Foreshadowing the Grid Theorem for Induced Subgraphs

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Doctor of Philosophy
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2024

Examining Committee Membership

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

The present content is based on:

(a) Six coauthored papers [2, 3, 5, 6, 7, 8], also listed below, which I played a major role in producing. All six papers are joint with Bogdan Alecu, Maria Chudnovsky and Sophie Spirkl, and the first two are also joint with Tara Abrishami.

1. Induced subgraphs and tree-decompositions
   VII. Basic obstructions in H-free graphs
   J. Comb. Theory Ser. B 164 (2024)
2. Induced subgraphs and tree-decompositions
   VIII. Excluding a forest in (theta, prism)-free graphs
   Combinatorica (2024)
3. Induced subgraphs and tree-decompositions
   IX. Grid theorem for perforated graphs
   Submitted. Manuscript available at `arxiv:2305.15615` (2023)
4. Induced subgraphs and tree-decompositions
   XI. Local structure for even-hole-free graphs of large treewidth
   Submitted. Manuscript available at `arxiv:2205.04420` (2023)
5. Induced subgraphs and tree-decompositions
   XII. Grid theorem for pinched graphs
   Submitted. Manuscript available at `arxiv:2309.12227` (2023)
6. Induced subgraphs and tree-decompositions
   XIII. Basic obstructions in $\mathcal{H}$-free graphs for finite $\mathcal{H}$
   Submitted. Manuscript available at `arxiv:2311.05066` (2023)

(b) Two single-author papers [36, 37], also listed below.

1. Induced subdivisions with pinned branch vertices
   Accepted in European Journal of Combinatorics.
   Manuscript available at `arxiv:2308.01502` (2023)
2. Chordal graphs, even-hole-free graphs, and sparse obstructions to bounded treewidth
   Submitted. Manuscript available at `arxiv:2401.01299` (2024)

Several passages are taken verbatim from the above papers, of which I am the sole writer.

Abstract

We prove several dichotomy theorems toward a complete description of the unavoidable induced subgraphs of graphs with large treewidth. This is motivated by the *Grid Theorem* of Robertson and Seymour (1986) which achieves the same goal for minors (and subgraphs).

Given a graph class $\mathcal{C}$, we say that $\mathcal{C}$ is *clean* if the only induced subgraph obstructions to bounded treewidth in $\mathcal{C}$ are the *basic* ones: complete graphs, complete bipartite graphs, subdivided walls and the line graphs of the subdivided walls. The analog of the Grid Theorem for induced subgraphs (still out of reach) is then observed to be equivalent to a characterization of all hereditary classes that are clean.

We characterize all clean classes that are defined by finitely many excluded induced subgraphs. Specifically, we identify a family of "non-basic" obstructions which, in this scenario, litmus-test the clean classes against the non-clean ones.

The analogous characterization remains elusive in the case of infinitely many forbidden induced subgraphs. Among the infinite sets of graphs whose exclusion is known to result in a non-clean class, the following four appear to expose a distinctive gap in our understanding of cleanness:

- Graphs which are the union of *three* cycles, all sharing a vertex and otherwise pairwise vertex-disjoint.
- Graphs which are the union of *two* vertex-disjoint cycles.
- Graphs consisting of two non-adjacent vertices with three pairwise internally disjoint paths between them, known as *thetas*.
- Cycles with an even number of vertices (at least four), known as *even holes*.

For $i = 1, 2, 3, 4$, let $\mathcal{C}_i$ be the class obtained by excluding the i^{th} set above. We prove a full "grid-type theorem" for each of $\mathcal{C}_1$ and $\mathcal{C}_2$. Both results extend to an arbitrary number of excluded cycles (instead of "three" and "two") of lower bounded lengths.

In $\mathcal{C}_3$ and $\mathcal{C}_4$, we characterize the "local" structure of graphs with large treewidth. Explicitly, given a graph H, we prove the following:

(a) Every (theta, K_3)-free graph of large enough treewidth has an induced subgraph isomorphic to H, if and only if H is a K_3-free chordal graph (that is, a forest).

(b) Every (even hole K_4)-free graph of large enough treewidth has an induced subgraph isomorphic to H, if and only if H is a K_4-free chordal graph.

We generalize both (a) and (b) to the "right" class of K_t-free graphs for all t. We also derive, from a very special case of (b), one of the two conjectures of Sintiari and Trotignon on even-hole-free graphs of large treewidth.

Acknowledgments

First and foremost, I would like to extend my gratitude and respect to my advisor Sophie Spirkl. Her dedicated supervision and support have made the past four years some of the most productive and fulfilling in my journey as a mathematician.

I thank Therese Biedl, Zdeněk Dvořák, Peter Nelson and Luke Postle for serving as my thesis examiners. Therese's comments were particularly thorough and resulted in numerous improvements.

I am thankful to Ramin Javadi for introducing me to the world of research, Joseph Cheriyan for funding my first couple of years at Waterloo, Maria Chudnovsky for her generosity with her time and experience, and Paul Seymour for never letting me down when I would ambush him with my questions and thoughts.

I am grateful to my parents, Sepideh and Majid, for setting me on the road and all the sacrifices they made to keep me on track, and my sister Setareh and my brother Sahab for never leaving me alone (even when I wanted them to).

Last but not least, my heartfelt appreciation goes to my wife and my soulmate, Hoda, for her unwavering patience and care. None of this would have been possible without her companionship.

Table of Contents

Preliminaries

Some basic (often customized) notations and definitions are given below. For the standard graph-theoretic terminology, the reader is referred to [22]. Further specialized definitions will appear in later chapters immediately before their first application.

1. Numbers and sets. For an integer n, we denote by $[n]$ the set of all positive integers that are less than or equal to n. In particular, we have $[n] = \varnothing$ if and only if $n \leq 0$. The set of all positive integers is denoted by $\mathbb{N}$.

Let X be a set. We denote by 2^X the set of all subsets of X. For $k \in \mathbb{N} \cup \{0\}$, by a *$k$-subset* of X we mean a subset of X with $|X| = k$. The set of all k-subsets of X is denoted by $X^{(k)}$.

2. Graphs. Throughout, by a *graph* we mean a finite graph with at least one vertex, and a *class* (or *set*) of graphs is always assumed to be an isomorphism quotient. The vertex set of a graph G is denoted by $V(G)$ and the edge set of G is denoted by $E(G)$. The edges of G are treated as 2-subsets $V(G)$; that is, we have $E(G) \subseteq V(G)^{(2)}$. In particular, graphs have no "loops" or "parallel edges." For $v \in V(G)$ and $e \in E(G)$, we say v *is an end of* e or v *is incident with* e if $v \in e$. For a subset $\mathcal{X} \subseteq 2^{V(G)}$, we define

$$V(\mathcal{X}) = \bigcup_{X \in \mathcal{X}} X.$$

The *line graph* of a graph G is the graph with vertex set $E(G)$ and edge set as follows (see Figure 2):

$$\{\{\{u, v\}, \{u', v'\}\} \in E(G)^{(2)} : \{u, v\} \cap \{u', v'\} \neq \emptyset\}.$$

3. Induced subgraphs. Let G be a graph. Recall that a *minor* of G is a graph obtained from G by a sequence of vertex-deletions, edge-deletions and edge-contractions, whereas a *subgraph* of G is a minor of G which has been obtained by only deleting vertices and edges. For an edge e of G, we write $G - e$ for the graph obtained from G by removing the edge e; that is, we have $V(G - e) = V(G)$ and $E(G - e) = E(G) \setminus \{e\}$.

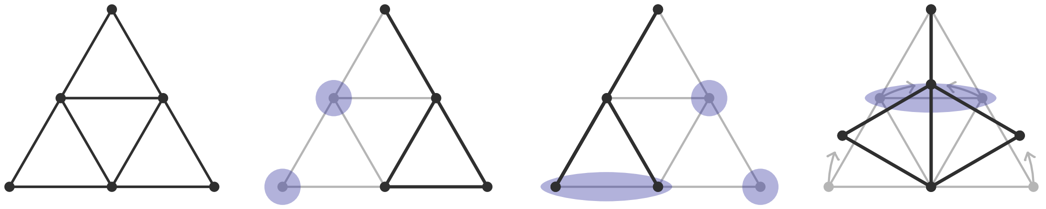

Figure 1: From Left to right: A graph G, an induced subgraph of G, a subgraph of G and a minor of G.

However, we will avoid deleting and contracting the edges of a graph almost entirely. Instead, we are interested in removing vertices: for $X \subseteq V(G)$, we denote by $G \setminus X$ the graph obtained from G by removing X. Therefore, we have $V(G \setminus X) = V(G) \setminus X$ and

$$E(G \setminus X) = E(G) \cap (V(G) \setminus X)^{(2)}.$$

By an *induced subgraph* of G, we mean the graph $G \setminus X$ for some $X \subseteq V(G)$. It follows that induced subgraphs of G are exactly the subgraphs that are obtained by only removing vertices from G (see Figure 1).

For $X \subseteq V(G)$, the *subgraph of G induced by X* is the induced subgraph $G \setminus (V(G) \setminus X)$ of G, for which the standard notation is $G[X]$. The discussion in this thesis revolves almost entirely around induced subgraphs; so much so that for a graph G and a subset X of $V(G)$, we use X and $G[X]$ interchangeably to denote the subgraph of G induced by X.

Given a graph H, we say G is *H-free* if G has no induced subgraph isomorphic to H. For a set $\mathcal{H}$ of graphs, we say a graph G is *$\mathcal{H}$-free* if G is H-free for every $H \in \mathcal{H}$. We say a graph class $\mathcal{C}$ is *hereditary* if $\mathcal{C}$ is closed under taking induced subgraphs. Equivalently, a graph class $\mathcal{C}$ is hereditary if $\mathcal{C}$ is the class of all $\mathcal{H}$-free graphs for some set $\mathcal{H}$ of graphs. In this context, we insist on calling $\mathcal{H}$ a "set" to distinguish it from the "class" $\mathcal{C}$ (but we emphasize there is *no* other technical difference between these two terms).

4. Adjacency. Let G be a graph. For distinct vertices $u, v \in V(G)$, we say u and v are *adjacent in G* if $\{u, v\} \in E(G)$. In this case, we often denote the edge $\{u, v\}$ by uv.

Let $x \in V(G)$. We say a vertex $y \in V(G)$ is a *neighbor of x in G* if x and y are adjacent in G. The set of all neighbor of x in G is denoted by $N_G(x)$, and the *degree of x in G* is the number $|N_G(x)|$ of the neighbors of x in G. If the degree of x in G is zero, one, or at least

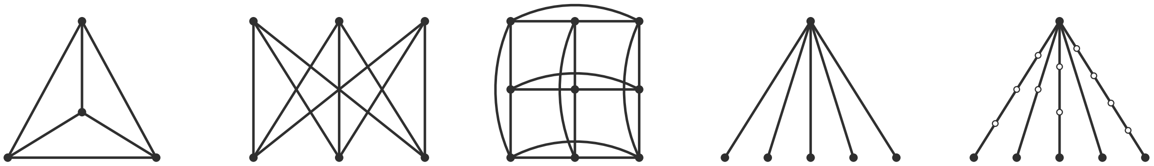

Figure 2: From left to right: the complete graph K_4, the complete bipartite graph $K_{3,3}$, the line graph of $K_{3,3}$, the star $K_{1,5}$ and a subdivision of $K_{1,5}$.

three, respectively, then we say that x is an *isolated vertex* in G, a *leaf* in G or a *branch vertex* in G. We also write $N_G[x] = N_G(x) \cup \{x\}$. For $Y \subseteq V(G)$ (which may or may not contain x), we write $N_Y(x) = N_G(x) \cap Y$ and $N_Y[x] = N_Y(x) \cup \{x\}$. We say *$x$ is complete to Y in G* if $N_Y(x) = Y$, and we say *x is anticomplete to Y in G* if $N_G[x] \cap Y = \emptyset$. In particular, if $x \in Y$, then x is neither complete nor anticomplete to Y in G.

Similarly, for a subset X of $V(G)$, we denote by $N_G(X)$ the set of all vertices in $V(G) \setminus X$ with at least one neighbor in X, and we write $N_G[X] = N_G(X) \cup X$. For $Y \subseteq V(G)$, we write $N_Y(X) = N_G(X) \cap Y$ and $N_Y[X] = N_Y(X) \cup X$. We say *X and Y are complete in G* if every vertex in X is complete to Y in G, and we say *X and Y are anticomplete in G* if every vertex in X is anticomplete to Y in G. In particular, if X and Y are either complete or anticomplete in G, then $X \cap Y = \emptyset$.

5. Special graphs and induced subgraphs. Let $t \in \mathbb{N}$. The *complete graph K_t* is the unique (up to isomorphism) t-vertex graph in which every two distinct vertices are adjacent (see Figure 2). For a graph G, a *clique* in G is a set of pairwise adjacent vertices in G, and a *stable set* in G is a set of pairwise non-adjacent vertices in G. We note that the empty set is both a clique and a stable set in G. For $t \in \mathbb{N} \cup \{0\}$, by a *$t$-clique* in G we mean a clique in G of cardinality t. A 3-clique is often called a *triangle*.

For $s, t \in \mathbb{N}$, the *complete bipartite graph $K_{s,t}$* is the unique (up to isomorphism) bipartite graph with a bipartition (S, T) where $|S| = s$, $|T| = t$ and S and T are complete in $K_{s,t}$. For every $t \geq 3$, we refer to every graph isomorphic to the complete bipartite graph $K_{1,t}$ as a *star* (see Figure 2).

Let P be a graph which is a path. Then we write, for $t \in \mathbb{N}$,

$$P = p_1\text{-}\cdots\text{-}p_t$$

to mean $V(P) = \{p_1, \ldots, p_t\}$, and for all $i, j \in [t]$, the vertices p_i and p_j are adjacent in P if and only if $|i - j| = 1$. We call the vertices p_1 and p_t the *ends of P*, and we say P is a

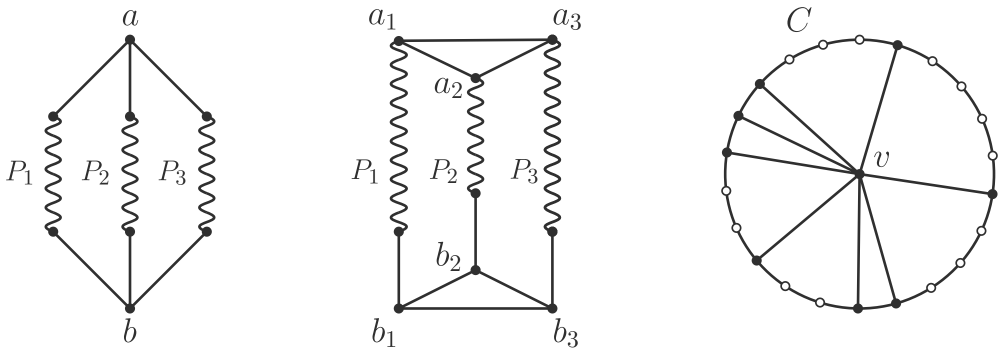

Figure 3: From left to right: A theta, a prism and an even wheel. Squiggly lines represent paths of arbitrary (possibly zero) length.

path from p_1 *to* p_t or a *path between* p_1 *and* p_t. We refer to $V(P) \setminus \{p_1, p_t\}$ as the *interior of* P and denote it by P^*. The *length* of a path is its number of edges. Given a graph G, by a *path in* G we mean an induced subgraph of G which is a path. If $\mathcal{P}$ is a set of paths in G, then we write $\mathcal{P}^* = \{P^* : \mathcal{P} \in \mathcal{P}\}$.

Similarly, for $t \in \mathbb{N} \setminus \{1, 2\}$, given a t-vertex graph C which is a cycle, we write

$$C = c_1\text{-}\cdots\text{-}c_t\text{-}c_1$$

to mean $V(C) = \{c_1, \ldots, c_t\}$, and for all $i, j \in [t]$, the vertices c_i and c_j are adjacent in G if and only if $|i - j| \in \{1, t - 1\}$. The *length* of a cycle is its number of edges (which is the same as its number of vertices). For a graph G, a *cycle in* G is a subgraph of G which is a cycle, and a *hole* in G is an induced subgraph of G which is a cycle of length at least four. An *even hole* in a graph is a hole in G of even length.

A *theta* is a graph Θ consisting of two non-adjacent vertices a, b, called the *ends of* Θ, and three pairwise internally disjoint paths P_1, P_2, P_3 of length at least two in Θ from a to b, called the *paths of* Θ, such that P_1^*, P_2^*, P_3^* are pairwise anticomplete in Θ (see Figure 3). A *prism* is a graph Π consisting of two disjoint triangles $\{a_1, a_2, a_3\}, \{b_1, b_2, b_3\}$ called the *triangles of* Π, and three pairwise disjoint paths P_1, P_2, P_3 in Π, called the *paths of* Π, where P_i has ends a_i, b_i for each $i \in \{1, 2, 3\}$, and for distinct $i, j \in \{1, 2, 3\}$, a_ia_j and b_ib_j are the only edges of Π with an end in P_i and an end in P_j (see Figure 3). Given a graph G, by a *theta in* G we mean an induced subgraph of G which is a theta, and a *prism in* G is an induced subgraph of G which is a prism. Observe that a graph is a prism if and only if it is the line graph of a theta. Also, thetas and prisms contain even holes.

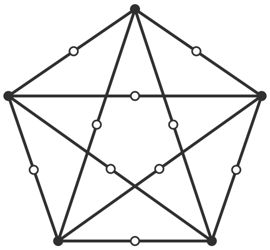

Figure 4: The 1-subdivision of K_5.

Let G be a graph. A *wheel* in G is a pair (C, v) where C is a hole in G and v is a vertex in $V(G) \setminus C$ with at least three neighbors in C. An *even wheel* in G is a wheel (C, v) where v has an even number of (and so at least four) neighbors in C (see Figure 3). One may observe that if a graph G contains an even wheel, then G contains an even hole. In conclusion, if G is an even-hole-free graph, then G is (C_4, theta, prism, even wheel)-free (where C_4 is the four-vertex cycle). A substantial part of this thesis, including the last four Chapters 7–10, will be occupied with results about the structural properties of even-hole-free graphs. We prove those results in the modestly larger class of (C_4, theta, prism, even wheel)-free graphs.

6. Subdivisions. Recall that a *subdivision* of a graph G is a graph obtained from G by replacing the edges of G with pairwise internally disjoint paths of non-zero length between the corresponding ends. For instance, subdivided stars are exactly the trees with one branch vertex (see Figure 2), and thetas are exactly the subdivisions of $K_{2,3}$.

Let G be a graph and let G' be a subdivision of G. We refer to the paths in G' replacing the edges of G as the *subdivision paths*. For $p \in \mathbb{N} \cup \{0\}$, we say G' is a *p-subdivision*, a *$(\leq p)$-subdivision* and a *$(\geq p)$-subdivision* of G, respectively, if all subdivision paths in G' have length exactly, at most and at least $p+1$. We also say that G' is a *proper subdivision* of H if all subdivision paths in G' have length at least two. It follows that G' is a proper subdivision of G if and only if G' is a (≥ 1)-subdivision of G, if and only if G' is a subdivision of the 1-subdivision of G (see Figure 4).

> *One cannot invent the structure of an object. The most we can do is to patiently bring it to the light of day, with humility.*
>
> – Alexander Grothendieck

Chapter 1

Introduction

Structural graph theory stands on two main pillars: graph minor theory and the theory of induced subgraphs. In this thesis, we pursue the analog of a foundational result of Robertson and Seymour on graph minors, known as the *Grid Theorem* [61], in the realm of induced subgraphs.

The Grid Theorem (see Theorem 1.6 below) is about the interplay between the graph minors and a graph parameter called the *treewidth*. The treewidth of a graph G measures the structural complexity of G, and so the smaller the treewidth of G the easier it is to study the properties of G from various perspectives. The Grid Theorem in turn tells us exactly when to expect a graph G to have small treewidth, provided G is coming from a graph class that is closed under taking minors (or subgraphs, as it turns out to be equivalent). The question we study is: What if the home class of G is closed under taking induced subgraphs? What can be said then about the obstructions preventing G from having small treewidth?

To make this precise, we begin by motivating and defining the notion of treewidth. Among multiple equivalent definitions for this parameter, there are two definitions due to Robertson and Seymour [60, 61] that are quite popular: one using the "tree decompositions" and the other using the "clique sums." We will hint at tree decompositions in Chapter 3. But we believe the question of "why is treewidth natural to define?" will be best answered if this notion is viewed as stemming from a classical result known as the *Helly property of subtrees*. We include a proof for the sake of completeness:

Proposition 1.1 (Folklore, see [32, 35, 39])**.** *Let T be a tree and let $\mathcal{T}$ be a non-empty set of subtrees of T. Assume that every pair of subtrees in $\mathcal{T}$ share at least one vertex. Then there is a vertex of T that belongs to all the subtrees in $\mathcal{T}$.*

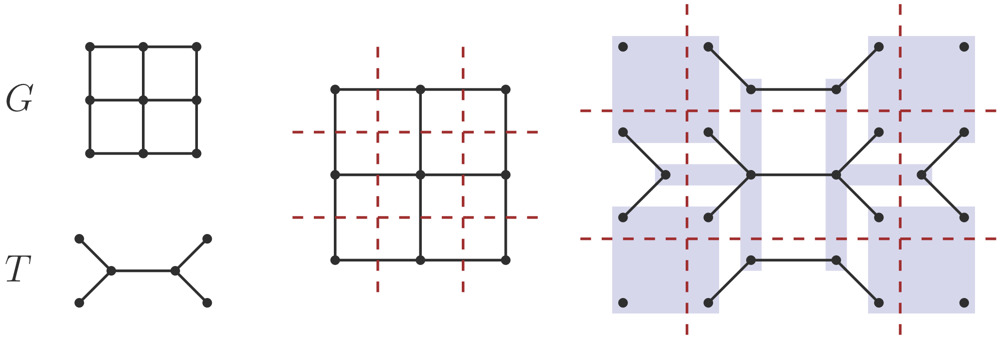

Figure 1.1: A graph G (of treewidth three), a tree T and a T-subtree representation of G where every vertex of T belongs to four subtrees.

Proof. We proceed by induction on $|V(T)|$. The result is trivial if $|V(T)| = 1$. Assume that $|V(T)| > 1$. Then we may choose a leaf x of T. Let y be the unique neighbor of x in T. If the subtree of T with vertex set $\{x\}$ belongs to $\mathcal{T}$, then by the assumption, the vertex x belongs to all the subtrees in $\mathcal{T}$. Otherwise, every subtree in $\mathcal{T}$ which has the vertex x should also contain the vertex y. But now $\mathcal{T}' = \{S \setminus \{x\} : S \in \mathcal{T}\}$ is a non-empty set of pairwise intersecting subtrees of the tree $T \setminus \{x\}$, and so the result follows from the induction hypothesis applied to $T \setminus \{x\}$ and $\mathcal{T}'$. ■

A natural question to ask is: What if only *some* pairs of subtrees in $\mathcal{T}$ are required to intersect (while other pairs are free to intersect or not)? Can we then arrange for every vertex of T to fall into fewer (than all) subtrees in $\mathcal{T}$? How many fewer?

This is precisely answered by the treewidth: for every choice of some 2-subsets of a set $\mathcal{T}$ declared as the "intersecting pairs," the "treewidth of that choice" is the smallest integer k (minus 1, to be accurate) such that $\mathcal{T}$ may be realized as a set of subtrees of some tree T where every vertex of T belongs to at most k subtrees in $\mathcal{T}$. On the other hand, every choice of some 2-subsets of a set $\mathcal{T}$ defines a graph with vertex set $\mathcal{T}$, and vice versa. This explains why treewidth is a graph parameter.

The formal definition should now be easy to guess. Given a graph G and a tree T, a *T-subtree representation of G* is a tuple $(T_v : v \in V(G))$ of subtrees of T such that for every edge $uv \in E(G)$, we have $V(T_u) \cap V(T_v) \neq \varnothing$. The *treewidth* of a graph G, denoted $\operatorname{tw}(G)$, is the smallest integer $w \in \mathbb{N}$ for which one may choose a tree T and a T-subtree representation $(T_v : v \in V(G))$ of G such that every vertex of T belongs to at most $w + 1$ subtrees among $(T_v : v \in V(G))$; see Figure 1.1.

In this language, Proposition 1.1 says that complete graphs have arbitrarily large treewidth. It turns out that the same is true for complete bipartite graphs.

Proposition 1.2 (Folklore; see [22])**.** *For every* $t \in \mathbb{N}$*, we have* $\mathrm{tw}(K_{t+1}) = \mathrm{tw}(K_{t,t}) = t$*.*

Proof. We first prove the upper bounds. Let T be a 1-vertex tree, and for every $v \in V(K_{t+1})$, let $T_v = T$. Then $(T_v : v \in V(K_{t+1}))$ is a T-subtree representation of K_t such that every vertex of T belongs to $t+1$ subtrees among $(T_v : v \in V(K_{t+1}))$. It follows that $\mathrm{tw}(K_{t+1}) \leq t$. In order to show that $\mathrm{tw}(K_{t,t}) \leq t$, let (A, B) be a bipartition of $K_{t,t}$, let $a \in A$ and let T be the subgraph of $K_{t,t}$ induced by $\{a\} \cup B$. Then T is a star isomorphic to $K_{1,t}$. In particular, T is a tree. For every $v \in A$, let $T_v = T$, and for every $v \in B$, let $T_v = \{v\}$. It is straightforward to check that $(T_v : v \in A \cup B)$ is a T-subtree representation of G where every vertex of T belongs to at most $t+1$ subtrees among $(T_v : v \in A \cup B)$. Hence, we have $\mathrm{tw}(K_{t,t}) \leq t$.

As for the lower bounds, note that $\mathrm{tw}(K_{t+1}) \geq t$ is immediate from Proposition 1.1. It remains to prove that $\mathrm{tw}(K_{t,t}) \geq t$. To that ends, choose a tree T and a T-subtree representation $(T_v : v \in V(K_{t,t}))$ of $K_{t,t}$ such that every vertex of T belongs to at most $\mathrm{tw}(K_{t,t})+1$ subtrees among $(T_v : v \in V(K_{t,t}))$. Let $\{a_ib_i : i \in [t]\}$ be a perfect matching in $K_{t,t}$. Let $U_0 = T_{a_1}$, let $U_1 = T_{b_1}$ and let $U_i = T_{a_i} \cup T_{b_i}$ for every $i \in [t] \setminus \{1\}$. Since $(T_v : v \in V(K_{t,t}))$ is a T-subtree representation of $K_{t,t}$, it follows that $\{U_0, U_1, \ldots, U_t\}$ is a set of $t+1$ pairwise intersecting subtrees of T. By Proposition 1.1, we have $U_0 \cap U_1 \cap \cdots \cap U_t \neq \emptyset$. It follows for some $x \in V(T)$, we have $x \in V(T_{a_1})$ and $x \in V(T_{b_1})$ and for every $i \in [t] \setminus \{1\}$, either $x \in V(T_{a_i})$ or $x \in V(T_{b_i})$. But then x belongs to at least $t+1$ subtrees among $(T_v : v \in V(K_{t,t}))$, and so $\mathrm{tw}(K_{t,t}) \geq t$. ■

The argument above showing that $\mathrm{tw}(K_{t,t}) \geq t$ can be adapted to prove a stronger fact:

Proposition 1.3 (Folklore; see [22])**.** *Let* G *and* H *be graphs where* G *has a minor isomorphic to* H*. Then we have* $\mathrm{tw}(G) \geq \mathrm{tw}(H)$*.*

Proof. Choose a tree T and a T-subtree representation $(T_v : v \in V(G))$ of G such that every vertex of T belongs to $\mathrm{tw}(G)+1$ subtrees among $(T_v : v \in V(G))$. Since G has a minor isomorphic to H, it follows that there is a minor model of H in G. Explicitly, there are connected and pairwise disjoint induced subgraphs $(G_v : v \in V(H))$ of G such that for every $uv \in V(H)$, there is an edge of G with an end in G_u and an end in G_v. For each $v \in V(H)$, let $U_v = \bigcup_{u \in V(G_v)} T_v$. From the choice of $(T_v : v \in V(G))$, it follows (and we leave it to the reader to check) that $(U_v : v \in V(H))$ is a T-subtree representation of H such that every vertex of T belongs to $\mathrm{tw}(G)+1$ subtrees among $(U_v : v \in V(H))$. But then $\mathrm{tw}(H) \leq \mathrm{tw}(G)$. ■

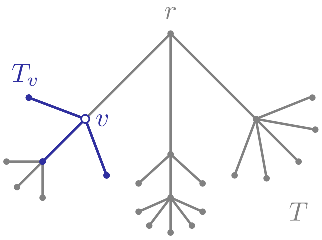

Figure 1.2: Proof of Proposition 1.4.

Treewidth and minors are a match made in heaven. The interplay between the two contributed fundamentally to the development of the graph minors project [59, 60, 61, 62, 63, 65]. The significance of treewidth as a standalone notion has also been unraveled over time. For instance, considerable work [38, 52, 53, 72] on "graph width parameters" has been inspired by the study of treewidth. Various definitions of "width" have also been examined for hypergraphs, matroids and submodular functions [9, 23, 40, 45].

As the name suggests, one may think of "width" parameters as measuring the "thickness" of a general graph compared to certain graphs declared as the "thinnest." Under this analogy, the treewidth displays all graphs on a spectrum of "tree-likeness," and trees are the only (connected) graphs of the smallest possible treewidth. The latter is thanks to the bothersome "+1" in the definition of the treewidth:

Proposition 1.4 (Folklore; see [22])**.** *A graph has treewidth* 1 *if and only if it is a forest.*

Proof. If a graph H is not a forest, then H has a minor isomorphic to K_3, and so $\mathrm{tw}(H) \geq \mathrm{tw}(K_3) = 2$. Assume that H is a forest. Then there is a tree T of which H is an induced subgraph (and so a minor). It follows that $\mathrm{tw}(H) \leq \mathrm{tw}(T)$. Pick a vertex $r \in V(T)$ as the root, and for every $v \in V(T)$, let T_v be the subgraph of T induced by v and its children (see Figure 1.2). Now $(T_v : v \in V(T))$ is a T-subtree representation of T such that every vertex of T belongs to at most two subtrees among $(T_v : v \in V(T))$, as required. ■

Said differently, the smaller the treewidth of a graph, the more tree-like the graph is. This brings on a wealth of results of striking generality [11, 20, 49, 60, 61, 63] on the properties of graphs with bounded treewidth. The following theorem of Courcelle [20] is an illuminating example:

Theorem 1.5 (Courcelle [20])**.** *Every graph property that is expressible in the "monadic second-order logic" can be tested in linear time on graphs of bounded treewidth.*

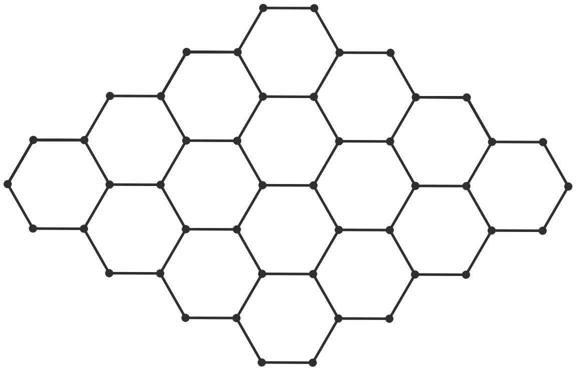

Figure 1.3: The graph $W_{5\times 5}$.

Similar results are also known for other width parameters such as “rankwidth” [54] and “twinwidth” [15]. Courcelle’s theorem has also given rise to a substantial literature at the intersection of mathematical logic and algorithmic combinatorics, occupied by the so-called “Meta Theorems.” See [46] for further details.

The other half of the world is about graphs of large treewidth. In this context, the most important result is Theorem 1.6 below due to Robertson and Seymour [61], known as the *Grid Theorem*. Recall that by Proposition 1.3, if a graph G has a minor isomorphic to a graph H, then the treewidth of G is at least as large as the treewidth of H. In effect, the Grid Theorem pursues the converse. The direct converse to Proposition 1.3 is not true even if $H \in \{K_5, K_{3,3}\}$ and the treewidth of G is much larger than that of H. This is because the t-by-t hexagonal grid (see Figure 1.3) is a planar graph of treewidth t [61, 70] for every $t \in \mathbb{N}$. It follows from Kuratowski’s theorem [48] that the converse to Proposition 1.3 remains false for every choice of H that is not planar. However, according to the Grid Theorem, the situation is different when H is planar:

Theorem 1.6 (Robertson and Seymour [61])**.** *If H is a planar graph, then every graph of sufficiently large treewidth has a minor isomorphic to H.*

The t-by-t hexagonal grid is also known as the *t-by-t wall*, denoted $W_{t\times t}$. One may observe [61, 62] that every planar graph is isomorphic to a minor of $W_{t\times t}$ for some $t \in \mathbb{N}$, and a graph G has a minor isomorphic to $W_{t\times t}$ if and only if G has a subgraph isomorphic to a subdivision of $W_{t\times t}$. Therefore, Theorem 1.6 may be restated as follows:

Theorem 1.7 (Robertson and Seymour [61])**.** *For every $t \in \mathbb{N}$, every graph of sufficiently large treewidth has a minor isomorphic to $W_{t\times t}$, or equivalently, a subgraph isomorphic to a subdivision of $W_{t\times t}$.*

This formulation of the Grid Theorem arms us with a useful view on large treewidth, that it can be blamed on a "local obstruction" with (relatively) large treewidth whose structure is much simpler than the host graph. The graph minors project already contains two remarkable applications of this perspective:

1. The main result of the project is the "Well-Quasi-Order Theorem" for minors: *there is no infinite antichain of (finite) graphs under the partial order of minors.* The proof handles graphs of bounded [62] and unbounded [66] treewidth separately. In the former case, since graphs of bounded treewidth are tree-like, the result follows from a careful adaptation [62] of Kruskal's well-quasi-order theorem for trees [47]. For graphs of large enough treewidth, the Grid Theorem guarantees the existence of a large planar subgraph. This signals a topological orderliness which is then exhaustively worked out in the "Graph Minors Structure Theorem" [58] as the main tool used to conclude the proof [65, 66].
2. Among the highlights of the graph minors project on the algorithmic front, the existence of polynomial-time algorithms for the DISJOINT PATHS PROBLEM and the MINOR TESTING PROBLEM [64] is rather striking from the point of view of computational complexity theory[1]. Both algorithms run in two phases depending on the treewidth of the input graph. In the case of small treewidth, the tree-like shape of the graph allows for an application of the so-called "Dynamic Programming" [11, 64]. For input graphs of large treewidth, it turns out that a sufficiently "central" vertex of the planar subgraph provided by Theorem 1.7 is "irrelevant" from the problem [67], meaning it may be removed without affecting the feasibility of the instance [64].

In yet another formulation, the Grid Theorem yields a characterization of all minor-closed and all subgraph-closed graph classes for which there is a universal upper bound on the treewidth of all graphs in the class. Given a graph class $\mathcal{C}$, we say that $\mathcal{C}$ *has bounded treewidth* if there is a constant $c \in \mathbb{N}$ such that for every $G \in \mathcal{C}$, we have $\mathrm{tw}(G) \leq c$. Using this terminology, the Grid Theorem may equivalently be stated in the following ways:

Theorem 1.8 (Robertson and Seymour [61]). *Let $\mathcal{C}$ be a graph class which is closed under taking minors. Then $\mathcal{C}$ has bounded treewidth if and only if $W_{t\times t} \notin \mathcal{C}$ for some $t \in \mathbb{N}$.*

Theorem 1.9 (Robertson and Seymour [61]). *Let $\mathcal{C}$ be a graph class which is closed under taking subgraphs. Then $\mathcal{C}$ has bounded treewidth if and only if for some $t \in \mathbb{N}$, no subdivision of $W_{t\times t}$ belongs to $\mathcal{C}$.*

---

[1] We leave the reader with following quote by Donald Knuth [43]: "This consequence of Robertson and Seymour's theorem definitely surprised me [$\cdots$] And it tipped the balance, in my mind, toward the hypothesis that P=NP."

In general, for a given width parameter and a given containment relation, by a "grid-type theorem" we mean a characterization of graph classes closed under the corresponding containment (preferably in terms of minimal excluded graphs under the same containment) for which there is a universal upper bound on the corresponding width of all graphs in the class. Specifically, Theorems 1.8 and 1.9, in order, achieve this goal for treewidth and minors and treewidth and subgraphs. Admitting a grid-type theorem is also a feature of several other well-known width parameters and containment relations [14, 29, 30, 31].

As mentioned earlier, our discussion in this thesis is centered around the search for a grid-type theorem for treewidth and induced subgraphs. Within the framework described above, this finally brings us to the overarching question of interest in this thesis: Which hereditary classes have bounded treewidth? Equivalently, we ask:

Question 1.10. *For which sets $\mathcal{H}$ of graphs do $\mathcal{H}$-free graphs have bounded treewidth?*

More precisely, by the ultimate "Grid Theorem for induced subgraph" (for instance, in the title of this thesis), we mean a description of all sets $\mathcal{H}$ of graphs for which the class of all $\mathcal{H}$-free graphs has bounded treewidth[2].

Unlike the case of minors and subgraphs, there is not even a conjecture predicting the exact answer to Question 1.10. On the other hand, some necessary conditions have long been known for a hereditary class to be of bounded treewidth. In fact, our results address more directly a question equivalent to 1.10, asked as Question 1.13 below, which refines Question 1.10 up to certain necessary conditions.

Building our way toward Question 1.13, the story begins with the following result of Lozin and Razgon [50]. It says that, although the full answer to Question 1.10 remains elusive, there is a pretty answer when $\mathcal{H}$ is finite:

Theorem 1.11 (Lozin and Razgon [50])**.** *Let $\mathcal{H}$ be a finite set of graphs. Then the class of all $\mathcal{H}$-free graphs has bounded treewidth if and only if $\mathcal{H}$ contains a complete graph, a complete bipartite graph, a forest each component of which is either a path or isomorphic to a subdivision of $K_{1,3}$, and the line graph of such a forest.*

Moreover, the same argument for the "only if" implication in Theorem 1.11 can be used to derive a necessary condition for bounded treewidth in general hereditary classes. So we might as well refine Question 1.10 up to this necessary condition.

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[2] There remains the question of how explicit of a description would count. This may sound quite subjective at first. However, as blunt as it may appear, we think the answer should be: *the more explicit the better!* Of course the reader is free to judge.

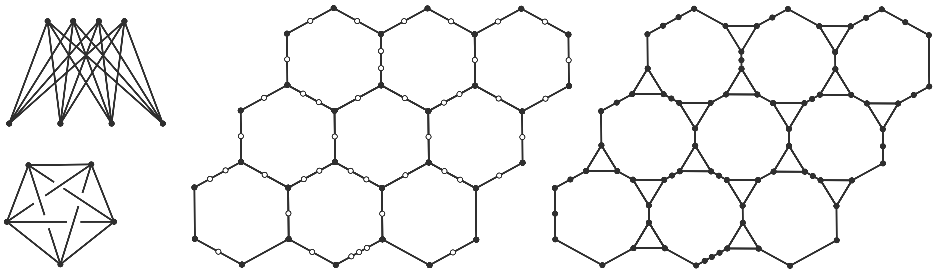

Figure 1.4: The 4-basic obstructions.

Let us elaborate. Recall that for every $t \in \mathbb{N}$, the complete graph K_{t+1}, the complete bipartite graph $K_{t,t}$ and the t-by-t-wall $W_{t\times t}$ all have treewidth t. From Theorem 1.3, it follows that every subdivision of $W_{t\times t}$ has treewidth at least t; in fact, the treewidth of a graph and all its subdivisions are the same (this is well-known and easy to prove; we omit the details). One can also observe (see Figure 1.4) that the line graph of every subdivision of $W_{t\times t}$ has a subgraph isomorphic to some subdivision of $W_{t\times t}$. It follows from Theorem 1.3 that the line graph of every subdivision of $W_{t\times t}$ has treewidth at least t.

We refer to these four types of graphs as the "basic obstructions." For every $t \in \mathbb{N}$, by a *t-basic obstruction* we mean one of the following graphs: K_{t+1}, $K_{t,t}$, all subdivision of $W_{t\times t}$ and the line graphs of all subdivision of $W_{t\times t}$ (see Figure 1.4). We say a graph G is *t-clean* if G has no induced subgraph isomorphic to any of the t-basic obstructions. As discussed above, the t-basic obstructions have treewidth at least t. It follows that if $\mathcal{C}$ is a graph class of bounded treewidth, then there exists $t \in \mathbb{N}$ such that $\mathcal{C}$ contains no t-basic obstructions. For hereditary classes of bounded treewidth, this yields the following necessary condition in line with Question 1.10:

Observation 1.12. *Let $\mathcal{C}$ be a hereditary class of bounded treewidth. Then there exists $t \in \mathbb{N}$ for which every graph in $\mathcal{C}$ is t-clean.*

In particular, it is straightforward to observe that the "only if" implication in the statement of Theorem 1.11 follows from Observation 1.12 (see Figure 1.5).

So it suffices to answer Question 1.10 for those classes which satisfy the condition in Observation 1.12: *for which hereditary classes $\mathcal{C}$ does the class of all t-clean graphs in $\mathcal{C}$ have bounded treewidth for every $t \in \mathbb{N}$?* We call such classes "clean." Given a graph class $\mathcal{C}$, we say $\mathcal{C}$ is *clean* if for every $t \in \mathbb{N}$, there is a constant $w = w(t) \in \mathbb{N}$ such that every t-clean graph $G \in \mathcal{C}$ satisfies $\operatorname{tw}(G) \leq w$. So Question 1.10 will now be superseded by the following refinement: Which hereditary classes are clean? Equivalently, we ask:

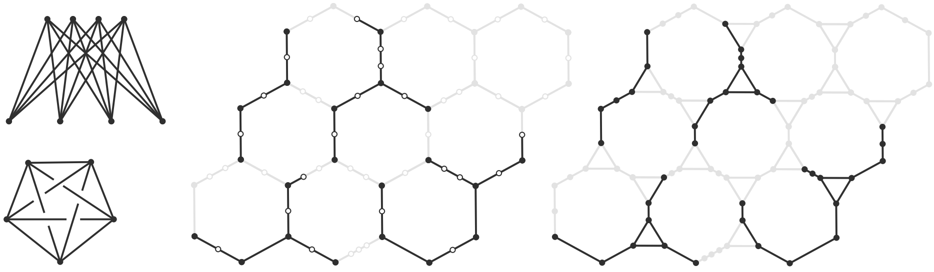

Figure 1.5: Outcomes of Theorem 1.11.

Question 1.13. *For which sets $\mathcal{H}$ of graphs is the class of all $\mathcal{H}$-free graphs clean?*

On par with Question 1.10, the complete answer to Question 1.13 remains out of reach. The contribution of this thesis is a number of dichotomy results approaching that complete answer. We will give a brief outline of our results in a moment. But first let us remark that some partial results on Question 1.13 already exist. For instance, it seems to have long been known in the graph minors community that the answer to Question 1.13 will be *all* classes if "hereditary" is replaced by "proper minor-closed." In particular, see [69] from September 2013 where Paul Seymour mentions this result. See also the mathoverflow post [55] from October 2013 where Cosmin Pohoata poses this as a question and Paul Wollan [74] sketches a proof using the "Flat Wall Theorem" [41].

The first published proof, and a different one at that, is due to Aboulker, Adler, Kim, Sintiari and Trotignon:

Theorem 1.14 (Aboulker, Adler, Kim, Sintiari and Trotignon [1])**.** *Every proper minor-closed class of graphs is clean. Equivalently, for all $k, t \in \mathbb{N}$, there is a constant $f_{1.14} = f_{1.14}(k,t) \in \mathbb{N}$ such that if G is a graph which has no minor isomorphic to K_k and no induced subgraph isomorphic to a subdivision of $W_{t\times t}$ or the line graph of a subdivision of $W_{t\times t}$, then* $\mathrm{tw}(G) \leq f_{1.14}$.

The same authors also conjectured [1] that every graph class of bounded maximum degree is clean, which was later proved by Korhonen [44]:

Theorem 1.15 (Korhonen [44])**.** *Every graph class of bounded maximum degree is clean. Equivalently, for all $d, t \in \mathbb{N}$, there is a constant $f_{1.15} = f_{1.15}(d,t) \in \mathbb{N}$ if G is a graph of maximum degree at most d which has no induced subgraph isomorphic to a subdivision of $W_{t\times t}$ or the line graph of a subdivision of $W_{t\times t}$, then* $\mathrm{tw}(G) \leq f_{1.15}$.

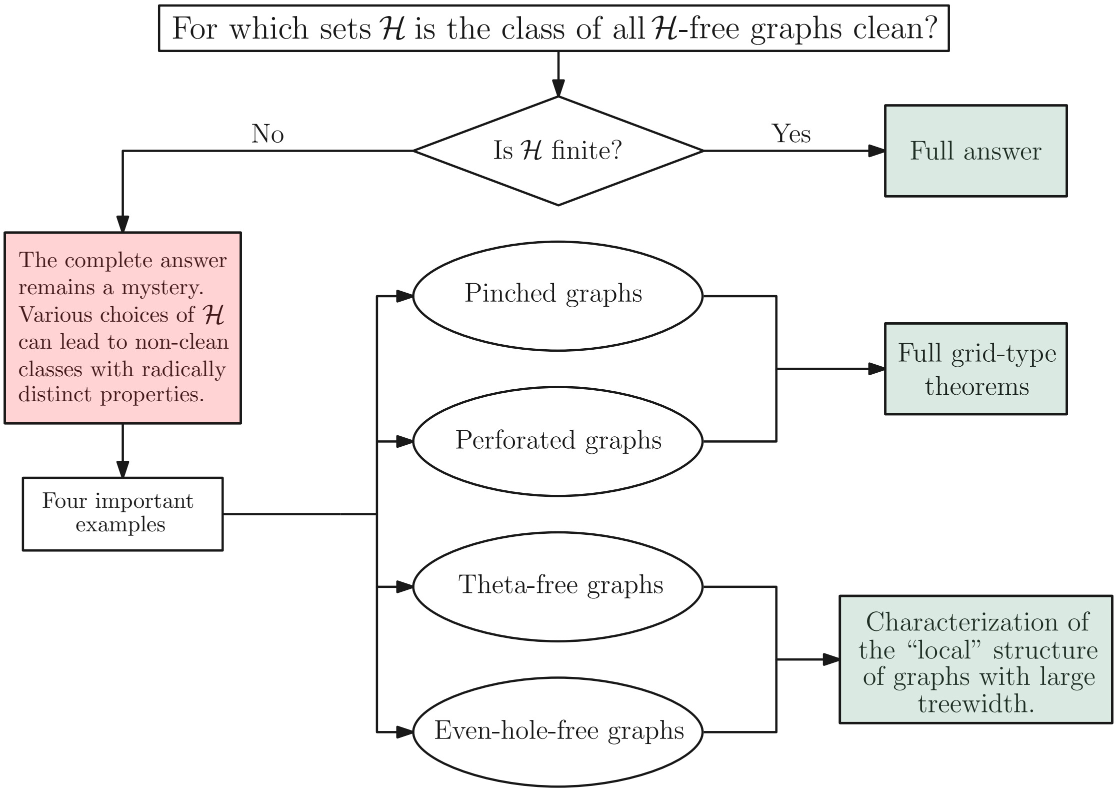

Figure 1.6: A summary of our contribution.

We conclude the introduction with a summary of our results on Question 1.13 (see also Figure 1.6). Then, in the next chapter, we give the formal statement of each individual result and discuss the motivations and the consequences in detail.

The analog of Theorem 1.11 for cleanness (instead of bounded treewidth) is a good starting point to attack Question 1.13. So we ask: *what if $\mathcal{H}$ is finite?* We managed to give a complete answer to Question 1.13 in this case. This result is stated under Theorem 2.4 in Chapter 2, Section 2.1, of which the proof will be given in Chapter 4.

The case where $\mathcal{H}$ is infinite appears to be much more difficult. In one way, this may be blamed on the vast variety of the so-called "non-basic" obstructions that have been discovered so far. These are constructions of 3- or 4-clean graphs with unbounded treewidth each manifesting unique structural properties, and the following four of them have somehow been closer to the center of attention:

- *Arrays* [21, 56];
- *Occultations* [13];
- *Theta-free Layered Wheels* [71]; and
- *Even-hole-free Layered Wheels* [71].

The above constructions, in order, come from the following four non-clean hereditary classes that are defined by an infinite set $\mathcal{H}$ of excluded graphs:

- *Pinched graphs*;
- *Perforated graphs*;
- (Theta, K_3)-free graphs; and
- (Even-hole, K_4)-free graphs.

(The exact definitions will appear in Chapter 2.)

We undertake an exclusive study of each class. Specifically, for pinched graphs and perforated graphs, we prove full grid-type theorems. This means we characterize exactly *all* sets $\mathcal{H}$ of graphs for which the class of $\mathcal{H}$-free pinched graphs has bounded treewidth, and similarly, we characterize *all* sets $\mathcal{H}$ of graphs for which the class of $\mathcal{H}$-free perforated graphs has bounded treewidth. These two results, in order, are stated under Theorems 2.7 and 2.10 in Chapter 2, Sections 2.2 and 2.3. The proofs of Theorems 2.7 and 2.10 (and in fact, of certain natural extensions of those theorems) will appear in Chapters 5 and 6, respectively.

For (theta, K_3)-free graphs and (even-hole, K_4)-free graphs, proving a full grid-type theorem seems impossibly difficult. Instead, we characterize the "local" structure of graphs with large treewidth in each class. More explicitly, we identify all graphs H for which every (theta, K_3)-free graph of sufficiently large treewidth has an induced subgraph isomorphic to H, and similarly, we identify all graphs H for which every (even-hole, K_4)-free graph of sufficiently large treewidth has an induced subgraph isomorphic to H. These two results, in order, are stated under Theorems 2.16 and 2.20 in Chapter 2, Section 2.4. The proof of Theorems 2.16 and 2.20 (and again, of certain natural extensions of those theorems) will occupy the last four chapters of this thesis, eventually finished in Chapter 9 (except Chapter 10 completes an important technical step of those proofs that we will postpone until the end to keep the rest of the proofs flowing).

Chapter 2

Overview of the results

2.1 A few graphs more than the basic obstructions

Our first main result is a characterization of all finite sets $\mathcal{H}$ of graphs for which the class of all $\mathcal{H}$-free graphs is clean. Perhaps we should step back and first ask the same question for $\mathcal{H} = \emptyset$: *Is the class of all graphs clean?*

The answer is negative, as there are several constructions [13, 21, 56, 71] of t-clean graphs with arbitrarily large treewidth for $t \in \{3, 4\}$. In this section and the one after, we will focus on "arrays" [21, 56] as the simplest example of 4-clean graphs with arbitrarily large treewidth. The definition is as follows. For $s \in \mathbb{N}$, an *s-array* is a graph A whose vertex set can be partitioned into $s+1$ subsets $S, L_1, \ldots, L_s$, where $|S| = s$ and S is a stable set, $L_1, \ldots, L_s$ are pairwise disjoint and anticomplete paths in A, and the following hold.

- For every $i \in [s]$, every vertex in S has *at least* one neighbor in L_i; and
- There is a linear order $x_1, \ldots, x_s$ of the vertices in S such for every $i \in [n]$, the neighbors of $x_1, \ldots, x_s$ in L_i appear along L_i in this order (in particular, each vertex of L_i is adjacent to at most of one $x_1, \ldots, x_s$).

See Figure 2.1. A slightly weaker notion of arrays was first introduced by Pohoata [56] in 2014, and rediscovered by Davies [21] in 2022 (we will define these graphs in Section 5.1). Both authors observed that their "arrays" are 4-clean with arbitrarily large treewidth. This remains true for our extended definition of arrays:

Theorem 2.1. *For every $s \in \mathbb{N}$, every s-array is a 4-clean graph of treewidth at least n.*

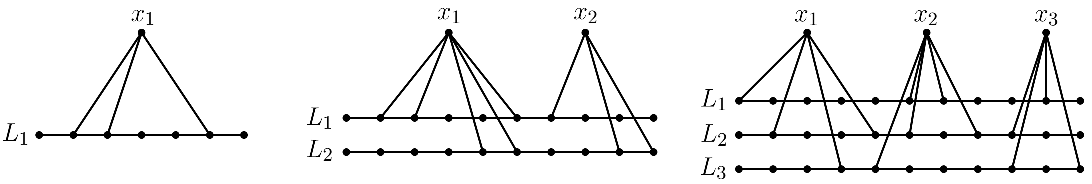

Figure 2.1: Left to right: examples of s-arrays for $s = 1, 2, 3$.

We will prove (a strengthening of) Theorem 2.1 in Chapter 5 under Proposition 5.2, where we will also re-define arrays in the more technical language of "constellations."

From Theorem 2.1, it follows that for every graph class $\mathcal{C}$ that is clean, there are only finitely many $s \in \mathbb{N}$ for which there is an s-array in $\mathcal{C}$. For hereditary classes, we may rephrase this necessary condition as follows:

Observation 2.2. *Let $\mathcal{H}$ be a set of graphs such that the class of $\mathcal{H}$-free graphs is clean. Then there is no $\mathcal{H}$-free s-array except possibly for finitely many $s \in \mathbb{N}$.*

Moreover, if $\mathcal{H}$ is a singleton, then we may describe $\mathcal{H}$ more explicitly:

Theorem 2.3. *Let H be a graph such that the class of all H-free graphs is clean. Then every component of H is either a path or a subdivided star.*

Proof. Suppose not. It follows that either there is a cycle in H, or there is a component of H which has at least two branch vertices. Choose $\ell \in \mathbb{N}$ such that there is no cycle of length more than ℓ in H, and every two branch vertices in the same component of H are at a distance less than ℓ in H.

By the assumption, there exists $w \in \mathbb{N}$ such that every 3-clean H-free graph has treewidth less than w. In particular, by Theorem 2.1, there is no H-free w-array.

For each $k \in \{1, 2\}$, let A_k be a w-array with $x_1, \ldots, x_w, L_1, \ldots, L_w$ as in the definition, such that

- for all $i, j \in [w]$, the set $N_{L_j}(x_i)$ is a k-clique in A_k; and
- for all $i \in [w-1]$ and $j \in [w]$, the unique path in A_k from x_i to x_{i+1} with interior in L_j has length ℓ.

See Figure 2.2. It follows that for each $k \in \{1, 2\}$, there is an induced subgraph H_k of A_k that is isomorphic to H. We claim that:

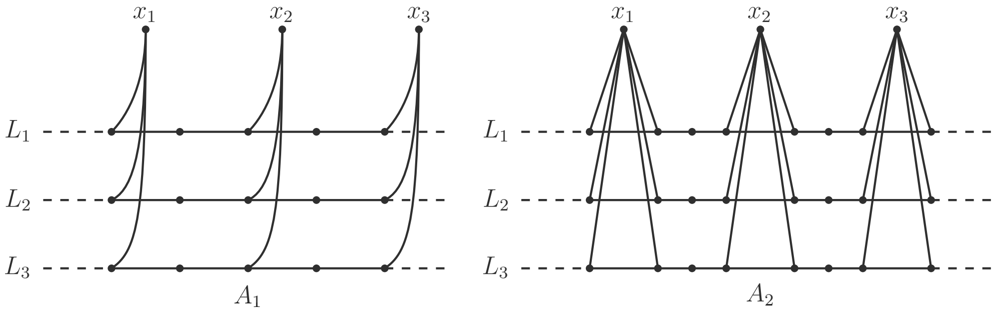

Figure 2.2: The arrays A_1 and A_2 from the proof of Theorem 2.3 (for $\ell = 4$ and $w = 3$).

(1) *H is a forest.*

Note that there is no cycle of length smaller than 2ℓ in A_1, and so there is no such cycle in H_1. Also, recall that there is no cycle of length more than ℓ in H. Therefore, there is no cycle in H at all. This proves (1).

By (1), there is a component of H which has at least two branch vertices. From the choice of ℓ, it follows that there are two branch vertices u, v in the same component of H_2 that are at distance less than ℓ. Consequently, there is a triangle T in A_2 such that $(T \cap \{u, v\}) \setminus \{x_1, \ldots, x_w\} \neq \emptyset$. This, along with the fact that both u and v have degrees more than two in H_2, implies that $T \subseteq V(H_2)$. But then there is a triangle in H, a contradiction with (1). This completes the proof of Theorem 2.3. ■

The necessary condition from Observation 2.2 is not sufficient in general. For instance, if $\mathcal{H}$ is the set of all thetas, then there is no $\mathcal{H}$-free s-array for any $s \geq 3$, whereas the class of all theta-free graphs is not clean [71] (further discussion will appear in Section 2.4).

It turns out, however, that the necessary condition from Observation 2.2 is in fact sufficient when $\mathcal{H}$ is finite:

Theorem 2.4. *Let $\mathcal{H}$ be a finite set of graphs. Then the class of all $\mathcal{H}$-free graphs is clean if and only if there is no $\mathcal{H}$-free s-array except possibly for finitely many $s \in \mathbb{N}$.*

In particular, we have:

Theorem 2.5. *Let H be a graph. Then the class of all H-free graphs is clean if and only if every component of H is either a path or a subdivided star.*

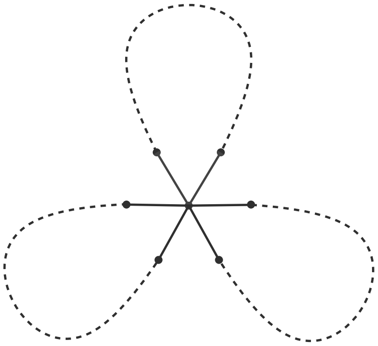

Figure 2.3: A graph in $\mathcal{H}_1$. Dashed lines represent paths of non-zero length.

Both Theorems 2.4 and 2.5 will be proved in Chapter 4. We remark that, unlike the case of a single graph (which admits Theorem 2.3), we were not able to find an explicit description of those finite sets $\mathcal{H}$ which satisfy the condition from Observation 2.2. Specifically, the following remains open as far as we know:

Question 2.6. *Is there an algorithm that, for a finite set $\mathcal{H}$ of graphs, decides in polynomial time (in terms of $\sum_{H \in \mathcal{H}} |V(H)|$) whether the class of all $\mathcal{H}$-free graphs is clean?*

2.2 Pinched graphs

We now turn to $\mathcal{H}$-free graphs for certain choices of an infinite set $\mathcal{H}$ of graphs. As we will argue in a moment, our first choice of $\mathcal{H}$ is inspired by the properties of the arrays: let $\mathcal{H}_1$ be the set of all graphs which consist of three induced cycles, all sharing a common vertex and otherwise pairwise disjoint and anticomplete (see Figure 2.3). We say a graph G is *pinched* if G is $\mathcal{H}_1$-free.

It follows that complete graphs and complete bipartite graphs are pinched. Moreover, every graph that is not pinched must have a vertex of degree at least six, so subdivided walls and their line graphs are also pinched. In other words, the class of all pinched graphs contains all the basic obstructions. It is therefore natural to ask: *Is the class of all pinched graphs clean?* The answer is negative, as certain arrays are also pinched.

The special arrays which we are interested in are the original ones due to Pohoata [56] and Davies [21]. For every integer $s \in \mathbb{N}$, we define PD_s to be the graph whose vertex set can be partitioned into $s+1$ subsets $S, L_1, \dots, L_s$, where $|S| = s$ and S is a stable set, $L_1, \dots, L_s$ are pairwise disjoint and anticomplete paths in A each on s vertices, and the following hold.

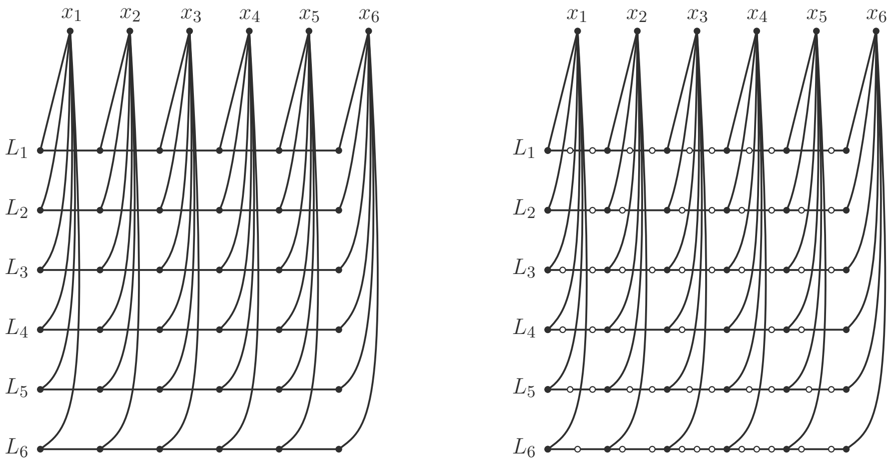

Figure 2.4: The graph PD_6 (left) and an expansion of PD_6 (right).

- For every $i \in [s]$, every vertex in S has *exactly* one neighbor in L_i; and
- There is a linear order $x_1, \dots, x_s$ of the vertices in S such for every $i \in [s]$, the neighbors of $x_1, \dots, x_s$ in L_i appear along L_i in this order (in particular, each vertex of L_i is adjacent to exactly of one $x_1, \dots, x_s$).

By an *expansion of* PD_s we mean a graph obtained from PD_s by subdividing the edges of the paths $L_1, \dots, L_s$ (see Figure 2.4). It follows that for every $s \in \mathbb{N}$, every expansion of PD_s is an s-array, and so by Theorem 2.1, every expansion of PD_s is a 4-clean graph of treewidth at least s.

Also, the expansions of PD_s are pinched for all $s \in \mathbb{N}$. We will prove (a strengthening of) this result in Chapter 5 under Proposition 5.2. Here we provide a brief version; hoping to demonstrate how being pinched follows quite naturally from the construction of the Pohoata-Davies graphs. Suppose for some $s \in \mathbb{N}$, there is an expansion G of PD_s which is not $\mathcal{H}_1$-free. Then there are three induced cycles C_1, C_2, C_3 in G, all sharing a single common vertex and otherwise pairwise disjoint and anticomplete in G. Since x has degree at least six and C_1, C_2, C_3 are cycles, it follows that there are distinct $i, i_1, i_2, i_3 \in [s]$ such that $V(C_1) \cap V(C_2) \cap V(C_3) = \{x_i\}$ and $x_{i_j} \in V(C_j) \setminus \{x_j\}$ for all $j \in [3]$. We may assume that $i_1 < i_2 < i$. But then by the construction of PD_s, x_{i_2} has a neighbor in $V(C_1) \setminus \{x_{i_1}\}$, a contradiction with and $V(C_1) \setminus \{x_{i_1}\}$ and $V(C_2) \setminus \{x_{i_2}\}$ being anticomplete.

The above discussion shows that, on top of the basic obstructions, the Pohoata-Davies graphs form a new family of obstructions to bounded treewidth in the class of all pinched graphs. We complement this observation by showing that the Pohoata-Davies graphs are *the only* "non-basic" obstructions in the class of all pinched graphs.

In fact, we prove more. For $c \in \mathbb{N}$, a graph G is *c-pinched* if no induced subgraph of G consists of c cycles, all sharing a vertex and otherwise pairwise disjoint and anticomplete (so G is pinched if and only if G is 3-pinched). Our main result on pinched graphs is the following grid-type theorem:

Theorem 2.7. *For all $c, s, t \in \mathbb{N}$, there is a constant $f_{2.7} = f_{2.7}(c, s, t) \in \mathbb{N}$ such that every t-clean c-pinched graph G with $\operatorname{tw}(G) > f_{2.7}$ has an induced subgraph isomorphic to an expansion of the graph PD_s.*

Note that, for $s \geq 3$, the expansions of PD_s are not 2-pinched. Therefore, it follows from Theorem 2.7 that:

Corollary 2.8. *The class of all 2-pinched graphs is clean.*

We will prove Theorem 2.7 in Chapter 5; in fact, we prove a strengthening where the length of the excluded cycles may be lower bounded. We will also derive the corresponding strengthening of Corollary 2.8.

2.3 Perforated graphs

Our second choice of an infinite set of excluded graphs is inspired by a construction from [13], which we will describe in a moment. Let $\mathcal{H}_2$ be the set of all 2-regular graphs with exactly two components. We say a graph G is *perforated* if G is $\mathcal{H}_2$-free. Equivalently, G is perforated if there are no two disjoint and anticomplete cycles in G.

Unlike the case of pinched graphs, complete graphs and complete bipartite graphs are the only basic obstructions which are perforated: for every $t \geq 4$, subdivisions of $W_{t\times t}$ and their line graphs are not perforated. One may also observe that n-arrays are not perforated for any $n \geq 4$. Therefore, it makes sense to ask: *for every $t \in \mathbb{N}$, does every perforated graph of large enough treewidth have an induced subgraph isomorphic to K_{t+1} or $K_{t,t}$?*

The answer is negative, provided by the following construction from [13]. For $s \in \mathbb{N}$, an *s-occultation*[1] is a graph $\mathfrak{o}$ whose vertex set can be partitioned into a stable set S in $\mathfrak{o}$ of cardinality s and a path L in $\mathfrak{o}$, such that the following hold (see Figure 2.5).

---

[1] A name by us, and not the authors of [13]. Also, from Wikipedia: "An *occultation* is an event that occurs when one object is hidden from the observer by another object that passes between them. The term is often used in astronomy, but can also refer to any situation in which an object in the foreground blocks from view (occults) an object in the background."

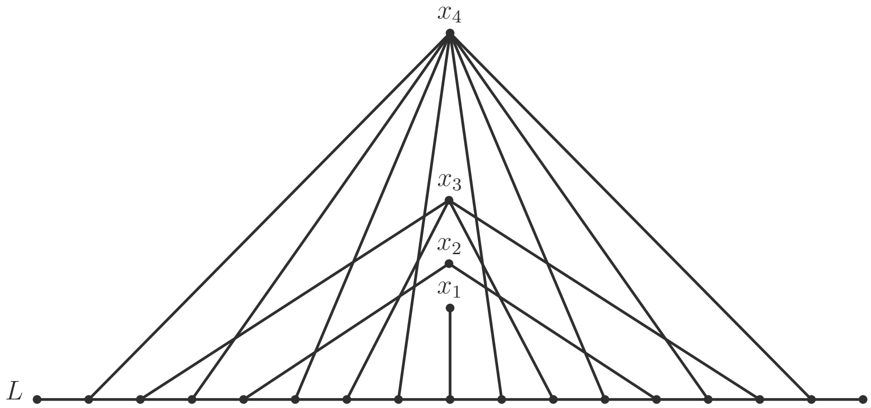

Figure 2.5: A 4-occultation.

- No two vertices in S have a common neighbor in $V(L)$.
- The ends of L are anticomplete to S.
- There is a linear order $x_1, \dots, x_s$ on the vertices in S such that x_1 has exactly one neighbor in L, and for $i \in [s]$ with $i > 2$, the vertex x_i has *exactly* one neighbour between every two successive vertices in L which are either an end of L or a neighbour of a vertex in $\{x_1, \dots, x_{i-1}\}$. In particular, x_i has 2^{i-1} neighbors in L for $i \in [s]$.
- No vertex in L has degree 2 in $\mathfrak{o}$. In particular, L has length 2^s.

It is proved in [13] that:

Theorem 2.9 (Bonamy, Bonnet, Déprés, Esperet, Geniet, Hilaire, Thomassé and Wesolek [13])**.** *For every $s \in \mathbb{N}$, every s-occultation is a $(K_{3,3}, K_3)$-free perforated graph of treewidth at least $s - 1$.*

However, a closer look at their proof shows that the assertion of Theorem 2.9 remains correct if (a) the edges of L are allowed to be subdivided, and (b) the word "exactly" in the third bullet is replaced by "at least." This leads us to the following definition. For $s \in \mathbb{N}$, a *full s-occultation* is a graph $\mathfrak{o}$ whose vertex set can be partitioned into a stable set S in $\mathfrak{o}$ of cardinality s and a path L in $\mathfrak{o}$, such that the following hold (see Figure 2.6).

- No two vertices in S have a common neighbor in $V(L)$.
- The ends of L are anticomplete to S.

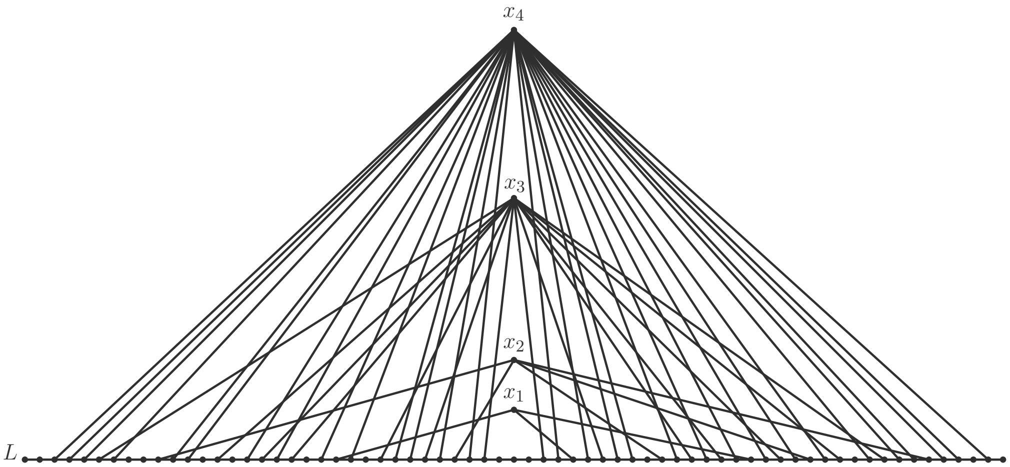

Figure 2.6: A full 4-occultation.

- There is a linear order $x_1, \dots, x_s$ on the vertices in S such that x_1 has exactly one neighbor in L, and for $i \in [s]$ with $i > 2$, the vertex x_i has *at least* one neighbor between every two successive vertices in L which are either an end of L or a neighbour of a vertex in $\{x_1, \dots, x_{i-1}\}$.

We will prove in Chapter 6 under Theorem 6.2 that Theorem 2.9 holds for full occultations, as well. We will also define full occultations once again in Section 6.1 in terms of "asterisms."

It turns out that complete graphs, complete bipartite graphs and full occultations are the only induced subgraph obstructions to bounded treewidth in perforated graphs. Moreover, the same holds for any number of excluded cycles rather than two. Given $c \in \mathbb{N}$, we say a graph G is *c-perforated* if there are no c pairwise disjoint and anticomplete induced subgraphs of G, each of which is a cycle.

Our main result on perforated graphs is the following. In Chapter 6, we will prove not only this result but also a strengthening to the case where the lengths of the excluded cycles are lower bounded.

Theorem 2.10. *For all $c, s, t \in \mathbb{N}$, there is a constant $f_{2.10} = f_{2.10}(c, s, t) \in \mathbb{N}$ such that every $(K_{t+1}, K_{t,t})$-free c-perforated graph G with $\operatorname{tw}(G) > f_{2.10}(c, s, t)$ has an induced subgraph which is a full s-occultation.*

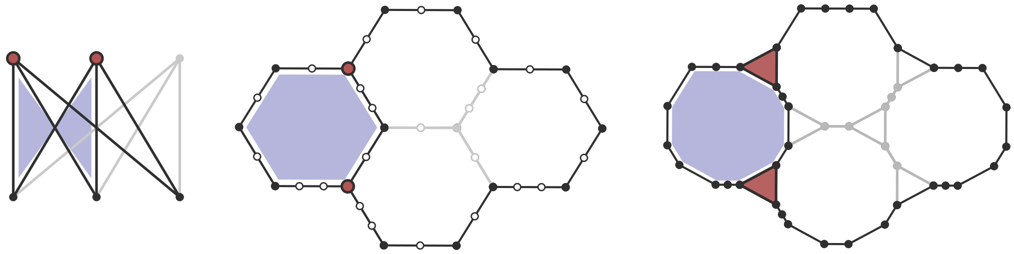

Figure 2.7: For $t \geq 3$, all non-complete t-basic obstructions contain even holes. Note, in thick lines, the theta on the left and the middle, and the prism on the right, and note the even holes highlighted in them.

