The r-coloring and maximum stable set problem in hypergraphs with bounded matching number and edge size

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Abstract

Motivated by the analogous questions in graphs, we study the complexity of coloring and stable set problems in hypergraphs with forbidden substructures and bounded edge size. Letting $\nu(G)$ denote the maximum size of a matching in H, we obtain complete dichotomies for the complexity of the following problems parametrized by fixed $r, k, s \in \mathbb{N}$:

- r-Coloring in hypergraphs G with edge size at most k and $\nu(G) \leq s$;
- r-Precoloring Extension in k-uniform hypergraphs G with $\nu(G) \leq s$;
- r-Precoloring Extension in hypergraphs G with edge size at most k and $\nu(G) \leq s$;
- Maximum Stable Set in k-uniform hypergraphs G with $\nu(G) \leq s$;
- Maximum Weight Stable Set in k-uniform hypergraphs with $\nu(G) \leq s$;

as well as partial results for r-Coloring in k-uniform hypergraphs $\nu(G) \leq s$. We then turn our attention to 2-Coloring in 3-uniform hypergraphs with forbidden induced subhypergraphs, and give a polynomial-time algorithm when restricting the input to hypergraphs excluding a fixed one-edge hypergraph. Finally, we consider linear 3-uniform hypergraphs (in which every two edges share at most one vertex), and show that excluding an induced matching in G implies that $\nu(G)$ is bounded by a constant; and that 3-coloring linear 3-uniform hypergraphs G with $\nu(G) \leq 532$ is NP-hard.

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1 Preliminaries

A *hypergraph* G is a pair (V, E) where V is a finite set, and $E \subseteq 2^V \setminus \{\emptyset\}$. V is called the set of *vertices* and E is called the set of *edges*. For a hypergraph $G = (V, E)$, we define $V(G) = V$ and $E(G) = E$. For $k \in \mathbb{N}$, we say that G is k*-uniform* if $|e| = k$ for all edges $e \in E$, and G is k*-bounded* if $|e| \leq k$ for all edges $e \in E$. A 2-uniform hypergraph is simply called a graph.

Given a set $X \subseteq V(G)$, $G[X] = (X, \{e \in E(G) : e \subseteq X\})$ is an *induced hypersubgraph* of G. A *matching* of G is a set of pairwise disjoint edges. A *maximal matching* of G is a matching which is maximal with respect to inclusion. For a hypergraph G, we denote $\nu(G)$ be the maximum integer s such that G contains a matching of size s. A set $S \subseteq V(G)$ of G is *stable* if $e \cap S \neq e$ for every $e \in E(G)$.

We use $[r]$ to denote the set $\{1, \dots, r\}$. Given a hypergraph G and a positive integer r, a function $c : V(G) \to [r]$ is an r*-coloring* of G if for all $i \in [r]$, $c^{-1}(i)$ is a stable set in G. G is r*-colorable* if there exists an r-coloring of G. The *chromatic number* of G, denoted $\chi(G)$, is the minimum integer r such that G is r-colorable.

A function $c : X \to [r]$ for some $X \subseteq V(G)$ is a *partial* r*-coloring* of G if c is an r-coloring of $G[X]$. For convenience, we also denote a partial coloring as (X, c). Given a partial r-coloring (X, c) of G, an r*-precoloring extension* of (X, c) is a partial r-coloring (X', c') with $c'(v) = c(v)$ for all $v \in X$, and $X \subset X'$. We say that a partial coloring (X, c) r*-extends* to G if there is an r-precoloring extension $(V(G), c')$ of (X, c).

For a fixed integer r, the Hypergraph r-Coloring Problem is to decide whether a given hypergraph G is r-colorable, and the Hypergraph r-Precoloring Extension Problem is to decide given a hypergraph G and a partial r-coloring (X, c), whether (X, c) r-extends to G.

The Graph r-Coloring Problem is a well-known NP-hard problem:

Theorem 1.1 (Karp [10])**.** *For every fixed integer r with $r \geq 3$, the* Graph r-Coloring Problem *is NP-complete.*

However, placing structural restrictions on the input graphs may make the problem easier. This is well-studied for graphs; see [8] for a survey of complexity results for coloring problems in graphs with forbidden induced subgraphs. Few graphs H have the property that r-coloring H-free graphs can be solved in polynomial time for all r. The following result shows that graphs of the form $H = sP_2$ (that is, matchings) have this property:

Theorem 1.2 (Dabrowski, Lozin, Raman, Ries [5] based on results of Balas and Yu [2] and Tsukiyama, Ide, Ariyoshi, Shirakawa [15]; explicitly stated in Golovach, Johnson, Paulusma and Song [8])**.** *For fixed positive integers r and s, the* Graph r-Coloring Problem *restricted to sP_2-free graphs is polynomial-time solvable.*

The hypergraph coloring problem is a natural extension of the graph coloring problem; see the survey [3]. The following result shows that the problem is NP-hard, even in uniform hypergraphs.

Theorem 1.3 (Phelps and Rödl [13]). *For all $k \geq 3$ and $r \geq 2$, the k-*Uniform Hypergraph *r-*Coloring Problem *is NP-complete.*

It is natural to ask which restrictions of the input hypergraphs make the Hypergraph r-Coloring Problem polynomial-time solvable. In this paper, we mainly focus on bounded or uniform hypergraphs. In view of Theorem 1.2, we are particularly interested in excluding a matching; but as it turns out, even bounding the maximum size of a matching (a much stronger condition than excluding a large matching as an induced sub(hyper)graph, as is the case in Theorem 1.2) does not always lead to a polynomial-time algorithm. We prove the following dichotomies:

Theorem 1.4. *Let k, r and s be positive integers with $k, r \geq 2$. The k-*Bounded Hypergraph *r-*Coloring Problem*, the k-*Bounded Hypergraph *r-*Precoloring Extension Problem *as well as the k-*Uniform Hypergraph *r-*Precoloring Extension Problem*, restricted to hypergraphs with $\nu(G) \leq s$, are polynomial-time solvable if*

- *$s \leq r - 1$, or*
- *$k = 3$ and $r = 2$, or*
- *$k = 2$,*

and NP-complete otherwise.

We also show the following result:

Theorem 1.5. *Let k, r and s be positive integers with $k, r \geq 2$. The k-*Uniform Hypergraph *r-*Coloring Problem *restricted to hypergraphs with $\nu(G) \leq s$ is polynomial-time solvable if*

- *$s \leq r - 1$, or*
- *$k = 3$ and $r = 2$, or*
- *$k = 2$,*

and is NP-complete if

- *$s \geq (r-1)k + 1$, and*
- *$k \geq 4$ or $r \geq 3$.*

Theorem 1.2 is based on a result of [2] that sP_2-free graphs have only polynomially many maximal (with respect to inclusion) stable sets. Using this, [2] gave a polynomial-time algorithm for finding a maximum (weight) stable set in an sP_2-free graph. We ask an analogous question in hypergraphs with bounded maximum matching size, and prove:

Theorem 1.6. *For fixed positive integers k and s with $k \geq 3$, the k-*UNIFORM HYPERGRAPH MAXIMUM STABLE SET PROBLEM *restricted to hypergraphs with $\nu(G) \leq s$ is polynomial-time solvable, and the k-*UNIFORM HYPERGRAPH MAXIMUM WEIGHT STABLE SET PROBLEM *restricted to hypergraphs with $\nu(G) \leq s$ is NP-complete.*

We also give a first result for excluding an induced subhypergraph:

Theorem 1.7. *Let $t \in \mathbb{N}$ be fixed, and let H be the 3-uniform hypergraph with $t+3$ vertices and one edge. Then there is a polynomial-time algorithm for the* 3-BOUNDED HYPERGRAPH 2-COLORING PROBLEM *restricted to H-free hypergraphs.*

Finally, we consider linear hypergraphs. A hypergraph G is *linear* if $|e \cap e'| \leq 1$ for every two distinct $e, e' \in E(G)$. The restriction to linear hypergraphs does not affect NP-hardness:

Theorem 1.8 (Phelps and Rödl [13]). *For every $r \geq 2$, the* 3-UNIFORM HYPERGRAPH r-COLORING PROBLEM *restricted to linear hypergraphs is NP-complete.*

The following result gives an algorithm for 2-coloring certain linear hypergraphs:

Theorem 1.9 (Chattopadhyay and Reed [4]). *There is a polynomial-time algorithm for the k-*UNIFORM HYPERGRAPH 2-COLORING PROBLEM *restricted to linear hypergraphs with maximum degree bounded by a function of k.*

We ask how our results extend to linear 3-uniform hypergraphs. For $s \in \mathbb{N}$, we let M_s denote the 3-uniform hypergraph with $3s$ vertices and s pairwise disjoint edges. We show that in linear hypergraphs, excluding a fixed induced matching implies bounded matching number, which immediately implies (assuming Theorems 1.4 and 1.6):

Theorem 1.10. *Let $s \in \mathbb{N}$. The* 3-UNIFORM HYPERGRAPH 2-COLORING PROBLEM, *the* 3-UNIFORM HYPERGRAPH 2-PRECOLORING EXTENSION PROBLEM, *and the* 3-UNIFORM HYPERGRAPH MAXIMUM STABLE SET PROBLEM *restricted to linear M_s-free hypergraphs are polynomial-time solvable.*

We prove:

Theorem 1.11. *The* 3-UNIFORM HYPERGRAPH 3-COLORING PROBLEM *restricted to linear hypergraphs G with $\nu(G) \leq s$ is NP-complete for all $s \geq 532$.*

We will give polynomial-time algorithms for the case $k = 3$ and $r = 2$ and the case $s \leq r - 1$ of Theorems 1.4 and 1.5 in Section 2 and Section 3 respectively. In Section 4, we will talk about some NP-hard cases and complete the proof of Theorems 1.4 and 1.5. In Section 5, we will prove Theorem 1.6. In Section 6, we will prove Theorem 1.7. Finally, in Section 7, we will prove results about linear hypergraphs, including Theorem 1.11.

