

Storey-based Stability Analysis of Unbraced Steel Frames at Ambient and Elevated Temperatures

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

A fundamental task in structural stability analysis is to ensure the safety of structures throughout their operational life so as to prevent catastrophic consequences either at ambient or elevated temperatures. This thesis concerns the stability of unbraced steel frames due to abnormal loadings or fire loads, and develops practical methods to evaluate the stability capacity of unbraced steel frames at ambient temperature or in fire.

The problem of determining the elastic buckling strengths of unbraced steel frames subjected to variable loadings can be expressed as an optimization problem with stability constraints based on the concept of storey-based buckling. The optimization problem can be solved by the linear programming method, which is considerably simpler and more suitable for engineering practice than the nonlinear programming method. However, it was found that the frame buckling strength obtained from the linear programming method based on Taylor series approximation on column stiffness may be overestimated in some cases. Thus, a secant approximation of the column stiffness was introduced, and a modified linear programming method based on the secant approximation was proposed. Numerical examples show that the linear programming method in light of the secant approximation can yield conservative results and maintain simplicity.

In spite of the convenience of the modified linear programming method, numerical examples show that the linear programming method cannot accurately detect the maximum and minimum frame buckling strengths in some cases. Therefore, an alternative method to assess the lateral stiffness of an axially loaded column derived by using two cubic Hermite elements to signify the column is proposed. Unlike the column stiffness obtained from the Euler-Bernoulli beam theory containing transcendental functions, the stiffness in the proposed method includes only polynomials. Thus, the column stiffness within the proposed method enables the minimization and maximization problems to be solved by efficient gradient-based nonlinear programming algorithms, which overcome the

inability of linear programming algorithm to detect the minimum frame buckling strength in some cases. The accuracy of the column stiffness associated with the proposed method was compared with that of the Euler-Bernoulli beam theory. Four unbraced steel frames were investigated to demonstrate the efficiency of the proposed method.

It is known that the evaluation of the lateral stability of steel frames subjected to elevated temperatures is different from that at ambient temperature due to the degradation of material strength. Thus, the storey-based buckling method at ambient temperature was extended to evaluating the stability of unbraced steel frames subjected to elevated temperature. To simulate a steel column exposed to the elevated temperature, an analytical model was proposed to examine the effects of axial loading, elevated temperature, and thermal boundary restraints on the lateral stiffness of steel columns in unbraced frames. The procedure of evaluating the stability capacity of unbraced steel frames at elevated temperature was then concluded. Numerical examples are presented to demonstrate the evaluation procedure of the proposed method.

The column model was then refined to evaluate the lateral stiffness of steel column subjected to non-uniform elevated temperature distributions along the longitudinal direction. The lateral stiffness equation of the column model was derived based on the Euler-Bernoulli beam theory. The procedure to evaluate the stability capacity of unbraced steel frames subjected to non-uniform elevated temperature distributions was then concluded. The numerical examples were investigated with the proposed method for non-uniform elevated temperature distributions.

Finally, initial attempts were made to evaluate the stability of unbraced steel frames with fire-protected columns at different fire scenarios. A degradation factor characterizing the variation of the Young's Modulus of steel at elevated temperature was introduced. The objective and constraint functions were constructed, and optimal tools were used to determine the buckling strength of an unbraced steel frame at different fire scenarios.

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TO

Li, My Wife

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Chapter 1

Introduction

1.1 Background

Steel frames are extensively used in buildings because of their structural efficiency and flexibility in construction. Apart from the advantages of steel as a construction material, structural stability is one of the primary concerns in designing steel frames. In design of steel frames, it is always a challenge for engineers to determine critical load patterns which may result in structural failures or collapses because of the structural instability. In addition, the problem of structural stability of steel frames in fire has attracted more attention in recent years. The investigation of the collapse of the twin towers of the World Trade Center on September 11, 2001 ([Kodur, 2003](#)) showed that the collapse of the two buildings was caused by the buckling of steel columns at high temperature rather than melting of steel columns. Therefore, it is absolutely imperative to develop a practical, effective and reliable approach to determine the stability of steel frames both at ambient and elevated temperatures.

The concept of the effective length factor has been well established and widely used in column design. With the effective-length based design concept, the alignment chart method ([Julian and Lawrence, 1959](#)) is the most extensively used method for design of

steel frames. While this method adopts certain simplifications which may result in the inaccuracy of the estimated column strengths in some cases. The storey-based buckling method, as an alternative to the effective length method, discard the simplification associated with the alignment chart method, and can yield more accurate results (Yura, 1971; LeMessurier, 1977; Xu and Liu, 2002a). This method is based on the concept that lateral sway instability of an unbraced frame is a storey phenomenon involving the interaction of lateral sway resistance of each column in the same storey and the total gravity load in the columns of that storey.

Based on the concept of storey-based buckling, a new approach that utilized mathematical programming methods, such as linear programming method or non-linear programming method, was proposed by Xu (2002, 2003) to evaluate the buckling strengths of unbraced steel frames subject to variable loadings. Unlike conventional methods of stability analysis which are based on the assumption of proportional loading, the variable loading methods enable structural engineers to predict the maximum and minimum buckling capacities of unbraced steel frames, as well as their associated load patterns. However, compared with the results obtained from the nonlinear programming method, it was found that the buckling capacities of the unbraced frames might be overestimated with the linear programming method as a result of linear approximation of the lateral stiffness modification factor of the axially loaded columns. The overestimation of frame buckling loads may consequently result in unconservative design. To maintain reasonable accuracy and simplicity for engineering practice, and more importantly, to avoid unconservative design, an improved linear approximation was proposed in this study. Although the linear programming problem can be solved by a manual calculation method, it is also found in this study that the linear programming based method cannot accurately detect the maximum and minimum frame buckling strength in the case in which the end-fixity factors of all columns in the same storey are identical. Considering the complexity and inefficiency of the non-linear programming based method proposed by Xu (2003) based

on the Euler-Bernoulli beam theory, an alternative method with use of two cubic Hermite elements to model the stability behaviour of an axially loaded column was developed in this study.

The evaluation of structural stability of unbraced steel frames subjected to elevated temperatures is vastly different from that at ambient temperature due to thermal effects and material deterioration associated with elevated temperature. Numerous studies have been carried out to investigate the behaviour of steel columns and frames in fire; however, most of them are based on experimental or numerical methods and yet to be practical for structural engineers to determine the stability of unbraced frames with analytical methods. In addition, most of researchers assumed that the columns and frames experienced uniform elevated temperature. The lateral stability of steel frames that experience different elevated temperatures has not been addressed well in current design practice. In reality, compartment fire is usually restrained in one or two rooms of a building. In such cases, unheated columns in the same storey may provide lateral supports for the heated columns to maintain the stability of the storey. Moreover, in real fires, hot air and smoke accumulate at the upper zone, while the cooler air stays at the lower zone of the compartment. The assumption of structural members experiencing uniform elevated temperature may result in inaccurate evaluations on fire-resisting capability of unbraced steel frames. From that point, the method proposed by [Xu and Liu \(2002a\)](#) based on the concept of storey-based buckling will be extended to unbraced steel frames subjected to uniform and non-uniform elevated temperature in this study. The stability of unbraced steel frames with fire-protected columns at different fire scenarios was also investigated by optimization methods in this study.

1.2 Research Objectives and Scope

Presented in this thesis is research regarding the structural stability of unbraced steel frames at ambient and elevated temperatures. The objectives of this research are as

follow:

- Improve the accuracy and avoid unconservative estimations of frame buckling capacities associated with the method proposed by [Xu \(2002\)](#) for evaluating the stability of unbraced steel frames subject to variable loading.
- Derive an alternative method to evaluate the lateral stiffness of an axially loaded column, and investigate the stability of unbraced steel frames subjected to variable loading with use of gradient-based optimization algorithm.
- Develop an approach to assess the stability of unbraced steel frames subjected to uniform elevated temperature.
- Develop an approach to assess the stability of unbraced steel frames subjected to non-uniform elevated temperature.
- Identify the stability behaviours of unbraced steel frames with fire-protected columns at different fire scenarios.

The scope of this research is the stability analysis of planar unbraced steel frames at ambient and elevated temperatures, to predict the structural instability and failure temperatures.

1.3 Thesis Organization

This thesis includes the development of analytical methods for evaluating the buckling strength of unbraced steel frames at ambient and elevated temperatures. The thesis is organized into eight chapters as follows:

- Chapter [2](#) gives a review of selected previous studies on the stability analysis of unbraced steel frames at ambient and elevated temperature.

- Chapter 3 develops an alternative linear approximation for the lateral stiffness modification factor of axially loaded columns. Such approximation will ensure that the linear programming method proposed by Xu (2002) will not yield unconservative buckling capacities for unbraced steel frames. Four unbraced steel frames are investigated to compare the results obtained from Taylor series approximation and the proposed approximation.
- Chapter 4 presents an alternative method to evaluate the lateral stiffness of an axially loaded column with use of two cubic Hermite elements to model the stability behaviour of the axially loaded column. The proposed method is then used to detect the minimum and maximum buckling strength of unbraced steel frames subjected to variable loadings using a gradient-based nonlinear programming algorithm.
- Chapter 5 presents a study which extends the storey-based buckling method developed by Xu and Liu (2002a) to the stability analysis of unbraced steel frames at uniform elevated temperature. The stability behaviours of two unbraced steel frames were investigated with the proposed method, and the results were verified by those obtained from the finite element method.
- Chapter 6 derives the lateral stiffness of an axially loaded column subjected to non-uniform elevated temperature. The procedure to evaluate frame buckling strength at the non-uniform elevated temperature based on the concept of storey-based buckling is presented. The stability behaviours of the two unbraced steel frames investigated in Chapter 5 are re-examined with the proposed method. The results are also verified by those obtained from the finite element method.
- Chapter 7 investigates the stability of unbraced steel frames with fire-protected columns at different fire scenarios. The stability problems for various fire scenarios are constructed as a pair of maximization and minimization problems with a stability constraint, and can be solved by a nonlinear programming method.

- Chapter 8 discusses the conclusions drawn from the study, and recommendations for future research work concerning stability of unbraced steel frames are outlined.

Chapter 2

Literature Survey

2.1 Introduction

In the past few decades, extensive research has been conducted on the stability of unbraced steel frames. The theoretical approach, commonly referred as the system buckling method, is generally considered impractical ([Galambos, 1998](#)) since it involves solving for the critical load multiplier of the structural system as the least non-negative eigenvalue of either a highly nonlinear or a transcendental equation ([Majid, 1972](#)). In design practice, practical approaches have been adopted in the design standards for the stability analysis and design of unbraced steel frames are effective length method, notional load method, and storey-based method. One of the commonalities of the foregoing approaches is the assumption of proportional loading. In reality, certain types of loads, such as live load may be extremely volatile by nature. The variability in both magnitude and location is the basic characteristic of live load that a structural engineer should always be reminded in his or her design practice. Unfortunately, the stability of unbraced frames subjected to variable loading is as yet unsolved. On the other hand, the stability of unbraced steel frames at elevated temperature is of importance in structural fire design as well. In contrast to that of steel frames at ambient temperature, aside from the deterioration of

material properties of steel, the stability of unbraced steel frames is affected by thermal restraints associated with the remaining unheated components of the structure. Recent studies of steel frames in fire mostly focus on the investigation of single member behaviour via either experimental testing or numerical simulation. The interactions between the heated and unheated members have not been well addressed in the evaluation of the stability of unbraced steel frames at elevated temperature.

2.2 Stability Analysis Approaches in Current Design Standards

2.2.1 Effective Length Factor Approach

The concept of effective length to evaluate buckling strength of compressive members is widely used in design practice. According to the concept of effective length, the buckling strength of a compressive member with length L is equated to the buckling strength of an equivalent pin-ended member, subject to axial force only, with length KL , in which K is the effective length factor. This concept is considered as an essential part of many analysis procedures and has been recommended by almost all of the current design specifications ([AISC, 2005](#); [CSA, 2009](#)). There are several methods to determine the K factors based on the concept of effective length and different idealizations of the composition. Among those methods, the alignment chart method that was proposed by [Julian and Lawrence \(1959\)](#) is the most widely used method in frame design. The method is based on the assumption that all individual columns in one storey buckle simultaneously under their proportionate shares of the total gravity load ([Duan and Chen, 1989](#)), and it also takes into account the rotational restraints provided by upper and lower beam column assemblages to provide a direct means to evaluate the K factors. However, since the alignment chart method involves several simplifications and assumptions that are not

realistic in general practice, the K factors evaluated by the method may be inaccurate when the assumptions are violated. [Bridge and Fraser \(1987\)](#) presented a modified G-factor method to improve the effectiveness of the alignment chart method. [Duan and Chen \(1988\)](#) proposed a procedure to evaluate the K factors of compressive members in both braced and unbraced frames, in which the far ends of the columns above and below are not necessarily continuous but can either be hinged or fixed.

2.2.2 Storey-based Buckling Approach

In contrast to the alignment-chart-based effective length methods, in which the interaction among columns in a frame is neglected, the concept of storey-based buckling for unbraced frames is based on the fact that an individual column cannot fail by lateral sway-buckling without all of the other columns in the same storey also buckling in the same sway mode ([Yura, 1971](#)). [LeMessurier \(1977\)](#) proposed a procedure of estimating the frame buckling from the storey-buckling manner, which took into account the lateral stiffness interaction among columns in a storey in resisting lateral sway-buckling of the frame. Compared with the alignment chart method, LeMessurier's method can yield a more accurate estimation of the effective length factors K . A simpler method that accounts for member stability (P - δ) and frame stability (P - Δ) in the calculation of effective length factor K was proposed by [Lui \(1992\)](#). The method only involves a first-order frame analysis and no special charts or iterative procedures are required. [Shanmugam and Chen \(1995\)](#) conducted an assessment of four approaches to determine K factors of columns in frames, namely, the alignment chart approach, LeMessurier's approach, Lui's approach, and the system buckling approach. The study concluded that Lui's method is the most appropriate for general use in design practice. The AISC LRFD Specification ([AISC, 2005](#)) adopted the concept of storey-based buckling since the alignment chart method did not consider destabilizing effects of lean-on columns in a frame. Unlike the former storey-based buckling methods that all require using either a first-order elastic

analysis or the alignment chart while evaluating the storey-based effective length factor, [Xu and Liu \(2002a\)](#) proposed a more efficient storey-based buckling method, which employed a lateral stiffness factor to characterize the lateral stiffness of columns in one storey without conducting a frame analysis nor using the alignment chart. Based on this method, the buckling strength of unbraced frame subject to variable loading was addressed by [Xu \(2002\)](#) with the use of a linear programming method.

2.2.3 Notional Load Approach

In the past two decades, an alternative approach called, the notional load approach, has been proposed to use the actual column length (i.e., $K=1$) in conjunction with “notional” internal loads acting at each storey level and a second-order elastic analysis is then conducted on the geometrically perfect structure. The notional load approach takes into account the imperfection of storey out-of-plumbness under gravity loads and it is widely adopted in the British Standard BS5950: Part 1 ([BSI, 1990](#)), the Australian Standard AS4100-1990([SA, 1990](#)), the Canadian Standard CAN/CSA-S16-09([CSA, 2009](#)) and the Eurocode 3 ([CEN, 2005](#)). A comprehensive discussion of the notional load approach and design procedure can be found in the 1995 Research Report from the University of Sydney ([Clarke and Bridge, 1995](#)). In this report, a detailed study of the calibration and verification of the notional load approach for the assessment of frame stability is presented.

2.3 Stability Research of Steel Frames in Fire

2.3.1 Stability of Structural Steel Columns in Fire

Compared with the buckling strength of steel frames at ambient temperature, the stability of steel frames in fire is vastly different because the buckling capacity of steel frames

at elevated temperature is affected by the degradation of the steel mechanical properties, thermal restraint, creep strain, and other factors. [Culver \(1972\)](#) investigated the influence of non-uniform elevated temperature on the buckling strength of wide-flanged steel columns. [Franssen et al. \(1998\)](#) conducted two series of full-scale tests for steel columns at elevated temperature. An analytical formula for the buckling coefficient was calibrated based on test results. [Wang \(1997\)](#) investigated the effect of frame continuity on the behaviour of steel columns under fire with finite element analysis. The study concluded that the rotational restraint from adjoining members of the column would increase the fire resistance capacity of the column, while restraining the column thermal expansion would reduce its limiting temperature. Similar conclusions had been presented in the study of fire resistance of steel columns with elastically restrained axial elongation and bending by [Valente and Neves \(1999\)](#), which showed that increasing the axial restraint would decrease the critical temperature while the rotational restraint would increase the critical temperature. [Huang and Tan \(2003\)](#) extended the traditional Rankine approach formula to predict the critical temperature of an axially restrained steel column with use of a linear spring at the column top end to simulate the axial restrained effect on an isolated heated column. The proposed Rankine approach incorporated both the axial restraint and creep strain, and yielded good agreements with the finite element predictions. [Tan and Yuan \(2008\)](#) studied the elastic buckling of a pin-ended column subject to a longitudinal non-uniform temperature distribution with consideration of zone modeling of a compartment fire. They found that the buckling strength of the column in compartment fire was underestimated when evaluated with a uniform temperature distribution conservatively based on the temperature at the top portion of the column.

2.3.2 Stability of Steel Frames in Fire

The foregoing researches discussed in Section [2.3.1](#) focused on the stability of steel columns in braced frames. However, the behaviour of columns in an unbraced steel

frame can be different due to the stiffness interaction among members in the frame. For columns in unbraced steel frames, numerous studies have also been conducted in the past several decades. A series of quarter-to-half scale fire tests on steel sub-assemblies under fire were carried out by [Rubert and Schaumann \(1986\)](#), in which the characteristic parameters that influenced the critical temperature of steel frames were investigated. [Wang et al. \(1995\)](#) conducted finite element analysis for steel frames subject to elevated temperature with accounting for both geometrical and material nonlinearity, in which a number of parametric studies were carried out to investigate various aspects of steel frame behaviour under fire conditions. [Bailey et al. \(1999\)](#) presented the results of two fire tests conducted on a full-scale eight-storey steel-framed building. The remaining integrity of the building after fire tests showed that the existing design codes are conservative, and the major contribution to the survival of the frame came from the composite floor. A simple analytical approach to determine the ultimate resistance of steel frames in fire based on the Rankine principle was developed by [Toh et al. \(2001\)](#). The approach presented an approximation of fire resistance of steel frames through a simple interaction between two idealized structural behaviour—strength and stability. [Chan and Chan \(2001\)](#) proposed an effective method based on refined plastic hinge method to predict the structural behaviour of steel frames at elevated temperature, in which numerical schemes accounting for the large deflection analysis of steel frames under elevated temperature with use of the bi-linear stress strain curve were implemented. The numerical results obtained from the proposed method were validated with experimental results from others. [Couto et al. \(2013\)](#) investigated the buckling strength of braced and unbraced frames exposed to different fire scenarios with their software CLoad. The results obtained from advanced calculation models were used to validate their studies.

It is noted that most of the foregoing studies were conducted through either experimental investigation or numerical simulations. In addition, the stiffness interaction among columns in a frame were hardly addressed in the studies. Therefore, it is desirable

to extend the storey-based buckling concept for the stability analysis of unbraced steel frames subjected to fire and develop a practical approach for design practice.

Chapter 3

Stability of Unbraced Steel Frames Subjected to Variable Loadings

3.1 Introduction

Current design practice concerning stability and integrity of framed structures is almost exclusively based on the assumption of proportional loading, in which predefined load patterns are assigned to the structure based on load combinations specified in design standards and designers' anticipation of the possible occurrences of various types of loads that may be encountered during the lifespan of the structure. However, the worst load patterns are not always guaranteed by the load combinations specified in the standards or by the engineers due to the unpredictable nature of varying types of loads, especially, for structures in fire or in other catastrophes. Thus, variable loading in both magnitudes and locations need to be accounted for when assessing the stability of structures; otherwise, public safety may be jeopardized by structural damages.

Based on the concept of storey-based buckling introduced by [Yura \(1971\)](#), a linear programming method to investigate stability strengths of unbraced steel frames subjected to variable loadings was proposed by [Xu \(2002\)](#), in which the stiffness interaction among

columns in resisting lateral buckling was characterized by a lateral stiffness modification factor. In contrast to current frame stability analysis involving only proportional loading, the proposed approach permits individual applied loads on the frame to vary independently. The approach also captures the load pattern that causes instability of unbraced steel frames at the maximum load level (the most favorable pattern) and the minimum (the worst load pattern), which enables structural engineers to evaluate the stability strengths of unbraced steel frames in extreme situations.

Considering that the linear programming method is derived from the linear simplification of the lateral stiffness modification factor, [Xu \(2003\)](#) proposed a nonlinear programming method without simplification of the modification factor, which can yield more accurate results, but requires significant computational efforts. Compared with the nonlinear programming method, the linear programming method can be facilitated with manual calculation and is simpler for structural engineers. However, it is found in this study that the stability strengths of the frames obtained from the linear programming method could be overestimated in some cases when compared to the results obtained by the nonlinear programming method due to the use of the linear approximation of Taylor series for the column lateral stiffness modification factor. For the design purpose, it is desirable to have a conservative design by underestimating rather than overestimating the frame buckling strength. Since the nonlinear programming method is not suitable for practical design purposes due to its complexity, a modified linear programming method which maintains simplicity and reasonable accuracy will be proposed in the following analysis herein.

3.2 Lateral Stiffness of Axially Loaded Semi-Rigid Column

3.2.1 End-Fixity Factor of Semi-Rigid Member

A semi-rigid member with a rotational spring attached at each end is shown in Figure 3.1. To account for the semi-rigid moment-rotation behaviour of beam-to-column connections in stability analysis of unbraced steel frames, an end-fixity factor should first be introduced.

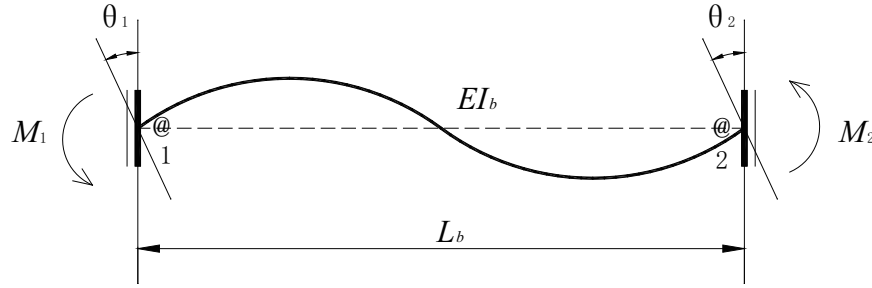


Figure 3.1: End rotation of a semi-rigid beam

The relationship between the end-fixity factor and connection stiffness can be characterized as (Monforton and Wu, 1963):

$$r_i = \frac{1}{(1 + 3EI_b/R_i L_b)} \quad (i = 1, 2) \quad (3.1)$$

where r_i is the end-fixity factor which defines the stiffness of each end connection relative to that of the attached member. R_i is the end-connection spring stiffness, and EI_b/L_b is the flexural stiffness of the attached member. For flexible, or so-called pinned connections, the rotational stiffness of the connection is zero; hence, the value of the end-fixity factor is zero. For fully-restrained or so-called rigid connections, the end-fixity factor is unity due to the rotational stiffness approaching infinit, thus, a semi-rigid connection has an end-fixity factor between zero and unity.

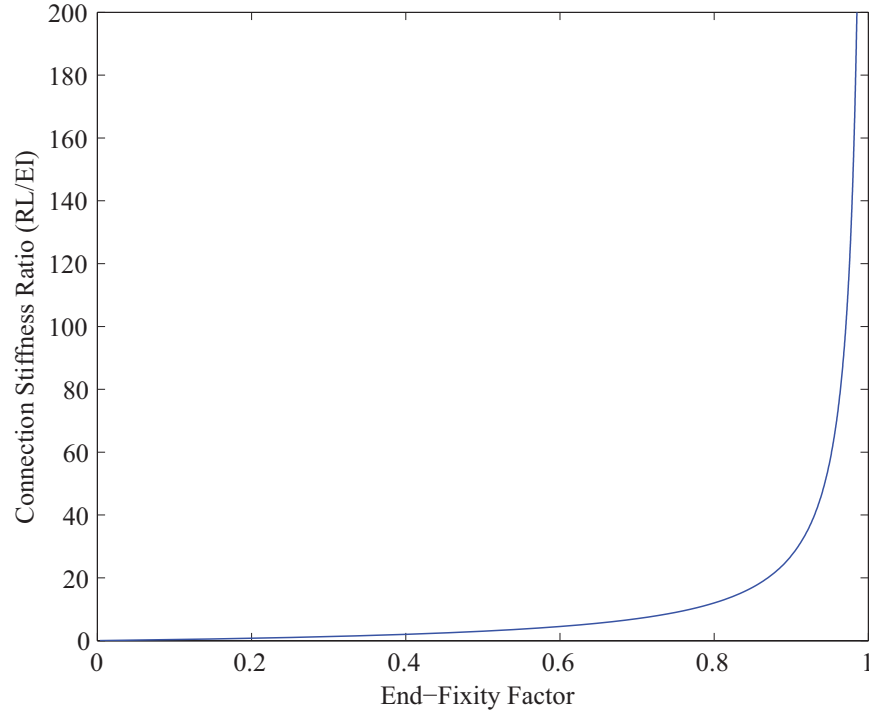


Figure 3.2: Relationship between the end-fixity factor and connection stiffness (Xu and Liu, 2002b)

According to Eq. (3.1), the relationship between the end-fixity factor and the connection stiffness ratio is nonlinear, as shown in Figure 3.2. It can be observed from Figure 3.2 that the relationship between the connection stiffness ratio and the end-fixity factor is almost linear when the connection is relatively flexible with a value of the end-fixity factor between 0.0 and 0.6; however, for a rigid connection, the value of connection stiffness ratio will have to approach infinity while the end-fixity factor becomes unity. Therefore, to facilitate structural analysis for semi-rigid frames, it is more convenient to use the end-fixity factor to characterize the semi-rigid behavior of the connections rather than the connection stiffness ratio.

Although the end-fixity factor was first introduced to characterize the behaviour of semi-rigid beams, it can be used to facilitate the lateral stiffness assessment of axially loaded columns while accounting for end rotational restrains imposed from adjoining

members in the frame as shown in the next section.

3.2.2 Lateral Stiffness of An Axially Loaded Column

An axially loaded semi-rigid column is illustrated in Figure 3.3. Let R_{il} and R_{iu} be the flexural rotational restraining stiffness provided by the connected members at the column lower and upper end, respectively.

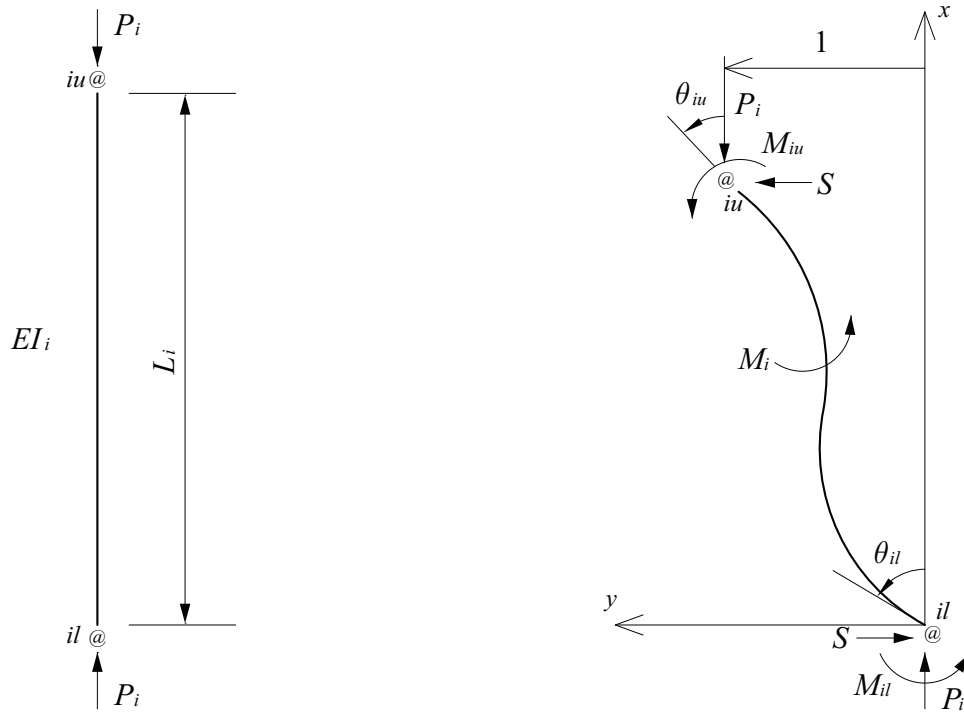


Figure 3.3: Axially loaded column of an unbraced frame with deformations and forces (Xu and Liu, 2002a)

The lateral stiffness of the column S_i in an unbraced frame was derived as (Xu and Liu, 2002a):

$$S_i = \beta_i(\phi_i, r_{il}, r_{iu}) \frac{12EI_i}{L_i^3} \quad (3.2)$$

where r_{il} and r_{iu} are the end-fixity factors for the lower and upper end of the column, respectively; E is the Young's modulus. I_i and L_i are the moment of inertia and the

length of column, respectively. The applied load ratio ϕ_i is defined as

$$\phi_i = \sqrt{\frac{P_i L_i^2}{EI_i}} = \pi \sqrt{\frac{P_i}{P_{ei}}} \quad (3.3)$$

in which P_i is the applied axial load and P_{ei} is the Euler buckling strength for the column with pinned connections.

The coefficient $\beta_i(\phi_i, r_{il}, r_{iu})$ is a modification factor of the column lateral stiffness accounting for both the effect of axial load and column end rotational restraints and can be expressed as

$$\beta_i(\phi_i, r_{il}, r_{iu}) = \frac{\phi_i^3}{12} \frac{a_1 \phi_i \cos \phi_i + a_2 \sin \phi_i}{18 r_{il} r_{iu} - a_3 \cos \phi_i + (a_1 - a_2) \phi_i \sin \phi_i} \quad (3.4)$$

where

$$a_1 = 3[r_{il}(1 - r_{iu}) + r_{iu}(1 - r_{il})] \quad (3.5a)$$

$$a_2 = 9r_{il}r_{iu} - (1 - r_{iu})(1 - r_{il})\phi_i^2 \quad (3.5b)$$

$$a_3 = 18r_{il}r_{iu} + 3[r_{il}(1 - r_{iu}) + r_{iu}(1 - r_{il})]\phi_i^2 \quad (3.5c)$$

It can be observed from Eq. (3.2) that the lateral stiffness of an axially loaded column can be characterized by the modification factor β . Consider the case that the lower end of the column is rigidly connected ($r_{il} = 1$) and let the end-fixity factor at the upper end vary from zero to unity, the relationship between the modification factor β and the axial load ration P_i/P_{ei} can be illustrated as Figure 3.4. It is shown in Figure 3.4 that the magnitude of β will decrease with the increase of axial load. The positive value of β indicates that the column is laterally stable and can provide lateral support to other columns. As the applied axial load increases, the value of β is diminished and reduced to zero, which signifies that the column loses its lateral stiffness and becomes laterally unstable. With further increase of axial load, β becomes negative, which means the column is no longer able to sustain the axial load and must rely upon the lateral support from other columns to maintain its stability.

Xu and Liu (2002b) compared Eq. (3.2) to the stability equations with the alignment chart method and concluded that the equations in the alignment chart method to compute the buckling strength of unbraced and braced frames, are the special cases of Eq. (3.2) with the value of β being zero and negative infinity, respectively. The conclusion can be demonstrated in Figure 3.4. The zero value of β indicates the unbraced column has reached its lateral buckling strength in the case that no external lateral bracing is provided. As further increase of the axial load, the lateral stiffness of the column decreases and the value of β eventually approaches negative infinity, which means the column reaches its rotational (non-sway) buckling strength in the case that fully lateral bracing is provided. Therefore, the column lateral stiffness modification factor β provides a quantitative measure of the stiffness interactions among the columns in a storey of the frame.

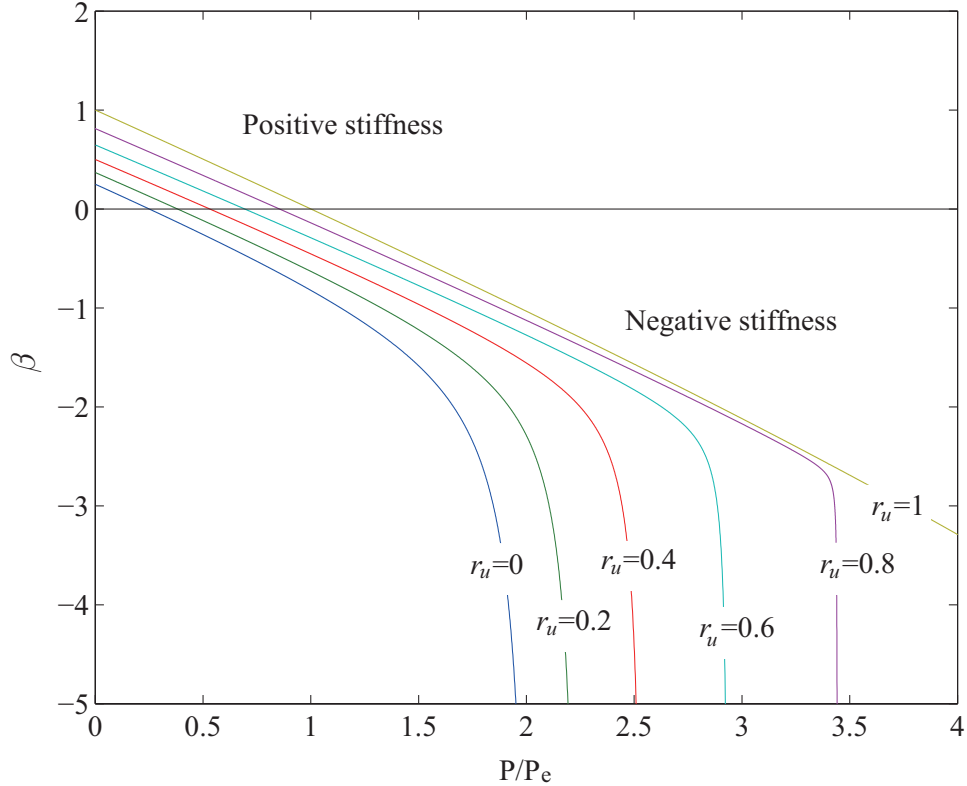


Figure 3.4: Relationship between modification factor and load ratio

3.3 Storey-Based Stability of Unbraced Frames Subjected to Variable Loadings

The concept of storey-based buckling indicates that lateral sway instability of an unbraced frame is a storey phenomenon involving the interaction of lateral stiffness among columns in the same storey. In resisting lateral instability, the weaker or heavily loaded columns are relied upon the lateral support provided by the stronger or lightly loaded columns in the same storey to sustain the applied loads. Thus, the condition for multi-column storey-based buckling in a lateral sway mode is that the lateral stiffness of the storey vanishes. The stability equation of a single storey semi-rigid frame buckling in a lateral sway mode, based on Eq. (3.2), was given by Xu and Liu (2002a)

$$\sum_{i=1}^n S_i = \sum_{i=1}^n \beta_i(\phi_i, r_{il}, r_{iu}) \frac{12EI_i}{L_i^3} = 0 \quad (3.6)$$

where n is the number of columns in a storey, and $\beta_i(\phi_i, r_{il}, r_{iu})$ is the column lateral stiffness modification factor expressed in Eq. (3.4).

However, it is inconvenient to evaluate a column buckling load in multi-column frames from Eq. (3.4) due to the transcendental relationship between β and ϕ . Thus, the Taylor series expansion was employed to simplify the equation as follow:

$$\beta_i(\phi_i, r_{il}, r_{iu}) = \beta_{0i}(r_{il}, r_{iu}) - \beta_{1i}(r_{il}, r_{iu})\phi_i^2 \quad (3.7)$$

where β_{0i} and β_{1i} are

$$\beta_{0i} = \frac{r_{il} + r_{ru} + r_{il}r_{iu}}{4 - r_{il}r_{iu}} \quad (3.8a)$$

$$\beta_{1i} = \frac{40 + 8(r_{il}^2 + r_{iu}^2) + r_{il}r_{iu}(r_{il} + r_{ru} + 3r_{il}r_{iu} - 34)}{30(4 - r_{il}r_{iu})^2} \quad (3.8b)$$

Substituting Eq. (3.2), Eq. (3.3) and Eq. (3.7) into Eq. (3.6), the simplified stability equation of a single storey semi-rigid frame can be obtained:

$$\sum_{i=1}^n S_i = 12 \sum_{i=1}^n \left[\frac{EI_i}{L_i^3} \beta_{0i}(r_{il}, r_{iu}) - \frac{P_i}{L_i} \beta_{1i}(r_{il}, r_{iu}) \right] = 0 \quad (3.9)$$

in which L_i and P_i are the length and axial force of the column, respectively.

There is no unique solution for Eq. (3.9) because the equation is associated with n variables of P_i . Thus, the stability of unbraced frames under variable loadings can be stated as the problems of finding the lower and upper bounds of buckling loads of the frames. The problem of determining the elastic buckling loads of the frames subjected to variable loadings is expressed as a pair of maximization and minimization problems with stability constraints (Xu, 2002):

$$\begin{matrix} \text{Minimum} \\ \text{Maximum} \end{matrix} : Z = \sum_{i=1}^n P_i \quad (3.10)$$

subject to

$$\sum_{i=1}^n S_i = 12 \sum_{i=1}^n \left(\frac{EI_i}{L_i^3} \beta_{0i} - \frac{P_i}{L_i} \beta_{1i} \right) = 0 \quad (3.11a)$$

$$P_{il} \leq P_i \leq P_{iu} = \frac{\pi^2 EI_i}{K_i^2 L_i^2} \quad (i = 1, 2, \dots, n) \quad (3.11b)$$

where the applied column load P_i is the variable to be solved in the problem. The objective function Eq. (3.10) corresponds to either the minimum or the maximum elastic frame buckling strength, as given by the sum of individual column loads. The storey-based stability condition imposed on the frame is defined by Eq. (3.11a), in which β_{0i} and β_{1i} are defined in Eq. (3.8a) and Eq. (3.8b). The lower bound imposed on each applied column load can be specified by the invariant portion of the applied load, while the upper bound, $\pi^2 EI_i / K_i^2 L_i^2$, is imposed to ensure that the magnitude of the applied load will not exceed the critical load associated with non-sway buckling of the individual column. The factor K_i is the effective length factor of the column associated with non-sway buckling, which is related to the rotational restraints at the column ends. In this study, K_i is evaluated as follow (Newmark, 1949; Xu, 2003)

$$K_i^2 = \frac{[\pi^2 + (6 - \pi^2)r_{iu}] \times [\pi^2 + (6 - \pi^2)r_{il}]}{[\pi^2 + (12 - \pi^2)r_{iu}] \times [\pi^2 + (12 - \pi^2)r_{il}]} \quad (3.12)$$

3.4 Taylor Series Approximation of Column Stiffness Modification Factor

It can be observed from Eq. (3.10) that the problem of seeking the maximum and minimum buckling loads is a pair of linear programming problems. Thus, the problems can be solved with the simplex method that can be easily adopted by engineers in design practice. However, the difference between the β defined in Eq. (3.4) and its Taylor series approximation defined in Eq. (3.7) needs to be investigated on whether the approximation yields a conservative solution or not. From the previous derivation, Eq. (3.7) is a second-order Taylor series approximation of the modification factor β defined in Eq. (3.4) at point $\phi = 0$. The Taylor series expression of Eq. (3.4) can be expressed as

$$\beta(\phi, r_l, r_u) = \beta_0(r_l, r_u) - \beta_1(r_l, r_u)\phi^2 + R_2(\phi) \quad (3.13)$$

where $R_2(\phi)$ is the remainder term and can be stated as

$$R_2(\phi) = \frac{\beta^{(3)}(\varepsilon)}{3!}\phi^3 \quad (3.14)$$

in which ε is a number between the approaching point 0 and ϕ . Obviously, the quality of the approximation is determined by the remainder term. According to the Taylor theorem, as ϕ moves further away from the zero point, the approximation becomes less accurate. That is, the absolute value of the remainder term R_2 defined in Eq. (3.14) will increase as ϕ increases. By analyzing Eq. (3.4), the value of the term $\beta^{(3)}$ computed with l'Hôpital's rule is zero for any given end-fixity factors when $\phi = 0$. Correspondingly, the remainder term R_2 is equal to zero at point $\phi = 0$. Shown in Figure 3.5 are the relationships between R_2 and ϕ for an axially loaded column with different end-restraining cases. It can be seen from Figure 3.5 that the absolute value of the remainder term R_2 increases as the applied load ratio ϕ increases. Because the value of R_2 is less than zero for $\phi > 0$, it can be concluded that the value of the modification factor β obtained

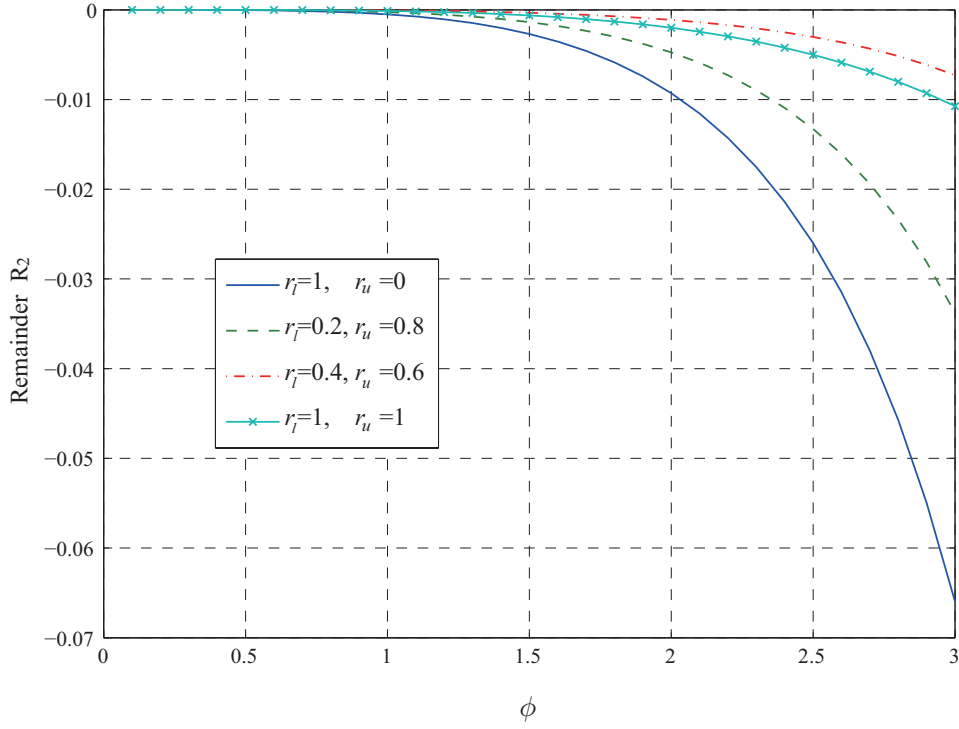


Figure 3.5: Relationship between remainder term and applied load ratio

from Eq. (3.7) is greater than the value calculated based on Eq. (3.4) as a result of the absence of the remainder term. Consequently, the lateral stiffness evaluated based on the approximation of β defined in Eq. (3.7) is overestimated.