2.4 Even-hole-free graphs

1. Motivation. The set of all cycles of even length is our next choice of an infinite set of excluded graphs. This is inspired by the properties of the basic obstruction in the first place. Specifically, there is a theta in the complete bipartite graph $K_{3,3}$, there is a theta in every subdivision of $W_{3\times 3}$, and there is a prism in the line graph of every subdivision of $W_{3\times 3}$ (see Figure 2.7). It follows that for every $t \geq 3$:

- there is either a theta or a triangle in every t-basic obstruction; and
- there is either an even hole or a t-clique in every t-basic obstruction.

Note that even holes are the only "interesting" induced subgraphs shared by all the basic obstructions except for the complete graphs. One may also observe that there is a theta in every array and full occultation of sufficiently large treewidth. Therefore, it is natural to ask if all the induced subgraph obstructions to bounded treewidth (basic or not) except for complete graphs contain even holes:

Question 2.11. *Does the class of all (theta, K_3)-free graphs have bounded treewidth?*

Question 2.12. *Is it true that for every $t \in \mathbb{N}$, the class of all (even hole, K_t)-free graphs has bounded treewidth?*

2. The difficulty with even-hole-free graphs. The latter two questions are not entirely hopeless at first glance. In particular, The answer to Question 2.12 is positive for $t \leq 3$: (even-hole, K_3)-free graphs have treewidth at most five [16].

However, in 2019, Sintiari and Trotignon [71] answered both Question 2.11 and Question 2.12 for $t \geq 4$ in the negative. They provided explicit examples of (theta, K_3)-free graphs and (even-hole, K_4)-free graphs with arbitrarily large treewidth, which they called the *layered wheels* (see Figure 2.8).

Unlike the arrays and the occultations, the layered wheels are rather involved in the construction. The full definition [71] takes a couple of pages, which is why we omit it. Nevertheless, despite their unwieldy global structure, the local structure of the layered wheels is as restricted as possible:

Theorem 2.13 (Sintiari and Trotignon [71]). *The following hold.*

(a) *There are theta-free graphs of arbitrarily large treewidth and arbitrarily large girth.*

(b) *There are (even-hole, K_4)-free graphs of arbitrarily large treewidth and arbitrarily long shortest holes.*

Graphs with no hole are called *chordal.* In particular, forests are exactly the graphs which are chordal and triangle-free. For general $k \in \mathbb{N}$, we say a graph H is a *k-forest* if H is chordal and K_{k+2}-free. This arranges for the forests to be the same as 1-forests. Using the above definition, Theorem 2.13 may be restated as follows:

Theorem 2.14 (Sintiari and Trotignon [71]). *The following hold for every $h \in \mathbb{N}$.*

(a) *There are (theta, K_3)-free graphs of arbitrarily large treewidth in which every induced subgraph on at most h vertices is a 1-forest.*

(b) *There are (even-hole, K_4)-free graphs of arbitrarily large treewidth in which every induced subgraph on at most h vertices is a 2-forest.*

We will refer to the graphs from Theorem 2.14(a) and (b), respectively, as the *theta-free layered wheels* and the *even-hole-free layered wheels.* It is well-known [28] that among the graphs of any given clique number, chordal graphs of the same clique number are exactly the graphs in which every induced subgraph has the smallest possible treewidth. In particular, Theorem 2.14 says that layered wheels are graphs of arbitrarily large treewidth in which every small induced subgraph – no matter how large our notion of "small" is chosen to be – has the smallest possible treewidth.

We were not able to prove a full grid-type theorem for theta-free graphs and even-hole-free graphs. But we will show that, despite their intricate, almost contrived-looking global structure, the layered wheels are quite canonical from a local perspective, in that they represent the local structure of all even-hole-free graphs of large treewidth.

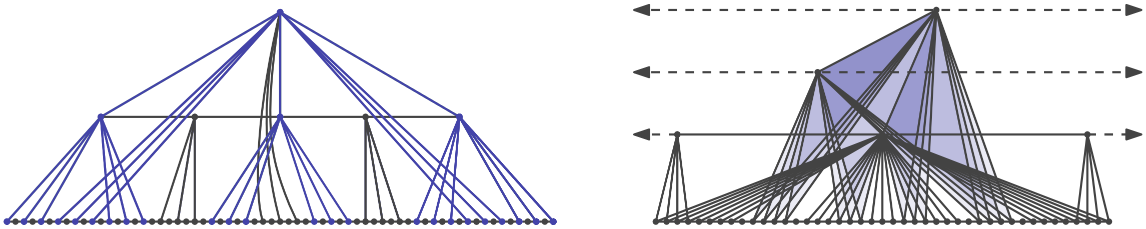

Figure 2.8: A bit of the theta-free layered wheel (left) and the even-hole-free layered wheel (right).

3. Forests in theta-free graphs. Suppose F is a graph such that every theta-free layered wheel of sufficiently large treewidth has an induced subgraph isomorphic to F. Then, by Theorem 2.14(a), F must be a forest. In fact, one may readily observe that Theorem 2.13(a), Theorem 2.14(a), and the following are all equivalent (see Figure 2.8):

Theorem 2.15 (Sintiari and Trotignon [71])**.** *Let F be a graph. Then every theta-free layered wheel of large enough treewidth has an induced subgraph isomorphic to F, if and only if F is a forest.*

We prove that the converse is also true:

Theorem 2.16. *Let F be a graph. Then every (theta, K_3)-free graph of large enough treewidth has an induced subgraph isomorphic to F, if and only if F is a forest.*

There is a vexatious asymmetry in Theorem 2.16: now that excluding thetas eliminates the subdivided walls (and complete bipartite graphs), why would we not rule out the line graphs of the subdivided walls by excluding the line graphs of thetas? Recall that the line graphs of thetas are prisms. We strengthen Theorem 2.16 to the larger class of (theta, prism)-free graphs of bounded clique number:

Theorem 2.17. *Let $t \in \mathbb{N}$. Let F be a graph. Then every (theta, prism, K_{t+1})-free graph of large enough treewidth has an induced subgraph isomorphic to F, if and only if F is a forest.*

Note also that the "only if" implication in Theorem 2.17 follows from Theorem 2.15. It remains to show that:

Theorem 2.18. *For all $t \in \mathbb{N}$ and every forest F, there is a constant $f_{2.18} = f_{2.18}(F, t) \in \mathbb{N}$ such that every (theta, prism, K_{t+1})-free graphs with $\mathrm{tw}(G) > f_{2.18}$ has an induced subgraph isomorphic to F.*

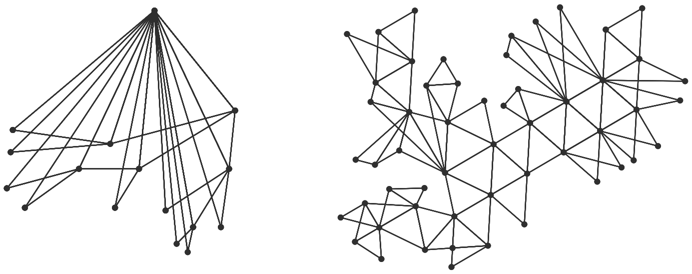

Figure 2.9: Examples of 2-forests (which are in fact 2-trees; see Chapter 9, Section 9.2.)

We will prove Theorem 9.1 in Chapter 9, Section 9.1.

4. 2-forests in even-hole-free graphs. Suppose H is a graph such that every even-hole-free layered wheel of sufficiently large treewidth has an induced subgraph isomorphic to H. Then, by Theorem 2.14(b), H must be a 2-forest (see Figure 2.9). Indeed, it is straightforward to check that Theorem 2.13(b), Theorem 2.14(b), and the following are all equivalent (see Figure 2.8):

Theorem 2.19 (Sintiari and Trotignon [71])**.** *Let H be a graph. Then every even-hole-free layered wheel of large enough treewidth has an induced subgraph isomorphic to H, if and only if H is a 2-forest.*

Once again, we prove that the converse is also true:

Theorem 2.20. *Let H be a graph. Then every (even hole, K_4)-free graph of large enough treewidth has an induced subgraph isomorphic to H, if and only if H is a 2-forest.*

Incidentally, Theorem 2.20 contains a conjecture of Sintiari and Trotignon [71] as a (very) special case: when H is obtained from a two-edge path by adding a universal vertex. Given a graph F, we denote by $\mathsf{cone}(F)$ the graph obtained from F by adding a universal vertex. The cone of a two-edge path is known as the *diamond* (see Figure 2.10):

Conjecture 2.21 (Sintiari and Trotignon)**.** *The class of all (even hole, diamond, K_4)-free graphs has bounded treewidth.*

Since the diamond is a 2-forest, Conjecture 2.21 follows immediately from the "if" implication of Theorem 2.20.

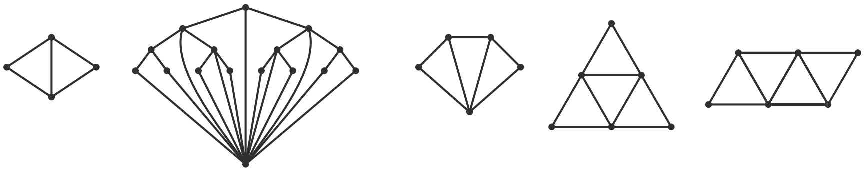

Figure 2.10: From left to right: the diamond, a coned tree, the gem, and the two smallest (2-connected) 2-forests that are not coned forests (the second from the right is commonly known as the *antinet*).

In general, the graph $\mathsf{cone}(F)$ is a 2-forest if and only if F is a forest. We prove yet another strengthening of Conjecture 2.21, where the "diamond" is replaced by "$\mathsf{cone}(F)$ for any forest F" and "K_4" is replaced by "K_t for any $t \geq 4$." In view of Theorem 2.14(b), this yields a characterization of all forests:

Theorem 2.22. *Let $t \geq 4$. Let F be a graph. Then every (even hole, K_t)-free graph of large enough treewidth has an induced subgraph isomorphic to $\mathsf{cone}(F)$, if and only if F is a forest.*

While on the topic of K_t-free graph for general $t \geq 4$, it is natural to ask whether Theorem 2.20 may be extended to even-hole-free graphs with larger clique numbers. The "only if" implication in Theorem 2.20 remains valid for (even-hole, K_t)-free graphs for all $t \geq 4$. We conjecture that the converse is also true:

Conjecture 2.23. *Let $t \geq 4$. Given a graph H, every (even hole, K_t)-free graph of large enough treewidth has an induced subgraph isomorphic to H if (and only if) H is a 2-forest.*

Note that Theorems 2.20 and 2.22 settle, in order, the following two special cases of Conjecture 2.23: (a) where $t = 4$ and H is arbitrary, and (b) where H is a coned forest and $t \geq 4$ is arbitrary. Instead of those results, we prove a common strengthening of Theorems 2.20 and 2.22 which characterizes forests and 2-forests simultaneously:

Theorem 2.24. *For all $t \in \mathbb{N}$, every forest F and every 2-forest $H \in \mathcal{H}$, there is a constant $f_{2.24} = f_{2.24}(F, H, t)$ such that every (even-hole, K_{t+1})-free graph G with $\mathrm{tw}(G) > f_{2.24}$ has an induced subgraph isomorphic to either $\mathsf{cone}(\mathsf{cone}(F))$ or H.*

We will prove Theorem 2.24 in Chapter 9, Section 9.3. In particular, Theorem 2.20 follows from Theorem 2.24 applied to $t = 3$ and $F = K_2$, whereas Theorem 2.22 follows from Theorem 2.24 by setting $H = \mathsf{cone}(F)$. Note that by Theorem 2.14(b), Theorem 2.24 will be false if neither F is a forest nor H is a 2-forest.

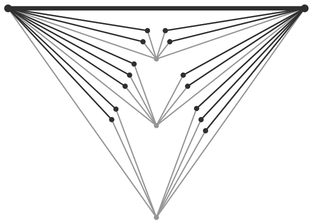

Figure 2.11: A crystal.

Alternatively, one may improve on each of Theorems 2.20 and 2.22 separately. The one we do have something to contribute to is extending Theorem 2.22 beyond coned forests. So we wish to identify more 2-forests H (other than coned forests) for which Conjecture 2.23 holds for all $t \geq 4$. One may check that among 2-forests that are not coned forests, there are only two (up to isomorphism and 2-connectivity) with the fewest number of vertices.

Both these graphs are obtained from a coned K_2 (that is, a triangle) and a coned three-edge path (commonly known as a *gem*, which is annoyingly more similar to a "diamond" than the diamond), by gluing them together along the unique edge of K_2 and some edge of the three-edge path (see Figure 2.10). We prove that for all $t \geq 4$, Conjecture 2.23 holds for both of these graphs as a choice of H.

In fact, we prove Theorem 2.25 below which is a much stronger result. A *double star* is a tree with exactly two vertices of degree more than one (which are necessarily adjacent). The *middle edge* of a double star is the unique edge between the two non-leaf vertices. By a *crystal* we mean a graph obtained from $\mathsf{cone}(F_1), \dots, \mathsf{cone}(F_k)$ for some $k \in \mathbb{N}$ and some choice of k double stars $F_1, \dots, F_k$, by identifying the middle edges of $F_1, \dots, F_k$ (see Figure 2.11). We will prove the following in Chapter 9, Section 9.4:

Theorem 2.25. *For all $t \in \mathbb{N}$ and every crystal H, there is a constant $f_{2.25} = f_{2.25}(H,t) \in \mathbb{N}$ such that every (even-hole, K_{t+1})-free graph of treewidth more than $f_{2.25}$ has an induced subgraph isomorphic to H.*

To conclude, we remark that another approach would be to generalize Theorem 2.20 from (even-hole, K_4)-free graphs to (even-hole, K_t)-free graphs for some large values of t. We do not have much to say about this, except it appears to us that we are somehow bound to prove Conjecture 2.23 for all values of $t > 4$ at once. More generally, we wonder if the following is true, which will imply Conjecture 2.23 by reducing it to Theorem 2.20:

Conjecture 2.26. *For every $t \geq 1$, every even-hole-free graph of large enough treewidth has an induced subgraph of treewidth t which is either complete or K_4-free.*

2.5 Proofs: the method of strong blocks

The proofs of all of our main results use the "method of strong blocks." Roughly, this means we begin all proofs by applying the following statement to the graphs we study: every t-clean graph of large enough treewidth has a large subset of vertices that are pairwise highly connected in the whole graph. To make this precise, we need the following definition. Let G be a graph and let $k, l \in \mathbb{N}$. A (k,l)-*block* in G is a pair $(B, \mathcal{P})$ where $B \subseteq V(G)$ with $|B| \geq k$ and $\mathcal{P} : B^{(2)} \to 2^{V(G)}$ is map such that $\mathcal{P}(\{x,y\})$, for each 2-subset $\{x,y\}$ of B, is a set of at least l pairwise internally disjoint paths in G from x to y. We say $(B, \mathcal{P})$ is *strong* if for all distinct 2-subsets $\{x,y\}, \{x',y'\}$ of B, we have $V(\mathcal{P}^*_{\{x,y\}}) \cap V(\mathcal{P}_{\{x',y'\}}) = \varnothing$; that is, each path $P \in \mathcal{P}(\{x,y\})$ is disjoint from each path $P' \in \mathcal{P}(\{x',y'\})$, except P and P' may share an end.

The following is a major tool that we will use in the proof of all our main results:

Theorem 2.27. *For all $k, l \in \mathbb{N}$, the class of all graphs with no strong (k,l)-block is clean. Equivalently, for all $k, l, t \in \mathbb{N}$, there is a constant $f_{2.27} = f_{2.27}(k,l,t) \in \mathbb{N}$ such that for every t-clean graph G with $\operatorname{tw}(G) > f_{2.27}$, there is a strong (k,l)-block in G.*

We prove Theorem 2.27 in Chapter 3, Section 3.1. In the proof of each of our main results, we first apply Theorem 2.27 to our t-clean graphs of large enough treewidth to obtain a sufficiently large strong block. The rest of the proof then will be to examine the structure of the subgraph induced by the paths of the strong block. This step often captures the main difficulty and relies on a menagerie of Ramsey-theoretic arguments that are delicately tailored to the specifications of each proof. Thus, we postpone a more detailed outline of those techniques to later chapters, immediately before the corresponding proofs.

Chapter 3

Blocks, subdivisions and minor models[1]

In this chapter, we put together some of the results that will serve as the common technical underpinning of the subsequent chapters. In particular, we prove Theorem 2.27. We will then combine the latter with several Ramsey-type arguments to show that t-clean graphs of sufficiently large treewidth contain large complete bipartite minor models in which every branching set induces a path, and the entire model induces a "sparse" subgraph.

3.1 Strong blocks in t-clean graphs

Our goal in this section is to show that:

Theorem 2.27. *For all $k, l \in \mathbb{N}$, the class of all graphs with no strong (k, l)-block is clean. Equivalently, for all $k, l, t \in \mathbb{N}$, there is a constant $f_{2.27} = f_{2.27}(k, l, t) \in \mathbb{N}$ such that for every t-clean graph G with $\text{tw}(G) > f_{2.27}$, there is a strong (k, l)-block in G.*

The proof of Theorem 2.27 needs some preparation. We begin with the definition of a "tree decomposition" which also yields the more commonly used definition of the treewidth. Let G be a graph. A *tree decomposition* for G is a pair (T, β) where T is a tree and $\beta : V(T) \to 2^{V(G)}$ is a map such that:

- we have $V(G) = \bigcup_{x \in V(T)} \beta(t)$ and $E(G) = \bigcup_{x \in V(T)} E(\beta(x))$; and
- $T[\{x \in V(T) : v \in \beta(x)\}]$ is a connected for all $v \in V(G)$.

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[1] This chapter is based on the coauthored papers [3, 8] and the single-author paper [37].

We leave it to the reader to check that (T, β) is a tree decomposition for G if and only if $(T[\{x \in V(T) : v \in \beta(x)\}] : v \in V(G))$ is a T-subtree representation of G [28]. It follows that the treewidth of G is equal to the minimum of $\max_{v \in V(G)} |\beta(v)| - 1$ over all tree decompositions (T, β) of G. This is in fact one of the most common ways to define treewidth in the literature [60, 61].

Let T be a tree. For an edge $xy \in E(T)$, we denote by $T_{x,y}$ the component of $T - xy$ containing x. Let G be a graph and let (T, β) be a tree decomposition for G. For every induced subgraph S of T, we write $\beta(S) = \bigcup_{x \in V(S)} \beta(x)$, and for every edge $xy \in E(T)$, we write $\beta(x, y) = \beta(x) \cap \beta(y) = \beta(T_{x,y}) \cap \beta(T_{y,x})$.

Given a vertex $x \in V(T)$, by the *torso of* (T, β) *at* x, denoted $\hat{\beta}(x)$, we mean the graph obtained from $G[\beta(x)]$ by, for each $y \in N_T(x)$, adding an edge between every two non-adjacent vertices in $\beta(x, y)$. It is a well-known observation that clique cutsets do not affect the treewidth. More precisely, the following holds (a proof can be worked out easily using Lemma 5 from [12]).

Theorem 3.1 (Folklore, see Lemma 5 in [12]). *Let G be a graph and let (T, β) be a tree decomposition for G. Then we have:*

$$\operatorname{tw}(G) = \max_{x \in V(T)} \operatorname{tw}\left(\hat{\beta}(x)\right).$$

There are two properties of a tree decomposition that matter to us in this section: that of being "k-atomic" (for some $k \in \mathbb{N} \cup \{0\}$) and that of being "tight." We omit the definition of the former [73] as it will not be used in our proofs, but we need to define the latter. A tree decomposition (T, β) of a graph G is *tight* if for each vertex $x \in V(T)$ and every vertex $y \in N_T(x)$, there is a component C of $\beta(T_{y,x}) \setminus \beta(T_{x,y})$ such that every vertex in $\beta(x, y)$ has at least one neighbor in C. The following is proved in [73].

Theorem 3.2 (Weißauer, Lemma 6 in [73]). *For every $k \in \mathbb{N} \cup \{0\}$ and every graph G, every k-atomic tree decomposition for G is tight.*

Let (T, β) be a tree decomposition for a graph G and let $\mathcal{S}$ be a set of pairwise vertex-disjoint subtrees of T. Let T' be the tree obtained from T by contracting each subtree $S \in \mathcal{S}$ into a new vertex v_S. Let $\beta' : V(T') \to 2^{V(G)}$ be defined as follows. Let $\beta'(v_S) = \beta(S)$ for every $S \in \mathcal{S}$, and let $\beta'(v) = \beta(v)$ for every $v \in V(T') \setminus \{v_S : S \in \mathcal{S}\} = V(T) \setminus (\bigcup_{S \in \mathcal{S}} S)$. It is straightforward to observe that (T', β'), called *contraction of* (T, β), is also a tree decomposition for G. The following result from [26] is a key ingredient in our proof of Theorem 2.27.

Theorem 3.3 (Erde and Weißauer [26], see also Grohe and Marx [34])**.** *Let $r \in \mathbb{N}$ and let G be a graph with no subgraph isomorphic to a subdivision of K_r. Then G admits a tree decomposition (T, β) for which the following hold.*

(a) *(T, β) is a contraction of an $(r^2 - r)$-atomic tree decomposition for G.*

(b) *For every edge $xy \in E(T)$, we have $|\beta(x, y)| \leq r^2$.*

(c) *For every $x \in V(T)$, either $\hat{\beta}(x)$ has fewer than r^2 vertices of degree at least $2r^4$, or $\hat{\beta}(x)$ has no minor isomorphic to K_{2r^2}.*

We remark that the corresponding statement from [26], namely Theorem 4 therein, does not explicitly mention that (T, β) is a contraction of a k-atomic tree decomposition. However, as the reader can check, the proof of this result, given in Section 3 of [26], starts with a $(r^2 - r)$-atomic tree decomposition "$(T, \mathcal{V})$" and concludes at the end that (T, β) is a specific contraction of $(T, \mathcal{V})$.

Observe that every contraction of a tight tree decomposition is tight. Also, one may easily check that for every $k \in \mathbb{N}$ and every graph G, if G has a subgraph isomorphic to a subdivision of $K_{k^2 l}$, then there is a strong (k, l)-block in G. So Theorem 3.4 below is immediate from combining Theorems 3.2 and 3.3.

Theorem 3.4. *Let $k, l \in \mathbb{N}$ and let G be a graph. Assume that there is no strong (k, l)-block in G. Then G admits a tight tree decomposition (T, β) such that for every $x \in V(T)$, either $\hat{\beta}(x)$ has fewer than $k^4 l^2$ vertices of degree at least $2k^8 l^4$, or $\hat{\beta}(x)$ has no minor isomorphic to $K_{2k^4 l^2}$.*

We also need the following well-known result (see Figure 3.1):

Theorem 3.5 (Folklore, see Lemma 5.1 in [4])**.** *Let G be a connected graph, let $S \subseteq V(G)$ with $|S| = 3$ and let H be a connected induced subgraph of G such that $S \subseteq V(H)$ and $V(H)$ is minimal with respect to inclusion. Then one of the following holds.*

(a) *There exists a vertex $a \in V(H)$ and a path P_x (possibly of length zero) from a to x for each $x \in S$, such that*
  - *$H = \bigcup_{x \in S} P_x$, and;*
  - *the sets $(P_x \setminus \{a\} : x \in S)$ are pairwise disjoint and anticomplete in H.*

(b) *There exists triangle $\{a_x : x \in S\}$ in H and a path P_x (possibly of length zero) from a_x to x for each $x \in S$, such that*

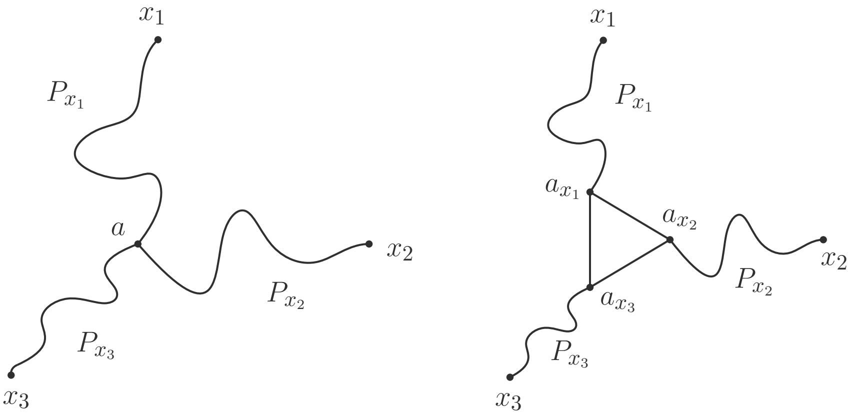

Figure 3.1: Outcomes of Theorem 3.5: (a) on the left and (b) on the right.

- *$H = \bigcup_{x \in S} P_x$;*
- *for all distinct $x, y \in S$, the sets $P_x \setminus \{a_x\}$ and P_y are disjoint and anticomplete in H.*

We can now prove Theorem 2.27, which we restate:

Theorem 2.27. *For all $k, l \in \mathbb{N}$, the class of all graphs with no strong (k,l)-block is clean. Equivalently, for all $k, l, t \in \mathbb{N}$, there is a constant $f_{2.27} = f_{2.27}(k,l,t) \in \mathbb{N}$ such that for every t-clean graph G with $\operatorname{tw}(G) > f_{2.27}$, there is a strong (k,l)-block in G.*

Proof. We use Theorems 1.14 and 1.15. Let

$$f_0 = f_{1.15}(3, t);$$

$$f_1 = \max\{f_{1.14}(K_{2k^4l^2}, 3f_0), f_{1.15}(2k^8l^4, 3f_0)\}.$$

We will prove that

$$f_{2.27} = f_{2.27}(k,l,t) = f_1 + k^4l^2$$

satisfies the theorem. Suppose for a contradiction that some t-clean graph G with $\operatorname{tw}(G) > f_{2.27}$ has no strong (k,l)-block. By Theorem 3.4, G admits a tight tree decomposition (T, β) where every torso either has fewer than k^4l^2 vertices of degree at least $2k^8l^4$ or has no minor isomorphic to $K_{2k^4l^2}$. For each $x \in V(T)$, let $K_x \subseteq V(\hat{\beta}(x)) = \beta(x)$ be the set of all vertices in $\hat{\beta}(x)$ whose degree in the graph $G[\hat{\beta}(x)]$ is at is least $2k^8l^4$, and define J_x as follows: if $|K_x| < k^4l^2$, then let $J_x = \hat{\beta}(x) \setminus K_x$; otherwise, let $J_x = \hat{\beta}(x)$. In particular, we have $\operatorname{tw}(\hat{\beta(x)}) \leq \operatorname{tw}(J_x) + k^4l^2$ for all $x \in V(T)$.

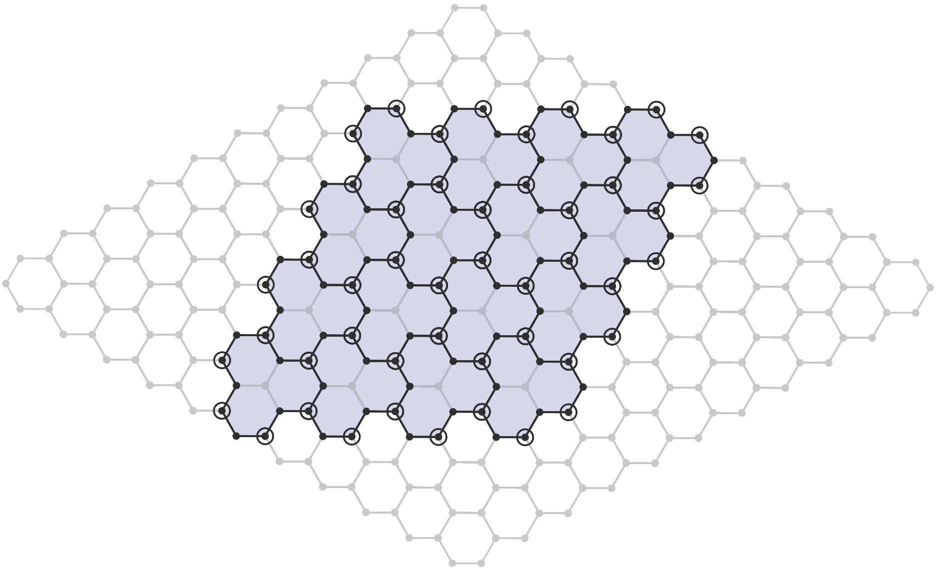

Figure 3.2: An induced subgraph of $W_{12\times 12}$ isomorphic to a 1-subdivision of $W_{5\times 5}$.

Since $\mathrm{tw}(G) > f_{2.27}$, it follows from Theorem 3.1 that there exists $x \in V(T)$ with $\mathrm{tw}(\hat{\beta}(x)) > f_{2.27}$, and so $\mathrm{tw}(J_x) > f_1$. We deduce that:

(2) *The graph J_x has an induced subgraph W_0 isomorphic to either a proper subdivision of the wall $W_{(f_0+1)\times(f_0+1)}$ or the line graph of a proper subdivision of the wall $W_{(f_0+1)\times(f_0+1)}$.*

By definition, either J_x has maximum degree less than $2k^8l^4$ or J_x has no minor that is isomorphic to $K_{2k^4l^2}$. Therefore, the choice of f_1 together with Theorems 1.14 and 1.15 implies that J_x has an induced subgraph W that is isomorphic to either a subdivision of $W_{3f_0\times 3f_0}$ or the line graph of a subdivision of $W_{3f_0\times 3f_0}$. It is straightforward to observe that for every $r \in \mathbb{N}$, every subdivision of $W_{3r\times 3r}$ has an induced subgraph isomorphic to a proper subdivision of $W_{(r+1)\times(r+1)}$ (see Figure 3.2). It follows that W, and so J_x, has an induced subgraph W_0 that is isomorphic to either a proper subdivision of $W_{(f_0+1)\times(f_0+1)}$ or the line graph of a proper subdivision of $W_{(f_0+1)\times(f_0+1)}$. This proves (2).

Let W_0 be as in (2). By a *blossom* we mean a non-empty subset K of $V(W_0)$ for which there exists $y \in N_T(x)$ such that $K \subseteq \beta(x, y)$, and subject to this property, K is maximal with respect to inclusion. It follows that every blossom K is a k-clique in W_0 for some $k \in [3]$. Moreover, since W_0 is K_4-free and by maximality, every two distinct blossoms intersect in at most one vertex, and blossoms of cardinality three are pairwise disjoint.

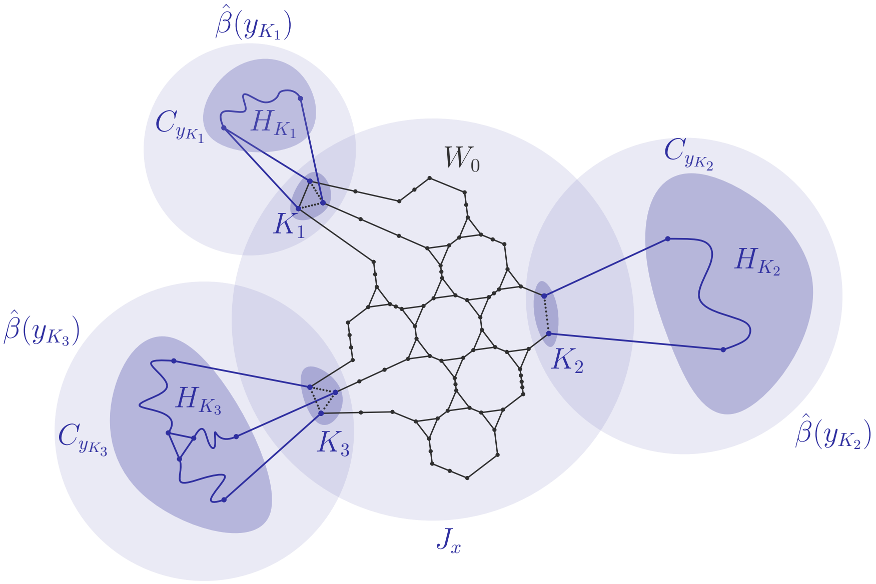

Figure 3.3: Proof of Theorem 2.27. Dotted segments represent edges in $E(W_0) \setminus E(\beta(x))$ with both ends in K_1, K_2 or K_3.

Let $\mathcal{K}$ be the set of all blossoms, and for each blossom $K \in \mathcal{K}$, fix $y_K \in N_T(x)$ such that $K \subseteq \beta(x, y_K)$. From the maximality of blossoms, it follows that the vertices $(y_K : K \in \mathcal{K})$ are all distinct. Recall that (T, β) is tight, and so for every $y \in N_T(x)$, there exists a component C_y of $\beta(T_{y,x}) \setminus \beta(T_{x,y})$ such that the every vertex in $\beta(x, y)$ has a neighbor in C_y. Since (T, β) is a tree decomposition, it follows that the sets $(C_y : y \in N_T(x))$ are pairwise distinct, disjoint and anticomplete in G.

For each $K \in \mathcal{K}$, let H_K be a connected induced subgraph of $G[(C_{y_K} \cup K)]$ which contains K, and subject to this property, assume that $V(H_K)$ is minimal with respect to inclusion. It follows that if $|K| = 1$, then $V(H_K) = V(K)$, if $|K| = 2$, then H_K is a path in G between the two vertices in K with $H_K^* \subseteq C_{y_K}$, and if $|K| = 3$, then H_K satisfies one of the two outcomes of Theorem 3.5 applied to $S = K$. Also, the sets $(H_K \setminus K : K \in \mathcal{K})$ are pairwise distinct, disjoint and anticomplete in G. See Figure 3.3.

Let

$$H = G\left[\left(W_0 \setminus \left(\bigcup_{K \in \mathcal{K}} K\right)\right) \cup \left(\bigcup_{K \in \mathcal{K}} H_K\right)\right].$$

We claim that:

(3) *The graph H has maximum degree at most three and* $\operatorname{tw}(H) > f_0$.

The first assertion is immediate from the fact that W_0 has maximum degree at most three. For the second assertion, let H' be the minor of H obtained by taking the following steps in order:

- First, for each blossom $K \in \mathcal{K}$ with $|K| = 3$, contract the connected induced subgraph H_K of H into a vertex.
- Second, for each blossom $K \in \mathcal{K}$ with $|K| = 2$ such that K is contained in a 3-clique in W_0, contract the path H_K in H to an edge between the two vertices in K.
- Third, contract each triangle of the resulting graph after Step 2 into a vertex.

Since W_0 is isomorphic to either a proper subdivision of $W_{(f_0+1)\times(f_0+1)}$ or the line graph of a proper subdivision of $W_{(f_0+1)\times(f_0+1)}$, it is readily observed that H' is isomorphic to a subdivision of $W_{(f_0+1)\times(f_0+1)}$. It follows that H has a minor isomorphic to $W_{(f_0+1)\times(f_0+1)}$, and so $\operatorname{tw}(H) \geq f_0 + 1$. This proves (3).

From (3), Theorem 1.15 and the choice of f_0, it follows that H has an induced subgraph isomorphic to either a subdivision of $W_{t\times t}$ or the line graph of a subdivision of $W_{t\times t}$. Since H is an induced subgraph of G, it follows that G has an induced subgraph isomorphic to either a subdivision of $W_{t\times t}$ or the line graph of a subdivision of $W_{t\times t}$, a contradiction with the assumption that G is t-clean. This completes the proof of Theorem 2.27. ■

3.2 Induced subdivisions with pinned branch vertices

From earlier works of Dvořák [25] and Lozin and Razgon [50], it follows almost immediately that for all $d \in \mathbb{N} \cup \{0\}$, $t \in \mathbb{N}$, and sufficiently large m, if G is a t-clean graph, then no subgraph of G is isomorphic to a $(\leq d)$-subdivision of K_m. Here we prove a strengthening, answering in the affirmative a question of Lozin and Razgon (see Problem 1 in [50]):

Theorem 3.6. *For all $d \in \mathbb{N} \cup \{0\}$ and $s, t \in \mathbb{N}$, there is a constant $f_{3.6} = f_{3.6}(d, s, t) \in \mathbb{N}$ with the following property. Let G be a $(K_{t+1}, K_{t,t})$-free graph which has a subgraph K isomorphic to a $(\leq d)$-subdivision of $K_{f_{3.6}}$. Then G has an induced subgraph isomorphic to a proper $(\leq d)$-subdivision of K_s whose branch vertices are also branch vertices of K.*

We will deduce Theorem 3.6 from an even stronger statement, which guarantees both the branch vertices and the subdivision paths between them to be preserved:

Theorem 3.7. *For all* $d \in \mathbb{N} \cup \{0\}$ *and* $s, t \in \mathbb{N}$*, there is a constant* $f_{3.7} = f_{3.7}(d, s, t) \in \mathbb{N}$ *with the following property. Let* G *be a* $(K_{t+1}, K_{t,t})$*-free graph and let* $(B, \mathcal{P})$ *be a strong* $(f_{3.7}, 1)$*-block in* G*. For each* $\{x, y\} \in B^{(2)}$*, let* $\mathcal{P}(\{x, y\}) = \{P_{x,y}\}$ *where* $P_{x,y}$ *has length at most* $d + 1$*. Then there is a stable set* $S \subseteq B$ *with* $|S| = s$ *such that the following hold.*

- (a) *For any three vertices* $x, y, z \in S$*, the vertex* x *is anticomplete to* $P^*_{y,z}$ *in* G*.*
- (b) *For all distinct* $\{x, y\}, \{x', y'\} \in S^{(2)}$*,* $P^*_{x,y}$ *and* $P^*_{x',y'}$ *are anticomplete in* G*.*

In other words,

$$G\left[\bigcup_{\{x,y\} \in S^{(2)}} P_{x,y}\right]$$

is isomorphic to a proper $(\leq d)$*-subdivision of* K_s *with* S *as its set of branch vertices (except for the case* $s \in \{1, 2\}$*, where there is no branch vertex).*

The proof of Theorem 3.7 will be in several steps. Let us begin with the multicolor version of Ramsey's Theorem for complete uniform hypergraphs:

Theorem 3.8 (Ramsey [57]). *For all* $l, m, n \in \mathbb{N}$*, there is a constant* $f_{3.8} = f_{3.8}(l, m, n) \in \mathbb{N}$ *with the following property. Let* U *be a set of cardinality at least* $f_{3.8}$ *and let* F *be a non-empty set of cardinality at most* l*. Let* $\Phi : U^{(m)} \to F$ *be a map. Then there exist* $i \in F$ *and an* n*-subset* Z *of* U *such that* $\Phi(X) = i$ *for all* $X \in Z^{(m)}$*.*

From Theorem 3.8, we deduce the next two lemmas:

Lemma 3.9. *For all* $a, b, s \in \mathbb{N}$*, there is a constant* $f_{3.9} = f_{3.9}(a, b, s) \in \mathbb{N}$ *with the following property. Let* G *be a graph and let* $(B, \mathcal{P})$ *be a strong* $(f_{3.9}, 1)$*-block in* G*. For each* $\{x, y\} \in B^{(2)}$*, let* $\mathcal{P}(\{x, y\}) = \{P_{x,y}\}$*. Then one of the following holds.*

- (a) *There exist* $A \subseteq B$ *with* $|A| = a$ *and* $\mathcal{B} \subseteq (B \setminus A)^{(2)}$ *with* $|\mathcal{B}| = b$ *such that every* $x \in A$ *has a neighbor in* $P_{y,z}$ *for every* $\{y, z\} \in \mathcal{B}$*.*
- (b) *There exists a stable set* $S \subseteq B$ *in* G *with* $|S| = s$ *such that for any three vertices* $x, y, z \in S$*, the vertex* x *is anticomplete to* $P^*_{y,z}$ *in* G*.*

Proof. We claim that

$$f_{3.9}(a, b, s) = f_{3.8}(8, 3, \max\{3a + 2b, s\})$$

satisfies the lemma.

Assume that 3.9(a) does not hold. Fix an enumeration $B = \{w_1, \dots, w_{f_{3.9}}\}$ of B. For every 3-subset $T = \{t_1, t_2, t_3\}$ of $[f_{3.9}]$ with $t_1 < t_2 < t_3$, let $\Phi(T) \in 2^{[3]}$ such that for each $i \in [3]$, we have $i \in \Phi(T)$ if and only if w_{t_i} has a neighbor in G in $P_{w_{t_j}, w_{t_k}}$, where $\{i, j, k\} = \{1, 2, 3\}$.

It follows that the map $\Phi : [f_{3.9}]^{(3)} \to 2^{[3]}$ is well-defined. By the choice of $f_{3.9}$, we can apply Theorem 3.8 and deduce that there exists $F \subseteq [3]$ as well as $Z \subseteq [f_{3.9}]$ with $|Z| = \max\{3a + 2b, s\}$ such that for every $T \in Z^{(3)}$, we have $\Phi(T) = F$. In particular, since $|Z| \geq 3a + 2b$, we may choose $I_1, I_2, I_3, J, K \subseteq Z$ with $|I_1| = |I_2| = |I_3| = a$ and $|J| = |K| = b$, such that

$$\max I_1 < \min J \leq \max J < \min I_2 \leq \max I_2 < \min K \leq \max K < \min I_3.$$

It follows that I_1, I_2, I_3, J and K are pairwise disjoint. Moreover:

(4) *We have $F = \varnothing$.*

Suppose not. Pick $f \in F \subseteq [3]$. Write $J = \{j_1, \dots, j_b\}$ and $K = \{k_1, \dots, k_b\}$. Note that for every $i \in I_f$ and every $t \in [b]$, the set $\{i, j_t, k_t\}$ is a 3-subset of Z. So we have $\Phi(\{i, j_t, k_t\}) = F$, which means $f \in \Phi(\{i, j_t, k_t\})$. This, along with the choices of Φ and the sets I_1, I_2, I_3, J, K implies that for every $i \in I_f$ and every $t \in [b]$, w_i has a neighbor in $P_{w_{j_t}, w_{k_t}}$. But then $A = \{w_i : i \in I_f\}$ and $\mathcal{B} = \{(w_{j_t}, w_{k_t}) : t \in [b]\}$ satisfy 3.9(a), a contradiction. This proves (4).

Since $|Z| \geq \max\{s, 3\}$, it follows from (4) that for every $Z' \subseteq Z$ with $|Z'| = s$, the set $S = \{w_i : i \in Z\} \subseteq B$ satisfies 3.9(b). This completes the proof of Lemma 3.9. ■

Lemma 3.10. *For all $c, s \in \mathbb{N}$, there is a constant $f_{3.10} = f_{3.10}(c, s) \in \mathbb{N}$ with the following property. Let G be a graph and let $(B, \mathcal{P})$ be a strong $(f_{3.9}, 1)$-block in G. For each $\{x, y\} \in B^{(2)}$, let $\mathcal{P}(\{x, y\}) = \{P_{x,y}\}$. Then one of the following holds.*

- (a) *There are disjoint c-subsets $\mathcal{C}$ and $\mathcal{C}'$ of $B^{(2)}$ such that for every $\{x, y\} \in \mathcal{C}$ and every $\{x', y'\} \in \mathcal{C}'$, there is an edge in G with an end in $P^*_{x,y}$ and an end in $P^*_{x',y'}$.*
- (b) *There exists $S \subseteq W$ with $|S| = s$ such that for all distinct $\{x, y\}, \{x', y'\} \in S^{(2)}$, the sets $P^*_{x,y}$ and $P^*_{x',y'}$ are anticomplete in G.*

Proof. We claim that

$$f_{3.10}(c, s) = f_{3.8}\left(2^{15}, 4, \max\{4c, s\}\right)$$

satisfies the lemma.

Assume that 3.10(a) does not hold. Fix an enumeration $B = \{w_1, \dots, w_{f_{3.10}}\}$ of B. For every 4-subset $T = \{t_1, t_2, t_3, t_4\}$ of $[f_{3.10}]$ with $t_1 < t_2 < t_3 < t_4$, let $\Phi(T)$ be the set of all 2-subsets $\{\{i,j\},\{i',j'\}\}$ of $[4]^{(2)}$ for which $P^*_{w_i,w_j}$ and $P^*_{w_{i'},w_{j'}}$ are not anticomplete in G. Then the map

$$\Phi : [f_{3.10}]^{(4)} \to 2^{\left(\left([4]^{(2)}\right)^{(2)}\right)}$$

is well-defined. Consequently, by the choice of $f_{3.10}$ and Theorem 3.8, there exists a set F of 2-subsets of $[4]^{(2)}$ as well as a subset Z of $[f_{3.10}]$ with $|Z| = \max\{4c, s\}$ such that for every $T \in [Z]^{(4)}$, we have $\Phi(T) = F$. In particular, since $|Z| \geq 4c$, we may choose $T_1, T_2, T_3, T_4 \subseteq Z$ with $|T_1| = |T_2| = |T_3| = |T_4| = c$ such that

$$\max T_1 < \min T_2 \leq \max T_2 < \min T_3 \leq \max T_3 < \min T_4.$$

It follows that T_1, T_2, T_3 and T_4 are pairwise disjoint. Moreover:

(5) *We have $F = \varnothing$.*

Suppose not. Let $\{P, P'\} \in F$ where P, P' are distinct 2-subsets of $[4]$. Then we may write $P = \{i,j\}$ and $P' = \{i',j'\}$ such that $j \notin P'$ and $j' \notin P$ (while $i = i'$ is possible). Pick an element $t_i \in T_i$ and an element $t_{i'} \in T_{i'}$ (this is doable as $c > 0$). It follows that for every $t \in T_j$ and every $t' \in T_{j'}$, $\{t_i, t_{i'}, t, t'\}$ is a 4-subset of Z. Therefore, we have $\Phi(\{t_i, t_{i'}, t, t'\}) = F$, and so $\{P, P'\} \in \Phi(\{t_i, t_{i'}, t, t'\})$. This, together with the choice of Φ and the sets T_1, T_2, T_3, T_4 implies that for every $t \in T_j$ and every $t' \in T_{j'}$, $P^*_{w_{t_i},w_t}$ and $P^*_{w_{t_{i'}},w_{t'}}$ are not anticomplete in G. But then $\mathcal{C} = \{(w_{t_i}, w_t) : t \in T_j\}$ and $\mathcal{C}' = \{(w_{t_{i'}}, w_{t'}) : t' \in T_{j'}\}$ satisfy 3.10(a), a contradiction. This proves (5).

Since $|Z| \geq s$, there exists $Z' \subseteq Z$ with $|Z'| = s$. Now by (5), $S = \{w_i : i \in Z\} \subseteq B$ satisfies 3.10(b). This completes the proof of Lemma 3.10. ■

Combining Lemmas 3.9 and 3.10, we deduce the following:

Theorem 3.11. *For all $a, b, c, s \in \mathbb{N}$, there is a constant $f_{3.11} = f_{3.11}(a,b,c,s) \in \mathbb{N}$ with the following property. Let G be a graph and let $(B, \mathcal{P})$ be a strong $(f_{3.11}, 1)$-block in G. For each $\{x,y\} \in B^{(2)}$, let $\mathcal{P}(\{x,y\}) = \{P_{x,y}\}$. Then one of the following holds.*

- (a) *There exist $A \subseteq B$ with $|A| = a$ and $\mathcal{B} \subseteq (B \setminus A)^{(2)}$ with $|\mathcal{B}| = b$ such that every $x \in A$ has a neighbor in $P_{y,z}$ for every $\{y,z\} \in \mathcal{B}$.*
- (b) *There are disjoint c-subsets $\mathcal{C}$ and $\mathcal{C}'$ of $B^{(2)}$ such that for every $\{x,y\} \in \mathcal{C}$ and every $\{x',y'\} \in \mathcal{C}'$, there is an edge in G with an end in $P^*_{x,y}$ and an end in $P^*_{x',y'}$.*

(c) *There exists a stable set* $S \subseteq B$ *in* G *with* $|S| = s$ *for which the following hold.*

- *For any three vertices* $x, y, z \in S$*, the vertex* x *is anticomplete to* $P^*_{y,z}$ *in* G*.*
- *The sets* $\left(P^*_{x,y} : \{x, y\} \in S^{(2)}\right)$ *are pairwise anticomplete in* G*.*

In other words,

$$G\left[\bigcup_{\{x,y\}\in S^{(2)}} P_{x,y}\right]$$

is isomorphic to a proper subdivision of K_s *with* S *as its set of branch vertices (except for the case* $s \in \{1, 2\}$*, where there is no branch vertex).*

Proof. We claim that

$$f_{3.11} = f_{3.11}(a, b, c, s) = f_{3.9}(a, b, f_{3.10}(c, s))$$

satisfies the lemma. The choice of $f_{3.11}$ calls for an application of Lemma 3.9 to $(B, \mathcal{P})$. Since 3.11(a) is identical to Lemma 3.9(a), we may assume that Lemma 3.9(b) holds. It follows that there exists $S_0 \subseteq B$ with $|S_0| = f_{3.10}(c, s)$ such that:

- S_0 is a stable set in G; and
- for any three vertices $x, y, z \in S_0$, the vertex x is anticomplete to $P^*_{y,z}$.

In particular, $\left(S_0, \mathcal{P}|_{S_0^{(2)}}\right)$ is a strong $(f_{3.10}(c, s), 1)$-block in G to which we can apply Lemma 3.10.

The result now follows from the fact that 3.11(b) is identical to Lemma 3.10(a) and 3.11(c) is identical to Lemma 3.10(b) combined with the above two bullets. ■

We also need Lemma 3.12 below which has been observed several times independently, for instance, in Lemma 2 from [50]. We will also use Lemma 3.12 extensively in subsequent chapters. For the sake of completeness, we include a short proof:

Lemma 3.12 (Folklore; see Lemma 2 from [50])**.** *For all* $q, r, s, t \in \mathbb{N}$*, there is a constant* $f_{3.12} = f_{3.12}(q, r, s, t) \in \mathbb{N}$ *with the following property. Let* G *be a* $(K_{s,s}, K_t)$*-free graph. Let* $\mathcal{X}$ *be a set of at least* $f_{3.12}$ *pairwise disjoint subsets of* $V(G)$*, each of cardinality at most* r*. Then there are* q *distinct sets in* $\mathcal{X}$ *which are pairwise anticomplete in* G*.*

Proof. The proof relies on Theorem 3.8. Let

$$f_{3.12} = f_{3.12}(q,r,s,t) = f_{3.8}\left(2^{r^2}, 2, \max\{q, 2s, t\}\right).$$

Choose $f_{3.12}$ distinct sets $X_1, \ldots, X_{f_{3.12}}$ from $\mathcal{X}$. For each $i \in [f_{3.12}]$, fix an enumeration $X_i = \{x_{i,1}, \ldots, x_{i,|X_i|}\}$. For every 2-subset $\{i,j\}$ of $[f_{3.12}]$, let

$$\Phi(\{i,j\}) = \{(f,f') : x_{i,f}x_{j,f'} \in E(G)\} \subseteq [|X_i|] \times [|X_j|] \subseteq [r]^2.$$

By the assumption of Lemma 3.12, the map $\Phi : [f_{3.12}]^{(2)} \to 2^{[r]^2}$ is well-defined. From the choice of $f_{3.12}$ and Theorem 3.8, it follows that there are subsets Z_0, Z_1, Z_2, Z_3 of $[f_{3.12}]$ with $Z_0 = q$, $|Z_1| = |Z_2| = s$ and $|Z_3| = t$ such that $Z_1 \cap Z_2 = \varnothing$, as well as $F \subseteq [r]^2$ such that for every 2-subset $\{i,j\}$ of $Z_0 \cup Z_1 \cup Z_2 \cup Z_3$, we have $\Phi(\{i,j\}) = F$. Moreover:

(6) *For every $f \in [r]$, we have $(f,f) \notin F$.*

For otherwise it follows from the definition of Φ that $G[\{x_{i,f} : i \in Z_3\}]$ is isomorphic to K_t, a contradiction. This proves (6).

(7) *We have $F = \varnothing$.*

Suppose not. Let $(f,f') \in F$. By (6), we have $f \neq f'$ and $(f,f),(f',f') \in F$. By the definition of Φ, for every $i \in Z_1$ and every $i' \in Z_2$, we have $x_{i,f}x_{i',f'} \in E(G)$, while $\{x_{i,f} : i \in Z_1\}$ and $\{x_{i',f'} : i' \in Z_2\}$ are both stable sets in G. But now the graph $G[\{x_{i,f} : i \in Z_1\} \cup \{x_{i',f'} : i' \in Z_2\}]$ is isomorphic to $K_{s,s}$, a contradiction. This proves (7).

From (7) and the choice of Φ, it follows that the q sets $(X_i : i \in Z_0)$ are pairwise anticomplete in G. This proves Lemma 3.12. ■

We now restate and prove Theorem 3.7:

Theorem 3.7. *For all $d \in \mathbb{N} \cup \{0\}$ and $s,t \in \mathbb{N}$, there is a constant $f_{3.7} = f_{3.7}(d,s,t) \in \mathbb{N}$ with the following property. Let G be a $(K_{t+1}, K_{t,t})$-free graph and let $(B, \mathcal{P})$ be a strong $(f_{3.7}, 1)$-block in G. For each $\{x,y\} \in B^{(2)}$, let $\mathcal{P}(\{x,y\}) = \{P_{x,y}\}$ where $P_{x,y}$ has length at most $d+1$. Then there is a stable set $S \subseteq B$ with $|S| = s$ such that the following hold.*

(a) *For any three vertices $x,y,z \in S$, the vertex x is anticomplete to $P^*_{y,z}$ in G.*

(b) *For all distinct* $\{x,y\}, \{x',y'\} \in S^{(2)}$, $P^*_{x,y}$ *and* $P^*_{x',y'}$ *are anticomplete in* G.

In other words,

$$G\left[\bigcup_{\{x,y\}\in S^{(2)}} P_{x,y}\right]$$

is isomorphic to a proper $(\leq d)$*-subdivision of* K_s *with* S *as its set of branch vertices (except for the case* $s \in \{1,2\}$*, where there is no branch vertex).*

Proof. The proof uses Theorem 3.11 and Lemma 3.12. Let

$$f = f_{3.12}(2, \max\{d+3, 2d\}, t, t+1).$$

We claim that $f_{3.7} = f_{3.7}(d,s,t) = f_{3.11}(f,f,f,s)$ satisfies the theorem. Suppose not. Then we have:

(8) *There are pairwise disjoint subsets* $X_1, \ldots, X_{f_{3.12}}$ *of* $V(G)$*, each of cardinality at most* $\max\{d+3, 2d\}$*, such that for all distinct* $i,j \in [f_{3.12}]$*,* X_i *and* X_j *are not anticomplete in* G*.*

To see this, note that by the choice of $f_{3.7}$, we can apply Theorem 3.11 to $(B, \mathcal{P})$. Since Theorems 3.11(c) implies 3.7, it follows that one of the following holds.

- There exist $A = \{x_1, \ldots, x_{f_{3.12}}\} \subseteq B$ and a set $\mathcal{B} = \{\{y_i, z_i\} : i \in [f_{3.12}]\} \subseteq (B \setminus A)^{(2)}$ such that for all $i,j \in [f_{3.12}]$, x_i has a neighbor in P_{y_j,z_j}.
- There are two disjoint $f_{3.12}$-subsets of $B^{(2)}$, say $\mathcal{C} = \{\{x_i, y_i\} : i \in [f_{3.12}]\}$ and $\mathcal{C}' = \{\{x'_i, y'_i\} : i \in [f_{3.12}]\}$, such that for all $i,j \in [f_{3.12}]$, the sets $P^*_{x_i,y_i}$ and $P^*_{x'_j,y'_j}$ are not anticomplete in G.

For each $i \in [f_{3.12}]$, in the former case above, let $X_i = \{x_i\} \cup P_{y_i,z_i}$, and in the latter case above, let $X_i = P^*_{x_i,y_i} \cup P^*_{x'_i,y'_i}$. Then for all distinct $i,j \in [f_{3.12}]$, X_i and X_j are disjoint, and there is an edge in G with an end in X_i and an end in X_j. Also, recall that $P_{x,y}$ has length at most $d+1$ for all $\{x,y\} \in B^{(2)}$, and so $|X_i| \leq \max\{d+3, 2d\}$ for all $i \in [f_{3.12}]$. This proves (8).

Now from (8), Lemma 3.12 and the choice of $f_{3.12}$, we deduce that G contains an induced subgraph isomorphic to K_{t+1} or $K_{t,t}$, a contradiction. ■

3.3 Bisets

In this section, we prove a technical Ramsey-type result about pairs of sets of vertices in a graph, which we will use in several later proofs. We need the following "product version" of Ramsey's Theorem:

Theorem 3.13 (Graham, Rothschild and Spencer [33])**.** *For all $n, q, r \in \mathbb{N}$, there is a constant $f_{3.13} = f_{3.13}(n, q, r) \in \mathbb{N}$ with the following property. Let $U_1, \dots, U_n$ be n sets, each of cardinality at least $f_{3.13}$ and let W be a non-empty set of cardinality at most r. Let Φ be a map from the Cartesian product $U_1 \times \cdots \times U_n$ to W. Then there exist $i \in W$ and a q-subset Z_j of U_j for each $j \in [n]$, such that for every $z \in Z_1 \times \cdots \times Z_n$, we have $\Phi(z) = i$.*

Let U be a set and let $a \in \mathbb{N} \cup \{0\}$. An *$a$-biset over U* is a pair (A, B) of subsets of U with $|A| \leq a$. Two bisets $(A, B), (A', B')$ are said to be *disjoint* if $B \cap B' = \varnothing$.

The main result of this section is the following:

Theorem 3.14. *For all $a, b \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{N}$, there is a constant $f_{3.14} = f_{3.14}(a, b, m) \in \mathbb{N}$ with the following property. Let G be a graph. Let $\mathcal{B}_1, \dots, \mathcal{B}_m$ be sets of pairwise disjoint a-bisets over $V(G)$, each of cardinality at least $f_{3.14}$. Then for every $i \in [m]$, there exists $\mathcal{B}'_i \subseteq \mathcal{B}_i$ with $|\mathcal{B}'_i| \geq b$ such that for all distinct $i, j \in [m]$, the following hold.*

- (a) *We have $A_i \cap B_j = \varnothing$ for all $(A_i, B_i) \in \mathcal{B}'_i$ and $(A_j, B_j) \in \mathcal{B}'_j$.*
- (b) *Either A_i and B_j are anticomplete in G for all $(A_i, B_i) \in \mathcal{B}'_i$ and $(A_j, B_j) \in \mathcal{B}'_j$, or for every $(A_i, B_i) \in \mathcal{B}'_i$, there exists $x_i \in A_i$ such that x_i has a neighbor in B_j for every $(A_j, B_j) \in \mathcal{B}'_j$.*
- (c) *Either B_i and B_j are disjoint and anticomplete in G for all $(A_i, B_i) \in \mathcal{B}'_i$ and $(A_j, B_j) \in \mathcal{B}'_j$, or for every $(A_i, B_i) \in \mathcal{B}'_i$ and every $(A_j, B_j) \in \mathcal{B}'_j$, either $B_i \cap B_j \neq \varnothing$ or there is an edge in G with an end in B_i and an end in B_j.*

Proof. We prove that

$$f_{3.14}(a, b, m) = f_{3.13}\left(m, \max\{b, 2\}, 2^{(2a+1)m^2}\right);$$

satisfies the theorem.

Let $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_m$. For each $(A, B) \in \mathcal{B}$, fix an enumeration $A = \{x^i_A : i \in [|A|]\}$ of the elements of A; recall that $|A| \leq a$. For every two a-bisets $(A, B), (A', B') \in \mathcal{B}$, let

$$I_1(A, B') = \{i \in [|A|] : x^i_A \in B'\} \subseteq [a];$$

$$I_2(A, B') = \{i \in [|A|] : N_G(x^i_A) \cap B' \neq \varnothing\} \subseteq [a].$$

Also, let $I_3(B, B') \in \{0, 1\}$ such that $I_3(B, B') = 0$ if B and B' are disjoint and anticomplete in G, and let $I_3(B, B') = 1$ if either $B_i \cap B_j \neq \varnothing$ or there is an edge in G with an end in B_i and an end in B_j.

Let $\mathbb{M}$ be the set of all m-by-m matrices whose entries are subsets of $[a]$ and let $\mathbb{B}$ be the set of all m-by-m binary matrices. So we have $|\mathbb{M}| = 2^{am^2}$ and $|\mathbb{B}| = 2^{m^2}$. Consider the product $\Pi = \mathcal{B}_1 \times \cdots \times \mathcal{B}_m$. For every $z = ((A_1, B_1), \cdots, (A_m, B_m)) \in \Pi$, define $M_1(z), M_2(z) \in \mathbb{M}$ and $M_3(z) \in \mathbb{B}$ such that for all $i, j \in [m]$, we have

$$[M_1(z)]_{ij} = I_1(A_i, B_j);$$

$$[M_2(z)]_{ij} = I_2(A_i, B_j);$$

$$[M_2(z)]_{ij} = I_3(B_i, B_j).$$

It follows that for every $z \in \Pi$, $M_1(z), M_2(z)$ and $M_3(z)$ are unique, and so the map $\Phi : \Pi \to \mathbb{M}^2 \times \mathbb{B}$ with $\Phi(z) = (M_1(z), M_2(z), M_3(z))$ is well-defined. This, along with the choice of $f_{3.14}$ and Theorem 3.13, implies that there exists $\mathcal{B}'_i \subseteq \mathcal{B}_i$ with $|\mathcal{B}_i| \geq \max\{b, 2\}$ for each $i \in [m]$, as well as $M_1, M_2 \in \mathbb{M}$, such that for every $z \in \mathcal{B}'_1 \times \cdots \times \mathcal{B}'_m$, we have $M_1(z) = M_1$ and $M_2(z) = M_2$. Moreover, we deduce:

(9) *Let $i, j \in [m]$ be distinct. Then we have $A_i \cap B_j = \varnothing$ for all $(A_i, B_i) \in \mathcal{B}'_i$ and $(A_j, B_j) \in \mathcal{B}'_j$*

Suppose for a contradiction that there are distinct $i, j \in [m]$ such that for some $(A_i, B_i) \in \mathcal{B}'_i$ and $(A_j, B_j) \in \mathcal{B}'_j$, we have $A_i \cap B_j \neq \varnothing$. Then we have $I_1(A_i, B_j) \neq \varnothing$. Also, since $|\mathcal{B}'_j| \geq 2$, we may choose $(A'_j, B'_j) \in \mathcal{B}'_j \setminus \{(A_j, B_j)\}$. It follows that $I_1(A_i, B_j) = [M_1]_{ij} = I_1(A_i, B'_j)$ is non-empty. But then $B_j \cap B'_j \neq \varnothing$, a contradiction with the assumption that $(A_j, B_j), (A'_j, B'_j) \in \mathcal{B}'_j \subseteq \mathcal{B}_j$ are disjoint. This proves (9).

(10) *Let $i, j \in [m]$ be distinct. Then either A_i is anticomplete to B_j in G for all $(A_i, B_i) \in \mathcal{B}'_i$ and $(A_j, B_j) \in \mathcal{B}'_j$, or for every $(A_i, B_i) \in \mathcal{B}'_i$, there exists $x_i \in A_i$ such that x_i has a neighbor in B_j for every $(A_j, B_j) \in \mathcal{B}'_j$.*

Note that for all $(A_i, B_i) \in \mathcal{B}'_i$ and $(A_j, B_j) \in \mathcal{B}'_j$, we have $I_2(A_i, B_j) = [M_2]_{ij} \subseteq [a]$. If $[M_2]_{ij} = \varnothing$, then A_i is anticomplete to B_j in G for all $(A_i, B_i) \in \mathcal{B}'_i$ and $(A_j, B_j) \in \mathcal{B}'_j$. Otherwise, one may choose $k \in [M_2]_{ij}$, and so for each $(A_i, B_i) \in \mathcal{B}'_i$, the vertex $x_i = x^k_{A_i} \in A_i$ has a neighbor in B_j for every $(A_j, B_j) \in \mathcal{B}'_j$. This proves (10).

(11) *Let $i, j \in [m]$ be distinct. Then either B_i and B_j are disjoint and anticomplete in G for all $(A_i, B_i) \in \mathcal{B}'_i$ and $(A_j, B_j) \in \mathcal{B}'_j$, or for every $(A_i, B_i) \in \mathcal{B}'_i$ and every $(A_j, B_j) \in \mathcal{B}'_j$, either $B_i \cap B_j \neq \varnothing$ or there is an edge in G with an end in B_i and an end in B_j.*

Note that for all $(A_i, B_i) \in \mathcal{B}'_i$ and $(A_j, B_j) \in \mathcal{B}'_j$, we have $I_3(B_i, B_j) = [M_3]_{ij} \in \{0, 1\}$. If $[M_3]_{ij} = 0$, then B_i and B_j are disjoint and anticomplete in G for all $(A_i, B_i) \in \mathcal{B}'_i$ and $(A_j, B_j) \in \mathcal{B}'_j$. If $[M_3]_{ij} = 1$, then for all $(A_i, B_i) \in \mathcal{B}'_i$ and $(A_j, B_j) \in \mathcal{B}'_j$, either $B_i \cap B_j \neq \varnothing$ or there is an edge in G with an end in B_i and an end in B_j. This proves (11).

Now the result follows from (9), (10) and (11). ■

3.4 Short blocks and long blocks

Given a graph G and $d, k, l \in \mathbb{N}$, we say a (k, l)-block $(B,\mathcal{P})$ in G is *d-short* if for every 2-subset $\{x, y\}$ of B, every path $P \in \mathcal{P}(\{x, y\})$ has length at most d. We say $(B,\mathcal{P})$ is *d-long* if for every 2-subset $\{x, y\}$ of B, every path $P \in \mathcal{P}(\{x, y\})$ has length at least $d + 1$.

We will prove three results concerning short and long blocks:

Theorem 3.15. *For all $d, k, l \in \mathbb{N}$, there is a constant $f_{3.15} = f_{3.15}(d, k, l) \in \mathbb{N}$ with the following property. Let G be a graph and let B be a d-short $(k, f_{3.15})$-block in G. Then there exists a d-short strong (k, l)-block $(B, \mathcal{P}')$ in G such that $\mathcal{P}'(\{x, y\}) \subseteq \mathcal{P}(\{x, y\})$ for all $\{x, y\} \in B^{(2)}$.*

Proof. The proof uses Theorem 3.14. Specifically, we claim that

$$f_{3.15} = f_{3.15}(d, k, l) = f_{3.14}\left(d + 1, l, \binom{k}{2}\right)$$

satisfies the theorem. For every 2-subset $\{x, y\}$ of B, let $\mathcal{B}_{\{x,y\}} = \{(P, P^*) : P \in \mathcal{P}(\{x, y\})\}$. Then $\mathcal{B}_{\{x,y\}}$ is a set of $f_{3.15}$ pairwise disjoint $(d + 1)$-bisets over $V(G)$. Thus, the choice of $f_{3.15}$ allows for an application of Theorem 3.14 to the sets $\left(\mathcal{B}_{\{x,y\}} : \{x, y\} \in B^{(2)}\right)$. We deduce that for every $\{x, y\} \in B^{(2)}$, there exists $\mathcal{P}'(\{x, y\}) \subseteq \mathcal{P}(\{x, y\})$ with $|\mathcal{P}'(\{x, y\})| \geq k$, such that for all distinct $\{x, y\}, \{x', y'\} \in B^{(2)}$, the sets

$$\mathcal{B}'_{\{x,y\}} = \{(P, P^*) : P \in \mathcal{P}'(\{x, y\})\} \subseteq \mathcal{B}_{\{x,y\}}$$

and

$$\mathcal{B}'_{\{x',y'\}} = \{(P, P^*) : P \in \mathcal{P}'(\{x', y'\})\} \subseteq \mathcal{B}_{\{x',y'\}}$$

satisfy the outcomes of Theorem 3.14. In particular, it follows from Theorem 3.14(a) that for every $P \in \mathcal{P}'(\{x, y\})$ and every $P' \in \mathcal{P}'(\{x', y'\})$, we have $P^* \cap P' = \varnothing$. Equivalently, we have $V(\mathcal{P}'^*(\{x, y\})) \cap V(\mathcal{P}'(\{x', y'\})) = \varnothing$. Hence, $(B, \mathcal{P}')$ is a d-short strong (k, l)-block in G. This completes the proof of Theorem 3.15. ■

Theorem 3.16. *For all $d, t \in \mathbb{N}$, there are constants $f_{3.16} = f_{3.16}(d, t) \in \mathbb{N}$ and $g_{3.16} = g_{3.16}(d, t) \in \mathbb{N}$ such that if G is a t-clean graph, then there is no d-short $(f_{3.16}, g_{3.16})$-block in G.*

Proof. The proof uses Theorems 3.7 and 3.15. Specifically, we show that

$$f_{3.16} = f_{3.16}(d, t) = f_{3.7}\left(d, 2t^2, t\right)$$

$$g_{3.16} = g_{3.16}(d, t) = f_{3.15}\left(d, f_{3.7}(d, 2t^2, t), 1\right)$$

satisfies the theorem. Let G be a t-clean graph, and suppose for a contradiction that there is a d-short $(f_{3.16}, g_{3.16})$-block B in G. By Theorem 3.15, there exists a d-short strong $(f_{3.16}, 1)$-block $(B, \mathcal{P}')$ in G such that $\mathcal{P}'(\{x, y\}) \subseteq \mathcal{P}(\{x, y\})$ for every 2-subset $\{x, y\}$ of B. Since G is $(K_{t+1}, K_{t,t})$-free, it follows that from Theorem 3.7 that G has an induced subgraph isomorphic to a proper subdivision of K_{2t^2}. Since $W_{t\times t}$ has $2t^2 - 2$ vertices, it follows that G has an induced subgraph isomorphic to a (proper) subdivision of $W_{t\times t}$, a contradiction. This completes the proof of Theorem 3.16. ■

We will often use a quantified version of Theorem 3.8 for graphs. The proof is straightforward, and we include it for the sake of completeness.