2 Algorithm for the case $k = 3$ and $r = 2$

In this section, we prove:

Theorem 2.1. *For every fixed positive integer s, the* 3-Bounded Hypergraph 2-Coloring Problem *restricted to hypergraphs with $\nu(G) \leq s$ is polynomial-time solvable.*

A common strategy for coloring algorithms is using an algorithm for 2-SAT as a subroutine: Given an instance I consisting of n Boolean variables and m clauses, each of which contains 2 literals, the 2-Satisfiability Problem (2-SAT) is to decide whether there exists a truth assignment for every variable such that every clause contains at least one true literal. We say I is *satisfiable* if it admits such an assignment.

Theorem 2.2 (Krom [11]; Aspvall, Plass and Tarjan [1])**.** *The* 2-SAT Problem *can be solved in time $O(n + m)$, where n is the number of variables and m is the number of clauses.*

Given two partial r-coloring collections $\mathcal{C}$, $\mathcal{C}'$ of a hypergraph G, we say $\mathcal{C}$ and $\mathcal{C}'$ are *r-equivalent* if $\mathcal{C}$ contains a partial r-coloring c_1 which r-extends to G if and only if $\mathcal{C}'$ contains a partial r-coloring c_2 which r-extends to G. We say (X, c) is r-equivalent to $\mathcal{C}$ if the collection $\{(X, c)\}$ is r-equivalent to $\mathcal{C}$. We say that $\mathcal{C}$ is *r-equivalent* to G if G is r-colorable if and only if $\mathcal{C}$ contains a partial r-coloring which r-extends to G.

Proof of Theorem 2.1. Let G be a 3-bounded hypergraph with $\nu(G) \leq s$. First, we create a collection $\mathcal{C}$ of partial 2-colorings as follows. We fix a maximal matching F of G. We define the set $X^F = \cup_{e \in F} e$. Let $\mathcal{C}$ be the set of all partial 2-colorings $(X^F, c : X^F \to [2])$ of G.

We claim that the collection $\mathcal{C}$ has the following three properties. The theorem follows immediately from these properties.

(1) *$\mathcal{C}$ is 2-equivalent to G.*

It suffices to show that if G has a 2-coloring c, then there is a partial 2-coloring in $\mathcal{C}$ which has a 2-precoloring extension. Let c be a 2-coloring of G. Consider the partial 2-coloring $(X^F, c|_{X^F}) \in \mathcal{C}$. Then c is a 2-precoloring extension of $c|_{X^F}$. Thus we have proved (1).

(2) *$\mathcal{C}$ can be computed in time $O(n^3)$.*

Let $|V(G)| = n$. Since G is 3-bounded, $|E(G)| \leq O(n^3)$. We can go through all edges and construct a maximal matching F in time $O(n^3)$. Checking whether (X, c) is a partial 2-coloring takes time $O(1)$, as the size of X is bounded. Since $|F| \leq \nu(G) \leq s$, we have $|\mathcal{C}| \leq 2^{3s} = O(1)$. Thus, $\mathcal{C}$ can be constructed from F in time $O(n^3)$.

(3) *For every partial 2-coloring c' in $\mathcal{C}$, whether c' has a 2-precoloring extension $(V(G), c)$ can be decided in polynomial time.*

Let $(X^F, c') \in \mathcal{C}$. Since F is a maximal matching and G is 3-uniform, for every edge $e \in E(G) \setminus F$, $|e \setminus X^F| \leq 2$.

We define a 2-precoloring extension (X, c) of c' as follows. We define the sets X_0, $X_1, \ldots$ iteratively. Let $X_0 = X^F$. Let $c(v) = c'(v)$ for all $v \in X^F$. Suppose that we have defined X_i. If there exists an edge $e \in E(G)$ such that $e \subseteq X_i$ and e is monochromatic, then c' does not have a 2-precoloring extension and we return this determination. If there exists an edge $e \in E(G)$ such that $|e \setminus X_i| = 1$ and $c(e \cap X_i) = \{j\}$ for some $j \in [2]$, we define $c(w)$ to be the unique element of $[2] \setminus \{j\}$ for $w \in e \setminus X_i$, and define $X_{i+1} = X_i \cup \{w\}$. Otherwise we stop and let $X = X_i$. This terminates within at most $O(n^3)$ steps. From the construction, clearly $\{(X_i, c)\}$ is equivalent to $\{(X_{i+1}, c)\}$ at every step; and it follows that $\{(X, c)\}$ is equivalent to $\{(X^F, c')\}$, and that if this step returns a determination that (X^F, c') does not 2-extend to G, then this determination is correct.

We define a 2-SAT instance as follows. For every $v \in V(G) \setminus X$, we have a variable x_v. Let $E' \subseteq E(G)$ be the set of edges such that $|e \setminus X| = 2$ for all $e \in E'$. For every edge $e \in E'$, we create a clause C_e. Let $e = \{v, u, w\}$ with $v \in X$ and $u, w \in V(G) \setminus X$. If $c(v) = 1$, we set $C_e = x_u \vee x_w$. Otherwise, let $C_e = \overline{x_u} \vee \overline{x_w}$.

If the 2-SAT instance has a solution x, where "true" and "false" are represented by 1 and 0 respectively, then we set $c(v) = x_v + 1$ for every $v \in V(G) \setminus X$. Take an edge $e \in E(G)$. If $|e \setminus X| \leq 1$, by the construction of X, e is not monochromatic. If $|e \setminus X| = 2$, the clause C_e of 2-SAT instance and the construction of c guarantees that at least one of the vertices in $e \setminus X$ receives the opposite color from the vertex in $e \cap X$. Since F is maximal, there is no edge e in $E(G)$ with $|e \setminus X| = 3$. Thus, c is a 2-precoloring extension of (X, c').

If there is a 2-precoloring extension d of (X, c), then we set $x_v = d(v) - 1$ for every $v \in V(G) \setminus X$. For every edge $e = \{v, u, w\} \in E'$ with $e \cap X = \{v\}$, if $d(v) = 1$, then $C_e = x_u \vee x_w$. Since e is not monochromatic, without loss of generality we may assume $d(u) = 2$, and so $x_u = d(v) - 1 = 1$. Thus, the clause C_e is satisfied. A similar argument applies for $d(v) = 2$. From the construction of clauses of this 2-SAT instance, we conclude that x is a solution to the 2-SAT instance.

Therefore, deciding whether (X, c) has a 2-coloring extension is equivalent to solving the 2-SAT instance defined above.

It remains to show that this can be done in polynomial-time. Let n be the number of vertices of G. Constructing the set X takes time $O(n^6)$. Constructing the equivalent 2-SAT instance takes time $O(n^3)$. Solving this 2-SAT instance takes time $O(n)$. So the total running time is $O(n^6)$. □

This immediately implies, for fixed r, a polynomial-time algorithm for 2-coloring tournaments with no r vertex-disjoint cyclic triangles, which was first proved by Hajebi [9].

3 Algorithm for the case $s \leq r - 1$

In this section, we prove:

Theorem 3.1. *For fixed positive integers r, k, s with $s \leq r-1$, the k-*BOUNDED HYPERGRAPH r-PRECOLORING EXTENSION PROBLEM *restricted to hypergraphs G with $\nu(G) \leq s$ is polynomial-time solvable.*

Lemma 3.2. *Let $r, k, s \in \mathbb{N}$ with $s \leq r-1$. Let G be a k-bounded hypergraph with $\nu(G) \leq s$. Given a partial r-coloring (X, c) of G, we define $E_i = \{e \in E(G) : e \cap X \subseteq c^{-1}(i)\}$. If $E_i \neq \emptyset$ for all $i \in [r]$, then there is a vertex set $X' \supset X$ and a collection of partial r-colorings $\mathcal{C}$ such that*

- *For every $(X^*, c^*) \in \mathcal{C}$, $X^* = X'$;*
- *$|\mathcal{C}| \leq r^{ks} = O(1)$, and $\mathcal{C}$ can be computed from (X, c) in time $O(n^k)$;*
- *There is a color $j \in [r]$ such that for every edge $e \in E_j$, $|e \cap X'| \geq |e \cap X| + 1$; and*
- *$\mathcal{C}$ is r-equivalent to (X, c).*

Proof. Let S be a matching in G such that $S \subseteq \bigcup_{i\in[r]} E_i$, and S is maximal with respect to this condition. Let $X^S = \cup_{e\in S} e$. Let $X' = X \cup X^S$. Let $\mathcal{C}$ be the set of all partial r-colorings $(X', c' : X' \to [r])$ such that $c'|_X = c$. The first property follows immediately from the construction. Since $|S| \leq s$, $|X^S| \leq ks$, and we have $|\mathcal{C}| \leq r^{ks} = O(1)$. Finding S takes time $O(n^k)$, and thus, $\mathcal{C}$ can be computed from (X, c) in time $O(n^k)$. This proves the second property.

