $v \in TM$ in some given fibre over $x \in M$ by its usual components with respect to the holonomic coordinates - i.e.

$$v = v^\lambda \partial_\lambda \tag{E.2}$$

If we were dealing with the associated bundle of $\mathcal{B}(M)$, i.e. $(\mathcal{B}(M)_{\Re^N}, \pi_{\Re^N}, M)$, where each point in each fibre represents the equivalence class $[b, a]$ defined in terms of the group $GL(N, R)$ this would be straightforward and we could use the "built-in" isomorphism (5.20) since *whatever* frame, b, I happened to choose for my point $[b, a]$, I can *always* use the equivalence class to rewrite this in terms of my holonomic coordinates using the tetrads. On the other hand, for the bundle associated with $SO(3,1)$, I might find myself choosing some $SO(3,1)$ frame s which can *not* be related to $\{\partial_\lambda\}$ via some $SO(3,1)$ transformation - and hence my relationship depends on which frame I choose to begin with for s (or, conversely, how I decide to represent my element of TM, v). In order to circumvent this, i.e. to make our isomorphism between $(SO(3,1)(M)_{\Re^N}, \pi_{\Re^N}, M)$ and TM well-defined, we must necessarily introduce the frame field, e to remove this frame-dependence and thereby enable *any* $[s, a]$ to be associated with *any* coordinate system I choose for TM (we will always choose, as mentioned previously, the holonomic coordinate system, where the frame is $\{\partial_\lambda\}$).

Having clarified (hopefully) this subtle point, we move on to establishing our consistency relationship of Section 6.2.

The aim is to write the connection coefficients, ${w_\alpha}^I{}_J$ on $(SO(3,1)(M)_{\Re^N}, \pi_{\Re^N}, M)$ in terms of those, ${\hat{\Gamma}_\alpha}{}^\gamma{}_\beta$ on TM using the isomorphism e alluded to above.

Let us assume that our covariant derivative, $\hat{\nabla}$, on the associated bundle gives, for some $x \in M$,

$$\hat{\nabla}_\alpha b = c \tag{E.3}$$

where

$$b \Leftrightarrow [s, b] \tag{E.4}$$

$$c \Leftrightarrow [s, c], \tag{E.5}$$

and therefore

$$c^I = b^I{}_{,\alpha} + {w_\alpha}^I{}_J\, b^J \tag{E.6}$$

and that e gives the isomorphism

$$e(s_I) = e_I^\lambda\, \partial_\lambda \tag{E.7}$$

If $v, z \in TM$ are associated with $b \Leftrightarrow [s, b], c \Leftrightarrow [s, c]$ respectively, we thus have:

$$v^\lambda = b^I e_I^\lambda \tag{E.8}$$

and

$$z^\lambda = c^I e_I^\lambda \tag{E.9}$$

with the associated TM-covariant derivative:

$$\hat{\nabla}_\alpha v = z \tag{E.10}$$

i.e.

$$z^\gamma = v^\gamma{}_{,\alpha} + {\hat{\Gamma}_\alpha}{}^\gamma{}_\sigma v^\sigma \tag{E.11}$$

Therefore (E.6),(E.8),(E.9) and (E.11) combine to give:

$$\hat{\Gamma}_{\alpha}{}^{\gamma}{}_{\lambda} = e_{\lambda}^{J}\, e_{I}^{\gamma}\, w_{\alpha}{}^{I}{}_{J} + \partial_{\alpha}(e_{\lambda}^{J})e_{J}^{\gamma}, \tag{E.12}$$

i.e. (6.11).

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