Consider an axially loaded column with pinned-fixed ends, the variations of the lateral stiffness modification factor β , calculated based on Eq. (3.4) and its Taylor series approximation defined in Eq. (3.7), versus the applied load ratio ϕ are plotted in Figure 3.6. As illustrated in Figure 3.6, as the ϕ value increases, the divergence between the two equations becomes significant. In addition, the greater magnitude of β associated with the approximation indicates that the corresponding lateral stiffness of the column will be overestimated. Overestimation of column lateral stiffness will lead to overestimating column and frame buckling strength in both proportional and variable loading cases, which may result in unconservative designs.

To achieve a better approximation with use of the Taylor series, a higher order ap-

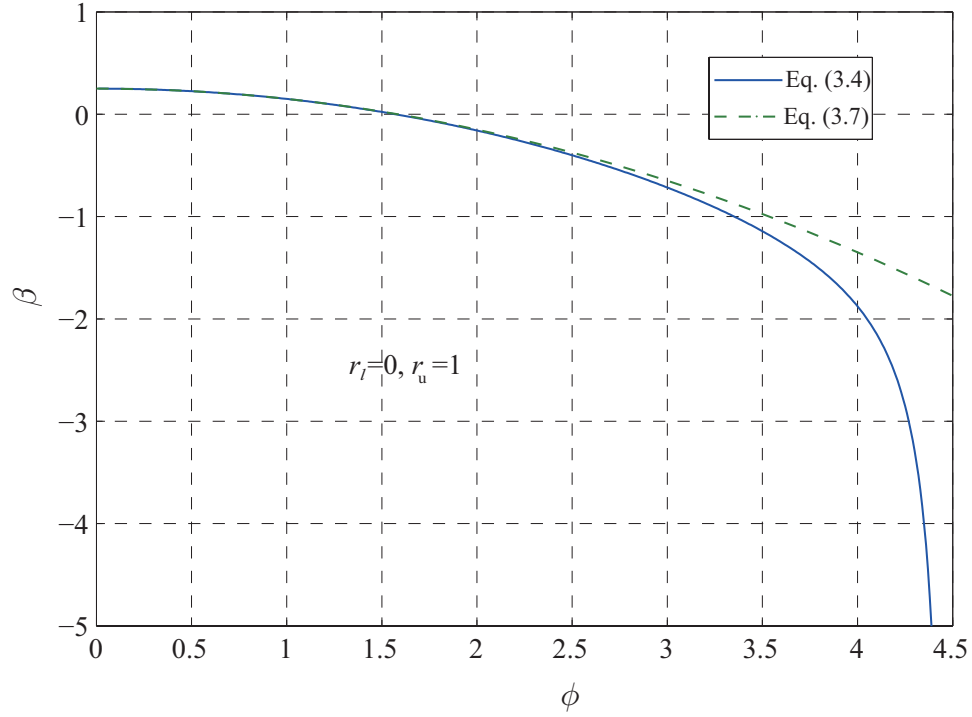


Figure 3.6: Difference of β values between Eq. (3.4) and Eq. (3.7)

proximation is required. However, a previous study conducted by [Xu and Liu \(2002a\)](#) suggested that the procedure based on the higher order Taylor series approximation would be too tedious for engineering practice. Thus, an alternative approximation of the modification factor β needs to be developed for the evaluation of lateral stiffness of axially loaded columns.

3.5 Secant Approximation Method

As discussed in the previous section, using a linear Taylor series approximation for the column stiffness modification factor β may overestimate the column lateral stiffness, and consequently overestimate the maximum and minimum buckling strength of steel frames. However, the maximization and minimization problems with use of the linear Taylor series approximation for the column stiffness modification factor β are a pair of linear

programming problems which can also be solved by a manual calculation approach proposed by Xu (2002). The manual procedure can be conveniently adopted by engineers in design practice. From that point, it is desirable to obtain an alternative linear approximation of the column stiffness modification factor to avoid overestimating the stiffness and buckling strength of the column and the frame as much as possible.

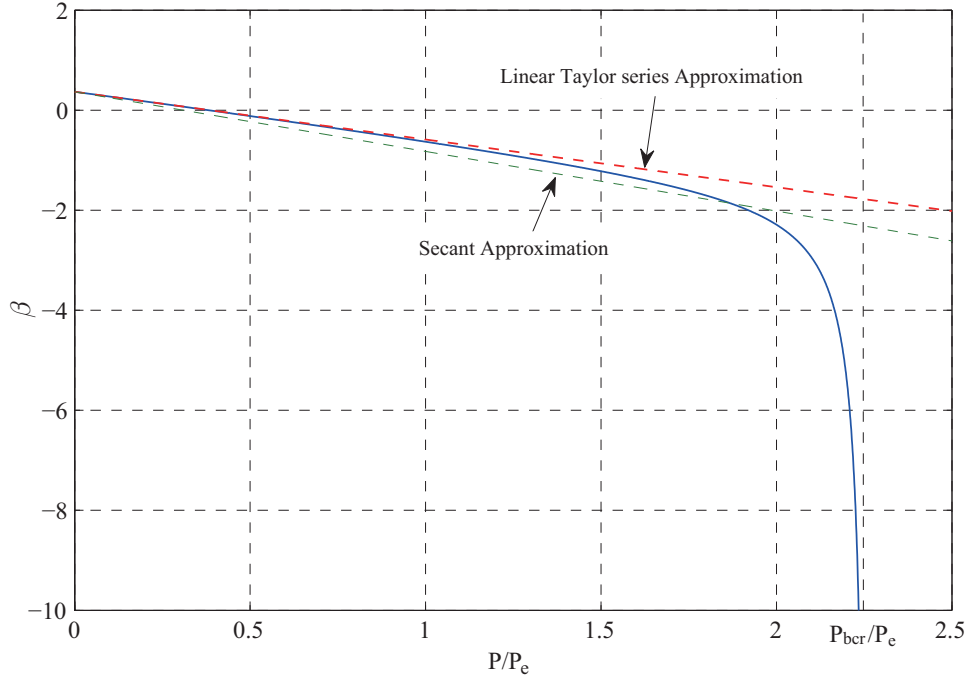


Figure 3.7: Stiffness modification factor and its approximations

From the relationship between the stiffness modification factor and the applied load ratio for the column with fixed-pinned ends shown in Figure 3.7, it can be observed that the value of the stiffness modification factor can be represented by a secant approximation which is generally conservative than the linear Taylor series approximation. It can also be observed that when the applied load approaches the buckling load of the fully braced column P_{bcr} , the lateral stiffness decreases significantly with little increase in applied load. Therefore, the secant approximation will also inevitably become unconservative when the applied load approaches P_{bcr} . Nevertheless, the secant approximation is preferable because it is conservative compared to the linear Taylor series approximation.

Observing the linear Taylor series approximation applied to Eq. (3.7), it can be found that Eq. (3.7) defines the tangent of the column stiffness modification factor β at the point $P/P_e = 0$. To maintain the linear approximation of β for practical purposes, the linear Taylor series approximation can be replaced by an appropriately selected secant approximation at the point $P/P_e = 0$, as illustrated in Figure 3.7. Apparently, the selection of the slope for the secant approximation is essential to achieve the balance between conservative and accurate results. Accordingly, the variation of the slope of the secant approximation with respect to the applied load ratio P/P_e should be investigated. In addition to the applied load, the column stiffness modification factor β is also a function of the column end-fixity factors r_l and r_u as demonstrated in Eq. (3.4). For the case of a column with the lower end fixed ($r_l = 1$), the slopes of β factors obtained from the first-order derivative of Eq. (3.4) and Eq. (3.7) with respect to the variation of the applied load ratio P/P_{bcr} and the upper end semi-rigid factors ($0 < r_u \leq 1$), are listed in Table 3.1.

Table 3.1: Value of the slope of column stiffness modification factor β

$r_u \backslash P/P_{bcr}$	Eq. (3.7)	Based on Eq. (3.4)										
	≥ 0	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
0	0.99	-0.99	-1.01	-1.04	-1.08	-1.14	-1.25	-1.43	-1.83	-2.93	-8.69	-30.32
0.2	-0.95	-0.95	-0.97	-0.99	-1.01	-1.05	-1.11	-1.23	-1.46	-2.11	-5.60	-19.39
0.4	-0.93	-0.93	-0.94	-0.95	-0.97	-0.99	-1.02	-1.08	-1.18	-1.48	-3.06	-9.35
0.6	-0.93	-0.93	-0.93	-0.94	-0.95	-0.96	-0.98	-1.00	-1.04	-1.13	-1.56	-3.19
0.8	-0.94	-0.94	-0.95	-0.96	-0.97	-0.98	-0.99	-1.01	-1.03	-1.07	-1.13	-1.26
1.0	-0.99	-0.99	-1.00	-1.02	-1.05	-1.09	-1.13	-1.20	-1.30	-1.48	-1.78	-2.04

It can be observed from Table 3.1 that the slope of β factor at the initial point ($P/P_{bcr} = 0$) calculated based on Eq. (3.4), for the given value of r_u , is the same as that calculated based on Eq. (3.7). The slope decreases as the applied load increases, resulting in that the theoretical value yielded from Eq. (3.4) is smaller than the linear

approximate value calculated based on Eq. (3.7). In addition, the variation of the slope evaluated based on Eq. (3.4) is relatively small before the applied load reaches 90% of the buckling strength of the fully braced column, P_{bcr} . Since the slope of the β factor is also a function of the column end-fixity factor, it can be observed from Table 3.1 that the magnitude of the slope change becomes smaller as the end-fixity factor r_u increases.

For any given values of end-fixity factors, the magnitude of β factors and the corresponding slopes computed based on Eq. (3.4) and (3.7) are identical at the point $P/P_{bcr} = 0$. Therefore, the secant approximation can be initiated at this point. The second point of the secant approximation is yet to be determined. For columns with one end fixed and the other end in the nearly fixed condition, say $r_u = 0.8$, little change occurs in slope even though P/P_{bcr} reaches 0.95, as shown in Table 3.1. Thus, a good linear approximation can be achieved in a range of $0 \leq P/P_{bcr} \leq 0.95$. However, if the other end of the column is in a nearly pinned condition, say $r_u = 0.2$, the range for the linear approximation may be reduced to $0 \leq P/P_{bcr} \leq 0.7$. Therefore, the accuracy, the conservative nature and applicable range of P/P_{bcr} ratio of a linear approximation are related to the column end-fixity factors. To that end, a slope modification factor $\alpha(r_l, r_u)$ is introduced to Eq. (3.7) to form the secant approximation for the lateral stiffness modification factor as

$$\beta_r(\phi, r_l, r_u) = \beta_0(r_l, r_u) - \alpha(r_l, r_u)\beta_1(r_l, r_u)\pi^2(P/P_e) \quad (3.15)$$

where $\alpha(r_l, r_u)$ is the slope modification factor. Apparently, the secant approximation requires that $\alpha(r_l, r_u) \geq 1$.

Considering the fact that the resistance factor for the axial compressive members in the current Limit States Design Standard (CSA, 2009) is 0.9, the maximum permissible applied axial loads for the column is $P = 0.9P_{bcr}$. To maintain a reasonable accuracy of secant approximation, the slope variation in the selected range of the applied load ratio should be small. By inspecting the slopes of β factors listed in Table 3.1, it is found that the most significant slope variation occurs when $r_u = 0$, which indicates the poor secant

approximation may occur when $r_l = 1$ and $r_u = 0$. From Table 3.1, for the case $r_u = 0$, the slopes at $P/P_{bcr} = 0.8$ and 0.9 are nearly three and nine times that of the initial point, respectively. For other values of r_u , magnitudes of the slopes at $P/P_{bcr} = 0.9$ are about three times as that of the initial point. Taking into account the fact that the selected upper bound of applied load ratio should be near $P/P_{bcr} = 0.9$, to maintain the reasonable accuracy and maximize the applicable range of the secant approximation, the upper bound of P/P_{bcr} ratio for the secant approximation was selected as the point where the slope is three times that of the initial point. Based on this criterion, the upper bounds of P/P_{bcr} ratio corresponding to different end-fixity factors can be determined and the value of column lateral stiffness modification factors β at the upper bounds can be obtained from Eq. (3.4). Consequently, the corresponding slope modification factor α can be computed with Eq. (3.15). The results of α for different values of r_l and r_u are listed in Table 3.2 and the distribution of α with respect to r_l and r_u is plotted in Figure 3.8.

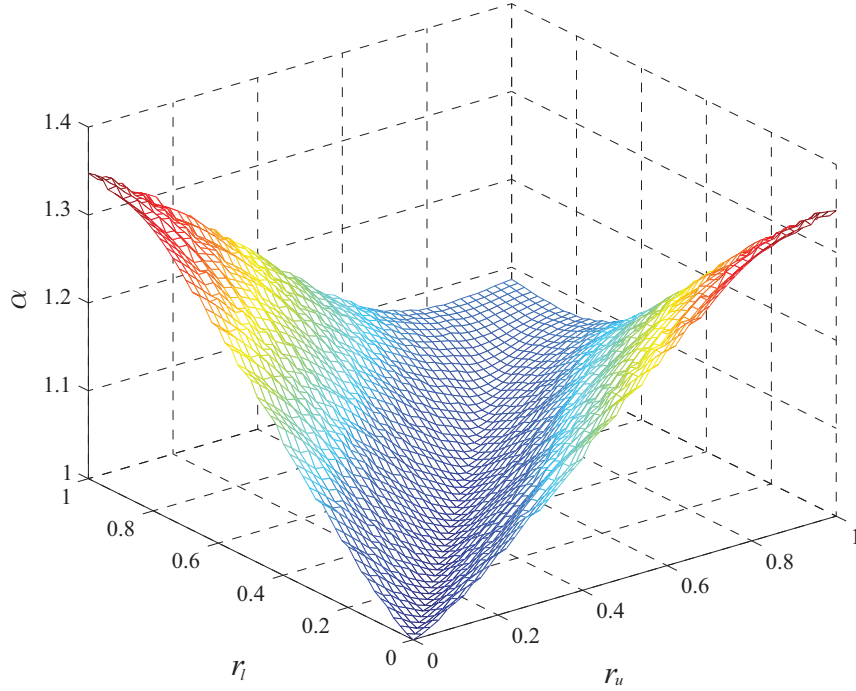


Figure 3.8: Revised factor α distribution in terms of r_l and r_u

Table 3.2: Value of revised slope modification factor α

$r_l \backslash r_u$	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	1	1.028	1.069	1.105	1.15	1.204	1.245	1.281	1.32	1.338	1.348
0.1	1.028	1.000	1.027	1.069	1.114	1.158	1.199	1.243	1.28	1.306	1.312
0.2	1.069	1.027	1.001	1.029	1.072	1.116	1.164	1.207	1.24	1.263	1.275
0.3	1.105	1.069	1.029	1.003	1.033	1.075	1.121	1.161	1.199	1.226	1.241
0.4	1.15	1.114	1.072	1.033	1.006	1.036	1.078	1.121	1.155	1.183	1.12
0.5	1.204	1.158	1.116	1.075	1.036	1.012	1.043	1.081	1.115	1.143	1.158
0.6	1.245	1.199	1.164	1.121	1.078	1.043	1.021	1.05	1.082	1.108	1.119
0.7	1.281	1.243	1.207	1.161	1.121	1.081	1.05	1.035	1.06	1.085	1.092
0.8	1.317	1.28	1.24	1.197	1.155	1.115	1.082	1.06	1.053	1.073	1.081
0.9	1.338	1.306	1.263	1.226	1.183	1.143	1.108	1.085	1.073	1.073	1.081
1	1.348	1.312	1.275	1.241	1.12	1.158	1.119	1.092	1.081	1.081	1.086

It can be seen that α is unity when both ends of the columns are pinned ($r_l = r_u = 0$), which means that there is no need to revise the slope in Eq. (3.7). As r_u and r_l increase, the slope modification factor α will increase. It can further be observed from Table 3.2 that the largest value of the slope modification factor can be obtained for columns with pinned-fixed or fixed-pinned ends, which agrees with the observation from Table 3.1. The valley shown in Figure 3.8 demonstrates that when r_u is equal to or close to r_l , the smaller value of α can be obtained, and the value of α is close to unity as shown in Table 3.2. That means the slope change of β for such case is relatively small and the secant approximation method will yield reasonably accurate value of β . Worthy of particular note from Figure 3.8 and Table 3.2 is that the variation of α is associated with the variation of end-fixity factors r_u and r_l . Thus, the slope modification factor can be expressed as the function with respect to end-fixity factors. To that end, a second-

order polynomial function with two variables is employed to fit the 3D surface shown in Figure 3.8. By surface fitting analysis, the function of the slope modification factor α can be obtained as

$$\alpha = 1 + 0.1425(r_l + r_u) + 0.2907(r_l^2 + r_u^2) - 0.8395r_l r_u \quad (3.16)$$

where r_l and r_u are the end-fixity factors of the column at the lower and upper end, respectively. Illustrated in Figure 3.9 is the relationship among the slope modification factor α , upper and lower end-fixity factors r_u and r_l . It should be noted that the value of α is greater than unity for all r_u and r_l except at the point $r_l = r_u = 0$, resulting in that the column lateral stiffness modification factor computed with the slope modification factor is smaller than that obtained from Eq. (3.4), which ensures that the lateral stiffness of the column will not be overestimated.

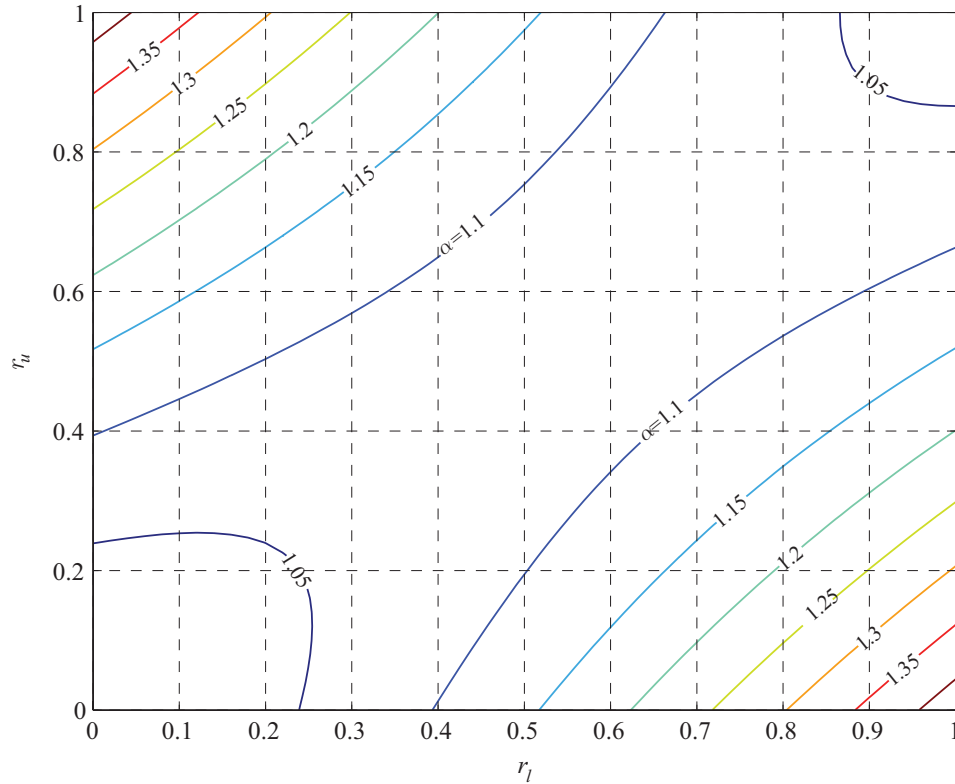


Figure 3.9: Relationship among revised factor α , end-fixity factors r_u and r_l

Moreover, it can be observed from Figure 3.9 that the slope modification factor α will increase if one end constraint increases and the other end constraint decreases. For example, if $r_l = 1$, the modification factor α decreases as r_u increases from zero to unity as shown in Figure 3.9. In addition, the value of α on the diagonal close to unity reveals that the secant approximation can yield accurate value of β when the constraints of the upper and lower ends of the column are identical.

3.6 Storey-Based Stability of Unbraced Frames with Secant Approximation

3.6.1 Problem Formulation with Secant Approximation

Having the slope modification factor α computed from Eq. (3.16), the corresponding modified lateral stiffness modification factor β can be obtained from Eq. (3.15). The maximization and minimization problems of Eqs. (3.10) and (3.11) can be expressed as follow

$$\begin{matrix} \text{Minimum} \\ \text{Maximum} \end{matrix} : Z = \sum_{i=1}^n P_i \quad (3.17)$$

subject to

$$\sum_{i=1}^n S_i = 12 \sum_{i=1}^n \left(\frac{EI_i}{L_i^3} \beta_{0i} - \frac{P_i}{L_i} \alpha_i \beta_{1i} \right) = 0 \quad (3.18a)$$

$$P_{il} \leq P_i \leq P_{iu} = \frac{\pi^2 EI_i}{K_i^2 L_i^2} \quad (i = 1, 2, \dots, n) \quad (3.18b)$$

where α_i is the slope modification factor for column i and defined in Eq. (3.16). It can be seen that the minimization and maximization problems defined in Eqs. (3.17) and (3.18) are almost the same as those of Eqs. (3.10) and (3.11) except the introduced modification factor α_i in Eq. (3.18a). Therefore, the manual calculation approach proposed by Xu (2002) is applicable for the modified linear programming problems defined in Eqs. (3.17) and (3.18), which is convenient for design practitioners. More importantly, the maximum

and minimum frame buckling strength obtained from Eqs. (3.17) and (3.18) will not be overestimated.

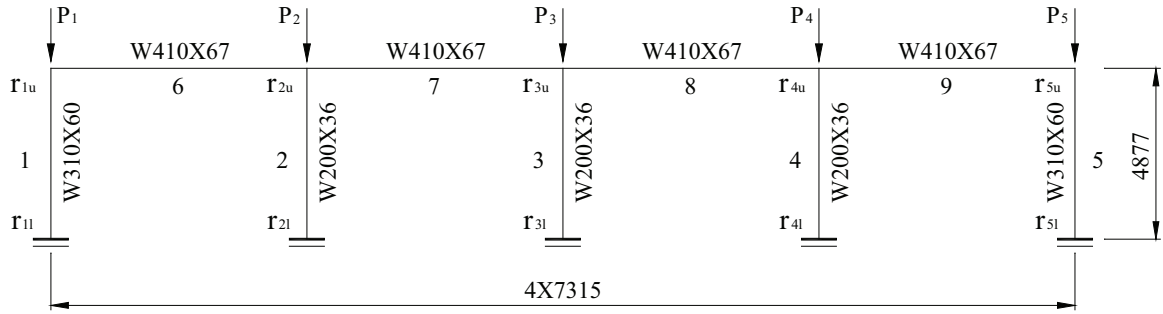
3.6.2 Numerical Example

The stability of four one-storey and four-bay unbraced steel frames with different beam-to-column and column-base connections as shown in Figure 3.10 were investigated by using non-linear programming method by Xu (2003). The frames are adopted in this study for comparing the results obtained by the nonlinear programming problems, the linear programming problems with the Taylor series approximation and the proposed secant approximation. Young's modulus of steel E is taken to be $200\,000\text{MPa}$. The moments of inertia for the exterior columns are $I_1 = I_5 = 129 \times 10^6 \text{mm}^4$, while $I_2 = I_3 = I_4 = 34.1 \times 10^6 \text{mm}^4$ for the interior columns. The moments of inertia of the beams are $I_6 = I_7 = I_8 = I_9 = 245 \times 10^6 \text{mm}^4$. All the connections in the four frames are either rigid or pinned connections. The lower and upper bounds for the applied column loads are $P_{il} = 0$ and $P_{iu} = \pi^2 EI_i / K_i^2 L_i^2$, respectively, where the column effective length factors K_i are given in Eq. (3.12). The end-fixity factors and upper bounds of the applied loads for each column of the four frames are listed in Table 3.3 and Table 3.4, respectively.

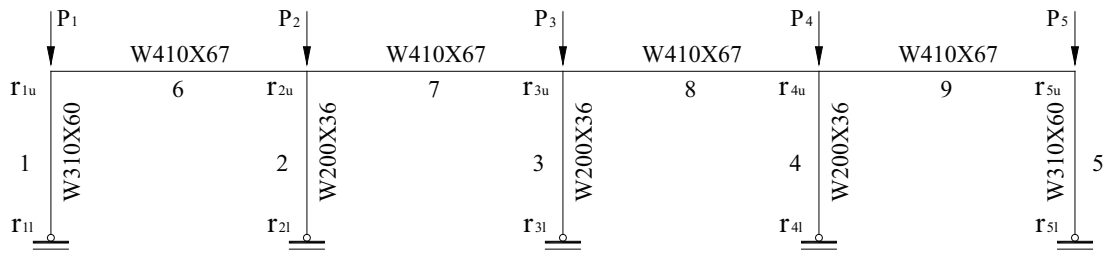
Based on Eq. (3.10), the linear programming problems with Taylor series approximation (LP-T) to find the minimum and maximum buckling strength of the four steel frames in Figure 3.10 can be expressed in the form of Eqs. (3.10) and (3.11) with the number of columns $n = 5$.

Similarly, the linear programming problems with the secant approximation (LP-S) can be expressed by the form of Eqs. (3.17) and (3.18) with the number of columns $n = 5$.

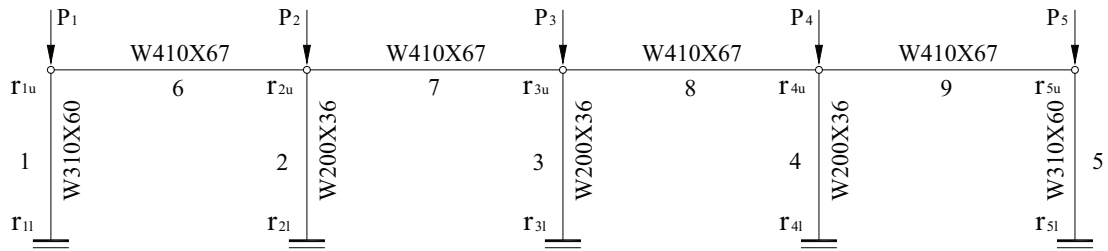
Following the manual calculation approach presented in Appendix A, the minimum and maximum buckling strengths of each frame are computed for the linear programming problems with the Taylor series approximation and the secant approximation, and the



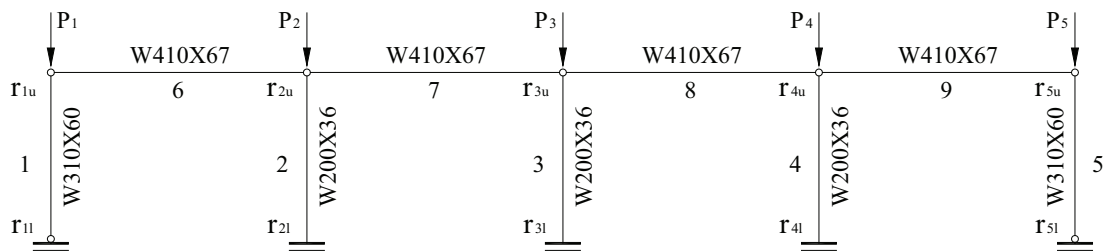
(a) Type-1 Frame



(b) Type-2 Frame



(c) Type-3 Frame



(d) Type-4 Frame

Figure 3.10: One-storey four-bay unbraced steel frames

Table 3.3: Column end-fixity factors

End-fixity Factor	Frame			
	Type-1	Type-2	Type-3	Type-4
r_{1l}	1.0	0.0	1.0	0.0
r_{1u}	0.717	0.717	0.0	0.0
r_{2l}	1.0	0.0	1.0	1.0
r_{2u}	0.95	0.95	0.0	0.0
r_{3l}	1.0	0.0	1.0	1.0
r_{3u}	0.95	0.95	0.0	0.0
r_{4l}	1.0	0.0	1.0	1.0
r_{4u}	0.95	0.95	0.0	0.0
r_{5l}	1.0	0.0	1.0	0.0
r_{5u}	0.717	0.717	0.0	0.0

Table 3.4: Upper bound on column loads

Maximum Load	Frame			
	Type-1	Type-2	Type-3	Type-4
$P_{1u}(\text{kN})$	34394	17197	21411	10706
$P_{2u}(\text{kN})$	10869	5434	5660	5660
$P_{3u}(\text{kN})$	10869	5434	5660	5660
$P_{4u}(\text{kN})$	10869	5434	5660	5660
$P_{5u}(\text{kN})$	34394	17197	21411	10706

resulting frame buckling loads and strengths are presented in Table 3.5. To have a better understanding of the effects of the linear approximations, the results from the nonlinear programming method (Xu, 2003), in which there is no approximation in the evaluation of column stiffness modification factor β , are also listed in Table 3.5. The relative differences

of the buckling strengths of the frames between the linear and nonlinear programming problems are listed in Table 3.6, in which the positive value denotes that the frame buckling strength is overestimated while the negative value indicates that the frame buckling strength is underestimated.

Comparing the results from Table 3.5 to those of the nonlinear programming problems, the buckling strengths obtained from the linear programming problems with Taylor series approximation are overestimated, while the buckling strengths computed by the linear programming problems with the secant approximation are underestimated. In practice, the underestimation of frame buckling strengths would result in a conservative design. Therefore, the linear programming problems with the secant approximation are more preferable than that with the Taylor approximation to be adopted in engineering practice.

On the other hand, comparing to the maximum buckling strength, the minimum buckling strengths of the steel frames are of more concern for design purpose. This is because the minimum buckling strength signifies the worst vulnerable state defined by the worst load pattern and the minimum magnitudes of the applied loads beyond which the steel frame will fail. From Table 3.6, although the differences of the maximum frame buckling strength between the linear programming problems with Taylor series approximation and the nonlinear programming problems are less than that between the linear programming problems with the secant approximation and the nonlinear programming problems, the secant approximation can yield more accurate results for the minimum frame buckling strength. It can also be observed from Table 3.6 that the minimum differences between the linear programming problems and the non-linear programming problems occurred in Type-1 frame, which suggests that the linear programming approximations achieve good approximation for rigid frames or frames in which the magnitudes of the end-fixity factors at the column lower and upper ends are similar.

As discussed previously, for frames with columns of which the magnitudes of the column lower and upper end-fixity factors are substantially different, such as cases of

Table 3.5: Frame buckling loads and strengths

Frame Type	Loading Pattern	Approach	P_1 (kN)	P_2 (kN)	P_3 (kN)	P_4 (kN)	P_5 (kN)	$\sum P_i$ (kN)
1	minimum	LP-T	0	8213.5	8213.5	8213.5	0	24641
		LP-S	11795	0	0	0	11795	23590
		NLP	717	7420	7420	7420	717	23694
	maximum	LP-T	12858	0	0	0	12858	25716
		LP-S	0	7875	7875	7875	0	23625
		NLP	12664	0	0	0	12664	25328
2	minimum	LP-T	0	2010	2010	2010	0	6030
		LP-S	0	1438	1438	1438	0	4314
		NLP	0	0	0	4485	0	4485
	maximum	LP-T	3227	0	0	0	3227	6454
		LP-S	2578	0	0	0	2578	5156
		NLP	3178	0	0	0	3178	6356
3	minimum	LP-T	1515	1515	1515	1515	1515	7575
		LP-S	1057	1057	1057	1057	1057	5285
		NLP	0	1244	0	4655	0	5899
	maximum	LP-T	1515	1515	1515	1515	1515	7575
		LP-S	1057	1057	1057	1057	1057	5285
		NLP	2695	695	695	695	2695	7475
4	minimum	LP-T	0	717	717	717	0	2151
		LP-S	0	500	500	500	0	1500
		NLP	0	0	0	2047	0	2047
	maximum	LP-T	1290	0	0	0	1290	2580
		LP-S	1290	0	0	0	1290	2580
		NLP	0	0	0	0	2580	2580

Table 3.6: Difference of minimum buckling strengths between different methods

Pattern	Difference	Type-1	Type-2	Type-3	Type-4
Maximum	LP-T to NLP	+1.5%	+1.5%	+1.3%	0%
	LP-S to NLP	-6.7%	-18.9%	-29%	0%
Minimum	LP-T to NLP	+4%	+34%	+28%	+5%
	LP-S to NLP	-0.4%	-4%	-10%	-27%

Type-2, Type-3 and Type-4 frames, increased discrepancies on frame buckling strengths were observed between the linear programming problems with secant approximations and the non-linear problems as that shown in Table 3.6. However, the frame buckling strengths were not overestimated by the linear programming problems with the secant approximation in any scenario.

Also noted from Table 3.5 and Table 3.6, in Type-4 frame, the maximum frame buckling strengths obtained from the linear programming problems are identical as that of non-linear problem no matter which linear approximation was employed to evaluate the column lateral stiffness modification factors. Further reviewing the results shown in Table 3.5, it is found that the maximum frame buckling strength occurred only when the exterior lean-on columns ($r_l=r_u=0$) are loaded while the interior columns which provide the lateral stiffness for the lean-on columns to sustain the loads are not loaded ($P_2=P_3=P_4=0$). It can be seen from Figure 3.7, at the initial point of $P/P_e=0$, the magnitude of the β factor is identical no matter if it was calculated based on Eq. (3.4) or linear approximation with Eq. (3.7) or Eq. (3.15).

For Type-3 frame, it is worthy of note from Table 3.5 that the linear programming problems with either Taylor series or the secant approximations failed to detect the difference between the maximum and minimum frame buckling loads as that did in the non-linear problems for this specific case. This issue will be investigated in the following study.

3.7 Conclusion

Based on the concept of storey-based buckling, stability of unbraced steel frames subjected to variable loading can be expressed as a pair of problems of seeking the maximum and minimum frame buckling strengths, which can be further simplified as a pair of linear programming problem (Xu, 2002). However, according to the parametric analysis carried out in this study, it was found that the linear programming problems, based on Taylor series approximation of the column lateral stiffness factor β , may overestimate the frame buckling strengths due to the overestimation of the column lateral stiffness. As the overestimated frame buckling strengths may result in unconservative designs, a secant approximation of the column lateral stiffness modification factor β was proposed in this chapter to generate a conservative estimation of the column stiffness and to maintain the simplicity. To this end, a slope modification factor, α , was introduced to modify the Taylor series approximation, and the corresponding equations of the linear programming problems were developed. Four unbraced steel frames investigated in this study showed that the linear programming problems based on the secant approximation yielded conservative results in comparison with the Taylor series approximation. Compared with the significant computational effort required by the nonlinear programming problems, the linear programming problems based on the secant approximation, which maintains reasonable accuracy and simplicity, is more practical for design engineers to evaluate the frame stability subjected to variable loadings. However, the example also revealed the limitation of the linear programming problems associated with the linear approximation, which will be investigated in the future study.

Chapter 4

Stability of Unbraced Steel Frames Subjected to Variable Loadings Evaluated by NLP Method

4.1 Introduction

Current researches on frame stability ([Xu and Wang, 2008](#); [Tong et al., 2009](#); [Hellesland, 2009](#); [Aristizabal-Ochoa, 2012](#)) are almost exclusively based on the assumption of proportional loading, in which predefined load patterns are assigned to the frame. Accordingly, the possible worst load pattern may not always be included in the specified load combinations due to the unpredictable nature of various types of loads. Thus, the variable loadings both in magnitudes and locations must be considered when assessing the stability of the structures. To this end, the buckling strength of unbraced steel frames subjected to variable loading was first investigated by [Xu et al. \(2001\)](#) based on the concept of storey-based buckling and with use of a linear programming algorithm. In contrast to conventional frame stability analysis which only accounts for proportional loading, the method proposed by [Xu et al. \(2001\)](#) can capture the frame buckling strengths and their

associated load patterns that cause instability of unbraced steel frames at the minimum load levels (the worst load pattern) and the maximum load levels (the most favorable load pattern). The method enables design practitioners to evaluate the stability of unbraced steel frames in extreme loading cases. Considering that the Taylor series approximation of the lateral stiffness of an axially loaded column was adopted in the linear programming based method, [Xu \(2003\)](#) subsequently proposed a nonlinear programming based approach in which the column stiffness was derived directly from the Euler-Bernoulli beam theory without any approximation. The problems to determine the minimum and maximum load levels (the most favourable pattern) and frame buckling strength were solved by a non-gradient-based algorithm due to the complexity of the column stiffness being a transcendental stability function. In addition, in Chapter [3](#), it was found that the linear programming based method ([Xu et al., 2001](#)) cannot accurately detect the maximum and minimum frame buckling strength in special cases such as the Type-3 frame shown in Section [3.6.2](#).

In conventional frame stability analysis, a cubic Hermite element was adopted to derive the flexural stiffness matrix of beam-column members accounting for the effect of axial loading on the bending stiffness of the members. The stiffness matrix was then used to determine the critical load multiplier of the frame subjected to the proportional loading. In this study, the stiffness matrix will be used to derive the lateral stiffness of an axially loaded column in an unbraced frame. Compared to the lateral stiffness based on the classical Euler-Bernoulli beam theory, the proposed lateral stiffness equation is in good accuracy and considerably simpler, which enables the gradient-based algorithm being used to evaluate the minimum and maximum frame buckling strengths of unbraced steel frames subjected to variable loading. Numerical examples are also presented to demonstrate that the accuracy and efficiency of the proposed method.