For every $e \in S$, there exists $i \in [r]$ such that $e \in E_i$, and therefore we have that $c(v) = i$ for all $v \in e \cap X$. Since $|S| \leq s \leq r-1$, there exists a color $j \in [r]$ such that $c(v) \neq j$ for all $v \in X \cap X^S$. Let e be an edge in E_j. We know that $e \cap X \subseteq c^{-1}(j)$, so $e \cap X^S \cap X = \emptyset$. But from the definition of S, we have $e \cap X^S \neq \emptyset$, as otherwise S is not maximal. Thus, $e \cap (X^S \setminus X) \neq \emptyset$. This proves the third property.

Suppose that there is a partial r-coloring $(X', c') \in \mathcal{C}$ which r-extends to G. Then by the construction of $\mathcal{C}$, $c'|_X = c$. Thus, every r-precoloring extension of (X', c') is also an r-precoloring extension of (X, c). Now suppose (X, c) r-extends to $V(G)$, that is, there is a coloring $c' : V(G) \to [r]$ with $c'|_X = c$. Then by the construction of $\mathcal{C}$, $(X', c'|_{X'}) \in \mathcal{C}$. Therefore, the last property holds. □

Theorem 3.3. *For fixed positive integers r, k, s with $s \leq r-1$, there is an algorithm with the following specifications:*

- *Input: A k-bounded hypergraph G with $\nu(G) \leq s$, and an r-precoloring (X, c).*
- *Output: one of*
  - *an r-precoloring extension of (X, c) to $V(G)$;*
  - *a determination that (X, c) does not r-extend to G.*
- *Running time: $O(|V(G)|^k)$.*

Proof. We define a sequence $\mathcal{C}_0, \dots$ of collections of partial r-colorings iteratively, as follows. Let $\mathcal{C}_0 = \{(X, c)\}$.

Suppose that we have defined $\mathcal{C}_t$. Given a partial r-coloring $(Y, d) \in \mathcal{C}_t$, let

$$E_{t,i}^{Y,d} = \{e \in E(G) : e \cap Y \subseteq d^{-1}(i)\}.$$

If $E_{t,i}^{Y,d} = \emptyset$ for some $i \in [r]$ and $(Y, d) \in \mathcal{C}_t$, then we define d' by setting $d'|_Y = d|_Y$ and $d'(v) = i$ for all $v \in V(G) \setminus Y$ and return d'. Note that d' is an r-coloring of G: Since (Y, d) is a partial r-coloring, it follows that no edge of $G[Y]$ is monochromatic. Therefore, if G contains an edge e which is monochromatic with respect to d', then $e \setminus Y \neq \emptyset$. It follows that $e \setminus Y \neq \emptyset$, and since $d'(v) = i$ for all $v \in V(G) \setminus Y$, it follows that every vertex of e is colored i by d'. But then $e \cap Y \subseteq d^{-1}(i)$, a contradiction. This shows that d' is an r-coloring of G.

Otherwise, for every $(Y, d) \in \mathcal{C}_t$, we have that $E_{t,i}^{Y,d} \neq \emptyset$, and so there is a collection of partial r-colorings $\mathcal{C}_{t+1}^{Y,d}$ which satisfies the properties in Lemma 3.2 applied to G and (Y, d). Let $\mathcal{C}_{t+1} = \cup_{(Y,d) \in \mathcal{C}_t} \mathcal{C}_{t+1}^{Y,d}$. By Lemma 3.2, $\mathcal{C}_{t+1}$ is r-equivalent to $\mathcal{C}_t$; and inductively, $\mathcal{C}_{t+1}$ is equivalent to $\mathcal{C}_0 = \{(X, c)\}$. Thus, if $\mathcal{C}_{t+1} = \emptyset$, then (X, c) does not r-extend to G and we return this.

It remains to show that this algorithm terminates and runs in polynomial time. To prove this, we define a potential function $\psi((Y, d)) = \sum_{i \in [r]} \max(\{0\} \cup \{|e \setminus Y| : e \cap Y \subseteq d^{-1}(i)\})$. We have $\psi((X, c)) \leq rk$ since each summand is at most k. We prove by induction on t that for every $(Y, d) \in \mathcal{C}_t$, we have $\psi((Y, d)) \leq rk - t$.

It suffices to show that if $(Y, d) \in \mathcal{C}_t$ and $(Y', d') \in \mathcal{C}_{t+1}^{Y,d}$ then $\psi((Y', d')) \leq \psi((Y, d)) - 1$. By the third property of Lemma 3.2, there is a color $j \in [r]$ such that for every edge $e \in E_{t,j}^{Y,d}$, $|e \cap Y'| \geq |e \cap Y| + 1$, which means that

$$\max(\{0\} \cup \{|e \setminus Y'| : e \cap Y' \subseteq d'^{-1}(j)\}) \leq \max(\{|e \setminus Y| : e \cap Y \subseteq d^{-1}(j)\}) - 1.$$

It follows that $\psi((Y', d')) \leq \psi((Y, d)) - 1$, as claimed.

Since $\psi((Y, d)) \geq 0$ for every partial r-coloring (Y, d) of G, it follows that this algorithm terminates in t' steps for some $t' \leq rk$. Since there are $O(1)$ iterations, and by Lemma 3.2, we have $|\mathcal{C}_t| = O(1)$ for all $t \leq t'$. Moreover, the set $\mathcal{C}_{t+1}$ can be computed from $\mathcal{C}_t$ in time $|\mathcal{C}_t| \cdot O(n^k) = O(n^k)$. Thus, each step takes time $O(n^k)$. So the total running time is $O(n^k)$. □

4 NP-hardness results for bounded matching number

Let G and H be two hypergraphs. We define an operation, $\ltimes$, via

$$G \ltimes H := (V(G) \cup V(H), E(G) \cup \{e \cup \{x\} : e \in E(H), x \in V(G)\}).$$

We have the following properties.

Lemma 4.1. *Let G and H be hypergraphs. Then $\nu(G \ltimes H) \leq |V(G)|$.*

Proof. This follows immediately from the fact that every edge in $G \ltimes H$ contains at least one vertex in $V(G)$. $\square$

Lemma 4.2. *Let H be a hypergraph. If G is a hypergraph with $\chi(G) = r$, then $G \ltimes H$ is r-colorable if and only if H is r-colorable.*

Proof. Suppose for a contradiction that $G \ltimes H$ has an r-coloring c and H is not r-colorable. Since $c|_{V(H)}$ is not an r-coloring of H, there exists an edge $e \in E(H)$ such that e is monochromatic with respect to $c|_{V(H)}$. Since $\chi(G) = r$ and $(G \ltimes H)[V(G)] = G$, there exist vertices $v_1, \ldots, v_r \in V(G)$ such that $c(v_i) = i$ for all $i \in [r]$. But then one of the edges $e \cup \{v_1\}, \ldots, e \cup \{v_r\}$ is monochromatic, which contradicts the fact that c is an r-coloring of $G \ltimes H$.

Now suppose that H has an r-coloring d. Since $\chi(G) = r$, G has an r-coloring d'. We define a new function $d^* : V(G \ltimes H) \to [r]$ with $d^*(v) = d(v)$ if $v \in V(H)$ and $d^*(v) = d'(v)$ otherwise. For every edge $e \in E(G \ltimes H)$, if $e \in E(G)$, then $d^*|_e = d'|_e$. So e is not monochromatic. Otherwise $e = e' \cup \{v\}$ for some $e' \in E(H)$ and $v \in V(G)$. Then $d^*|_{e'} = d|_{e'}$, and since the edge e' is not monochromatic, it follows that e is not monochromatic. Thus d^* is an r-coloring of $G \ltimes H$. $\square$

Theorem 4.3. *Given fixed integers k and r with $k, r \geq 2$, if the k-*Bounded Hypergraph *r-*Coloring Problem *is NP-complete, then the $(k+1)$-*Bounded Hypergraph *r-*Coloring Problem *restricted to hypergraphs with $\nu(G) \leq r$ is NP-complete.*

Proof. Let H be a k-bounded hypergraph. We set the hypergraph $G = K_r$, a complete graph on r vertices. We have $\chi(G) = r$. The hypergraph $G \ltimes H$ can be constructed from G and H in time $O(n^{k+1})$, where $n = |V(G \ltimes H)|$. By the construction, $G \ltimes H$ is $(k+1)$-bounded. The remaining part of the proof follows immediately from Lemmas 4.1 and 4.2. $\square$

Theorem 4.4. *Given fixed integers k and r with $k, r \geq 2$, if the k-*Uniform Hypergraph *r-*Coloring Problem *is NP-complete, then the $(k+1)$-*Uniform Hypergraph *r-*Coloring Problem *restricted to hypergraphs with $\nu(G) \leq (r-1)k + 1$ is NP-complete.*

Proof. Let H be a k-uniform hypergraph and let G be the complete $(k+1)$-uniform hypergraph with $(r-1)k+1$ vertices. The hypergraph $G \ltimes H$ can be constructed from G and H in time $O(n^{k+1})$, where $n = |V(G \ltimes H)|$. By the construction, $G \ltimes H$ is $(k+1)$-uniform.

We want to show that $\chi(G) = r$. We choose k vertices to color i for every $i \in [r-1]$, and color the remaining vertex r. Since G is $(k+1)$-uniform, every edge of G receives at least two colors. Thus, $\chi(G) \leq r$. Suppose for a contradiction that $\chi(G) \leq r - 1$. Then take an $(r-1)$-coloring c of G. There exists one color i with $|c^{-1}(i)| \geq \lceil \frac{(r-1)k+1}{r-1} \rceil \geq \lceil k + \frac{1}{r-1} \rceil = k + 1$. This means that there is a monochromatic edge in G, which contradicts the fact that c is an $(r-1)$-coloring of G.

