Horospherical geometry: combinatorial algebraic stacks and approximating rational points

by

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Examining Committee Membership

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Chapter 1 is a summary of the thesis, written solely by Sean Monahan. This writing draws from the introductions of [Mon23a] (solo-authored by Sean Monahan), [Mon23b] (solo-authored by Sean Monahan), and [MS23b] (co-authored by Sean Monahan and Matthew Satriano).

Chapter 2 is simply a declaration of notation, written solely by Sean Monahan.

Chapter 3 is background material which is based on the survey article [Mon23a], which was written solely by Sean Monahan.

Chapter 4 is original research based on [Mon23b], which was written solely by Sean Monahan.

Chapter 5 is original research based on [MS23b], which was joint work between Sean Monahan and Matthew Satriano.

Appendix A is background material, written solely by Sean Monahan.

Abstract

The purpose of this thesis is to explore and develop several aspects of the theory of horospherical geometry. Horospherical varieties are equipped with the action of a reductive algebraic group such that there is an open orbit whose points are stabilized by maximal unipotent subgroups. This includes the well-known classes of toric varieties and flag varieties. Using this orbit structure and representation-theoretic condition on the stabilizer, one can classify horospherical varieties using combinatorial objects called coloured fans. We give an overview of the main features of this classification through a new, accessible notational framework.

There are two main research themes in this thesis. The first is the development of a combinatorial theory for horospherical stacks, vastly generalizing that for horospherical varieties. We classify horospherical stacks using combinatorial objects called stacky coloured fans, extending the theory of coloured fans. As part of this classification, we describe the morphisms of horospherical stacks in terms of maps between the stacky coloured fans, we completely describe the good moduli space of a horospherical stack, and we introduce a special, hands-on class of horospherical stacks called coloured fantastacks.

The second major theme is using horospherical varieties to probe a conjecture in arithmetic geometry. In 2007, McKinnon conjectured that, for a given point on a projective variety, there is a sequence, lying on a curve, which best approximates this point. We verify a version of this conjecture for horospherical varieties, contingent on Vojta's Main Conjecture, which says that there is a sequence, lying on a curve, which approximates the given point better than any Zariski dense sequence.

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Lastly, thank you to my close friends – the lads know who they are – and my family. Especially a big thank you to my parents, Tom and Cindy, without whom I would not have made it to this point. I am very grateful for your constant support. While having a stressful time with research or teaching, I would always look forward to going home to spend time with you (and getting to pass by the big apple).

Dedication

I dedicate this thesis to my parents, Tom and Cindy, and to my aunt, Susan.

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Chapter 1

Introduction

An essential step in mathematical research is the testing of one's proposed results on a small, well-understood class of examples. Such a class of examples should have an explicit, hands-on theory that lends itself to easy computations. In the context of varieties in algebraic geometry, we are interested in the class of horospherical varieties. These varieties are equipped with an algebraic action by a reductive group G with two essential properties: there is an open G-orbit and there are finitely many G-orbits. This simple orbit structure gives rise to a combinatorial structure, called a coloured fan, that encodes the variety. There is a rich correspondence between horospherical varieties and coloured fans wherein the algebro-geometric properties of the varieties are described by the tangible combinatorial properties of the coloured fans. Moreover, this class of varieties is fairly broad, encompassing toric varieties and flag varieties (e.g. Grassmannians), so one can use this class to test a wide range of conjectures.

Following the pioneering work by Luna, Vust, Brion, and Pauer [LV83; BL87; BLV86; Pau81; Pau83; BP87], horospherical varieties have garnered significant interest, particularly in the last 15 years. Much is known about the birational geometry of horospherical varieties, e.g. their minimal models [Bri93; Pas15b; Pas18]; many types of singularities (e.g. canonical, terminal, and log terminal) for horospherical varieties have been characterized [Lan17; Pas16; Pas15a]; the generalized Mukai conjecture has been proven for horospherical varieties [Pas10]; smooth horospherical varieties with Picard number 1 have been classified and studied [Pas09b; Gon+22; Hon16]; and the Cox ring and Cox GIT construction (as introduced in [Cox95] for toric varieties) were described for horospherical varieties [Bri07; Gag14; Gag19]. The aforementioned combinatorial theory plays a crucial role in all these works.

The focus of this thesis is to explore and develop several aspects of the theory of

horospherical geometry. We begin with an overview of horospherical varieties and coloured fans using a new, accessible framework of notation and terminology. Then we present the main research of this thesis which is divided into two parts: the first is developing a novel combinatorial theory for horospherical stacks, vastly generalizing the theory of horospherical varieties; and the second is using the combinatorial machinery of horospherical varieties to probe a well-known conjecture in arithmetic geometry regarding approximating rational points on varieties. In what follows, we provide a summary of this content.

Horospherical variety background. In Chapter 3, we review the necessary algebraic group theory, provide more detail on horospherical varieties, and give an overview of the dictionary between coloured fans and horospherical varieties. Let k be an algebraically closed field of characteristic 0 (e.g. $\mathbb{C}$), and let G be a connected reductive algebraic group over k (e.g. GL_n, SL_n, or an algebraic torus $\mathbb{G}_m^n$).

Firstly, a *horospherical variety* X is a normal G-variety such that there is an open G-orbit $G \cdot x_0 \subseteq X$ where the stabilizer H of x_0 contains a maximal unipotent subgroup of G; note that $G \cdot x_0$ is the homogeneous space G/H. This condition on H is advantageous from a representation-theoretic perspective. There are two extreme examples to mention: when $G = \mathbb{G}_m^n$ is a torus, we recover toric varieties; and on the other hand, when $H = P$ is a parabolic subgroup of G, we recover the flag variety G/P. Horospherical varieties are also part of a larger class of varieties called *spherical varieties*, which are G-varieties with an open orbit under the action of a Borel subgroup of G.

Much of the combinatorial theory is encoded in the horospherical homogeneous space G/H. Let $B \subseteq G$ be a Borel subgroup (e.g. the group of upper triangular matrices in SL_n). Using the Bruhat decomposition, there are finitely many B-invariant prime divisors in G/H, which we call *colour divisors*. The colour divisors can be indexed by a finite *universal colour set* $\mathcal{C}$, the elements of which we regard as abstract *colours* (for those familiar with the Bruhat decomposition, $\mathcal{C}$ is a subset of the set of simple roots for G relative to B). The space of B-eigenfunctions forms a lattice, and by evaluating the order of vanishing valuations for these colour divisors on B-eigenfunctions, one can associate points u_α in the dual lattice N to each colour $\alpha \in \mathcal{C}$, which we call *colour points*.

A *coloured cone* on N is a pair $\sigma^c = (\sigma, \mathcal{F})$ where σ is a strongly convex polyhedral cone in the vector space $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathcal{F} \subseteq \mathcal{C}$ is a set of colours such that $u_\alpha \in \sigma \setminus \{0\}$ for each $\alpha \in \mathcal{F}$. Then a *coloured fan* Σ^c on N is a finite collection of coloured cones on N which is closed under taking faces and intersections of coloured cones. The main result in this combinatorial theory is that there is an explicit bijective correspondence between horospherical varieties X with open orbit G/H and coloured fans Σ^c on N, extending the well-known correspondence between toric varieties and fans; given Σ^c, we let X_{Σ^c} denote the corresponding horospherical variety. This correspondence was first developed by Luna

and Vust [LV83] for the more general class of spherical varieties, although there is more to keep track of in this general setting.

One way to intuit the correspondence between Σ^c and X_{Σ^c} is as follows. The coloured cones in Σ^c correspond to the G-orbits in X_{Σ^c}; in fact, the containment of coloured cones in Σ^c has the same poset structure (in reverse order) as containment of G-orbit closures in X_{Σ^c}. In the correspondence between toric varieties and fans, this poset structure on the torus orbits, as captured by the fan, is (essentially) enough to determine the variety. This relies on the fact that the open torus orbit is affine, and therefore the irreducible components of its complement are the torus-invariant divisors. However, for horospherical varieties, the presence of colour divisors prevents the open orbit G/H from being affine, so the complement $X_{\Sigma^c} \setminus G/H$ does not simply decompose into the G-invariant divisors. Therefore, one needs to keep track of this fan poset structure *and* the interaction between the colour divisors and the G-orbits. For a coloured cone $\sigma^c = (\sigma, \mathcal{F}) \in \Sigma^c$, a colour α being in the colour set $\mathcal{F}$ is precisely indicating that the G-orbit closure corresponding to σ^c is contained in the colour divisor corresponding to α.

The following are some of the main features of the correspondence between Σ^c and X_{Σ^c}:

- G-equivariant morphisms of horospherical varieties correspond to linear maps of the coloured fans.
- X_{Σ^c} is affine if and only if Σ^c has a unique maximal coloured cone $(\sigma, \mathcal{F})$ and $\mathcal{F} = \mathcal{C}$. Moreover, there is a local description for horospherical varieties in terms of affine horospherical varieties.
- Local properties of X_{Σ^c} such as $\mathbb{Q}$-factoriality, factoriality, and smoothness can be checked using combinatorial properties of Σ^c.
- Elements of $\mathrm{Pic}(X_{\Sigma^c})$ can be described using piecewise linear functionals on Σ^c.

Research theme #1: horospherical stacks. The first main research theme of this thesis is on extending the above combinatorial theory from varieties to the much larger world of stacks. The classical approach of algebraic geometry is to study systems of polynomial equations using algebraic sets, while the modern approach is an abstraction of this idea using schemes. Schemes offer a "higher-resolution" view of the algebro-geometric properties that one studies in algebraic sets, and as a result, schemes have become an indispensable object in this field. However, as with algebraic sets, there are some problems that schemes are not properly calibrated to tackle. The primary instance of this is in the study of moduli spaces, which are spaces that parametrize certain algebro-geometric objects up to equivalence (e.g. the well-known $\mathcal{M}_{g,n}$ which parametrizes genus g curves with n marked points). Schemes

are too "blurry" to provide a proper picture for these moduli spaces, so we need an even higher-resolution view. To achieve this goal, one typically uses a generalization of schemes called stacks, which have a structure that is rich enough to capture all the fine details of moduli spaces.

In full generality, stacks are very abstract, so it is helpful to build intuition on well-behaved classes of stacks. For the same reasons that horospherical varieties are helpful in understanding general varieties, we can use the class of horospherical stacks to help us understand general stacks. We are aware of only one theory of horospherical stacks, which is by Javanpeykar, Langlois, and Terpereau in [JLT19]. They focus on developing the structural theory of these objects, generalizing the work of Geraschenko and Satriano on toric stacks [GS15b]. In particular, they consider *abstract* horospherical stacks, which are algebraic stacks with a reductive group action and a horospherical homogeneous space as an open substack, and they present conditions for when these objects are "concrete" horospherical stacks, meaning that they are stack quotients of horospherical varieties.

These horospherical stacks are an extension of toric stacks, which have been keenly studied for about 20 years [BCS05; FMN10; GM15; GS15a; GS15b; Iwa09; Tyo12; Laf02]. The primary advantage which toric stacks possess is their combinatorial description using so-called stacky fans, generalizing the correspondence between toric varieties and fans. In Chapter 4, we introduce a combinatorial theory for horospherical stacks, primarily motivated by the work of Geraschenko and Satriano on toric stacks [GS15a]. Since horospherical stacks are in their infancy, there is much to discover about them, and we believe that this combinatorial theory will be an invaluable tool in their development.

For a horospherical variety X, the group $\mathrm{Aut}^G(X)$ of G-equivariant automorphisms is a torus, and the lattice of one-parameter subgroups (dual to characters) is N; we denote the torus associated to N by T_N. Note that this N is the same as the one above constructed via B-eigenfunctions. Now any closed subgroup $K \subseteq T_N$ acts on X by G-equivariant automorphisms, and we can consider the quotient stack $[X/K]$. A stack of this form is called a *horospherical G-stack*. Avoiding details, one can think of $[X/K]$ as a formal quotient of X by the action of K; for the details, we provide a review of algebraic stacks in Appendix A.

To describe $[X/K]$ combinatorially, we can handle X and K separately; see Section 4.1.3. As we know, X corresponds to a coloured fan Σ^c on N. We can view K as the kernel of a surjection of tori $T_N \to T_L$ where T_L is some torus corresponding to a lattice L, and this surjection corresponds to a linear map $\beta : N \to L$ with finite cokernel; in this setting we write K_β for K. Therefore, horospherical stacks correspond to *stacky coloured fans*, which are pairs (Σ^c, β) where Σ^c is a coloured fan on N and $\beta : N \to L$ is a lattice map with finite cokernel; given (Σ^c, β), we let $\mathcal{X}_{\Sigma^c,\beta} = [X_{\Sigma^c}/K_\beta]$ denote the corresponding horospherical

stack. Note that the stacky coloured fan $(\Sigma^c, \mathrm{id} : N \to N)$ corresponds to the horospherical variety $[X_{\Sigma^c}/\{1\}] = X_{\Sigma^c}$, so this correspondence extends the one between horospherical varieties and coloured fans.

We develop several aspects of the dictionary between horospherical stacks and stacky coloured fans; namely we establish the following:

- Equivariant morphisms of horospherical stacks correspond to maps of the associated stacky coloured fans.
- We introduce a class of easy-to-compute horospherical stacks called *coloured fantastacks*, generalizing the toric fantastacks from [GS15a].
- We discuss *toroidal* horospherical stacks, which closely resemble toric stacks in a certain local sense; in particular, we completely describe the *decolouration* of a horospherical stack.
- We completely describe good moduli spaces (in the sense of [Alp13]) for horospherical stacks using an explicit combinatorial construction.

To expand on the items listed above, we provide more detail on the main results of Chapter 4. First, we delve into the morphisms. A *map of stacky coloured fans*

$$(\Phi, \phi) : (\Sigma_1^c, \beta_1 : N_1 \to L_1) \to (\Sigma_2^c, \beta_2 : N_2 \to L_2)$$

consists of a linear map $\Phi : N_1 \to N_2$ compatible with the coloured fans Σ_1^c and Σ_2^c and a linear map $\phi : L_1 \to L_2$ such that $\beta_2 \circ \Phi = \phi \circ \beta_1$; see Section 4.2.2. We can capture this notation in the following diagram:

$$\begin{array}{ccc} \Sigma_1^c & \longrightarrow & \Sigma_2^c \\ N_1 & \xrightarrow{\Phi} & N_2 \\ {\scriptstyle\beta_1}\downarrow & & \downarrow{\scriptstyle\beta_2} \\ L_1 & \xrightarrow{\phi} & L_2 \end{array}$$

It is easy to see how a map of stacky coloured fans $(\Sigma_1^c, \beta_1) \to (\Sigma_2^c, \beta_2)$ induces a morphism of the associated horospherical stacks $\mathcal{X}_{\Sigma_1^c, \beta_1} \to \mathcal{X}_{\Sigma_2^c, \beta_2}$; see Section 4.2.3. However, it is not exactly true that *every* morphism of horospherical stacks comes from a map of stacky coloured fans (this has to do with the fact that there can be different presentations for the same quotient stack; see Remark 4.1.11 and Section 4.2.4). The following theorem

addresses this by essentially saying that, given a morphism of horospherical stacks, one can change the quotient presentation of the domain stack (i.e. change the stacky coloured fan) so that the morphism is induced by a map of stacky coloured fans.

Theorem (Theorem 4.2.8)**.** *Let $\mathcal{X}_{\Sigma_i^c,\beta_i}$ be horospherical G-stacks for $i = 1, 2$, and suppose that $\pi : \mathcal{X}_{\Sigma_1^c,\beta_1} \to \mathcal{X}_{\Sigma_2^c,\beta_2}$ is a morphism of horospherical stacks. Then there exists a stacky coloured fan (Σ_0^c, β_0) and a tower of stacky coloured fans (below on the left) which induces the following commutative diagram of horospherical stacks (below on the right)*

$$\begin{array}{ccc} (\Sigma_0^c,\beta_0) & & \\ \downarrow & \searrow & \\ (\Sigma_1^c,\beta_1) & & (\Sigma_2^c,\beta_2) \end{array} \qquad \begin{array}{ccc} \mathcal{X}_{\Sigma_0^c,\beta_0} & & \\ \wr\downarrow & \searrow & \\ \mathcal{X}_{\Sigma_1^c,\beta_1} & \xrightarrow[\pi]{} & \mathcal{X}_{\Sigma_2^c,\beta_2} \end{array}$$

where $\mathcal{X}_{\Sigma_0^c,\beta_0} \xrightarrow{\sim} \mathcal{X}_{\Sigma_1^c,\beta_1}$ is an isomorphism.

We also provide combinatorial criteria for checking when a morphism of horospherical stacks is an isomorphism.

Theorem (Theorem 4.2.12)**.** *Let $\pi : \mathcal{X}_{\Sigma_1^c,\beta_1} \to \mathcal{X}_{\Sigma_2^c,\beta_2}$ be a morphism of horospherical stacks which is induced by a map of stacky coloured fans $(\Phi, \phi) : (\Sigma_1^c, \beta_1) \to (\Sigma_2^c, \beta_2)$. Then π is an isomorphism if and only if ϕ is an isomorphism of coloured lattices and, for each $(\sigma_2, \mathcal{F}_2) \in \Sigma_2^c$, the preimage under Φ (see Notation 4.2.10) is a coloured cone $(\sigma_1, \mathcal{F}_1) \in \Sigma_1^c$ such that $\mathcal{F}_1 = \mathcal{F}_2$ and the map Φ induces an isomorphism of monoids $\sigma_1 \cap N_1 \to \sigma_2 \cap N_2$.*

In addition to these results on general horospherical morphisms, we give combinatorial criteria for when a morphism of horospherical stacks is a good moduli space morphism in the sense of [Alp13]; see Theorem 4.5.5. As a corollary to this criteria, we are able to describe the good moduli space of a horospherical stack (when it exists), which one should think of as a scheme approximation of the stack. In Notation 4.5.7, we detail an explicit construction of a coloured fan Σ_{gms}^c depending on a given stacky coloured fan (Σ^c, β).

Theorem (Corollary 4.5.8)**.** *Let $\mathcal{X}_{\Sigma^c,\beta}$ be a horospherical G-stack, and let Σ_{gms}^c be the coloured fan constructed in Notation 4.5.7. Then the good moduli space of $\mathcal{X}_{\Sigma^c,\beta}$ exists if and only if (Σ^c, β) is compatible with Σ_{gms}^c (in the sense of Notation 4.5.7). Moreover, in this case, the horospherical variety corresponding to Σ_{gms}^c is the good moduli space of $\mathcal{X}_{\Sigma^c,\beta}$.*

The last significant aspect of this combinatorial theory is the development of a new hands-on class of horospherical stacks which generalizes the fantastacks in [GS15a]. Starting

with a horospherical variety X_{Σ^c} corresponding to a coloured fan Σ^c on N, we outline an explicit procedure in Section 4.3.1 (with several degrees of freedom) to create a coloured fan $\widehat{\Sigma}^c$ on a new coloured lattice $\widehat{N}$ and a linear map $\beta : \widehat{N} \to N$ compatible with $\widehat{\Sigma}^c$ and Σ^c. Any horospherical stack $\mathscr{F}_{\Sigma^c,\beta} := \mathcal{X}_{\widehat{\Sigma}^c,\beta}$ arising through this construction is called a *coloured fantastack*.

The map of stacky coloured fans (β, id_N) induces a morphism of horospherical stacks $\mathscr{F}_{\Sigma^c,\beta} \to X_{\Sigma^c}$. Using our theory, it is easy to check that this is a good moduli space morphism, i.e. X_{Σ^c} is the good moduli space of the associated coloured fantastack. From this perspective, one should think of the coloured fantastack $\mathscr{F}_{\Sigma^c,\beta}$ as a way of putting "stacky structure" on top of X_{Σ^c}.

One special case of this coloured fantastack construction recovers the Cox GIT construction for horospherical varieties: when Σ^c is simplicial (i.e. X_{Σ^c} is $\mathbb{Q}$-factorial), the map $X_{\widehat{\Sigma}^c} \to X_{\Sigma^c}$ induced by β is a geometric quotient for the action of $K_\beta \subseteq T_{\widehat{N}}$; see Section 4.3.2.

Research theme #2: approximating rational points. The second main research theme of this thesis involves probing a well-known conjecture in arithmetic geometry by restricting to the class of horospherical varieties and exploiting their combinatorial tools. This conjecture has its roots going back to Dirichlet's famous Approximation Theorem from 1842, which says that for every irrational number x, there are infinitely many rational numbers $\frac{a}{b}$ such that $|x - \frac{a}{b}| < \frac{1}{b^2}$.

To describe later generalizations of this result, it is helpful to re-frame it in the following way. Let $\tau_x \in (0, \infty]$ denote the supremum over all $\gamma \in (0, \infty)$ for which

$$\left| x - \frac{a}{b} \right| \leq \frac{1}{b^\gamma}$$

has infinitely many solutions $\frac{a}{b} \in \mathbb{Q}$ in reduced form. One should think of τ_x as measuring how well x can be approximated by rational numbers: the larger the value of τ_x, the better the approximation. Note that the bound on this approximation depends on the denominator of the rational number used to approximate x; we think of $\frac{1}{b}$ as measuring the *complexity* of $\frac{a}{b}$. In this framework, Dirichlet's Theorem says that $\tau_x \geq 2$ for all irrational x. Shortly after, Liouville [Lio44] proved that, if $x \in \mathbb{R}$ is algebraic of degree d over $\mathbb{Q}$, then $\tau_x \leq d$. Subsequently, much work was done on obtaining sharper upper bounds for τ_x for algebraic x [Thu09; Sie21; Dys47], culminating in Roth's celebrated theorem [Rot55] which says that $\tau_x \leq 2$ for all algebraic $x \in \mathbb{R}$. Therefore, these combined works show that $\tau_x = 2$ for all irrational and algebraic $x \in \mathbb{R}$.

In [McK07], McKinnon gave a novel perspective, generalizing the above framework to

study the approximation of rational points on an arbitrary projective variety X defined over a number field k. Much of the formalism was later developed by McKinnon and Roth [MR15], their goal being to generalize τ_x to points $p \in X(k)$. To do this, they adapt the setting from $\mathbb{R}$ to X by replacing rational numbers $\frac{a}{b}$ with sequences of rational points $\{x_i\} \subseteq X(k)$, replacing $|\cdot|$ with a distance function $\mathrm{dist}_v(\cdot,\cdot)$ on X depending on a place v of k, and replacing the measure of complexity $\frac{1}{b}$ with a height function $H_D(\cdot)$ on X depending on an ample divisor D. Given a sequence $\{x_i\}$ approximating p (i.e. converging to p under dist_v), they define the *approximation constant* $\alpha_{p,\{x_i\}}(D)$ in an analogous way to τ_x, although they use the reciprocal because this behaves better with respect to changes in D. That is, $\alpha_{p,\{x_i\}}(D)$ is the infimum over all $\gamma \in (0,\infty)$ such that

$$\mathrm{dist}_v(p,x_i)^\gamma < \frac{1}{H_D(x_i)}.$$

To account for different sequences, let $\alpha_p(D)$ denote the infimum of $\alpha_{p,\{x_i\}}(D)$ as we range over all sequences $\{x_i\}$ approximating p; if one restricts to only consider sequences contained in a subvariety $Z \subseteq X$, then we denote the corresponding infimum by $\alpha_{p,Z}(D)$. Note that, when $X = \mathbb{P}^1$ and D corresponds to $\mathscr{O}_{\mathbb{P}^1}(1)$, we have $\alpha_x(D) = \frac{1}{\tau_x}$ for all $x \in \mathbb{P}^1(\overline{\mathbb{Q}}) \setminus \mathbb{P}^1(\mathbb{Q})$, so we recover the classical case.

In McKinnon's original paper [McK07], he conjectured that a sequence of best approximation, i.e. a sequence of rational points $\{x_i\}$ which minimizes $\alpha_{p,\{x_i\}}(D)$, exists and can be chosen to lie on a curve; this is known as the "Curve of Best Approximation" (COBA) Conjecture.

Conjecture (COBA Conjecture, [McK07, Conjecture 2.7])**.** *Let X be a projective variety over a number field k, let D be an ample divisor on X, and fix a point $p \in X(k)$ for which there is a rational curve, defined over k, passing through p. Then there exists a curve $C \subseteq X$ (necessarily rational) such that $\alpha_{p,C}(D) = \alpha_p(D)$.*

This conjecture is known in some special cases, primarily in dimension 2: it was shown for split rational surfaces of Picard rank at most four in [McK07], cubic surfaces in [MR16], blow-ups of the n-th Hirzebruch surface at special configurations of at most $2n$ points in [San19], and special classes of toric varieties in [Hua21], namely when p is in the open torus and the pseudo-effective cone $\overline{\mathrm{Eff}}(X)$ is simplicial. This conjecture was also recently proved by Lehmann, McKinnon, and Satriano [LMS24] for smooth split projective surfaces, assuming Vojta's Main Conjecture [Voj87, Conjecture 3.4.3]; this was inspired by earlier work of McKinnon and Satriano [MS21] which handled the case of split toric surfaces X.

In addition to their work in the surface case, McKinnon and Satriano also proved higher-dimensional analogues of the COBA Conjecture in [MS21]. In Chapter 5, we generalize their

higher-dimensional results to the horospherical setting. With $\dim(X) > 2$, it is difficult to address the COBA Conjecture directly. Instead, our aim is to achieve bounds for $\alpha_{p,C}(D)$, similar to the individual works in the classical τ_x setting. Specifically, we aim to prove existence of a curve C for which $\alpha_{p,C}(D) \leq \alpha_{p,\{x_i\}}(D)$ for all *Zariski dense* sequences $\{x_i\}$. In the case of $\dim(X) = 2$, a sequence is either Zariski dense or it lies on a curve, so this bound is equivalent to existence of a curve C for which $\alpha_{p,C}(D) = \alpha_p(D)$ as in the COBA Conjecture.

To achieve this bound, we require the following technical assumption regarding the approximation constant at p, which was introduced in [MS21, Definition 1.2]. If X is a general projective $\mathbb{Q}$-Gorenstein variety over a number field k, then we say that X is *canonically bounded at* $p \in X(k)$ if $\alpha_{p,\{x_i\}}(-K_X) \geq \dim(X)$ for all Zariski dense sequences $\{x_i\} \subseteq X(k)$. This assumption is useful because the minimal model program (MMP) is instrumental in proving our main result, and the (anti)canonical divisor is used heavily in the MMP.

Under the presence of Vojta's Main Conjecture, we show that canonical boundedness is automatic for smooth horospherical varieties. Note that, by *split* horospherical variety, we mean that the acting group G is split reductive over k.

Proposition (Proposition 5.2.3)**.** *Let X be a smooth projective split horospherical variety over a number field k. Then Vojta's Main Conjecture for X implies that all $p \in X(k)$ are canonically bounded.*

With the assumption of canonical boundedness, or Vojta's Main Conjecture, we are able to prove the following weaker version of the COBA Conjecture for projective horospherical varieties; intuitively, this result is saying that there is a sequence, lying on a curve $C \subseteq X$, that approximates p better than all Zariski dense sequences. Note that the following theorem applies not only to ample divisors D, but to all $\mathbb{Q}$-Cartier nef $\mathbb{Q}$-divisors.

Theorem (Theorem 5.2.1 and Theorem 5.2.4)**.** *Let X be a split projective horospherical variety over a number field k, let $p \in X(k)$, and let D be a $\mathbb{Q}$-Cartier nef $\mathbb{Q}$-divisor on X. Assume that there exists a resolution $X' \to X$ where either X' is canonically bounded at p, or p is a smooth point and Vojta's Main Conjecture holds for X'. Then there exists a rational curve $C \subseteq X$ through p such that*

$$\alpha_{p,C}(D) \leq \alpha_{p,\{x_i\}}(D)$$

for all Zariski dense sequences $\{x_i\} \subseteq X(k)$.

Chapter 2

Notation and conventions

Throughout this thesis, we use the following notation and conventions.

- Let k denote a field of characteristic 0. In all chapters except for Chapter 5, we assume that k is algebraically closed; in Chapter 5, k denotes a number field. All schemes (including varieties and algebraic groups) and stacks are assumed to be defined over this base field k (except in Appendix A where we work over a general scheme S).
- $\mathbb{G}_m$ denotes the multiplicative group of the field k. Recall that a *torus* (or *algebraic* torus) is an algebraic group of the form $\mathbb{G}_m^n$ for some n.
- Let G denote a connected reductive (linear) algebraic group defined over k. We fix a Borel subgroup $B \subseteq G$ and a maximal torus $T \subseteq B$, and let $U \subseteq B$ be the unipotent radical of B; see Section 3.1.1.
- By *variety*, we mean an integral separated scheme of finite type over our field k. In particular, varieties are irreducible.

For ease of reference, we list some of the main notations used in the theory of horospherical varieties which appear frequently in this thesis. Below, G/H denotes a horospherical homogeneous space, and X denotes a G/H-horospherical variety.

- $P = N_G(H)$ denotes the parabolic subgroup of G associated to H.
- S denotes the set of simple roots of G (except in Appendix A).
- I denotes the subset of S corresponding to P.

- $\mathcal{C} = \mathcal{C}(G/H) = S \setminus I$ is the universal set of abstract colours for G/H.
- $\mathcal{D}(X)$ is the set of B^--invariant prime divisors in X.
- $D_\alpha = D_\alpha^X$ is the colour divisor in X associated to $\alpha \in \mathcal{C}$.
- $N = N(G/H)$ denotes the coloured lattice associated to G/H.
- $T_N = \mathrm{Aut}^G(X)$ is the torus with one-parameter subgroup lattice $N = N(G/H)$.
- u_α denotes the colour point in N associated to $\alpha \in \mathcal{C}$.
- $\mathcal{F}(X)$ denotes the colour set of X.
- $\Sigma^c(X)$ denotes the coloured fan corresponding to X.

Chapter 3

An overview of horospherical varieties and coloured fans

A comprehensive resource on horospherical varieties and coloured fans is largely absent from the literature, and we feel that the notation and presentation of this theory is often overwhelming and sometimes unintuitive. Therefore, the purpose of this chapter is to provide a detailed overview of this topic – drawing primarily from [Kno91; Pas09a; Per14; Per18; Tim11] – with more accessible notation; the inspiration for the notational framework is the book [CLS11] on toric varieties.

3.1 Review of algebraic groups and roots

First, we briefly review the necessary algebraic group theory. For more details on algebraic groups, we recommend [Hum75]. Recall that G is a connected reductive algebraic group over k (and k is algebraically closed with characteristic 0).

3.1.1 Borel subgroups

Recall that the radical $\mathrm{Rad}(G)$ of G is the identity component of the maximal normal solvable subgroup of G, and the unipotent radical $\mathrm{Rad}_u(G)$ of G is the subgroup of $\mathrm{Rad}(G)$ consisting of all unipotent elements; recall that an element of an algebraic group is *unipotent* if all of its eigenvalues are 1. Then G being reductive means that $\mathrm{Rad}_u(G)$ is trivial. For example, the radical $\mathrm{Rad}(GL_n) \cong \mathbb{G}_m$ of GL_n is the subgroup of diagonal matrices where all diagonal entries are equal, so the unipotent radical $\mathrm{Rad}_u(GL_n)$ is trivial, and hence GL_n

is reductive.

Fix a Borel subgroup $B \subseteq G$ (i.e. a maximal solvable closed connected subgroup) and a maximal torus $T \subseteq G$ which is contained in B. There are two main examples to keep in mind: (1) when $G = T$ is a torus, in which case $G = B = T$; and (2) when $G = SL_n$, in which case we can take $B = B_n$ to be the subgroup of upper triangular matrices and $T = T_n$ to be the subgroup of diagonal matrices.

Let $U \subseteq B$ denote the subgroup consisting of all unipotent elements in B. We can describe U for our two main examples: (1) when $G = T$ is a torus, we have $U = \{1\}$; and (2) when $G = SL_n$, we have $U = U_n \subseteq B_n$, the subgroup of upper triangular matrices with all 1's on the diagonal. This U is a maximal unipotent subgroup of G, so U is a solvable closed connected subgroup of G.

Note that all Borel subgroups of G are conjugate to B. Moreover, all maximal tori (resp. maximal unipotent subgroups) of G are those of the Borel subgroups of G, and all maximal tori (resp. maximal unipotent subgroups) are conjugate. In particular, the *Weyl group* $W = W(G,T) := N_G(T)/T$ acts freely and transitively by conjugation on the set of Borel subgroups of G which contain T.

There exists a unique Borel subgroup $B^- \subseteq G$, called the *opposite* of B (relative to T), such that $B \cap B^- = T$. Note that B^- is conjugate to B by an element of W. In our example with $G = SL_n$, the opposite Borel B_n^- (relative to T_n) is the subgroup of lower triangular matrices.

3.1.2 Characters

For characters and one-parameter subgroups, we use much of the notation and conventions from [CLS11, Section 1.1]. Recall that a *character* of an algebraic group G' is an algebraic group homomorphism $\chi : G' \to \mathbb{G}_m$. The dual objects, i.e. algebraic group homomorphisms $\lambda : \mathbb{G}_m \to G'$, are called *one-parameter subgroups*. We denote the abelian groups of characters and one-parameter subgroups of G' by $\mathfrak{X}(G')$ and $\mathfrak{X}^*(G')$, respectively. We use additive notation for $\mathfrak{X}(G')$ and $\mathfrak{X}^*(G')$. However, the group operation on characters (and one-parameter subgroups) is naturally multiplication, so for $m \in \mathfrak{X}(G')$ we use χ^m to denote the corresponding multiplicative map $\chi^m : G' \to \mathbb{G}_m$. Note that $m_1 + m_2$ corresponds to $\chi^{m_1+m_2} = \chi^{m_1}\chi^{m_2}$.

We are mostly interested in characters and one-parameter subgroups of diagonalizable (algebraic) groups, i.e. closed subgroups of tori. There is a contravariant equivalence of categories $K \mapsto \mathfrak{X}(K)$ between diagonalizable groups and finitely generated abelian groups. In particular, this restricts to a contravariant equivalence of categories between tori and

lattices (i.e. free abelian groups of finite rank). If K is a torus, then the groups $\mathfrak{X}(K)$ and $\mathfrak{X}^*(K)$ are dual lattices. After identifying $\mathfrak{X}(K)$ and $\mathfrak{X}^*(K)$ with $\mathbb{Z}^n$ for some n, we can view the dual pairing as the usual dot product on $\mathbb{Z}^n$; note that the $\mathbb{R}$-vector spaces $\mathfrak{X}(K)_{\mathbb{R}} := \mathfrak{X}(K) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathfrak{X}^*(K)_{\mathbb{R}}$ can be identified with $\mathbb{R}^n$ with the usual dot product.

Given a lattice N, we let $T_N \cong \mathbb{G}_m^{\mathrm{rk}(N)}$ denote the torus whose one-parameter subgroup lattice is N; there is a natural isomorphism $N \otimes_{\mathbb{Z}} \mathbb{G}_m \xrightarrow{\sim} T_N$ via $\lambda \otimes t \mapsto \lambda(t)$, where $\lambda : \mathbb{G}_m \to T_N$ denotes a one-parameter subgroup. If $\Phi : N_1 \to N_2$ is a $\mathbb{Z}$-linear map of lattices, then the dual map $\Phi^\vee : N_2^\vee \to N_1^\vee$ induces a map of tori $T_{N_1} \to T_{N_2}$ via the contravariant equivalence mentioned above (viewing $N_i^\vee$ as the character lattice of T_{N_i}); we denote this map by $\Phi_{\mathbb{G}_m} : T_{N_1} \to T_{N_2}$.

Remark 3.1.1 (Characterizations of finite cokernel)**.** If $\beta : N_1 \to N_2$ is a $\mathbb{Z}$-linear map of lattices, then the following conditions are equivalent:

(1) $\beta : N_1 \to N_2$ has finite cokernel.

(2) The $\mathbb{R}$-linear extension $\beta_{\mathbb{R}} : (N_1)_{\mathbb{R}} \to (N_2)_{\mathbb{R}}$ is surjective.

(3) $\beta^\vee : N_2^\vee \to N_1^\vee$ is injective.

(4) $\beta_{\mathbb{G}_m} : T_{N_1} \to T_{N_2}$ is surjective.

If $H' = \cap_{\chi \in \mathfrak{X}(G')} \ker(\chi)$, then G'/H' is diagonalizable and $\mathfrak{X}(G') = \mathfrak{X}(G'/H')$. Thus $\mathfrak{X}(G')$ is finitely generated, and if G' is connected, then $\mathfrak{X}(G')$ is a lattice.

Remark 3.1.2. For our Borel subgroup B and maximal unipotent $U \subseteq B$, we have $\cap_{\chi \in \mathfrak{X}(B)} \ker(\chi) = U$ and $B/U \cong T$, so $\mathfrak{X}(B) = \mathfrak{X}(B/U) = \mathfrak{X}(T)$.

Example 3.1.3 (Characters of $B_n \subseteq SL_n$)**.** Consider $G = SL_n$ with the Borel subgroup B_n and maximal torus T_n. We have $\mathfrak{X}(B_n) = \mathfrak{X}(T_n) \cong \mathbb{Z}^{n-1}$. Throughout this thesis, we implicitly use the basis $\{e_1, \ldots, e_{n-1}\}$ for $\mathfrak{X}(T_n)$ where e_i corresponds to the character $\chi_i : T_n \to \mathbb{G}_m$ sending a diagonal matrix to the product of the first i diagonal entries. Said another way, χ_i is the character $B_n \to \mathbb{G}_m$ sending an upper triangular matrix to the determinant of the upper-left $i \times i$-submatrix. Note that, if we allowed for $i = n$, then χ_n would be the full determinant, which is trivial in SL_n. This basis will be particularly nice in Section 3.3. ♢

3.1.3 Roots

Throughout this subsection we assume that G is semisimple, i.e. $\mathrm{Rad}(G)$ is trivial. For example, $G = SL_n$ is semisimple, but $G = GL_n$ or $G = T$ a torus are not semisimple. We

explain how to treat the case where G is reductive at the end in Remark 3.1.5. Let $\mathfrak{g}$, $\mathfrak{b}$, and $\mathfrak{t}$ denote the Lie algebras of G, B, and T, respectively.

The adjoint representation of $\mathfrak{g}$ in $GL(\mathfrak{g})$ is defined by $\mathrm{ad}(x) : y \mapsto [x, y]$ for $x, y \in \mathfrak{g}$, where $[\cdot, \cdot]$ is the Lie bracket. An element $\alpha \in \mathfrak{X}(T)$ is called a *root* of (G, T) if the space of α-eigenvectors for the adjoint representation

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : [t, x] = \chi^\alpha(t)x, \ \forall t \in \mathfrak{t}\}$$

is nonzero. Let $R = R(G, T)$ denote the set of roots of (G, T), which is a finite set. Also let $R^+ := \{\alpha \in R : \mathfrak{g}_\alpha \subseteq \mathfrak{b}\}$ denote the subset of *positive roots* (relative to B), and let $R^- := R \setminus R^+$ denote the subset of *negative roots*. Note that $R^- = \{-\alpha : \alpha \in R^+\}$. Then

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha = \underbrace{\mathfrak{t} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha}_{\mathfrak{b}} \oplus \bigoplus_{\alpha \in R^-} \mathfrak{g}_\alpha$$

and $\dim_k(\mathfrak{g}_\alpha) = 1$ for each $\alpha \in R$.

Let $S \subseteq R^+$ denote the set of *simple roots*, i.e. positive roots which cannot be written as a sum of distinct positive roots. Every positive (resp. negative) root is a unique $\mathbb{Z}_{\geq 0}$-linear (resp. $\mathbb{Z}_{\leq 0}$-linear) combination of elements from S; although, not *every* $\mathbb{Z}$-linear combination of elements from S yields a root. It follows that S is an $\mathbb{R}$-basis for $\mathfrak{X}(T)_\mathbb{R} = \mathfrak{X}(T) \otimes_\mathbb{Z} \mathbb{R}$, and in particular $\#S = \dim_\mathbb{R}(\mathfrak{X}(T)_\mathbb{R}) = \dim(T)$. For $I \subseteq S$, we let R_I denote the subset of R generated by I, i.e. R_I consists of the roots which are $\mathbb{Z}$-linear combinations of elements in I. Also let $R_I^+ := R_I \cap R^+$ and $R_I^- := R_I \cap R^-$.

The Weyl group $W = N_G(T)/T$ acts on $\mathfrak{X}(T)$ by $w \cdot m$ being the character defined as $(w \cdot \chi^m) := \chi^m(w^{-1}tw)$ for all $w \in W$, $m \in \mathfrak{X}(T)$, and $t \in T$. This action permutes the roots, so W is a subgroup of the symmetric group of the set R. In particular, W is finite.

Given $\alpha \in R$, let G_α denote the subgroup of G whose Lie algebra is $\mathfrak{t} \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$. Each G_α contains T, and the Weyl group of G_α, i.e. $N_{G_\alpha}(T)/T$, has order 2. Therefore, each $\alpha \in R$ determines a unique nontrivial element of W which has a representative in $N_{G_\alpha}(T)$; we call this element the *reflection* associated to α, and we denote it $r_\alpha \in W$. Note that $r_\alpha = r_{-\alpha}$ and $r_\alpha \cdot \alpha = -\alpha$.

The Weyl group W is generated by $\{r_\alpha : \alpha \in R\}$, and is even generated by $\{r_\alpha : \alpha \in S\}$. For any $I \subseteq S$, we let W_I denote the subgroup of W generated by $\{r_\alpha : \alpha \in I\}$. In particular, $W_\varnothing$ is trivial and $W_S = W$.

Example 3.1.4 (Roots of SL_n)**.** Consider $G = SL_n$. Then $\mathfrak{g} = \mathfrak{sl}_n$ is the traceless $n \times n$ matrices (over k), and $[x, y] = xy - yx$. Consider $T = T_n$ and $B = B_n$. Throughout this

thesis, we implicitly use the following notation for the roots and simple roots of SL_n.

If $\alpha \in R$ is a root, then there exists $0 \neq x \in \mathfrak{g}$ such that $tx - xt = \chi^\alpha(t)x$ for all $t \in \mathfrak{t}$. Since $t \in \mathfrak{t}$ is diagonal, we can write $t = \operatorname{diag}(t_1, \ldots, t_n)$ for some $t_1, \ldots, t_n \in k$. Hence, for all $1 \leq i, j \leq n$ we have

$$\chi^\alpha(t)x_{i,j} = (tx - xt)_{i,j} = t_i x_{i,j} - t_j x_{i,j} = (t_i - t_j)x_{i,j}$$

(the subscript i, j is denoting the (i, j)-entry). Thus $\chi^\alpha(t) = t_i - t_j$ as a map $\mathfrak{t} \to k$; or if we view χ^α as a character of T then $\chi^\alpha(t) = t_i t_j^{-1}$. Therefore, the roots are

$$R = \{\alpha_{i,j} : 1 \leq i \neq j \leq n\}, \qquad \chi^{\alpha_{i,j}} : T \to \mathbb{G}_m \quad \operatorname{diag}(t_1, \ldots, t_n) \mapsto t_i t_j^{-1}.$$

It is easy to check that $\mathfrak{g}_{\alpha_{i,j}}$ is the subalgebra generated by $E_{i,j}$ (i.e. the standard basis matrix with 1 in the (i, j) entry and 0 elsewhere). So we have $\mathfrak{g}_{\alpha_{i,j}} \subseteq \mathfrak{b}$ if and only if $j > i$. Therefore, the positive roots are $R^+ = \{\alpha_{i,j} : 1 \leq i < j \leq n\}$, and it is easy to see that the simple roots are $S = \{\alpha_i : 1 \leq i < n\}$ where $\alpha_i := \alpha_{i,i+1}$.

Note that $N_{SL_n}(T)$ is the group of generalized permutation matrices in SL_n, so the Weyl group $W(SL_n, T) = N_{SL_n}(T)/T$ is isomorphic to $\mathfrak{S}_n$ (the symmetric group). The reflection r_{α_i} for $\alpha_i \in S$ corresponds to the transposition $(i \;\; i+1) \in \mathfrak{S}_n$. ♢

Remark 3.1.5 (Roots of reductive G)**.** If G is reductive, then $[G, G]$ (the commutator) and $G/\operatorname{Rad}(G)$ are semisimple, and there is a natural bijective correspondence between the roots of G, $[G, G]$, and $G/\operatorname{Rad}(G)$. The only difference in the root data between G and these associated semisimple groups is in the rank of the maximal torus. For example, the roots of $G = GL_n$ are the same as the roots of $[G, G] = G/\operatorname{Rad}(G) = SL_n$, but the maximal torus of GL_n has rank n and the maximal torus of SL_n has rank $n - 1$.

Example 3.1.6 (Roots of torus)**.** Using Remark 3.1.5, we see that a torus has a trivial root system, i.e. the root set is empty and the Weyl group is trivial. ♢

Remark 3.1.7. If G is reductive, then there always exists an isogeny (i.e. a surjective algebraic group homomorphism with finite kernel) $G^{ss} \times T_0 \to G$ where G^{ss} is semisimple simply connected (i.e. every isogeny from a connected algebraic group to G^{ss} is an isomorphism) and T_0 is a torus. We can choose a Borel subgroup and a maximal torus of $G^{ss} \times T_0$ that map onto B and T, respectively. Then the roots of $G^{ss} \times T_0$ (relative to this Borel and maximal torus) are the same as the roots of G, and the maximal tori have the same rank.

3.1.4 Parabolic subgroups

We no longer restrict to the assumption that G is semisimple, i.e. G can be reductive. A *parabolic subgroup* of G is a subgroup of G which contains a Borel subgroup of G. We say that a parabolic subgroup $P \subseteq G$ is *standard* if it contains our fixed B. If P is a parabolic subgroup of G, then P is closed and connected, we have $N_G(P) = P$, and the homogeneous space G/P is a smooth projective variety. Any variety of the form G/P with P parabolic is called a *flag variety*.

For $I \subseteq S$, set $P_I := BW_IB \subseteq G$. For example, $P_\varnothing = B$ and $P_S = G$. Each P_I is a standard parabolic subgroup of G whose Lie algebra is

$$\mathfrak{p}_I = \underbrace{\mathfrak{t} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha}_{\mathfrak{b}} \oplus \bigoplus_{\alpha \in R_I^-} \mathfrak{g}_\alpha.$$

A *Levi subgroup* of an algebraic group G' is a closed subgroup $L' \subseteq G'$ such that the natural map $L' \to G'/\operatorname{Rad}_u(G')$ is an isomorphism. Note that, if G' is connected, then L' is a connected reductive algebraic group. Every standard parabolic subgroup $P \subseteq G$ has a unique Levi subgroup which contains T; we call this the *standard Levi subgroup* of P. If $P = P_I$ for some $I \subseteq S$, then we denote this unique Levi subgroup by L_I. For example, $L_\varnothing = T = B/U$ and $L_S = G = G/\operatorname{Rad}_u(G)$.

Since L_I is connected and reductive, we can consider its roots. Relative to the Borel subgroup $B \cap L_I \subseteq L_I$ and the maximal torus $T \cap L_I \subseteq L_I$, the set of roots for L_I is R_I, the set of simple roots is I, and the Weyl group of L_I is W_I. The Lie algebra of L_I is

$$\mathfrak{l}_I = \mathfrak{t} \oplus \bigoplus_{\alpha \in R_I} \mathfrak{g}_\alpha = \underbrace{\mathfrak{t} \oplus \bigoplus_{\alpha \in R_I^+} \mathfrak{g}_\alpha}_{\mathfrak{b} \cap \mathfrak{l}_I} \oplus \bigoplus_{\alpha \in R_I^-} \mathfrak{g}_\alpha.$$

Proposition 3.1.8 (Classification of parabolic subgroups)**.** *The map $I \mapsto P_I$ is a bijection between subsets $I \subseteq S$ and standard parabolic subgroups of G. This bijection satisfies the following properties:*

(1) If P_I and P_J are conjugate, then $P_I = P_J$.

(2) $P_I \subseteq P_J$ (resp. $L_I \subseteq L_J$) if and only if $I \subseteq J$.

(3) $P_{I \cap J} = P_I \cap P_J$ and $L_{I \cap J} = L_I \cap L_J$.

Proof. See [Hum75, Theorem 29.3]. □

Example 3.1.9 (Parabolics of SL_n)**.** Consider $G = SL_n$ with the Borel subgroup B_n. We describe the standard parabolic and Levi subgroups of SL_n using Proposition 3.1.8. Recall that $S = \{\alpha_1, \dots, \alpha_{n-1}\}$.

Since B_n is the subgroup of upper triangular matrices, the parabolics of SL_n containing B_n are subgroups consisting of block upper triangular matrices. For the parabolic P_I, the set $I \subseteq S$ describes the blocks on the diagonal. Specifically, if $S \setminus I = \{\alpha_{i_1}, \dots, \alpha_{i_s}\}$ with $i_1 < i_2 < \cdots < i_s$, and we set $i_0 := 0$ and $i_{s+1} := n$, then P_I is the subgroup of SL_n consisting of block upper triangular matrices where the blocks on the diagonal have sizes $i_1 - i_0, i_2 - i_1, \dots, i_{s+1} - i_s$.

The standard Levi subgroups consist of block diagonal matrices: the set I describes the blocks on the diagonal for L_I. Specifically, L_I is the subgroup of block diagonal matrices where the blocks have sizes $i_1 - i_0, i_2 - i_1, \dots, i_{s+1} - i_s$.

As some specific examples in SL_4, we have

$$P_{\{\alpha_1,\alpha_2\}} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix}, \quad P_{\{\alpha_1,\alpha_3\}} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}, \quad P_{\{\alpha_2\}} = \begin{bmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix}$$

and

$$L_{\{\alpha_1,\alpha_2\}} = \begin{bmatrix} * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}, \quad L_{\{\alpha_1,\alpha_3\}} = \begin{bmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{bmatrix}, \quad L_{\{\alpha_2\}} = \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}.$$

With this description, one can understand the flag varieties SL_n/P_I. In particular, when $I = \varnothing$, we have the prototypical flag variety SL_n/B which is the space of complete flags in the vector space k^n. Another key example is the Grassmannian $\mathrm{Gr}(\ell, k^n)$ of ℓ-dimensional subspaces of k^n, which we can write as SL_n/P_I where $S \setminus I = \{\alpha_\ell\}$. For example, $SL_4/P_{\{\alpha_1,\alpha_3\}}$ is $\mathrm{Gr}(2, k^4)$. ♢

Remark 3.1.10. If P is a standard parabolic subgroup of G, then we have a surjection of tori $T = B/U \to P/(\cap_{\chi \in \mathfrak{X}(P)} \ker(\chi))$, so we always have an inclusion of character lattices $\mathfrak{X}(P) \subseteq \mathfrak{X}(T)$; compare with Remark 3.1.2.

A final word about parabolic subgroups is that, for any parabolic $P \subseteq G$, there exists a unique *opposite* parabolic subgroup $P^- \subseteq G$ containing B^-. If $P = P_I$ is standard,

then $P^- = P_I^-$ is the unique parabolic subgroup of G which has the same standard Levi subgroup L_I as P_I and satisfies $P_I \cap P_I^- = L_I$. In Example 3.1.9 for $G = SL_n$, the opposite parabolic P_I^- is the appropriate subgroup of block *lower* triangular matrices, where the blocks are described by I as discussed in this example.

3.1.5 Bruhat decomposition

To finish this section, we discuss the Borel orbits of flag varieties using the well-known Bruhat decomposition. This is very useful in describing the Borel orbits of horospherical homogeneous spaces, which is a key ingredient in the study of horospherical (and spherical) varieties.

Proposition 3.1.11 (Bruhat decomposition)**.** *If $P_I \subseteq G$ is a standard parabolic subgroup (with $I \subseteq S$), then we have*

$$G = \bigsqcup_{w \in W/W_I} B^- w P_I$$

(here w is an element of W/W_I and is also a representative in G). Furthermore, $B^- w P_I = B^- w' P_I$ if and only if $w = w'$ in W/W_I.

Proof. See [Hum75, Theorem 28.3]. □

This proposition tells us the B^--orbit structure of flag varieties because we have

$$G/P_I = \bigsqcup_{w \in W/W_I} B^- w P_I / P_I. \tag{3.1.12}$$

The $B^- w P_I / P_I$ appearing above are called the *Schubert cells* of the flag variety G/P_I,[1] which are exactly the B^--orbits of G/P_I. The closures of the Schubert cells are called *Schubert varieties*; these are closed subvarieties of G/P_I.

If $w \in W/W_I$ is written as a product of generators $w = r_{\alpha_1} \cdots r_{\alpha_\ell}$ with $\alpha_i \in S$ and ℓ is minimal, then we call $\ell = \ell(w)$ the *length* of w; see [Hum75, Section 29.2] or [Bri04, pg. 8] for more details. The identity $w = e$ in W/W_I is the unique shortest element, i.e. $\ell(e) = 0$. The codimension of the Schubert cell $B^- w P_I / P_I$ (or of the corresponding Schubert variety) in G/P_I is equal to $\ell(w)$. Thus, the only Schubert cell of full dimension (i.e. dimension

[1] Based on our convention of using the opposite Borel B^- in the cell $B^- w P_I$, these are typically known as the *opposite* cells; e.g. see [Bri04, Definition 1.2.4]

$\dim(G/P_I)$) is when $w = e$: B^-P_I/P_I; we call this the *big Schubert cell*. This big Schubert cell is an open B^--orbit in G/P_I, and the corresponding Schubert variety is G/P_I.

3.2 Horospherical objects

Before studying the combinatorial description of horospherical varieties, we use this section to define the various horospherical terms: horospherical subgroups, horospherical homogeneous spaces, horospherical varieties, and horospherical morphisms.

3.2.1 Horospherical subgroups

We first define horospherical subgroups, which act as the general stabilizer groups for horospherical varieties.

Definition 3.2.1 (Horospherical subgroup)**.** A closed subgroup $H \subseteq G$ is *horospherical* if it contains a maximal unipotent subgroup of G. If H contains our fixed U, then we say that H is *standard*.

Example 3.2.2 (Basic examples of horospherical subgroups)**.**

- If $G = T$ is a torus, then $U = \{1\}$, so any closed subgroup of T is horospherical.
- For any G, maximal unipotent subgroups are horospherical.
- Any parabolic subgroup $P \subseteq G$ is horospherical.
- If G_i are two connected reductive algebraic groups and $H_i \subseteq G_i$ are horospherical subgroups for $i = 1, 2$, then $H_1 \times H_2$ is a horospherical subgroup of $G_1 \times G_2$. ♢

The following proposition describes the normalizer of a horospherical subgroup, which is used heavily in our study of horospherical varieties.

Proposition 3.2.3 (Normalizer of horospherical subgroup)**.** *Let $H \subseteq G$ be a (resp. standard) horospherical subgroup. Then $N_G(H)$ is a (resp. standard) parabolic subgroup of G and $N_G(H)/H$ is a torus.*

For a horospherical subgroup $H \subseteq G$, we call $P = N_G(H)$ the *associated parabolic* to H, and we call P/H the *associated torus* (to H or G/H). Let $N = N(G/H) := \mathfrak{X}^*(P/H)$ be

the one-parameter subgroup lattice of the torus P/H associated to G/H; so $N^\vee = \mathfrak{X}(P/H)$ is the dual lattice of characters. As in Section 3.1.2, we can write $P/H = T_N$.

Assuming H is standard, one can quickly verify that $P = TH$ and thus $T_N = T/(T \cap H)$; recall that $T \subseteq B$ is the maximal torus. So we have a surjection of tori and a corresponding inclusion of character lattices:

$$T \twoheadrightarrow T_N = T/(T \cap H), \qquad N^\vee = \mathfrak{X}(T_N) \hookrightarrow \mathfrak{X}(T). \tag{3.2.4}$$

Example 3.2.5 (Basic examples of associated parabolics and tori)**.** Let $H \subseteq G$ be a horospherical subgroup. For the examples in Example 3.2.2, we have:

- If $G = T$ is a torus, then the associated parabolic to H is T, and the associated torus is T/H.
- The associated parabolic to the maximal unipotent U is the Borel B, and the associated torus is the maximal torus $B/U = T$.
- If $H = P \subseteq G$ is parabolic, then the associated parabolic is P itself, and the associated torus is the trivial group P/P.
- If G_i are two connected reductive algebraic groups and $H_i \subseteq G_i$ are horospherical subgroups for $i = 1, 2$, then the associated parabolic to $H_1 \times H_2$ is $P_1 \times P_2$, where P_i is the associated parabolic to H_i, and the associated torus is $(P_1 \times P_2)/(H_1 \times H_2) = (P_1/H_1) \times (P_2/H_2)$. ♢

Now we give a classification of horospherical subgroups. Consider a pair (I, M) where $I \subseteq S$ and $M \subseteq \mathfrak{X}(P_I) \subseteq \mathfrak{X}(T)$. The pair (I, M) determines a standard horospherical subgroup $H_{(I,M)}$ defined below whose associated parabolic is P_I:

$$H_{(I,M)} := \bigcap_{\chi \in M} \ker(\chi).$$

Indeed, this is a standard horospherical subgroup because every character $\chi : P_I \to \mathbb{G}_m$ is trivial on U, so $H_{(I,M)} \supseteq U$. Notice that, if M and M' generate the same sublattice of $\mathfrak{X}(P_I)$, then $H_{(I,M)} = H_{(I,M')}$. So we can just look at pairs (I, M) where $I \subseteq S$ and M is a sublattice of $\mathfrak{X}(P_I) \subseteq \mathfrak{X}(T)$. The following proposition shows that every standard horospherical subgroup arises in this way.