4.2 Element Stiffness Matrix for Beam Column

Consider a beam-column subject to an axial load P and a set of loads q , as shown in Figure 4.1a. The corresponding displacements δ are depicted in Figure 4.1b. It is assumed that the deformations δ of the element are determined by a two stage loading. Only the axial load P is applied in the first stage, and during the second stage, the element is subjected to bending by the forces q while P remains constant. Since the element is in

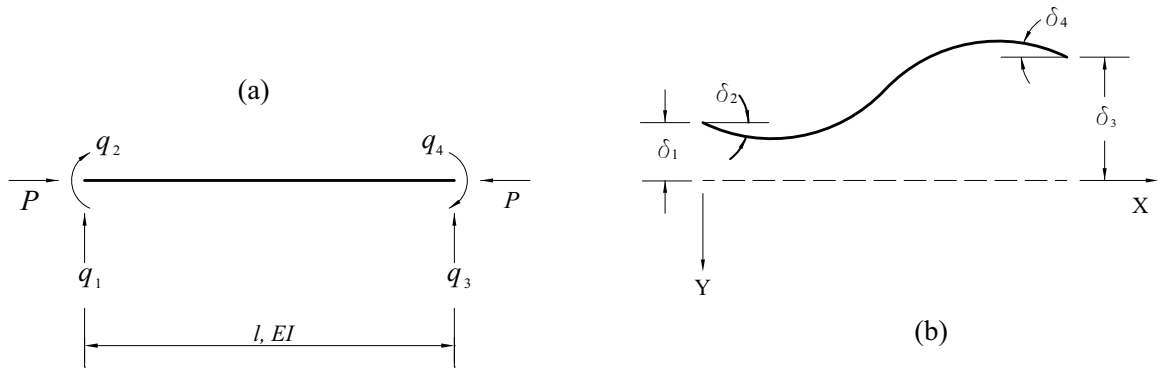


Figure 4.1: Axially loaded beam-column

equilibrium at the end of stage one and two, the external work done by the applied forces must be equal to the strain energy stored in the member not only for the entire loading process but also for stage two by itself. The external work corresponding to the second loading stage is

$$W_e = \frac{1}{2} \delta^T q + \frac{P}{2} \int_0^l (y')^2 dx \quad (4.1)$$

in which the first term represents the work of the q forces and the second term represents the work done by the axial load P . The strain energy stored in the member during stage two is only due to bending. Thus

$$U = \frac{EI}{2} \int_0^l (y'')^2 dx \quad (4.2)$$

Equating the strain energy to the external work gives

$$\frac{1}{2} \delta^T q + \frac{P}{2} \int_0^l (y')^2 dx = \frac{EI}{2} \int_0^l (y'')^2 dx \quad (4.3)$$

Since the member end forces $\mathbf{q}$ can be determined by the relationship $\mathbf{q} = \mathbf{k}\boldsymbol{\delta}$, in which $\mathbf{k}$ is the element stiffness matrix, Eq. (4.3) can be rewritten as

$$\boldsymbol{\delta}^T \mathbf{k} \boldsymbol{\delta} = EI \int_0^l (y'')^2 dx - P \int_0^l (y')^2 dx \quad (4.4)$$

To obtain the stiffness matrix $\mathbf{k}$, a relationship between deflection y and member end displacement $\boldsymbol{\delta}$ is needed. A cubic shape function whose coordinate system shown in Figure 4.1 is selected to represent the deflection

$$y = A + Bx + Cx^2 + Dx^3 \quad (4.5)$$

The choice of the cubic shape function is based on the flexural behavior of the beam and satisfies the conditions of constant shear and linearly varying bending moment that exists in the beam element (Chajes, 1974). Substituting the boundary conditions $y(0) = -\delta_1$, $y(0)' = \delta_2$, $y(l) = -\delta_3$, and $y(l)' = \delta_4$ into Eq. (4.5), the following expression for y is obtained:

$$y = \begin{bmatrix} \left(\frac{3x^2}{l^2} - \frac{2x^3}{l^3} - 1\right) & \left(x - \frac{2x^2}{l} + \frac{x^3}{l^3}\right) & \left(\frac{2x^3}{l^3} - \frac{3x^2}{l^2}\right) & \left(\frac{x^3}{l^2} - \frac{x^2}{l}\right) \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \mathbf{A}\boldsymbol{\delta} \quad (4.6)$$

By taking derivatives of the shape function of Eq. (4.6) with respect to x , the expressions for y' and y'' are obtained in Eq. (4.7) and Eq. (4.8), respectively.

$$y' = \begin{bmatrix} \left(\frac{6x}{l^2} - \frac{6x^2}{l^3}\right) & \left(1 - \frac{4x}{l} + \frac{3x^2}{l^2}\right) & \left(\frac{6x^2}{l^3} - \frac{6x}{l^2}\right) & \left(\frac{3x^2}{l^2} - \frac{2x}{l}\right) \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \mathbf{C}\boldsymbol{\delta} \quad (4.7)$$

$$y'' = \begin{bmatrix} \left(\frac{6}{l^2} - \frac{12x}{l^3}\right) & \left(-\frac{4}{l} + \frac{6x}{l^2}\right) & \left(\frac{12x^2}{l^3} - \frac{6}{l^2}\right) & \left(\frac{6x}{l^2} - \frac{2}{l}\right) \end{bmatrix} \begin{bmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{bmatrix} = \mathbf{D}\boldsymbol{\delta} \quad (4.8)$$

Eq. (4.7) and Eq. (4.8) can be expressed as follow:

$$(y')^2 = \boldsymbol{\delta}^T \mathbf{C}^T \mathbf{C} \boldsymbol{\delta} \quad (4.9)$$

$$(y'')^2 = \boldsymbol{\delta}^T \mathbf{D}^T \mathbf{D} \boldsymbol{\delta} \quad (4.10)$$

Substituting Eq. (4.9) and Eq. (4.10) into Eq. (4.4), the matrix form in Eq. (4.11) can be obtained:

$$\boldsymbol{\delta}^T \mathbf{k} \boldsymbol{\delta} = \boldsymbol{\delta}^T \left\{ EI \int_0^l \mathbf{D}^T \mathbf{D} dx - P \int_0^l \mathbf{C}^T \mathbf{C} dx \right\} \boldsymbol{\delta} \quad (4.11)$$

from which the integral form of the element stiffness matrix $\mathbf{k}$ is given in Eq. (4.12)

$$\mathbf{k} = EI \int_0^l \mathbf{D}^T \mathbf{D} dx - P \int_0^l \mathbf{C}^T \mathbf{C} dx \quad (4.12)$$

The first integral on the right side of Eq. (4.12) corresponds to the conventional stiffness matrix of a flexural element $\mathbf{k}_e$ and the second term represents the geometric stiffness matrix $\mathbf{k}_g$ induced by the axial load P acting on the deformed geometry of the member. By integrating both expressions, the element stiffness matrix is obtained:

$$\mathbf{k} = \mathbf{k}_e + \mathbf{k}_g = \left\{ EI \begin{bmatrix} \frac{12}{l^3} & -\frac{6}{l^2} & -\frac{12}{l^3} & -\frac{6}{l^2} \\ -\frac{6}{l^2} & \frac{4}{l} & \frac{6}{l^2} & \frac{2}{l} \\ -\frac{12}{l^3} & \frac{6}{l^2} & \frac{12}{l^3} & \frac{6}{l^2} \\ -\frac{6}{l^2} & \frac{2}{l} & \frac{6}{l^2} & \frac{4}{l} \end{bmatrix} - P \begin{bmatrix} \frac{6}{5l} & -\frac{1}{10} & -\frac{6}{5l} & -\frac{1}{10} \\ -\frac{1}{10} & \frac{2l}{15} & \frac{1}{10} & -\frac{1}{30} \\ -\frac{6}{5l} & \frac{1}{10} & \frac{6}{5l} & \frac{1}{10} \\ -\frac{1}{10} & -\frac{l}{30} & \frac{1}{10} & \frac{2l}{15} \end{bmatrix} \right\} \quad (4.13)$$

The decrease in flexural stiffness due to the axial compressive force can be observed through the diagonal elements of the matrix $\mathbf{k}_g$. The matrix was derived for an axial compression force. It could serve equally well for a member with an initial tension force if the sign preceding the second matrix were changed from minus to plus. In addition, P - Δ effect is included in the $\mathbf{k}_g$ matrix.

4.3 One Element Model

For an axially loaded column, the lateral stiffness of the column can be defined as the lateral force applied at the upper end under which the lateral deflection of the upper

end is unity. Thus, the lateral stiffness can be determined by assessing the value of the lateral force. To this end, the axially loaded semi-rigid column shown in Figure 3.3 is modeled by one beam-column element to compute the lateral force S corresponding to a unit lateral displacement. As shown in Figure 3.3, the corresponding end displacements of the columns are $\boldsymbol{\delta} = [1 \ \theta_u \ 0 \ \theta_l]^T$, and the end force is $\mathbf{q} = [S \ M_u \ -S \ M_l]^T$. Let R_l and R_u be the end rotational restraining stiffnesses provided by the adjoining members at the column lower and upper ends as shown in Figure 3.3; therefore, the moments can be calculated as

$$M_u = -R_u \theta_u \quad (4.14a)$$

$$M_l = -R_l \theta_l \quad (4.14b)$$

The relationship between $\mathbf{q}$ and $\boldsymbol{\delta}$ can be represented as

$$\mathbf{q} = \mathbf{k}\boldsymbol{\delta} \quad (4.15)$$

where $\mathbf{k}$ is the element stiffness matrix given in Eq. (4.13). Since there is no translational displacement at the lower end, the 4×4 stiffness matrix $\mathbf{k}$ can be reduced to a 3×3 matrix, and the following three equations can be obtained

$$S = EI \left[\frac{12}{l^3} + \left(\frac{P}{10EI} - \frac{6}{l^2} \right) (\theta_u + \theta_l) - \frac{6P}{5EI} \right] \quad (4.16)$$

$$M_u = -R_u \theta_u = EI \left[-\frac{6}{l^2} + \left(\frac{4}{l} - \frac{2Pl}{15EI} \right) \theta_u + \left(\frac{2}{l} + \frac{Pl}{20EI} \right) \theta_l + \frac{P}{10EI} \right] \quad (4.17)$$

$$M_l = -R_l \theta_l = EI \left[-\frac{6}{l^2} + \left(\frac{2}{l} - \frac{Pl}{30EI} \right) \theta_u + \left(\frac{4}{l} - \frac{2Pl}{15EI} \right) \theta_l + \frac{P}{10EI} \right] \quad (4.18)$$

To express the lateral stiffness S with respect to the end-fixity factor r , the end flexural restraints R_l and R_u can be expressed as functions in terms of end-fixity factors r_l and r_u by rewriting Eq. (3.1) as

$$R_u = \frac{3EI}{l} \frac{r_u}{(1 - r_u)} \quad (4.19a)$$

$$R_l = \frac{3EI}{l} \frac{r_l}{(1 - r_l)} \quad (4.19b)$$

Substituting Eqs. (4.19) into Eqs. (4.16~4.18), three variables S , θ_u , and θ_l can be determined by solving Eqs. (4.16~4.18). The lateral stiffness S can be presented as

$$S = \beta(\phi, r_l, r_u) \frac{12EI}{l^3} \quad (4.20)$$

where the coefficient β is a modification factor of the column lateral stiffness accounting for both the effect of axial load and column end rotational restraints, and it can be expressed as:

$$\beta = -\frac{1}{12} \frac{a_0 + a_1\phi^2 + a_2\phi^4 + a_3\phi^6}{b_0 + b_1\phi^2 + b_2\phi^4} \quad (4.21)$$

where

$$\phi = \sqrt{\frac{Pl^2}{EI}} = \pi \sqrt{\frac{P}{P_e}} \quad (4.22a)$$

$$a_0 = -2160(r_l + r_u + r_l r_u) \quad (4.22b)$$

$$a_1 = 720 - 504r_l r_u + 216(r_l + r_u) \quad (4.22c)$$

$$a_2 = 45(r_l + r_u) - 18r_l r_u - 72 \quad (4.22d)$$

$$a_3 = r_l r_u - (r_l + r_u) + 1 \quad (4.22e)$$

$$b_0 = 720 - 180r_l r_u \quad (4.22f)$$

$$b_1 = 48(r_l + r_u) - 24r_l r_u - 72 \quad (4.22g)$$

$$b_2 = 1 + r_l r_u - (r_l + r_u) \quad (4.22h)$$

The stiffness modification factor β characterizes the relationship between the column lateral stiffness and applied axial load. Similar to the discussion in Chapter 3, for the stiffness modification factor β given in Eq. (4.21), an increase of the axial load results in a decrease of the magnitude of β , which consequently reduces the lateral stiffness of the column. A zero value of β denotes the column has completely lost its lateral stiffness and will buckle in a lateral sway mode if no external support is provided. A negative value of β signifies that the column completely relies upon the external lateral support to sustain the applied load. With an adequate external support, the column can sustain

the applied load up to the level where the magnitude of β approaches the negative infinity value, which signifies that the column has reached its rotational non-sway buckling strength. Two methods of evaluating β were proposed by Xu et al. (2001) and were discussed in Chapter 3. First, based on Euler-Bernoulli beam theory, a transcendental relationship (EB) between β and ϕ was derived. However, the transcendental relationship between β and ϕ resulted in the complexity of having to use mathematical programming methods to obtain the maximum and minimum frame buckling strength when frames are subjected to variable loading. Subsequently, the Taylor series approximation of the transcendental relationship (TEB) was adopted for the reason of simplicity so that the linear programming method can be employed to calculate the maximum and minimum frame buckling strength. To verify the proposed method with use of one beam-column element, an axially loaded column with pin-fixed ends example is employed to demonstrate the differences of β values among the proposed method in this section, EB method and TEB method. The variations of β based on the three methods versus the applied load ratio ϕ are plotted in Figure 4.2. Since the EB method is theoretically derived without any approximation, it can serve as the benchmark in this comparison.

It can be observed from Figure 4.2 that the results obtained from Eq. (4.21) are in good agreement with those obtained from the other two methods when only the lateral sway-buckling strength of the column is of concern, and the differences between them are less than 1%. However, the values of β obtained by the proposed method with one beam-column element and the Taylor series approximation deviate from those obtained from the nonlinear method based on the Euler-Bernoulli theory as the applied load ratio ϕ increases. The considerable difference of β between the proposed method and EB method for ϕ greater than 4 demonstrates that the proposed method with use of one beam-column element cannot accurately predict the rotational non-sway buckling strength of the column. It is because that the column modelled by one beam-column element can only be used to compute the end displacement yet the displacement between two ends.

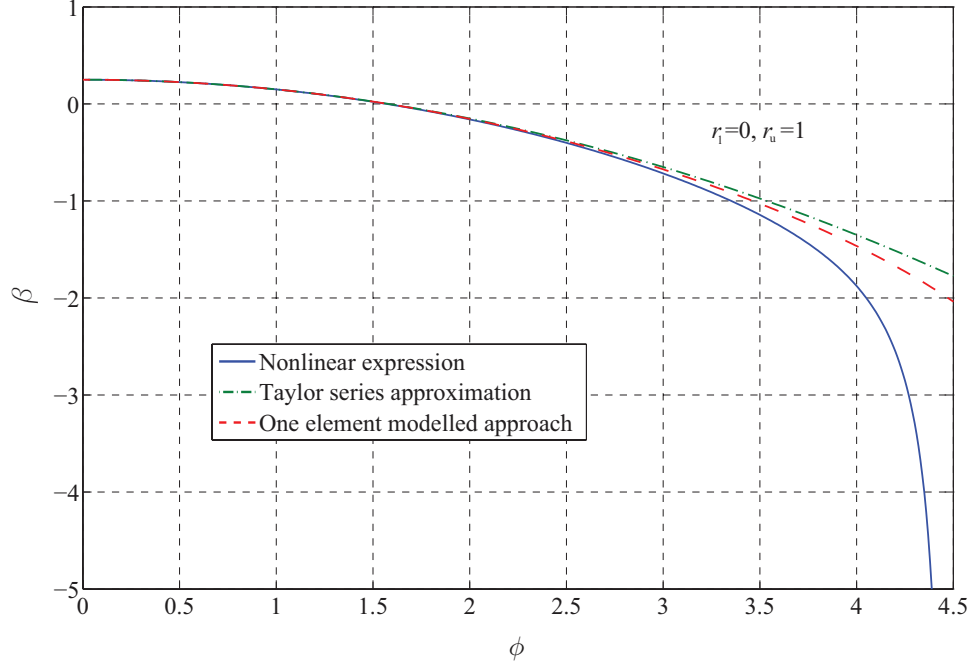


Figure 4.2: β calculated by three methods

For non-sway buckling, buckle shapes show that the location of the maximum deflection is close to or at the column mid-point. Accordingly, to obtain more accurate results, the number of beam-column elements that are used to model the stability behavior of unbraced steel columns must be increased.

4.4 Two Element Model

In the conventional effective length method, each column in unbraced frames is always assumed to buckle in a lateral sway mode. However, in the concept of storey-based buckling, the lateral sway buckling of weak or heavily loaded columns may be restrained by the lateral bracing provided by those strong or lightly loaded columns in the same storey. Accordingly, axially loaded columns in unbraced frames may buckle in two modes: rotational non-sway buckling and lateral sway buckling as shown in Figure 4.3.

As discussed in Section 4.3, for lateral sway buckling, the buckling strength assessed by

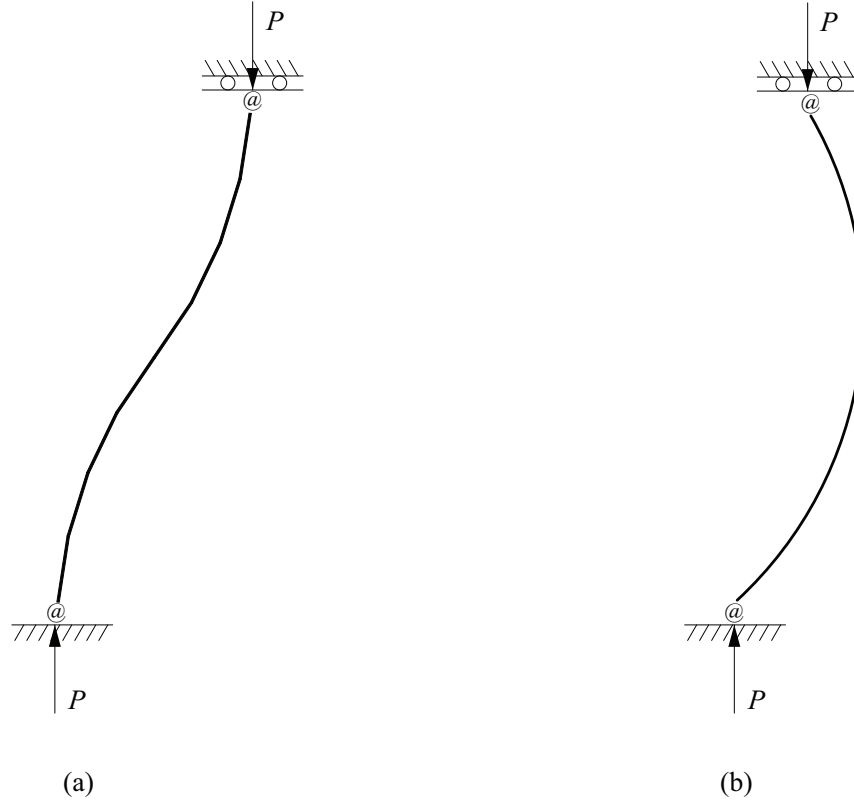


Figure 4.3: Two kinds of deformed shapes of lateral sway columns

the proposed method with use of one beam-column element can be accurately predicted with an error less than 1%. However, the error associated with using single beam-column element to evaluate rotational non-sway buckling strength can be considerably increased, which is not acceptable. Previous studies showed that the error can be reduced to 2% when two beam-column elements were used to model a column in rotational non-sway buckling (Teh, 2001; So and Chan, 1991). To this end, in this section, the column shown in Figure 4.3 with a length L and a uniform flexural rigidity EI is evenly subdivided into two elements. The positive deformations and the corresponding forces for the column and individual elements are defined in Figure 4.4. In this figure, Δ is the nodal displacement of the column, and W is the nodal force applied at the column. δ is the end deformation of one element and q is the end force of one element.

The structure stiffness matrix for the entire column can be obtained by transform-

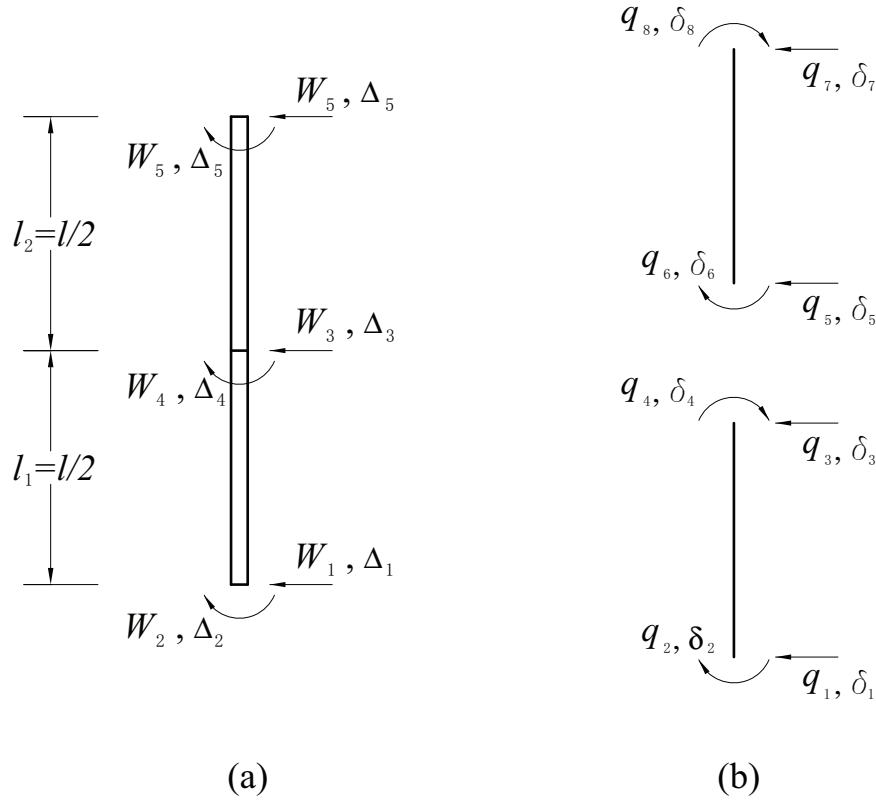


Figure 4.4: Nodal forces and displacements for a column

ing the individual element stiffness matrices from element to structure coordinates and then combining the resulting matrices. Therefore, based on the stiffness matrix of one element in Eq. (4.13), the structure stiffness matrix $\mathbf{K}$ for the column can be obtained by superimposing the matrices of two elements as follow:

$$\mathbf{K} = \left\{ EI \begin{bmatrix} \frac{12}{l^3} & -\frac{6}{l^2} & -\frac{12}{l^3} & -\frac{6}{l^2} & 0 & 0 \\ & \frac{4}{l} & \frac{6}{l^2} & \frac{2}{l} & 0 & 0 \\ & & \frac{24}{l^3} & 0 & -\frac{12}{l^3} & -\frac{6}{l^2} \\ & & & \frac{8}{l} & \frac{6}{l^2} & \frac{2}{l} \\ Sym & & & & \frac{12}{l^3} & \frac{6}{l^2} \\ & & & & & \frac{4}{l} \end{bmatrix} - P \begin{bmatrix} \frac{6}{5l} & -\frac{1}{10} & -\frac{6}{5l} & -\frac{1}{10} & 0 & 0 \\ & \frac{2l}{15} & \frac{1}{10} & -\frac{l}{30} & 0 & 0 \\ & & \frac{12}{5l} & 0 & -\frac{6}{5l} & -\frac{1}{10} \\ & & & \frac{4l}{15} & \frac{1}{10} & -\frac{l}{30} \\ Sym & & & & \frac{6}{5l} & \frac{1}{10} \\ & & & & & \frac{2l}{15} \end{bmatrix} \right\} \quad (4.23)$$

Since there is no external force at the mid-point of the column, the nodal forces can

be stated as $\mathbf{W} = [-S \ M_l \ 0 \ 0 \ S \ M_u]^T$, in which the end moment M_u and M_l can be expressed as Eq. (4.14a) and Eq. (4.14), respectively. The nodal displacements of the column can be found from Figure as $\Delta = [0 \ \theta_l \ \Delta_3 \ \Delta_4 \ 1 \ \theta_u]^T$. Thus, the force-displacement relationship for the column in Figure 4.4 can be stated as

$$\mathbf{W} = \mathbf{K} \Delta \quad (4.24)$$

in which $\mathbf{K}$ is the structure stiffness matrix defined in Eq. (4.23). Substituting the relationship between end flexural restraint R and end-fixity factor r defined in Eq. (4.19) into Eq. (4.24), and solving five equations for five variables with the previous procedure (The procedure is presented in Appendix B), the lateral stiffness S can be as well expressed as the same equation of Eq. (4.20) in terms of the lateral stiffness modification factor β . However, the factor β here is changed to the following equation:

$$\beta = -\frac{1}{12} \frac{a_0 + a_1\phi^2 + a_2\phi^4 + a_3\phi^6 + a_4\phi^8 + a_5\phi^{10}}{b_0 + b_1\phi^2 + b_2\phi^4 + b_3\phi^6 + b_4\phi^8} \quad (4.25)$$

where

$$a_0 = -132710400(r_i + r_j + r_i r_j) \quad (4.26a)$$

$$a_1 = 44236800 + 17694720(r_i + r_j) - 26542080r_i r_j \quad (4.26b)$$

$$a_2 = 2474496(r_i + r_j) - 440064r_i r_j - 5895240 \quad (4.26c)$$

$$a_3 = 158208 - 11218(r_i + r_j) + 732824r_i r_j \quad (4.26d)$$

$$a_4 = 1127(r_i + r_j) - 947r_i r_j - 1280 \quad (4.26e)$$

$$a_5 = 1 - (r_i + r_j) + r_i r_j \quad (4.26f)$$

$$b_0 = 44236800 - 1105920r_i r_j \quad (4.26g)$$

$$b_1 = 2949120(r_i + r_j) - 1105920r_i r_j - 5898240 \quad (4.26h)$$

$$b_2 = 158208 - 115968(r_i + r_j) + 80640r_i r_j \quad (4.26i)$$

$$b_3 = 1136(r_i + r_j) - 992r_i r_j - 1280 \quad (4.26j)$$

$$b_4 = 3(1 + r_i r_j - r_i - r_j) \quad (4.26k)$$

and ϕ in the Eq. (4.25) is the applied load ratio defined in Eq. (4.22a). Shown in Figure 4.5 are the relationships between β and ϕ obtained from Eq. (4.25) and that of EB and TEB methods for different column end rotational restraints. Similar to that discussed in Section 4.3, EB method is served as the benchmark in this comparison.

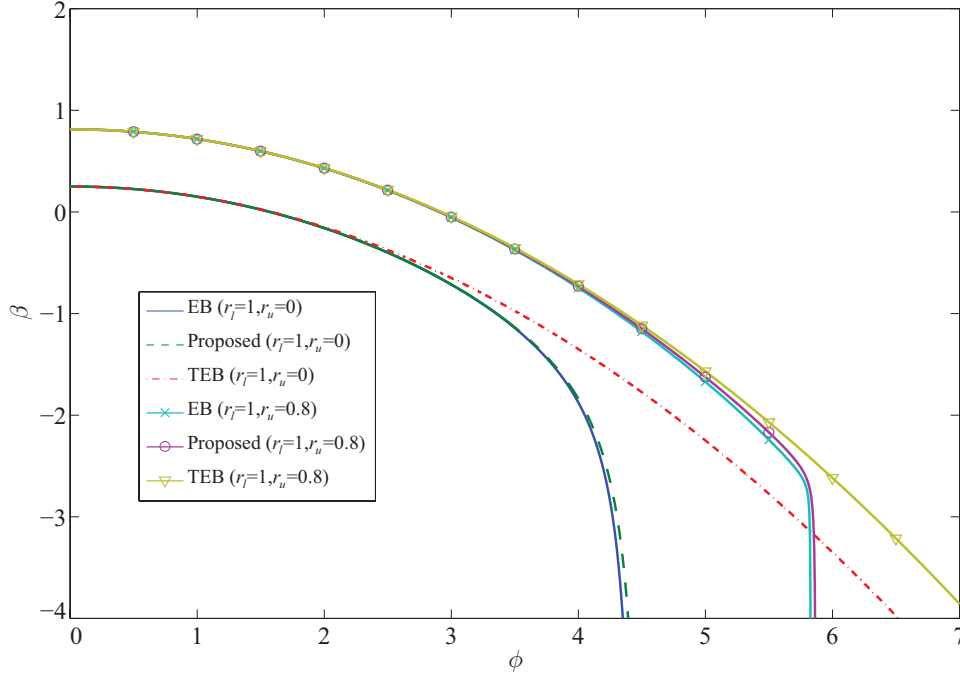


Figure 4.5: Relationships between β and ϕ of different methods

It can be found from Figure 4.5 that the sway lateral buckling strengths ($\beta=0$) predicted by all three methods are close to each other. However, as the increase of the magnitude of ϕ , β values predicted by TEB method gradually deviate from that of EB method and the difference of β values between the two methods becomes significant. On the contrary, the result obtained from the proposed method is in good agreement with that from EB method even with substantial increase of the magnitude of ϕ . As the axially loaded columns in an unbraced frame can buckle either in lateral sway or rotational non-sway mode depends on lateral stiffness of the frames, the predicted frame buckling strength would be inaccurate if the rotational non-sway buckling strength of a

column cannot be accurately evaluated. Therefore, the proposed method need to predict the accurate results of critical value of ϕ for not only at $\beta = 0$ but also at $\beta = -\infty$ (negative infinity) which are associated with the sway and non-sway buckling mode, respectively. Once a critical value of ϕ is calculated, the corresponding buckling strength can be obtained from Eq. (4.20).

Table 4.1: Critical value of ϕ for different column end restraints

Column end restraints	Approach	Critical value of ϕ	
		Non-sway buckling	Sway buckling
$r_l=1, r_u=0$	EB	4.49	1.57
	TEB	NA	1.58
	Proposed	4.55	1.57
$r_l=1, r_u=0.2$	EB	4.72	1.94
	TEB	NA	1.95
	Proposed	4.78	1.94
$r_l=1, r_u=0.8$	EB	5.83	2.90
	TEB	NA	2.92
	Proposed	5.87	2.91

Shown in Table 4.1 are the critical values of ϕ associated with rotational non-way buckling and lateral sway buckling for columns with different end restraints evaluated based on the three methods. Similar to the observation in Section 4.3, it can be seen that results obtained from both TEB method and the proposed two beam-column element method are in good agreement with that of EB method while assessing the sway buckling strength of the columns. However, significant differences are observed when it comes to evaluate the buckling strength of braced columns. TEB method based on Taylor series approximation to evaluate β value is not applicable (NA) to calculate the rotational non-sway buckling strength of the columns. Consequently, TEB based method may result in

erroneous predictions of the minimum and maximum unbraced frame buckling strength in the case when weak or heavily loaded columns tend to fail in rotational non-sway buckling mode when they are laterally braced adequately by strong or lightly loaded columns in the same storey. As buckling strength evaluated based on the proposed two beam-column element method are in good agreement with that of the EB for both the sway and non-sway columns. That being said, in evaluation of frame buckling strength subjected to variable loading, the proposed two beam-column element method would be able to overcome the deficiency associated with TEB method and the proposed method with using only one beam-column element as shown in Figure 4.2. In addition, the proposed method is simpler than EB method which would be a great advantage in terms of efficiency while evaluating the maximum and minimum frame buckling strength by means of mathematic programing algorithms.

4.5 The Maximum and Minimum Buckling Strength of Unbraced Frames

Considering the balance of accuracy and efficiency, the proposed method with use of two beam-column elements to model the stability behaviour of an axially loaded column will be used to solve the optimization problem of determining the buckling strength of unbraced steel frames subjected to variable loadings, and the maximization and minimization problems with stability constraints in Chapter 3 can be rewritten as

$$\begin{matrix} \text{Minimum} \\ \text{Maximum} \end{matrix} : Z = \sum_{i=1}^n P_i \quad (4.27)$$

subject to

$$\sum_{i=1}^n S_i = 12E \sum_{i=1}^n \beta_i(\phi_i, r_{li}, r_{iu}) \frac{I_i}{L_i^3} \quad (4.28a)$$

$$P_{il} \leq P_i \leq P_{iu} = \frac{\pi^2 EI_i}{K_i^2 L_i^2} \quad (i = 1, 2, \dots, n) \quad (4.28b)$$

where the applied column load P_i is the variable to be solved in the problem. The objective function Eq. (4.27) corresponds to either the minimum or the maximum elastic frame buckling strength, as given by the sum of individual column loads. The storey-based lateral instability condition imposed on the frame is defined by Eq. (4.28a), in which $\beta_i(\phi_i, r_{li}, r_{ru})$ is the lateral stiffness modification factor defined in Eq. (4.25) and is a function of the applied load. The lower and upper bounds imposed on each applied column load are the same as that discussed in Section 3.3. Unlike the maximization and minimization problem discussed in Chapter 3, which can be solved by linear programming method because there is a linear relationship between the applied load P_i and the stiffness modification factor β_i . The relationship defined in Eq. (4.28a), however, is nonlinear but contains polynomials only. Thus, the minimization and maximization problems stated in Eqs. (4.27) and (4.28) will be able to be solved by efficient gradient-based nonlinear programming algorithm. In this study, sequential quadratic programming algorithm is employed to solve the problems.

4.6 Numerical Examples

The stability of four single-storey unbraced steel frames subjected to variable loading studied in Chapter 3 and shown in Figure 3.10 were investigated by Xu (2003) based on expressions similar to that of Eqs. (4.27) and (4.28). However, because the column stiffness was evaluated by EB method which resulted in the constraint shown in Eq. (4.28a) containing transcendental functions, it was too complicated to directly use gradient-based nonlinear programming algorithms to solve the problems. Consequently, the problems of Eqs. (4.27) and (4.28) were reformulated by combining the objective function Eq. (4.27) and the stability constraint Eq. (4.28a) in which an artificial penalty coefficient was applied to Eq. (4.28a). The reformulated problems were then solved by a less efficient non-gradient-based algorithm.

In this section, the problems defined in Eqs. (4.27) and (4.28) for the four frames are

solved by the sequential quadratic programming method for the steel frames shown in Figure 3.10. The material and dimensional properties of structural members are list in Section 3.6.2. The lower and upper bounds for the applied column loads are $P_{il} = 0$ and $P_{iu} = \pi^2 EI_i / K_i^2 L_i^2$, respectively, where the column effective length factors K_i are defined in Eq. (3.12). The end-fixity factors for each column of the four frames are listed in Table 3.4.

The minimum and maximum buckling strengths of the four steel frames obtained from the sequential quadratic programming in which the lateral stiffness of columns are evaluated by the proposed method are presented in Table 4.2. For the reason of comparison, the results obtained from a non-gradient-based algorithm (NLP-NG) by Xu (2003) and linear programming (LP) in Chapter 3 in which the stiffness are calculated based on EB and TEB method, respectively, are also listed in Table 4.2. The upper bounds of the column applied loads of the frames can be found in Table 3.4. The relative differences among methods are presented in Table 4.3.

Comparing the results of Table 4.2 and 4.3, it can be observed that the minimum and maximum frame buckling strengths obtained from the nonlinear programming problems of Eqs. (4.27) and (4.28) based on the proposed gradient-based method with use of two beam-column elements, are close to the results of the non-gradient-based algorithm by Xu (2003). The results also suggest that the proposed method is reliable in evaluating the minimum and maximum buckling strength of unbraced steel frames subjected to variable loadings as the difference between the results of NLP-G and NLP-NG are less than 2%. It is also noted that the frame buckling strength evaluated based on the proposed method (NLP-G) are slightly greater than that of NLP-NG method except one case, which indicates that column stiffness modification factors β_i are slightly overestimated by the proposed method compared to that of EB method. The only one case that frame buckling strength yield by the proposed method is less than that of NLP-NG method is the minimum frame buckling strength of Type-3 frame, in which the strength

Table 4.2: Critical Frame buckling strengths

Frame Type	Loading Pattern	Approach	P_1 (kN)	P_2 (kN)	P_3 (kN)	P_4 (kN)	P_5 (kN)	$\sum P_i$ (kN)
1	minimum	LP	0	8213.5	8213.5	8213.5	0	24641
		NLP-G	0	2144	10869	10869	0	23882
		NLP-NG	717	7420	7420	7420	717	23694
	maximum	LP	12858	0	0	0	12858	25716
		NLP-G	12735	0	0	0	12735	25470
		NLP-NG	12664	0	0	0	12664	25328
2	minimum	LP	0	2010	2010	2010	0	6030
		NLP-G	0	0	4523	0	0	4523
		NLP-NG	0	0	0	4485	0	4485
	maximum	LP	3227	0	0	0	3227	6454
		NLP-G	3179	0	0	0	3179	6454
		NLP-NG	3178	0	0	0	3178	6356
3	minimum	LP	1515	1515	1515	1515	1515	7575
		NLP-G	0	0	0	5043	0	5043
		NLP-NG	0	1244	0	4655	0	5899
	maximum	LP	1515	1515	1515	1515	1515	7575
		NLP-G	2679	708	708	708	2679	7482
		NLP-NG	2695	695	695	695	2695	7475
4	minimum	LP	0	717	717	717	0	2151
		NLP-G	0	0	0	2050	0	2050
		NLP-NG	0	0	0	2047	0	2047
	maximum	LP	1290	0	0	0	1290	2580
		NLP-G	1290	0	0	0	1290	2580
		NLP-NG	0	0	0	0	2580	2580

Table 4.3: Difference of minimum buckling strengths between two methods

Pattern	Difference	Type-1	Type-2	Type-3	Type-4
Maximum	LP to NLP-NG	+1.5%	+1.5%	+1.3%	0%
	NLP-G to NLP-NG	+0.56%	+1.5%	+0.09%	+0.15%
Minimum	LP to NLP-NG	+4%	+34%	+28%	+5%
	NLP-G to NLP-NG	+0.79%	+0.85%	-0.15%	0%

obtained from the proposed method (NLP-G: $\sum P_i = 5043$ kN) is less than that obtained from the nonlinear programming problem solved by non-gradient-based algorithm (NLP-NG: $\sum P_i = 5899$ kN). In this case, it appears that the non-gradient-based algorithm did not find a global minimum other than a local one.

It is also observed that the load patterns associated with the frame buckling strengths obtained from the proposed and NLP-NG methods may not be the same, which indicates there may be more than one load pattern associated with the minimum or maximum frame buckling strength in the case of variable loading. As discussed by Xu (2003), the load patterns associated with the minimum and maximum frame buckling strength depend on the presence of strong and weak columns, as defined by large and small lateral stiffness capacity, respectively. The load patterns associated with the minimum frame buckling strength tends to maximize loads on the weak columns and minimize the loads on the strong columns and vice versa for that associated with the maximum frame buckling loads. The results shown in Table 4.2 are in agreement in with that observation.

Meanwhile, it is observed from Table 4.3, the relative differences between the results of LP and NLP-NG methods are much greater than that between the results of NLP-N and proposed (NLP-G) methods. That being said, the results obtained from the proposed method are more accurate than that obtained from the linear programming method. Also, it can be found that for all these four type frames, the LP method overestimates the minimum frame buckling strength considerably. In addition, the LP method failed

to detect the minimum frame buckling strength of the Type-3 frame. In practice, the minimum frame buckling strength is far more important to engineering practitioners than the maximum one since the minimum frame buckling strength associated with the worst load pattern under which the framed structure will fail with the minimum amount of applied load. Therefore, despite the simplicity, the LP method in which column lateral stiffness are evaluated based on the Taylor series approximation of the Euler-Bernoulli theory (TEB) may not be suitable for engineering practice as it may result in unsafe designs. From this perspective, the proposed method is preferable and recommended for the investigation of frame stability subjected to variable loading.