The remaining part of the proof follows immediately from Lemmas 4.1 and 4.2. $\square$

Theorem 4.5. *Given fixed integers k and r with $k, r \geq 2$, if the k-*UNIFORM HYPERGRAPH r-COLORING PROBLEM *is NP-complete, then the* $(k+1)$-UNIFORM HYPERGRAPH r-PRECOLORING EXTENSION PROBLEM *restricted to hypergraphs with $\nu(G) \leq r$ is NP-complete.*

Proof. Let H be a k-uniform hypergraph and let G be a graph with a set of vertices $\{v_1, \ldots, v_r\}$ and no edges. Define the precoloring of $G \ltimes H$ to be $(V(G), c')$ with $c'(v_i) = i$ for all $i \in [r]$. The hypergraph $G \ltimes H$ can be constructed from G and H in time $O(n^{k+1})$, and the precoloring $(V(G), c')$ of $G \ltimes H$ can be constructed in time $O(n)$, where $n = |V(G \ltimes H)|$. The graph H is k-uniform and $E(G) = \emptyset$, so $G \ltimes H$ is $(k+1)$-uniform.

It remains to show that $G \ltimes H$ has an r-precoloring extension with respect to the precoloring $(V(G), c')$ if and only if H is r-colorable.

Suppose $G \ltimes H$ has an r-precoloring extension c. Assume for a contradiction that H is not r-colorable. Since $c|_{V(H)}$ is not an r-coloring of G, there exists an edge $e \in E(H)$ such that e is monochromatic. By the definition of c', one of the vertices $v_1, \ldots, v_r$ receives the same color as e, which contradicts the fact that c is an r-precoloring extension of $G \ltimes H$ and $(V(G), c')$.

Now suppose that H has an r-coloring d. We define a new function $d^* : V(G \ltimes H) \to [r]$ with $d^*(v) = d(v)$ if $v \in V(H)$ and $d^*(v) = c'(v)$ otherwise. For every edge $e \in E(G \ltimes H)$, $e = e' \cup \{v\}$ for some $e' \in E(H)$ and $v \in V(G)$. Then $d^*|_{e'} = d|_{e'}$. The edge e' is not monochromatic, so e is not monochromatic. Thus d^* is an r-coloring of $G \ltimes H$ which r-extends $(V(G), c')$. □

Theorem 4.6. *Given fixed integers k and r with $k, r \geq 2$, the k-*UNIFORM HYPERGRAPH r-COLORING PROBLEM *is NP-complete if $k + r \geq 5$.*

Proof. The statement holds for the cases $k = 3$ and $r = 2$ by Theorem 1.3, and $k = 2$ and $r \geq 3$ by Theorem 1.1. By Theorem 4.4, if the k-UNIFORM HYPERGRAPH r-COLORING PROBLEM is NP-complete, then the $(k+1)$-UNIFORM HYPERGRAPH r-COLORING PROBLEM is NP-complete. □

Now we are ready to prove our main results.

Proof of Theorem 1.4. The first and second polynomial-time solvable cases follow from Theorem 3.1 and Theorem 2.1 respectively. The third polynomial-time solvable case follows from Theorem 1.2, as a graph G with $\nu(G) \leq s$ is guaranteed to be $(s+1)P_2$-free. Combining Theorem 4.6 with either Theorem 4.3 or Theorem 4.5, we have completed the dichotomies. □

Proof of Theorem 1.5. The first and second polynomial-time solvable cases follow from Theorem 3.1 and Theorem 2.1 respectively. The third polynomial-time solvable case follows from Theorem 1.2, as a graph G with $\nu(G) \leq s$ is guaranteed to be $(s+1)P_2$-free. The NP-completeness result comes from Theorems 4.6 and 4.4. □

5 Stable Set

In this section, we consider the complexity of stable set problems in hypergraphs with bounded matching number. The k-Uniform Hypergraph Maximum Weight Stable Set Problem is the following: Given a k-uniform hypergraph G and a weight function $w : V(G) \to \mathbb{R}_{\geq 0}$, compute a stable set $S \subseteq V(G)$ with $w(S)$ maximized. When all weights are 1, this is called the k-Uniform Hypergraph Maximum Stable Set Problem.

For graphs, the Graph Maximum Weight Stable Set Problem can be solved in polynomial time if the maximum size of an induced matching is bounded:

Theorem 5.1 (Balas and Yu [2])**.** *For a fixed positive integer s, the* Graph Maximum Weight Stable Set Problem *restricted to sP_2-free graphs can be solved polynomial time.*

For hypergraphs, we notice that:

Theorem 5.2. *For fixed positive integers k and s, the* k-Uniform Hypergraph Maximum Stable Set Problem *restricted to hypergraphs with $\nu(G) \leq s$ is polynomial-time solvable.*

Proof. Let G be a k-uniform hypergraph with $\nu(G) \leq s$, and let $F \subseteq E(G)$ be a maximal matching. Let n be the number of vertices of G. We have $|F| \leq s$. The set $V(G) \setminus (\cup_{e \in F} e)$ is stable as F is maximal, and $|V(G) \setminus (\cup_{e \in F} e)| \geq n - ks$. Thus, a maximum stable set of G is of size at least $n - ks$.

Therefore, to find a maximum stable set, we can simply enumerate all choices of a set $U \subseteq V(G)$ with $|U| \leq ks$, and check if the set $V(G) \setminus U$ is stable, and return the largest stable set found this way. There are n^{ks} choices of the set U, and for each U, it takes time $O(n^k)$ to verify stability. Thus, the total running time is $O(n^{k(s+1)})$. ☐

In contrast, we show the following result for the weighted version of the problem:

Theorem 5.3. *For a fixed positive integer $k \geq 3$, the* k-Uniform Hypergraph Maximum Weight Stable Set Problem *restricted to hypergraphs with $\nu(G) \leq 1$ is NP-complete.*

In order to prove Theorem 5.3, we need the following results.

Theorem 5.4 (Garey and Johnson [7])**.** *The* Maximum Stable Set Problem *is NP-complete.*

Lemma 5.5. *For a fixed positive integers $k \geq 3$, if the* $(k-1)$-Uniform Hypergraph Maximum Weight Stable Set Problem *is NP-complete, then the* k-Uniform Hypergraph Maximum Weight Stable Set Problem *restricted to hypergraphs with $\nu(G) \leq 1$ is NP-complete.*

Proof. Suppose the $(k-1)$-Uniform Hypergraph Maximum Weight Stable Set Problem is NP-complete. Let G be a $(k-1)$-uniform hypergraph with weight function w. We construct a new k-uniform hypergraph H with $V(H) = \{v\} \cup V(G)$ and $E(H) = \{\{v\} \cup e : e \in E(G)\}$. We define the weight function $w' : V(G) \to \mathbb{R}_{\geq 0}$ such that $w'(u) = w(u)$ for each $u \in V(G)$ and $w'(v) = \sum_{u \in V(G)} w(u) + 1$. From the construction, since v is contained in every edge of H, it follows that the hypergraph H satisfies $\nu(H) \leq 1$.

For a set $T \subseteq V(G)$, T is a stable set of G if and only if $T \cup \{v\}$ is a stable set of H. Let S be a maximum weight stable set of H with respect to the weight function w'. By the construction, the vertex v is in S. It follows that $S \setminus \{v\}$ is a maximum weight stable set of G, and thus, to find a maximum weight stable set of G, it suffices to find a maximum weight stable set of H.

Since the construction can be done in polynomial time, we have proved this lemma. □

Proof of Theorem 5.3. We prove this by induction on k. When $k = 2$, by Theorem 5.4, the Graph Maximum Stable Set Problem is NP-complete. Thus, the Graph Maximum Weight Stable Set Problem is NP-complete.

Suppose that the k-Uniform Hypergraph Maximum Weight Stable Set Problem is NP-complete. By Lemma 5.5, the $(k+1)$-Uniform Hypergraph Maximum Weight Stable Set Problem restricted to hypergraphs with $\nu(G) \leq 1$ is NP-complete. Moreover, the $(k+1)$-Uniform Hypergraph Maximum Weight Stable Set Problem is NP-complete. □

6 Excluding an induced subhypergraph with one edge

For $t \in \mathbb{N}$ with $t \geq 3$, let H_t be the 3-bounded hypergraph with $t+3$ vertices and one edge. In this section, we give a polynomial-time algorithm for 2-coloring 3-bounded H_t-free hypergraphs.

Lemma 6.1. *Let $t \in \mathbb{N}$, and let G be a 3-bounded H_t-free hypergraph. There is a polynomial-time algorithm to test if G has a 2-coloring with at least t vertices of each color.*

Proof. We may assume that $|e| \geq 2$ for all $e \in E(G)$, since G is not 2-colorable otherwise. Let $\mathcal{C}$ be a partial 2-coloring collection containing all partial 2-colorings $(X \cup Y, c')$ for every pair of disjoint sets $X, Y \subseteq V(G)$ with $|X| = |Y| = t$ and X, Y stable, and $c' : X \cup Y \to [2]$ with $c'(v) = 1$ for all $v \in X$ and $c'(v) = 2$ otherwise. It suffices to show that $\mathcal{C}$ has the following three properties.

(1) *$\mathcal{C}$ is 2-equivalent to the collection of all 2-colorings of G with at least t vertices of each color.*

We only need to show that if G has a 2-coloring c with at least t vertices of each color, then there exists a partial 2-coloring in $\mathcal{C}$ which 2-extends to G. Let c be a

2-coloring of G such that $|c^{-1}(i)| \geq t$ for all $i \in [2]$. Let X and Y be subsets of $c^{-1}(1)$ and $c^{-1}(2)$ respectively, with $|X| = |Y| = t$. We have $(X \cup Y, c|_{X\cup Y}) \in \mathcal{C}$, and c is a 2-precoloring extension of $(X \cup Y, c|_{X\cup Y})$ to $V(G)$. This proves (1).