Proposition 3.2.6 (Classification of horospherical subgroups)**.** *There is a bijection*

$$\{(I, M) : I \subseteq S,\ M \subseteq \mathfrak{X}(P_I) \text{ a sublattice}\} \longrightarrow \{\text{Standard horospherical subgroups of } G\}$$

$$(I, M) \longmapsto H_{(I,M)}.$$

In particular, for a fixed standard parabolic subgroup $P \subseteq G$, this restricts to an inclusion reversing bijection

$$\{\textit{Sublattices } M \subseteq \mathfrak{X}(P)\} \longrightarrow \{\textit{Horospherical subgroups with associated parabolic } P\}.$$

Proof. See [Pas08, Proposition 2.4]. □

Notation 3.2.7 (Minimal horospherical subgroups)**.** Given a standard parabolic subgroup $P = P_I$ of G for $I \subseteq S$, there is a unique minimal horospherical subgroup $H(P) := H_{(I,\mathfrak{X}(P))}$ whose associated parabolic is P. In particular, $H(B) = U$.

Example 3.2.8. Consider $G = SL_2$. Then $S = \{\alpha\}$ and $\mathfrak{X}(T_2) \cong \mathbb{Z}$. We use Proposition 3.2.6 to determine all standard horospherical subgroups of SL_2.

If $I = S$, then $\mathfrak{X}(P_S) = \mathfrak{X}(SL_2)$ is trivial, so $H_{(S,\{0\})} = SL_2$ is the only (standard) horospherical subgroup of SL_2 with associated parabolic $P_S = SL_2$.

The other possibility for the subset $I \subseteq S$ is $I = \varnothing$. In this case, $\mathfrak{X}(P_\varnothing) = \mathfrak{X}(B_2) = \mathfrak{X}(T_2) \cong \mathbb{Z}$. Therefore, the standard horospherical subgroups of SL_2 with associated parabolic $P_\varnothing = B_2$ are $H_{(\varnothing,M)}$ where $M \subseteq \mathbb{Z}$ is a sublattice. Since sublattices of $\mathbb{Z}$ are of the form $a\mathbb{Z}$ for $a \in \mathbb{Z}_{\geq 0}$, we can write these horospherical subgroups explicitly as

$$H_{(\varnothing,a\mathbb{Z})} = \left\{ \begin{bmatrix} \zeta_a & * \\ 0 & \zeta_a^{-1} \end{bmatrix} : \zeta_a^a = 1 \right\}.$$

Note that, when $a = 0$, this yields the Borel $P_\varnothing = B_2$. ♢

3.2.2 Horospherical homogeneous spaces

If $H \subseteq G$ is a horospherical subgroup, then we call the homogeneous space G/H a *horospherical homogeneous space*. Since G/H is the same (up to G-equivariant isomorphism) after conjugating H, we may assume that H is standard; going forward, we always assume that this is the case for horospherical homogeneous spaces.

Remark 3.2.9 (G-equivariant automorphisms of G/H)**.** Let G/H be a horospherical homogeneous space. The associated torus P/H, with $P = N_G(H) \supseteq B$, is naturally isomorphic to the group of G-equivariant automorphisms of G/H, denoted $\mathrm{Aut}^G(G/H)$.

Given a horospherical homogeneous space G/H, we have an *associated flag variety* G/P, where $P = N_G(H)$ is the associated parabolic, and we have a natural projection map

$G/H \to G/P$. The fibre over $eP \in G/P$ is the associated torus P/H, and every other fibre is isomorphic to this one. That is, $G/H \to G/P$ is a principal P/H-bundle. The following proposition shows that this structure characterizes horospherical homogeneous spaces.

Proposition 3.2.10 (Characterization of horospherical homogeneous spaces)**.** *A homogeneous space G/H is horospherical if and only if it is a principal torus bundle over a flag variety.*

Proof. The forward direction is shown above. Conversely, if there exists a flag variety G/P such that $G/H \to G/P$ is a principal torus bundle, then we may assume that $H \subseteq P$ and this map is the natural projection. The fibre P/H is a torus, so H is normal in P and it follows that H is horospherical with associated parabolic P. □

Example 3.2.11 (Primary horospherical homogeneous space)**.** Since U is the smallest standard horospherical subgroup of G, we see that G/U is the "biggest" horospherical G-homogeneous space. What we mean by this is that, for any other horospherical homogeneous space G/H, we have a G-equivariant surjection $G/U \to G/H$, i.e. the natural projection induced by $U \subseteq H$. We call G/U the *primary horospherical homogeneous space* (with respect to G).

Now consider a standard parabolic subgroup $P \subseteq G$. As in Notation 3.2.7, $H(P)$ is the smallest horospherical subgroup with associated parabolic P, so $G/H(P)$ is the "biggest" horospherical G-homogeneous space with associated flag variety G/P. That is, for any other horospherical homogeneous space G/H with associated flag variety G/P, we have a G-equivariant surjection $G/H(P) \to G/H$, i.e. the natural projection induced by $H(P) \subseteq H$. We call $G/H(P)$ the *P-primary horospherical homogeneous space* (with respect to G). ♢

Again, consider a horospherical homogeneous space G/H with associated flag variety G/P; so $P = N_G(H) \supseteq B$. Recall that the Bruhat decomposition gives the B^--orbit structure of G/P; see Eq. (3.1.12). This can be used to obtain the B^--orbit structure for G/H:

$$G/H = \bigsqcup_{w \in W/W_I} B^- wH/H \tag{3.2.12}$$

(note that $B^- wP/H = B^- wH/H$ since $P = TH$ and $wT = w$). That is, the B^--orbits of G/H are the preimages of the B^--orbits of G/P under the projection $G/H \to G/P$.

Remark 3.2.13 (Open B^--orbit of G/H)**.** Every horospherical homogeneous space has an affine open B^--orbit, namely the big cell $B^- H/H$, i.e. the B^--orbit of the point $eH \in G/H$. One way to see that $B^- H/H$ is affine is using Section 3.4.1: we have $B^- H/H = X_{G/H,B^-}$.

3.2.3 Horospherical varieties

Now we define the main algebro-geometric object of this chapter: horospherical varieties. Let G/H be a horospherical homogeneous space.

Definition 3.2.14 (Horospherical variety)**.** A *G/H-horospherical variety* is a normal G-variety X together with a base point $x \in X$ such that the orbit $G \cdot x \subseteq X$ is open and the stabilizer G_x is the horospherical subgroup H.

We call X a *horospherical G-variety* if it is a G/H-horospherical variety for some horospherical $H \subseteq G$. In a G/H-horospherical variety X, the open orbit $G \cdot x$ is G-equivariantly isomorphic to the horospherical homogeneous space G/H. So we regard $G/H \subseteq X$ as the open orbit; in this notation, the base point is simply $eH \in G/H \subseteq X$.

Remark 3.2.15 (Horospherical varieties are spherical)**.** Horospherical varieties fit into the larger class of spherical varieties; a *spherical G-variety* is a normal G-variety X with an open orbit under the action of a Borel subgroup of G. Indeed, by Remark 3.2.13, every G/H-horospherical variety X has an open B^--orbit, namely $B^-H/H \subseteq X$ (the orbit of the base point).

We briefly discuss spherical varieties in Section 3.3.1, and we see an important way of viewing horospherical varieties as a subclass of spherical varieties (Proposition 3.3.5).

Proposition 3.2.16 (Rationality of horospherical varieties)**.** *Every horospherical variety is rational and has rational singularities (hence is Cohen-Macaulay).*

Proof. See [Per14, Corollary 2.1.3, Corollary 2.3.4]. □

Example 3.2.17 (Basic examples of horospherical varieties)**.**

- Any toric variety with open torus T is a $T/\{1\}$-horospherical variety.
- G/H is itself a G/H-horospherical variety.
- As a special case of the previous example, any flag variety G/P is the *unique* G/P-horospherical variety.
- If X_i are G_i/H_i-horospherical varieties for $i = 1, 2$, then $X_1 \times X_2$ is a $(G_1/H_1) \times (G_2/H_2) = (G_1 \times G_2)/(H_1 \times H_2)$-horospherical variety. ♢

Example 3.2.18. Consider $G = SL_2$. There is an SL_2-action on $\mathbb{A}^2$ via matrix multiplication. Choosing the base point $(1,0) \in \mathbb{A}^2$, the orbit $SL_2 \cdot (1,0) = \mathbb{A}^2 \setminus \{0\}$ is open in $\mathbb{A}^2$, and the stabilizer is U_2. Therefore, $\mathbb{A}^2$ is a SL_2/U_2-horospherical variety.

We have an open embedding $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$ using the affine chart $\{(1 : y : z)\} \subseteq \mathbb{P}^2$. The SL_2-action on $\mathbb{A}^2$ above yields an action on $\mathbb{P}^2$, and with the base point $(1 : 1 : 0) \in \mathbb{P}^2$ we can view $\mathbb{P}^2$ as a SL_2/U_2-horospherical variety using the previous paragraph. ♢

The varieties in Example 3.2.18 are viewed as SL_2/U_2-horospherical varieties, but both are intrinsically toric varieties so they can be studied using toric geometry. To get a sense of which varieties are horospherical but not toric, we prove the following proposition.

Proposition 3.2.19 (Non-toric condition)**.** *Let X be a G/H-horospherical variety. If the associated flag variety G/P is not a product of projective spaces, then X is not a toric variety. Moreover, when X is affine or complete (e.g. projective), the converse is also true.*

Proof. Without loss of generality, we may assume that $\mathscr{O}^*_X(X) = k^*$. Indeed, this uses Remark 3.4.40 in which we can write $X = X_1 \times T_1$ where $\mathscr{O}^*_{X_1}(X_1) = k^*$ and T_1 is a torus, and it is clear that X is a toric variety if and only if X_1 is a toric variety. By reducing to this case, we can consider the Cox ring of X (see [Gag14] for the horospherical case).

We prove the contrapositive, so suppose that X is a toric variety. Then the Cox ring $\mathrm{Cox}(X)$ is a polynomial ring over k. By [Gag14, Theorem 3.8], this implies that $\mathrm{Cox}(G/P)$ is a polynomial ring over k. Thus, G/P is a toric variety by [GH17, Proposition 6.1] (since G/P is complete), so G/P is a product of projective spaces by [Tho97, Theorem 1].

Now we tackle the converse statement in the case where X is affine or complete. In general, each implication in the above argument is reversible except for the first one, i.e. it is *not* known to be true that if $\mathrm{Cox}(X)$ is a polynomial ring over k, then X is a toric variety. However, this is known to be true when X is complete (see [Arz+15, Exercise 3.9]), so the converse statement holds in this case. Below we explain why this also works when X is affine.

If X is affine and $\mathrm{Cox}(X)$ is a polynomial ring over k, then $\mathrm{Spec}(\mathrm{Cox}(X))$ is $\mathbb{A}^n$ for some n, and we have a good quotient map $\mathbb{A}^n \to X$ for the action of a diagonalizable group K (we discuss this in Section 4.3.2). Moreover, there is a connected reductive algebraic group G' such that $\mathbb{A}^n$ is a horospherical G'-variety, and K acts by G'-equivariant automorphisms. As in the argument of [GH17, Proposition 6.1], G' acts linearly on $\mathbb{A}^n$, and therefore K acts linearly on $\mathbb{A}^n$. It follows that the good quotient X of $\mathbb{A}^n$ by K is a toric variety. □

Example 3.2.20. Consider $G = SL_3$. There is an action of SL_3 on $\mathbb{A}^6 = \mathbb{A}^3_{\vec{x}} \times \mathbb{A}^3_{\vec{y}}$ by matrix multiplication on the first $\mathbb{A}^3_{\vec{x}}$ factor and contragredient (i.e. inverse transpose)

matrix multiplication on the second $\mathbb{A}^3_{\vec{y}}$ factor; here $\vec{x}$ and $\vec{y}$ represent coordinate vectors in 3-space. That is, $A \in SL_3$ acts by $A \cdot (\vec{x}, \vec{y}) = (A\vec{x}, (A^T)^{-1}\vec{y})$. Notice that, if $\vec{x} \cdot \vec{y} = 0$ (usual dot product), then $(A\vec{x}) \cdot ((A^T)^{-1}\vec{y}) = 0$. It follows that, choosing the base point $p := (1, 0, 0, 0, 0, 1)$, the orbit

$$SL_3 \cdot p = \{\vec{x} \cdot \vec{y} = 0 : \vec{x}, \vec{y} \neq 0\}$$

is open in $X := \{\vec{x} \cdot \vec{y} = 0\} \subseteq \mathbb{A}^3_{\vec{x}} \times \mathbb{A}^3_{\vec{y}}$, and the stabilizer is U_3. Therefore, X is a SL_3/U_3-horospherical variety. Notice that X is not toric because the associated flag variety SL_3/B_3 is not a product of projective spaces (see Proposition 3.2.19). ♢

Example 3.2.21. Consider $G = SL_n$ for $n \geq 3$ and a horospherical subgroup $H \subseteq SL_n$ whose associated parabolic P is not maximal, i.e. $P = P_I$ corresponds to a subset $I \subseteq S$ with $\#I < n - 2$ (e.g. $P = B_n$). Then SL_n/P is not a (product of) projective space(s), so Proposition 3.2.19 tells us that every SL_n/H-horospherical variety is not a toric variety. ♢

Remark 3.2.22. As in Remark 3.1.7, we have an isogeny $\varepsilon : G^{ss} \times T_0 \to G$ where G^{ss} is semisimple simply connected and T_0 is a torus. A horospherical homogeneous space G/H is ε-equivariantly isomorphic to the horospherical homogeneous space $G^{ss} \times T_0/\varepsilon^{-1}(H)$. Moreover, these horospherical homogeneous spaces have the same equivariant embeddings. Therefore, G/H-horospherical varieties are the same as $G^{ss} \times T_0/\varepsilon^{-1}(H)$-horospherical varieties.

We end this section by defining the morphisms which preserve the horospherical structure.

Definition 3.2.23 (Horospherical morphism)**.** Let X_i be G/H_i-horospherical varieties for $i = 1, 2$. A *horospherical morphism* is a morphism of varieties $X_1 \to X_2$ which is G-equivariant and sends the base point of X_1 to the base point of X_2.

From the definition of horospherical morphism, the map $X_1 \to X_2$ restricts to a map of open orbits $G/H_1 \to G/H_2$ and sends $eH_1 \mapsto eH_2$. This means that we must have $H_1 \subseteq H_2$ and the restricted map $G/H_1 \to G/H_2$ is the natural projection.

Therefore, all horospherical morphisms are dominant. This seems like a somewhat restrictive condition, but there are many important types of morphisms that are horospherical, such as the following: open embeddings of G-invariant subsets, G-equivariant isomorphisms, G-equivariant surjections, and blow-ups of G-invariant closed subsets.

Remark 3.2.24. Consider two horospherical homogeneous spaces G_i/H_i where these G_i are a connected reductive algebraic groups for $i = 1, 2$, and suppose that there is a surjective algebraic group homomorphism $\varepsilon : G_1 \to G_2$. If X_i are G_i/H_i-horospherical varieties for

$i = 1, 2$, then we can view X_2 as a $G_1/\varepsilon^{-1}(H_2)$-horospherical variety where G_1 acts via ε. This perspective is useful because now we can view both X_1 and X_2 as horospherical G_1-varieties, and in particular we can study horospherical morphisms $X_1 \to X_2$.

3.3 Colours

Throughout this section, we fix a horospherical homogeneous space G/H, where $H \supseteq U$ is horospherical with associated parabolic $P = N_G(H) \supseteq B$. Write $P = P_I$ for a unique $I \subseteq S$, and let $\mathcal{C} = \mathcal{C}(G/H) := S \setminus I$. Recall that we have the lattice $N = N(G/H)$ associated to the torus $P/H = T_N$.

In this section we put a "colour structure" on N by connecting $\mathcal{C}$ and N via the B^--invariant divisors of G/H. We should think of $\mathcal{C}$ as an abstract set of *colours* for G/H. Note that we prefer to consider B^--invariant divisors instead of B-invariant divisors, which is not common in the literature. We choose to do this because the open B-orbit in G/H does not contain the base point eH, but the open B^--orbit does contain eH, so it is more convenient to work with the open B^--orbit, in which case one should study the complementary B^--invariant divisors. One can simply swap the roles of B and B^- (since $(B^-)^- = B$) throughout this thesis to instead consider B-invariant divisors, if desired.

3.3.1 Eigenfunctions

We start with some technical details regarding eigenfunctions and valuations. For any algebraic group G' and any G'-module V, the set of G'-eigenvectors in V is denoted $V^{(G')} := \{v \in V \setminus \{0\} : g \cdot v = \chi(g)v \text{ for some } \chi \in \mathfrak{X}(G'),\ \forall g \in G'\}$.

In particular, if X is a normal G-variety, then the *B^--eigenfunctions* of $k(X)$ are

$$k(X)^{(B^-)} = \{f \in k(X)^* : b \cdot f = \chi(b)f \text{ for some } \chi \in \mathfrak{X}(T),\ \forall b \in B^-\}.$$

Note that we used characters of the maximal torus T above since $\mathfrak{X}(B) = \mathfrak{X}(B^-) = \mathfrak{X}(T)$. The B^--eigenfunctions are $\mathfrak{X}(T)$-graded:

$$k(X)^{(B^-)} \cup \{0\} = \bigoplus_{m \in \mathfrak{X}(T)} k(X)^{(B^-)}_{-m}$$

where

$$k(X)^{(B^-)}_{-m} := \{f \in k(X) : b \cdot f = \chi^{-m}(b)f\ \forall b \in B^-\}$$

$$= \{f \in k(X) : f(b \cdot x) = \chi^m(b) f(x) \ \forall b \in B^-, x \in X\}$$

where χ^m denotes the character function associated to the lattice point $m \in \mathfrak{X}(T)$. Recall that $(b \cdot f)(x) := f(b^{-1} \cdot x)$, which explains the minus sign used above.

Given $f \in k(X)^{(B^-)}$, let m_f denote the element of $\mathfrak{X}(T)$ such that $f \in k(X)^{(B^-)}_{-m_f}$. The assignment $f \mapsto m_f$ is a group homomorphism $k(X)^{(B^-)} \to \mathfrak{X}(T)$; let $\Lambda(X) \subseteq \mathfrak{X}(T)$ denote the image of this map. If X is spherical, then the kernel of this map is exactly the constant rational functions, i.e. we have an exact sequence of groups

$$\begin{aligned} 1 \longrightarrow k^* \longrightarrow k(X)^{(B^-)} &\longrightarrow \Lambda(X) \longrightarrow 0 \\ f &\longmapsto m_f. \end{aligned} \tag{3.3.1}$$

Lemma 3.3.2. *If X is a G/H-horospherical variety, then $\Lambda(X) = N^\vee$. Therefore, we have $N^\vee \cong k(X)^{(B^-)}/k^*$.*

Proof. See the paragraph immediately before Section 7.3 in [Tim11]; this is a corollary of [Tim11, Theorem 4.7]. □

For a normal G-variety X, let $\mathrm{Val}(X)$ denote the set of valuations of X, i.e. group homomorphisms $\nu : k(X)^* \to \mathbb{R}$ such that the image is a discrete subgroup of $\mathbb{R}$ (i.e. isomorphic to $\mathbb{Z}$), $\nu(k^*) = 0$, and $\nu(f_1 + f_2) \geq \min\{\nu(f_1), \nu(f_2)\}$ for all $f_1, f_2 \in k(X)^*$. Let $\mathcal{V}(X) \subseteq \mathrm{Val}(X)$ denote the set of G-invariant valuations.

If X is spherical, then Eq. (3.3.1) and the fact that valuations are trivial on constant functions imply that we have a set map

$$\begin{aligned} \xi_{\mathrm{Val}} : \mathrm{Val}(X) &\longrightarrow \Lambda(X)^\vee_{\mathbb{R}} = \mathrm{Hom}_{\mathbb{Z}}(\Lambda(X), \mathbb{R}) \\ \nu &\longmapsto (m_f \mapsto \nu(f)). \end{aligned} \tag{3.3.3}$$

For simplicity, we denote $u_\nu := \xi_{\mathrm{Val}}(\nu)$ for $\nu \in \mathrm{Val}(X)$. If $D \subseteq X$ is a prime divisor then it has a corresponding order of vanishing valuation $\nu_D : k(X)^* \to \mathbb{Z}$; when D is G-invariant, we have $\nu_D \in \mathcal{V}(X)$. In this case, we write u_D to denote u_{ν_D}, which is a lattice point in $\Lambda(X)^\vee \subseteq \Lambda(X)^\vee_{\mathbb{R}}$.

Example 3.3.4. Let X be a toric variety with open torus T, i.e. X is a $T/\{1\}$-horospherical variety. Then X corresponds to a fan Σ on the lattice $N = N(T/\{1\})$. Each ray ρ of Σ corresponds to a T-invariant prime divisor $D_\rho \subseteq X$. In the notation above, the point u_{D_ρ} is exactly the minimal ray generator for ρ; see [CLS11, Proposition 4.1.1]. A generalization of this to all horospherical varieties is stated in Corollary 3.4.26. ♢

The map ξ_{Val} is usually not injective. However, for a spherical variety X, we see below that this map is injective when restricted to $\mathcal{V}(X)$. Moreover, horospherical varieties can be characterized as the subclass of spherical varieties for which $\xi_{\mathrm{Val}}|_{\mathcal{V}(X)}$ is surjective.

Proposition 3.3.5 (Valuation cone)**.** *Let X be a normal G-variety. If X is spherical, then the restricted map $\xi_{\mathrm{Val}}|_{\mathcal{V}(X)} : \mathcal{V}(X) \to \Lambda(X)^\vee_{\mathbb{R}}$ is injective, and the image is a convex polyhedral cone called the* valuation cone*. Moreover, this restricted map is a bijection if and only if X is horospherical.*

Proof. See [Kno91, Corollary 1.8], and see [Kno91, Corollary 6.2] for the "moreover" claim. □

Let X be a G/H-horospherical variety. Combining Lemma 3.3.2 and Proposition 3.3.5, we see that the valuation cone $\mathcal{V}(X)$ is in bijection with $N_{\mathbb{R}}$, so points in $N_{\mathbb{R}}$ are all of the form u_ν for some unique $\nu \in \mathcal{V}(X)$. However, some points in $N_{\mathbb{R}}$ may correspond to multiple valuations when we use *non-G-invariant* valuations. Also, recall that each prime divisor $D \subseteq X$ yields a lattice point $u_D \in N$.

Remark 3.3.6. If X is a G/H-horospherical variety, then $k(X)^{(B^-)} = k(G/H)^{(B^-)}$, $\Lambda(X) = \Lambda(G/H)$, $\mathrm{Val}(X) = \mathrm{Val}(G/H)$, and $\mathcal{V}(X) = \mathcal{V}(G/H)$. Thus, everything developed in this subsection is the same for every G/H-horospherical variety because it only depends on G/H.

3.3.2 Colour divisors

In a G/H-horospherical variety X, the open B^--orbit $B^-H/H \subseteq X$ of the base point is affine, and the boundary $X \setminus (B^-H/H)$ is equal to the union of the B^--invariant prime (i.e. irreducible) divisors of X. These B^--invariant prime divisors come in two flavours: either they are G-invariant or they are not. The divisors of the latter flavour are key in developing the combinatorics of horospherical varieties.

Definition 3.3.7 (Colour divisor)**.** Let X be a G/H-horospherical variety. A *colour divisor* of X is a prime divisor which is B^--invariant but not G-invariant.

For a G/H-horospherical variety X, let $\mathcal{D}(X)$ denote the (finite) set of B^--invariant prime divisors of X. This breaks down as $\mathcal{D}(X) = \mathcal{D}^G(X) \sqcup \mathcal{D}^{\mathcal{C}}(X)$ where $\mathcal{D}^G(X)$ and $\mathcal{D}^{\mathcal{C}}(X)$ are the sets of G-invariant prime divisors and colour divisors of X, respectively.

Example 3.3.8 (Basic examples of colour divisors)**.**

- Consider the case where $G = T$ is a torus and X is a toric variety with open torus T, i.e. X is a $T/\{1\}$-horospherical variety. Then $B^- = T$, so there are no colour divisors for X. Thus, the set $\mathcal{D}(X)$ is just $\mathcal{D}^G(X) = \mathcal{D}^T(X)$.

- Consider a flag variety G/P. Since G/P is a single G-orbit, there are no G-invariant prime divisors. Thus, the set $\mathcal{D}(G/P)$ is just $\mathcal{D}^{\mathcal{C}}(G/P)$. Note that the colour divisors are exactly the Schubert divisors.

- If X_i are horospherical G_i-varieties for $i = 1, 2$, then the sets $\mathcal{D}^{\mathcal{C}}(X_1 \times X_2)$ and $\mathcal{D}^{G_1 \times G_2}(X_1 \times X_2)$ can be naturally identified with $\mathcal{D}^{\mathcal{C}}(X_1) \sqcup \mathcal{D}^{\mathcal{C}}(X_2)$ and $\mathcal{D}^{G_1}(X_1) \sqcup \mathcal{D}^{G_2}(X_2)$, respectively. ♢

In the rest of this subsection we describe the B^--invariant prime divisors of horospherical varieties. First, we compute the B^--invariant prime divisors of G/H. Since G/H is a single G-orbit, there are no G-invariant divisors, i.e. $\mathcal{D}^G(G/H) = \varnothing$. So we just need to compute the colour divisors, which are the B^--orbit closures of codimension 1 in G/H.

Recall that from Eq. (3.2.12), the B^--orbits of G/H are the preimages of the B^--orbits of G/P under the natural projection $G/H \to G/P$. In particular, the colour divisors of G/H are the pullbacks of the colour divisors of G/P. The colour divisors of G/P are exactly the codimension 1 Schubert varieties, i.e. the $\overline{B^- wP/P} \subseteq G/P$ where the length of $w \in W/W_I$ is 1, i.e. $w = r_\alpha$ for some $\alpha \in \mathcal{C} = S \setminus I$. Therefore, the colour divisors of G/H are

$$\mathcal{D}^{\mathcal{C}}(G/H) = \{\overline{B^- r_\alpha H/H} \subseteq G/H : \alpha \in \mathcal{C}\}. \tag{3.3.9}$$

The following proposition tells us about the B^--invariant divisors of G/H-horospherical varieties. In particular, this proposition says that the colour divisors are determined by the colour divisors of G/H given above.

Proposition 3.3.10. *Let X be a G/H-horospherical variety. Every colour divisor of X is the closure of a colour divisor of G/H, and conversely every colour divisor of G/H is the restriction of a colour divisor of X. That is, we have a bijective correspondence*

$$\begin{aligned} \mathcal{D}^{\mathcal{C}}(G/H) &\longleftrightarrow \mathcal{D}^{\mathcal{C}}(X) \\ D &\longmapsto \overline{D} \\ D \cap G/H &\longmapsfrom D. \end{aligned}$$

Proof. The map $\mathcal{D}^{\mathcal{C}}(G/H) \to \mathcal{D}^{\mathcal{C}}(X)$ is clearly well-defined. To show that the reverse map is well-defined, we just need to check that $D \cap G/H \neq \varnothing$ for each $D \in \mathcal{D}^{\mathcal{C}}(X)$. If

$D \cap G/H = \varnothing$, then D is an irreducible component of the boundary $X \setminus G/H$, so D must be a G-invariant prime divisor, which is a contradiction. □

Let X be a G/H-horospherical variety. As a consequence of Proposition 3.3.10, we have a bijection between abstract colours and colour divisors:[1]

$$\mathcal{C} \longrightarrow \mathcal{D}^{\mathcal{C}}(X) \qquad \alpha \longmapsto D_\alpha^X := \overline{B^- r_\alpha H/H} \subseteq X. \tag{3.3.11}$$

When there is no confusion, we simply write D_α rather than D_α^X for the colour divisor in X corresponding to $\alpha \in \mathcal{C}$.

From Example 3.3.8 we know that toric varieties (viewed as $T/\{1\}$-horospherical varieties) have no colour divisors, so Eq. (3.3.11) implies that they are "colourless". The following corollary tells us that toric varieties are the *only* "colourless" horospherical varieties; so we can think of colours (or colour *divisors*) as the main difference between horospherical varieties and toric varieties.

Corollary 3.3.12. *The horospherical homogeneous space G/H is a torus if and only if $\mathcal{C} = \varnothing$ (i.e. G/H is "colourless").*

Proof. The forward direction follows from Example 3.3.8 and Eq. (3.3.11). For the reverse direction, $\mathcal{C} = \varnothing$ means $I = S$, so $G/H = P/H$ is a torus. □

3.3.3 Coloured lattices

Now we use the algebro-geometric data developed in the previous subsections to put a "colour structure" on $N = N(G/H)$. Recall that $\mathcal{C} = \mathcal{C}(G/H) = S \setminus I$ is in bijection with $\mathcal{D}^{\mathcal{C}}(G/H)$; namely, the colour $\alpha \in \mathcal{C}$ corresponds to the colour divisor $D_\alpha \subseteq G/H$. Let $\xi : \mathcal{C} \to N$ be the map sending

$$\alpha \mapsto \xi(\alpha) := \xi_{\mathrm{Val}}(D_\alpha) = u_{D_\alpha}. \tag{3.3.13}$$

We also denote $\xi(\alpha)$ by $u_\alpha \in N$, or sometimes $u_\alpha^N \in N$ to signify that u_α is in N; these points are called *colour points*. These colour points can be difficult to compute directly from ξ_{Val}, but there are other ways to determine them quickly; see Example 3.3.18 and Remark 3.3.23.

[1] Most sources do not make this distinction, but we find it helpful to distinguish between the combinatorial abstract colours and the geometric colour divisors.

Note that the lattice N depends on the torus P/H, the abstract set of colours $\mathcal{C}$ only depends on the associated flag variety G/P, and the colour map ξ depends on the divisors in $\mathcal{D}(G/H) = \mathcal{D}^{\mathcal{C}}(G/H)$.

Definition 3.3.14 (Coloured lattice)**.** The triple $(N, \mathcal{C}, \xi)$ constructed above is called the *coloured lattice associated to G/H*. We call $\mathcal{C}$ the *universal colour set* and ξ the *colour map*. We usually just denote $(N, \mathcal{C}, \xi)$ by $N = N(G/H)$, and the colour structure $\xi : \mathcal{C} = \mathcal{C}(G/H) \to N$ is implicit.

Remark 3.3.15 (FAQs about coloured lattices)**.**

- Can a colour point u_α be at the origin in N? The answer is *yes*, e.g. see the flag variety example in Example 3.3.16, or Example 3.4.31.
- Can two colours $\alpha_1, \alpha_2 \in \mathcal{C}$ yield the same colour point $u_{\alpha_1} = u_{\alpha_2}$ in N? The answer is *yes*, e.g. consider flag varieties, or see Example 3.5.14. This is the reason why we keep track of the abstract colours rather than just the colour points.
- If $u_\alpha \neq 0$, then does u_α have to be a primitive lattice point in N? The answer is *no*, e.g. see Example 3.4.9.
- Does the coloured lattice uniquely determine the horospherical homogeneous space? In general, the answer is *no* if one allows for different full groups G, but if one is only looking at a specific full group G and keeps track of the data appropriately, then the answer *can be yes*; this is made precise in Theorem 3.3.19.

Example 3.3.16 (Basic examples of coloured lattices)**.**

- For a torus T, the coloured lattice $N = N(T/\{1\})$ is the usual one-parameter subgroup lattice for T, and there are no colour points. By Corollary 3.3.12, a coloured lattice $N(G/H)$ has no colour points if and only if G/H is a torus.
- The coloured lattice $N = N(G/P)$ for G/P is trivial, i.e. rank 0, and all colour points are at the origin. It is easy to see that a coloured lattice $N(G/H)$ is trivial if and only if $H = P$ is parabolic.
- If G_i/H_i are two horospherical G_i-homogeneous spaces for $i = 1, 2$, then we have $N((G_1 \times G_2)/(H_1 \times H_2)) = N(G_1/H_1) \times N(G_2/H_2)$ and the universal colour set is $\mathcal{C}(G_1/H_1) \sqcup \mathcal{C}(G_2/H_2)$. ♢

Example 3.3.17 (Colours of SL_n/U_n)**.** Consider the horospherical homogeneous space $G/H = SL_n/U_n$. We describe $N = N(SL_n/U_n)$ as follows. Recall the basis $\{e_1, \dots, e_{n-1}\}$ for $\mathfrak{X}(T_n) \cong \mathbb{Z}^{n-1}$ from Example 3.1.3. Since $N^\vee = \mathfrak{X}(T_n)$, this gives us a basis for $N^\vee$, and in turn we get a dual basis for N which we also denote $\{e_1, \dots, e_{n-1}\}$. With this basis, we have $e_i = u_{\alpha_i}$ for each $1 \leq i \leq n-1$, where $\mathcal{C} = \{\alpha_1, \dots, \alpha_{n-1}\}$ is the universal colour set for N.

Indeed, to see why this is true, we need to check that there is a basis of B_n^--eigenfunctions $\{f_1, \dots, f_{n-1}\}$ for $k(SL_n/U_n)^{(B_n^-)}/k^* \cong N^\vee$ such that $m_{f_j} = e_j$ (in the notation of Section 3.3.1) and $\nu_{D_{\alpha_i}}(f_j) = \delta_{i,j}$ (Kronecker delta) for each $1 \leq i, j \leq n-1$. Choose f_j to be the rational function on SL_n/U_n which sends a matrix to the determinant of its upper-left $j \times j$-submatrix. Then clearly $m_{f_j} = e_j$. Moreover, each f_j is regular and $D_{\alpha_i} = \{f_i = 0\}$, so it follows that $\nu_{D_{\alpha_i}}(f_j) = \delta_{i,j}$. ♢

The previous example shows that the colour points of $N(SL_n/U_n)$ form a basis for the lattice. The following example generalizes this to similar coloured lattices.

Example 3.3.18 (Primary coloured lattice)**.** Recall from Example 3.2.11 that G/U is the primary horospherical homogeneous space (with respect to G). The associated coloured lattice $N(G/U)$ is called the *primary coloured lattice* (again, with respect to G). Note that the universal colour set is $\mathcal{C}(G/U) = S$. In this case, the colour points u_α with $\alpha \in \mathcal{C}(G/U)$ form part of a $\mathbb{Z}$-basis for $N(G/U)$; this is summarized in the introduction of [Gag14] since the divisor class group $\mathrm{Cl}(G/U)$ is trivial.

More generally, if $P \subseteq G$ is a standard parabolic subgroup, then $G/H(P)$ is the P-primary horospherical homogeneous space. The associated coloured lattice $N(G/H(P))$ is called the *P-primary coloured lattice*, and the colour points u_α with $\alpha \in \mathcal{C}(G/H(P))$ form part of a $\mathbb{Z}$-basis for $N(G/H(P))$; again, this uses the fact that $\mathrm{Cl}(G/H(P))$ is trivial. ♢

We end this subsection by stating a uniqueness result from [Los09] for horospherical homogeneous spaces in terms of their coloured lattices. Let $G_{D_\alpha} \subseteq G$ denote the stabilizer of a colour divisor D_α; note that this is the parabolic subgroup $P^-_{S\setminus\{\alpha\}} \supseteq B^-$.

Theorem 3.3.19 (Uniqueness of G/H)**.** *Let $H_1, H_2 \subseteq G$ be any horospherical subgroups. Then G/H_1 and G/H_2 are G-equivariantly isomorphic if and only if the following holds: there is an isomorphism of lattices $\theta : N(G/H_1) \to N(G/H_2)$, and there exists a bijection $\iota : \mathcal{C}(G/H_1) \to \mathcal{C}(G/H_2)$ such that $G_{D_\alpha} = G_{D_{\iota(\alpha)}}$ and $\theta(u_\alpha) = u_{\iota(\alpha)}$ for each $\alpha \in \mathcal{C}(G/H_1)$.*

Proof. See [Los09, Theorem 1]. □

3.3.4 Maps of coloured lattices

Now let G/H_i be two horospherical homogeneous spaces for $i = 1, 2$ and assume that $H_1 \subseteq H_2$. We use the same notation developed previously for G/H but now with subscripts $i = 1, 2$; e.g. $P_i = N_G(H_i)$ and $N_i = N(G/H_i)$.

Consider the natural projection $\pi : G/H_1 \to G/H_2$. For each colour divisor D of G/H_1 there are two cases: either (1) $\overline{\pi(D)}$ is a colour divisor of G/H_2, or (2) $\overline{\pi(D)} = G/H_2$ (i.e. D maps dominantly via π). We relate these possibilities to the combinatorics as follows.

Since $P_i = TH_i$ for $i = 1, 2$, we have $P_1 = P_{I_1} \subseteq P_2 = P_{I_2}$, so $I_1 \subseteq I_2$, and thus $\mathcal{C}_2 = S \setminus I_2 \subseteq \mathcal{C}_1 = S \setminus I_1$. Denote the colour divisor in G/H_i corresponding to $\alpha \in \mathcal{C}_i$ by D^i_α. The inclusion $\mathcal{C}_2 \subseteq \mathcal{C}_1$ is viewed in terms of colour divisors as follows: each $D^2_\alpha \in \mathcal{D}^{\mathcal{C}}(G/H_2)$ pulls back to $D^1_\alpha = \pi^{-1}(D^2_\alpha) \in \mathcal{D}^{\mathcal{C}}(G/H_1)$. Thus, the D^1_α with $\alpha \in \mathcal{C}_2$ are precisely the colour divisors of G/H_1 which map to colour divisors of G/H_2, i.e. these are the colour divisors in case (1) listed in the previous paragraph. On the other hand, the D^1_α with $\alpha \in \mathcal{C}_1 \setminus \mathcal{C}_2$ are the colour divisors of G/H_1 which map dominantly to G/H_2 via π, i.e. these are the colour divisors in case (2) listed above.

For $i = 1, 2$, we have natural projection maps $G/H_i \to G/P_i$. Therefore, we have a commutative diagram of horospherical morphisms (natural projection maps):

$$\begin{array}{ccc} G/H_1 & \xrightarrow{\pi} & G/H_2 \\ \downarrow & & \downarrow \\ G/P_1 & \longrightarrow & G/P_2 \end{array}$$

The vertical maps are principal P_i/H_i-bundles, and we get an induced map of the fibres $T_{N_1} = P_1/H_1 \to T_{N_2} = P_2/H_2$ which is a surjective homomorphism of tori. This induces an inclusion of character lattices and a dual map of one-parameter subgroup lattices:

$$\Phi^\vee : N_2^\vee = \mathfrak{X}(T_{N_2}) \hookrightarrow N_1^\vee = \mathfrak{X}(T_{N_1}), \qquad \Phi : N_1 = \mathfrak{X}^*(T_{N_1}) \to N_2 = \mathfrak{X}^*(T_{N_2}). \tag{3.3.20}$$

Note that $\Phi_{\mathbb{R}}$ is surjective since $\Phi^\vee$ is injective.

We call the map $\Phi : N_1 \to N_2$ the *associated map of coloured lattices* (associated to $\pi : G/H_1 \to G/H_2$). This map preserves the colour structure of these lattices in the following way. Recall that $\mathcal{C}_2 \subseteq \mathcal{C}_1$. Denote $\mathcal{C}_\Phi := \mathcal{C}_1 \setminus \mathcal{C}_2 = I_2 \setminus I_1$, which we call the set of *dominantly mapped colours* (the usage of *dominant* is explained above). Note that $\mathcal{C}_1 = \mathcal{C}_\Phi \sqcup \mathcal{C}_2$. If u^i_α denotes the colour point in N_i corresponding to α, then it is easy to

check that

$$\Phi(u_\alpha^1) = u_\alpha^2 \ \ \forall \alpha \in \mathcal{C}_2 \quad \text{and} \quad \Phi(u_\alpha^1) = 0 \ \ \forall \alpha \in \mathcal{C}_\Phi. \tag{3.3.21}$$

Remark 3.3.22. Following [Kno91, Section 4], there is an equivalent way of thinking about the associated map of coloured lattices $\Phi : N_1 \to N_2$ constructed above. Since $\pi : G/H_1 \to G/H_2$ is dominant, we get an induced inclusion $k(G/H_2) \hookrightarrow k(G/H_1)$ of function fields. Since π is G-equivariant, it is easy to check that this induces an inclusion $k(G/H_2)^{(B^-)} \hookrightarrow k(G/H_1)^{(B^-)}$. Therefore by Lemma 3.3.2, we have an inclusion $N_2^\vee \hookrightarrow N_1^\vee$; this is the same as the inclusion in Eq. (3.3.20). Then the dual map is exactly $\Phi : N_1 \to N_2$ given above.

Remark 3.3.23. Recall from Example 3.3.18 that we have the primary coloured lattice $N(G/U)$. For any other coloured lattice $N = N(G/H)$, we have an associated map of coloured lattices $N(G/U) \to N$ coming from the projection $G/U \to G/H$. On the other hand, if H has associated parabolic P, then we have a map of coloured lattices $N(G/H(P)) \to N$ associated to the projection $G/H(P) \to G/H$. Combining these two maps we get a commutative diagram of coloured lattice maps:

$$N(G/U) \longrightarrow N(G/H(P)) \longrightarrow N$$

The dual maps are the inclusions of character lattices $N^\vee \subseteq \mathfrak{X}(P) \subseteq \mathfrak{X}(T)$. By Eq. (3.3.21), the colour points of N are exactly the images of the non-dominantly mapped colour points in $N(G/U)$ or $N(G/H(P))$.

Example 3.3.24 (Quotient coloured lattice)**.** Let $N' \subseteq N = N(G/H)$ be a saturated sublattice and let $\mathcal{C}' \subseteq \mathcal{C} = \mathcal{C}(G/H)$ be a subset of colours such that $u_\alpha \in N'$ for each $\alpha \in \mathcal{C}'$; we call N' a *coloured sublattice* of N. Then the quotient lattice N/N' is the coloured lattice associated to some horospherical homogeneous space G/H' with $H' \supseteq H$; see [Kno91, Theorem 4.4]. The colour structure of N/N' is described as follows. The quotient map $N \to N/N'$ is a map of coloured lattices (associated to the projection $G/H \to G/H'$) whose set of dominantly mapped colours is $\mathcal{C}'$. So the universal colour set of N/N' is $\mathcal{C} \setminus \mathcal{C}'$, and the colour points of N/N' are the images of the colour points of N from $\mathcal{C} \setminus \mathcal{C}'$.

We can determine H' as follows. Since $\mathcal{C} = S \setminus I$ and $\mathcal{C}' \subseteq \mathcal{C}$, we can write $\mathcal{C}' = S \setminus I'$ for a unique $I' \subseteq S$. The dual map $(N/N')^\vee \hookrightarrow N^\vee$ is an inclusion of character lattices $M' := \mathfrak{X}(T_{N/N'}) \hookrightarrow \mathfrak{X}(T_N)$. Then $H' = H_{(I',M')}$. ♢

3.4 Classification of horospherical varieties

Now we describe the combinatorial classification of horospherical varieties via coloured fans. This section is self-contained regarding the necessary polyhedral geometry, but we recommend [CLS11] (mainly Section 1.2 and Section 3.1) for more on the relevant definitions and properties.

Throughout this section, we fix a horospherical homogeneous space G/H and use the notation from the previous sections. In particular, $P = P_I = N_G(H) \supseteq B$ is the associated parabolic, $N = N(G/H)$ is the associated coloured lattice with universal colour set $\mathcal{C} = \mathcal{C}(G/H) = S \setminus I$, and $\xi : \mathcal{C} \to N$ is the colour map.

3.4.1 Simple open cover

In toric geometry, a general toric variety admits an open cover by affine toric varieties. However, horospherical G-varieties usually do not admit an open cover by affine horospherical G-varieties. Shifting perspective, one can think of affine toric varieties as being the toric varieties with a unique closed torus orbit. This perspective proves useful for generalizing the ideas in toric geometry to horospherical geometry. In this subsection, we show that every horospherical G-variety admits an open cover by horospherical G-varieties with a unique closed G-orbit.

Definition 3.4.1 (Simple horospherical variety)**.** A G/H-horospherical variety is called *simple* if it contains a unique closed G-orbit.

Example 3.4.2 (Basic examples of simple horospherical varieties)**.**

- The horospherical homogeneous space G/H is simple because it has a single G-orbit. In particular, flag varieties are simple.
- Any affine toric variety is simple because it has a unique closed torus orbit (this follows from the orbit-cone correspondence [CLS11, Theorem 3.2.6]).
- If X_i are two simple horospherical G_i-varieties for $i = 1, 2$, then $X_1 \times X_2$ is a simple horospherical $G_1 \times G_2$-variety. ♢

Let X be a G/H-horospherical variety and let $\mathcal{O}$ be any G-orbit. Set

$$\mathcal{D}_{\mathcal{O}}(X) := \{D \in \mathcal{D}(X) : \mathcal{O} \subseteq D\},$$
$$\mathcal{D}^G_{\mathcal{O}}(X) := \{D \in \mathcal{D}^G(X) : \mathcal{O} \subseteq D\},$$

$$\mathcal{D}^{\mathcal{C}}_{\mathcal{O}}(X) := \{D \in \mathcal{D}^{\mathcal{C}}(X) : \mathcal{O} \subseteq D\}.$$

Note that, if X is simple and $\mathcal{O}$ is the unique closed G-orbit, then $\mathcal{D}^G_{\mathcal{O}}(X) = \mathcal{D}^G(X)$ since every G-invariant prime divisor contains $\mathcal{O}$.

For any G-orbit $\mathcal{O} \subseteq X$, let

$$X_{\mathcal{O},B^-} := X \setminus \bigcup (\mathcal{D}(X) \setminus \mathcal{D}_{\mathcal{O}}(X))$$

(so the union is over all B^--invariant prime divisors of X which do *not* contain $\mathcal{O}$). Set $X_{\mathcal{O},G} := G \cdot X_{\mathcal{O},B^-}$. Clearly $X_{\mathcal{O},B^-}$ is open in X, so $X_{\mathcal{O},G}$ is also open in X.

Proposition 3.4.3. *Let X be a G/H-horospherical variety and let $\mathcal{O}$ be any G-orbit.*

(1) $X_{\mathcal{O},B^-}$ is a B^--invariant affine open subset of X and $\mathcal{O} \cap X_{\mathcal{O},B^-}$ is a B^--orbit.

(2) $X_{\mathcal{O},B^-} = \{x \in X : \overline{B^- \cdot x} \supseteq \mathcal{O}\}$ and $X_{\mathcal{O},G} = \{x \in X : \overline{G \cdot x} \supseteq \mathcal{O}\}$.

(3) $\mathcal{O}$ is the unique closed G-orbit in $X_{\mathcal{O},G}$. In particular, $X_{\mathcal{O},G}$ is a simple G/H-horospherical variety. Moreover, if X is simple and $\mathcal{O}$ is the unique closed G-orbit, then $X = X_{\mathcal{O},G}$.

Proof. See [Kno91, Theorem 2.1] for (1) and (3). Part (2) is straightforward to check. □

Proposition 3.4.4. *Let X be a G/H-horospherical variety. Then X has a finite open cover consisting of simple G/H-horospherical varieties. Specifically, X is covered by the open subvarieties $X_{\mathcal{O},G}$ as $\mathcal{O}$ ranges over the G-orbits of X.*

Proof. This follows from the previous work and the fact that X has finitely many G-orbits. □

3.4.2 Coloured cone of a simple horospherical variety

In the correspondence between toric varieties and fans, the simple toric varieties correspond to cones. We illustrate how this works for simple horospherical varieties using *coloured* cones, where the colour structure comes from Section 3.3.

For us, a *cone* on N is a rational polyhedral cone in $N_{\mathbb{R}}$, i.e. a set of the form $\mathrm{Cone}(A) := \mathbb{R}_{\geq 0}A$ for some finite set $A \subseteq N$. Recall that a cone is said to be *strongly convex* if it does not contain any positive-dimensional subspaces of $N_{\mathbb{R}}$ (see [CLS11, Proposition 1.2.12] for equivalent characterizations).

Definition 3.4.5 (Coloured cone)**.** A *coloured cone* on the coloured lattice N is a pair $\sigma^c = (\sigma, \mathcal{F})$ where $\sigma \subseteq N_{\mathbb{R}}$ is a cone and $\mathcal{F} \subseteq \mathcal{C}$ satisfies $\xi(\mathcal{F}) \subseteq \sigma$ and $0 \notin \xi(\mathcal{F})$. We say that σ^c is *strongly convex* if σ is strongly convex.

Given a coloured cone σ^c on N, we let σ denote the underlying cone, and we let $\mathcal{F}(\sigma^c) \subseteq \mathcal{C}$ denote the *colour set*; so we have $\sigma^c = (\sigma, \mathcal{F}(\sigma^c))$.

For a G/H-horospherical variety X and any G-orbit $\mathcal{O}$, let

$$\mathcal{F}_{\mathcal{O}}(X) := \{\alpha \in \mathcal{C} : D_\alpha \in \mathcal{D}^{\mathcal{C}}_{\mathcal{O}}(X)\}$$

(so $\mathcal{F}_{\mathcal{O}}(X)$ consists of the $\alpha \in \mathcal{C}$ such that D_α contains $\mathcal{O}$). Note that $\mathcal{F}_{\mathcal{O}}(X)$ is in bijection with $\mathcal{D}^{\mathcal{C}}_{\mathcal{O}}(X)$ by Eq. (3.3.11). This notation just allows us to explicitly view $\mathcal{D}^{\mathcal{C}}_{\mathcal{O}}(X)$ as a subset of $\mathcal{C}$. We also let

$$\mathcal{F}(X) := \bigcup_{\mathcal{O} \subseteq X} \mathcal{F}_{\mathcal{O}}(X)$$

where the union is over all G-orbits $\mathcal{O} \subseteq X$. In words, $\mathcal{F}(X)$ is the set of $\alpha \in \mathcal{C}$ such that D_α contains a G-orbit of X. Note that, when X is simple and $\mathcal{O}$ is the unique closed G-orbit, we have $\mathcal{F}(X) = \mathcal{F}_{\mathcal{O}}(X)$.

Definition 3.4.6 (Coloured cone of a simple horospherical variety)**.** Let X be a simple G/H-horospherical variety with unique closed G-orbit $\mathcal{O}$. We define the *coloured cone associated to X* as $\sigma^c(X) := (\sigma(X), \mathcal{F}(X))$ where $\sigma(X)$ is the cone in $N_{\mathbb{R}}$ given by

$$\sigma(X) := \mathrm{Cone}(u_D : D \in \mathcal{D}_{\mathcal{O}}(X)) = \mathrm{Cone}(u_\alpha, u_D : \alpha \in \mathcal{F}(X), D \in \mathcal{D}^G(X)).$$

This $\sigma^c(X)$ is a strongly convex coloured cone on N (see [Kno91, Theorem 3.1]).

Example 3.4.7 (Basic examples of associated coloured cones)**.**

- The horospherical homogeneous space G/H is a simple G/H-horospherical variety, and its associated coloured cone is the *trivial coloured cone* $0^c = (0, \varnothing)$ on N; note that this is the unique coloured cone whose underlying cone is trivial. In particular, 0^c is the coloured cone associated to any flag variety.
- If X is an affine toric variety, then it corresponds to a strongly convex cone σ by [CLS11, Theorem 1.3.5]. Viewing X as a $T/\{1\}$-horospherical variety, the coloured cone associated to X is $(\sigma, \varnothing)$.

- If X_i are simple G_i/H_i-horospherical varieties for $i = 1, 2$, then $\sigma^c(X_1 \times X_2) = \sigma^c(X_1) \times \sigma^c(X_2)$, where the product of these coloured cones is defined componentwise as $(\sigma(X_1) \times \sigma(X_2), \mathcal{F}(X_1) \sqcup \mathcal{F}(X_2))$. ♢

When drawing a coloured cone $\sigma^c = (\sigma, \mathcal{F})$, we draw the cone σ, we use "un-filled" circular points to denote the colour points in the coloured lattice (the point u_α is marked with α), and we "fill in" (with the colour orange) the colour point u_α to indicate that $\alpha \in \mathcal{F}$.

Example 3.4.8. Consider $G = SL_2$ and $H = U_2$. Then the coloured lattice N is isomorphic to $\mathbb{Z}$, we have $\mathcal{C} = \{\alpha\}$, and $u_\alpha = e_1$.

Recall from Example 3.2.18 that $\mathbb{A}^2$ is a SL_2/U_2-horospherical variety where SL_2 acts by ordinary matrix multiplication. The unique closed SL_2-orbit is the origin $\{0\} \subseteq \mathbb{A}^2$, so $\mathbb{A}^2$ is simple. The colour divisor D_α is $\{(0, y)\} \subseteq \mathbb{A}^2$, which clearly contains the closed SL_2-orbit, so $\mathcal{F}(\mathbb{A}^2) = \{\alpha\}$. One can easily check that $\mathbb{A}^2$ has no SL_2-invariant divisors. Therefore $\sigma(\mathbb{A}^2) = \mathrm{Cone}(u_\alpha) = \mathrm{Cone}(e_1)$, so $\sigma^c(\mathbb{A}^2) = (\mathrm{Cone}(e_1), \{\alpha\})$; see the coloured cone on the left in the diagram below.

Now consider the blow-up $\mathrm{Bl}_0 \mathbb{A}^2$ of $\mathbb{A}^2$ at the origin. This has an SL_2-action which lifts the action on $\mathbb{A}^2$, and by choosing a base point which maps to the base point of $\mathbb{A}^2$ we can view $\mathrm{Bl}_0 \mathbb{A}^2$ as a SL_2/U_2-horospherical variety. Since $0 \in \mathbb{A}^2$ is fixed by SL_2, the exceptional divisor E is an SL_2-invariant prime divisor. One can check that the corresponding lattice point is $u_E = e_1 \in N = \mathbb{Z}$. It is easy to see that E is the unique closed SL_2-orbit of $\mathrm{Bl}_0 \mathbb{A}^2$, so $\mathrm{Bl}_0 \mathbb{A}^2$ is simple. Now the colour divisor $D_\alpha \subseteq \mathrm{Bl}_0 \mathbb{A}^2$ does *not* contain E, so $\mathcal{F}(\mathrm{Bl}_0 \mathbb{A}^2) = \varnothing$. Therefore $\sigma(\mathrm{Bl}_0 \mathbb{A}^2) = \mathrm{Cone}(u_E) = \mathrm{Cone}(e_1)$, so $\sigma^c(\mathrm{Bl}_0 \mathbb{A}^2) = (\mathrm{Cone}(e_1), \varnothing)$; see the coloured cone on the right in the diagram below.



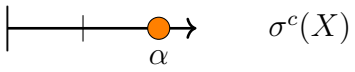
♢

Example 3.4.9. Consider $G = SL_2$ and $H = H_{(\varnothing, 2\mathbb{Z})}$ from Example 3.2.8. Then the coloured lattice N is isomorphic to $\mathbb{Z}$, we have $\mathcal{C} = \{\alpha\}$, and $u_\alpha = 2e_1$. Note that $N^\vee$ is regarded as the sublattice $2\mathbb{Z}$ of $\mathfrak{X}(T_2) = \mathbb{Z}$.

Let X be the affine variety $\{xz = y^2\} \subseteq \mathbb{A}^3$, where x, y, z are the $\mathbb{A}^3$-coordinates. So X is the affine cone over the projective variety $C \subseteq \mathbb{P}^2$ given by the image of $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$ via $(u : v) \mapsto (u^2 : uv : v^2)$. There is a natural action of SL_2 on C: we treat $(u : v) \in \mathbb{P}^1$ as a vector with two coordinates and SL_2 acts by matrix multiplication. Then SL_2 acts

on X by lifting the action on C. The point $(1,0,0) \in X$ has stabilizer H and the orbit $SL_2/H = X \setminus \{0\}$ is open in X, so X is a SL_2/H-horospherical variety.