4.7 Conclusion

In this chapter, an alternative method was proposed to evaluate the lateral stiffness of an axially loaded column by modeling a column with two beam-column elements. The proposed method was first derived with an axially loaded column modelled by one beam-column element. However, the investigation showed that the error of the rotational buckling strength by the proposed method with use of one beam-column element was too large to be used in design practice. The proposed method using two beam-column elements was then studied. The accuracy of the proposed method with use of two beam-column elements on column rotational non-sway and lateral sway buckling strength are investigated for columns with different end rotational restraints and bracing conditions. The results showed that the proposed method with use of two beam-column elements are in good agreement with that obtained from the method based on Euler-Bernoulli beam theory proposed by [Xu et al. \(2001\)](#). Unlike the method based on Euler-Bernoulli beam theory in which the expression of lateral stiffness of axially loaded column involves transcendental functions of applied load, the proposed equation only contains polynomials, which is certainly simpler than the transcendental functions especially when the nonlinear programming algorithms are employed to find the maximum and minimum buckling

strength of unbraced frames subjected to variable loading.

Four single storey unbraced steel frames subjected to variable loadings were investigated in this study. The investigation demonstrated that the maximum and minimum frame buckling strengths obtained based on the proposed method with use a gradient-based algorithm are can yield accurate results that are in good agreement with those evaluated based on the Euler-Bernoulli theory with use of a non-gradient-based algorithm (Xu, 2003). It is also found in the examples that the lateral stiffness of axially loaded column may be overestimated when the Taylor series approximation of Euler-Bernoulli beam theory was employed. Consequently, the frame buckling strength evaluated based on the linear programming formulation may be overestimated which can result in unsafe design. In that respect, stability capacity of unbraced frame subjected to variable loading should be evaluated in accordance with the nonlinear programming based formulation and solved by the proposed method.

Chapter 5

Storey-based Evaluation of Stability of Unbraced Steel Frames at Elevated Temperature

5.1 Introduction

Conventionally, steel framed structures are designed to sustain the various loads and maintain the stability at ambient temperature. In the case of a fire, the structure's integrity is assumed to be immune from elevated temperature because of the presence of the fire protection materials. However, the recent adoption of Annex K "Structural design for fire conditions" in CAN/CSA S16-09 ([CSA, 2009](#)) provides an innovative way of designing steel framed structures for fire conditions instead of solely relying on traditional fire protection techniques which are quite costly. In the mean time, the adoption of Annex K in CAN/CSA S16-09 ([CSA, 2009](#)) brings certain challenges for structural engineers since there is no specific design guide available at this time. Therefore, assessing the stability performance of steel frames in fire conditions is of importance for the engineering design community.

The evaluation of the stability of a compressive member in steel frames subject to elevated temperatures is different from that at ambient temperature due to thermal effects. In recent years, the stability of steel columns at elevated temperature has been extensively studied through experimental investigations and theoretical developments. Two series of experimental full-scale tests have been performed on steel columns at elevated temperature by [Franssen et al. \(1998\)](#). [Ali and O'Connor \(2001\)](#) conducted a series of tests on axially-restrained steel columns subjected to quasi-standard fire. An experimental study to investigate the failure time of unprotected steel columns subjected to axial restraint at elevated temperature has been carried out by [Tan et al. \(2007\)](#). On the other hand, the restraints affect the stability of steel columns at elevated temperature were widely investigated by [Neves \(1995\)](#), [Shepherd et al. \(1997\)](#), [Wang \(1997\)](#), and [Tang et al. \(2001\)](#) with theoretical derivations. However, the foregoing researches all focused on the isolated column at elevated temperature without consideration of the interaction between members. One of the fire protection measures in buildings is to contain a fire in one compartment. Therefore, while assessing the stability of columns that are experiencing elevated temperature, the restraints provided by the members which are not experiencing the elevated temperature should not be ignored. Column stability at elevated temperature should be evaluated in the context of the steel frame rather than as an isolated column. The behavior of steel frames subjected to elevated temperature has been studied by several researchers ([Cheng and Mak, 1975](#); [Furumura and Shinohara, 1978](#); [Rubert and Schaumann, 1986](#); [Najjar and Burgess, 1996](#)) with numerical simulations, which often requires significant computational efforts. [Toh et al. \(2001\)](#) proposed a simple Rankie method based on an empirical approach to approximate the strength and stability of steel frames in fire. However, as the empirical approximation was established for simple portal frames, the applicability of the method to steel frames in practice had not yet been validated. Considering the fact that current design methods endeavor to warrant the stability and integrity of building structures at ambient temperature, there

is an urgent need to develop practical approaches to evaluate the strength and stability of steel frames subjected to the elevated temperature.

For unbraced steel frames at ambient temperature, a practical approach to evaluate the frame buckling strength was proposed by [Xu and Liu \(2002a\)](#) based on the concept of storey-based buckling introduced by [Yura \(1971\)](#). The approach accounted for both beam-to-column rotational restraints and lateral stiffness interactions among columns in the same storey of the frame, and is comprehensive and efficient for the stability capacity of unbraced steel frames. In this study, this approach will be extended to evaluate the stability capacity of unbraced steel frames in fire.

Based on the concept of storey-based buckling, a frame becomes laterally unstable due to diminishing of its lateral stiffness contributed by the increase of the magnitude of the applied loads. In fire analysis, the frame may become unstable at elevated temperature even though there is no increase in applied loads. This is because the degradation of the Young's Modulus of steel associated with elevated temperature will lead to the loss of column lateral stiffness. Similar to that at ambient temperature, the lateral stiffness and stability of a column subject to the elevated temperature is also affected by the boundary restraints provided by the adjoining members in the frame. Thus, the effects of axial loading, elevated temperature, and thermal boundary restraints on the lateral stiffness of steel columns in unbraced frames are first investigated in this study. Then, the approach proposed by [Xu and Liu \(2002a\)](#) is extended to evaluate the stability capacity of unbraced steel frames in fire. Numerical examples are presented to demonstrate the evaluation procedure of the proposed method and investigate the frame stability subjected to different scenarios of frame members exposed to the elevated temperature. To simplify the analysis, the connection stiffness is assumed to remain constant at elevated temperature and the thermal expansion of beam at elevated temperature is neglected.

5.2 Storey-based Stability of Unbraced Steel Frames in Fire

In Chapter 3, the stability of unbraced steel frames at ambient temperature has been analyzed using the storey-based buckling method. However, when the steel frames are subjected to fire, the behavior of the steel frames is different from that at ambient temperature. In reality, compartment fire occurs randomly and is difficult to predict in advance. Chances are that a fire takes place in one compartment of a building and may not spread to other compartments because buildings are constructed to be complied with building fire code. Consequently, elevated temperature not only results in the deterioration of the steel material but also induces additional compressive stress to columns due to the presence of the thermal axial restraint provided by adjoining members which are not exposed to elevated temperature.

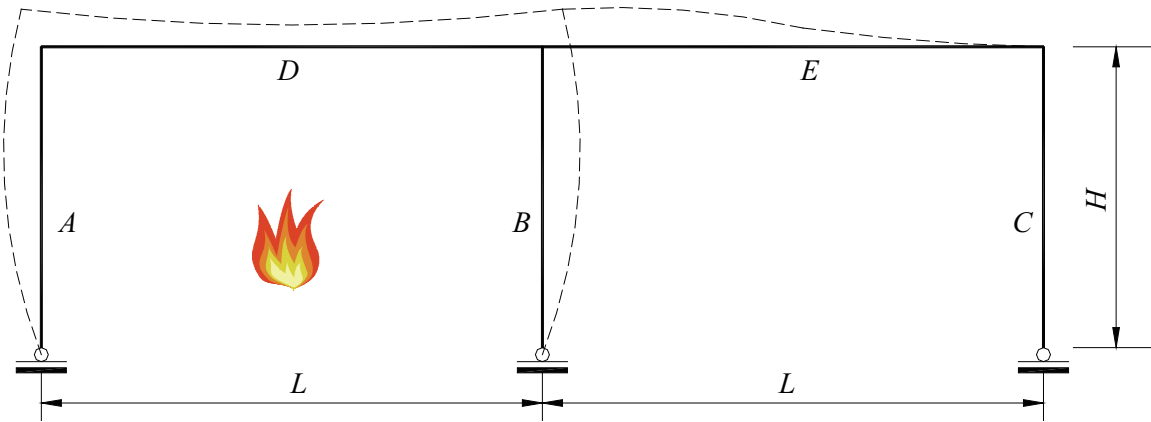


Figure 5.1: Steel frame subject to compartment fire

Taking the fire scenario of a single-storey steel frame shown in Figure 5.1 as an example, an axial restraint induced by the unheated Beam E is imposed on the upper end of Column B. The analytical model for the axially loaded Column B in the unbraced frame subjected to elevated temperature is shown in Figure 5.2, where R_u is the stiffness of the rotational restraint provided by Beam D and E, and k is the stiffness associated

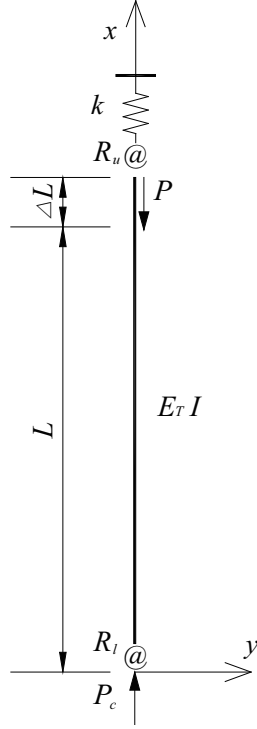


Figure 5.2: Column with thermal restraints

with the axial restraint related to the unheated Beam E.

5.2.1 Thermal Axial Restraint

To evaluate the axial restraining stiffness k associated with the unheated Beam E, the rotational stiffnesses of the connected columns were assumed to be infinite when compared to that of the beam. The assumption tends to be conservative, since the infinitely rigid column will result in an overestimated axial restraint, which consequently underestimates the buckling capacity of the column at elevated temperature (Valente and Neves, 1999). In this study, a steel beam with semi-rigid connections as shown in Figure 5.3 is adopted to compute the stiffness k of the beam-to-column axial restraint.

For the semi-rigid member shown in Figure 3.1, the slope-deflection equation for the

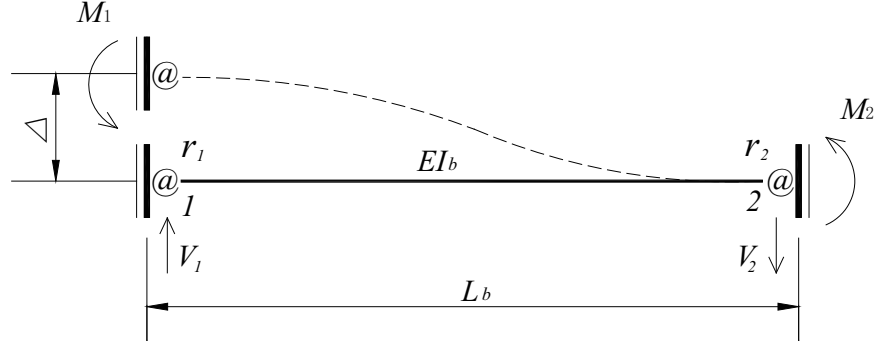


Figure 5.3: Model of semi-rigid steel beam

member can be expressed in terms of the end-fixity factor r_1 and r_2 as (Xu, 1994):

$$M_1 = \frac{3r_1}{4 - r_1r_2} \frac{EI_b}{L_b} (4\theta_1 + 2r_2\theta_2) \quad (5.1)$$

where M_1 is the restraining moment at end 1. EI_b/L_b is the flexural stiffness of the beam, r_1 and r_2 are the end-fixity factors. θ_1 and θ_2 are the connection rotations of the beam at the ends 1 and 2, respectively. Similarly, the slope-deflection equation for the beam with relative movement Δ at one end as shown in Figure 5.3 can be expressed as

$$M_1 = \frac{3r_1}{4 - r_1r_2} \frac{EI_b}{L_b} \left[\left(4\theta_1 - \frac{\Delta}{L_b} \right) + 2r_2 \left(\theta_2 - \frac{\Delta}{L_b} \right) \right] \quad (5.2)$$

Having the moment obtained from Eq. (5.2), the shear force at end 1 can be obtained by summing up moment about end 2, and then the vertical stiffness k can be computed as

$$k = \frac{V_1}{\Delta} = \frac{12(r_1 + r_2 + r_1r_2)}{4 - r_1r_2} \frac{EI_b}{L_b^3} \quad (5.3)$$

where V_1 is the shear force at end 1, and Δ is the relative translational movement between end 1 to end 2.

5.2.2 Column Internal Axial Force at Elevated Temperature

For the analytical model shown in Figure 5.2, P is the external applied load and P_c is the internal axial force of the column. R_u and R_l are the beam-to-column rotational

stiffnesses at the upper and lower end of the column, respectively. E_T is the elastic Young's Modulus at elevated temperature T . ΔL denotes the shortening of spring k due to the thermal expansion of the column. The strain and stress are defined as positive when the column expands and the column is in tension, respectively. The static equilibrium of the column can be stated as

$$P + k\Delta L = -\sigma A = P_c \quad (5.4)$$

in which A is the cross-sectional area of the column, and σ is the column internal normal stress. From deformation compatibility, the spring shortening ΔL in Eq. (5.4) is equal to the column expansion which can be expressed in terms of column strain as

$$\Delta L = \varepsilon L = (\varepsilon_T + \varepsilon_e)L \quad (5.5)$$

where ε is the total normal strain of the column, and, ε_e and ε_T are the mechanical elastic strain and the thermal strain of the column, respectively. Considering that the axial forces induced by different axial shortenings among columns in the same storey of the frame are ignored in the derivation of the lateral stiffness of columns at ambient temperature, the mechanical elastic strain associated with ambient temperature was not taken into account in this study as well. Therefore, these two strains can be computed as

$$\varepsilon_e = \frac{\sigma}{E_T} - \frac{\sigma}{E_{20}} \quad (5.6)$$

$$\varepsilon_T = \int \alpha(T) dT \quad (5.7)$$

where E_{20} is the elastic Young's Modulus at ambient temperature, and $\alpha(T)$ is the coefficient of thermal expansion and can be stated as (Kodur and Harmathy, 2002)

$$\alpha(T) = (0.004T + 12) \times 10^{-6} (^\circ\text{C}^{-1}); \quad (5.8)$$

Substituting Eqs. (5.5~5.8) into Eq. (5.4), the internal axial force P_c of the axially

loaded column at elevated temperature shown in Figure 5.2 can be obtained

$$P_c = \frac{P + kL \int \alpha(T) dT}{1 + \frac{kL(E_{20} - E_T)}{E_{20}E_TA}} \quad (5.9)$$

The stress-strain relationship at elevated temperature used in this study is adopted from ASTM STP 464 (Harmathy and Stanzak, 1970) and the variation of Young's Modulus of steel with respect to elevated temperature is defined as follow

for $0^\circ\text{C} < T \leq 600^\circ\text{C}$

$$E_T = \left[1.0 + \frac{T}{2000 \ln \left(\frac{T}{1100} \right)} \right] E_{20} \quad (5.10a)$$

for $600^\circ\text{C} < T < 1100^\circ\text{C}$

$$E_T = \frac{690 - 0.69T}{T - 53.5} E_{20} \quad (5.10b)$$

5.2.3 Lateral Stiffness of Columns at Elevated Temperature

Based on the derivation of lateral stiffness of axially loaded steel columns at ambient temperature in Chapter 3, the lateral stiffness of the column with consideration of thermal restraint at elevated temperature can be similarly expressed as:

$$S_T = \beta_T(\phi_T, r_l, r_u) \frac{12E_TI}{L^3} \quad (5.11)$$

where r_l and r_u are the end-fixity factors for the lower and upper ends of the column at elevated temperature, respectively; E_T is Young's modulus at temperature T and is defined in Eqs. (5.10). I and L are the moment of inertia and the length of column, respectively. The applied load ratio ϕ_T is defined as

$$\phi_T = \sqrt{\frac{P_c L^2}{E_T I}} = \pi \sqrt{\frac{P_c}{P_{eT}}} \quad (5.12)$$

in which P_c is the column internal axial force defined in Eq. (5.9) and P_{eT} is the Euler buckling strength of the column with pinned ends at temperature T . In a corresponding manner, the modification factor of the column lateral stiffness β_T can be expressed as

$$\beta_T = \frac{\phi_T^3}{12} \frac{a_1 \phi_T \cos \phi_T + a_2 \sin \phi_T}{18r_l r_u - a_3 \cos \phi_T + (a_1 - a_2) \phi_T \sin \phi_T} \quad (5.13)$$

where a_1 , a_2 and a_3 are coefficients in terms of end-fixity factor r_l and r_u , defined in Eqs. (3.5). It can be found that the expression of Eq. (5.13) has no difference with Eq. (3.4) but the applied load ratio ϕ_T . Once the specified temperature is given, the lateral stiffness of the column at that temperature can then be computed.

5.3 Evaluation Procedure of Frame Buckling Strength at Elevated Temperature

According to the concept of storey-based buckling, a storey in an unbraced steel frame at elevated temperature will buckle in a sway mode when the lateral stiffness of the storey diminishes. Thus, the buckling strength of the frame at a specific temperature can be determinate by proportionally increasing the loadings to the level that the lateral stiffness of the storey vanishes. Since the transcendental relationship between β_T and ϕ_T expressed in Eq. (5.13) is complicated and inconvenient to solve the buckling strength of the frame, a trial-and-error procedure of evaluating the buckling strength of steel frames subject to proportional loadings and elevated temperature is proposed as following:

1. Compute the internal axial force P_c for columns at a specified temperature T from Eq. (5.9).
2. Calculate the end-fixity factors r_l and r_u from Eq. (3.1) for each column in the frame, in which the Young's modulus at ambient temperature E in Eq. (3.1) will be replaced by the Young's modulus at the elevated temperature E_T defined in Eqs. (5.10).

3. Determine the storey stiffness $\sum S_i$ by summing up the lateral stiffnesses S_i of each individual column, which can be evaluated based on Eq. (5.11).
4. If the storey stiffness is equal to zero or within an acceptable tolerance, the buckling strength of the frame at temperature T is obtained. Otherwise, proportionally increase the applied loads on the frame, and then repeat step(1).

5.4 Illustrated Column Example

Valente and Neves (1999) investigated the influence of axial and rotational restrains to the critical temperature of fully braced steel columns subjected to fire by using a numerical approach. A set of columns made of the same profile HEB260 (equivalent to W250X89) but with different slenderness ratios, axial and rotational restraints were studied. As analyzed in Chapter 3, the column with fully lateral bracing will reach its rotational buckling strength when the value of the lateral stiffness modification factor approaches negative infinity. In other words, for the column with fully lateral bracing in fire, the buckling strength is such that the value of the denominator of β_T in Eq. (5.13) becomes zero. Thus, numerical examples of the fully braced column in fire illustrated by Valente and Neves (1999) can be used to verify the proposed approach of evaluating the column buckling strength at elevated temperature.

Considering the proposed approach only focused on the elastic buckling strength of steel columns in fire, the column with the slenderness ratio $\lambda = 120$ and no eccentricity subject to the axial restraint $k = 5$ kN/cm and different rotational stiffnesses R are selected to compare the proposed analytical approach with the numerical investigation conducted by Valente and Neves (1999). A constant load $P = 0.7N_{RD}$ was applied on the column, where N_{RD} is the design value of the buckling strength of a pinned end column at ambient temperature calculated in accordance with Eurocode 3 (CEN, 2005). In addition, the axial and rotational restraining stiffnesses of the column, namely, k and

R as shown in Table 5.1, are assumed to be unaffected by the elevated temperature in this example (Valente and Neves, 1999).

Table 5.1: Comparison of critical temperatures

$k(kN/cm)$	$R(kNm)$	Numerical (Valente and Neves, 1999)	Proposed	Difference
		$T_{cr}^N(^{\circ}C)$	$T_{cr}^A(^{\circ}C)$	$(T_{cr}^N - T_{cr}^A)/T_{cr}^N$
5	0	586	502	-14.3%
	927.39	631	618	-2.1%
	9273.9	667	682	+2.2%
	92739	670	686	+2.4%
	∞	671	687	+2.4%

The column critical temperatures T_{cr} at which the axially loaded column buckles as evaluated by the numerical approach by Valente and Neves (1999) and the proposed approach are listed in Table 5.1. The comparison between the two approaches shows that the proposed approach has good agreement with the numerical values, and the results validate the proposed approach in evaluating the lateral stiffness and buckling strength of steel columns subjected to axial and rotational restraints at elevated temperature.

5.5 Illustrated One-bay Frame Examples

Consider a one-storey and one-bay frame as shown in Figure 5.4 which has pinned column base and rigid beam-to-column connections. The column height H is $4877mm$ and the beam span length L is $7315mm$. Young's modulus of the steel at ambient temperature is $2 \times 10^5 MPa$. The moments of inertia for the columns and beam are $I_1 = I_2 = 34.1 \times 10^6 mm^4$ and $I_3 = 245 \times 10^6 mm^4$, respectively. The cross-sectional areas for the

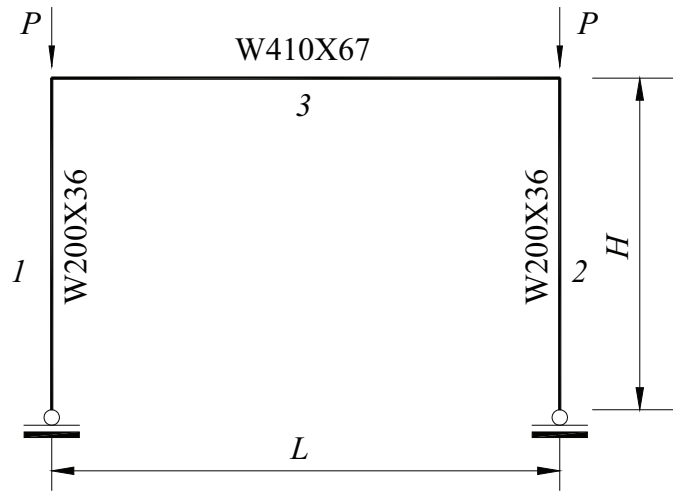


Figure 5.4: Example: One bay and one storey frame

columns are $A_1 = A_2 = 4568mm^2$. The axial loads are applied on the top of the columns and are to be increased proportionally.

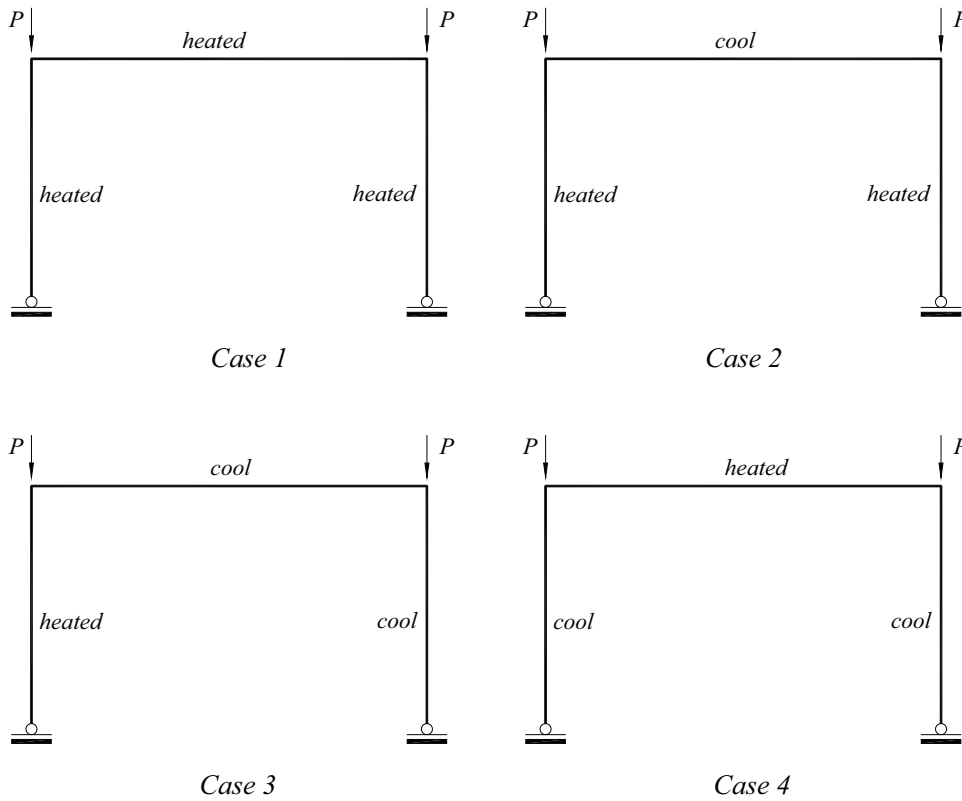


Figure 5.5: Different cases of frame members subjected to elevated temperature

To investigate the behavior and the buckling strength of the frame in fire, four cases in which different members in the frame experiencing elevated temperature are studied. For all four cases as shown in Figure 5.5, the temperature distribution within the heated members was assumed to be uniform and the temperature identical for the heated members.

5.5.1 Case 1

In case 1, the beam and both columns of the frame are experiencing the elevated temperature. Since the properties of both columns are identical and the columns are heated at the same temperature, the axial deformation associated with the elevated temperature in the columns should be identical. Thus, the corresponding axial thermal restraints are relatively small and can be neglected. Consequently, the internal forces P_c in Column 1 and 2 are equal to the external applied load P . Moreover, since there is no differential temperature between the columns and beam, the beam-to-column stiffness ratio remains constant as the increase of the temperature, and the end-fixity factors for each column stay the same during the fire. The values of r_l and r_u can be evaluated at the ambient temperature and they are 0 and 0.9057, respectively. From the expression of lateral stiffness in Eq. (5.11), it can be concluded that when all the members in the frame are experiencing the same elevated temperature, the storey buckling is only affected by the degradation of Young's Modulus of steel at elevated temperature.

Based on the procedure proposed in Section 5.3, the relationship between frame buckling strength and elevated temperature can be obtained and plotted in Figure 5.6. It can be seen that the frame buckling strength decrease as the temperature increase but not in a linear manner. Furthermore, since Column 1 and 2 are identical and experiencing the same elevated temperature, the lateral stiffness for these two columns decrease simultaneously at the same rate, and no stiffness interaction exists between columns occurs in case 1. Therefore, the buckling strengths of column 1 (P_{1cr}) and 2 (P_{2cr}) are half of that

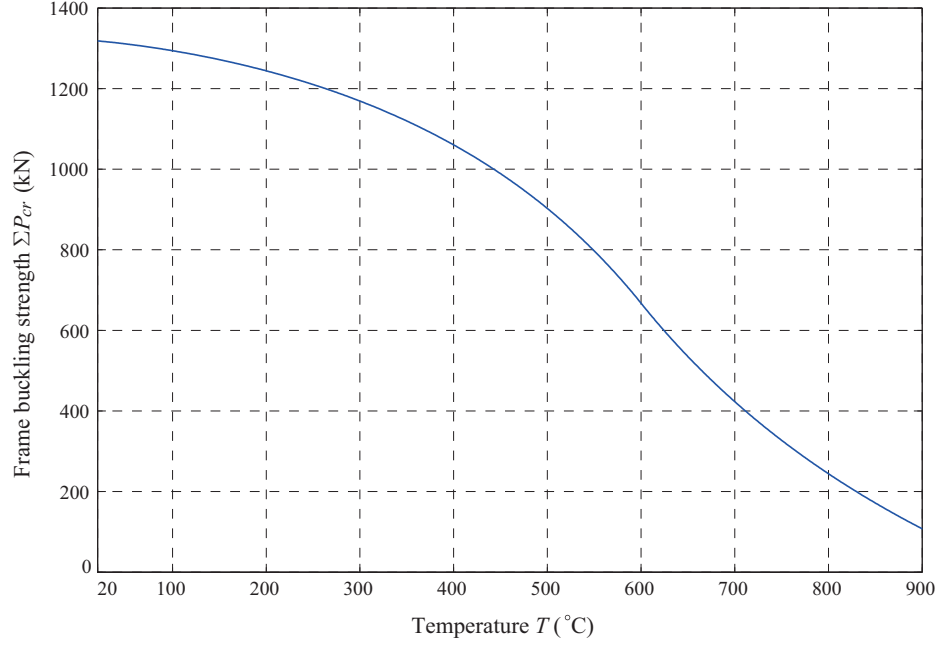


Figure 5.6: Case 1: Variation of frame buckling strength subject to elevated temperature

of the frame $(\sum_{i=1}^2 P_{icr})$.

5.5.2 Case 2

As shown in Figure 5.5, both columns of the frame are experiencing the elevated temperature, but not the beam. Similar to that of case 1, as both columns are identical, the column axial thermal restraint is small as explained in case 1 and can be ignored in this case. Since the beam is unheated, the Young's modulus of the beam will remain as that of the ambient temperature. Therefore, unlike case 1, the beam-to-column rotational stiffness ratio will increase when only columns are experiencing the elevated temperature due to the degradation of the Young's modulus of columns. Shown in Figure 5.7 is the relationship between the end-fixity factor r_u for the upper end of the column and temperature T for case 2. It can be seen that the column end-fixity factor r_u increases as the temperature increases. It should be pointed out that increasing the magnitude of col-

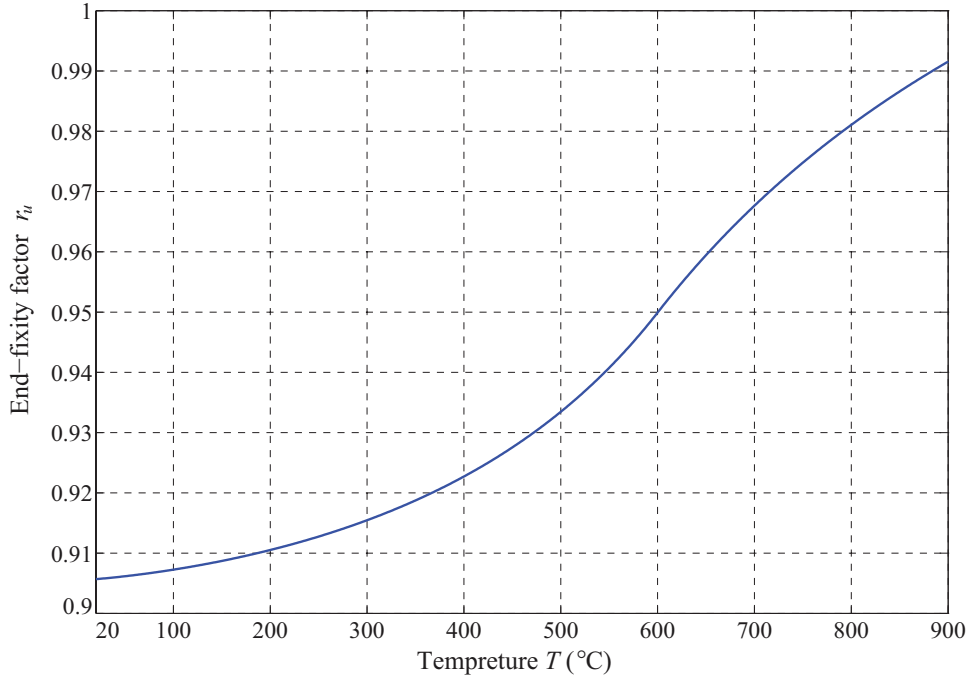


Figure 5.7: Case 2: Variation of column end-fixity factor r_u vs. temperature

umn end fixity signifies the increase of rotational restraint at the end of a column, which normally leads to the increasing buckling strength of the column. However, in this case, the increase of the rotational restraint at the upper end of the column is not contributed by the increase of the stiffness of the beam. Instead, it is due to the degradation of the Young's modulus of the column. Consequently, the strength of the frame decreases as the temperature increases.

Since the thermal axial restraint does not play a role in this case, based on Eq. (5.9), the internal force P_c of Column 1 and 2 is equal to the external force P . In addition, similar to that of case 1, the buckling strengths for both columns are identical and equal to one half of the frame buckling strength. The relationship between frame buckling strength and elevated temperature for case 2 is plotted in Figure 5.8. For the reason of comparison, the frame buckling strength in case 1 is also plotted in Figure 5.8. It can be observed from Figure 5.8 that the buckling strength in case 2 coincides with that in case 1 at the very beginning. Thereafter, as the temperature increases, the buckling

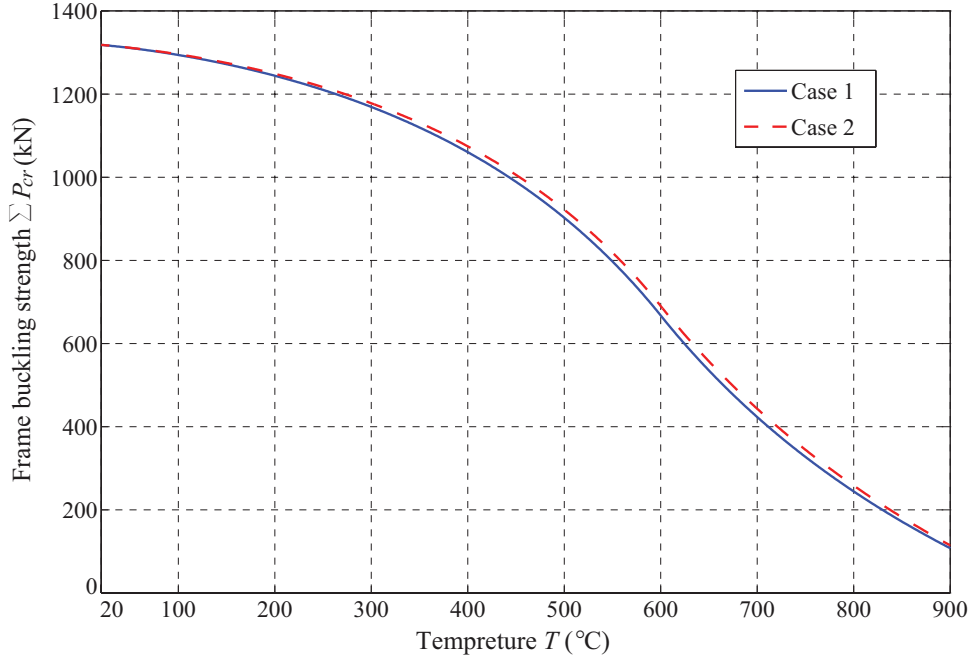


Figure 5.8: Case 2: Variation of frame buckling strength subject to elevated temperature

strength in case 2 becomes slightly greater than that in case 1, which is contributed by the unheated beam providing the stronger rotational restraint to the upper ends of the columns. When the temperature in columns is greater than 800°C, the difference of buckling strength between two cases diminishes because the degradation of materials dominates the deterioration of the frame buckling strength. The comparison between two cases suggests that the increase of rotational restraints is beneficial for columns to maintain the stability in fire, which agrees with the conclusion made by [Wang \(1997\)](#). However, it should be pointed out that the difference of buckling strength between case 1 and case 2 is relatively small, the greatest difference in absolute value between two cases is found as 22.8kN (3.65%) when $T=612^{\circ}\text{C}$, and the greatest percentage difference is found as 6.57% when $T = 900^{\circ}\text{C}$.

5.5.3 Case 3

In case 3, only Column 1 is experiencing the elevated temperature and the other members of the frame are unheated. As the temperature increases, Column 1 elongates but the length of Column 2 remains the same, therefore, the axial thermal restraint cannot be omitted in this case. Also, similar to that of case 2, the variation of rotational thermal restraint induced by the unheated beam on the upper end of Column 1 should be taken into account in the evaluation of the buckling strength of the column. In cases 1 and 2, there is no axial thermal restraints imposed on either of the columns because there is no differential elongation between the two columns at elevated temperature. However, the differential elongation does exist in case 3 because only Column 1 is experiencing the elevated temperature. To investigate the effect of thermal axial restraint on column buckling strength, a sub-assembly model of Column 1 in case 3 shown in Figure 5.9 is employed to eliminate the possible lateral stiffness interaction between two columns.

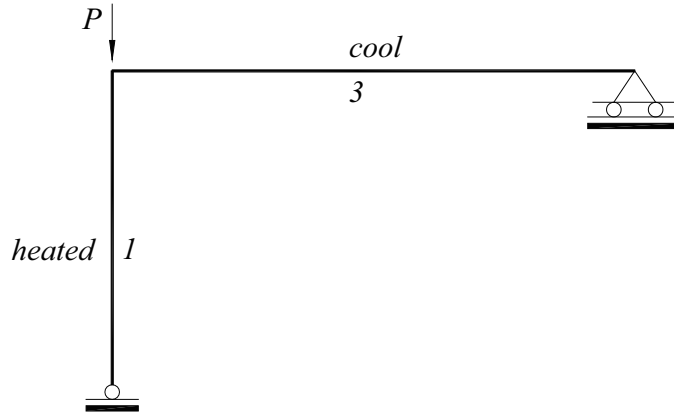


Figure 5.9: Case 3: A sub-assembly model of Column 1

The buckling strength for Column 1 of the sub-assembly and that of the previous two cases are plotted in Figure 5.10. It can be seen from Figure 5.10 that the buckling strength of Column 1 of the sub-assembly is considerably less than that of case 1 and 2. As previously discussed, the difference of buckling strength between case 1 and 2 is contributed by the thermal rotational restraint induced by the unheated beam in case

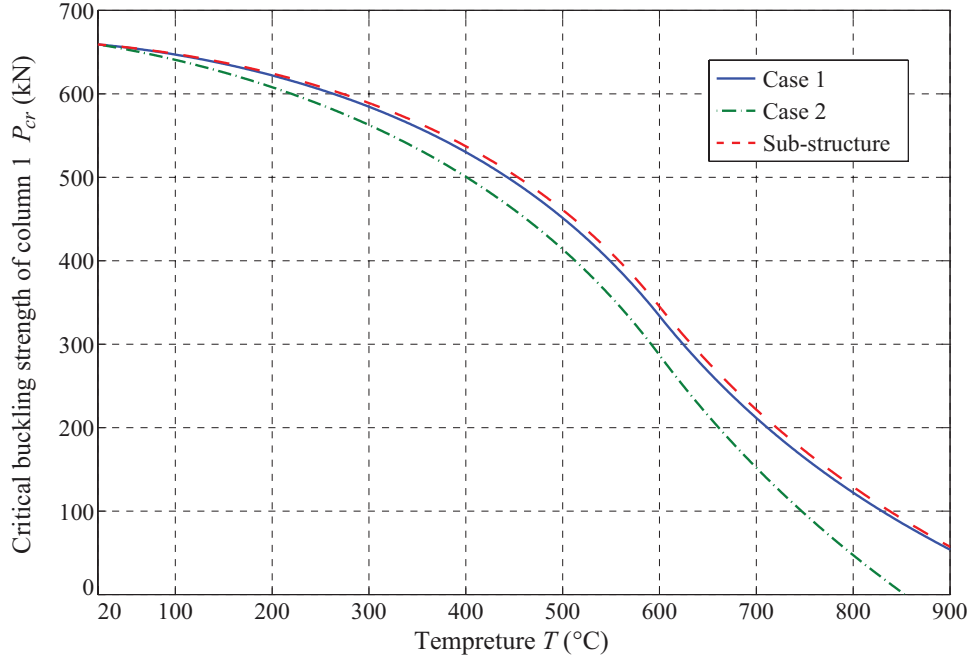


Figure 5.10: Case 3: Comparison of bucking strength of Column 1

2. The reason of the column buckling strength of the sub-assembly is considerably less than that in case 1 and 2 is primarily due to the thermal axial restraint imposed by the unheated beam. Since no axial thermal restraint is considered in case 1 and 2, the internal forces of Column 1 in case 1 and 2 are equal to the external applied load, but for the sub-assembly, an additional compressive force will be induced in the column by the restraint imposed on thermal expansion of column. As a result, the internal force of Column 1 of the sub-assembly is greater than the external applied load. To demonstrate the effect of the thermal axial restraint on column internal force, considering a case that an external load $P=100$ kN is applied on Column 1 of the sub-assembly, the relationship between the internal force of Column 1 and elevated temperature can be obtained from Eq. (5.9) and is plotted in Figure 5.11. It can be seen from the figure that despite the external load remains the same, the internal force of the column increases almost linearly as the temperature increases, which consequently reduces the buckling strength of the column. Therefore, it can be concluded that the thermal axial restraint

is detrimental to the stability of steel columns at elevated temperature.