(2) *$\mathcal{C}$ can be computed in time $O(n^{2t+3})$, where $n = |V(G)|$.*

Since G is 3-bounded, $|E(G)| \leq O(n^3)$. By construction, we have $|\mathcal{C}| \leq O(n^{2t})$. Constructing the sets X, Y and the corresponding partial 2-coloring c takes time $O(n^{2t})$. Checking whether $(X \cup Y, c)$ is a partial 2-coloring takes time $O(n^3)$. Thus, $\mathcal{C}$ can be constructed in time $O(n^{2t+3})$. This proves (2).

(3) *For every partial 2-coloring $(X \cup Y, c)$ in $\mathcal{C}$, whether c 2-extends to G can be decided in polynomial time.*

For convenience, let us denote $S = X \cup Y$. We define a 2-SAT instance as follows. For every $v \in V(G) \setminus S$, we have a variable x_v. Let $E' \subseteq E(G)$ be the set of edges $e \in E(G)$ with $|c(e \cap S)| = 1$. Note that for every edge $e \in E'$, we have $e \cap S \neq \emptyset$ and $e \setminus S \neq \emptyset$ (since (S, c) is a partial 2-coloring). Thus, $|e \setminus S| \in \{1, 2\}$. For every edge $e \in E'$, we create a clause C_e. Let $u, w \in e \setminus S$ with $u \neq w$ with $|e \setminus S| = 2$. If $c(e \cap S) = \{1\}$, we set $C_e = x_u \vee x_w$. Otherwise, let $C_e = \overline{x_u} \vee \overline{x_w}$. Next, let E'' be the set of edges $e \in E(G)$ with $|e| = 2$ and $e \cap S = \emptyset$. For every $e \in E''$, say $e = \{u, w\}$, we add two clauses $C'_e = x_u \vee x_w$ and $C''_e = \overline{x_u} \vee \overline{x_w}$.

If the 2-SAT instance has a solution $(s_v)_{v \in V(G) \setminus S}$, where "true" and "false" are represented by 1 and 0 respectively, then we set $d(v) = s_v + 1$ for every $v \in V(G) \setminus S$, and $d(v) = c(v)$ for all $v \in S$. We claim that d is a 2-coloring of G. Consider an edge $e \in E(G)$. If $|c(e \cap S)| > 1$, then e is not monochromatic. If $|c(e \cap S)| = 1$, then $e \in E'$. It follows that the clause C_e of 2-SAT instance and the construction of d guarantees that at least one vertex in $e \setminus S$ receives the opposite color from the vertices in $e \cap S$. Since both sets are non-empty, it follows that e is not monochromatic. It remains to consider the case that $e \cap S = \emptyset$. If $|e| = 2$, then the clauses C'_e and C''_e guarantee that the two vertices of e receive different colors. Therefore, we may assume that $|e| = |e \setminus S| = 3$. Suppose for a contradiction that e is monochromatic. Without loss of generality, assume $d(v) = 1$ for all $v \in e$. Let $X = (S \cap c^{-1}(1))$, and consider the set $X \cup e$. Since all edges with a non-empty intersection with S and all edges of size 2 are non-monochromatic, there is no edge $e' \in E(G)$ with $e' \subseteq X \cup e$ and $e' \neq e$. Thus, $G[X \cup e]$ is an induced copy of H_t in G, which contradicts the fact that G is H_t-free. Therefore, d is a 2-precoloring extension of (S, c).

If there is a 2-precoloring extension d of (S, c), then we set $x_v = d(v) - 1$ for every $v \in V(G) \setminus S$. For every edge $e \in E'$, if $C_e = x_u \vee x_w$, then $e \cap S$ contains only vertices colored 1, and so $d(u) = 2$ or $d(w) = 2$; it follows that C_e is satisfied. If $C_e = \overline{x_u} \vee \overline{x_w}$, then $e \cap S$ contain only vertices colored 2, and so $d(u) = 1$ or $d(w) = 1$; it follows that C_e is satisfied. For every edge $e = \{u, w\} \in E''$, it follows that $d(u) \neq d(w)$, and hence one of x_u, x_w is "true" and the other is "false." It follows that C'_e and C''_e are satisfies. From the construction of clauses of this 2-SAT instance, we conclude that this assignment is a solution to the 2-SAT instance. Therefore, deciding whether (S, c) has a 2-coloring extension is equivalent to solving the 2-SAT instance defined above.

It remains to show that this can be done in polynomial time. Constructing the 2-SAT instance takes time $O(n^3)$. Solving this 2-SAT instance takes time $O(n^3)$. So the total running time is $O(n^3)$. This proves (3) and concludes the proof. □

Theorem 6.2. *Let $t \in \mathbb{N}$, and let G be a 3-bounded H_t-free hypergraph. There is a polynomial-time algorithm which takes G as input, and outputs either a 2-coloring of G, or a determination that G is not 2-colorable.*

Proof. If G satisfies the conditions of Lemma 6.1, then we are done. Otherwise we can go through every possible coloring such that less than t vertices receive color i for some $i \in [2]$, and check whether it is a 2-coloring, in time $O(n^{t+3})$. □

Note that the proof of Lemma 6.1 can be modified to work for the precoloring extension version of the problem, and so can Theorem 6.2.

7 Linear Hypergraphs

7.1 The polynomial-time algorithm

We say a k-uniform hypergraph is *complete* if its edge set is the set of all k-vertex subsets of its vertex set. The *hypergraph Ramsey number*, $R_k(n_1, \dots, n_t)$, is the smallest integer N such that for every function $f : E(G) \to [t]$ for a complete k-uniform hypergraph G with at least N vertices, there exists $i \in [t]$ and a set $S \subseteq V(G)$ with $|S| \geq n_i$ such that all edges $e \subseteq S$ satisfy $f(e) = i$.

Theorem 7.1 (Ramsey [14])**.** *For all positive integers k, n_1, …, n_t, the hypergraph Ramsey number $R_k(n_1, \dots, n_t)$ exists.*

Lemma 7.2. *For every positive integer s, there exists a positive integer s' such that every 3-uniform linear hypergraph G which contains a matching of size s' contains an induced matching of size s.*

Proof. We may assume that $s \geq 4$. Let $\{G_1, \dots, G_t\}$ be the set of all linear 3-uniform hypergraphs with vertex set $\{x_1, \dots, x_9\}$. Since there at most $2^{\binom{9}{3}}$ distinct 3-uniform (labelled) hypergraphs on 9 vertices, it follows that $t \leq 2^{\binom{9}{3}}$. Let $s' = R_3(n_1, \dots, n_t)$ with $n_1 = \cdots = n_t = s$.

Let $\{e_1, \dots, e_{s'}\}$ be a matching of size s' in G. For $i \in [s']$, let $e_i = \{u_i, v_i, w_i\}$. Let H be a complete 3-uniform hypergraph $V(H) = \{1, \dots, s'\}$. We define $f : E(H) \to [t]$ as follows. For $e = \{i, j, k\} \subseteq [s']$ with $i < j < k$, we define $f(e) = m$ if $G[e_i \cup e_j \cup e_k]$ is isomorphic to G_m via the isomorphism $u_i \mapsto x_1$, $v_i \mapsto x_2$, $w_i \mapsto x_3$, $u_j \mapsto x_4$, $v_j \mapsto x_5$, $w_j \mapsto x_6$, $u_k \mapsto x_7$, $v_k \mapsto x_8$ and $w_k \mapsto x_9$.

From Theorem 7.1, it follows that there is a set $S \subseteq [s']$ with $|S| = s$ and $m \in [t]$ such that $f(e) = m$ for all $e \subseteq S$. We claim that

$$E(G_m) = \{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_7, x_8, x_9\}\}.$$

Let $i, j, k, l \in S$ with $i < j < k < l$. Since $G[e_i, e_j, e_k]$ contains the edges e_i, e_j, e_k, it follows that $\{\{x_1, x_2, x_3\}, \{x_4, x_5, x_6\}, \{x_7, x_8, x_9\}\} \subseteq E(G_m)$. Suppose for a contradiction that $E(G_m)$ contains a fourth edge $\{x_a, x_b, x_c\}$. Then, since G_m is linear, we may assume that $a \in \{1, 2, 3\}$, $b \in \{4, 5, 6\}$, and $c \in \{7, 8, 9\}$. The graphs $G[e_i \cup e_j \cup e_k]$ and $G[e_i \cup e_j \cup e_l]$ are isomorphic to G_m via isomorphisms φ, φ', say; and from the definition of f it follows that $\varphi^{-1}(x_a) = \varphi'^{-1}(x_a)$, $\varphi^{-1}(x_b) = \varphi'^{-1}(x_b)$, and $\varphi^{-1}(x_c) \neq \varphi'^{-1}(x_c)$ (since $\varphi^{-1}(x_c) \in e_k$ and $\varphi'^{-1}(x_c) \in e_l$ and $e_k \cap e_l = \emptyset$). But this implies that G contains the edges $\varphi^{-1}(\{x_a, x_b, x_c\})$ and $\varphi'^{-1}(\{x_a, x_b, x_c\})$ which have exactly two vertices in common, contrary to the assumption that G is linear. This proves our claim.