Theorem 3.17 (Ramsey [57])**.** *For all $c, s \in \mathbb{N}$, every graph G on at least c^s vertices contains either a c-clique or a stable set of cardinality s.*

Proof. The proof is by induction on s for fixed c. The cases $c = 1$ and $s = 1$ are easily seen to hold. So we may assume that $c, s > 1$. Let G be a graph on at least c^s vertices with no c-clique and no stable set of cardinality s. Let K be a maximum clique in G; thus, we have $|K| \leq c - 1$. For each $x \in K$, let M_x be the set of all vertices in G which are non-adjacent to x. It follows from the maximality of K that $V(G) = \bigcup_{x\in K}(M_x \cup \{x\})$. Now, for every $x \in K$, $G[M_x]$ contains no c-clique (as neither does G) and no stable set of cardinality $s-1$ (or otherwise $M_x \cup \{x\}$, and so G, contains a stable set of cardinality t). Consequently, by the induction hypothesis, we have $|M_x| < c^{s-1}$. But then $|V(G)| \leq (c-1)c^{s-1} < c^s$, a contradiction. This proves Theorem 3.17. ■

Theorem 3.18. *For all $d, k, l, t \in \mathbb{N}$, there are constants $f_{3.18} = f_{3.18}(d, k, l, t) \in \mathbb{N}$ and $g_{3.18} = g_{3.18}(d, k, l, t) \in \mathbb{N}$ with the following property. Let G be a t-clean graph and let $(B, \mathcal{P})$ be a $(f_{3.18}, g_{3.18})$-block in G. Then there is a d-long (k, l)-block $(B', \mathcal{P}')$ in G with $B' \subseteq B$ and $\mathcal{P}'(\{x, y\}) \subseteq \mathcal{P}(\{x, y\})$ for every 2-subset $\{x, y\}$ of B'. In particular, if $(B, \mathcal{P})$ is strong, then so is $(B', \mathcal{P}')$.*

Proof. We show that

$$f_{3.18} = f_{3.18}(d, k, l, t) = f_{3.16}(d, t)^k;$$

$$g_{3.18} = g_{3.18}(d, k, l, t) = g_{3.16}(d, t) + l$$

satisfies the theorem. For each 2-subset $\{x, y\}$ of B, let $\mathcal{Q}(\{x, y\})$ be the set of all paths of length at most d in $\mathcal{P}(\{x, y\})$, and let $\mathcal{P}'(\{x, y\}) = \mathcal{P}(\{x, y\}) \setminus \mathcal{Q}(\{x, y\})$. Let Γ be the graph with vertex set B such that $xy \in E(\Gamma)$ for distinct $x, y \in B$, if and only if $|\mathcal{Q}(\{x, y\})| \geq g_{3.16}(d, t)$. Since $|V(\Gamma)| = |B| = (f_{3.16}(d, t))^k$, it follows from Theorem 3.17 that Γ contains either a $f_{3.16}(d, t)$-clique C or a stable set B' of cardinality k. In the former case, $(C, \mathcal{Q})$ is a d-short $(f_{3.16}(d, t), g_{3.16}(d, t))$-block in G, which along with the assumption that G is t-clean violates Theorem 3.16. In the latter case, for every 2-subset $\{x, y\}$ of B', we have $|\mathcal{P}'(\{x, y\})| \geq g_{3.18} - g_{3.16}(d, t) = l$, and so $(B', \mathcal{P}')$ is a d-long block in G. This completes the proof of Theorem 3.18. ■

3.5 Sparse complete bipartite minor models

We begin with a couple of definitions, which will be of extensive use in the remainder of the thesis.

Let G be a graph. A *polypath in G* is a set $\mathcal{W}$ of pairwise disjoint paths in G, and we say $\mathcal{W}$ is a *w-polypath in G* if $|\mathcal{W}| = w$ for some $w \in \mathbb{N}$. For $d \in \mathbb{N}$, we say $\mathcal{W}$ is *d-loose* if for every $W \in \mathcal{W}$, each vertex $v \in W$ has neighbors (in G) in less than d paths in $\mathcal{W} \setminus \{W\}$. Also, for $w' \in \mathbb{N}$ with $w' \leq w$, we say $\mathcal{W}$ is *w'-fancy* if there exists $\mathcal{W}' \subseteq \mathcal{W}$ with $|\mathcal{W}'| = w'$ such that for every $W' \in \mathcal{W}'$ and every $W \in \mathcal{W} \setminus \mathcal{W}'$, there is an edge in G with an end in W' and an end in W. It follows that if $\mathcal{W}$ is a w-polypath in G which is w'-fancy, then $G[V(\mathcal{W})]$ has a minor isomorphic to $K_{w', w-w'}$; see Figure 3.4.

For $s, l \in \mathbb{N}$, by an *(s, l)-constellation in G* we mean a pair $\mathfrak{c} = (S_{\mathfrak{c}}, \mathcal{L}_{\mathfrak{c}})$ where $S_{\mathfrak{c}} \subseteq V(G)$ is a stable set of cardinality s and $\mathcal{L}_{\mathfrak{c}}$ is an l-polypath in $G \setminus S_{\mathfrak{c}}$, such that each vertex $x \in S_{\mathfrak{c}}$ has at least one neighbor in each path $L \in \mathcal{L}_{\mathfrak{c}}$. We write $V(\mathfrak{c}) = S_{\mathfrak{c}} \cup (\cup_{L \in \mathcal{L}_{\mathfrak{c}}} V(L))$. If $l = 1$, say $\mathcal{L}_{\mathfrak{c}} = \{L_{\mathfrak{c}}\}$, then we say $\mathfrak{c}$ is an *s-constellation in G* and denote it by $(S_{\mathfrak{c}}, L_{\mathfrak{c}})$. For $d \in \mathbb{N}$, we say $\mathfrak{c}$ is *d-ample* if for every two distinct vertices $x, y \in S_{\mathfrak{c}}$ and every $L \in \mathcal{L}_{\mathfrak{c}}$,

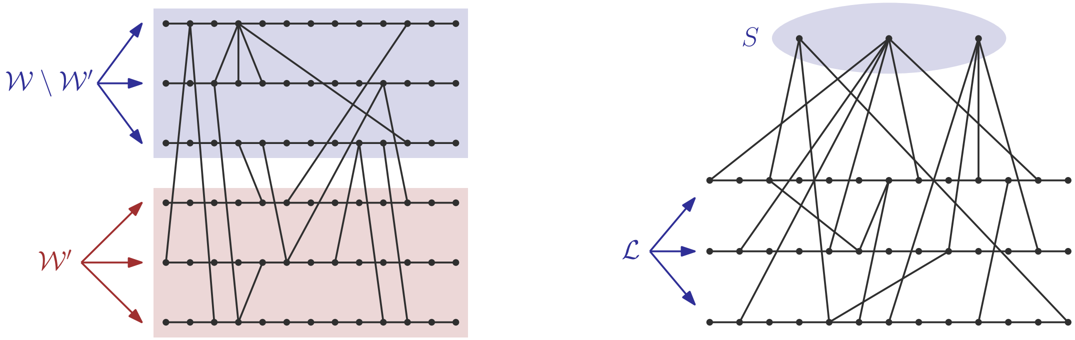

Figure 3.4: A 6-polypath $\mathcal{W}$ which is 3-fancy and 2-loose (left) and a $(3,3)$-constellation $(S, \mathcal{L})$ which is 2-ample (right)

there is no path R of length at most $d+1$ in G from x to y with $R^* \subseteq V(L)$. We also say $\mathfrak{c}$ is *ample* if $\mathfrak{c}$ is 1-ample. It follows that $\mathfrak{c}$ is ample if and only if no two vertices in $S_\mathfrak{c}$ have a common neighbor in $V(\mathcal{L}_\mathfrak{c})$. Also, it is easily seen that if $\mathfrak{c}$ is an (s,l)-constellation in G, then $G[V(\mathfrak{c})]$ has a minor isomorphic to $K_{s,l}$; see Figure 3.4.

The main result of this section is the following:

Theorem 3.19. *For all $d, l, s, t, w \in \mathbb{N}$, there are constants $f_{3.19} = f_{3.19}(d, l, s, t, w) \in \mathbb{N}$ and $g_{3.19} = g_{3.19}(d, l, s, t) \in \mathbb{N}$ with the following property. Let G be a t-clean graph with* $\operatorname{tw}(G) > f_{3.19}$. *Then one of the following holds.*

(a) *There is a d-ample (s,l)-constellation in G.*

(b) *There is a $2w$-polypath in G which is both w-fancy and $g_{3.19}$-loose.*

Note in particular that $g_{3.19}$ does not depend on w.

First, we prove a few lemmas.

Lemma 3.20. *For all $d, l, s, t \in \mathbb{N}$, there are constants $f_{3.20} = f_{3.20}(d, l, s, t) \in \mathbb{N}$ and $g_{3.20} = g_{3.20}(d, l, s, t) \in \mathbb{N}$ with the following property. Let G be a t-clean graph and let $\mathfrak{c}$ be a $(f_{3.20}, g_{3.20})$-constellation in G. Then there exists $S \subseteq S_\mathfrak{c}$ and $\mathcal{L} \subseteq \mathcal{L}_\mathfrak{c}$ with $|S| = s$ and $|\mathcal{L}| = l$ such that $(S, \mathcal{L})$ is a d-ample (s,l)-constellation in G.*

Proof. The proof relies on Theorems 3.8 and 3.16. Specifically, assuming $f = f_{3.16}(d+1, t)$ and $g = g_{3.16}(d+1, t)$, we prove that

$$f_{3.20} = f_{3.20}(d, l, s, t) = f_{3.8}\left(2^{g\binom{f}{2}+l}, 2, \max\{f, s\}\right)$$

$$g_{3.20} = g_{3.20}(d, l, s, t) = g\binom{f}{2} + l.$$

satisfy the lemma. For every 2-subset $\{x, y\}$ of $S_{\mathfrak{c}}$, let $\Phi(\{x, y\})$ be the set of all paths $L \in \mathcal{L}_{\mathfrak{c}}$ for which there is a path R of length at most $d+1$ in G from x to y with $R^* \subseteq L$. It follows that the map $\Phi : S_{\mathfrak{c}}^{(2)} \to 2^{\mathcal{L}_{\mathfrak{c}}}$ is well-defined. From Theorem 3.8 and the choice of $f_{3.20}$, we deduce that there exists $\mathcal{W} \subseteq \mathcal{L}_c$ as well as $Z \subseteq S_{\mathfrak{c}}$ with $|Z| = \max\{f, s\}$, such that for every 2-subset $\{x, y\}$ of Z, we have $\Phi(\{x, y\}) = \mathcal{W}$.

First, assume that $|\mathcal{W}| \geq g\binom{f}{2}$. Pick $M \subseteq Z$ with $|M| = f$. Then one may choose a set $\mathcal{L}(\{x, y\}) \subseteq \mathcal{W}$ of g paths for each 2-subset $\{x, y\}$ of M, such that the $g\binom{f}{2}$ paths in $\bigcup_{\{x,y\} \in M^{(2)}} \mathcal{L}(\{x, y\})$ are pairwise distinct, and so disjoint. Also, from the definition of Φ, it follows that for every 2-subset $\{x, y\}$ of M and each path $L \in \mathcal{L}(\{x, y\})$, there exists a path R_L of length at most $d+1$ in G from x to y with $R_L^* \subseteq L$; let $\mathcal{R}(\{x, y\}) = \{R_L : L \in \mathcal{L}(\{x, y\})\}$. Then $(M, \mathcal{R})$ is a $(d+1)$-short (strong) (f, g)-block in G. Since G is t-clean, this is a contradiction with Theorem 3.16.

We conclude that $|\mathcal{W}| < g\binom{f}{2}$. As a result, there exists $\mathcal{L} \subseteq \mathcal{L}_{\mathfrak{c}} \setminus \mathcal{W}$ with $|\mathcal{L}| = l$. Pick a subset S of Z with $|S| = s$. For every 2-subset $\{x, y\}$ of S and every path $L \in \mathcal{L}$, since we have $\Phi(\{x, y\}) = \mathcal{W}$, it follows from the definition of Φ that there is no path R in G of length at most $d+1$ from x to y with $R^* \subseteq L$. Then $(S, \mathcal{L})$ is a d-ample (s, l)-constellation in G. This completes the proof of Lemma 3.20. ■

Lemma 3.21. *For all $l, q, s, t \in \mathbb{N}$, there is a constant $f_{3.21} = f_{3.21}(l, q, s, t) \in \mathbb{N}$ with the following property. Let G be a K_{t+1}-free graph and let $\mathcal{P}$ be a $f_{3.21}$-polypath in G. Then one of the following holds.*

(a) *There exists an (s, l)-constellation $(S, \mathcal{L})$ in G such that $S \subseteq V(\mathcal{P})$ and $\mathcal{L} \subseteq \mathcal{P}$.*

(b) *There exists an l-loose q-polypath $\mathcal{Q}$ in G with $\mathcal{Q} \subseteq \mathcal{P}$.*

Proof. We claim that

$$f_{3.21} = f_{3.21}(l, q, s, t) = f_{3.8}\left(\max\{l + (t+1)^s, q\}, l+1, 2^{l+1}\right)$$

satisfies the lemma. Assume that 3.21(a) does not hold. Fix an enumeration $\mathcal{P} = \{P_1, \ldots, P_{f_{3.21}}\}$ of the elements of $\mathcal{P}$. For every $(l+1)$-subset $T = \{t_1, \ldots, t_{l+1}\}$ of $[f_{3.21}]$ with $t_1 < \cdots < t_{l+1}$, let $\Phi(T) \subseteq [l+1]$ be the set of all $i \in [l+1]$ for which there exists a vertex $x_{t_i} \in P_{t_i}$ that has a neighbor in P_{t_j} for every $j \in [l+1] \setminus \{i\}$. It follows that the map $\Phi : [f_{3.21}]^{(l+1)} \to 2^{[l+1]}$ is well-defined. Therefore, by the choice of $f_{3.21}$, we can apply Theorem 3.8 and obtain $Z \subseteq [f_{3.21}]$ with $|Z| = \max\{l + (t+1)^s, q\}$ as well as $F \subseteq [l+1]$ such that for every $I \in Z^{(l+1)}$, we have $\Phi(I) = F$. Furthermore:

(12) *We have $F = \varnothing$.*

Suppose not. Then we may choose $f \in F \subseteq [l+1]$. Since $|Z| \geq l + (t+1)^s$, it follows that there exist $I, J, K \subseteq Z$ with $|I| = f-1$, $|J| = (t+1)^s$ and $|K| = l - f + 1$, such that $\max I < \min J$ and $\max J < \min K$. Then, for each $j \in J$, we have $T_j = I \cup \{j\} \cup K \in Z^{(l+1)}$, and so $\Phi(T_j) = F$. It follows that for every $j \in J$, we have $f \in \Phi(T_j)$, which in turn implies that there exists a vertex $x_j \in P_j$ that has a neighbor in P_t for every $t \in T_j \setminus [j] = I \cup K$. Also, since G is K_{t+1}-free, it follows from Theorem 3.17 that there is stable set $S \subseteq \{x_j : j \in J\}$ in G with $|S| = s$. Let $\mathcal{L} = \{P_t : t \in I \cup K\}$. Then $\mathcal{L}$ is an l-polypath in $G \setminus S$, and every vertex in S has a neighbor in every path in $\mathcal{L}$. But now $(S, \mathcal{L})$ is an (s,l)-constellation in G satisfying 3.21(a), a contradiction. This proves (12).

Since $|Z| \geq q$, we may choose $Q \subseteq Z$ with $|Q| = q$. Let $\mathcal{Q} = \{P_i : i \in Q\}$. Then $\mathcal{Q}$ is a q-polypath in G with $\mathcal{Q} \subseteq \mathcal{P}$, and by (12), $\mathcal{Q}$ is l-loose. Hence, $\mathcal{Q}$ satisfies 3.21(b). This completes the proof of Lemma 3.21. ■

Lemma 3.22. *Let $l, s, t, w \in \mathbb{N}$. Let G be a K_{t+1}-free graph and assume that $\mathcal{Q}, \mathcal{Q}'$ be two $((t+1)^s + 1)^{l+1} w^{l^2}$-polypaths in G with $V(\mathcal{Q}) \cap V(\mathcal{Q}') = \varnothing$. Then one of the following holds.*

- (a) *There is an (s,l)-constellation $(S, \mathcal{L})$ in G such that either $S \subseteq V(\mathcal{Q})$ and $\mathcal{L} \subseteq \mathcal{Q}'$, or $S \subseteq V(\mathcal{Q}')$ and $\mathcal{L} \subseteq \mathcal{Q}$.*
- (b) *There exist $\mathcal{W} \subseteq \mathcal{W}$ and $\mathcal{W}' \subseteq \mathcal{Q}'$ with $|\mathcal{W}| = |\mathcal{W}'| = w$ such that every vertex in $V(\mathcal{W})$ has neighbors in fewer than l paths in $\mathcal{W}'$, and every vertex in $V(\mathcal{W}')$ has neighbors in fewer than l paths in $\mathcal{W}$.*

Proof. Suppose that 3.22(a) does not hold. Let $r = (t+1)^s w^l + w$. Then we have

$$\begin{aligned} |\mathcal{Q}| = |\mathcal{Q}'| &= ((t+1)^s + 1)^{l+1} w^{l^2} \\ &= ((t+1)^s + 1)((t+1)^s + 1)^l w^{l^2} \\ &= (t+1)^s \left((t+1)^s w^l + w^l\right) + ((t+1)^s + 1)^l w^{l^2} \\ &\geq (t+1)^s r^l + w \\ &\geq r. \end{aligned}$$

In particular, one may choose $\mathcal{R}' \subseteq \mathcal{Q}'$ with $|\mathcal{R}'| = r$. For every $Q \in \mathcal{Q}$, let r_Q be the largest integer in $[r]$ for which there exists a vertex $p_Q \in Q$ which has neighbors in at least r_Q paths in $\mathcal{R}'$. It follows that:

(13) *We have* $|\{Q \in \mathcal{Q} : r_Q \geq l\}| < (t+1)^s r^l$.

Suppose not. Let $\mathcal{P} \subseteq \{Q \in \mathcal{Q} : r_Q \geq l\}$ with $|\mathcal{P}| = (t+1)^s r^l$. Then for every $Q \in \mathcal{P}$, we may choose $p_Q \in Q$ and $\mathcal{L}_Q \subseteq \mathcal{R}'$ such that $|\mathcal{L}_Q| = l$ and p_Q has a neighbor in every path in $\mathcal{L}_Q$. Since $|\mathcal{R}'| = r$, it follows that there exist $\mathcal{L} \subseteq \mathcal{R}'$ and $\mathcal{T} \subseteq \mathcal{P}$ such that $|\mathcal{L}| = l$, $|\mathcal{T}| = (t+1)^s$, and for every $Q \in \mathcal{S}$, we have $\mathcal{L}_Q = \mathcal{L}$. Also, since G is K_{t+1}-free, it follows from Theorem 3.17 that there exists $\mathcal{S} \subseteq \mathcal{T}$ with $|\mathcal{S}| = s$ such that $S = \{q_R : R \in \mathcal{S}\}$ is a stable set in G with $|S| = s$. But now $(S, \mathcal{L})$ is an (s,l)-constellation in G with $S \subseteq V(\mathcal{S}) \subseteq V(\mathcal{P}) \subseteq V(\mathcal{Q})$ and $\mathcal{L} \subseteq \mathcal{R}' \subseteq \mathcal{Q}'$, contrary to the assumption that 3.22(a) does not hold. This proves (13).

By (13) and since $|\mathcal{Q}| \geq (t+1)^s r^l + w$, we may choose $\mathcal{W} \subseteq \mathcal{Q}$ with $|\mathcal{W}| = w$ such that every vertex in $V(\mathcal{W})$ has neighbors in fewer than l paths in $\mathcal{R}'$.

Next, for every $R \in \mathcal{R}'$, let w_R be the largest integer in $[w]$ for which there exists a vertex $q_R \in R$ which has neighbors in at least w_R paths in $\mathcal{W}$.

(14) *We have* $|\{R \in \mathcal{R}' : w_R \geq l\}| < (t+1)^s w^l$.

Suppose not. Let $\mathcal{P} \subseteq \{R \in \mathcal{R}' : w_R \geq l\}$ with $|\mathcal{P}| = (t+1)^s w^l$. Then for every $R \in \mathcal{P}$, we may choose $p_R \in R$ and $\mathcal{L}_R \subseteq \mathcal{W}$ such that $|\mathcal{L}_R| = l$ and p_R has a neighbor in every path in $\mathcal{L}_R$. Since $|\mathcal{W}| = w$, it follows that there exist $\mathcal{L} \subseteq \mathcal{W}$ and $\mathcal{T} \subseteq \mathcal{P}$ such that $|\mathcal{L}| = l$, $|\mathcal{T}| = (t+1)^s$, and for every $R \in \mathcal{T}$, we have $\mathcal{L}_R = \mathcal{L}$. Also, since G is K_{t+1}-free, it follows from Theorem 3.17 that there exists $\mathcal{S} \subseteq \mathcal{T}$ with $|\mathcal{S}| = s$ such that $S = \{q_R : R \in \mathcal{S}\}$ is a stable set in G with $|S| = s$. But then $(S, \mathcal{L})$ is an (s,l)-constellation in G with $S \subseteq V(\mathcal{S}) \subseteq V(\mathcal{P}) \subseteq V(\mathcal{R}') \subseteq V(\mathcal{Q}')$ and $\mathcal{L} \subseteq \mathcal{W} \subseteq \mathcal{Q}$, contrary to the assumption that 3.22(a) does not hold. This proves (14).

By (14) and since $|\mathcal{R}'| = r = (t+1)^s w^l + w$, we may choose $\mathcal{W}' \subseteq \mathcal{R}' \subseteq \mathcal{Q}'$ with $|\mathcal{W}'| = w$ such that every vertex in $V(\mathcal{W}')$ has neighbors in fewer than l paths in $\mathcal{W}$. Now $\mathcal{W}, \mathcal{W}'$ satisfy 3.22(b). This completes the proof of Lemma 3.22. ■

We are now in a position to prove Theorem 3.19, which we restate:

Theorem 3.19. *For all* $d, l, s, t, w \in \mathbb{N}$, *there are constants* $f_{3.19} = f_{3.19}(d,l,s,t,w) \in \mathbb{N}$ *and* $g_{3.19} = g_{3.19}(d,l,s,t) \in \mathbb{N}$ *with the following property. Let* G *be a* t*-clean graph with* $\mathrm{tw}(G) > f_{3.19}$. *Then one of the following holds.*

(a) *There is a* d*-ample* (s,l)*-constellation in* G.

(b) *There is a $2w$-polypath in G which is both w-fancy and $g_{3.19}$-loose.*

Note in particular that $g_{3.19}$ does not depend on w.

Proof. The proof relies on Theorems 2.27 and 3.11 as well as Lemmas 3.20, 3.21 and 3.22. Specifically, let $f = f_{3.20}(d, l, s, t)$, let $g = g_{3.20}(d, l, s, t)$ and let

$$f_1 = f_{3.21}(g, ((t+1)^f + 1)^{g+1} w^{g^2}, f);$$

$$f_2 = f_{3.11}(f, g, f_1, 2t^2).$$

We claim that

$$f_{3.19} = f_{3.19}(d, l, s, t, w) = f_{2.27}(f_2, 1, t)$$

and

$$g_{3.19} = g_{3.19}(d, l, s, t) = g$$

satisfy the theorem. Suppose that 3.19(a) does not hold, that is, there is no d-ample (s, l)-constellation in G. Since G is t-clean and by the choices of f and g, it follows from Lemma 3.20 that:

(15) *There is no (f, g)-constellation in G.*

Also, by Theorem 2.27, there is a strong $(f_2, 1)$-block $(B, \mathcal{P})$ in G. By the choice of f_2, we may apply Theorem 3.11 to G and $(B, \mathcal{P})$. By (15), Theorem 3.11(a) does not hold. Also, if Theorem 3.11(c) holds, then G contains a proper subdivision of K_{2t^2} (as an induced subgraph), which in turn contains a proper subdivision of every graph on $2t^2$ vertices. But this violates the assumption that G is t-clean because $|V(W_{t\times t})| = 2t^2 - 2$.

We deduce that Theorem 3.11(b) holds. Explicitly, there are two f_1-polypaths $\mathcal{C}, \mathcal{C}'$ in G such that for every $L \in \mathcal{C}$ and every $L' \in \mathcal{C}'$, L is not anticomplete to L' in G. Now, from (15), the choice of f_1 and Lemma 3.21, it follows that there are two g-loose $((t+1)^{(g+1)f} w^{g^2})$-polypaths $\mathcal{Q} \subseteq \mathcal{C}$ and $\mathcal{Q}' \subseteq \mathcal{C}'$ in G. Furthermore, from (15) and Lemma 3.22, it follows that there exist $\mathcal{W} \subseteq Q$ and $\mathcal{W}' \subseteq Q'$ with $|\mathcal{W}| = |\mathcal{W}'| = w$ such that every vertex in $V(\mathcal{W})$ has neighbors in fewer than g paths in $\mathcal{W}'$, and every vertex in $V(\mathcal{W}')$ has neighbors in fewer than g paths in $\mathcal{W}$. But now $\mathcal{W} \cup \mathcal{W}'$ is a $2w$-polypath in G which is both w-fancy and g-loose, and so 3.19(b) holds. This completes the proof of Theorem 3.19. ■

Chapter 4

Cleanness in $\mathcal{H}$-free graphs for finite $\mathcal{H}$[1]

4.1 Warm-up: excluding one graph

The goal of this chapter is to prove Theorem 2.4:

Theorem 2.4. *Let* $\mathcal{H}$ *be a finite set of graphs. Then the class of all* $\mathcal{H}$*-free graphs is clean if and only if there is no* $\mathcal{H}$*-free* s*-array except possibly for finitely many* $s \in \mathbb{N}$.

The proof will use the bulk of the machinery developed in Chapter 3, demonstrating our first application of the method of strong blocks. To provide the reader with a taste of how this method is applied, we first prove the special case of Theorem 2.4 when $|\mathcal{H}| = 1$:

Theorem 2.5. *Let* H *be a graph. Then the class of all* H*-free graphs is clean if and only if every component of* H *is either a path or a subdivided star.*

Proof. The "only if" implication is proved in Theorem 2.3. To prove the "if" implication, we need to show that for every graph H each component of which is a path or a subdivided star, and for every $t \in \mathbb{N}$, there is a constant $w = w(H, t) \in \mathbb{N}$ such that every H-free t-clean graph has treewidth at most w. The definition of w uses Theorems 2.27 and 3.18 as well as Lemma 3.12. Choose $\kappa, \Delta, \rho \in \mathbb{N}$ such that the number of components of H is at most κ, the maximum degree of H is at most Δ, and the radius of every component of H is at most ρ. It follows that H is isomorphic to an induced subgraph of the graphs which has exactly κ components, each isomorphic to the ρ-subdivision of $S_{1,\Delta}$.

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[1] This chapter is based on the coauthored paper [8].

Let

$$m_1 = f_{3.12}(\Delta, \rho+1, t, t+1);$$
$$m_2 = f_{3.12}(\kappa, \Delta(\rho+1)+1, t, t+1).$$

Let

$$f = f_{3.18}(\rho+1, 2m_2, m_1, t);$$
$$g = g_{3.18}(\rho+1, 2m_2, m_1, t).$$

We define

$$w = w(H,t) = f_{2.27}(f, g, t).$$

Suppose for a contradiction that there is an H-free t-clean graph G with $\operatorname{tw}(G) > w$. From Theorem 2.27 and the choice of w, it follows that there is a strong (f,g)-block in G. Then, from Theorem 3.18 and the choices of f and g, it follows that there is a $(\rho+1)$-long strong $(2m_2, m_1)$-block in G. In particular, one may choose $2m_2$ vertices $(x_i, y_i : i \in [m_2])$ of G, as well as a set $\mathcal{P}_i$, for each $i \in [m_2]$, of m_1 pairwise internally disjoint paths of length at least $\rho+2$ in G from x_i to y_i, such that the sets $(V(\mathcal{P}_i) : i \in [m_2])$ are pairwise disjoint. We deduce that:

(16) *For each $i \in [m_2]$, there exists $S_i \subseteq V(\mathcal{P}_i)$ such that $G[S_i]$ is isomorphic to the ρ-subdivision of $K_{1,\Delta}$.*

To see this, note that for each path $P \in \mathcal{P}_i$, since P has length at least $\rho+2$, there is path L_P on $\rho+2$ vertices in G with $V(L_P) \subsetneq V(P)$ such that x_i is an end of L_P (so the other end of L_P is different from y_i). It follows that the sets $(V(L_P) \setminus \{x_i\} : P \in \mathcal{P}_i)$ are pairwise disjoint $(\rho+1)$-subsets of $V(G)$. By the choice of $m_1 = |\mathcal{P}_i|$ and the assumption that G is $(K_{t,t}, K_{t+1})$-free, we can apply Lemma 3.12 to obtain a Δ-subset $\mathcal{D}_i$ of $\mathcal{P}_i$ such that the sets $(V(L_P) \setminus \{x_i\} : P \in \mathcal{D}_i)$ are pairwise anticomplete in G. Let $S_i = \cup_{P \in \mathcal{D}_i} V(L_P)$. Then $G[S_i]$ is isomorphic to the ρ-subdivision of $S_{1,\Delta}$. This proves (16).

From (16), it follows in particular that the sets $(S_i : i \in [m_2])$ are pairwise disjoint $(\Delta(\rho+1)+1)$-subsets of $V(G)$. By the choice of m_2 and the assumption that G is $(K_{t,t}, K_{t+1})$-free, we can apply Lemma 3.12 to obtain a κ-subset I of $[m_2]$ such that the sets $(S_i : i \in I)$ are pairwise anticomplete in G. Let

$$S = \bigcup_{i \in I} S_i.$$

By (16), the graph $G[S]$ has exactly κ components, each isomorphic to the ρ-subdivision of $S_{1,\Delta}$. But now H is isomorphic to an induced subgraph of $G[S]$, a contradiction with the assumption that G is H-free. This completes the proof of Theorem 2.5. ■

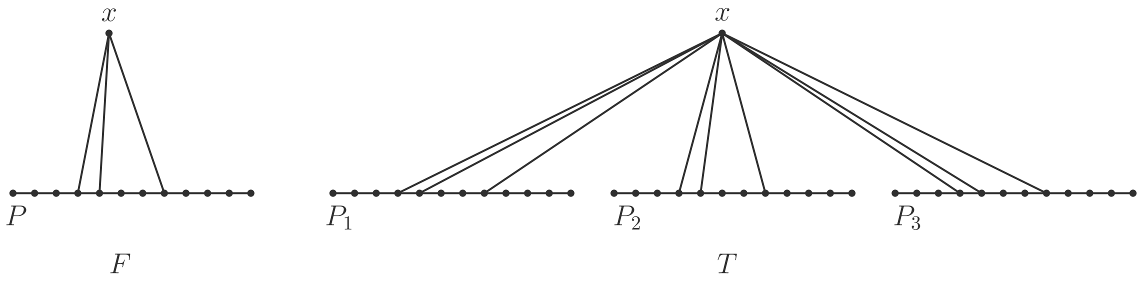

Figure 4.1: Left: A 3-strand F with neck x and path P. Right: A 3-tassel T with neck x and paths P_1, P_2, P_3, obtained from three copies of F as its strands.

The proof of Theorem 2.4 is much more complicated, and we will complete it in several steps, as follows. Note that, in view of Observation 2.2, we only need to prove the "if" implication. To that end, we introduce two properties for a general set $\mathcal{H}$ of graphs: that of being "tasselled" and that of "hassled." These two properties, compared to "there are finitely many $s \in \mathbb{N}$ for which an $\mathcal{H}$-free s-array exists," reveal the structure of $\mathcal{H}$ somewhat more explicitly, but they are also more technical. Indeed, it follows that these notions are successively stronger than the necessary condition from Observation 2.2: every hassled set of graphs is also a tasselled set, and for a tasselled set $\mathcal{H}$ of graphs, there is no $\mathcal{H}$-free s-array except possibly for finitely many $s \in \mathbb{N}$. Surprisingly, it turns out that when $\mathcal{H}$ is finite, all three properties are equivalent. We prove this in the next section under Theorems 4.1 and 4.2. It therefore remains to show that for every finite set $\mathcal{H}$ of graphs which is hassled, the class of all $\mathcal{H}$-free graphs is clean. This will be done in Sections 4.3 and 4.4, making extensive use of the material from Chapter 3.

4.2 Tassels and Hassles

Here we perform a two-step reduction of the "if" implication in Theorem 2.4: first to a statement in terms of "the columns of the arrays," which we refer to as "tassels," and then to a statement in terms of an even more technical notion called a "hassle."

Let $c \in \mathbb{N}$. A *c-strand* is a graph F obtained from a path P by adding a new vertex x with at least one neighbor in P, such that x is not adjacent to the first and last c vertices of P. We call x and P the *neck of* F and the *path of* F, respectively. By a *c-tassel* we mean a graph T obtained from at least c pairwise disjoint copies of a c-strand F by identifying their necks into a single vertex, called the *neck of* T. We also refer to each copy of F in T as a *strand of* T, and to the path of each copy of F as a *path of* T (see Figure 4.1).

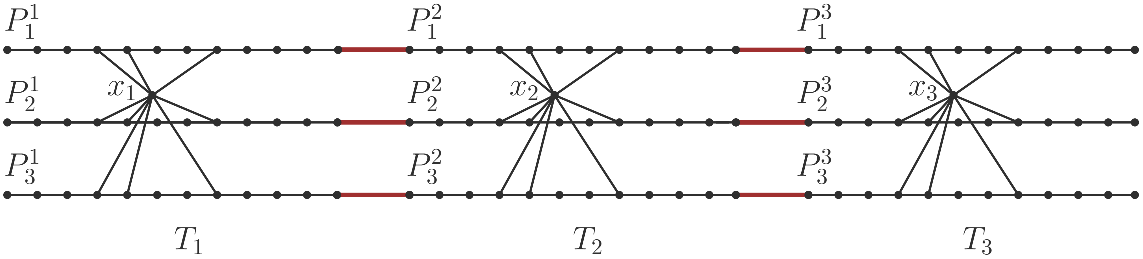

Figure 4.2: The 3-array A as described in the proof of Theorem 4.1, obtained from three copies T_1, T_2, T_3 of the 3-tassel T in Figure 4.1.

We say that a set $\mathcal{H}$ of graphs is *tasselled* if there is a constant $c = c(\mathcal{H}) \in \mathbb{N}$ with the property that for every c-tassel T, there exists a graph $H \in \mathcal{H}$ such that each component of H is isomorphic to an induced subgraph of T.

For a finite set $\mathcal{H}$ of graphs, we define $||\mathcal{H}||_1 = \sum_{H\in\mathcal{H}} |V(H)|$. We prove that:

Theorem 4.1. *Let $\mathcal{H}$ be a finite set of graphs such that there is no $\mathcal{H}$-free s-array except possibly for finitely many $s \in \mathbb{N}$. Then $\mathcal{H}$ is tasselled.*

Proof. Choose $s_0 \in \mathbb{N}$ such that there is no $\mathcal{H}$-free s-array for any $s \geq s_0$. Let $c = \max\{s_0, ||\mathcal{H}||_1\}$. In order to prove that $\mathcal{H}$ is tasselled, we show that for every c-tassel T, there is a graph $H \in \mathcal{H}$ each component of which is isomorphic to an induced subgraph T.

Let $d \geq c$ be the number of paths of T. Construct a d-array A as follows. Start with d pairwise disjoint copies $T_1, \ldots, T_d$ of T. For each $i \in [d]$, let x_i be the neck of T_i, fix an enumeration $P_1^i, \ldots, P_d^i$ of the paths of T_i, and for every $j \in [d]$, fix a labeling u_j^i, v_j^i of the ends of P_j^i. Then, for every $i \in [d-1]$ and every $j \in [d]$, add an edge between u_j^i and v_j^{i+1} (see Figure 4.2).

Since A is an d-array with $d \geq c \geq s_0$, it follows that A is not $\mathcal{H}$-free, and so we may choose a graph $H \in \mathcal{H}$ which is isomorphic to an induced subgraph of A.

Let K be a component of H. Our goal is to show that K is isomorphic to an induced subgraph of T. This is immediate if K is a path, because $|K| \leq ||\mathcal{H}||_1 \leq c$ and T is a c-tassel. Thus, we may assume that K is not a path. Since H is isomorphic to an induced subgraph of A, it follows that some induced subgraph K' of A is isomorphic to K. In particular, K' is a connected graph on at most $||\mathcal{H}||_1$ vertices which is not a path. Also, from the construction of A, it is readily observed that the necks $x_1, \ldots, x_d$ of $T_1, \ldots, T_d$ are pairwise at distance at least $2c + 3 > ||\mathcal{H}||_1 \geq |K'|$ in A. Consequently, there exists exactly one $i \in [d]$ for which x_i belongs to $V(K')$.

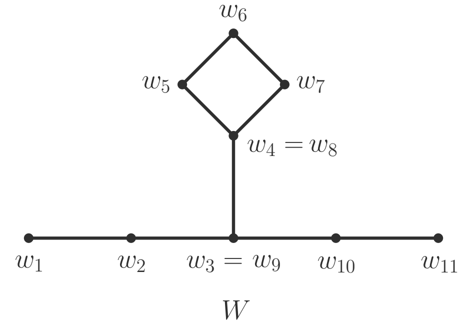

Figure 4.3: A 3-stretched walk W with $n_W = 11$ and $\varphi_W(i) = w_i$ for all $i \in [11]$.

Now, since K' is connected, it follows that for every component P of $K' \setminus \{x_i\}$, we have $P \subseteq V(G) \setminus \{x_1, \ldots, x_d\}$ and x_i has a neighbor in P. This, along with the fact that P is connected and $|P| < ||\mathcal{H}||_1 \leq c$, implies that $P \subseteq P^i_j \subseteq T_i \setminus \{x_i\}$ for some $j \in [d]$. In conclusion, we have shown that $K' \setminus \{x_i\} \subseteq T_i \setminus \{x_i\}$, and so $K' \subseteq T_i$. Hence, K is isomorphic to an induced subgraph of T, as desired. ■

The notion of a "c-hassle" is similar to that of a c-tassel, except its paths are replaced by walks that are "locally" isomorphic to a path, and, up to "weak sparsity," there may be additional edges between these walks.

Let us make this precise. Given $n \in \mathbb{N} \cup \{0\}$, by an *$n$-segment* we mean a set S of at most n consecutive integers. For instance, $[n]$ is an n-segment. A graph W is said to be a *walk* if there exists $n_W \in \mathbb{N}$ and a surjective map $\varphi_W : [n_W] \to V(W)$ such that for every $i \in [n_W - 1]$, the vertices $\varphi_W(i)$ and $\varphi_W(i+1)$ are adjacent in W (one may observe that W is a walk if and only if W is connected). Given a walk W along with the choices of n_W and ϕ_W as in the definition, for every $c \in \mathbb{N}$, we refer to $\varphi_W([c])$ and $\varphi_W([n_W] \setminus [n_W - c])$ as the *first c vertices of W* and the *last c vertices of W*, respectively. We also say W is *c-stretched* if $\varphi_W(S)$ is a path in W for every c-segment $S \subseteq [n_W]$ (see Figure 4.3 – playfully put, this means a snake of length $c - 1$ can traverse through W and never see/hit itself.)

A *c-hassle*, where $c \in \mathbb{N}$, is a graph Ξ obtained from at least c pairwise disjoint c-stretched walks, called the *walks of* Ξ, by adding edges arbitrarily between the walks, and then adding a vertex x, called the *neck of* Ξ, which has a neighbor in each walk and which is anticomplete to the first and last c vertices of each walk (see Figure 4.4).

For a set $\mathcal{H}$ of graphs, we say $\mathcal{H}$ is *hassled* if for every integer $t \in \mathbb{N}$, there is a constant $c = c(\mathcal{H}, t) \in \mathbb{N}$ with the property that for every $(K_{t,t}, K_{t+1})$-free c-hassle T there is a graph

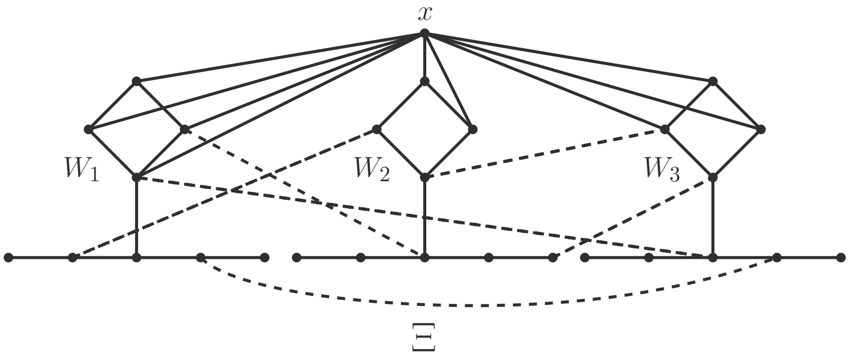

Figure 4.4: A 3-hassle Ξ with neck x, where the walks W_1, W_2, W_3 of Ξ are three copies of the 3-stretched walk W from Figure 4.3. Dashed lines represent possible edges.

$H \in \mathcal{H}$ each component of which is isomorphic to an induced subgraph of T. In particular, every c-tassel is a $(K_{3,3}, K_4)$-free c-hassle, and so every hassled set is also tasselled. More importantly, for finite sets of graphs, the converse is also true:

Theorem 4.2. *Let $\mathcal{H}$ be a finite set of graphs which is tasselled. Then $\mathcal{H}$ is hassled.*

The proof of Theorem 4.2 relies on the next two lemmas.

Lemma 4.3. *Let $b, c, k \in \mathbb{N}$ be integers, let $\mathcal{T}$ be a set of c-tassels, each with exactly c paths, such that $|\mathcal{T}| \geq bkc^k$. For each $T \in \mathcal{T}$, let x_T be the neck of T and fix an enumeration $P_1^T, \dots, P_c^T$ of the paths of T. Let K be a connected graph on at most k vertices which is not a path, and assume that for every $T \in \mathcal{T}$, there is an isomorphism f_T from K to an induced subgraph of T; in particular, K is isomorphic to an induced subgraph of T. Then there exist $x' \in V(K)$ and $\mathcal{T}' \subseteq \mathcal{T}$ with $|\mathcal{T}'| = b$ for which the following hold.*

(a) *For every $T \in \mathcal{T}'$, we have $f_T(x') = x_T$.*

(b) *For every component L of $K \setminus \{x'\}$, there exists $i(L) \in \{1, \dots, c\}$ such that for every $T \in \mathcal{T}'$, we have $f_T(L) \subseteq P_{i(L)}^T$.*

Proof. For each $T \in \mathcal{T}$, let $X'_T = f_T^{-1}(\{x_T\}) \subseteq K$. So we have $|X'_T| = 1$ because K is not a path; say $X'_T = \{x'_T\}$. Also, since K has at most k vertices, it follows that:

(17) *There exist $x' \in V(K)$ and $\mathcal{T}_1 \subseteq \mathcal{T}$ with $|\mathcal{T}_1| = bc^k$ such that for every $T \in \mathcal{T}_1$, we have $f_T(x') = x_T$.*

By (17), for each $T \in \mathcal{T}_1$ and every component L of $K \setminus \{x'\}$, we have $f_T(L) \subseteq T \setminus \{x_T\}$, which in turn implies that there exists $i(L, T) \in [c]$ for which we have $f_T(L) \subseteq P_{i(L,T)}^T$.

Now, since $K \setminus \{x'\}$ has at most k components and since $|\mathcal{T}_1| \geq bc^k$, it follows that there exists $\mathcal{T}' \subseteq \mathcal{T}_1 \subseteq \mathcal{T}$ with $|\mathcal{T}'| = b$, as well as $i(L) \in [c]$ for every component L of $K \setminus \{x'\}$, such that for each $T \in \mathcal{T}'$, we have $i(L, T) = i(L)$. Hence, x' and $\mathcal{T}'$ satisfy 4.3(b). Moreover, from (17), it follows that x' and $\mathcal{T}'$ satisfy 4.3(a), as desired. ■

Lemma 4.4. *For every finite and tasselled set $\mathcal{H}$ of graphs, there is a constant $f_{4.4} = f_{4.4}(\mathcal{H}) \in \mathbb{N}$ with the following property. For every $f_{4.4}$-hassle Ξ with neck x, there is a graph $H \in \mathcal{H}$ such that for every component K of H, one of the following holds.*

(a) *K is a path.*

(b) *There exists $x' \in V(K)$ and a map $f : V(K) \to V(\Xi)$ with $f^{-1}(\{x\}) = \{x'\}$, such that for every component L of $K \setminus \{x'\}$, the restriction of f to $\{x'\} \cup L$ is an isomorphism from $K[\{x'\} \cup L]$ to $\Xi[\{x\} \cup f(L)]$.*

Proof. Since $\mathcal{H}$ is tasselled, it follows that there is a constant $c = c(\mathcal{H}) \in \mathbb{N}$ with the property that for every c-tassel T there is a graph $H \in \mathcal{H}$ each component of which is isomorphic to an induced subgraph of T.

We claim that
$$f_{4.4} = f_{4.4}(\mathcal{H}) = ||\mathcal{H}||_1^2 \cdot c^{||\mathcal{H}||_1+1}$$
satisfies the lemma. To see this, let Ξ be a $f_{4.4}$-hassle with neck x, and let $\mathcal{W}$ be the set of all walks of Ξ. Recall that each walk in Ξ is $f_{4.4}$-stretched.

For each $W \in \mathcal{W}$, let n_W and φ_W be as in the definition of a walk, and construct a c-tassel T_W as follows. Let $P_1^W, \dots, P_c^W$ be c pairwise disjoint and anticomplete paths, each on n_W vertices, and for every $i \in [c]$, choose a bijection $g_i^W : V(P_i^W) \to [n_W]$ such that $p, p' \in V(P_i^W)$ are adjacent in P_i^W if and only if $|g_i^W(p) - g_i^W(p')| = 1$ (note that there are only two such bijections). Let T_W be the graph obtained from $P_1^W, \dots, P_c^W$ by adding a vertex x_W such that for every $i \in [c]$ and every $p \in V(P_i^W)$, the vertex x_W is adjacent to p in T_W if and only if x is adjacent to $\varphi_W(g_i^W(p))$ in Ξ (see Figure 4.5).

Note that for every $W \in \mathcal{W}$, in the graph Ξ, the vertex x has a neighbor in W and no neighbor among the first and last $f_{4.4}$ vertices of W. In particular, from the construction and the fact that $f_{4.4} \geq c$, it follows that for every $i \in [c]$, x has a neighbor in P_i^W and no neighbor among the first and last c vertices of P_i^W. Thus, for every $W \in \mathcal{W}$, the graph T_W is a c-tassel with neck x_W and paths $P_1^W, \dots, P_c^W$.

Also, since $\mathcal{H}$ is tasselled, it follows from the choice of c that for every $W \in \mathcal{W}$ there exist $H \in \mathcal{H}$ each component of which is isomorphic to an induced subgraph of the c-tassel T_W. Consequently, since $|\mathcal{W}| \geq f_{4.4}$ and $|\mathcal{H}| \leq ||\mathcal{H}||_1$, it follows that there exists $H \in \mathcal{H}$

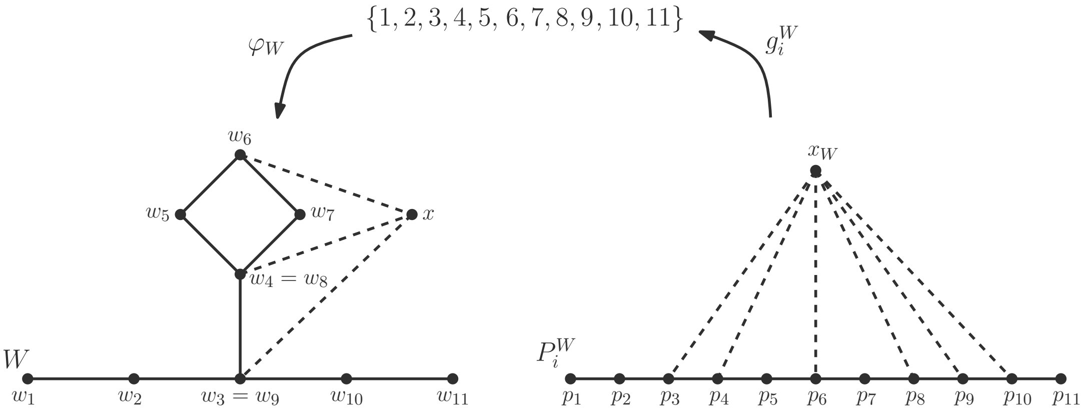

Figure 4.5: Left: the walk W with $n_W = 11$ and $\varphi_W(j) = w_j$ for all $i \in [11]$. Right: the path P_i^W with $g_i^W(p_j) = j$ for all $j \in [11]$. For every $j \in [11]$, we have $\varphi_W(g_i^W(p_j)) = w_j$, and x_W is adjacent to p_j in T_W if and only if x is adjacent to w_j in Ξ.

and $\mathcal{W}' \subseteq \mathcal{W}$ with $|\mathcal{W}'| = ||\mathcal{H}||_1 c^{||\mathcal{H}||_1+1}$ such that for every $W \in \mathcal{W}'$, each component of H is isomorphic to an induced subgraph of T_W.

We now prove that H satisfies 4.4. Let K be a component of H which is not a path. We wish to show that K satisfies 4.4(b). Note that for every $W \in \mathcal{W}'$, the c-tassel T_W has an induced subgraph isomorphic to K, and so there is an isomorphism f_W from K to an induced subgraph of T_W. This allows for an application of Lemma 4.3 to $\mathcal{T} = \{T_W : W \in \mathcal{W}'\}$ and K. Since $|V(K)| \leq |V(H)| \leq ||\mathcal{H}||_1$, it follows that there exist $x' \in V(K)$ and $W_1, \dots, W_c \in \mathcal{W}'$ such that x' and $\mathcal{T}' = \{T_{W_1}, \dots, T_{W_c}\}$ satisfy Lemma 4.3(a) and (b).

Henceforth, for all $i, j \in [c]$, we write

$$T_j = T_{W_j}; \quad x_j = x_{W_j}; \quad f_j = f_{W_j}; \quad n_j = n_{W_j}; \quad \varphi_j = \varphi_{W_j};$$

$$P_{i,j} = P_i^{W_j}; \quad g_{i,j} = g_i^{W_j}.$$

The outcomes of Lemma 4.3 can now be rewritten as:

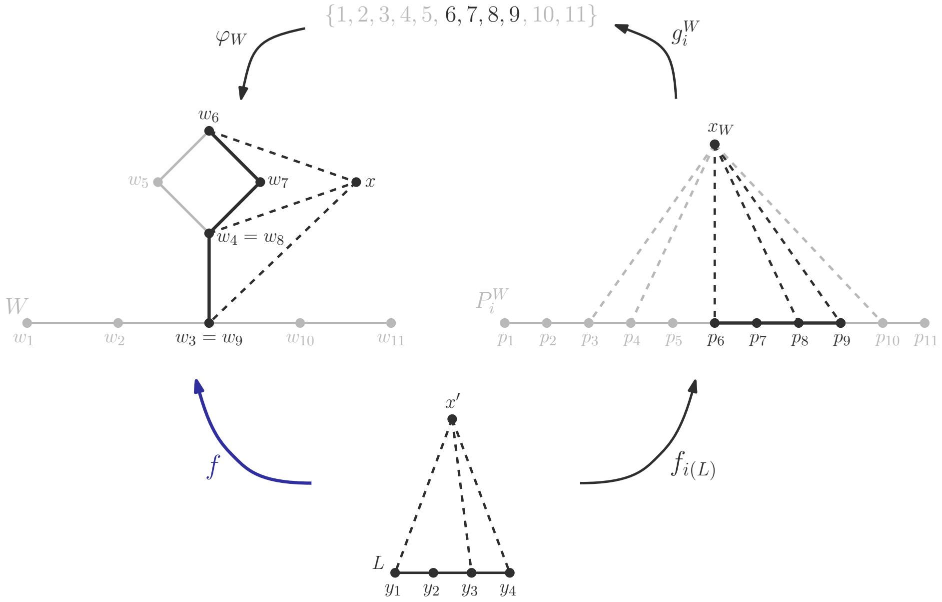

Figure 4.6: The map $f : V(K) \to V(\Xi)$. For each $j \in [4]$, we have $f_{i(L)}(y_j) = p_{j+5}$, which yields $f(y_j) = w_{j+5}$.

(18) *The following hold.*

- *For every $j \in [c]$, we have $f_j(x') = x_j$.*
- *For every component L of $K \setminus \{x'\}$, there exists $i(L) \in \{1, \dots, c\}$ such that for every $j \in [c]$, we have $f_j(L) \subseteq P_{i(L),j}$.*

Since $|K| \leq |H| \leq ||\mathcal{H}||_1 \leq f_{4.4}$, it follows that every component of $K \setminus \{x'\}$ is a path on less than $f_{4.4}$ vertices. This, combined with the second bullet of (18), yields the following:

(19) *For every $j \in [c]$ and every component L of $K \setminus \{x'\}$, there exists a $f_{4.4}$-segment $S_{j,L} \subseteq [n_j]$ for which we have $f_j(L) = g^{-1}_{i(L),j}(S_{j,L}) \subseteq P_{i(L),j}$.*

Define $f : V(K) \to V(\Xi)$ as follows (see Figure 4.6). Let $f(x') = x$, and for every component L of $K \setminus \{x'\}$ and every $y \in L$, let

$$f(y) = \varphi_{i(L)} \left(g_{i(L),i(L)}(f_{i(L)}(y)) \right).$$

We prove that f satisfies 4.4(b). Let L be a component of $K \setminus \{x'\}$. By (19), we have $f(L) = \varphi_{i(L)}(S_{i(L),L}) \subseteq W_{i(L)} \subseteq V(\Xi) \setminus \{x\}$, and so we have $f^{-1}(\{x\}) = \{x'\}$. This, along with the assumption that $W_{i(L)}$ is a $f_{4.4}$-stretched walk, implies that $f(L)$ is a path in $W_{i(L)}$. In particular, the restriction of f to L is an isomorphism from $K[L]$ to $\Xi[f(L)]$.

It remains to show that for every $y \in L$, the vertices x', y are adjacent in K if and only if $x, f(y)$ are adjacent in Ξ. To that end, note that since $f_{i(L)}$ is an isomorphism from K to an induced subgraph of $T_{i(L)}$, it follows from the first bullet of (18) that x' is adjacent to y in K if and only if $f_{i(L)}(x') = x_{i(L)}$ is adjacent to $f_{i(L)}(y) \in P_{i(L),i(L)}$ in $T_{i(L)}$. In addition, from the definition of T_i, it follows that $x_{i(L)}$ is adjacent to $f_{i(L)}(y) \in P_{i(L),i(L)}$ in $T_{i(L)}$ if and only if x is adjacent to $\varphi_{i(L)}(g_{i(L),i(L)}(f_{i(L)}(y))) = f(y)$ in Ξ. This completes the proof of Lemma 4.4. ■

We can now prove Theorem 4.2, which we restate:

Theorem 4.2. *Let $\mathcal{H}$ be a finite set of graphs which is tasselled. Then $\mathcal{H}$ is hassled.*

Proof. We use Lemmas 3.12 and 4.4. Let $f_1 = f_1(\mathcal{H}) = f_{4.4}(\mathcal{H})$. For every $t \in \mathbb{N}$, let

$$f_2 = f_2(\mathcal{H}, t) = f_{3.12}(||\mathcal{H}||_1, ||\mathcal{H}||_1, t, t+1);$$

and let

$$c = c(\mathcal{H}, t) = f_1 f_2 \cdot ||\mathcal{H}||_1^2.$$

We prove that for every $(K_{t,t}, K_{t+1})$-free c-hassle Ξ, there is a graph $H \in \mathcal{H}$ each component of which is isomorphic to an induced subgraph of Ξ. This will show that $\mathcal{H}$ is hassled.

Let x be the neck of Ξ. By the choice of c, we may choose a set $\mathfrak{W}$ of $f_2 \cdot ||\mathcal{H}||_1^2$ pairwise disjoint families of walks of Ξ, each of cardinality f_1. It follows that for every $\mathcal{W} \in \mathfrak{W}$, the graph $\Xi_{\mathcal{W}} = \Xi[\{x\} \cup (\bigcup_{W \in \mathcal{W}} V(W))]$ is a f_1-hassle with neck x, and with $\mathcal{W}$ as its set of walks.

Since $\mathcal{H}$ is tasselled, and by the choice of f_1, for each $\mathcal{W} \in \mathfrak{W}$, we can apply Lemma 4.4 to $\mathcal{H}$ and $\Xi_{\mathcal{W}}$, and obtain a graph $H_{\mathcal{W}} \in \mathcal{H}$ satisfying Lemma 4.4(a) and (b). Moreover, since $|\mathcal{H}| \leq ||\mathcal{H}||_1$ and $|\mathfrak{W}| = f_2 ||\mathcal{H}||_1^2$, it follows that there exist $H \in \mathcal{H}$ and $\mathfrak{X} \subseteq \mathfrak{W}$ with $|\mathfrak{X}| = f_2 ||\mathcal{H}||_1$ such that for every $\mathcal{W} \in \mathfrak{X}$, we have $H_{\mathcal{W}} = H$. More explicitly, for every component K of H, one of the following holds.

- K is a path.
- For every $\mathcal{W} \in \mathfrak{X}$, there exists $x'_{\mathcal{W}} \in V(K)$ and an injective map $f_{\mathcal{W}} : V(K) \to V(\Xi)$ with $f^{-1}(\{x\}) = \{x'_{\mathcal{W}}\}$, such that for every component L of $K \setminus \{x'_{\mathcal{W}}\}$, the restriction of $f_{\mathcal{W}}$ to $\{x'_{\mathcal{W}}\} \cup L$ is an isomorphism from $K[\{x'_{\mathcal{W}}\} \cup L]$ to $\Xi[\{x\} \cup f_{\mathcal{W}}(L)]$.

To conclude the proof, it suffices to show that each component K of H is isomorphic to some induced subgraph of Ξ. Assume that K is a path. Since Ξ is a c-hassle, it follows that Ξ has a path on c vertices. But now we are done because $|K| \leq ||\mathcal{H}||_1 \leq c$. Consequently, we may assume that K is not a path, and so the second bullet above holds for K and every $\mathcal{W} \in \mathfrak{X}$. In addition, from $|K| \leq ||\mathcal{H}||_1$ and $|\mathfrak{X}| = f_2||\mathcal{H}||_1$, it follows that there exist $x' \in V(K)$ and $\mathfrak{Y} \subseteq \mathfrak{X}$ with $|\mathfrak{Y}| = f_2$ such that for every $\mathcal{W} \in \mathfrak{Y}$, we have $x'_{\mathcal{W}} = x'$.

On the other hand, $\{f_{\mathcal{W}}(K \setminus \{x'\}) : \mathcal{W} \in \mathfrak{Y}\}$ is a set of f_2 pairwise disjoint subsets of $\Xi \setminus \{x\}$, each of cardinality less than $|K| \leq ||\mathcal{H}||_1$. This, along with the choice f_2 and the assumption that Ξ is $(K_{t,t}, K_{t+1})$-free, allows for an application of Lemma 3.12. We deduce that:

(20) *There exist* $\mathcal{W}_1, \dots, \mathcal{W}_{||\mathcal{H}||_1} \in \mathfrak{Y}$ *for which the sets*

$$f_{\mathcal{W}_1}(K \setminus \{x'\}), \dots, f_{\mathcal{W}_{||\mathcal{H}||_1}}(K \setminus \{x'\}) \subseteq \Xi \setminus \{x\}$$

are pairwise anticomplete in Ξ.

Now, let $L_1, \dots, L_k$ be an enumeration of the components of $K \setminus \{x'\}$; then we have $k < |K| \leq ||\mathcal{H}||_1$. Let $\mathcal{W}_1, \dots, \mathcal{W}_k \in \mathfrak{Y}$ be as in (20). By the second bullet above, for each $i \in [k]$, the restriction of $f_{\mathcal{W}_i}$ to $\{x'\} \cup L_i$ is an isomorphism from $K[\{x'\} \cup L_i]$ and $\Xi[\{x\} \cup f_{\mathcal{W}_i}(L_i)]$ with $f^{-1}_{\mathcal{W}_i}(\{x\}) = \{x'\}$. Moreover, by (20), the sets $(f_{\mathcal{W}_i}(L_i) : i \in [k])$ are pairwise disjoint and anticomplete in Ξ. Hence, K is isomorphic to

$$\Xi\left[\{x\} \cup \left(\bigcup_{i=1}^{k} f_{\mathcal{W}_i}(L_i)\right)\right].$$

This completes the proof of Theorem 4.2. ■

In view of Observation 2.2 and Theorems 4.1 and 4.2, in order to prove Theorem 2.4, it remains to show that:

Theorem 4.5. *Let $\mathcal{H}$ be a finite set of graphs which is hassled. Then the class of all $\mathcal{H}$-free graphs is clean.*

We will devote the rest of this chapter to the proof of Theorem 4.5, which will be completed at the end of Section 4.4. We also remark that, combining Observation 2.2 with Theorems 4.1, 4.2 and 4.5, one may summarize the cornerstone results of this chapter as follows (see Figure 4.7).

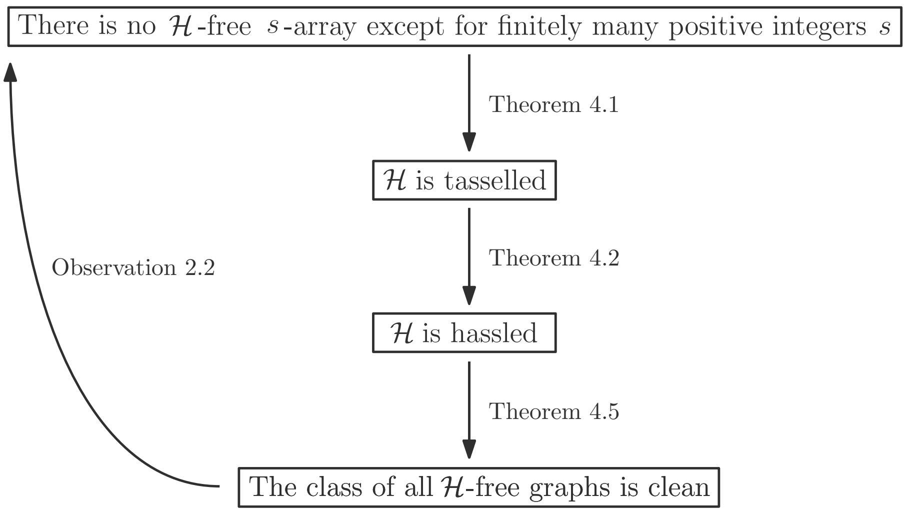

Figure 4.7: Corollary 4.6

Corollary 4.6. *The following are equivalent for every finite set $\mathcal{H}$ of graphs.*

- *There is no $\mathcal{H}$-free s-array except possibly for finitely many $s \in \mathbb{N}$.*
- *$\mathcal{H}$ is tasselled.*
- *$\mathcal{H}$ is hassled.*
- *The class of all $\mathcal{H}$-free graphs is clean.*

4.3 Getting hassled

Our goal in this section is to prove the following:

Theorem 4.7. *For all $c, t \in \mathbb{N}$, there is a constant $f_{4.7} = f_{4.7}(c, t)$ such that every t-clean graph G with $\mathrm{tw}(G) > f_{4.7}$ has an induced subgraph which is a c-hassle.*

From Theorem 3.19, we know that every t-clean graph of sufficiently large treewidth contains, omitting the corresponding parameters, either an ample constellation or a polypath which is both loose and fancy. Accordingly, it suffices to prove Theorem 4.7 in each of these two cases. First, we show that:

Theorem 4.8. *Let $c \in \mathbb{N}$, let G be a graph and assume that there is a c-ample $(3, 6c)$-constellation in G. Then G has an induced subgraph which is a c-hassle.*

Proof. Let $(S, \mathcal{L})$ be a c-ample $(3, 6c)$-constellation in G. Since $|S| = 3$, $|\mathcal{L}| = 6c$ and $(S, \mathcal{L})$ is ample, it follows that there is an enumeration $S = \{x, y, z\}$ as well as a c-subset $\mathcal{L}'$ of $\mathcal{L}$ where for every $L \in \mathcal{L}'$, there is a path P_L in L with one end adjacent to x and one end adjacent to z, such that y has a neighbor in P_L and x, z are anticomplete to P_L^*. Moreover, since $(S, \mathcal{L})$ is c-ample, it follows that for every $L \in \mathcal{L}'$, the vertex y is anticomplete to the first and last c vertices of P_L. Now $\Xi = G[\{y\} \cup V(\{P_L : L \in \mathcal{L}'\})]$ is a c-hassle with neck y and with $\{P_L : L \in \mathcal{L}'\}$ as its set of walks (each of which is a path in Ξ). This completes the proof of Theorem 4.8. ■

Unlike the case of ample constellations, handling the second outcome of Theorem 3.19 takes quite a bit of work. We begin with a definition. Let G be a graph, let $x, y \in V(G)$ be distinct and non-adjacent, and let $\mathcal{P}$ be a set of pairwise internally disjoint paths in G from x to y. An *x-slash for $\mathcal{P}$ in G* is a path W in $G \setminus \{x, y\}$ such that for every $P \in \mathcal{P}$, the unique neighbor of x in P belongs to W. Our first lemma says that:

Lemma 4.9. *Let $c, p, q \in \mathbb{N}$, let G be a graph, let $x, y \in V(G)$ be distinct and non-adjacent, and let $\mathcal{P}_0$ be a set of $cq(p+1)$ pairwise internally disjoint paths in G from x to y. Let W_0 be an x-slash for $\mathcal{P}_0$ in G. Then one of the following holds.*

(a) *There is a p-subset $\mathcal{P}$ of $\mathcal{P}_0$ and a path W in G with $V(W) \subseteq V(W_0)$, such that:*
  - *W is an x-slash for $\mathcal{P}$ in G; and*
  - *there is no path of length at most $c+1$ in $G[V(\mathcal{P}) \cup W]$ from x to y.*

(b) *There is a set $\mathcal{Q}$ of q pairwise internally disjoint paths in G from x to y, each of length at most $c+1$.*

Proof. Let $\mathcal{C}$ be a maximal set of pairwise internally disjoint paths in G from x to y, each of length at most $c+1$. It follows that Q^* is a path in G on at most c vertices for every $Q \in \mathcal{C}$.

Note that if $|\mathcal{C}| \geq q$, then 4.9(b) holds, as required. Thus, we may assume that $|\mathcal{C}| < q$, and so $|\mathcal{C}^*| < cq$. In particular, $W_0 \setminus \mathcal{C}^*$ has at most cq components, and since the paths in $\mathcal{P}_0$ are pairwise internally disjoint, it follows that there are at most $|\mathcal{C}^*| < cq$ paths P in $\mathcal{P}_0$ for which $P \cap \mathcal{C}^* \neq \varnothing$. This, along with the assumption that $|\mathcal{P}_0| = cq(p+1)$ and W_0 is an x-slash for $\mathcal{P}_0$ in G, implies that there exist $\mathcal{P} \subseteq \mathcal{P}_0$ with $|\mathcal{P}| = p$ and $V(\mathcal{P}) \cap \mathcal{C}^* = \varnothing$, as well as a component W of $W_0 \setminus \mathcal{C}^* \subseteq G \setminus \{x, y\}$, such that for every $P \in \mathcal{P}$, the unique neighbor of x in P belongs to W. In other words, W is an x-slash for $\mathcal{P}$. Moreover, we have $(V(\mathcal{P}) \cup W) \cap \mathcal{C}^* = \varnothing$, and so by the maximality of $\mathcal{C}$, there is no path Q of length at most $c+1$ in $G[V(\mathcal{P}) \cup W]$ from x to y. Now $\mathcal{P}$ and W satisfy 4.9(a), as desired. ■

Most of the technicality in this section goes into the proof of our second lemma:

Lemma 4.10. *For all $c, q, t \in \mathbb{N}$, there is a constant $f_{4.10} = f_{4.10}(c, q, t) \in \mathbb{N}$ with the following property. Let G be a t-clean graph, let $x, y \in V(G)$ be distinct and non-adjacent, and let $\mathcal{P}_0$ be a set of $f_{4.10}$ pairwise internally disjoint paths in G from x to y. Assume that there is an x-slash for $\mathcal{P}_0$ in G. Then one of the following holds.*

- (a) *G has an induced subgraph which is a c-hassle.*
- (b) *There is a set $\mathcal{Q}$ of q pairwise internally disjoint paths in G from x to y, each of length at most $c+1$.*

Proof. The proof relies on Theorem 3.14 and Lemma 3.20. Specifically, let

$$f_1 = f_{3.14}(2c+2, 1, c);$$

$$m = cf_1;$$

$$b = \max\{f_{3.20}(c, 6c, 3, t), g_{3.20}(c, 6c, 3, t)\};$$

$$f_2 = f_{3.14}(c, b, 3).$$

We prove that

$$f_{4.10} = f_{4.10}(c, q, t) = c((c+2)mf_2 + 1)q$$

satisfies the lemma. Suppose for a contradiction that none of the two outcomes of 4.10 hold. Apply Lemma 4.9 to $x, y, \mathcal{P}_0$ and W_0. Since 4.10(b) and Lemma 4.9(b) are identical, it follows from the choice of $f_{4.10}$ that:

(21) *There is a subset $\mathcal{P}$ of $\mathcal{P}_0$ with $|\mathcal{P}| = (c+2)mf_2$ and a path W in G with $V(W) \subseteq V(W_0)$, such that:*

- *W is an x-slash for $\mathcal{P}$ in G; and*
- *there is no path of length at most $c+1$ in $G[V(\mathcal{P}) \cup W]$ from x to y.*

From now on, let $\mathcal{P}$ and W be as in (21). For every vertex $v \in N_{V(\mathcal{P})}(x)$, we denote by P_v the unique path in $\mathcal{P}$ for which v is the unique neighbor of x in P_v. Since W is an x-slash for $\mathcal{P}$, it follows that $N_{V(\mathcal{P})}(x) \subseteq W$. Let w_1 and w_2 be the ends of W. Since $|\mathcal{P}| = cmf_2 + 2mf_2$, it follows that one may choose $3m$ pairwise disjoint f_2-subsets $(U_{1,i}, V_i, U_{2,i} : i \in [m])$ of $N_{V(\mathcal{P})}(x) \subseteq W$, such that the following hold.

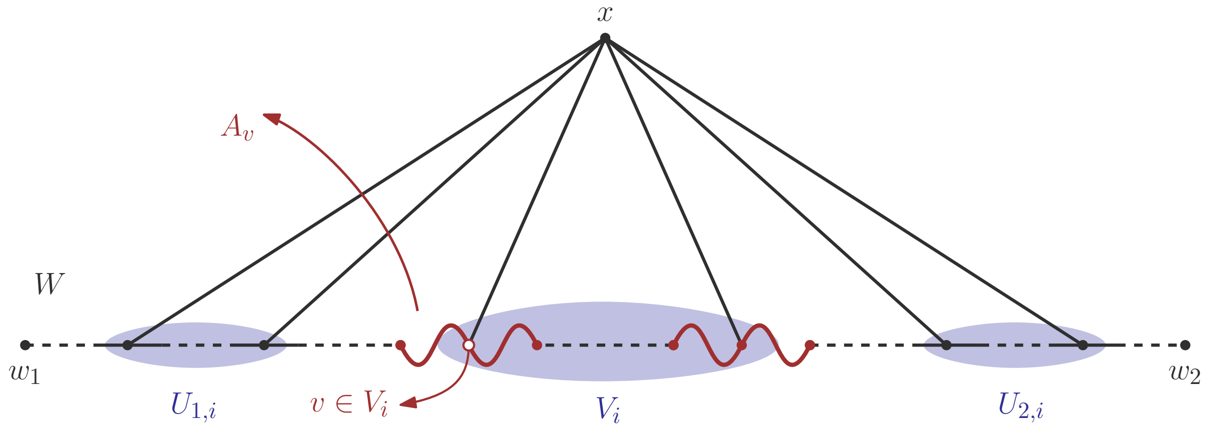

Figure 4.8: The subsets $U_{1,i}, V_i, U_{2,i}$ of $N_{V(\mathcal{P})}(x) \subseteq W$, and the paths $(A_v : v \in V_i)$ depicted by squiggly lines.

- For each $i \in [m]$, there are f_2 pairwise disjoint paths $(A_v : v \in V_i)$ in W, each on c vertices, such that for every $v \in V_i$
  - A_v contains v; and
  - Traversing W from w_1 to w_2, every vertex in $U_{1,i}$ appears before every vertex in A_v, and every vertex in A_v appears before every vertex in $U_{2,i}$.

  In particular, traversing W from w_1 to w_2, every vertex in $U_{1,i}$ appears before every vertex in V_i, and every vertex in V_i appears before every vertex in $U_{2,i}$ (see Figure 4.8).

- For every $i \in [m-1]$, traversing W from w_1 to w_2, every vertex in $U_{2,i}$ appears before every vertex in $U_{1,i+1}$ (see Figure 4.9).