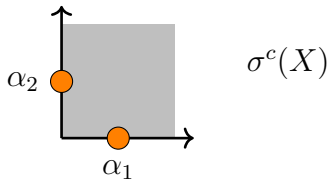
The unique closed SL_2-orbit of X is the origin $\{0\} \subseteq X$, so X is simple. The colour divisor $D_\alpha \subseteq X$ is $\{x = 0\} \subseteq X$. This clearly contains the closed SL_2-orbit, so $\mathcal{F}(X) = \{\alpha\}$. One can check that X has no SL_2-invariant divisors. Therefore $\sigma(X) = \text{Cone}(u_\alpha) = \text{Cone}(2e_1) = \text{Cone}(e_1)$, so $\sigma^c(X) = (\text{Cone}(e_1), \{\alpha\})$; see the diagram below.



♢

Example 3.4.10. Consider $G = SL_3$ and $H = U_3$. Then the coloured lattice N is isomorphic to $\mathbb{Z}^2$, we have $\mathcal{C} = \{\alpha_1, \alpha_2\}$, and $u_{\alpha_i} = e_i$ are the standard basis vectors for $i = 1, 2$.

Let X be the affine variety $\{ux + vy + wz = 0\} \subseteq \mathbb{A}^6$ from Example 3.2.20; recall that X is a SL_3/U_3-horospherical variety. The unique closed SL_3-orbit of X is the origin $\{0\} \subseteq X$, which is contained in both colour divisors D_{α_1} and D_{α_2}, so $\mathcal{F}(X) = \{\alpha_1, \alpha_2\}$. One can check that X has no SL_3-invariant divisors. Therefore $\sigma(X) = \text{Cone}(u_{\alpha_1}, u_{\alpha_2}) = \text{Cone}(e_1, e_2)$, so $\sigma^c(X) = (\text{Cone}(e_1, e_2), \{\alpha_1, \alpha_2\})$; see the diagram below.



♢

The following is the main theorem of this subsection, which says that simple horospherical varieties correspond to strongly convex coloured cones. This generalizes the correspondence between affine toric varieties and strongly convex cones (see [CLS11, Theorem 1.3.5]).

Theorem 3.4.11 (Classification of simple horospherical varieties)**.** *The following map is a bijection:*

$$\{\textit{Simple horospherical } G/H\textit{-varieties}\}/\cong \longrightarrow \{\textit{Strongly convex coloured cones on } N\}$$
$$X \longmapsto \sigma^c(X).$$

(Here, $\cong$ refers to G-equivariant isomorphisms).

Proof. See [Kno91, Theorem 3.1]. □

Notation 3.4.12. Given a strongly convex coloured cone σ^c on N, we let X_{σ^c} denote the simple G/H-horospherical variety corresponding to σ^c via Theorem 3.4.11 (this is defined up to G-equivariant isomorphism).

3.4.3 Coloured fan of a horospherical variety

The classification of general toric varieties uses fans, which are collections of cones that glue together along boundaries. The way that the cones piece together in the fan is reflected geometrically by the simple open cover of the corresponding toric variety. We illustrate how this works for horospherical varieties: we define *coloured* fans, which are collections of *coloured* cones, and we see how the data of a coloured fan encodes the simple open cover of a horospherical variety developed in Section 3.4.1.

Recall that a *face* of a cone σ is a "boundary cone" of σ, i.e. a subset of the form $\sigma \cap m^\perp$ for some $m \in \sigma^\vee$, where $\sigma^\vee := \{m \in N^\vee_{\mathbb{R}} : \langle m, u\rangle \geq 0 \ \forall u \in \sigma\}$ is the *dual cone* of σ and $m^\perp \subseteq N^\vee_{\mathbb{R}}$ denotes the hyperplane orthogonal to m.

Definition 3.4.13 (Coloured face)**.** Let σ^c be a coloured cone on N. A *coloured face* of σ^c is a coloured cone τ^c on N whose underlying cone $\tau \subseteq \sigma$ is a face of σ, and whose colour set is given by $\mathcal{F}(\tau^c) = \{\alpha \in \mathcal{F}(\sigma^c) : u_\alpha \in \tau\}$.

Note that σ^c is always a coloured face of itself. Also, σ^c is strongly convex if and only if the trivial coloured cone 0^c is a coloured face of σ^c. For each $l \in \mathbb{Z}_{\geq 0}$, let $\sigma^c(l)$ denote the set of l-dimensional coloured faces of σ^c; in particular, $\sigma^c(1)$ is the set of coloured rays. Note that each coloured ray $\rho^c \in \sigma^c(1)$ has a minimal ray generator $u_\rho \in \rho \cap N$ (i.e. this is a primitive element in N), and σ is generated by $\{u_\rho : \rho^c \in \sigma^c(1)\}$.

Lemma 3.4.14. *Let X be a simple G/H-horospherical variety. For each G-orbit $\mathcal{O}$, the coloured cone $\sigma^c(X_{\mathcal{O},G})$ is a coloured face of $\sigma^c(X)$. Moreover, every coloured face of $\sigma^c(X)$ is of this form.*

Proof. See [Kno91, Lemma 3.2]. □

Before defining coloured fans, we note that containment and intersections of coloured cones are defined componentwise, i.e. $(\sigma_1, \mathcal{F}_1) \subseteq (\sigma_2, \mathcal{F}_2)$ if and only if $\sigma_1 \subseteq \sigma_2$ and $\mathcal{F}_1 \subseteq \mathcal{F}_2$, and $(\sigma_1, \mathcal{F}_1) \cap (\sigma_2, \mathcal{F}_2) = (\sigma_1 \cap \sigma_2, \mathcal{F}_1 \cap \mathcal{F}_2)$.

Definition 3.4.15 (Coloured fan)**.** A *coloured fan* Σ^c on the coloured lattice N is a finite collection of strongly convex coloured cones on N which satisfies the following:

(1) Every coloured face of any coloured cone in Σ^c is also in Σ^c.

(2) The intersection of any two coloured cones in Σ^c is a coloured face of both.

To obtain the definition of an ordinary *fan* in polyhedral geometry, simply remove the word "coloured" from Definition 3.4.15.

Given a coloured fan Σ^c on N, we let $\Sigma := \{\sigma : \sigma^c \in \Sigma^c\}$ denote the underlying fan, and we let $\mathcal{F}(\Sigma^c) := \cup_{\sigma^c \in \Sigma^c} \mathcal{F}(\sigma^c)$ denote the *colour set*. For each $l \in \mathbb{Z}_{\geq 0}$, let $\Sigma^c(l)$ denote the set of l-dimensional coloured cones in Σ^c; in particular, $\Sigma^c(1)$ is the set of coloured rays. We also let $\Sigma^c_{\max}$ denote the set of maximal coloured cones in Σ^c.

Remark 3.4.16. A coloured cone σ^c on N can be viewed as a coloured fan on N by taking the (finite) collection of all coloured faces of σ^c. When we say that a coloured fan is "generated by a single coloured cone", we mean that it has a unique maximal coloured cone in this sense.

Recall that, in a G/H-horospherical variety X, any G-orbit $\mathcal{O}$ determines an open simple G/H-horospherical variety $X_{\mathcal{O},G} \subseteq X$ whose unique closed G-orbit is $\mathcal{O}$. These $X_{\mathcal{O},G}$'s form an open cover for X. By Theorem 3.4.11, $X_{\mathcal{O},G}$ corresponds to a strongly convex coloured cone $\sigma^c(X_{\mathcal{O},G})$ on N.

Definition 3.4.17 (Coloured fan of a horospherical variety)**.** Let X be a G/H-horospherical variety. We define the *coloured fan associated to X* as

$$\Sigma^c(X) := \{\sigma^c(X_{\mathcal{O},G}) : \mathcal{O} \subseteq X \text{ is a } G\text{-orbit}\}.$$

This $\Sigma^c(X)$ is a coloured fan on N (see [Kno91, Theorem 3.3]).

Note that, for a G/H-horospherical variety X, we have $\mathcal{F}(X) = \mathcal{F}(\Sigma^c(X))$.

Example 3.4.18 (Basic examples of associated coloured fans)**.**

- If X is a simple G/H-horospherical variety, then $\Sigma^c(X)$ is generated by the single coloured cone $\sigma^c(X)$ from Definition 3.4.6; here we are using the terminology of Remark 3.4.16.
- If X is a toric variety, then it corresponds to a fan Σ by [CLS11, Corollary 3.1.8]. Viewing X as a $T/\{1\}$-horospherical variety, the coloured fan associated to X is $\{(\sigma, \varnothing) : \sigma \in \Sigma\}$.

- If X_i are G_i/H_i-horospherical varieties for $i = 1, 2$, then $\Sigma^c(X_1 \times X_2) = \Sigma^c(X_1) \times \Sigma^c(X_2)$, where this product coloured fan is the collection of all product coloured cones $\sigma_1^c \times \sigma_2^c$ with $\sigma_i^c \in \Sigma^c(X_i)$ (see Example 3.4.7). ♢

The following result is the main theorem of this section, which says that horospherical varieties correspond to coloured fans. This generalizes the correspondence between toric varieties and fans (see [CLS11, Theorem 3.1.5, Corollary 3.1.8]).

Theorem 3.4.19 (Classification of horospherical varieties)**.** *The following map is a bijection:*

$$\begin{aligned} \{G/H\text{-horospherical varieties}\}/\cong &\longrightarrow \{\text{Coloured fans on } N\} \\ X &\longmapsto \Sigma^c(X). \end{aligned}$$

(Here, $\cong$ refers to G-equivariant isomorphisms).

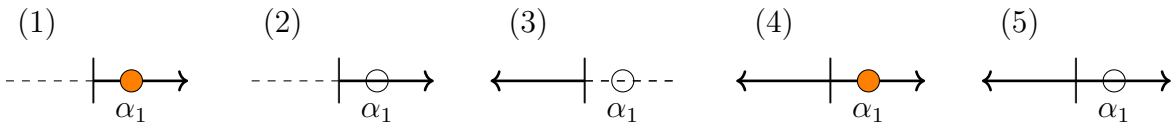
Proof. See [Kno91, Theorem 3.3]. □

Notation 3.4.20. Given a coloured fan Σ^c on N, we let X_{Σ^c} denote the G/H-horospherical variety corresponding to Σ^c via Theorem 3.4.19 (this is defined up to G-equivariant isomorphism).

Remark 3.4.21. Let Σ^c be a coloured fan on N. Then X_{Σ^c} is covered by the open simple G/H-horospherical varieties X_{σ^c} as σ^c ranges through the coloured cones in Σ^c; compare with Proposition 3.4.4. In fact, X_{Σ^c} is covered by the X_{σ^c} for $\sigma^c \in \Sigma^c_{\max}$.

Example 3.4.22. Consider $G = SL_2$ and $H = U_2$. Then the coloured lattice N is isomorphic to $\mathbb{Z}$, we have $\mathcal{C} = \{\alpha\}$, and $u_\alpha = e_1$. We describe all SL_2/U_2-horospherical varieties by describing all coloured fans on N.

Of course, the trivial coloured cone 0^c yields the horospherical variety $SL_2/U_2 \cong \mathbb{A}^2 \setminus \{0\}$. Note that the colour divisor D_α of SL_2/U_2 is $\{(0, y)\} \subseteq \mathbb{A}^2 \setminus \{0\}$. It is easy to see that the following are all of the possible nontrivial coloured fans on N:



The following table says what the corresponding variety is in each case:

Σ^c	(1)	(2)	(3)	(4)	(5)
X_{Σ^c}	$\mathbb{A}^2$	$\mathrm{Bl}_0\,\mathbb{A}^2$	$\mathbb{P}^2 \setminus \{(1:0:0)\}$	$\mathbb{P}^2$	$\mathrm{Bl}_{(1:0:0)}\,\mathbb{P}^2$

The action of SL_2 on $\mathbb{A}^2$ and $\mathbb{P}^2$ is described in Example 3.2.18, and the actions on the other varieties are induced by these actions. Recall that examples (1) and (2) are treated in more detail in Example 3.4.8. ♢

Given a coloured fan Σ^c on N, the *support* of Σ^c is the support of the underlying fan, i.e. $|\Sigma^c| := |\Sigma| := \cup_{\sigma\in\Sigma}\sigma \subseteq N_{\mathbb{R}}$. We say that Σ^c is *complete* if $|\Sigma^c| = N_{\mathbb{R}}$.

Proposition 3.4.23 (Characterization of complete)**.** *A G/H-horospherical variety X is complete (i.e. proper over $\mathrm{Spec}(k)$) if and only if $\Sigma^c(X)$ is complete.*

Proof. See [Kno91, Theorem 4.2]. □

3.4.4 G-orbits

Just like the orbit-cone correspondence for toric varieties (see [CLS11, Theorem 3.2.6]), we have an orbit-*coloured* cone correspondence for horospherical varieties. Before stating this theorem, we give a remark concerning the stabilizers of collections of colour divisors, which is relevant for calculating the dimensions of orbits in the theorem (and for the local analysis in Section 3.5.2).

Remark 3.4.24 (Stabilizers of colour divisors)**.** The stabilizer G_{D_α} of a colour divisor D_α is the parabolic subgroup $P^-_{S\setminus\{\alpha\}} \supseteq B^-$. Hence, if $\mathcal{F} \subseteq \mathcal{C}$, then the stabilizer of all colour divisors D_α with $\alpha \in \mathcal{C}\setminus\mathcal{F} = S\setminus(I\cup\mathcal{F})$ is the parabolic subgroup $\cap_{\alpha\in\mathcal{C}\setminus\mathcal{F}} P^-_{S\setminus\{\alpha\}} = P^-_{I\cup\mathcal{F}} \supseteq B^-$.

Theorem 3.4.25 (Orbit-coloured cone correspondence)**.** *Let X be a G/H-horospherical variety. Then there is a bijection*

$$\begin{aligned}\{G\text{-orbits in } X\} &\longrightarrow \Sigma^c(X)\\ \mathcal{O} &\longmapsto \sigma^c(X_{\mathcal{O},G})\end{aligned}$$

which satisfies the following properties. Let $\mathcal{O}(\sigma^c)$ denote the G-orbit corresponding to $\sigma^c \in \Sigma^c(X)$.

(1) For each $\sigma^c = (\sigma,\mathcal{F}) \in \Sigma^c(X)$, we have

$$\dim(\mathcal{O}(\sigma^c)) = \dim_{\mathbb{R}}(N_{\mathbb{R}}) - \dim(\sigma) + \dim(G/P_{I\cup\mathcal{F}}).$$

(2) For each $\sigma^c \in \Sigma^c(X)$, we have

$$X_{\sigma^c} = \bigcup_{\tau^c \subseteq \sigma^c} \mathcal{O}(\tau^c)$$

where the union is over all coloured faces τ^c of σ^c.

(3) For each $\sigma^c, \tau^c \in \Sigma^c(X)$, we have that τ^c is a coloured face of σ^c if and only if $\mathcal{O}(\sigma^c) \subseteq \overline{\mathcal{O}(\tau^c)}$. Moreover, for each $\tau^c \in \Sigma^c(X)$, we have

$$\overline{\mathcal{O}(\tau^c)} = \bigcup_{\tau^c \subseteq \sigma^c} \mathcal{O}(\sigma^c)$$

where the union is over all coloured cones $\sigma^c \in \Sigma^c(X)$ which contain τ^c.

Proof. The bijective correspondence follows from Lemma 3.4.14. The proof in [Kno91, Lemma 3.2] also gives (3). The formula in (1) is shown in [Kno91, Theorem 6.6]. Finally, (2) follows from the description of simple horospherical varieties and their associated coloured cones. □

In Theorem 3.4.25, the trivial coloured cone $0^c \in \Sigma^c$ corresponds to the open orbit $G/H \subseteq X_{\Sigma^c}$. On the other hand, the maximal coloured cones $\sigma^c \in \Sigma^c_{\max}$ correspond to the closed G-orbits $\mathcal{O}(\sigma^c) = \overline{\mathcal{O}(\sigma^c)}$ by part (3).

We can use Theorem 3.4.25 to identify the G-invariant prime divisors (which are closures of codimension 1 G-orbits) of a horospherical variety through the associated coloured fan. Given a coloured fan Σ^c on N, a coloured ray of the form $\rho^c = (\rho, \varnothing) \in \Sigma^c(1)$ with empty colour set is called a *non-coloured ray*.

Corollary 3.4.26. *Let X be a G/H-horospherical variety. There is a bijection*

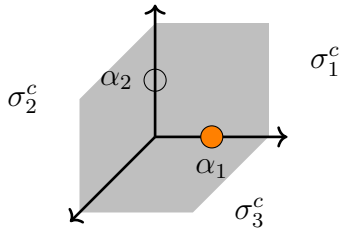
$$\begin{aligned} \{\textit{Non-coloured rays of } \Sigma^c(X)\} &\longrightarrow \mathcal{D}^G(X) \\ \rho^c = (\rho, \varnothing) &\longmapsto D_\rho := \overline{\mathcal{O}(\rho^c)}. \end{aligned}$$

Furthermore, the minimal ray generator $u_\rho \in N$ of $\rho^c = (\rho, \varnothing) \in \Sigma^c(1)$ equals u_D.

Proof. This follows from Theorem 3.4.25, and [Kno91, Lemma 2.4] for the final claim. □

Example 3.4.27. Consider $G = SL_3$ and $H = U_3$. The coloured lattice N is isomorphic to $\mathbb{Z}^2$, we have $\mathcal{C} = \{\alpha_1, \alpha_2\}$, and $u_{\alpha_i} = e_i$ are the standard basis vectors for $i = 1, 2$. Let $X = X_{\Sigma^c}$ be the SL_3/U_3-horospherical variety where Σ^c is given in the diagram below. The

maximal coloured cones in Σ^c are $\sigma_1^c := (\mathrm{Cone}(e_1, e_2), \{\alpha_1\})$, $\sigma_2^c := (\mathrm{Cone}(e_2, -e_1 - e_2), \varnothing)$, and $\sigma_3^c := (\mathrm{Cone}(e_1, -e_1 - e_2), \{\alpha_1\})$.



We use Theorem 3.4.25 to give the dimensions of the SL_3-orbits of X. Recall that $I = \varnothing$ since $B_3 = P_\varnothing$ is the parabolic associated to $H = U_3$. Note that $\dim(SL_3/P_\varnothing) = 3$ and $\dim(SL_3/P_{\{\alpha_1\}}) = \dim(SL_3/P_{\{\alpha_2\}}) = 2$. There are seven coloured cones in Σ^c, so there are seven SL_3-orbits. The following table shows the dimensions of the orbits.

$\sigma^c \in \Sigma^c$	σ_1^c	σ_2^c	σ_3^c	$\sigma_1^c \cap \sigma_2^c$	$\sigma_1^c \cap \sigma_3^c$	$\sigma_2^c \cap \sigma_3^c$	0^c
$\dim(\mathcal{O}(\sigma^c))$	2	3	2	4	3	4	5

♢

Let X be a G/H-horospherical variety with associated coloured fan Σ^c. The closure of a G-orbit in X has the structure of a horospherical G-variety, and we can describe the associated coloured fan as follows. Let $\mathcal{O}(\tau^c) \subseteq X$ be a G-orbit for some $\tau^c \in \Sigma^c$. The sublattice $\mathbb{Z}(\tau \cap N)$ is saturated in N, so using Example 3.3.24 we can form the quotient coloured lattice, which we denote N/τ^c. The underlying lattice is $N/\mathbb{Z}(\tau \cap N)$, the universal colour set is $\mathcal{C} \setminus \mathcal{F}(\tau^c)$, and the quotient map $\phi : N \to N/\tau^c$ is a map of coloured lattices. For each $\sigma^c \in \Sigma^c$ such that $\sigma \supseteq \tau$, let $(\sigma/\tau)^c$ be the coloured cone on N/τ^c whose underlying cone is σ/τ, the image of σ under $\phi_{\mathbb{R}}$, and whose colour set is $\mathcal{F}(\sigma^c) \setminus \mathcal{F}(\tau^c)$. It is easy to check that $(\sigma/\tau)^c$ is a strongly convex coloured cone, and the collection

$$(\Sigma/\tau)^c := \{(\sigma/\tau)^c : \tau^c \subseteq \sigma^c \in \Sigma^c\} \tag{3.4.28}$$

is a coloured fan on N/τ^c.

Proposition 3.4.29 (Orbit closures)**.** *Let X_{Σ^c} be a G/H-horospherical variety, and let $\mathcal{O}(\tau^c) \subseteq X_{\Sigma^c}$ be a G-orbit for some $\tau^c \in \Sigma^c$. The orbit closure $\overline{\mathcal{O}(\tau^c)} \subseteq X_{\Sigma^c}$ is a horospherical G-variety whose associated coloured fan is $(\Sigma/\tau)^c$.*

Proof. This follows from Theorem 3.4.25. □

Remark 3.4.30. Let X be a G/H-horospherical variety and take $\tau^c \in \Sigma^c(X)$. Then the G-orbit $\mathcal{O}(\tau^c)$ is G-equivariantly isomorphic to the horospherical homogeneous space $G/H_{(J,M)}$ determined by the pair $(J, M) = (\mathcal{F}(\tau^c), (N/\tau^c)^\vee)$.

Example 3.4.31. Consider $G = SL_3$ and $H = U_3$. The coloured lattice N is isomorphic to $\mathbb{Z}^2$, we have $\mathcal{C} = \{\alpha_1, \alpha_2\}$, and $u_{\alpha_i} = e_i$ are the standard basis vectors for $i = 1, 2$. Let $X = X_{\Sigma^c}$ be the SL_3/U_3-horospherical variety from Example 3.4.27.

Using Proposition 3.4.29, the coloured fans for the SL_3-orbit closures $\overline{\mathcal{O}(\sigma_1^c)}$ (left), $\overline{\mathcal{O}(\sigma_1^c \cap \sigma_2^c)}$ (middle), and $\overline{\mathcal{O}(\sigma_1^c \cap \sigma_3^c)}$ (right) are given in the diagram below.



Note that $\overline{\mathcal{O}(\sigma_1^c)}$ is the flag variety $SL_3/H_{(\{\alpha_1\},\{0\})} = SL_3/P_{\{\alpha_1\}}$, $\overline{\mathcal{O}(\sigma_1^c \cap \sigma_2^c)}$ is a $SL_3/H_{(\varnothing,0\times\mathbb{Z})}$-horospherical variety, and $\overline{\mathcal{O}(\sigma_1^c \cap \sigma_3^c)}$ is a $SL_3/H_{(\varnothing,\mathbb{Z}\times 0)}$-horospherical variety. ♢

3.4.5 Horospherical morphisms and maps of coloured fans

In this subsection, we consider two horospherical homogeneous spaces G/H_i for $i = 1, 2$, and we assume that $H_1 \subseteq H_2$. We use the same notation developed previously for G/H but now with subscripts $i = 1, 2$; e.g. $N_i = N(G/H_i)$ and $\mathcal{C}_i = \mathcal{C}(G/H_i)$.

As in Section 3.3.4, the natural projection map $G/H_1 \to G/H_2$ induces an associated map of coloured lattices $\Phi : N_1 \to N_2$. Recall that $\mathcal{C}_\Phi \subseteq \mathcal{C}_1$ is the set of dominantly mapped colours. The following theorem says that the map $G/H_1 \to G/H_2$ extends to a horospherical morphism of G/H_i-horospherical varieties precisely when Φ respects the structure of the associated coloured fans.

Definition 3.4.32 (Compatible coloured fan map)**.** Given coloured fans Σ_i^c on N_i for $i = 1, 2$, we say that $\Phi : N_1 \to N_2$ is *compatible with Σ_1^c and Σ_2^c* if, for each $\sigma_1^c = (\sigma_1, \mathcal{F}_1) \in \Sigma_1^c$, there exists $\sigma_2^c = (\sigma_2, \mathcal{F}_2) \in \Sigma_2^c$ such that $\Phi_{\mathbb{R}}(\sigma_1) \subseteq \sigma_2$ and $\mathcal{F}_1 \setminus \mathcal{C}_\Phi \subseteq \mathcal{F}_2$.

Theorem 3.4.33 (Classification of horospherical morphisms)**.** *Let $X_{\Sigma_i^c}$ be G/H_i-horospherical varieties for $i = 1, 2$. Then the natural projection $G/H_1 \to G/H_2$ extends to a horospherical morphism $X_{\Sigma_1^c} \to X_{\Sigma_2^c}$ if and only if Φ is compatible with Σ_1^c and Σ_2^c.*

Proof. See [Kno91, Theorem 4.1]. □

Remark 3.4.34. There is a method for studying G/H_i-horospherical varieties without referencing the second G/H_2; this uses *generalized coloured fans*, which are the same as a coloured fans except that we do not require the elements to be strongly convex (compare with *generalized fans* in toric geometry [CLS11, Definition 6.2.2]). We do not go into detail on this topic; see the discussion of integral submersions in [Kno91, Section 4] for details. This approach is used in [Bri93] to combinatorially describe the minimal model program for (horo)spherical varieties.

Example 3.4.35 (Open horospherical subvarieties)**.** Let $X = X_{\Sigma^c}$ be a G/H-horospherical variety. If $X' \subseteq X$ is a G-invariant open subvariety, then X' is also a G/H-horospherical variety. The coloured fan $\Sigma^c(X')$ is a *sub-coloured fan* of Σ^c, i.e. it is a coloured fan which is a subset of Σ^c. Conversely, any sub-coloured fan $(\Sigma')^c$ of Σ^c yields an open horospherical subvariety $X_{(\Sigma')^c} \subseteq X_{\Sigma^c}$. Indeed, the identity map $N \to N$ is compatible with $(\Sigma')^c$ and Σ^c, and the induced map $X_{(\Sigma')^c} \to X_{\Sigma^c}$ is a G-equivariant open embedding. ♢

Recall from Remark 3.2.9 that $T_N = P/H$ is naturally isomorphic to $\mathrm{Aut}^G(G/H)$, the group of G-equivariant automorphisms of the horospherical homogeneous space G/H. The following result says that all G-equivariant automorphisms of G/H extend to any G/H-horospherical variety. In particular, T_N acts on any G/H-horospherical variety by G-equivariant automorphisms.

Proposition 3.4.36. *Let X be a G/H-horospherical variety. Then $\mathrm{Aut}^G(X) = \mathrm{Aut}^G(G/H)$.*

Proof. See [AKP15, Lemma 4.1]. □

Lastly, we give a combinatorial characterization of proper morphisms, which generalizes Proposition 3.4.23.

Proposition 3.4.37 (Characterization of proper)**.** *Let $X_i = X_{\Sigma_i^c}$ be G/H_i-horospherical varieties for $i = 1, 2$, and let $\pi : X_1 \to X_2$ be a horospherical morphism. Then π is proper if and only if $\Phi_{\mathbb{R}}^{-1}(|\Sigma_2^c|) = |\Sigma_1^c|$.*

Proof. See [Kno91, Theorem 4.2]. □

3.4.6 Torus factors

To finish this section, we examine the notion of "torus factors" for horospherical varieties, extending [CLS11, Proposition 3.3.9] for toric varieties.

Definition 3.4.38 (Torus factor)**.** Let X be a G/H-horospherical variety with coloured fan Σ^c. We say that X has a *torus factor* if X is G-equivariantly isomorphic to $X_1 \times T_1$ where X_1 is a horospherical G-variety of smaller dimension than X and T_1 is a nontrivial torus.

The following proposition gives a nice combinatorial interpretation of torus factors. Note that, for a G/H-horospherical variety X, the set $\{u_D : D \in \mathcal{D}(X)\}$ is the set of all colour points in N together with all the minimal generators for the non-coloured rays of $\Sigma^c(X)$.

Proposition 3.4.39 (Characterizations of torus factors)**.** *Let X be a G/H-horospherical variety with coloured fan Σ^c. The following conditions are equivalent:*

(1) X has a torus factor.

(2) $\mathscr{O}^*_X(X) \neq k^*$, i.e. X has a nonconstant function $X \to k^*$.

(3) $\{u_D : D \in \mathcal{D}(X)\}$ does not span $N_{\mathbb{R}}$.

Proof. (1)⇒(2) If $X \cong X_1 \times T_1$ in the notation of Definition 3.4.38, where T_1 is a nontrivial torus, then $k^* \neq \mathscr{O}^*_{T_1}(T_1) \hookrightarrow \mathscr{O}^*_X(X)$.

(2)⇒(3) We prove the contrapositive, so suppose that $\{u_D : D \in \mathcal{D}(X)\}$ does not span $N_{\mathbb{R}}$. Using Lemma 3.3.2, the character lattice $N^\vee$ of P/H is isomorphic to $k(G/H)^{(B^-)}/k^*$. So a point $m \in N^\vee$ determines a function $f_m \in k(G/H)^{(B^-)}$ up to an invertible constant. Our supposition is equivalent to the map

$$N^\vee \longrightarrow \mathrm{Div}_{B^-}(X), \qquad f_m \mapsto \mathrm{div}(f_m)$$

being non-injective; see Section 3.6.1 for the group of B^--invariant Weil divisors $\mathrm{Div}_{B^-}(X)$. Thus, there exist $f_{m_1}, f_{m_2} \in k(G/H)^{(B^-)}$ which are not constant multiples of each other such that $\mathrm{div}(f_{m_1}) = \mathrm{div}(f_{m_2})$. This implies that $\mathrm{div}(f_{m_1}/f_{m_2}) = 0$, so f_{m_1}/f_{m_2} is a nonconstant element of $\mathscr{O}^*_X(X)$, which is what we wanted to show.

(3)⇒(1) Suppose that $\{u_D : D \in \mathcal{D}(X)\}$ does not span $N_{\mathbb{R}}$, so $\mathrm{Span}_{\mathbb{R}}\{u_D : D \in \mathcal{D}(X)\} \cap N$ is a proper saturated sublattice N' of N. Let N'' be a nontrivial complementary sublattice, i.e. $N = N' \times N''$. Then we have a projection $N \to N'$, which induces an inclusion $(N')^\vee \subseteq N^\vee$, which in turn induces a containment of horospherical subgroups $H = H_{(I,N^\vee)} \subseteq H_{(I,(N')^\vee)} =: H'$ (using Proposition 3.2.6). Notice that Σ^c can be viewed as a coloured fan on either N or N' because Σ^c is contained in $N'_{\mathbb{R}}$. In the former case,

Σ^c corresponds the the G/H-horospherical variety $X = X_{\Sigma^c,N}$, but in the latter case, it corresponds to a G/H'-horospherical variety $X_{\Sigma^c,N'}$. Finally, we have

$$X = X_{\Sigma^c,N} = X_{\Sigma^c,N'} \times T_{N''}$$

where $T_{N''}$ is a nontrivial torus. Thus, X has a torus factor. □

Remark 3.4.40 (Removing torus factors)**.** In the notation of Proposition 3.4.39, let $N(\Sigma^c) := \mathrm{Span}_{\mathbb{R}}\{u_D : D \in \mathcal{D}(X)\} \cap N$, which is a saturated sublattice of N; note that $N(\Sigma^c)_{\mathbb{R}} = \mathrm{Span}_{\mathbb{R}}\{u_D : D \in \mathcal{D}(X)\}$ is the subspace of $N_{\mathbb{R}}$ spanned by all rays of Σ together with all colour points in N. Then Proposition 3.4.39 says that we have $N(\Sigma^c) = N$ if and only if $X = X_{\Sigma^c}$ has no torus factors.

Assuming that $N(\Sigma^c) \neq N$, the proof of Proposition 3.4.39 shows us that

$$X = X_{\Sigma^c,N} = X_{\Sigma^c,N(\Sigma^c)} \times T_{N/N(\Sigma^c)} \tag{3.4.41}$$

where $T_{N/N(\Sigma^c)}$ is the nontrivial torus with one-parameter subgroup lattice $N/N(\Sigma^c)$. Now $X_{\Sigma^c,N(\Sigma^c)}$ is a $G/H_{(I,N(\Sigma^c)^\vee)}$-horospherical variety which has no torus factors.

3.5 Affine horospherical varieties and local structure

In this section we characterize affine horospherical varieties, give a way of computing the coordinate ring of an affine horospherical variety, illustrate a certain local description for a general horospherical variety using affine horospherical varieties, and state combinatorial characterizations of factoriality and smoothness.

Throughout this section, we fix a horospherical homogeneous space G/H and use the notation from the previous sections. In particular, $P = P_I = N_G(H) \supseteq B$ is the associated parabolic, and $N = N(G/H)$ is the associated coloured lattice with universal colour set $\mathcal{C} = \mathcal{C}(G/H) = S \setminus I$.

3.5.1 Affine horospherical varieties

As mentioned in Section 3.4.1, every simple *toric* variety is affine, but not every simple *horospherical* variety is affine. The following proposition tells us exactly which horospherical varieties are affine.

Proposition 3.5.1 (Characterization of affine)**.** *Let X be a G/H-horospherical variety. Then the following are equivalent:*

(1) X is affine.

(2) X is simple and $X = X_{\mathcal{O},B^-}$, where $\mathcal{O}$ denotes the unique closed G-orbit of X.

(3) X is simple and $\mathcal{F}(X) = \mathcal{C}$.

Proof. See [Kno91, Theorem 6.7] for the equivalence of (1) and (3). Recall that $\mathcal{V}(X) = N_{\mathbb{R}}$ since X is horospherical, so the only functional which is nonpositive on $\mathcal{V}(X)$ is the zero functional. The equivalence between (2) and (3) follows from the definition of $X_{\mathcal{O},B^-}$. □

Consider an affine G/H-horospherical variety X; so X corresponds to a coloured cone $\sigma^c = (\sigma, \mathcal{F})$ on N such that $\mathcal{F} = \mathcal{C}$. Due to the condition on the colour set for an affine horospherical variety, we see that X is completely determined by the cone σ. We can use σ to describe the coordinate ring of X as follows. Similar to Section 3.3.1, we can consider the regular B^--eigenfunctions $k[G]^{(B^-)}$ of $k[G]$, which has a $\mathfrak{X}(T)$-grading:

$$k[G]^{(B^-)} \cup \{0\} = \bigoplus_{m \in \mathfrak{X}(T)} k[G]^{(B^-)}_{-m}$$

where

$$\begin{aligned} k[G]^{(B^-)}_{-m} &:= \{f \in k[G] : b \cdot f = \chi^{-m}(b) f \ \forall b \in B^-\} \\ &= \{f \in k[G] : f(b \cdot x) = \chi^m(b) f(x) \ \forall b \in B^-, x \in G\}. \end{aligned}$$

Since $N^\vee \subseteq \mathfrak{X}(T)$, we can consider the graded pieces $k[G]^{(B^-)}_{-m}$ for $m \in N^\vee$.

For example, if $G = T$ is a torus, then $k[G] = k[T]$ is the ring of Laurent monomials (in the number of variables equal to $\dim(T)$) and $k[G]^{(B^-)}_{-m} = k[T]^{(T)}_{-m}$ consists of Laurent monomials of degree m.

Proposition 3.5.2 (Coordinate ring)**.** *Let $X = X_{\sigma^c}$ be an affine G/H-horospherical variety. The coordinate ring of X is given by*

$$k[X] = \bigoplus_{m \in \sigma^\vee \cap N^\vee} k[G]^{(B^-)}_{-m}.$$

Proof. See [Tim11, Theorem 28.3]. □

Example 3.5.3. Let X be an affine toric variety with open torus T, i.e. X is an affine $T/\{1\}$-horospherical variety. Then X corresponds to a (strongly convex) cone σ and the associated coloured cone is $\sigma^c(X) = (\sigma, \varnothing)$. As in [CLS11, Theorem 1.3.5], the coordinate

ring of X is the monoid algebra $k[X] = k[\sigma^\vee \cap N^\vee]$. This agrees with Proposition 3.5.2 since $k[T]^{(T)}_{-m}$ is the space of Laurent monomials of degree $m \in N^\vee$. ♢

Example 3.5.4. Consider $G = SL_3$ and $H = U_3$. Then the coloured lattice N is isomorphic to $\mathbb{Z}^2$, we have $\mathcal{C} = \{\alpha_1, \alpha_2\}$, and $u_{\alpha_i} = e_i$ are the standard basis vectors for $i = 1, 2$. Let X be the affine SL_3/U_3-horospherical variety whose associated coloured cone is $(\mathrm{Cone}(e_1, e_2), \mathcal{C})$; compare with Example 3.4.10. We compute the coordinate ring $k[X]$ using Proposition 3.5.2 as follows.

First, we can view SL_3 as a closed subvariety of $\mathbb{A}^9$ where the nine coordinates represent the matrix entries. Thus $k[SL_3]$ is a quotient of $k[r, s, t, u, w, v, x, y, z]$; here (r, s, t) represents the first row of entries in SL_3, and so on.

To compute the direct sum in Proposition 3.5.2, we mainly need to compute $k[SL_3]^{(B_3^-)}_{-m}$ for m in a generating set of the monoid $\sigma^\vee \cap N^\vee$. Note that $\sigma^\vee \cap N^\vee = \mathbb{Z}^2_{\geq 0}$ where $N^\vee = \mathbb{Z}^2$, so $\{e_1, e_2\}$ is a generating set. The action of B_3^- on SL_3 tells us that

$$k[SL_3]^{(B_3^-)}_{-e_1} = k \cdot \{r, s, t\} \quad \text{and} \quad k[SL_3]^{(B_3^-)}_{-e_2} = k \cdot \{rv - su, rw - tu, sw - tv\}$$

(note that these quantities $rv - su$, $wr - tu$, and $sw - tv$ are the upper 2×2-minors of the matrices in SL_3). Therefore, we have

$$k[X] = k[r, s, t, rv - su, rw - tu, sw - tv] \cong k[a, b, c, d, e, f]/\langle ad + be + cf \rangle. \qquad ♢$$

3.5.2 Affine local structure

Consider a simple G/H-horospherical variety $X = X_{\sigma^c}$. As discussed in the previous subsection, X is generally not affine. In this subsection we present a local description for X in terms of affine horospherical varieties.

Let Q be the parabolic subgroup $P_{I \cup \mathcal{F}(X)}$; recall from Remark 3.4.24 that Q^- is the stabilizer of the colour divisors coming from $\mathcal{C} \setminus \mathcal{F}(X)$. Then $P = P_I \subseteq Q$, so we have a natural projection $G/H \to G/P \to G/Q$. The universal colour set $\mathcal{C}(G/Q)$ for G/Q is $\mathcal{C} \setminus \mathcal{F}(X)$, so this projection $G/H \to G/Q$ extends to a horospherical morphism $X \to G/Q$ by Theorem 3.4.33. Let Z denote the fibre of the base point eQ; so Z is a closed subvariety of X. Note that every fibre of $X \to G/Q$ is isomorphic to Z since the map is G-equivariant and the base is a single G-orbit. In particular, we have $X = G \times^Q Z$; recall that the associated bundle $G \times^Q Z$ is $(G \times Z)/Q$ where $q \cdot (g, z) := (gq^{-1}, q \cdot z)$. Since eQ is a Q-fixed point in G/Q, we see that Z is Q-invariant. Moreover, $\mathrm{Rad}_u(Q)$ acts on Z trivially, so Z is an L-variety where $L = L_{I \cup \mathcal{F}(X)}$ is the standard Levi subgroup of Q.

Notice that the coloured lattice $N(L/(L\cap H))$ satisfies the following properties. As a lattice, it is the same as $N(G/H)$ because $L/(L\cap H)$ and G/H have the same associated torus. However, the colour structure is different: the universal colour set is $\mathcal{C}(L/(L\cap H)) = \mathcal{F}(X)$. That is, using the terminology of Example 3.3.24, $N(L/(L\cap H))$ is a coloured sublattice of $N(G/H)$, and the coloured quotient lattice $N(G/H)/N(L/(L\cap H))$ is precisely $N(G/Q)$.

Theorem 3.5.5 (Affine local structure)**.** *Let $X = X_{\sigma^c}$ be a simple G/H-horospherical variety. In the notations immediately above, we have the following:*

(1) Z is an affine $L/(L\cap H)$-horospherical variety whose associated coloured cone is σ^c on the coloured lattice $N(L/(L\cap H))$.

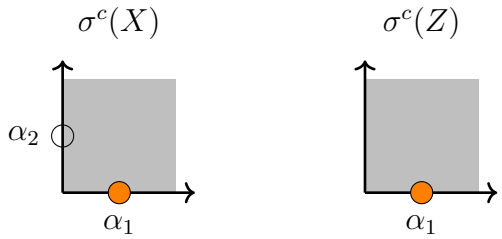
(2) If $\mathcal{O}\subseteq X$ denotes the unique closed G-orbit, then we have $Z\subseteq X_{\mathcal{O},B^-}$ and the map $\mathrm{Rad}_u(Q^-)\times Z\to X_{\mathcal{O},B^-}$ via $(q,z)\mapsto q\cdot z$ is a Q^--equivariant isomorphism.

Proof. See [Tim11, Theorem 28.2] for (1) and [Per18, Theorem 3.2.2] for (2). □

Remark 3.5.6. One can get a similar result to Theorem 3.5.5 where the affine closed subvariety $Z\subseteq X$ has a fixed point; see [Tim11, Theorem 28.2] for details.

Example 3.5.7. Consider $G = SL_3$ and $H = U_3$. Then the coloured lattice N is isomorphic to $\mathbb{Z}^2$, we have $\mathcal{C} = \{\alpha_1,\alpha_2\}$, and $u_{\alpha_i} = e_i$ are the standard basis vector for $i = 1,2$. Let X be the simple SL_3/U_3-horospherical variety whose associated coloured cone is $\sigma^c(X) = (\mathrm{Cone}(e_1,e_2),\{\alpha_1\})$; see the coloured cone on the left in the diagram below.

In the notation at the beginning of this subsection, we have $Q = P_{\{\alpha_1\}}$ since $I = \varnothing$ and $\mathcal{F}(X) = \{\alpha_1\}$, so $L = L_{\{\alpha_1\}}$. Note that, using Remark 3.2.22, we can think of $L/(L\cap U_3)$ as $(SL_2\times\mathbb{G}_m)/(U_2\times\{1\})$. Then Z is the $L/(L\cap U_3)$-horospherical variety whose associated coloured cone is $\sigma^c(Z) = (\mathrm{Cone}(e_1,e_2),\{\alpha_1\})$ on the coloured lattice $\mathbb{Z}^2$ with universal colour set $\{\alpha_1\}$; see the coloured cone on the right in the diagram below.



We can describe Z as $\mathbb{A}^2\times\mathbb{A}^1 = \mathbb{A}^3$ as a $(SL_2\times\mathbb{G}_m)/(U_2\times\{1\})$-horospherical variety, where SL_2 acts on the $\mathbb{A}^2$ by matrix multiplication and $\mathbb{G}_m$ acts on the $\mathbb{A}^1$ by multiplication. Then $X = SL_3\times^Q\mathbb{A}^3$. ♢

We end this subsection by looking at a special class of horospherical varieties whose affine local structure (from Theorem 3.5.5) uses toric varieties. These varieties are particularly nice because one can study all of their local properties (e.g. singularities) using toric geometry.

Definition 3.5.8 (Toroidal)**.** A G/H-horospherical variety X is called *toroidal* if $\mathcal{F}(X) = \varnothing$.

For example, toric varieties and flag varieties are toroidal. Notice that being toroidal means that $Q = P$ and $L/(L \cap H) = T_N$, where Q and L are defined at the beginning of this subsection. In particular, X is toroidal if and only if the projection $G/H \to G/P$ extends to a horospherical morphism $X \to G/P$.

Corollary 3.5.9. *Let $X = X_{\sigma^c}$ be a simple toroidal G/H-horospherical variety. In the notation at the beginning of this subsection, the affine closed subvariety $Z \subseteq X$ is an affine toric variety with open torus T_N whose associated cone is σ.*

Example 3.5.10. Let $X = X_{\Sigma^c}$ be a G/H-horospherical variety. We describe a canonical open toroidal subvariety of X as follows. Let $X' \subseteq X$ denote the open subset where we remove all G-orbits in X of codimension greater than 1. Based on Theorem 3.4.25, X' is a G/H-horospherical variety whose coloured fan consists of the trivial coloured cone and the non-coloured rays in Σ^c. In particular, X' is toroidal. ♢

Example 3.5.11 (Decolouration)**.** Let $X = X_{\Sigma^c}$ be a G/H-horospherical variety. We describe a canonical toroidal resolution of X as follows. Consider the coloured fan $\widetilde{\Sigma}^c$ whose underlying fan is exactly Σ and whose colour set is empty, i.e. $\mathcal{F}(\widetilde{\Sigma}^c) = \varnothing$. This yields a toroidal G/H-horospherical variety $\widetilde{X} = X_{\widetilde{\Sigma}^c}$, which we call the *decolouration* of X. Moreover, the identity map of coloured lattices $N \to N$ is compatible with $\widetilde{\Sigma}^c$ and Σ^c, so we have a horospherical morphism $\widetilde{X} \to X$; note that this map is surjective and proper.

For example, $\mathrm{Bl}_0 \mathbb{A}^2$ is the decolouration of $\mathbb{A}^2$ where both are viewed as SL_2/U_2-horospherical varieties; see Example 3.4.8. ♢

3.5.3 Factoriality and smoothness

We finish this section with combinatorial descriptions of two local properties: factoriality and smoothness. Given a coloured fan Σ^c on N and $\sigma^c \in \Sigma^c$, consider the multiset

$$\{u_\rho : (\rho, \varnothing) \in \sigma^c(1)\} \cup \{u_\alpha : \alpha \in \mathcal{F}(\sigma^c)\} \tag{3.5.12}$$

of lattice points in N; recall that these u_ρ are the minimal ray generators for the non-coloured rays $(\rho, \varnothing) \in \sigma^c(1)$. Note that we say *multiset* here because we want to count

colour points with multiplicity. We say that σ^c is *simplicial* (resp. *regular*) if the multiset Eq. (3.5.12) is $\mathbb{R}$-linearly independent (resp. is part of a $\mathbb{Z}$-basis for N). We say that Σ^c is *simplicial* (resp. *regular*) if each coloured cone in Σ^c is simplicial (resp. regular). Clearly regular implies simplicial.

Proposition 3.5.13 (Factoriality criteria)**.** *Let X_{Σ^c} be a G/H-horospherical variety.*

(1) X_{Σ^c} is factorial if and only if Σ^c is regular.

(2) X_{Σ^c} is $\mathbb{Q}$-factorial if and only if Σ^c is simplicial.

Proof. See [Per18, Theorem 4.2.3]. □

Example 3.5.14. Consider $G = SL_3$ and $H = H_{(\varnothing,M)}$ where $M = \mathbb{Z}$ is the diagonal sublattice inside $\mathfrak{X}(T_3) = \mathbb{Z}^2$. Then the coloured lattice N is isomorphic to $\mathbb{Z}$, we have $\mathcal{C} = \{\alpha_1, \alpha_2\}$, and $u_{\alpha_i} = e_1$ for both $i = 1, 2$.

Let $\sigma^c = (\sigma, \mathcal{F}) = (\mathrm{Cone}(e_1), \{\alpha_1, \alpha_2\})$; see the left part of the diagram below. The multiset Eq. (3.5.12) consists of two copies of e_1, so σ^c is neither regular nor simplicial. Therefore, the corresponding simple SL_3/H-horospherical variety X_{σ^c} is not $\mathbb{Q}$-factorial by Proposition 3.5.13. Note that, if $\mathcal{F}$ consisted of only one colour, say α_1, then σ^c would be regular; see the right part of the diagram below.



Since X_{σ^c} is affine (by Proposition 3.5.1), one can use Proposition 3.5.2 to check that

$$\begin{aligned} k[X_{\sigma^c}] &= \frac{k[ad, ae, af, bd, be, bf, cd, ce, cf]}{\langle ad + be + cf \rangle} \\ &\cong \frac{k[r, s, t, u, v, w, x, y, z]}{\left\langle \begin{matrix} rv - su, rw - tu, ry - sx, rz - tx, sw - tv, \\ sz - ty, uy - vx, vz - wx, uz - wx, r + v + z \end{matrix} \right\rangle} \end{aligned}$$

where the a, b, c, d, e, f coordinates are the ones used in Example 3.5.4. ♢

For those familiar with toric geometry, they will recognize that a regular fan (not a *coloured* fan) is a smooth fan; see [CLS11, Definition 3.1.18]. In [CLS11, Theorem 3.1.19], smooth fans are used to characterize smooth toric varieties. It is known that a toric variety

is smooth if and only if it is factorial; so smooth fans really characterize *factorial* toric varieties, which is more in-line with Proposition 3.5.13. The following theorem gives criteria for smoothness of general horospherical G-varieties, which uses regularity of coloured fans but also uses properties of the Dynkin diagram for G.

Theorem 3.5.15 (Smoothness criteria)**.** *Let X_{Σ^c} be a G/H-horospherical variety. Then X is smooth if and only if, for each $\sigma^c \in \Sigma^c$, the following are satisfied:*

(1) σ^c is regular.

(2) All $\alpha \in \mathcal{F}(\sigma^c)$ are isolated from each other in the Dynkin diagram of G (i.e. in separate connected components), and each $\alpha \in \mathcal{F}(\sigma^c)$ is connected (i.e. directly adjacent) with at most one component I_α of I such that $I_\alpha \cup \{\alpha\}$ has type $\mathbf{A}_l$ or $\mathbf{C}_l$ in which α is the first simple root.

Proof. See [Tim11, Theorem 28.10]. □

Example 3.5.16. Consider $G = SL_5$ and $H = H(P_{\{\alpha_2,\alpha_4\}})$. Then the coloured lattice N is isomorphic to $\mathbb{Z}^2$, we have $\mathcal{C} = \{\alpha_1, \alpha_3\}$, and $u_{\alpha_1} = e_1$ and $u_{\alpha_3} = e_2$ are the standard basis vectors. In the table below we consider three simple SL_5/H-horospherical varieties given by coloured cones σ_i^c for $i = 1, 2, 3$. We test smoothness of each variety $X_{\sigma_i^c}$ using Theorem 3.5.15.

i	σ_i^c	smoothness
1	$(\mathrm{Cone}(e_1, -e_1 + e_2), \mathcal{C})$	NOT smooth
2	$(\mathrm{Cone}(e_1, e_2), \{\alpha_1\})$	smooth
3	$(\mathrm{Cone}(e_1, e_2), \{\alpha_3\})$	NOT smooth

For $i = 1$, the variety is not smooth because σ_1^c is not regular. Indeed, the multiset Eq. (3.5.12) is $\{e_1, e_2, -e_1 + e_2\}$, which clearly cannot be extended to a $\mathbb{Z}$-basis for N. Moreover, condition (2) is not satisfied since the elements of $\mathcal{F}(\sigma_1^c) = \mathcal{C}$ are in the same connected component of the Dynkin diagram of SL_5.

For $i = 2, 3$, it is clear that each σ_i^c is regular since the multiset Eq. (3.5.12) is $\{e_1, e_2\}$. It remains to check condition (2) for each of these σ_i^c.

For $i = 2$, condition (2) is satisfied since α_1 is the only colour in $\mathcal{F}(\sigma_2^c)$ and it is connected with the component $\{\alpha_2\}$ of $I = \{\alpha_2, \alpha_4\}$ so that $\{\alpha_1\} \cup \{\alpha_2\}$ has type $\mathbf{A}_2$ where α_1 is the first simple root.

For $i = 3$, condition (2) is not satisfied. Indeed, α_3 is connected to two components of I, namely $\{\alpha_2\}$ and $\{\alpha_4\}$. ♢

For toroidal horospherical varieties, condition (2) in Theorem 3.5.15 is vacuous, so we recover the smoothness criteria that one would expect from toric geometry.

Corollary 3.5.17. *Let X_{Σ^c} be a toroidal G/H-horospherical variety. Then X_{Σ^c} is smooth if and only if Σ^c is regular.*

3.6 Weil and Cartier divisors

To end this chapter, we discuss Weil and Cartier divisors on horospherical varieties. In particular, we show how to compute the class group and Picard group using the B^--invariant divisors and the coloured fan combinatorics.

Throughout this section, we fix a horospherical homogeneous space G/H and use the notation from the previous sections. In particular, $P = P_I = N_G(H) \supseteq B$ is the associated parabolic, and $N = N(G/H)$ is the associated coloured lattice with universal colour set $\mathcal{C} = \mathcal{C}(G/H) = S \setminus I$.

3.6.1 The class group

Given a G/H-horospherical variety X, let $\mathrm{Div}(X)$ denote the group of Weil divisors on X, i.e. the free abelian group generated by the prime divisors of X. Recall that the class group of X, denoted $\mathrm{Cl}(X)$, is the quotient of $\mathrm{Div}(X)$ by the subgroup generated by principal divisors, i.e. divisors of the form $\mathrm{div}(f)$ for $f \in k(X)^*$.

Let $\mathrm{Div}_{B^-}(X)$ denote the subgroup of $\mathrm{Div}(X)$ consisting of B^--invariant Weil divisors. It is easy to see that $\mathrm{Div}_{B^-}(X)$ is freely generated by $\mathcal{D}(X)$, i.e. the B^--invariant prime divisors of X. Thus, any $\delta \in \mathrm{Div}_{B^-}(X)$ can be written as

$$\delta = \sum_{D \in \mathcal{D}(X)} a_D D = \sum_{(\rho,\varnothing)} a_\rho D_\rho + \sum_{\alpha \in \mathcal{C}} a_\alpha D_\alpha \tag{3.6.1}$$

for some coefficients $a_D \in \mathbb{Z}$ or $a_\rho, a_\alpha \in \mathbb{Z}$; here the first sum in the last expression is implicitly over all non-coloured rays $(\rho, \varnothing) \in \Sigma^c(1)$, these D_ρ are the G-invariant prime divisors of X, and these D_α are the colour divisors of X.

Since $N^\vee \cong k(X)^{(B^-)}/k^*$ (see Lemma 3.3.2), an element $m \in N^\vee$ determines a B^--eigenfunction f_m which is unique up to a nonzero constant multiple. The principal divisors

of the form $\mathrm{div}(f_m)$ are precisely the B^--invariant principal divisors. One can check that

$$\mathrm{div}(f_m) = \sum_{D\in\mathcal{D}(X)} \langle m, u_D\rangle D = \sum_{(\rho,\varnothing)} \langle m, u_\rho\rangle D_\rho + \sum_{\alpha\in\mathcal{C}} \langle m, u_\alpha\rangle D_\alpha. \tag{3.6.2}$$

The following Theorem tells us that $\mathrm{Cl}(X)$ is generated by the B^--invariant prime divisors, and the relations come from the principal divisors of the form $\mathrm{div}(f_m)$.

Theorem 3.6.3 (Class group)**.** *Let $X = X_{\Sigma^c}$ be a G/H-horospherical variety. There is an exact sequence of groups*

$$\begin{aligned} N^\vee &\longrightarrow \mathrm{Div}_{B^-}(X) \longrightarrow \mathrm{Cl}(X) \longrightarrow 0 \\ m &\longmapsto \mathrm{div}(f_m) \end{aligned}$$

where the surjection sends $\delta \in \mathrm{Div}_{B^-}(X)$ to its class in $\mathrm{Cl}(X)$. Moreover, the following are equivalent:

(1) The above exact sequence is left exact, i.e. $N^\vee \to \mathrm{Div}_{B^-}(X)$ is injective.

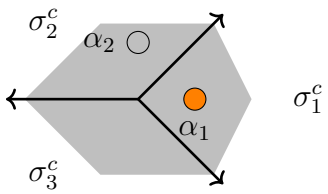
(2) X has no torus factors.

(3) $\mathscr{O}_X^*(X) = k^*$.

(4) $\{u_D : D \in \mathcal{D}(X)\}$ spans $N_\mathbb{R}$ (e.g. Σ^c has a coloured cone of full dimension).

Proof. See [Per18, Theorem 4.2.1] for the main claim and the equivalence between (1) and (4). The equivalence between (2), (3), and (4) is contained in Proposition 3.4.39. □

Example 3.6.4. Consider $G = SL_3$ and $H = U_3$. Then the coloured lattice N is isomorphic to $\mathbb{Z}^2$, we have $\mathcal{C} = \{\alpha_1, \alpha_2\}$, and $u_{\alpha_i} = e_i$ are the standard basis vectors for $i = 1, 2$. Let $X = X_{\Sigma^c}$ be the SL_3/U_3-horospherical variety whose coloured fan is given in the diagram below. The maximal coloured cones in Σ^c are $\sigma_1^c := (\mathrm{Cone}(e_1 + e_2, e_1 - e_2), \{\alpha_1\})$, $\sigma_2^c := (\mathrm{Cone}(-e_1, e_1 + e_2), \varnothing)$, and $\sigma_3^c := (\mathrm{Cone}(-e_1, e_1 - e_2), \varnothing)$. We use Theorem 3.6.3 to compute $\mathrm{Cl}(X)$.