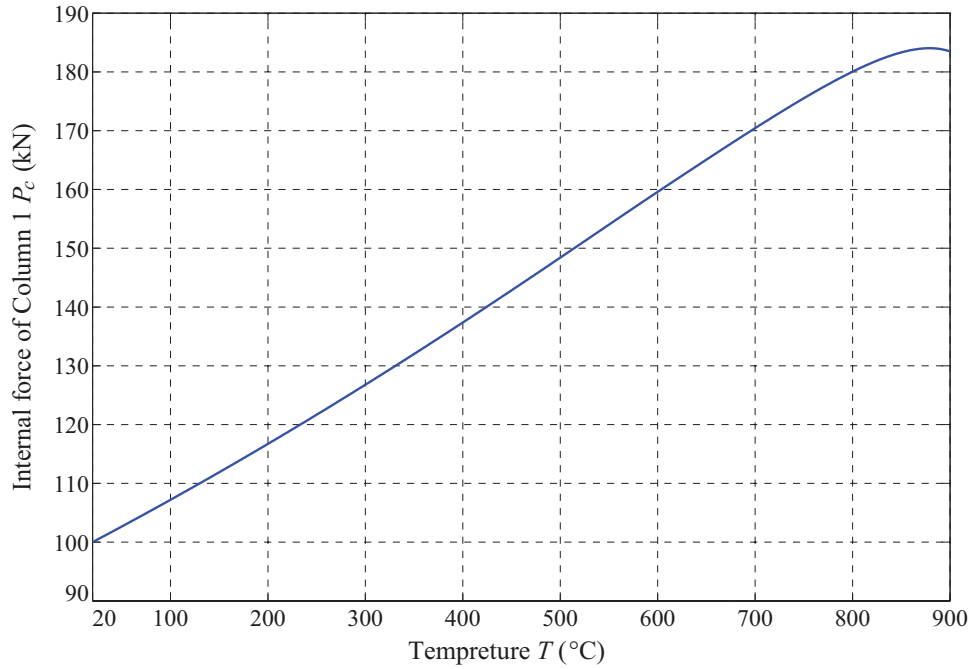


Figure 5.11: Case 3: Effect of axial thermal restraint on column internal force at elevated temperature

However, when it comes to the frame buckling strength, the frame in case 3 shows greater buckling capacity than that of case 1 and 2. The frame buckling strengths associated with elevated temperature for the three cases are plotted in Figure 5.12 and it can be seen from the figure that the frame buckling strength of case 3 is considerably greater than that of case 1 and 2. The increasing buckling strength in case 3 comes from the lateral stiffness interaction between the two columns. In case 3, as the temperature elevates, stiffness degradation occurs in Column 1 as the decrease of Young's modulus and the increase of internal force induced by thermal axial restraint. Column 1 becomes a weak column which relies upon Column 2 to provide the required lateral support in order to sustain the applied load. Column 2, on the other hand, is not exposed to the elevated temperature; therefore, it is not experiencing the material deterioration and is capable of sustaining external load and as well as providing the lateral support to Column

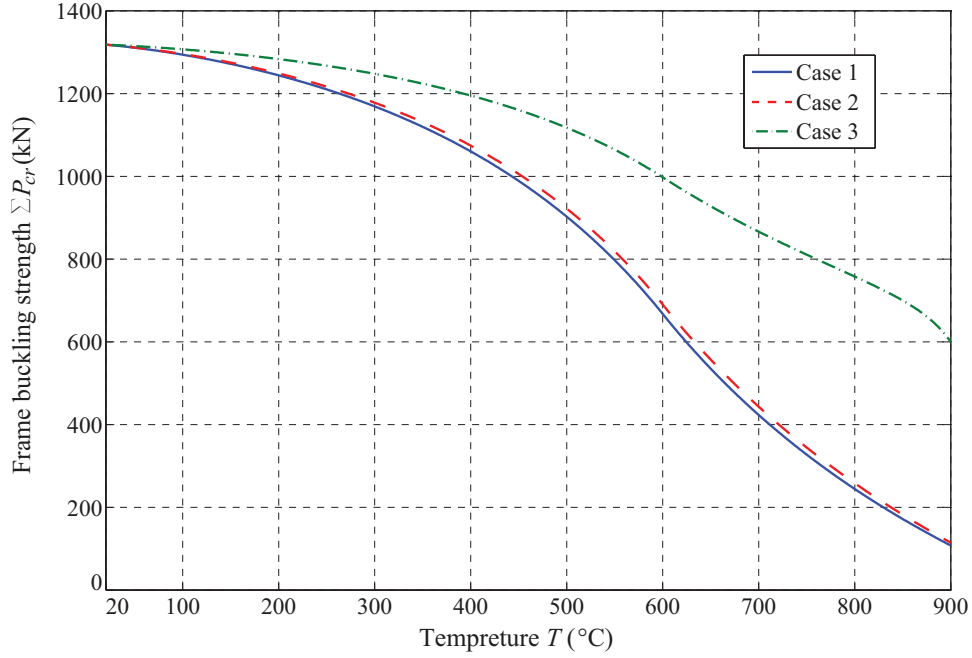


Figure 5.12: Frame buckling strength in case 1, 2 and 3

1 to maintain the stability of the frame, consequently, resulting in a considerably large capacity of the frame buckling strength. The stiffness interaction can be observed from Figure 5.13, which illustrates that the relationship between the lateral stiffness of Column 1 and 2 in case 3 must be satisfied at the time when the frame lateral instability occurs at different elevated temperatures. It is observed that the lateral stiffness of Column 1 illustrated by the solid line is always negative, which signifies that the demand for lateral stiffness to be provided by Column 2 to sustain the applied axial load. On the other hand, the dashed line associated with Column 2 represents the value of the stiffness that Column 2 must provide to sustain its applied load as well as to support Column 1 to maintain the stability of the frame. In the case that Column 2 failed to provide such stiffness, the instability of the frame will occur. Thus, if only the isolated column is considered in assessing the frame stability subjected to elevated temperature without accounting for the lateral stiffness interaction among columns in the same storey, the stability capacity of the frame would be significantly underestimated.

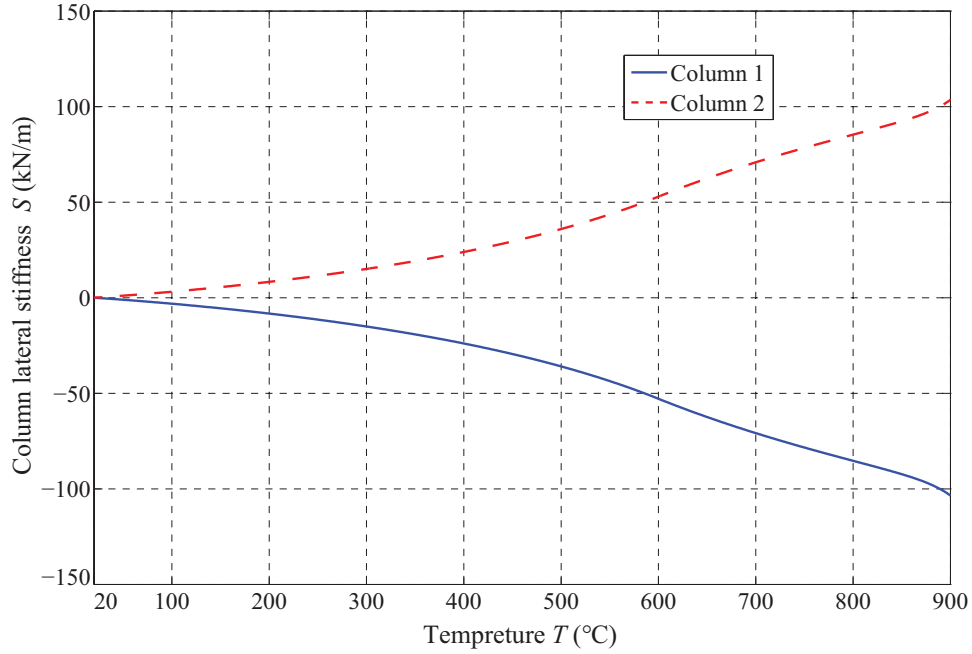


Figure 5.13: Case 3: Lateral stiffness demand for Column 1

5.5.4 Case 4

In reality, columns are often fire protected but beams may be experiencing the elevated temperature. Case 4 presented herein investigates such fire scenario. As only the beam is exposed to fire but not the columns in this case, the beam-to-column stiffness ratio will decrease as the temperature increases, which subsequently results in the decrease of the end-fixity factors of the columns.

Illustrated in Figure 5.14 is the relationship between the column end-fixity factor r_u and the elevated temperature T . Since both Column 1 and 2 remain at ambient temperature, thermal axial restraint is not occurred in the columns; thus, no additional compressive forces are induced in the columns. The frame buckling strength in case 4 is only affected by the decrease of rotational restraints at upper end of the columns.

Shown in Figure 5.15 is the relationship between frame buckling strength and elevated temperature for case 4. For the reason of comparison, the frame buckling strength of case 2, in which both columns are exposed to the elevated temperature but not the

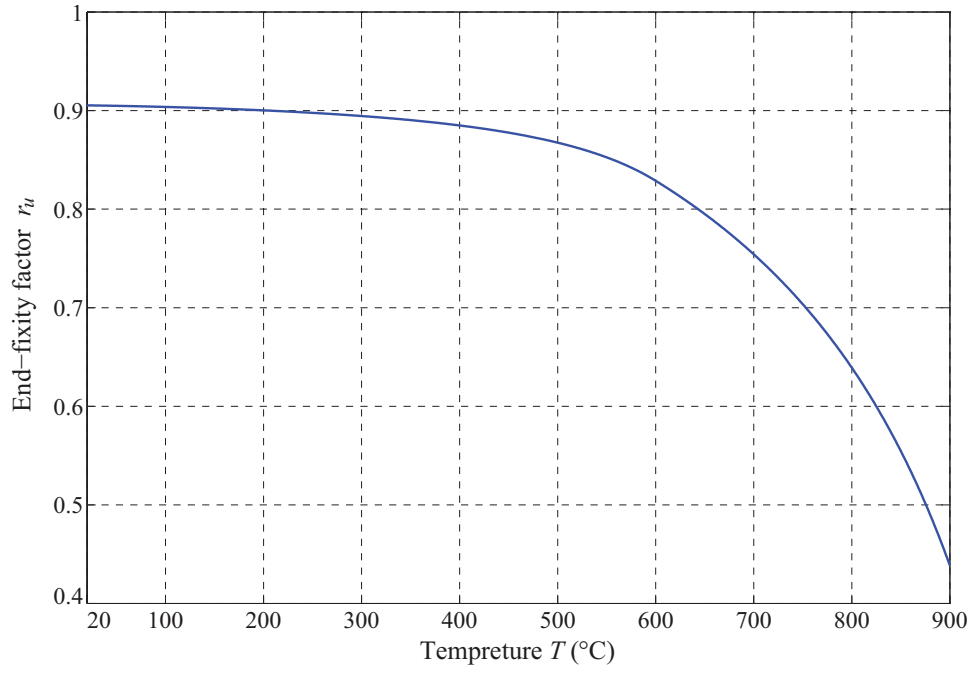


Figure 5.14: Case 4: End-fixity factor r_u of upper ends of columns

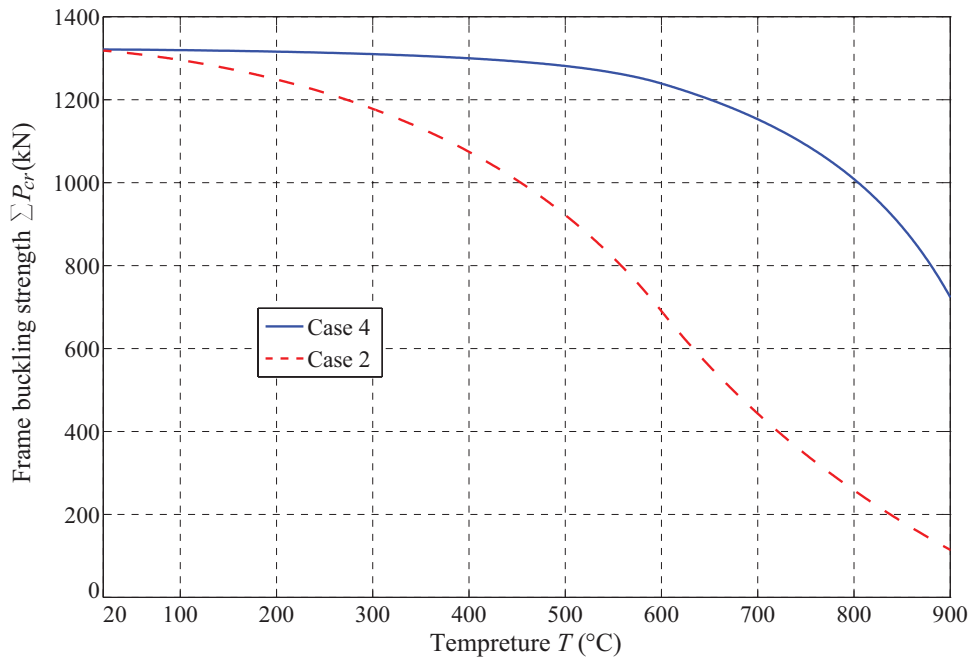


Figure 5.15: Case 4: Comparison of frame buckling strength between case 2 and 4

beam, is also presented in the Figure 5.15. As illustrated by the solid line, due to the decrease of rotational restraint at the upper ends of the columns in case 4, the frame buckling strength decreases as the temperature increases as shown in Figure 5.15. However, the frame buckling strength of case 4 is significantly greater than that in case 2. Particularly, when the temperature reaches higher than 600°C, the frame buckling strength in case 2 decreases substantially, while the frame in case 4 can still sustain considerable amount of applied loads. The comparison between case 2 and case 4 indicates that protecting columns from fire is more crucial and efficient to maintain the frame stability than protecting beams.

5.6 Illustrated Two-bay Frame Example

In this section, a one-storey and two-bay unbraced steel frame as shown in Figure 5.16 will be investigated with the proposed method. The dimensions and cross sections of columns and beams are the same as those of one-bay frames in the aforementioned examples. The axial loads applied on the top of columns are assumed to be identical and increased proportionally. Consider compartment fire only occurs in the left compartment, and thus only Column 1, 2 and Beam 4 are experiencing elevated temperature. As Beam

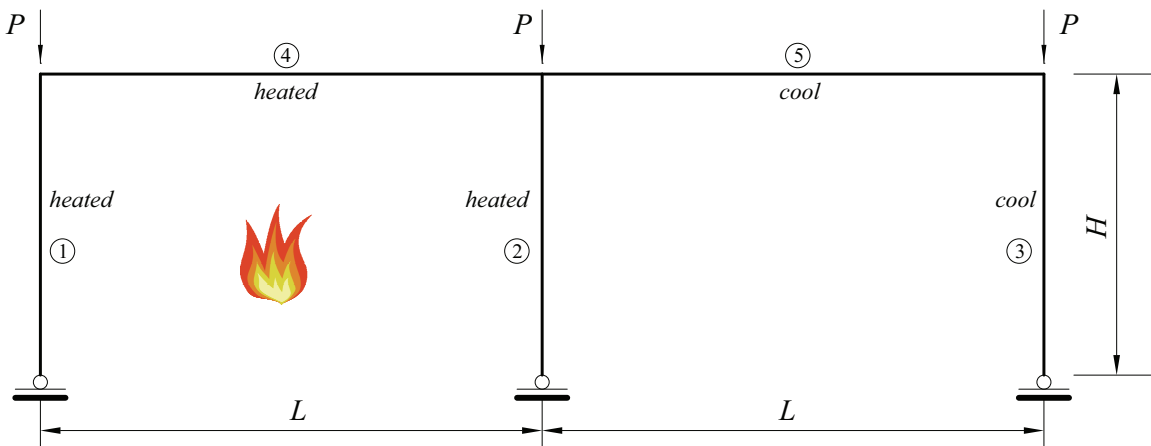


Figure 5.16: Example: Two-bay and one storey frame

5 is not experiencing elevated temperature, it can be conceived that the internal axial force in Column 2 will be greater than Column 1 as the temperature of compartment fire increases because of the restraint provided by Beam 5. To simplify the analytical process and consider that the thermal expansion rather than mechanical shortening caused by axial loads dominates the change of column length, the axial forces in Column 1 and 2 induced by unequal end displacements of Beam 4 are neglected. That is, no additional internal force will be generated in Column 1, and the additional load in Column 2 only comes from Beam 5. Following the same procedure described in Section 5.3, the buckling strength of the frame can be plotted in Figure 5.17. It can be seen that the buckling strength of the two-bay frame in a decreased trend similar to that of the one-bay frame in case 3.

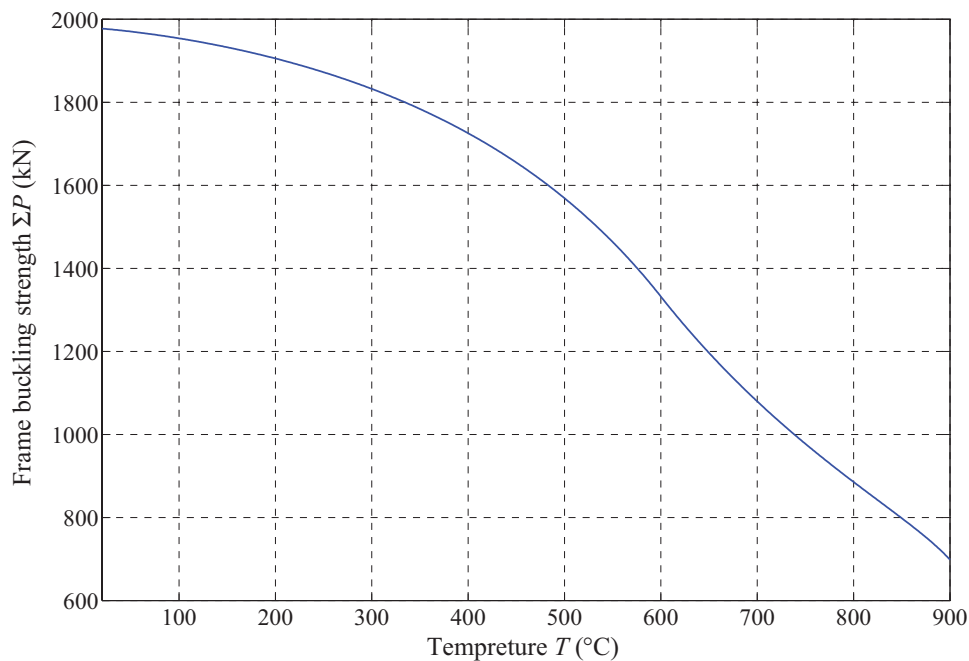


Figure 5.17: Two-bay frame: Variation of frame buckling strength subjected to elevated temperature

To compare with the one-bay frame in case 3 in the previous example, the external loads applied on Column 2 in the two-bay frame and on Column 1 in the one-bay frame

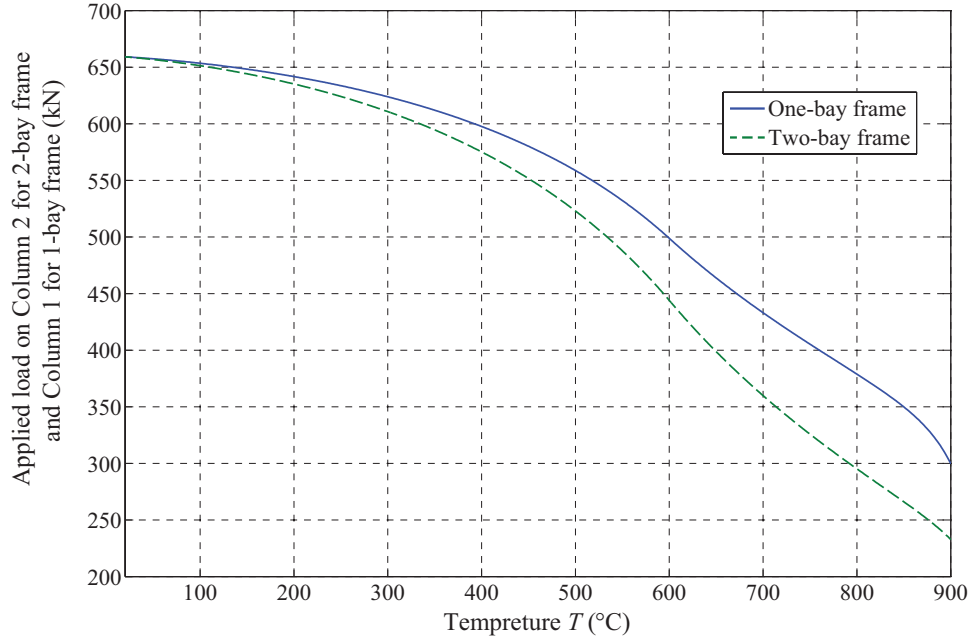


Figure 5.18: Two-bay frame: Comparison of column buckling strength between two-bay frame and case 3 of one-bay frame

of case 3 are plotted on Figure 5.18. In the two-bay frame example, expansion of Column 2 is restrained by unheated Beam 5 and rotation is restrained by heated Beam 4 and unheated Beam 5. In case 3 of the one-bay frame example, expansion and rotation of Column 1 are both restrained by unheated Beam 3. Thus, the rotational restraint on Column 2 in the two-bay frame example is greater than that on Column 1 in the one-bay frame example. From the perspective of single column, the buckling strength of Column 2 in the two-bay frame example should be greater than that of Column 1 in the one-bay frame example. However, it can be observed from Figure 5.18 that the applied load P on Column 2 in the two-bay frame is actually less than that on Column 1 in the one-bay frame when both frames buckle and fail to sustain external loads. It is because that heated columns lose their lateral stiffness as the temperature of the compartment fire increases and have to rely on the unheated columns to provide lateral support to maintain the stability. In the one-bay frame example, heated Column 1 is laterally supported by unheated Column 2. However, in the two-bay frame example, unheated

Column 3 not only needs to support Column 2 but also provide lateral support to Column 1. Consequently, in terms of lateral stiffness, Column 2 in the two-bay frame receives less lateral support than Column 1 in the one-bay frame, and the applied load that Column 2 in the two-bay frame can sustain is less than that of Column 1 in the one-bay frame. The comparison between two examples further proves the necessity of evaluation of frame buckling strength at elevated temperature based on storey-buckling concept rather than sole column buckling.

5.7 Verification with Numerical Analysis

5.7.1 Introduction

The finite element method (FEM) has been proven as an efficient and powerful approach to calculate the elastic buckling strength of steel frames. The previous examples of single-bay frames and two-bay frames will be investigated by the finite element based software ABAQUS to verify the results obtained from the analytical method proposed in this chapter.

The buckling problems can be seen as the problems of identifying the transition from a stable to an unstable structure without changing the applied load, which can be determined from an incremental analysis. That is, the buckling problem can be expressed by the following equation:

$$[\mathbf{K}_E + \lambda \mathbf{K}_g] \Delta = \mathbf{0} \quad (5.14)$$

in which $\mathbf{K}_E$ is the initial stiffness matrix, $\mathbf{K}_g$ is the geometric stiffness matrix. Apparently, the buckling problem expressed in Eq. (5.14) is an eigenvalue problem. The eigenvalue represents the ratio of the elastic critical load to the reference load, and the eigenvector $\{\Delta\}$ defines the buckling mode shape. ABAQUS provides the Lanczos and subspace iteration methods to solve the eigenvalue. In this study, Lanczos method is selected to solve the buckling strength.

5.7.2 FEM analysis

Since the study only focused on planar unbraced steel frames, three-node, quadratic beam elements (B23) were used to model the columns and beams. Columns and beams in the single-bay frames and the two-bay frames are all divided into 8 elements with considering the balance between accuracy and efficiency. Elevated temperatures are set up in predefined temperature field. For all the frames analyzed in previous sections, an axial load $P=500\text{kN}$ is applied on the top of each column. The corresponding failure temperatures T_{cr} of all the frames are investigated by using numerical method and the proposed analytical method, respectively, and the results are presented in Table 5.2.

Table 5.2: Comparison of frame critical temperature

$P(\text{kN})$	Example	Numerical	Analytical	Difference
		(FEM) $T_{cr}^N(^{\circ}\text{C})$	(Proposed) $T_{cr}^A(^{\circ}\text{C})$	$(T_{cr}^A - T_{cr}^N)/T_{cr}^N$
500	Case 1	443	437	-1.35%
	Case 2	454	448	-1.32%
	Case 3	599	587	-2.00%
	Case 4	804	799	-0.62%
	Two bay	520	515	-0.96%

It can be seen from the table, the results obtained from the proposed method are in good agreement with that from the numerical analysis. The difference between the proposed method and numerical method for two-bay frame is only -0.96% , which suggests that the assumption of neglecting the axial load induced by unequal end displacement of Beam 4 is reasonable. Furthermore, it can be observed from the table that the maximum difference between proposed method and numerical method for one-bay frame is only -2% . It should be pointed out that the finite element results for failure temperature are

always smaller than the results obtained by the proposed method, which is because the proposed method neglects the expansion of the beam at elevated temperature. However, the insignificant difference between the finite element method and the proposed method shows that the results based on the assumption for beam at elevated temperature is reliable and can considerably simplify the derivation procedure of the proposed method. In general, the comparison suggests that the proposed method can be used to evaluate the buckling strength of unbraced steel frames in fire.

5.8 Conclusion

Discussed in this chapter is the lateral stability of unbraced steel frames subjected to uniform elevated temperature based on the concept of storey-based buckling. To simulate a steel column exposed to the elevated temperature, an analytical model shown in Figure 5.2 was proposed to obtain the column internal force and lateral stiffness with consideration to the effects of thermal axial and rotational restraints. Having the equations of evaluating the column internal force and lateral stiffness with accounting for uniform elevated temperature established on the proposed model, a procedure of computing the frame buckling strength at the elevated temperature and based on the concept of storey-based buckling was then presented. The behavior and buckling strength of an unbraced one-bay steel frame with four scenarios of fire and a two-bay frame subjected to compartment fire were investigated. For the one-bay frame, the investigations showed that the buckling strength of a steel column at the elevated temperature is slightly enhanced by thermal rotational restraint but considerably reduced by thermal axial restraint. In addition, the frame buckling strength will be significantly increased if the columns were protected from fire. The results obtained from case 3 further indicate that the evaluation of steel frame stability at elevated temperature solely based on the behavior of individual column may considerably underestimate the frame buckling strength since unheated columns in the same storey may be able to provide lateral support for columns experi-

enced the elevated temperature. The investigation of the two-bay frame example also confirms the observation from case 3 of one-bay frame. Finally, finite element method was employed to verify the proposed method by assessing the critical temperature of one-bay frame and two-bay frame example. The results obtained from the proposed method are in good agreement with that from numerical method, which suggests that the proposed method is reliable. Thus, the proposed method accounts for lateral stiffness interaction among columns is appropriate for assessing the frame buckling strength in fire.

Chapter 6

Storey-based Evaluation of Stability of Unbraced Steel Frames Subjected to Non-uniform Elevated Temperature Distribution

6.1 Introduction

The stability of steel columns and frames subjected to elevated temperature has been extensively studied since the 1970s ([Wang, 1997](#); [Huang and Tan, 2006](#); [Shepherd and Burgess, 2011](#)). Among these studies, investigations based on the assumption that steel frames or members are subjected to uniform elevated temperature have prevailed. Presented in Chapter [5](#) is an analysis of unbraced steel frames exposed to fire, based on the storey-buckling concept. The analysis was based on the assumption that steel frames experience a uniform elevated temperature. However, steel columns may experience a non-uniform temperature distribution in the longitudinal directions during compartment fires in a building. In current research, the one-zone fire model is primarily used to

represent the assumption of a uniform temperature during compartment fires, which is only suitable for modelling post-flashover fires ([Kawagoe, 1958](#)). In a real building fire, however, hot air and smoke accumulate at the upper zone and beneath the ceiling, while the cooler air stays at the lower zone of the compartment. Therefore, a more accurate two-zone fire model was developed ([Thomas et al., 1963](#)). In the two-zone fire model, the compartment fire is separated into two fire zones: a hot upper zone and a cool lower zone. Each zone has a uniform temperature, and the temperature that deteriorates a whole structural column along the longitudinal direction in a compartment fire is thus non-uniform. Therefore, the conclusion for stability of unbraced steel frames drawn in the previous chapter may not be accurate when considering that the frame is subjected to a non-uniform temperature distribution during the pre-flashover stage of a compartment fire.

Few studies have been conducted on the behaviour of steel structures subjected to non-uniform elevated temperature. [Culver \(1972\)](#) investigated the buckling loads of wide-flange steel columns subjected to non-uniform temperature distribution along the length of the member, but the influence of end restraints was not considered. [Becker \(2002\)](#) studied the effects of longitudinally non-uniform temperature distribution on fire protected steel structures, but the study was accomplished by a numerical method and the longitudinal non-uniform temperature distribution was induced by the heat sinks at floor levels. [Tan and Yuan \(2008, 2009\)](#) developed an analytical method to study the elastic and inelastic buckling strength of a pin-ended steel column subjected to longitudinal non-uniform elevated temperature distribution. Nevertheless, the beam-to-column rotational restraints were not considered, and the studies were focused on the stability of a single column, neglecting the interactions between the columns in a frame.

Investigated in this chapter was the stability of unbraced steel frames subjected to a two-zone fire. The term “non-uniform elevated temperature distribution” in the following denotes two different temperatures along the longitudinal directions of columns

in a compartment fire. The lateral stiffness of steel columns experiencing non-uniform temperature is first derived, and the procedure to evaluate the frame buckling strength, based on the storey-based buckling concept is then summarized. Numerical examples are presented to demonstrate the difference between the frame buckling strength at uniform and non-uniform elevated temperature distributions. To simplify the analysis, the rotational stiffness of beam-to-column connection is assumed to remain constant at elevated temperature and, as well, the thermal expansion of beam at elevated temperature is neglected.

6.2 Storey-based Stability of Unbraced Steel Frames in Two-Zone Fire

In the two-zone fire model, structural columns experience non-uniform temperature distribution, which results in material deterioration. Therefore, the column model proposed in Chapter 5 and shown in Figure 5.2 is not appropriate for the calculation of the lateral stiffness of columns subjected to the two-zone fire model. Taking the frame shown in Figure 6.1 as an example, the temperature of the hotter upper zone is T_u , while the temperature of the cooler lower zone is T_l . Correspondingly, the height of the upper layer is L_u , and the height of the lower layer is L_l . Obviously, the expansion of Column

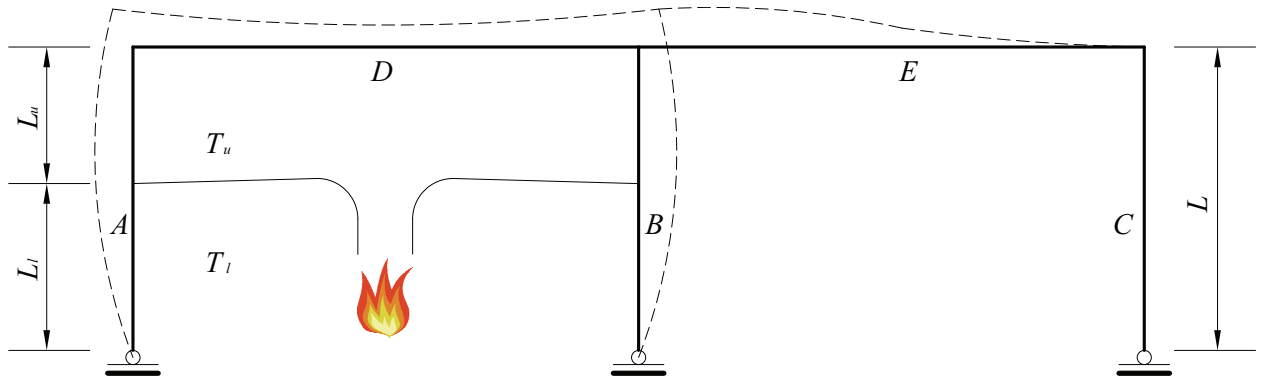


Figure 6.1: Steel frame subject to two-zone compartment fire

B will be constrained by the unheated Beam E. To better investigate the behaviour of such columns subjected to non-uniform fire and restricted by their adjoining members, an analytical model, shown in Figure 6.2, is proposed, where R_u and R_l are the stiffness of the rotational restraints at the upper end and lower end of the column, respectively, and k is the stiffness associated with the axial restraint related to the unheated adjoining beams and is defined in Eq. (5.3).

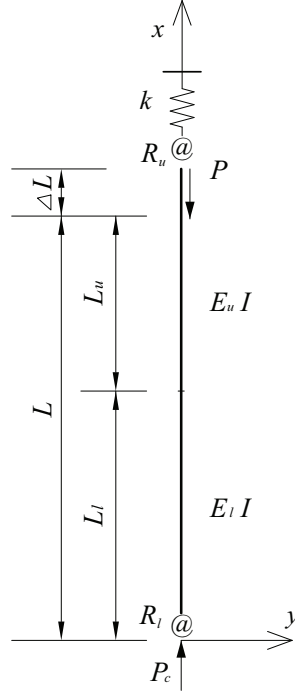


Figure 6.2: Column with thermal restraints subjected to non-uniform elevated temperature distribution

6.2.1 Column Internal Axial Force at Non-uniform Elevated Temperature Distribution

Compared with the column model in Chapter 5, the column internal axial force of the column subjected to non-uniform elevated temperature distribution in Figure 6.2 must take into account the additional force induced by thermal expansion of different column

segments. For the analytical model in this figure, ΔL denotes the shortening of spring k due to the thermal expansion of both segments of the column. E_u and E_l , the elastic Young's modulus of the upper segment at temperature T_u and the lower segment at temperature T_l , respectively, can be defined as

$$E_u = \lambda_u E_{20} \quad (6.1a)$$

$$E_l = \lambda_l E_{20} \quad (6.1b)$$

where λ_u and λ_l are the material degradation factors associated with temperature T_u and T_l , respectively, and E_{20} is the Young's modulus at ambient temperature. According to the stress-strain relationship of Eqs. (5.10) adopted in this study, the material degradation factor λ can be stated as

for $0^\circ\text{C} < T \leq 600^\circ\text{C}$

$$\lambda_T = \frac{E_T}{E_{20}} = \left[1.0 + \frac{T}{2000 \ln \left(\frac{T}{1100} \right)} \right] \quad (6.2a)$$

for $600^\circ\text{C} < T < 1100^\circ\text{C}$

$$\lambda_T = \frac{E_T}{E_{20}} = \frac{690 - 0.69T}{T - 53.5} \quad (6.2b)$$

where T is the elevated temperature. For the upper and lower segments, T is denoted as T_u and T_l , respectively. Other notations are defined the same as those defined in Chapter 5. The strain and stress are defined as positive when the column expands and the column is in tension. The static equilibrium of the column can be stated as

$$P + k\Delta L = -\sigma A = P_c \quad (6.3)$$

The spring shortening Δ_L in Eq. (6.3) is equal to the whole column's elongation, which can be expressed in terms of the axial strain of the column as

$$\Delta L = \varepsilon L = \varepsilon(L_u + L_l) = (\varepsilon_{Tu} + \varepsilon_{eu})L_u + (\varepsilon_{Tl} + \varepsilon_{el})L_l \quad (6.4)$$

where ε is the total normal strain of the column. ε_{eu} and ε_{el} are the mechanical elastic strains for the upper and lower segments, while ε_{Tu} and ε_{Tl} are the thermal strain of the column for its upper and lower segments, respectively. As defined in Eqs. (5.6~5.7) Chapter 5, these two strain values can be computed as

$$\varepsilon_{eu} = \frac{\sigma}{E_u} - \frac{\sigma}{E_{20}} \quad (6.5a)$$

$$\varepsilon_{el} = \frac{\sigma}{E_l} - \frac{\sigma}{E_{20}} \quad (6.5b)$$

$$\varepsilon_{Tu} = \int_{20}^{T_u} \alpha(T) dT \quad (6.6a)$$

$$\varepsilon_{Tl} = \int_{20}^{T_l} \alpha(T) dT \quad (6.6b)$$

in which $\alpha(T)$ is the coefficient of thermal expansion and defined at Eq. (5.8). Substituting Eqs. (6.1~6.2) and Eqs. (6.4~6.6) into Eq. (6.3), the internal axial force P_c of the axially loaded column subjected to non-uniform elevated temperature distribution shown in Figure 6.2 can be obtained

$$P_c = \frac{P + kL \left[L_l \int_{20}^{T_l} \alpha(T) dT + L_u \int_{20}^{T_u} \alpha(T) dT \right]}{1 + \frac{kL}{E_m A}} \quad (6.7)$$

in which E_m is introduced to simplify the equation. The factor E_m can be referred to as the general Young's modulus of the whole column, and the reciprocal of E_m can be expressed as

$$\frac{1}{E_m} = \frac{E_{20} - E_l}{E_{20} E_l} \frac{L_l}{L} + \frac{E_{20} - E_u}{E_{20} E_u} \frac{L_u}{L} \quad (6.8)$$

where L_l/L and L_u/L are the lower and upper segment length ratios of the column, respectively.

6.2.2 Lateral Stiffness of Columns Subjected to Non-uniform Elevated Temperature Distribution

To express the relative stiffness of the beam and the rotational end-spring connections, the concept of an end-fixity factor was introduced (Monforton and Wu, 1963; Cunningham, 1990) . The end-fixity factor defines the stiffness of each end-connection relative to the attached member and can be described as the ratio of the rotation of the member to the combined rotation of the member and the connection, which arises from a unit end-moment.

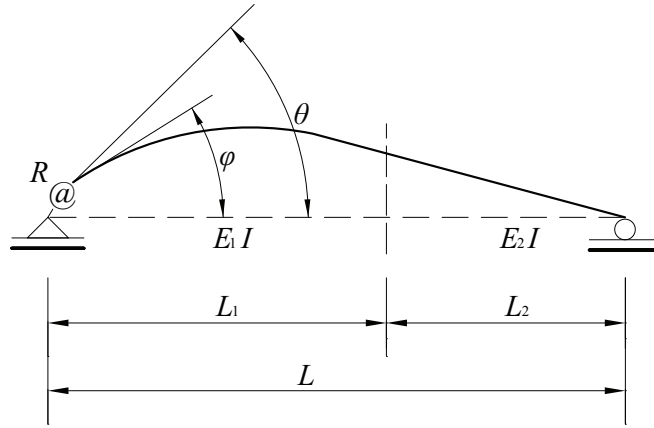


Figure 6.3: Definition of end-fixity factor

However, the equation of the end-fixity factor was introduced only for members with uniform section properties along the length. Based on the definition, the end-fixity factor of members with non-uniform section properties due to experiencing non-uniform temperature distribution as shown in Figure 6.3 can be calculated by using a virtual work principle:

$$r = \frac{1}{1 + \frac{3EI}{\tau RL}} \quad (6.9)$$

where R is the stiffness of the end rotational spring, and E is the Young's modulus at ambient temperature. E_1 and E_2 are the Young's modulus of segment 1 and 2, while L_1 and L_2 are the length of segment 1 and 2, respectively. L is the total length of the

beam. τ is the factor accounting for the effect of the different Young's modulus in two segments, and can be defined as

$$\tau = \frac{\left(\frac{L_2}{L}\right)^3}{\frac{E_2}{E}} + \frac{\frac{L_1}{L} \left[\left(\frac{L_2}{L}\right)^2 + \frac{L_2}{L} + 1\right]}{\frac{E_1}{E}} \quad (6.10)$$

Upon the introduction of the end-fixity factor, the effect of beam-to-column end rotational restraint for the column model in Figure 6.2 can be characterized by end-fixity factors:

$$r_l = \frac{1}{1 + \frac{3EI}{\lambda_l RL}} \quad (6.11a)$$

$$r_u = \frac{1}{1 + \frac{3EI}{\lambda_u RL}} \quad (6.11b)$$

in which

$$\tau_l = \frac{\left(\frac{L_u}{L}\right)^3}{\lambda_u} + \frac{\frac{L_l}{L} \left[\left(\frac{L_u}{L}\right)^2 + \frac{L_u}{L} + 1\right]}{\lambda_l} \quad (6.12a)$$

$$\tau_u = \frac{\left(\frac{L_l}{L}\right)^3}{\lambda_l} + \frac{\frac{L_u}{L} \left[\left(\frac{L_l}{L}\right)^2 + \frac{L_l}{L} + 1\right]}{\lambda_u} \quad (6.12b)$$

where λ_l and λ_u are the material degradation factors of lower and upper segments, respectively, and can be calculated by Eqs. (6.2).