It follows that $G\left[\bigcup_{q \in S} e_q\right]$ is an induced matching of size s in G. □

Theorem 7.3. *For all s, the* 2-Precolouring Extension Problem *restricted to 3-uniform linear hypergraphs with no induced matching of size at least s can be solved in polynomial time.*

Proof. By Theorem 2.1 and Lemma 7.2. □

Note that from Theorem 3.1 and Lemma 7.2, it also follows that for all s there exists s' such that for all $S > s'$, the S-Precolouring Extension Problem restricted to 3-uniform linear hypergraphs with no induced matching of size at least s can be solved in polynomial time.

7.2 NP-hardness of 3-coloring with bounded matching number

In this section, we prove the following result.

Theorem 7.4. *The* 3-Uniform Hypergraph 3-coloring Problem *restricted to linear hypergraphs with $\nu(G) \leq s$ is NP-complete for all $s \geq 532$.*

We use the following theorems.

Theorem 7.5 (Garey, Johnson and Stockmeyer [6])**.** *The* 3-Coloring Problem *restricted to graphs with maximum degree at most 4, is NP-complete.*

Theorem 7.6 (Vizing [16], Misra and Gries [12])**.** *There is a $O(mn)$-algorithm for edge-coloring a graph G with $D+1$ colors, where D is the maximum degree of G, m is the number of edges and n is the number of vertices.*

Let us introduce a new way to describe 3-uniform hypergraphs. Instead of using edges with three vertices, we use 2-edges with labeled vertices. Given a graph G, we say a function $l : E(G) \to V(G)$ with $l(e) \notin e$ for all $e \in E(G)$ is a *labeling* of G. The vertex $l(e)$ is called the *label* of e, and the edge e is a *labeled edge*.

For a linear 3-uniform hypergraph G, let $l : E(G) \to V(G)$ be a function with $l(e) \in e$ for all $e \in E(G)$. Let G' be the graph with vertex set $V(G)$ and edge set $\{\{e \setminus \{l(e)\} : e \in E(G)\}$, and let $l'(e \setminus \{l(e)\}) = l(e)$. Since G is linear, each edge of G' corresponds to a unique edge of G, and thus l' is well-defined. We call (G', l') a *labeled graph representation* of G. Notice that with a labeled graph representation, we can reconstruct the corresponding linear 3-uniform hypergraph.

In this section, all of the figures of 3-uniform hypergraphs are drawn using the labeled graph representation.

The following two lemmas give constructions for gadgets we use in our NP-hardness reduction. The existence of similar gadgets in 3-uniform linear hypergraphs was first proved in [13]. Here we give an explicit construction to obtain a precise bound for the matching number. The construction is shown in Figure 1.

Lemma 7.7. *There is a linear 3-uniform hypergraph G_1 with three specified vertices a, b, c with the following properties:*

- *For every 3-coloring f of G_1, either $f(a), f(b), f(c)$ are all distinct, or $f(a) = f(b) = f(c)$.*
- *There is a 3-coloring f' of G_1 with $f'(a), f'(b), f'(c)$ all distinct.*
- *There is a set $Z \subseteq V(G_1)$ with $|Z| \leq 19$ such that $G_1 \setminus Z$ has no edges, and $a, b, c \in Z$.*
- *No edge e of G_1 contains more than one of the vertices a, b, c.*

Proof. We want to define G_1 using the labeled graph representation (G_1', l). First, we create three vertices a, b, c. Then we create 4 copies of K_4, say H_1, H_2, H_3, H_4. For $i \in [4]$, let $V(H_i) = \{s_i, t_i, u_i, v_i\}$. We define the labeling $l(s_i t_i) = l(u_i v_i) = a$, $l(s_i u_i) = l(t_i v_i) = b$, and $l(s_i v_i) = l(t_i u_i) = c$.

Let $S = V(H_1) \times V(H_2) \times V(H_3) \times V(H_4)$. For every 4-tuple $T = (x, y, z, w) \in S$, we create 5 new copies of K_4, say $H_0^T, H_1^T, H_2^T, H_3^T, H_4^T$. Let $V(H_i^T) = \{s_i^T, t_i^T, u_i^T, v_i^T\}$ for $i \in [4]$, and $V(H_0^T) = \{r_1^T, r_2^T, r_3^T, r_4^T\}$. We define the labeling $l(s_i^T t_i^T) = l(u_i^T v_i^T) = a$, $l(s_i^T u_i^T) = l(t_i^T v_i^T) = b$ and $l(s_i^T v_i^T) = l(t_i^T u_i^T) = c$ for $i \in [4]$, and $l(r_1^T r_2^T) = l(r_3^T r_4^T) = a$, $l(r_1^T r_3^T) = l(r_2^T r_4^T) = b$ and $l(r_1^T r_4^T) = l(r_2^T r_3^T) = c$. For each $i \in [4]$, we add edges $s_i^T r_i^T$ with $l(s_i^T r_i^T) = x$, $t_i^T r_i^T$ with $l(t_i^T r_i^T) = y$, $u_i^T r_i^T$ with $l(u_i^T r_i^T) = z$ and $v_i^T r_i^T$ with $l(v_i^T r_i^T) = w$.

Let $V(G_1') = \{a, b, c\} \cup (\cup_{i \in [4]} V(H_i)) \cup (\cup_{T \in S} \cup_{i=0}^4 V(H_i^T))$, and $E(G_1')$ be the set of all labeled edges defined above. By the construction, the function l defined above is a labeling of G_1'. Notice that there is no edge incident to more than one of the vertices a, b, c, and $l(V(G_1')) = \{a, b, c\} \cup (\cup_{i \in [4]} V(H_i))$. Thus, by taking $Z = l(V(G_1'))$, we have $|Z| \leq 19$ and $a, b, c \in Z$; so Z satisfies the third property of the lemma. We now prove the other properties.

(1) *The 3-uniform hypergraph G_1 is linear.*

Let $X_1 = \{a, b, c\}$, $X_2 = (\cup_{i\in[4]} V(H_i))$ and $X_3 = (\cup_{T\in S} \cup_{i=0}^{4} V(H_i^T))$. From the construction, it follows that for every edge e of G_1, there exist $i, j \in [3]$ with $i < j$ such that e contains one vertex of X_i and two vertices of X_j and with $e \cap X_j \in E(G'_1)$ (and therefore, $\{l(e \cap X_j)\} = e \cap X_i$).

Suppose for a contradiction that there exist distinct $e, e' \in E(G_1)$ with $|e \cap e'| = 2$. Let $j, j' \in [3]$ such that $|e \cap X_j| = 2$ and $|e' \cap X_{j'}| = 2$. It follows that $j = j'$. Since G'_1 is simple, we have $e \cap X_j \neq e' \cap X_j$, and so $e \setminus X_j = e' \setminus X_j = \{l(e \cap X_j)\} = \{l(e' \cap X_j)\}$. But in G'_1, every two edges with the same label are not incident to a common vertex, a contradiction. We conclude that G_1 is linear. This proves (1).

(2) *There is a 3-coloring f' of G_1 with $f'(a), f'(b), f'(c)$ all distinct.*

We define a function $f' : (V(G_1)) \to [3]$ as follows. Let $f'(a) = 1$, $f'(b) = 2$ and $f'(c) = 3$. For each $i \in [4]$, let $f'(s_i) = f'(t_i) = 2$ and $f'(u_i) = f'(v_i) = 3$. Since $l(s_i t_i) = l(u_i v_i) = a$, the edges of G_1 corresponding to labeled edges in $G'_1[V(H_i)]$ are not monochromatic.

For each $T \in S$ and each $i \in [4]$, let $f'(s_i^T) = f'(u_i^T) = 1$, $f'(t_i^T) = f'(v_i^T) = 3$, $f'(r_1^T) = f'(r_4^T) = 1$, and $f'(r_2^T) = f'(r_3^T) = 2$. For $i \in [4]$, no vertex $v \in V(H_i^T)$ has $f'(v) = 2$, and no edge between $V(H_i^T)$ and $V(H_0^T)$ is labeled a. So there is no monochromatic edge e in G_1 with $e \cap \cup_{i=0}^{4} V(H_i^T) \neq \emptyset$. Therefore, the function f' is a 3-coloring of G_1. This proves (2).

(3) *For each 3-coloring f of G_1, either $f(a), f(b), f(c)$ are all distinct, or $f(a) = f(b) = f(c)$.*

Assume for a contradiction that, without loss of generality, there is a 3-coloring f of G_1 such that $f(a) = f(b)$. Without loss of generality, we may assume that $f(a) = f(b) = 1$ and $f(c) = 2$.

We claim that there exists $x_0 \in V(H_1)$ such that $f(x_0) = 3$. Assume for a contradiction that every vertex $v \in V(H_1)$ has $f(v) \neq 3$. Since $l(s_1 v_1) = c$ and $f(c) = 2$, without loss of generality let $f(s_1) \neq 2$. So $f(s_1) = 1$. Since $l(s_1 t_1) = a$, $l(s_1 u_1) = b$ and $f(a) = f(b) = 1$, we have $f(t_1) = f(u_1) = 2$. But the edge $t_1 u_1$ is labeled c and $f(c) = 2$, the corresponding edge $\{t_1, u_1, c\}$ of G_1 is monochromatic, which violates the condition that f is a 3-coloring of G_1.