Let $D_{e_1+e_2}$, D_{-e_1}, and $D_{e_1-e_2}$ denote the SL_3-invariant prime divisors corresponding to the non-coloured rays generated by e_1+e_2, $-e_1$, and e_1-e_2, respectively. Then $\mathrm{Cl}(X)$ is generated by the elements of $\mathcal{D}(X)$: $D_{e_1+e_2}$, D_{-e_1}, $D_{e_1-e_2}$, D_{α_1}, and D_{α_2}. The relations are given by

$$0 \sim \sum_{D\in\mathcal{D}(X)} \langle e_1, u_D\rangle D = 1\cdot D_{e_1+e_2} + (-1)\cdot D_{-e_1} + 1\cdot D_{e_1-e_2} + 1\cdot D_{\alpha_1} + 0\cdot D_{\alpha_2},$$
$$0 \sim \sum_{D\in\mathcal{D}(X)} \langle e_2, u_D\rangle D = 1\cdot D_{e_1+e_2} + 0\cdot D_{-e_1} + (-1)\cdot D_{e_1-e_2} + 0\cdot D_{\alpha_1} + 1\cdot D_{\alpha_2}.$$

Hence, $\mathrm{Cl}(X)$ is isomorphic to $\mathbb{Z}^3$ and can be generated by $D_{e_1+e_2}$, D_{α_1}, and D_{α_2}. ♢

3.6.2 The Picard group

Given a G/H-horospherical variety X, we let $\mathrm{CDiv}(X)$ denote the subgroup of $\mathrm{Div}(X)$ consisting of Cartier divisors on X, and we let $\mathrm{CDiv}_{B^-}(X)$ denote the subgroup of $\mathrm{CDiv}(X)$ consisting of B^--invariant Cartier divisors on X. Recall that the Picard group of X, denoted $\mathrm{Pic}(X)$, is the quotient of $\mathrm{CDiv}(X)$ by the subgroup generated by principal divisors. As a consequence of Theorem 3.6.3, we have an exact sequence

$$\begin{aligned} N^\vee &\longrightarrow \mathrm{CDiv}_{B^-}(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow 0 \\ m &\longmapsto \mathrm{div}(f_m) \end{aligned} \tag{3.6.5}$$

(and the same conditions from Theorem 3.6.3 hold for left exactness). In particular, the Picard group $\mathrm{Pic}(X)$ is generated by B^--invariant prime divisors.

Similar to [CLS11, Section 4.2], we can define piecewise linear functions (or "support functions") on a coloured fan Σ^c. A *piecewise linear function* on Σ^c is a function $\varphi : |\Sigma^c| \to \mathbb{R}$ which is linear on each cone of Σ and satisfies $\varphi(|\Sigma^c| \cap N) \subseteq \mathbb{Z}$. Since φ is linear on each cone $\sigma \in \Sigma$, we can write $\varphi(u) = \langle m_\sigma, u\rangle$ for some $m_\sigma \in N^\vee$. So we can view φ as a collection of piecewise linear data $\{m_\sigma\}_{\sigma\in\Sigma}$ where these $m_\sigma \in N^\vee$ satisfy $\langle m_\sigma, u\rangle = \langle m_{\sigma'}, u\rangle$ for all $u \in \sigma\cap\sigma'$. Note that φ is completely determined by the collection $\{m_\sigma\}_{\sigma\in\Sigma_{\max}}$.

If a piecewise linear function φ on Σ^c is given by two sets of piecewise linear data $\{m_\sigma\}_\sigma$ and $\{m'_\sigma\}_\sigma$ then, for each $\sigma \in \Sigma$, the points m_σ and m'_σ have the same image in $N^\vee/(\sigma^\perp \cap N^\vee)$. In particular, if Σ^c is complete, then the collection $\{m_\sigma\}_{\sigma\in\Sigma_{\max}}$ is unique to φ.

Proposition 3.6.6 (Cartier divisors)**.** *Let $X = X_{\Sigma^c}$ be a G/H-horospherical variety. Then*

$$\delta = \sum_{(\rho,\varnothing)} a_\rho D_\rho + \sum_{\alpha\in\mathcal{C}} a_\alpha D_\alpha \in \mathrm{Div}_{B^-}(X)$$

is Cartier if and only if there exists a piecewise linear function φ on Σ^c such that, for each $\sigma^c = (\sigma, \mathcal{F}) \in \Sigma^c$, we have $\varphi(u_\rho) = a_\rho$ for all non-coloured rays $(\rho, \varnothing) \in \sigma^c(1)$ and $\varphi(u_\alpha) = a_\alpha$ for all $\alpha \in \mathcal{F}$.

Proof. See [Per18, Lemma 4.3.1]. □

Given $\delta \in \mathrm{CDiv}_{B^-}(X)$ written in the form Eq. (3.6.1), we denote the associated piecewise linear function from Proposition 3.6.6 by φ_δ; note that the assignment $\delta \mapsto \varphi_\delta$ is indeed well-defined. We call piecewise linear data $\{m_\sigma\}_\sigma$ for φ_δ *Cartier data* for δ. Note that, if $\delta = \mathrm{div}(f_m)$ is principal for some $m \in N^\vee$, then φ_δ is globally linear because it is given by $\varphi(u) = \langle m, u\rangle$ for all $u \in |\Sigma^c|$.

Remark 3.6.7. Let $X = X_{\Sigma^c}$ be a G/H-horospherical variety. Any Weil divisor of the form $\delta = \sum_{\alpha\in\mathcal{C}\setminus\mathcal{F}(\Sigma^c)} a_\alpha D_\alpha$ is always Cartier. Indeed, this follows from Proposition 3.6.6 since the corresponding Cartier data is trivial, i.e. $\varphi_\delta = 0$. In particular, each colour divisor $D_\alpha \subseteq X$ with $\alpha \in \mathcal{C} \setminus \mathcal{F}(\Sigma^c)$ is Cartier.

Denote the abelian group of piecewise linear functions on Σ^c by $\mathrm{PLF}(\Sigma^c)$, and denote the subgroup of global linear functions by $\mathrm{LF}(\Sigma^c)$. The following theorem tells us how to compute $\mathrm{Pic}(X_{\Sigma^c})$ in a way that is similar to [CLS11, Theorem 4.2.12] for toric varieties.

Theorem 3.6.8 (Picard group)**.** *Let $X = X_{\Sigma^c}$ be a G/H-horospherical variety. There is an exact sequence*

$$\begin{array}{ccccccccc} \mathrm{Span}_{\mathbb{R}}(|\Sigma^c|)^\perp & \longrightarrow & \mathbb{Z}(\mathcal{C}\setminus\mathcal{F}(\Sigma^c)) & \longrightarrow & \mathrm{Pic}(X) & \longrightarrow & \mathrm{PLF}(\Sigma^c)/\,\mathrm{LF}(\Sigma^c) & \longrightarrow & 0 \\ m & \longmapsto & \mathrm{div}(f_m) & & [D] & \longmapsto & [\varphi_D] & & \end{array}.$$

In particular, if $\mathrm{Span}_{\mathbb{R}}(|\Sigma^c|)$ is full-dimensional (e.g. if Σ^c has a coloured cone of full dimension), then the above sequence becomes

$$0 \longrightarrow \mathbb{Z}(\mathcal{C}\setminus\mathcal{F}(\Sigma^c)) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{PLF}(\Sigma^c)/\,\mathrm{LF}(\Sigma^c) \longrightarrow 0.$$

Proof. See [Per18, Theorem 4.3.4]. □

Corollary 3.6.9. *Let $X = X_{\Sigma^c}$ be a G/H-horospherical variety.*

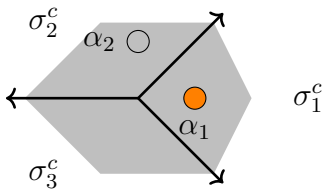
(1) If Σ contains a cone of full dimension, then $\mathrm{Pic}(X)$ *is a free abelian group.*

(2) If X is simple, i.e. $\Sigma^c = \sigma^c$ is generated by a single coloured cone, and if σ has full dimension, then $\mathrm{Pic}(X) = \mathbb{Z}(\mathcal{C} \setminus \mathcal{F}(\Sigma^c))$.

(3) If X is affine, then $\mathrm{Pic}(X) = 0$.

Proof. Clearly (2) and (3) follow immediately from Theorem 3.6.8 and Proposition 3.5.1. If Σ contains a cone of full dimension, then $\mathrm{PLF}(\Sigma^c)/\mathrm{LF}(\Sigma^c)$ is a free abelian group, so (1) also follows from Theorem 3.6.8. □

Example 3.6.10. Consider $G = SL_3$ and $H = U_3$. Then the coloured lattice N is isomorphic to $\mathbb{Z}^2$, we have $\mathcal{C} = \{\alpha_1, \alpha_2\}$, and $u_{\alpha_i} = e_i$ are the standard basis vectors for $i = 1, 2$. Let $X = X_{\Sigma^c}$ be the SL_3/U_3-horospherical variety from Example 3.6.4; we recall the diagram for Σ^c below. We use Theorem 3.6.8 to compute $\mathrm{Pic}(X)$.



We first compute $\mathrm{PLF}(\Sigma^c)/\mathrm{LF}(\Sigma^c)$. Since we are looking at the piecewise linear function $\{m_\sigma\}_\sigma$ up to adding a linear function, we may assume that $m_{\sigma_1} = 0$. Then we require

$$\begin{cases} \langle m_{\sigma_2}, e_1 + e_2 \rangle = 0 \\ \langle m_{\sigma_3}, e_1 - e_2 \rangle = 0 \\ \langle m_{\sigma_2} - m_{\sigma_3}, -e_1 \rangle = 0 \end{cases} \implies \begin{array}{l} m_{\sigma_2} = (a, -a) \\ m_{\sigma_3} = (a, a) \end{array} \quad \text{for some } a \in \mathbb{Z}.$$

In particular, $\mathrm{PLF}(\Sigma^c)/\mathrm{LF}(\Sigma^c) \cong \mathbb{Z}$. Since $\mathbb{Z}(\mathcal{C} \setminus \mathcal{F}(\Sigma^c)) = \mathbb{Z}D_{\alpha_2} \cong \mathbb{Z}$, we find that $\mathrm{Pic}(X)$ is isomorphic to $\mathbb{Z}^2$ and can be generated by D_{-e_1} and D_{α_2}. Note that when $a = -1$ above, $\{m_\sigma\}_\sigma$ is Cartier data for D_{-e_1}. ♢

3.6.3 Basepoint free and ample

Similar to [CLS11, Section 6.1], we can consider convex and strictly convex piecewise linear functions, and see how they correspond to basepoint free (i.e. globally generated) and ample Cartier divisors.

Let $\varphi : |\Sigma^c| \to \mathbb{R}$ be a piecewise linear function on Σ^c, and let $\{m_\sigma\}_\sigma$ be piecewise linear data for φ. We say that φ is *convex* (resp. *strictly convex*) if, for each $\sigma \in \Sigma$, we have $\langle m_{\sigma'}, u\rangle \leq \langle m_\sigma, u\rangle$ (resp. $\langle m_{\sigma'}, u\rangle < \langle m_\sigma, u\rangle$) for all $\sigma' \in \Sigma \setminus \{\sigma\}$ and all $u \in \sigma \setminus \sigma'$.

Proposition 3.6.11 (Basepoint free and ample criteria)**.** *Let $X = X_{\Sigma^c}$ be a complete G/H-horospherical variety, and let*

$$\delta = \sum_{(\rho,\varnothing)} a_\rho D_\rho + \sum_{\alpha\in\mathcal{C}} a_\alpha D_\alpha \in \mathrm{CDiv}_{B^-}(X).$$

Then δ is basepoint free (resp. ample) if and only if φ_δ is convex (resp. strictly convex) and, for each $\alpha \in \mathcal{C} \setminus \mathcal{F}(\Sigma^c)$, we have $\varphi_\delta(u_\alpha) \leq a_\alpha$ (resp. $\varphi_\delta(u_\alpha) < a_\alpha$).

Proof. See [Per18, Corollary 4.3.8]. □

Remark 3.6.12. Let $X = X_{\Sigma^c}$ be a G/H-horospherical variety. As in Remark 3.6.7, any Weil divisor of the form $\delta = \sum_{\alpha\in\mathcal{C}\setminus\mathcal{F}(\Sigma^c)} a_\alpha D_\alpha$ is Cartier and has trivial Cartier data, i.e. $\varphi_\delta = 0$. Hence, any δ of this form is always basepoint free by Proposition 3.6.11. In particular, each colour divisor $D_\alpha \subseteq X$ with $\alpha \in \mathcal{C} \setminus \mathcal{F}(\Sigma^c)$ is basepoint free.

Corollary 3.6.13 (Characterization of projective)**.** *Let X_{Σ^c} be a complete G/H-horospherical variety. Then the following are equivalent:*

(1) X_{Σ^c} is projective.

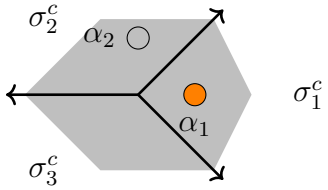
(2) There exists a strictly convex piecewise linear function on Σ^c.

(3) The toric variety X_Σ is projective.

In particular, if $\dim_{\mathbb{R}}(N_{\mathbb{R}}) \leq 2$, then X_{Σ^c} is projective.

Proof. (1) and (2) are equivalent by Proposition 3.6.11. Since (2) only depends on the underlying fan Σ, it follows that (2) and (3) are equivalent. □

Example 3.6.14. Consider $G = SL_3$ and $H = U_3$. Then the coloured lattice N is isomorphic to $\mathbb{Z}^2$, we have $\mathcal{C} = \{\alpha_1, \alpha_2\}$, and $u_{\alpha_i} = e_i$ are the standard basis vectors for $i = 1, 2$. Let $X = X_{\Sigma^c}$ be the SL_3/U_3-horospherical variety from Example 3.6.4; we recall the diagram for Σ^c below. We use Proposition 3.6.11 to check which Cartier divisors on X are basepoint free or ample.



Recall from Example 3.6.10 that $\mathrm{Pic}(X)$ is generated by D_{-e_1} and D_{α_2}. For a general Cartier divisor of the form $\delta = aD_{-e_1} + bD_{\alpha_2}$, with $a, b \in \mathbb{Z}$, we showed that the Cartier data $\{m_\sigma\}_\sigma$ is given by $m_{\sigma_1} = 0$, $m_{\sigma_2} = (-a, a)$, and $m_{\sigma_3} = (-a, -a)$. One can check that φ_δ is convex (resp. strictly convex) if and only if $a \geq 0$ (resp. $a > 0$). Therefore, δ is basepoint free (resp. ample) if and only if $a \geq 0$ and $a \leq b$ (resp. $a > 0$ and $a < b$). ♢

3.6.4 The canonical divisor

To end this section we examine the canonical divisor K_X of a horospherical G-variety X. The coefficients are given in terms of the pairing between roots and dual roots via the Dynkin diagram of G. For convenience, we state the following proposition for the *anti*canonical divisor $-K_X$.

Proposition 3.6.15 (Anticanonical divisor)**.** *Let X be a G/H-horospherical variety. The anticanonical divisor of X is given by*

$$-K_X = \sum_{(\rho,\varnothing)} D_\rho + \sum_{\alpha\in\mathcal{C}} b_\alpha D_\alpha$$

where the coefficient b_α in front of the colour divisor $D_\alpha \subseteq X$ is given by

$$b_\alpha = \sum_{\gamma\in R^+\setminus R_I} \langle \gamma, \alpha^\vee \rangle_R$$

where $\langle \gamma, \alpha^\vee \rangle_R$ is the pairing between roots and dual roots in R.[1] *Moreover, $b_\alpha \in \mathbb{Z}_{\geq 2}$ for each $\alpha \in \mathcal{C}$.*

Proof. This follows from Theorem [Per18, Theorem 6.1.2] and [Per18, Proposition 6.1.1] (see the following remark). □

Remark 3.6.16. One should think of the anticanonical divisor $-K_X$ in Proposition 3.6.15 as follows. First, $-K_X = -K_{X'}$ where $X' \subseteq X$ is the open toroidal subvariety from

[1] See [Pas09a, Page 6] for a quick reference on this root pairing.

Example 3.5.10. By Section 3.5.2, the fibre of the map $X' \to G/P$ is the toric variety whose fan is the underlying fan associated to X'. Thus, $\sum_{(\rho,\varnothing)} D_\rho$ is the anticanonical divisor of the fibre, and $\sum_{\alpha\in\mathcal{C}} b_\alpha D_\alpha$ is the anticanonical divisor of the base G/P.

Example 3.6.17. Consider $G = SL_n$. If H is any (standard) horospherical subgroup of SL_n with associated parabolic B_n (e.g. $H = U_n$), then $I = \varnothing$ so $\mathcal{C} = S \setminus I = S = \{\alpha_1, \ldots, \alpha_{n-1}\}$. Using the notation of Proposition 3.6.15, one can check that $b_\alpha = 2$ for each $\alpha \in \mathcal{C}$; for example, in SL_3 we have

$$b_{\alpha_1} = \langle \alpha_1, \alpha_1^\vee \rangle_R + \langle \alpha_2, \alpha_1^\vee \rangle_R + \langle \alpha_1 + \alpha_2, \alpha_1^\vee \rangle_R = 2 + (-1) + (2 + (-1)) = 2.$$

Therefore, if $X = X_{\Sigma^c}$ is any SL_n/H-horospherical variety, then

$$-K_X = \sum_{(\rho,\varnothing)} D_\rho + \sum_{\alpha\in\mathcal{C}} 2D_\alpha.$$

For example, if X is the SL_3/U_3-horospherical variety from Example 3.6.4, then

$$-K_X = D_{e_1+e_2} + D_{-e_1} + D_{e_1-e_2} + 2D_{\alpha_1} + 2D_{\alpha_2}. \qquad \diamondsuit$$

Chapter 4

Horospherical stacks and stacky coloured fans

The first main research theme of this thesis is on developing a novel combinatorial theory for horospherical stacks, extending the theory for horospherical varieties discussed in Chapter 3. For details on algebraic stacks, see Appendix A, although one does not need to be familiar with algebraic stacks to appreciate this chapter.

Recall that G is a connected reductive algebraic group over an algebraically closed field k of characteristic 0. Throughout this chapter, fix a Borel subgroup $B \subseteq G$, a maximal torus $T \subseteq B$, and let $U \subseteq B$ denote the maximal unipotent subgroup of B. Fix a horospherical subgroup $H \subseteq G$ containing U and let $P := N_G(H)$ denote the associated parabolic. Let S be the set of simple roots, and let $I \subseteq S$ be such that $P = P_I$. Let $N = N(G/H)$ be the associated coloured lattice, and let $\mathcal{C} = \mathcal{C}(G/H)$ be the universal colour set.

4.1 Horospherical stacks and stacky coloured fans

In this section we define the central objects of this chapter: horospherical stacks and stacky coloured fans. These objects generalize the toric stacks and stacky fans defined in [GS15a, Section 2]. We describe the connection between the two objects, give some basic properties, and give many examples at the end.

For the reader who is unfamiliar with stacks, we suggest treating the quotient stack $[X/K]$ as a formal quotient of a variety X by the action of an algebraic group K; for more explicit details, see Appendix A.7.

4.1.1 Horospherical stacks

The idea is to take a quotient stack of a G/H-horospherical variety X by a closed subgroup K of the associated torus $P/H = T_N = \mathrm{Aut}^G(X)$. For any subgroup K of $\mathrm{Aut}^G(X)$, we let $\overline{K}$ denote the corresponding subgroup of G, i.e. so that $K = \overline{K}/H \subseteq P/H = \mathrm{Aut}^G(X)$. Note that $\overline{K}$ is a horospherical subgroup since it contains the horospherical subgroup H.

Definition 4.1.1 (Horospherical stack)**.** A *$G/\overline{K}$-horospherical stack* is an algebraic G-stack of the form $[X/K]$ where X is a horospherical G-variety and K is a closed subgroup of $\mathrm{Aut}^G(X)$.

A *horospherical G-stack* is a stack which is a $G/\overline{K}$-horospherical stack for some horospherical subgroup $\overline{K} \subseteq G$. A *horospherical stack* is a stack which is a horospherical G-stack for some connected reductive algebraic group G.

Remark 4.1.2 (Open G-orbit of a horospherical stack)**.** If $\mathcal{X}$ is a $G/\overline{K}$-horospherical stack, then $G/\overline{K}$ is its open G-orbit.

The natural base point of the G/H-horospherical variety X is $eH \in G/H \subseteq X$. We use this to define the *base point* of the $G/\overline{K}$-horospherical stack $[X/K]$ as the point $e\overline{K} \in G/\overline{K}$ of the open G-orbit of $[X/K]$.

Remark 4.1.3 (Horospherical varieties are horospherical stacks)**.** A G/H-horospherical variety X can be viewed as the G/H-horospherical stack $X = [X/\{1\}]$.

Remark 4.1.4 (Toric stacks are horospherical stacks)**.** We use the definition of *toric stack* from [GS15a, Definition 1.1]. If $[X/K]$ is a toric stack where X is a toric variety with torus T, then $[X/K]$ is a T/K-horospherical stack since $K \subseteq T = \mathrm{Aut}^T(X)$. In this case, $K = \overline{K}$.

Note, however, that we do not give an analogue of *non-strict* toric stacks, which were defined to allow for toric stacks with non-trivial generic stabilizer.

Remark 4.1.5 (Expanding the group action)**.** Let X be a G/H-horospherical variety. Suppose that G' is a connected reductive algebraic group, T' is a torus, and we have a homomorphism of algebraic groups $\varepsilon : G' \times T' \to G$ which restricts to an isogeny $G' \to G$. Then $G' \times T'$ acts on X via ε and we can view X as a $G' \times T'/\varepsilon^{-1}(H)$-horospherical variety. As in Remark 3.2.22, the combinatorics of X is the same whether we view it as a horospherical G-variety or a horospherical $G' \times T'$-variety. In particular, the coloured lattice N is unchanged, and the coloured fan for X is unchanged.

This can be applied to studying horospherical stacks: we can view $[X/K]$ as a horospherical G-stack, or as a horospherical $G' \times T'$-stack.

Remark 4.1.6 (Alternative definition of horospherical stack)**.** Note that our definition of horospherical stack is slightly different than [JLT19, Definition 3.6]. They define a horospherical G-stack as $[X/K]$ where X is a horospherical $G \times T_0$-variety, where T_0 is a torus acting faithfully on X, and $K \subseteq \mathrm{Aut}^{G\times T_0}(X)$ is a closed subgroup containing T_0; this definition just allows for a torus factor to be added to the group G, which is then "cancelled out" in the quotient by K. Using the idea from Remark 4.1.5, one can view a horospherical G-stack $\mathcal{X}$ (as per Definition 4.1.1) as a horospherical $G \times T_0$-stack for any torus T_0, where we let T_0 act trivially.

4.1.2 Stacky coloured fans

Now we introduce stacky coloured fans, the combinatorial counterpart to horospherical stacks.

Definition 4.1.7 (Stacky coloured fan)**.** A *stacky coloured fan* is a pair (Σ^c, β) satisfying the following:

(SCF1) Σ^c is a coloured fan on N; recall that N is the coloured lattice associated to some horospherical homogeneous space G/H.

(SCF2) $\beta : N \to L$ is a $\mathbb{Z}$-linear map of lattices with finite cokernel.

Remark 4.1.8 (Base of a stacky coloured fan is a coloured lattice)**.** If $(\Sigma^c, \beta : N \to L)$ is a stacky coloured fan, then the base lattice L inherits a colour structure through β: the universal colour set for L is the same as N (i.e. $\mathcal{C}$) and the colour points of L are the images of the colour points of N under β. Moreover, L is the coloured lattice associated to some horospherical homogeneous space.

Indeed, since $\beta : N \to L$ has finite cokernel, it induces a surjection of tori $\beta_{\mathbb{G}_m} : T_N \to T_L$. Letting $K_\beta := \ker(\beta_{\mathbb{G}_m})$, which we can write as $K_\beta = \overline{K_\beta}/H \subseteq P/H$, we have $T_L = P/\overline{K_\beta}$. Since $H \subseteq \overline{K_\beta}$, we see that $\overline{K_\beta}$ is horospherical, so the homogeneous space $G/\overline{K_\beta}$ is horospherical. It is easy to see that β is the associated map to the natural projection $G/H \to G/\overline{K_\beta}$, so L is the coloured lattice associated to $G/\overline{K_\beta}$, i.e. L is $N(G/\overline{K_\beta})$.

Throughout this chapter, we use the notation of Definition 4.1.7 and Remark 4.1.8 for stacky coloured fans. If we want to highlight the full group, then we sometimes call (Σ^c, β) a *G-stacky coloured fan*.

Remark 4.1.9 (Stacky fans are stacky coloured fans)**.** We use the definition of *stacky fan* from [GS15a, Definition 2.4]. If (Σ, β) is a stacky fan, then we can view this as a T-stacky coloured fan (the lattices N and L have no colour points in this case).

Since we do not define an analogue of *non-strict* toric stacks, we also do not define an analogue of *non-strict* stacky fans.

4.1.3 Stacky coloured fan → horospherical stack

We describe how a stacky coloured fan determines a horospherical stack. Let $(\Sigma^c, \beta : N \to L)$ be a stacky coloured fan. Then the coloured fan Σ^c determines a G/H-horospherical variety X_{Σ^c}. The map $\beta : N \to L$ induces a homomorphism of tori $\beta_{\mathbb{G}_m} : T_N \to T_L$. Since $\operatorname{coker}(\beta)$ is finite, $\beta_{\mathbb{G}_m}$ is surjective. As in Remark 4.1.8, consider $K_\beta = \ker(\beta_{\mathbb{G}_m}) \subseteq T_N = \operatorname{Aut}^G(X)$, which is a closed subgroup.

Notation 4.1.10. Using the notation above, if (Σ^c, β) is a stacky coloured fan, we define the $G/\overline{K_\beta}$-horospherical stack $\mathcal{X}_{\Sigma^c,\beta}$ to be the quotient stack $[X_{\Sigma^c}/K_\beta]$.

Remark 4.1.11. One can consider two stacky coloured fans to be "the same" if they are isomorphic in the sense of Definition 4.2.3. The stacky coloured fan (Σ^c, β) determines the pair (X_{Σ^c}, K_β), which provides a presentation for the horospherical stack $\mathcal{X}_{\Sigma^c,\beta}$. It is important to note that different stacky coloured fans can yield different presentations of the same horospherical stack; see Remark 4.1.16. We describe exactly when different stacky coloured fans yield isomorphic horospherical stacks in Theorem 4.2.12.

4.1.4 Horospherical stack → stacky coloured fan

Now we describe how every horospherical stack arises from a stacky coloured fan. Any $G/\overline{K}$-horospherical stack is of the form $[X/K]$ where X is a G/H-horospherical variety (for some horospherical subgroup $H \subseteq G$) and $K = \overline{K}/H$ is a closed subgroup of $\operatorname{Aut}^G(X) = T_N$. Then the G/H-horospherical variety X corresponds to a coloured fan Σ^c on N. The surjection of tori $T_N \twoheadrightarrow T_N/K =: T_L$ induces a map of one-parameter subgroup lattices $\beta : N \to L$. Since this map of tori is surjective, $\operatorname{coker}(\beta)$ is finite. As in Remark 4.1.8, L is the coloured lattice associated to the horospherical homogeneous space $G/\overline{K}$. Thus (Σ^c, β) is a stacky coloured fan, and we have $[X/K] = \mathcal{X}_{\Sigma^c,K_\beta}$.

4.1.5 Examples

We give many examples of stacky coloured fans and their corresponding horospherical stacks. Diagrams for stacky coloured fans are drawn as follows: the coloured fan Σ^c on the coloured lattice N is on the left, the map $\beta : N \to L$ is shown at the top, and the coloured

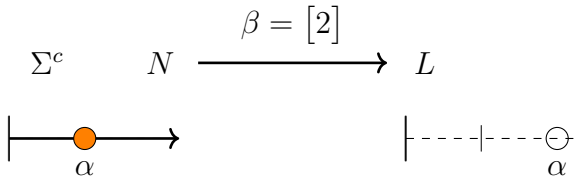
lattice L is shown on the right. For a colour α, we draw an "un-filled" circle point marked α to indicate u_α in the coloured lattice; if we want to indicate that $\alpha \in \mathcal{F}(\Sigma^c)$, then we "fill in" this circle point (with the colour orange).

Example 4.1.12. Let Σ^c be a coloured fan on N. If $\beta : N \to N$ is an isomorphism, then the stacky coloured fan (Σ^c, β) yields the G/H-horospherical stack $\mathcal{X}_{\Sigma^c,\beta} = X_{\Sigma^c}$. ♢

Example 4.1.13. Given a $\mathbb{Z}$-linear map $\beta : N \to L$ with finite cokernel, the stacky coloured fan $(0^c, \beta)$ yields the $G/\overline{K_\beta}$-horospherical stack $[(G/H)/K_\beta] = G/\overline{K_\beta}$. ♢

Recall that for colour points of horospherical SL_n-varieties, we use the convention of Example 3.3.17. In particular, for $N(SL_n/U_n) \cong \mathbb{Z}^{n-1}$, the colour point u_{α_i} is the standard basis vector e_i for $1 \leq i \leq n-1$, where the dual basis vector $e_i \in N(SL_n/U_n)^\vee$ corresponds to the character $B_n \to \mathbb{G}_m$ sending a matrix to the determinant of its upper-left $i \times i$-submatrix.

Example 4.1.14. Consider $G = SL_2$ and the horospherical subgroup $H = U_2$. The lattice N is isomorphic to $\mathbb{Z}$, the universal colour set is $\{\alpha\}$, and $u_\alpha = e_1$. Let (Σ^c, β) be the stacky coloured fan given in the diagram below.



In this case, Σ^c is generated by a single coloured cone $(\mathrm{Cone}(e_1), \{\alpha\})$. We have $X_{\Sigma^c} = \mathbb{A}^2$, where SL_2 acts by matrix multiplication; we can take the base point to be $(1, 0) \in \mathbb{A}^2$.

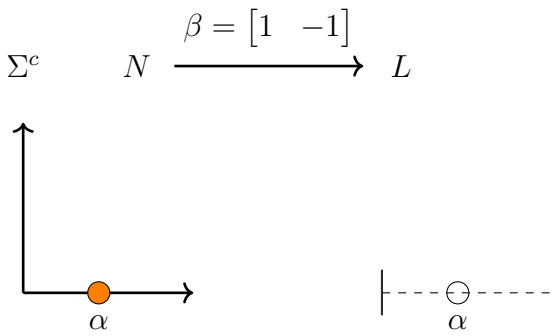
The corresponding map of tori $\beta_{\mathbb{G}_m} : T_N = \mathbb{G}_m \to T_L = \mathbb{G}_m$ sends $t \mapsto t^2$. Thus, $K_\beta = \{\zeta : \zeta^2 = 1\} \subseteq \mathbb{G}_m$. This yields the subgroup

$$\overline{K_\beta} = \left\{ \begin{bmatrix} \zeta & * \\ 0 & \zeta^{-1} \end{bmatrix} \ : \ \zeta^2 = 1 \right\} \subseteq SL_2.$$

The action of T_N on $X_{\Sigma^c} = \mathbb{A}^2$ is via SL_2-equivariant automorphisms. Thus, $T_N = \mathbb{G}_m$ acts by scaling the coordinates of $\mathbb{A}^2$, so $K_\beta \cong \mathbb{Z}/2\mathbb{Z}$ acts by scaling (changing the sign of) the coordinates of $\mathbb{A}^2$.

Therefore, we have $\mathcal{X}_{\Sigma^c,\beta} = [\mathbb{A}^2/(\mathbb{Z}/2\mathbb{Z})]$, which is a $SL_2/\overline{K_\beta}$-horospherical stack. Note that this is different from the singular cone $\{xy = z^2\} \subseteq \mathbb{A}^3_{(x,y,z)}$ (although this variety is the good moduli space of $\mathcal{X}_{\Sigma^c,\beta}$; see Section 4.5.2). ♢

Example 4.1.15. Consider $G = SL_2 \times \mathbb{G}_m$ and the horospherical subgroup $H = U_2 \times \{1\}$. The lattice N is isomorphic to $\mathbb{Z}^2$, the universal colour set is $\mathcal{C} = \{\alpha\}$, and $u_\alpha = e_1$. Let (Σ^c, β) be the stacky coloured fan given in the diagram below.



In this case, Σ^c has two maximal coloured cones: $(\mathrm{Cone}(e_1), \{\alpha\})$ and $(\mathrm{Cone}(e_2), \varnothing)$. We have $X_{\Sigma^c} = \mathbb{A}^3 \setminus \{0\}$, where SL_2 acts by matrix multiplication on the first two coordinates and $\mathbb{G}_m$ acts by scaling the last coordinate; we can take the base point to be $(1, 0, 1) \in \mathbb{A}^3$.

The corresponding map of tori $\beta_{\mathbb{G}_m} : T_N = \mathbb{G}_m^2 \to T_L = \mathbb{G}_m$ sends $(s, t) \mapsto st^{-1}$. Thus $K_\beta = \{(s, s) : s \in \mathbb{G}_m\}$, which is the diagonal $\mathbb{G}_m$ inside of $T_N = \mathbb{G}_m^2$. This yields the subgroup

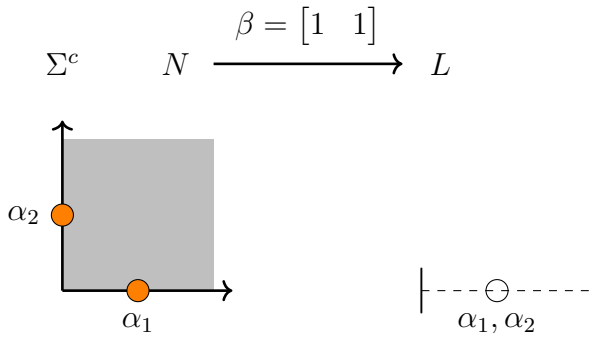
$$\overline{K_\beta} = \left\{ \left(\begin{bmatrix} s & * \\ 0 & s^{-1} \end{bmatrix} , \ s \right) \ : \ s \in \mathbb{G}_m \right\} \subseteq SL_2 \times \mathbb{G}_m.$$

The action of T_N on $X_{\Sigma^c} = \mathbb{A}^3 \setminus \{0\}$ is via $SL_2 \times \mathbb{G}_m$-equivariant automorphisms. Thus, $(s, t) \in T_N = \mathbb{G}_m^2$ acts by scaling the first two coordinates of $\mathbb{A}^3 \setminus \{0\}$ by s and scaling the last coordinate by t. So $K_\beta = \mathbb{G}_m$ acts by scaling the coordinates of $\mathbb{A}^3 \setminus \{0\}$.

Therefore, we have $\mathcal{X}_{\Sigma^c, \beta} = [(\mathbb{A}^3 \setminus \{0\})/\mathbb{G}_m] = \mathbb{P}^2$. Note that this is a $SL_2 \times \mathbb{G}_m/\overline{K_\beta} \cong SL_2/U_2$-horospherical stack. ♢

Remark 4.1.16 (Horospherical stacks can arise from different covers)**.** From Example 4.1.12 and Example 4.1.15, we see that the SL_2/U_2-horospherical stack $\mathbb{P}^2$ can be written in (at least) two different ways as the quotient stack of a horospherical variety by a diagonalizable group. In particular, these two different ways of writing $\mathbb{P}^2$ use different (non-isomorphic) stacky coloured fans; recall that Example 3.4.22 gives the coloured fan for $\mathbb{P}^2$.

Example 4.1.17. Consider $G = SL_3$ and the horospherical subgroup $H = U_3$. The lattice N is isomorphic to $\mathbb{Z}^2$, the universal colour set is $\mathcal{C} = \{\alpha_1, \alpha_2\}$, and we have $u_{\alpha_1} = e_1$ and $u_{\alpha_2} = e_2$. Let (Σ^c, β) be the stacky coloured fan given in the diagram below.



In this case, Σ^c is generated by a single coloured cone $(\mathrm{Cone}(e_1, e_2), \{\alpha_1, \alpha_2\})$. The variety X_{Σ^c} is $\{ux + vy + wz = 0\} \subseteq \mathbb{A}^3_{(u,v,w)} \times \mathbb{A}^3_{(x,y,z)}$, where SL_3 acts by matrix multiplication on the (u, v, w)-coordinates and acts by contragredient matrix multiplication (i.e. the inverse transpose action) on the (x, y, z)-coordinates; we can take the base point to be $(1, 0, 0, 0, 0, 1)$ in the (u, v, w, x, y, z)-coordinates.

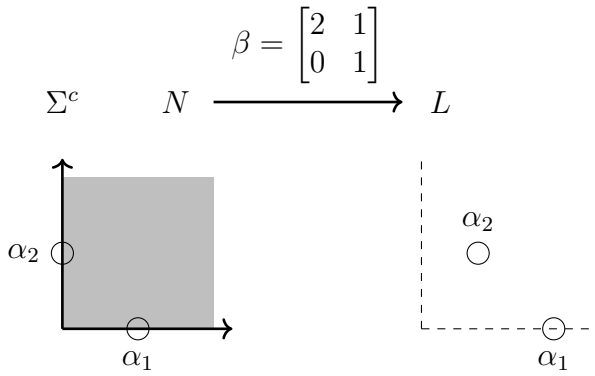
The corresponding map of tori $\beta_{\mathbb{G}_m} : T_N = \mathbb{G}_m^2 \to T_L = \mathbb{G}_m$ sends $(s, t) \mapsto st$. Thus, $K_\beta = \{(s, s^{-1}) : s \in \mathbb{G}_m\}$ inside of $T_N = \mathbb{G}_m^2$. This yields the subgroup

$$\overline{K_\beta} = \left\{ \begin{bmatrix} s & * & * \\ 0 & * & * \\ 0 & 0 & s \end{bmatrix} \ : \ s \in \mathbb{G}_m \right\} \subseteq SL_3.$$

The action of T_N on X_{Σ^c} is via SL_3-equivariant automorphisms. Thus, $(s, t) \in T_N = \mathbb{G}_m^2$ acts by scaling the (u, v, w)-coordinates by s and scaling the (x, y, z)-coordinates by t. So $s \in K_\beta = \mathbb{G}_m$ acts by scaling the (u, v, w)-coordinates by s and scaling the (x, y, z)-coordinates by s^{-1}.

This gives the $SL_3/\overline{K_\beta}$-horospherical stack $\mathcal{X}_{\Sigma^c,\beta} = [X_{\Sigma^c}/K_\beta]$. ♢

Example 4.1.18. Consider $G = SL_3$ and the horospherical subgroup $H = U_3$. The lattice N is isomorphic to $\mathbb{Z}^2$, the universal colour set is $\mathcal{C} = \{\alpha_1, \alpha_2\}$, and we have $u_{\alpha_1} = e_1$ and $u_{\alpha_2} = e_2$. Let (Σ^c, β) be the stacky coloured fan given in the diagram below.



In this case, Σ^c is generated by a single coloured cone $(\mathrm{Cone}(e_1, e_2), \varnothing)$. Using the affine local structure from Section 3.5.2, we can describe X_{Σ^c} as $SL_3 \times^{B_3} \mathbb{A}^2$.

The corresponding map of tori $\beta_{\mathbb{G}_m} : T_N = \mathbb{G}_m^2 \to T_L = \mathbb{G}_m^2$ sends $(s, t) \mapsto (s^2 t, t)$. Thus, $K_\beta = \{(\zeta, 1) : \zeta^2 = 1\}$ inside of $T_N = \mathbb{G}_m^2$. This yields the subgroup

$$\overline{K_\beta} = \left\{ \begin{bmatrix} \zeta & * & * \\ 0 & \zeta^{-1} & * \\ 0 & 0 & 1 \end{bmatrix} \ : \ \zeta^2 = 1 \right\} \subseteq SL_3.$$

The action of T_N on X_{Σ^c} is via SL_3-equivariant automorphisms; this can be viewed as usual $\mathbb{G}_m^2$ multiplication on the $\mathbb{A}^2$ in $SL_3 \times^{B_3} \mathbb{A}^2$. So $K_\beta \cong \mathbb{Z}/2\mathbb{Z}$ acts by swapping the sign of the first coordinate in this $\mathbb{A}^2$.

This gives the $SL_3/\overline{K_\beta}$-horospherical stack $\mathcal{X}_{\Sigma^c,\beta} = [X_{\Sigma^c}/K_\beta]$. ♢

Remark 4.1.19 (Toric vs. non-toric examples)**.** The horospherical stacks in Example 4.1.14 and Example 4.1.15 are both toric, so they can also be studied using the theory of toric stacks. However, the other examples in this subsection are not toric; see Proposition 4.3.8.

4.2 Horospherical morphisms and maps of stacky coloured fans

In this section, we discuss morphisms of horospherical stacks and how they relate to maps of the corresponding stacky coloured fans (one direction is Section 4.2.3, and the other is described in Theorem 4.2.8). In particular, we give a combinatorial description of horospherical isomorphisms (in Theorem 4.2.12).

4.2.1 Horospherical morphisms

We first define the types of morphisms that we want to consider between two horospherical stacks; these should preserve the group action and the base points.

Definition 4.2.1 (Horospherical morphism)**.** Let $\mathcal{X}_i = [X_i/K_i]$ be $G/\overline{K_i}$-horospherical stacks for $i = 1, 2$. A *horospherical morphism* $\pi : \mathcal{X}_1 \to \mathcal{X}_2$ is a G-equivariant morphism of stacks which preserves the base point (i.e. sends the base point of $\mathcal{X}_1$ to the base point of $\mathcal{X}_2$).

Note that, for a horospherical morphism $\pi : \mathcal{X}_1 \to \mathcal{X}_2$, the condition of preserving the base point means that $\overline{K_1} \subseteq \overline{K_2}$ and π restricts to a map of open orbits $G/\overline{K_1} \to G/\overline{K_2}$ which is the natural projection map. Clearly horospherical morphisms of horospherical varieties are horospherical morphisms in the sense of Definition 4.2.1, when viewing the horospherical varieties as horospherical stacks via Remark 4.1.3.

Remark 4.2.2 (Dominant toric morphisms are horospherical morphisms)**.** A morphism of toric stacks $\pi : \mathcal{X}_1 \to \mathcal{X}_2$ restricts to a group homomorphism $T_1 \to T_2$ of the open tori such that π is equivariant with respect to this group homomorphism; see [GS15a, Definition 3.1]. If this restricted map $T_1 \to T_2$ is surjective, then we can view $\mathcal{X}_1$ and $\mathcal{X}_2$ as horospherical T_1-stacks (since T_2 is a quotient of T_1), and π is a horospherical morphism.

4.2.2 Maps of stacky coloured fans

The combinatorial counterpart to horospherical morphisms are maps of stacky coloured fans.

Definition 4.2.3 (Map of stacky coloured fans)**.** Let (Σ_i^c, β_i) be stacky coloured fans for $i = 1, 2$; we use the notation of Section 4.1 but with subscripts $i = 1, 2$. A *map of stacky coloured fans* is a pair of $\mathbb{Z}$-linear maps $\Phi : N_1 \to N_2$ and $\phi : L_1 \to L_2$ which satisfy the following conditions:

(MSCF1) Φ is a map of coloured lattices (in the sense of Section 3.3.4).

(MSCF2) Φ is compatible with Σ_1^c and Σ_2^c.

(MSCF3) We have $\beta_2 \circ \Phi = \phi \circ \beta_1$.

We can draw a map of stacky coloured fans $(\Phi, \phi) : (\Sigma_1^c, \beta_1) \to (\Sigma_2^c, \beta_2)$ as the following commutative diagram:

$$\begin{array}{ccc} \Sigma_1^c & \longrightarrow & \Sigma_2^c \\ N_1 & \xrightarrow{\Phi} & N_2 \\ {\scriptstyle\beta_1}\downarrow & & \downarrow{\scriptstyle\beta_2} \\ L_1 & \xrightarrow{\phi} & L_2 \end{array}$$

Remark 4.2.4 (Base map in a map of stacky coloured fans)**.** Let $(\Phi, \phi) : (\Sigma_1^c, \beta_1) \to (\Sigma_2^c, \beta_2)$ be a map of stacky coloured fans using the notation in Definition 4.2.3. By Remark 4.1.8, we know that L_i is the coloured lattice associated to $G/\overline{K_i}$. We necessarily have $\overline{K_1} \subseteq \overline{K_2}$ and the map $\phi : L_1 \to L_2$ is the map associated to the natural projection $G/\overline{K_1} \to G/\overline{K_2}$.

Throughout this chapter, we use the notation of Definition 4.2.3 and Remark 4.2.4 for maps of stacky coloured fans.

Remark 4.2.5 (Dominantly mapped colours for a map of stacky coloured fans)**.** Given a map of stacky coloured fans $(\Phi, \phi) : (\Sigma_1^c, \beta_1) \to (\Sigma_2^c, \beta_2)$, the commuting condition (MSCF3) and the fact that $\mathcal{C}_{\beta_1} = \mathcal{C}_{\beta_2} = \varnothing$ imply $\mathcal{C}_\Phi = \mathcal{C}_\phi$.

Remark 4.2.6 (Map of stacky coloured fans has finite cokernel)**.** If (Φ, ϕ) is a map of stacky coloured fans, then Φ and ϕ both have finite cokernel. Indeed, this follows from Remark 3.1.1 and because $\Phi : N_1 \to N_2$ and $\phi : L_1 \to L_2$ correspond to surjections of tori $\Phi_{\mathbb{G}_m} : T_{N_1} \to T_{N_2}$ and $\phi_{\mathbb{G}_m} : T_{L_1} \to T_{L_2}$, respectively.

4.2.3 Map of stacky coloured fans → horospherical morphism

We describe how a map of stacky coloured fans determines a horospherical morphism of the corresponding horospherical stacks. Let $\mathcal{X}_{\Sigma_i^c, \beta_i}$ be $G/\overline{K_i}$-horospherical stacks coming from stacky coloured fans (Σ_i^c, β_i) for $i = 1, 2$. Let $(\Phi, \phi) : (\Sigma_1^c, \beta_1) \to (\Sigma_2^c, \beta_2)$ be a map of the stacky coloured fans. By (MSCF2), we know that Φ is compatible with Σ_1^c and Σ_2^c. Thus, the map $\Phi : N_1 \to N_2$ induces a horospherical morphism $X_{\Sigma_1^c} \to X_{\Sigma_2^c}$ of horospherical G-varieties. By the commuting condition (MSCF3), the map of tori $\Phi_{\mathbb{G}_m} : T_{N_1} \to T_{N_2}$ restricts to a map $K_{\beta_1} \to K_{\beta_2}$. It follows that the map $X_{\Sigma_1^c} \to X_{\Sigma_2^c}$ induces a map on the quotient stacks

$$\mathcal{X}_{\Sigma_1^c, \beta_1} = [X_{\Sigma_1^c}/K_{\beta_1}] \to [X_{\Sigma_2^c}/K_{\beta_2}] = \mathcal{X}_{\Sigma_2^c, \beta_2}.$$

By construction, this map is G-equivariant and preserves the base point (since the map $X_{\Sigma_1^c} \to X_{\Sigma_2^c}$ is horospherical), so this map is horospherical.

Notation 4.2.7. Using the notation above, if $(\Phi, \phi) : (\Sigma_1^c, \beta_1) \to (\Sigma_2^c, \beta_2)$ is a map of stacky coloured fans, we let $\pi_{\Phi,\phi}$ denote the associated horospherical morphism $\mathcal{X}_{\Sigma_1^c,\beta_1} \to \mathcal{X}_{\Sigma_2^c,\beta_2}$.

4.2.4 Horospherical morphism → map of stacky coloured fans

Now we describe how any horospherical morphism of horospherical stacks relates to a map of stacky coloured fans. This process is not as simple as the other direction above. The reason for this is that a morphism $X_1 \to X_2$ can determine a morphism of the quotient stacks $[X_1/K_1] \to [X_2/K_2]$, but a morphism of the quotient stacks may not correspond to a morphism of *these specific covers* X_1 and X_2. The following theorem tells us that, up to changing the presentation of the domain quotient stack, a horospherical morphism of horospherical stacks comes from a map of the corresponding stacky coloured fans; this is a natural generalization of [GS15a, Theorem 3.4].

Theorem 4.2.8 (Horospherical morphisms)**.** *Let $\mathcal{X}_i := \mathcal{X}_{\Sigma_i^c,\beta_i}$ be $G/\overline{K_i}$-horospherical stacks coming from stacky coloured fans $(\Sigma_i^c, \beta_i : N_i \to L_i)$ for $i = 1, 2$. Note that we can regard each $\mathcal{X}_i$ as a horospherical $G \times K_{\beta_2}^\circ$-stack by letting the torus $K_{\beta_2}^\circ$ act trivially. Suppose that we have a horospherical morphism $\pi : \mathcal{X}_1 \to \mathcal{X}_2$. Then there exists a stacky coloured fan $(\Sigma_0^c, \beta_0 : N_0 \to L_0)$, yielding a horospherical $G \times K_{\beta_2}^\circ$-stack $\mathcal{X}_0 := \mathcal{X}_{\Sigma_0^c,\beta_0}$, and there exist maps of stacky coloured fans $(\Phi_i : N_0 \to N_i, \phi_i : L_0 \to L_i)$ for $i = 1, 2$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathcal{X}_0 & & \\ {\scriptstyle \pi_{\Phi_1,\phi_1}}\downarrow & \searrow^{\pi_{\Phi_2,\phi_2}} & \\ \mathcal{X}_1 & \xrightarrow[\pi]{} & \mathcal{X}_2 \end{array}$$

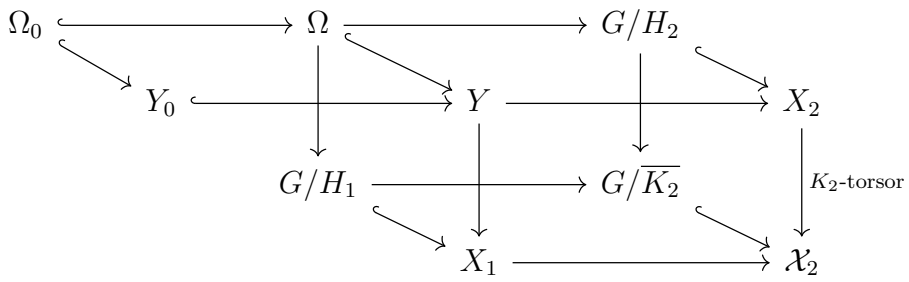
and π_{Φ_1,ϕ_1} is an isomorphism.

Remark 4.2.9. Before proving Theorem 4.2.8, we note a general fact about quotient stacks. If a group K is an extension of a quotient K_2 by a normal subgroup K_1, then $[X/K]$ is identified with $[[X/K_1]/K_2]$.

Proof of Theorem 4.2.8. Set $X_i := X_{\Sigma_i^c}$, which is a G/H_i-horospherical variety for $i = 1, 2$, and let $K_i := K_{\beta_i}$. By the definition of horospherical morphism, we have a natural projection map $G/\overline{K_1} \to G/\overline{K_2}$. Thus, we can compose natural projections to get a map $G/H_1 \to G/\overline{K_1} \to G/\overline{K_2}$.

Let $Y := X_1 \times_{\mathcal{X}_2} X_2$ and consider the open subset $\Omega := (G/H_1) \times_{G/\overline{K_2}} (G/H_2) \subseteq Y$. Let $\Omega_0 \subseteq \Omega$ be the connected component of the base point (eH_1, eH_2), and let $Y_0 \subseteq Y$ be

the connected component of Y containing Ω_0. We have the following commutative diagram (in which the hook arrows are open embeddings):



Notice that Ω is a K_2-torsor over G/H_1. Based on the K_2-action and [GS15a, Lemma 3.3], it follows that the connected component Ω_0 is a K_2°-torsor over G/H_1. In particular, Ω_0 is a $G \times K_2^\circ$-homogeneous space, so it is of the form $(G \times K_2^\circ)/H_0$ for some closed subgroup H_0 of the connected reductive algebraic group $G \times K_2^\circ$. In fact, this H_0 can be taken to be the stabilizer of the base point (eH_1, eH_2). Then H_0 is clearly horospherical since the maximal unipotent subgroup $U \times \{1\} \subseteq G \times K_2^\circ$ stabilizes the base point. Furthermore, for $i = 1, 2$ we have $H_0 \subseteq H_i \times K_2^\circ$, and the map

$$\Omega_0 = (G \times K_2^\circ)/H_0 \longrightarrow G/H_i = (G \times K_2^\circ)/(H_i \times K_2^\circ)$$

is the natural projection, so these are horospherical morphisms of horospherical $G \times K_2^\circ$-homogeneous spaces.

Since X_1 is normal and separated, and Y is a K_2-torsor over X_1, we see that Y is also normal and separated. Thus, Y_0 is a normal, separated, and connected scheme. This implies that Y_0 is irreducible, so it is a normal variety. Hence, Y_0 is a horospherical $G \times K_2^\circ$-variety with open horospherical homogeneous space Ω_0. Let Σ_0^c denote the coloured fan associated to Y_0, which is on the coloured lattice N_0 associated to Ω_0.

For $i = 1, 2$, let $\Phi_i : N_0 \to N_i$ be the coloured lattice maps associated to the projections $\Omega_0 \to G/H_i$. The morphisms $Y_0 \to X_i$ for $i = 1, 2$ are base point preserving and $G \times K_2^\circ$-equivariant (where K_2° acts trivially on the X_i), so they are horospherical morphisms. Thus, the maps of coloured lattices $\Phi_i : N_0 \to N_i$ are compatible with the coloured fans Σ_0^c and Σ_i^c. Set $\phi_1 := \mathrm{id}_{L_1}$, $\beta_0 := \beta_1 \circ \Phi_1$, and $K_0 := K_{\beta_0}$. We have the following morphisms of

stacky coloured fans:

$$\begin{array}{ccccc}
\Sigma_1^c & \longleftarrow & \Sigma_0^c & \longrightarrow & \Sigma_2^c \\
N_1 & \xleftarrow{\Phi_1} & N_0 & \xrightarrow{\Phi_2} & N_2 \\
{\scriptstyle\beta_1}\downarrow & & \downarrow{\scriptstyle\beta_0} & & \downarrow{\scriptstyle\beta_2} \\
L_1 & \xlongequal{\phi_1} & L_1 & \xrightarrow{\phi_2} & L_2
\end{array}$$

Note that, since Ω_0 is a K_2°-torsor over G/H_1, we see that $\overline{\Omega_0} = Y_0$ is a K_2°-torsor over $\overline{G/H_1} = X_1$. Thus $Y_0/K_2^\circ \cong X_1$.

Since each Φ_i is a map of coloured lattices, they both have finite cokernel. Now [GS15a, Lemma A.1] implies that K_0 is an extension of K_1 by K_2°. Therefore, the map of stacky coloured fans $(\Phi_1, \phi_1) : (\Sigma_0^c, \beta_0) \to (\Sigma_1^c, \beta_1)$ induces the isomorphism

$$\pi_{\Phi_1,\phi_1} : \mathcal{X}_0 = [Y_0/K_0] \xrightarrow{\sim} [(Y_0/K_2^\circ)/K_1] = \mathcal{X}_1.$$

On the other hand, the map of stacky coloured fans $(\Phi_2, \phi_2) : (\Sigma_0^c, \beta_0) \to (\Sigma_2^c, \beta_2)$ induces the morphism $\pi_{\Phi_2,\phi_2} : \mathcal{X}_0 \to \mathcal{X}_2$. □

4.2.5 Isomorphisms

In Theorem 4.2.12 (stated below) we give a combinatorial characterization for a horospherical morphism (induced by a map of stacky coloured fans) to be an isomorphism; this is a natural generalization of [GS15a, Theorem B.3]. The proof is in Section 4.6.

Notation 4.2.10 (Preimage coloured cone)**.** Suppose that $\Phi : N_1 \to N_2$ is a map of coloured lattices which is compatible with coloured fans Σ_1^c and Σ_2^c. Given $\sigma_2^c \in \Sigma_2^c$, we let $\Phi^{-1}(\sigma_2^c)$ denote the sub-coloured fan of Σ_1^c consisting of all coloured cones in Σ_1^c which map into σ_2^c. Similarly, we let $\Phi^{-1}(\sigma_2)$ denote the underlying fan for $\Phi^{-1}(\sigma_2^c)$, i.e. this is the sub-fan of Σ_1 consisting of all cones in Σ_1 which map into σ_2.