The lateral stiffness of an axially loaded two-segment column subjected to two zone elevated temperature can be obtained from the equilibrium of the deformed column model shown in Figure 6.4. The internal moment in the two segments of the column can be expressed as

$$M_{il} = Py_l + Sx - M_l \quad (6.13a)$$

$$M_{iu} = Py_u + Sx - M_l \quad (6.13b)$$

where M_{il} , M_{iu} are the internal moment of the lower and upper segments of the column, respectively. M_l is the end moment at the lower end. S is the lateral force associated with the unit lateral deflection at column upper end, and defined as the lateral stiffness

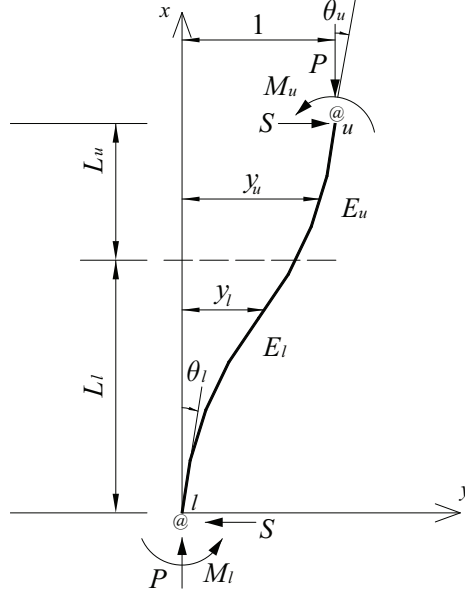


Figure 6.4: Axially loaded column with deformations and forces

of the column. y_l and y_u are the lateral deflection of the lower and upper segments of the column, respectively. Thus, the equilibrium condition of the column yields

$$EIy_l'' = M_l - Py_l - Sx \quad (6.14a)$$

$$EIy_u'' = M_l - Py_u - Sx \quad (6.14b)$$

In addition, the end moment at the upper end can also be obtained from Eq. (6.13b) as

$$M_u = P + SL - M_l \quad (6.15)$$

Let θ_u and θ_l be the upper and lower end rotations of the column, respectively. The end moments then can be expressed as

$$M_l = R_l\theta_l \quad (6.16a)$$

$$M_u = R_u\theta_u \quad (6.16b)$$

where the rotational stiffness of column end restraints R_l and R_u are contributed by beams connected to the lower and upper ends of the column, respectively. To solve the

differential equations of Eqs. (6.14), the following boundary conditions of the column are required:

$$y_l|_{x=0} = 0 \quad (6.17a)$$

$$y_u|_{x=L} = 1 \quad (6.17b)$$

$$y_l|_{x=L_l} = y_u|_{x=L_l} \quad (6.17c)$$

$$y'_u|_{x=l} = \theta_u \quad (6.17d)$$

$$y'_l|_{x=0} = \theta_l \quad (6.17e)$$

$$y'_l|_{x=L_l} = y'_u|_{x=L_l} \quad (6.17f)$$

solving Eqs. (6.14) and Eq. (6.15) with the boundary conditions in Eqs. (6.17), the lateral stiffness of an axially loaded column subjected to non-uniform temperature distribution can be obtained as follows (the derivation is presented in Appendix C):

$$S = \beta(\phi, \phi_l, \phi_u, r_l, r_u, \tau_l, \tau_u) \frac{12EI}{L^3} \quad (6.18)$$

in which the axial load ratio ϕ , ϕ_l and ϕ_u are defined as

$$\phi = \sqrt{\frac{P_c L^2}{EI}} \quad (6.19a)$$

$$\phi_l = \sqrt{\frac{P_c L^2}{E_l I}} = \frac{\phi}{\sqrt{\lambda_l}} \quad (6.19b)$$

$$\phi_u = \sqrt{\frac{P_c L^2}{E_u I}} = \frac{\phi}{\sqrt{\lambda_u}} \quad (6.19c)$$

where P_c is the axial compressive load accounting for the effect of non-uniform temperature and is defined in Eq. (6.7). The coefficient $\beta(\phi, \phi_l, \phi_u, r_l, r_u, \tau_l, \tau_u)$ is the modification factor of the column lateral stiffness with consideration of non-uniform temperature, and can be expressed as:

$$\beta =$$

$$\frac{\frac{\phi^2}{12}(A_1 \phi_l \phi_u \cos \alpha_l \cos \alpha_u + A_{21} \phi_l \cos \alpha_l \sin \alpha_u + A_{22} \phi_u \sin \alpha_l \cos \alpha_u + A_3 \sin \alpha_l \sin \alpha_u)}{18 \phi_l \phi_u r_l r_u + \phi_l \cos \alpha_l \sin \alpha_u (A_1 - A_{21}) + \phi_u \sin \alpha_l \cos \alpha_u (A_1 - A_{22}) - \phi_l \phi_u (18 r_l r_u + A_1) + \sin \alpha_l \sin \alpha_u A_4} \quad (6.20)$$

where

$$\alpha_l = \phi_l \frac{L_l}{L} \quad (6.21a)$$

$$\alpha_u = \phi_u \frac{L_u}{L} \quad (6.21b)$$

$$A_1 = 3\phi^2[\tau_l r_u + \tau_u r_l - (\tau_l + \tau_u)r_l r_u] \quad (6.21c)$$

$$A_{21} = \phi^4[\tau_l \tau_u(r_u + r_l) - \tau_l \tau_u(1 + r_l r_u)] + 9\phi_u^2 r_l r_u \quad (6.21d)$$

$$A_{22} = \phi^4[\tau_l \tau_u(r_u + r_l) - \tau_l \tau_u(1 + r_l r_u)] + 9\phi_l^2 r_l r_u \quad (6.21e)$$

$$A_3 = 3\phi^2[\phi_l^2 r_l r_u(r_u - 1) + \phi_u^2 \tau_l \tau_u(r_l - 1)] \quad (6.21f)$$

$$A_4 = 9r_l r_u(\phi_l^2 + \phi_u^2) - A_3 \quad (6.21g)$$

It can be seen that Eq. (6.20) is the same as the column stiffness modification factor presented in Eq. (5.13) in Chapter 5 when $L_l=L$ or $L_u=L$. In addition to accounting for the effects of the axial load and end-fixity factor, the stiffness modification factor in Eq. (6.20) also takes into account the effect of non-uniform temperature distribution by including the material degradation factors λ_u and λ_l , and the segment length ratios.

Consider the case in which the lower end of the column is rigidly connected ($r_l=1$) and the upper end is a pinned end ($r_u=0$); the applied load is a constant load with an applied load ratio $P/P_e=0.5$; the temperature of the lower segment is set as 200°C. The effect of the beam-to-column rotational restraint is taken into account but not for the effect of axial thermal restraint. Then, the relationship between the lateral stiffness modification factor β and the temperature of the upper segment T_u for different segment length ratios, L_l/L , are plotted as shown in Figure 6.5. It can be observed that the magnitude of β will decrease with an increase of T_u for all the segment length ratios, and the magnitude of β will increase as the segment length ratio L_l/L increases. In addition, the magnitude of β for L_l/L equal to 0.3 is only slightly higher than that for L_l/L equal to 0.1, while the increased magnitude of β between segment length ratio 0.3 and 0.5 is higher than that between 0.3 and 0.1. When the segment length ratio L_l/L increases from 0.5 to

0.75, the lateral sway resistant capacity of the column will be significantly improved, as shown in the dramatically increased value of β . Thus, it can be concluded that the step temperature distribution has considerable effect on the lateral stiffness of the column. If the lower temperature zone occupies a larger portion of the compartment fire, the lateral stiffness of the column will be underestimated if the uniform temperature assumption is adopted.

However, as the axial beam-to-column thermal restraint is not considered herein, the effect of the beam-to-column axial restraint on the column lateral stiffness cannot be observed in Figure 6.5. Presented in the following section is an investigation regarding the effect of the axial restraint on frame stability when subjected to a non-uniform elevated temperature distribution.

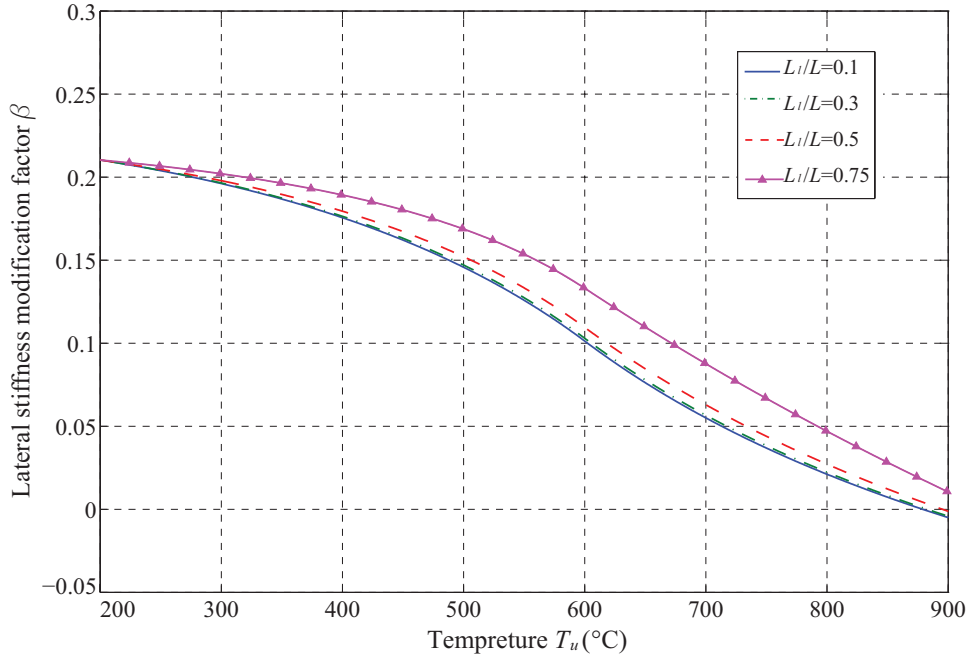


Figure 6.5: Lateral stiffness modification factor β for columns with different segment length ratios

6.3 Evaluation Procedure of Frame Buckling Strength at Non-uniform Elevated Temperature

Based on the concept of storey-buckling, the buckling strength of unbraced frames subjected to non-uniform temperature and proportional loadings can be evaluated with a similar trial-and-error procedure as presented in Section 5.3. For a two-zone fire model, the volume of the upper zone height will increase as the compartment fire develops; meanwhile, the volume of the lower zone will decrease (Quintiere, 2002). In addition, one fire experiment (Remesh and Tan, 2006) showed that the temperature difference between two zones increased before a compartment fire developed to the flashover stage. Hence, to simplify the analysis procedure, the lower zone temperature and the segment length ratio are first assumed to be constant in this study. Once the segment length ratio and the lower zone temperature T_l have been determined, the procedure of evaluating the frame buckling strength for the specified non-uniform temperature distribution based on the two-zone fire model can be stated as follow:

1. Compute the internal axial force P_c for columns at a specified temperature of the upper segment T_u from Eq. (6.7), and calculate the applied load ratio ϕ for each column.
2. Calculate the end-fixity factors r_l and r_u from Eqs. (6.11) and τ_l and τ_u from Eqs. (6.12) for each column in the frame.
3. Determine the storey stiffness $\sum S_i$ by summing up the lateral stiffness S_i of each individual column, which can be evaluated based on Eq. (6.18).
4. If the storey stiffness is equal to zero or within an acceptable tolerance, the buckling strength of the frame at the specified temperature T_u and T_l are obtained. Otherwise, proportionally increase the applied loads on the frame, and then go back to

step (1).

Therefore, the frame buckling strength within the range specified for T_u can be obtained by increasing T_u with a constant step size and repeating the foregoing procedure. The frame buckling strength curve for a specified temperature range can then be plotted, and the critical temperature for a specified applied load P_c can be determined from the curve.

6.4 Illustrated One-bay Frame Examples

The frame example previously investigated in Section 5.5 of Chapter 5 is employed herein to study its behaviour when the frame is subjected to different scenarios of step temperature distribution. Members and configuration of the frame shown in Figure 6.6 are the same as that in Chapter 5. The two cases shown in Figure 6.7 are studied in this investigation. In case 1, the upper part of the frame, including the beam, is subjected to temperature T_u , while the lower part of the frame is subjected to temperature T_l . To investigate the interaction between columns, in case 2, only Column 1 is experiencing the non-uniform temperature distribution.

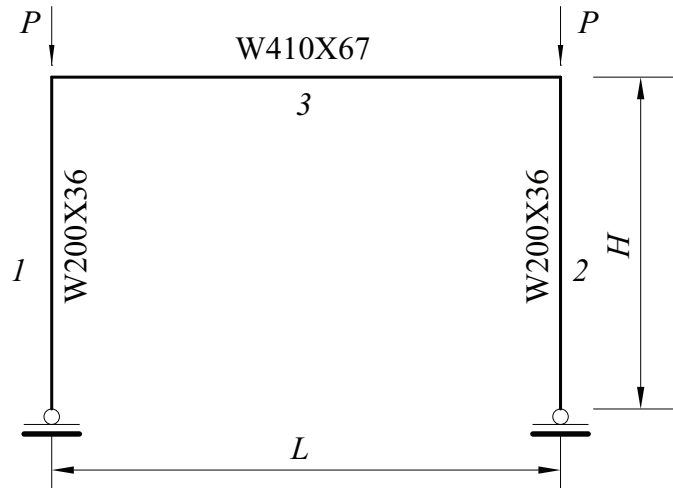


Figure 6.6: Example: One bay and one storey frame

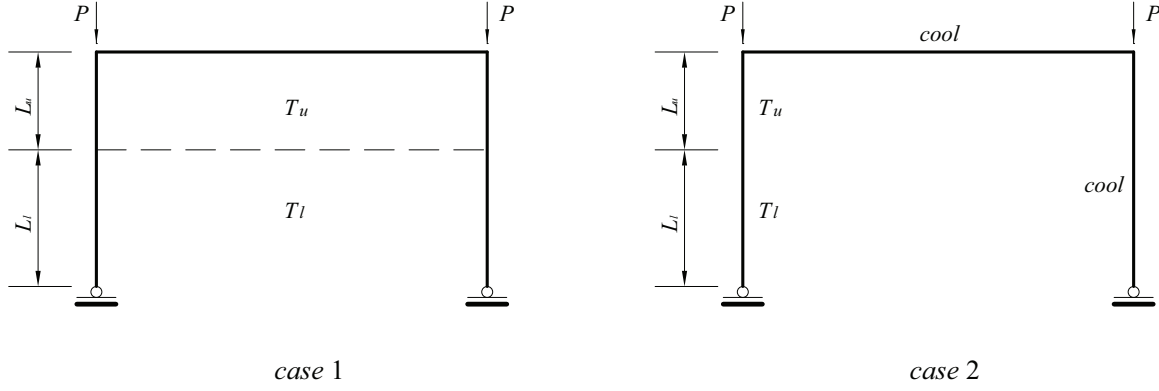


Figure 6.7: Different cases of frame members subjected to non-uniform elevated temperature

For the two cases shown in Figure 6.7, the temperature distribution within the heated columns is assumed to be uniform across the section and only non-uniform in the longitudinal direction.

6.4.1 Case 1

For case 1, two segment length ratios $L_l/L=0.3$ and $L_l/L=0.5$ were investigated. The temperature of the lower segment T_l is assumed to be constant at 200°C . Since the properties of both columns are identical and subjected to the same temperature distribution, the axial deformation associated with the elevated temperature in the columns will be identical. Thus, the corresponding axial thermal restraints can be neglected. Accordingly, the internal forces P_c in Column 1 and 2 are the same as the external applied load P . As discussed in Chapter 5, when the columns and the beam of the frame are exposed to a uniform temperature, there will be no differential temperature between the columns and beam. Thus, the beam-to-column stiffness ratio remains constant as the temperature increases, and the end-fixity factors for each column remain unchanged during the fire. For the temperature distribution in case 1, the temperature of the beam is the same as the temperature of the upper segment of both columns but the temperature of the lower segment of both columns stays unchanged, and hence, the rotational stiffness of

the beam is decreased when compared to the lower segment of the columns. Presented in Figure 6.8 is the relationship between the end-fixity factor r_u for the upper end of the column and the temperature T_u of upper segment of the columns for case 1. As shown in the figure, the end-fixity factors r_u for two segment length ratios, which characterize the beam-to-column rotational restraints of the column upper end, are decreased as the temperature T_u increases. Moreover, the larger segment ratio has more impact on the decrease of the beam-to-column rotational restraints, as can be observed in Figure 6.8 that r_u for segment length ratio 0.5 decreases at a faster rate than that of segment length ratio 0.3.

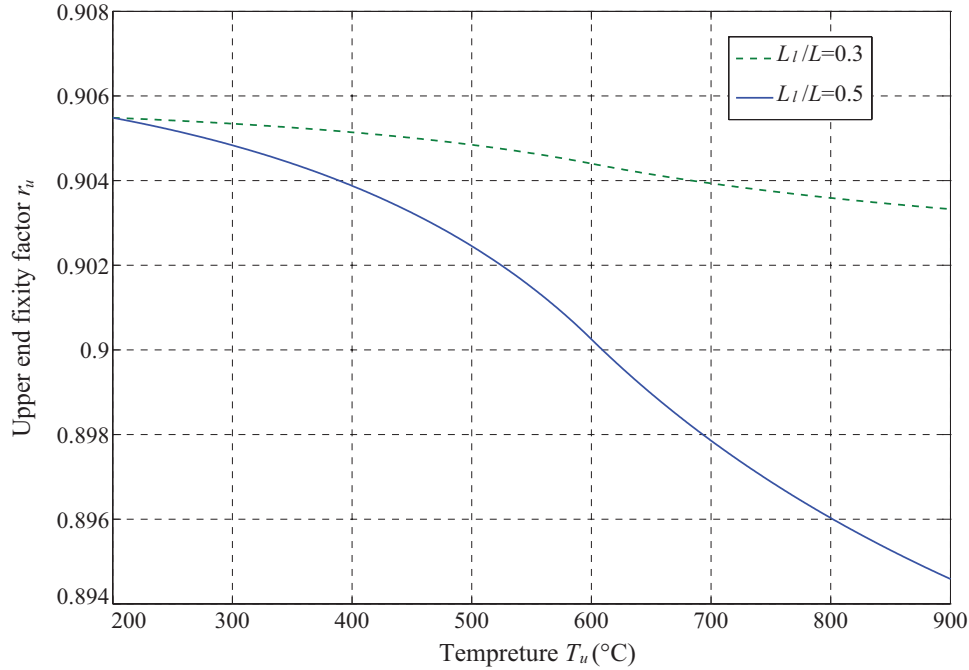


Figure 6.8: Variation of end-fixity factor r_u vs. temperature T_u for different segment length ratios

Based on the procedure proposed in Section 6.3, the relationship between the frame buckling strength and elevated temperature for two segment length ratios was obtained and plotted as shown in Figure 6.9. To compare with the results obtained at uniform elevated temperature, the buckling strength of the frame subjected to uniform tempera-

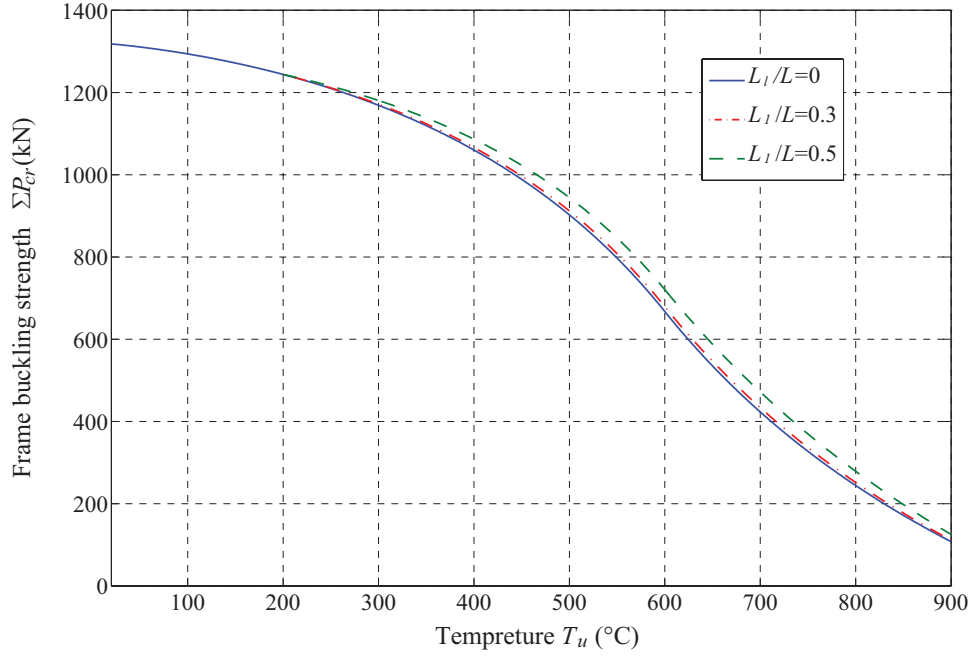


Figure 6.9: Case 1: Variation of frame buckling strength subject to non-uniform elevated temperature distribution

ture, with lower segment length ratio $L_l/L = 0$ and its columns and beam experiencing only uniform temperature T_u , is plotted in Figure 6.9 as well. In Figure 6.9, the buckling strengths for the frames subjected to non-uniform elevated temperature distribution have a decrease similar to that for the frames experiencing only uniform elevated temperature. In addition, as the segment length ratio that signifies the height of the lower cool zone of the compartment fire increases, the frame buckling strength will increase as well. It is known that increasing the beam-to-column rotational restraint will result in higher column buckling strength. Figure 6.8 demonstrates that the magnitude of end-fixity factor r_u for $L_l/L=0.5$ is less than that for $L_l/L=0.3$ as the temperature T_u increases. However, as shown in Figure 6.9, the increase of frame buckling strength from $L_l/L=0$ to $L_l/L=0.5$ is greater than that from $L_l/L=0$ to $L_l/L=0.3$, even though the steel frame for $L_l/L=0.3$ has stronger beam-to-column rotational restraints than that for $L_l/L=0.5$. The maximum difference between the frame buckling strength when $L_l/L=0.5$ and when

$L_1/L=0$ is 53.6kN, while the maximum difference between the frame buckling strength when $L_l/L=0.3$ and when $L_l/L=0$ is only 11.8kN. The comparison clearly suggests that the frame buckling strength is primarily affected by the length of the lower cool zone of the compartment fire. Thus, the greater the portion of the compartment fire the lower the cool zone occupies, the more conservative the frame buckling strength evaluated based on the assumption of uniform elevated temperature is.

The greatest difference in the frame buckling strength in case 1 between the non-uniform and uniform elevated temperature can be found to be 2.13% at temperature $T_u=634^\circ\text{C}$ between $L_l/L=0.3$ and $L_l/L=0$, and 9.28% at temperature $T_u=643^\circ\text{C}$ between $L_l/L=0.5$ and $L_l/L=0$, respectively. When the differences reach their peak value, they gradually diminish as temperature T_u increases since the degradation of materials dominates the deterioration of the frame buckling strength. Furthermore, since Column 1 and 2 are identical and subjected to the same elevated temperatures, the lateral stiffness for these two columns decrease at the same rate simultaneously, and no lateral stiffness interaction between columns occurs in case 1. Therefore, the buckling strengths of Column 1 (P_{1cr}) and 2 (P_{2cr}) are half that of the frame buckling strength ($\sum_{i=1}^n P_{icr}$) for the three lower-segment ratios discussed.

6.4.2 Case 2

In case 2, as shown in Figure 6.7, only Column 1 experiences non-uniform elevated temperatures and other members of the frame are assumed to be unheated. Therefore, the beam-to-column thermal axial and rotational restraints on Column 1 must be considered, because the elongation of column 1 at elevated temperature is confined by unheated Beam 2. To demonstrate the effect of stiffness interactions between the two columns on the frame buckling strength, the sub-assembly model for Column 1 shown in Figure 6.10 is also used to investigate the column buckling strength of Column 1. Unlike Column 1 of the unbraced frame shown in Figure 6.7, Column 1 of the sub-assembly model in Fig-

ure 6.10 has no lateral support from other unheated columns, and thus column stiffness interaction is not considered in the sub-assembly model. The temperature T_l of Column 1 in the sub-assembly model and in the frame is assumed to be constant at 200°C. Three lower segment length ratios that are investigated are $L_l/L = 0.3$, $L_l/L = 0.5$, and $L_l/L = 0.75$.

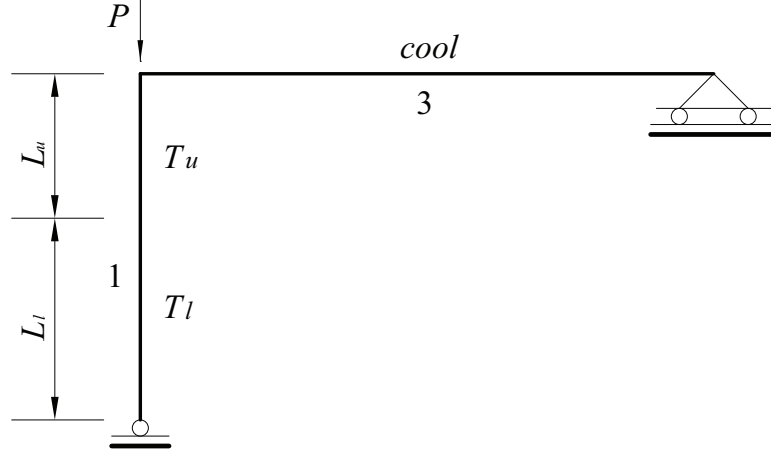


Figure 6.10: Case 2: A sub-assembly model for Column 1

The relationship between the column buckling strength of column 1 in the sub-assembly model and temperature T_u investigated based on the procedure described in Section 6.3 are plotted in Figure 6.11. To compare with the frame buckling strength, the critical loads P_{cr} of Column 1 for case 2 frame and sub-assembly model versus T_u are plotted in Figure 6.11 as well. It can be seen from the figure that the column buckling strength of Column 1 in the sub-assembly model is considerably less than the critical load of Column 1 in the frame of case 2 shown in Figure 6.7. In addition, the difference between the critical loads of column 1 for each segment length ratio will increase as T_u increase, which indicates that Column 1 in the frame relies upon more lateral support from unheated Column 2 as T_u increases. Thus, if the steel frame stability is solely evaluated based on the behaviour of an individual column, the capacity of unbraced steel frames to resist lateral instability will be significantly underestimated.

To compare the stability behaviour of the unbraced steel frame subjected to uni-

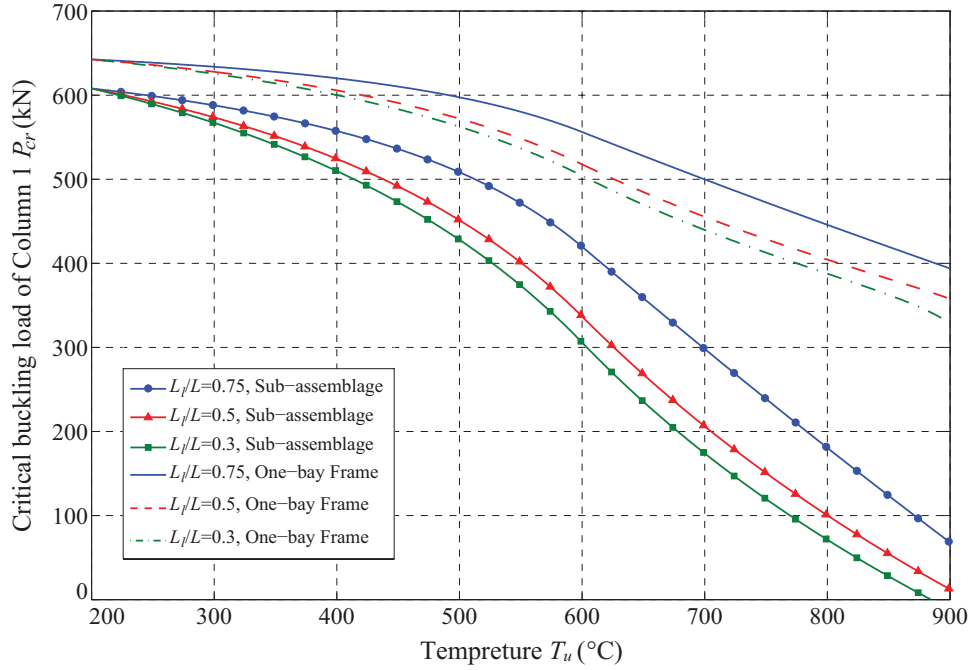


Figure 6.11: Case 2: Comparison of buckling strength of Column 1

form and non-uniform temperature distributions, the relationships between temperature T_u and the frame buckling strengths corresponding to four segment length ratios are plotted in Figure 6.12, in which the segment length ratio $L_l/L=0$ represents the frame subjected to uniform temperature distribution. For the non-uniform temperature distribution, temperature T_l is assumed to be constant at 200°C for all the segment length ratios. As shown in Figure 6.12, the frame buckling strength at uniform temperature is less than that subjected to non-uniform temperatures distribution for each segment length ratio. The difference of the frame buckling strength between uniform and non-uniform temperatures will increase as temperature T_u rises. It is noted that the frame buckling strength for $L_l/L=0.75$ is significantly greater than the frame buckling strength at uniform temperature.

Since Column 2 and Beam 3 are unheated, the primary factors that influence the frame buckling strength are the elevated temperature and axial thermal force on Column 1. The axial thermal force results from column elongation confined by axial thermal

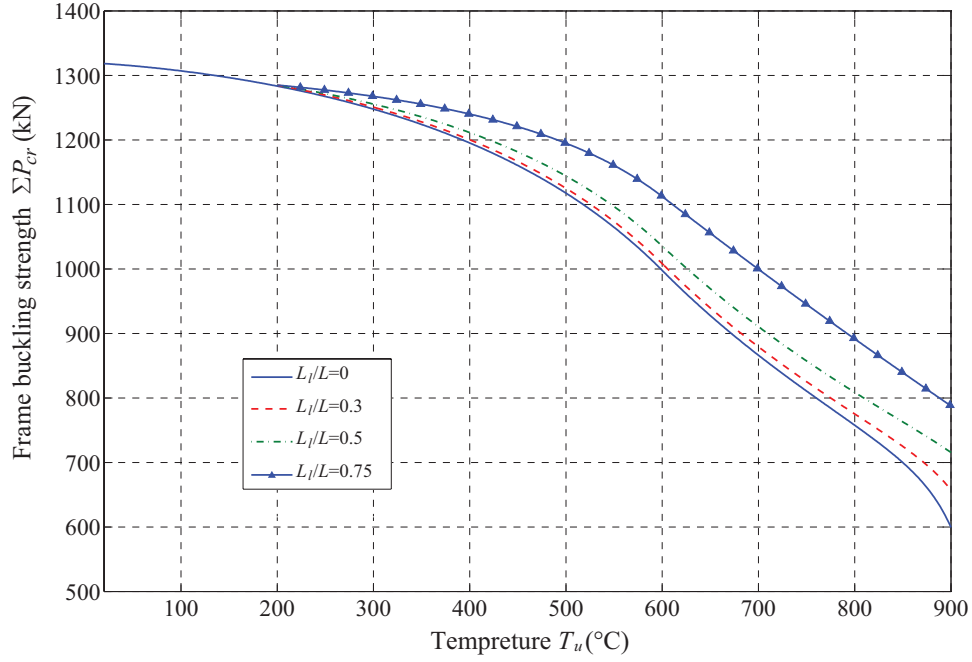


Figure 6.12: Case 2: Frame buckling strength vs temperature T_u

restraint and equal to the column internal force P_c minus the column external force P . Exhibited in Figure 6.13 is the variation of the axial thermal force on Column 1 when the frame buckles as temperature T_u increases. It can be observed that the axial thermal force corresponding to the uniform elevated temperature is greater than that for the non-uniform elevated temperature distribution. Therefore, in addition to the external applied load, the axial thermal force further reduces the lateral stiffness of the columns, which consequently reduce the buckling strength of the frame.

In the foregoing discussion, temperature T_l for the lower segment is assumed to be constant. In the next study, the lower segment length ratio L_l/L is fixed at 0.5, and the frame buckling strength will be investigated when T_l is taken different values. The relationships between frame buckling strength and temperature of $T_l=200^\circ\text{C}$ and $T_l=400^\circ\text{C}$ are plotted in Figure 6.14.

As shown in Figure 6.14, the frame buckling strength at $T_l=400^\circ\text{C}$ is less than that of $T_l=200^\circ\text{C}$ when upper portion temperature, T_u , is the same. The difference can be

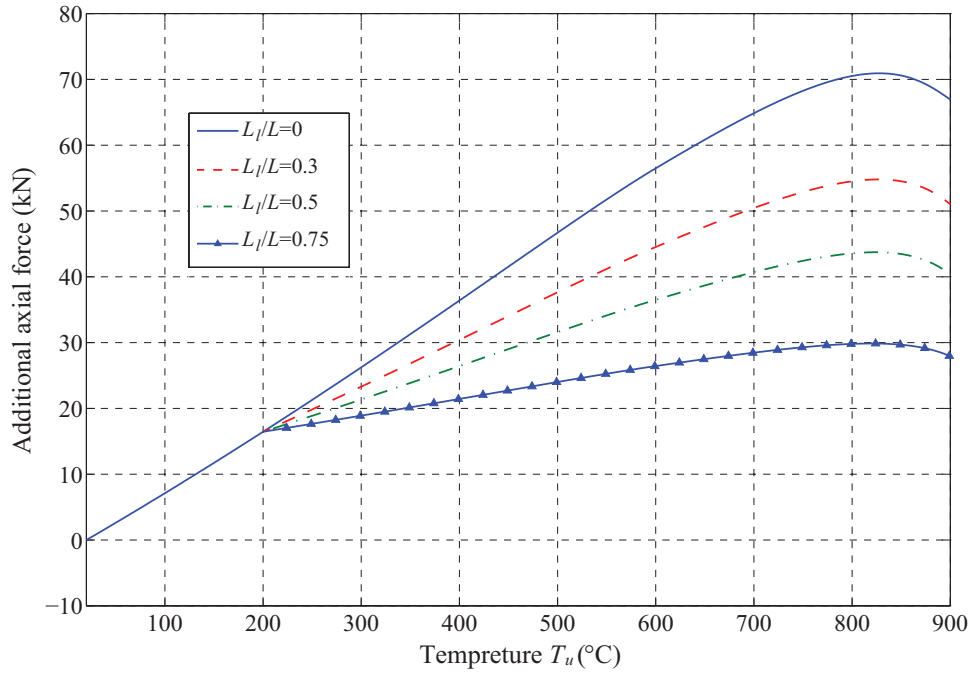


Figure 6.13: Case 2: frame buckling strength vs temperature T_u

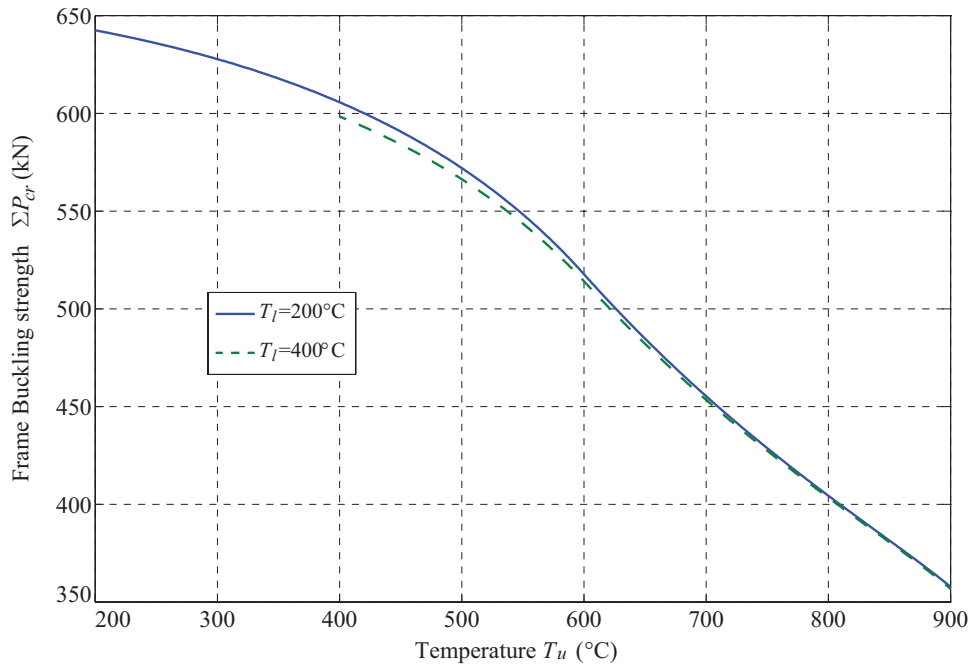


Figure 6.14: Case 2: additional load on column 1

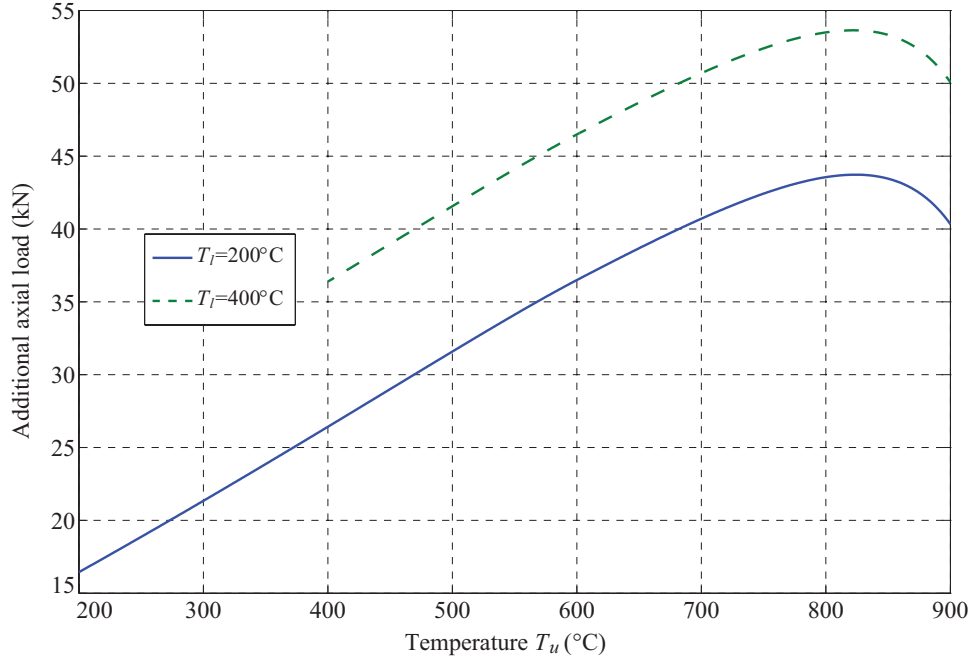


Figure 6.15: Case 2: additional axial load on Column 1

explained by the axial thermal force in Column 1. The relationship between axial thermal force on Column 1 associated with column elongation and temperature T_u is plotted in Figure 6.15. It is noted that the axial thermal force at $T_l=400^\circ\text{C}$ is considerably greater than that at $T_l=200^\circ\text{C}$, and the difference between them from 400°C to 900°C remains unchanged. Obviously, when the lower segment of Column 1 is subjected to a higher temperature, the column will experience the larger elongation, which consequently yields greater axial thermal force in the column and reduces the frame buckling strength. Nevertheless, it can be found from Figure 6.14 that the difference between two curves will gradually diminish. Therefore, it is the degradation of materials that eventually controls the deterioration of the frame buckling strength as temperature T_u rises.