A similar argument holds for every H_i with $i \in \{2, 3, 4\}$, and H_j^T with $T \in S$ and $j \in \{0, 1, \ldots, 4\}$. There exist vertices $y_0 \in V(H_2)$, $z_0 \in V(H_3)$, $w_0 \in V(H_4)$ such that $f(y_0) = f(z_0) = f(w_0) = 3$. Let $T = (x_0, y_0, z_0, w_0)$. By the argument above, there is a $j \in [4]$ such that $f(r_j) = 3$. Since there is a vertex $v \in V(H_j^T)$ with $f(v) = 3$, and $f(l(vr_j)) = 3$ (because $l(vr_j) \in \{x_0, y_0, z_0, w_0\}$), the edge $\{v, r_j, l(vr_j)\}$ of G_1 is monochromatic, which contradicts the condition that f is a 3-coloring of G_1. This proves (3). ∎

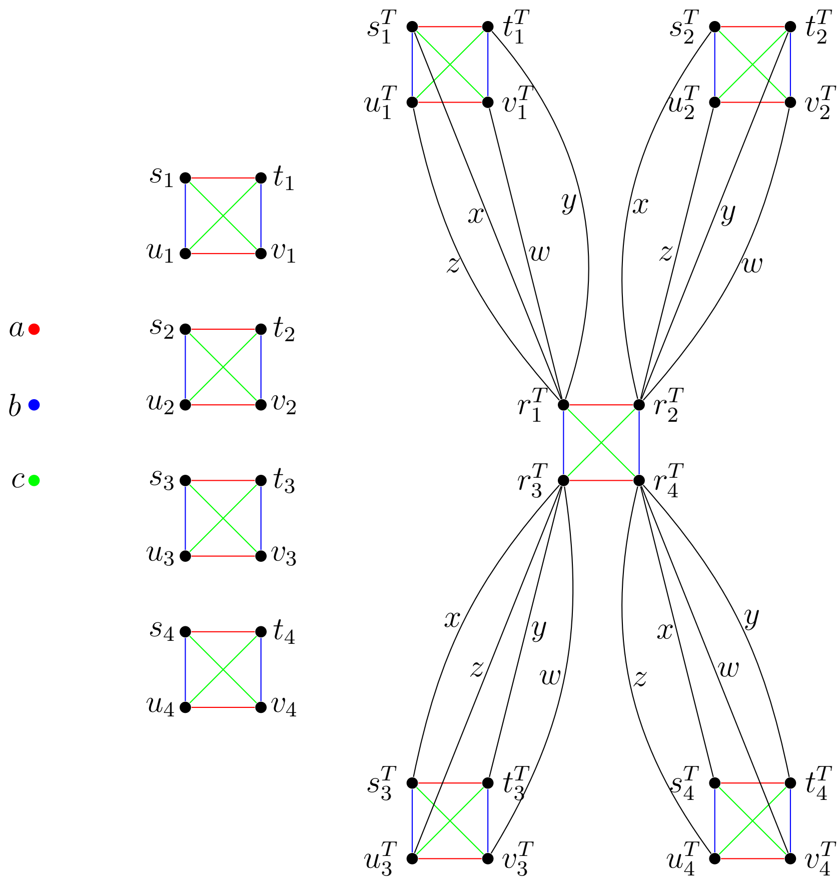

Figure 1: The construction from Lemma 7.7. The colored edge means the label of this edge is the vertex of the corresponding color. The right-hand side shows $H_0^T, \ldots, H_4^T$ for $T = (x, y, z, w)$.

Lemma 7.8. *There is a linear 3-uniform hypergraph G_2 with specified vertices a, b, c with the following properties:*

- *For every 3-coloring f of G_2, we have $f(a), f(b), f(c)$ all distinct.*
- *G_2 is 3-colorable.*
- *There is a set $Z \subseteq V(G_2)$ with $|Z| \leq 19$ such that $G_2 \setminus Z$ has no edges, and $a, b, c \in Z$.*
- *At most one edge of G_2 contains more than one of the vertices a, b, c.*

Proof. Let G_2 be obtained from G_1 defined in Lemma 7.7 by adding the edge $\{a, b, c\}$. The result follows immediately from Lemma 7.7. ∎

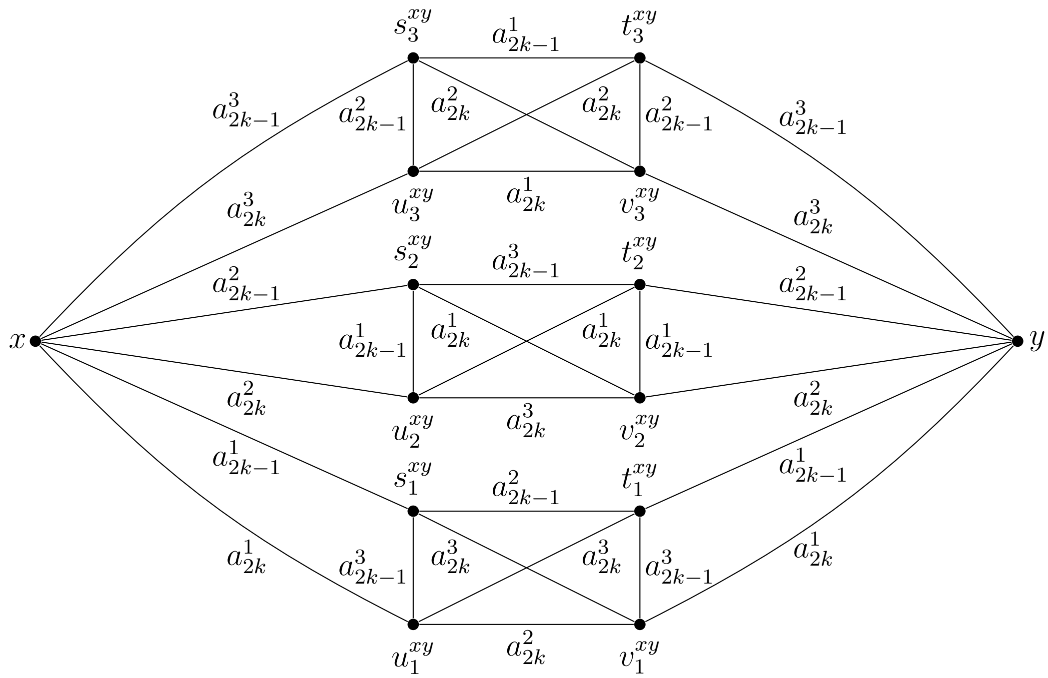

Figure 2: The construction of H_3^{xy}, H_2^{xy}, H_1^{xy} (top to bottom) for an edge xy with $f'(xy) = k$.

Now we are ready to prove Theorem 7.4.

Proof of Theorem 7.4. We give an NP-hardness reduction from the Graph 3-Coloring Problem restricted to graphs with maximum degree at most 4, which is NP-hard by Theorem 7.5.

Let G^* be a graph with maximum degree at most 4. Let $f' : E(G^*) \to [5]$ be an edge-coloring of G^*. We construct a labeled graph representation (G', l) of a 3-uniform linear hypergraph G as follows.

We create three sets of vertices $A = \{a_1^1, \dots, a_{10}^1\}$, $B = \{a_1^2, \dots, a_{10}^2\}$ and $C = \{a_1^3, \dots, a_{10}^3\}$. For the vertices a_1^1, a_1^2, a_1^3, we create a new copy of G_2 as defined in Lemma 7.8, denoted G^1, with a_1^1, a_1^2, a_1^3 as its specified vertices. For every $i \in \{2, \dots, 10\}$, we create three new copies of G_1 as defined in Lemma 7.7, one with specified vertices a_i^1, a_1^2, a_1^3, one with specified vertices a_1^1, a_i^2, a_1^3 and one with specified vertices a_1^1, a_1^2, a_i^3, respectively. We denote these three hypergraphs $G^{i,1}$, $G^{i,2}$ and $G^{i,3}$ respectively. For convenience, we also define $G^{1,1} = G^{1,2} = G^{1,3} = G^1$.

Next, for all $k \in [5]$ and for each edge $e = xy \in E(G^*)$ with $f'(xy) = k$, we create three copies of K_4, say H_1^e, H_2^e, H_3^e; see Figure 2 for a picture of the construction described below. Let $V(H_i^e) = \{s_i^e, t_i^e, u_i^e, v_i^e\}$ for $i \in [3]$. Let $l(s_i^e t_i^e) = a_{2k-1}^{(i+1)}$, $l(s_i^e u_i^e) = l(t_i^e v_i^e) = a_{2k-1}^{(i+2)}$, $l(u_i^e v_i^e) = a_{2k}^{(i+1)}$ and $l(s_i^e v_i^e) = l(t_i^e u_i^e) = a_{2k}^{(i+2)}$ for all $i \in [3]$, where superscripts are read modulo 3, so $a_j^4 = a_j^1$ and $a_j^5 = a_j^2$ for all $j \in [10]$. We also add edges xs_i^e, yt_i^e with $l(xs_i^e) = l(yt_i^e) = a_{2k-1}^i$, and edges xu_i^e, yv_i^e with $l(xu_i^e) = l(yv_i^e) = a_{2k}^i$ for all $i \in [3]$.

Let $\mathcal{G} = \{G^1\} \cup \{G^{i,j} : i \in \{2, \ldots, 10\}, j \in [3]\}$. Let

$$U = (\cup_{G'' \in \mathcal{G}} V(G'')) \setminus (A \cup B \cup C), W = \cup_{e \in E(G^*)} \cup_{i \in [3]} V(H_i^e).$$

Let $V(G') = A \cup B \cup C \cup U \cup W \cup V(G^*)$ and let $E(G')$ be the set of all labeled edges defined above. By the construction, the function l defined above is a labeling of G'. Let G be the corresponding 3-uniform hypergraph of (G', l).