We deduce that:

(22) *For every $i \in [m]$, there exist $u_{1,i} \in U_{1,i}$, $v_i \in V_i$ and $u_{2,i} \in U_{2,i}$, such that A_{v_i} is disjoint from $P_{u_{1,i}} \cup P_{u_{2,i}}$ and anticomplete to $(P_{u_{1,i}} \cup P_{u_{2,i}}) \setminus \{x\}$.*

To see this, let

$$\mathcal{A}_i = \{(A_v, \varnothing) : v \in V_i\};$$
$$\mathcal{B}_{1,i} = \{(\varnothing, P^*_u) : u \in U_{1,i}\};$$
$$\mathcal{B}_{2,i} = \{(\varnothing, P^*_u) : u \in U_{2,i}\}.$$

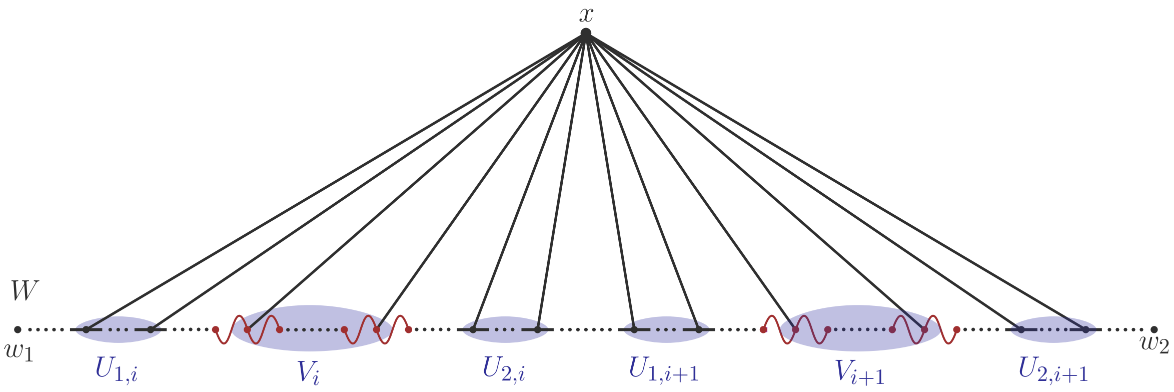

Figure 4.9: The subsets $U_{1,i}, V_i, U_{2,i}, U_{1,i+1}, V_{i+1}, U_{2,i+1}$ of $N_{V(\mathcal{P})}(x) \subseteq W$.

Then $\mathcal{A}_i, \mathcal{B}_{1,i}, \mathcal{B}_{2,i}$ are three sets of pairwise disjoint c-bisets over $V(G)$, each of cardinality f_2. By the choice of f_2, we can apply Theorem 3.14 to $\mathcal{A}_i, \mathcal{B}_{1,i}, \mathcal{B}_{2,i}$. It follows that there exist $U'_{1,i} \subseteq U_{1,i}$, $V'_i \subseteq V_i$ and $U'_{2,i} \subseteq U_{2,i}$ with $|U'_{1,i}| = |V'_i| = |U'_{2,i}| = b$ such that the sets $\mathcal{A}'_i = \{(A_v, \varnothing) : v \in V'_i\}$, $\mathcal{B}'_{1,i} = \{(\varnothing, P^*_u) : u \in U'_{1,i}\}$ and $\mathcal{B}'_{2,i} = \{(\varnothing, P^*_u) : u \in U'_{2,i}\}$ satisfy Theorem 3.14(a) and 3.14(b). In fact, Theorem 3.14(a) along with the assumption that $x, y \notin W$, implies that for every $u_1 \in U'_{1,i}$, every $v \in V'_i$ and every $u_2 \in U'_{2,i}$, we have $A_v \cap (P_{u_1} \cup P_{u_2}) = \varnothing$. It remains to show that there exist $u_{1,i} \in U'_{1,i}$, $v_i \in V'_i$ and $u_{2,i} \in U'_{2,i}$, for which A_{v_i} is anticomplete to $(P_{u_{1,i}} \cup P_{u_{2,i}}) \setminus \{x\}$. Suppose not. Note that for every $v \in V'_i$, since $v \in A_v$ is a neighbor of x, it follows from the second bullet of (21) that y is anticomplete to A_v. Consequently, by Theorem 3.14(b), there exists $j \in \{1, 2\}$ such that for every $v \in V_i$, there exists a vertex $x_v \in A_v$ which has a neighbor in P^*_u for every $u \in U'_{j,i}$. On the other hand, it follows from the choice of b that there exists $V \subseteq V'_i$ with $|V| = f_{3.20}(c, 6c, 3, t)$ and there exists $U \subseteq U'_{j,i}$ with $|U| = g_{3.20}(c, 6c, 3, t)$. Let $S = \{x_v : v \in V\}$ and let $\mathcal{L} = \{P^*_u : u \in U\}$. Then $(S, \mathcal{L})$ is a $(f_{3.20}(c, 6c, 3, t), g_{3.20}(c, 6c, 3, t))$-constellation in G. This, combined with Lemma 3.20 and the assumption that G is t-clean, implies that there is a c-ample $(3, 6c)$-constellation in G. But now by Theorem 4.8, G has an induced subgraph which is a c-hassle, a contradiction with the assumption that 4.10(a) does not hold. This proves (22).

Henceforth, for each $i \in [m]$, let $u_{1,i} \in U_{1,i}$, $v_i \in V_i$ and $u_{2,i} \in U_{2,i}$ be as in (22). We write $A_i = A_{v_i}$, $P_{1,i} = P_{u_{1,i}}$ and $P_{2,i} = P_{u_{2,i}}$. Also, we denote by $a_{1,i}, a_{2,i}$ the ends of A_i, such that W traverses the vertices $w_1, a_{1,i}, a_{2,i}, w_2$ in this order. It follows that W traverses the vertices $w_1, u_{1,i}, a_{1,i}, a_{2,i}, u_{2,i}, w_2$ in this order, and $a_{1,i}, a_{2,i}$ are the only vertices among $u_{1,i}, a_{1,i}, a_{2,i}, u_{2,i}$ which can be the same (only if $c = 1$).

Let $i \in [m]$ be fixed. For each $j \in \{1,2\}$, traversing $a_{j,i}$-W-$u_{j,i}$ starting at $a_{j,i}$, let $w'_{j,i}$ be the first vertex in $a_{j,i}$-W-$u_{j,i}$ with a neighbor in $P_{j,i} \setminus \{x\}$ (note that $w'_{j,i}$ exists because $u_{j,i}$ has a neighbor in $P_{j,i} \setminus \{x\}$). From (22), we know that A_i is disjoint from and anticomplete to $(P_{1,i} \cup P_{2,i}) \setminus \{x\}$; in particular, for every $j \in \{1,2\}$, $u_{j,i} \in P_{j,i} \setminus \{x\}$ has a neighbor in the interior of $a_{j,i}$-w-$u_{j,i}$. It follows that for every $j \in \{1,2\}$, the vertex $w'_{j,i}$ belongs to the interior of $a_{j,i}$-W-$u_{j,i}$, and there is a path $R_{j,i}$ in G from $w'_{j,i}$ to y whose interior is contained in $P^*_{j,i}$.

For every $i \in [m]$ and each $j \in \{1,2\}$, let $R'_{j,i}$ be the longest path of length at most $c+1$ in $R_{j,i} \setminus \{y\}$ containing $w'_{j,i}$. It follows that:

(23) *For every $i \in [m]$ and each $j \in \{1,2\}$, we have $R'_{j,i} \setminus \{w'_{j,i}\} \subseteq P^*_{j,i}$. Consequently, the sets $(R'_{1,i} \cup R'_{2,i} : i \in [m])$ are pairwise disjoint in G.*

By the choice of $R'_{j,i}$, either $R'_{j,i} = R_{j,i} \setminus \{y\}$, in which case (23) follows immediately from $R^*_{j,i} \subseteq P^*_{j,i}$, or $R'_{j,i}$ is a $(c+2)$-vertex path with $R'_{j,i} \setminus \{w'_{j,i}\} \subseteq P^*_{j,i}$. This proves (23).

Now, for each $i \in [m]$, let $W'_i = w'_{1,i}$-W-$w'_{2,i}$, and let $W_i = G[R'_{1,i} \cup W'_i \cup R'_{2,i}]$ be the walk such that traversing W_i from $\varphi_{W_i}(1)$ to $\varphi_{W_i}(n_{W_i})$, we first traverse the path $R'_{1,i}$ starting at the end that is distinct from $w'_{1,i}$ and stopping at $w'_{1,i}$, then we traverse the path W'_i from $w'_{1,i}$ to $w'_{2,i}$, and then we traverse $R'_{2,i}$ starting at $w'_{2,i}$ (so we have $n_{W_i} = |R'_{1,i}| + |W'_i| + |R'_{2,i}| - 2$). In particular, we have $W_i \subseteq V(\mathcal{P}^*) \cup W$.

(24) *The following hold.*

- *For all $i \in [m]$, the walk W_i is c-stretched and x has a neighbor in W_i.*
- *For all $i \in [m]$, the vertex x is anticomplete to the first and the last c vertices of W_i.*
- *The paths $(W'_i : i \in [m])$ are pairwise disjoint.*
- *The sets $(W_i \setminus W'_i : i \in [m])$ are pairwise disjoint.*

The first bullet is immediate from $|A_i| = c$ and the observation that both $R'_{1,i}$-$w'_{1,i}$-W-$a_{2,i}$ and $R'_{2,i}$-$w'_{1,i}$-W-$a_{1,i}$ are paths in G containing A_i. Also, the third bullet is trivial, and the fourth is immediate from (23). It remains to prove the second bullet. Note that by the definition of $R'_{1,i}$ and $R'_{2,i}$, either y has a neighbor among the first or the last c vertices of W_i, or the first $c+1$ vertices of W_i are contained in $P^*_{1,i}$, and the last $c+1$ vertices of W_i are contained in $P^*_{2,i}$. In the former case, the result follows directly from the second bullet of (21), and in the latter case, the result follows from the fact that x has exactly one neighbor in $P^*_{1,i}$ and exactly one neighbor in $P^*_{2,i}$. This proves (24).

We can now finish the proof. For each $k \in [c]$, let

$$I_k = \{i \in [m] : i = k \pmod{c}\};$$

$$\mathcal{B}_k = \{(W_i \setminus W_i', W_i') : i \in I_k\}.$$

From the choice of m and the third bullet of (24), it follows that $\mathcal{B}_1, \dots, \mathcal{B}_c$ are sets of pairwise disjoint $(2c+2)$-bisets over $V(G)$, each of cardinality f_1. The choice of f_1 in turn allows for an application of Theorem 3.14 to $\mathcal{B}_1, \dots, \mathcal{B}_c$. In particular, Theorem 3.14(a) implies that for every $k \in [c]$, there exists $i_k \in I_k$ such that $W_{i_k} \setminus W_{i_k}'$ and W_{i_l} are disjoint for all distinct $k, l \in [c]$. This, combined with the third and the fourth bullet of (24), implies that $W_{i_1}, \dots, W_{i_c}$ are pairwise disjoint. But now by the first two bullets of (24), the subgraph of G induced on $\{x\} \cup W_{i_1} \cup \dots \cup W_{i_c}$ is a c-hassle with neck x and walks $W_{i_1}, \dots, W_{i_c}$, contradicting the assumption that 4.10(a) does not hold. This completes the proof of Lemma 4.10. ■

From Lemma 4.10, we deduce the following:

Lemma 4.11. *For all $c, d, q, t \in \mathbb{N}$, there is a constant $f_{4.11} = f_{4.11}(c, d, q, t) \in \mathbb{N}$ with the following property. Let G be a t-clean graph and let $\mathcal{W}$ be a d-loose polypath in G. Let $x, y \in V(\mathcal{W})$ be distinct and non-adjacent, and let $\mathcal{P}$ be a set of $f_{4.11}$ pairwise internally disjoint paths in G from x to y with $V(\mathcal{P}) \subseteq V(\mathcal{W})$. Then one of the following holds.*

- (a) *G has an induced subgraph which is a c-hassle.*
- (b) *There is a set $\mathcal{Q}$ of q pairwise internally disjoint paths in G from x to y, each of length at most $c+1$.*

Proof. Let $f_{4.10} = f_{4.10}(c, q, t)$. We show that

$$f_{4.11} = f_{4.11}(c, d, q, t) = (2d)f_{4.10}$$

satisfies the lemma. Let $W \in \mathcal{W}$ such that $x \in W$. Then x has at most two neighbors in W. Also, since $\mathcal{W}$ is d-loose, it follows that x has neighbors in at most $d-1$ paths in $\mathcal{W} \setminus \{W\}$. This, along with the fact that $|\mathcal{P}| = f_{4.11} \geq 2(d-1)f_{4.10} + 2$, implies that there exists $\mathcal{P}_1 \subseteq \mathcal{P}$ with $|\mathcal{P}_1| = 2f_{4.10}$ and a path $W_1 \in \mathcal{W} \setminus \{W\}$, such that for every $P \in \mathcal{P}_1$, the unique neighbor of x in P belongs to W_1. So we have $x \notin W_1$. In addition, since $W_1 \setminus \{y\}$ has one or two components (depending on whether $y \in W_1^*$ or not), it follows that there exist $\mathcal{P}_0 \subseteq \mathcal{P}_1$ with $|\mathcal{P}_0| = f_{4.10}$ as well as a component W_0 of $W_1 \setminus \{y\}$, such that for every $P \in \mathcal{P}_0$, the unique neighbor of x in P belongs to W_0. Therefore, $\mathcal{P}_0$ is a set of $f_{4.10}$ pairwise internally disjoint paths in G from x to y, and W_0 is an x-slash for $\mathcal{P}_0$ in G. Now the result follows from Lemma 4.10 applied to $\mathcal{P}_0$. ■

We can now tackle the second outcome of Theorem 3.19:

Theorem 4.12. *For all* $c, d, t \in \mathbb{N}$*, there is a constant* $f_{4.12} = f_{4.12}(c, d, t) \in \mathbb{N}$ *with the following property. Let* G *be a* t*-clean graph. Assume that there is a* $2f_{4.12}$*-polypath in* G *which is both* $f_{4.12}$*-fancy and* d*-loose. Then there is a* c*-hassle in* G*.*

Proof. The proof uses Theorems 2.27 and 3.16 as well as Lemma 4.11. Specifically, let

$$f = f_{3.16}(c + 1, t);$$

$$g = g_{3.16}(c + 1, t);$$

$$h = f_{4.11}(c, d, g, t).$$

We claim that

$$f_{4.12} = f_{4.12}(c, d, t) = f_{2.27}\left((t + 1)^f, h, t\right) + 1$$

satisfies the theorem. Let G be a t-clean graph and let $\mathcal{W}$ be a $2f_{4.12}$-polypath in G which is both $f_{4.12}$-fancy and d-loose. Suppose for a contradiction that G does not contain a c-hassle.

Let $H = G[V(\mathcal{W})]$. Then H is a t-clean graph, and since there is a $2f_{4.12}$-polypath in G which is $f_{4.12}$-fancy, it follows that H has a minor isomorphic to $K_{f_{4.12},f_{4.12}}$. In particular, we have $\text{tw}(H) \geq f_{4.12} > f_{2.27}((t + 1)^f, h, t)$, and so by Theorem 2.27, there is a strong $((t + 1)^f, h)$-block $(B, \mathcal{P})$ in G. Furthermore, since G is K_{t+1}-free, it follows from Theorem 3.17 that there exists a stable set $A \subseteq B$ in H of cardinality f.

For each 2-subset $\{x, y\}$ of A, the vertices x, y are non-adjacent in G. By the choice of f, we can apply Lemma 4.11 to $\mathcal{W}$ and $\mathcal{P}(\{x, y\})$. Note that Lemma 4.11(a) would violate the assumption that G has no induced subgraph which is a c-hassle. It follows that Lemma 4.11(b) holds; that is, there exists a set $\mathcal{Q}(\{x, y\})$ of g pairwise internally disjoint paths in G from x to y, each of length at most $c + 1$. But now $(A, \mathcal{Q})$ is a $(c + 1)$-short (not necessarily strong) (f, g)-block in the t-clean graph G. Combined with the choice of f and g, this violates Theorem 3.16, hence completing the proof of Theorem 4.12. ∎

Theorem 4.7 is now immediate from Theorems 3.19, 4.8 and 4.12:

Theorem 4.7. *For all* $c, t \in \mathbb{N}$*, there is a constant* $f_{4.7} = f_{4.7}(c, t)$ *such that every* t*-clean graph* G *with* $\text{tw}(G) > f_{4.7}$ *has an induced subgraph which is a* c*-hassle.*

Proof. Let

$$g = g_{3.19}(c, 6c, 3, t)$$

and let

$$f = f_{4.12}(c, g, t).$$

We claim that

$$f_{4.7} = f_{4.7}(c, t) = f_{3.19}(c, 6c, 3, t, f);$$

satisfies the theorem. Let G be a t-clean graph with $\operatorname{tw}(G) > f_{4.7}$. By Theorem 3.19, either there exists a c-ample $(3, 6c)$-constellation in G, or there is a $2f$-polypath in G which is both f-fancy and g-loose. In the former case, by Theorem 4.8, G has an induced subgraph which is a c-hassle. Also, in the latter case, the choice of f along with Theorem 4.12 implies that G has an induced subgraph which is a c-hassle. This completes the proof of Theorem 4.7. ■

4.4 Being hassled

Finally, we bring everything together and prove Theorem 4.5:

Theorem 4.5. *Let $\mathcal{H}$ be a finite set of graphs which is hassled. Then the class of all $\mathcal{H}$-free graphs is clean.*

Proof. It is enough to prove, for every $t \in \mathbb{N}$, that there is a constant $w = w(\mathcal{H}, t) \in \mathbb{N}$ such that every t-clean graph G with $\operatorname{tw}(G) > w$ has an induced subgraph isomorphic to some graph $H \in \mathcal{H}$.

We begin with setting the value of w, using Lemma 3.12 and Theorem 4.7. Since $\mathcal{H}$ is finite, it follows that $||\mathcal{H}||_1 \in \mathbb{N}$. Also, since $\mathcal{H}$ is hassled, it follows that for every $t \in \mathbb{N}$, there is a constant $c = c(\mathcal{H}, t) \in \mathbb{N}$ such that for every $(K_{t,t}, K_{t+1})$-free c-hassle Ξ, there is a graph $H \in \mathcal{H}$ each component of which is isomorphic to an induced subgraph of Ξ. Let $f_1 = f_1(\mathcal{H}, t) = f_{4.7}(c, t)$ and let $f_2 = f_2(\mathcal{H}, t) = f_{3.12}(||\mathcal{H}||_1, ||\mathcal{H}||_1, t + 1, t)$. Define

$$w = w(\mathcal{H}, t) = f_1 + f_2||\mathcal{H}||_1^2.$$

Let G be a t-clean graph with $\operatorname{tw}(G) > w$. Since $w \geq f_1$, by Theorem 4.7, there is an induced subgraph Ξ of G which is a c-hassle. This, combined with the assumption that $\mathcal{H}$ is hassled and G is $(K_{t,t}, K_{t+1})$-free, implies that there exists $W \subseteq V(\Xi) \subseteq V(G)$ with $|W| \leq ||\mathcal{H}||_1$, as well as a graph $H \in \mathcal{H}$, such that each component of H is isomorphic to an induced subgraph of $G[W]$.

Let $m \geq 1$ be maximum with the property that there are m pairwise disjoint subsets $W_1, \ldots, W_m$ of $V(G)$, each of cardinality at most $||\mathcal{H}||_1$, where for every $i \in [m]$, there is a graph $H_i \in \mathcal{H}$ such that each component of H_i is isomorphic to an induced subgraph of G_i. We claim that:

(25) $m \geq f_2||\mathcal{H}||_1$.

Suppose not. Let $G' = G \setminus (W_1 \cup \cdots \cup W_m)$. Then we have

$$\text{tw}(G') \geq \text{tw}(G) - m||\mathcal{H}||_1 > \text{tw}(G) - f_2||\mathcal{H}||_1^2 > f_1;$$

and so by Theorem 4.7, there is an induced subgraph Ξ' of G' which is a c-hassle. Since $\mathcal{H}$ is hassled and G is $(K_{t,t}, K_{t+1})$-free, it follows that there exists $W_{m+1} \subseteq V(\Xi') \subseteq V(G')$ with $|W_{m+1}| \leq ||\mathcal{H}||_1$ as well as a graph $H_{m+!} \in \mathcal{H}$, such that each component of H_{m+1} is isomorphic to an induced subgraph of $G[W_{m+1}]$, a contradiction with the choice of m. This proves (25).

By (25), we can choose pairwise disjoint subsets $W_1, \ldots, W_{f_2||\mathcal{H}||_1}$ of $V(G)$, each of cardinality at most $||\mathcal{H}||_1$, such that for every $i \in [f_2||\mathcal{H}||_1]$, there is a graph $H_i \in \mathcal{H}$ each component of which is isomorphic to an induced subgraph of $G[W_i]$. Since $|\mathcal{H}| \leq ||\mathcal{H}||_1$, it follows that there is a graph $H \in \mathcal{H}$ as well as an f_2-subset I of $[f_2||\mathcal{H}||_1]$ such that $H_i = H$ for all $i \in I$. Moreover, since G is $(K_{t,t}, K_{t+1})$-free, from the choice of f_2 and Lemma 3.12, it follows that there exists $i_1, \ldots, i_{||\mathcal{H}||_1} \in I$ such that $W_{i_1}, \ldots, W_{i_{||\mathcal{H}||_1}}$ are pairwise anticomplete in G.

To finish the proof, let $K_1, \ldots, K_h$ be the components of H. Then we have $h \leq ||\mathcal{H}||_1$ and for each $j \in [h]$, there exists $X_j \subseteq W_{i_j}$ such that $G[X_j]$ is isomorphic to K_j. But now $G[X_1 \cup \cdots \cup X_h]$ is isomorphic to H, as required. ■

Chapter 5

Grid Theorem for pinched graphs[1]

5.1 Alignments and hollow constellations

In this chapter, we prove the following, which extends Theorem 2.7 to the case where the lengths of the excluded cycles are lower bounded:

Theorem 5.1. *For all $c, o, s, t \in \mathbb{N}$, there is a constant $f_{5.1} = f_{5.1}(c, o, s, t) \in \mathbb{N}$ such that for every t-clean (c, o)-pinched graph G with $\operatorname{tw}(G) > f_{5.1}$, there is an (s, o)-array in G.*

We need to define "(c, o)-pinched graphs" and "(s, o)-arrays." For $c, o \in \mathbb{N}$, we say a graph G is *(c, o)-pinched* if there are no c induced subgraphs $H_1, \dots, H_c$ of G, each being a cycle of length at least $o+2$, such that for some $x \in V(G)$, we have $V(H_1) \cap \dots \cap V(H_c) = \{x\}$ and $V(H_1) \setminus \{x\}, \cdots, V(H_c) \setminus \{x\}$ are pairwise disjoint and anticomplete in G. In particular, a graph G is $(c, 1)$-pinched if and only if G is c-pinched.

The definition of an (s, o)-array relies on several notions associated with constellations, which we will introduce below. In what follows, up to Proposition 5.2, let G be a graph and let $s, l \in \mathbb{N}$.

Let $\mathfrak{c}$ be an (s, l)-constellation in G. By an *order on* $\mathfrak{c}$ we mean a bijection $\pi : [s] \to S_{\mathfrak{c}}$. For every non-empty subset X of $S_{\mathfrak{c}}$, we write $\mathfrak{c}|X$ for the $(|X|, l)$-constellation $(X, \mathcal{L}_{\mathfrak{c}})$. Note that for an order π on $\mathfrak{c}$, the map $\pi|_X$ is an order on $\mathfrak{c}|X$. For every $L \in \mathcal{L}_{\mathfrak{c}}$, we denote by $\mathfrak{c}_L$ the s-constellation $(S_{\mathfrak{c}}, L)$. Again, observe that an order π on $\mathfrak{c}$ is also an order on $\mathfrak{c}_L$ for every $L \in \mathcal{L}_{\mathfrak{c}}$. For a vertex $x \in S_{\mathfrak{c}}$, by an *x-gap in* $\mathfrak{c}$ we mean a path P in

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[1] This chapter is based on the coauthored papers [5, 7].

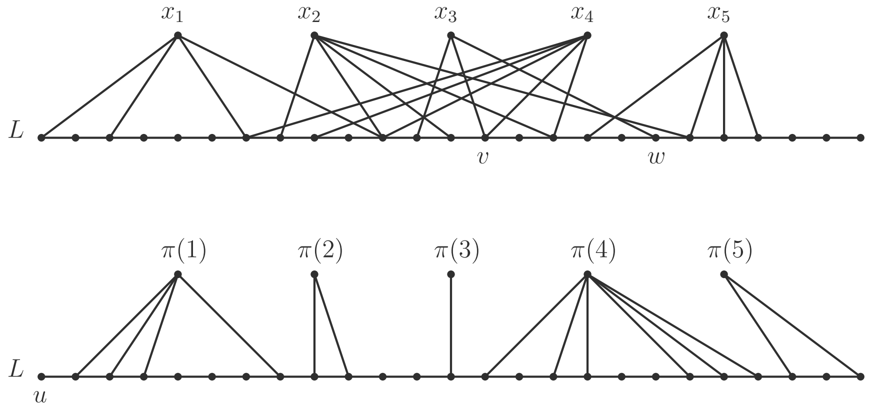

Figure 5.1: Top: A 5-constellation $\mathfrak{c}$ with $S_{\mathfrak{c}} = \{x_1, x_2, x_3, x_4, x_5\}$ and $L_{\mathfrak{c}} = L$, where $\mathfrak{c}$ is 6-hollow (with the x_3-gap v-L-w of length five being the longest). Bottom: A 5-alignment.

G (possibly of length zero) with $V(P) \subseteq V(L)$ for some $L \in \mathcal{L}_{\mathfrak{c}}$, such that x is adjacent to the ends of P and anticomplete to P^*. For $d \in \mathbb{N}$, we say $\mathfrak{c}$ is *d-hollow* if for every $x \in S_{\mathfrak{c}}$, every x-gap in $\mathfrak{c}$ has length less than d (see Figure 5.1). For instance, if $\mathfrak{c}$ is 1-hollow, then every vertex in $S_{\mathfrak{c}}$ has exactly one neighbor in each path $L \in \mathcal{L}_{\mathfrak{c}}$.

Given an s-constellation $\mathfrak{s}$ in G and an order π on $\mathfrak{s}$, we say that $\mathfrak{s}$ is an *s-alignment in $\mathfrak{c}$ in G with respect to π* if for some end u of $L_{\mathfrak{s}}$ and all $i, j \in [s]$ with $i < j$, traversing $L_{\mathfrak{s}}$ starting at u, the neighbors of $\pi(i)$ appear "before" the neighbors of $\pi(j)$. More precisely, for all $i, j \in [s]$ with $i < j$, every neighbor $v_i \in L_{\mathfrak{s}}$ of $\pi(i)$ and every neighbor $v_j \in L_{\mathfrak{s}}$ of $\pi(j)$, the path in $L_{\mathfrak{s}}$ from u to v_j contains v_i in its interior (see Figure 5.1). We say $\mathfrak{s}$ is an *s-alignment in G* if $\mathfrak{s}$ is an s-alignment in G with respect to some order π on $\mathfrak{s}$. It follows that if $\mathfrak{s}$ is an s-alignment in G, then $\mathfrak{s}$ is ample, and for every non-empty subset X of $S_{\mathfrak{s}}$, $\mathfrak{s}|X$ is a $|X|$-alignment in G.

For a polypath $\mathcal{P}$ in G, we say $\mathcal{P}$ is *plain* if every two paths $P, P' \in \mathcal{P}$ are anticomplete in G. We also say a constellation $\mathfrak{c}$ in G is *plain* if $\mathcal{L}_{\mathfrak{c}}$ is a plain polypath in G.

We are now ready to give the main definition. For $o \in \mathbb{N}$, an *(s, o)-array* in G is a plain, o-hollow (s, s)-constellation $\mathfrak{a}$ in G for which there is an order π on $\mathfrak{a}$ such that for every $L \in \mathcal{L}_{\mathfrak{a}}$, $\mathfrak{a}_L$ is an s-alignment in G with respect to π (see Figure 5.2). We deduce that:

Proposition 5.2. *Let $o, s \in \mathbb{N}$ and let G be a graph. Let $\mathfrak{a}$ be an (s, o)-array in G. Then $G[V(\mathfrak{a})]$ is a 4-clean $(3, o)$-pinched graph of treewidth at least s.*

Proof. Let $J = G[V(\mathfrak{a})]$. Note that J contains a $K_{s,s}$-minor, which implies that $\text{tw}(J) \geq s$.

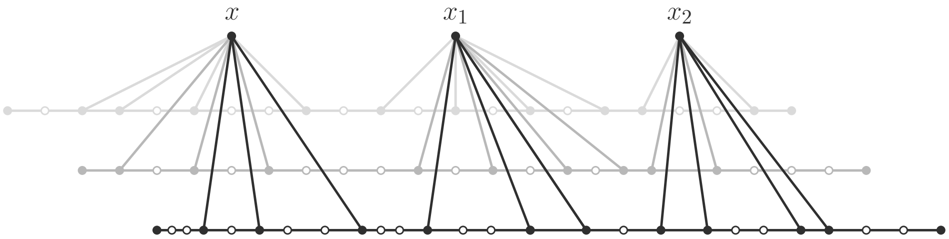

Figure 5.2: A $(3,4)$-array (the labels x, x_1 and x_2 are as in the proof of Proposition 5.2).

Observe that J does not have an induced subgraph isomorphic to a subdivision of K_4 with at most one edge unsubdivided, whereas for every $t \geq 4$, every subdivision of $W_{t\times t}$ does have such an induced subgraph. It follows that for every $t \geq 4$, J has no induced subgraph isomorphic to any subdivision of $W_{t\times t}$.

Let us say that a connected graph H is *fragile* if either H has a vertex v such that $H \setminus N_H[v]$ is not connected, or H has a set S of at most two vertices such that $H \setminus S$ has maximum degree at most two. Then every connected induced subgraph of J is fragile. On the other hand, assuming W to be the line-graph of a subdivision of $W_{t\times t}$ for some integer $t \geq 4$, then one may observe (see, for instance, Figure 1.4) that $W \setminus N_W[v]$ is connected for every vertex $v \in V(W)$, and W contains a stable set S with $|S| \geq 3$ all of whose vertices have degree three in W. It follows that W is not fragile, and so W is not isomorphic to an induced subgraph of J. Also, J is easily seen to be $\{K_4, K_{3,3}\}$-free. So J is 4-clean.

It remains to show that J is $(3,o)$-pinched. Suppose for a contradiction that there are three induced subgraphs C_1, C_2, C_3 of J, each of which is a cycle of length at least $o+2$, such that for some $x \in V(G)$, we have $V(C_1) \cap V(C_2) \cap V(C_3) = \{x\}$ and the sets $V(C_1) \setminus \{x\}, V(C_2) \setminus \{x\}, V(C_3) \setminus \{x\}$ are pairwise disjoint and anticomplete in G. Let π be the order on $\mathfrak{a}$ satisfying the definition of an (s,o)-array. Since x has degree at least six in J, it follows that $x \in S_{\mathfrak{a}}$. Also, since $\mathfrak{a}$ is o-hollow, it follows that there is no cycle of length at least $o+2$ in $J[V(\mathcal{L}_{\mathfrak{a}}) \cup \{x\}]$. Thus, for each $\ell \in [3]$, we may pick a vertex $x_\ell \in C_\ell \cap (S_{\mathfrak{a}} \setminus \{x\})$. By symmetry, we may assume that there are distinct $\ell_1, \ell_2 \in [3]$ for which $\pi^{-1}(x)$ is smaller than $\pi^{-1}(x_{\ell_1})$ and $\pi^{-1}(x_{\ell_2})$. In particular, we may assume without loss of generality that $\pi^{-1}(x) < \pi^{-1}(x_1) < \pi^{-1}(x_2)$. But then $x, x_2 \in C_2$ are in different components of $J \setminus N_J[x_1]$ (see Figure 5.2), which violates the fact that $C_1 \setminus \{x\}$ and $C_2 \setminus \{x\}$ are disjoint and anticomplete. This completes the proof of Proposition 5.2. ■

By Proposition 5.2, for all $c, o \geq 1$, Theorem 5.1 yields a complete description of the unavoidable induced subgraphs of (c,o)-pinched graphs with large treewidth. Moreover, it is easily observed that there is an $(s, 1)$-array in a graph G if and only if G has an induced subgraph isomorphic to an expansion of the graph PD_s. Therefore, Theorem 2.7 is the special case of Theorem 5.1 for $o = 1$. We also point out that for all $o, s \in \mathbb{N}$ with $s \geq 4$, if a graph G is $(2, o)$-pinched, then there is no (s, o)-array in G. Hence, Theorem 5.1 implies a strengthening of Corollary 2.8, that for every $o \in \mathbb{N}$, the class of all $(2, o)$-pinched graphs is clean.

5.2 Outline of the main proof

The rest of this chapter is devoted to the proof of Theorem 5.1, which consists of two steps. First, we prove that every t-clean (c, o)-pinched graph of sufficiently large treewidth contains a large plain constellation:

Theorem 5.3. *For all $c, l, o, s, t \in \mathbb{N}$, there is a constant $f_{5.3} = f_{5.3}(c, l, o, s, t) \in \mathbb{N}$ such that if G is a t-clean, (c, o)-pinched graph with $\mathrm{tw}(G) > f_{5.3}$, then there is a plain (s, l)-constellation in G.*

Then, we extract our desired array from the plain constellation obtained in Theorem 5.3:

Theorem 5.4. *For all $c, o, s, t \in \mathbb{N}$, there are constants $f_{5.4} = f_{5.4}(c, o, s, t) \in \mathbb{N}$ and $g_{5.4} = g_{5.4}(c, o, s, t) \in \mathbb{N}$ with the following property. Let G be a t-clean (c, o)-pinched graph. Suppose there is a plain $(f_{5.4}, g_{5.4})$ constellation in G. Then there is an (s, o)-array in G.*

Theorem 5.1 would then be immediate:

Theorem 5.1. *For all $c, o, s, t \in \mathbb{N}$, there is a constant $f_{5.1} = f_{5.1}(c, o, s, t) \in \mathbb{N}$ such that for every t-clean (c, o)-pinched graph G with $\mathrm{tw}(G) > f_{5.1}$, there is an (s, o)-array in G.*

Proof. Let $f = f_{5.4}(c, o, s, t)$ and let $g = g_{5.4}(c, o, s, t)$. We will show that

$$f_{5.1} = f_{5.1}(c, o, s, t) = f_{5.3}(c, g, o, f, t)$$

satisfies 5.1. Let G be a t-clean (c, o)-pinched graph with $\mathrm{tw}(G) > f_{5.1}$. From the choice of $f_{5.1}$ and Theorem 5.3, it follows that there is a plain (f, g)-constellation in G. The choices of f and g in turn combined with Theorem 5.4 imply that there is an (s, o)-array in G. ■

It remains to prove Theorems 5.3 and 5.4, which we will do in Sections 5.5 and 5.6, respectively. In Sections 5.3 and 5.4, we establish the technical results that will be used in the proofs of both Theorems 5.3 and 5.4, as well as several proofs in subsequent chapters.

5.3 Connectifiers

A vertex v of a graph G is said to be *simplicial* if $N_G(v)$ is a clique of G. The set of all simplicial vertices of G is denoted by $\mathcal{Z}(G)$. For instance, if G is a tree, the $\mathcal{Z}(G)$ is the set of all leaves of G. Also, if G is the line graph of a tree T, the $\mathcal{Z}(G)$ is the set of all leaf edges of T.

In this section, we are interested in the following question: Given $\eta \in \mathbb{N}$, a connected graph G and a sufficiently large subset S of $V(G)$, what can be said about a minimal connected induced subgraph H of G that contains at least η vertices from S? For instance, Theorem 3.1 tells us the answer when $\eta \leq 3$, that H is a path, a subdivided star of maximum degree three or the line graph of a subdivided star of maximum degree three, and $\mathcal{Z}(H) \subseteq S$.

Also, it is well-known that when $S = V(G)$, then H is either a path, a star or a complete graph (which is the line graph of a star):

Theorem 5.5 (Folklore; see [4]). *Let $d, q \in \mathbb{N}$ and let G be a connected graph on more than d^q vertices. Then either G has a vertex of degree at least d or there is a path in G on $q+1$ vertices.*

Proof. The assertion is trivial for $d = 1$. Suppose $d > 1$ and G is a connected graph on more than d^q such that neither G has a vertex of degree at least d nor is there a path in G on $q+1$ vertices. Since G is connected and $|V(G)| > 1$, we may choose a vertex $x \in V(G)$ and an integer $k > 1$ for which there is a partition $(N_1, \ldots, N_k)$ of $V(G)$ where N_i, for each $i \in [k]$, is the set of all vertices in G at distance $i - 1$ from x. In particular, $N_1 = \{x\}$ and $N_2 = N_G(x)$. Since there is no path on $q + 1$ vertices in G, it follows that $k \leq q$. Also, since G has no vertex of degree at least d, it follows that $|N_i| \leq d^{i-1}$ for all $i \in [k]$. But now

$$|V(G)| \leq \sum_{i=1}^{k} d^{i-1} = \frac{d^k - 1}{d - 1} < d^k \leq d^q;$$

a contradiction. ■

The main result of this section, Theorem 5.6, answers the aforementioned question in full generality. It turns out that H is always either a path, a subdivided star or its line graph, or a "caterpillar" or its line graph, where a *caterpillar* is a tree C of maximum degree three in which all vertices of degree three lie on a path (we remark that our definition of a "caterpillar" is non-standard).

Explicitly, we prove the following (see Figure 5.3):

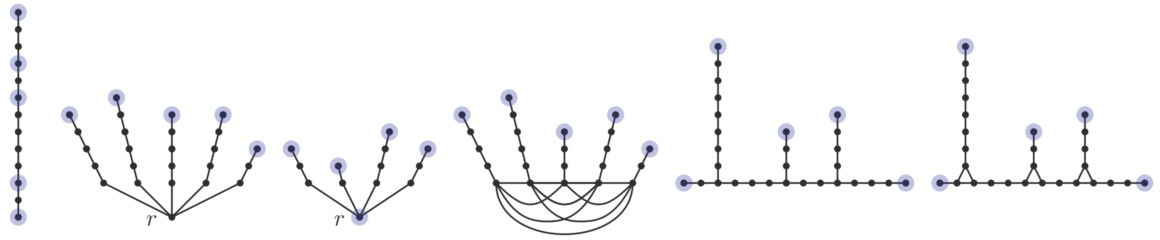

Figure 5.3: Outcomes of Theorem 5.6 when $\eta = 5$. From left to right: the first for (a), the second for (b) when $r \notin S$, the third for (b) when $r \in S$, the fourth for (c), and the fifth and the sixth for (d).

Theorem 5.6. *Let $\eta \in \mathbb{N}$ and let G be a graph. Let $S \subseteq V(G)$ with $|S| \geq \eta^{8\eta^4}$ such that S is contained in a connected component of G. Then there is an induced subgraph H of G with $|V(H) \cap S| = \eta$ for which one of the following holds.*

- (a) *H is a path with ends in S.*
- (b) *H is a subdivided star with root r such that $\mathcal{Z}(H) \subseteq V(H) \cap S \subseteq \mathcal{Z}(H) \cup \{r\}$.*
- (c) *H is the line graph of a subdivided star with $V(H) \cap S = \mathcal{Z}(H)$.*
- (d) *H is either a caterpillar or the line graph of a caterpillar, and $V(H) \cap S = \mathcal{Z}(H)$.*

Proof. By an *S-bump* we mean a vertex $v \in V(G) \setminus S$ of degree two in G, say $N_G(v) = \{v_1, v_2\}$, such that $v_1v_2 \notin E(G)$. An S-bump v is said to be *round* if v belongs to the vertex set of some cycle in G. By *suppressing* an S-bump v we mean removing the vertex v and adding the edge v_1v_2.

Let G_0 be the minor of G obtained by first successively removing round S-bumps until there are no round S-bumps, and then successively suppressing the remaining S-bumps until there are no S-bumps at all. It follows that G_0 is connected and $S \subseteq V(G_0)$. Moreover, there is no S-bump in G_0 and every cycle in G_0 is also a cycle in G. Let G_1 be a connected induced subgraph of G_0 with $|V(G_1)|$ as small as possible such that $S \subseteq V(G_1)$. In particular, there is no S-bump in G_1 and every cycle in G_1 is also a cycle in G.

It is straightforward to check that if G_1 has an induced subgraph H satisfying 5.6, then G has an induced subgraph H satisfying 5.6. Therefore, we only need to show that G_1 has such an induced subgraph H, which we will do in the rest of the proof. We begin with the following:

(26) *Every vertex in $V(G_1) \setminus S$ is a cut vertex of G_1. Moreover, for every $C \subseteq V(G_1)$ where C is connected and every component Q of $G_1 \setminus C$, we have $Q \cap S \neq \varnothing$. In particular, we have $\mathcal{Z}(G_1) \subseteq S$.*

The first assertion is immediate from the minimality of $|V(G_1)|$. Now suppose for a contradiction that for some $C \subseteq V(G_1)$ where C is connected, there is a component Q of $G_1 \setminus C$ such that $Q \cap S = \varnothing$. Since C is connected, it follows that $G_1 \setminus Q$ is connected with $S \subseteq V(G_1 \setminus Q)$, which contradicts the minimality of $|V(G_1)|$. This proves (26).

For a subset X of $V(G_1)$, we say a vertex $x \in X$ is *X-safe* if either $x \in S$ or there is a component $M_{X,x}$ of $G_1 \setminus \{x\}$ which is disjoint from X; in the latter case, it follows that $N_{G_1}(M_{X,x}) = \{x\}$. We denote by $\mathcal{S}_X$ the set of all X-safe vertices in X which are not in S. We deduce that:

(27) *Let $X \subseteq V(G_1)$ and let $x \in X \setminus S$ such that $X \setminus \{x\}$ is connected. Then $x \in \mathcal{S}_X$.*

Since $X \setminus \{x\}$ is connected, it follows that there is a component Q of $G_1 \setminus \{x\}$ which contains $X \setminus \{x\}$. Since $x \in V(G_1) \setminus S$, it follows from (26) that x is a cut vertex of G_1, and so there is a component Q' of $G_1 \setminus \{x\}$ which is distinct from Q. But then Q' is disjoint from X, and so we may choose $M_{X,x} = Q'$. This proves (27).

(28) *For every $X \subseteq V(G_1)$, the sets $(M_{X,x} : x \in \mathcal{S}_X)$ are pairwise disjoint and anticomplete in G_1, each containing at least one vertex from S.*

Suppose that there are two vertices $x, y \in \mathcal{S}_X$ for which $M_{X,x} \cup M_{X,y}$ is connected. Since G_1 is connected and $M_{X,x}$ is a component of $G_1 \setminus \{x\}$, it follows that x has a neighbor in $M_{X,x}$. Also, since $M_{X,y}$ is a component of $G_1 \setminus \{y\}$, since $M_{X,x} \cup M_{X,y}$ is connected and since $y \notin M_{X,x}$, it follows that $M_{X,x} \subseteq M_{X,y}$. But then $x \notin M_{X,y}$ has a neighbor in $M_{X,y}$, a contradiction with the fact that $N_G(M_{X,y}) = \{y\}$. Moreover, for each $x \in \mathcal{S}_X$, since $\{x\}$ is connected and $M_{X,x}$ is a component of $G_1 \setminus \{x\}$, it follows from (26) that $M_{X,x} \cap S \neq \varnothing$. This proves (28).

Suppose that $\eta \in \{1, 2\}$. Since G_1 is connected and $|S| \geq \eta$, it follows that there is a path H in G_1 with η ends, where the ends of H are contained in S and $H^* \cap S = \varnothing$. But then H satisfies 5.6(a).

So we assume that $\eta \geq 3$, and in particular:

$$|V(G_1)| \geq |S| \geq (\eta^\eta)^{8\eta^3} > (\eta^\eta)^{8\eta^3-1}.$$

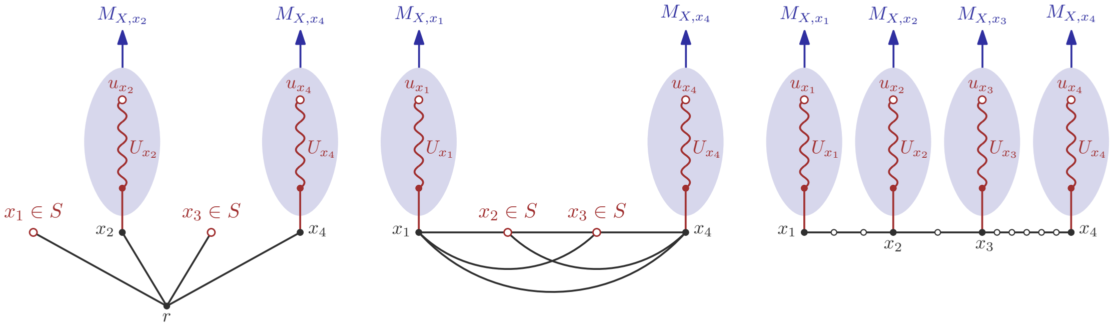

Figure 5.4: Left: X is a star (and r may or may not belong to S). Middle: X is a clique. Right: there is a path X'' in $X \setminus S$ with at least η vertices that are X-safe. Squiggly lines represent paths of arbitrary (possibly zero) length.

Since G_1 is connected, we deduce from Theorem 5.5 that either G_1 has a vertex of degree at least η^η, or there is a path on $8\eta^3$ vertices in G_1. This, along with Theorem 3.17, implies that there is an induced subgraph X of G_1 which is either a star with η leaves, a complete graph on η vertices, or a path on $8\eta^3$ vertices. From (27), it follows that every simplicial vertex of X is X-safe. In particular,

- if X is a star, then every leaf of X is X-safe; and
- if X is complete, then every vertex in X is X-safe.

By (28), for every $x \in \mathcal{S}_X$, there is a path U_x in $M_{X,x} \cup \{x\}$ from x to a vertex $u_x \in M_{X,x} \cap S$, such that $S \cap (U_x \setminus \{u_x\}) = \varnothing$. It also follows from (28) that the sets $(U_x \setminus \{x\} : x \in \mathcal{S}_x)$ are pairwise disjoint and anticomplete in G_1. Let

$$H = G_1 \left[X \cup \left(\bigcup_{x \in \mathcal{S}_X} U_x \right) \right].$$

If X is a star, then H satisfies 5.6(b), and if X is complete, then H satisfies 5.6(c) (see Figure 5.4). So we may assume that X is a path on $8\eta^3$ vertices. If there are at least η vertices $X \cap S$, then there is a path H in X (and so in G_1 that satisfies 5.6(a). Thus, we may assume that there is a subpath X' of X on $8\eta^2$ vertices such that $X' \cap S = \emptyset$. Suppose further that there are at least η vertices in $X' \cap \mathcal{S}_X$. Choose a minimal path X'' in X' containing exactly η vertices from $\mathcal{S}_X$; thus, the ends of X'' belong to $\mathcal{S}_X$. Then

$$H = G_1 \left[X'' \cup \left(\bigcup_{x \in X'' \cap \mathcal{S}_X} U_x \right) \right]$$

is a caterpillar satisfying 5.6(d) (see Figure 5.4).

Therefore, we may assume that there are fewer than η vertices in X' that are X-safe. It follows that there is a subpath P of X' (and so of X) on 8η vertices such that no vertex in P is X-safe. In particular, we have $P \subseteq X^*$ and $P \cap S = \varnothing$.

Let a, c be the ends of X. For every vertex $p \in P$, let A_p be the component of $G_1 \setminus \{p\}$ containing $V(a\text{-}X\text{-}p) \setminus \{p\}$ and let C_p be the component of $G_1 \setminus \{p\}$ containing $V(p\text{-}X\text{-}c) \setminus \{p\}$. We deduce that:

(29) *Let $p \in P$. Then A_p and C_p are the only distinct components of $G_1 \setminus \{p\}$. In particular, A_p and C_p are disjoint.*

Since $p \in V(P)$, it follows that p is not X-safe. This means $p \notin S$, and there is no component of $G_1 \setminus \{p\}$ disjoint from X. By (26), p is a cut vertex of G_1. So $G_1 \setminus \{p\}$ has exactly two components, namely A_p and C_p. This proves (29).

For an edge $pp' \in E(P)$, we say pp' is *heavy* if there is a component $K_{pp'}$ of $G_1 \setminus \{p, p'\}$ which is disjoint from X.

(30) *Let $p\text{-}p'\text{-}p''$ be a path in P. Then either pp' or $p'p''$ is heavy.*

Since p' is neither in S nor an S-bump, it follows that p' has a neighbor z in $V(G_1) \setminus X$. It follows that $z \in C_p \setminus \{p'\}$ and $z \in A_{p''} \setminus \{p\}$. By (31), either $z \in A_{p'} \setminus \{p\}$ or $z \in C_{p'} \setminus \{p''\}$. Assume that $z \in A_{p'} \setminus \{p\}$. Then we have $z \in A_{p'} \cap C_p$, and so by (29), we have

$$z \in V(G_1) \setminus (\{p, p'\} \cup A_p \cup C_{p'}).$$

Note also that A_p and $C_{p'}$ are the only components of $G_1 \setminus \{p, p'\}$ which intersect X. Therefore, the component of $G_1 \setminus \{p, p'\}$ containing z is disjoint from X, and so pp' is heavy. A similar argument shows that $p'p''$ is heavy if $z \in C_{p'} \setminus \{p''\}$. This proves (30).

(31) *Let pp' and qq' be two heavy edges of P where $\{p, p'\}$ and $\{q, q'\}$ are disjoint and anticomplete in G_1. Then $K_{pp'}$ and $K_{qq'}$ are disjoint and anticomplete in G_1.*

Suppose for a contradiction that $K_{pp'} \cup K_{qq'}$ is connected. We may assume that there is a vertex $v \in V(P)$ such that a, p, p', v, q, q', c appear on X in this order. Since G_1 is connected, it follows that $N_{G_1}(K_{pp'}) \subseteq \{p, p'\}$ and $N_{G_1}(K_{qq'}) \subseteq \{q, q'\}$ are both nonempty. As a result, there is a path R in G_1 with an end in $\{p, p'\} \subseteq A_v$ and an end in $\{q, q'\} \subseteq C_v$

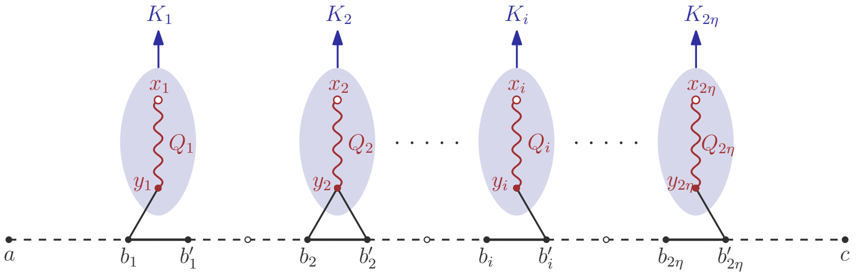

Figure 5.5: Proof of Theorem 5.6. Squiggly lines represent paths of arbitrary (possibly zero) length, and dashed lines depict paths of non-zero length.

such that $R^* \subseteq K_{pp'} \cup K_{qq'} \subseteq V(G_1) \setminus X$. But now R is a path in $G_1 \setminus \{v\}$ with an end in A_v and an end in C_v, a contradiction with (29). This proves (31).

Since $|V(P)| = 8\eta$, it follows from (30) that there is a 2η-subset $\{b_i b'_i : i \in [2\eta]\}$ of $E(P)$ such that

- for each $i \in [2\eta]$, the edge $b_i b'_i$ is heavy;
- the sets $\{\{b_i, b'_i\} : i \in [2\eta]\}$ are pairwise disjoint and anticomplete in G_1; and
- the vertices $a, b_1, b'_1, \dots, b_{2\eta}, b'_{2\eta}, c$ appear on X in this order.

For each $i \in [2\eta]$, let $K_i = K_{b_i b'_i}$ be the component of $G_1 \setminus \{b_i, b'_i\}$ as in the definition of a heavy edge. Since G_1 is connected, it follows that $N_{G_1}(K_i) \subseteq \{b_i, b'_i\}$ is non-empty. Also, from (26), it follows that $K_i \cap S \neq \varnothing$. So there is a path Q_i in K_i from a vertex $x_i \in K_i \cap S$ to a vertex $y_i \in N_{K_i}(\{b_i, b'_i\})$ such that $(V(Q_i) \setminus \{x_i\}) \cap S = \varnothing$, and $V(Q_i) \setminus \{y_i\}$ and $\{b_i, b'_i\}$ are anticomplete in G_1 (see Figure 5.5).

By (31), the sets $(Q_i : i \in [2\eta])$ are pairwise disjoint anticomplete in G_1. Let I_1 be the set of all $i \in [2\eta]$ with $|N_{\{b_i,b'_i\}}(y_i)| = 1$ and let I_2 be the set of all $i \in [2\eta]$ with $|N_{\{b_i,b'_i\}}(y_i)| = 2$. Then $I_1 = [2\eta] \setminus I_2$, and so there is an η-subset I of $[2\eta]$ such that either $I \subseteq I_1$ or $I \subseteq I_2$. Let i_1 and i_η be the smallest and the largest element of I, respectively. Since $\eta \geq 3$, it follows that i_1 and i_η are distinct and $i_\eta \geq 3$. Let Z_1 be a path in $G_1[V(Q_{i_1}) \cup \{b_{i_1} b'_{i_1}\}]$ from b'_{i_1} to x_{i_1} and let Z_η be a path in $G_1[V(Q_{i_\eta}) \cup \{b_{i_\eta} b'_{i_\eta}\}]$ from b_{i_η} to x_{i_η}. Let

$$H = G_1\left[V(b'_{i_1}\text{-}P\text{-}b_{i_\eta}) \cup Z_1 \cup Z_\eta \cup \left(\bigcup_{i \in I \setminus \{i_1, i_\eta\}} V(Q_i)\right)\right].$$

If $I \subseteq I_1$, then H is a caterpillar satisfying 5.6(d), and if $I \subseteq I_2$, then H is the line graph of a caterpillar and H satisfies 5.6(d). This completes the proof of Theorem 5.6. ■

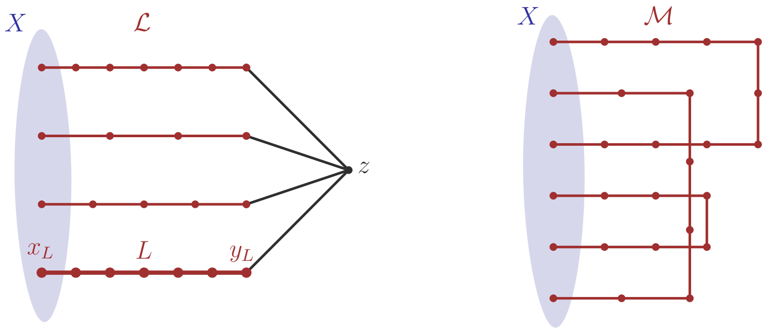

Figure 5.6: Left: A $(3, 4)$-patch $(\{z\}, \mathcal{L})$ for X in G. Right: A $(6, 3)$-match $\mathcal{M}$ for X in G.

5.4 Patch and Match

Let $d \in \mathbb{N} \cup \{0\}$, let $r \in \mathbb{N}$, let G be a graph and let $X \subseteq V(G)$. By a *(d, r)-patch for X in G* we mean a $(1, r)$-constellation $\mathfrak{p}$ in G, say $S_\mathfrak{p} = \{z_\mathfrak{p}\}$, such that for every path $L \in \mathcal{L}_\mathfrak{p}$, the following hold.

(P1) L has length at least d.

(P2) We have $L \cap X = \{x_L\}$ and $N_L(z_\mathfrak{p}) = \{y_L\}$ where x_L, y_L are the end of L.

Note that $z_\mathfrak{p}$ may or may not belong to X.

By a *(d, r)-match for X in G* we mean an r-polypath $\mathcal{M}$ in G with the following properties.

(M1) Every path $L \in \mathcal{M}$ has length at least d.

(M2) We have $V(\mathcal{M}) \cap X = V(\mathcal{M}) \setminus \mathcal{M}^*$.

See Figure 5.6. In this section, we prove the following result, which is the main technical step is the proof of Theorem 5.3 as well as its counterpart for perforated graphs in Chapter 6, namely Theorem 6.4.

Theorem 5.7. *For every $d \in \mathbb{N} \cup \{0\}$ and all $l, r, r', s, t \in \mathbb{N}$, there is a constant $f_{5.7} = f_{5.7}(d, l, r, r', s, t) \in \mathbb{N}$ with the following property. Let G be a t-clean graph, let $X \subseteq V(G)$ and let $\mathfrak{p}$ be a $(d, f_{5.7})$-patch for X in G. Then one of the following holds.*

(a) *There exists a plain (s,l)-constellation in G.*

(b) *There exists a plain $(2d+1,r)$-match $\mathcal{M}$ for X in G such that $V(\mathcal{M}) \subseteq V(\mathfrak{p})$.*

(c) *There exists a plain (d,r')-patch $\mathfrak{q}$ for X in G such that $V(\mathfrak{q}) \subseteq V(\mathfrak{p})$.*

The proof of Theorem 5.7 is by induction on d. The base case $d=0$ is straightforward to derive from Theorem 5.6:

Lemma 5.8. *Let $r, r', t \in \mathbb{N}$ and let $\eta = \max\{r+1, 3r', t+1\}$. Let G be a K_{t+1}-free graph, let $X \subseteq V(G)$ and let $\mathfrak{p}$ be a $\left(0, \eta^{8\eta^4}\right)$-patch for X in G. Then one of the following holds.*

(a) *There exists a plain $(1,r)$-match $\mathcal{M}$ for X in G such that $V(\mathcal{M}) \subseteq V(\mathfrak{p})$.*

(b) *There exists a plain $(0,r')$-patch $\mathfrak{q}$ for X in G such that $V(\mathfrak{q}) \subseteq V(\mathfrak{p})$.*

Proof. For every $L \in \mathcal{L}_{\mathfrak{p}}$, let x_L and y_L be the ends of L such that (P2) holds (where $x_L = y_L$ is possible). Let $G' = G[V(\mathfrak{p})]$ and let $S = \{x_L : L \in \mathcal{L}_{\mathfrak{p}}\}$. Then G' is connected and $S \subseteq V(G')$ with $|S| = f_{5.8}$. By the choice of $f_{5.8}$, we can apply Theorem 5.6 to G' and S to deduce that there is an induced subgraph H of G' with $|V(H) \cap S| = \eta = \max\{r+1, 3r', t+1\}$ for which one of the following holds.

- H is a path with ends in S.
- H is a subdivided star with root u such that $\mathcal{Z}(H) \subseteq V(H) \cap S \subseteq \mathcal{Z}(H) \cup \{u\}$.
- H is the line graph of a subdivided star with $V(H) \cap S = \mathcal{Z}(H)$.
- H is either a caterpillar or the line graph of a caterpillar, and $V(H) \cap S = \mathcal{Z}(H)$.

Since G is K_{t+1}-free and $|S| \geq t+1$, it follows that the third bullet above does not happen. Since $|S| \geq r+1$, it follows that the second bullet above implies 5.8(b). Moreover, it is straightforward to observe that since $|S| \geq 3r'$, the first and last bullet above each imply 5.8(b). This completes the proof of Lemma 5.8. ■

Let $d \in \mathbb{N}$ and let G be graph. We say a constellation $\mathfrak{c}$ in G is *d-meager* if every vertex $x \in V(\mathcal{L}_{\mathfrak{c}})$ has at most d neighbors in $S_{\mathfrak{c}}$. It follows that $\mathfrak{c}$ is 1-meager if and only if $\mathfrak{c}$ is ample.

The following is a key tool in several proofs in the rest of this chapter as well as the subsequent ones:

Theorem 5.9. *Let $a, d, l, s \in \mathbb{N}$ and let G be a graph. Assume that there exists a d-meager $a^{l-1}(s+2dl)$-constellation $\mathfrak{a}$ in G. Then one of the following holds.*

(a) *There exists an a-alignment $\mathfrak{s}$ in G with $S_{\mathfrak{s}} \subseteq S_{\mathfrak{a}}$ and $L_{\mathfrak{s}} \subseteq L^*_{\mathfrak{a}}$. In particular, if $\mathfrak{a}$ is d'-ample for $d' \in \mathbb{N}$, then $\mathfrak{s}$ is also d'-ample.*

(b) *There exists a plain (s, l)-constellation $\mathfrak{c}$ in G such that $S_{\mathfrak{c}} \subseteq S_{\mathfrak{a}}$ and $L \subseteq L^*_{\mathfrak{a}}$ for every $L \in \mathcal{L}_{\mathfrak{c}}$. In particular, if $\mathfrak{a}$ is d'-ample for $d' \in \mathbb{N}$, then $\mathfrak{c}$ is also d-ample.*

Proof. For fixed a and s, we proceed by induction on l. Let u_1 and u_2 be the ends of $L_{\mathfrak{a}}$. Since $\mathfrak{a}$ is ample, it follows that there exists $S \subseteq S_{\mathfrak{a}}$ with $|S| = a^{l-1}(s + 2dl) - 2d$ such that S and $\{u_1, u_2\}$ are anticomplete in G. In particular, we have $|S| \geq s \geq 1$, and so 5.9 holds if either $a = 1$ or $l = 1$.

Therefore, we may assume that $a, l \geq 2$. For every vertex $x \in S$, traversing $L_{\mathfrak{a}}$ from u_1 to u_2, let v_x and w_x be the first and the last neighbor of x in $L_{\mathfrak{a}}$, and let P_x be the unique path in $L_{\mathfrak{a}}$ with ends v_x, w_x. It follows that $P_x \subseteq L^*_{\mathfrak{a}}$.

Define Γ to be the graph with vertex set S such that for distinct $x, y \in S_{\mathfrak{a}}$, we have $xy \in E(\Gamma)$ if and only if $P_x \cap P_y \neq \varnothing$. It is readily seen that Γ is an interval graph, and Γ is perfect [10]. Also, assuming $b = a^{l-2}(s + 2d(l-1))$, it follows from $a, l \geq 2$ that

$$|V(\Gamma)| = |S| = a^{l-1}(s + 2dl) - 2d = ab + (2a^{l-1} - 2)d \geq ab + (2a - 2)d \geq a(b + d).$$

In summary, Γ is a perfect graph with $|V(\Gamma)| \geq a(b + d)$ for $a, b + d \geq 1$. Thus, Γ contains either a stable set A of cardinality a or a clique B of cardinality $b + d$. In the former case, we may write $A = \{x_1, \ldots, x_a\}$ such that, for all distinct $i, j \in [a]$, the path in $L_{\mathfrak{a}}$ from u_1 to v_{x_j} contains v_{x_i} if and only if $i < j$. But then the ordered s-constellation $\mathfrak{s} = (A, L_{\mathfrak{a}})$ in G with $\pi(i) = x_i$ for every $i \in [a]$ is an a-alignment in G which satisfies 5.9(a), as desired.

Next, assume that Γ contains a clique B of cardinality $b + d$. Then there exists a vertex $u \in L^*_{\mathfrak{a}}$ such that for every $x \in B$, we have $u \in P_x$. Let $L_i = u_i\text{-}L_{\mathfrak{a}}\text{-}u$ for $i \in \{1, 2\}$. Since $\mathfrak{a}$ is d-meager, it follows that $|N_B(u)| \leq d$, and so there exists $B' \subseteq B$ with $|B'| = b = a^{l-2}(s + 2d(l-1))$ such that B' and $\{u_1, u, u_2\}$ are anticomplete in G, and every vertex in B' has a neighbor in L^*_1 and a neighbor in L^*_2. It follows that, for every $i \in \{1, 2\}$, $\mathfrak{a}_i = (B', L_i)$ is an $a^{l-2}(s + 2d(l-1))$-constellation in G which is d-meager, as so is $\mathfrak{a}$. From the induction hypothesis applied to $\mathfrak{a}_1$, we deduce that either there exists an a-alignment $\mathfrak{s}$ in G with $S_{\mathfrak{s}} \subseteq S_{\mathfrak{a}_1} = B' \subseteq S_{\mathfrak{a}}$ and $L_{\mathfrak{s}} \subseteq L^*_{\mathfrak{a}_1} = L^*_1 \subseteq L^*_{\mathfrak{a}}$, or there exists a plain $(s, l-1)$-constellation $\mathfrak{c}_1$ in G such that $S_{\mathfrak{c}_1} \subseteq S_{\mathfrak{a}_1} = B' \subseteq S_{\mathfrak{a}}$ and $L \subseteq L^*_{\mathfrak{a}_1} = L^*_1 \subseteq L^*_{\mathfrak{a}}$ for every $L \in \mathcal{L}_{\mathfrak{c}_1}$. In the former case, $\mathfrak{s}$ satisfies 5.9(a). In the latter case, $\mathfrak{c} = (S_{\mathfrak{c}_1}, \mathcal{L}_{\mathfrak{c}_1} \cup \{L^*_2\})$ is a plain (s, l)-constellation in G such that $S_{\mathfrak{c}} = S_{\mathfrak{c}_1} \subseteq S_{\mathfrak{a}}$ and $L \subseteq L^*_1 \cup L^*_2 \subseteq L^*_{\mathfrak{a}}$ for every $L \in \mathcal{L}_{\mathfrak{c}}$, and so 5.9(b) holds. This completes the proof of Theorem 5.9. ■

We can now prove Theorem 5.7, which we restate:

Theorem 5.7. *For every $d \in \mathbb{N} \cup \{0\}$ and all $l, r, r', s, t \in \mathbb{N}$, there is a constant $f_{5.7} = f_{5.7}(d, l, r, r', s, t) \in \mathbb{N}$ with the following property. Let G be a t-clean graph, let $X \subseteq V(G)$ and let $\mathfrak{p}$ be a $(d, f_{5.7})$-patch for X in G. Then one of the following holds.*

(a) *There exists a plain (s, l)-constellation in G.*
(b) *There exists a plain $(2d + 1, r)$-match $\mathcal{M}$ for X in G such that $V(\mathcal{M}) \subseteq V(\mathfrak{p})$.*
(c) *There exists a plain (d, r')-patch $\mathfrak{q}$ for X in G such that $V(\mathfrak{q}) \subseteq V(\mathfrak{p})$.*

Proof. Let $s \in \mathbb{N} \cup \{0\}$ and $l, r, r', t \in \mathbb{N}$ be fixed. We begin with recursively defining a sequence $\{\zeta_i : i \in \mathbb{N}\}$ of positive integers. The definition relies on Theorem 3.14 and Lemmas 3.20 and 5.8. Specifically, for every $i \in \mathbb{N}$, let

$$f_i = f_{3.20}(1, 1, (4ir)^{l-1}(s + 2l), t);$$

$$g_i = g_{3.20}(1, 1, (4ir)^{l-1}(s + 2l), t).$$

Let $\eta = \max\{r + 1, 3r', t + 1\}$ and let

$$\zeta_1 = \eta^{8\eta^4}.$$

For every $i \in \mathbb{N} \setminus \{1\}$, let

$$\zeta_i = \zeta_{i-1} f_{3.14}(1, \max\{(t + 1)^{f_{i-1}}, g_{i-1}\}, \zeta_{i-1}).$$

This concludes the definition of ζ_i for all $i \in \mathbb{N}$.

We prove by induction on $d \in \mathbb{N} \cup \{0\}$ that

$$f_{5.7} = f_{5.7}(d, l, r, r', s, t) = \zeta_{d+1}.$$

satisfies the theorem. Let G be a t-clean graph, let $X \subseteq V(G)$ and let $\mathfrak{p}$ be a (d, ζ_{d+1})-patch for X in G. If $d = 0$, then by the definition of ζ_0 and the assumption that G is t-clean, 5.7 follows directly from Lemma 5.8. So we may assume that $d \geq 1$.

For every $L \in \mathcal{L}_{\mathfrak{p}}$, let x_L and y_L be the ends of L such that (P2) holds. Since $d \in \mathbb{N}$ and from the definition of ζ_{d+1}, it follows that there exists a partition $(\mathcal{L}_1, \ldots, \mathcal{L}_{\zeta_d})$ of $\mathcal{L}_{\mathfrak{p}}$ with

$$|\mathcal{L}_1| = \cdots = |\mathcal{L}_{\zeta_d}| = f_{3.14}(1, \max\{(t + 1)^{f_d}, g_d\}, \zeta_d).$$

For each $i \in [\zeta_d]$, let

$$\mathcal{B}_i = \{(\{x_L\}, L) : L \in \mathcal{L}_i\}.$$

Then $\mathcal{B}_i$ is a set of pairwise disjoint 1-bisets over $V(G)$ with

$$|\mathcal{B}_i| = f_{3.14}(1, \max\{(t+1)^{f_d}, g_d\}, \zeta_d).$$

Therefore, we may apply Theorem 3.14 to the sets $\mathcal{B}_1, \ldots, \mathcal{B}_{\zeta_d}$ and obtain, for each $i \in [\zeta_d]$, a $\max\{(t+1)^{f_d}, g_d\}$-subset $\mathcal{L}'_i$ of $\mathcal{L}_i$ such that for all distinct $i, j \in [\zeta_d]$, the sets $\{(x_L, L) : L \in \mathcal{L}'_i\}$ and $\{(x_L, L) : L \in \mathcal{L}'_j\}$ satisfy the three outcomes of Theorem 3.14. In particular, from Theorem 3.14(b), we deduce that:

(32) *One of the following holds.*

- *There exists distinct $i, j \in [\zeta_d]$ such that for every $L_i \in \mathcal{L}'_i$ and every $L_j \in \mathcal{L}'_j$, the vertex x_{L_i} has a neighbor in L_j in G.*
- *For all distinct $i, j \in [\zeta_d]$, every $L_i \in \mathcal{L}'_i$ and every $L_j \in \mathcal{L}'_j$, the vertex x_{L_i} is anticomplete to L_j in G.*

Now, suppose that the first outcome of (32) holds for some $i, j \in [\zeta_d]$. Since G is t-clean and $|\mathcal{L}'_i| \geq (t+1)^{f_d}$, it follows from Theorem 3.17 that there exists $\mathcal{S} \subseteq \mathcal{L}'_i$ with $|\mathcal{S}| = f_d$ such that $S = \{x_L : L \in \mathcal{S}\}$ is a stable set in G. Also, we have $|\mathcal{L}'_j| \geq g_d$ and so one may choose a g_d-subset $\mathcal{L}$ of $\mathcal{L}'_j$. Since $i, j \in [\zeta_d]$ satisfy the first outcome of (32), it follows that $(S, \mathcal{L})$ is an (f_d, g_d)-constellation in G. This, combined with the choice of f_d and g_d and the assumption that G is t-clean, allows us to apply Lemma 3.20 to G and $(S, \mathcal{L})$. It follows that there exists an ample $(4dr)^{l-1}(s+2l)$-constellation $\mathfrak{a}$ in G with $S_{\mathfrak{a}} \subseteq S \subseteq X$ and $L_{\mathfrak{a}} \in \mathcal{L} \subseteq \mathcal{L}'_j \subseteq \mathcal{L}_j \subseteq \mathcal{L}_{\mathfrak{p}}$. Consequently, we can apply Theorem 5.9 to $\mathfrak{a}$. Note that if Theorem 5.9(b) holds, then there is a plain (s, l)-constellation in G, and so 5.7(a) holds, as desired. Therefore, we may assume that Theorem 5.9(a) holds, that is, there exists a $4dr$-alignment $\mathfrak{s}$ in G with $S_{\mathfrak{s}} \subseteq S_{\mathfrak{a}} \subseteq X$ and $L_{\mathfrak{s}} \subseteq L^*_{\mathfrak{a}} \subseteq \mathcal{L}^*_{\mathfrak{p}} \subseteq V(\mathfrak{p}) \setminus X$. Let π be the order on $S_{\mathfrak{s}}$ as in the definition of an alignment. It follows that for each $i \in [2r-1]$, there is a unique path M_i in G from $\pi(2di) \in X$ to $\pi(2d(i+1)) \in X$ with $M^*_i \subseteq L_{\mathfrak{s}} \subseteq V(\mathfrak{p}) \setminus X$. Moreover, since $\mathfrak{s}$ is ample, it follows that the paths $(M_{2i-1} : i \in [r])$ are pairwise disjoint and anticomplete in G, each of length at least $2d+2 > 2d+1$ (see Figure 5.7). But now $\mathcal{M} = \{M_i : i \in [r]\}$ is a plain $(2d+1, r)$-match for X in G with $V(\mathcal{M}) \subseteq V(\mathfrak{p})$, showing that 5.7(b) holds.