6.5 Illustrated Two-bay Frame Example

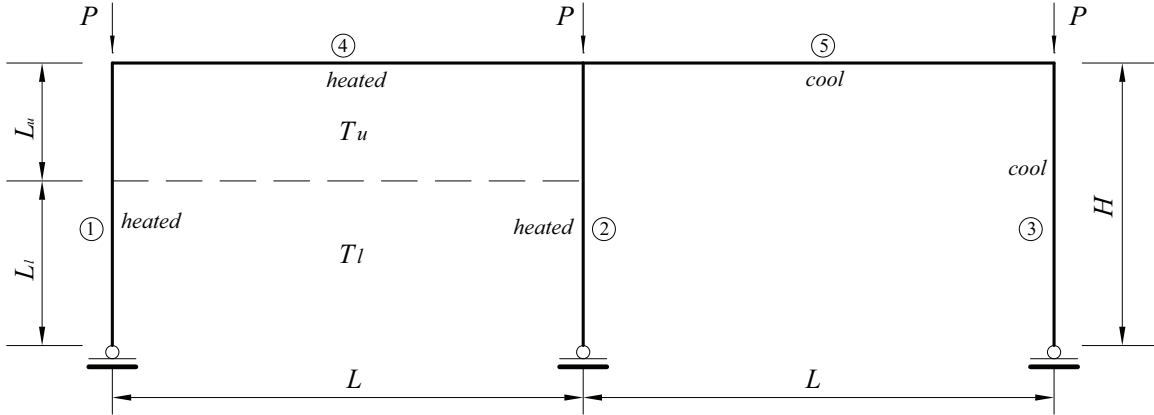


Figure 6.16: Example: Two-bay and one storey frame

In this section, the one-storey and two-bay unbraced steel frame shown in Figure 5.16 will be investigated for its stability behavior in two-zone fire. The dimensions and cross sections of columns and beams are the same as the example in Section 5.6. In this example, the axial forces in Column 1 and 2 induced by unequal end displacements of Beam 4 are neglected because of the same reason discussed in Section 5.6. The lower zone temperature T_l is assumed to be the constant at 200°C, and the frame buckling strength of three lower segment ratios L_l/L , 0.3, 0.5, and 0.75, will be investigated by following the procedure presented in Section 6.3. To compare with the one-zone fire, the frame buckling strength subjected to uniform elevated temperature is assessed as well, with the lower segment ratio equal to 0.

The frame buckling strengths of four lower segment ratios affected by the increasing temperature are plotted in Figure 6.17. The decreasing trends of these four lower segment ratios are similar to that of the one-bay frame example depicted in Figure 6.12. The difference in frame buckling strength between uniform and non-uniform temperature distribution increases as the temperature and lower segment ratio increase. This observa-

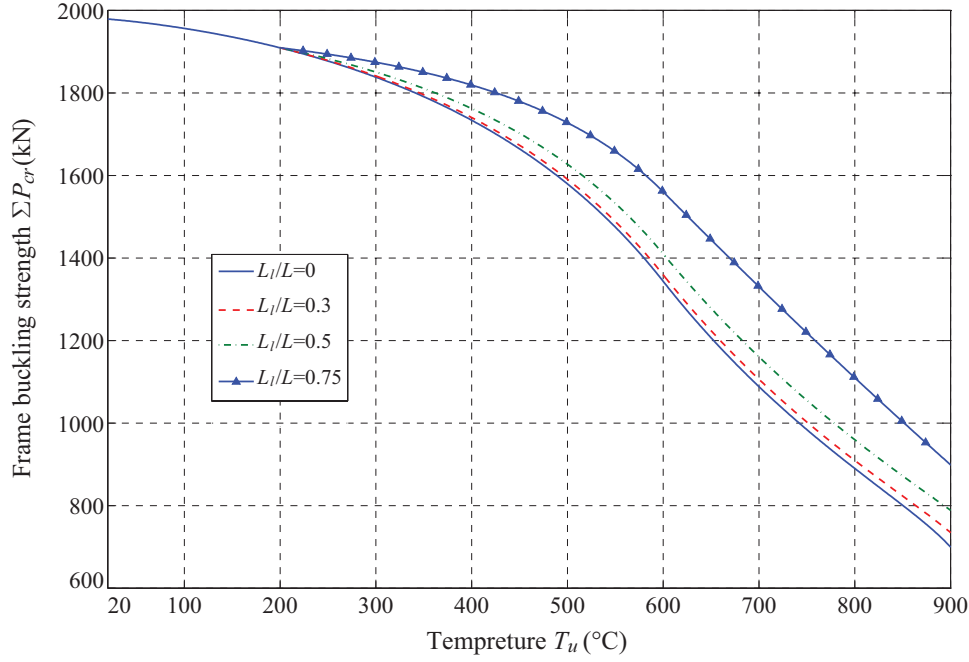


Figure 6.17: Two bay frame example: frame buckling strength vs temperature T_u

tion further suggests that the frame buckling strength subjected to non-uniform elevated temperature distribution but evaluated based on the method of uniform temperature will be more conservative when lower segment ratio is relatively large.

6.6 Verification with Numerical Analysis

The finite element method was used to assess the buckling strength of steel frame subjected to fire, which has been discussed in Section 5.7. The finite element package ABAQUS is also used in this section to verify the proposed method by analyzing the stability of the frames shown in Figure 6.6 and 6.16. Columns and beams in the examples are modelled by three-node quadratic beam elements (B23) and divided into 8 elements with considering the balance between accuracy and efficiency. Elevated temperatures are set up in predefined temperature field. A constant axial load $P = 500kN$ was applied on the top of each column for each example. The failure temperatures T_{cr} of

Table 6.1: Comparison of frame critical temperature

$P(\text{kN})$	Example	$\frac{L_l}{L}$	T_l (°C)	Numerical (FEM) $T_{cr}^N(\text{°C})$	Analytical (Proposed) $T_{cr}^A(\text{°C})$	Difference $(T_{cr}^A - T_{cr}^A)/T_{cr}^N$	
500	One-bay frame	Case 1	0.3	200	440	448	-1.8%
			0.5	200	454	466	-2.6%
		Case 2	0.3	200	591	606	-2.53%
			0.5	200	607	626	-3.13%
			0.75	200	667	700	-4.95%
			0.5	400	602	621	-3.16%
	Two bay frame		0.3	200	525	545	-3.81%
			0.5	200	540	564	-4.44%
			0.75	200	592	626	-5.74%

each frame were investigated by finite element analysis and the proposed approach, respectively. The results are listed in Table 6.1, and it can be concluded that the results of the proposed approach for one-bay and two-bay examples are in good agreement with the finite element results. The maximum difference between the finite element method and the proposed method are only -4.95% for the one-bay example and -5.74% for the two-bay example, respectively. Similar to the results for uniform temperature in Section 5.7, the finite element results for non-uniform temperature cases are also less than the results obtained by the proposed method, which is resulted from the assumption of neglecting the expansion of beam at elevated temperature in the proposed method. However, the insignificant difference between the proposed method and the finite element method shows that the expansion of beam at elevated temperature is not the primary factor that affects the frame buckling strength of unbraced steel frame subjected to elevated temperature. In general, the comparison suggests that the proposed method can be used to assess the

stability of steel frames subjected to non-uniform elevated temperature distribution.

6.7 Conclusion

Discussed in this chapter was the lateral stability of unbraced steel frames subjected to non-uniform elevated temperature distributions based on the concept of storey-based buckling. An analytical model was proposed to calculate the column internal force under non-uniform temperature distribution accounting for the effects of thermal axial and rotational restraints. The lateral stiffness of axially loaded columns exposed to non-uniform elevated temperature distribution was derived based on the Euler-Bernoulli theory. The stiffness reduction associated with the non-uniform elevated temperature distribution and axial loadings were expressed in terms of the column lateral stiffness modification factor. Upon the adoption of the proposed model and the derived lateral stiffness equations, the procedure to evaluate frame buckling strength for the non-uniform elevated temperature distribution based on the concept of storey-buckling was presented. The behaviour and buckling strengths of a one-bay unbraced steel frame with two fire scenarios and a two-bay unbraced steel frame with one fire scenarios were investigated. The study shows that the frame buckling strength evaluated based on non-uniform temperature distribution is greater than that evaluated based on a uniform elevated temperature, and the stability capacity of unbraced steel frame will be improved if the height of the lower cool segment of the compartment fire increases. Thus, considering the nature of a real fire in a building, it is more appropriate to evaluate frame buckling strength based on the proposed method in this chapter than the method presented in Chapter 5. In addition, the examples also demonstrated the necessity of evaluating frame buckling strength subjected to non-uniform elevated temperature distribution based on the concept of storey-buckling rather than individual column failure; otherwise, frame buckling strength will be considerably underestimated as stiffness interaction between weak columns due to fire and strong columns at ambient temperature in the same storey are neglected while consider-

ing individual column alone. Therefore, the proposed study can help engineers to obtain more economic frame design without sacrificing safety in case of fire. At the end, the finite element programme ABAQUS was used to verify the proposed method. The results assure that the proposed method can be used to evaluate the stability of unbraced steel frame at non-uniform elevated temperature distribution.

Chapter 7

Stability of Unbraced Steel Frames with Fire-Protected Columns Subjected to Fire

7.1 Introduction

Fire resistance of steel frames has become of a serious concern to structural designers after several structural-collapse disasters caused by the buckling of steel columns at elevated temperature, such as the World Trade Centre Building ([Kodur, 2003](#)). In the traditional design method, designers employ fire protective materials to help the steel members survive the fire period. More specifically, many steel-framing structures are now designed such that only the columns are protected by fire protective materials and the beams are left exposed to fire, since floors are made of composite slab systems that will provide partial fire protection to beams. The investigation of buckling strength of Type-4 frame in the example of Chapter [5](#) suggested that the fire protection of steel columns in frames will considerably increase the frame's fire-resisting capability. In addition, the fire test conducted on a full-scale steel frame building by the Building Research Establishment

at Cardington ([Bailey et al., 1999](#)) demonstrated that the steel frame could withstand the compartment fire in spite of the fact that only the columns were fire protected. On the other hand, most current studies on steel frames subjected to elevated temperature focused on compartment fires that localized fire in a single room. However, in real situations, it is possible that a compartment fire might spread to adjacent rooms due to the opening of fire doors or balance of fuel supply, or extinction and ventilation conditions. Accordingly, it is necessary to take different fire scenarios into account, for example, fire occurring in different compartments of the same storey but not at the same time.

The structural behaviors of steel frames with one-storey completely exposed to fire have been intensely investigated in recent research. [Bailey et al. \(1996\)](#) proposed an analytical model predicting the behaviour of steel-framed buildings during both the heating and cooling phases of a fire. The effect of fire spread on the structural behaviour of a braced steel frame is investigated but the fire in the investigation was assumed to occur in the same compartment first rather than in any possible compartment randomly. [Lu et al. \(2005\)](#) proposed a fire analysis procedure for predicting thermal and cooling effects on an isolated element and a multi-storey frame, and the structural behavior of an eight-storey steel frame with one storey totally exposed to fire was studied. [Couto et al. \(2013\)](#) studied the elastic critical load of braced and unbraced frames exposed to storey fire, and developed the buckling lengths as a function of the temperature. However, all the fire scenarios investigated in these studies are based on the assumption that a fire occurs in the specified compartments. In this study, the nature of fire occurring randomly in locations is of primary concern. In real fires, external loadings are assumed to be constant for single-storey steel frame since live loads and dead load may not change in such a short time when fire occurs. A single storey and two-bay unbraced steel frames was investigated in this chapter, in which only columns protected against fire and the magnitudes of external loadings were set as constants. Initial attempts were made to assess the fire resisting capability of unbraced steel frames with only columns being fire

protected subjected to different fire scenarios. That being said, only beams experience elevated temperature, and thus, there is no need to make a distinction between uniform and non-uniform elevated temperatures that unbraced frames investigated in this chapter are subjected to. In this study, a degradation factor that characterized the variation of the Young's modulus of steel at elevated temperature was introduced. Then, the objective and constraint functions were constructed, and optimization methods were used to find the buckling strength of an unbraced steel frame at different fire scenarios based on the storey-buckling concept.

7.2 Material Degradation Factor of Young's Modulus at Elevated Temperature

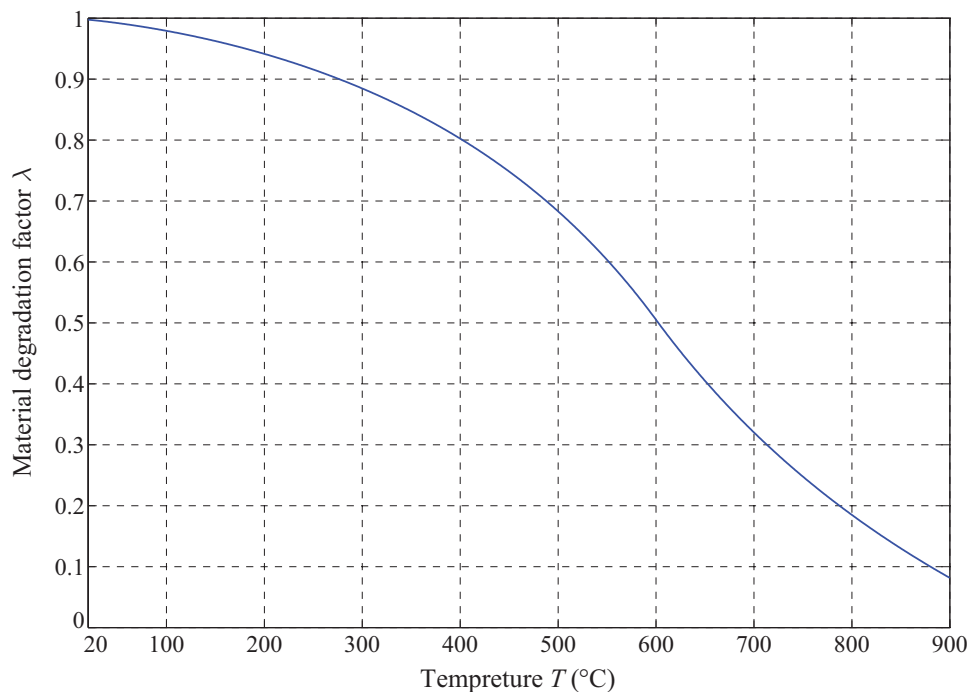


Figure 7.1: Variation of material degradation factor λ at elevated temperature

It is known that steel will lose its strength when heated. As the stress-strain re-

relationship at elevated temperature adopted in this study, the variation of the Young's modulus of steel at elevated temperature can be characterized by the Young's modulus multiplying a degradation factor λ , which can be defined in Eqs. (6.2).

The relationship between λ and T is plotted as shown in Figure 7.1, from which it can be seen that the value of λ will monotonically decrease as the temperature T increases from 20 to 1000°C, and the upper and lower bounds of λ are 1 and 0, respectively. As only beams experience elevated temperature rather than columns in this chapter, λ is applied to the Young's modulus of beams only.

7.3 Storey-based Stability of Unbraced Steel Frames Subjected to Different Fire Scenarios

Since only beams are considered to be exposed to fire, the strength of columns are assumed not to be affected by the elevated temperature, which means the Young's modulus of the columns remains as a constant during a fire. However, the rotational restraint to the column induced by adjoining beams will decrease with the deterioration of the stiffness of steel beams. The reduced rotational stiffness of the adjoining beam with semi-rigid connections R_T at elevated temperature can be calculated by substituting the Young's modulus at elevated temperature into the equation derived by Xu and Liu (2002a)

$$R_T = \left(\frac{6r_1}{4 - r_1r_2} \right) \left(\frac{\lambda E_{20} I_b}{L_b} \right) (2 + vr_2) \quad (7.1)$$

in which r_1 and r_2 are the end-fixity factors of the beam, and λ is the degradation factor defined in Eqs. (6.2). I_b and L_b are the moment of inertia and the length of the beam, respectively. v is the ratio of joint rotations at the two ends of the beam. For unbraced steel frames, which buckle in an asymmetrically lateral sway mode, Xu and Liu (2002a) suggested that $v = 1$, and this value is adopted in this study. By substituting R_T into Eq. (3.1), the end-fixity factor of the column related to the adjoining beam at elevated temperature can be obtained.

As the rotational stiffness decreases due to the increasing temperature, the lateral stiffness of the column will be reduced, although the external loading remains as a constant. The lateral stiffness of a column then can be expressed as an equation in terms of the degradation factor λ :

$$S_T(\lambda) = \beta(\lambda) \frac{12E_{20}I_c}{L_c^3} \quad (7.2)$$

where I_c and L_c are the moment of inertia and the length of the beam, respectively, and β is the stiffness modification factor and only affected by the elevated temperature. Similar to Eq. (5.13), it can be expressed as

$$\beta(\lambda) = \frac{\phi^3}{12} \frac{a_1 \phi \cos \phi + a_2 \sin \phi}{18r_l(\lambda)r_u(\lambda) - a_3 \cos \phi + (a_1 - a_2)\phi \sin \phi} \quad (7.3)$$

where

$$a_1 = 3\{r_l(\lambda)[1 - r_u(\lambda)] + r_u(\lambda)[1 - r_l(\lambda)]\} \quad (7.4a)$$

$$a_2 = 9r_l(\lambda)r_u(\lambda) - [1 - r_u(\lambda)][1 - r_l(\lambda)]\phi^2 \quad (7.4b)$$

$$a_3 = 18r_l(\lambda)r_u(\lambda) + 3\{r_l(\lambda)[1 - r_u(\lambda)] + r_u(\lambda)[1 - r_l(\lambda)]\}\phi^2 \quad (7.4c)$$

in which r_l and r_u are the end-fixity factors for the lower and upper ends of the column, respectively, and they are affected by the elevated temperature which can be characterized by the factor λ . For columns with one adjoining beam exposed to fire, the equation of the lateral stiffness S is only related to one variable of λ . In contrast, for columns with two adjoining beams exposed to fire, the lateral stiffness equation is associated with two variables λ as each beam might be subjected to different elevated temperatures. For instance, for the one storey two-bay frame shown in Figure 7.2, the temperatures T_1 and T_2 represent different fire scenarios that may occur in the frame. The decrease of the strength and stiffness of the beams can be characterized by the degradation factors λ_1 and λ_2 . It can be seen from Figure 7.2 that Columns 1 and 3 are connected to Beams 4 and 5, respectively; their lateral stiffness S_{T1} and S_{T3} are only associated with factors λ_1 and λ_2 , respectively. As Column 2 is connected to Beam 4 and 5, its lateral stiffness is thus affected by the strength decrease of both Beam 4 and 5. Therefore, the lateral

stiffness S_{T2} of Column 2 can be expressed as an equation in terms of factors λ_1 and λ_2 . In an analogous manner, the expression of lateral stiffness in Eq. (7.2) can be extended to more general multi-bay steel frame. The lateral stiffness of the whole storey can then be expressed as

$$\sum_{i=1}^n S_{Ti}(\lambda_i) = 12E_{20} \left[\beta_1(\lambda_1) \frac{I_{c1}}{L_{c1}^3} + \beta_2(\lambda_1, \lambda_2) \frac{I_{c2}}{L_{c2}^3} + \dots + \beta_{n-1}(\lambda_{n-2}, \lambda_{n-1}) \frac{I_{cn-1}}{L_{cn-1}^3} + \beta_n(\lambda_{n-1}) \frac{I_{cn}}{L_{cn}^3} \right] \quad (7.5)$$

where n is the number of columns in a storey. Based on the concept of storey-buckling, when the lateral stiffness of the whole storey is reduced to zero by a fire, the storey will buckle in a lateral sway mode, and the buckling can be expressed by the zero value of Eq. (7.5).

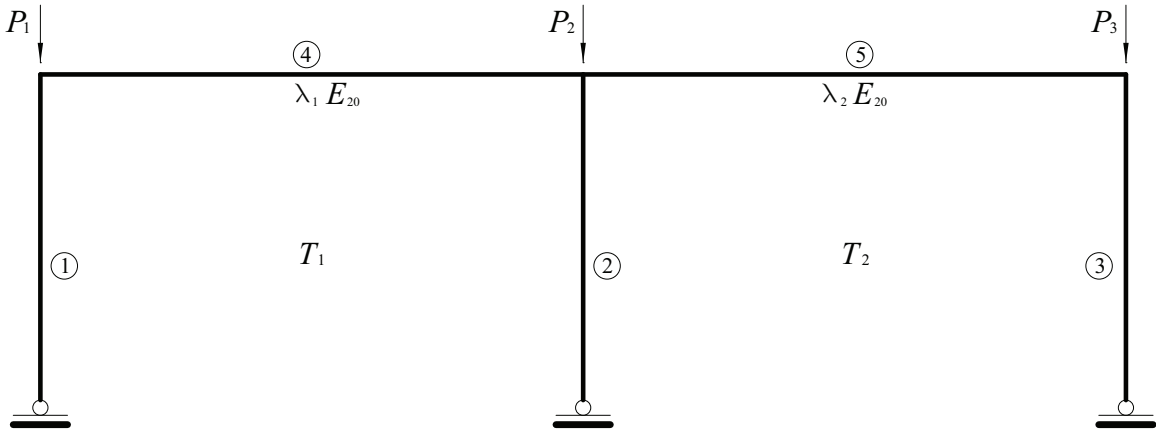


Figure 7.2: Two-bay and one storey frame

For simplicity, the variables used in the equation are the degradation factors λ instead of temperature. As shown in Figure 7.1 of the relationship between λ and T , the changing combinations of factors λ_i signify different fire scenarios because of each factor λ corresponding to a specific temperature T . Moreover, it can be observed that no unique solution exists for Eq. (7.5) due to one equation being related to $n-1$ variables of λ_i . Thus, the stability of unbraced steel frames subjected to different fire scenarios can be stated as a problem of finding the lower and upper bounds of the summation of factors λ_i . By determining maximum and minimum summations of λ_i , two different fire

scenarios for the corresponding frame buckling can be obtained. Since $\lambda(T)$ in Eqs (6.2) monotonically decreases as the increase of temperature T , the maximum summation of λ_i represents the minimum summations of the temperature each compartment, while the minimum summation of λ_i denotes the maximum summations of the temperature of each compartment. The problem can now be stated as a pair of maximization and minimization problems with stability constraints:

$$\begin{array}{l} \text{Minimum} \\ \text{Maximum} \end{array} : Z = \sum_{i=1}^n \lambda_i \quad (7.6)$$

subject to

$$\sum_{i=1}^n S_{Ti} = 12E_{20} \left[\beta_1(\lambda_1) \frac{I_1}{L_1^3} + \sum_{i=2}^{n-1} \beta_i(\lambda_{i-1}, \lambda_i) \frac{I_i}{L_i^3} + \beta_n(\lambda_{n-1}) \frac{I_n}{L_n^3} \right] = 0 \quad (7.7a)$$

$$0 \leq \lambda_i \leq 1 \quad (i = 1, 2, \dots, n) \quad (7.7b)$$

where the factor λ_i is the variable to be solved, and n is the total number of columns in the storey. The objective function Eq. (7.6) corresponds to two fire scenarios under which the unbraced steel frame will buckle laterally. The storey-based stability condition imposed on the frame is defined by Eq. (7.7a). The lower bound of factor λ_i in Eq. (7.7b) denotes that the temperature of the compartment fire reaches as high as 1000°C and the beam has totally lost its strength, while the upper bound of factor λ_i in Eq. (7.7b) means no fire in that compartment.

7.4 Numerical Example

The stability of a single storey and two-bay unbraced steel frame shown in Figure 7.3 was investigated by LeMessurier (1977), Shanmugam and Chen (1995), Schmidt (1999), Xu and Liu (2002a). The frame is adopted in this study to investigate its stability when subjected to different fire scenarios. Young's modulus of steel is taken to be $E = 200\,000\text{MPa}$, the moment of inertia for columns and beams are $I_1 = 3.134 \times 10^{-5}\text{m}^4$,

$I_2 = 9.4485 \times 10^{-5} m^4$, $I_3 = 7.6586 \times 10^{-5} m^4$, and $I_4 = I_5 = 136.12 \times 10^{-5} m^4$, respectively. Since the fire protection of columns significantly increases the frame's stability capability against fire as discussed in Chapter 5, the external loadings are proportionally increased to the values shown in Figure 7.3 to avoid the case in which both beams lose their strength in a fire without having the frame buckle. The temperature in the left room is T_1 , corresponding to the degradation factor λ_1 of Beam 4, while the temperature in the right room is T_2 , corresponding to the degradation factor of λ_2 of Beam 5.

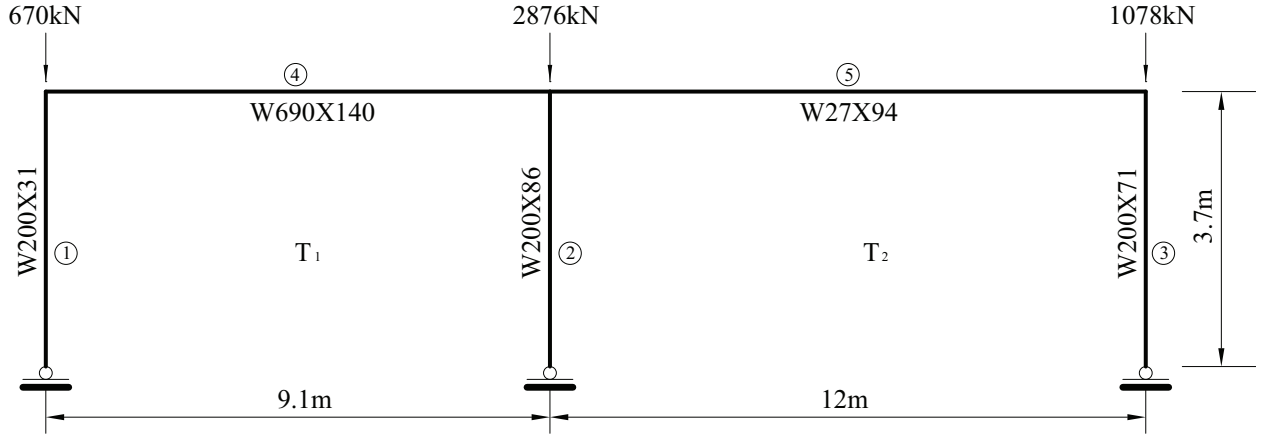


Figure 7.3: Two-bay and one storey frame design example

By calculating the end-fixity factors of each column and based on Eq. (7.6) and Eqs. (7.7), the optimization problem can be constructed as

$$\begin{array}{l} \text{Minimum} \\ \text{Maximum} \end{array} : Z = \lambda_1 + \lambda_2$$

subject to

$$\sum_{i=1}^3 S_{Ti} = 12 \frac{E_{20}}{L^3} [\beta_1(\lambda_1)I_1 + \beta_2(\lambda_1, \lambda_2)I_2 + \beta_3(\lambda_2)I_3] = 0$$

$$0 \leq \lambda_i \leq 1 \quad (i = 1, 2)$$

in which

$$\beta_1(\lambda_1) = \frac{14.186\lambda_1 - 0.429}{138.299\lambda_1 + 3.517}$$

$$\beta_2(\lambda_1, \lambda_2) = \frac{1.25\lambda_1 + 0.948\lambda_2 - 0.401}{31.732\lambda_1 + 24.063\lambda_2 + 2.308}$$

$$\beta_3(\lambda_2) = \frac{1.935\lambda_2 - 0.086}{2.628\lambda_2 + 1.076}$$

It is noted that the optimum solution for the problem stated above can be found by the sequential quadratic programming algorithm, and the results are listed in Table 7.1. The corresponding temperatures are obtained through Eqs. (6.2) and also listed in Table 7.1.

Table 7.1: Critical temperature at fire scenarios

Fire Scenarios	λ_1	$T_1(^{\circ}\text{C})$	λ_2	$T_2(^{\circ}\text{C})$	$\sum \lambda_i$
Case 1 (Min $\Sigma\lambda$)	0.0446	943	0.0961	885	0.1407
Case 2 (Max $\Sigma\lambda$)	1	20	0.0037	995	1.0037

The results in Table 7.1 clearly demonstrate the difference between two fire scenarios in which the steel frame buckles laterally. In Case 1 with the minimum value of $\Sigma\lambda$, the temperature of the fire in the left room is 943°C, and the temperature in the right room is 885°C, lower than that in the left room. The fire scenarios in Case 1 can thus be seen as a fire which occurs in the left room and spreads to the right room shortly after the fire is ignited. From this perspective, this fire scenario can be seen as a case in which fire is ignited in two rooms almost at the same time. On the other hand, in Case 2 with the maximum value of $\Sigma\lambda$, only the right room is subjected to an elevated temperature 995°C, while the left room has no fire in itself and remains at ambient temperature. Therefore, the fire scenarios in Case 2 can be seen as a case in which a fire only occurs in the right room.

It can be observed from both cases that the frame does not buckle until the beams are subjected to high temperature, which further assures that protecting columns from fire is

a more efficient and economical way to improve the fire-resisting capability of steel frames. Nevertheless, differences can still be found between the two cases. Upon comparison of Case 1 and Case2, the steel frame suffers a longer time in the fire scenarios of Case 2, as the fire temperature in Case 2 is greater than that in Case 1. In other words, the fire scenario of Case 1 is worse than that of Case 2. From this perspective, the maximum value of $\Sigma\lambda$ can determine the preferable fire scenario in which the frame would buckle, while the minimum value of $\Sigma\lambda$ can find the worst fire scenario. The difference between the two cases can also be explained by the concept of storey-buckling. The columns in the compartment subjected to lower or no elevated temperature will maintain greater lateral stiffness and provide lateral support to the columns in the compartment of the same storey but subjected to higher elevated temperature; as a result, the steel frame can survive longer. Take Case 2 as an example, the unheated left room enable Column 1 and 2 provide strong lateral support to Column 3, although the temperature in the right room of Case 2 is higher than that in both rooms of Case 1.

Since Case 2 represents the fire scenario that fire only occurs in the right room, the case in which fire only occurs in the left room is studied as well. The investigation of the case shows that the frame will not buckle even though the elevated temperature in the left room reaches 1000°C and the rotational restraints from Beam 4 to Column 1 and 2 has completely diminished. The case with only the left room subjected to fire further suggests that lateral stiffness interaction between columns cannot be neglected when assessing the frame stability in fire. Considering that the temperature difference between the two rooms in Case 2 is much more significant than that in Case 1, it can be concluded that fire spreading to an adjacent room shortly after fire occurs will be detrimental to the integrity of steel frames subjected to fire. From the perspective of fire safety, effective fire compartmentalization that confines fire in one compartment will enhance steel frames' capability to resist the fire.

However, it must be noticed that Beam 4 in Case 1 and Beam 5 in Case 2 almost

lost its axial strength at such high elevated failure temperatures. In such cases, whether the lateral support between columns can be transferred from the strong columns to the weak columns is doubtful, and it is possible that local collapse would thus occur in the frame before the whole steel frame buckles. On the other hand, the presence of concrete slab may help to transfer the required lateral support. Therefore, the interaction between columns provided by adjoining beams and concrete slab subjected to fire should be investigated in future studies to better understand the structural behaviour of unbraced steel frames subjected to fire.

7.5 Conclusion

In this chapter, the stability of unbraced steel frames whose columns were protected from fire subjected to various fire scenarios was investigated. The degradation factor was introduced to characterize the material degradation at elevated temperature. Based on the concept of storey-buckling, the stability of unbraced steel frames subjected to various fire scenarios can be constructed as a pair of maximization and minimization problems with a stability constraint, and solved by a nonlinear programming method. A single-storey two-bay unbraced steel frame subjected to different fire scenarios was investigated by the proposed method. The results suggested that fire spreading from one room to another shortly after the fire was ignited in one room will do more harm to frame stability. It is also found from the results that effective fire compartment will increase the fire resistance of unbraced steel frames. Thus, the proposed method can be further developed to facilitate fire safety engineers to assess fire risk to and implement fire protection on unbraced steel frames.

Chapter 8

Conclusions and Recommendations

8.1 Summary

As one of the primary concerns in designing unbraced steel frames, stability for unbraced frames at ambient and elevated temperatures has been extensively studied. In current design practice, the buckling strength of unbraced frames is primarily determined by the stability capacity of single columns, particularly at elevated temperature. As such, the stiffness interaction among columns in the same storey is neglected, which may result in the inaccurate estimation of the frame buckling strength. The concept of storey-based buckling proposed by [Yura \(1971\)](#) was developed from the basis that an individual column cannot fail by lateral sway-buckling without all of the other columns in the same storey also buckling in the same sway mode. Based on this concept, [Xu et al. \(2001\)](#) proposed a more efficient method by introducing a lateral stiffness modification factor to characterize the relationship between the column lateral stiffness and applied axial load. In this study, the method proposed by [Xu et al. \(2001\)](#) will improve the buckling strength of unbraced steel frame subjected to variable loadings at ambient temperature, and extended to investigate the stability behavior of unbraced steel frame in fire. Chapter [2](#) briefly reviewed the existing literatures concerning the stability of unbraced steel frames

at ambient and elevated temperature. The investigations of stability of unbraced steel frames subjected to variable loadings are presented in Chapter 3 and 4. The stability behaviors of unbraced steel frames in fire are studied from Chapter 5 to Chapter 7.

8.1.1 Stability of Unbraced Steel Frames Subjected to Variable Loadings

In conventional frame stability analysis, the axial loads are assumed to be proportionally applied on the columns, which cannot guarantee that the load pattern is the worst load pattern during the life span of the structure. On the contrary, the variable-loading stability analysis proposed by Xu (2002), characterized by a pair of minimization and maximization problems, permits individual applied loads on the frame to vary independently so as to capture the load patterns that cause elastic instability of frames at minimum and maximum load levels. However, the parametric analysis performed in Chapter 3 pointed out that the linear programming problems based on the Taylor series approximation of the column lateral stiffness factor β may overestimate the frame buckling strength due to the overestimation of the column lateral stiffness. To avoid the possibility that the overestimated frame buckling strength may result in unconservative design, a secant approximation of the column lateral stiffness modification factor β was proposed in Chapter 3 to yield a conservative estimation of column stiffness and maintain the simplicity of linear programming method. The stabilities of four unbraced steel frames subjected to variable loadings were investigated in Chapter 3 to demonstrate that the results generated by the secant approximation are conservative. However, the example also reveals the limitation of the linear programming problems related to the linear approximation, which were addressed by the alternative method proposed in Chapter 4.

On one hand, the application of evaluating variable-loading stability problems as a linear programming problem is affected by the limitation revealed in Chapter 3. On the other hand, the stability evaluation method proposed by Xu et al. (2001) based on

the Euler-Bernoulli beam theory, which includes transcendental functions, may result in difficulties of applying nonlinear programming algorithm to solve the variable-loading stability problems. Therefore, an alternative method was proposed in Chapter 4 to access the lateral stiffness of an axially loaded column by modelling the column with two cubic Hermite elements. Unlike the Euler-Bernoulli beam theory method, the proposed equation only contains polynomials, which enable the variable-loading stability problems to be solved by a more efficient gradient-based nonlinear programming algorithm. The accuracy of the proposed method on the column rotational non-sway and lateral sway buckling strengths were compared with that of Euler-Bernoulli beam theory by investigation for columns with different end rotational restraints and bracing conditions. The example of four unbraced steel frames investigated in Chapter 3 was also employed in this chapter to demonstrate the efficiency of the proposed method.

8.1.2 Stability of Unbraced Steel Frames in Fire

Compared with the stability of unbraced steel frames at ambient temperature, the stability of unbraced steel frames in fire is different, since the buckling strength of steel frames at elevated temperature is affected by the degradation of the steel mechanical properties, thermal restraint, creep strain, and other factors. In recent years, the stability of steel frames at elevated temperature has been extensively studied through both experimental investigations and theoretical developments. However, many studies only focused on isolated columns or braced frames. The stability of unbraced steel frame at elevated temperature was primarily investigated by numerical methods which require significant computational effort. Thus, there is an urgent need to develop practical methods to evaluate the stability behavior of unbraced steel frames subjected to elevated temperature. In Chapter 5, the storey-buckling method proposed by Xu et al. (2001) for unbraced steel frames at ambient temperature was extended herein to evaluate the frame stability at elevated temperature. An analytical model was proposed in this chapter to obtain the

column internal force and the column lateral stiffness with accounting for the effects of thermal axial and rotational restraints. Upon the introduction of the proposed model, the evaluation procedure for frame buckling strength was then presented based on the concept of storey-based buckling. At the end of Chapter 5, the stability behaviours of a one-bay frame with four fire scenarios and a two-bay frame with one fire scenario were investigated with the proposed method, and the results are verified by the finite element analysis. The investigations of the examples show that the evaluation of stability of unbraced steel frame at elevated temperature solely based on the behaviour of individual columns may considerably underestimate the frame buckling strength. From that respect, the proposed method accounts for lateral stiffness interaction among columns is more appropriate to evaluate the frame buckling strength in fire.

Although the stability of unbraced steel frames at elevated temperature has been studied in Chapter 5, the proposed method is based on the assumptions that steel frames are subjected to uniform elevated temperature, which is primarily used in one-zone fire and suitable for post-flashover fire. In real fire, hot air and smoke accumulate at the upper layer and beneath the ceiling, while the cooler air stay at the lower layer of the compartment; thus, the temperature along the longitudinal direction in compartment fire is non-uniform. To reflect the actual stability behavior of unbraced steel frames in real fire, a more accurate two-zone fire model was used in Chapter 6, and an analytical column model corresponding to the two-zone fire was proposed. The lateral stiffness of the column model was derived based on the Euler-Bernoulli beam theory, and the procedure to evaluate the frame buckling strength subjected to a two-zone fire based on the concept of storey-based buckling was presented in Chapter 6. The behaviour and buckling strengths of a one-bay unbraced steel frame with two fire scenarios and a two-bay unbraced steel frame with one fire scenarios are investigated with the proposed method, and the results are verified by finite element methods. The study in this chapter showed that evaluating the frame buckling strength subjected to non-uniform fire but evaluated with the use

of the method of uniform fire would be conservative. Therefore, the method proposed in this chapter will be of great help to structural engineer to find economical design for unbraced steel frame subjected to non-uniform elevated temperature distribution without compromising the structural safety.

In Chapter 7, initial attempts were made to study the stability behavior of unbraced steel frames with fire-protected columns subjected to different fire scenarios. The frame stability problem at different fire scenarios was characterized as a pair of maximization and minimization problem with stability constraint in this chapter, and solved by a non-linear programming algorithm. The investigation for an unbraced steel frames suggested that effective fire compartment was of significant importance for unbraced steel frame to resist lateral buckling in fire.

8.2 Future Research

To improve the robustness of the proposed methods for the stability analysis of unbraced steel frames at ambient and elevated temperature, the following aspects of research concerning the frame stability are required to further investigated:

- The current study of the stability of unbraced steel frames at ambient and elevated temperatures was based on the assumption that the steel behaves elastically. For engineering design practice, the inelastic stability behavior of steel frames must be taken into account.
- The calculation of end-fixity factors in this study was based on the assumption that unbraced frames will buckle in an asymmetric buckling mode, which may result in inaccurate assessments of end-fixity factors. By incorporating the improved method proposed in Appendix E, a more accurate evaluation of the end-fixity factors and frame buckling strengths can be developed in the future research.

- The proposed methods in this study are only applicable for single-storey frames, and initial geometric imperfections of columns were not considered. Since the storey-based method for assessing the stability of unbraced steel frames at ambient temperature has been successfully applied on multi-storey unbraced frames ([Liu and Xu, 2005](#)), and has also accounted for the initial geometric imperfection of columns (Appendix D), the proposed methods need to be expanded to include multi-storey unbraced steel frames with initial imperfections at ambient and elevated temperatures.
- The current study only focused on the loading type that external loads are axially applied on columns. The effects of more complex types of loadings, such as distributed loads or concentrated loads on the beams, to the stability of unbraced steel frames should be investigated in future research.
- In this study, the connection stiffness was assumed to remain constant at elevated temperature. However, the moment-rotation relationship of a connection is highly nonlinear at elevated temperature, and it is not only related to elevated temperature but is also affected by beam axial force. To accurately predict the frame buckling strength, the computational model of the connection should be carefully selected and incorporated in the proposed methods in future study.
- The lateral stiffness interaction between columns in a storey relies on the axial strength of the adjoining beams. However, steel beams will gradually lose its strength in fire, and the transmission of lateral support from strong columns to weak columns will be in question. In addition, the presence of concrete slab may contribute to transfer the required lateral support. Thus, the interaction between columns provided by adjoining beams and concrete slab subjected to fire should be investigated in the future research.