Notice that from the construction, there is no other edge $e \in E(G)$ with $e \cap U \neq \emptyset$ and $e \cap (W \cup V(G^*)) \neq \emptyset$. Furthermore, except for the edge $\{a_1^1, a_1^2, a_1^3\}$, there is no edge $e \in E(G)$ with $e \subseteq A \cup B \cup C \cup V(G^*)$. Moreover, for every edge $e \in E(G) \setminus \{a_1^1, a_1^2, a_1^3\}$, we have $|e \cap (A \cup B \cup C)| \leq 1$. Thus, for each edge $e \in E(G) \setminus \{a_1^1, a_1^2, a_1^3\}$, exactly one of the conditions $|e \cap U| \geq 2$ and $|e \cap (W \cup V(G^*))| = 2$ holds. Moreover, for all $e \in E(G)$, we have that $|e \cap V(G^*)| \leq 1$.

(1) *The 3-uniform hypergraph G is linear.*

We take two edges $e, e' \in E(G)$ with $e \neq e'$. Assume for a contradiction that $|e \cap e'| = 2$. It follows that $e, e' \neq \{a_1^1, a_1^2, a_1^3\}$, since no edge except $\{a_1^1, a_1^2, a_1^3\}$ contains more than one vertex of $A \cup B \cup C$.

If $|e \cap U| \geq 2$, then $e \subseteq G^{a,b}$ for some $a \in [10]$ and $b \in [3]$. Since $|e \cap e'| = 2$, we have that $e' \cap U \neq \emptyset$, and so $|e' \cap U| \geq 2$. It follows that $e' \subseteq V(G^{c,d})$ for some $c \in [\{2, \ldots, 10\}]$ and $d \in [3]$. By Lemma 7.7, $(a, b) \neq (c, d)$. But then $V(G^{a,b}) \cap V(G^{c,d}) \subseteq \{a_1^1, a_1^2, a_1^3\}$ and so $e \cap e' \subseteq \{a_1^1, a_1^2, a_1^3\}$. But $|e \cap \{a_1^1, a_1^2, a_1^3\}| \leq 1$, so $|e \cap e'| \leq 1$, which is a contradiction.

If $|e \cap (W \cup V(G^*))| = 2$, then $|e \cap (A \cup B \cup C)| = 1$. Since $|e \cap e'| = 2$ and exactly one of $e' \cap U \neq \emptyset$ and $e' \cap (W \cup V(G^*)) \neq \emptyset$ holds, we have $e' \cap (W \cup V(G^*)) \neq \emptyset$. It follows that $|e' \cap (W \cup V(G^*))| = 2$. Consider the labeled graph G'. Notice that by the construction above, for each $e^* \in E(G^*)$, no two edges of $G'[e^* \cup (\cup_{i \in [3]} V(H_i^{e^*}))]$ with the same label are incident to one common vertex. Thus e and e' are both incident to a common vertex $x \in V(G^*)$. For every $xy_1, xy_2 \in E(G^*)$, since f' is an edge coloring of G^*, $f'(xy_1) \neq f'(xy_2)$. Thus, for every two edges e_1, e_2 of G' incident to x, $|e_1 \cap e_2| = 1$. Hence, we have proved $|e \cap e'| \leq 1$, which leads to a contradiction. This proves (1).

(2) *We have $\nu(G) \leq 532$.*

By Lemmas 7.7 and 7.8, for every graph $G'' = G^{i,j} \in \mathcal{G}$, there is a set $S_{G''}$ of size at most 19 which contains a_i^j such that $G'' \setminus S_{G''}$ has no edges; for G^1, the set S_{G^1} contains all of a_1^1, a_1^2, a_1^3. Each edge which is not a subset of $A \cup B \cup C \cup U$ contains a vertex in $A \cup B \cup C$. Thus, the set $X = \cup_{G'' \in \mathcal{G}} S_{G''}$ meets all edges of G. Since $\mathcal{G}$ is a set of 28 graphs, it follows that $|X| \leq 19 \cdot 28$. So $\nu(G) \leq 19 \cdot 28 = 532$. This proves (2).

(3) *The graph G^* is 3-colorable if and only if G is 3-colorable.*

Let c' be a 3-coloring of G. By Lemma 7.8, $c'(a_1^1)$, $c'(a_1^2)$ and $c'(a_1^3)$ are all distinct.

Without loss of generality let $c'(a_1^1) = 1$, $c'(a_1^2) = 2$ and $c'(a_1^3) = 3$. From the construction, by Lemma 7.7, $c'(a_i^1) = 1$, $c'(a_i^2) = 2$ and $c'(a_i^3) = 3$ for all $i \in [10]$. We want to prove that $c'|_{V(G^*)}$ is a 3-coloring of G^*.

Suppose for a contradiction that there exists an edge $xy \in E(G^*)$ with $c'(x) = c'(y)$. Let $k = f'(xy)$. Without loss of generality, let $c'(x) = c'(y) = 1$. Then consider the graph H_1^{xy}. Because of the edges $\{x, s_1^{xy}, a_{2k-1}^1\}$, $\{x, u_1^{xy}, a_{2k}^1\}$, $\{y, t_1^{xy}, a_{2k-1}^1\}$ and $\{y, v_1^{xy}, a_{2k}^1\}$, all of the vertices $s_1^{xy}, t_1^{xy}, u_1^{xy}, v_1^{xy}$ are colored 2 or 3. Since $c'(a_{2k-1}^3) = 3$, from the edge $\{s_1^{xy}, u_1^{xy}, a_{2k-1}^3\}$, it follows that one of the vertices s_1^{xy}, u_1^{xy} is not colored 3. Without loss of generality let $c'(s_1^{xy}) = 2$. Because of the edge $\{s_1^{xy}, t_1^{xy}, a_{2k-1}^2\}$, we have $c'(t_1^{xy}) = 3$. Consider the edges $\{t_1^{xy}, u_1^{xy}, a_{2k}^3\}$ and $\{t_1^{xy}, v_1^{xy}, a_{2k-1}^3\}$. Since $c'(a_{2k}^3) = c'(a_{2k-1}^3) = 3$, we have $c'(u_1^{xy}) = c'(v_1^{xy}) = 2$. But then the edge $\{u_1^{xy}, v_1^{xy}, a_{2k}^2\}$ is monochromatic, which contradicts the fact that c' is a 3-coloring of G. This proves that if G is 3-colorable, then so is G^*.

For the converse direction, let c be a 3-coloring of G^*. We want to define a 3-coloring d of G. Let $d(v) = 1$ for all $v \in A$, $d(v) = 2$ for all $v \in B$, and $d(v) = 3$ for all $v \in C$. By Lemmas 7.7 and 7.8, there is a way to extend d to $G[A \cup B \cup C \cup U]$.

Let $d(v) = c(v)$ for all $v \in V(G^*)$. For each edge $xy \in E(G^*)$ and each $i \in [3]$, since c is a 3-coloring of G^*, one of the vertices x, y is not colored i. If $c(x) \neq i$, then for the set $V(H_i^{xy})$, we set $d(s_i^{xy}) = d(u_i^{xy}) = i$ and $d(t_i^{xy}) = d(v_i^{xy}) = i + 1$, reading colors modulo 3 (so if this would assign color 4, we assign color 1 instead). If $c(x) = i$, then $c(y) \neq i$, and for the set $V(H_i^{xy})$, we set $d(s_i^{xy}) = d(u_i^{xy}) = i + 1$, again reading colors modulo 3; and $d(t_i^{xy}) = d(v_i^{xy}) = i$. Thus, we have defined the function d for all vertices of G.

We then want to show that d is a 3-coloring of G. From the construction, all edges e with $e \cap U \neq \emptyset$ are contained in $G[A \cup B \cup C \cup U]$ and hence not monochromatic. It remains to consider edges $e \in E(G)$ with $e \cap W \neq \emptyset$. It follows that there is an edge $xy \in E(G^*)$ and $i \in [3]$ such that $\emptyset \neq e \cap V(H_i^{xy}) = e \cap W$. If $x \in e$, then either $s_i^{xy} \in e$ or $t_i^{xy} \in e$ and from the construction of d, we have that $d(e \cap (A \cup B \cup C)) = \{i\}$, and either $d(x) \neq i$ or $d(s_i^{xy}), d(t_i^{xy}) \neq i$. The case $y \in e$ follows analogously. Therefore, we may assume that $|e \cap V(H_i^{xy})| = 2$. Now either the two vertices in $e \cap V(H_i^{xy})$ receive different colors, or they receive the same color in $\{i, i+1\}$ and $d(e \cap (A \cup B \cup C)) = \{i+2\}$. Thus, the edge e is not monochromatic. This proves (3).

(4) *The 3-hypergraph G can be constructed from G^* in time $O(n^3)$, where $n = |V(G^*)|$.*

Since $|V(G_1)| = O(1)$ and $|E(G_1)| = O(1)$, the 3-uniform hypergraph G_1 can be constructed in time $O(1)$. Similarly, the 3-uniform hypergraph G_2 can be constructed in time $O(1)$. We create $3 \cdot (10) - 2 = 28$ copies of the gadgets G_1 or G_2. This step can be done in time $O(1)$.

Let $n = |V(G^*)|$, and $m = |E(G^*)|$. The edge coloring f' of G^* can be computed in time $O(mn) \leq O(n^3)$ by Theorem 7.6. For each edge $e \in E(G^*)$, we create 12 new vertices and 30 edges. Thus, constructing the vertex set W and all edges incident to W takes time $O(n^2)$.

□

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