It remains to consider the case where the second outcome of (32) holds, that is, for all distinct $i, j \in [\zeta_d]$, every $L_i \in \mathcal{L}'_i$ and every $L_j \in \mathcal{L}'_j$, the vertex x_{L_i} is anticomplete to L_j in G. For each $i \in [\zeta_d]$, fix a path $L_i \in \mathcal{L}'_i$ (this is possible because since $|\mathcal{L}'_i| = \max\{(t+1)^{f_d}, g_d\} \geq 1$). Since $d \geq 1$, it follows that L_i has at least two vertices (and so x_{L_i} and y_{L_i} are distinct). Let x'_i be the unique neighbor of x_i is L_i and let $P_i = L_i \setminus x_i$. Then P_i

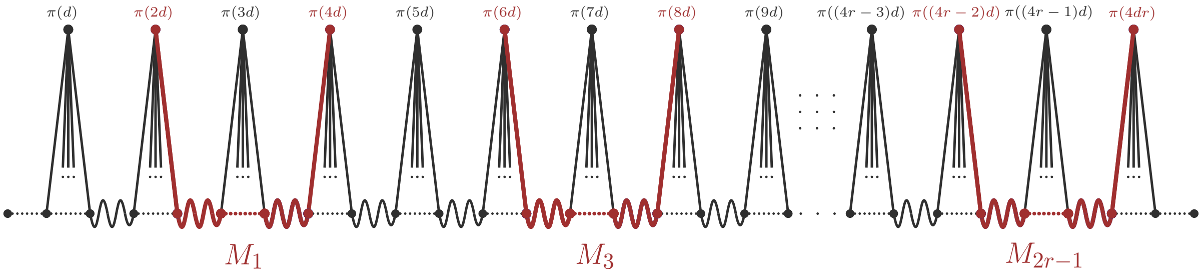

Figure 5.7: Proof of Theorem 5.7. Dotted lines represent paths of arbitrary (possibly zero) lengths, and squiggly lines represent paths of length at least d.

is a path of length at least $d-1$ in G with ends x'_i, y_{L_i} such that $V(P_i) \cap X = \varnothing$. Moreover, x_{L_i} is the only vertex in $\{x_{L_i} : i \in [\zeta_d]\} \subseteq X$ with a neighbor in P_i. Let $X' = \{x'_i : i \in [\zeta_d]\}$ and let

$$\mathfrak{p}' = (\{z_{\mathfrak{p}}\}, \{P_i : i \in [\zeta_d]\}).$$

Then $\mathfrak{p}'$ is a $(1, \zeta_d)$-constellation in G such that for every $i \in [\zeta_d]$, the path P_i has length at least $d-1$, and we have $P_i \setminus P_i^* = \{x'_i, y_{L_i}\}$ where $P_i \cap X' = \{x'_i\}$. It follows that $\mathfrak{p}'$ is a $(d-1, \zeta_d)$-patch for X' in G with $V(\mathfrak{p}') \subseteq V(\mathfrak{p}) \setminus X$. Moreover, for each $i \in [\zeta_d]$, we have $N_{V(\mathfrak{p}')}(x_{L_i}) = \{x'_i\}$. Now, we may apply the induction hypothesis to $\mathfrak{p}'$ to deduce that one of the following holds:

- There exists a plain (s, l)-constellation in G.
- There exists a plain $(2d-1, r)$-match $\mathcal{M}'$ for X' in G such that $V(\mathcal{M}') \subseteq V(\mathfrak{p}')$.
- There exists a plain $(d-1, r')$-patch $\mathfrak{q}'$ for X' in G such that $V(\mathfrak{q}') \subseteq V(\mathfrak{p}')$.

The first bullet above is identical to 5.7(a). Assume that the second bullet above holds. Since $2d-1 \geq 1$, it follows that for each paths $M \in \mathcal{M}'$, the ends $x^1_M, x^2_M \in X'$ of M are distinct. Moreover, for each $i \in \{1,2\}$, there is a vertex $u^i_M \in \{x_{L_i} : i \in [\zeta_d]\} \subseteq X$ with $N_{V(\mathfrak{p})}(u^i_M) = \{x^i_M\}$. Let $M^+ = u^1_M\text{-}x^1_M\text{-}M\text{-}x^2_M\text{-}u^2_M$ and $\mathcal{M} = \{M^+ : M \in \mathcal{M}'\}$. Then $\mathcal{M}$ is a plain $(2d+1, r)$-match for X in G such that $V(\mathcal{M}) \subseteq V(\mathfrak{p}') \cup \{x_{L_i} : i \in [\zeta_d]\} \subseteq (V(\mathfrak{p}) \setminus X) \cup X = V(\mathfrak{p})$. This means $\mathcal{M}$ satisfies 5.7(b). Finally, assume that the third bullet above holds; say $S_{\mathfrak{q}'} = \{z_{\mathfrak{q}'}\}$. Since $d-1 \geq 0$, it follows that for each path $L \in \mathcal{L}_{\mathfrak{q}'}$, one may write $L \setminus L^* = \{x'_L, y'_L\}$ where $L \cap X' = \{x'_L\}$ and $N_L(z_{\mathfrak{q}'}) = \{y'_L\}$ (where $x'_L = y'_L$ is possible). Moreover, there is a vertex $u_L \in \{x_{L_i} : i \in [\zeta_d]\} \subseteq X$ with $N_{V(\mathfrak{p})}(u_L) = \{x'_L\}$. Let $L^+ = u_L\text{-}x'_L\text{-}L\text{-}y'_L$ and let $\mathcal{L}^+ = \{L^+ : L \in \mathcal{L}(\mathfrak{q}')\}$. Then $\mathfrak{q} = (\{z_{\mathfrak{q}'}\}, \mathcal{L}^+)$ is a plain (d, r')-match for X in G such that $V(\mathfrak{q}) \subseteq V(\mathfrak{p}') \cup \{x_{L_i} : i \in [\zeta_d]\} \subseteq (V(\mathfrak{p}) \setminus X) \cup X = V(\mathfrak{p})$. Hence, $\mathfrak{q}$ satisfies 5.7(c). This completes the proof of Theorem 5.7. ■

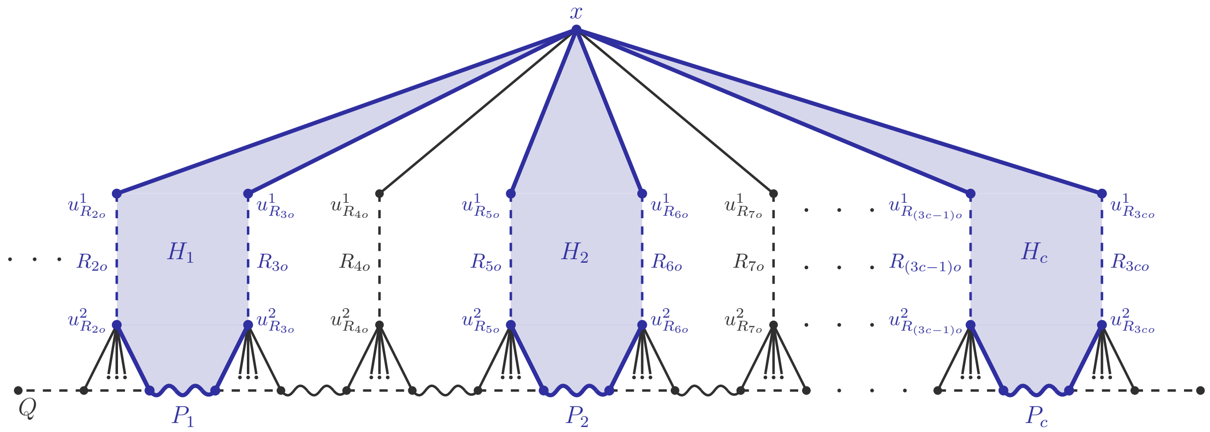

Figure 5.8: Proof of Lemma 5.10. Dashed lines represent paths of arbitrary (possibly zero) length, and squiggly lines represent paths of length at least o.

5.5 From large treewidth to a plain constellation

Here we complete the first step in the proof of Theorem 5.1 by proving Theorem 5.3. We begin with a lemma:

Lemma 5.10. *Let $c, d, o, s, l \in \mathbb{N}$ and let G be a (c, o)-pinched graph. Let $x \in V(G)$, let Q_0 be a path in $G \setminus \{x\}$ such that x is anticomplete to Q_0^* in G, and let $\mathcal{R}$ be a plain $(3co)^{(l-1)}(s+2dl)$-polypath in $G \setminus \{x\}$ such that $V(Q_0) \cap V(\mathcal{R}) = \varnothing$. Assume that the ends of each path $R \in \mathcal{R}$ may be called u^1_R and u^2_R (which could be the same) such that u^1_R is the only neighbor of x in R and u^2_R is the only vertex in R with a neighbor in Q_0. Assume also that $(3co)^{(l-1)}(s+2dl)$-constellation $(\{u^2_R : R \in \mathcal{R}\}, Q_0)$ in G is a d-meager. Then there is a plain (s, l)-constellation.*

Proof. Suppose not. Then applying Theorem 5.9 to the d-meager $(3co)^{(l-1)}(s+2dl)$-constellation $(\{u^2_R : R \in \mathcal{R}\}, Q_0)$ in G, it follows that Theorem 5.9(a) holds, that is, there are $R_1, \dots, R_{3co} \in \mathcal{R}$ and $L \subseteq Q_0^*$ such that

$$\left(\{u^2_{R_i} : i \in [3co]\}, L\right)$$

is a $3co$-alignment in G with respect to the order $\pi(u^2_{R_i}) = i$ on $\{u^2_{R_i} : i \in [3co]\}$. In particular, x is anticomplete to L, and for every $i \in [c]$, there is a unique path P_i in G from $u^2_{R_{(3i-1)o}}$ to $u_{R_{3io}}$ whose interior is contained in L. Now, for each $i \in [c]$, let

$$H_i = x\text{-}u^1_{R_{(3i-1)o}}\text{-}R_{(3i-1)o}\text{-}u^2_{R_{(3i-1)o}}\text{-}P_i\text{-}u^2_{R_{3io}}\text{-}R_{3io}\text{-}u^1_{R_{3io}}\text{-}x.$$

See Figure 5.8. Then $H_1, \dots, H_c$ are c induced cycles in G, each of length at least $o+4 > o+2$, such that $x \in V(H_1) \cap \dots \cap V(H_c)$ and the sets $\{V(H_i) \setminus \{x\} : i \in [c]\}$ are pairwise disjoint and anticomplete in G. This violates the assumption that G is (c, o)-pinched, hence completing the proof of Lemma 5.10. ■

The following result is immediate from Theorem 3.17, and we will also use it in future chapters. We define a *digraph* D to be a pair $(V(D), E(D))$ where $V(D)$ is a finite and non-empty set of *vertices* and $E(D)$ is a subset of $V(D) \times V(D)$ such that for every $(u, v) \in E(D)$, we have $u \neq v$. By a *stable set* in D we mean a subset S of $V(D)$ such that for all $u, v \in S$, neither (u, v) nor (v, u) belongs to $E(D)$.

Theorem 5.11. *Let $c, s \in \mathbb{N}$ and let D be a digraph on at least c^{cs} vertices. Then either there is a stable set S in D with $|S| = s$, or there are c vertices $u_1, \dots, u_c$ in D such that for all distinct $i, j \in [c]$ with $i < j$, we have $(i, j) \in E(D)$.*

Proof. By Theorem 3.17, either there is a stable set S of D with $|S| = s$, or there are c^c vertices $v_1, \dots, v_{c^c}$ in D such that for all distinct $i, j \in [c^c]$, at least one of (v_i, v_j) or (v_j, v_i) belongs to $E(D)$. Let $D^\natural$ be the graph with $V(D^\natural) = \{v_1, \dots, v_{c^c}\}$ such that for all distinct $i, j \in [c^c]$ with $i < j$, we have $v_i v_j \in E(D^\natural)$ if and only if $(v_i, v_j) \in E(D)$. By Theorem 3.17, there exists $i_1, \dots, i_c \in [c^c]$ such that $i_1 < \dots < i_c$ and $\{v_{i_1}, \dots, v_{i_c}\}$ is either a clique or a stable set in $D^\natural$. In the former case, let $u_j = v_{i_j}$ for every $j \in [c]$, and in the latter case, let $u_j = v_{i_{c-j+1}}$ for every $j \in [c]$. Then from the definition of $D^\natural$, it follows that $u_1, \dots, u_c$ satisfy Theorem 5.11, as required. ■

Theorem 5.3. *For all $c, l, o, s, t \in \mathbb{N}$, there is a constant $f_{5.3} = f_{5.3}(c, l, o, s, t) \in \mathbb{N}$ such that if G is a t-clean, (c, o)-pinched graph with $\operatorname{tw}(G) > f_{5.3}$, then there is a plain (s, l)-constellation in G.*

Proof. The definition of $f_{5.3}(c, l, o, s, t)$ relies on multiple earlier results, including Theorems 2.27, 3.8, 3.14, 3.18 and 5.7 and Lemmas 3.12 and 6.9. Specifically, let

$$b_1 = (3co)^{(l-1)}(s + 2l^2);$$

$$b_2 = ((t+1)^s + 1)^{l+1} b_1^{l^2}.$$

Observe that $b_2 > b_1 > l + 1 \geq 2$. Let

$$c' = (t+1)^{(s+1)^{c(s+1)}};$$

$$f_0 = f_{3.14}(2, b_2, c');$$

$$f_1 = f_{5.7}(o, l, c, f_0, s, t).$$

Let

$$f_2 = f_{3.18}(o, f_1, c' + 1, t);$$
$$g_2 = g_{3.18}(o, f_1, c' + 1, t).$$

We will show that

$$f_{5.3} = f_{5.3}(c, l, s, o, t) = f_{2.27}(f_2, g_2, t)$$

satisfies the theorem.

Let G be a t-clean (c, o)-pinched graph with $\mathrm{tw}(G) > f_{5.3}$, and suppose for a contradiction that there is no plain (s, l)-constellation in G. By Theorem 2.27, there is a strong (f_2, g_2)-block in G. Since G is t-clean and by the choices of f_2 and g_2, it follows from Theorem 3.18 that there is an o-long strong $(c' + 1, f_1)$-block $(B, \mathcal{P})$ in G. In particular, B is a stable set, and one may choose a set $c' + 1$ pairwise distinct and non-adjacent vertices $x, z_1, \dots, z_{c'}$ in B.

Let $i \in [c']$. Define $\mathcal{L}_i = \{P^* : P \in \mathcal{P}(\{x_i, z_i\})\}$. Then $|\mathcal{L}_i| = f_1$. Since $(B, \mathcal{P})$ is o-long, it follows that every path $L \in \mathcal{L}_i$ has length at least $o - 1 \geq 0$, and so x and z_i each have a unique (possibly the same) neighbor in L, which we call x_L and y_L, respectively. Let $X_i = \{x_L : L \in \mathcal{L}_i\}$. Then $\mathfrak{p}_i = (\{z_i\}, \mathcal{L}_i)$ is a $(1, f_1)$-contellation in G such that for every path $L \in \mathcal{L}_{\mathfrak{p}_i} = \mathcal{L}_i$, we know that L has length at least $o-1$, and we have $L \setminus L^* = \{x_L, y_L\}$ such that $L \cap X_i = \{x_L\}$ and $N_L(z_{\mathfrak{p}}) = \{y_L\}$.

We deduce that $\mathfrak{p}_i$, for each $i \in [c']$, is an $(o-1, f_1)$-patch for X_i in G with $X_i \subseteq V(\mathfrak{p}_i)$. Moreover, since $(B, \mathcal{P})$ is strong, it follows that the sets

$$V(\mathfrak{p}_i) = V(\mathcal{P}(\{x, z_i\})) \setminus \{x\}$$

are pairwise disjoint for all $i \in [c']$. Therefore, since G is t-clean and by the choice of f_1, we can apply Theorem 5.7 to $\mathfrak{p}_1, \dots, \mathfrak{p}_{c'}$. From the assumption that there is no plain (s, l)-constellation in G, it follows that for each $i \in [c']$, one of the following holds.

- There exists a plain $(2o - 1, c)$-match $\mathcal{M}_i$ for X_i in G such that $V(\mathcal{M}_i) \subseteq V(\mathfrak{p}_i)$. In particular, $G[V(M) \cup \{x\}]$ is an cycle of length at least $2o + 1 \geq o + 2$ for every $M \in \mathcal{M}_i$.
- There exists a plain $(o - 1, f_0)$-patch $\mathfrak{q}_i$ for X_i in G such that $V(\mathfrak{q}_i) \subseteq V(\mathfrak{p}_i)$. In particular, if $S_{\mathfrak{q}_i} \subseteq X_i$, then $G[\{x\} \cup S_{\mathfrak{q}_i} \cup V(L)]$ is a cycle of length at least $o + 2$ for every $L \in \mathcal{L}_{\mathfrak{q}_i}$, and if $S_{\mathfrak{q}_i} \cap X_i = \varnothing$, then $G[\{x\} \cup S_{\mathfrak{q}_i} \cup V(L) \cup V(L')]$ is a cycle of length at least $2o + 2 > o + 2$.

Indeed, since G is (c,o)-pinched, It follows that the second bullet above holds.

For each $i \in [c']$, let $A_i = S_{\mathsf{q}_i}$ and let $\mathcal{Q}_i = \mathcal{L}_{\mathsf{q}_i}$. It follows that $\mathcal{Q}_i$ is a plain f_0-polypath in G, and:

(33) *For every $i \in [c']$ and all distinct $Q, Q' \in \mathcal{Q}_i$, there is an induced subgraph of G which is a cycle of length at least $o+2$ containing x, whose all vertices but x are contained in $A_i \cup V(Q) \cup V(Q')$.*

For each $i \in [c']$, let

$$\mathcal{B}_i = \{(A_i, Q) : Q \in \mathcal{Q}_i\}.$$

Then $\mathcal{B}_1, \dots, \mathcal{B}_{c'}$ are c' sets of pairwise disjoint 1-bisets in G, each of cardinality f_0. By Theorem 3.14 and the choice of f_0, it follows that for each $i \in [c']$, there exists $\mathcal{Q}'_i \subseteq \mathcal{Q}_i$ with $|\mathcal{Q}'_i| = b_2$ such that for all distinct $i, j \in [c']$, the sets

$$\{(A_i, Q) : Q \in \mathcal{Q}'_i\}$$

and

$$\{(A_i, Q) : Q \in \mathcal{Q}'_j\}$$

satisfy the three outcomes of Theorem 3.14. Let us now prove that:

(34) *There exists $I \subseteq [c']$ with $|I| = c$ such that for all distinct $i, j \in I$ and every $Q_j \in \mathcal{Q}'_j$, the sets A_i and $A_j \cup V(Q_j)$ are anticomplete in G.*

Note that since G is K_{t+1}-free, it follows from Lemma 3.17 there is subset I_0 of $[c']$ with $|I_0| = (s+1)^{c(s+1)}$ for which the sets $(A_i : i \in I)$ are pairwise anticomplete in G. It remains to show that there exists $I \subseteq I_0$ with $|I| = c$ such that for all distinct $i, j \in I$ and every $Q_j \in \mathcal{Q}'_j$, the sets A_i and $V(Q_j)$ are anticomplete in G. Let D be the digraph with $V(D) = I_0$ such that for all distinct $(i, j) \in I_0$, we have $(i, j) \in E(D)$ if the unique vertex in A_i has a neighbor in every path $Q \in \mathcal{Q}'_j$. By Theorem 5.11, either there is a stable set $I \subseteq V(D) = I_0$ in D of cardinality c, or there there are $s+1$ elements $i_1, \dots, i_{s+1}$ of $V(D) = I_1$ such that for all distinct $i, j \in [s+1]$, we have $(i, j) \in E(D)$. In the former case, by Theorem 3.14(b), I satisfies (34), and we are done. Also, in the latter case, it follows in particular that for each $j \in [s]$, the unique vertex in A_{i_j} has a neighbor in every path $Q \in \mathcal{Q}'_{i_{s+1}}$. Since $(A_i : i \in I_0)$ are pairwise anticomplete in G, it follows that $S = A_{i_1} \cup \dots \cup A_{i_s}$ is a stable set of cardinality s in G. Also, $\mathcal{Q}'_{i_{s+1}}$ is a plain polypath in G with $|\mathcal{Q}'_{i_{s+1}}| = b_2 > l$, and so there is a plain l-polypath $\mathcal{L} \subseteq \mathcal{Q}'_{i_{s+1}}$ in G. But now $(S, \mathcal{L})$ is a plain (s, l)-constellation in G, a contradiction. Therefore, we have $k = 1$, and so one may choose a c-subset I of I_1 with the desired property. This proves (34).

Henceforth, let I be as in (34). We deduce that:

(35) *For each $i \in I$, there are distinct paths $Q_i, Q'_i \in \mathcal{Q}'_i$, such that the sets*

$$(A_i \cup V(Q_i) \cup V(Q'_i) : i \in I)$$

are pairwise anticomplete in G.

By (34) and since $|\mathcal{Q}'_i| = b_2 > 2$ for each $i \in I$, in order to prove (35), it suffices to show that for all distinct $i, j \in I$, every $Q_i \in \mathcal{Q}'_i$ and every $Q_j \in \mathcal{Q}'_j$, the sets $V(Q_i)$ and $V(Q_j)$ are anticomplete in G. Suppose not. Then by Theorem 3.14(c), there distinct $i, j \in I$ such that for every $Q_i \in \mathcal{Q}'_i$ and every $Q_j \in \mathcal{Q}'_j$, there is an edge in G with an end in $V(Q_i)$ and an end in $V(Q_j)$. Since $|\mathcal{Q}'_i| = |\mathcal{Q}'_j| = b_2$, G is K_{t+1}-free and $V(\mathcal{Q}'_i) \cap V(\mathcal{Q}'_j) = \varnothing$, it follows from the choice of b_2 that we can apply Lemma 3.22 to $\mathcal{Q}'_i$ and $\mathcal{Q}'_j$. Assume that Lemma 3.21(a) holds. Then there is an (s, l)-constellation $\mathfrak{c}$ in G such that either $S_{\mathfrak{c}} \subseteq V(\mathcal{Q}'_i)$ and $\mathcal{L}_{\mathfrak{c}} \subseteq \mathcal{Q}'_j$, or $S_{\mathfrak{c}} \subseteq V(\mathcal{Q}'_i)$ and $\mathcal{L}_{\mathfrak{c}} \subseteq \mathcal{Q}'_j$. But $\mathcal{Q}'_i$ and $\mathcal{Q}'_j$ are both plain, and so in either case $\mathfrak{c}$ is a plain (s, l)-constellation in G, a contradiction. It follows that Lemma 3.21(a) holds, that is, there exist $\mathcal{Q}''_i \subseteq \mathcal{Q}'_i$ and $\mathcal{Q}''_j \subseteq \mathcal{Q}'$ with $|\mathcal{Q}''_i| = |\mathcal{Q}''_j| = b_1$ such that every vertex in $V(\mathcal{Q}''_i)$ has neighbors in fewer than l paths in $\mathcal{Q}''_j$, and every vertex in $V(\mathcal{Q}''_j)$ has neighbors in fewer than l paths in $\mathcal{Q}''_i$. Now, fix a path $Q_0 \in \mathcal{Q}''_i$. By the choice of i, j, for every $Q \in \mathcal{Q}''_j$, one may choose a path $R_Q \subseteq Q$ in G with (not necessarily distinct) ends u^1_Q and u^2_Q such that u^1_Q is the unique neighbor of x in Q (and so in R_Q) and u^2_Q is the only vertex in R_Q with a neighbor in Q_2. Since $\mathcal{Q}''_j$ is a plain polypath in G, it follows that $\mathcal{R} = \{R_Q : Q \in \mathcal{Q}''_j\}$ is a plain b_1-polypath in G. Moreover, by the choice of $\mathcal{Q}''_i$ and $\mathcal{Q}''_j$, the b_1-constellation $(\{u^2_Q : Q \in \mathcal{Q}''_j\}, Q_0)$ is l-meager. But then by the choice of b_1, we can apply Lemma 5.10 to x, Q_0 and $\mathcal{R}$ to deduce that there is a plain (s, l)-constellation in G, a contradiction. This proves (35).

Finally, the fact that $I \subseteq [c']$ with $|I| = c$ combined with (33) and (35) yields a contradiction with the assumption that G is (c, o)-pinched. This completes the proof of Theorem 5.3. ■

5.6 Dealing with a plain constellation

We now complete our proof of Theorem 5.1 by proving Theorem 5.4, restated below. But let us first show that:

Lemma 5.12. *Let $a, c, o \in \mathbb{N}$ and let G be a (c, o)-pinched graph. Assume that there exists an ample $a^{2co-1}(o + 2co)$-constellation $\mathfrak{a}$ in G. Then there exists an a-alignment $\mathfrak{s}$ in G with $S_{\mathfrak{s}} \subseteq S_{\mathfrak{a}}$ and $L_{\mathfrak{s}} \subseteq L^*_{\mathfrak{a}}$.*

Proof. Suppose not. Then from Theorem 5.9 applied to $\mathfrak{a}$, it follows that there is a plain and ample $(o, 2co)$-constellation $\mathfrak{c}$ in G such that $S_{\mathfrak{c}} \subseteq S_{\mathfrak{a}}$ and $L \subseteq L^*_{\mathfrak{a}}$ for every $L \in \mathcal{L}_{\mathfrak{c}}$. Let u be an end of $L_{\mathfrak{a}}$. Let $\mathcal{L}_{\mathfrak{c}} = \{L_i : i \in [2co]\}$ and for each $i \in [2co]$, let u_i, v_i be the ends of L_i, such that traversing $L_{\mathfrak{a}}$ starting at u, the vertices $u_1, v_1, \dots, u_{2co}, v_{2co}$ appear on $L_{\mathfrak{a}}$ in this order. For each $i \in [2co - 1]$, let v'_i be the neighbor of v_i in $L_{\mathfrak{a}} \setminus L_i$. Since $\mathfrak{c}$ is plain, it follows that v_i-$L_{\mathfrak{a}}$-u_{i+1} is a path of length at least two in L_0 whose interior is disjoint from $\{u_i, v_i : i \in [2co]\}$ and contains v'_i.

For every $i \in [2co]$, since $\mathfrak{c}$ is a constellation, every vertex in $S_{\mathfrak{c}}$ has a neighbor in L_i. Let R_i be the shortest path in L_i containing v_i such that every vertex in $S_{\mathfrak{c}}$ has a neighbor in R_i. Since $\mathfrak{c}$ is ample and $|S_{\mathfrak{c}}| = o$, it follows that for $|R_i| \geq o$ for all $i \in [2co]$. Let w_i be the end of R_i distinct from v_i. Then the minimality of R_i implies that there exists a vertex $x_i \in S_{\mathfrak{c}}$ which is adjacent to w_i and anticomplete to $R_i \setminus \{w_i\}$. Since $|S_{\mathfrak{c}}| = o$, it follows that there exist $x \in S_{\mathfrak{c}}$ as well as a $2c$-subset $\{i_k, j_k : k \in [c]\}$ of $[2co]$, such that $i_1 < j_1 < \dots < i_c < j_c$ and we have $x_{i_k} = x_{j_k} = x$ for all $k \in [c]$.

For each $k \in [c]$, traversing v'_{i_k}-$L_{\mathfrak{a}}$-w_{j_k} starting at v'_{i_k}, let w'_k be the first neighbor of x in v'_{i_k}-$L_{\mathfrak{a}}$-w_{j_k} (see Figure 5.9; note that w'_k exists because x is adjacent to w_{j_k}). Let

$$H_k = x\text{-}w_{i_k}\text{-}L_{\mathfrak{a}}\text{-}w'_k\text{-}x.$$

Then H_k is an induced cycle of length at least $o + 2$ in G because $R_k \cup \{v'_{i_k}\} \subseteq H_k$ and $|R_k| \geq o$. Now $H_1, \dots, H_c$ are c induced cycles of length at least $o + 2$ in G such that $V(H_1) \cap \dots \cap V(H_c) = \{x\}$. Also, note that $V(H_1) \setminus \{x\}, \dots, V(H_c) \setminus \{x\}$ are contained in pairwise distinct component of $L_{\mathfrak{a}} \setminus \{v'_{j_k} : k \in [c]\}$, and so $V(H_1) \setminus \{x\}, \dots, V(H_c) \setminus \{x\}$ are pairwise disjoint and anticomplete in G. This violates the assumption that G is (c, o)-pinched, hence completing the proof of Lemma 5.12. ■

Theorem 5.4. *For all $c, o, s, t \in \mathbb{N}$, there are constants $f_{5.4} = f_{5.4}(c, o, s, t) \in \mathbb{N}$ and $g_{5.4} = g_{5.4}(c, o, s, t) \in \mathbb{N}$ with the following property. Let G be a t-clean (c, o)-pinched graph. Suppose there is a plain $(f_{5.4}, g_{5.4})$ constellation in G. Then there is an (s, o)-array in G.*

Proof. Let

$$\sigma = s^{2co-1}(o + 2co);$$

$$\lambda = cs(s!)\sigma^s.$$

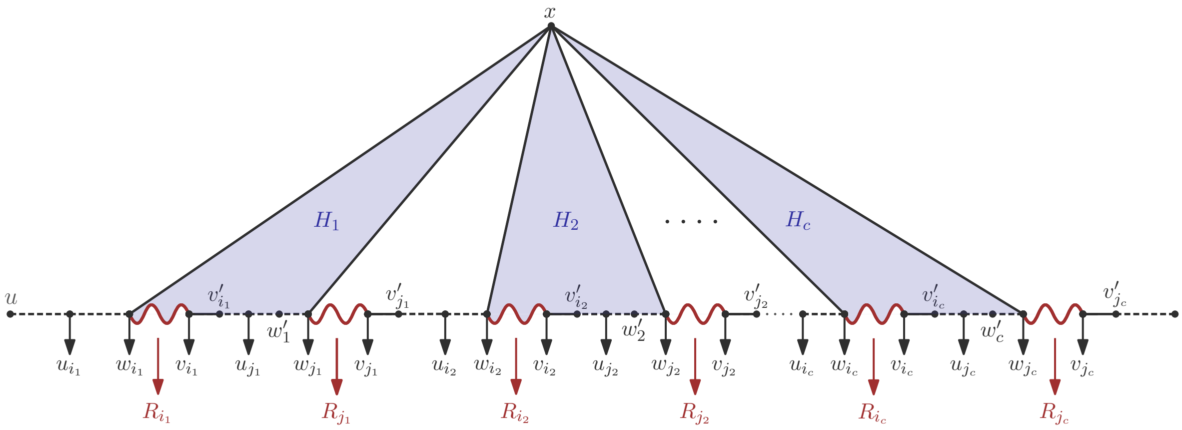

Figure 5.9: Proof of Lemma 5.12. Squiggly lines represent paths of length at least $o-1$, and the dashed lines depict paths of arbitrary (possibly zero) lengths.

We claim that:

$$f_{5.4} = f_{5.4}(c, o, s, t) = f_{3.20}(1, \lambda, \sigma, t);$$

$$g_{5.4} = g_{5.4}(c, o, s, t) = g_{3.20}(1, \lambda, \sigma, t);$$

satisfy the theorem.

Let G be a t-clean (c, o)-pinched graph and and assume that there is a plain $(f_{5.4}, g_{5.4})$ constellation $\mathfrak{c}$ in G. Since G is t-clean, it follows from Lemma 3.20 and the choices of $(f_{5.4}$ and $g_{5.4}$ that there exist $S_0 \subseteq S_{\mathfrak{c}}$ and $\mathcal{L}_0 \subseteq \mathcal{L}_{\mathfrak{c}}$ such that $(S_0, \mathcal{L}_0)$ is an ample (σ, λ)-constellation in G. Moreover, $(S_0, \mathcal{L}_0)$ is plain because $\mathfrak{c}$ is.

By the choice of σ, for each $L \in \mathcal{L}_0$, we can apply Lemma 5.12 to (S_0, L) to show that:

(36) *For every $L \in \mathcal{L}_0$, there exist $S_L \subseteq S_0$, $Q_L \subseteq L^*$ and an order π_L on S_L such that (S_L, Q_L) is an s-alignment in G with respect to π_L as in the definition.*

Moreover, since $|S_L| = s$ for all $L \in \mathcal{L}$, and since $|S_{\mathfrak{c}}| = \sigma$ and $|\mathcal{L}_0| = \lambda$, the choices of σ and λ along with (36) imply that:

(37) *There exists $S \subseteq S_0$ with $|S| = s$, $\mathcal{L} \subseteq \mathcal{L}_0$ with $|\mathcal{L}| = cs$ and an order π on S, such that for every $L \in \mathcal{L}$, (S, Q_L) is an s-alignment in G with respect to π as in the definition.*

We further deduce that:

(38) *There exists $\mathcal{S} \subseteq \mathcal{L}$ with $|\mathcal{S}| = s$ such that for every $L \in \mathcal{S}$, the s-constellation (S, Q_L) is o-hollow.*

By (37), we have $|S| = s$ and $|\mathcal{L}| = cs$. Therefore, in order to prove (38), it suffices to show that for every $x \in S$, there are fewer than c paths $L \in \mathcal{L}$ for which there is a path $P_L \subseteq Q_L$ of length at least o such that x is complete to $P_L \setminus P_L^*$ and anticomplete to P_L^*. Suppose for a contradiction that for some $x \in S$, there are c distinct paths $L_1, \dots, L_c \in \mathcal{L}$ such that for each $i \in [c]$, there is there is a path $P_{L_i} \subseteq Q_{L_i}$ as described above; let y_i, z_i be the ends of P_i. It follows that $H_i = x\text{-}y_i\text{-}P_{L_i}\text{-}z_i\text{-}x$ is an induced cycle of length at least $o + 2$ in G. Now $H_1, \dots, H_c$ are c induced cycles of length at least $o + 2$ in G with $V(H_1) \cap \dots \cap V(H_c) = \{x\}$. Moreover, since $\mathcal{L} \subseteq \mathcal{L}_\mathfrak{c}$ is a plain polypath in G, it follows that the sets

$$V(H_1) \setminus \{x\} \subseteq V(L_i)$$

for $i \in [c]$ are pairwise disjoint and anticomplete in G. This violates the assumption that G is (c, o)-pinched, and so proves (38).

Let $\mathfrak{a} = (S, \mathcal{S})$ where S comes from (37) and $\mathcal{S}$ comes from (38). Let π be as in (37). It follows from (37) and (38) that $\mathfrak{a}$ is an (s, o)-array in G. This completes the proof of Theorem 5.4. ■

Chapter 6

Grid Theorem for perforated graphs[1]

6.1 Asterisms

In this chapter, we give a proof of Theorem 2.10, or more exactly, the following strengthening to the case where the lengths of the excluded cycles are lower bounded:

Theorem 6.1. *For all* $c, o, s, t \in \mathbb{N}$*, there is a constant* $f_{6.1} = f_{6.1}(c, o, s, t) \in \mathbb{N}$ *such that for every* $(K_{t,t}, K_{t+1})$*-free* (c, o)*-perforated graph* G *with* $\operatorname{tw}(G) > f_{6.1}$*, there is a full* (s, o)*-occultation in* G*.*

There are two terms in the above statement that we need to define: "(c, o)-perforated graphs" and "full (s, o)-occultations." For $c, o \in \mathbb{N}$, we say a graph G is (c, o)*-perforated* if there are no c pairwise disjoint and anticomplete induced subgraphs $H_1, \dots, H_c$ of G, each being a cycle of length at least $o + 2$. It follows that a graph G is $(c, 1)$-perforated if and only if G is c-perforated.

Recall the definition of full s-occultations from Chapter 2, Section 2.3, and observe that a full s-occultations is in fact an s-constellation along with a specific order. The same is also the case for a "full (s, o)-occultation." However, we believe this definition, as well as most proofs in this chapter, will be presented best if we switch to a slightly restricted notion of an s-constellation which we call an "s-asterism." Let $s \in \mathbb{N}$ and let G be a graph. An s*-asterism*[2] *in* G is an s-constellation $\mathfrak{a}$ in G with the additional property that the ends of $L_{\mathfrak{a}}$ are anticomplete to $S_{\mathfrak{a}}$ in G. An *asterism* in G is an s-asterism in G for some $s \in \mathbb{N}$.

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[1] This chapter is based on the coauthored paper [5].

[2] From Wikipedia: An *asterism* is an observed pattern or group of stars in the sky.

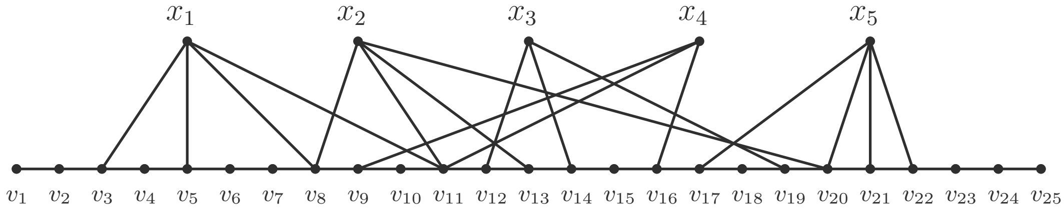

Figure 6.1: A 5-asterism $\mathfrak{a}$ with $S_{\mathfrak{a}} = \{x_1, \dots, x_5\}$ and $L_{\mathfrak{a}} = v_1\text{-}\cdots\text{-}v_{25}$. Note, for instance, that x_2-v_{13}-v_{14}-v_{15}-v_{16}-x_4 is an $\mathfrak{a}$-route that is not minimal, and x_2-v_{13}-v_{14}-x_3 is an $\mathfrak{a}$-route that is minimal. Also, there are exactly 15 $\mathfrak{a}$-pieces; 13 of which internal and v_1-v_2-v_3 and v_{22}-v_{23}-v_{24}-v_{25} are the two external $\mathfrak{a}$-pieces. For instance, v_{17}-v_{18}-v_{19} is an internal $\mathfrak{a}$-piece which is open, and v_5-v_6-v_7-v_8 is a closed $\mathfrak{a}$-piece (which is necessarily internal).

We emphasize the fact that every s-asterism is an s-constellation, and so all the notions and properties associated with s-constellations remain valid for s-asterisms.

But we need to add a few more. Let $\mathfrak{a}$ be an s-asterism in a graph G. By an *$\mathfrak{a}$-route* we mean a path in G with ends in $S_{\mathfrak{a}}$ and interior contained in $L_{\mathfrak{a}}$. An $\mathfrak{a}$-route R is *minimal* if there is no $\mathfrak{a}$-route Q with Q^* properly contained in R^*. By an *$\mathfrak{a}$-piece* we mean a path P in $L_{\mathfrak{a}}$ of non-zero length such that every end of P that is not an end of $L_{\mathfrak{a}}$ has a neighbor in $S_{\mathfrak{a}}$, while P^* and $S_{\mathfrak{a}}$ are anticomplete in G. An $\mathfrak{a}$-piece P is *internal* if P is contained in $L^*_{\mathfrak{a}}$; otherwise P is *external* (so there are exactly two external $\mathfrak{a}$-pieces). An $\mathfrak{a}$-piece P is said to be *open* if the ends of P have no common neighbor in $S_{\mathfrak{a}}$, and *closed* if the ends of P have a common neighbor in $S_{\mathfrak{a}}$. It follows that every external $\mathfrak{a}$-piece is open. In fact, one may observe that an $\mathfrak{a}$-piece P is open if and only if either P is external or P is the interior of a minimal $\mathfrak{a}$-route (see Figure 6.1).

For a fixed order π on $\mathfrak{a}$ and every $i \in \mathbb{N}$, we write $\mathfrak{a}^i = \mathfrak{a}|\pi([i])$. We say $\mathfrak{a}$ is *interrupted with respect to* π if for every $i \in [s-1]$:

(INT) $\pi(i+1)$ has at least one neighbor in every open $\mathfrak{a}^i$-piece.

Similarly, we say $\mathfrak{a}$ is *invaded with respect to* π if for every $i \in [s-1]$:

(INV) $\pi(i+1)$ has at least one neighbor in every closed $\mathfrak{a}^i$-piece.

When the order π is not specified, we also say $\mathfrak{a}$ is *interrupted* to mean $\mathfrak{a}$ is interrupted with respect to some order π on $\mathfrak{a}$, and we say $\mathfrak{a}$ is *invaded* to mean $\mathfrak{a}$ is invaded with respect to some order π.

We are almost ready to define "full (s, o)-occultations." For the sake of clarity, let us first re-define full s-occultations in terms of asterisms. Given $s \in \mathbb{N}$ and a graph G, by a *full s-occultation in G* we mean an ample s-asterism $\mathfrak{o}$ in G for which there is an order π on $\mathfrak{o}$ such that $\mathfrak{o}$ is both interrupted and invaded with respect to π.

Equivalently, a full s-occultation in G is an ample s-asterism $\mathfrak{o}$ in G for which there is an order π on $\mathfrak{o}$ such that for every $i \in [s-1]$, $\pi(i+1)$ has at least one neighbor in (the interior of) every $\mathfrak{o}^i$-piece. It is straightforward to check that this definition is equivalent to the one mentioned in Section 2.3, modulo the convenient nuance of viewing a full occultation as an induced subgraph of a fixed graph G with its "(S, L) partition" given, rather than independently defined graphs.

In this language, it turns out that "full (s, o)-occultations" are nothing but full s-occultation with "relaxed invadedness." Let $o, s \in \mathbb{N}$ and let G be a graph. For an s-asterism $\mathfrak{a}$ in G along with an order π on $\mathfrak{a}$, we say $\mathfrak{a}$ is *o-invaded with respect to π* if for every $i \in [s-1]$:

(OI) $\pi(i+1)$ has at least one neighbor in each closed $\mathfrak{a}^i$-piece of length at least o.

When the order π is not specified, we also say $\mathfrak{a}$ is *o-invaded* if $\mathfrak{a}$ is o-invaded with respect to some order π on $\mathfrak{a}$. In particular, $\mathfrak{a}$ is 1-invaded if and only if $\mathfrak{a}$ is invaded.

We now give the main definition. Let $o, s \in \mathbb{N}$ and let G be a graph. By a *full (s, o)-occultation in G*, we mean an ample ordered s-asterism $\mathfrak{o}$ in G for which there is an order π on $\mathfrak{o}$ such that $\mathfrak{o}$ is both interrupted and o-invaded with respect to π. In particular, $\mathfrak{o}$ is a full $(s, 1)$-occultation in G if and only if $\mathfrak{o}$ is a full s-occultation in G. There is also an analogue of Theorem 2.9 for full (s, o)-occultations:

Theorem 6.2. *Let $g, o, s \in \mathbb{N}$ and let G be a graph. Then the following hold.*

(a) *Let $\mathfrak{o}$ be a full $(g+s-1, o)$-occultation in G. Then $\mathfrak{o}^s$ is a full (s, o)-occultation in G and $G[V(\mathfrak{o}^s)]$ has girth at least $g+2$.*

(b) *Let $\mathfrak{o}$ be a full (s, o)-occultation in G. Then $G[V(\mathfrak{o})]$ is a $(K_{3,3}, K_4)$-free $(2, o)$-perforated graph of treewidth least $s-1$.*

Proof. We leave the proof of 6.2(a) to the reader as it is easy, and will only give a proof of 6.2(b), which is almost identical to the proof of Theorem 2.9 in [13]. Let $G' = G[V(\mathfrak{o})]$ and let π be an order on $\mathfrak{o}$ with such that $\mathfrak{o}$ is both interrupted and o-invaded. Then G' contains a subdivision of an s-occultation as a subgraph, and so by Theorem 2.9, G' has treewidth at least $s-1$. Now suppose for a contradiction that there are two disjoint and anticomplete induced subgraphs H_1 and H_2 of G', each being a cycle of length at least

$o+2$. It follows that for each $i \in \{1,2\}$, neither $V(H_i) \cap S_{\mathfrak{o}}$ nor $V(H_i) \cap L_{\mathfrak{o}}$ is empty. Let $i \in [s]$ be maximum with $\pi(i) \in V(H_1) \cup V(H_2)$, and without loss of generality, assume that $\pi(i) \in H_2$. Let P be a connected component of $H_1[V(H_1) \cap V(L_{\mathfrak{o}})]$. Then we may write $\partial P = \{u, v\}$ such that for some choice of $j, k \in [i-1]$, u is adjacent $\pi(j)$ in G and v is adjacent $\pi(k)$ in G. It follows that either P contains the interior of minimal $\mathfrak{o}^{i-1}$-route, that is, P contains an open $\mathfrak{o}^{i-1}$-piece P', or $j = k$ and P is a closed $\mathfrak{o}^{i-1}$-piece. In the former case, since $\mathfrak{o}$ is interrupted, by (INT), $\pi(i) \in V(H_2)$ has a neighbor in $P' \subseteq P \subseteq V(H_1)$. But this violates the assumption that H_1 and H_2 are anticomplete in G. Also, in the latter case, we have $H_1 = \pi(j)$-u-P-v-$\pi(j)$. Since H_1 has length at least $o+2$, it follows that P has length at least o. More precisely, P is a closed $\mathfrak{o}^{i-1}$-piece of length at least o. But then since $\mathfrak{o}$ is o-invaded, by (OI), the vertex $\pi(i) \in V(H_2)$ has a neighbor in $P \subseteq V(H_1)$, again a contradiction with H_1 and H_2 being anticomplete. This completes the proof of Theorem 6.2. ■

We remark that Theorem 2.10 is the special case of Theorem 6.1 when $o = 1$. By Theorem 6.2(b), for every $c, o \in \mathbb{N}$, Theorem 6.1 gives a grid-type theorem for the class of (c, o)-perforated graphs. Indeed, to make sure that all three outcomes of Theorem 6.1 are necessary, we may combine Theorem 6.1 and Theorem 6.2(a) to obtain the following more "efficient" version of Theorem 6.1:

Corollary 6.3. *For all $c, g, o, s, t \in$, there is a constant $f_{6.3} = f_{6.3}(c, g, o, s, t) \in \mathbb{N}$ such that for every $(K_{t,t}, K_{t+1})$-free (c, o)-perforated graph G with $\operatorname{tw}(G) > f_{6.3}$, there is a full (s, o)-occultation $\mathfrak{o}$ in G such that $G[V(\mathfrak{o})]$ has girth more than $g+2$.*

6.2 Outline of the main proof

The proof of Theorem 6.1 follows the same trajectory as that of the proof of Theorem 5.1, except the second step will be split into two sub-steps.

Similar to the proof of Theorem 5.1, our first step is to prove that every $(K_{t,t}, K_{t+1})$-free (c, o)-perforated graph of sufficiently large treewidth contains a large plain constellation:

Theorem 6.4. *For all $c, l, o, s, t \in \mathbb{N}$, there is a constant $f_{6.4} = f_{6.4}(c, l, o, s, t) \in \mathbb{N}$ such that if G is a $(K_{t,t}, K_{t+1})$-free (c, o)-perforated graph with $\operatorname{tw}(G) > f_{5.3}$, then there is a plain (s, l)-constellation in G.*

For the second step, we begin with showing that a sufficiently large constellation in a $(K_{t,t}, K_{t+1})$-free (c, o)-perforated graph G guarantees the existence of a large 2-ample and interrupted asterism in G:

Theorem 6.5. *For all integers $c, o, s, t \in \mathbb{N}$, there are constants $f_{6.5} = f_{6.5}(c, o, s, t) \in \mathbb{N}$ and $g_{6.5} = g_{6.5}(c, o, s, t) \in \mathbb{N}$ with the following property. Let G be a $(K_{t,t}, K_{t+1})$-free (c, o)-perforated graph. Assume that there exists a plain $(f_{6.5}, g_{6.5})$-constellation $\mathfrak{c}$ in G. Then there is an $(o + 1)$-ample, interrupted s-asterism $\mathfrak{a}$ in G such that $S_{\mathfrak{a}} \subseteq S_{\mathfrak{c}}$, and for some $L \in \mathcal{L}_{\mathfrak{c}}$, we have $L_{\mathfrak{a}} \subseteq L$.*

Then, we prove that a sufficiently large 2-ample and interrupted asterism in G yields the desired full occultation:

Theorem 6.6. *Let $c, o, s \in \mathbb{N}$ and let G be a (c, o)-perforated graph. Assume that there is an interrupted cs-asterism $\mathfrak{a}$ in G which is 2-ample. Then there is a full (s, o)-occultation $\mathfrak{o}$ in G with $S_{\mathfrak{o}} \subseteq S_{\mathfrak{a}}$ and $L_{\mathfrak{o}} \subseteq L_{\mathfrak{a}}$.*

With the above three results, Theorem 6.1 becomes immediate:

Theorem 6.1. *For all $c, o, s, t \in \mathbb{N}$, there is a constant $f_{6.1} = f_{6.1}(c, o, s, t) \in \mathbb{N}$ such that for every $(K_{t,t}, K_{t+1})$-free (c, o)-perforated graph G with $\mathrm{tw}(G) > f_{6.1}$, there is a full (s, o)-occultation in G.*

Proof. Let $f = f_{6.5}(c, o, cs, t)$ and let $g = g_{6.5}(c, o, cs, t)$. We will show that

$$f_{6.1} = f_{6.1}(c, o, s, t) = f_{6.4}(c, g, o, f, t)$$

satisfies the theorem. Let G be a $(K_{t,t}, K_{t+1})$-free (c, o)-perforated graph with $\mathrm{tw}(G) > f_{6.1}$. From the choice of $f_{6.1}$ and Theorem 6.4, it follows that there is a plain (f, g)-constellation in G. The choices of f and g in turn combined with Theorem 6.5 imply that there is an $(o+1)$-ample, interrupted s^c-asterism $\mathfrak{a}$ in G. In particular, since $o \geq 1$, it follows that $\mathfrak{a}$ is 3-ample. But now by Theorem 6.6, there is a full (s, o)-occultation in G. This completes the proof Theorem 6.1. ■

It remains to prove Theorems 6.4, 6.5 and 6.6, which we will, in the reverse order, in Sections 6.3 and 6.4 and 6.5.

6.3 The cherry on top

Recall that by definition, every full occultation is both ample and interrupted. Our goal in this section is to prove that in perforated graphs, a qualitative converse holds, too:

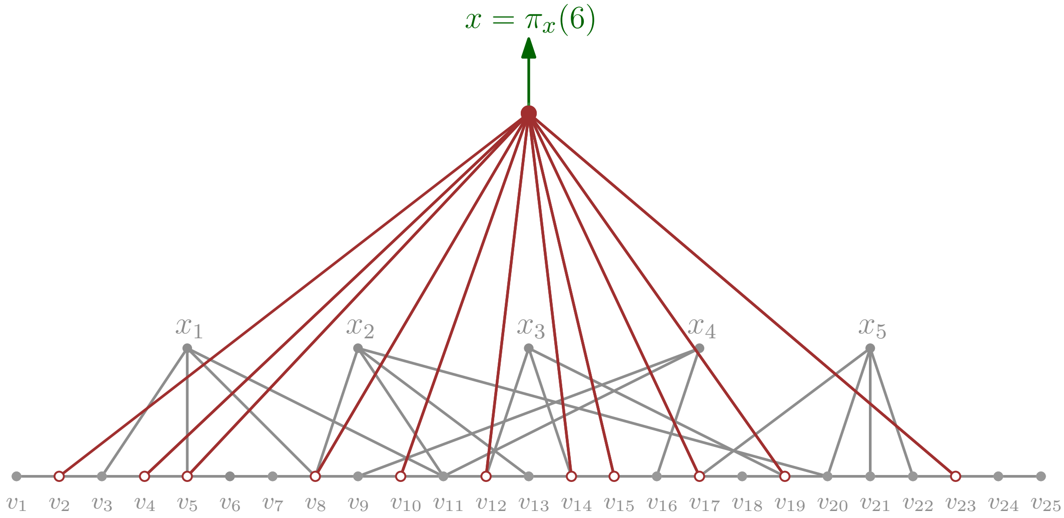

Figure 6.2: The vertex x is a cherry on top of the asterism $\mathfrak{a}$ from Figure 6.1.

Theorem 6.6. *Let $c, o, s \in \mathbb{N}$ and let G be a (c, o)-perforated graph. Assume that there is an interrupted cs-asterism $\mathfrak{a}$ in G which is 2-ample. Then there is a full (s, o)-occultation $\mathfrak{o}$ in G with $S_\mathfrak{o} \subseteq S_\mathfrak{a}$ and $L_\mathfrak{o} \subseteq L_\mathfrak{a}$.*

The proof of Theorem 6.6 calls for a few more definitions and lemmas. Let $s' \in \mathbb{N}$, let G be a graph and let $x \in V(G)$. Let $\mathfrak{a}'$ be an s'-asterism in G with $V(\mathfrak{a}') \subseteq V(G) \setminus \{x\}$. We say x is a *cherry on top of* $\mathfrak{a}'$ *in* G if:

(CH1) x is anticomplete in G to $S_{\mathfrak{a}'}$, and to the ends of $L_{\mathfrak{a}'}$; and

(CH2) x has a neighbor in every open $\mathfrak{a}'$-piece (and so x has a neighbor in $L_{\mathfrak{a}'}$).

See Figure 6.2. In this case, $(S_{\mathfrak{a}'} \cup \{x\}, L_{\mathfrak{a}'})$ is an $(s'+1)$-asterism in G, which we will denote by $\mathsf{Cher}(\mathfrak{a}', x)$. Moreover, if π is an order on $\mathfrak{a}$, we denote by π_x the order on $\mathsf{Cher}(\mathfrak{a}', x)$ where $\pi_x(s'+1) = x$ and $\pi_x(i) = \pi(i)$ for all $i \in [s']$. Note that by (CH2), $\mathsf{Cher}(\mathfrak{a}', x)$ is interrupted with respect to π_x if and only if $\mathfrak{a}'$ is interrupted with respect to π.

Observe that the notion of a "cherry on top" arises naturally from viewing interrupted ordered asterisms as ordered asterisms which can be constructed by "successively adding cherries on top." Said more carefully, it is straightforward to observe that given $s \in \mathbb{N}$ and a graph G, an s-asterism $\mathfrak{a}$ in G is interrupted with respect to an order π on $\mathfrak{a}$, if and only if for every $i \in [s-1]$, $\pi(i+1)$ is a cherry on top of $\mathfrak{a}^i$ in G.

Accordingly, our first lemma, which we will use both here and in Section 6.4, provides a tool for growing interrupted "sub-asterisms" of a given asterism by adding one cherry on top at a time. It roughly says the following: Let G be a graph, let $\mathfrak{a}$ be an asterism in G which is 2-ample and let $x \in S_{\mathfrak{a}}$. Let $S' \subseteq S_{\mathfrak{a}} \setminus \{x\}$ such that x is a cherry on top of $\mathfrak{a}|S' = (S', L_{\mathfrak{a}})$ in G. Then the same property is also inherited by an "optimally" chosen subpath $L' \subseteq L_{\mathfrak{a}}$, that is, x is a cherry on top of the asterism (S', L') in G.

This notion of "optimality" is defined as follows. Let $s' \in \mathbb{N}$, let G be a graph, let $\mathfrak{a}$ be an asterism in G and let $x \in S_{\mathfrak{a}}$. By an $(\mathfrak{a}, x, s')$-*candidate in G* we mean an interrupted s'-asterism $\mathfrak{a}'$ in G with $S_{\mathfrak{a}'} \subseteq S_{\mathfrak{a}} \setminus \{x\}$ and $L_{\mathfrak{a}'} \subseteq L_{\mathfrak{a}}$, such that for some order π on $\mathfrak{a}'$, $\mathfrak{a}'$ is interrupted with respect to π, and the following hold.

(CA) Let L be a path in G with $V(L_{\mathfrak{a}'}) \subsetneq V(L) \subseteq V(L_{\mathfrak{a}})$ such that $(S_{\mathfrak{a}'}, L)$ is an asterism in G which is interrupted with respect to π. Then there exists $i \in [s'-1]$ such that $\pi(i)$ has a neighbor in the interior of L (and so we have $s' \geq 2$).

We deduce that:

Lemma 6.7. *Let G be a graph and let $\mathfrak{a}$ be a 2-ample s-asterism in G for some integer $s \geq 2$. Let $x \in S_{\mathfrak{a}}$ and let $\mathfrak{a}'$ be an $(\mathfrak{a}, x, s-1)$-candidate in G with the order π on $\mathfrak{a}'$ as in (CA). If x is a cherry on top of $\mathfrak{a}|S_{\mathfrak{a}'}$ in G, then x is a cherry on top of $\mathfrak{a}'$ in G.*

Proof. We need to show that $\mathfrak{a}'$ and x satisfy (CH1) and (CH2). Note that $S_{\mathfrak{a}'} = S_{\mathfrak{a}} \setminus \{x\}$. First, we show that:

(39) *Let u be an end of $L_{\mathfrak{a}'}$ that is not an end of $L_{\mathfrak{a}}$. Then the unique neighbor of u in $L_{\mathfrak{a}} \setminus L_{\mathfrak{a}'}$ has a neighbor in $\pi([s-2])$.*

Suppose not. Let v be the ends of $L_{\mathfrak{a}'}$ other than u and let u' be the unique neighbor of u in $L_{\mathfrak{a}} \setminus L_{\mathfrak{a}'}$. Then u' is anticomplete to $\pi_{\mathfrak{a}'}([s-2])$. Since $\mathfrak{a}'$ is an $(\mathfrak{a}, x, s-1)$-candidate in G and u is not an end of $L_{\mathfrak{a}}$, applying (CA) to $L = L_{\mathfrak{a}}$ implies that $s \geq 3$. Also, u' is adjacent to $x' = \pi(s-1)$, as otherwise $L = v\text{-}L_{\mathfrak{a}'}\text{-}u\text{-}u'$ violates (CA). Let u'' be the end of $L_{\mathfrak{a}}$ for which $u\text{-}L_{\mathfrak{a}}\text{-}u''$ contains u'. Since x' is adjacent to u' and x' is not adjacent to u'', traversing $u'\text{-}L_{\mathfrak{a}}\text{-}u''$ from u' to u'', we may choose the first vertex w which is not adjacent to x'. It follows that $u' \neq w$ but u'' and w might be the same. Let w' be the unique neighbor of w in $u'\text{-}L_{\mathfrak{a}}\text{-}w$ (so u' and w' might be the same). Since $\mathfrak{a}$ is 2-ample and x' is complete to $u'\text{-}L_{\mathfrak{a}}\text{-}w'$, it follows that $\pi([s-2])$ is anticomplete to $u'\text{-}L_{\mathfrak{a}}\text{-}w$. But now $L = v\text{-}L_{\mathfrak{a}'}\text{-}u\text{-}u'\text{-}L_{\mathfrak{a}}\text{-}w$ violates (CA). This proves (39).

From (39) and the fact that $\mathfrak{a}$ is 2-ample, it follows that x is anticomplete to the ends of $L_{\mathfrak{a}'}$, and so $\mathfrak{a}'$ and x satisfy (CH1). Also, we claim that:

(40) *Let P be an open $\mathfrak{a}'$-piece. Then x has a neighbor in P.*

First, assume that P is an internal open $\mathfrak{a}'$-piece. Then P is an open $\mathfrak{a}|S_{\mathfrak{a}'}$-piece. Since x is a cherry on top of $\mathfrak{a}|S_{\mathfrak{a}'}$ in G, it follows from (CH2) that x has a neighbor in P, as desired. Next, assume that P is an external $\mathfrak{a}'$-piece. Then P and $L_{\mathfrak{a}'}$ share at least one end, say u. By (39), either u is an end of $L_{\mathfrak{a}}$, or the unique neighbor u' of u in $L_{\mathfrak{a}} \setminus L_{\mathfrak{a}'}$ is adjacent to $\pi(i)$ for some $i \in [s-2]$. In the former case, P is an external $\mathfrak{a}|S_{\mathfrak{a}'}$-piece, and so P is an open $\mathfrak{a}|S_{\mathfrak{a}'}$-piece. Again, since x is a cherry on top of $\mathfrak{a}|S_{\mathfrak{a}'}$ in G, it follows from (CH2) that x has a neighbor in P. In the latter case, traversing $L_{\mathfrak{a}'}$ starting at u, let u'' be the first vertex with a neighbor in $S_{\mathfrak{a}'}$. Since $\mathfrak{a}'$ is interrupted, it follows that u'' is a neighbor of $\pi(s-1)$, and so there exists an $\mathfrak{a}|S_{\mathfrak{a}'}$-route R from $\pi(s-1)$ to $\pi(i)$ such that $P = R^* \setminus \{u'\}$. Note that since $\mathfrak{a}$ is d-ample, the ends of R^* have no common neighbor in $S_{\mathfrak{a}'}$, and so R^* is an open $\mathfrak{a}|S_{\mathfrak{a}'}$-piece. Therefore, since x is a cherry on top of $\mathfrak{a}|S_{\mathfrak{a}'}$ in G, it follows from (CH2) that x has a neighbor in R^*. On the other hand, since $x'' \in S_{\mathfrak{a}} \setminus \{x\}$ is adjacent to u' and $\mathfrak{a}$ is d-ample, it follows that x is not adjacent to u'. But now x has a neighbor in P. This proves (40).

By (40), $\mathfrak{a}'$ and x satisfy (CH2). This completes the proof of Lemma 6.7. ■

We also need the following, which is an application of Lemma 6.7:

Lemma 6.8. *Let $o, s, \mu \in \mathbb{N}$ such that $\mu > s-1 \geq 1$. Let G be a graph and let $\mathfrak{a}$ be a 2-ample μ-asterism in G which is interrupted with respect to some order π on $\mathfrak{a}$. Let $S' \subseteq \pi([\mu-1])$ such that $\pi(\mu)$ has a neighbor in every closed $\mathfrak{a}|S'$-piece of length at least o. Assume that there is a full $(s-1, o)$-occultation $\mathfrak{o}'$ in G with $S_{\mathfrak{o}'} \subseteq S'$ and $L_{\mathfrak{o}'} \subseteq L_{\mathfrak{a}}$. Then there exists a full (s, o)-occultation $\mathfrak{o}$ in G with $S_{\mathfrak{o}} \subseteq S_{\mathfrak{a}}$ and $L_{\mathfrak{o}} \subseteq L_{\mathfrak{a}}$.*

Proof. We write $x = \pi(\mu)$. Choose a full $(s-1, o)$-occultation $\mathfrak{o}'$ in G with $S_{\mathfrak{o}'} \subseteq S'$ and $L_{\mathfrak{o}'} \subseteq L_{\mathfrak{a}}$, such that $L_{\mathfrak{o}'}$ is maximal with respect to inclusion. In particular, $\mathfrak{o}'$ is ample, interrupted and o-invaded with respect to some order π' on $\mathfrak{o}'$, and we have $S_{\mathfrak{o}'} \subseteq S' \subseteq S_{\mathfrak{a}} \setminus \{x\}$. We further deduce that:

(41) *$\mathfrak{o}'$ is an $(\mathfrak{a}, x, s-1)$-candidate in G and x is a cherry on top of $\mathfrak{a}|S_{\mathfrak{o}'}$ in G.*

From the maximality of $L_{\mathfrak{o}'}$, it follows immediately that $\mathfrak{a}$, x, $\mathfrak{o}'$ and π' satisfy (CA). So $\mathfrak{o}'$ is an $(\mathfrak{a}, x, s-1)$-candidate in G. It remains to show that x is a cherry on top of $\mathfrak{a}|S_{\mathfrak{o}'}$.

To that end, we need to argue that $\mathfrak{a}|S_{\mathfrak{o}'}$ and x satisfy (CH1) and (CH2). Observe that (CH1) follows immediately from the fact that $L_{\mathfrak{a}|S_{\mathfrak{o}'}} = L_{\mathfrak{a}}$. For (CH2), let P be an open $\mathfrak{a}|S_{\mathfrak{o}'}$-piece. Since $S_{\mathfrak{o}'} \subseteq S' \subseteq S_{\mathfrak{a}^{\mu-1}}$, it follows that P contains an open $\mathfrak{a}^{\mu-1}$-piece P'. But now since $\mathfrak{a}$ is interrupted with respect to π, it follows that $x = \pi(\mu)$ has a neighbor in the open $\mathfrak{a}^{\mu-1}$-piece P', and so x has a neighbor in P. This proves (41).

In view of (41), we can apply Lemma 6.7 to $\mathfrak{a}, \mathfrak{o}'$ and x, and deduce that x is a cherry on top of $\mathfrak{o}'$. It follows that $\mathfrak{o} = \mathsf{Cher}(\mathfrak{o}', x)$ is a 2-ample ordered s-asterism in G, interrupted with respect to π'_x, such that $S_{\mathfrak{o}} \subseteq S_{\mathfrak{a}}$ and $L_{\mathfrak{o}} \subseteq L_{\mathfrak{a}}$. Also, we have:

(42) $\mathfrak{o}$ *is o-invaded with respect to* π'_x.

We need to prove that $\mathfrak{o}$ and π'_x satisfy (OI) for every $i \in [s-1]$. This is immediate for $i \in [s-1]$ as $\mathfrak{o}'$ is o-invaded with respect to π'. For $i = s$, let P be a closed $\mathfrak{o}^{s-1}$-piece of length at least o. Our goal is to show that $\pi'_x(s) = x$ has a neighbor in P. Since $S_{\mathfrak{o}^{s-1}} = S_{\mathfrak{o}'} \subseteq S' \subseteq S_{\mathfrak{a}^{\mu-1}}$, it follows that either P is a closed $\mathfrak{a}|S'$-piece, or P contains an open $\mathfrak{a}|S'$-piece, which in turn implies that P contains an open $\mathfrak{a}^{\mu-1}$-piece P'. In the former case, x has a neighbor in P due to the assumption that $\pi(\mu) = x$ has a neighbor in every closed $\mathfrak{a}|S'$-piece of length at least o. In the latter case, since $\mathfrak{a}$ is interrupted, it follows that $x = \pi(\mu)$ has a neighbor in the open $\mathfrak{a}^{\mu-1}$-piece P', and so x has a neighbor in P. This proves (42).

In conclusion, we have shown that $\mathfrak{o}$ is a 2-ample s-asterism in G, interrupted and o-invaded with respect to π'_x, such that $S_{\mathfrak{o}} \subseteq S_{\mathfrak{a}}$ and $L_{\mathfrak{o}} \subseteq L_{\mathfrak{a}}$. Hence, $\mathfrak{o}$ is a full (s, o)-occultation in G with $S_{\mathfrak{o}} \subseteq S_{\mathfrak{a}}$ and $L_{\mathfrak{o}} \subseteq L_{\mathfrak{a}}$. This completes the proof of Lemma 6.8. ■

We can now prove the main result of this section, which we restate:

Theorem 6.6. *Let $c, o, s \in \mathbb{N}$ and let G be a (c, o)-perforated graph. Assume that there is an interrupted cs-asterism $\mathfrak{a}$ in G which is 2-ample. Then there is a full (s, o)-occultation $\mathfrak{o}$ in G with $S_{\mathfrak{o}} \subseteq S_{\mathfrak{a}}$ and $L_{\mathfrak{o}} \subseteq L_{\mathfrak{a}}$.*

Proof. For fixed c, we proceed by induction on s. The result is immediate when $s = 1$. Assume that $s \geq 2$. Write $\mu = cs$; then $\mu > s - 1 \geq 1$. Let us first show that:

(43) *Let $i, i' \in [\mu - 1]$ be distinct, let P_i be a $\pi(i)$-gap in $\mathfrak{a}$ and let $P_{i'}$ be a $\pi(i')$-gap in $\mathfrak{a}$ such that x is anticomplete to $P_i \cup P_{i'}$ in G. Then $P_i \cup \{\pi(i)\}$ and $P_{i'} \cup \{\pi(i')\}$ are disjoint and anticomplete in G.*

Suppose not. Then, since $\mathfrak{a}$ is 2-ample, it follows that there is a path R (of non-empty interior) in G from $\pi(i)$ to $\pi(i')$ such that either $R^* \subseteq P_i$ or $R^* \subseteq P_{i'}$. As a result, there is an (internal) open $\mathfrak{a}^{\mu-1}$-piece P such that either $P \subseteq P_i$ or $P \subseteq P_{i'}$. On the other hand, since $\mathfrak{a}$ is interrupted with respect to π, it follows that $x = \pi(\mu)$ has a neighbor in the open $\mathfrak{a}^{\mu-1}$-piece P. But then x has a neighbor in $P_i \cup P_{i'}$, a contradiction. This proves (43).

Let I be the set of all $i \in [\mu - 1]$ for which there is a $\pi(i)$-gap P_i of length at least o in $\mathfrak{a}$ such that x is anticomplete to P_i in G. From (43), we deduce that:

(44) $|I| < c$.

Suppose not. Then we may choose a c-subset I' of $[\mu - 1]$ where for each $i \in [\mu - 1]$, there is a $\pi(i)$-gap P_i of length at least o in $\mathfrak{a}$ such that x is anticomplete to P_i in G. But now by (43), $(P_i \cup \{\pi(i)\} : i \in I')$ are c pairwise disjoint and anticomplete induced subgraphs of G, each of which is a cycle of length at least $o + 2$, a contradiction with the assumption that G is (c, o)-perforated. This proves (44).

We are almost done. By (44), there exists $S' \subseteq \pi(\mu - 1)$ with $|S'| = \mu - 1 - (c - 1) = c(s - 1)$ such that for every $x' \in S'$, the vertex $x = \pi(\mu)$ has at least one neighbor in every x'-gap in $\mathfrak{a}$. In particular, $\pi(\mu)$ has a neighbor in every closed $\mathfrak{a}|S'$-piece of length at least o. Moreover, since $\mathfrak{a}$ is interrupted and 2-ample, it follows that $\mathfrak{a}|S'$ is a $c(s - 1)$-asterism in G which is interrupted and 2-ample. Therefore, by the induction hypothesis, there is a full $(s - 1, o)$-occultation $\mathfrak{o}'$ in G with $S_{\mathfrak{o}'} \subseteq S_{\mathfrak{a}|S'} = S'$ and $L_{\mathfrak{o}'} \subseteq L_{\mathfrak{a}|S'} = L_{\mathfrak{a}}$. But now we can apply Lemma 6.8 to $\mathfrak{a}, S'$ and $\mathfrak{o}$ to deduce that there is a full (s, o)-occultation $\mathfrak{o}$ in G with $S_{\mathfrak{o}} \subseteq S_{\mathfrak{a}}$ and $L_{\mathfrak{o}} \subseteq L_{\mathfrak{a}}$. This completes the proof of Theorem 6.6. ■

6.4 The transition graph

The following definition is central to our discussion in this section. Let G be a graph and let $\mathfrak{a}$ be an asterism in G. By the transition graph of $\mathfrak{a}$, denoted $\mathsf{T}_{\mathfrak{a}}$, we mean the graph with vertex set $S_{\mathfrak{c}}$ such that $x, y \in S_{\mathfrak{c}}$ are adjacent in $\mathsf{T}_{\mathfrak{c}}$ if and only if there is an $\mathfrak{a}$-route R in G from x to y such that every vertex in $S_{\mathfrak{a}} \setminus \{x, y\}$ is anticomplete to R^* (and so to R) in G (see Figure 6.3).