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Appendix A

The Manual Approach for The Linear Programming Problems

The manual calculation approach for solving the linear programming problems stated in Eq. (3.10) and Eq. (3.11) was proposed by Xu (2002). The manual procedure for evaluating the upper and lower bounds of buckling load subject to the variable loading is described as follows:

1. Rewrite Eq. (3.9) as

$$\sum_{i=1}^n \beta_{1i} \frac{P_i}{L_i} = \sum_{i=1}^n \frac{12EI_i}{L_i^3} \beta_{0i} \quad (\text{A.1})$$

2. Evaluate the end-fixity factors r_l and r_u of each column from Eq. (3.1).
3. Calculate the coefficients β_{0i} and β_{1i} of each column from Eq. (3.8a) and Eq. (3.8b), respectively.
4. Formulate the laterally buckling constraint Eq. (A.1) for the frame, in which $P_i (i = 1, 2, \dots, n)$ are the applied loads to be decided. Note that the right side of the equation is a constant.

5. To obtain the maximum buckling load of the frame, first set all applied loads $P_i (i = 1, 2, \dots, n)$ to their associated lower bounds P_{1i} , then assign the largest permissible load $P_{ui} (P_{ui} \leq \pi^2 EI_i / L_i^2)$ to the column with the least value of β_{1i} / L_i .
6. If the value of left-hand side of Eq. (A.1) is greater than that of the right-hand side of the equation, then reduce the magnitude of P_i so that the equation can be satisfied. The maximum buckling load of the frame is obtained by summing the load on each column and the process is terminated; otherwise go to Step 7.
7. Assign the largest permissible load $P_{ui} (P_{ui} \leq \pi^2 EI_i / L_i^2)$ to the next column with the least value of β_{1i} / L_i among columns whose permissible loads have not been assigned, and go back to Step 6.

To obtain the minimum buckling load of the frames, follow Step 1 to 7 except assigning the load to the column with the largest value of β_{1i} / L_i instead of the least ones.

Appendix B

Lateral Stiffness of Axially Loaded Column Modelled with Cubic Hermite Elements

In Section 4.4, the lateral stiffness of an axially loaded column based on two element model is discussed. The details to evaluate the lateral stiffness modification factor given in Eq. (4.25) are presented in this Appendix.

The force-displacement relationship in Eq. (4.24) can be expressed as

$$\left\{ \begin{matrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \\ W_6 \end{matrix} \right\} = \left\{ \begin{matrix} EI \\ -P \end{matrix} \right\} \left[\begin{array}{cccccc} \frac{12}{l^3} & -\frac{6}{l^2} & -\frac{12}{l^3} & -\frac{6}{l^2} & 0 & 0 \\ & \frac{4}{l} & \frac{6}{l^2} & \frac{2}{l} & 0 & 0 \\ & & \frac{24}{l^3} & 0 & -\frac{12}{l^3} & -\frac{6}{l^2} \\ & & & \frac{8}{l} & \frac{6}{l^2} & \frac{2}{l} \\ Sym & & & & \frac{12}{l^3} & \frac{6}{l^2} \\ & & & & & \frac{4}{l} \end{array} \right] \left[\begin{array}{cccccc} \frac{6}{5l} & -\frac{1}{10} & -\frac{6}{5l} & -\frac{1}{10} & 0 & 0 \\ & \frac{2l}{15} & \frac{1}{10} & -\frac{l}{30} & 0 & 0 \\ & & \frac{12}{5l} & 0 & -\frac{6}{5l} & -\frac{1}{10} \\ & & & \frac{4l}{15} & \frac{1}{10} & -\frac{l}{30} \\ Sym & & & & \frac{6}{5l} & \frac{1}{10} \\ & & & & & \frac{2l}{15} \end{array} \right] \left\{ \begin{matrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{matrix} \right\} \quad (B.1)$$

Substituting $W = [-S \ M_l \ 0 \ 0 \ S \ M_u]^T$ and $\Delta = [0 \ \theta_l \ \theta_3 \ \theta_4 \ 1 \ \theta_u]^T$, Eqs. (4.14) into

Eq. (B.1) and reducing the 6×6 matrix to a 5×5 matrix, Eq. (B.1) can be rewritten as

$$\begin{pmatrix} -R_l\theta_l \\ 0 \\ 0 \\ S \\ R_u\theta_u \end{pmatrix} = EI \begin{pmatrix} \frac{4}{l} & \frac{6}{l^2} & \frac{2}{l} & 0 & 0 \\ & \frac{24}{l^3} & 0 & -\frac{12}{l^3} & -\frac{6}{l^2} \\ & & \frac{8}{l} & \frac{6}{l^2} & \frac{2}{l} \\ Sym & & & \frac{12}{l^3} & \frac{6}{l^2} \\ & & & & \frac{4}{l} \end{pmatrix} - P \begin{pmatrix} \frac{2l}{15} & \frac{1}{10} & -\frac{l}{30} & 0 & 0 \\ & \frac{12}{5l} & 0 & -\frac{6}{5l} & -\frac{1}{10} \\ & & \frac{4l}{15} & \frac{1}{10} & -\frac{l}{30} \\ Sym & & & \frac{6}{5l} & \frac{1}{10} \\ & & & & \frac{2l}{15} \end{pmatrix} \begin{pmatrix} \theta_1 \\ \Delta_3 \\ \Delta_4 \\ 1 \\ \theta_u \end{pmatrix} \quad (B.2)$$

Then five equations can be obtained:

$$-\frac{R_l\theta_l}{l_1} = \frac{EI}{l_1^3}(-6\Delta_3 + 2\Delta_4l_1 + 4\theta_l l_1) - \frac{P}{l_1} \left(-\frac{\Delta_3}{10} - \frac{\Delta_4l_1}{30} + \frac{2\theta_l l_1}{15} \right) \quad (B.3)$$

$$0 = \frac{EI}{l_1^3}(-12 + 6\theta_u l_1 + 24\Delta_3 - 6\theta_l l_1) - \frac{P}{l_1} \left(-\frac{6}{5} + \frac{\theta_u l_1}{10} + \frac{12\Delta_3}{5} - \frac{\theta_l l_1}{10} \right) \quad (B.4)$$

$$0 = \frac{EI}{l_1^3}(-6 + 2\theta_u l_1 + 8\Delta_4l_1 + 2\theta_l l_1) - \frac{P}{l_1} \left(-\frac{1}{10} - \frac{\theta_u l_1}{30} + \frac{4\Delta_4l_1}{15} - \frac{\theta_l l_1}{30} \right) \quad (B.5)$$

$$S = \frac{EI}{l_1^3}(12 - 6\theta_u l_1 - 12\Delta_3 - 6\Delta_4l_1) - \frac{P}{l_1} \left(-\frac{6}{5} - \frac{\theta_u l_1}{10} - \frac{6\Delta_3}{5} - \frac{\Delta_4l_1}{10} \right) \quad (B.6)$$

$$-\frac{R_u\theta_u}{l_1} = \frac{EI}{l_1^3}(-6 + 4\Delta_u l_1 + 6\Delta_3 + 2\Delta_4l_1) - \frac{P}{l_1} \left(-\frac{1}{10} + \frac{2\theta_u l_1}{15} + \frac{\Delta_3}{10} - \frac{\Delta_4l_1}{30} \right) \quad (B.7)$$

Simplifying Eq. (B.3)~(B.7), and assume $k = P/EI$:

$$\left(\frac{2}{15}kl_1^2 - 4 - \frac{R_l l_1}{EI} \right) \theta_l = \frac{kl_1^2 - 60}{10l_1} \Delta_3 + \frac{60 + kl_1^2}{30} \Delta_4 \quad (B.8)$$

$$\Delta_3 = \frac{1}{2} + \frac{(60 - kl_1^2)l_1}{24(kl_1^2 - 30)}(\theta_u + \theta_l) \quad (B.9)$$

$$\Delta_4 = \frac{3(kl_1^2 - 60)}{8l_1(kl_1^2 - 30)} + \frac{(60 + kl_1^2)l_1}{8(kl_1^2 - 30)}(\theta_u + \theta_l) \quad (B.10)$$

$$S = \frac{12EI}{l_1^3} \left(\frac{10 - kl_1^2}{10} + \frac{kl_1^2 - 60}{120}\theta_u l_1 + \frac{kl_1^2 - 10}{10}\Delta_3 + \frac{kl_1^2 - 60}{120}\Delta_4 l_1 \right) \quad (B.11)$$

$$\left(\frac{2}{15}kl_1^2 - 4 - \frac{R_u l_1}{EI} \right) \theta_u = \frac{kl_1^2 - 60}{10l_1} + \frac{60 - kl_1^2}{10l_1}\Delta_3 + \frac{60 + kl_1^2}{30}\Delta_4 \quad (B.12)$$

Substituting Eqs. (B.9) and (B.10) into Eqs. (B.8) and (B.12), the equations of θ_u and θ_l in terms of k and R_u, R_l can be found:

$$\theta_u = \frac{(4DB + BC) \left(E_l - \frac{B^2}{120A} \right)}{l \left[\left(E_l - \frac{B^2}{120A} \right) \left(E_u - \frac{C^2}{120D} \right) + \left(E_u - \frac{B^2}{120A} \right) \left(E_l - \frac{C^2}{120D} \right) \right]} \quad (\text{B.13})$$

$$\theta_l = \frac{(4DB + BC) \left(E_u - \frac{B^2}{120A} \right)}{l \left[\left(E_l - \frac{B^2}{120A} \right) \left(E_u - \frac{C^2}{120D} \right) + \left(E_u - \frac{B^2}{120A} \right) \left(E_l - \frac{C^2}{120D} \right) \right]} \quad (\text{B.14})$$

where

$$A = kl_1^2 - 10 \quad (\text{B.15a})$$

$$B = kl_1^2 - 60 \quad (\text{B.15b})$$

$$C = kl_1^2 + 60 \quad (\text{B.15c})$$

$$D = kl_1^2 - 30 \quad (\text{B.15d})$$

$$E_u = \frac{2}{15}kl_1^2 - 4 - \frac{R_u l_1}{EI} \quad (\text{B.15e})$$

$$E_l = \frac{2}{15}kl_1^2 - 4 - \frac{R_l l_1}{EI} \quad (\text{B.15f})$$

$$\frac{R_u l_1}{EI} = \frac{3r_u}{2(1 - r_u)} \quad (\text{B.15g})$$

$$\frac{R_l l_1}{EI} = \frac{3r_l}{2(1 - r_l)} \quad (\text{B.15h})$$

Substituting Eqs. (B.13), (B.14) and (B.9), (B.10) into Eq. (B.11), the equation of β with respect to r_l and r_u can be obtained:

$$\beta = -\frac{1}{12} \frac{a_0 + a_1\phi^2 + a_2\phi^4 + a_3\phi^6 + a_4\phi^8 + a_5\phi^{10}}{b_0 + b_1\phi^2 + b_2\phi^4 + b_3\phi^6 + b_4\phi^8} \quad (\text{B.16})$$

where

$$a_0 = -132710400(r_i + r_j + r_i r_j) \quad (\text{B.17a})$$

$$a_1 = 44236800 + 17694720(r_i + r_j) - 26542080r_i r_j \quad (\text{B.17b})$$

$$a_2 = 2474496(r_i + r_j) - 440064r_i r_j - 5895240 \quad (\text{B.17c})$$

$$a_3 = 158208 - 11218(r_i + r_j) + 732824r_ir_j \quad (\text{B.17d})$$

$$a_4 = 1127(r_i + r_j) - 947r_ir_j - 1280 \quad (\text{B.17e})$$

$$a_5 = 1 - (r_i + r_j) + r_ir_j \quad (\text{B.17f})$$

$$b_0 = 44236800 - 1105920r_ir_j \quad (\text{B.17g})$$

$$b_1 = 2949120(r_i + r_j) - 1105920r_ir_j - 5898240 \quad (\text{B.17h})$$

$$b_2 = 158208 - 115968(r_i + r_j) + 80640r_ir_j \quad (\text{B.17i})$$

$$b_3 = 1136(r_i + r_j) - 992r_ir_j - 1280 \quad (\text{B.17j})$$

$$b_4 = 3(1 + r_ir_j - r_i - r_j) \quad (\text{B.17k})$$

Appendix C

Lateral Stiffness of Axially Loaded Columns subjected to Non-uniform Fire

For the axially loaded column subjected to non-uniform elevated temperature as shown in Figure 6.4, the derivation process of the lateral stiffness modification factors shown in Eq. (6.20) are as follow.

The general solution of Eqs. (6.14) can be obtained as

$$y_l = C_1 \cos \frac{\phi_l x}{L} + C_2 \sin \frac{\phi_l x}{L} - \frac{Sx}{P} + \frac{M_l}{P} \quad (\text{C.1})$$

$$y_u = C_3 \cos \frac{\phi_u x}{L} + C_4 \sin \frac{\phi_u x}{L} - \frac{Sx}{P} + \frac{M_l}{P} \quad (\text{C.2})$$

where ϕ_u and ϕ_l are the applied load ratio accounting for temperature T_u and T_l , and defined in Eq. (6.19b) and Eq. (6.19c), respectively. C_1 , C_2 , C_3 and C_4 are coefficients to be determined by the boundary conditions given in Eqs. (6.17).

Applying the boundary conditions given in Eqs. (6.17) to Eq. (C.1) and Eq. (C.2)

and combine with Eq. (6.15), the following equations can be obtained:

$$C_1 + \frac{M_l}{P} = 0 \quad (\text{C.3})$$

$$\frac{C_2 \phi_l}{L} - \frac{S}{P} = \theta_l \quad (\text{C.4})$$

$$C_3 \cos \phi_u + C_4 \sin \phi_u - \frac{M_u}{P} = 0 \quad (\text{C.5})$$

$$- \frac{C_3 \phi_u}{L} \sin \phi_u + \frac{C_4 \phi_u}{L} \cos \phi_u - \frac{S}{P} = \phi_u \quad (\text{C.6})$$

$$C_1 \cos \frac{\phi_l L_l}{L} + C_2 \sin \frac{\phi_l L_l}{L} = C_3 \cos \frac{\phi_u L_u}{L} + C_4 \sin \frac{\phi_u L_u}{L} \quad (\text{C.7})$$

$$- \frac{C_1 \phi_l}{L} \sin \frac{\phi_l L_l}{L} + \frac{C_2 \phi_l}{L} \cos \frac{\phi_l L_l}{L} = \frac{C_3 \phi_u}{L} \sin \frac{\phi_u L_u}{L} + \frac{C_4 \phi_u}{L} \cos \frac{\phi_u L_u}{L} \quad (\text{C.8})$$

Solving Eq. (C.7) and Eq. (C.8), C_3 and C_4 can be expressed by C_1 and C_2 as

$$C_3 = \frac{C_1 \phi_l \sin \sigma_l \sin \sigma_u - C_2 \phi_l \cos \sigma_l \sin \sigma_u + C_1 \phi_u \cos \sigma_l \cos \sigma_u + C_2 \phi_u \sin \sigma_l \cos \sigma_u}{\phi_u} \quad (\text{C.9})$$

$$C_4 = \frac{C_1 \phi_l \sin \sigma_l \cos \sigma_u + C_2 \phi_l \cos \sigma_l \cos \sigma_u + C_1 \phi_u \cos \sigma_l \sin \sigma_u + C_2 \phi_u \sin \sigma_l \sin \sigma_u}{\phi_u} \quad (\text{C.10})$$

in which $\sigma_l = \phi_l L_l / L$, and $\sigma_u = \phi_u L_u / L$. Let

$$C = \frac{SL}{P} \quad (\text{C.11})$$

$$C_u = \frac{PL}{R_u} = \frac{\tau_u(1 - r_u)}{3r_u} \phi^2 \quad (\text{C.12})$$

$$C_l = \frac{PL}{R_l} = \frac{\tau_l(1 - r_l)}{3r_l} \phi^2 \quad (\text{C.13})$$

Substituting Eqs. (C.9~C.13) and Eq. (6.16) to Eq. (C.3~C.6) and Eq. (6.15), the following equations can be obtained

$$C_1 C_l = -\theta_l L \quad (\text{C.14})$$

$$C_2 \phi_l = C + \theta_l L \quad (\text{C.15})$$

$$C_3 \phi_u \sin \phi_u + C_4 \phi_u \cos \phi_u - C = \theta_u L \quad (\text{C.16})$$

$$-C_3\phi_u \sin \phi_u + C_4\phi_u \cos \phi_u - C = \theta_u L \quad (\text{C.17})$$

$$1 + C = \frac{\theta_u L}{C_u} + \frac{\theta_l L}{C_l} \quad (\text{C.18})$$

C_2 can be expressed with respect to C_1 by solving Eq. (C.14) and Eq. (C.15).

$$C_2 = \frac{C - C_1 C_l}{\phi_l} \quad (\text{C.19})$$

Substituting Eq. (C.14), Eq. (C.15) and Eq. (C.19) into Eq. (C.18), C_1 can be solved as

$$C_1 = \frac{\phi_l \phi_u (1 + C) - C(\phi_l \cos \alpha_l \sin \alpha_u)}{\phi_l \phi_u (\cos \alpha_l \cos \alpha_u - 1) - \phi_l^2 \sin \alpha_l \sin \alpha_u - C_l (\phi_l \cos \alpha_l \sin \alpha_u + \phi_2 \sin \alpha_l \cos \alpha_u)} \quad (\text{C.20})$$

where α_l and α_u are defined in Eq. (6.21a) and Eq. (6.21b), respectively. Substituting Eq. (C.9), Eq. (C.9), and Eq. (C.19) into Eq. (C.16) and Eq. (C.17), and solving this two equations, another form of C_1 can be obtained

$$C_1 = \frac{C(-\phi_2^2 \sin \alpha_l \sin \alpha_u - \phi_l \phi_u - C_u \phi_l \cos \alpha_l \sin \alpha_u - C_u \phi_2 \sin \alpha_l \cos \alpha_u + \phi_l \phi_u \cos \alpha_l \cos \alpha_u)}{(C_l + C_u) \phi_l \phi_u \cos \alpha_l \cos \alpha_u - C_l C_u (\phi_l \cos \alpha_l \sin \alpha_u + \phi_u \sin \alpha_l \cos \alpha_u) - (C_u \phi_l^2 + C_l \phi_u^2) \sin \alpha_l \sin \alpha_u + \phi_l \phi_u (\phi_l \sin \alpha_l \cos \alpha_u + \phi_u \cos \alpha_l \sin \alpha_u)} \quad (\text{C.21})$$

Solving Eq. (C.20) and Eq. (C.20) by substituting Eq. (C.12) and Eq. (C.12) into these two equations, C can be determined as

$$C = \frac{A_1 \phi_l \phi_u \cos \alpha_l \cos \alpha_u + A_{21} \phi_l \cos \alpha_l \sin \alpha_u + A_{22} \phi_u \sin \alpha_l \cos \alpha_u + A_3 \sin \alpha_l \sin \alpha_u}{18 \phi_l \phi_u r_l r_u + \phi_l \cos \alpha_l \sin \alpha_u (A_1 - A_{21}) + \phi_u \sin \alpha_l \cos \alpha_u (A_1 - A_{22}) - \phi_l \phi_u (18 r_l r_u + A_1) + \sin \alpha_l \sin \alpha_u A_4} \quad (\text{C.22})$$

where A_1 , A_{21} , A_{22} , A_3 , and A_4 are defined in Eqs. (6.21c~6.21g), respectively. Knowing the coefficient C , the lateral stiffness of the column accounting for non-uniform elevated temperature can be obtained from Eq. (C.11) as

$$S = \beta(\phi, r_l, r_u, \tau_l, \tau_u) \frac{12EI}{L^3} \quad (\text{C.23})$$

where coefficient $\beta(\phi, r_l, r_u, \tau_l, \tau_u)$ is the modification factor of the column lateral stiffness with consideration of non-uniform temperature and can be expressed as

$$\beta(\phi, r_l, r_u, \tau_l, \tau_u) = C \frac{\phi^2}{12} \quad (\text{C.24})$$

Appendix D

Lateral Stiffness Derivation of Axially Loaded Columns with Initial Geometric Imperfections

For the axially loaded column with initial geometric imperfection as shown in Figure D.1, the derivation process of the lateral stiffness modification factors are discussed as follow.

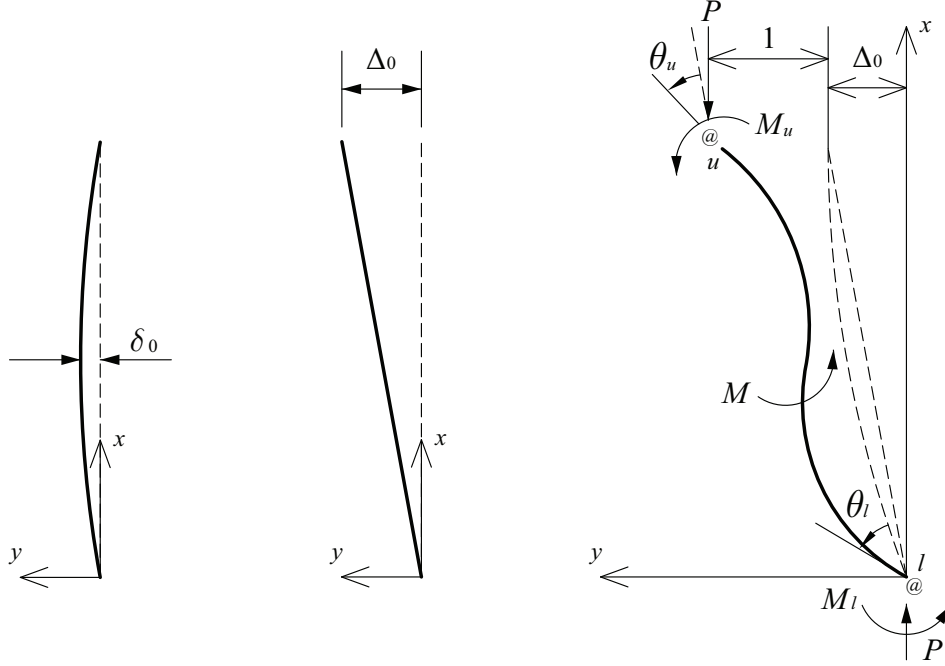
As shown in Figure D.1, the imperfection function related to out-of-straightness of column is

$$y_1 = \delta_0 \sin \frac{\pi x}{L} \quad (\text{D.1})$$

where δ_0 is the initial out-of-straightness at the middle of the column. The imperfection associated with out-of-plumbness of column is

$$y_2 = \frac{x\Delta_0}{L} \quad (\text{D.2})$$

in which Δ_0 is the initial out-of-plumbness at the upper joint of the column shown in Figure D.1. With a lateral deflection Δ at the upper end as shown in Figure D.1, the internal moment of the column with both column out-of-straightness and frame out-of-



(a) Out-of-straightness (b) Out-of-plumbness (c) Rotationally restrained sway column

Figure D.1: An axially loaded column with initial geometric imperfections

plumbness can be expressed as

$$M = -EIy'' = P(y + y_1 + y_2) + Sx - M_l \quad (\text{D.3})$$

where M is the internal moment of the column; M_l is the end moment at the column upper end; P is the applied axial load; L is the column length; S is the lateral force applied at the column upper end, which is related to the lateral deflection Δ ; y is the lateral deflection of the column. The end moment at the upper end can be obtained from Eq. (D.3)

$$M_u = P(\Delta + \Delta_0) + SL - M_l \quad (\text{D.4})$$

Substituting Eqs. (D.1) and (D.2) into Eq. (D.3), and solving the differential equation Eq. (D.3), the lateral deflection y can be obtained

$$y = C_1 \cos \frac{\phi x}{L} + C_2 \sin \frac{\phi x}{L} - \left(\frac{\phi^2 \delta_0 \sin(\frac{\pi x}{L})}{\phi^2 - \pi^2} + \frac{\Delta_0 x}{L} \right) - \frac{Sx}{P} + \frac{M_l}{P} \quad (\text{D.5})$$

where ϕ is the applied load ratio and defined in Eq. (3.3). C_1 and C_2 are unknown factors.

Let θ_l and θ_u be the end rotations of the column at the lower and upper ends, respectively, the boundary conditions of the column are described as follow:

$$y|_{x=0} = 0 \quad (\text{D.6a})$$

$$y|_{x=L} = \Delta \quad (\text{D.6b})$$

$$y'|_{x=0} = \theta_l \quad (\text{D.6c})$$

$$y'|_{x=L} = \theta_u \quad (\text{D.6d})$$

Applying the boundary conditions described in Eqs. (D.6) into Eq. (D.5), the following equations can be obtained

$$C_1 + \frac{M_l}{P} = 0 \quad (\text{D.7})$$

$$C_1 \cos \phi + C_2 \sin \phi - \frac{M_u}{P} = 0 \quad (\text{D.8})$$

$$\frac{C_2 \phi}{L} - \left(\frac{\pi \phi^2 \delta_0}{\phi^2 - \pi^2} \right) - \frac{\Delta_0}{L} - \frac{S}{P} = \theta_l \quad (\text{D.9})$$

$$- \frac{C_1 \phi}{L} \sin \phi + \frac{C_2 \phi}{L} \cos \phi - \left(\frac{\pi \phi^2 \delta_0}{\phi^2 - \pi^2} \right) - \frac{\Delta_0}{L} - \frac{S}{P} = \theta_u \quad (\text{D.10})$$

As shown in the figure, the end moment can be expressed as

$$M_l = R_l \theta_l \quad (\text{D.11a})$$

$$M_u = R_u \theta_u \quad (\text{D.11b})$$

where R_l and R_u are the rotational restraining stiffness at the lower and upper ends of column, respectively. By rewriting Eq. (3.1), R_l and R_u can be expressed as functions with respect to end-fixity factors r_l and r_u as

$$R_l = \frac{3EI}{L} \frac{r_l}{(1 - r_l)} \quad (\text{D.12a})$$

$$R_u = \frac{3EI}{L} \frac{r_u}{(1 - r_u)} \quad (\text{D.12b})$$

To related the end-fixity factors r with the applied load ratio ϕ , two factors C_u and C_l are defined as

$$C_l = \frac{PL}{R_l} = \frac{1 - r_l}{3r_l} \phi^2 \quad (\text{D.13a})$$

$$C_u = \frac{PL}{R_u} = \frac{1 - r_u}{3r_u} \phi^2 \quad (\text{D.13b})$$

Substituting Eqs. (D.11~D.13) into Eqs. (D.7~D.10) and Eq. (D.4), the following equations can be obtained:

$$C_1 C_l = -\theta_l L \quad (\text{D.14})$$

$$C_1 \cos \phi + C_2 \sin \phi = \frac{\theta_u L}{C_u} \quad (\text{D.15})$$

$$C_2 \phi = \frac{\pi \phi^2 \delta_0}{\phi^2 - \pi^2} + \Delta_0 + C + \theta_l L \quad (\text{D.16})$$

$$-C_1 \phi \sin \phi + C_2 \phi \cos \phi = \frac{\pi \phi^2 \delta_0}{\phi^2 - \pi^2} + \Delta_0 + C + \theta_u L \quad (\text{D.17})$$

$$\Delta + \Delta_0 + C = \frac{R_u \theta_u}{P} + \frac{R_l \theta_l}{P} = \frac{\theta_u L}{C_u} + \frac{\theta_l L}{C_l} \quad (\text{D.18})$$

C_2 can be expressed with respect to C_1 by solving Eq. (D.14) and Eq. (D.15)

$$C_2 = \frac{\frac{\pi \phi^2 \delta_0}{\phi^2 - \pi^2} + \Delta_0 + C - C_1 C_l}{\phi} \quad (\text{D.19})$$

Substituting Eq. (D.14), (D.15) and (D.19) into Eq. (D.18), the factors C_1 can be expressed in terms of C as

$$C_1 = \frac{\phi(C + \Delta + \Delta_0) - \left(\frac{\pi \phi^2 \delta_0}{\phi^2 - \pi^2} + C + \Delta_0 \right) \sin \phi}{\phi(\cos \phi - 1) - C_l \sin \phi} \quad (\text{D.20})$$

C_1 can also be obtained by solving Eq. (D.15), Eq. (D.17) and Eq. (D.19) as

$$C_1 = \frac{\left(\frac{\pi \phi^2 \delta_0 + C + \Delta_0}{\phi^2 - \pi^2} + C + \Delta_0 \right) (\phi \cos \phi - \phi - C_u \sin \phi)}{\phi^2 \sin \phi + (C_l + C_u) \phi \cos \phi - C_l C_u \sin \phi} \quad (\text{D.21})$$

Thus, the lateral stiffness S/Δ can be determined by solving Eq. (D.20) and Eq. (D.21) as

$$\frac{S}{\Delta} = \frac{n}{\frac{m_1 L^3}{\phi^2 EI} + \frac{m_1 \Delta_0}{S} + \frac{m_2}{S} \left(\frac{\pi \phi^2 \delta_0}{\phi^2 - \pi^2} \right)} \quad (\text{D.22})$$

in which

$$n = \phi(a_1 \phi \cos \phi + a_2 \sin \phi) \quad (\text{D.23a})$$

$$m_1 = 18r_l r_u - a_3 \cos \phi + (a_1 - a_2)\phi \sin \phi \quad (\text{D.23b})$$

$$m_2 = a_1 \phi \sin \phi + 18r_l r_u (1 - \cos \phi) \quad (\text{D.23c})$$

$$a_1 = 3[r_l(1 - r_u) + r_u(1 - r_l)] \quad (\text{D.23d})$$

$$a_2 = 9r_l r_u - (1 - r_u)(1 - r_l)\phi^2 \quad (\text{D.23e})$$

$$a_3 = 18r_l r_u + 3[r_l(1 - r_u) + r_u(1 - r_l)]\phi^2 \quad (\text{D.23f})$$

Appendix E

Improved Method of Calculating End-fixity Factor of Columns

E.1 Revised End-fixity Factor

To calculate the end-fixity factor of a column accounting for the rotational restraints attributed to the attached beams, the rotational restraint stiffness of the attached beams must first be determined. For a semi-rigid member shown in Figure 3.1, the equation to calculate the rotational stiffness can be derived from the slope-deflection equation for such a member as

$$R_1 = \frac{6r_1}{4 - r_1r_2} \frac{EI_b}{L_b} (2 + vr_2) \quad (\text{E.1})$$

in which v is the ratio of θ_2 and θ_1 , and θ_1 and θ_2 are the joint rotations associated with the column to which the beam ends 1 and 2 are connected, respectively. EI_b/L_b is the flexural stiffness of the beam, and r_1 and r_2 are the end-fixity factors defined in Eq. (3.1), characterizing the rotational capability of the end-connection spring at the beam ends.

In current design practice, braced frames are assumed to buckle in a symmetric buckling mode with $v = -1$, while unbraced frames are assumed to buckle in an asymmetric

buckling mode with $v = 1$. However, using the storey-based buckling concept, unbraced frames might not buckle in asymmetric buckling mode due to the stiffness interaction between columns in the same storey. [Xu and Liu \(2002a\)](#), by investigating the variation of R as v changes from -1 to 1, concluded that the inaccuracy associated with the assumption of $v = 1$ for unbraced frames with semi-rigid members is insignificant. Therefore, $v = 1$ was adopted in their study and also in this thesis. The results were convincing if the value of v is limited between -1 and 1. Nevertheless, it was found that the value of v may be out of the range of -1 and 1 for some cases. Rearranging Eq. (E.2), the ratio v can be obtained as

$$v = \frac{4 - r_1 r_2}{6 r_1 r_2} \frac{R_1 L_b}{EI_b} - \frac{2}{r_2} \quad (\text{E.2})$$

For a fixed stiffness ratio of $R_1 L_b / EI_b$, say $R_1 L_b / EI_b = 3$, the relationship between v and r_1 when $r_2 = 1$ can be plotted in Figure E.1. It can be seen from the figure that v is greater than 1 when r_1 is less than 0.8. Moreover, the value of v is considerably increased when r_1 approaches 0. That being said, the assumption that v falls into the range of -1 and 1 is inaccurate for some cases, which will result in inaccurate evaluation of the rotational stiffness R of the beam and the end-fixity factor of the attaching columns. Therefore, it is preferable to evaluate v rather than assume $v = 1$ so as to accurately assess the value of R .

For the semi-rigid member shown in Figure 3.1, the moments at the two ends can be obtained from the slop-deflection equation as

$$M_1 = \frac{3r_1}{4 - r_1 r_2} \frac{EI_b}{L_b} (4\theta_1 + 2r_2\theta_2) \quad (\text{E.3a})$$

$$M_2 = \frac{3r_2}{4 - r_1 r_2} \frac{EI_b}{L_b} (4\theta_2 + 2r_1\theta_1) \quad (\text{E.3b})$$

where M_1 and M_2 are the restrained moments at end 1 and 2, respectively. Combining Eqs. (E.3), the ratio v can be determined as

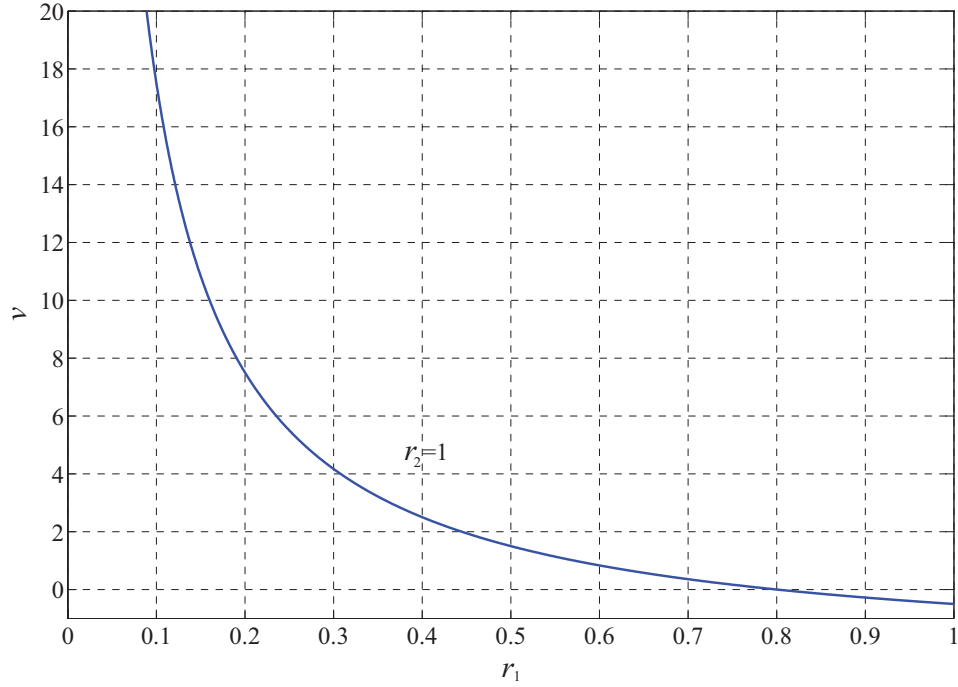


Figure E.1: Relationship between ratio v and end-fixity factor r_1

$$v = \frac{2 - \frac{M_1}{M_2} r_2}{2 \frac{M_1}{M_2} \frac{r_2}{r_1} - r_2} \quad (\text{E.4})$$

However, it is difficult to determine the end moments when the frame buckles. It is known that the buckled shape of unbraced frames can be assembled by the displaced configuration of the frame which is applied with a small disturbing lateral force. In addition, it can be found from Eq. (E.4) that the value of v can be determined once the ratio of M_1 and M_2 has been found. Thus, a small disturbing force, whose value is taken as 0.005 times the storey gravity load, is first applied on the frame as shown in Figure E.2. The end moments developed in the beam calculated using a first-order elastic analysis under the action of the lateral disturbing force are used to determine the moment ratio, and the value of v then can be determined.

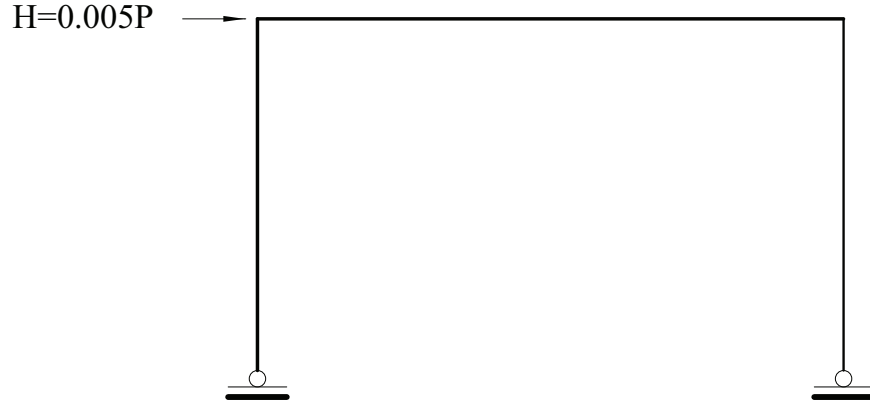


Figure E.2: A small disturbing force applied on a unbraced frame

E.2 Numerical Example

Shown in Figure E.3 is a single storey and single bay steel frame, in which the lower ends of two columns are pinned connections and two columns are rigidly connected to the beam. The sections of Column 1 and Beam 3 are W200X36 and W410X67, respectively. To investigate the effects of different buckle configurations to the end-fixity factors and the buckling strength, the sections of column 2 will vary from the following sections: W200X36, W250X45, W310X60, W410X67. P is the axial load applied on the columns, and it will be increased proportionally to the level in which the frame buckles. Young's Modulus of E is taken to be 200 000MPa.

Two methods, including $v = 1$ and the improved method proposed in this section, will be used to calculate the end-fixity factors of two columns, and the frame buckling strength will then be computed by the storey-based buckling method proposed by Xu and Liu (2002a). To verify the proposed method, the structural analysis software MASTAN2 (McGuire et al., 1999) is also used to investigate the frame stability. The results calculated with these three methods and the comparisons are listed in Table E.1.

It can be found from the table that the results obtained from the storey-based buckling method based on the assumption of $v = 1$ are acceptable when the sections of Column 2 are close to the section of Column 1. The differences for W200X36 and W250X45 between

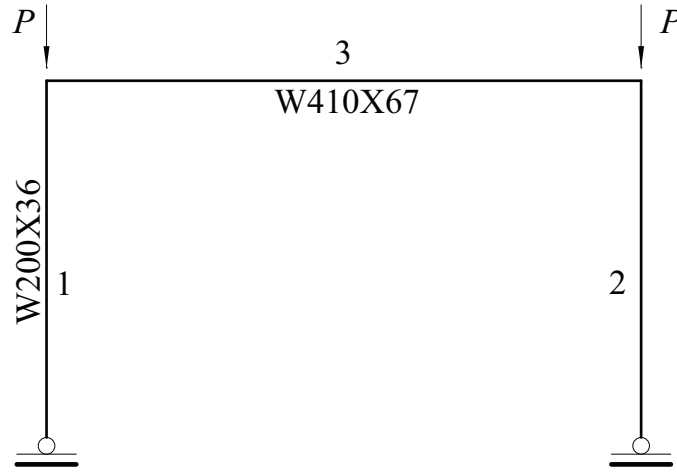


Figure E.3: One-bay and One-storey steel frame

Table E.1: Axial buckling load P

Column 2 sections	Axial buckling load P (kN)			Difference	
	MASTAN2	$v = 1$	improved v	$v = 1$	improved v
W410X67	1572.04	1884	1575	19.8%	0.2%
W310X60	1242.63	1349	1243	8.6%	0.03%
W250X45	943.31	964	946	2.2%	0.3%
W200X36	666.33	661	661	-0.8%	-0.8%

MASTAN2 and the method based on the assumption of $v = 1$ are only -0.8% and 2.2%. This is because the lateral stiffness of Column 1 and 2 are the same or close to each other, and little column stiffness interaction occurs between columns when the frame buckles. In such cases, the buckle configurations can be deemed as the asymmetrical configurations, and the assumption of $v = 1$ is therefore reasonable. However, the difference increases as the sections of Column 2 increase. Especially, the difference is increased to 19.8% when the section of Column 2 is W410X67. In that case, Column 2 becomes the strong column and provides lateral support to the relative weak column of Column 1; thus, the buckle configuration cannot be seen as asymmetrical configuration, and the assumption

of $v = 1$ will result in inaccurate evaluation of frame buckling strength. On the other hand, the difference between MASTAN 2 and the storey-based buckling method based on the improved results of v are always less than 1% no matter how the section of Column 2 changes. It can be concluded that the improved method of evaluating ratio v has overcome the defect of the simple assumption of $v = 1$ in the assessment of the frame buckling strength.