Remark 4.2.11 (Preimage of simple opens)**.** Suppose that $\Phi : N_1 \to N_2$ is a map of coloured lattices which is compatible with coloured fans Σ_1^c and Σ_2^c. Then we have a horospherical morphism $X_{\Sigma_1^c} \to X_{\Sigma_2^c}$. Given $\sigma_2^c \in \Sigma_2^c$, we have an open simple horospherical subvariety $X_{\sigma_2^c} \subseteq X_{\Sigma_2^c}$. The sub-coloured fan $\Phi^{-1}(\sigma_2^c) \subseteq \Sigma_1^c$ yields the open horospherical subvariety $X_{\Phi^{-1}(\sigma_2^c)} \subseteq X_{\Sigma_1^c}$ which is naturally the preimage of $X_{\sigma_2^c}$, i.e. $X_{\Sigma_1^c} \times_{X_{\Sigma_2^c}} X_{\sigma_2^c}$.

Theorem 4.2.12 (Horospherical isomorphisms)**.** *Let $(\Sigma_i^c, \beta_i : N_i \to L_i)$ be stacky coloured fans for $i = 1, 2$, and let $(\Phi, \phi) : (\Sigma_1^c, \beta_1) \to (\Sigma_2^c, \beta_2)$ be a map of stacky coloured fans. Then*

the induced horospherical morphism $\pi_{\Phi,\phi} : \mathcal{X}_{\Sigma_1^c,\beta_1} \to \mathcal{X}_{\Sigma_2^c,\beta_2}$ *is an isomorphism if and only if the following conditions hold:*

(Iso1) ϕ *is an isomorphism of coloured lattices. Thus* $\mathcal{C}_\Phi = \mathcal{C}_\phi = \varnothing$ *and* $\mathcal{C}_1 = \mathcal{C}_2$.

(Iso2) For each $\sigma_2^c \in \Sigma_2^c$, $\Phi^{-1}(\sigma_2^c)$ *is generated by a single coloured cone and* $\mathcal{F}(\Phi^{-1}(\sigma_2^c)) = \mathcal{F}(\sigma_2^c)$.

(Iso3) For each $\sigma_2^c = (\sigma_2, \mathcal{F}_2) \in \Sigma_2^c$, *the map* Φ *induces an isomorphism of monoids* $\Phi^{-1}(\sigma_2) \cap N_1 \to \sigma_2 \cap N_2$, *where here we can view* $\Phi^{-1}(\sigma_2)$ *as a cone.*

4.2.6 Products of horospherical morphisms

We finish this section by examining products of horospherical morphisms. Recall that, if X_i are G_i/H_i-horospherical varieties for $i = 1, 2$, then $X_1 \times X_2$ is a $(G_1 \times G_2)/(H_1 \times H_2)$-horospherical variety. Then the associated coloured lattice to a product of two horospherical homogeneous spaces is the product of the individual associated coloured lattices, and the product of two coloured fans corresponds to the product of the two corresponding horospherical varieties, i.e. $X_{\Sigma_1^c \times \Sigma_2^c} = X_{\Sigma_1^c} \times X_{\Sigma_2^c}$. The following proposition is an immediate generalization of [GS15a, Proposition 3.6].

Proposition 4.2.13 (Products of horospherical morphisms)**.** *Let* $\mathbf{P}$ *be a property of morphisms (of algebraic stacks) which is stable under composition and base change. Consider two morphisms of stacky coloured fans:*

$$\begin{array}{ccc} \Sigma_1^c & \longrightarrow & \Sigma_2^c \\ N_1 & \xrightarrow{\Phi} & N_2 \\ {\scriptstyle\beta_1}\downarrow & & \downarrow{\scriptstyle\beta_2} \\ L_1 & \xrightarrow{\phi} & L_2 \end{array} \qquad \begin{array}{ccc} (\Sigma_1')^c & \longrightarrow & (\Sigma_2')^c \\ N_1' & \xrightarrow{\Phi'} & N_2' \\ {\scriptstyle\beta_1'}\downarrow & & \downarrow{\scriptstyle\beta_2'} \\ L_1' & \xrightarrow{\phi'} & L_2' \end{array}$$

Then the product morphism $\pi_{\Phi\times\Phi',\phi\times\phi'} = \pi_{\Phi,\phi} \times \pi_{\Phi',\phi'}$ *induced by the map of stacky coloured fans*

$$\begin{array}{ccc} \Sigma_1^c \times (\Sigma_1')^c & \longrightarrow & \Sigma_2^c \times (\Sigma_2')^c \\ N_1 \times N_1' & \xrightarrow{\Phi\times\Phi'} & N_2 \times N_2' \\ {\scriptstyle\beta_1\times\beta_1'}\downarrow & & \downarrow{\scriptstyle\beta_2\times\beta_2'} \\ L_1 \times L_1' & \xrightarrow{\phi\times\phi'} & L_2 \times L_2' \end{array}$$

has property $\mathbf{P}$ *if and only if both* $\pi_{\Phi,\phi}$ *and* $\pi_{\Phi',\phi'}$ *have property* $\mathbf{P}$.

Proof. The reverse direction holds because $\pi_{\Phi,\phi} \times \pi_{\Phi',\phi'} = (\pi_{\Phi,\phi} \times \mathrm{id}) \circ (\mathrm{id} \times \pi_{\Phi',\phi'})$, and $\pi_{\Phi,\phi} \times \mathrm{id}$ (resp. $\mathrm{id} \times \pi_{\Phi',\phi'}$) is a base change of $\pi_{\Phi,\phi}$ (resp. $\pi_{\Phi',\phi'}$).

For the forward direction, suppose that $\pi_{\Phi\times\Phi',\phi\times\phi'} = \pi_{\Phi,\phi} \times \pi_{\Phi',\phi'}$ has property $\mathbf{P}$. Note that the base point of $\mathcal{X}_{\Sigma_2^c,\beta_2}$ induces a horospherical morphism $\mathcal{X}_{(\Sigma_2')^c,\beta_2'} \to \mathcal{X}_{\Sigma_2^c,\beta_2} \times \mathcal{X}_{(\Sigma_2')^c,\beta_2'}$. Base changing $\pi_{\Phi,\phi} \times \pi_{\Phi',\phi'}$ by this morphism yields $\pi_{\Phi',\phi'}$. Thus $\pi_{\Phi',\phi'}$ has property $\mathbf{P}$. A similar argument shows that $\pi_{\Phi,\phi}$ has property $\mathbf{P}$. □

4.3 Coloured fantastacks

In this section, we introduce a specific class of horospherical stacks called *coloured fantastacks*. These (in some ways) generalize toric fantastacks from [GS15a, Section 4]. Coloured fantastacks are created through a very hands-on, "bottom-up" combinatorial construction; this contrasts the "top-down" approach in Section 4.1.

4.3.1 Construction of coloured fantastacks

Let $\ell := \#\mathcal{C}$ be the number of universal colours, so we can write $\mathcal{C} = \{\alpha_1, \ldots, \alpha_\ell\}$. For a chosen $n \geq \ell$, consider the lattice $\widehat{N} := \mathbb{Z}^n = \mathbb{Z}^\ell \oplus \mathbb{Z}^{n-\ell}$ equipped with the universal colour set $\mathcal{C}$ where the colour points are $\widehat{u}_{\alpha_i} = e_i \in \widehat{N}$ for $1 \leq i \leq \ell$.

Let Σ^c be a coloured fan on N and let $\beta : \widehat{N} \to N$ be a $\mathbb{Z}$-linear map which satisfy the following properties:

(CF1) $\{u_\alpha \in N : \alpha \in \mathcal{C}\} \cup |\Sigma^c|$ spans $N_\mathbb{R}$.

(CF2) $\beta(e_i) \in |\Sigma^c|$ for all $\ell + 1 \leq i \leq n$.

(CF3) For each $1 \leq i \leq \ell$, we have $\beta(e_i) = u_{\alpha_i}$.

(CF4) For each non-coloured ray $\rho^c = (\rho, \varnothing) \in \Sigma^c$, there exists $\ell + 1 \leq i \leq n$ such that $\beta(e_i) \in \rho \setminus \{0\}$.

We make some observations regarding these conditions: first, (CF1) is a property of just Σ^c, not β; second, β necessarily has finite cokernel because $\beta_\mathbb{R} : \widehat{N}_\mathbb{R} \to N_\mathbb{R}$ is surjective by (CF1), (CF3), and (CF4); and third, we must have $n \geq \#\mathcal{D}(X_{\Sigma^c})$ by (CF3) and (CF4).

Now for each $\sigma^c = (\sigma, \mathcal{F}) \in \Sigma^c$, let $\widehat{\sigma}^c := (\widehat{\sigma}, \mathcal{F})$ be the coloured cone on $\widehat{N}$ where

$$\widehat{\sigma} := \mathrm{Cone}(e_i : 1 \leq i \leq \ell \text{ and } \alpha_i \in \mathcal{F}, \text{ or } \ell + 1 \leq i \leq n \text{ and } \beta(e_i) \in \sigma).$$

Then each $\widehat{\sigma}^c$ is a strongly convex coloured cone on $\widehat{N}$. Finally, let $\widehat{\Sigma}^c$ be the coloured fan on $\widehat{N}$ generated by all $\widehat{\sigma}^c$, i.e. $\widehat{\Sigma}^c$ is the collection of all $\widehat{\sigma}^c$ together with all of their coloured faces.

Using Remark 3.1.7, we may assume that $G = G^{ss} \times T_0$ where G^{ss} is semisimple simply connected and T_0 is a torus. Let T_1 be the torus of rank $n - \mathrm{rk}(N)$, and let $\widehat{G} := G \times T_1 = G^{ss} \times (T_0 \times T_1)$. Note that we can identify the root system of $\widehat{G}$ with that of G using an appropriate maximal torus and Borel subgroup of $\widehat{G}$ which map onto T and B, respectively, via the projection $\widehat{G} \to G$. Then $\widehat{N}$ is the coloured lattice associated to the horospherical homogeneous space $\widehat{G}/\widehat{H}$ where $\widehat{H} = H(P \times T_1)$ is the minimal horospherical subgroup of $\widehat{G} = G \times T_1$ with associated parabolic $P \times T_1$ (see Example 3.3.18 and Notation 3.2.7).

The coloured fan $\widehat{\Sigma}^c$ yields a $\widehat{G}/\widehat{H}$-horospherical variety $X_{\widehat{\Sigma}^c}$, and $\beta : \widehat{N} \to N$ induces a horospherical morphism $X_{\widehat{\Sigma}^c} \to X_{\Sigma^c}$ extending the projection $\widehat{G}/\widehat{H} \to G/H$. Therefore, we have a map of stacky coloured fans

$$\begin{array}{ccc} \widehat{\Sigma}^c & \longrightarrow & \Sigma^c \\ \widehat{N} & \xrightarrow{\beta} & N \\ {\scriptstyle\beta}\downarrow & & \| \\ N & = & N \end{array}$$

which induces a horospherical morphism $\mathcal{X}_{\widehat{\Sigma}^c,\beta} \to X_{\Sigma^c}$.

Definition 4.3.1 (Coloured fantastack)**.** Given Σ^c and β as above which satisfy (CF1) to (CF4), the horospherical $\widehat{G}$-stack $\mathscr{F}_{\Sigma^c,\beta} := \mathcal{X}_{\widehat{\Sigma}^c,\beta} = [X_{\widehat{\Sigma}^c}/K_\beta]$ is called a *coloured fantastack*.

Remark 4.3.2 (Some fantastacks are coloured fantastacks)**.** We use the definition of *fantastack* from [GS15a, Definition 4.1]. Let $(\Sigma, \beta : N \to L)$ be a stacky fan which satisfies the following: $|\Sigma|$ spans $N_{\mathbb{R}}$, and each ray of Σ contains a *nonzero* image point under β. Then the associated fantastack $\mathscr{F}_{\Sigma,\beta}$ is a coloured fantastack in the setting where $G = T$ is a torus and $H = \{1\}$ (in this case, N and L have no colour points).

Remark 4.3.3 (Cover for coloured fantastack is quasiaffine)**.** Let $\mathscr{F}_{\Sigma^c,\beta} = [X_{\widehat{\Sigma}^c,\beta}/K_\beta]$ be a coloured fantastack. Then the cover $X_{\widehat{\Sigma}^c}$ is a quasiaffine horospherical variety, however

it might not be an open subvariety of affine space. This is different from the case of fantastacks: the cover of a fantastack is an open subvariety of affine space. In particular, if $\Sigma^c = \sigma^c$ is generated by a single coloured cone, then the cover of the fantastack $\mathscr{F}_{\sigma,\beta}$ is affine space, however the cover of the coloured fantastack $\mathscr{F}_{\sigma^c,\beta}$ might not even be affine, e.g. see Example 4.3.11.

Remark 4.3.4 (Removal of (CF1))**.** The condition (CF1) guarantees that X_{Σ^c} has no torus factors; see Section 3.4.6. To construct a coloured fantastack without assuming (CF1), one can use Remark 3.4.40 to remove the torus factors from X_{Σ^c}, i.e. write $X_{\Sigma^c} = X_{(\Sigma')^c} \times T'$ where T' is a torus and $(\Sigma')^c$ satisfies (CF1), and then perform the coloured fantastack construction on $X_{(\Sigma')^c}$; compare with "The General Case" in [CLS11, Section 5.1].

4.3.2 Good quotient Cox construction

In [Cox95], Cox developed the *homogeneous coordinate ring* of a toric variety, and used this to express a given toric variety as the good quotient of a quasiaffine toric variety by a subgroup of its torus. In [GS15a, Section 5], it is shown how this construction by Cox fits into the framework of fantastacks for toric stacks.

There is a similar "Cox construction" that works for horospherical varieties. We give an outline of this construction and show how it fits into the framework of coloured fantastacks. Let X be a G/H-horospherical variety with associated coloured fan Σ^c on N for which (CF1) holds (to remove this condition, see Remark 4.3.4).

We define a map β to use the construction in Section 4.3.1. Choose n to be $\#\mathcal{D}(X)$; note that to achieve (CF3) and (CF4), this is the smallest possible value for n. Then $n-\ell$ is the number of G-invariant prime divisors, so Σ^c has $n-\ell$ non-coloured rays, say $\rho_i^c = (\rho_i, \varnothing)$ for $\ell+1 \leq i \leq n$. Let $\beta : \widehat{N} = \mathbb{Z}^\ell \oplus \mathbb{Z}^{n-\ell} \to N$ be the map given by

$$\beta(e_i) = \begin{cases} u_{\alpha_i} & 1 \leq i \leq \ell \\ u_{\rho_i} & \ell+1 \leq i \leq n \end{cases} \tag{4.3.5}$$

where u_{ρ_i} is the minimal ray generator for ρ_i. It is easy to check that (CF1) to (CF4) hold for Σ^c and β. As in Section 4.3.1, we have a horospherical homogeneous space $\widehat{G}/\widehat{H}$ whose associated coloured lattice is $\widehat{N}$, and the map of coloured lattices associated to the projection $\widehat{G}/\widehat{H} \to G/H$ is β.

Now we follow [Gag19, Section 2] to summarize the Cox construction. Let

$$\mathrm{Cox}(X) := \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathscr{O}_X(D)(X)$$

denote the *Cox ring* of X. When X is toric, this is the same as the homogeneous coordinate ring constructed in [Cox95]. Since X is horospherical, $\mathrm{Cl}(X)$ is a finitely generated abelian group, and since $\mathscr{O}_X^*(X) = k^*$ (by (CF1) and Proposition 3.4.39), $\mathrm{Cox}(X)$ is a finitely generated k-algebra. Thus, $\mathbb{K} := \mathrm{Spec}(k[\mathrm{Cl}(X)])$ is a diagonalizable group and $\mathrm{Spec}(\mathrm{Cox}(X))$ is an affine variety. With respect to β from Eq. (4.3.5), $\mathrm{Spec}(\mathrm{Cox}(X))$ is an affine $\widehat{G}/\widehat{H}$-horospherical variety. Moreover, there exists a $\widehat{G}$-invariant open subset $\widehat{X} \subseteq \mathrm{Spec}(\mathrm{Cox}(X))$ such that there is a $\widehat{G}$-equivariant good quotient $\pi : \widehat{X} \to X$ for the action of $\mathbb{K}$; note that the short exact sequence in Theorem 3.6.3 gives a surjection $\widehat{N} \to \mathrm{Cl}(X)$, which induces an inclusion $\mathbb{K} \subseteq T_{\widehat{N}}$, so $\mathbb{K}$ acts on $\mathrm{Spec}(\mathrm{Cox}(X))$ by $\widehat{G}$-equivariant automorphisms. This map π restricts to the projection $\widehat{G}/\widehat{H} \to G/H$, so the associated map of coloured lattices is $\beta : \widehat{N} \to N$.

The following lemma shows that the coloured fan associated to the $\widehat{G}/\widehat{H}$-horospherical variety $\widehat{X}$ is precisely $\widehat{\Sigma}^c$, as constructed in Section 4.3.1 based on our choice of β.

Lemma 4.3.6 (Combinatorics of Cox construction)**.** *Let X be a G/H-horospherical variety with coloured fan Σ^c which satisfies (CF1), and let β be the map constructed in Eq. (4.3.5). Then we have the following:*

(1) The coloured fan associated to the $\widehat{G}/\widehat{H}$-horospherical variety $\mathrm{Spec}(\mathrm{Cox}(X))$ is generated by the single coloured cone $(\mathrm{Cone}(e_i : 1 \le i \le n), \mathcal{C})$ on $\widehat{N}$.

(2) The coloured fan associated to the $\widehat{G}/\widehat{H}$-horospherical variety $\widehat{X}$ is $\widehat{\Sigma}^c$ (constructed in Section 4.3.1 using Σ^c and β).

(3) $\mathbb{K} = K_\beta$.

Proof. Since $\mathrm{Spec}(\mathrm{Cox}(X))$ is an *affine* horospherical variety, it corresponds to a coloured cone whose colour set is equal to the universal colour set $\mathcal{C} = \{\alpha_1, \dots, \alpha_\ell\}$. The fact that the underlying cone is equal to $\mathrm{Cone}(e_i : 1 \le i \le n)$ is shown in [Gag19, Proposition 2.1]. This proves (1).

Let $(\Sigma')^c$ denote the coloured fan associated to $\widehat{X}$, which lives on $\widehat{N}$. To prove (2), we show that $(\Sigma')^c = \widehat{\Sigma}^c$. Since $\widehat{X}$ is an open horospherical subvariety of $\mathrm{Spec}(\mathrm{Cox}(X))$, it follows that $(\Sigma')^c$ is a sub-coloured fan of $(\mathrm{Cone}(e_i : 1 \le i \le n), \mathcal{C})$.

It is clear that $(\Sigma')^c$ is a sub-coloured fan of $\widehat{\Sigma}^c$. Indeed, by construction of $\widehat{\Sigma}^c$, it is the maximal sub-coloured fan of $(\mathrm{Cone}(e_i : 1 \leq i \leq n), \mathcal{C})$ with the property that β is compatible with it and Σ^c. Since β is compatible with $(\Sigma')^c$ and Σ^c, we must have $(\Sigma')^c \subseteq \widehat{\Sigma}^c$.

It remains to show that $\widehat{\Sigma}^c \subseteq (\Sigma')^c$. It suffices to show that, for each $\sigma^c \in \Sigma^c$, we have $\widehat{\sigma}^c \in (\Sigma')^c$. Fix $\sigma^c \in \Sigma^c$ and let X_{σ^c} be the corresponding simple horospherical variety, which is open in X. Let $\mathcal{O}$ be the unique closed G-orbit in X_{σ^c}.

We first show that $\pi^{-1}(X_{\sigma^c})$ is simple. By way of contradiction, assume that $\mathcal{O}_1, \mathcal{O}_2$ are distinct closed $\widehat{G}$-orbits in $\pi^{-1}(X_{\sigma^c})$. Then each $\mathcal{O}_i$ is $T_{\widehat{N}}$-invariant, so they are $\mathbb{K}$-invariant. Thus, $\pi(\mathcal{O}_1)$ and $\pi(\mathcal{O}_2)$ are closed, disjoint, and $\mathbb{K}$-invariant, so π being a good quotient for the action of $\mathbb{K}$ implies that $\pi(\mathcal{O}_1)$ and $\pi(\mathcal{O}_2)$ are disjoint. But $\pi(\mathcal{O}_1) = \pi(\mathcal{O}_2) = \mathcal{O}$ by $\widehat{G}$-equivariance, which is a contradiction. Thus, $\pi^{-1}(X_{\sigma^c})$ has a unique closed G-orbit, say $\widehat{\mathcal{O}}$.

Now let $\alpha \in \mathcal{C}$ be a colour, and let D and $\widehat{D}$ denote the corresponding colour divisors in X and $\widehat{X}$, respectively. We show that $D \supseteq \mathcal{O}$ if and only if $\widehat{D} \supseteq \widehat{\mathcal{O}}$. The reverse direction holds by compatibility. For the forward direction, suppose that $\mathcal{O} \subseteq D$. Then $\widehat{\mathcal{O}} \subseteq \pi^{-1}(\mathcal{O}) \subseteq \pi^{-1}(D) = \mathbb{K} \cdot \widehat{D}$. Note that $T_{\widehat{N}}$ acts by $\widehat{G}$-equivariant automorphisms of $\widehat{X}$, so by connectedness it must preserve the (finitely many) B^--invariant prime divisors. Thus $T_{\widehat{N}} \cdot \widehat{D} = \widehat{D}$, which implies $\mathbb{K} \cdot \widehat{D} = \widehat{D}$, which gives what we want.

To finish the proof of (2), we need to show that $\widehat{\sigma}^c \in (\Sigma')^c$. Note that the coloured fan for $\pi^{-1}(X_{\sigma^c})$ is a sub-coloured fan of $(\Sigma')^c$ (see Remark 4.2.11). Since $\widehat{X}$ has complementary codimension at least 2 inside $\mathrm{Spec}(\mathrm{Cox}(X))$ (see [Gag19, Section 2]), and using the result in the previous paragraph, we deduce that all rays of $\widehat{\sigma}^c$ must be contained in the coloured fan for $\pi^{-1}(X_{\sigma^c})$. Since we showed that $\pi^{-1}(X_{\sigma^c})$ is simple, its coloured fan is generated by a single coloured cone, so the previous paragraph and the previous sentence tell us that it must equal $\widehat{\sigma}^c$. Therefore $\widehat{\sigma}^c \in (\Sigma')^c$.

Finally, note that we have an exact sequence of abelian groups

$$0 \longrightarrow N^\vee \longrightarrow \widehat{N}^\vee \cong \bigoplus_{D \in \mathcal{D}(X)} \mathbb{Z}D \longrightarrow \mathrm{Cl}(X) \longrightarrow 0$$

as described in Theorem 3.6.3; exactness on the left is guaranteed by (CF1). Thus, (3) follows from an easily-adapted proof of [CLS11, Lemma 5.1.1(c)]. □

As a result of Lemma 4.3.6, if we are given Σ^c satisfying (CF1) and β from Eq. (4.3.5), then we have a good quotient $\widehat{X} = X_{\widehat{\Sigma}^c} \to X = X_{\Sigma^c}$ for the action of $\mathbb{K} = K_\beta$.

Remark 4.3.7 (Geometric quotient / free action)**.** By [Arz+15, Corollary 1.6.2.7] and Proposition 3.5.13, we can make the following two remarks:

(1) Σ^c being simplicial (equivalently, X^c_Σ being $\mathbb{Q}$-factorial) is equivalent to the good quotient $\widehat{X} \to X$ described above being a geometric quotient.

(2) Σ^c being regular (equivalently, X_{Σ^c} being factorial) is equivalent to the good quotient $\widehat{X} \to X$ described above being a geometric quotient wherein $\mathbb{K} = K_\beta$ acts freely on $\widehat{X}$.

To finish this subsection, we observe that the Cox construction above fits into the framework of coloured fantastacks. The map β from Eq. (4.3.5) satisfies (CF2) to (CF4), so we get a coloured fantastack $\mathscr{F}_{\Sigma^c,\beta} = [X_{\widehat{\Sigma}^c}/K_\beta] = [\widehat{X}/\mathbb{K}]$. When Σ^c is regular, the quotient $\widehat{X}/\mathbb{K}$ is free and geometric, so we have $[\widehat{X}/\mathbb{K}] = X$ in this case, i.e. $\mathscr{F}_{\Sigma^c,\beta} = X_{\Sigma^c}$.

4.3.3 Non-toric condition

Before finishing this section with examples of coloured fantastacks, we use the Cox ring from Section 4.3.2 to give a condition for a horospherical stack to be non-toric, extending Proposition 3.2.19. These conditions are useful for finding examples of horospherical stacks that cannot be described using the theory of toric stacks.

Proposition 4.3.8 (Non-toric condition)**.** *Let $\mathcal{X} = [X/K]$ be a $G/\overline{K}$-horospherical stack. If the associated flag variety G/P (with $P = N_G(\overline{K}) = N_G(H)$) is not a product of projective spaces, then $\mathcal{X}$ is not a toric stack.*

Proof. By Proposition 3.2.19, X is not a toric variety. Now we use this to show that $\mathcal{X}$ is not a toric stack. By way of contradiction, suppose otherwise. Then we can write $\mathcal{X} = [V/K_V]$ for some toric variety V with open torus T_V and some closed subgroup $K_V \subseteq T_V$. Consider the following pullback diagram:

$$\begin{array}{ccc} Y := V \times_{\mathcal{X}} X & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle K\text{-torsor}} \\ V & \xrightarrow[K_V\text{-torsor}]{} & \mathcal{X} \end{array}$$

Then $Y \to V$ is a K-torsor. In particular, this implies that Y is a normal, separated scheme of finite type over k.

Since K is a diagonalizable group, we can write

$$Y = \operatorname{Spec}_V \left(\bigoplus_{m \in \mathfrak{X}(K)} \mathscr{L}_m \right)$$

for some line bundles $\mathscr{L}_m$ on V indexed by m in the character group $\mathfrak{X}(K)$ of K; this is a well-known result, e.g. this is referenced in [GS15a, Proposition 7.1]. Since V is a toric variety, all of its line bundles admit a T_V-equivariant structure. Therefore, the sheaf of $\mathscr{O}_V$-algebras $\oplus_{m \in \mathfrak{X}(K)} \mathscr{L}_m$ admits a T_V-equivariant structure, i.e. we have a T_V-action on Y such that $Y \to V$ is T_V-equivariant. Furthermore, this commutes with the K-action on Y, so we have a $T_V \times K$-action on Y.

Now consider the following pullback diagram where the hook arrows are open embeddings:

$$\begin{array}{ccc} T_V \times_V Y & \hookrightarrow & Y \\ \downarrow & & \downarrow{\scriptstyle K\text{-torsor}} \\ T_V & \hookrightarrow & V \end{array}$$

Then $T_V \times_V Y \to T_V$ is a K-torsor. Since T_V is a torus and K is diagonalizable, this implies that $T_V \times_V Y$ is a diagonalizable group containing K and the map $T_V \times_V Y \to T_V$ is the quotient map by K. Furthermore, we have an embedding $T_V \times_V Y \hookrightarrow T_V \times K$, which gives a $T_V \times_V Y$-action on Y.

Set $T_0 := (T_V \times_V Y)^\circ$, which is a torus, and let Y_0 be the connected component of Y containing T_0. Note that Y_0 is a normal, separated, connected scheme, so it is irreducible, which means that Y_0 is a normal variety. The torus T_0 acts on Y, and since it is connected it must preserve Y_0, i.e. we have a T_0-action on Y_0. It follows that Y_0 is a toric variety with open torus T_0.

Lastly, since $Y \to X$ is a K_V-torsor, we see that $Y_0 \hookrightarrow Y \to X$ is a K_V'-torsor for some $K_V' \subseteq K_V$ (see [GS15a, Lemma 3.3]). Since K_V' is diagonalizable, we conclude that $Y_0/K_V' \cong X$ is a toric variety. This yields the contradiction, so we are done. □

Example 4.3.9. Consider $G = SL_n$. The condition $\#\mathcal{C} \leq 1$ is necessary for G/P to be a (product of) projective space(s); recall that $P = P_I$ and $\mathcal{C} = S \setminus I$. Therefore, if $[X/K]$ is a $G/\overline{K}$-horospherical stack such that $\#\mathcal{C} > 1$, then $[X/K]$ is not toric. For example, if $n > 2$, then we have $\#\mathcal{C} > 1$ when the associated parabolic P is B_n. ♢

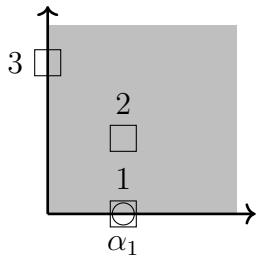
Remark 4.3.10 (Toric stacks must use toric presentations)**.** The proof of Proposition 4.3.8 shows that every presentation of a toric stack must use a toric variety. That is, if $\mathcal{X}$ is a

toric stack and $[X/K]$ is a presentation for $\mathcal{X}$ with K a diagonalizable group, then X must be a toric variety.

4.3.4 Examples

In the following examples, we draw a stacky coloured fan Σ^c on N with marked "square points" to indicate the map β: a square point marked with i is the image of $e_{i+\ell}$ under β, for $1 \leq i \leq n-\ell$; recall that $\ell = \#\mathcal{C}$ is the number of universal colours. Note that we automatically require $\beta(e_i) = u_{\alpha_i}$ for all $1 \leq i \leq \ell$ by (CF3). To ensure that (CF2) and (CF4) are satisfied, we need at least one "square point" on each non-coloured ray of Σ^c, and we need all "square points" to be in $|\Sigma^c|$.

Example 4.3.11. Consider $G = SL_2 \times \mathbb{G}_m$ and the horospherical subgroup $H = U_2 \times \{1\}$. The lattice N is isomorphic to $\mathbb{Z}^2$, the universal colour set is $\mathcal{C} = \{\alpha\}$, and we have $u_\alpha = e_1$. Let Σ^c be the coloured fan on N given in the diagram below.



In this case, Σ^c is generated by a single coloured cone $(\mathrm{Cone}(e_1, e_2), \varnothing)$. We have $X_{\Sigma^c} = \mathrm{Bl}_0 \mathbb{A}^2 \times \mathbb{A}^1$, where SL_2 acts on $\mathrm{Bl}_0 \mathbb{A}^2$ (lifting the matrix action of SL_2 on $\mathbb{A}^2$) and $\mathbb{G}_m$ acts by scaling $\mathbb{A}^1$. There are three "square points" in the diagram above, so

$$\beta = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} : \widehat{N} = \mathbb{Z}^4 \to N = \mathbb{Z}^2.$$

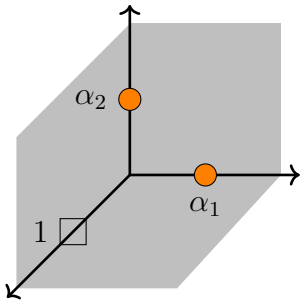
Note that (CF1) to (CF4) are satisfied.

The group $\widehat{G}$ is $SL_2 \times \mathbb{G}_m^3$, and we have $\widehat{G}/\widehat{H} \cong (SL_2/U_2) \times \mathbb{G}_m^3$. The coloured fan $\widehat{\Sigma}^c$ is generated by a single coloured cone $(\mathrm{Cone}(e_2, e_3, e_4), \varnothing)$ on $\widehat{N} = \mathbb{Z}^4$. Then $X_{\widehat{\Sigma}^c} = (SL_2/U_2) \times \mathbb{A}^3 = (\mathbb{A}^2 \setminus \{0\}) \times \mathbb{A}^3$.

By construction, $\overline{K_\beta}$ is a subgroup of $\widehat{G} = SL_2 \times \mathbb{G}_m^3$ such that $\widehat{G}/\overline{K_\beta} \cong (SL_2/U_2) \times \mathbb{G}_m$. The subgroup $K_\beta = \{(s, t, (st)^{-1}, \zeta_{st}) : s, t \in \mathbb{G}_m,\ \zeta_{st}^2 = st\}$ of $T_{\widehat{N}} = \mathbb{G}_m^4$ acts by scaling the $\mathbb{A}^2 \setminus \{0\}$ coordinates by s and multiplying the $\mathbb{A}^3$ coordinates by $(t, (st)^{-1}, \zeta_{st})$.

All this yields the coloured fantastack $\mathscr{F}_{\Sigma^c,\beta} = [((\mathbb{A}^2 \setminus \{0\}) \times \mathbb{A}^3)/K_\beta]$. ♢

Example 4.3.12. Consider $G = SL_3$ and the horospherical subgroup $H = U_3$. The lattice N is isomorphic to $\mathbb{Z}^2$, the universal colour set is $\mathcal{C} = \{\alpha_1, \alpha_2\}$, and we have $u_{\alpha_1} = e_1$ and $u_{\alpha_2} = e_2$. Let Σ^c be the coloured fan on N given in the diagram below.



This Σ^c has three maximal coloured cones: $(\mathrm{Cone}(e_1, e_2), \{\alpha_1, \alpha_2\})$, $(\mathrm{Cone}(e_2, -e_1 - e_2), \{\alpha_2\})$, and $(\mathrm{Cone}(e_1, -e_1 - e_2), \{\alpha_1\})$. Since there is one "square point" in the diagram above, we get

$$\beta = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} : \widehat{N} = \mathbb{Z}^3 \to N = \mathbb{Z}^2.$$

Note that (CF1) to (CF4) are satisfied.

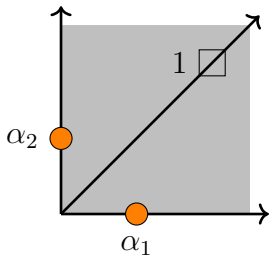
The group $\widehat{G}$ is $SL_3 \times \mathbb{G}_m$, and we have $\widehat{G}/\widehat{H} \cong (SL_3/U_3) \times \mathbb{G}_m$. The coloured fan $\widehat{\Sigma}^c$ has three maximal coloured cones on $\widehat{N} = \mathbb{Z}^3$: $(\mathrm{Cone}(e_1, e_2), \{\alpha_1, \alpha_2\})$, $(\mathrm{Cone}(e_2, e_3), \{\alpha_2\})$, and $(\mathrm{Cone}(e_1, e_3), \{\alpha_1\})$. The variety $X_{\widehat{\Sigma}^c}$ is $(Y \times \mathbb{A}^1) \setminus \{0\}$ where Y is $\{ux + vy + wz = 0\} \subseteq \mathbb{A}^3_{(u,v,w)} \times \mathbb{A}^3_{(x,y,z)}$.

By construction, $\overline{K_\beta}$ is a subgroup of $\widehat{G} = SL_3 \times \mathbb{G}_m$ such that $\widehat{G}/\overline{K_\beta} \cong SL_3/U_3$. The subgroup $K_\beta = \{(t, t, t) : t \in \mathbb{G}_m\}$ of $T_{\widehat{N}} = \mathbb{G}_m^3$ acts by scaling $Y \times \mathbb{A}^1$ by t.

All this yields the coloured fantastack $\mathscr{F}_{\Sigma^c,\beta} = [((Y \times \mathbb{A}^1) \setminus \{0\})/\mathbb{G}_m]$. This is exactly the coloured fantastack which captures the Cox construction (Section 4.3.2) for X_{Σ^c}. By Remark 4.3.7, we have

$$\mathscr{F}_{\Sigma^c,\beta} = X_{\Sigma^c} = \{ux + vy + wz = 0\} \subseteq \mathbb{P}^6_{(u:v:w:x:y:z:z')}.$$ ♢

Example 4.3.13. Consider $G = SL_3$ and the horospherical subgroup $H = U_3$. The lattice N is isomorphic to $\mathbb{Z}^2$, the universal colour set is $\mathcal{C} = \{\alpha_1, \alpha_2\}$, and we have $u_{\alpha_1} = e_1$ and $u_{\alpha_2} = e_2$. Let Σ^c be the coloured fan on N given in the diagram below.



This Σ^c has two maximal coloured cones: $(\mathrm{Cone}(e_1, e_1+e_2), \{\alpha_1\})$ and $(\mathrm{Cone}(e_2, e_1+e_2), \{\alpha_2\})$. There is one "square point" at $(2,2)$ in the diagram above, so

$$\beta = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix} : \widehat{N} = \mathbb{Z}^3 \to N = \mathbb{Z}^2.$$

Note that (CF1) to (CF4) are satisfied.

As in the previous example, let Y be $\{ux + vy + wz = 0\} \subseteq \mathbb{A}^3_{(u,v,w)} \times \mathbb{A}^3_{(x,y,z)}$. Then X_{Σ^c} is the blow-up of Y at the origin; the exceptional divisor E is the SL_3-invariant divisor of X_{Σ^c} corresponding to the non-coloured ray $(\mathrm{Cone}(e_1+e_2), \varnothing)$ in Σ^c.

The group $\widehat{G}$ is $SL_3 \times \mathbb{G}_m$, and $\widehat{G}/\widehat{H} \cong (SL_3/U_3) \times \mathbb{G}_m$. The coloured fan $\widehat{\Sigma}^c$ has two maximal coloured cones on $N = \mathbb{Z}^3$: $(\mathrm{Cone}(e_1, e_3), \{\alpha_1\})$ and $(\mathrm{Cone}(e_2, e_3), \{\alpha_2\})$. The variety $X_{\widehat{\Sigma}^c}$ is $(Y \setminus \{0\}) \times \mathbb{A}^1$.

By construction, $\overline{K_\beta}$ is a subgroup of $\widehat{G} = SL_3 \times \mathbb{G}_m$ such that $\widehat{G}/\overline{K_\beta} \cong SL_3/U_3$. The subgroup $K_\beta = \{(t, t, \zeta_t^{-1}) : t \in \mathbb{G}_m,\ \zeta_t^2 = t\}$ of $T_{\widehat{N}} = \mathbb{G}_m^3$ acts by scaling the coordinates of Y by t and scaling the coordinate of $\mathbb{A}^1$ by ζ_t^{-1}.

All this yields the coloured fantastack $\mathscr{F}_{\Sigma^c,\beta} = [((Y \setminus \{0\}) \times \mathbb{A}^1)/K_\beta]$. This is *almost* the coloured fantastack which captures the Cox construction (Section 4.3.2) for X_{Σ^c}, except that β sends $e_3 \mapsto (2,2)$ rather than $e_3 \mapsto (1,1)$. The effect here is that $\mathscr{F}_{\Sigma^c,\beta}$ adds a $\mathbb{Z}/2\mathbb{Z}$ stabilizer to the exceptional divisor E, i.e. $\mathscr{F}_{\Sigma^c,\beta}$ is the 2-root stack over E. ♢

Remark 4.3.14 (Root stacks)**.** The root stack construction in Example 4.3.13 works more generally. Let X_{Σ^c} be a G/H-horospherical variety satisfying (CF1), let $r \in \mathbb{Z}_{\geq 1}$, and let D be a G-invariant prime divisor in X_{Σ^c}. For D, we have a corresponding non-coloured ray, say $\rho_i^c = (\rho_i, \varnothing)$ for some $\ell+1 \leq i \leq n$ (see Section 4.3.2). Consider the coloured fantastack $\mathcal{F}_{\Sigma^c,\beta}$ where β is *almost* the map Eq. (4.3.5) except that $\beta(e_i) = ru_{\rho_i}$. Then $\mathcal{F}_{\Sigma^c,\beta}$ is the r-root stack over D.

Remark 4.3.15 (Toric vs. non-toric examples)**.** The horospherical stack in Example 4.3.11 is toric, so it can also be studied using the theory of toric stacks. However, the other examples in this subsection are not toric; see Proposition 4.3.8.

4.4 Toroidal horospherical stacks and decolouration

In this section, we study a certain class of *toroidal* horospherical stacks. These are meant to be horospherical stacks which are "close" to toric stacks (in a certain local sense; see Section 3.5.2).

4.4.1 Toroidal horospherical stacks

We first give a very simple combinatorial definition of these stacks and discuss some obvious properties.

Definition 4.4.1 (Colour set)**.** Given a horospherical G-stack $\mathcal{X} = [X/K]$, we define its *colour set* as $\mathcal{F}(\mathcal{X}) := \mathcal{F}(X)$. This definition is independent of the choice of horospherical G-variety X by Theorem 4.2.12.

Definition 4.4.2 (Toroidal horospherical stack)**.** We say that a horospherical stack $\mathcal{X}$ is *toroidal* if $\mathcal{F}(\mathcal{X}) = \varnothing$.

Proposition 4.4.3 (Characterizations of toroidal)**.** *Let $\mathcal{X} = [X/K]$ be a $G/\overline{K}$-horospherical stack, where X is a G/H-horospherical variety. Then the following are equivalent:*

(1) $\mathcal{X}$ is toroidal.

(2) X is toroidal.

(3) The G-equivariant rational map $\mathcal{X} \dashrightarrow G/P$, induced by the open embedding $G/\overline{K} \hookrightarrow \mathcal{X}$, is a horospherical morphism defined on all of $\mathcal{X}$.

Proof. Based on the definition of the colour set of a horospherical stack, it is clear that (1) and (2) are equivalent.

Based on the definition of coloured fan compatibility, the projection $G/H \to G/P$ extends to a horospherical morphism $X \to G/P$ if and only if $\mathcal{F}(X) = \varnothing$. It follows that $G/\overline{K} \to G/P$ extends to a horospherical morphism $\mathcal{X} \to G/P$ if and only if $\mathcal{F}(\mathcal{X}) = \varnothing$, so (1) and (3) are equivalent. □

Remark 4.4.4 (Alternative definition of toroidal horospherical stack)**.** Condition (3) in Proposition 4.4.3 is the definition of toroidal horospherical stacks given in [JLT19, Definition 3.4]. Therefore, the results in [JLT19] regarding toroidal horospherical stacks are applicable in our situation (e.g. [JLT19, Sections 4, 5]).

Remark 4.4.5 (Toric stacks are toroidal)**.** If $\mathcal{X}$ is a toric stack, then $\mathcal{F}(\mathcal{X}) = \varnothing$ since the universal colour set associated to a torus is empty. Therefore, toric stacks are toroidal.

Remark 4.4.6 (Open toroidal substack)**.** Consider a horospherical stack $\mathcal{X}_{\Sigma^c,\beta} = [X_{\Sigma^c}/K_\beta]$. As in Example 3.5.10, if $X' \subseteq X_{\Sigma^c}$ is the complement of all G-orbits of codimension at least 2, then $\mathcal{F}(X') = \varnothing$. Therefore, this construction gives an open toroidal horospherical substack $[X'/K_\beta] \hookrightarrow \mathcal{X}_{\Sigma^c,\beta}$.

4.4.2 Decolouration

Given a G/H-horospherical variety X, recall from Example 3.5.11 that we can form its decolouration $\widetilde{X}$ by "taking the colours out of X".

Definition 4.4.7 (Decolouration of a horospherical stack)**.** Given a $G/\overline{K}$-horospherical stack $\mathcal{X} = [X/K]$, we define its *decolouration* $\widetilde{\mathcal{X}}$ to be the $G/\overline{K}$-horospherical stack $[\widetilde{X}/K]$, where $\widetilde{X}$ is the decolouration of X described above.

Note that the *construction* of the decolouration $\widetilde{\mathcal{X}}$ of $\mathcal{X}$ depends on the presentation of the quotient stack $\mathcal{X}$. However, we show in Corollary 4.4.11 that the decolourations of isomorphic stacks are also isomorphic.

The decolouration of $\mathcal{X}_{\Sigma^c,\beta}$ is $\mathcal{X}_{\widetilde{\Sigma}^c,\beta}$. We have a map of stacky coloured fans

$$\begin{array}{ccc} \widetilde{\Sigma}^c & \longrightarrow & \Sigma^c \\ N & = & N \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta} \\ L & = & L \end{array}$$

which induces a *decolouration morphism* $\widetilde{\pi} : \mathcal{X}_{\widetilde{\Sigma}^c,\beta} \to \mathcal{X}_{\Sigma^c,\beta}$. Note that $\widetilde{\pi}$ is a surjective proper horospherical morphism.

If $(\Phi, \phi) : (\Sigma_1^c, \beta_1) \to (\Sigma_2^c, \beta_2)$ is a map of stacky coloured fans, then Φ is compatible with Σ_1^c and Σ_2^c. It is easy to see that Φ is also compatible with $\widetilde{\Sigma}_1^c$ and $\widetilde{\Sigma}_2^c$, so (Φ, ϕ) is also a map of stacky coloured fans $(\widetilde{\Sigma}_1^c, \beta_1) \to (\widetilde{\Sigma}_2^c, \beta_2)$. Therefore, (Φ, ϕ) yields the following

commutative diagram of horospherical morphisms:

$$\begin{array}{ccc} \mathcal{X}_{\widetilde{\Sigma}_1^c,\beta_1} & \longrightarrow & \mathcal{X}_{\widetilde{\Sigma}_2^c,\beta_2} \\ {\scriptstyle\widetilde{\pi}_1}\downarrow & & \downarrow{\scriptstyle\widetilde{\pi}_2} \\ \mathcal{X}_{\Sigma_1^c,\beta_1} & \xrightarrow[\pi_{\Phi,\phi}]{} & \mathcal{X}_{\Sigma_2^c,\beta_2} \end{array}$$

Remark 4.4.8 (Decolouration of toroidal stack)**.** If $\mathcal{X}$ is a toroidal horospherical stack, then the decolouration $\widetilde{\mathcal{X}}$ is just $\mathcal{X}$, i.e. decolouration does nothing to toroidal stacks. In particular, decolouration does nothing to toric stacks.

Remark 4.4.9 (Decolourations are toroidal)**.** For any horospherical stack $\mathcal{X}$, its decolouration $\widetilde{\mathcal{X}}$ is a toroidal horospherical stack.

Proposition 4.4.10 (Universal property of decolouration)**.** *Let $\mathcal{X}_{\Sigma^c,\beta}$ be a horospherical G-stack, and let $\mathcal{Y}$ be a toroidal horospherical G-stack. If we have a horospherical morphism $\mathcal{Y} \to \mathcal{X}_{\Sigma^c,\beta}$, then there exists a unique horospherical morphism $\mathcal{Y} \to \mathcal{X}_{\widetilde{\Sigma}^c,\beta}$ which makes the following diagram commute:*

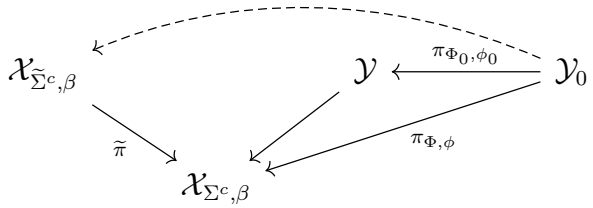
$$\begin{array}{ccc} \mathcal{X}_{\widetilde{\Sigma}^c,\beta} & \xleftarrow{\exists!} & \mathcal{Y} \\ & {\scriptstyle\widetilde{\pi}}\searrow \quad \swarrow & \\ & \mathcal{X}_{\Sigma^c,\beta} & \end{array}$$

Proof. We can write $\mathcal{Y} = \mathcal{X}_{(\Sigma')^c,\beta'}$ for some stacky coloured fan $((\Sigma')^c, \beta')$. By Theorem 4.2.8, we can create the following commutative diagram for some horospherical stack $\mathcal{Y}_0 = [Y_0/K_0]$ and some maps of stacky coloured fans (Φ_0, ϕ_0) and (Φ, ϕ), where π_{Φ_0,ϕ_0} is an isomorphism:

$$\begin{array}{ccc} \mathcal{Y}_0 & & \\ {\scriptstyle\pi_{\Phi_0,\phi_0}}\downarrow\wr & \searrow^{\pi_{\Phi,\phi}} & \\ \mathcal{Y} & \longrightarrow & \mathcal{X}_{\Sigma^c,\beta} \end{array}$$

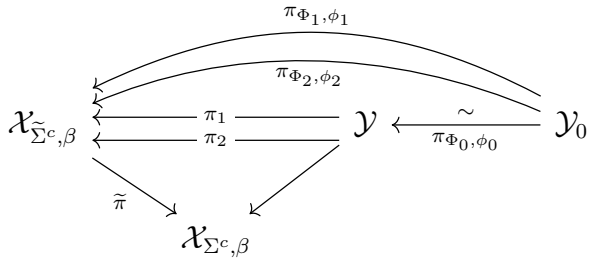
Here, Y_0 is a horospherical variety with coloured fan Σ_0^c on a coloured lattice N_0. Note that, since $\mathcal{Y}$ is toroidal, $(\Sigma')^c$ must have empty colour set, so by compatibility of Φ_0 we see that Σ_0^c must also have empty colour set.

Now we have the following commutative diagram of horospherical morphisms:



Since Φ is compatible with Σ_0^c and Σ^c, and since Σ_0^c has empty colour set, it follows that Φ is compatible with Σ_0^c and $\widetilde{\Sigma}^c$. Therefore, we get a dotted arrow in the above commutative diagram, i.e. a horospherical morphism $\mathcal{Y}_0 \to \mathcal{X}_{\widetilde{\Sigma}^c,\beta}$ induced by (Φ, ϕ). The composition of π_{Φ_0,ϕ_0}^{-1} followed by this map gives a horospherical morphism $\mathcal{Y} \to \mathcal{X}_{\widetilde{\Sigma}^c,\beta}$ making the diagram commute, which proves the existence part of the proposition.

Now we prove uniqueness, so suppose that there are two morphisms $\pi_1, \pi_2 : \mathcal{Y} \to \mathcal{X}_{\widetilde{\Sigma}^c,\beta}$ which make the original diagram commute. We can apply Theorem 4.2.8 three times to get a common tower $\mathcal{Y}_0$ in the following diagram:



where π_{Φ_0,ϕ_0} is an isomorphism and $\pi_i \circ \pi_{\Phi_0,\phi_0} = \pi_{\Phi_i,\phi_i}$ for $i = 1, 2$.

Since $\widetilde{\pi} \circ \pi_{\Phi_1,\phi_1} = \widetilde{\pi} \circ \pi_{\Phi_2,\phi_2}$, we get a commuting condition on the maps of stacky fans, which simply implies $(\Phi_1, \phi_1) = (\Phi_2, \phi_2)$ based on the decolouration map. Hence $\pi_{\Phi_1,\phi_1} = \pi_{\Phi_2,\phi_2}$, and it follows that $\pi_1 = \pi_2$. □

Corollary 4.4.11 (Uniqueness of decolouration)**.** *Let $\mathcal{X}_{\Sigma_1^c,\beta_1}$ and $\mathcal{X}_{\Sigma_2^c,\beta_2}$ be horospherical G-stacks. If we have a horospherical isomorphism $\mathcal{X}_{\Sigma_1^c,\beta_1} \to \mathcal{X}_{\Sigma_2^c,\beta_2}$, then the induced horospherical morphism $\mathcal{X}_{\widetilde{\Sigma}_1^c,\beta_1} \to \mathcal{X}_{\widetilde{\Sigma}_2^c,\beta_2}$ is also an isomorphism.*

4.5 Good moduli spaces

The notion of a good moduli space was first introduced by Alper [Alp13]. Given a stack $\mathcal{X}$, its good moduli space (when it exists) is a scheme (or more generally, an algebraic space) X which is meant to be a "scheme approximation" of $\mathcal{X}$; there is a morphism $\mathcal{X} \to X$ which transports much the algebro-geometric data of $\mathcal{X}$ to X, such as the structure sheaf, but it omits the "stacky structure". Affine locally, the good moduli space of $[\mathrm{Spec}(A)/K]$ is the GIT quotient $\mathrm{Spec}(A^K)$.

In this section, we give a combinatorial description of when a morphism of horospherical stacks is a good moduli space morphism, extending the criteria for toric good moduli space morphisms from [GS15a, Section 6]. With this description, we are able to combinatorially describe the good moduli space of a horospherical stack, when it exists (in Corollary 4.5.8).

4.5.1 Good moduli space morphisms

We first state the definition of a good moduli space ("good moduli space" is sometimes abbreviated "GMS"), as developed by Alper [Alp13, Definition 4.1]; see [Alp13, Section 4] for general properties of good moduli spaces.

Definition 4.5.1 (Good moduli space)**.** A quasi-compact and quasi-separated morphism of algebraic stacks $\pi : \mathcal{X} \to \mathcal{Y}$ is a *good moduli space morphism* if the following conditions hold:

(1) (π is *Stein*) The morphism $\mathscr{O}_{\mathcal{Y}} \to \pi_* \mathscr{O}_{\mathcal{X}}$ is an isomorphism.

(2) (π is *cohomologically affine*) The pushforward functor $\pi_* : \mathrm{QCoh}(\mathscr{O}_{\mathcal{X}}) \to \mathrm{QCoh}(\mathscr{O}_{\mathcal{Y}})$ on quasicoherent sheaves is exact.

Given an algebraic stack $\mathcal{X}$ and an algebraic space Y, we say that Y is the *good moduli space* of $\mathcal{X}$ if there exists a good moduli space morphism $\mathcal{X} \to Y$.[1]

Remark 4.5.2 (Cohomologically affine stack)**.** An algebraic stack $\mathcal{X}$ is *cohomologically affine* if the structure morphism $\mathcal{X} \to \mathrm{Spec}(k)$ is cohomologically affine. A horospherical stack $[X/K]$ is cohomologically affine if and only if X is an affine variety.

Indeed, we can see this by an argument made in [GS15a, Remark B.6]. If X is affine, then $[X/K] \to \mathrm{Spec}(k)$ is cohomologically affine by [Alp13, Proposition 3.13] since K affine.

[1] Note that good moduli spaces are unique (when they exist), so we can say "*the* good moduli space" of $\mathcal{X}$; see [Alp13, Section 6].

Conversely, suppose that $[X/K] \to \mathrm{Spec}(k)$ is cohomologically affine. Since $X \to [X/K]$ is a K-torsor, we see that X is affine over $[X/K]$, so $X \to \mathrm{Spec}(k)$ is cohomologically affine. Thus, X is affine by Serre's criteria (see [GD71, Chapter II, Corollary 5.2.2]).

Definition 4.5.3 (Unstable cone; cf. [GS15a, Defintion 6.2])**.** Let $(\Sigma^c, \beta : N \to L)$ be a stacky coloured fan. We say that a coloured cone $\tau^c \in \Sigma^c$ (or simply the underlying cone τ) is *unstable* if any of the following equivalent conditions hold:

(1) Every linear functional $L \to \mathbb{Z}$ which is nonnegative on $\beta_{\mathbb{R}}(\tau)$ vanishes on $\beta_{\mathbb{R}}(\tau)$. Said another way: for each $l \in L^\vee$, if $\beta^\vee(l) \in \tau^\vee$, then $\beta^\vee(l) \in \tau^\perp$.

(2) The relative interior of $\beta_{\mathbb{R}}(\tau)$ contains 0.

(3) $\tau \cap \ker(\beta)$ is not contained in any proper face of τ.

Notation 4.5.4. Given a cone τ on N (possibly not strongly convex), we let τ^{gp} denote the group $\mathbb{Z}(N \cap \tau)$. This is a saturated sublattice of N.

The following is the main theorem of this section, which we prove in Section 4.7; this is a natural generalization of [GS15a, Theorem 6.3].

Theorem 4.5.5 (GMS map criteria)**.** *Let (Σ_i^c, β_i) be stacky coloured fans for $i = 1, 2$, and let $(\Phi, \phi) : (\Sigma_1^c, \beta_1) \to (\Sigma_2^c, \beta_2)$ be a map of stacky coloured fans. Then the induced morphism $\pi_{\Phi,\phi} : \mathcal{X}_{\Sigma_1^c,\beta_1} \to \mathcal{X}_{\Sigma_2^c,\beta_2}$ is a good moduli space morphism if and only if the following conditions hold:*

(GMS1) *For each $\sigma_2^c = (\sigma_2, \mathcal{F}_2) \in \Sigma_2^c$, the preimage $\Phi^{-1}(\sigma_2^c)$ is generated by a single coloured cone $\sigma_1^c = (\sigma_1, \mathcal{F}_1) \in \Sigma_1^c$ such that $\Phi_{\mathbb{R}}(\sigma_1) = \sigma_2$ and $\mathcal{F}_1 = \mathcal{F}_2 \cup \mathcal{C}_\Phi$. In particular, the preimage of the trivial coloured cone is generated by a single coloured cone $\tau^c \in \Sigma_1^c$.*

(GMS2) *τ^c is unstable.*

(GMS3) *ϕ is surjective.*

(GMS4) $\ker(\phi) = (\beta_1)_{\mathbb{R}}(\tau)^{\mathrm{gp}}$.[1]

Remark 4.5.6. By [GS15a, Lemma 6.9] and [Alp13, Proposition 4.7(i)], it follows that good moduli space morphisms are stable under composition and base change, so we can use Proposition 4.2.13 for products of good moduli space morphisms of horospherical stacks.