Our goal is to prove Theorem 6.5, restated below:

Theorem 6.5. *For all integers $c, o, s, t \in \mathbb{N}$, there are constants $f_{6.5} = f_{6.5}(c, o, s, t) \in \mathbb{N}$ and $g_{6.5} = g_{6.5}(c, o, s, t) \in \mathbb{N}$ with the following property. Let G be a $(K_{t,t}, K_{t+1})$-free (c, o)-perforated graph. Assume that there exists a plain $(f_{6.5}, g_{6.5})$-constellation $\mathfrak{c}$ in G. Then there is an $(o + 1)$-ample, interrupted s-asterism $\mathfrak{a}$ in G such that $S_{\mathfrak{a}} \subseteq S_{\mathfrak{c}}$, and for some $L \in \mathcal{L}_{\mathfrak{c}}$, we have $L_{\mathfrak{a}} \subseteq L$.*

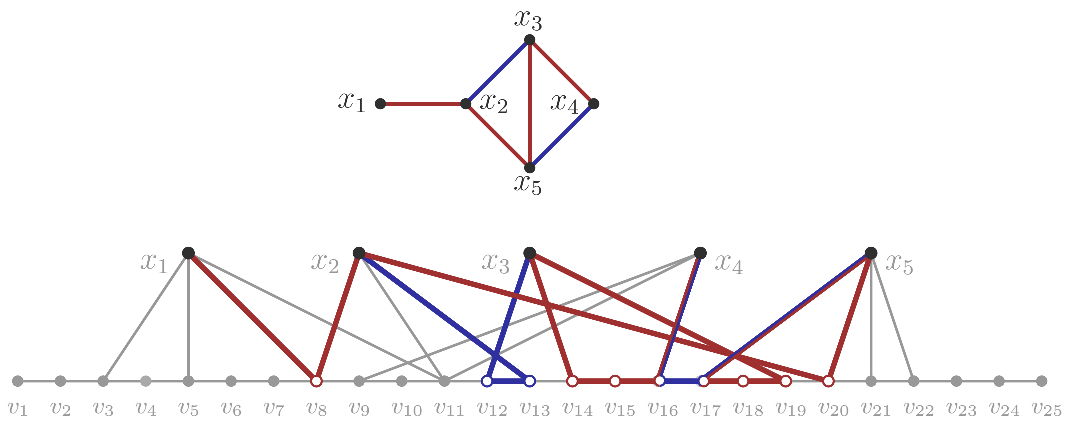

Figure 6.3: The transition graph $\mathsf{T}_{\mathfrak{a}}$ of the the 5-asterism $\mathfrak{a}$ from Figure 6.1.

Proof. Let $c, o, t \in \mathbb{N}$ be fixed. The definitions of $f_{6.5}$ and $g_{6.5}$ rely on Theorems 3.8 and 3.14 and Lemma 3.20. First, we define two sequences $\{\sigma_i : i \in \mathbb{N}\}$ and $\{\lambda_i : i \in \mathbb{N}\}$ of positive integers recursively, as follows. Let

$$t' = \max\{c(o+1), t\};$$

$$\sigma_1 = f_{3.20}(1, 1, 3, t');$$

$$\lambda_1 = g_{3.20}(1, 1, 3, t').$$

For $i \in \mathbb{N} \setminus \{1\}$, assuming σ_{i-1} and λ_{i-1} is defined, let

$$f_3 = (\sigma_{i-1} + 1)^{c(\sigma_{i-1}+1)}$$

$$f_2 = f_{3.14}(2, \lambda_{i-1} + 1, f_3);$$

$$f_1 = f_{3.8}(2f_3, 2, 2^{2c} - 1);$$

let

$$g_2 = 2^{2c} f_1^{2f_3} f_2 f_3;$$

$$g_1 = \lambda_1(c, o, s, t) = 2^{(2c+f_1)^2} g_2;$$

and let

$$\sigma_i = f_{3.20}(o + 1, g_1, 2c + f_1 + 2g_1, t');$$

$$\lambda_i = g_{3.20}(o + 1, g_1, 2c + f_1 + 2g_1, t').$$

This concludes the recursive definition of $\{\sigma_i : i \in \mathbb{N}\}$ and $\{\lambda_i : i \in \mathbb{N}\}$.

We will prove, by induction on $s \in \mathbb{N}$ for fixed $c, o, t \in \mathbb{N}$, that

$$f_{6.5} = f_{6.5}(c, o, s, t) = \sigma_s;$$

$$g_{6.5} = g_{6.5}(c, o, s, t) = \lambda_s;$$

satisfy the theorem.

First, observe that neither a subdivision of the wall $W_{c(o+1)\times c(o+1)}$ nor the line graph of a subdivision of the $W_{c(o+1)\times c(o+1)}$ is (c, o)-perforated. Now, let $\mathfrak{c}$ be a plain (σ_s, λ_s)-constellation in a $(K_{t,t}, K_{t+1})$-free (c, o)-perforated graph G. Then from the choice of t', it follows that G is t'-clean, and so by the choices of σ_s and λ_s, we can apply Lemma 3.20 to $\mathfrak{c}$.

If $s = 1$, then it follows that there exists an ample 3-constellation (S, L_1) in G such that $S \subseteq S_{\mathfrak{c}}$, and for some $L \in \mathcal{L}_{\mathfrak{c}}$, we have $L_1 \subseteq L$. In particular, each end of L_1 has at most one neighbor in S, and there exists $x \in S$ such that $\mathfrak{a} = (\{x\}, L_1)$ is a 1-asterism in G. But now we are done because a 1-asterism is automatically $(o+1)$-ample and interrupted.

Consequently, we may assume that $s \geq 2$, and there exist $S_1 \subseteq S_{\mathfrak{c}}$ with $|S| = 2c+f_1+2g_1$ and $\mathcal{L}_1 \subseteq \mathcal{L}_{\mathfrak{c}}$ with $|\mathcal{L}_1| = g_1$ such that $(S_1, \mathcal{L}_1)$ is an $(o+1)$-ample $(2c+f_1+2g_1, g_1)$-constellation in G. In particular, each end of every path in $\mathcal{L}_1$ has at most one neighbor in S_1. Thus, there exists $S \subseteq S_1 \subseteq S_{\mathfrak{c}}$ with $|S| = 2c + f_1$ such that $\mathfrak{f} = (S, \mathcal{L}_1)$ is an $(o+1)$-ample $(2c + f_1, g_1)$-constellation in G where each end of every path in $\mathcal{L}_1$ is anticomplete to S. In other words, $\mathfrak{f}_L$ is an $(o+1)$-ample $(2c + f_1)$-asterism in G for every $L \in \mathcal{L}_1$.

Note that $\mathfrak{f}$ is plain because $\mathfrak{c}$ is. Moreover, since $|\mathcal{L}_1| = g_1 = 2^{|S|^2} g_2$, it follows that:

(45) *There exists $\mathcal{L}_2 \subseteq \mathcal{L}_1$ with $|\mathcal{L}_2| = g_2$ as well as $E_2 \subseteq S^{(2)}$, such that for every $L \in \mathcal{L}_2$, we have $E(\mathsf{T}_{\mathfrak{f}_L}) = E_2$.*

Henceforth, let $\mathcal{L}_2$ and E_2 be as in (45). Let T be the graph with $V(\mathsf{T}) = S$ and $E(\mathsf{T}) = E_2$. Then (45) implies that $\mathsf{T}_{\mathfrak{f}_L} = \mathsf{T}$ for every $L \in \mathcal{L}_2$. We prove that:

(46) *There is no matching of cardinality c in T.*

Suppose for a contradiction that there exists a matching $\{x_1x'_1, \dots, x_cx'_c\} \subseteq E(\mathsf{T})$ of cardinality c in T. Since $|\mathcal{L}_2| = g_2 \geq 2$, we may choose two distinct paths $L_1, L_2 \in \mathcal{L}_2$. Then we have $\{x_1x'_1, \dots, x_cx'_c\} \subseteq E(\mathsf{T}_{\mathfrak{f}_{L_1}}) = E(\mathsf{T}_{\mathfrak{f}_{L_2}})$. Since $\mathfrak{f}_{L_1}$ and $\mathfrak{f}_{L_2}$ are both $(o+1)$-ample, $\mathfrak{f}$ is plain, and from the definition of the transition graph, it follows that for each

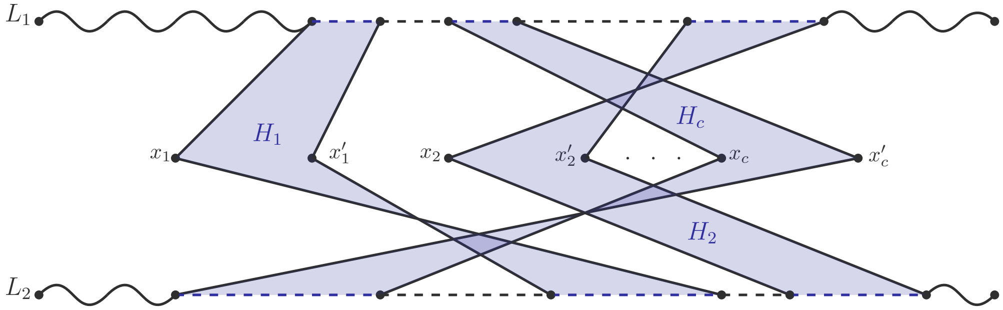

Figure 6.4: Proof of (46) (each dashed segment represents a path of length at least $o+1$ and each squiggly segment depicts a path of arbitrary yet non-zero length).

$i \in \{1,2\}$ and each $j \in [c]$, there is an $\mathfrak{f}_{L_i}$-route $R_{i,j}$ of length at least $o+4$ in G from x_j to x'_j such that for all distinct $j, j' \in [c]$, $H_j = R_{1,j} \cup R_{2,j}$ and $H_{j'} = R_{1,j'} \cup R_{2,j'}$ are disjoint and anticomplete cycles in G (see Figure 6.4). But then $(H_j : j \in [c])$ are c pairwise disjoint and anticomplete induced subgraphs of G, each of which is a cycle of length at least $2o+8$, a contradiction with the assumption that G is (c,o)-perforated. This proves (46).

By (46), T has a vertex cover $X \subseteq S = V(\mathsf{T})$ with $|X| = 2c$. It follows that $S \setminus X = V(\mathsf{T}) \setminus X$ is a stable set in T. Moreover, we have:

(47) *Let $L \in \mathcal{L}_2$. Then for every $\mathfrak{f}_L$-route R, there exists a vertex $x \in X$ which has a neighbor in R^*.*

Suppose not. Then we may pick $L \in \mathcal{L}_2$ as well as an $\mathfrak{f}_L$-route R such that X and R^* are anticomplete in G, and subject to this property, R^* is minimal with respect to inclusion. Let $z, z' \in S$ be the ends of R. Then $z, z' \in S \setminus X = V(\mathsf{T}) \setminus X$. Since $\mathfrak{f}_L$ is $(o+1)$-ample and from the minimality of R, it follows that $S \setminus (X \cup \{z, z'\})$ is anticomplete to R^*. We conclude that $S \setminus \{z, z'\} = V(\mathsf{T}_{\mathfrak{f}_L}) \setminus \{z, z'\}$ is anticomplete to R^*. But then from the definition of the transition graph, it follows that $zz' \in E(\mathsf{T}_{\mathfrak{f}_L}) = E(\mathsf{T})$, a contradiction with the fact that $V(\mathsf{T}) \setminus X$ is a stable set in T. This proves (47).

In view of (47), for every $L \in \mathcal{L}_2$ and all distinct $z, z' \in S \setminus X$, we can choose a minimal non-empty subset $\Phi_L(\{z, z'\})$ of X such that for every $\mathfrak{f}_L$-route R from z to z', there is a vertex $x \in \Phi_L(\{z, z'\})$ which has a neighbor in R^*. It follows that, for each $L \in \mathcal{L}_2$, the map $\Phi_L : (S \setminus X)^{(2)} \to 2^X \setminus \{\varnothing\}$ is well-defined. On the other hand, we have $|S \setminus X| = f_1$, which along with the choice of f_1 and Theorem 3.8 yields the following:

(48) *For every $L \in \mathcal{L}_2$, there exists $Y_L \subseteq X$ and $Z_L \subseteq S \setminus X$ such that:*

- *we have $Y_L \neq \varnothing$ and $|Z_L| = 2f_3$; and*
- *for all distinct $z, z' \in Z_L$, we have $\Phi_L(\{z, z'\}) = Y_L$.*

Combining (48) with the fact that $|\mathcal{L}_2| = g_2 = 2^{2c} f_1^{2f_3} f_2 f_3$, it follows from a pigeonhole argument that for several choices of $L \in \mathcal{L}_2$ for which the sets Y_L associated with them are all the same, and so are the sets Z_L. More precisely, we deduce:

(49) *There exist $\mathcal{L}_3 \subseteq \mathcal{L}_2$, $Y \subseteq X$ and $Z \subseteq S \setminus X$, such that:*

- *we have $|\mathcal{L}_3| = f_2 f_3$, $Y \neq \varnothing$ and $|Z| = 2f_3$; and*
- *for every $L \in \mathcal{L}_3$ and all distinct $z, z' \in Z$, we have $\Phi_L(\{z, z'\}) = Y$.*

Now, by the first bullet of (49), we may fix a partition

$$(\mathcal{P}_i : i \in [f_3])$$

of $\mathcal{L}_3$ into f_2-subsets. We can also pick a vertex $y \in Y$ and fix an enumeration

$$\{z_i, z'_i : i \in [f_3]\}$$

of the elements of Z. By the second bullet of (49), for each $L \in \mathcal{L}_3$ and every $i \in [f_3]$, we have $y \in Y = \Phi_L(\{z_i, z'_i\})$. This, together with the minimality of $\Phi_L(\{z_i, z'_i\})$, implies that for each $L \in \mathcal{L}_3$ and every $i \in [f_3]$, there exists an $\mathfrak{f}_L$-route $Q_{i,L}$ from z_i to z'_i, where we write $R_{i,L} = Q^*_{i,L}$, such that y is the only vertex in Y with a neighbor in $R_{i,L}$.

With this notation, we claim that:

(50) *There exists a plain $(\sigma_{s-1}, \lambda_{s-1})$-constellation $\mathfrak{c}'$ in G such that for some $i_0 \in [f_3]$, we have $S_{\mathfrak{c}'} \subseteq Z \setminus \{z_{i_0}, z'_{i_0}\}$ and $\mathcal{L}_{\mathfrak{c}'} \subseteq \{R_{i_0,L} : L \in \mathcal{P}_{i_0}\}$. In particular, we have $S_{\mathfrak{c}'} \subseteq S_{\mathfrak{c}} \setminus \{y, z_{i_0}, z'_{i_0}\}$, and for every $L' \in \mathcal{L}_{\mathfrak{c}'}$, we have $L' \subseteq L$ for some $L \in \mathcal{L}_{\mathfrak{c}}$.*

To see this, for each $i \in [f_3]$, let

$$\mathcal{B}_i = \{(\{z_i, z'_i\}, R_{i,L}) : L \in \mathcal{P}_i\}.$$

Then $\mathcal{B}_i$ is a set of f_2 pairwise disjoint 2-bisets in G. From the choice of f_2 and Theorem 3.14, it follows that for each $i \in [f_3]$, there exists a $(\lambda_{s-1} + 1)$-subset $\mathcal{P}'_i$ of $\mathcal{P}_i$, such that for all distinct $i, j \in [f_3]$, the sets

$$\{(\{z_i, z'_i\}, R_{i,L}) : L \in \mathcal{P}'_i\} \subseteq \mathcal{B}_i$$

and

$$\{(\{z_j, z'_j\}, R_{j,L}) : L \in \mathcal{P}'_j\} \subseteq \mathcal{B}_j$$

satisfy the three outcomes of Theorem 3.14. Let D be the digraph with $V(D) = [f_3]$ such that all distinct $i, j \in [f_3]$, we have $(i, j) \in E(D)$ if there is a vertex $w_i \in \{z_i, z'_i\}$ which has has a neighbor in $R_{j,L}$ for every path $L \in \mathcal{P}'_j$. By the choice of f_3 and Theorem 5.11, either there is a stable set $I \subseteq V(D) = [f_3]$ in D with $|I| = c$, or there are $\sigma_{s-1} + 1$ elements $i_1, \dots, i_{\sigma_{s-1}}, i_{\sigma_{s-1}+1} \in I_0$ such that for each $j \in [\sigma_{s-1}]$, there is a vertex $w_{i_j} \in \{z_{i_j}, z'_{i_j}\}$ which has a neighbor $R_{i_{\sigma_{s-1}+1},L}$ for every $L \in \mathcal{P}'_{i_{\sigma_{s-1}+1}}$. In the former case, for each $i \in I$, since $|\mathcal{P}'_i| = \lambda_{s-1} + 1 \geq 2$, it follows that one may choose two paths $L^i_1, L^i_2 \in \mathcal{P}'_i$, such that for all distinct $i, j \in I$, the sets $\{z_i, z'_i\}$ and $R_{j,L^j_1} \cup R_{j,L^j_2}$ are anticomplete in G. Since $\mathcal{L}_3$ is a plain polypath and $\mathfrak{f}_L$ is $(o+1)$-ample for every $L \in \mathcal{L}_3$, it follows that for every $i \in I$, the induced subgraph $H_i = Q_{i,L^i_1} \cup Q_{i,L^i_2}$ is a cycle of length at least $2o + 8$, and the sets $(H_i : i \in I)$ are pairwise disjoint and anticomplete cycles in G. But this violates the assumption that G is (c, o)-perforated. We deduce that the latter holds. Now, let $W = \{w_{i_j} : j \in [\sigma_{s-1}]\}$ and let $\mathcal{W}$ be a λ_{s-1}-subset of $\mathcal{P}'_{i_{\sigma_{s-1}+1}} \subseteq \mathcal{P}_{i_{\sigma_{s-1}+1}}$. Then

$$\mathfrak{c}' = \left(W, \left\{R_{i_{\sigma_{s-1}+1},L} : L \in \mathcal{W}\right\}\right)$$

is a plain $(\sigma_{s-1}, \lambda_{s-1})$-constellation in G such that $S_{\mathfrak{c}'} \subseteq Z \setminus \{z_{i_0}, z'_{i_0}\}$ and $\mathcal{L}_{\mathfrak{c}'} \subseteq \{R_{i_0,L} : L \in \mathcal{P}_{i_0}\}$. This proves (50).

Henceforth, let us fix $\mathfrak{c}'$ and $i_0 \in [f_3]$ as give by (50). It follows from the induction hypothesis applied to $\mathfrak{c}'$ that:

(51) *For some $L \in \mathcal{P}_{i_0}$, there is an $(o+1)$-ample, interrupted $(s-1)$-asterism $\mathfrak{a}'$ in G such that $S_{\mathfrak{a}'} \subseteq S_{\mathfrak{c}'} \subseteq S_{\mathfrak{c}} \setminus \{y, z_{i_0}, z_{i'_0}\}$ and $L_{\mathfrak{a}'} \subseteq R_{i_0,L}$.*

From now on, let $L \in \mathcal{P}_{i_0}$ be as in (51). Then we may choose an $(o+1)$-ample, interrupted $(s-1)$-asterism $\mathfrak{a}'$ in G such that $S_{\mathfrak{a}'} \subseteq S_{\mathfrak{c}'} \subseteq S_{\mathfrak{c}} \setminus \{y, z_{i_0}, z_{i'_0}\}$, $L_{\mathfrak{a}'} \subseteq R_{i_0,L}$, and subject to these properties, $V(L_{\mathfrak{a}'})$ is maximal with respect to inclusion. By (50), every vertex in $S_{\mathfrak{c}'}$ has a neighbor in $R_{i_0,L}$ in G. Also, recall the choice of $y \in Y = \Phi(\{z_{i_0}, z'_{i_0}\})$, which implies that

$$y \in X \subseteq S \setminus Z \subseteq S \setminus (S_{\mathfrak{c}'} \cup \{z_{i_0}, z'_{i_0}\})$$

and y has a neighbor in $R_{i_0,L} = Q^*_{i_0,L}$. In conclusion, every vertex in $S_{\mathfrak{c}'} \cup \{y\}$ has a neighbor in $R_{i_0,L}$. In fact, since $\mathfrak{f}_L$ is $(o+1)$-ample and each end of $R_{i_0,L}$ is adjacent to one of z_{i_0} or z'_{i_0}, it follows that every vertex in $S_{\mathfrak{c}'} \cup \{y\}$ has a neighbor in $R^*_{i_0,L}$ and the ends of $R_{i_0,L}$ are anticomplete to $S' \cup \{y\}$ in G. In other words, $\mathfrak{a}^+ = (S_{\mathfrak{c}'} \cup \{y\}, R_{i_0,L})$ is a

$(\sigma_{s-1}+1)$-asterism in G where $S_{\mathfrak{a}'} \subseteq S_{\mathfrak{c}'} = S_{\mathfrak{a}^+} \setminus \{y\}$ and $L_{\mathfrak{a}'} \subseteq R_{i_0,L} = L_{\mathfrak{a}^+}$. Furthermore, we deduce that:

(52) $\mathfrak{a}'$ *is a* $(\mathfrak{a}^+, y, s-1)$*-candidate in* G *and* y *is a cherry on top of* $\mathfrak{a}^+|S_{\mathfrak{a}'}$ *in* G*.*

From the maximality of $V(L_{\mathfrak{a}'}) \subseteq R_{i_0,L}$ in the choice of $\mathfrak{a}'$, it follows immediately that $\mathfrak{a}^+$, $\mathfrak{a}'$ and y satisfy (CA), and so $\mathfrak{a}'$ is a $(\mathfrak{a}^+, y, s-1)$-candidate in G. It remains to prove that y is a cherry on top of $\mathfrak{a}^+|S_{\mathfrak{a}'}$ in G. To that end, we need to show that $\mathfrak{a}^+|S_{\mathfrak{a}'}$ and y satisfy (CH1) and (CH2). Observe that (CH1) follows from the fact that $L_{\mathfrak{a}^+|S_{\mathfrak{a}'}} = L_{\mathfrak{a}^+}$. To see (CH2), let P be an open $\mathfrak{a}^+|S_{\mathfrak{a}'}$-piece (which may be internal of external). It follows that $P \subseteq R_i$ is the interior an $\mathfrak{f}_L$-route between two vertices in $S_{\mathfrak{c}'} \cup \{z_{i_0}, z'_{i_0}\}$. On the other hand, since $S_{\mathfrak{c}'} \cup \{z_{i_0}, z'_{i_0}\} \subseteq Z$ and $L \in \mathcal{P}_{i_0} \subseteq \mathcal{L}_3$, it follows from the second bullet of (49) that some vertex in Y has a neighbor in P. But then from $P \subseteq R_{i_0,L} = Q^*_{i_0,L}$ and the choice of $Q_{i_0,L}$, we conclude that y has a neighbor in P. This proves (52).

We are almost done. Note that $\mathfrak{a}^+$ is $(o+1)$-ample, because $\mathfrak{f}_L$ is. Therefore, since $o+1 \geq 2$, in view of (52), we can apply Lemma 6.7 to $\mathfrak{a}^+, \mathfrak{a}'$ and y, and deduce that $\mathfrak{a} = \mathsf{Cher}(\mathfrak{a}', y)$ is an interrupted s-asterism in G with $S_{\mathfrak{a}} \subseteq S_{\mathfrak{a}^+} = S_{\mathfrak{c}'} \cup \{y\} \subseteq S \subseteq S_{\mathfrak{c}}$ and $L_{\mathfrak{a}} = L_{\mathfrak{a}^+} = R_{i_0,L} \subseteq L$ where $L \in \mathcal{P}_{i_0} \subseteq \mathcal{L}_3 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}_1 \subseteq \mathcal{L}_{\mathfrak{c}}$. Moreover, $\mathfrak{a}$ is $(o+1)$-ample because $\mathfrak{a}^+$ is. This completes the proof of Theorem 6.5. ■

6.5 Plain constellations in perforated graphs

In this section, we give a proof of Theorem 6.4, hence concluding our proof of Theorem 6.1. Similar to Section 5.5, we start with a lemma:

Lemma 6.9. *Let* $c, d, o, s, l \in \mathbb{N}$ *and let* G *be a* (c, o)*-perforated graph. Let* $\{Q_1, Q_2\}$ *be a plain* 2*-polypath in* G *and let* $\mathcal{R}$ *be a plain* $(3co)^{2(l-1)^2}(s+2dl)^l$*-polypath in* G *such that* $V(\{Q_1, Q_2\}) \cap V(\mathcal{R}) = \varnothing$*. Assume that the ends of each path* $R \in \mathcal{R}$ *(which could be the same) are called* u^1_R *and* u^2_R *such that* u^1_R *is the only vertex in* R *with a neighbor in* Q_1 *and* u^2_R *is the only vertex in* R *with a neighbor in* Q_2*. Assume also that for each* $k \in \{1, 2\}$*, the* $(3co)^{2(l-1)^2}(s+2dl)^l$*-constellation* $(\{u^1_R : R \in \mathcal{R}\}, Q_k)$ *in* G *is a* d*-meager. Then there is a plain* (s, l)*-constellation.*

Proof. Suppose not. Note that

$$(3co)^{2(l-1)^2}(s+2dl)^l = \left((3co)^{2(l-1)}(s+2dl)\right)^{l-1}(s+2dl).$$

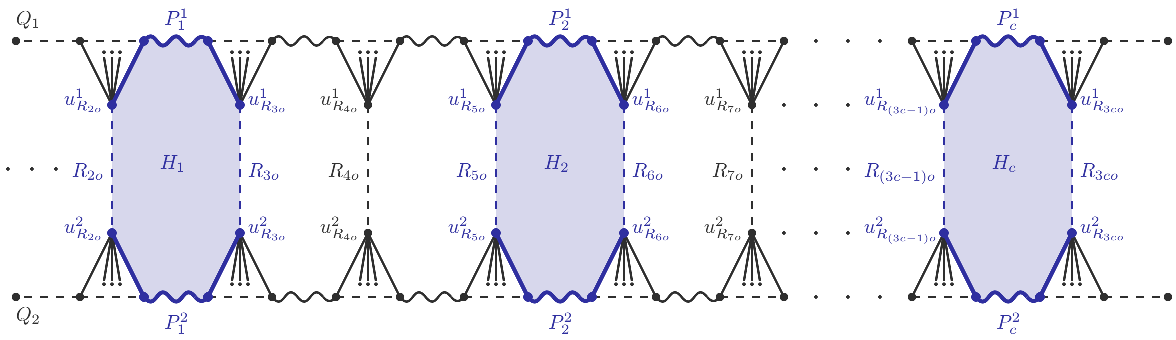

Figure 6.5: Proof of Lemma 6.9. Dashed lines represent paths of arbitrary (possibly zero) length, and squiggly lines represent paths of length at least o.

Therefore, we can apply Theorem 5.9 to the d-meager $(3co)^{2(l-1)^2}(s+2dl)^l$-constellation $(\{u^1_R : R \in \mathcal{R}\}, Q_1)$ in G. Since there is no plain (s,l)-constellation in G, it follows that Theorem 5.9(a) holds, that is, there exists $\mathcal{S} \subseteq \mathcal{R}$ as well as a path $L_1 \subseteq Q_1^*$ such $(\{u^1_R : R \in \mathcal{S}_1\}, L_1)$ is a $(3co)^{2(l-1)}(s+2dl)$-alignment in G. In particular, since $|\mathcal{S}| = (3co)^{2(l-1)}(s+2dl)$, we can apply Theorem 5.9 to d-meager $(3co)^{2(l-1)}(s+2dl)$-constellation $(\{u^2_R : R \in \mathcal{S}\}, Q_2)$ in G. Again, since there is no plain (s,l)-constellation in G, it follows that Theorem 5.9(a) holds, that is, there exists $\mathcal{S}' \subseteq \mathcal{S}$ as well as a path $L_2 \subseteq Q_2^*$ such $(\{u^2_R : R \in \mathcal{S}'_1\}, L_2)$ is a $(3co)^2$-alignment in G. On the other hand, $(\{u^1_R : R \in \mathcal{S}'_1\}, L_1)$ is also a $(3co)^2$-alignment in G. Since $(3co)^2 > (3co-1)^2$, it follows from the Erdős-Szekeres Theorem [27] that there exist $R_1, \dots, R_{3co} \in \mathcal{S}' \subseteq \mathcal{R}$ such for each $k \in \{1,2\}$,

$$\left(\{u^k_{R_i} : i \in [3co]\}, L_k\right)$$

is a $3co$-alignment in G with respect to the order $\pi(u^k_{R_i}) = i$ on $\{u^k_{R_i} : i \in [3co]\}$. In particular, for every $i \in [c]$ and $k \in \{1,2\}$, there is a unique path P^k_i in G from $u^k_{R_{(3i-1)o}}$ to $u^k_{R_{3io}}$ whose interior is contained in L_k. Now, for each $i \in [c]$, let

$$H_i = u^1_{R_{(3i-1)o}}\text{-}P^1_i\text{-}u^1_{R_{3io}}\text{-}R_{3io}\text{-}u^2_{R_{3io}}\text{-}P^2_i\text{-}u^2_{R_{(3i-1)o}}\text{-}R_{(3i-1)o}\text{-}u^1_{R_{(3i-1)o}}.$$

See Figure 6.5. Then $H_1, \dots, H_c$ are c induced cycles in G, each of length at least $2o + 4 > o + 2$, and the sets $\{V(C_i) : i \in [c]\}$ are pairwise disjoint and anticomplete in G. This violates the assumption that G is (c,o)-perforated, hence completing the proof of Lemma 6.9. ■

Theorem 6.4. *For all $c, l, o, s, t \in \mathbb{N}$, there is a constant $f_{6.4} = f_{6.4}(c,l,o,s,t) \in \mathbb{N}$ such that if G is a $(K_{t,t}, K_{t+1})$-free (c,o)-perforated graph with $\operatorname{tw}(G) > f_{5.3}$, then there is a plain (s,l)-constellation in G.*

Proof. The definition of $f_{6.4}(c, l, o, s, t)$ relies on multiple earlier results, including Theorems 2.27, 3.8, 3.14, 3.18 and 5.7 and Lemmas 3.12 and 6.9. Specifically, let

$$b_1 = (3co)^{2(l-1)^2}(s + 2l^2)^l;$$

$$b_2 = ((t + 1)^s + 1)^{l+1} b_1^{l^2}.$$

Observe that $b_2 > b_1 > l + 1 \geq 2$. Let

$$c' = f_{3.12}((s + 1)^{c(s+1)}, 2, t, t + 1);$$

$$f_0 = f_{3.14}(2, b_2, c').$$

Let

$$t' = \max\{c(o + 1), t\};$$

$$f_1 = f_{5.7}(o, l, f_0, f_0, s, t').$$

Let

$$f_2 = f_{3.18}(o, f_1, 2c', t');$$

$$g_2 = g_{3.18}(o, f_1, 2c', t').$$

We will show that

$$f_{6.4} = f_{6.4}(c, l, s, o, t) = f_{2.27}(f_2, g_2, t')$$

satisfies 6.4. Let G be a $(K_{t,t}, K_{t+1})$-free (c, o)-perforated graph with $\mathrm{tw}(G) > f_{6.4}$, and suppose for a contradiction that there is no plain (s, l)-constellation in G.

From the choice of t' and the assumption that G is both $(K_{t,t}, K_{t+1})$-free and (c, o)-perforated, it follows that G is t'-clean. Thus, by Theorem 2.27, there is a strong (f_2, g_2)-block in G. Since G is t'-clean and by the choices of f_2 and g_2, it follows from Theorem 3.18 that there is an o-long strong $(2c', f_1)$-block $(B, \mathcal{P})$ in G. In particular, B is a stable set, and one may choose a set $\{x_i, z_i : i \in [c']\}$ of $2c'$ pairwise distinct and non-adjacent vertices in B.

Let $i \in [c']$. Define $\mathcal{L}_i = \{P^* : P \in \mathcal{P}(\{x_i, z_i\})\}$. Then $|\mathcal{L}_i| = f_1$. Since $(B, \mathcal{P})$ is o-long, it follows that every path $L \in \mathcal{L}_i$ has length at least $o - 1 \geq 0$, and so x_i and z_i each have a unique (possibly the same) neighbor in L, which we call x_L and y_L, respectively. Let $X_i = \{x_L : L \in \mathcal{L}_i\}$. Then $\mathfrak{p}_i = (\{z_i\}, \mathcal{L}_i)$ is a $(1, f_1)$-constellation in G such that for every path $L \in \mathcal{L}_{\mathfrak{p}_i} = \mathcal{L}_i$, we know that L has length at least $o-1$, and we have $L \setminus L^* = \{x_L, y_L\}$ such that $L \cap X_i = \{x_L\}$ and $N_L(z_{\mathfrak{p}}) = \{y_L\}$.

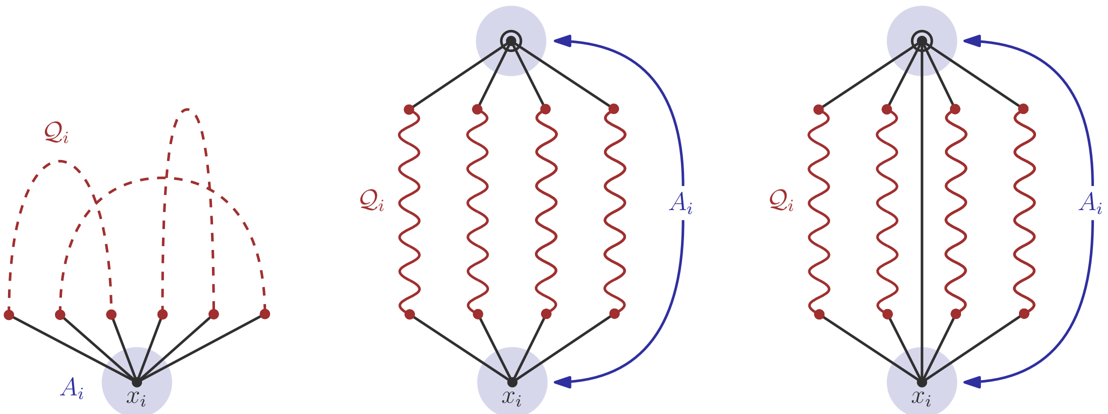

Figure 6.6: Proof of Theorem 6.4. Dashed lines represent paths of length at least $2o-1$, squiggly lines represent paths of length at least $o-1$, and circled nodes in the middle and the left figure depict the unique vertex in $S_{\mathfrak{q}_i}$.

We deduce that $\mathfrak{p}_i$, for each $i \in [c']$, is an $(o-1, f_1)$-patch for X_i in G with $X_i \subseteq V(\mathfrak{p}_i)$. Moreover, since $(B, \mathcal{P})$ is strong, it follows that the sets

$$V(\mathfrak{p}_i) \cup \{x_i\} = V(\mathcal{P}(\{x_i, z_i\}))$$

are pairwise disjoint for all $i \in [c']$. Therefore, since G is t'-clean and by the choice of f_1, we can apply Theorem 5.7 to $\mathfrak{p}_1, \dots, \mathfrak{p}_{c'}$. From the assumption that there is no plain (s,l)-constellation in G, it follows that for each $i \in [c']$, one of the following holds.

- There exists a plain $(2o-1, f_0)$-match $\mathcal{M}_i$ for X_i in G such that $V(\mathcal{M}_i) \subseteq V(\mathfrak{p}_i)$. In particular, $G[V(M) \cup \{x_i\}]$ is an cycle of length at least $2o+1 \geq o+2$ for every $M \in \mathcal{M}_i$.
- There exists a plain $(o-1, f_0)$-patch $\mathfrak{q}_i$ for X_i in G such that $V(\mathfrak{q}_i) \subseteq V(\mathfrak{p}_i)$. In particular, if $S_{\mathfrak{q}_i} \subseteq X_i$, then $G[\{x_i\} \cup S_{\mathfrak{q}_i} \cup V(L)]$ is a cycle of length at least $o+2$ for every $L \in \mathcal{L}_{\mathfrak{q}_i}$, and if $S_{\mathfrak{q}_i} \cap X_i = \varnothing$, then $G[\{x_i\} \cup S_{\mathfrak{q}_i} \cup V(L) \cup V(L')]$ is a cycle of length at least $2o+2 > o+2$.

We define A_i and $\mathcal{Q}_i$ as follows: if the first bullet above holds for i, then let $A_i = \{x_i\}$ and let $\mathcal{Q}_i = \mathcal{M}_i$, and if the second bullet above holds for i, then let $A_i = \{x_i\} \cup S_{\mathfrak{q}_i}$ and let $\mathcal{Q}_i = \mathcal{L}_{\mathfrak{q}_i}$ (see Figure 6.6). It follows that $\mathcal{Q}_i$ is a plain f_0-polypath in G, and:

(53) *For every $i \in [c']$ and all distinct $Q, Q' \in \mathcal{Q}_i$, the graph $G[A_i \cup V(Q) \cup V(Q)]$ has an induced subgraph which is a cycle of length at least $o + 2$.*

For each $i \in [c']$, let

$$\mathcal{B}_i = \{(A_i, Q) : Q \in \mathcal{Q}_i\}.$$

Then $\mathcal{B}_1, \ldots, \mathcal{B}_{c'}$ are c' sets of pairwise disjoint 2-bisets in G, each of cardinality f_0. By Theorem 3.14 and the choice of f_0, it follows that for each $i \in [c']$, there exists $\mathcal{Q}'_i \subseteq \mathcal{Q}_i$ with $|\mathcal{Q}'_i| = b_2$ such that for all distinct $i, j \in [c']$, the sets

$$\{(A_i, Q) : Q \in \mathcal{Q}'_i\}$$

and

$$\{(A_i, Q) : Q \in \mathcal{Q}'_j\}$$

satisfy the three outcomes of Theorem 3.14. Let us now prove that:

(54) *There exists $I \subseteq [c']$ with $|I| = c$ such that for all distinct $i, j \in I$ and every $Q_j \in \mathcal{Q}'_j$, the sets A_i and $A_j \cup V(Q_j)$ are anticomplete in G.*

Note that since G is $(K_{t,t}, K_{t+1})$-free and since $|A_i| \leq 2$ for all $i \in [c']$, it follows from Lemma 3.12 and the choice of c' that there exists a subset I_0 of $[c']$ with $|I_0| = (s+1)^{c(s+1)}$ for which the sets $(A_i : i \in I)$ are pairwise anticomplete in G. It remains to show that there exists $I \subseteq I_0$ with $|I| = c$ such that for all distinct $i, j \in I$ and every $Q_j \in \mathcal{Q}'_j$, the sets A_i and $V(Q_j)$ are anticomplete in G. Let D be the digraph with $V(D) = I_0$ such that for all distinct $i, j \in I_0$, we have $(i, j) \in E(D)$ if there is a vertex $w_i \in A_i$ which has a neighbor in every path $Q \in \mathcal{Q}'_j$. Then, by Theorem 5.11, either there is a stable set $I \subseteq V(D) = I_0$ in D with $|I| = c$, or there are $i_1, \ldots, i_{s+1} \in I_0$ such that for each $j \in [s]$, the vertex $w_{i_j} \in A_{i_j}$ has a neighbor in every path $Q \in \mathcal{Q}'_{i_{s+1}}$. In the former case, by Theorem 3.14(b), I satisfies (54), and we are done. In the latter case, since $(A_i : i \in I_0)$ are pairwise anticomplete in G, it follows that $S = \{w_{i_j}, j \in [s]\}$ is a stable set of cardinality s in G. Also, $\mathcal{Q}'_{i_{s+1}}$ is a plain polypath in G with $|\mathcal{Q}'_{i_{s+1}}| = b_2 > l$, and so there is a plain l-polypath $\mathcal{L} \subseteq \mathcal{Q}'_{i_{s+1}}$ in G. But now $(S, \mathcal{L})$ is a plain (s, l)-constellation in G, a contradiction. This proves (54).

Henceforth, let I be as in (54). We deduce that:

(55) *For each $i \in I$, there are distinct paths $Q_i, Q'_i \in \mathcal{Q}'_i$, such that the sets*

$$(A_i \cup V(Q_i) \cup V(Q'_i) : i \in I)$$

are pairwise anticomplete in G.

By (54) and since $|\mathcal{Q}'_i| = b_2 > 2$ for each $i \in I$, in order to prove (55), it suffices to show that for all distinct $i, j \in I$, every $Q_i \in \mathcal{Q}'_i$ and every $Q_j \in \mathcal{Q}'_j$, the sets $V(Q_i)$ and $V(Q_j)$ are anticomplete in G. Suppose not. Then by Theorem 3.14(c), there distinct $i, j \in I$ such that for every $Q_i \in \mathcal{Q}'_i$ and every $Q_j \in \mathcal{Q}'_j$, there is an edge in G with an end in $V(Q_i)$ and an end in $V(Q_j)$. Since $|\mathcal{Q}'_i| = |\mathcal{Q}'_j| = b_2$, G is K_{t+1}-free and $V(\mathcal{Q}'_i) \cap V(\mathcal{Q}'_j) = \varnothing$, it follows from the choice of b_2 that we can apply Lemma 3.22 to $\mathcal{Q}'_i$ and $\mathcal{Q}'_j$. Assume that Lemma 3.21(a) holds. Then there is an (s, l)-constellation $\mathfrak{c}$ in G such that either $S_{\mathfrak{c}} \subseteq V(\mathcal{Q}'_i)$ and $\mathcal{L}_{\mathfrak{c}} \subseteq \mathcal{Q}'_j$, or $S_{\mathfrak{c}} \subseteq V(\mathcal{Q}'_i)$ and $\mathcal{L}_{\mathfrak{c}} \subseteq \mathcal{Q}'_j$. But $\mathcal{Q}'_i$ and $\mathcal{Q}'_j$ are both plain, and so in either case $\mathfrak{c}$ is a plain (s, l)-constellation in G, a contradiction. It follows that Lemma 3.21(a) holds, that is, there exist $\mathcal{Q}''_i \subseteq \mathcal{Q}'_i$ and $\mathcal{Q}''_j \subseteq \mathcal{Q}'$ with $|\mathcal{Q}''_i| = |\mathcal{Q}''_j| = b_1$ such that every vertex in $V(\mathcal{Q}''_i)$ has neighbors in fewer than l paths in $\mathcal{Q}''_j$, and every vertex in $V(\mathcal{Q}''_j)$ has neighbors in fewer than l paths in $\mathcal{Q}''_i$. Now, fix two path $Q_1, Q_2 \in \mathcal{Q}''_i$ (again, this is possible because $|\mathcal{Q}'_i| = b_2 > 2$). Since $\mathcal{Q}''_i$ is a plain polypath, it follows that $\{Q_1, Q_2\}$ is a plain 2-polypath in G as well. By the choice of i, j, for every $Q \in \mathcal{Q}''_j$, one may choose a path $R_Q \subseteq Q$ in G with (not necessarily distinct) ends u^1_Q and u^2_Q such that u^1_Q is the only vertex in R_Q with a neighbor in Q_1 and u^2_Q is the only vertex in R_Q with a neighbor in Q_2. Since $\mathcal{Q}''_j$ is a plain polypath in G, it follows that $\mathcal{R} = \{R_Q : Q \in \mathcal{Q}''_j\}$ is a plain b_1-polypath in G. Moreover, by the choice of $\mathcal{Q}''_i$ and $\mathcal{Q}''_j$, for each $k \in \{1, 2\}$, the b_1-constellation $(\{u^k_Q : Q \in \mathcal{Q}''_j\}, Q_k)$ is l-meager. But then by the choice of b_1, we can apply Lemma 6.9 to $\{Q_1, Q_2\}$ and $\mathcal{R}$ to deduce that there is a plain (s, l)-constellation in G, a contradiction. This proves (55).

Finally, the fact that $I \subseteq [c']$ with $|I| = c$ combined with (53) and (55) yields a contradiction with the assumption that G is (c, o)-perforated. This completes the proof of Theorem 6.4. ■

Chapter 7

Even-hole-free graphs I. Constellating neighbors[1]

7.1 Kaleidoscopes

This is the first chapter in a series of three, each building on the material from the preceding ones, eventually leading to the proofs of Theorems 2.18, 2.24 and 2.25. The proof of Theorem 2.18 will only use the results from this chapter. The proofs of Theorems 2.24 and 2.25, however, will consume the entire machinery we will develop in the rest of this thesis. We will complete all three of those proofs in Chapter 9.

The statement of our main result in this chapter uses the following definitions. Given $w \in \mathbb{N}$ and a graph G, a *w-kaleidoscope* in G is a 4-tuple $(a, x, y, \mathcal{W})$ where:

- (K1) we have $a, x, y \in V(G)$, and x-a-y is a path in G;
- (K2) $\mathcal{W}$ is a set of w pairwise internally disjoint paths in $G \setminus \{a\}$ from x to y; and
- (K3) the vertex a is anticomplete to $\mathcal{W}^*$ in G.

Given a subset $Z \subseteq V(G)$ and $d \in \mathbb{N}$, we say that *Z is d-mirrored by* $(a, x, y, \mathcal{W})$ (or rather *z is d-mirrored by* $(a, x, y, \mathcal{W})$, if $Z = \{z\}$), if:

- (M1) Z is disjoint from $(\bigcup_{W \in \mathcal{W}} V(W)) \cup \{a\}$;
- (M2) the vertex a has at most one neighbor in Z; and

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[1] This chapter is based on the coauthored papers [2, 6] and the single-author paper [36].

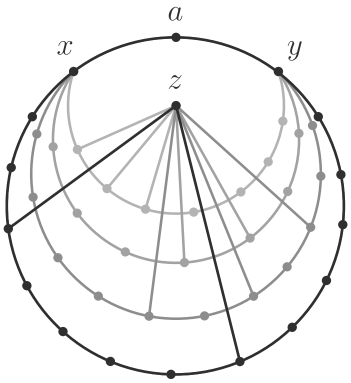

Figure 7.1: A 4-kaleidoscope $(a, x, y, \mathcal{W})$ by which the vertex z is 2-mirrored.

(M3) for all $z \in Z$ and $W \in \mathcal{W}$, z is anticomplete to $N_W[x] \cup N_W[y]$, and z has at least d distinct neighbors in W. In particular, z is anticomplete to $\{x, y\}$.

See Figure 7.1.

Our goal is to prove the following, which forms the core of our inductive arguments in the proofs of Theorems 2.24 and 2.25.

Theorem 7.1. *For all $d, t, w \in \mathbb{N}$, there is a constant $f_{7.1} = f_{7.1}(d, t, w) \in \mathbb{N}$ with the following property. Let G be a (theta, prism, even wheel, K_{t+1})-free graph, let $(a, x, y, \mathcal{W})$ be an $f_{7.1}$-kaleidoscope in G, and let $z \in V(G)$ be 1-mirrored by $(a, x, y, \mathcal{W})$. Then there exists $\mathcal{W}' \subseteq \mathcal{W}$ with $|\mathcal{W}'| = w$ such that z is d-mirrored by $(a, x, y, \mathcal{W}')$.*

We will complete the proof of Theorem 7.1 in Section 7.3.

7.2 Apexed constellations

In this section, we prepare for the proof of Theorem 7.1 by establishing a number of technical results, many of which will also be used in future chapters.

The first result concerns the local structure imposed by a pair of highly connected vertices in a (theta, prism)-free graph of bounded clique number:

Theorem 7.2. *For all $l, s, t \in \mathbb{N}$, there is a constant $f_{7.2} = f_{7.2}(l, s, t) \in \mathbb{N}$ with the following property. Let G be a (theta, prism, K_{t+1})-free graph, let $a, b \in V(G)$ be distinct* *and non-adjacent and let $\mathcal{P}$ be a collection of pairwise internally disjoint paths in G from a to b with $|\mathcal{P}| \geq f_{7.2}$. Then there is a subset $\mathcal{N}$ of $\mathcal{P}$ where $N_{V(\mathcal{N})}(a)$ is a stable set in G, and there is an (s,l)-constellation $\mathfrak{c}$ in G such that $S_{\mathfrak{c}} \subseteq N_{V(\mathcal{N})}(a)$ and $\mathcal{L}_{\mathfrak{c}} \subseteq \mathcal{N}^*$.*

The proof of Theorem 7.2 relies on yet another technical result which roughly says that in a (theta, prism)-free graph G of bounded clique number, if some vertex $a \in V(G)$ is "trapped" in an induced subgraph H of G where $H \setminus \{a\}$ is isomorphic to the line graph of a proper subdivision of a tree, then a may be separated from the rest of G be the removal of a small cutset. We will give the exact statement of this result as Theorem 7.3 below. Nevertheless, to avoid the digression caused by its length and technicality, we postpone the proof of Theorem 7.3 to Chapter 10. It is important to note that the proof does *not* use the material from Chapters 7, 8 and 9.

Back to the statement of Theorem 7.3, we need two definitions. Let G be a graph. Given an induced subgraph H of G and a vertex $a \in V(H)$, we say a is *trapped in H* if we have $N_G[N_G[a]] \subseteq V(H)$, and every vertex in $N_H(a) = N_G(a)$ has degree two in H (and so in G). By a *separation* in G we mean a triple (L, M, R) of pairwise disjoint subsets $V(G)$ with $L \cup M \cup R = V(G)$, such that neither L nor R is empty and L and R are anticomplete in G. Let $x, y \in G$ be distinct. We say a set $M \subseteq G \setminus \{x, y\}$ *separates x and y in G* if there exists a separation (L, M, R) in G with $x \in L$ and $y \in R$. Also, for disjoint sets $X, Y \subseteq V(G)$, we say a set $M \subseteq V(G) \setminus (X \cup Y)$ *separates X and Y* if there exists a separation (L, M, R) in G with $X \subseteq L$ and $Y \subseteq R$. If $X = \{x\}$, we say that M *separates x and Y* to mean M separates X and Y.

Theorem 7.3. *Let $t \in \mathbb{N}$ and let G be a (theta, prism, K_{t+1})-free graph. Let H be an induced subgraph of G that is isomorphic to the line graph of a proper subdivision of a tree which is not a path. Let $a \in V(G) \setminus V(H)$ such that a is trapped in $G[V(H) \cup \{a\}]$. Then for every vertex $x \in G \setminus N_G[a]$, there exists $S_x \subseteq V(G) \setminus \{a, x\}$ with $|S_x| < 4t^6$ such that S_x separates a and x in G.*

This enables us to prove Theorem 7.2:

Proof of Theorem 7.2. Let $\eta = 8t^6 + 1$ and let $c = \eta^{8\eta^4}$. We will show that

$$f_{7.2} = f_{7.2}(l, s, t) = (t+1)^{(s+l)^{c(s+l)}}$$

satisfies the theorem. For each $P \in \mathcal{P}$, let x_P be the unique neighbor of a in P (so $x_P \neq b$). Since G is K_{t+1}-free, it follows from Theorem 3.17 and the choice of $f_{7.2}$ that there exists $\mathcal{N} \subseteq \mathcal{P}$ with $|\mathcal{N}| = (s+l)^{c(s+l)}$ such that $N = \{x_P : P \in \mathcal{N}\}$ is a stable set in G.

Let D be the digraph with $V(D) = \mathcal{N}$ such that for all distinct paths $P, Q \in \mathcal{N}$, we have $(P, Q) \in E(D)$ if and only if x_P has a neighbor in Q^* in G. We deduce that:

(56) *D has no stable set of cardinality c.*

Suppose not. Let $\mathcal{K} \subseteq \mathcal{N}$ be a stable set in D with $|\mathcal{K}| = c$. Let $G_1 = G[V(\mathcal{K}) \setminus \{a\}]$. By the definition of D, for every $P \in \mathcal{K}$, we have $N_{G_1}(x_P) = N_P(x_P) \setminus \{a\}$, and in particular $|N_{G_1}(x_P)| = 1$. Since G_1 is connected and by the choice of $c = |\mathcal{K}|$, we can apply Theorem 5.6 to G_1 and $K = \{x_P : P \in \mathcal{K}\} \subseteq N \setminus \{b\}$. Recall that G_1 is K_{t+1}-free. Also, every vertex in K has a unique neighbor in G_1, and so no path in G_1 contains more than two vertices from K. Since $\eta = 8t^6 + 1 > t + 1 \geq 2$, it follows that there is an induced subgraph H_1 of G_1 with $|V(H_1) \cap K| = 8t^6 + 1$ for which one of the following holds.

- H_1 is either a caterpillar or the line graph of a caterpillar with $V(H_1) \cap K = \mathcal{Z}(H_1)$.
- H_1 is a subdivided star with root r_1 such that $\mathcal{Z}(H_1) \subseteq V(H_1) \cap K \subseteq \mathcal{Z}(H_1) \cup \{r_1\}$.

If H_1 is a caterpillar, then $G[H_1 \cup \{a\}]$ contains a theta with ends a and a' for every branch vertex a' of H_1, a contradiction. Also, if the second bullet above holds, then since every vertex in K has degree one in G_1, it follows that $V(H_1) \cap K = \mathcal{Z}(H_1)$, and so r_1 is not adjacent to a. But then $G[H_1 \cup \{a\}]$ contains a theta with ends a and r_1, again a contradiction. This implies that H_1 is the line graph of a caterpillar with $|V(H_1) \cap K| = 8t^6 + 1$ and $V(H_1) \cap K = \mathcal{Z}(H_1)$. Moreover, every vertex in $V(H_1) \cap K \subseteq K$ has a unique neighbor in G_1, and so in H_1. Thus, there is an induced subgraph H_2 of H_1 which is isomorphic to the line graph of a proper subdivision of a caterpillar, such that $|V(H_2) \cap K| = 4t^6$ and $V(H_2) \cap K = \mathcal{Z}(H_2)$. In particular, $\mathcal{Z}(H_2) \subseteq K \subseteq N$ is the set of all vertices of degree one in H_2, and so there is a $4t^6$-subset $\mathcal{H}$ of $\mathcal{K}$ such that $\mathcal{Z}(H_2) = \{x_P : P \in \mathcal{H}\}$. Let

$$G_2 = G\left[V(H_2) \cup \left(\bigcup_{P \in \mathcal{H}} V(P)\right)\right].$$

Then G_2 is (theta, prism, K_{t+1})-free (because G is) and H_2 is an induced subgraph of G_2 which is isomorphic to the line graph of a proper subdivision of a caterpillar. Moreover, we have $a \in V(G_2) \setminus V(H_2)$, and $N_{G_2}(a) = \mathcal{Z}(H_2)$, which in turn implies that a is trapped in $G_2[V(H_2) \cup \{a\}]$. Hence, we can apply Theorem 7.3 to G_2, H_2 and a. Since a and $b \in V(G_2)$ are distinct and non-adjacent, it follows that there exists $S_b \subseteq G_2 \setminus \{a, b\}$ such that $|S_b| < 4t^6$ such that S_b separates a and b in G_2. But there are $4t^6$ pairwise internally disjoint paths in $\mathcal{H}$ from a to b in G_2, a contradiction. This proves (56).

From (56), Theorem 5.11 and the definition of D, it follows that there exist $P_1, \dots, P_{s+1} \in V(D) = \mathcal{N}$ such that for all distinct $i, j \in [s+l]$ with $i < j$, the vertex x_{P_i} has a neighbor in P_j^*. Let

$$S = \{x_{P_i} : i \in [s]\}$$

and let

$$\mathcal{L} = \{P_{s+i} : i \in [l]\}.$$

Then $\mathfrak{c} = (S, \mathcal{L}^*)$ is an (s,l)-constellation in G with $S_\mathfrak{c} = S \subseteq N_{V(\mathcal{N})}(a)$ and $\mathcal{L}_\mathfrak{c} = \mathcal{L}^* \subseteq \mathcal{N}^*$. This completes the proof of Theorem 7.2. ■

We continue with another definition. Let G be a graph, let H be an induced subgraph of G and let $v \in V(G) \setminus V(H)$. We say that:

- (G) v is *H-good* if $|N_H(v)| = 1$;
- (B) v is *H-bad* if $N_H(v)$ is a clique in H on at least two vertices; and
- (U) v is *H-ugly* if $N_H(v)$ is not a clique in H.

It follows that each vertex in $N_G(H) \subseteq V(G) \setminus V(H)$ is exactly one of H-good, H-bad, or H-ugly. We deduce that:

Lemma 7.4. *For all $m, n, p, q, t \in \mathbb{N}$, there is a constant $g_{7.4} = g_{7.4}(m, n, p, q, t) \in \mathbb{N}$ with the following property. Let G be a (theta, K_{t+1})-free graph. Let $\mathfrak{c}$ be an $(3(p+q+3), g_{7.4})$-constellation in G and let $a \in V(G) \setminus V(\mathfrak{c})$ such that a is complete to $S_\mathfrak{c}$ and anticomplete to $V(\mathcal{L}_\mathfrak{c})$ in G. Then there is a constellation $\mathfrak{m}$ in G with $S_\mathfrak{m} \subseteq S_\mathfrak{c}$ and $\mathcal{L}_\mathfrak{m} \subseteq \mathcal{L}_\mathfrak{c}$, as well as an order π on $\mathfrak{m}$, with the following specifications.*

- (a) *One of the the following holds.*
  - $|S_\mathfrak{m}| = p$, $|\mathcal{L}_\mathfrak{m}| = m$ *and every vertex* $x \in S_\mathfrak{m}$ *is L-bad for every* $L \in \mathcal{L}_\mathfrak{m}$.
  - $|S_\mathfrak{m}| = q$, $|\mathcal{L}_\mathfrak{m}| = n$ *and every vertex* $x \in S_\mathfrak{m}$ *is L-ugly for every* $L \in \mathcal{L}_\mathfrak{m}$.
- (b) *For each $L \in \mathcal{L}_\mathfrak{m}$, $\mathfrak{m}_L$ is both an asterism in G and an alignment in G with respect to π.*

Proof. Let

$$\lambda_2 = m(p+q+1)^p + n(p+q+1)^q;$$

$$\lambda_1 = \lambda_2(3(p+q+3))^{p+q+3}(p+q+3)!.$$

We claim that

$$g_{7.4} = g_{7.4}(m, n, p, q, t) = 9(p+q+3)^2(t+1) + \lambda_1$$

satisfies the lemma.

(57) *There is a* λ_1*-subset* $\mathcal{L}_1$ *of* $\mathcal{L}_\mathfrak{c}$ *such that every vertex in* $V(\mathcal{L}_1)$ *has at most one neighbor in* $S_\mathfrak{c}$*.*

Suppose not. Then, since $|S_\mathfrak{c}| = 3(p+q+3)$ and by the choice $g_{7.4}$, it follows that there is a subset $\mathcal{X}$ of $\mathcal{L}_\mathfrak{c}$ with $|\mathcal{X}| = 9(p+q+3)^2(t+1) = |S_\mathfrak{c}|^2(t+1)$, such that for each $L \in \mathcal{X}$, there is a vertex $x_L \in V(L)$ and two vertices $y_L, z_L \in S_\mathfrak{c}$ such that x_L is complete to $\{y_L, z_L\}$. It follows that there is a $(t+1)$-subset $\mathcal{T}$ of $\mathcal{X}$ as well as two vertices $y, z \in S_\rfloor$ such that for every $L \in \mathcal{T}$, we have $\{y_L, z_L\} = \{y, z\}$. Since G is K_{t+1}-free, it follows from Theorem 3.17 that there are two paths $L_1, L_2 \in \mathcal{T}$ such that x_{L_1} and x_{L_2} are not adjacent in G. But then there is a theta in G with ends y, z and paths y-a-z, y-x_{L_1}-z, y-x_{L_2}-z, a contradiction. This proves (57).

Henceforth, let $\mathcal{L}_1$ be as (57).

(58) *For every* $L \in \mathcal{L}_1$*, there exists a* $(p+q+3)$*-subset* S_L *of* $S_\mathfrak{c}$ *as well as a bijection* $\psi_L : [p+q+3] \rightarrow S_L$*, such that the constellation* (S_L, L) *is an alignment in* G *with respect to* ψ_L*.*

To see this, for each vertex $x \in S_\mathfrak{c}$, let P_x be the (unique) shortest path in L which contains all neighbors of x in L (so x is adjacent to the ends of P_x and anticomplete to $V(L) \setminus V(P_x)$. Let Γ be the graph with $V(\Gamma) = S_\mathfrak{c}$ where $x, y \in S_\mathfrak{c}$ are adjacent in Γ if and only if P_x and P_y are not disjoint. We claim that Γ has no 3-clique. Suppose for a contradiction that there is a 3-subset X of $S_\mathfrak{c}$ for which the sets $(V(P_x) : x \in X)$ are pairwise intersecting. Then there exists a vertex $u \in V(L)$ such that for every $x \in X$, no component of $L \setminus \{u\}$ contains all neighbors of x in L. Also, by (57), u has at most one neighbor in X. It follows that $L \setminus \{x\}$ has two distinct components L_1, L_2, and there are distinct vertices $x, y \in X$ such that both x and y have a neighbor in both L_1 and L_2. It follows that for each $i \in \{1, 2\}$, there is a path R_i in G from x to y with interior contained in L_i. In particular, R_1^* and R_2^* are anticomplete in G. But now there is a theta in G with ends x, y and paths x-a-y, R_1, R_2, a contradiction. The claim follows. Observe that Γ is an interval graph, and so Γ is perfect [10]. Thus, since $|V(\Gamma)| = |S_\mathfrak{c}| = 3(p+q+3)$ and Γ has no 3-clique, it follows that there is a $(p+q+3)$-subset S_L of $V(\Gamma) = S_\mathfrak{c}$ which is a stable set in Γ. This, along with the definition of Γ, implies that there exists a bijection $\psi_L : [p+q+3] \rightarrow S_L$ for which (S_L, L) is an alignment in G with respect to ψ_L. This proves (58).

From the choice of $|\mathcal{L}_1| = \lambda_1 = \lambda_2 |S_\mathfrak{c}|^{p+q+3}(p+q+3)!$ and (58), we deduce that:

(59) *There exists a* $(p+q+3)$*-subset* S_0 *of* $S_{\mathfrak{c}}$*, a bijection* $\psi_0 : [p+q+3] \to S_0$*, and a* λ_2*-subset* $\mathcal{L}_2$ *of* $\mathcal{L}_1$*, such that for every* $L \in \mathcal{L}_2$*, the constellation* (S_0, L) *is an alignment in* G *with respect to* ψ_0*.*

Henceforth, let S_0, ψ_0 and $\mathcal{L}_2$ be as in (59). Let $S = S_0 \setminus \{\psi_0(1), \psi_0(p+q+3)\}$. Then $|S| = p+q+1$. Let $\psi : [p+q+1] \to S$ be the bijection given by $\psi(i) = \psi_0(i+1)$ for each $i \in [p+q+1]$. Then the following is immediate from (59):

(60) *For each* $L \in \mathcal{L}_2$*,* (S, L) *is both an asterism in* G *and an alignment in* G *with respect to* ψ*.*

Moreover, we deduce that:

(61) *For every* $L \in \mathcal{L}_2$*, one of the following holds.*

- *There is a* p*-subset* B_L *of* S *such that every* $x \in B_L$ *is* L*-bad.*
- *There is a* q*-subset* U_L *of* S *such that every* $x \in U_L$ *is* L*-ugly.*

Suppose not. Then, since $|S| = p+q+1$, it follows that there are $i, j, k \in [p+q+1]$ with $i < j < k$ such that $\psi(i), \psi(j)$ and $\psi(k)$ are all L-good. Let u_i, u_j and u_k be the unique neighbors of $\psi(i), \psi(j)$ and $\psi(k)$ in L, respectively. Then u_j belongs to the interior of the path u_i-L-u_k. But then there is a theta in G with ends a, u_j and paths a-$\psi(i)$-u_i-L-u_j, a-$\psi(j)$-u_j and a-$\psi(k)$-u_k-L-u_k, a contradiction. This proves (61).

By (61) and the choice of λ_2, it follows that either there exists $\mathcal{L} \subseteq \mathcal{L}_2$ such that either $|\mathcal{L}| = m|S|^p$ and every path in $\mathcal{L}$ satisfies the first outcome of (61), or $|\mathcal{L}| = n|S|^q$ and every path in $\mathcal{L}$ satisfies the second outcome of (61). In the former case, there is a p-subset B of S and an m-subset $\mathcal{M}$ of $\mathcal{L}$ such that every vertex $x \in B$ is L-bad for every $L \in \mathcal{M}$. Therefore, $\mathfrak{m} = (B, \mathcal{M})$ satisfies the first outcome of 7.4(a). In the latter case, there is a q-subset U of S and an n-subset $\mathcal{N}$ of $\mathcal{L}$ such that every vertex $x \in U$ is L-ugly for every $L \in \mathcal{N}$. Hence, $\mathfrak{m} = (U, \mathcal{N})$ satisfies the second outcome of 7.4(a). Moreover, in both cases, by (59), the corresponding choice of $\mathfrak{m}$ along with $\pi = \psi|_{S_{\mathfrak{m}}}$ satisfies 7.4(b). This completes the proof of Lemma 7.4. ■

The last result in this section shows that, if G is (prism, even wheel)-free as well, then Lemma 7.4 may be strengthened so that the second outcome of 7.4(a) always holds. To that end, the following will be our key tool for using the even-wheel-free assumption:

Theorem 7.5. *Let G be a (theta, prism, even wheel)-free graph, let C be a hole in G and let $z_1, z_2 \in G \setminus V(C)$ be distinct and adjacent, each with at least one neighbor in C. Assume that z_1 and z_2 have no common neighbor in C. Then either both z_1 and z_2 are C-good and their (unique) neighbors in C are distinct and adjacent, or exactly one of z_1 and z_2 is C-bad. Consequently, if G is C_4-free as well, then exactly one of z_1 and z_2 is C-bad.*

Proof. Note that if both z_1 and z_2 are C-bad, then since z_1 and z_2 have no common neighbor in C, it follows that $C \cup \{z_1, z_2\}$ is a prism in G, a contradiction. So we may assume without loss of generality that z_1 is either C-good or C-ugly. If z_2 is C-bad, then we are done. So we can consider the case that z_2 is also either C-good or C-ugly; in particular, since neither (C, z_1) nor (C, z_2) is an even wheel in G, it follows that for every $i \in \{1, 2\}$, $|N_C(z_i)|$ is an odd integer. Assume first that both z_1 and z_2 are C-good, say $N_C(z_i) = \{x_i\}$ for $i \in \{1, 2\}$. Then since z_1 and z_2 have no common neighbor in C, and $C \cup \{z_1, z_2\}$ is not a theta in G, it follows that x_1 and x_2 are distinct and adjacent in G, as required.

This leaves the case where one of z_1 and z_2, say the former, is C-ugly. Since z_1 and z_2 have no common neighbor in C, it follows that $N_C(z_2) \subseteq C \setminus N_C(z_1)$. Note that every component of $C \setminus N_C(z_1)$ is a path in C (and so in G). Moreover, for every component P of $C \setminus N_C(z_1)$, $C_P = N_C[P] \cup \{z_1\}$ is a hole in G. Since $C_P \cup \{z_2\}$ is not a theta in G, and (C_P, z_2) is not even wheel in G, and z_1 and z_2 have no common neighbor in C, it follows that z_2 has an even number of neighbors in P. In conclusion, we have shown that z_2 has an even number of neighbors in each component of $C \setminus N_C(z_1)$. But then z_2 has an even number of neighbors in C, a contradiction. We conclude that either both z_1 and z_2 are C-good and their neighbors in C are distinct and adjacent, or exactly one of z_1 and z_2 is C-bad. In addition, if G is C_4-free, then the first outcome does not hold, as otherwise $G[N_C[z_1] \cup N_C[z_2]]$ is isomorphic to C_4, a contradiction. This completes the proof of Theorem 7.5. ■

We conclude this section with the following:

Theorem 7.6. *For all $l, s, t \in \mathbb{N}$, there is a constant $g_{7.6} = g_{7.6}(l, s, t) \in \mathbb{N}$ with the following property. Let G be a (theta, prism, even wheel, K_{t+1})-free graph.Let $\mathfrak{c}$ be an $(3(s+6), g_{7.6})$-constellation in G and let $a \in V(G) \setminus V(\mathfrak{c})$ such that a is complete to $S_{\mathfrak{c}}$ and anticomplete to $V(\mathcal{L}_{\mathfrak{c}})$ in G. Then there is a constellation $\mathfrak{m}$ in G with $S_{\mathfrak{m}} \subseteq S_{\mathfrak{c}}$ and $\mathcal{L}_{\mathfrak{m}} \subseteq \mathcal{L}_{\mathfrak{c}}$, as well as an order π on $\mathfrak{m}$, such that the following hold for every $L \in \mathcal{L}_{\mathfrak{m}}$.*

(a) *Every vertex $x \in S_{\mathfrak{m}}$ is L-ugly.*

(b) *$\mathfrak{m}_L$ is an s-asterism in G which is also an s-alignment in G with respect to π.*

Proof. We use Lemmas 3.12 and 7.4 and Theorem 7.2. Specifically, let

$$\Lambda_2 = f_{7.2}(1, 1, t);$$

$$\lambda_2 = f_{3.12}(\Lambda_2, 4, 3, t + 1);$$
$$\Lambda_1 = f_{7.2}(\lambda_2, 1, t);$$
$$\lambda_1 = f_{3.12}(\Lambda_1, 2, 3, t + 1).$$

We claim that

$$g_{7.6} = g_{7.6}(l, s, t) = g_{7.4}(\lambda_1, l, 3, s, t)$$

satisfies the theorem. By the choice of $g_{7.6}$, one can apply Lemma 59 to $\mathfrak{c}$ for $m = \lambda_1$, $n = l$, $p = 3$, $q = s$ and the same choice of t. We obtain a constellation $\mathfrak{m}$ in G with $S_{\mathfrak{m}} \subseteq S_{\mathfrak{c}}$ and $\mathcal{L}_{\mathfrak{m}} \subseteq \mathcal{L}_{\mathfrak{c}}$, as well as an order π on $\mathfrak{m}$, satisfying 7.4(a) and 7.4(b).

Assume that $\mathfrak{m}$ satisfies the second outcome of 7.4(a). Then $\mathfrak{m}$ is an (s, l)-constellation in G satisfying 7.6(b). Furthermore, since $\mathfrak{m}$ satisfies 7.4(b), it follows that 7.6(a) hold for $\mathfrak{m}$, and we are done.

For the rest of the proof, we assume that $\mathfrak{m}$ satisfies the first outcome of 7.4(a) and we will derive a contradiction. In particular, $\mathfrak{m}$ is a $(3, \lambda_1)$-constellation in G.

Let $x = \pi(1), z = \pi(2)$ and $y = \pi(3)$. For every $L \in \mathcal{L}_3$, traversing L starting at x_L, let u_L be the last neighbor of x in L, let z^1_L, z^2_L be the first and the last neighbor of z in L, respectively, and let v_L be first neighbor of y in L. Since $\mathfrak{m}$ an alignment with respect to π, it follows that u_L, z^1_L, z^2_L and v_L are all distinct, appearing on L in this order. Also, since z is L-bad, it follows that $N_L(z) = \{z^1_L, z^2_L\}$ is a clique of G. Since G is $(K_{3,3}, K_{t+1})$-free and from the choice of λ_1, we can apply Lemma 3.12 to the sets $(\{z^1_L, z^2_L\} : L \in \mathcal{L}_{\mathfrak{m}})$ to deduce the following:

(62) *There is a* Λ_1*-subset* $\mathcal{L}_1$ *of* $\mathcal{L}_{\mathfrak{m}}$ *for which the sets* $(\{z^1_L, z^2_L\} : L \in \mathcal{L}_1)$ *are pairwise anticomplete in* G*.*

Next, we launch the first application of Theorem 7.2. Note that $\{z\text{-}z^2_L\text{-}L\text{-}v_L\text{-}y : L \in \mathcal{L}_1\}$ is a set of Λ_2 pairwise internally disjoint paths in G between non-adjacent vertices z and y. Consequently, by to the choice of ψ_2, we can apply Theorem 7.2 to this collection, and deduce that there exists $L_3 \in \mathcal{L}'_1$ as well as $\mathcal{L}'_2 \subseteq \mathcal{L}_1 \setminus \{L_3\}$ with $|\mathcal{L}'_2| = \lambda_2$ such that:

- $\{z^2_L : L \in \mathcal{L}'_2\} \cup \{z^2_{L_3}, y\}$ is a stable set in G (though this already holds by (62)); and
- for all $L \in \mathcal{L}'_2$, the vertex $z^2_{L_3}$ has a neighbor in the interior of $z^2_L\text{-}L\text{-}v_L\text{-}y$.

For each $L \in \mathcal{L}'_2$, traversing z^2_L-L-v_L-y from z^2_L to y, let w^1_L, w^2_L be the first and the last neighbors of $z^2_{L_3}$ in z^2_L-L-v_L-y, respectively; it follows that $\{w^1_L, w^2_L\} \cap \{z^2_L, y\} = \varnothing$ and $z^1_L, z^2_L, w^1_L, w^2_L$ appear on L in this order. Let C_L denote the hole a-x-u_L-L-v_L-y-a in G. We deduce that:

(63) *The vertex $z^2_{L_3}$ is C_L-bad for every $L \in \mathcal{L}'_2$. Explicitly, w^1_L and w^2_L are distinct and adjacent, and we have $N_{C_L}(z^2_{L_3}) = \{w^1_L, w^2_L\}$.*

To see this, note that z and $z^2_{L_3}$ are two adjacent vertices in $G \setminus C_L$, each with at least one neighbor in C_L. Indeed, we have $N_{C_L}(z) = \{a, z^1_L, z^2_L\}$, and so z is C_L-ugly. Since a is anticomplete to L_3 and from (62), it follows that $z^2_{L_3}$ is anticomplete to $\{a, z^1_L, z^2_L\} = N_{C_L}(z)$. Therefore, z and $z^2_{L_3}$ have no common neighbor in C_L, and so by Theorem 7.5, $z^2_{L_3}$ is C_L-bad. This proves (63).

Since G is $(K_{3,3}, K_{t+1})$-free and from the choice of λ_2, we can apply Lemma 3.12 to the sets $\{\{w^1_L, w^2_L, z^1_L, z^2_L\} : L \in \mathcal{L}'_2\}$, and deduce that:

(64) *There is a Λ_2-subset $\mathcal{L}_2$ of $\mathcal{L}'_2$ for which the sets $(\{w^1_L, w^2_L, z^1_L, z^2_L\} : L \in \mathcal{L}_2)$ are pairwise anticomplete in G.*

It is time for the second application of Theorem 7.2. Note that $\{z^2_{L_3}$-w^2_L-L-v_L-y : $L \in \mathcal{L}_2\}$ is a set of Λ_2 pairwise internally disjoint paths in G between non-adjacent vertices $z^2_{L_3}$ and y. Together with the choice of Λ_2, this allows us to apply Theorem 7.2 to $\{z^2_{L_3}$-w^2_L-L-v_L-$y : L \in \mathcal{L}_1\}$, and obtain two paths $L_1, L_2 \in \mathcal{L}_1$ such that

- $\{w^2_{L_1}, w^2_{L_2}, y\}$ is a stable set in G (though this already follows from (64)); and
- the vertex $w^2_{L_2}$ has a neighbor in the interior of $w^2_{L_1}$-L_1-v_L-y.

Traversing $w^2_{L_1}$-L_1-v_L-y from $w^2_{L_1}$ to y, let w, w' be the first and the last neighbors of $w^2_{L_2}$ in $w^2_{L_1}$-L_1-v_L-y, respectively; it follows that $\{w, w'\} \cap \{w^2_{L_1}, y\} = \varnothing$ and $w^1_{L_1}, w^2_{L_1}, w, w'$ appear on L_1 in this order. In addition, we have:

(65) *The vertex $w^2_{L_2}$ is C_{L_1}-bad. More precisely, w and w' are distinct and adjacent, and we have $N_{C_{L_1}}(w^2_{L_2}) = \{w, w'\}$.*

Let $C_1 = a$-z-$z^2_{L_1}$-L_1-v_L-y-a; then C_1 is a hole in G. Note that $w^2_{L_2}$ and $z^2_{L_3}$ are two adjacent vertices in $G \setminus C_1$, each with at least one neighbor in C_1. In fact, we have $N_{C_1}(z^2_{L_3}) = \{z, w^1_{L_1}, w^2_{L_1}\}$, and so $z^2_{L_3}$ is C_1-ugly. This, along with (64), implies that $w^2_{L_2}$

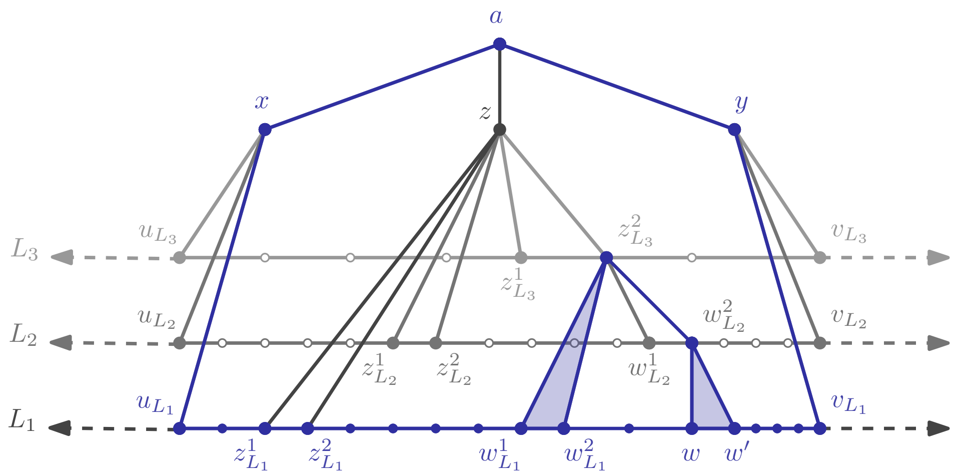

Figure 7.2: Proof of Theorem 7.6.

is anticomplete to $\{z, w^1_{L_1}, w^2_{L_1}\} = N_{C_1}(z^2_{L_3})$. It follows that $w^2_{L_2}$ and $z^2_{L_3}$ have no common neighbor in C_1. But then by Theorem 7.5, $w^2_{L_2}$ is C_1-bad. More precisely, w and w' are distinct and adjacent, and we have $N_{C_1}(w^2_{L_2}) = \{w, w'\}$. Since $C_{L_1} \setminus C_1 = x\text{-}u_{L_1}\text{-}L_1\text{-}z^1_{L_1}$, it remains to show that $w^2_{L_2}$ is anticomplete to $u_{L_1}\text{-}L_1\text{-}z^1_{L_1}$. Suppose not. Recall that by (64), a and $w^2_{L_2}$ are not adjacent in G. Consequently, there is a path Q of length at least two in G from a to $w^2_{L_2}$ such that Q^* is contained in the interior of $a\text{-}x\text{-}u_{L_1}\text{-}L_1\text{-}z^1_{L_1}$. But then by (63), there is a theta in G with ends $a, w^2_{L_2}$ and paths $a\text{-}z\text{-}z^2_{L_3}\text{-}w^2_{L_2}$, $a\text{-}y\text{-}v_{L_1}\text{-}L_1\text{-}w'\text{-}w^2_{L_2}$ and Q, a contradiction. This proves (65).

Finally, by (63) and (65), there is a prism in G with triangles $\{z^2_{L_3}, w^1_{L_1}, w^2_{L_1}\}$ and $\{w^2_{L_2}, w, w'\}$ and paths $z^2_{L_3}\text{-}w^2_{L_2}$, $w^2_{L_1}\text{-}L_1\text{-}w$ and $w^1_{L_1}\text{-}L_1\text{-}u_{L_1}\text{-}x\text{-}a\text{-}y\text{-}v_{L_1}\text{-}L_1\text{-}w'$ (see Figure 7.2), a contradiction. This completes the proof of Theorem 7.6. ■

7.3 Enhanced mirroring

Using the material from Section 7.2, here we give a proof of Theorem 7.1, restated below:

Theorem 7.1. *For all $d, t, w \in \mathbb{N}$, there is a constant $f_{7.1} = f_{7.1}(d, t, w) \in \mathbb{N}$ with the following property. Let G be a (theta, prism, even wheel, K_{t+1})-free graph, let $(a, x, y, \mathcal{W})$ be an $f_{7.1}$-kaleidoscope in G, and let $z \in V(G)$ be 1-mirrored by $(a, x, y, \mathcal{W})$. Then there exists $\mathcal{W}' \subseteq \mathcal{W}$ with $|\mathcal{W}'| = w$ such that z is d-mirrored by $(a, x, y, \mathcal{W}')$.*

Proof. We will be using Lemma 3.12 and Theorems 7.2 and 7.6. Specifically, let

$$\lambda = g_{7.6}(1, 4, t);$$

$$\sigma = f_{7.2}(\lambda, 30, t);$$

$$\varphi = f_{3.12}(\sigma, d, 3, t + 1).$$

We claim that

$$f_{7.1} = f_{7.1}(d, t, w) = \varphi + w$$

satisfies the theorem.

Let $\mathcal{W}' \subseteq \mathcal{W}$ be the set of all paths $W \in \mathcal{W}$ for which z has at least d neighbors in W. If $|\mathcal{W}'| \geq w$, the we are done. For the rest of the proof, we assume that $|\mathcal{W}'| < w$, and will derive a contradiction. From the choice of $f_{7.1} = |\mathcal{W}|$, it follows that there exists $\mathcal{W}_0 \subseteq \mathcal{W}$ with $|\mathcal{W}_0| = \varphi$ such that z has less than d neighbors in each path $W \in \mathcal{W}_0$.

For every $W \in \mathcal{W}_0$, traversing W from x to y, let x_W be the last neighbor of x in W, let u^1_W, u^2_W be the first and the last neighbor of z in W, respectively, and let y_W be the first neighbor of y in W. It follows that the vertices $x, x_W, u^1_W, u^2_W, y_W, y$ appear on W in this order, and u^1_W, u^2_W are the only two vertices among them which may be the same. Since G is $(K_{3,3}, K_{t+1})$-free and by the choice of $\varphi = |\mathcal{W}_0|$, it follows from Lemma 3.12 applied to the sets $(N_W(z) : W \in \mathcal{W}_0)$ that:

(66) *There is a σ-subset $\mathcal{W}_1$ of $\mathcal{W}_0$ for which the sets $(N_W(z) : W \in \mathcal{W}_1)$ are pairwise anticomplete in G.*

Note that $\{z\text{-}u^1_W\text{-}W\text{-}x_W\text{-}x : W \in \mathcal{W}_1\}$ is a set of σ pairwise internally disjoint paths in G between non-adjacent vertices z and x. Consequently, by the choice of σ, we can apply Theorem 7.2 and deduce that there are two disjoint subsets $\mathcal{W}_2$ and $\mathcal{W}_3$ of $\mathcal{W}_1$ with $|\mathcal{W}_2| = \sigma$ and $|\mathcal{W}_3| = \lambda$, such that:

- $\{u^1_W : W \in \mathcal{W}_2 \cup \mathcal{W}_3\} \cup \{x\}$ is a stable set in G (though this is already guaranteed by (66) and (M3) as z is 1-mirrored by $(a, x, y, \mathcal{W})$); and
- for every $W \in \mathcal{W}_2$ and every $W' \in \mathcal{W}_3$, the vertex u^1_W has a neighbor in the interior of the path $L_{W'} = u^1_{W'}\text{-}W'\text{-}x_{W'}\text{-}x$.

Let $S = \{u^1_W : W \in \mathcal{W}_2\}$ and let $\mathcal{L} = \{L^*_{W'} : W' \in \mathcal{W}_3\}$. Then we have $|S| = 30$ and $|\mathcal{L}| = \lambda$, and so $(S, \mathcal{L})$ is a $(3(4 + 6), g_{7.6}(1, 4, t))$- constellation in G, where $z \in V(G) \setminus S \cup V(\mathcal{L})$ is complete to S and anticomplete to $V(\mathcal{L})$ in G. This allows for an application of Theorem 7.6 (with $l = 1$, $s = 4$ and the same choice of t). It follows that there exist $W_1, W_2, W_3, W_4 \in \mathcal{W}_2$ and $W' \in \mathcal{W}_3$ such that the following hold.

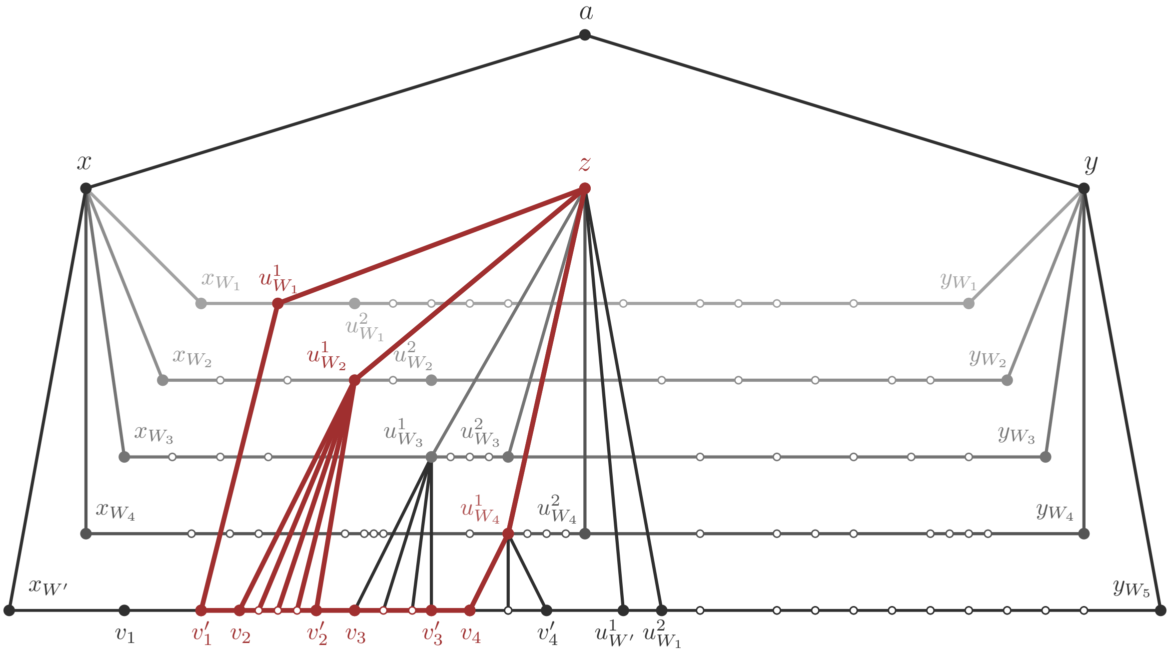

Figure 7.3: Proof of (67).

- For every $i \in [4]$, $u^1_{W_i}$ is $L_{W'}$-ugly; and
- $(\{u^1_{W_i} : i \in [4]\}, L^*_{W'})$ is a 4-asterism in G which is also a 4-alignment in G with respect to the order π given by $\pi(u^1_{W_i}) = i$ for each $i \in [4]$, and traversing $L^*_{W'}$ starting at x, the vertex $u^1_{W_1}$ is the first with a neighbor in $L^*_{W'}$.

Next, for each $i \in [4]$, traversing $L^*_{W'}$ starting at $x_{W'}$, let v_i, v'_i be the first and the last neighbors of $u^1_{W_i}$ in $L^*_{W'}$, respectively; it follows that $\{v_i, v'_i\} \cap \{x_{W'}, u^1_{W'}\} = \varnothing$ and the vertices $x_{W'}, v_1, v'_1, v_2, v'_2, v_3, v'_3, v_4, v'_4, u^1_{W'}, u^2_{W'}, y_{W'}$ appear on W' in this order. Let $C = a\text{-}x\text{-}x_{W'}\text{-}W'\text{-}y_{W'}\text{-}y\text{-}a$. Then C is a hole in G and $u^1_{W_1}$ and z are two adjacent vertices in $G \setminus C$, each with a neighbor in C. Also, $u^1_{W_1}$ is $L_{W'}$-ugly, and so C-ugly, and by (66), $u^1_{W_1}$ and z have no common neighbor in C. This, combined with Theorem 7.5, implies that z is C-bad. More precisely, a and z are not adjacent (though we do not use this), and $N_C(z) = N_{W'}(z) = \{u^1_{W'}, u^2_{W'}\}$ is a 2-clique in G. We further deduce that:

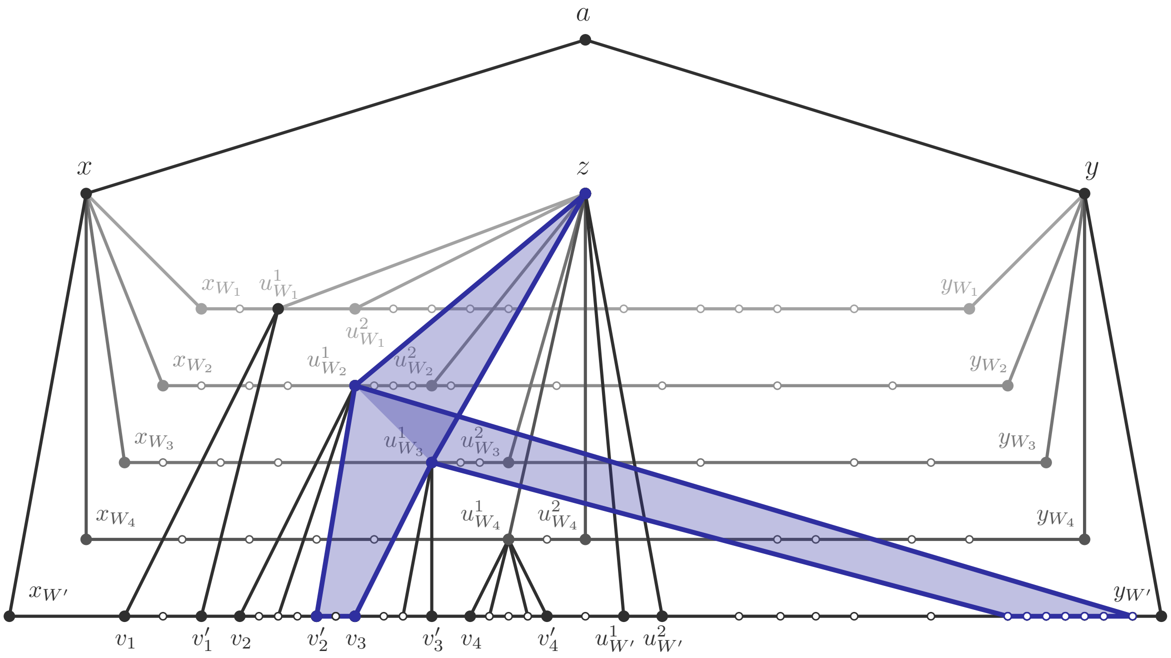

Figure 7.4: Proof of Theorem 7.1.

(67) *The vertices* $u^1_{W_2}$ *and* $u^1_{W_3}$ *each have at least one neighbor in the interior of* $u^2_{W'}$-W'-$y_{W'}$-y.

Suppose not. Then by (66), $u^1_{W_i}$ is anticomplete to $u^1_{W'}$-$u^2_{W'}$-W'-$y_{W'}$. Let $C' =$ z-$u^1_{W_1}$-v'_1-W'-v_4-$u^1_{W_4}$-z. Then C' is a hole in G and we have $N_{C'}(u^1_{W_i}) = N_{L_{W'}}(u^1_{W_i}) \cup \{z\} = N_C(u^1_{W_i}) \cup \{z\}$. Also, $u^1_{W_i}$ is C-ugly, because it is $L_{W'}$-ugly. This, along with the fact that $G[V(C) \cup \{u^1_{W_i}\}]$ is not a theta in G and $(C, u^1_{W_i})$ is not an even wheel in G, implies that $|N_C(u^1_{W_i})|$ is an odd integer which is at least three. But then $(C', u^1_{W_i})$ is an even wheel in G, a contradiction (see Figure 7.3). This proves (67).

To finish the proof, note that by (67), there exists a path P in G from $u^1_{W_2}$ to $u^1_{W_3}$ such that P^* is contained in the interior of $u^2_{W'}$-W'-$y_{W'}$-y. But now there is a theta in G with ends $u^1_{W_2}, u^1_{W_3}$ and paths $u^1_{W_2}$-z-$u^1_{W_3}$, $u^1_{W_2}$-v'_2-W'-v_3-$u^1_{W_3}$ and P, a contradiction (see Figure 7.4). This completes the proof of Theorem 7.1. ■

Chapter 8

Even-hole-free graphs II. Phantom layered wheels[1]

Roughly, the proofs of our main results on even-hole-free graphs, namely Theorems 2.24 and 2.25, are in two steps. First, we prove that every even-hole-free graph of bounded clique number and sufficiently large treewidth has an induced subgraph which is an "approximate version" of a layered wheel of large treewidth. We call that induced subgraph a "phantom" in G. The second step, then, is to take the phantom on a roller coaster ride of Ramsey-type arguments, extracting more and more similarities with the structure of the layered wheels until the desired chordal induced subgraph is obtained.

Our goal in this section is to complete the first step, and the main result is Theorem 8.1 below. Before giving the formal statement, we must clarify what we mean by the "approximate version" of the layered wheels. This needs a slightly clearer picture of the structure of these graphs [71].

For an integer $r \in \mathbb{N}$, a layered wheel L of treewidth (at least) r is a K_{r+1}-minor model in which each branching set induces a path. This means that $V(L)$ can be partitioned into $r+1$ pairwise disjoint paths $L_0, \dots, L_r$ in L such that for all distinct $i, j \in [r] \cup \{0\}$, there is an edge of L with an end in $V(L_i)$ and an end in $V(L_j)$.

In fact, a lot more happens: *for each $i \in [r]$, every two vertices $z_1, z_2 \in V(L_0) \cup \dots \cup V(L_{i-1})$ that are adjacent in L have (many) "private" common neighbors in L_i.* This is crucial for the layered wheels to be even-hole-free (the reason can be traced back to Theorem 7.5), and our "approximate" layered wheels are exactly meant to capture this property.

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[1] This chapter is based on the coauthored paper [6] and the single-author paper [36].

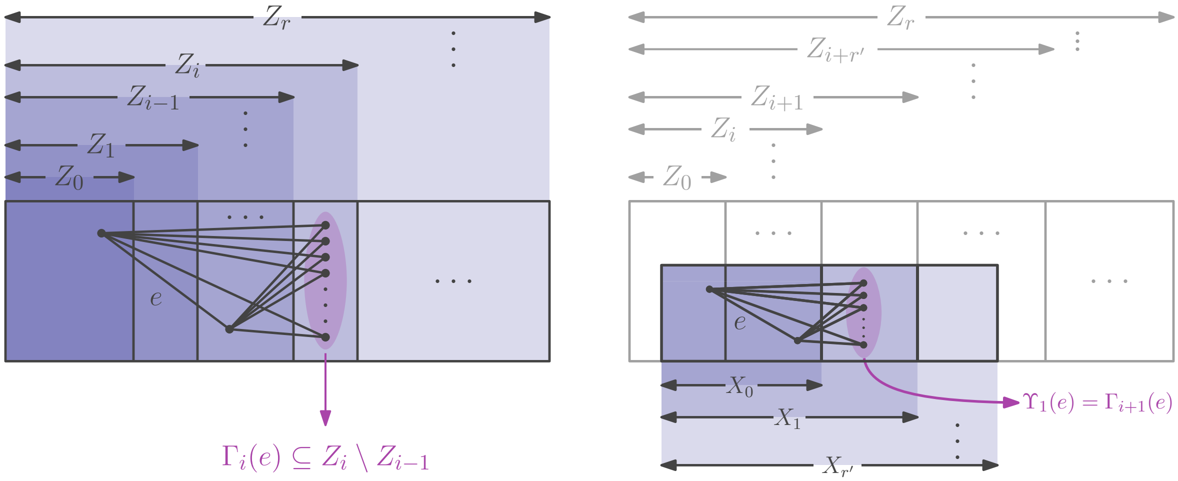

Figure 8.1: Left: A (Z_0, d, r)-phantom $\mathfrak{p}$. Right: The (X_0, d, r')-phantom $\mathfrak{p}[X_0; i, r']$ for $X_0 \subseteq Z_i$.

The precise definition is as follows. Let $d \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$, let G be a graph and let $Z_0 \subseteq V(G)$. A *(Z_0, d, r)-phantom in G* is a $(2r+1)$-tuple $\mathfrak{p} = (Z_0, \dots, Z_r; \Gamma_i : i \in [r])$ where the following hold for every $i \in [r]$ (see Figure 8.1).

(PH1) Z_i is a subset of $V(G)$ containing Z_{i-1}.

(PH2) $\Gamma_i : E(G[Z_{i-1}]) \to (Z_i \setminus Z_{i-1})^{(d)}$ is a map with the following specifications.

- For every $e \in E(G[Z_{i-1}])$, the ends of e are complete to $\Gamma_i(e)$.
- The sets $(\Gamma_i(e) : e \in E(G[Z_{i-1}]))$ are pairwise disjoint.

In particular, we have $Z_0 \subseteq \cdots \subseteq Z_r \subseteq V(G)$.

For a phantom $\mathfrak{p}$ as above, we will often use a natural notion of a "sub-phantom" of $\mathfrak{p}$, which we define next. Let $i, r' \in \{0, \dots, r\}$ such that $i + r' \leq r$ and let $X_0 \subseteq Z_i$. We denote by $\mathfrak{p}[X_0; i, r']$ the (X_0, d, r')-phantom $(X_0, \dots, X_{r'}; \Upsilon_i : i \in [r'])$ which is defined recursively, as follows (again, see Figure 8.1). For each $j \in [r']$ with X_{j-1} already defined, let

$$X_j = X_{j-1} \cup \left(\bigcup_{e \in E(G[X_{j-1}])} \Gamma_{i+j}(e) \right)$$

and let

$$\Upsilon_j = \Gamma_{i+j}\Big|_{E(G[X_{j-1}])}.$$

We leave it to the reader to check that the above definition does yield a (X_0, d, r')-phantom. Also, for each $j \in \{0, \ldots, r'\}$, we have $X_j \subseteq Z_{i+j}$, and for each $j \in [r']$, we have $X_i \setminus X_{i-1} \subseteq Z_{i+j} \setminus Z_{i+j-1}$.

As mentioned earlier, we would like to show that every even-hole-free graph of bounded clique number and huge treewidth contains a large phantom. We need this to be true both when Z_0 is a 2-clique and when Z_0 is a 3-clique. Accordingly, our main result in this chapter says that:

Theorem 8.1. *For all $d, t \in \mathbb{N}$ and $r_1 \in \mathbb{N} \cup \{0\}$, there is a constant $f_{8.1} = f_{8.1}(d, r_1, t) \in \mathbb{N}$ with the following property. Let G be a (theta, prism, even wheel, C_4, K_{t+1})-free graph with $\operatorname{tw}(G) > f_{8.1}$. Then there is a (Z_0^1, d, r_1)-phantom in G for some 2-clique Z_0^1. Moreover, for every $r_2 \in \mathbb{N}$ with $r_2 < r_1$, there is a (Z_0^2, d, r_2)-phantom in G for some 3-clique Z_0^2.*

8.1 Securing a kaleidoscope

We will prove Theorem 8.1 in Section 8.3. The proof will be by induction on r_1. For the induction step to work, we will have to prove a strengthening of Theorem 8.1, namely, Theorem 8.8, asserting that the desired phantom may always be found such that its vertices are "sufficiently mirrored" by a relatively large kaleidoscope. The base case $r_1 = 0$, however, needs to be worked out separately, and that is what we will do in this section. Explicitly, we show that:

Theorem 8.2. *For all integers $d, t, w \in \mathbb{N}$, there is a constant $f_{8.2} = f_{8.2}(d, t, w) \in \mathbb{N}$ with the following property. Let G be a (theta, prism, even wheel, K_{t+1})-free graph with $\operatorname{tw}(G) > f_{8.2}$. Then there is a w-kaleidoscope $(a, x, y, \mathcal{W})$ in G as well as a 2-clique Z_0 in G such that Z_0 is d-mirrored by $(a, x, y, \mathcal{W})$.*

Proof. The proof relies on three earlier results: Theorems 2.27, 7.1 and 7.2. Specifically, let

$$f_1 = f_{7.1}(d, t, w);$$

$$\varphi_1 = f_{7.2}(1, f_1 + (t+1)^3, t);$$

$$f_2 = f_{7.1}(d, t, \varphi_1);$$

$$\lambda = g_{7.6}(f_2, 3, t);$$

$$\varphi_2 = f_{7.2}(\lambda, 27, t).$$

We will show that

$$f_{8.2} = f_{8.2}(d, t, w) = f_{2.27}(\varphi_2, 2, t+1)$$

satisfies the theorem. Let G be a (theta, prism, even wheel, K_{t+1})-free graph with $\operatorname{tw}(G) > f_{8.2}$. Then G is $(t+1)$-clean, and by Theorem 2.27 and the choice of $f_{8.2}$, there are distinct and non-adjacent vertices $a, b \in V(G)$ and a set $\mathcal{P}$ of φ_2 pairwise internally disjoint paths in G from a to b. From the choice of φ_2 and Theorem 7.2, it follows that there is a subset $\mathcal{N}$ of $\mathcal{P}$ where $N_{V(\mathcal{N})}(a)$ is stable set in G, and an $(30, l)$-constellation $\mathfrak{c}$ in G such that $S_{\mathfrak{c}} \subseteq N_{V(\mathcal{N})}(a)$ and $\mathcal{L}_{\mathfrak{c}} \subseteq \{P^* : P \in \mathcal{N}\}$.

For every $R \in \mathcal{L}_{\mathfrak{c}}$, let $L_R = R \setminus N_R(a)$, and let x'_{L_R} be the unique vertex in $N^2_R(a)$. Then x'_{L_R} is an end of L_R. Moreover, since $N_{V(\mathcal{N})}(a)$ is stable set in G, it follows that every vertex $x \in S_{\mathfrak{c}}$ has a neighbor in L_R for every $R \in \mathcal{L}_{\mathfrak{c}}$. Consequently, $(S_{\mathfrak{c}}, \{L_R : R \in R \in \mathcal{L}_{\mathfrak{c}}\})$ is a $(3(3+6), \lambda)$-constellation in G. Note also that a is complete to $S_{\mathfrak{c}}$, by the definition, a is anticomplete to L_R for every $R \in R \in \mathcal{L}_{\mathfrak{c}}$. This, along with the choice of λ and Theorem 7.6 (for $l = f_2$, $s = 3$ and the same choice of t), implies that there exist $x, y, z \in S_{\mathfrak{c}}$ and and f_2-subset $\mathcal{L}'$ of $\{L_R : R \in R \in \mathcal{L}_{\mathfrak{c}}\}$ with such that The following holds for every $L \in \mathcal{L}'$.

- The vertices x, y, z are all L-ugly.
- The pair $(\{x, y, z\}, L)$ is a 3-asterism in G which is also a 3-alignment in G with respect to the order π given by $\pi(x) = 1$, $\pi(y) = 2$ and $\pi(z) = 3$, where traversing L_R starting at x'_L, the vertex x is the first with a neighbor in L.

For every $L \in \mathcal{L}'$, traversing L starting at x'_L, let u_L be the last neighbor of x in L, let z_L be the last neighbor of z_1 in L and let v_L be the first neighbor of y in L. Let $W_L = x\text{-}u_L\text{-}L\text{-}v_L\text{-}y$. Then W_L is a path in G from x to y and we have $z_L \in W_L \setminus (N_{W_L}[x] \cup N_{W_L}[y])$. In particular:

(68) *For every $L \in \mathcal{L}'$, the vertex z_1 is anticomplete to $N_{W_L}[x] \cup N_{W_L}[y]$ and z_1 has a neighbor in W_L (namely z_L).*

From (68), it follows that $(a, x, y, \{W_L : L \in \mathcal{L}'\})$ is a f_2-kaleidoscope in G by which z_1 is 1-mirrored. This, along with the choice of f_2 and Theorem 7.1, implies that there is a φ_1-subset $\mathcal{L}_1$ of $\mathcal{L}'$ such that z_1 is d-mirrored by the φ_1-kaleidoscope $(a, x, y, \{W_L : L \in \mathcal{L}_1\})$.

For every path $L \in \mathcal{L}_1$, let $P_L = z_1\text{-}z_L\text{-}L\text{-}v_L\text{-}y$. Then $\mathcal{P}' = \{P_L : L \in \mathcal{L}_1\}$ is a set of φ_1 pairwise internally disjoint paths in G between the two non-adjacent vertices z_1 and y. By Theorem 7.2, this time applied to z_1, y and $\mathcal{P}'$, there are $L_0, L_1, \dots, L_{f_1+(t+1)^3} \in \mathcal{L}'$ such that $\{z_{L_i} : i \in [f_1 + (t+1)^3] \cup \{0\}\}$ is a stable set in G, and for every $j \in [f_1 + (t+1)^3]$, the vertex z_{L_0} has a neighbor in $P^*_{L_j} \subseteq W_{L_j}$.

Let $z_2 = z_{L_0}$. It follows that z_2 is anticomplete to $\{a, x, y\}$, and for each $j \in [f_1+(t+1)^3]$, z_2 has a neighbor in W_{L_j}. Furthermore, we have:

(69) *The vertex z_2 is anticomplete to $\{u_{L_j} : j \in [f_1 + (t + 1)^3]\}$. Also, there exists $I \subseteq [f_1 + (t+1)^3]$ with $|I| = f_1$ for which z_2 is anticomplete to $\{v_{L_j} : j \in I\}$. Consequently, for every $j \in I$, z_2 is anticomplete to $\{a\} \cup N_{W_{L_j}}[x] \cup N_{W_{L_j}}[y]$, and z_2 has a neighbor in W_{L_j}.*

To see the first assertion, note that for all $j \in [f_1 + (t + 1)^3]$, there is a path Q_j in G from z_2 to y with $Q^*_j \subseteq P^*_{L_j} \setminus \{z_{L_j}\}$; in particular, u_{L_j} is anticomplete to Q^*_j. Therefore, if z_2 is adjacent to u_{L_j} for some $j \in [f_1 + (t + 1)^3]$, then there is a theta in G with ends a, z_2 and paths a-x-u_{L_j}-z_2, a-z_1-z_2 and a-y-Q_j-z_2, a contradiction. Now we prove the second assertion. Suppose for a contradiction that there is a $(t + 1)^3$-subset I' of $[f_1 + (t + 1)^3]$ such that z_2 is complete to $V' = \{v_{L_j} : j \in I'\}$. Since G is K_{t+1}-free, it follows from Theorem 3.17 applied to $G[V']$ that there is a stable set $\{v, v', v''\}$ in G contained in V'. But then there is a theta in G with ends z_2, y and paths z_2-v-y, z_2-v'-y and z_2-v''-y, a contradiction. This proves (69).

Let I be as in (69). Then $(a, x, y, \{W_{L_j} : j \in I\})$ is a f_1-kaleidoscope in G by which z_2 is 1-mirrored. From the choice of f_1 and Theorem 7.1 applied to $(a, x, y, \{W_{L_j} : j \in I\})$ and z_2, we conclude that there is a w-subset $\mathcal{W} \subseteq \{W_{L_j} : j \in I\} \subseteq \{W_L : L \in \mathcal{L}'\}$ such that z_2 is d-mirrored by the w-kaleidoscope $(a, x, y, \mathcal{W})$. Moreover, z_1 is d-mirrored $(a, x, y, \mathcal{W})$ because it is d mirrored by $(a, x, y, \{W_L : L \in \mathcal{L}'\}$. Hence, $Z_0 = \{z_1, z_2\}$ is d-mirrored by $(a, x, y, \mathcal{W})$. This completes the proof of Theorem 8.2. ■

8.2 Deus ex machina

In this section, we will show how the most critical challenge in the proof of Theorem 8.1 is surmounted through an utterly unexpected "trick." As said in the previous section, the proof of Theorem 8.1 is by induction on r_1. In the induction step, we are given a $(Z_0, d, i - 1)$-phantom for some $i \in [r_1]$, and our goal is, for each edge $e \in E(G[Z_{i-1}])$, to find a set $\Gamma_i(e)$ of d common neighbors of the ends of e (outside Z_{i-1}) and then define Z_i as the union of these sets and Z_{i-1}. To that end, since we know the phantom is sufficiently mirrored by a kaleidoscope, all we need is, for each edge $e \in E(G[Z_{i-1}])$, to find a large constellation $(S, \mathcal{L})$ where the ends of e are complete to S and anticomplete to $\mathcal{L}$.

A natural tool to try on this problem would therefore be Theorem 7.2. But there are two major difficulties: first, here we are dealing with paths coming out of the common

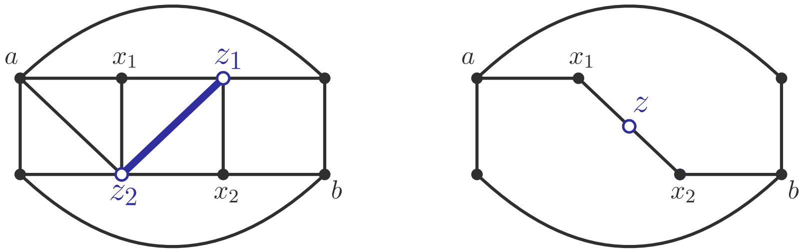

Figure 8.2: Left: a (theta, prism, even wheel)-free graph G containing a clique $\{z_1, z_2\}$ for which $N_G(z_1) \cap N_G(z_2) = \{x_1, x_2\}$ is a stable set of vertices of degree three in G (observe that G does contain C_4). Right: the z_1z_2-contraption of G which is a theta with ends a, b.

neighbors of two adjacent vertices (the ends of e) rather than a single vertex (called a in Theorem 7.2), and second, the said paths are not necessarily induced paths in G because the end of e may have further (private) neighbors along each path.

Surprisingly, it turns out we can surmount the above obstacles by simply "pretending" they do not exist. This is formally stated in Theorem 8.3 below, which we will prove in this section.

In essence, Theorem 8.3 says that certain minors of (theta, prism, even wheel, C_4)-free graphs are also (theta, prism, even wheel, C_4)-free. Recall that a minor of G is a graph that is obtained from G by a sequence of vertex deletions, edge deletions and *edge contractions*, where for an edge xy of a graph G, the *xy-contraption* of G is the graph obtained from G by identifying the two vertices x and y into a single vertex.

For a graph G and two adjacent vertices $z_1, z_2 \in V(G)$, we define the *z_1z_2-contraption* of G to be the minor of G (without parallel edges) obtained by first contracting the edge z_1z_2 into a new vertex z, and then removing every edge in the resulting graph between z and a vertex in the symmetric difference of $N_G(z_1)$ and $N_G(z_2)$. In other words, the z_1z_2-contraption of G is the graph $\widetilde{G}$ with the following specifications:

- $V(\widetilde{G}) = (V(G) \setminus \{z_1, z_2\}) \cup \{z\}$;
- $\widetilde{G}[V(G) \setminus \{z_1, z_2\}] = G \setminus \{z_1, z_2\}$; and
- $N_{\widetilde{G}}(z) = N_G(z_1) \cap N_G(z_2)$.

See Figure 8.2. The main result of this section says that:

Theorem 8.3. *Let G be a (theta, prism, even wheel, C_4)-free graph and let $z_1, z_2 \in V(G)$ be distinct and adjacent such that $N_G(z_1) \cap N_G(z_2)$ is a stable set of vertices of degree at most three in G. Then the z_1z_2-contraption of G is also (theta, prism, even wheel, C_4)-free.*

It is worth noting that, as far as our application of Theorem 8.3 is concerned, it suffices to show that $\widetilde{G}$ is (theta, prism)-free (under the same assumptions). This is thanks to Theorem 7.2 holding true for the larger class of (theta, prism)-free graphs. Also, the proof of Theorem 8.3, hence its application in the proof of Theorem 8.1, is the only place in the proof of Theorem 2.24 and 2.25 where we use the assumption that G is C_4-free. As unfortunate as it may appear, however, excluding C_4 is necessary even if we ask for the contraption not to "be" a theta; see Figure 8.2.

We now plunge into the proof of Theorem 8.3, beginning with the following lemma.

Lemma 8.4. *Let G be a (theta, prism, even wheel, C_4)-free graph and let $z_1, z_2 \in V(G)$ be distinct and adjacent such that $N_G(z_1) \cap N_G(z_2)$ is a stable set of vertices of degree at most three in G. Let $\widetilde{G}$ be the z_1z_2-contraption of G with $z \in V(\widetilde{G})$ as in the definition, and let W be an induced subgraph of $\widetilde{G}$ which is either a theta, or a prism, or an even wheel. Then there is a path P in W with ends a, b for which the following hold.*

- (a) *We have $z \in P \setminus (N_W[a] \cup N_W[b])$, and so $V(W) \setminus P^* \subseteq V(G) \setminus (N_G[z_1] \cap N_G[z_2])$.*
- (b) *In W, the vertices in P^* (including z) have degree two, and a, b have degree three.*
- (c) *In G, both z_1 and z_2 have a neighbor in $V(W) \setminus V(P)$.*

Proof. First, assume that there is no path P in W satisfying 8.4(a) and 8.4(b). Since G is (theta, prism, even wheel, C_4)-free, it follows that $z \in W$. Also, since $N_G(z_1) \cap N_G(z_2)$ is a stable set of vertices of degree at most three in G, it follows that $N_{\widetilde{G}}(z)$ is a stable set of vertices of degree at most two in $\widetilde{G}$. In particular, there is no wheel (C, v) in $\widetilde{G}$ where $z \in N_C[v]$, and z does not belong to a triangle of a prism in $\widetilde{G}$. Moreover, from the assumption that there is no path in W satisfying 8.4(a) and 8.4(b), it follows that there is no wheel (C, v) in $\widetilde{G}$ where $z \in C \setminus N_C(v)$, and z does not belong the interior of a path of a theta or a prism in $\widetilde{G}$. We deduce that W is a theta in $\widetilde{G}$ and z is an end of W. Let $z' \in V(\widetilde{G}) \setminus N_{\widetilde{G}}[z] = V(G) \setminus (N_G[z_1] \cap N_G[z_2])$ be the other end of W and let P_1, P_2, P_3 be the paths of W. Then for every $i \in [3]$, P_i has ends z, z', and for some $j \in \{1, 2\}$, z_j is not adjacent to z' in G. On the other hand, for every $i \in [3]$, we have $N_{P_i}(z) \subseteq N_G(z_j) \cap (P_i \setminus \{z\})$. Thus, traversing $P_i \setminus \{z\}$ starting at z', we may choose x_i to be the first vertex in $N_G(z_j) \cap (P_i \setminus \{z\})$; it follows that $x_i \in P_i^*$. But then there is a theta in G with ends z_j, z' and paths z_j-x_i-P_i-z' for $i \in [3]$, which violates the assumption that G is theta-free. This proves that there exists a path P in W with ends a, b for which 8.4(a) and 8.4(b) hold.

It remains to show that P satisfies 8.4(c). Suppose for a contradiction, and without loss of generality, that z_1 is anticomplete to $W \setminus P$ in G. Then $U = (P \setminus \{z\}) \cup \{z_1\}$

is a connected induced subgraph of G with $a, b \in U$ such that $U \setminus \{a, b\} \subseteq P^* \cup \{z_1\}$ is anticomplete to $W \setminus P$ in G. Consequently, there exists a path P_1 in U from a to b where P_1^* is anticomplete to $W \setminus P$ in G. But now $(W \setminus P) \cup P_1$ is a theta, a prism or an even wheel in G (depending on whether W is a theta, a prism or an even wheel in G, respectively), which is impossible because G is (theta, prism, even wheel)-free. This completes the proof of Lemma 8.4. ■

The next two results, in turn, show that under the assumptions of Theorem 8.3, the $z_1 z_2$-contraption is theta-free and prism-free.

Theorem 8.5. *Let G be a (theta, prism, even wheel, C_4)-free graph and let $z_1, z_2 \in V(G)$ be distinct and adjacent such that $N_G(z_1) \cap N_G(z_2)$ is a stable set of vertices of degree at most three in G. Then the $z_1 z_2$-contraption of G is theta-free.*

Proof. Suppose for a contradiction that there is a theta W in the $z_1 z_2$-contraption $\widetilde{G}$ of G. Let $z \in V(\widetilde{G})$ be as in the definition of a contraption. Let P be the path in W with ends a, b satisfying Lemma 8.4. It follows from Lemma 8.4(a) and (b) that a, b are the ends of W, P is a path of W, and we have $z \in P \setminus (N_P[a] \cup N_P[b])$. Let Q_1, Q_2 be the paths of W distinct from P; so Q_1 and Q_2 both have ends a, b, as well. Let $C = Q_1 \cup Q_2$. Then C is a hole in $G \setminus \{z_1, z_2\}$ and we have $C = W \setminus P^*$.

From the definition of $\widetilde{G}$, it follows that $W \setminus \{z\} \subseteq G \setminus \{z_1, z_2\}$ and $\{z_1, z_2\}$ is complete to $N_P(z)$. As a result, for every $i \in \{1, 2\}$, there are two paths $P_{a,i}, P_{b,i}$ in $(P \setminus \{z\}) \cup \{z_i\}$ from a to z_i and from b to z_i, respectively, such that $P_{a,i} \setminus \{z_i\}$ and $P_{b,i} \setminus \{z_i\}$ are disjoint and anticomplete in G, and $P_{a,i}^* \cup P_{b,i}^*$ and $C \setminus \{a, b\}$ are disjoint and anticomplete to in G. We claim that:

(70) *Let $i \in \{1, 2\}$. Then either a is adjacent to z_i in G, or $N_C(z_i) \subseteq N_{Q_j}[b]$ for some $j \in \{1, 2\}$. Similarly, either b is adjacent to z_i in G, or $N_C(z_i) \subseteq N_{Q_j}[a]$ for some $j \in \{1, 2\}$.*

We only need to show that for every $i \in \{1, 2\}$, either a is adjacent to z_i in G, or $N_C(z_i) \subseteq N_{Q_j}[b]$ for some $j \in \{1, 2\}$. Suppose not. Then we may assume without loss of generality, that a is not adjacent to z_1 in G, and there is a vertex in $Q_1^* \setminus N_{Q_1}(b)$ which is adjacent to z_1 in G. It follows that $P_{a,1}$ has length at least two, and that there is a path R of length at least two in G from a to z_1 such that $R^* \subseteq Q_1^* \setminus N_{Q_1}(b)$. Also, since $P_{b,1} \cup Q_2$ is a connected induced subgraph of G containing the two non-adjacent vertices a and z_1, it follows that there exists a path S of length at least two in $P_{b,1} \cup Q_2$ from a to z_1. But then there is theta in G with ends a, z_1 and paths $P_{a,1}, R, S$, contrary to the assumption that G is theta-free. This proves (70).

From (70), it follows immediately that:

(71) *Let $i \in \{1,2\}$ such that z_i has a neighbor in C. Then the following hold.*

- *If z_i is C-good, then we have $N_C(z_i) \subseteq (N_C(a) \cap N_C(b)) \cup \{a,b\}$.*
- *If z_i is C-bad, then for some $j \in \{1,2\}$, either $N_C(z_i) = N_{Q_j}[a]$ or $N_C(z_i) = N_{Q_j}[b]$.*
- *If z_i is C-ugly, then we have $a, b \in N_C(z_i)$.*

Now, since $C \setminus \{a,b\} = W \setminus P$ and $W \setminus P \subseteq W \setminus P^* = C$, from Lemma 8.4(a) and the definition of $\widetilde{G}$ it follows that z_1 and z_2 have no common neighbor in C, and from Lemma 8.4(c), it follows that z_1 and z_2 each have at least one neighbor in $C \setminus \{a,b\}$. Consequently, by Theorem 7.5 and without loss of generality, we may assume that z_1 is C-bad and z_2 is either C-good or C-ugly. It follows from the second bullet of (71) that for some $j \in \{1,2\}$, we have either $N_C(z_1) = N_{Q_j}[a]$ or $N_C(z_1) = N_{Q_j}[b]$. We may exploit the symmetry between a, b and between Q_1, Q_2, and assume that $N_C(z_1) = N_{Q_1}[a]$. Since $a \in V(H)$ is not a common neighbor of z_1 and z_2, we deduce from the third bullet of (71) that z_2 is C-good. This, together with the first bullet of (71), the fact that z_2 has a neighbor in $C \setminus \{a,b\}$ and the fact that z_1 and z_2 have no common neighbor in C, implies that Q_2 has length two, say $Q_2 = a\text{-}q\text{-}b$, and we have $N_C(z_2) = \{q\}$. But then $G[\{a, q, z_1, z_2\}]$ is isomorphic to C_4, contrary to the assumption that G is C_4-free. This completes the proof of Theorem 8.5. ∎

Theorem 8.6. *Let G be a (theta, prism, even wheel, C_4)-free graph and let $z_1, z_2 \in V(G)$ be distinct and adjacent such that $N_G(z_1) \cap N_G(z_2)$ is a stable set of vertices of degree at most three in G. Then the z_1z_2-contraption of G is prism-free.*

Proof. Suppose for a contradiction that there is a prism W in the z_1z_2-contraption $\widetilde{G}$ of G. Let $z \in V(\widetilde{G})$ be as in the definition. Let P be the path in W with ends a, b satisfying Lemma 8.4. It follows from Lemma 8.4(a) and (b) that P is a path of W, a and b belong to distinct triangles of W and we have $z \in P \setminus (N_P[a] \cup N_P[b])$. Let $\{a, a_1, a_2\}$ and $\{b, b_1, b_2\}$ be the triangles of W and let Q_1, Q_2 be the paths of W distinct from P such that Q_i has ends a_i, b_i for $i \in \{1,2\}$. Let $C = Q_1 \cup Q_2$. Then C is a hole in $G \setminus \{z_1, z_2\}$ and we have $C = W \setminus P$.

From the definition of $\widetilde{G}$, it follows that $W \setminus \{z\} \subseteq G \setminus \{z_1, z_2\}$ and $\{z_1, z_2\}$ is complete to $N_P(z)$. As a result, for every $i \in \{1,2\}$, there are two paths $P_{a,i}, P_{b,i}$ in $(P \setminus \{z\}) \cup \{z_i\}$

from a to z_i and from b to z_i, respectively, such that $P_{a,i} \setminus \{z_i\}$ and $P_{b,i} \setminus \{z_i\}$ are disjoint and anticomplete in G, and both $P^*_{a,i}$ and $P^*_{b,i}$ are disjoint from and anticomplete to C in G.

Now, since $C = W \setminus P \subseteq W \setminus P^*$, it follows from Lemma 8.4(a) and the definition of $\widetilde{G}$ that z_1 and z_2 have no common neighbor in C, and it follows from Lemma 8.4(c) that z_1 and z_2 each have at least one neighbor in $C \setminus \{a, b\}$, Consequently, by Theorem 7.5, one of z_1 and z_2 is C-bad; say z_1 is C-bad. Let us write $N_C(z_1) = \{q_1, q_2\}$ where q_1 and q_2 are adjacent. Due to the symmetry between $\{a_1, a_2\}$ and $\{b_1, b_2\}$, we may assume, without loss of generality, that $\{a_1, a_2\} \cap \{q_1, q_2\} \subseteq \{a_1\}$, and there are disjoint paths R_1 and R_2 in C from a_1 to q_1 and from a_2 to q_2, respectively. It follows that either $\{a_1, a_2\} \cap \{q_1, q_2\} = \varnothing$ or $\{a_1, a_2\} \cap \{q_1, q_2\} = \{a_1\} = \{q_1\}$. In the former case, there is a prism in G with triangles $\{a, a_1, a_2\}, \{z_1, q_1, q_2\}$ and paths $P_{a,1}$, R_1 and R_2, a contradiction. Also, in the latter case, $C' = a\text{-}P_{a,1}\text{-}z_1\text{-}q_2\text{-}R_2\text{-}a_2\text{-}a$ is a hole in G and $a_1 = q_1 \in G \setminus C'$ has exactly four neighbors in C', namely a, a_2, q_2 and z_1. But then (C', z_1) is an even wheel in G, again a contradiction. This completes the proof of Theorem 8.6. ■

We can now restate and prove Theorem 8.3:

Theorem 8.3. *Let G be a (theta, prism, even wheel, C_4)-free graph and let $z_1, z_2 \in V(G)$ be distinct and adjacent such that $N_G(z_1) \cap N_G(z_2)$ is a stable set of vertices of degree at most three in G. Then the z_1z_2-contraption of G is also (theta, prism, even wheel, C_4)-free.*

Proof. Suppose not. Let $\widetilde{G}$ be the z_1z_2-contraption of G and let $z \in V(\widetilde{G})$ be as in the definition of $\widetilde{G}$. First, we show that:

(72) *$\widetilde{G}$ is C_4-free.*

To see this, suppose there is a hole C of length four in $\widetilde{G}$. Since G is C_4-free, it follows that $z \in V(C)$. So we have $N_C(z) = C \cap N_G(z_1) \cap N_G(z_2)$ and there exists exactly one vertex z' in C with $z' \in V(\widetilde{G}) \setminus N_{\widetilde{G}}[z] = V(G) \setminus (N_G[z_1] \cap N_G[z_2])$. As a result, for some $j \in \{1, 2\}$, z_j is not adjacent to z' in G. But now $(C \setminus \{z\}) \cup \{z_j\}$ is a hole of length four in G, a contradiction. This proves (72).

From (72) and Theorems 8.5 and 8.6, we deduce that there exists an even wheel (C, v) in $\widetilde{G}$. Let $W = G[V(C) \cup \{v\}]$ and let P be the path in W with ends a, b satisfying Lemma 8.4. It follows from Lemma 8.4(a) and (b) that $z \in P \setminus (N_P[a] \cup N_P[b])$ and P is a path of length at least four in C such that $a, b \in N_C(v) \subseteq N_G(v)$ and v is anticomplete to

P^* in $\widetilde{G}$. Let $Q = C \setminus P^*$. Then Q is a path in G from a to b. Let a' and b' be the neighbors of a and b in Q, respectively. Since C is an even wheel in $\widetilde{G}$, it follows that Q has length at least three, and so a, a', b, b' are all distinct. In addition, we have $W \setminus P^* = Q \cup \{v\}$, $W \setminus P = Q^* \cup \{v\}$, and $|N_Q(v)| = |N_C(v)| \geq 4$ is an even integer.

From the definition of $\widetilde{G}$, it follows that $W \setminus \{z\} \subseteq G \setminus \{z_1, z_2\}$ and $\{z_1, z_2\}$ is complete to $N_P(z)$. As a result, for every $i \in \{1, 2\}$, there are two paths $P_{a,i}, P_{b,i}$ in $(P \setminus \{z\}) \cup \{z_i\}$ from a to z_i and from b to z_i, respectively, such that $P_{a,i} \setminus \{z_i\}$ and $P_{b,i} \setminus \{z_i\}$ are disjoint and anticomplete in G, and both $P^*_{a,i}$ and $P^*_{b,i}$ are disjoint from and anticomplete to $Q^* \cup \{v\}$ in G. We claim that:

(73) *The vertex v has a neighbor in $Q \setminus \{a, a', b, b'\}$.*

Suppose not. Then we have $N_Q(v) = \{a, a', b, b'\}$. Then $C' = Q^* \cup \{v\} = W \setminus P$ is a hole in G. By Lemma 8.4(a) and the definition of $\widetilde{G}$, z_1 and z_2 have no common neighbor in C', and by Lemma 8.4(c), z_1 and z_2 each have at least one neighbor in C'. Therefore, by Theorem 7.5, one of z_1 and z_2, say the former, is C'-bad in G. Let $N_{C'}(z_1) = \{q, q'\}$ where q and q' are adjacent. The symmetry between a' and b' and between q and q' allows us to assume that $|\{a', v\} \cap \{q, q'\}| \leq 1$, and there are disjoint paths R and R' in C' from v to q and from a' to q', respectively. It follows that either

- $\{a', v\} \cap \{q, q'\} = \varnothing$; or
- $\{a', v\} \cap \{q, q'\} = \{a'\} = \{q'\}$; or
- $\{a', v\} \cap \{q, q'\} = \{v\} = \{q\}$.

If the first bullet above holds, then there is a prism in G with triangles $\{a, a', v\}, \{q, q', z_1\}$ and paths $P_{a,1}, R$ and R', a contradiction. Also, if the second bullet above holds, then $C'' = a\text{-}P_{a,1}\text{-}z_1\text{-}q\text{-}R\text{-}v\text{-}a$ is a hole in G and $a' = q' \in G \setminus C''$ has exactly four neighbors in C'', namely a, v, q and z_1, which in turn implies that (C'', a') is an even wheel in G. Similarly, if the third bullet above holds, then $C'' = a\text{-}P_{a,1}\text{-}z_1\text{-}q'\text{-}R'\text{-}a'\text{-}a$ is a hole in G and $v = q \in G \setminus C''$ has exactly four neighbors in C'', namely a, a', q' and z_1. It follows that (C'', v) is an even wheel in G. Each of the last three conclusions goes against the assumption that G is even-wheel-free. This proves (73).

(74) *Let $i \in \{1, 2\}$ such that v is not adjacent to z_i in G. Then $N_{W \setminus P}(z_i)$ is a non-empty subset of $\{a', b'\}$.*

Suppose not. By Lemma 8.4(c), z_1 and z_2 each have at least one neighbor in $W \setminus P = Q^* \cup \{v\}$, and so $N_{W \setminus P}(z_i) \neq \varnothing$. It follows that there exists a vertex $q \in Q^* \setminus \{a', b'\} =$

$Q \setminus (N_G[a] \cup N_G[b])$ which is adjacent to z_i in G. This, together with (73), implies that $(Q^* \setminus \{a', b'\}) \cup \{v, z_i\}$ is connected, and so there exists a path S of length at least two in $(Q^* \setminus \{a', b'\}) \cup \{v, z_i\}$ from v to z_i. But now there is a theta in G with ends v, z_i and paths v-a-$P_{a,i}$-z_i, v-b-$P_{b,i}$-z_i and S, contrary to the assumption that G is theta-free. This proves (74).

(75) *There exists $i \in \{1, 2\}$ for which v is adjacent to z_i in G.*

Suppose for a contradiction that v is anticomplete $\{z_1, z_2\}$. By (74), both $N_{W \setminus P}(z_1)$ and $N_{W \setminus P}(z_2)$ are non-empty subset of $\{a', b'\}$. By Lemma 8.4(a) and the definition of $\widetilde{G}$, in the graph G, z_1 and z_2 do not have a common neighbor in $W \setminus P^* = Q \cup \{v\}$. Therefore, due to the symmetry between z_1 and z_2 and between a' and b', it is safe to assume that $N_{Q^* \cup \{v\}}(z_1) = \{a'\}$ and $N_{Q^* \cup \{v\}}(z_2) = \{b'\}$. In particular, $D = a'$-z_1-z_2-b'-Q-a' is a hole in G and $N_D(v) = N_Q(v) \setminus \{a, b\}$. Recall that $|N_Q(v)|$ is an even integer which is at least four, and so $|N_D(v)|$ is a non-zero even integer. Since (D, v) is not an even wheel in G and $D \cup \{v\}$ is not a theta in G, it follows that v is D-bad. From this combined with (73), and without loss of generality, we may assume that v is not adjacent to a' and there is a path R in G from a' to v with $R^* \subseteq Q^* \setminus \{a', b'\}$. Moreover, again by Lemma 8.4(a) and the definition of $\widetilde{G}$, in the graph G, z_1 and z_2 do not have a common neighbor in $W \setminus P^* = Q \cup \{v\}$. Specifically, there exists $i \in \{1, 2\}$ for which z_i is not adjacent to a. But now there is a theta in G with ends a, z_i and paths a-a'-z_i, $P_{a,i}$ and a-v-b-$P_{b,i}$-z_i, a contradiction. This proves (75).

(76) *Let $i \in \{1, 2\}$ such that v is not adjacent to z_i in G. Then either z_i is anticomplete to $\{a, b\}$ in G, or v is anticomplete to $\{z_1, z_2\}$ in G.*

Suppose not. By Lemma 8.4(a), z_1 and z_2 do not have a common neighbor in $\{a, b, v\} \subseteq Q \cup \{v\} = W \setminus P^*$. But now either either $G[\{a, v, z_1, z_2\}]$ or $G[\{b, v, z_1, z_2\}]$ is isomorphic to C_4, a contradiction. This proves (76).

Let us now finish the proof. In view of (75), we may assume, without loss of generality, that v is adjacent to z_2 in G. Thus, by (76), z_1 is anticomplete to $\{a, b\}$ in G. Since $N_{W \setminus P}(z_1)$ is a non-empty subset of $\{a', b'\}$, we may assume that a' is adjacent to z_1 in G. Moreover, since $G[\{a', v, z_1, z_2\}]$ is not isomorphic to C_4, it follows that a' and v are not adjacent in G. But then there is a theta in G with ends a, z_1 and paths a-a'-z_1, $P_{a,1}$ and a-v-b-$P_{b,1}$-z_1, a contradiction. This completes the proof of Theorem 8.3. ■

8.3 Conjuring a phantom

In this section, we complete the proof of Theorem 8.1, of which the bulk of the difficulty is captured in the following lemma:

Lemma 8.7. *For all integers $d, h, t, w \in \mathbb{N}$, there is a constant $f_{8.7} = f_{8.7}(d, h, t, w) \in \mathbb{N}$ with the following property. Let G be a (theta, prism, even wheel, C_4, K_{t+1})-free graph and let $Z_0 \subseteq V(G)$ with $|Z_0| \leq h$. Assume that Z_0 is 3-mirrored by a ξ-kaleidoscope $(a, x, y, \mathcal{W})$ in G. Then there exists a $(Z_0, d, 1)$-phantom (Z_0, Z_1, Γ_1) in G with $|Z_1| \leq 2dh^2$ and a w-subset $\mathcal{W}'$ of $\mathcal{W}$, such that Z_1 is 3-mirrored by the w-kaleidoscope $(a, x, y, \mathcal{W}')$.*

Proof. For fixed $d, t, w \in \mathbb{N}$, we define the sequence $\{\xi_j : j = 0, \dots, h^2\}$ of positive integers recursively, as follows. The definition will rely on Lemma 3.12 and Theorems 7.1 and 7.2, respectively. Let $\xi_0 = \xi_0(d, t, w) = w$. For each $j \in [h^2]$, assuming ξ_{j-1} is defined, let

$$f_j = f_j(d, t, w) = df_{7.1}(3, t, \xi_{j-1})^{\xi_{j-1}} + f_{7.1}(3, t, \xi_{j-1});$$

$$g_j = g_j(d, t, w) = f_j^{f_j f_{7.2}(1,1,t)};$$

$$\xi_j = \xi_j(d, t, w) = f_{3.12}(g_j, 3, 2, t+1).$$

In particular, observe that the sequence $\{\xi_j : j = 0, \dots, h^2\}$ is increasing.

We prove that

$$f_{8.7} = f_{8.7}(d, h, t, w) = \xi_{h^2}$$

satisfies the lemma. Indeed, we prove a stronger statement tailored to an inductive proof. Let $|E(G[Z_0])| = m$; then we have $0 \leq m \leq h^2$. Pick an enumeration $e_1, \dots, e_m$ of the edges of $G[Z_0]$. Our goal is to show that:

(77) *For each $j \in \{0, 1, \dots, m\}$, there exist j pairwise disjoint d-subsets $(\Gamma_k : k \in [j])$ of $V(G) \setminus Z_0$ as well as a ξ_{m-j}-subset $\mathcal{W}_j$ of $\mathcal{W}$, such that the following hold.*

- *For every $k \in [j]$, the ends of e_k are complete to Γ_k.*
- *The set $Z_0 \cup (\bigcup_{k \in [j]} \Gamma_k)$ is 3-mirrored by the ξ_{m-j}-kaleidoscope $(a, x, y, \mathcal{W}_j)$.*

Let us first prove that (77) implies 8.7. Let $\{\Gamma_k : k \in [m]\}$ and $\mathcal{W}' = \mathcal{W}_m$ be as in (77) for $j = m$. Let $Z_1 = Z_0 \cup (\cup_{k \in [m]} \Gamma_k)$. Then we have $|Z_1| = |Z_0| + dm \leq 2dh^2$ and $|\mathcal{W}'| = \xi_0 = w$. Define the map $\Gamma_1 : E(G[Z_0]) \to (Z_1 \setminus Z_0)^{(d)}$ such that $\Gamma_1(e_k) = \Gamma_k$ for every $k \in [m]$. Then from (77), and specifically the first bullet of (77), it follows that (Z_0, Z_1, Γ_1) is a $(Z_0, d, 1)$-phantom in G with $|Z_1| \leq 2dh^2$, and from the second bullet of (77), it follows that Z_1 is 3-mirrored by the w-kaleidoscope $(a, x, y, \mathcal{W}')$, as desired.

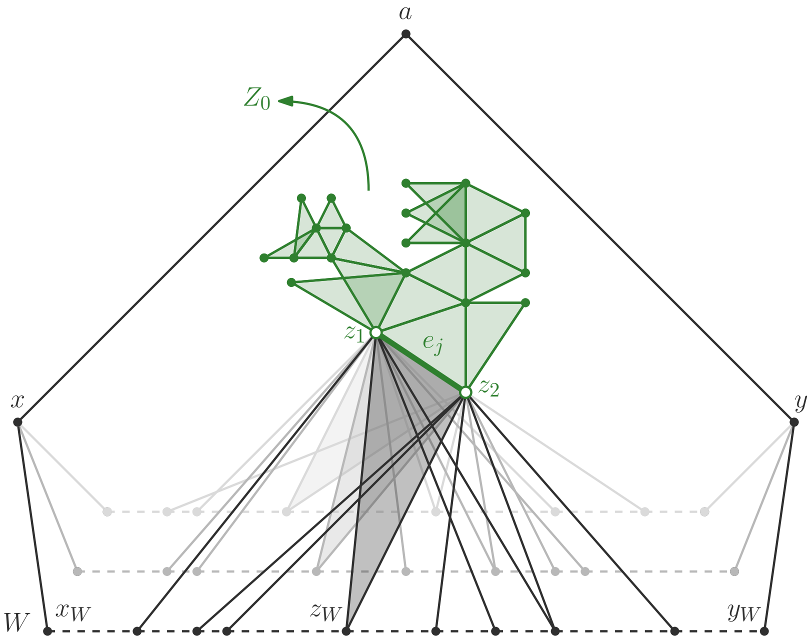

Figure 8.3: Proof of Theorem 8.7 (dashed lines represents paths of undetermined length).

We now turn to the proof of (77), which is by induction on j. For $j = 0$, the result follows from the fact that $\xi_{h^2} \geq \xi_m$ as well as the assumption that Z_0 is 3-mirrored by the ξ_{h^2}-kaleidoscope $(a, x, y, \mathcal{W})$. So we may assume that $j \geq 1$. By the induction hypothesis, there exist $j-1$ pairwise disjoint d-subsets $(\Gamma_k : k \in [j-1])$ of $V(G) \setminus Z_0$ as well as a ξ_{m-j+1}-subset $\mathcal{W}_{j-1}$ of $\mathcal{W}$, such that:

- for every $k \in [j-1]$, the ends of e_k are complete to Γ_k; and
- the set $Z_0 \cup (\cup_{k\in[j-1]}\Gamma_k)$ is 3-mirrored by the ξ_{m-j+1}-kaleidoscope $(a, x, y, \mathcal{W}_{j-1})$.

Let $z_1, z_2 \in Z_0$ be the ends of e_j and let $Z^+ = Z_0 \cup (\cup_{k\in[j-1]}\Gamma_k)$. Let $W \in \mathcal{W}_{j-1}$ be fixed. Let x_W, y_W be the neighbors of x and y in W, respectively. Consider the hole $C_W = a\text{-}x\text{-}W\text{-}y\text{-}a$ in G. Since Z^+ is 3-mirrored by $(a, x, y, \mathcal{W}_{j-1})$, it follows from (M3) that z_1 and z_2 each have at least three neighbors in $W \subseteq C_W$, and from (M2) that a is adjacent to at most one of z_1 and z_2. Thus, by Theorem 7.5, the vertices z_1 and z_2 have a common neighbor in W. Traversing W from x to y, let z_W be the first common neighbor of z_1 and z_2 in W. By (M3), $\{z_1, z_2\}$ is anticomplete to $\{x, x_W, y, y_W\}$, and in particular we have $z_W \in W \setminus \{x, x_W, y, y_W\}$ (see Figure 8.3).

Since G is $(K_{2,2}, K_t)$-free and since $|\mathcal{W}_{j-1}| = \xi_{m-j+1} = f_{3.12}(g_{m-j+1}, 3, 2, t)$, it follows from Lemma 3.12 applied to the sets $(\{x_W, z_W, y_W\} : W \in \mathcal{W}_{j-1})$ that there exists $\mathcal{W}'_{j-1} \subseteq \mathcal{W}_{j-1}$ with $|\mathcal{W}'_{j-1}| = g_{m-j+1}$ such that the sets $(x_W, z_W, y_W\} : W \in \mathcal{W}'_{j-1})$ are pairwise anticomplete in G.

Next, we define a digraph D with vertex set $\mathcal{W}'_{j-1}$ such that for distinct paths $W_1, W_2 \in \mathcal{W}'_{j-1}$, the arc (W_1, W_2) is present in D if and only if z_{W_1} has a neighbor in W_2^*. We claim that:

(78) *D has no stable set of cardinality $f_{7.2}(1, 1, t)$.*

Suppose not. Then there is an $f_{7.2}(1, 1, t)$-subset $\mathcal{S}$ of $\mathcal{W}'_{j-1}$ such that for all distinct $W_1, W_2 \in \mathcal{S}$, the vertex z_{W_1} is anticomplete to W_2^*. Let

$$J = G\left[\left(\bigcup_{W \in \mathcal{S}} V(z_W\text{-}W\text{-}x)\right) \cup \{z_1, z_2\}\right].$$

Then J is (theta, prism, even wheel, C_4, K_{t+1})-free, because G is. Note that the vertices $z_1, z_2 \in V(J)$ are distinct and adjacent with $N_J(z_1) \cap N_J(z_2) = \{z_W : W \in \mathcal{S}\}$ being a stable set of vertices of degree three in J. Consequently, by Theorem 8.3, we have the z_1z_2-contraption $\widetilde{J}$ of J is (theta, prism, even wheel, C_4)-free. Moreover, $\widetilde{J}$ is K_{t+1}-free (because J is). Let $z \in V(\widetilde{J})$ be as in the definition of the z_1z_2-contraption. Then $N_{\widetilde{J}}(z) = N_J(z_1) \cap N_J(z_2) = \{z_W : W \in \mathcal{S}\}$ is a stable set of vertices of degree two in J. Moreover, $\mathcal{P} = \{z\text{-}z_W\text{-}W\text{-}x : W \in \mathcal{S}\}$ is a collection of $f_{7.2}(1, 1, t)$ pairwise internally disjoint paths in $\widetilde{J}$ between the non-adjacent vertices z and x. Applying Theorem 7.2 to $\mathcal{P}$, we deduce that there exist $W_1, W_2 \in \mathcal{S}$ such that, in the graph $\widetilde{J}$, the vertex z_{W_1} has a neighbor in the interior of $z_{W_2}\text{-}W_2\text{-}x$. But then $z_{W_1} \in N_{\widetilde{J}}(z)$ has degree at least three in $\widetilde{J}$, a contradiction. This proves (78).