[1] $(\beta_1)_{\mathbb{R}}(\tau)$ is a cone in $(L_1)_{\mathbb{R}}$. Also, $(\beta_1)_{\mathbb{R}}(\tau)^{\mathrm{gp}}$ is equal to the saturation of $\beta_1(\tau^{\mathrm{gp}})$ inside L_1.

4.5.2 Good moduli space of a horospherical stack

Now we use Theorem 4.5.5 to give a combinatorial characterization of when a horospherical stack has a horospherical variety as a good moduli space, and in this case we describe its coloured fan; this generalizes [GS15a, Corollary 6.5].

Recall that Example 3.3.24 describes the coloured lattice structure of the quotient of a coloured lattice N by a saturated sublattice with a distinguished colour set. If τ^c is any coloured cone on N (possibly not strongly convex), then the saturated sublattice $\tau^{\mathrm{gp}} \subseteq N$ is naturally equipped with the universal colour set $\mathcal{F}(\tau^c)$. Now we can form the coloured lattice N/τ^c: the underlying lattice is N/τ^{gp} and the universal colour set is $\mathcal{C} \setminus \mathcal{F}(\tau^c)$; compare with Eq. (3.4.28).

Notation 4.5.7 (GMS coloured fan)**.** Let $(\Sigma^c, \beta : N \to L)$ be a stacky coloured fan. Suppose that, among unstable coloured cones of Σ^c, there is a unique maximal one τ^c. Let $L_{\mathrm{gms}} := L/\beta_{\mathbb{R}}(\tau)^{\mathrm{gp}}$, which is a coloured lattice with universal colour set $\mathcal{C} \setminus \mathcal{F}(\tau^c)$. Let $\phi : L \to L_{\mathrm{gms}}$ be the quotient map and set $\Phi := \phi \circ \beta$. Let Σ^c_{gms} be the set of coloured cones $\sigma^c_{\mathrm{gms}} = (\sigma_{\mathrm{gms}}, \mathcal{F}_{\mathrm{gms}})$ on L_{gms} which satisfy the following conditions:

(1) $\Phi^{-1}(\sigma^c_{\mathrm{gms}})$ is generated by a single coloured cone.

(2) $\sigma_{\mathrm{gms}} = \Phi_{\mathbb{R}}(\Phi^{-1}(\sigma_{\mathrm{gms}}))$.

(3) $\mathcal{F}(\Phi^{-1}(\sigma^c_{\mathrm{gms}})) = \mathcal{F}_{\mathrm{gms}} \cup \mathcal{C}_\Phi$, or equivalently, $\mathcal{F}_{\mathrm{gms}} = \mathcal{F}(\Phi^{-1}(\sigma^c_{\mathrm{gms}})) \setminus \mathcal{C}_\Phi$.

Then Σ^c_{gms} is a coloured fan on L_{gms}.[1]

Corollary 4.5.8 (GMS of a horospherical stack)**.** *Let $(\Sigma^c, \beta : N \to L)$ be a stacky coloured fan. Then $\mathcal{X}_{\Sigma^c,\beta}$ has a variety as a good moduli space if and only if the following conditions are satisfied:*

(1) Among unstable coloured cones of Σ^c, there is a unique maximal one, denoted τ^c.

(2) Using Notation 4.5.7, Φ is compatible with Σ^c and Σ^c_{gms}.

Moreover, if these two conditions are satisfied, then the good moduli space of $\mathcal{X}_{\Sigma^c,\beta}$ is $X_{\Sigma^c_{\mathrm{gms}}}$, and the good moduli space morphism is the horospherical morphism $\mathcal{X}_{\Sigma^c,\beta} \to X_{\Sigma^c_{\mathrm{gms}}}$ induced by (Φ, ϕ) from Notation 4.5.7.

[1] See the footnote of [GS15a, Notation 4] for why Σ_{gms} is a fan. The fact that Σ^c_{gms} is a *coloured* fan follows immediately from this.

Proof. If conditions (1) and (2) are satisfied, then $X_{\Sigma^c_{\mathrm{gms}}} = \mathcal{X}_{\Sigma^c_{\mathrm{gms}},\mathrm{id}_{L_{\mathrm{gms}}}}$ is a horospherical variety and, by Theorem 4.5.5, (Φ, ϕ) induces a good moduli space morphism.

Now we prove the forward direction, so suppose that $\mathcal{X}_{\Sigma^c,\beta}$ has a good moduli space which is a variety, say X'. So we have a good moduli space morphism $\pi : \mathcal{X}_{\Sigma^c,\beta} \to X'$. By [Alp13, Theorem 4.16(viii)], X' is normal. By [Alp13, Theorem 6.6], this good moduli space morphism π induces a G-action on X' making π a G-equivariant morphism. We choose the base point of X' to be the image of the base point of $\mathcal{X}_{\Sigma^c,\beta}$. By G-equivariance, it follows that the stabilizer of the base point of X' is horospherical. Since π is surjective (see [Alp13, Theorem 4.16(i)]) and G-equivariant, the image of the open G-orbit in $\mathcal{X}_{\Sigma^c,\beta}$ is a dense G-orbit in X'. Since orbits are open in their closure, we deduce that X' has an open G-orbit (the orbit of the base point). Therefore, X' is a horospherical G-variety, so we can write $X' = X_{(\Sigma')^c}$ for some coloured fan $(\Sigma')^c$ on a coloured lattice N'.

By applying Theorem 4.2.8 to π, there exists maps of stacky coloured fans

$$\begin{array}{ccc} \Sigma_0^c & \longrightarrow & \Sigma^c \\ N_0 & \xrightarrow{\Phi_0} & N \\ {\scriptstyle\beta_0}\downarrow & & \downarrow{\scriptstyle\beta} \\ L_0 & \xrightarrow{\phi_0} & L \end{array} \qquad \begin{array}{ccc} \Sigma_0^c & \longrightarrow & (\Sigma')^c \\ N_0 & \xrightarrow{\Phi'} & N' \\ {\scriptstyle\beta_0}\downarrow & & \| \\ L_0 & \xrightarrow{\phi'} & N' \end{array}$$

such that π_{Φ_0,ϕ_0} is an isomorphism and $\pi = \pi_{\Phi',\phi'} \circ \pi^{-1}_{\Phi_0,\phi_0}$. By Theorem 4.2.12, we can see that (Σ^c, β) satisfies conditions (1) and (2) if and only if (Σ^c_0, β_0) does. Therefore, we may assume that π is induced by a map of stacky coloured fans $(\Psi, \psi) : (\Sigma^c, \beta) \to ((\Sigma')^c, \mathrm{id}_{N'})$.

By Theorem 4.5.5, properties (GMS1) and (GMS2) imply that $\tau^c := \Psi^{-1}(0^c)$ is the unique maximal unstable coloured cone in Σ^c, which establishes (1). By (GMS3) and (GMS4), we must have $\psi = \phi$ and $\Psi = \psi \circ \beta = \phi \circ \beta = \Phi$. Combining this with (GMS1) implies that (2) holds. □

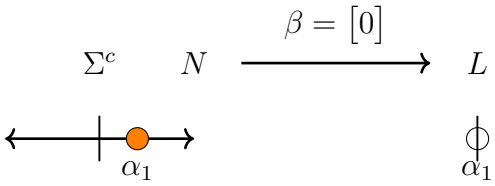
4.5.3 Examples

Our first example highlights a fundamental property of coloured fantastacks.

Example 4.5.9 (GMS of coloured fantastack)**.** Let $\mathscr{F}_{\Sigma^c,\beta}$ be a coloured fantastack. Using Theorem 4.5.5, one can check that the horospherical morphism $\mathscr{F}_{\Sigma^c,\beta} \to X_{\Sigma^c}$ is a good moduli space morphism. Therefore, X_{Σ^c} is the good moduli space of $\mathscr{F}_{\Sigma^c,\beta}$. ♢

In the following examples, we use Corollary 4.5.8 with Notation 4.5.7 to construct the good moduli space $X_{\Sigma^c_{\mathrm{gms}}}$ of a given $\mathcal{X}_{\Sigma^c,\beta}$.

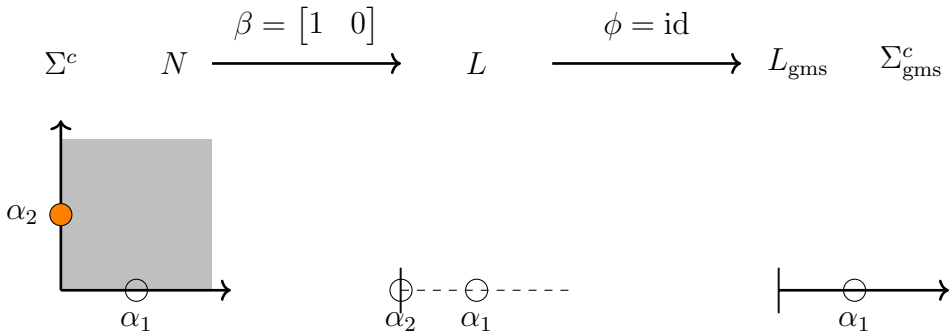
Example 4.5.10. Consider $G = SL_2$ and the horospherical subgroup $H = U_2$. The lattice N is isomorphic to $\mathbb{Z}$, the universal colour set is $\mathcal{C} = \{\alpha\}$, and we have $u_\alpha = e_1$. Let (Σ^c, β) be the stacky coloured fan given in the diagram below.



In this case, Σ^c has two maximal coloured cones: $(\mathrm{Cone}(e_1), \{\alpha\})$ and $(\mathrm{Cone}(-e_1), \varnothing)$. The variety X_{Σ^c} is $\mathbb{P}^2$, where SL_2 acts by matrix multiplication on a standard open $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$. The subgroup $K_\beta = \mathbb{G}_m$ of $T_N = \mathbb{G}_m$ is acting by scaling the coordinates of this open $\mathbb{A}^2$.

Observe that both maximal coloured cones in Σ^c are unstable, so there is not a unique maximal unstable coloured cone. Therefore, Corollary 4.5.8 tells us that $\mathcal{X}_{\Sigma^c,\beta}$ does not have a variety good moduli space. ♢

Example 4.5.11. Consider $G = SL_3$ and the horospherical subgroup $H = U_3$. The lattice N is isomorphic to $\mathbb{Z}^2$, the universal colour set is $\mathcal{C} = \{\alpha_1, \alpha_2\}$, and we have $u_{\alpha_1} = e_1$ and $u_{\alpha_2} = e_2$. Let (Σ^c, β) be the stacky coloured fan given in the diagram below.



In this case, Σ^c is generated by a single coloured cone $(\mathrm{Cone}(e_1, e_2), \{\alpha_2\})$. Using the affine local structure from Section 3.5.2, we can describe X_{Σ^c} as $SL_3 \times^{P_{\{\alpha_2\}}} \mathbb{A}^3$. An element $(1, t)$ of the subgroup $K_\beta = \{(1, t) : t \in \mathbb{G}_m\}$ of $T_N = \mathbb{G}_m^2$ acts on $SL_3 \times^{P_{\{\alpha_2\}}} \mathbb{A}^3$ by scaling the last two coordinates of this $\mathbb{A}^3$ by t.

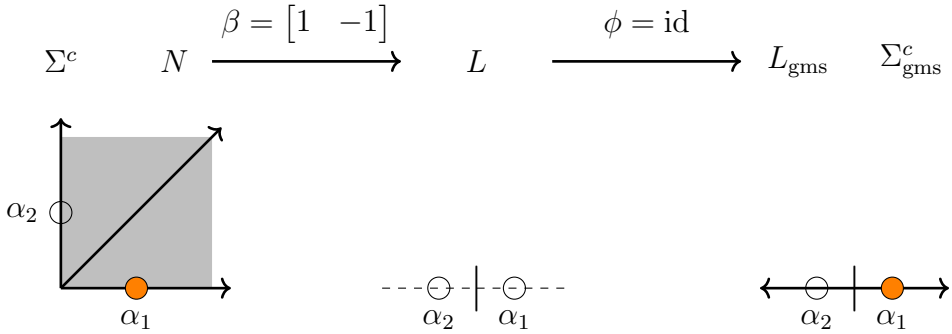
Note that $\tau^c = (\mathrm{Cone}(e_2), \{\alpha_2\})$ is the unique maximal unstable coloured cone in Σ^c. Since $\beta_{\mathbb{R}}(\tau)^{\mathrm{gp}} = 0$ and $\mathcal{F}(\tau^c) = \{\alpha_2\}$, we have $L_{\mathrm{gms}} = L = \mathbb{Z}$ with universal colour set $\{\alpha_1\}$. Then Σ^c_{gms} is generated by a single coloured cone $(\mathrm{Cone}(e_1), \varnothing)$ on L_{gms}; see the above diagram.

The conditions of Corollary 4.5.8 are satisfied, so $X_{\Sigma^c_{\mathrm{gms}}}$ is the good moduli space of $\mathcal{X}_{\Sigma^c,\beta} = [X_{\Sigma^c}/K_\beta]$. Note that $X_{\Sigma^c_{\mathrm{gms}}}$ is a $SL_3/\overline{K_\beta}$-horospherical variety where

$$\overline{K_\beta} = \left\{ \begin{bmatrix} 1 & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix} \right\} \subseteq SL_3.$$

It is easy to see that $\overline{K_\beta}$ is the stabilizer of the base point $(1,0,0)$ in $\mathbb{A}^3$, where SL_3 acts by usual matrix multiplication. It follows that $X_{\Sigma^c_{\mathrm{gms}}}$ is the blow-up of $\mathbb{A}^3$ at the origin. ♢

Example 4.5.12. Consider $G = SL_3$ and the horospherical subgroup $H = U_3$. The lattice N is isomorphic to $\mathbb{Z}^2$, the universal colour set is $\mathcal{C} = \{\alpha_1, \alpha_2\}$, and we have $u_{\alpha_1} = e_1$ and $u_{\alpha_2} = e_2$. Let (Σ^c, β) be the stacky coloured fan given in the diagram below.



This Σ^c has two maximal coloured cones: $(\mathrm{Cone}(e_1, e_1+e_2), \{\alpha_1\})$ and $(\mathrm{Cone}(e_2, e_1+e_2), \varnothing)$. Using the affine local structure from Section 3.5.2, we can describe X_{Σ^c} as $SL_3 \times^{P_{\{\alpha_1\}}} \mathrm{Bl}_0 \mathbb{A}^3$. The subgroup $K_\beta = \{(t,t) : t \in \mathbb{G}_m\} = \mathbb{G}_m$ of $T_N = \mathbb{G}_m^2$ acts by lifting the diagonal action of $\mathbb{G}_m$ on $\mathbb{A}^3$ to $\mathrm{Bl}_0 \mathbb{A}^3$.

Note that $\tau^c = (\mathrm{Cone}(e_1+e_2), \varnothing)$ is the unique maximal unstable coloured cone in Σ^c. Since $\beta_{\mathbb{R}}(\tau)^{\mathrm{gp}} = 0$ and $\mathcal{F}(\tau^c) = \varnothing$, we have $L_{\mathrm{gms}} = L = \mathbb{Z}$ with universal colour set $\{\alpha_1, \alpha_2\}$. Then Σ^c_{gms} has two maximal coloured cones: $(\mathrm{Cone}(e_1), \{\alpha_1\})$ and $(\mathrm{Cone}(-e_1), \varnothing)$ on L_{gms}; see the above diagram.

The conditions of Corollary 4.5.8 are satisfied, so $X_{\Sigma^c_{\mathrm{gms}}}$ is the good moduli space of $\mathcal{X}_{\Sigma^c,\beta} = [X_{\Sigma^c}/\mathbb{G}_m]$. Note that $X_{\Sigma^c_{\mathrm{gms}}}$ is a $SL_3/\overline{K_\beta}$-horospherical variety where

$$\overline{K_\beta} = \left\{ \begin{bmatrix} * & * & * \\ 0 & 1 & * \\ 0 & 0 & * \end{bmatrix} \right\} \subseteq SL_3.$$

Using the Cox construction for $X_{\Sigma^c_{\mathrm{gms}}}$ (see Section 4.3.2 and Remark 4.3.7) and the affine local structure from Section 3.5.2, we can describe this good moduli space as $SL_3 \times^{P_{\{\alpha_1\}}} \mathbb{P}^2$. $\diamondsuit$

Remark 4.5.13 (Toric vs. non-toric examples)**.** The horospherical stack in Example 4.5.10 is toric, so it can also be studied using the theory of toric stacks. However, the other examples in this subsection are not toric; see Proposition 4.3.8.

4.6 Proof of Theorem 4.2.12

In this section, we give the proof of Theorem 4.2.12. The idea is to first consider the case where the stacks are cohomologically affine (i.e. the cover varieties are affine, by Remark 4.5.2); for this case we use some ideas from [GS15a, Appendix B]. Then we reduce the general case to this case using the affine local structure from Section 3.5.2; recall that this allows us to write a simple horospherical variety X as $G \times^Q Z$ where Q is parabolic and Z is an affine horospherical $\mathcal{L}$-variety, where $\mathcal{L}$ is the standard Levi subgroup of Q.

Throughout this section, we assume that $(\Phi, \phi) : (\Sigma^c_1, \beta_1) \to (\Sigma^c_2, \beta_2)$ is a map of stacky coloured fans. So we have an induced horospherical morphism $\pi_{\Phi,\phi} : \mathcal{X}_{\Sigma^c_1,\beta_1} \to \mathcal{X}_{\Sigma^c_2,\beta_2}$.

Lemma 4.6.1. *Suppose that $\Sigma^c_i = \sigma^c_i$ is generated by a single coloured cone and that $X_{\sigma^c_i}$ is affine for $i = 1, 2$. Suppose that the following conditions hold:*

(1) ϕ is an isomorphism of coloured lattices. Thus $\mathcal{C}_\Phi = \mathcal{C}_\phi = \varnothing$ and $\mathcal{C}_1 = \mathcal{C}_2$.

(2) $\Phi_{\mathbb{R}}(\sigma_1) = \sigma_2$ and Φ induces an isomorphism of monoids $\sigma_1 \cap N_1 \to \sigma_2 \cap N_2$.

Then $\pi_{\Phi,\phi} : \mathcal{X}_{\sigma^c_1,\beta_1} \to \mathcal{X}_{\sigma^c_2,\beta_2}$ is an isomorphism.

Proof. Let $(N_1)_{\sigma_1}$ be the sublattice of N_1 generated by $\sigma_1 \cap N_1$, and let $(N_1)'_{\sigma_1}$ be a direct complement inside N_1. By [Tim11, Theorem 28.2], there exists an affine horospherical $\mathcal{L}_0$-variety $Z_{\sigma^c_1}$ with a fixed point under $\mathcal{L}_0$ such that $X_{\sigma^c_1} = Z_{\sigma^c_1} \times T'_1$; here $\mathcal{L}_0$ is between $[G, G]$ (the commutator) and G, and $T'_1 = G/\mathcal{L}_0$ is a torus whose associated lattice of one-parameter subgroups is $(N_1)'_{\sigma_1}$. Thus $\mathcal{X}_{\sigma^c_1,\beta_1} = [(Z_{\sigma^c_1} \times T'_1)/K_{\beta_1}]$.

By condition (2), the sublattice of N_2 generated by $\sigma_2 \cap N_2$ is isomorphic to $(N_1)_{\sigma_1}$, and this isomorphism identifies σ_1 with σ_2. Let $(N_2)'_{\sigma_1}$ be a direct complement of this copy of $(N_1)_{\sigma_1}$ inside N_2, and let T'_2 be the torus corresponding to $(N_2)'_{\sigma_1}$, i.e. $(N_2)'_{\sigma_1}$ is the one-parameter subgroup lattice of T'_2. By Proposition 3.5.2 and (1), Φ induces an isomorphism between $Z_{\sigma^c_1}$ and $Z_{\sigma^c_2}$; here $Z_{\sigma^c_2}$ is similarly constructed to $Z_{\sigma^c_1}$. Hence $X_{\sigma^c_2} = Z_{\sigma^c_1} \times T'_2$, which implies $\mathcal{X}_{\sigma^c_2,\beta_2} = [(Z_{\sigma^c_1} \times T'_2)/K_{\beta_2}]$.

Notice that $T_{N_i} = T_{(N_i)_{\sigma_1}} \times T_i'$ for $i = 1, 2$. By [GS15a, Lemma A.1] and (1), the following diagram has exact rows:

$$\begin{array}{ccccccccc} \{1\} & \longrightarrow & K_\Phi & \longrightarrow & K_{\beta_1} & \longrightarrow & K_{\beta_2} & \longrightarrow & \{1\} \\ & & \| & & \downarrow & & \downarrow & & \\ \{1\} & \longrightarrow & K_\Phi \times \{1\} & \longrightarrow & T_1' \times T_{(N_1)_{\sigma_1}} & \longrightarrow & T_2' \times T_{(N_1)_{\sigma_1}} & \longrightarrow & \{1\} \end{array}$$

Therefore, $\pi_{\Phi,\phi}$ is the isomorphism

$$\mathcal{X}_{\sigma_1^c,\beta_1} = [(Z_{\sigma_1^c} \times T_1')/K_{\beta_1}] \xrightarrow{\sim} \left[[(Z_{\sigma_1^c} \times T_1')/K_\Phi]/K_{\beta_2}\right] = [(Z_{\sigma_1^c} \times T_2')/K_{\beta_2}] = \mathcal{X}_{\sigma_2^c,\beta_2}. \quad \square$$

Lemma 4.6.2. *Suppose that $\Sigma_i^c = \sigma_i^c$ are generated by single coloured cones for $i = 1, 2$. Let $X_i := X_{\sigma_i^c}$, let $\mathcal{O}_i \subseteq X_i$ be the unique closed G-orbit, and let $\pi : X_1 \to X_2$ be the horospherical morphism induced by Φ. For each colour divisor $D_2 \subseteq X_2$, if $D_2 \not\supseteq \mathcal{O}_2$, then $\pi^{-1}(D_2)$ is a colour divisor of X_1.*

Proof. We know that $D_2 := \overline{\pi(D_1)}$ for some colour divisor $D_1 \subseteq X_1$. Then $D_1 \subseteq \pi^{-1}(D_2)$, and thus D_1 is the unique colour divisor in $\pi^{-1}(D_2)$. Indeed, the preimage of any colour divisor $D_2 \subseteq X_2$ must contain some colour divisor of X_1, and since the number of colour divisors of X_1 which are not mapped dominantly under π is equal to the number of colour divisors of X_2, the claim follows from the pigeonhole principle.

Since D_2 is B^--invariant and Cartier (see Remark 3.6.7), we know that $\pi^{-1}(D_2)$ is a B^--invariant effective Cartier divisor. The irreducible components of $\pi^{-1}(D_2)$ must be B^--invariant since B^- is connected, so every irreducible component aside from D_1 must be G-invariant. In particular, every irreducible component aside from D_1 must contain $\mathcal{O}_1$. If D_1 is not the only irreducible component, then $\mathcal{O}_1 \subseteq \pi^{-1}(D_2)$ and thus $\mathcal{O}_2 = \pi(\mathcal{O}_1) \subseteq D_2$, which is a contradiction. Therefore $\pi^{-1}(D_2) = D_1$. $\square$

Lemma 4.6.3. *Suppose that $\Sigma_i^c = \sigma_i^c$ are generated by single coloured cones for $i = 1, 2$. Let $X_i := X_{\sigma_i^c}$, let $\mathcal{O}_i \subseteq X_i$ be the unique closed G-orbit, and let $\pi : X_1 \to X_2$ denote the horospherical morphism induced by Φ. If $\mathcal{F}_1 = \mathcal{F}_2 \cup \mathcal{C}_\Phi$, then $\pi^{-1}(Z_2) = Z_1$ using the notation of Section 3.5.2.*

Proof. Since $\mathcal{C}_1 = \mathcal{C}_2 \sqcup \mathcal{C}_\Phi$ and $\mathcal{F}_1 = \mathcal{F}_2 \sqcup \mathcal{C}_\Phi$, it follows that $\mathcal{C}_1 \setminus \mathcal{F}_1 = \mathcal{C}_2 \setminus \mathcal{F}_2$. Moreover, Lemma 4.6.2 implies that $\pi^{-1}(D_\alpha) = D_\alpha$ for $\alpha \in \mathcal{C}_2 \setminus \mathcal{F}_2$, where D_α is regarded as the colour divisor corresponding to α in both X_1 and X_2. It follows that $Q_1 = Q_2$ in the notation of Section 3.5.2, which implies $\pi^{-1}(Z_2) = Z_1$. $\square$

Proposition 4.6.4. *Suppose that $\Sigma_i^c = \sigma_i^c = (\sigma_i, \mathcal{F}_i)$ are generated by single coloured cone for $i = 1, 2$. Suppose that the following conditions hold:*

(1) ϕ is an isomorphism of coloured lattices. Thus $\mathcal{C}_\Phi = \mathcal{C}_\phi = \varnothing$ and $\mathcal{C}_1 = \mathcal{C}_2$.

(2) $\mathcal{F}_1 = \mathcal{F}_2$.

(3) $\Phi_{\mathbb{R}}(\sigma_1) = \sigma_2$ and Φ induces an isomorphism of monoids $\sigma_1 \cap N_1 \to \sigma_2 \cap N_2$.

Then $\pi_{\Phi,\phi} : \mathcal{X}_{\sigma_1^c,\beta_1} \to \mathcal{X}_{\sigma_2^c,\beta_2}$ is an isomorphism.

Proof. Let Z_i be the closed affine horospherical subvariety of $X_{\sigma_i^c}$ for $i = 1, 2$ in the notation of the affine local structure from Section 3.5.2. By condition (2) and Lemma 4.6.3, we can base change $\pi_{\Phi,\phi}$ to $[Z_1/K_{\beta_1}] \to [Z_2/K_{\beta_2}]$. Now we are in the situation of Lemma 4.6.1 from conditions (1) and (3), so we know that $[Z_1/K_{\beta_1}] \to [Z_2/K_{\beta_2}]$ is an isomorphism.

Since $X_{\sigma_i^c} \cong G \times^Q Z_i$ (where $Q = Q_1 = Q_2$ is in the notation of the affine local structure from Section 3.5.2), it follows that the isomorphism $[Z_1/K_{\beta_1}] \to [Z_2/K_{\beta_2}]$ extends G-equivariantly to an isomorphism $[X_{\sigma_1^c}/K_{\beta_1}] \to [X_{\sigma_2^c}/K_{\beta_2}]$, which is what we wanted to show. □

Proof of Theorem 4.2.12. For the reverse direction, suppose that conditions (Iso1) to (Iso3) hold. To show that $\pi_{\Phi,\phi} : \mathcal{X}_{\Sigma_1^c,\beta_1} \to \mathcal{X}_{\Sigma_2^c,\beta_2}$ is an isomorphism, it suffices to look locally on the base, so we may assume that $\Sigma_2^c = \sigma_2^c$ is generated by a single coloured cone. Then the result follows from Proposition 4.6.4.

Now we prove the forward direction, so suppose that $\pi_{\Phi,\phi} : \mathcal{X}_{\Sigma_1^c,\beta_1} \to \mathcal{X}_{\Sigma_2^c,\beta_2}$ is an isomorphism. Fix $\sigma_2^c \in \Sigma_2^c$. We must have that $\Phi^{-1}(\sigma_2^c)$ is generated by a single coloured cone, say $\sigma_1^c \in \Sigma_1^c$. Indeed, $\pi_{\Phi,\phi}$ restricts to an isomorphism $\mathcal{X}_{\Phi^{-1}(\sigma_2^c),\beta_1} \to \mathcal{X}_{\sigma_2^c,\beta_2}$, so because the codomain has a unique closed G-orbit, the domain must as well, which means that $\Phi^{-1}(\sigma_2^c)$ is generated by a single coloured cone.

To finish the proof, we may assume that $\Sigma_i^c = \sigma_i^c$ for $i = 1, 2$, and we show that conditions (1)-(3) in Proposition 4.6.4 hold.

The isomorphism $\pi_{\Phi,\phi}$ restricts to an isomorphism of the open G-orbits $G/\overline{K_{\beta_1}} \to G/\overline{K_{\beta_2}}$. Thus ϕ is an isomorphism of coloured lattices, so condition (1) holds. In particular, we have $\mathcal{C}_1 = \mathcal{C}_2 =: \mathcal{C}$.

By compatibility of Φ, we must have $\mathcal{F}_1 \subseteq \mathcal{F}_2$. So to show that condition (2) holds, it suffices to show the other inclusion via contraposition, i.e. we show that $\mathcal{C} \setminus \mathcal{F}_1 \subseteq \mathcal{C} \setminus \mathcal{F}_2$. Let $\mathcal{O}_i$ be the unique closed G-orbit in $X_{\sigma_i^c}$ for $i = 1, 2$. Let $\pi : X_{\sigma_1^c} \to X_{\sigma_2^c}$ denote the horospherical morphism induced by Φ. Take a colour divisor $D_1 \subseteq X_{\sigma_1^c}$ with $D_1 \not\supseteq \mathcal{O}_1$,

i.e. this corresponds to an element of $\mathcal{C} \setminus \mathcal{F}_1$. The corresponding colour divisor in $X_{\sigma_2^c}$ is $D_2 := \overline{\pi(D_1)}$. Since $\pi_{\Phi,\phi}$ is an isomorphism and $\pi(\mathcal{O}_1) = \mathcal{O}_2$, it follows that $D_2 \not\supseteq \mathcal{O}_2$. Hence D_2 corresponds to an element of $\mathcal{C} \setminus \mathcal{F}_2$, which gives $\mathcal{C} \setminus \mathcal{F}_1 \subseteq \mathcal{C} \setminus \mathcal{F}_2$.

It remains to show that condition (3) holds. Let $\widetilde{\sigma}_i^c := (\sigma_i, \varnothing)$ be the decolourization of σ_i^c for $i = 1, 2$. From Section 4.4.2, we have the following commutative diagram of horospherical morphisms:

$$\begin{array}{ccc} \mathcal{X}_{\widetilde{\sigma}_1^c,\beta_1} & \longrightarrow & \mathcal{X}_{\widetilde{\sigma}_2^c,\beta_2} \\ \downarrow{\scriptstyle \widetilde{\pi}_1} & & \downarrow{\scriptstyle \widetilde{\pi}_2} \\ \mathcal{X}_{\sigma_1^c,\beta_1} & \xrightarrow[\pi_{\Phi,\phi}]{} & \mathcal{X}_{\sigma_2^c,\beta_2} \end{array}$$

Since $\pi_{\Phi,\phi}$ is an isomorphism, Corollary 4.4.11 implies that $\mathcal{X}_{\widetilde{\sigma}_1^c,\beta_1} \to \mathcal{X}_{\widetilde{\sigma}_2^c,\beta_2}$ is an isomorphism. The colour sets of these decolouration stacks are empty, so Lemma 4.6.3 tells us that we can base change to an isomorphism $[\widetilde{Z}_1/K_{\beta_1}] \to [\widetilde{Z}_2/K_{\beta_2}]$, where $\widetilde{Z}_i$ is the closed affine horospherical subvariety of $X_{\widetilde{\sigma}_i^c}$ for $i = 1, 2$ in the notation of the affine local structure from Section 3.5.2. In this case, the $\widetilde{Z}_i$ are affine toric varieties with corresponding cones σ_i, so we have an isomorphism of toric stacks induced by the map of stacky fans (Φ, ϕ) in the sense of [GS15a, Section 3]. Therefore, [GS15a, Theorem B.3] implies that Φ induces an isomorphism of monoids $\sigma_1 \cap N_1 \to \sigma_2 \cap N_2$, which shows that (3) holds. □

4.7 Proof of Theorem 4.5.5

In this section, we give the proof of Theorem 4.5.5. The idea is to first consider the case where the stacks are cohomologically affine, and then reduce the general case to this case. For the cohomologically affine case, we follow the proof strategy developed in [GS15a, Section 6.2].

4.7.1 Cohomologically affine case

Throughout this subsection, we use the following setup. We have a map of stacky coloured fans $(\Phi, \phi) : (\sigma_1^c, \beta_1) \to (\sigma_2^c, \beta_2)$ where $\sigma_i^c = (\sigma_i, \mathcal{F}_i)$ are coloured cones on N_i for $i = 1, 2$. Moreover, we assume that $\Phi_{\mathbb{R}}(\sigma_1) = \sigma_2$, and that $\mathcal{F}_i = \mathcal{C}_i$ for each $i = 1, 2$. The latter assumption implies that the associated horospherical varieties $X_{\sigma_i^c}$ are affine, which is equivalent to saying that the horospherical stacks $\mathcal{X}_{\sigma_i^c,\beta_i}$ are cohomologically affine (see Remark 4.5.2).

Lemma 4.7.1. *If β_2 is an isomorphism (of lattices), then the induced map*

$$\pi_{\Phi,\phi} = \mathcal{X}_{\sigma_1^c,\beta_1} \to \mathcal{X}_{\sigma_2^c,\beta_2} = X_{\sigma_2^c}$$

is a good moduli space morphism if and only if, for every $l_1 \in L_1^\vee$ such that $\beta_1^\vee(l_1) \in \sigma_1^\vee$, there exists a unique $l_2 \in L_2^\vee$ such that $\beta_2^\vee(l_2) \in \sigma_2^\vee$ and $\phi^\vee(l_2) = l_1$.[1]

Proof. Since $X_{\sigma_i^c}$ are affine for both $i = 1, 2$, we can use Proposition 3.5.2 to write

$$k[X_{\sigma_i^c}] = \bigoplus_{m \in \sigma_i^\vee \cap N_i^\vee} k[G]_{-m}^{(B^-)}$$

where $k[G]_{-m}^{(B^-)}$ is the $-m$-th graded piece of the B^--eigenfunctions $k[G]^{(B^-)}$ of the coordinate ring $k[G]$. Since K_{β_1} is linearly reductive, the induced map $\pi_{\Phi,\phi}$ is a good moduli space morphism if and only if the map of coordinate rings

$$k[X_{\sigma_2^c}] = \bigoplus_{m \in \sigma_2^\vee \cap N_2^\vee} k[G]_{-m}^{(B^-)} \to k[X_{\sigma_1^c}] = \bigoplus_{m \in \sigma_1^\vee \cap N_1^\vee} k[G]_{-m}^{(B^-)}$$

is the inclusion of ring invariants $k[X_{\sigma_1^c}]^{K_{\beta_1}}$. Notice that this map of coordinate rings corresponds to the map of monoids $\Phi^\vee|_{\sigma_2^\vee \cap N_2^\vee} : \sigma_2^\vee \cap N_2^\vee \to \sigma_1^\vee \cap N_1^\vee$.

Let $\theta : N_1^\vee \to \mathfrak{X}(K_{\beta_1})$ be the map of character groups corresponding to the action of K_{β_1} on T_{N_1}. Then the previous paragraph tells us that $\pi_{\Phi,\phi}$ is a good moduli space morphism if and only if $\sigma_1^\vee \cap \ker(\theta) = \Phi^\vee(\sigma_2^\vee \cap N_2^\vee)$ and $\Phi^\vee|_{\sigma_2^\vee \cap N_2^\vee}$ is injective. Consider the following commutative diagram of abelian groups:

$$\begin{array}{ccccc} \mathfrak{X}(K_{\beta_1}) & \xleftarrow{\theta} & N_1^\vee & \xleftarrow{\Phi^\vee} & N_2^\vee \\ & \nwarrow^{0} & \uparrow{\scriptstyle \beta_1^\vee} & & \wr \uparrow{\scriptstyle \beta_2^\vee} \\ & & L_1^\vee & \xleftarrow[\phi^\vee]{} & L_2^\vee \end{array}$$

(note that $\beta_1^\vee$ is injective because β_1 has finite cokernel; and $\beta_2^\vee$ is an isomorphism because β_2 is).

For the forward direction, suppose that $\pi_{\Phi,\phi}$ is a good moduli space morphism. Given $l_1 \in L_1^\vee$ such that $\beta_1^\vee(l_1) \in \sigma_1^\vee$, we have that $\beta_1^\vee(l_1) \in \sigma_1^\vee \cap \ker(\theta) = \Phi^\vee(\sigma_2^\vee \cap N_2^\vee)$. Since

[1] This condition is saying that: every linear functional $L_1 \to \mathbb{Z}$ which is nonnegative on $(\beta_1)_\mathbb{R}(\sigma_1)$ is the pullback under ϕ of a unique linear functional $L_2 \to \mathbb{Z}$ which is nonnegative on $(\beta_2)_\mathbb{R}(\sigma_2)$.

$\Phi^\vee|_{\sigma_2^\vee \cap N_2^\vee}$ is injective and $\beta_2^\vee$ is an isomorphism, there exists a unique $l_2 \in L_2^\vee$ such that $\beta_2^\vee(l_2) \in \sigma_2^\vee$ and $\Phi^\vee(\beta_2^\vee(l_2)) = \beta_1^\vee(l_1)$. Thus $\beta_1^\vee(\phi^\vee(l_2)) = \beta_1^\vee(l_1)$, which gives the desired result since $\beta_1^\vee$ is injective.

Now we prove the reverse direction. The inclusion $\sigma_1^\vee \cap \ker(\theta) \supseteq \Phi^\vee(\sigma_2^\vee \cap N_2^\vee)$ is easy because the image of $\Phi^\vee$ is in $\ker(\theta)$, and since $\Phi_{\mathbb{R}}(\sigma_1) = \sigma_2$ we see that $\Phi^\vee$ sends elements of $\sigma_2^\vee$ to elements of $\sigma_1^\vee$. Now we show that $\sigma_1^\vee \cap \ker(\theta) \subseteq \Phi^\vee(\sigma_2^\vee \cap N_2^\vee)$. Since $\ker(\theta) = \mathrm{im}(\beta_1^\vee)$, any element of $\sigma_1^\vee \cap \ker(\theta)$ is of the form $\beta_1^\vee(l_1)$ for some $l_1 \in L_1^\vee$; in fact, l_1 is unique since $\beta_1^\vee$ is injective. By hypothesis, we can write $l_1 = \phi^\vee(l_2)$ for a unique $l_2 \in L_2^\vee$ such that $\beta_2^\vee(l_2) \in \sigma_2^\vee$. So any element of $\sigma_1^\vee \cap \ker(\theta)$ is of the form $\beta_1^\vee(\phi^\vee(l_2)) = \Phi^\vee(\beta_2^\vee(l_2))$ with $\beta_2^\vee(l_2) \in \sigma_2^\vee \cap N_2^\vee$, which gives the desired inclusion. Finally, $\Phi^\vee|_{\sigma_2^\vee \cap N_2^\vee}$ is injective because of the uniqueness conditions. □

Lemma 4.7.2. *If ϕ is an isomorphism (of lattices), then $\pi_{\Phi,\phi} : \mathcal{X}_{\sigma_1^c,\beta_1} \to \mathcal{X}_{\sigma_2^c,\beta_2}$ is a good moduli space morphism.*

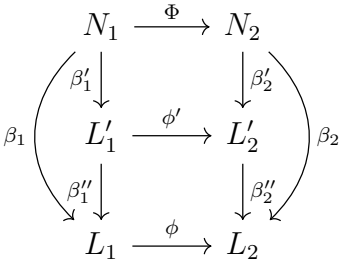
Proof. By [GS15a, Lemma A.1], K_{β_1} is an extension of K_{β_2} by $K_\Phi = \ker(\Phi_{\mathbb{G}_m})$. So the map $\pi_{\Phi,\phi}$ is

$$[X_{\sigma_1^c}/K_{\beta_1}] = [(X_{\sigma_1^c}/K_\Phi)/K_{\beta_2}] \to [X_{\sigma_2^c}/K_{\beta_2}].$$

By Lemma 4.7.1, $[X_{\sigma_1^c}/K_\Phi] \to X_{\sigma_2^c}$ is a good moduli space morphism. Since the property of being a good moduli space morphism can be checked locally on the base in the smooth topology (see [Alp13, Proposition 4.7]), we deduce that $[X_{\sigma_1^c}/K_{\beta_1}] \to [X_{\sigma_2^c}/K_{\beta_2}]$ is a good moduli space morphism. □

Lemma 4.7.3. *For each $i = 1, 2$, suppose that β_i factors as $N \xrightarrow{\beta_i'} L_i' \xrightarrow{\beta_i''} L_i$ where β_i' has finite cokernel. Suppose that $\phi' : L_1' \to L_2'$ makes the following commutative diagram*

(4.7.4)



Suppose that ϕ' induces an isomorphism between $\ker(\beta_1'')$ and $\ker(\beta_2'')$, and ϕ induces an isomorphism between $\mathrm{coker}(\beta_1'')$ and $\mathrm{coker}(\beta_2'')$. Then $\pi_{\Phi,\phi} : \mathcal{X}_{\sigma_1^c,\beta_1} \to \mathcal{X}_{\sigma_2^c,\beta_2}$ is a good moduli space morphism if and only if $\pi_{\Phi,\phi'} : \mathcal{X}_{\sigma_1^c,\beta_1'} \to \mathcal{X}_{\sigma_2^c,\beta_2'}$ is a good moduli space morphism.

Proof. By [GS15a, Lemma A.1], the rows in the following diagram are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_{\beta_1'} & \longrightarrow & K_{\beta_1} & \longrightarrow & K_{\beta_1''} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & K_{\beta_2'} & \longrightarrow & K_{\beta_2} & \longrightarrow & K_{\beta_2''} & \longrightarrow & 0 \end{array}$$

Moreover, the last supposition implies that the vertical map $K_{\beta_1''} \to K_{\beta_2''}$ is an isomorphism. Thus, the morphism

$$\pi_{\Phi,\phi} : \mathcal{X}_{\sigma_1^c,\beta_1} \cong [\mathcal{X}_{\sigma_1^c,\beta_1'}/K_{\beta_1''}] \to [\mathcal{X}_{\sigma_2^c,\beta_2'}/K_{\beta_2''}] \cong \mathcal{X}_{\sigma_2^c,\beta_2}$$

is the quotient of the morphism $\pi_{\Phi,\phi'} : \mathcal{X}_{\sigma_1^c,\beta_1'} \to \mathcal{X}_{\sigma_2^c,\beta_2'}$ by $K_{\beta_1''} \cong K_{\beta_2''}$. We conclude that $\pi_{\Phi,\phi}$ is a good moduli space morphism if and only if $\pi_{\Phi,\phi'}$ is a good moduli space morphism, because the property of being a good moduli space morphism can be checked locally on the base in the smooth topology (see [Alp13, Proposition 4.7]). □

Proposition 4.7.5. *Let τ^c be an unstable coloured face of σ_1^c. Suppose that $N_2 = N_1/\tau^{\mathrm{gp}}$ and $L_2 = L_1/(\beta_1)_{\mathbb{R}}(\tau)^{\mathrm{gp}}$, and suppose that Φ and ϕ are the quotient maps. Then $\pi_{\Phi,\phi} : \mathcal{X}_{\sigma_1^c,\beta_1} \to \mathcal{X}_{\sigma_2^c,\beta_2}$ is a good moduli space morphism.*

Proof. Let $\mathfrak{K} := \ker(\beta_1) \cap \tau^{\mathrm{gp}}$. Note that $\mathfrak{K}$ is saturated in N_1 (it is an intersection of saturated subgroups), so $N_1/\mathfrak{K}$ is a lattice. Moreover, $N_1/\mathfrak{K}$ is a coloured lattice for a horospherical homogeneous space as in Example 3.3.24 by considering the universal colour set on $\mathfrak{K}$ as the set of colours for τ^c whose colour points lie in $\mathfrak{K}$. Consider the following commutative diagram:

$$\begin{array}{ccc} N_1 & \xrightarrow{\Phi} & N_1/\tau^{\mathrm{gp}} \\ {\scriptstyle\beta_1'}\downarrow & & \beta_2'\| \\ N_1/\mathfrak{K} & \xrightarrow{\phi'} & N_1/\tau^{\mathrm{gp}} \\ {\scriptstyle\beta_1''}\downarrow & & \beta_2''\downarrow \\ L_1 & \xrightarrow{\phi} & L_1/(\beta_1)_{\mathbb{R}}(\tau)^{\mathrm{gp}} \end{array} \qquad \beta_1 \text{ (left arc)}, \quad \beta_2 \text{ (right arc)}$$

This is a rectangle in the form of Eq. (4.7.4). The top square induces a good moduli space morphism $\pi_{\Phi,\phi'} : \mathcal{X}_{\sigma_1^c,\beta_1'} \to \mathcal{X}_{\sigma_2^c,\beta_2'}$ by Lemma 4.7.1. Indeed, if $l_1 \in (N_1/\mathfrak{K})^\vee$ satisfies $(\beta_1')^\vee(l_1) \in \sigma_1^\vee \subseteq \tau^\vee$, then $(\beta_1')^\vee(l_1) \in \tau^\perp$ because τ^c is an unstable coloured face of σ_1^c, so $l_1 = (\phi')^\vee(l_2)$ for a unique $l_2 \in (N_1/\tau^{\mathrm{gp}})^\vee$.

Notice that ϕ' induces an isomorphism between $\ker(\beta_1'') = \ker(\beta_1)/\mathfrak{K}$ and $\ker(\beta_2'') =$

$(\ker(\beta_1)+\tau^{\mathrm{gp}})/\tau^{\mathrm{gp}}$ via the second isomorphism theorem. Similarly, ϕ induces an isomorphism between $\operatorname{coker}(\beta_1'')$ and $\operatorname{coker}(\beta_2'')$ via the third isomorphism theorem.

Therefore, we can apply Lemma 4.7.3 to deduce that $\pi_{\Phi,\phi} : \mathcal{X}_{\sigma_1^c,\beta_1} \to \mathcal{X}_{\sigma_2^c,\beta_2}$ is a good moduli space morphism. □

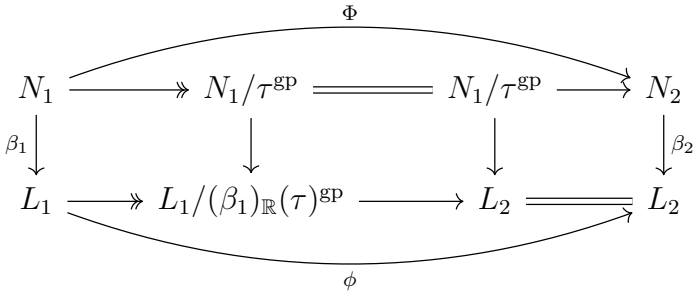
Proposition 4.7.6. *Let $\tau^c := \Phi^{-1}(0^c)$. Then $\pi_{\Phi,\phi} : \mathcal{X}_{\sigma_1^c,\beta_1} \to \mathcal{X}_{\sigma_2^c,\beta_2}$ is a good moduli space morphism if and only if the following conditions hold:*

(1) τ^c is unstable.

(2) ϕ is surjective.

(3) $\ker(\phi) = (\beta_1)_{\mathbb{R}}(\tau)^{\mathrm{gp}}$.

Proof. For the reverse direction, suppose that conditions (1)-(3) hold. Since a composition of good moduli space morphisms is a good moduli space morphism, it suffices to factor (Φ, ϕ) into three maps of stacky coloured fans as follows, and show that each square induces a good moduli space morphism:



By condition (1) and Proposition 4.7.5, the left square induces a good moduli space morphism. By conditions (2) and (3), the map $L_1/(\beta_1)_{\mathbb{R}}(\tau)^{\mathrm{gp}} \to L_2$ is an isomorphism (of lattices). Thus, Lemma 4.7.2 implies that the middle and right squares induce good moduli space morphisms.

Now we prove the forward direction, so suppose that $\pi_{\Phi,\phi}$ is a good moduli space morphism. By [Alp13, Proposition 4.7(i)], base changing $\pi_{\Phi,\phi}$ by the open G-orbit $\mathcal{X}_{0^c,\beta_2}$ of $\mathcal{X}_{\sigma_2^c,\beta_2}$ is a good moduli space morphism. Thus, we may replace σ_2^c with the trivial coloured

cone 0^c on N_2, replace σ_1^c by τ^c, and assume that (Φ, ϕ) is of the form

$$\begin{array}{ccc} \tau^c & \longrightarrow & 0^c \\ N_1 & \xrightarrow{\Phi} & N_2 \\ {\scriptstyle\beta_1}\downarrow & & \downarrow{\scriptstyle\beta_2} \\ L_1 & \xrightarrow{\phi} & L_2 \end{array}$$

Consider the following composition of maps of stacky coloured fans:

$$\begin{array}{ccccc} \tau^c & \longrightarrow & 0^c & \longrightarrow & 0^c \\ N_1 & \xrightarrow{\Phi} & N_2 & \xrightarrow{\beta_2} & L_2 \\ {\scriptstyle\beta_1}\downarrow & & \downarrow{\scriptstyle\beta_2} & & \| \\ L_1 & \xrightarrow{\phi} & L_2 & = & L_2 \end{array}$$

By our supposition, the left square induces a good moduli space morphism, and by Lemma 4.7.2 the right square induces a good moduli space morphism. Therefore, the full rectangle $(\beta_2 \circ \Phi, \phi)$ induces a good moduli space morphism. By applying Lemma 4.7.1 to the full rectangle, we see that every linear functional on L_1 which is nonnegative on $(\beta_1)_{\mathbb{R}}(\tau)$ must be induced uniquely from L_2, and thus is zero on $(\beta_1)_{\mathbb{R}}(\tau)$. Therefore τ^c is unstable, i.e. condition (1) is satisfied.

Now consider the following factorization of (Φ, ϕ) into two maps of stacky coloured fans:

$$\begin{array}{ccccc} \tau^c & \longrightarrow & 0^c & \longrightarrow & 0^c \\ N_1 & \twoheadrightarrow & N_1/\tau^{\mathrm{gp}} & \xrightarrow{\Phi'} & N_2 \\ {\scriptstyle\beta_1}\downarrow & & \downarrow{\scriptstyle\beta_1'} & & \downarrow{\scriptstyle\beta_2} \\ L_1 & \twoheadrightarrow & L_1/(\beta_1)_{\mathbb{R}}(\tau)^{\mathrm{gp}} & \xrightarrow{\phi'} & L_2 \end{array}$$

$$\phi$$

Since τ^c is unstable, Proposition 4.7.5 shows that the left square induces a good moduli space morphism. This combined with the supposition that (Φ, ϕ) induces a good moduli space morphism allows us to apply [GS15a, Lemma 6.9] to deduce that the right square (Φ', ϕ') induces a good moduli space morphism.

To show that conditions (2) and (3) are satisfied, it suffices to show that ϕ' is an isomorphism (of lattices). Since $\pi_{\Phi',\phi'}$ is a good moduli space morphism, it induces an

isomorphism of the good moduli spaces of $\mathcal{X}_{0^c,\beta_1'}$ and $\mathcal{X}_{0^c,\beta_2}$. By Lemma 4.7.1, these good moduli spaces are $G/\overline{K_1'}$ (the horospherical homogeneous space whose associated coloured lattice is $L_1/(\beta_1)_{\mathbb{R}}(\tau)^{\mathrm{gp}}$) and $G/\overline{K_2}$ (the horospherical homogeneous space whose associated coloured lattice is L_2), respectively. Therefore, the natural projection map $G/\overline{K_1'} \to G/\overline{K_2}$, whose associated lattice map is ϕ', is an isomorphism, so ϕ' must be an isomorphism. □

4.7.2 General case

Now we prove the full theorem. The idea is to reduce the general case to the previous case where the stacks are cohomologically affine. For the remainder of this section, the coloured fans Σ_i^c in the map of stacky coloured fans $(\Phi, \phi) : (\Sigma_1^c, \beta_1) \to (\Sigma_2^c, \beta_2)$ are not necessarily generated by single coloured cones.

Lemma 4.7.7. *Suppose that $\Sigma_i^c = \sigma_i^c$ are generated by single coloured cones for $i = 1, 2$. Let $X_i := X_{\sigma_i^c}$, let $\mathcal{O}_i \subseteq X_i$ be the unique closed G-orbit, and let $\pi : X_1 \to X_2$ denote the horospherical morphism induced by Φ. If $\pi_{\Phi,\phi}$ is a good moduli space morphism, then $\pi^{-1}(Z_2) = Z_1$ using the notation of the affine local structure from Section 3.5.2.*

Proof. In the notation of the affine local structure from Section 3.5.2, it suffices to show that $Q_1 = Q_2$. Since we always have $Q_1 \subseteq Q_2$, we just have to show that $Q_2 \subseteq Q_1$. To do this, it suffices to show that every colour divisor $D_1 \subseteq X_1$ such that $D_1 \not\supseteq \mathcal{O}_1$ is of the form $\pi^{-1}(D_2)$ for some colour divisor $D_2 \subseteq X_2$ such that $D_2 \not\supseteq \mathcal{O}_2$. For such a colour divisor D_1, we consider two cases: (1) when $\overline{\pi(D_1)}$ is a colour divisor of X_2, and (2) when $\overline{\pi(D_1)}$ is all of X_2.

(1) In the first case, let $D_2 := \overline{\pi(D_1)}$ be the colour divisor in X_2. By Lemma 4.6.2, we know that $\pi^{-1}(D_2) = D_1$, which proves what we want since $\mathcal{O}_1 \not\subseteq D_1 = \pi^{-1}(D_2)$ implies $\mathcal{O}_2 = \pi(\mathcal{O}_1) \not\subseteq D_2$.

(2) We show that this case is impossible by way of contradiction, so assume that $\overline{\pi(D_1)} = X_2$. Let $\alpha \in \mathcal{C}_1$ be the colour corresponding to D_1. Since the property of being a good moduli space morphism is local on the base, we have a good moduli space morphism $\mathcal{X}_{\Phi^{-1}(0^c),\beta_1} \to \mathcal{X}_{0^c,\beta_2} = G/\overline{K_2}$. Since α is mapped dominantly, α is not in the universal colour set for $G/\overline{K_2}$. That is, $\alpha \in \mathcal{C}_1$ but $\alpha \notin \mathcal{C}_2$. This combined with $I_1 \subseteq I_2$ implies $I_1 \subseteq I_2 \setminus \{\alpha\}$. Thus, we have a projection $G/\overline{K_1} \to G/P_{I_1} \to G/P_{I_2\setminus\{\alpha\}}$. The associated map of coloured lattices is compatible with $\Phi^{-1}(0^c)$ and the coloured fan for $G/P_{I_2\setminus\{\alpha\}}$ since $\alpha \notin \mathcal{F}(\Phi^{-1}(0^c))$. Therefore, we get a horospherical morphism $\mathcal{X}_{\Phi^{-1}(0^c),\beta_1} \to G/P_{I_2\setminus\{\alpha\}}$.

By [Alp13, Theorem 4.16(vi)], good moduli space morphisms are universal to schemes. Thus, there exists a unique dotted arrow which makes the following diagram commute:

$$\begin{array}{ccc} \mathcal{X}_{\Phi^{-1}(0^c),\beta_1} & & \\ \downarrow & \searrow & \\ G/\overline{K_2} & \dashrightarrow & G/P_{I_2\setminus\{\alpha\}} \end{array}$$

Since the solid arrows are horospherical morphisms, the dotted arrow is as well. This implies that $P_{I_2} \subseteq P_{I_2\setminus\{\alpha\}}$, which is a contradiction. □

Remark 4.7.8. Under the conditions of Lemma 4.7.7, the proof of the lemma shows that all dominantly mapped colours are in the colour set of σ_1^c, i.e. $\mathcal{C}_\Phi \subseteq \mathcal{F}(\sigma_1^c)$.

Lemma 4.7.9. *Let (σ_1^c, β_1) and (Σ_2^c, β_2) be stacky coloured fans, where σ_1^c is generated by a single coloured cone. Let $(\Phi, \phi) : (\sigma_1^c, \beta_1) \to (\Sigma_2^c, \beta_2)$ be a map of stacky coloured fans. If every G-orbit of $\mathcal{X}_{\Sigma_2^c,\beta_2}$ is in the image of $\pi_{\Phi,\phi}$ (e.g. if $\pi_{\Phi,\phi}$ is surjective), then $\Phi_{\mathbb{R}}(\sigma_1) = |\Sigma_2|$. In particular, this assumption implies that Σ_2^c is generated by a single coloured cone.*

Proof. The coloured cones of Σ_2^c correspond to the G-orbits of $X_{\Sigma_2^c}$ and of $\mathcal{X}_{\Sigma_2^c,\beta_2}$. Thus, the cones of Σ_2 correspond to these G-orbits as well. By the assumption, we see that every G-orbit of $X_{\Sigma_2^c}$ is in the image of $X_{\sigma_1^c}$ (under the horospherical morphism of varieties induced by Φ). It follows that the relative interior of every cone in Σ_2 contains the image of some face of σ_1. Hence, $\Phi_{\mathbb{R}}(\sigma_1)$ intersects all rays of Σ_2. Since $\Phi_{\mathbb{R}}(\sigma_1)$ is a cone, it must contain all the rays of Σ_2. Therefore $\Phi_{\mathbb{R}}(\sigma_1) \supseteq |\Sigma_2|$, and the other inclusion holds by compatibility of Φ.

Since Φ is compatible with the coloured fans, the equality $\Phi_{\mathbb{R}}(\sigma_1) = |\Sigma_2|$ implies that Σ_2 has one maximal cone, i.e. Σ_2 is the fan generated by a single cone. Thus, Σ_2^c is generated by a single coloured cone. □

Proof of forward direction of Theorem 4.5.5. We first prove the forward direction of Theorem 4.5.5, so suppose that $\pi_{\Phi,\phi} : \mathcal{X}_{\Sigma_1^c,\beta_1} \to \mathcal{X}_{\Sigma_2^c,\beta_2}$ is a good moduli space morphism. Fix $\sigma_2^c = (\sigma_2, \mathcal{F}_2) \in \Sigma_2^c$. Since the property of being a good moduli space morphism is local on the base, the map $\mathcal{X}_{\Phi^{-1}(\sigma_2^c),\beta_1} \to \mathcal{X}_{\sigma_2^c,\beta_2}$ is also a good moduli space morphism.

We first show that $\Phi^{-1}(\sigma_2^c)$ is generated by a single coloured cone in Σ_1^c. To do this, it suffices to show that $X_{\Phi^{-1}(\sigma_2^c)}$ has a unique closed G-orbit. Suppose that $\mathcal{O}_1, \mathcal{O}_1' \subseteq X_{\Phi^{-1}(\sigma_2^c)}$ are closed G-orbits. Let $\mathcal{O}_2 \subseteq X_{\sigma_2^c}$ be the unique closed G-orbit. Then, since the morphism

is horospherical, we have that $\pi_{\Phi,\phi}$ maps $\mathcal{O}_1/K_{\beta_1}$ and $\mathcal{O}_1'/K_{\beta_1}$ onto $\mathcal{O}_2/K_{\beta_2}$. So we can take closed points $x \in \mathcal{O}_1$ and $x' \in \mathcal{O}_1'$ which map to the same closed point in $\mathcal{O}_2/K_{\beta_2}$. We can apply [Alp13, Theorem 4.16(iv)] to deduce that the K_{β_1}-orbit closures of x and x' intersect. Thus, since $\mathcal{O}_1$ and $\mathcal{O}_1'$ are closed and K_{β_1}-invariant, $\mathcal{O}_1$ and $\mathcal{O}_1'$ intersect, which implies that $\mathcal{O}_1 = \mathcal{O}_1'$. Therefore, $\Phi^{-1}(\sigma_2^c)$ is generated by a single coloured cone, say $\sigma_1^c = (\sigma_1, \mathcal{F}_1) \in \Sigma_1^c$.

Let Z_i be the closed horospherical affine subvariety of $X_{\sigma_i^c}$ for $i = 1, 2$ in the notation of the affine local structure from Section 3.5.2. By Lemma 4.7.7 and the fact that the property of being a good moduli space morphism is stable under base change, we see that $[Z_1/K_{\beta_1}] \to [Z_2/K_{\beta_2}]$ is a good moduli space morphism. Now we are in the situation where the horospherical stacks are cohomologically affine. Note that $\Phi_{\mathbb{R}}(\sigma_1) = \sigma_2$ by Lemma 4.7.9 and since good moduli space morphisms are surjective (see [Alp13, Theorem 4.16(i)]). Thus, the conditions at the start of Section 4.7.1 are satisfied (see Remark 4.5.2), so we can apply Proposition 4.7.6 to obtain (GMS2) to (GMS4) since the coloured cone for Z_i is σ_i^c for $i = 1, 2$. Lastly, Remark 4.7.8 and the fact that we are in the cohomologically affine setting imply that $\mathcal{F}_2 \cup \mathcal{C}_\Phi \subseteq \mathcal{F}_1$. The reverse inclusion holds because Φ is compatible with the coloured fans, so we have $\mathcal{F}_2 \cup \mathcal{C}_\Phi = \mathcal{F}_1$, which establishes (GMS1). □

Proof of reverse direction of Theorem 4.5.5. We now prove the reverse direction of Theorem 4.5.5, so suppose that conditions (GMS1) to (GMS4) hold. Since the property of being a good moduli space morphism can be checked locally on the base, it suffices to show that the map $[X_{\Phi^{-1}(\sigma_2^c)}/K_{\beta_1}] \to [X_{\sigma_2^c}/K_{\beta_2}]$ is a good moduli space morphism, where $\sigma_2^c = (\sigma_2, \mathcal{F}_2) \in \Sigma_2^c$ is an arbitrary coloured cone. By (GMS1), $\Phi^{-1}(\sigma_2^c)$ is generated by a single coloured cone, say $\sigma_1^c = (\sigma_1, \mathcal{F}_1) \in \Sigma_1^c$.

Let Z_i be the closed affine horospherical subvariety of $X_{\sigma_i^c}$ for $i = 1, 2$ in the notation of the affine local structure from Section 3.5.2. By the assumption on $\mathcal{F}_2$ in (GMS1), we can apply Lemma 4.6.3 to base change the morphism to $[Z_1/K_{\beta_1}] \to [Z_2/K_{\beta_2}]$. Now we are in a situation where the horospherical stacks are cohomologically affine. Thus, the conditions at the start of Section 4.7.1 are satisfied, so (GMS2) to (GMS4) allow us to apply Proposition 4.7.6 to conclude that $[Z_1/K_{\beta_1}] \to [Z_2/K_{\beta_2}]$ is a good moduli space morphism since the coloured cone for Z_i is σ_i^c for $i = 1, 2$.

Since $X_{\sigma_i^c} \cong G \times^Q Z_i$ (where $Q = Q_1 = Q_2$ is in the notation of the affine local structure from Section 3.5.2), it follows that the good moduli space morphism $[Z_1/K_{\beta_1}] \to [Z_2/K_{\beta_2}]$ extends G-equivariantly to a good moduli space morphism $[X_{\sigma_1^c}/K_{\beta_1}] \to [X_{\sigma_2^c}/K_{\beta_2}]$, which is what we wanted to show. □

Chapter 5

Approximating rational points on horospherical varieties

The second main research theme of this thesis is on probing the COBA conjecture for horospherical varieties. We begin by providing the technical construction of the approximation constant discussed briefly in Chapter 1, and then we focus on proving the main approximation results, which are outlined in Section 5.2.

Throughout this chapter, k denotes a number field, and all varieties and algebraic groups are defined over k. Recall that G is a connected reductive algebraic group over k. Fix a Borel subgroup $B \subseteq G$, a maximal torus $T \subseteq B$, and let $U \subseteq B$ denote the maximal unipotent subgroup of B. We assume that G is *split* reductive; specifically, we assume that our fixed maximal torus T is split. Fix a horospherical subgroup $H \subseteq G$ containing U and let $P := N_G(H)$ denote the associated parabolic. Let S be the set of simple roots, and let $I \subseteq S$ be such that $P = P_I$. Let $N = N(G/H)$ be the associated coloured lattice, and let $\mathcal{C} = \mathcal{C}(G/H)$ be the universal colour set.

If X is a horospherical G-variety, where G is *split*, then we call X a *split* horospherical variety; see Remark 5.3.1. Throughout this chapter, since we are assuming that our fixed G is split, *we are implicitly only considering split horospherical varieties.*

5.1 Defining the approximation constant

In this section, we provide the definition of the approximation constant. For a more detailed discussion, see [MR15].

Definition 5.1.1 (Approximation constant $\alpha_{p,\{x_i\}}(D)$)**.** Let X be a projective variety, v be a place of k, $p \in X(\overline{k})$, and D be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. For any sequence $\{x_i\} \subseteq X(k)$ of distinct points with $\mathrm{dist}_v(p, x_i) \to 0$, we set

$$A(\{x_i\}, D) := \{\gamma \in \mathbb{R} \mid \mathrm{dist}_v(p, x_i)^\gamma H_D(x_i) \text{ is bounded from above}\}.$$

We define the *approximation constant* of $\{x_i\}$ with respect to D to be

$$\alpha_{p,\{x_i\}}(D) := \inf A(\{x_i\}, D).$$

Note that, by convention, $\inf \varnothing = \infty$.

Remark 5.1.2. Note that, if $\{x_i'\}$ is a subsequence of $\{x_i\}$, then $A(\{x_i\}, D) \subseteq A(\{x_i'\}, D)$. Thus, $\alpha_{p,\{x_i'\}}(D) \leq \alpha_{p,\{x_i\}}(D)$, so we may freely replace a sequence with a subsequence when establishing lower bounds on $\alpha_{p,\{x_i\}}(D)$.

As $i \to \infty$ we have $\mathrm{dist}_v(p, x_i) \to 0$. So, one expects that $\mathrm{dist}_v(p, x_i)^\gamma H_D(x_i)$ tends to 0 for large γ and to ∞ for small γ. The number $\alpha_{p,\{x_i\}}(D)$ marks the transition point between these two behaviours.

Definition 5.1.3 (Approximation constant $\alpha_p(D)$)**.** Let X be a projective variety, D a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, and $p \in X(\overline{k})$. Define $\alpha_p(D)$ to be the infimum of all $\alpha_{p,\{x_i\}}(D)$ as we range over sequences of distinct points $\{x_i\} \subseteq X(k)$ converging to p.

In Definition 5.1.3, if p is contained in a closed subvariety $Z \subseteq X$ and we only range over sequences $\{x_i\} \subseteq Z(k)$, then we denote the infimum by $\alpha_{p,Z}(D)$.

To finish this section, we recall the definition of canonically bounded, and we introduce the notion of *extrinsically* canonically bounded to simplify the statement of our main results. Recall that terminal singularities are the natural class one considers when running the minimal model program (MMP).

Definition 5.1.4 (Canonically bounded)**.** Let X be a projective $\mathbb{Q}$-Gorenstein variety defined over a number field k and let $p \in X(k)$. We say that X is *canonically bounded at p* if $\alpha_{p,\{x_i\}}(-K_X) \geq \dim(X)$ for all Zariski dense sequences $\{x_i\} \subseteq X(k)$.

Definition 5.1.5 (Extrinsically canonically bounded)**.** Suppose that X is a projective horospherical G-variety, and let $p \in X(k)$. We say that X is *extrinsically canonically bounded at p* if there exists a horospherical birational map $f : X' \to X$ where X' is a projective terminal $\mathbb{Q}$-factorial horospherical G-variety, f is an isomorphism at p, and X' is canonically bounded at p (in the sense of Definition 5.1.4).

5.2 Main results

The main result of this chapter is the following theorem, the proof of which is dealt with in the remaining sections. Recall that, when we say "horospherical variety", we implicitly mean "*split* horospherical variety".

Theorem 5.2.1. *Let X be a projective horospherical variety over a number field k, and let $p \in X(k)$. If X is extrinsically canonically bounded at p, then for all $\mathbb{Q}$-Cartier nef $\mathbb{Q}$-divisors D on X, there is an irreducible rational curve C through p such that C is unibranch at p and*

$$\alpha_{p,C}(D) \leq \alpha_{p,\{x_i\}}(D)$$

for all Zariski dense sequences $\{x_i\} \subseteq X(k)$.

Remark 5.2.2. Note that while the COBA conjecture is only stated for ample divisors, this Theorem 5.2.1 applies to the more general class of $\mathbb{Q}$-Cartier nef $\mathbb{Q}$-divisors.

As mentioned, for smooth horospherical varieties, the notion of canonically bounded is automatic under the presence of Vojta's Main Conjecture [Voj87, Conjecture 3.4.3], which we prove below.

Proposition 5.2.3. *Let X be a smooth projective horospherical variety over a number field k. Then Vojta's Main Conjecture for X implies that all $p \in X(k)$ are canonically bounded.*

Proof. By [MS23a, Proposition 0.1], we need only show $-K_X$ is big. This is well known. Choose an ample divisor A on X. We have $-K_X = \sum_{D \in \mathcal{D}(X)} b_D D$ where $b_D \in \mathbb{Z}_{>0}$ (see Proposition 3.6.15), and we can write $A = \sum_{D \in \mathcal{D}(X)} a_D D$ for some $a_D \in \mathbb{Z}$. By choosing $m \in \mathbb{Z}_{>0}$ large enough, we can achieve $\frac{a_D}{m} < b_D$ for each $D \in \mathcal{D}(X)$. Hence, $-K_X - \frac{1}{m}A$ is effective, so $-K_X$ is big. □

Combining Theorem 5.2.1 with Proposition 5.2.3, we obtain the following theorem.

Theorem 5.2.4. *Let X be a projective horospherical variety over a number field k. Assume that Vojta's Main Conjecture holds for some projective horospherical (strong) resolution of singularities of X. Then, for all smooth points $p \in X(k)$ and all $\mathbb{Q}$-Cartier nef $\mathbb{Q}$-divisors D on X, there exists an irreducible rational curve C through p such that C is unibranch at p and*

$$\alpha_{p,C}(D) \leq \alpha_{p,\{x_i\}}(D)$$

for all Zariski dense sequences $\{x_i\} \subseteq X(k)$.

5.3 Review of horospherical MMP

We provide a brief review some aspects of the minimal model program (MMP) for horospherical varieties. Much of this section involves specializing the notation and results of Brion's paper [Bri93] on spherical MMP to the horospherical subclass, which allows us to properly formulate the combinatorial arguments that are essential to proving Theorem 5.2.1.

Throughout this section, let X be a projective $\mathbb{Q}$-factorial G/H-horospherical variety. Denote its coloured fan by Σ^c, which is on the coloured lattice N; note that Σ^c is simplicial.

Remark 5.3.1. Since T is split, the simple roots via the Weyl group, and by extension the colour divisors, are $\mathrm{Gal}(\overline{k}/k)$-invariant. Furthermore, since $\mathrm{Aut}^G(X)$ is split, the lattice N, and by extension Σ^c, are $\mathrm{Gal}(\overline{k}/k)$-invariant. Every Cartier divisor on X can be written in terms of colour divisors and piecewise linear functions on Σ^c (see Section 5.3.1 below), so these facts imply that $\mathrm{Gal}(\overline{k}/k)$ acts trivially on $\mathrm{Pic}(X)$, and thus all MMP contractions and flips of X are well-defined over k. This is why we refer to X as *split*: the Picard group has trivial Galois action.

Let $\mathrm{cont}_{\mathcal{R}} : X \to X_{\mathcal{R}}$ denote the contraction of an extremal ray $\mathcal{R}$ of the Mori cone of X. From [Bri93, Theorem 3.1] we know that $X_{\mathcal{R}}$ is also a projective horospherical G-variety, and the map is a horospherical morphism. Thus, $X_{\mathcal{R}}$ corresponds to a coloured fan $\Sigma^c_{\mathcal{R}}$ on a coloured lattice $N_{\mathcal{R}}$. Recall that there are three possibilities for $\mathrm{cont}_{\mathcal{R}}$: it can be a Mori fibre space, a divisorial contraction, or a flipping contraction. In the last case, $X_{\mathcal{R}}$ is not $\mathbb{Q}$-factorial, but we have a flip $\psi : X \dashrightarrow X_+$ where X_+ is a projective $\mathbb{Q}$-factorial G/H-horospherical variety; see [Bri93, Proposition 4.3]. Let Σ^c_+ denote the coloured fan corresponding to X_+, which lives on the same coloured lattice N.

5.3.1 Cartier divisors

We quickly recall some notation for Cartier divisors from Section 3.6

The G-invariant prime divisors of X correspond to the non-coloured rays of Σ^c, and we let D_ρ denote the divisor corresponding to $(\rho, \varnothing) \in \Sigma^c(1)$.

Up to linear equivalence, every Cartier divisor δ on X can be written in terms of the B^--invariant divisors. Thus, we can write $\delta = \sum_{D \in \mathcal{D}(X)} a_D D$ for some $a_D \in \mathbb{Z}$; or we can write $\delta = \sum_{(\rho,\varnothing)} a_\rho D_\rho + \sum_{\alpha \in \mathcal{C}} a_\alpha D_\alpha$ for some $a_\rho, a_\alpha \in \mathbb{Z}$, where the first sum is over all non-coloured rays $(\rho, \varnothing)$ in Σ^c.

We can describe such a δ via a piecewise linear function $\varphi_\delta : N_{\mathbb{R}} \to \mathbb{R}$ on Σ^c. That is, for each $\sigma^c = (\sigma, \mathcal{F}) \in \Sigma^c$, we have $\varphi_\delta(u_\alpha) = a_\alpha$ for each $\alpha \in \mathcal{F}$ and $\varphi_\delta(u_\rho) = a_\rho$ for

each primitive generator u_ρ of a non-coloured ray $(\rho, \varnothing)$ contained in σ^c. This function has associated Cartier data $\{m_\sigma\}_{\sigma\in\Sigma} \subseteq N^\vee$, i.e. $\varphi_\delta(u) = \langle m_\sigma, u\rangle$ for each $u \in \sigma$. Note that this notation also works for $\mathbb{Q}$-Cartier δ by allowing rational coefficients and rational Cartier data.

5.3.2 The Mori cone

Following [Bri93, Section 3.2], the Mori cone $\mathrm{NE}(X)$ is generated by a finite set of curve classes, which we describe as follows. Let δ be a Cartier divisor on X with the notation above.

Given a wall $\mu \in \Sigma$, write $\mu = \sigma_+ \cap \sigma_-$ for unique maximal cones $\sigma_+, \sigma_- \in \Sigma$. Choose $m^\mu \in N^\vee$ to be a primitive lattice element such that the functional $\langle m^\mu, -\rangle$ is positive on σ^+, zero on μ, and negative on σ_-. Then $m_{\sigma_+} - m_{\sigma_-} \in N^\vee$ is an integer multiple of m^μ, which we denote $\frac{m_{\sigma_+} - m_{\sigma_-}}{m^\mu} \in \mathbb{Z}$. Define $C_\mu \in N_1(X) = N^1(X)^\vee$ via

$$\delta \cdot C_\mu := \frac{m_{\sigma_+} - m_{\sigma_-}}{m^\mu}. \tag{5.3.2}$$

We call C_μ a *wall curve class*. Note that there exists an irreducible B^--invariant curve (isomorphic to $\mathbb{P}^1$) whose class in $N_1(X)$ is C_μ; see [Bri93, Proposition 3.3].

On the other hand, given a pair (α, σ^c) where $\sigma^c \in \Sigma^c$ is maximal and $\alpha \in \mathcal{C} \setminus \mathcal{F}(\sigma^c)$, we define $C_{\alpha,\sigma^c} \in N_1(X) = N^1(X)^\vee$ via

$$\delta \cdot C_{\alpha,\sigma^c} := a_\alpha - \langle m_\sigma, u_\alpha\rangle. \tag{5.3.3}$$

We call C_{α,σ^c} a *colour curve class*. Note that, if the ray $\mathbb{R}_{\geq 0}C_{\alpha,\sigma^c} \subseteq N_1(X)_\mathbb{R}$ does not contain any wall curve classes, then there exists an irreducible B^--invariant curve (isomorphic to $\mathbb{P}^1$) whose class in $N_1(X)$ is C_{α,σ^c}; see [Bri93, Proposition 3.6].

Then $\mathrm{NE}(X)$ is generated by all these wall curve classes and colour curve classes, i.e.

$$\mathrm{NE}(X) = \mathrm{Cone}\left(C_\mu, C_{\alpha,\sigma^c} \ : \ \begin{array}{l} \mu \in \Sigma \text{ is a wall,} \\ (\alpha, \sigma^c) : \ \sigma^c \in \Sigma^c \text{ maximal}, \alpha \in \mathcal{C} \setminus \mathcal{F}(\sigma^c) \end{array}\right) \subseteq N_1(X)_\mathbb{R}.$$

By [Bri93, Theorem 3.2], every extremal ray of $\mathrm{NE}(X)$ which does not contain any wall curve classes is generated by a colour curve class C_{α,σ^c} with $\alpha \notin \mathcal{F}(X)$ and $u_\alpha \in \sigma$. Therefore, when we consider colour curve classes, we always assume that these conditions are satisfied.

5.3.3 Colour curve contractions

Suppose that $\mathcal{R}$ is an extremal ray of $\mathrm{NE}(X)$ which does not contain any wall curve classes. Then $\mathcal{R}$ is generated by some colour curve class, say C_{α,σ^c}. Under the contraction $\mathrm{cont}_{\mathcal{R}} : X \to X_{\mathcal{R}}$, we follow [Bri93, Section 3.4] to describe the coloured fan $\Sigma^c_{\mathcal{R}}$ associated to $X_{\mathcal{R}}$.

If $u_\alpha \neq 0$, then $N_{\mathcal{R}}$ is the coloured lattice N (with the same universal colour set); and if $u_\alpha = 0$, then $N_{\mathcal{R}}$ has the same underlying lattice as N, but the universal colour set is $\mathcal{C}_{\mathcal{R}} = \mathcal{C} \setminus \{\alpha\}$. The underlying fan $\Sigma_{\mathcal{R}}$ is the same as Σ, and there are two cases for the colour set: if $u_\alpha \neq 0$, then $\mathcal{F}(\Sigma^c_{\mathcal{R}}) = \mathcal{F}(\Sigma^c) \cup \{\alpha\}$; and if $u_\alpha = 0$, then $\mathcal{F}(\Sigma^c_{\mathcal{R}}) = \mathcal{F}(\Sigma^c)$.

The contraction $\mathrm{cont}_{\mathcal{R}}$ is a Mori fibre space if and only if $u_\alpha = 0$. Otherwise, $u_\alpha \neq 0$ and $\mathrm{cont}_{\mathcal{R}}$ is birational. In this case, $\mathrm{cont}_{\mathcal{R}}$ is divisorial if and only if u_α generates a non-coloured ray of Σ^c. When $\mathrm{cont}_{\mathcal{R}}$ is not divisorial, we can use [Bri93, Section 4.4] to describe the flip. Note that Σ^c_+ lives on the coloured lattice N since $u_\alpha \neq 0$ in this case. The underlying fan Σ_+ is obtained from Σ by star subdividing through the point u_α; see [CLS11, Section 11.1] for this star subdivision. For the colour set, there are two cases: if u_α is not on a ray of Σ, then $\mathcal{F}(\Sigma^c_+) = \mathcal{F}(\Sigma^c) \cup \{\alpha\}$; or if u_α is on a ray $\rho \in \Sigma$, then $\mathcal{F}(\rho^c) = \{\alpha'\}$ for some $\alpha' \neq \alpha$, and we have $\mathcal{F}(\Sigma^c_+) = (\mathcal{F}(\Sigma^c) \cup \{\alpha\}) \setminus \{\alpha'\}$.

5.3.4 Wall curve contractions

Now suppose that $\mathcal{R}$ is an extremal ray of $\mathrm{NE}(X)$ which is generated by a wall curve class C_μ. Under the contraction $\mathrm{cont}_{\mathcal{R}} : X \to X_{\mathcal{R}}$, we follow [Bri93, Section 4.6] to describe the coloured fan $\Sigma^c_{\mathcal{R}}$ associated to $X_{\mathcal{R}}$.

If we completely forget about the colour structure, then the underlying fan Σ of Σ^c corresponds to a toric variety X_Σ. The wall $\mu \in \Sigma$ corresponds to an irreducible torus-invariant curve in X_Σ, and the corresponding curve class generates an extremal ray in $\mathrm{NE}(X_\Sigma)$. Contracting this extremal ray yields a toric variety $X_{\Sigma'}$ corresponding to a fan Σ' which is built explicitly from Σ; see [CLS11, Section 15.4] for a description of the toric contraction and the fan Σ' (in their notation, this fan is denoted Σ_0). The punchline is that Σ' is precisely the underlying fan $\Sigma_{\mathcal{R}}$ of $\Sigma^c_{\mathcal{R}}$.

The lattice $N_{\mathcal{R}}$ is described via the toric MMP (it is the lattice on which Σ' above lives), and the universal colour set for $N_{\mathcal{R}}$ is $\mathcal{C}_{\mathcal{R}} = \mathcal{C} \setminus \mathcal{C}_\Phi$ where $\Phi : N \to N_{\mathcal{R}}$ is the coloured lattice map associated to $\mathrm{cont}_{\mathcal{R}} : X \to X_{\mathcal{R}}$; so $\mathcal{C}_\Phi$ consists of colours in $\mathcal{F}(\Sigma^c)$ whose colour points map to zero under Φ. The colour set $\mathcal{F}(\Sigma^c_{\mathcal{R}})$ is inherited from $\mathcal{F}(\Sigma^c)$, i.e. it is $\mathcal{F}(\Sigma^c) \cap \mathcal{C}_{\mathcal{R}}$. Now $\Sigma^c_{\mathcal{R}}$ is completely described by the underlying fan $\Sigma_{\mathcal{R}}$ and the colour set $\mathcal{F}(\Sigma^c_{\mathcal{R}})$.

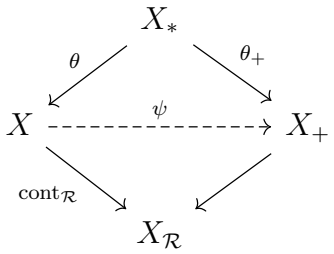
The contraction $\mathrm{cont}_{\mathcal{R}}$ is birational (i.e. not a Mori fibre space) if and only if the corresponding toric contraction of X_Σ is birational; see [CLS11, Proposition 15.4.5] for the different types of toric MMP contractions. In this case, $\mathrm{cont}_{\mathcal{R}}$ is divisorial if and only if the corresponding toric contraction of X_Σ is divisorial. When $\mathrm{cont}_{\mathcal{R}}$ is not divisorial, the underlying fan Σ_+ is the same as the fan corresponding to the flip of the toric variety X_Σ, and the colour set $\mathcal{F}(\Sigma_+^c)$ is inherited from $\mathcal{F}(\Sigma^c)$ so that Σ_+^c is simplicial. That is, $\mathcal{F}(\Sigma_+^c)$ is the same as $\mathcal{F}(\Sigma^c)$ except we remove all $\alpha \in \mathcal{F}(\Sigma^c)$ such that u_α is not on a ray of Σ_+.

5.3.5 Flip tower

Suppose that $\mathrm{cont}_{\mathcal{R}} : X \to X_{\mathcal{R}}$ is a flipping contraction, and let $\psi : X \dashrightarrow X_+$ denote the flip. Let Σ_*^c be the coloured fan on N defined as follows. If $\mathrm{cont}_{\mathcal{R}}$ is a colour curve contraction of C_{α,σ^c}, then let Σ_*^c be the same as Σ_+^c except that we remove α from the colour set. If $\mathrm{cont}_{\mathcal{R}}$ is a wall curve contraction of C_μ, then let Σ_*^c be given as follows: the underlying fan is the fan from [CLS11, Lemma 15.5.7], and the colour set is that of Σ_+^c.

In any case, Σ_* is the star subdivision of Σ and of Σ_+ through some primitive lattice point $u_* \in N$. If $\mathrm{cont}_{\mathcal{R}}$ is a colour curve contraction corresponding to the colour α, then $du_* = u_\alpha$ for some $d \in \mathbb{Z}_{>0}$; for notation, let $d := 1$ if $\mathrm{cont}_{\mathcal{R}}$ is a wall curve contraction. Note that $(\mathrm{Cone}(u_*), \varnothing) \in \Sigma_*^c$ is a non-coloured ray. It follows that the identity map on N is compatible with Σ_*^c and Σ^c, and with Σ_*^c and Σ_+^c. Therefore, if X_* denotes the G/H-horospherical variety corresponding to Σ_*^c, then we have a commutative diagram of horospherical morphisms

(5.3.4)



Moreover, θ and θ_+ are isomorphisms away from the exceptional locus $\mathrm{Exc}(\psi)$.

We prove the following result which generalizes [CLS11, Lemma 15.5.7] for toric varieties. Let $C \subseteq X$ be a curve which is contracted by $\mathrm{cont}_{\mathcal{R}}$.

Lemma 5.3.5. *Consider the setup above for Eq. (5.3.4). Let D_* denote the G-invariant prime divisor in X_* which corresponds to the new non-coloured ray $(\mathrm{Cone}(u_*), \varnothing) \in \Sigma_*^c$.*

Then, for all $\mathbb{Q}$-divisors δ on X and $\delta_+ := \psi_*\delta$, we have

$$\theta^*\delta = \theta_+^*\delta_+ - \frac{\delta \cdot C}{d} D_*.$$

Proof. Both X_* and X are G/H-horospherical varieties, and the map $\theta : X_* \to X$ is a proper birational horospherical morphism which is induced by the identity $N \to N$. Therefore, $\theta^*\delta$ and δ have the same associated piecewise linear function $\varphi : N_{\mathbb{R}} \to \mathbb{R}$. Let $\varphi_+ : N_{\mathbb{R}} \to \mathbb{R}$ denote the piecewise linear function for δ_+.

We may write $\delta = \sum_{(\rho,\varnothing)} a_\rho D_\rho + \sum_{\alpha\in\mathcal{C}} a_\alpha D_\alpha$ for some $a_\rho, a_\alpha \in \mathbb{Z}$, where the first sum is over all non-coloured rays $(\rho, \varnothing)$ in Σ^c. Note that the rays of Σ and Σ_* are the same, except that Σ_* has a new ray generated by u_*. For each non-coloured ray $(\rho, \varnothing) \in \Sigma^c(1)$, let D_ρ^* denote the G-invariant prime divisor in X_* corresponding to ρ; and for each colour $\alpha \in \mathcal{C}$, let D_α^* denote the colour divisor in X_* corresponding to α. Then

$$\theta^*\delta = \sum_{(\rho,\varnothing)} a_\rho D_\rho^* + \varphi(u_*)D_* + \sum_{\alpha\in\mathcal{C}} a_\alpha D_\alpha^*$$

and

$$\theta_+^*\delta_+ = \sum_{(\rho,\varnothing)} a_\rho D_\rho^* + \varphi_+(u_*)D_* + \sum_{\alpha\in\mathcal{C}} a_\alpha D_\alpha^*.$$

Thus, if suffices to prove that $\varphi_+(u_*) - \varphi(u_*) = \frac{\delta\cdot C}{d}$.

If $\mathrm{cont}_{\mathcal{R}}$ is a wall curve contraction, then we take $d = 1$, and the equality $\varphi_+(u_*) - \varphi(u_*) = \delta \cdot C$ follows from the argument in [CLS11, Lemma 15.5.7]. On the other hand, if $\mathrm{cont}_{\mathcal{R}}$ is a colour curve contraction corresponding to α, then $du_* = u_\alpha$ and $\varphi_+(u_\alpha) = a_\alpha$ because $\alpha \in \mathcal{F}(\Sigma_+^c)$, so we have

$$\varphi_+(u_*) - \varphi(u_*) = \frac{\varphi_+(u_\alpha) - \varphi(u_\alpha)}{d} = \frac{a_\alpha - \varphi(u_\alpha)}{d} = \frac{\delta \cdot C}{d}$$

where the last equality follows from Eq. (5.3.3). □

5.4 Preliminary reductions in the proof of Theorem 5.2.1

Our main goal is to prove the results in Section 5.2, i.e. we need to prove Theorem 5.2.1. In this section, we reduce the proof to a simpler case where p is in the exceptional locus of an MMP contraction. This simpler case is stated in Proposition 5.4.6 and proved in Section 5.6.

By [MS21, Proposition 4.1] (which is stated for toric varieties, but holds for horospherical varieties), to prove Theorem 5.2.1, we may assume that the horospherical variety X is projective, $\mathbb{Q}$-factorial, and has at worst terminal singularities. Thus, it suffices to prove the following theorem.

Theorem 5.4.1. *Let X be a projective terminal $\mathbb{Q}$-factorial horospherical variety, let $p \in X(k)$, and let D be a nef $\mathbb{Q}$-divisor on X. If p is canonically bounded, then there exists an irreducible curve C through p which is unibranch at p and*

$$\alpha_{p,C}(D) \leq \alpha_{p,\{x_i\}}(D)$$

for all Zariski dense sequences $\{x_i\} \subseteq X(k)$. Moreover, if X is not isomorphic to projective space, then we may choose C so that $-K_X \cdot C \leq \dim(X)$.

Let X be a projective $\mathbb{Q}$-factorial G/H-horospherical variety which is canonically bounded at $p \in X(k)$. Let D be a nef $\mathbb{Q}$-divisor on X. Let $[C_0], \dots, [C_\ell]$ be the curve classes which generate the K_X-negative extremal rays of $\mathrm{NE}(X)$. Set

$$a := \min_{0 \leq i \leq \ell} \frac{D \cdot C_i}{-K_X \cdot C_i} \tag{5.4.2}$$

and, without loss of generality, say $a = \frac{D \cdot C_0}{-K_X \cdot C_0}$. By construction, $D + aK_X$ is nef. By [MS21, Lemma 4.3], to prove Theorem 5.4.1 for D, it suffices to prove the theorem for $D + aK_X$.

The benefit to working with $D + aK_X$ is that $(D + aK_X) \cdot C_0 = 0$. Let $\pi : X \to Y$ denote the contraction corresponding to the extremal ray $\mathbb{R}_{\geq 0} C_0 \subseteq \mathrm{NE}(X)$. If π is a Mori fibre space or a divisorial contraction, then there exists a nef $\mathbb{Q}$-divisor D' on Y such that $D + aK_X = \pi^* D'$. On the other hand, if π is a flipping contraction, then let $\psi : X \dashrightarrow X_+$ denote the associated flip. If we let $D' := \psi_*(D + aK_X)$, then Lemma 5.3.5 tells us that $\theta^*(D + aK_X) = \theta_+^* D'$, where we are using the notation of Eq. (5.3.4). Since θ and θ_+ are proper and surjective, $D + aK_X$ being nef implies that $\theta^*(D + aK_X)$ is nef, which implies that D' is nef.

To simplify the notation, we let $\psi : X \dashrightarrow X'$ denote the elementary MMP step corresponding to the extremal ray $\mathbb{R}_{\geq 0}C_0$, i.e. if π is a Mori fibre space or a divisorial contraction, then we let $X' := Y$ and $\psi := \pi$; on the other hand, if π is a flipping contraction, then we let ψ denote the associated flip and $X' := X_+$. In any case, we have shown that there is a nef $\mathbb{Q}$-divisor D' on X' for which $D + aK_X = \psi^* D'$.

In Proposition 5.4.5 below, we show that the proof of Theorem 5.4.1 reduces to the case where $p \in \mathrm{Exc}(\psi)$. Before proving this reduction, we require the following two preliminary results.

Proposition 5.4.3. *Let X be a projective terminal $\mathbb{Q}$-factorial horospherical variety, and let $\psi\colon X \dashrightarrow X'$ be a birational elementary MMP step. If $p \in X(k) \setminus \mathrm{Exc}(\psi)$ is a canonically bounded point of X, then $\psi(p)$ is a canonically bounded point of X'.*

Proof. This follows from an almost identical proof of [MS21, Proposition 4.8]. Note that this proof relies on [MS21, (4.5), (4.6)], which are generalized to horospherical varieties in Eq. (5.3.4) and Lemma 5.3.5. □

Proposition 5.4.4. *Let X be a projective terminal $\mathbb{Q}$-factorial horospherical variety, and let $\psi\colon X \dashrightarrow X'$ be a birational elementary MMP step corresponding to the extremal ray $\mathcal{R}$. Let D and D' be nef $\mathbb{Q}$-divisors on X and X', respectively, such that $D \in \mathcal{R}^\perp$ and $D = \psi^* D'$. If $p \in X(k) \setminus \mathrm{Exc}(\psi)$ is a canonically bounded point and Theorem 5.4.1 holds for $(X', \psi(p), D')$, then it holds for (X, p, D).*

Proof. The proof is nearly the same as [MS21, Proposition 4.9], which is handled in [MS21, Section 5]. Note that [MS21, Lemma 5.1] is a statement about projective space; [MS21, Lemma 5.2] holds for horospherical varieties; and [MS21, Lemma 5.3] holds for horospherical varieties by Eq. (5.3.4) and Lemma 5.3.5. □

Proposition 5.4.5. *Suppose that Theorem 5.4.1 holds for all (X, p, D) where X is a projective terminal $\mathbb{Q}$-factorial horospherical variety, $\psi\colon X \dashrightarrow X'$ is an elementary MMP step corresponding to the extremal ray $\mathcal{R}$, D is a nef $\mathbb{Q}$-divisor on X such that $D \in \mathcal{R}^\perp$, and $p \in X(k) \cap \mathrm{Exc}(\psi)$ is a canonically bounded point. Then Theorem 5.4.1 holds for all triples.*

Proof. Start with a projective terminal $\mathbb{Q}$-factorial horospherical variety X_1, let D_1 be a nef $\mathbb{Q}$-divisor on X_1, and $p_1 \in X_1(k)$ a canonically bounded point. Let a_1 be as in Eq. (5.4.2), so that $D_1 + a_1 K_{X_1} \in \mathcal{R}_1^\perp$ for some K_{X_1}-negative extremal ray $\mathcal{R}_1$ of $\mathrm{NE}(X_1)$. Now let $\psi\colon X_1 \dashrightarrow X_2$ be the elementary MMP step associated to $\mathcal{R}_1$, and let D_2 be the

nef $\mathbb{Q}$-divisor on X_2 such that $D_1 + a_1 K_{X_1} = \psi_1^* D_2$. Repeating this procedure, we arrive at the following data: we have a sequence of elementary MMP steps

$$X_1 \overset{\psi_1}{\dashrightarrow} X_2 \overset{\psi_2}{\dashrightarrow} \cdots \overset{\psi_m}{\dashrightarrow} X_{m+1}$$

and a sequence of canonically bounded points $p_i \in X_i(k)$ such that $p_m \in \mathrm{Exc}(\psi_m)$, and $p_i \notin \mathrm{Exc}(\psi_i)$ and $p_{i+1} = \psi_i(p_i)$ for all $1 \leq i < m$. Furthermore, for each $1 \leq i \leq m$, we have a nef $\mathbb{Q}$-divisor D_i on X_i, and we have $a_i \in \mathbb{Q}_{\geq 0}$ such that $D_i + a_i K_{X_i} = \psi_i^* D_{i+1}$ is a nef $\mathbb{Q}$-divisor perpendicular to the K_{X_i}-negative extremal ray corresponding to ψ_i.

By Proposition 5.4.4, the case of $(X_i, p_i, D_i + a_i K_{X_i})$ follows from that of $(X_{i+1}, p_{i+1}, D_{i+1})$. Therefore, to prove Theorem 5.4.1 for the triple (X_1, p_1, D_1), it suffices to prove it for the triple (X_m, p_m, D_m) where we have $p_m \in X(k) \cap \mathrm{Exc}(\psi_m)$. □

In conclusion, we have reduced the proof of Theorem 5.4.1 to proving the following.

Proposition 5.4.6. *Let X be a projective terminal $\mathbb{Q}$-factorial horospherical variety, and let $\psi \colon X \dashrightarrow X'$ be an elementary MMP step corresponding to the extremal ray $\mathcal{R}$. If D is a nef $\mathbb{Q}$-divisor on X such that $D \in \mathcal{R}^\perp$ and $p \in X(k) \cap \mathrm{Exc}(\psi)$ is a canonically bounded point, then Theorem 5.4.1 holds for (X, p, D).*

5.5 Results on Picard number 1 horospherical varieties

This section is dedicated to proving Proposition 5.5.2, stated below, on projective $\mathbb{Q}$-factorial horospherical varieties with Picard number 1, which is the main input in the proof of Proposition 5.4.6.

We begin by classifying these horospherical varieties with Picard number 1.

Lemma 5.5.1. *Suppose that X is a projective $\mathbb{Q}$-factorial G/H-horospherical variety. Then X has Picard number 1 if and only if one of the following holds:*

(1) $\mathrm{rk}(N) = 0$, $\#\mathcal{C} = 1$, *and* $\Sigma^c = 0^c$. *Said another way, X is a flag variety with exactly one colour divisor.*

(2) $\mathrm{rk}(N) > 0$, $\mathcal{F}(\Sigma^c) = \mathcal{C}$, *and* $\#\Sigma^c(1) = \mathrm{rk}(N) + 1$.

Proof. Since X is $\mathbb{Q}$-factorial, the Picard number $\mathrm{rk}(\mathrm{Pic}(X))$ is equal to $\mathrm{rk}(\mathrm{Cl}(X))$, which is equal to $\#\mathcal{D}(X) - \mathrm{rk}(N)$. Since Σ^c is simplicial, we have an injection $\mathcal{F}(\Sigma^c) \hookrightarrow \Sigma^c(1)$ which sends $\alpha \mapsto (\mathrm{Cone}(u_\alpha), \{\alpha\})$. It follows that $\#\mathcal{D}(X) = \#\Sigma^c(1) + \#(\mathcal{C} \setminus \mathcal{F}(\Sigma^c))$. Since

X is projective, Σ^c is complete, and thus $\#\Sigma^c(1) \geq \mathrm{rk}(N) + 1$ or $\mathrm{rk}(N) = 0$ (in which case $\#\Sigma^c(1) = 0$ and $\mathcal{F}(\Sigma^c) = \varnothing$). Hence

$$\mathrm{rk}(\mathrm{Pic}(X)) = 1 \iff (\#\Sigma^c(1) - \mathrm{rk}(N)) + \#(\mathcal{C} \setminus \mathcal{F}(\Sigma^c)) = 1$$

and the final equation is satisfied if and only if conditions (1) or (2) are met. □

The following is the main result of this section. It yields a strong version of Theorem 5.4.1 for these horospherical varieties with Picard number 1.

Proposition 5.5.2. *Let W be a projective $\mathbb{Q}$-factorial horospherical variety with Picard number 1, and let $p \in W(k)$. Then there exists a unibranch rational curve $C \subseteq W$ through p which satisfies $-K_W \cdot C \leq 1 + \dim(W)$. Furthermore, if W has terminal singularities and is not isomorphic to projective space, then C can be chosen to additionally satisfy $-K_W \cdot C \leq \dim(W)$.*

From Lemma 5.5.1, we know that there are two types of projective $\mathbb{Q}$-factorial horospherical varieties with Picard number 1. The first is flag varieties with a single colour divisor. In this case, Proposition 5.5.2 becomes the following.

Proposition 5.5.3. *Proposition 5.5.2 holds when W is a flag variety with Picard number 1.*

Remark 5.5.4. Recall that $P \subseteq G$ corresponds uniquely to a subset $I \subseteq S$, where S is the set of simple roots for G; then the universal colour set is $\mathcal{C} = S \setminus I$. Let R^+ denote the set of positive roots in G (relative to B and T), and let R_I denote the set of roots generated by I.

Then [Pas08, Lemme 4.8] shows that we have

$$\sum_{\alpha \in \mathcal{C}} \sum_{\beta \in R^+ \setminus R_I} \langle \beta, \alpha^\vee \rangle \leq \#\mathcal{C} + \#(R^+ \setminus R_I). \tag{5.5.5}$$

where $\langle \cdot, \cdot \rangle$ denotes the dual pairing between roots and dual roots. Moreover, Eq. (5.5.5) is equality if and only if G/P is a product of projective spaces (see [Pas10, pg. 1153]).

From Proposition 3.6.15, the left-hand side of Eq. (5.5.5) is the sum of the coefficients of $-K_{G/P}$ when written as a linear combination of the colour divisors of G/P.

Proof of Proposition 5.5.3. Since $W = G/P$ is a single G-orbit, we may translate our curve by a $G(k)$-point to ensure that it passes through any particular point in W. Thus, we may assume that p is the base point $eP \in W = G/P$. By assumption, W has exactly one

colour divisor D_α with $\mathcal{C} = \{\alpha\}$. We may choose the dual Schubert curve $C \subseteq W$ so that $D_\alpha \cdot C = 1$. This curve C is unibranch, rational, and passes through p. As in Remark 5.5.4, $-K_W = aD_\alpha$ where a equals the left-hand side of Eq. (5.5.5). Therefore, Remark 5.5.4 and the fact that the right-hand side of Eq. (5.5.5) is equal to $\dim(W) + 1$ imply that

$$-K_W \cdot C = aD_\alpha \cdot C = a \leq \dim(W) + 1.$$

If W is not isomorphic to projective space, then the above inequality is a strict inequality of integers, so we can strengthen it to $-K_W \cdot C \leq \dim(W)$. □

Now we turn to Proposition 5.5.2 for varieties of the second type in Lemma 5.5.1. Together with Proposition 5.5.3, this implies Proposition 5.5.2.

Proposition 5.5.6. *Let W be as in Lemma 5.5.1(2), and let $p \in W(k)$. Then there is a unibranch rational curve $C \subseteq W$ through p which satisfies the following properties:*

(1) There exists a G-orbit closure $Z \subseteq W$ and a curve $C_0 \subseteq Z$ which is the closure of the orbit of a $Z(k)$-point under the action of a one-parameter subgroup of $\mathrm{Aut}^G(Z)$ such that C is the translate of C_0 by a $G(k)$-point.

(2) $-K_W \cdot C \leq 1 + \dim(W)$.

(3) If W has terminal singularities and is not isomorphic to projective space, then C can be chosen to additionally satisfy $-K_W \cdot C \leq \dim(W)$.

In the case where W is a toric variety, Proposition 5.5.6 follows from [MS21, Proposition 6.1]. Therefore, to complete the proof, we may choose to assume that W is not toric. The rest of this section is dedicated to handling this case.

5.5.1 Divisor-curve pairing on orbit closures

We begin with a preliminary result about divisor-curve intersection pairing on orbit closures of horospherical varieties; compare with [CLS11, Lemma 12.5.2] for toric varieties.

Suppose that X is a projective $\mathbb{Q}$-factorial G/H-horospherical variety, and suppose that $n := \mathrm{rk}(N) > 0$. Let $v_0, \ldots, v_l$ denote the primitive generators for the rays in $\Sigma(X)$. For each $0 \leq i \leq l$, if the coloured ray in $\Sigma^c(X)$ generated by v_i has colour set $\{\alpha\}$, then set $w_i := u_\alpha$; otherwise set $w_i := v_i$. Note that w_i is a positive integer multiple of v_i, which we denote by $c_i \in \mathbb{Z}_{>0}$. Each w_i corresponds to a B^--invariant prime divisor $D_i \subseteq X$ in the following way: if w_i generates a non-coloured ray in $\Sigma^c(X)$, then D_i denotes the

corresponding G-invariant prime divisor; and if w_i generates a coloured ray whose colour set consists of a single colour α, then $D_i := D^X_\alpha$. In total, we have

$$\mathcal{D}(X) = \{D_0, \dots, D_l\} \cup \{D^X_\alpha : \alpha \in \mathcal{C} \setminus \mathcal{F}(X)\}.$$

Now consider a G-orbit $\mathcal{O} \subseteq X$ corresponding to some $\tau^c \in \Sigma^c(X)$, and let $Z := \overline{\mathcal{O}}$; we assume that $\tau^c \neq 0^c$, i.e. $Z \neq X$. Let N' denote the coloured lattice on which $\Sigma^c(Z)$ lives, and let $\phi : N \to N'$ denote the quotient map; from Section 3.4.4, we have $N' = N/\tau^c$ which is the lattice $N/(\mathbb{Z}(\tau \cap N))$ with universal colour set $\mathcal{C} \setminus \mathcal{F}(\tau^c)$. The coloured fan $\Sigma^c(Z)$ is $(\Sigma(X)/\tau)^c$, which consists of the images under ϕ of all $\sigma^c \in \Sigma^c(X)$ which contain τ^c. Let $\mathrm{Star}(\tau^c)$ denote the sub-coloured fan of $\Sigma^c(X)$ consisting of coloured cones containing τ^c (this is similar to the star fan from [CLS11, (3.2.8)], except we do not pass through the quotient map ϕ); so we obtain $\Sigma^c(Z)$ by applying ϕ to $\mathrm{Star}(\tau^c)$.

The coloured rays of $\Sigma^c(Z)$ correspond to the coloured rays of $\mathrm{Star}(\tau^c)$. Without loss of generality, the rays of the underlying fan $\mathrm{Star}(\tau)$ are generated by $v_0, \dots, v_b$, and the rays of τ are generated by $v_0, \dots, v_a$ (so $0 \leq a \leq b \leq l$). Then the rays of $\Sigma(Z)$ are generated by the images of $v_{a+1}, \dots, v_b$ under ϕ. Similar to the previous paragraph, let $D'_i \subseteq Z$ denote the B^--invariant prime divisor corresponding to the i-th coloured ray in $\Sigma^c(Z)$. Then

$$\mathcal{D}(Z) = \{D'_{a+1}, \dots, D'_b\} \cup \{D^Z_\alpha : \alpha \in \mathcal{C} \setminus \mathcal{F}(\mathrm{Star}(\tau^c))\}.$$

Note that Z is a projective $\mathbb{Q}$-factorial horospherical G-variety; indeed, it is $\mathbb{Q}$-factorial because $\mathrm{Star}(\tau^c)$ is simplicial.

Lemma 5.5.7. *Consider the notation immediately above. Choose $a+1 \leq i \leq b$ and $\alpha \in \mathcal{C} \setminus \mathcal{F}(\mathrm{Star}(\sigma^c))$. If C is a curve contained in Z, then we have*

$$D_i \cdot C \leq D'_i \cdot C \quad \textit{and} \quad D^X_\alpha \cdot C \leq D^Z_\alpha \cdot C.$$

Proof. Let $\iota : Z \to X$ be the inclusion map. For a B^--invariant $\mathbb{Q}$-Cartier divisor D on X (resp. D' on Z), let $\varphi_D : N_\mathbb{R} \to \mathbb{R}$ (resp. $\varphi_{D'} : N'_\mathbb{R} \to \mathbb{R}$) denote the associated piecewise linear function.

First consider $D := D_i$; let $D' := D'_i$. Then $\iota^* D$ has the associated piecewise linear function $\varphi_{\iota^* D} : N'_\mathbb{R} \to \mathbb{R}$ which sends $\phi(w_j) \mapsto \varphi_D(w_j)$ for each $a+1 \leq j \leq b$. Note that $\phi(w_j) = d_j w'_j$ for some $d_j \in \mathbb{Z}_{>0}$, where $w'_j \in N'$ is the analogue of w_j for Z. Thus

$$\varphi_{\iota^* D}(w'_j) = \frac{1}{d_j}\varphi_D(w_j) = \frac{1}{d_j}\varphi_{D'}(w'_j)$$

for each $a+1 \leq j \leq b$, so $\iota^* D$ and $\frac{1}{d_i} D'$ have the same piecewise linear function. Hence

$$D \cdot C = \iota^* D \cdot C = \frac{1}{d_i} D' \cdot C \leq D' \cdot C.$$

To finish the proof, we consider the case where $D := D^X_\alpha$; let $D' := D^Z_\alpha$. In this case, $\iota^* D$ and D' have the same piecewise linear function (the trivial function), so

$$D \cdot C = \iota^* D \cdot C = D' \cdot C. \qquad \square$$

5.5.2 Proving Proposition 5.5.6

The first main step in the proof is the following essential inequality.

Lemma 5.5.8. *Let W be a G/H-horospherical variety as in Lemma 5.5.1(2), and let $p \in W(k)$. Then there exists a curve $C \subseteq W$ through p which satisfies Proposition 5.5.6(1), and satisfies $D \cdot C \leq 1$ for all $D \in \mathcal{D}(W)$.*

Proof. We prove the result via induction on $n := \mathrm{rk}(N) > 0$. We use the notation of Section 5.5.1 for the vectors $v_i, w_i \in N$ and the corresponding B^--invariant prime divisors $D_i \subseteq W$; note that $l = n$ and $\mathcal{D}(W) = \{D_0, \ldots, D_n\}$. There are relatively prime positive integers $a_0, \ldots, a_n$ such that $\sum_i a_i w_i = 0$. Without loss of generality, say $a_0 = \max_i\{a_i\}$.

Let $\lambda : \mathbb{G}_m \to \mathrm{Aut}^G(W)$ be the one-parameter subgroup corresponding to $v_0 \in N$, and let $C_\lambda := \overline{\{\lambda \cdot eH\}} \subseteq W$ be the closure of the λ-orbit of the base point $eH \in W$. Note that $\{\lambda \cdot eH\} \cong \mathbb{A}^1 \setminus \{0\}$ is contained in the open B^--orbit $B^- H/H$, so $C_\lambda \cong \mathbb{P}^1$ has two points which are in the boundary $W \setminus B^- H/H$. These two points are precisely the λ-invariant points, which are the base points of the G-orbits corresponding to the coloured ray generated by v_0 and the maximal coloured cone which contains $-v_0$. Since the former point is the only intersection of C_λ with D_0, we obtain

$$D_0 \cdot C_\lambda = \varphi_{D_0}(v_0) = \frac{1}{c_0} \varphi_{D_0}(w_0) = \frac{1}{c_0} \leq 1$$

where $\varphi_{D_0} : N_{\mathbb{R}} \to \mathbb{R}$ is the piecewise linear function associated to D_0. Since $\frac{1}{a_0} D_0$ and $\frac{1}{a_i} D_i$ are linearly equivalent for all i, we see that

$$D_i \cdot C_\lambda = \frac{a_i}{a_0} D_0 \cdot C_\lambda \leq 1 \qquad \forall i.$$

To start the induction argument, suppose that $n = 1$. Take $C := C_\lambda$ as above. Note that $D \cdot C \leq 1$ is satisfied for all $D \in \mathcal{D}(W)$, so we just need to check that a G-translate of C passes through any point in W. This holds because C contains the base points of all three G-orbits in W.

Now suppose that $n > 1$. To start, we assume that p is in the open G-orbit $G/H \subseteq W$, or that p is in the G-orbit $\mathcal{O}_0$ corresponding the the coloured ray generated by v_0. Again, take $C := C_\lambda$. Note that $D \cdot C \leq 1$ is satisfied for all $D \in \mathcal{D}(W)$. As above, since C contains the base points of G/H and $\mathcal{O}_0$, we can obtain a G-translate of C which contains p.

Now we finish the proof by handling the case where p is not in G/H or $\mathcal{O}_0$. There exists a G-orbit closure $Z = \overline{\mathcal{O}} \subseteq W$, where $\mathcal{O}$ is the G-orbit corresponding to a coloured ray generated by v_j (with $j \neq 0$), such that $p \in Z$. Now Z is as in Lemma 5.5.1(2) and has coloured fan $\Sigma^c(Z)$ on the coloured lattice $N' := N/\mathbb{Z}v_j$. Note that $0 < \mathrm{rk}(N') < n$, so the induction hypothesis implies that there exists a curve $C \subseteq Z$ which is the translate of the closure of a one-parameter subgroup curve by a $G(k)$-point. By construction, we have $D' \cdot C \leq 1$ for all $D' \in \mathcal{D}(Z)$. Similarly to Section 5.5.1, we can write $\mathcal{D}(Z) = \{D'_i : 0 \leq i \leq n,\ i \neq j\}$ where $D'_i \subseteq Z$ is the prime divisor corresponding to the image of w_i under the quotient $N \to N'$.

By Lemma 5.5.7, we have

$$D_i \cdot C \leq D'_i \cdot C \leq 1 \qquad \forall i \neq j.$$

Lastly, we have

$$D_j \cdot C = \frac{a_j}{a_0} D_0 \cdot C \leq D'_0 \cdot C \leq 1. \qquad \square$$

The above lemma implies that, for non-toric horospherical varieties as in Lemma 5.5.1(2), there exists a curve satisfying Proposition 5.5.6(1). The following result shows that this curve satisfies Proposition 5.5.6(2) and (3). Therefore, the following result concludes the proof of Proposition 5.5.6, which concludes the proof of Proposition 5.5.2.

Lemma 5.5.9. *Let W be a non-toric G/H-horospherical variety as in Lemma 5.5.1(2). Let C be a curve satisfying the conditions in Lemma 5.5.8. Then $-K_W \cdot C \leq \dim(W)$.*

Proof. Let $\mathcal{D}^G(W) \subseteq \mathcal{D}(W)$ denote the set of G-invariant prime divisors in W. By Proposition 3.6.15, we have $-K_W = \sum_{D \in \mathcal{D}(W)} a_D D$ where $a_D = 1$ if $D \in \mathcal{D}^G(W)$, and if D

is a colour divisor then we have

$$a_D = \sum_{\beta \in R^+ \setminus R_I} \langle \beta, \alpha^\vee \rangle.$$

Therefore, Remark 5.5.4 implies

$$\begin{aligned} -K_W \cdot C &= \sum_{D \in \mathcal{D}^G(W)} D \cdot C + \sum_{\alpha \in \mathcal{C}} \sum_{\beta \in R^+ \setminus R_I} \langle \beta, \alpha^\vee \rangle D_\alpha \cdot C \\ &\leq \#\mathcal{D}^G(W) + (\#\mathcal{C} + \#(R^+ \setminus R_I)) \\ &= (\#\mathcal{D}^G(W) + \#\mathcal{C}) + \#(R^+ \setminus R_I) \\ &= (\mathrm{rk}(N) + 1) + \dim(G/P) \\ &= \dim(W) + 1. \end{aligned}$$

Note that we used $\#\mathcal{D}^G(W) + \#\mathcal{C} = \mathrm{rk}(N) + 1$, which holds because both sides of the equality are the number of coloured rays in $\Sigma^c(W)$; and we used Theorem 3.4.25 for the final equality.

Lastly, since G/P is not a product of projective spaces (since W is not toric; see Proposition 3.2.19), Eq. (5.5.5) is a strict inequality of integers, so the above inequality can be strengthened to $-K_W \cdot C \leq \dim(W)$. □

5.6 Proof of Proposition 5.4.6

In this section, we prove Proposition 5.4.6, and in turn, Theorem 5.4.1 (and Theorem 5.2.1). Recall that Proposition 5.4.6 assumes the point p is in the exceptional locus of an elementary MMP step. The following Lemma shows that the fibre F containing p has Picard number 1; compare with [Pas15b, Section 4.6].

Lemma 5.6.1. *Suppose that X is a projective $\mathbb{Q}$-factorial G/H-horospherical variety. Let $\pi := \mathrm{cont}_{\mathcal{R}} : X \to X_{\mathcal{R}}$ be an MMP contraction of an extremal ray $\mathcal{R} \subseteq \mathrm{NE}(X)$, let $x \in X_{\mathcal{R}}$ be a point in the image of the exceptional locus $\mathrm{Exc}(\pi)$, and let F denote the reduction of the fibre $\pi^{-1}(x)$. Then F is a projective $\mathbb{Q}$-factorial horospherical variety with Picard number 1.*

Proof. Let Σ^c be the coloured fan for X and let $\Sigma^c_{\mathcal{R}}$ be the coloured fan for $X_{\mathcal{R}}$, as in Section 5.3.

First suppose that π is a wall curve contraction. In this case, F is the projective horospherical variety (for a Levi subgroup of G) corresponding to the coloured fan Σ_0^c whose underlying fan Σ_0 is described in [CLS11, Lemma 15.4.2] (in their notation, this fan is denoted $\Sigma_{\mathcal{R}}$). Let N_0 denote the coloured lattice on which Σ_0 lives, which has a universal colour set $\mathcal{C}_0$. Note that Σ_0 is a complete simplicial fan and has exactly $\mathrm{rk}(N_0)+1$ rays (see [CLS11, Lemma 15.4.2]).

If π is a Mori fibre space, then N_0 is the kernel of $\Phi : N \to N_{\mathcal{R}}$ with universal colour set $\mathcal{C}_0 = \mathcal{C}_\Phi$ in the notation of Section 5.3.4. Since Σ_0^c consists of the part of Σ^c which is contained in N_0, it follows that Σ_0^c is simplicial, so F is $\mathbb{Q}$-factorial. Furthermore, $\mathcal{F}(\Sigma_0^c) = \mathcal{F}(\Sigma^c) \cap \mathcal{C}_0 = \mathcal{C}_0$, so it follows from Lemma 5.5.1 that F has Picard number 1.

On the other hand, if π is not a Mori fibre space, then $\mathcal{C} = \mathcal{C}_{\mathcal{R}}$ and $\mathcal{F}(\Sigma^c) = \mathcal{F}(\Sigma_{\mathcal{R}}^c)$. There is some $\tau^c \in \Sigma^c$ for which Σ_0^c is the part of $(\Sigma/\tau)^c$ which lives in $N_0 \subseteq N/\tau^c$; compare with the proof of [CLS11, Proposition 15.4.5]. Since $(\Sigma/\tau)^c$ is simplicial, Σ_0^c is also simplicial, so F is $\mathbb{Q}$-factorial. Furthermore, since $\mathcal{C}_0 \subseteq \mathcal{F}(\mathrm{Star}(\tau^c)) \setminus \mathcal{F}(\tau^c) = \mathcal{F}((\Sigma/\tau)^c)$, we have $\mathcal{F}(\Sigma_0^c) = \mathcal{F}((\Sigma/\tau)^c) \cap \mathcal{C}_0 = \mathcal{C}_0$, so it follows from Lemma 5.5.1 that F has Picard number 1.

Finally, we consider the case where π is a colour curve contraction. In this case, F is the projective horospherical variety corresponding to $\Sigma_0^c = 0^c$ on the coloured lattice $N_0 = 0$ with $\mathcal{C}_0 = \{\alpha\}$, where α is the colour appearing in the colour curve contraction π (see Section 5.3.3). Thus, F is a flag variety with Picard number 1. □

With the above lemma, we can use the results in Section 5.5 to say that Theorem 5.4.1 holds for the fibre F containing p. The following lemma provides an inequality which helps us reduce the general problem on X to F.

Lemma 5.6.2. *Let X be a projective terminal $\mathbb{Q}$-factorial G/H-horospherical variety, and let $\pi : X \to Y$ be an MMP contraction. Suppose that $C \subseteq X$ is a curve contracted by π. If F is the reduction of the fibre of π containing C, then $-K_X \cdot C \leq -K_F \cdot C$.*

Proof. Let $y := \pi(F)$. Consider the unique G-orbit in Y containing y, and let V denote the closure of this orbit. Since the fibres of π are irreducible, $\pi^{-1}(V)$ is irreducible, so its reduction Z is a G-orbit closure in X. Let $\tau^c \in \Sigma^c(X)$ denote the coloured cone corresponding to Z. Since F is positive-dimensional and is a general fibre of $\pi|_Z : Z \to V$, we see that Z is contained in the exceptional locus $\mathrm{Exc}(\pi)$. Furthermore, since F is a general fibre of $\pi|_Z$, we have $K_Z|_F = K_F$, so it suffices to show that $-K_X \cdot C \leq -K_Z \cdot C$. We use the notation of Section 5.5.1 for the B^--invariant prime divisors of X and Z.

If C is a wall curve then, using the proof of [MS21, Lemma 7.1], we have $D_i \cdot C \leq 0$ for all $0 \leq i \leq a$. On the other hand, if C is a colour curve corresponding to $\gamma \in \mathcal{C}$, then τ is

the smallest cone in $\Sigma(X)$ containing u_γ, and $D_i \cdot C \leq 0$ for all $0 \leq i \leq a$ by Eq. (5.3.3). Therefore, in any case, we have $D_i \cdot C \leq 0$ for all $0 \leq i \leq a$, and it is easy to check that $D_i \cdot C = 0$ for all $b+1 \leq i \leq l$. By Proposition 3.6.15, we can write

$$-K_X = \sum_{D \in \mathcal{D}(X)} a(D)D \quad \text{and} \quad -K_Z = \sum_{D' \in \mathcal{D}(Z)} a(D')D'$$

for some $a(D), a(D') \in \mathbb{Z}_{>0}$, where $a(D) = a(D') = 1$ for all G-invariant prime divisors $D \subseteq X$, $D' \subseteq Z$.

If we use the notation of Remark 5.5.4 and let $I' \subseteq S$ be the set of simple roots such that $\mathcal{C}' = S \setminus I'$ is the universal colour set for the coloured lattice containing $\Sigma^c(Z)$, then we have

$$a(D^X_\alpha) = a(D^Z_\alpha) + \sum_{\beta \in R^+_{I'} \setminus R^+_I} \langle \beta, \alpha^\vee \rangle.$$

for all colours α in both $\mathcal{C}$ and $\mathcal{C}'$. Since $\alpha \notin I'$, we have $\langle \beta, \alpha^\vee \rangle \leq 0$ for each $\beta \in R^+_{I'} \setminus R^+_I$, so the above equation implies $a(D^X_\alpha) \leq a(D^Z_\alpha)$. It is easy to check that $a(D_i) \leq a(D'_i)$ for all G-invariant divisors with $a+1 \leq i \leq b$, so in total we have $a(D_i) \leq a(D'_i)$ for all $a+1 \leq i \leq b$ and $a(D^X_\alpha) \leq a(D^Z_\alpha)$ for all $\alpha \in \mathcal{C} \setminus \mathcal{F}(\mathrm{Star}(\tau^c))$.

Putting all this and Lemma 5.5.7 together, we obtain

$$\begin{aligned} -K_X \cdot C = \sum_{D \in \mathcal{D}(X)} a(D)D \cdot C &\leq \sum_{i=a+1}^{b} a(D_i)D_i \cdot C + \sum_{\alpha \in \mathcal{C} \setminus \mathcal{F}(\mathrm{Star}(\tau^c))} a(D^X_\alpha)D^X_\alpha \cdot C \\ &\leq \sum_{i=a+1}^{b} a(D'_i)D'_i \cdot C + \sum_{\alpha \in \mathcal{C} \setminus \mathcal{F}(\mathrm{Star}(\tau^c))} a(D^Z_\alpha)D^Z_\alpha \cdot C \\ &= \sum_{D' \in \mathcal{D}(Z)} a(D')D' \cdot C = -K_Z \cdot C. \end{aligned}$$

□

In the case where X is isomorphic to projective space, Theorem 5.4.1 follows from [McK07, Theorem 2.6]. Therefore, the following result concludes the proof of Proposition 5.4.6.

Proposition 5.6.3. *Let X be a projective terminal $\mathbb{Q}$-factorial horospherical variety, let $\pi : X \to Y$ be an MMP contraction corresponding to an extremal ray $\mathcal{R}$, let D be a nef $\mathbb{Q}$-divisor on X such that $D \in \mathcal{R}^\perp$, and let $p \in X(k) \cap \mathrm{Exc}(\pi)$ be a canonically bounded point.*

If X is not isomorphic to projective space, then there exists an irreducible rational curve C through p such that C is unibranch at p, $-K_X \cdot C \leq \dim(X)$, and $\alpha_{p,C}(D) \leq \alpha_{p,\{x_i\}}(D)$ for all Zariski dense sequences $\{x_i\} \subseteq X(k)$.

Proof. This follows from an almost identical proof of [MS21, Proposition 7.2] by using Lemma 5.6.2 (the analogue of [MS21, Lemma 7.1]), Lemma 5.6.1, and Proposition 5.5.2 (the analogue of [MS21, Proposition 6.1]). □

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APPENDICES

Appendix A

Brief review of stacks

We provide a very brief review of algebraic stacks following [Ols16], ending with Appendix A.7 where we define quotient stacks which are important in Chapter 4. Loosely speaking, one can think of an algebraic stack as follows. For a scheme X, we can completely describe it (via the Yoneda lemma) using its functor of points sheaf $h_X = \mathrm{Hom}(-, X)$, which takes in a scheme Y and outputs a *set* $\mathrm{Hom}(Y, X)$. Through this correspondence, we view X as being *represented* by the sheaf h_X. In the case of stacks, we want to allow for such a sheaf to be valued in *groupoids* rather than just sets, which means that we keep track of the elements *together with isomorphisms between elements*; then a stack is an object which is represented by such a sheaf. This framework is very natural in the study of moduli spaces where one wants to parametrize a certain class of objects *up to isomorphism*.

In this chapter, fix a scheme S, and let Sch/S denote the category of S-schemes; recall that an S-scheme is a scheme X equipped with a structure morphism $X \to S$. We always work with Sch/S, meaning that all schemes and stacks are defined over S. One should mainly keep in mind the case where $S = \mathrm{Spec}(k)$.

A.1 Grothendieck topologies on schemes

Topology is a fundamental aspect of geometry wherein one studies open subsets U of a space X. By shifting perspective, we can think of $U \subseteq X$ as a morphism $U \to X$ which has the property of being an open embedding. This leads to the notion of a Grothendieck topology, where the topology is defined by morphisms rather than subsets.

Definition A.1.1 (Grothendieck topology)**.** A *Grothendieck topology* on Sch/S consists of the following: for each scheme X, we have a set $\mathrm{Cov}(X)$ of collections of morphisms

$\{X_i \to X\}_i$ called *coverings of X*, such that the following holds:

(1) If $Y \to X$ is an isomorphism, then $\{Y \to X\} \in \mathrm{Cov}(X)$.

(2) If $\{X_i \to X\}_i \in \mathrm{Cov}(X)$ and $Y \to X$ is any morphism, then $\{X_i \times_X Y \to Y\}_i \in \mathrm{Cov}(Y)$.

(3) If $\{X_i \to X\}_i \in \mathrm{Cov}(X)$ and for each i we have $\{X_{i,j} \to X_i\}_j \in \mathrm{Cov}(X_i)$, then $\{X_{i,j} \to X_i \to X\}_{i,j} \in \mathrm{Cov}(X)$.

We say that a single morphism $f : X \to Y$ of schemes is a *cover* if $\{f\} \in \mathrm{Cov}(Y)$.

Remark A.1.2. Definition A.1.1 works if we replace Sch/S by any (locally small) category, however we are only interested in the category Sch/S. A category together with a Grothendieck topology is called a *site*.

Example A.1.3 (Classical topology)**.** Let X be a topological space, and let $\mathrm{Top}(X)$ be the category whose objects are open subsets of X and the only morphisms are inclusions of open subsets. For any open $U \subseteq X$, let $\mathrm{Cov}(U)$ consist of collections $\{U_i \to U\}_i$ where $\cup_i U_i = U$. Then this defines a Grothendieck topology on $\mathrm{Top}(X)$, which recovers the ordinary topology on X in this more general framework. ♢

Example A.1.4 (Zariski topology)**.** For a scheme U, let $\mathrm{Cov}(U)$ consist of collections $\{U_i \to U\}_i$ of morphisms such that each $U_i \to U$ is an open embedding (in the Zariski topology) and $\cup_i U_i = U$. The category Sch/S together with this Grothendieck topology is called the *big Zariski site* of S. We refer to this as the *Zariski topology* on Sch/S (or on S). ♢

Example A.1.5 (Étale/fppf/smooth topology)**.** We first recall some special types of scheme morphisms $f : X \to Y$:

(1) f is *fppf* if it is flat and locally of finite presentation.

(2) f is *smooth* if it is fppf and all geometric fibres of X are smooth.

(3) f is *étale* if it is smooth and of relative dimension zero (or equivalently, smooth and locally quasifinite).

These types of morphisms are all closed under composition and base change.

For a scheme X, let $\mathrm{Cov}(X)$ consist of collections $\{X_i \to X\}_i$ of morphisms such that each $X_i \to X$ is étale (resp. smooth, resp. fppf) and $\coprod_i X_i \to X$ is surjective. The category

Sch/S together with this Grothendieck topology is called the *big étale site* (resp. *smooth site*, resp. *fppf site*) of S. We refer to this as the *étale topology* (resp. *smooth topology*, resp. *fppf topology*) on Sch/S (or on S).

Note that a single scheme morphism $f : Y \to X$ is an étale cover (resp. smooth cover, resp. fppf cover) of X if f is surjective and étale (resp. smooth, resp. fppf). ♢

Remark A.1.6 (Inclusions of Grothendieck topologies)**.** It is clear that all open embeddings are étale, all étale morphisms are smooth, and all smooth morphisms are fppf. Therefore, we have the following inclusions of Grothendieck topologies:

$$(\text{Zariski topology}) \subseteq (\text{étale topology}) \subseteq (\text{smooth topology}) \subseteq (\text{fppf topology}).$$

For clarity, by *inclusion* here, we mean that $\tau_1 \subseteq \tau_2$ if every cover in τ_1 is a cover in τ_2.

Let $\mathbf{P}$ be a property of morphisms of scheme. Consider an arbitrary fibre diagram of schemes

$$\begin{array}{ccc} X \times_Y X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & Y \end{array}$$

We say that $\mathbf{P}$ is *stable under base change* if, for all fibre diagrams above in which f has $\mathbf{P}$, f' has $\mathbf{P}$. Now let [-] denote a placeholder for any Grothendieck topology on Sch/S (e.g. [-] could be "étale"). We say that $\mathbf{P}$ is *local on the target for the [-] topology* if, for all fibre diagrams above in which g is a cover in the [-] topology, f has $\mathbf{P}$ if and only if f' has $\mathbf{P}$. On the other hand, we say that $\mathbf{P}$ is *local on the source for the [-] topology* if, for all covers $h : X' \to X$ in the [-] topology, $f : X \to Y$ has $\mathbf{P}$ if and only if $f \circ h : X' \to Y$ has $\mathbf{P}$.

We often want to combine *stable under base change* and *local on the target*, so we say that $\mathbf{P}$ is *stable for the [-] topology* if it is both stable under base change and local on the target for the [-] topology. For example, the following properties are stable for the fppf topology: surjective, (locally) of finite type (resp. presentation), separated, proper, unramified, smooth, étale, flat, affine, isomorphism, open (resp. closed) embedding, finite, (locally) quasifinite, quasicompact, quasiseparated, etc.; see The Stacks Project tag 02WE.

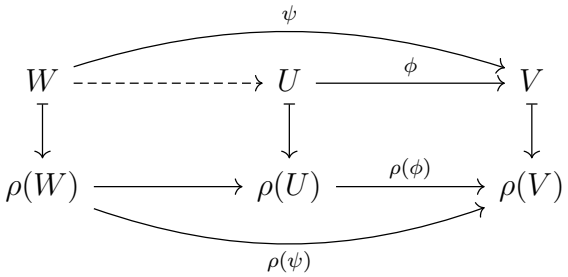
On the other hand, if $\mathbf{P}$ is a property of schemes, then we say that $\mathbf{P}$ *is stable for the [-] topology* if, for every scheme X and every $\{X_i \to X\}_i \in \mathrm{Cov}(X)$, the object X has property $\mathbf{P}$ if and only if each X_i has property $\mathbf{P}$.

A.2 Fibred categories

In this section we define fibred categories. As an analogy, one can think of a map $E \to B$ where E and B are sets and for each $x \in B$ we have the fibre E_x. We want to allow for E, B, and the fibres E_x to be categories rather than just sets. Furthermore, we want to ensure that pullbacks always exist in these categories, because pullbacks and fibred diagrams are instrumental in the construction of algebraic stacks.

Firstly, a *category over* Sch/S is a pair $(\mathfrak{F}, \rho)$ where $\mathfrak{F}$ is a category and $\rho : \mathfrak{F} \to \mathrm{Sch}/S$ is a functor. We say that a morphism $\phi : U \to V$ in $\mathfrak{F}$ is *cartesian* if, for all morphisms $\psi : W \to V$ in $\mathfrak{F}$ such that $\rho(\psi)$ factors through $\rho(\phi)$, there exists a unique dotted arrow $W \to U$ which makes the following diagram commute:

(A.2.1)



If $\phi : U \to V$ is a cartesian morphism in $\mathfrak{F}$, then we call U a *pullback* of V along $\rho(\phi)$. Note that, if we have two pullbacks $\phi : U \to V$ and $\phi' : U' \to V$ of V along $\rho(\phi)$, then there exists a unique $\lambda : U \to U'$ such that $\rho(\lambda) = \mathrm{id}_{\rho(U)}$ and $\phi' \circ \lambda = \phi$.

For a scheme X, we let $\mathfrak{F}(X)$ denote the *fibre over* X, i.e. the category with:

- Objects: $x \in \mathfrak{F}$ such that $\rho(x) = X$.
- Morphisms: $x \to x'$ in $\mathfrak{F}$ such that $\rho(x \to x') = \mathrm{id}_X$.

Definition A.2.2 (Fibred category)**.** A *fibred category over* Sch/S is a category $\rho : \mathfrak{F} \to \mathrm{Sch}/S$ over Sch/S such that pullbacks always exist, i.e. for each scheme morphism $f : X \to Y$ and each $y \in \mathfrak{F}(Y)$, there exists a cartesian morphism $\phi : x \to y$ such that $\rho(\phi) = f$.

Remark A.2.3. Definition A.2.2 works if we replace Sch/S by any site (see Remark A.1.2).

If $\rho_{\mathfrak{F}} : \mathfrak{F} \to \mathrm{Sch}/S$ and $\rho_{\mathfrak{G}} : \mathfrak{G} \to \mathrm{Sch}/S$ are fibred categories over Sch/S, then a *morphism of fibred categories* is a functor $F : \mathfrak{F} \to \mathfrak{G}$ such that $\rho_{\mathfrak{G}} \circ F = \rho_{\mathfrak{F}}$ and F sends cartesian morphisms to cartesian morphisms. If $F, G : \mathfrak{F} \to \mathfrak{G}$ are morphisms of

fibred categories, then a *base preserving natural transformation* is a natural transformation $\alpha : F \to G$ such that, for each $U \in \mathfrak{F}$, the morphism $\alpha_U : F(U) \to G(U)$ in $\mathfrak{G}$ is a morphism in $\mathfrak{G}(\rho_{\mathfrak{F}}(U))$. Let $\mathrm{Hom}_{\mathrm{Sch}/S}(\mathfrak{F}, \mathfrak{G})$ denote the category whose objects are morphisms of fibred categories $\mathfrak{F} \to \mathfrak{G}$ and whose morphisms are base preserving natural transformations.

Example A.2.4 (Localized category)**.** Fix a scheme X. Then Sch/X is a fibred category over Sch/S where the structure functor $\mathrm{Sch}/X \to \mathrm{Sch}/S$ is forgetful, i.e. it sends an X-scheme to itself viewed as an S-scheme. For a scheme Y, the fibre $(\mathrm{Sch}/X)(Y)$ is the category of all X-scheme structures for Y, i.e. the objects are really morphisms $Y \to X$. That is, $(\mathrm{Sch}/X)(Y) = \mathrm{Hom}(Y, X)$. ♢

Example A.2.5 (Fibred category of quasicoherent sheaves)**.** Let QCoh be the category with:

- Objects: $(X, \mathcal{F})$ with $\mathcal{F}$ a quasicoherent sheaf on a scheme X (*sheaf* here means a sheaf in the usual sense for the Zariski topology).
- Morphisms: $(X, \mathcal{F}) \to (X', \mathcal{F}')$ is a scheme morphism $f : X \to X'$ and a morphism of sheaves $\mathcal{F} \to f^*\mathcal{F}'$.

Then $\rho : \mathrm{QCoh} \to \mathrm{Sch}/S$ via $(X, \mathcal{F}) \mapsto X$ is a fibred category. For a scheme X, the fibre $\mathrm{QCoh}(X)$ is the category of quasicoherent sheaves on X. ♢

Example A.2.6 (Fibre product as a fibred category)**.** The fibre product of two schemes is a scheme *which is defined up to unique isomorphism*. In the setting of schemes, this is a minor technicality that is disregarded, but in the theory of stacks this is an important technicality. In this example, we see how fibred products can keep track of this technicality, i.e. keep track of the different choices of fibre products.

Let $f : X \to Y$ be a scheme morphism. Consider the category $\mathfrak{F}$ with:

- Objects: (T, P, a, b) where T and P are schemes, and $a : P \to T$ and $b : P \to X$ are morphisms such that

$$\begin{array}{ccc} P & \xrightarrow{b} & X \\ {\scriptstyle a}\downarrow & & \downarrow{\scriptstyle f} \\ T & \longrightarrow & Y \end{array}$$

  is cartesian.
- Morphisms: $(T, P, a, b) \to (T', P', a', b')$ is a pair of morphisms (α, β) where $\alpha : T \to T'$ is an scheme morphism and $\beta : P \to P'$ satisfies $a' \circ \beta = \alpha \circ a$ and $b' \circ \beta = b$.

We can make this into a fibred category $\rho : \mathfrak{F} \to \mathrm{Sch}/S$ by sending (T, P, a, b) to T (one can check that this satisfies the axioms of being a fibred category). Then, for a scheme T, the fibre $\mathfrak{F}(T)$ is the category of cartesian diagrams of the form above, i.e. the objects are choices of fibre product $T \times_Y X$. Any two objects in $\mathfrak{F}(T)$ are uniquely isomorphic. ♢

Proposition A.2.7 (2-Yoneda lemma)**.** *Let $\rho : \mathfrak{F} \to \mathrm{Sch}/S$ be a fibred category and pick a scheme X. The functor*

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sch}/S}(\mathrm{Sch}/X, \mathfrak{F}) &\longrightarrow \mathfrak{F}(X) \\ (F : \mathrm{Sch}/X \to \mathfrak{F}) &\longmapsto F(\mathrm{id}_X) \end{aligned}$$

is an equivalence of categories. In particular, for any other scheme Y, we have an equivalence of categories

$$\mathrm{Hom}_{\mathrm{Sch}/S}(\mathrm{Sch}/X, \mathrm{Sch}/Y) \longrightarrow \mathrm{Hom}(Y, X).$$

Proof. See [Ols16, Proposition 3.2.2]. □

Notation A.2.8. Let $\mathfrak{F} \to \mathrm{Sch}/S$ be a fibred category and pick a scheme X. We write $X \to \mathfrak{F}$ to denote a morphism of fibred categories $\mathrm{Sch}/X \to \mathfrak{F}$. By Proposition A.2.7, we can also think of $X \to \mathfrak{F}$ as an object in $\mathfrak{F}(X)$.

Recall that a *groupoid* is a category in which every morphism is an isomorphism. We can think of a set as a special kind of groupoid: every element in the set is represented by an object in the category, and the only morphisms are the identity maps.

Definition A.2.9 (Fibred in groupoids/sets)**.** A *category fibred in groupoids* (resp. *category fibred in sets*) is a fibred category $\rho : \mathfrak{F} \to \mathrm{Sch}/S$ such that, for each scheme X, the fibre $\mathfrak{F}(X)$ is a groupoid (resp. a set, in the sense mentioned above).

Remark A.2.10 (Fibred products of categories fibred in groupoids)**.** If $\mathfrak{F}_i$ are categories fibred in groupoids over Sch/S for $i = 1, 2, 3$, and we have morphisms of fibred categories $\mathfrak{F}_1 \to \mathfrak{F}_3$ and $\mathfrak{F}_2 \to \mathfrak{F}_3$, then we can form the fibred product $\mathfrak{F}_1 \times_{\mathfrak{F}_3} \mathfrak{F}_2$ which is a category fibred in groupoids over Sch/S. We do not go through the details here, but one can read [Ols16, Section 3.4.9].

A.3 Presheaves and sheaves on sites

Now we define presheaves and sheaves in the context of categories with Grothendieck topologies, which is very similar to the usual definitions in the case of classical topology.

Throughout this section, consider Sch/S equipped with a fixed Grothendieck topology (typically we consider the étale topology).

Definition A.3.1 (Presheaf/sheaf)**.** A functor $F : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$ is called a *presheaf* on Sch/S. We say that F is a *sheaf* if it further satisfies the following: for each scheme X and covering $\{X_i \to X\}_i \in \mathrm{Cov}(X)$, the sequence

$$F(X) \longrightarrow \prod_i F(X_i) \rightrightarrows \prod_{i,j} F(X_i \times_X X_j) \tag{A.3.2}$$

is exact,[1] where the two right-most maps are induced by the projections $X_i \times_X X_j \to X_i$ and $X_i \times_X X_j \to X_j$.

A fibred category $\mathfrak{F} \to \mathrm{Sch}/S$ is similar to a presheaf $F : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$, except that $F(X)$ is a set for each $X \in \mathrm{Sch}/S$, whereas $\mathfrak{F}(X)$ is a category. In the case where this category $\mathfrak{F}(X)$ is really a set, we expect fibred categories to just be presheaves.

Let $\rho : \mathfrak{F} \to \mathrm{Sch}/S$ be a category fibred in sets. For a morphism $f : X \to Y$ of schemes, we have a well-defined pullback map $f^* : \mathfrak{F}(Y) \to \mathfrak{F}(X)$ which is compatible with composition (this is well-defined since, for a category fibred in *sets*, the pullback is unique). This means we have a presheaf $F_{\mathfrak{F}} : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$ via $X \mapsto \mathfrak{F}(X)$.

Conversely, let $F : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$ be a presheaf. We construct an associated fibred category $\rho_F : \mathfrak{F}_F \to \mathrm{Sch}/S$ as follows:

- Objects in $\mathfrak{F}_F$: (X, x) where X is a scheme and $x \in F(X)$.
- Morphisms in $\mathfrak{F}_F$: $(X, x) \to (X', x')$ is a morphism $f : X \to X'$ of schemes such that $f^* x' = x$ in $F(X)$.

The map ρ_F sends (X, x) to X, so the fibre over X is the set $F(X)$.

The following proposition tells us that, in the manner described above, presheaves on Sch/S are the same as categories fibred in sets over Sch/S. For this reason, we often use the two objects interchangeably, and do not distinguish between a presheaf F and its corresponding fibred category.

Proposition A.3.3 (Presheaves $\leftrightarrow$ categories fibred in sets)**.** *The functor*

$$(\textit{presheaves on } \mathrm{Sch}/S) \longrightarrow (\textit{categories fibred in sets over } \mathrm{Sch}/S)$$

[1] By *exact* here, we mean that the first map (i.e. the left-most map) is the equalizer of the two right-most maps. Note that this implies the left-most map is injective.

$$F \longmapsto \mathfrak{F}_F$$

defined above is an equivalence of categories.

Proof. See [Ols16, Proposition 3.2.8]. □

Example A.3.4 (Representable presheaves)**.** For any scheme X, we have a presheaf

$$h_X : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set} \qquad Y \mapsto \mathrm{Hom}(Y, X).$$

called the *functor of points* for X. By the Yoneda lemma, the assignment $X \mapsto h_X$ is a fully faithful functor from Sch/S to the category of presheaves on Sch/S. As in Notation A.2.8, we often do not distinguish between X and h_X.

We say that a presheaf F on Sch/S is *representable* if $F \cong h_X$ (isomorphism of functors) for some scheme X. ♢

Theorem A.3.5 (h_X is an fppf sheaf)**.** *For any scheme X, the functor of points h_X is a sheaf in the fppf topology on* Sch/S*. Thus, h_X is a sheaf in the smooth, étale, and Zariski topologies on* Sch/S*.*

Proof. See [Ols16, Section 4.1]. □

Example A.3.6 (Isom presheaf)**.** Let $\rho : \mathfrak{F} \to \mathrm{Sch}/S$ be a category fibred in groupoids. Let X be a scheme and $x, x' \in \mathfrak{F}(X)$. Define a presheaf

$$\underline{\mathrm{Isom}}(x, x') : (\mathrm{Sch}/X)^{\mathrm{op}} \to \mathrm{Set}$$

as follows.

Given an object of Sch/X, which is really a scheme morphism $f : Y \to X$, choose two pullbacks f^*x and f^*x' in $\mathfrak{F}(Y)$. Then let

$$\underline{\mathrm{Isom}}(x, x')(f : Y \to X) := \mathrm{Isom}_{\mathfrak{F}(Y)}(f^*x, f^*x') = \mathrm{Hom}_{\mathfrak{F}(Y)}(f^*x, f^*x')$$

where $\mathrm{Isom}_{\mathfrak{F}(Y)}$ denotes isomorphisms in the category $\mathfrak{F}(Y)$, which is equal to $\mathrm{Hom}_{\mathfrak{F}(Y)}$ because $\mathfrak{F}(Y)$ is a groupoid. It is easy to check that $\underline{\mathrm{Isom}}(x, x')$ is indeed a presheaf. Note that, up to canonical isomorphism (of presheaves), $\underline{\mathrm{Isom}}(x, x')$ is independent of the choice of pullbacks.

In particular, when $x = x'$, this $\underline{\mathrm{Isom}}$ presheaf becomes a presheaf of groups denoted

$$\underline{\mathrm{Aut}}(x) : (\mathrm{Sch}/X)^{\mathrm{op}} \to \mathrm{Group}.$$ ♢

A.4 Categorical stacks

To begin defining algebraic stacks we first define "categorical" stacks, which are built from the category theory and Grothendieck topology that we have seen thus far, but lack the algebro-geometric structure that will be put on algebraic stacks. Throughout this section, consider Sch/S equipped with a fixed Grothendieck topology (typically we consider the étale topology).

Let $\rho : \mathfrak{F} \to \mathrm{Sch}/S$ be a fibred category. In what follows, we aim to replicate a fundamental notion in topology known as gluing. As an example, one can construct a vector bundle E on a space X by defining a collection $\{E_i\}_i$ of vector bundles on an open cover $\{U_i\}_i$ of X which glue along the intersections and satisfy the cocycle conditions. We extend this idea to objects in our fibred category $\mathfrak{F}$. For any collection of scheme morphisms $\{X_i \to Y\}_i$, we define a category $\mathfrak{F}(\{X_i \to Y\}_i)$ as follows:

- Objects: $(\{E_i\}_i, \{\sigma_{i,j}\}_{i,j})$ where each $E_i \in \mathfrak{F}(X_i)$ and each $\sigma_{i,j} : \mathrm{pr}_1^* E_i \to \mathrm{pr}_2^* E_j$ is an isomorphism in $\mathfrak{F}(X_i \times_Y X_j)$ such that the following diagram commutes in $\mathfrak{F}(X_i \times_Y X_j \times_Y X_l)$:

$$\begin{array}{ccccc} \mathrm{pr}_{1,2}^* \mathrm{pr}_1^* E_i & \xrightarrow{\mathrm{pr}_{1,2}^* \sigma_{i,j}} & \mathrm{pr}_{1,2}^* \mathrm{pr}_2^* E_j & = & \mathrm{pr}_{2,3}^* \mathrm{pr}_1^* E_j \\ \| & & & & \downarrow{\scriptstyle \mathrm{pr}_{2,3}^* \sigma_{j,l}} \\ \mathrm{pr}_{1,3}^* \mathrm{pr}_1^* E_i & \xrightarrow[\mathrm{pr}_{1,3}^* \sigma_{i,l}]{} & \mathrm{pr}_{1,3}^* \mathrm{pr}_2^* E_l & = & \mathrm{pr}_{2,3}^* \mathrm{pr}_2^* E_l \end{array} \tag{A.4.1}$$

  Here the equality maps are the canonical isomorphisms, and we are using the $\mathrm{pr}__$ maps for the various projections. We call $\{\sigma_{i,j}\}_{i,j}$ the *descent data* for $\{E_i\}_i$.

- Morphisms: $(\{E_i\}_i, \{\sigma\}_{i,j}) \to (\{E_i'\}_i, \{\sigma_{i,j}'\}_{i,j})$ is a collection of morphisms $\{g_i : E_i \to E_i'\}_i$ in $\mathfrak{F}(X_i)$ such that the following diagram commutes for all i, j:

$$\begin{array}{ccc} \mathrm{pr}_1^* E_i & \xrightarrow{\mathrm{pr}_1^* g_i} & \mathrm{pr}_1^* E_i' \\ {\scriptstyle \sigma_{i,j}}\downarrow & & \downarrow{\scriptstyle \sigma_{i,j}'} \\ \mathrm{pr}_2^* E_j & \xrightarrow{\mathrm{pr}_2^* g_j} & \mathrm{pr}_2^* E_j' \end{array}$$

There is a functor

$$\epsilon : \mathfrak{F}(Y) \longrightarrow \mathfrak{F}(\{X_i \to Y\}_i) \tag{A.4.2}$$

described as follows. Denote the map $X_i \to Y$ by f_i. Given $E_0 \in \mathfrak{F}(Y)$, there is a canonical isomorphism $\sigma_{i,j}^{\text{can}} : \text{pr}_1^* f_i^* E_0 \to \text{pr}_2^* f_j^* E_0$ since both objects are pullbacks of E_0 along $X_i \times_Y X_j \to Y$. Then ϵ sends E_0 to $(\{f_i^* E_0\}_i, \{\sigma_{i,j}^{\text{can}}\}_{i,j})$.

Definition A.4.3 (Effective descent)**.** Consider the setup above in which $\rho : \mathfrak{F} \to \text{Sch}/S$ is a fibred category.

(1) We say that descent data $\{\sigma_{i,j}\}_{i,j}$ for $\{E_i\}_i$ above is *effective* if $(\{E_i\}_i, \{\sigma_{i,j}\}_{i,j})$ is in the essential image of ϵ.

(2) The collection of morphisms $\{X_i \to Y\}_i$ is said to be of *effective descent for* $\mathfrak{F}$ if the functor ϵ in Eq. (A.4.2) is an equivalence of categories.

Remark A.4.4. Consider the setup above. We have that $\{X_i \to Y\}_i$ is of effective descent for $\mathfrak{F}$ if and only if the single morphism $\coprod_i X_i \to Y$ is of effective descent for $\mathfrak{F}$; see [Ols16, Lemma 4.2.7]. Thus, the notion of effective descent in general boils down to effective descent for single morphisms.

Remark A.4.5 (Morphisms with a section are of effective descent)**.** In the setup above, consider a single morphism $f : X \to Y$ of schemes. If f admits a section $Y \to X$, then f is of effective descent for $\mathfrak{F}$. See [Ols16, Proposition 4.2.10] for a proof.

We define a sheaf category, denoted Sh, as follows:

- Objects: (X, E) with X a scheme and E a sheaf on X.
- Morphisms: $(X, E) \to (X', E')$ is a map of schemes $f : X \to X'$ together with a morphism $E \to f^* E'$ of sheaves on Sch/X.

Then $\text{Sh} \to \text{Sch}/S$ via $(X, E) \mapsto X$ makes Sh into a fibred category. The fibre $\text{Sh}(X)$ is the category of sheaves on Sch/X.

Theorem A.4.6 (Sheaves have effective descent)**.** *If $f : X \to Y$ is a cover in* Sch/S*, then f is of effective descent for* Sh*.*

Proof. See [Ols16, Theorem 4.2.12]. □

Now we define the purely "categorical" version of stacks, which are simply called *stacks*. Later we will add some conditions to this definition to obtain *algebraic* stacks.

Definition A.4.7 ("Categorical" stack)**.** A category fibred in groupoids $\rho : \mathfrak{F} \to \mathrm{Sch}/S$ is a *stack* if, for each scheme X and $\{X_i \to X\}_i \in \mathrm{Cov}(X)$, this collection of morphisms is of effective descent for $\mathfrak{F}$.

A *morphism of stacks* $f : \mathfrak{F}_1 \to \mathfrak{F}_2$ is morphism of fibred categories over Sch/S.

Remark A.4.8 (Fibred products of "categorical" stacks)**.** If $\mathfrak{F}_i$ are stacks for $i = 1, 2, 3$, and we have morphisms of stacks $\mathfrak{F}_1 \to \mathfrak{F}_3$ and $\mathfrak{F}_2 \to \mathfrak{F}_3$, then from Remark A.2.10 we can form the fibred product $\mathfrak{F}_1 \times_{\mathfrak{F}_3} \mathfrak{F}_2$ as a category fibred in groupoids over Sch/S. This fibred product is a stack; see [Ols16, Proposition 4.6.4].

Proposition A.4.9 (Characterization of "categorical" stack)**.** *A category fibred in groupoids $\mathfrak{F} \to \mathrm{Sch}/S$ is a stack if and only if the following conditions hold:*

(1) For each scheme X and $x, y \in \mathfrak{F}(X)$, the presheaf $\underline{\mathrm{Isom}}(x, y)$ on Sch/X (see Example A.3.6) is a sheaf.

(2) For each scheme X and $\{X_i \to X\}_i \in \mathrm{Cov}(X)$, all descent data with respect to this cover is effective.

Proof. See [Ols16, Proposition 4.6.2]. □

A.5 Algebraic spaces

As a precursor to algebraic stacks we define algebraic spaces. One can think of an algebraic space as a natural generalization of a scheme where we replace the Zariski topology with the étale topology. Throughout this section, unless otherwise specified, we consider Sch/S equipped with the étale topology.

Given a morphism $f : F \to G$ of sheaves on Sch/S, we say that f is *representable by schemes* if, for each scheme T and morphism $T \to G$, the fibre product $F \times_G T$ is a scheme. Recall that here we are viewing T as the sheaf $h_T = \mathrm{Hom}(-, T)$ so that $T \to G$ is a map of sheaves (see Notation A.2.8), and $F \times_G T$ being a scheme means that it is representable by a scheme (see Example A.3.4).

If $f : F \to G$ is representable by schemes and $\mathbf{P}$ is a property of morphisms of schemes which is stable for the étale topology, then we say that *f has property* $\mathbf{P}$ if, for each scheme T and $T \to G$, the projection $F \times_G T \to T$ (which is a morphism of schemes by representability) has property $\mathbf{P}$.

Example A.5.1 (Maps of representable sheaves are representable by schemes)**.** If F and G are representable sheaves on Sch/S, i.e. they are schemes, then every morphism $f : F \to G$ is representable by schemes. Indeed, if $F = h_X$ and $G = h_Y$, then f is induced by a scheme morphism $X \to Y$, so for any scheme T and morphism $T \to G$ (induced by a scheme morphism $T \to Y$) we have $F \times_G T \cong h_{T \times_Y X}$.

Furthermore, in this situation, if $\mathbf{P}$ is a property of morphisms of schemes that is stable for the étale topology, then the map of schemes $X \to Y$ has $\mathbf{P}$ if and only if the induced map of sheaves $F = h_X \to G = h_Y$ has $\mathbf{P}$ in the sense defined above; see [Ols16, Lemma 5.1.8]. ♢

Definition A.5.2 (Algebraic space)**.** An *algebraic space* (over S) is a sheaf for the étale topology $X : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$ such that the following conditions hold:

(1) The diagonal map $\Delta : X \to X \times_S X$ is representable by schemes.

(2) There exists a scheme U and an étale cover $U \to X$.

A *morphism of algebraic spaces* $f : X \to Y$ is a morphism of presheaves on $(\mathrm{Sch}/S)^{\mathrm{op}}$; using Proposition A.3.3, this is the same as a morphism of fibred categories when X and Y are regarded as categories fibred in sets over Sch/S.

Remark A.5.3. In Definition A.5.2, condition (1) is equivalent to the following property: every morphism $T \to X$ from a scheme T is representable by schemes; see [Ols16, Lemma 5.1.9]. This explains why condition (2) makes sense: the map $U \to X$ is representable by schemes, so the property of "étale surjection" can be applied here.

Example A.5.4 (Schemes are algebraic spaces)**.** Any scheme X is an algebraic space. Indeed, the scheme X can be viewed as the presheaf $h_X : (\mathrm{Sch}/S)^{\mathrm{op}} \to \mathrm{Set}$, which is a sheaf in the étale topology by Theorem A.3.5. Condition (1) of Definition A.5.2 is satisfied since X and $X \times_S X$ are schemes. Lastly, condition (2) is satisfied since the identity $X \to X$ is an étale cover. ♢

Consider an algebraic space X. If $\mathbf{P}$ is a property of schemes which is local in the étale topology, then we say that *X has property* $\mathbf{P}$ if there exists an étale cover $U \to X$ from a scheme U which has property $\mathbf{P}$.

We can also phrase the previous definition for morphism properties in a similar way. That is, let $\mathbf{P}$ be a property of morphisms of schemes which is stable in the étale topology, and let $f : X \to Y$ be a morphism of algebraic spaces which is representable by schemes. Then it is easy to check that f has property $\mathbf{P}$ if and only if there exists an étale cover $V \to Y$ such that $V \times_Y X \to V$ has property $\mathbf{P}$.

If we drop the assumption that f is representable by schemes, and if we strengthen the property $\mathbf{P}$ to assume that it is local on the source for the étale topology (in addition to being stable for the étale topology), then we say that *f has property $\mathbf{P}$* if there exist étale covers $V \to Y$ and $U \to X$ such that $U \times_Y V \to V$ (which is a morphism of schemes) has property $\mathbf{P}$. In the case where f is representable by schemes, this definition agrees with the previous one.

A.6 Algebraic stacks

In this section, we use the combination of all our previous work to define an algebraic stack. Throughout this section, unless otherwise specified, we consider Sch/S equipped with the étale topology. By *stack* we mean a stack in the sense of Definition A.4.7 with respect to the étale topology.

A morphism of stacks $f : \mathcal{X} \to \mathcal{Y}$ is *representable* (by algebraic spaces) if, for each algebraic space X and morphism $X \to \mathcal{Y}$, the fibre product $\mathcal{X} \times_{\mathcal{Y}} X$ is an algebraic space. Note that f is representable if and only if, for each scheme X and morphism $X \to \mathcal{Y}$, the fibre product $\mathcal{X} \times_{\mathcal{Y}} X$ is an algebraic space; see [Ols16, Lemma 8.1.3].

If $f : \mathcal{X} \to \mathcal{Y}$ is a representable morphism of stacks and $\mathbf{P}$ is a property of morphisms of algebraic spaces which is stable for the smooth topology (on the category of algebraic spaces), then we say that *f has property $\mathbf{P}$* if, for each morphism $Y \to \mathcal{Y}$ from an algebraic space, the morphism of algebraic spaces (by representability) $\mathcal{X} \times_{\mathcal{Y}} Y \to Y$ has property $\mathbf{P}$.

Again, we can define properties of morphisms of stacks which are *not* representable. To do this, it helps to define the following notational package. Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks. A *chart for f* is a commutative diagram of the form:

(A.6.1)
$$\begin{array}{ccccc} & & h & & \\ & \overset{g}{\nearrow} & & \searrow & \\ X & & \longrightarrow & & Y \\ & \searrow & \downarrow & \lrcorner & \downarrow{\scriptstyle p} \\ & & \mathcal{X} & \xrightarrow{f} & \mathcal{Y} \end{array}$$

where X and Y are algebraic spaces, the square is cartesian, and g and p are smooth covers. If we require X and Y to be schemes, then this is a *chart for f by schemes*.

Let $\mathbf{P}$ be a property of morphisms of stacks which is local on the source and is stable in the smooth topology. We say that *$f : \mathcal{X} \to \mathcal{Y}$ has property $\mathbf{P}$* if there exists a chart for f by schemes such that h in Eq. (A.6.1) has property $\mathbf{P}$. In the case where f is representable,

this definition agrees with the previous one.

Definition A.6.2 (Algebraic stack)**.** A stack $\mathcal{X}$ (over S) is an *algebraic stack* (or *Artin* stack) if the following conditions hold:

(1) The diagonal map $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable.

(2) There exists a smooth cover $X \to \mathcal{X}$ with X a scheme.

A *morphism of algebraic stacks* $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of stacks, i.e. a morphism of categories fibred over Sch/S.

Consider an algebraic stack $\mathcal{X}$. If $\mathbf{P}$ is a property of schemes which is stable in the smooth topology, then we say that *$\mathcal{X}$ has property* $\mathbf{P}$ is there exists a smooth cover $X \to \mathcal{X}$ from a scheme X which has property $\mathbf{P}$.

Remark A.6.3. In Definition A.6.2, condition (1) implies that every morphism $T \to \mathcal{X}$ from a scheme (or even an algebraic space) is representable; see [Ols16, Remark 8.1.6]. In particular, the morphism $X \to \mathcal{X}$ in condition (2) makes sense.

Remark A.6.4 (Fibred products of algebraic stacks)**.** If $\mathcal{X}_i$ are stacks for $i = 1, 2, 3$, and we have morphisms of stacks $\mathcal{X}_1 \to \mathcal{X}_3$ and $\mathcal{X}_2 \to \mathcal{X}_3$, then from Remark A.4.8 we can form the fibred product $\mathcal{X}_1 \times_{\mathcal{X}_3} \mathcal{X}_2$ as a stack. This fibred product is an algebraic stack; see [Ols16, Proposition 8.1.16].

Proposition A.6.5. *Let $\mathcal{X}$ be an algebraic stack. Then, for every scheme U and $x, x' \in \mathcal{X}(U)$, the sheaf $\underline{\mathrm{Isom}}(x, x')$ is an algebraic space over U. In particular, for every scheme U and $x \in \mathcal{X}(U)$, the sheaf $\underline{\mathrm{Aut}}(x)$ is an algebraic space.*

Proof. [Ols16, Lemma 8.1.8]. □

Remark A.6.6. Suppose that $S = \mathrm{Spec}(k)$ for a field k. Any group algebraic space (i.e. an algebraic space which is a sheaf of groups) over k which is quasiseparated is a scheme. In particular, if $\mathcal{X}$ is an algebraic stack over k which is quasiseparated, then for every scheme U and $x \in \mathcal{X}(U)$, the sheaf $\underline{\mathrm{Aut}}(x)$ is a group scheme over k (by Proposition A.6.5).

Example A.6.7 (Algebraic spaces are algebraic stacks)**.** Any algebraic space X is an algebraic stack. Indeed, X is a presheaf over Sch/S, which is a category fibred in sets over Sch/S (see Proposition A.3.3), so X is a category fibred in groupoids over Sch/S. By Theorem A.4.6, X is a stack. It is easy to see that X satisfies conditions (1) and (2) of Definition A.6.2, meaning that X is an algebraic stack.

In particular, schemes are algebraic stacks (see Example A.5.4). Thus, we have the following diagram of inclusions which relates our three main algebro-geometric objects:

$$(\text{schemes}) \subseteq (\text{algebraic spaces}) \subseteq (\text{algebraic stacks}) \qquad \diamondsuit$$

Proposition A.6.8 (Which algebraic stacks are algebraic spaces)**.** *Let $\mathcal{X}$ be an algebraic stack. Then $\mathcal{X}$ is an algebraic space if and only if, for each scheme U and $x \in \mathcal{X}(U)$, the sheaf of groups $\underline{\mathrm{Aut}}(x)$ is trivial (i.e. $\underline{\mathrm{Aut}}(x)$ is the trivial group scheme over U). In fact, if the latter property holds whenever U is a geometric point (i.e. the spectrum of an algebraically closed field), then $\mathcal{X}$ is an algebraic space.*

Proof. See [Ols16, Corollary 8.3.5] and [Ols16, Remark 8.3.6]. □

A.7 Quotient stacks

Finally, we finish by defining quotient stacks, which are a specific kind of algebraic stack that arise through quotienting a scheme (or algebraic space) by a group action. Throughout this section, unless otherwise specified, we consider Sch/S equipped with the étale topology. We fix a group scheme K which is smooth and affine over S, and we fix an algebraic space X on which K acts.

Define a category $[X/K]$ fibred over Sch/S as follows:

- Objects: (T, P, π) where T is a scheme, $P \to T$ is a principal K-bundle,[1] and $\pi : P \to X$ is a K-equivariant morphism.
- Morphisms: $(T, P, \pi) \to (T', P', \pi')$ is a scheme morphism $T \to T'$ and an equivariant map of principal K-bundles $P \to P'$ such that the following diagram commutes and the square is cartesian:

$$\begin{array}{ccccc} & & \overset{\pi}{\frown} & & \\ P & \longrightarrow & P' & \xrightarrow{\pi'} & X \\ \downarrow & \ulcorner & \downarrow & & \\ T & \longrightarrow & T' & & \end{array}$$

Note that the fibre $[X/K](T)$, where T is a scheme, is the category whose objects are principal K-bundles $P \to T$ together with a K-equivariant map $P \to X$. These fibres are

[1] Since K is affine over S, a principal K-bundle is equivalent to a K-torsor; see [Ols16, Proposition 4.5.6].

groupoids because, in the case where $T \to T' = T$ is the identity map in the above diagram, the map $P \to P'$ is an isomorphism.

By [Ols16, Example 8.1.12], $[X/K]$ is an algebraic stack. In particular, we have a smooth cover $X \to [X/K]$ which is defined as the object of $[X/K](X)$ given by the trivial principal K-bundle $K \times X$ together with the K-equivariant map $K_X \to X$ which is the K-action map on X. One should think of the map $X \to [X/K]$ as the quotient map.

Definition A.7.1 (Quotient stack)**.** The algebraic stack $[X/K]$ constructed above is called a *quotient stack*.

Remark A.7.2. Since $X \to [X/K]$ is a smooth cover, if $\mathbf{P}$ is a property of schemes which is stable in the smooth topology, then $[X/K]$ has $\mathbf{P}$ if and only if X has $\mathbf{P}$. For example, $[X/K]$ is smooth if and only if X is smooth.

Example A.7.3 (Classifying stack)**.** The *classifying stack* of K is the quotient stack $BK := [S/K]$ where K acts trivially on S. For any scheme T, the objects in $BK(T)$ are just principal K-bundles $P \to T$, since the equivariant map $P \to S$ is always the structure morphism.

For example, $B\mathbb{G}_m = [S/\mathbb{G}_m]$ parametrizes principal $\mathbb{G}_m$-bundles, which correspond naturally to line bundles, so one can view $B\mathbb{G}_m$ as the moduli space of line bundles. ♢

Example A.7.4 (Free group action)**.** Suppose that X is a scheme and K acts on X freely. Then the stabilizer groups are all trivial, which means that $\underline{\mathrm{Aut}}(x)$ is trivial for every geometric point x of $[X/K]$; see [Ols16, Section 8.4.1]. Thus, Proposition A.6.8 implies that $[X/K]$ is an algebraic space, which we denote X/K. ♢

Example A.7.5 (Horospherical stacks)**.** Consider $S = \mathrm{Spec}(k)$ where k is an algebraically closed field of characteristic 0. If X is a horospherical variety, then any closed subgroup K of the equivariant automorphism group of X is affine and smooth over k, so we can form the quotient stack $[X/K]$. This is a horospherical stack, as in Chapter 4. ♢

Let $s : \mathrm{Spec}(k) \to S$ be a k-point of S, where k is any algebraically closed field, and let K_s be the pullback of K along s, which is a smooth affine group scheme over k. For any object (P, π) in $[X/K](k)$, the principal K-bundle $P \to \mathrm{Spec}(k)$ is trivial since k is algebraically closed. If we fix a trivialization $p \in P(k)$, i.e. a section $p : \mathrm{Spec}(k) \to P$, and let X_s be the pullback of X along s, then we have a map $K_s \to X_s$ sending $g \mapsto g \cdot \pi(p)$.

Now the fibre product $K_{\pi(p)}$ of the following diagram

$$\begin{array}{ccc} K_{\pi(p)} & \longrightarrow & \mathrm{Spec}(k) \\ \downarrow & \ulcorner & \downarrow{\scriptstyle \pi(p)} \\ K_s & \longrightarrow & X_s \end{array}$$

is isomorphic to the automorphism group scheme $\underline{\mathrm{Aut}}((P,\pi))$. By construction, $K_{\pi(p)}$ is really the $g \in K_s$ satisfying $g \cdot \pi(p) = \pi(p)$. Importantly, different trivializations of P result in conjugate subgroup schemes of K_s. Thus, for $x \in [X/K](k)$, it makes sense to talk about the *stabilizer* group scheme $K_x \subseteq K_s$, which is defined up to conjugation by elements of $K_s(k)$.

In an intuitive sense, this is what distinguishes the quotient stack $[X/K]$ from the usual quotient space X/K: as a set, X/K only cares about the K-orbits in X, but the stack structure on $[X/K]$ is also keeping track of these stabilizer groups.

Example A.7.6 ($[\mathbb{A}^2/(\mathbb{Z}/2\mathbb{Z})]$)**.** For simplicity, let $S = \mathrm{Spec}(\mathbb{C})$. Consider the action of $\mathbb{Z}/2 := \mathbb{Z}/2\mathbb{Z}$ on $\mathbb{A}^2$ via $x \mapsto -x$. The scheme quotient is the affine variety

$$\mathbb{A}^2/(\mathbb{Z}/2) = \mathrm{Spec}\left(\mathbb{C}[x,y]^{\mathbb{Z}/2}\right) = \mathrm{Spec}(\mathbb{C}[a,b,c]/\langle ac - b^2\rangle).$$

The $\mathbb{C}$-points of this scheme correspond to the orbits, which are of the form $\{x,-x\}$ for $0 \neq x \in \mathbb{A}^2$ and the one special orbit $\{0\}$.

On the other hand, a $\mathbb{C}$-point of the stack $[\mathbb{A}^2/(\mathbb{Z}/2)]$ is the category whose sole object corresponds to one of the above orbits, and whose morphisms are given by the stabilizer group of that orbit. In particular, all "non-special" orbits have trivial stabilizer groups, so these $\mathbb{C}$-points are the same between $\mathbb{A}^2/(\mathbb{Z}/2)$ and $[\mathbb{A}^2/(\mathbb{Z}/2)]$, but the special orbit $\{0\}$ has stabilizer $\mathbb{Z}/2$, so this point is different between the scheme and the stack.

Using the terminology of Section 4.5, it is easy to verify that $\mathbb{A}^2/(\mathbb{Z}/2)$ is the good moduli space of $[\mathbb{A}^2/(\mathbb{Z}/2)]$, i.e. one should think of $\mathbb{A}^2/(\mathbb{Z}/2)$ as the scheme approximation of the stack $[\mathbb{A}^2/(\mathbb{Z}/2)]$. The key difference between this stack and its scheme approximation is at the origin; for instance, the origin in $\mathbb{A}^2/(\mathbb{Z}/2)$ is a singular point, whereas in $[\mathbb{A}^2/(\mathbb{Z}/2)]$ it is smooth by Remark A.7.2. One can think of the good moduli space morphism $[\mathbb{A}^2/(\mathbb{Z}/2)] \to \mathbb{A}^2/(\mathbb{Z}/2)$ as a stack-version of a resolution of singularities. ♢