By (78) and the definition of f_{m-j+1}, it follows from Theorem 5.11 that there are two disjoint subsets $\mathcal{U}$ and $\mathcal{V}$ of $V(D) = \mathcal{W}'_{j-1}$ with $|\mathcal{U}| = df_{7.1}(3, t, \xi_{m-j})^{\xi_{m-j}}$ and $|\mathcal{V}| = f_{7.1}(3, t, \xi_{m-j})$, such that for all $W \in \mathcal{U}$ and $W' \in \mathcal{V}$, we have $(W, W') \in E(D)$. It follows that for every $W \in \mathcal{U}$, the vertex z_W is 1-mirrored by the $f_{7.1}(3, t, \xi_{m-j})$-kaleidoscope $(a, x, y, \mathcal{V})$. Therefore, by Theorem 7.1, for each $W \in \mathcal{U}$, there is a ξ_{m-j}-subset $\mathcal{V}_W$ of $\mathcal{V}$ such that the vertex z_W is 3-mirrored by the ξ_{m-j}-kaleidoscope $(a, x, y, \mathcal{V}_W)$. From this, combined with the fact that $|\mathcal{U}| = d(f_{7.1}(3, t, \xi_{m-j}))^{\xi_{m-j}}$, we deduce the following:

(79) *There exists a d-subset $\mathcal{D}$ of $\mathcal{U}$ and a ξ_{m-j}-subset of $\mathcal{W}_j$ of $\mathcal{V}$ such that for every $W \in \mathcal{D}$, we have $\mathcal{V}_W = \mathcal{W}_j$.*

Let $\Gamma_j = \{z_W : W \in \mathcal{D}\}$. Then Γ_j is a d-subset of $V(G) \setminus Z^+$ which is complete to $\{z_1, z_2\}$. In particular, $(\Gamma_k : k \in [j])$ are j pairwise disjoint d-subsets of $V(G) \setminus Z_0$, and for each $k \in [j]$, the ends of e_k are complete to Γ_k. By (79), Γ_j is 3-mirrored by the ξ_{m-j}-kaleidoscope $(a, x, y, \mathcal{W}_j)$. Moreover, Z^+ is 3-mirrored by the ξ_{m-j}-kaleidoscope $(a, x, y, \mathcal{W}_j)$, because $\mathcal{W}_j \subseteq \mathcal{V} \subseteq \mathcal{W}'_{j-1} \subseteq \mathcal{W}_{j-1}$. In conclusion, we have shown that $Z^+ \cup \Gamma_j = Z_0 \cup (\cup_{k\in[j]}\Gamma_k)$ is 3-mirrored by the ξ_{m-j}-kaleidoscope $(a, x, y, \mathcal{W}_j)$. Hence, the sets $(\Gamma_k : k \in [j])$ satisfy the two bullet conditions of (77). This completes the inductive proof of (77), and so finishes the proof of Lemma 8.7 ■

From Lemma 8.7, we deduce that:

Theorem 8.8. *For all $d, h, t \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$, there exist $(r + 1)$ positive integers $(\Xi_i = \Xi_i(d, h, t) : i = 0, \dots, r)$ with $\Xi_0 = 1$ for which the following holds. Let G be a (theta, prism, even wheel, C_4, K_{t+1})-free graph and let $Z_0 \subseteq V(G)$ with $|Z_0| \leq h$. Assume that Z_0 is 3-mirrored by a Ξ_r-kaleidoscope $(a, x, y, \mathcal{W}_0)$ in G. Then, for each $i \in \{0, \dots, r\}$, there exists a (Z_0, d, i)-phantom $(Z_0, \dots, Z_i, \Gamma_j : j \in [i])$ in G with $|Z_i| \leq (2d)^{2^i-1}h^{2^i}$ and a Ξ_{r-i}-subset $\mathcal{W}_i$ of $\mathcal{W}$, such that Z_i is 3-mirrored by the Ξ_{r-i}-kaleidoscope $(a, x, y, \mathcal{W}_i)$. In particular, there exists a (Z_0, d, r)-phantom $(Z_0, \dots, Z_r, \Gamma_j : j \in [r])$ in G.*

Proof. The definition of $(\Xi_i = \Xi_i(d, h, t) : i = 0, \dots, r)$ is recursive, as follows. We already know that $\Xi_0 = \Xi_0(d, h, t) = 1$. For each $i \in [r]$, assuming Ξ_{i-1} is defined, let

$$\Xi_i = \Xi_i(d, h, t) = f_{8.7}\left(d, (2d)^{2^{r-i}-1}h^{2^{r-i}}, t, \Xi_{i-1}\right).$$

We prove, by induction on i, that there exists a (Z_0, d, i)-phantom in G which satisfies Theorem 8.8.

The case $i = 0$ is immediate from the assumptions that $|Z_0| \leq h$ and Z_0 is 3-mirrored by the Ξ_r-kaleidoscope $(a, x, y, \mathcal{W}_0)$. Assume that $i \geq 1$. By the induction hypothesis, there exists a $(Z_0, d, i-1)$-phantom $(Z_0, \dots, Z_{i-1}, \Gamma_j : j \in [i-1])$ in G with $|Z_{i-1}| \leq (2d)^{2^{i-1}-1}h^{2^{i-1}}$ and a Ξ_{r-i+1}-subset $\mathcal{W}_{i-1}$ of $\mathcal{W}$, such that Z_{i-1} is 3-mirrored by the Ξ_{r-i+1}-kaleidoscope $(a, x, y, \mathcal{W}_{i-1})$.

Since $\Xi_{r-i+1} = f_{8.7}(d, (2d)^{2^{i-1}-1}h^{2^{i-1}}, t, \Xi_{r-i})$, we can apply Lemma 8.7 to Z_{i-1} and $(a, x, y, \mathcal{W}_{i-1})$ to deduce that there is a $(Z_{i-1}, d, 1)$-phantom (Z_{i-1}, Z_i, Γ_j) in G with $|Z_i| \leq 2d((2d)^{2^{i-1}-1}h^{2^{i-1}})^2 = (2d)^{2^i-1}h^{2^i}$ and a Ξ_{r-i}-subset $\mathcal{W}_i$ of $\mathcal{W}_{i-1} \subseteq \mathcal{W}$, such that Z_i is 3-mirrored by the Ξ_{r-i}-kaleidoscope $(a, x, y, \mathcal{W}_i)$. In particular, since $(Z_0, \dots, Z_{i-1}, \Gamma_j :$

$j \in [i-1]$) is a $(Z_0, d, i-1)$-phantom in G and (Z_{i-1}, Z_i, Γ_i) is a $(Z_{i-1}, d, 1)$-phantom in G, it follows that $(Z_0, \dots, Z_i, \Gamma_j : j \in [i])$ is a (Z_0, d, i)-phantom in G. The induction is completed, and so is the proof of Theorem 8.8. ■

Finally, we give a proof of Theorem 8.1, which we restate:

Theorem 8.1. *For all* $d, t \in \mathbb{N}$ *and* $r_1 \in \mathbb{N} \cup \{0\}$*, there is a constant* $f_{8.1} = f_{8.1}(d, r_1, t) \in \mathbb{N}$ *with the following property. Let* G *be a (theta, prism, even wheel,* C_4, K_{t+1}*)-free graph with* $\mathrm{tw}(G) > f_{8.1}$*. Then there is a* (Z_0^1, d, r_1)*-phantom in* G *for some* 2*-clique* Z_0^1*. Moreover, for every* $r_2 \in \mathbb{N}$ *with* $r_2 < r_1$*, there is a* (Z_0^2, d, r_2)*-phantom in* G *for some* 3*-clique* Z_0^2*.*

Proof. Let $\{\Xi_i(d, 3, t) : i = 0, \dots, r_1\}$ be as in Theorem 8.8, and let $\Xi = \Xi_{r_1}(d, 3, t)$. Let

$$f_{8.1} = f_{8.1}(d, r_1, t) = f_{8.2}(3, t, \Xi).$$

Let G be a (theta, prism, even wheel, C_4, K_{t+1})-free graph with $\mathrm{tw}(G) > f_{8.1}$. Since G is $(t+1)$-clean, and from the choice of $f_{8.1}$ combined with Theorem 8.2, it follows that there exists a Ξ-kaleidoscope $(a, x, y, \mathcal{W})$ in G as well as a 2-clique Z_0^1 in G such that Z_0^1 is 3-mirrored by $(a, x, y, \mathcal{W})$. Therefore, by Theorem 8.8 and the choice of Ξ, there exists a (Z_0^1, d, r_1)-phantom $\mathfrak{p}_1 = (Z_0^1, \dots, Z_{r_1}^1; \Gamma_i^1 : i \in [r_1])$ in G, and so Z_0^1 satisfies 8.1.

Now, let $r_2 \in [r_1 - 1]$. Assuming $Z_0^1 = \{z_1, z_2\}$, since $d, r_1 \geq 1$, we may choose a vertex $z \in \Gamma_1^1(z_1 z_2)$. Let $Z_0^2 = \{z_1, z_2, z\} \subseteq Z_1^1$ and let $\mathfrak{p}_2 = \mathfrak{p}_1[Z_0^2; 1, r_2]$. Then $\mathfrak{p}_2$ is a (Z_0^2, d, r_2)-phantom in G, and so Z_0^2 is a 3-clique in G which satisfies 8.1. This completes the proof of Theorem 8.1. ■

Chapter 9

Even-hole-free graphs
III. Chordal induced subgraphs[1]

In this chapter, we prove Theorems 2.18, 2.24 and 2.25. This will be done in Sections 9.1, 9.3 and 9.4. Section 9.2 is devoted to some preliminary results about chordal graphs of bounded clique number, which we will use in the proofs of Theorems 2.24 and 2.25.

9.1 Forests in (theta, prism)-free graphs

We begin with a definition. For $d \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$, let $T_{d,r}$ be the rooted tree of radius r such that if $r \geq 1$, then the root has degree d, and every vertex that is neither the root nor a leaf has degree $d+1$ (see Figure 9.1). It is easy to observe that $T_{d,r}$ contains every tree of maximum degree d and radius r. There is also a well-known result of Kierstead and Penrice [42], that in sparse graphs, we may pass from a non-induced tree to an induced one:

Theorem 9.1 (Kierstead and Penrice [42]). *For all $d, s, t \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$, there is a constant $f_{9.1} = f_{9.1}(d, r, s, t) \in \mathbb{N}$ with the following property. Let G be a $(K_{s,s}, K_t)$-free graph and U be a subgraph of G which is isomorphic to $T_{f_{9.1}, f_{9.1}}$. Then there is an induced subgraph U' of G with $V(U') \subseteq V(U)$ such that U' is isomorphic to $T_{d,r}$.*

The next lemma is the core of our proof of Theorem 2.18:

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[1] This chapter is based on the coauthored paper [6] and the single-author paper [36].

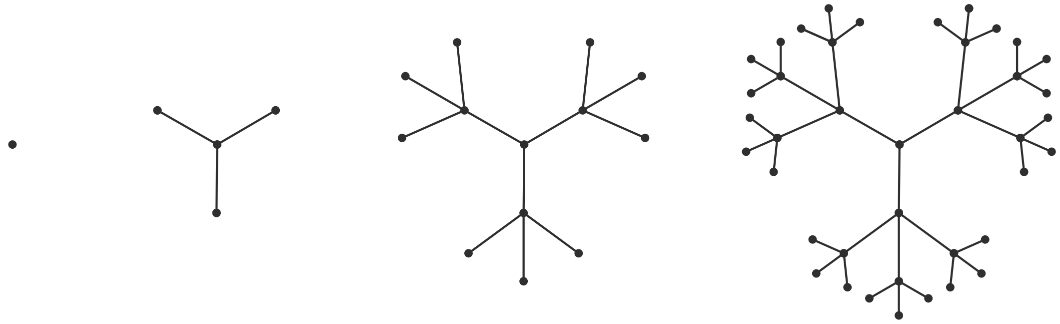

Figure 9.1: From left to right: the trees $T_{3,r}$ for $r = 0, 1, 2, 3$.

Lemma 9.2. *For all $d, t \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$, there is a constant $f_{9.2} = f_{9.2}(d, r, t) \in \mathbb{N}$ with the following property. Let G be a (theta, prism, K_{t+1})-free graph, let $a, b \in V(G)$ be non-adjacent, and let $\mathcal{P}$ be a set of $f_{9.2}$ pairwise internally disjoint paths in G from a to b. Then $G[V(\mathcal{P})]$ has a subgraph J isomorphic to T_d^r such that $a \in V(J)$ is the root of J, and $b \notin V(J)$.*

Proof. For fixed $d, t \in \mathbb{N}$, we define a sequence $\{m_i = m_i(d, t) : i \in \mathbb{N} \cup \{0\}\}$ recursively, as follows. Let $m_0 = 1$, and for $i > 1$, assuming m_{i-1} is defined, let

$$m_i = f_{7.2}(dm_{i-1}, d + 2, t).$$

We prove by induction on $r \geq 0$ that

$$f_{9.2} = f_{9.2}(d, r, t) = m_r$$

satisfies the lemma. The base case $r = 0$ is immediate by choosing $J = \{a\}$. Assume that $r \geq 1$. By the choice of $f_{9.2} = m_r = f_{7.2}(dm_{r-1}, d + 2, t)$, we can apply Theorem 7.2 to a, b and $\mathcal{P}$ to deduce that there is a $(d+2, dm_{r-1})$-constellation $\mathfrak{c}$ in G such that $S_\mathfrak{c} \subseteq N_{V(\mathcal{N})}(a)$ and $\mathcal{L}_\mathfrak{c} \subseteq \mathcal{N}^*$.

Moreover, there are at most two vertices in $S_\mathfrak{c}$ which are adjacent to b, as otherwise there is a theta in G with ends a, b and paths $(a\text{-}x_i\text{-}b : i \in [3])$ where $x_1, x_2, x_3 \in S_\mathfrak{c}$, a contradiction. It follows that we may choose d vertices $a_1, \ldots, a_d \in S_\mathfrak{c} \setminus N_G(b)$.

Since $\mathcal{L}_\mathfrak{c} = dm_{r-1}$, we can partition $\mathcal{L}_\mathfrak{c}$ into d pairwise disjoint subsets $(\mathcal{L}_i : i \in [d])$, each of cardinality m_{r-1}. Since $\mathfrak{c}$ is a constellation in G, it follows that for each $i \in [d]$ and every path $L \in \mathcal{L}_i \subseteq \{P^* : P \in \mathcal{P}\}$, the vertex a_i has a neighbor in L. Consequently, for each $i \in [d]$ and every $L \in \mathcal{L}_i$, there is a path in P_L in G from a_i to b with $P_L^* \subseteq V(L)$.

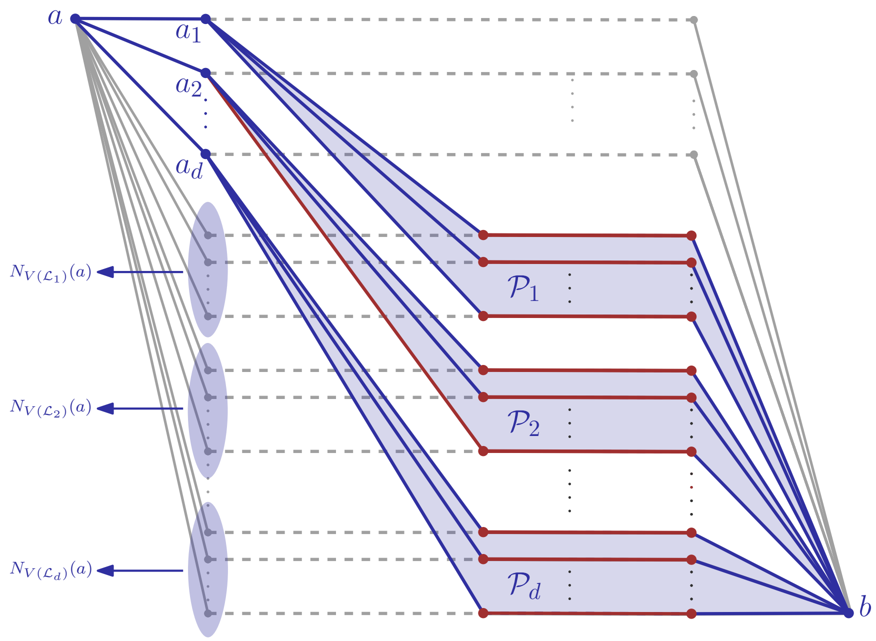

Figure 9.2: Proof of Lemma 9.2. Dashed lines represent paths of non-zero length, and red lines depict paths of arbitrary (possibly zero) length.

For each $i \in [d]$, let

$$\mathcal{P}_i = \{P_L : L \in \mathcal{L}_i\}.$$

See Figure 9.2. Then $\mathcal{P}_i$ is a set of m_{r-1} pairwise internally disjoint paths in G between the non-adjacent vertices a_i and b. It follows from the induction hypothesis that $G[V(\mathcal{P}_i)]$ has a subgraph J_i isomorphic to T_d^{r-1} such that $a_i' \in V(J_i)$ is the root of J_i, and $b \notin V(J_i)$. Furthermore, observe that the sets $(V(\mathcal{P}_i) \setminus \{b\} : i \in [d])$ are pairwise disjoint in G, and for each $i \in [d]$, we have $N_{V(\mathcal{P}_i)}(a) = \{a_i\}$. It follows that the sets $(V(J_i) : i \in [d])$ are pairwise disjoint in G, and for each $i \in [d]$, we have $N_{V(J_i)}(a) = \{a_i\}$. But now assuming

$$U = \left(\bigcup_{i=1}^{d} V(J_i)\right) \cup \{a\} \subseteq V(\mathcal{P}),$$

we deduce that $G[U]$ contains a (spanning) subgraph J isomorphic to T_d^r such that $a \in V(J)$ is the root of J, and $b \notin V(J)$. This completes the proof of Lemma 9.2. ■

We are now ready to proved Theorem 2.18, which we restate:

Theorem 2.18. *For all* $t \in \mathbb{N}$ *and every forest* F*, there is a constant* $f_{2.18} = f_{2.18}(F,t) \in \mathbb{N}$ *such that every (theta, prism,* K_{t+1}*)-free graphs with* $\operatorname{tw}(G) > f_{2.18}$ *has an induced subgraph isomorphic to* F*.*

Proof. Observe that there is tree T on $|V(F)|+1$ vertices which has an induced subgraph isomorphic to F. Let d and r be the maximum degree and the radius of T, respectively. Then T is isomorphic to an induced subgraph of $T_{d,r}$ and so F is isomorphic to an induced subgraph of $T_{d,r}$.

Let $f = f_{9.1}(d,r,3,t+1)$ and let $l = f_{9.2}(f,f,t)$. We claim that

$$f_{2.18} = f_{2.18}(F,t) = f_{2.27}(t+1,l,t+1)$$

satisfies the theorem. Let G be a (theta, prism, K_{t+1})-free graph with $\operatorname{tw}(G) > f_{2.18}$. Then G is $(t+1)$-clean, and from Theorem 2.27 and the choice of $f_{2.18}$, it follows that there is a strong $(t+1,l)$-block in G. Consequently, since G is K_{t+1}-free, it follows that there are two distinct and non-adjacent vertices $a, b \in V(G)$, and a set $\mathcal{P}$ of l pairwise internally disjoint paths in G from a to b. This, along with the choice of l and Lemma 9.2, implies that G has a subgraph isomorphic to T^f_f. Since G is $(K_{3,3}, K_{t+1})$-free, it follows from Theorem 9.1 and the choice of f that G has an induced subgraph isomorphic to T^r_d. Hence, F is isomorphic to an induced subgraph of G, as desired. ■

9.2 k-trees

For $k \in \mathbb{N}$, a *k-tree* is a graph ∇ with $|V(\nabla)| = h \geq k$ such that if $h = k$, then ∇ is complete, and if $h > k$, then there is a bijection $\varpi_\nabla : V(\nabla) \to [h]$ such that for every $i \in [h-k]$, the set of all *forward neighbors* of $\varpi_\nabla(i)$, that is, the set of all neighbors of $\varpi_\nabla(i)$ in $V(\nabla) \setminus \varpi_\nabla([i])$ is a k-clique in ∇ (see Figure 2.9 where both 2-forests are in fact 2-trees). In other words, k-trees are the connected graphs obtained from the k-vertex complete graph by successively adding a vertex whose neighborhood forms a k-clique. It follows that 1-trees are the same as trees. Recall also that forests are exactly the graphs which are induced subgraphs of the trees. This generalizes easily to k-forests and k-trees for all $k \in \mathbb{N}$. We give a proof for the sake of completeness:

Theorem 9.3. *Let* $k \in \mathbb{N}$ *and let* H *be a graph. Then* H *is a* k*-forest if and only if* H *is an induced subgraph of a* k*-tree.*

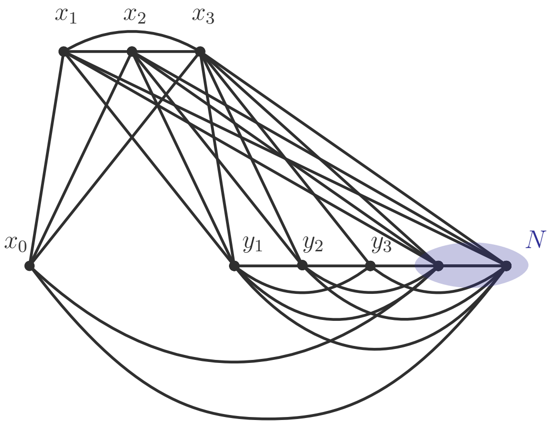

Figure 9.3: Proof of Theorem 9.3 (when $k = 5$ and $|N| = 2$).

Proof. The "if" implication is straightforward to check. Note also that every k-forest H is an induced subgraph of a connected k-forest (which, for instance, may be obtained from H by adding a new vertex with exactly one neighbor in each component of H). Therefore, to prove the "only if" implication, we only need to show that for every connected k-forest H, there exists a k-tree ∇ such that H is an induced subgraph of ∇.

We prove the above assertion by induction on $|V(H)| = h$. The case $h = 1$ is trivial. Assume that $h > 1$. Since H is chordal and K_{k+2}-free, it follows from a well-known result of Dirac [24] that there exists a bijection $\pi : V(H) \to [h]$ such that for every $i \in [h-1]$, the set of all neighbors of $\pi(i)$ in $V(H) \setminus \pi([i])$ is a clique of H on at most k vertices. Let $\pi(1) = x_0$, let $H^- = H \setminus \{x_0\}$, and let $N = N_H(x_0)$. Then N is a non-empty clique on at most k vertices in H, which in turn implies that H^- is a connected K_{k+2}-free chordal graph on $h - 1$ vertices and N is a non-empty clique on at most k vertices in H^-. By the induction hypothesis, there is a k-tree ∇^- such that H^- is an induced subgraph of ∇^-. In particular, N is a non-empty clique on at most k vertices in ∇^-. We deduce that:

(80) *There is a k-clique K in ∇^- such that $N \subseteq K$.*

This is immediate if ∇^- is a complete graph. So we may assume that ∇^- is a k-tree that is not complete. Choose $x \in N$ with $\varpi^{-1}_{\nabla^-}(x) \in [h-1]$ as small as possible. Let M be the set of forward neighbors of x in ∇^-. Then $M \cup \{x\}$ is a $(k+1)$-clique in ∇^- which contains the clique N of cardinality at most k. This proves (80).

Let K be as in (80). Fix an enumeration $\{y_i : i \in [k - |N|]\}$ of the elements of $K \setminus N$.

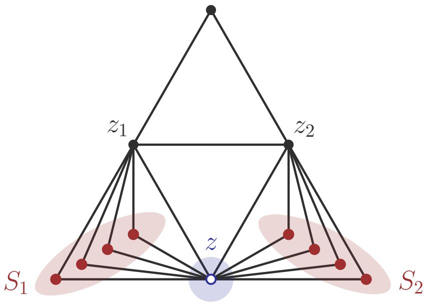

Figure 9.4: A crystallized vertex z in a 12-vertex 2-tree.

We define the graph ∇ as follows. Let

$$V(\nabla) = V(\nabla^-) \cup \{x_0\} \cup \{x_i : i \in [k - |N|]\}$$

and let

$$E(\nabla) = E(\nabla^-) \cup \left(\bigcup_{i=0}^{k-|N|} (\{x_i y, x_i y_j, x_i x_{j'} : y \in N, j \in [i], j' \in [k-|N|] \setminus [i]\}) \right).$$

We also define the bijection $\varpi_\nabla : V(\nabla) \to [|V(\nabla)|]$ so that $\varpi_\nabla(x_i) = i + 1$ for all $i \in \{0, \ldots, k - |N|\}$ and let $\varpi_\nabla(z) = \varpi_{\nabla^-}(v) + k - |N| + 1$ for all $v \in V(\nabla^-)$ (see Figure 9.3). Then one may check that ∇ is k-tree and H is an induced subgraph of ∇; we omit the details. ■

Another example of 2-trees are crystals defined in Section 2.4. Note that, by definition, every 2-tree ∇ on three or more vertices has a degree-two simplicial vertex, that is, a vertex v whose neighborhood is a 2-clique. There is a useful strengthening of this fact: if ∇ has four or more vertices, then we can arrange for v to be a "side vertex" from a "side crystal" in ∇.

To make this precise, given a graph H and a vertex $z \in V(H)$, we say z is *crystallized* if there is a 2-clique $\{z_1, z_2\} \subseteq N_H(z)$ of G with the following specifications.

(C1) $N_H(z) \setminus \{z_1, z_2\}$ is a non-empty stable in G (and this is why z cannot be a degree-two simplicial vertex).

(C2) There exists a partition (S_1, S_2) of $N_H(z) \setminus \{z_1, z_2\}$ such that for each $i \in \{1, 2\}$ and every vertex $x \in S_i$, we have $N_H(x) = \{z_i, z\}$.

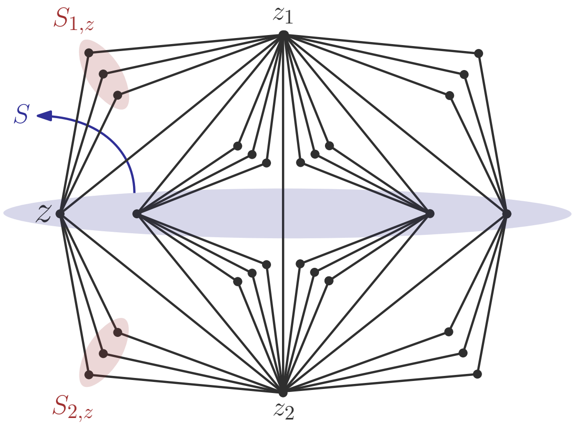

Figure 9.5: A $(z_1, z_2, 4, 3)$-crystal

See Figure 9.4. It follows that every vertex in $N_H(z) \setminus \{z_1, z_2\}$ is a degree-two simplicial vertex in H. Furthermore, we deduce that:

Theorem 9.4. *Every* 2*-tree* ∇ *with* $|V(\nabla)| \geq 4$ *has a crystallized vertex.*

Proof. We proceed by induction on $|V(\nabla)| = h \geq 4$. If $h = 4$, then ∇ is a diamond and both degree-three vertices of H are crystallized. Assume that $h > 4$. Let ϖ_∇ be as in the definition of a 2-tree, let $v = \varpi_\nabla(1)$ and let $\nabla' = \nabla \setminus \{v\}$. Then $N_\nabla(v) \subseteq V(\nabla')$ is 2-clique and ∇' is a 2-tree on $h-1 \geq 4$ vertices. By the induction hypothesis, ∇' has a crystallized z in ∇'. In particular, there exist a 2-clique $\{z_1, z_2\} \subseteq N_{\nabla'}(z)$ and a partition (S_1, S_2) of $N_{\nabla'}(z) \setminus \{z_1, z_2\}$ satisfying (C1) and (C2). Now, if $N_\nabla(v) \cap (N_{\nabla'}[z] \setminus \{z_1, z_2\}) = \varnothing$, then z is a crystallized vertex in ∇ (still with $\{z_1, z_2\} \subseteq N_\nabla(z)$ and (S_1, S_2) satisfying (C1) and (C2)). Otherwise, since $N_\nabla(v)$ is a 2-clique in ∇', there are two possibilities:

- There exists $i \in \{1, 2\}$ and $x \in S_i$ such that either $N_\nabla(v) = \{x, z_i\}$ or $N_\nabla(v) = \{x, z\}$.
- There exists $i \in \{1, 2\}$ such that $N_\nabla(v) = \{z_i, z\}$.

In the former case, x is a crystallized vertex in ∇ with the 2-clique $\{z_i, z\}$ contained in $N_\nabla(x)$ and the partition $(\{v\}, \varnothing)$ of $N_\nabla(x) \setminus \{z_i, z\}$ satisfying (C1) and (C2). In the latter case, let $\{1, 2\} \setminus \{i\} = \{i'\}$. Then z is a crystallized vertex in ∇ with the 2-clique $\{z_1, z_2\}$ contained in $N_\nabla(z)$ and the partition $(S_i \cup \{v\}, S_{i'})$ of $N_\nabla(x) \setminus \{z_1, z_2\}$ satisfying (C1) and (C2). This completes the proof of Theorem 9.4. ■

Our last result in this section is an analog of Theorem 9.1 for crystals. We begin with defining a "regular" version of crystals. Let $f, g \in \mathbb{N}$, let G be a graph and let

$z_1, z_2 \in V(G)$ be adjacent. A (z_1, z_2, f, g)-*crystal in* G is a tuple $\mathfrak{C} = (S_{1,z}, z, S_{2,z} : z \in S)$ with the following specifications (see Figure 9.5).

- (CR1) S is a subset of $V(G) \setminus \{z_1, z_2\}$ with $|S| = f$ (and thus $\mathfrak{C}$ is a $3f$-tuple).
- (CR2) $(S_{1,z}, S_{2,z} : z \in S)$ are $2f$ pairwise disjoint g-subsets of $V(G) \setminus (S \cup \{z_1, z_2\})$.
- (CR3) For each $i \in \{1, 2\}$ and all $z \in S$ and $x \in S_{i,z}$, we have $N_{\{z_1,z_2,z\}}(x) = \{z_i, z\}$.

We write $V(\mathfrak{C}) = S \cup (\bigcup_{z \in S}(S_{1,z} \cup S_{2,z}))$. Also, by an (f, g)-*crystal in* G, we mean a (z_1, z_2, f, g)-crystal in G for some pair z_1, z_2 of adjacent vertices in G.

Note that $\mathfrak{C}$ in G may or may not identify an *induced* subgraph of G which is a crystal. But if it does, then we say $\mathfrak{C}$ is "clear." More precisely, we say a (z_1, z_2, f, g)-crystal $\mathfrak{C} = (S_{1,z}, z, S_{2,z} : z \in S)$ in G is *clear* if S is a stable set and $(S_{1,z}, S_{2,z} : z \in S)$ are pairwise anticomplete stable sets in G. Just like Theorem 9.1, we need to pass, in sparse graphs, from crystals to clear crystals. This is attained in Theorem 9.5 below:

Theorem 9.5. *For all* $f, g, s, t \in \mathbb{N}$, *there are constants* $f_{9.5} = f_{9.5}(f, g, s, t) \in \mathbb{N}$ *and* $g_{9.5} = g_{9.5}(f, g, s, t) \in \mathbb{N}$ *with the following property. Let* G *be a* $(K_{s,s}, K_t)$*-free graph, let* $z_1, z_2 \in V(G)$ *be adjacent, and let* $\mathfrak{C} = (\Lambda_{1,z}, z, \Lambda_{2,z} : z \in \Lambda)$ *be a* $(z_1, z_2, f_{9.5}, g_{9.5})$*-crystal in* G*. Then there is a clear* (z_1, z_2, f, g)*-crystal in* G*.*

Proof. The proof relies on Lemma 3.12, Specifically, we will show that:

$$f_{9.5} = f_{9.5}(f, g, s, t) = f_{3.12}(f, 2g + 1, s, t);$$

$$g_{9.5} = g_{9.5}(f, g, s, t) = f_{3.12}(2g, 2, s, t)$$

satisfy the theorem. First, we deduce that:

(81) *For each* $i \in \{1, 2\}$ *and every* $z \in \Lambda$, *there exists a stable set* $S_{i,z} \subseteq \Lambda_{i,z}$ *of cardinality* g *such that* $S_{1,z}$ *and* $S_{2,z}$ *are anticomplete in* G*.*

Let $z \in \Lambda$ be fixed. Since $|\Lambda_{1,z}| = |\Lambda_{2,z}| = f_{9.5}$, it follows follows that we may choose a bijection $u : \Lambda_{1,z} \to \Lambda_{2,z}$. Therefore, the set $(\{x, u(x)\} : x \in \Lambda_{1,z})$ are pairwise disjoint 2-subsets of $V(G)$. Since G is $(K_{s,s}, K_t)$-free, it follows from the choice of $f_{9.5}$ and Lemma 3.12 that there are $2g$ vertices $x^1_1 \dots, x^1_g, x^2_1 \dots, x^2_g \in \Lambda_{1,z}$ for which the sets $(\{x^i_j, u(x^i_j)\} : i \in \{1, 2\}, j \in [g])$ are pairwise anticomplete in G. Let $S_{1,z} = \{x^1_j : j \in [g]\} \subseteq \Lambda_{1,z}$ and let $S_{2,z} = \{u(x^2_j) : j \in [g]\} \subseteq \Lambda_{2,z}$. Then $S_{1,z}$ and $S_{2,z}$ are anticomplete stable sets in G, each of cardinality g. This proves (81).

Now, $(S_{1,z} \cup S_{2,z} \cup \{z\} : z \in \Lambda)$ are $|\Lambda| = g_{9.5}$ pairwise disjoint $(2g+1)$-subsets of $V(G)$. Since G is $(K_{s,s}, K_t)$-free, it follows from the choice of $g_{9.5}$ and Lemma 3.12 that there exists an f-subset S of Λ for which the sets $(S_{1,z} \cup S_{2,z} \cup \{z\} : z \in S)$ are pairwise anticomplete in G. In particular, S is a stable set, and from (81), it follows that $\{S_{1,z}, S_{2,z} : z \in S\}$ are pairwise anticomplete stable sets in G. Hence, $\mathfrak{C}' = (S_{1,z}, z, S_{2,z} : z \in S)$ is a clear (z_1, z_2, f, g)-crystal in G. This completes the proof of Theorem 9.5. ■

9.3 Double coned forests

In this section, we complete the proof of Theorem 2.24. For the sake of an inductive argument, we need to begin with the following technical result:

Theorem 9.6. *Let $d, g, h, t \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$. Let G be a $(K_{2,3}, K_{t+1})$-free graph and let $Z \subseteq V(G)$ with $|Z| \leq h$. Let $Z_0 = \{z_1, z_2, z\} \subseteq Z$ be a 3-clique such that $N_Z(z) = \{z_1, z_2\}$. Let $\mathfrak{p} = (Z_0, \cdots, Z_r; \Gamma_i : i \in [r])$ be a $(Z_0, d + g + 2^h t^3, r)$-phantom in G such that $Z_r \cap Z = Z_0$. Then one of the following holds.*

(a) *There is a $(z_1, z_2, 1, g)$-crystal $\mathfrak{C}$ in $G[Z_r]$ with $V(\mathfrak{C})$ and $Z \setminus Z_0$ anticomplete in G.*

(b) *G has a subgraph U isomorphic to $T_{d,r}$ which satisfies the following.*
   - *U contains z as its root.*
   - *$\{z_1, z_2\}$ and $V(U)$ are disjoint and complete in G.*
   - *Let $i \in [r]$, let $u \in V(U)$ be at distance i in U from z and let u^- be the parent of u in U. Then $u \in \Gamma_i(u^- z_1) \cup \Gamma_i(u^- z_2) \subseteq Z_i \setminus Z_{i-1}$. In particular, we have $V(U) \cap Z = \{z\}$.*

Proof. The proof is by induction on r. If $r = 0$, then the subgraph U of G with $V(U) = \{z\}$ satisfies 9.6(b). Assume that $r \geq 1$. For each $i \in \{1, 2\}$, let K_i be the set of all vertices in $\Gamma_1(z_i z)$ with at least one neighbor in $Z \setminus Z_0$.

(82) *We have $|K_1|, |K_2| < 2^h t^3$.*

For suppose $|K_i| \geq 2^h t$ for some $i \in \{1, 2\}$. Since $|Z \setminus Z_0| < h$, it follows that $|K_i| > 2^{|Z \setminus Z_0|} t$, and so there is a t^3-subset K of K_i and a non-empty subset Y of $Z \setminus Z_0$ such that for every $x \in K$, we have $N_{Z \setminus Z_0}(x) = Y$. Pick a vertex $y \in Y$; then y, z are not adjacent in G. Since G is K_{t+1}-free, it follows that there is no t-clique of G contained in $K \subseteq \Gamma_i(z_i z) \subseteq N_G(z) \cap N_G(z_i)$. Thus, by Theorem 3.17, there are three pairwise non-adjacent vertices $x, x', x'' \in S$. But now $G[\{x, x', x'', y, z\}]$ is isomorphic to $K_{2,3}$, a contradiction. This proves (82).

By (82), there exist $L_1 \subseteq \Gamma_1(z_2z)$ and $L_2 \subseteq \Gamma_1(z_1z)$ with $|L_1| = |L_2| = d + g$ such that $L_1 \cup L_2$ and $Z \setminus Z_0$ are anticomplete in G. It follows that either

- for each $i \in \{1, 2\}$, there is a g-subset M_i of L_i to which z_i is anticomplete in G; or
- there exist $j \in \{1, 2\}$ and $N_j \subseteq L_j$ with $|N_j| = d$ such that z_i is complete to N_j in G.

In the former case, $\mathfrak{C} = (M_1, z, M_2)$ is a $(z_1, z_2, 1, g)$-crystal in $G[Z_1]$ such that $V(\mathfrak{C}) \subseteq L_1 \cup L_2 \cup \{z\}$ and $Z \setminus Z_0$ are anticomplete in G because $V(\mathfrak{C}) \subseteq L_1 \cup L_2 \cup \{z\}$, and so 9.6(a) holds. Therefore, we may assume that the second bullet above is the case, and by symmetry, we may also assume that $j = 1$. Since $N_1 \subseteq \Gamma_1(z_2z)$, it follows that N_1 is a d-subset of $V(G) \setminus (Z \cup \Gamma_1(z_1z_2))$ where $\{z_1, z_2\}$ is complete to N_1.

For every $z' \in N_1$, let $Z_{z'} = (Z \setminus \{z\}) \cup \{z'\}$ and let $Z_{0,z'} = \{z_1, z_2, z'\}$; then we have $Z_{0,z'} \subseteq Z_1$. Since $r \geq 1$, it follows that

$$\mathfrak{p}_{z'} = \mathfrak{p}[Z_{0,z'}; 1, r-1] = (Z_{0,z'}, \cdots, Z_{r-1,z'}; \Gamma_{i,z'} : i \in [r-1])$$

is a $(Z_{0,z'}, d + g + 2^h t^3, r-1)$-phantom in G such that $Z_{i,z'} \subseteq Z_{i+1}$ for all $i \in [r-1]$. Moreover, since $Z_{r-1,z'} \subseteq Z_r$ and $Z_r \cap Z = Z_0$, it follows that:

$$Z_{r-1,z'} \cap Z_{z'} \subseteq Z_r \cap ((Z \setminus \{z\}) \cup \{z'\}) = \{z_1, z_2, z'\} = Z_{0,z'};$$

and thus $Z_{r-1,z'} \cap Z_{z'} = Z_{0,z'}$ (see Figure 9.6).

Now, from the induction hypothesis, we deduce that:

(83) *For every vertex $z' \in N_1$, one of the following holds.*

- *There is a $(z_1, z_2, 1, g)$-crystal $\mathfrak{C}$ in $G[Z_{r-1,z'}]$ with $V(\mathfrak{C})$ and $Z_{z'} \setminus Z_{0,z'}$ anticomplete in G.*
- *G has a subgraph $U_{z'}$ isomorphic to $T_{d,r-1}$ which satisfies the following.*
  - *$U_{z'}$ contains z' as its root.*
  - *$\{z_1, z_2\}$ and $V(U_{z'})$ are disjoint and complete in G.*
  - *Let $i \in [r-1]$, let $u \in V(U_{z'})$ be at distance i in $U_{z'}$ from z' and let u^- be the parent of u in $U_{z'}$. Then $u \in \Gamma_{i,z'}(u^-z_1) \cup \Gamma_{i,z'}(u^-z_2) \subseteq Z_{i,z'} \setminus Z_{i-1,z'}$. In particular, we have $V(U_{z'}) \cap Z_{z'} = \{z'\}$.*

Note that if some vertex $z' \in N_1$ satisfies the first bullet of (83), then 9.6(a) holds because $Z_{r-1,z'} \subseteq Z_r$ and $Z_{z'} \setminus Z_{0,z'} = Z \setminus Z_0$. Therefore, we may assume that every vertex $z' \in N_1$ satisfies the second bullet of (83) (again, see Figure 9.6). We also claim that:

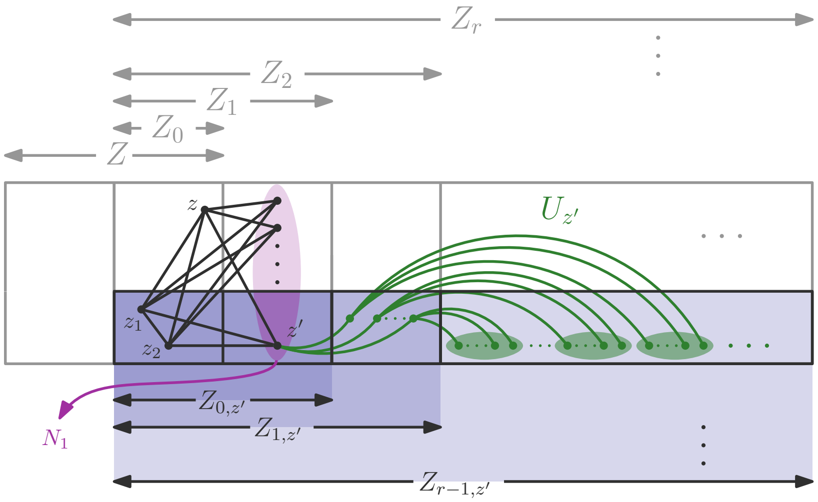

Figure 9.6: Proof of Theorem 9.6.

(84) *The sets* $(V(U_{z'}) : z' \in N_1)$ *are pairwise disjoint.*

Suppose for a contradiction that there are distinct vertices $z', z'' \in N_1$ for which $U_{z'}$ and $U_{z''}$ share some vertex u. Then u, z, z'' are all distinct, and so there are $i', i'' \in [r-1]$ such that u is at distance i' in $U_{z'}$ from z', and u is at distance i'' in $U_{z''}$ from z''. Let u be chosen such that $\min\{i', i''\}$ is as small as possible. Let u' and u'' be the parents of u in $U_{z'}$ and $U_{z''}$, respectively. Then, by the choice of u, the vertices u' and u'' are distinct. From the definition of $\mathfrak{p}_{z'}$ and the second bullet of (83), we deduce that:

$$u \in \Gamma_{i'+1}(u'z_1) \cup \Gamma_{i'+1}(u'z_2) \subseteq Z_{i'+1} \setminus Z_{i'}$$

and

$$u \in \Gamma_{i''+1}(u''z_1) \cup \Gamma_{i''+1}(u''z_2) \subseteq Z_{i''+1} \setminus Z_{i''}.$$

Particularly, it follows that $(Z_{i'+1} \setminus Z_{i'}) \cap (Z_{i''+1} \setminus Z_{i''}) \neq \varnothing$, which yields $i' = i''$. But then $u'z_1, u'z_2, u''z_1, u''z_2$ are pairwise distinct edges of $G[Z_{i'}]$ (because u', u'', z_1, z_2 are pairwise distinct vertices), for which we have

$$(\Gamma_{i'+1}(u'z_1) \cup \Gamma_{i'+1}(u'z_2)) \cap (\Gamma_{i'+1}(u''z_1) \cup \Gamma_{i'+1}(u''z_2)) \neq \varnothing.$$

This violates the second bullet condition in (PH2) from the definition of a phantom, hence completing the proof of (84).

Now, let U be the subgraph of G with

$$V(U) = \left(\bigcup_{z' \in N_1} V(U_{z'}) \right) \cup \{z\}$$

and

$$E(U) = \left(\bigcup_{z' \in N_1} E(U_{z'}) \right) \cup \{zz' : z' \in N_1\}.$$

Since z is complete to N_1 and $|N_1| = d$, it follows from the second bullet of (83) (specifically, the first dash) that U is isomorphic to $T_{d,r}$ and U contains z as the root. Also, from the second dash of the second bullet of (83) and the fact that $\{z_1, z_2\}$ is complete to z, it follows that $\{z_1, z_2\}$ is complete to $V(U)$ in G. Hence, we have shown that U satisfies the first and the second bullets of 9.6(b).

It remains to show that U satisfies the third bullet of 9.6(b). Let $i \in [r]$, let $u \in V(U)$ be at distance i in U from z and let u^- be the parent of u in U. Then we wish to prove that $u \in \Gamma_i(u^- z_1) \cup \Gamma_i(u^- z_2)$. To see this, note that if $i = 1$, then we have $u^- = z$ and $u \in N_1 \subseteq L_1 \subseteq \Gamma_1(u^- z_2)$, as required. Otherwise, there exists $z' \in N_1$ such that $u, u^- \in V(U_{z'})$, where u is at distance $i - 1 \in [r-1]$ from z' is $U_{z'}$ and u^- is the parent of u in $U_{z'}$. But now from the second bullet of (83) (specifically, the second dash) along with the definition of $\mathfrak{p}_{z'}$, we deduce that:

$$u \in \Gamma_{i-1,z'}(u^- z_1) \cup \Gamma_{i-1,z'}(u^- z_2) = \Gamma_i(u^- z_1) \cup \Gamma_i(u^- z_2).$$

This completes the proof of Theorem 9.6. ■

From Theorem 9.6, we deduce that:

Theorem 9.7. *For all $f, t, h \in \mathbb{N}$, there is a constant $f_{9.7} = f_{9.7}(f, h, t) \in \mathbb{N}$ such that for every (theta, prism, even wheel, C_4, K_{t+1})-free graph G of treewidth more than $f_{9.7}$, one of the following holds.*

- (a) *For every 2-tree ∇ on $h + 1$ vertices, G has an induced subgraph isomorphic to ∇.*
- (b) *G has a subgraph isomorphic to $\mathsf{cone}(\mathsf{cone}(T_{f,f}))$.*

Proof. The definition of $f_{9.7}$ relies on Theorem 8.1. Specifically, for each $i \in [h]$, let $r_i = (h - i)f$. We will prove that

$$f_{9.7} = f_{9.7}(f, h, t) = f_{8.1}\left(f + h + 2^h t^3, r_1, t\right)$$

satisfies the theorem. From Theorem 8.1 and the choices of $f_{9.7}, r_1$ and r_2, it follows that:

(85) *Then there is a $(Z_0^1, f + h + 2^h t^3, r_1)$-phantom in G for some 2-clique Z_0^1. Also, if $h \geq 2$, then there is a $(Z_0^2, f + h + 2^h t^3, r_2)$-phantom in G for some 3-clique Z_0^2.*

Now, instead of 9.7, we prove the following stronger statement by induction on i:

(86) *Assume that G has no subgraph isomorphic to* $\mathsf{cone}(\mathsf{cone}(T_{f,f}))$*. Then for each $i \in [h]$ and every 2-tree ∇ on $i+1$ vertices, there is an induced subgraph Z_0^i of G isomorphic to ∇ as well as a $(Z_0^i, f+h+2^ht^3, r_i)$-phantom $\mathfrak{p}^i$ in G.*

To launch the induction, note that if $i \in \{1,2\}$, then ∇ is an $(i+1)$-vertex complete graph, and so the (86) follows directly from (85). Thus, we may assume that $3 \leq i \leq h$, and in particular, ∇ has at least four vertices. By Theorem 9.4, there is a crystallized vertex z in ∇. Let $\{z_1, z_2\} \subseteq N_\nabla(z)$ and the partition (S_1, S_2) of $N_\nabla(z) \setminus \{z_1, z_2, z\}$ be as in (C1) and (C2) from the definition of a crystallized vertex. Let $\nabla' = \nabla \setminus (S_1 \cup S_2)$. It is straightforward to observe that ∇' is a 2-tree. Let $|V(\nabla')| = j+1$. Then $j \geq 2$. In fact, since $S_1 \cup S_2 \neq \varnothing$, it follows that $|V(\nabla')| < |V(\nabla)|$, and so $2 \leq j \leq i-1 \leq h-1$. Consequently, we may apply the induction hypothesis to ∇', obtaining an induced subgraph Z_0^j of G isomorphic to ∇' and a $(Z_0^j, f+h+2^ht^3, r_j)$-phantom $\mathfrak{p}^j = (Z_0^j, \dots, Z_{(h-j)f}^j; \Gamma_k^j : k \in [(h-j)f])$ in G.

Let $Z = Z_0^j$; so $|Z| = j+1 \leq i \leq h$. Let $\varphi : V(\nabla') \to V(Z)$ be an isomorphism from ∇' into Z. Let $Z_0 = \{\varphi(z_1), \varphi(z_2), \varphi(z)\} \subseteq Z$; then we have $N_Z(\varphi(z)) = \{\varphi(z_1), \varphi(z_2)\}$. Since $Z_0 \subseteq Z = Z_0^j$ and $(h-j) \geq 1$, it follows that $\mathfrak{p} = \mathfrak{p}^j[Z_0; 0, f]$ is a $(Z_0, f+h+2^ht^3, f)$-phantom $(Z_0, \dots, Z_f; \Gamma_k : k \in [f])$ in G. Moreover, since $Z_f \setminus Z_0 \subseteq Z_f^j \setminus Z_0^j = Z_f^j \setminus Z$, it follows that $Z_f \cap Z = Z_0$. This, combined with the assumption that G is (theta, K_{t+1})-free, allows for an application of Theorem 9.6 to G, Z, Z_0 and $\mathfrak{p}$.

From the assumption that G has no subgraph isomorphic to $\mathsf{cone}(\mathsf{cone}(T_{f,f}))$, we conclude that Theorem 9.6(a) holds. Explicitly, there is a $(\varphi(z_1), \varphi(z_2), 1, h)$-crystal $\mathfrak{C}$ in $G[Z_f]$ such that $V(\mathfrak{C})$ and $Z \setminus Z_0$ are anticomplete in G. Now, since $V(\mathfrak{C})$ is anticomplete to $Z \setminus Z_0$, and since $|S_1|, |S_2| \leq i \leq h$, it follows that there is an induced subgraph Z_0^i of G isomorphic to ∇ such that $Z \setminus \{\varphi(z)\} \subseteq Z_0^i \subseteq (Z \setminus \{\varphi(z)\}) \cup V(\mathfrak{C})$. In particular, we have $Z_0^i \subseteq Z \cup Z_f \subseteq Z_f^j$. Recall also that $j \leq i-1 \leq h-1$, and so we have $f, r_i \in \{0, \dots, r_j\}$ with $f + r_i \leq r_j$. As a result, $\mathfrak{p}^j = \mathfrak{p}^j[Z_0^i; f, r_i]$ is a $(Z_0^i, f+h+2^ht^3, r_i)$-phantom in G. In conclusion, we have shown that there is an induced subgraph Z_0^i of G isomorphic to ∇ as well as a $(Z_0^i, f+h+2^ht^3, r_i)$-phantom $\mathfrak{p}^i$ in G. This proves (86).

Observe that the assertion of 9.7 is equivalent to (86) for $i = h$. This completes the proof of Theorem 9.7. ■

Finally, we give a proof of Theorem 2.24, which we restate:

Theorem 2.24. *For all $t \in \mathbb{N}$, every forest F and every 2-forest $H \in \mathcal{H}$, there is a constant $f_{2.24} = f_{2.24}(F, H, t)$ such that every (even-hole, K_{t+1})-free graph G with $\operatorname{tw}(G) > f_{2.24}$ has an induced subgraph isomorphic to either* $\mathsf{cone}(\mathsf{cone}(F))$ *or H.*

Proof. Observe that there is tree T on $|V(F)|+1$ vertices which has an induced subgraph isomorphic to F. Let d and r be the maximum degree and the radius of T, respectively. Then T is isomorphic to an induced subgraph of $T_{d,r}$ and so F is isomorphic to an induced subgraph of $T_{d,r}$. Let $f = f_{9.1}(d, r, 3, t+1)$.

By Theorem 9.3, H is an induced subgraph of a 2-tree ∇. Let $|V(\nabla)| = h$.

We claim that

$$f_{2.24} = f_{2.24}(F, H, t) = f_{9.7}(f, h, t)$$

satisfies the theorem. Let G be an (even hole, K_{t+1})-free graph of treewidth more than $f_{2.24}$. Then G is (theta, prism, even wheel, C_4, K_{t+1})-free. From the choice of $f_{2.24}$ and Theorem 9.7, it follows that either G has an induced subgraph isomorphic to ∇, or G has a subgraph isomorphic to $\mathsf{cone}(\mathsf{cone}(T_{f,f}))$. In the former case, it follows from the choice of ∇ that G has an induced subgraph isomorphic to H. In the latter case, it follows from the choice of f and Theorem 9.1 that G has an induced subgraph isomorphic $\mathsf{cone}(\mathsf{cone}(T_{d,r}))$. Since F is isomorphic to an induced subgraph of $T_{d,r}$, we deduce that G has an induced subgraph isomorphic $\mathsf{cone}(\mathsf{cone}(F))$, as required. ■

9.4 Crystals

We conclude this chapter by proving Theorem 2.25. The approach is similar to that of Section 9.3: we begin with a technical result which is tailored to an inductive argument.

Theorem 9.8. *Let $f, g \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{0\}$ be integers, let G be a graph and let Z_0 be a 2-clique in G. Let $\mathfrak{p} = (Z_0, \cdots, Z_r; \Gamma_i : i \in [r])$ be a $(Z_0, f+g, r)$-phantom in G. Then one of the following holds.*

- (a) *There is an (f, g)-crystal in $G[Z_r]$.*
- (b) *There are g pairwise disjoint r-cliques $K_1, \ldots, K_g \subseteq Z_r \setminus Z_0$ of G such that Z_0 and $K_1 \cup \cdots \cup K_g$ are complete in G.*

Proof. The proof is by induction on r. Note that 9.8(b) holds trivially for $r = 0$. Assume that $r \geq 1$. Let $Z_0 = \{z_1, z_2\}$.

Let $j \in \{1, 2\}$ and $z \in \Gamma_1(z_1 z_2)$ be fixed. Let $Z^j_{0,z} = (Z_0 \setminus \{z_j\}) \cup \{z\}$; then $Z^j_{0,z} \subseteq Z_1$ is a 2-clique in G. Since $r \geq 1$, it follows that

$$\mathfrak{p}^j_z = \mathfrak{p}[Z_{0,z}; 1, r-1] = (Z_{0,z}, \cdots, Z_{r-1,z}; \Gamma_{i,z} : i \in [r-1])$$

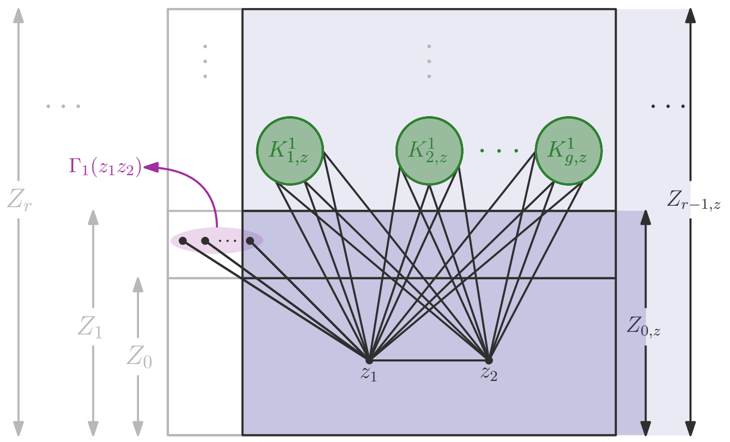

Figure 9.7: Proof of Theorem 9.8.

is a $(Z^j_{0,z}, f+g, r-1)$-phantom in G such that $Z^j_{i,z} \subseteq Z_{i+1}$ for all $i \in [r-1]$. We deduce from the induction hypothesis that:

(87) *For every* $j \in \{1,2\}$ *and every* $z \in \Gamma_1(z_1z_2)$, *one of the following holds.*

- *There exists an* (f,g)*-crystal in* $G[Z^j_{r-1,z}]$.
- *There are* g *pairwise disjoint* $(r-1)$*-cliques* $K^j_{1,z}, \dots, K^j_{g,z} \subseteq Z^j_{r-1,z} \setminus Z^j_{0,z} \subseteq Z_r \setminus Z_0$ *in* G *such that* $Z^j_{0,z}$ *and* $K^j_{1,z} \cup \dots \cup K^j_{g,z}$ *are complete in* G.

Now, if there exist $j \in \{1,2\}$ and $z \in \Gamma_1(z_1z_2)$ which satisfy the first bullet of (87), then 9.8(a) holds because $Z^j_{r-1,z} \subseteq Z_r$. Consequently, we may assume that for all $j \in \{1,2\}$ and every $z \in \Gamma_1(z_1z_2)$, the second outcome of (87) holds (see Figure 9.7).

Moreover, the following is immediate from the definition $\mathfrak{p}^j_z$ and the fact that $\mathfrak{p}$ satisfies (the second bullet of) (P2) from the definition of a phantom:

(88) *The* $2(f+g)$ *sets* $\left(Z^j_{r-1,z} \setminus Z^j_{0,z} : j \in \{1,2\}, z \in \Gamma_1(z_1z_2)\right)$ *are pairwise disjoint.*

Since $|\Gamma_1(z_1z_2)| = f+g$, it follows that either

- there is an f-subset X of $\Gamma_1(z_1z_2)$ such that for every $x \in X$, every $j \in \{1,2\}$ and every $k \in [g]$, the vertex z_j has a non-neighbor $u^j_{k,x} \in K^j_{k,x}$; or
- there is a g-subset $\{y_1, \dots, y_g\}$ of $\Gamma_1(z_1z_2)$ such that for every $i \in [g]$, there exists $j_i \in \{1,2\}$ and $k_i \in [g]$ for which z_{j_i} is complete to $K^{j_i}_{k_i,y_i}$.

In the former case, for each $x \in X$, let

$$U_{1,x} = \{u^1_{2,x} \dots , u^2_{g,x}\};$$

$$U_{2,x} = \{u^1_{1,x} \dots , u^1_{g,x}\}.$$

Then, the second bullet of (87) combined with (88) implies that $(U_{1,x}, x, U_{2,x} : x \in X)$ is a (z_1, z_2, f, g)-crystal in $G[Z_r]$, and so 9.8(a) holds.

In the latter case, for each $i \in [g]$, let

$$K_i = K^{j_i}_{k_i,y_i} \cup \{y_i\}.$$

Then, it follows from the second bullet of (87) combined with (88) that $K_1, \dots , K_g \subseteq Z_r \setminus Z_0$ are g pairwise disjoint r-cliques of G where Z_0 and $K_1 \cup \dots \cup K_g$ are complete in G. Hence, 9.8(b) holds. This completes the proof of Theorem 9.8. ■

We are ready to prove Theorem 2.25, restated below:

Theorem 2.25. *For all $t \in \mathbb{N}$ and every crystal H, there is a constant $f_{2.25} = f_{2.25}(H, t) \in \mathbb{N}$ such that every (even-hole, K_{t+1})-free graph of treewidth more than $f_{2.25}$ has an induced subgraph isomorphic to H.*

Proof. From the definition of crystal, it follows that for some $f, g \in \mathbb{N}$, there are f double stars $H_1, \dots , H_f$, each of maximum degree at most g, such that H is obtained from

$$\mathsf{cone}(H_1), \dots , \mathsf{cone}(H_f)$$

by identifying their middle edges. Let

$$f_1 = f_{9.5}(f, g, 3, t + 1);$$

$$g_1 = g_{9.5}(f, g, 3, t + 1).$$

We claim that

$$f_{2.25} = f_{2.25}(H, t) = f_{8.1}(f_1 + g_1, t + 1, t)$$

satisfies the theorem. Let G be an (even hole, K_{t+1})-free graph of treewidth more than $f_{2.25}$. Then G is (theta, prism, even wheel, C_4, K_{t+1})-free. From the choice of $f_{2.25}$ and Theorem 8.1, it follows that there is a $(Z_0, f_1 + g_1, t + 1)$-phantom $\mathfrak{p}$ in G for some 2-clique Z_0 of G. Therefore, we can apply Theorem 9.8 to G, Z_0 and $\mathfrak{p}$. Since G is K_{t+1}-free, it follows that Theorem 9.8(b) does not hold. We deduce that there is an (f_1, g_1)-crystal in G. This, along with the choices of f_1 and g_1, the assumption that G is $(K_{3,3}, K_{t+1})$-free, and Theorem 9.5, implies that there is a clear (f, g)-crystal in G. But now from the choices of f and g, it follows that G has an induced subgraph isomorphic to H, as desired. ■

Chapter 10

Even-hole-free graphs
IV. Separating a trapped vertex[1]

In this last chapter, we prove Theorem 7.3:

Theorem 7.3. *Let* $t \in \mathbb{N}$ *and let* G *be a (theta, prism,* K_{t+1}*)-free graph. Let* H *be an induced subgraph of* G *that is isomorphic to the line graph of a proper subdivision of a tree which is not a path. Let* $a \in V(G) \setminus V(H)$ *such that* a *is trapped in* $G[V(H) \cup \{a\}]$*. Then for every vertex* $x \in G \setminus N_G[a]$*, there exists* $S_x \subseteq V(G) \setminus \{a, x\}$ *with* $|S_x| < 4t^6$ *such that* S_x *separates* a *and* x *in* G*.*

The main technicality in the proof of Theorem 7.3 is isolated in Theorem 10.2, which gives an explicit decomposition theorem for all (theta, prism)-free graphs containing a vertex a as described in the statement of Theorem 7.3. It turns out that the entire graph $G \setminus \{a\}$ can be partitioned into two induced subgraphs L and L', such that L admits a line-graph-like structure imposed by H, and every vertex $v \in V(L')$ is either anticomplete to $V(L)$ or attaches to $V(L)$ in a highly restricted way; we refer to the latter type of vertices as the "jewels."

The formal statement of Theorem 10.1, as well as its proof, will appear in Section 10.1. In Section 10.2, we will examine the adjacency between the jewels and the line-graph-like part L more closely to show that in the case of bounded clique number, the jewels cluster into subsets of bounded cardinality, attaching at pairwise "far-apart" zones of L.

Combining the latter result and the decomposition theorem from Section 10.1, we will derive Theorem 7.3 in Section 10.3.

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[1] This chapter is based on the coauthored paper [2].

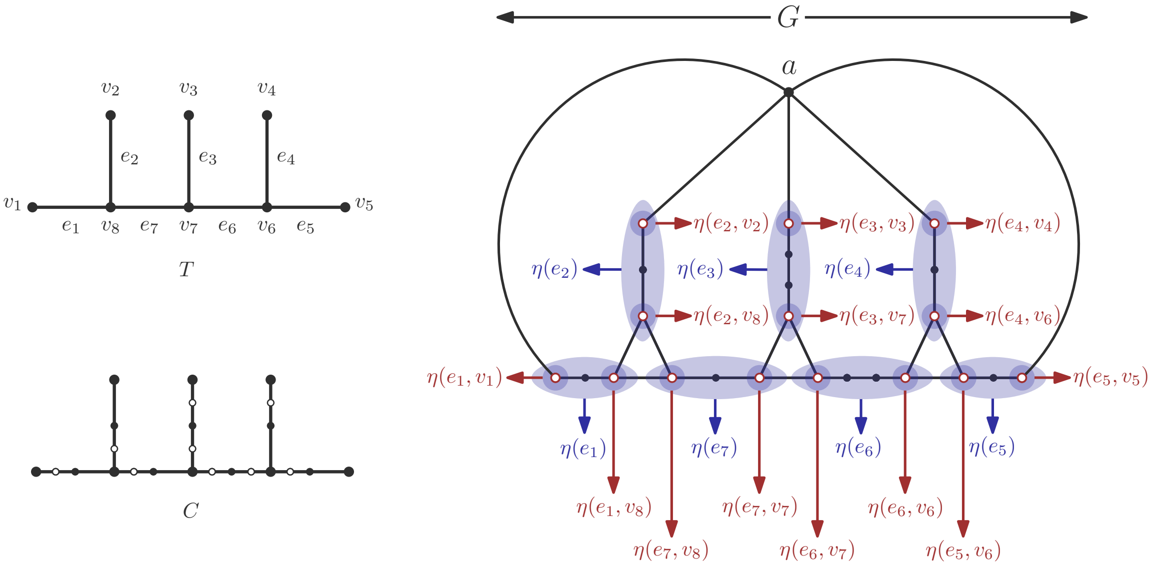

Figure 10.1: A smooth tree T (top left – note that T is a caterpillar), a proper subdivision C of T (bottom left – note that C is also a caterpillar), and a graph G in whivh there is a (T, a)-strip-structure η (right – note that $G[\eta(T)]$ is isomorphic to the line graph of C).

10.1 Jeweled strip structures

A tree T is said to be *smooth* if T has at least three vertices and no vertex of T has degree two. Let G be a graph, let $a \in V(G)$, let T be a smooth tree, and let

$$\eta : V(T) \cup E(T) \cup (E(T) \times V(T)) \to 2^{V(G) \setminus \{a\}}$$

be a function. For a subset S of $V(T)$, we define

$$\eta(S) = \bigcup_{v \in S,\ e \in E(T[S])} (\eta(v) \cup \eta(e))$$

and

$$\eta^+(S) = \eta(S) \cup \{a\}.$$

Also, for a vertex $v \in V(T)$, we define $B_\eta(v)$ as follows (although we will often omit the subscript η unless there is ambiguity).

$$B_\eta(v) = \bigcup_{v \in e \in E(T)} \eta(e, v).$$

With the notation as above, the function η is said to be a *(T, a)-strip-structure in G* if the following conditions are satisfied (see Figure 10.1; see also [17], [18] and [19] for similar notions).

- (S1) For all distinct $o, o' \in V(T) \cup E(T)$, we have $\eta(o) \cap \eta(o') = \varnothing$.
- (S2) If $l \in V(T)$ is a leaf of T, then $\eta(l)$ is empty.
- (S3) For all $e \in E(T)$ and $v \in V(T)$, we have $\eta(e, v) \subseteq \eta(e)$, and $\eta(e, v) \neq \varnothing$ if and only if e is incident with v.
- (S4) For all distinct edges $e, f \in E(T)$ and every vertex $v \in V(T)$, the sets $\eta(e, v)$ and $\eta(f, v)$ are complete in G, and there are no other edges in G between $\eta(e)$ and $\eta(f)$. In particular, if e and f share no end, then $\eta(e)$ and $\eta(f)$ are anticomplete in G.
- (S5) For every $e \in E(T)$ with ends u, v, define $\eta^{\circ}(e) = \eta(e) \setminus (\eta(e, u) \cup \eta(e, v))$. Then for every vertex $x \in \eta(e)$, either
  - we have $x \in \eta(e, u) \cap \eta(e, v)$; or
  - there is a path in $\eta(e)$ from x to a vertex in $\eta(e, u) \setminus \eta(e, v)$ with interior contained in $\eta^{\circ}(e)$, and there is a path in $\eta(e)$ from x to a vertex in $\eta(e, v) \setminus \eta(e, u)$ with interior contained in $\eta^{\circ}(e)$.
- (S6) For all $v \in V(T)$ and $e \in E(T)$, the sets $\eta(v)$ and $\eta(e) \setminus \eta(e, v)$ are anticomplete in G. Equivalently, we have $N_{\eta(T)}(\eta(v)) \subseteq B_{\eta}(v)$.
- (S7) For every $v \in V(T)$ and every component D of $\eta(v)$, we have $N_{B_{\eta(v)}}(D) \neq \varnothing$.
- (S8) For every leaf l of T with $e \in E(T)$ being the leaf-edge of T incident with l, the vertex a is complete to $\eta(e, l)$. Moreover, a has no other neighbors in $\eta(T)$.

A *club* is a 4-tuple (G, a, T, η) where G is a graph, $a \in V(G)$, T is a smooth tree and η is a (T, a)-strip-structure in G.

We say a subset C of $\eta(T)$ is *local in η* if $C \subseteq \eta(e)$ for some $e \in E(T)$ or $C \subseteq B_{\eta}(v) \cup \eta(v)$ for some $v \in V(T)$. The following lemma is a characterization of all local subsets:

Theorem 10.1. *Let (G, a, T, η) be a club. Then a subset C of $\eta(T)$ is local in η if and only if every 2-subset of C is local in η.*

Proof. The "only if" implication is trivial. For the "if" implication, suppose for a contradiction that there exists $C \subseteq \eta(T)$ which is not local in η, but every 2-subset of C is local in η. We claim that:

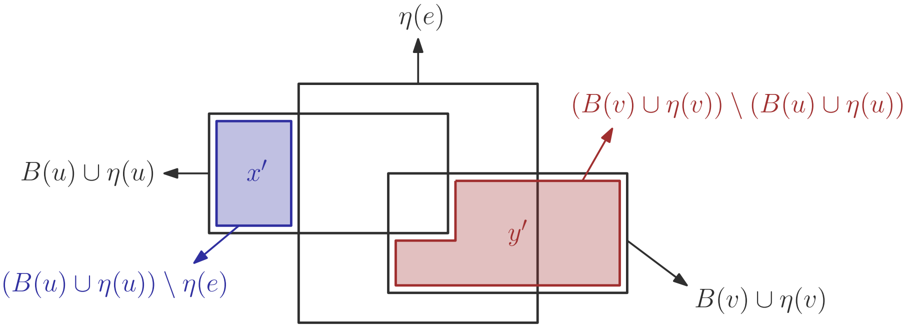

Figure 10.2: Proof of Theorem 10.1.

(89) $C \subseteq \bigcup_{v \in V(T)} (B(v) \cup \eta(v))$.

For suppose there exists a vertex $x \in C \cap \eta^\circ(e)$ for some $e \in E(T)$. Then by (S1), we have $x \notin \eta(f)$ for all $f \in E(T) \setminus \{e\}$, and also we have $x \notin B(v) \cup \eta(v)$ for every $v \in V(T)$. Thus, since C is not local, it follows that there exists a vertex $y \in C \setminus \eta(e)$. But then $\{x, y\}$ is a 2-subset of C which is not local in η, a contradiction. This proves (89).

By (89), and since the empty set is local in η, we may pick $x \in C$, $v \in V(T)$ and $e = uv \in E(T)$ such that $x \in \eta(e, v) \cup \eta(v)$. We deduce:

(90) $C \subseteq B(u) \cup \eta(u) \cup B(v) \cup \eta(v)$.

Suppose for a contradiction that there exists a vertex $y \in C \setminus (B(u) \cup \eta(u) \cup B(v) \cup \eta(v))$. By (89), we have $C \setminus (B(u) \cup \eta(u) \cup B(v) \cup \eta(v)) = C \setminus (\eta(e) \cup B(u) \cup \eta(u) \cup B(v) \cup \eta(v))$, and so $y \in C \setminus (\eta(e) \cup B(u) \cup \eta(u) \cup B(v) \cup \eta(v))$. But then by (S1), $\{x, y\}$ is a 2-subset of C which is not local in η, a contradiction. This proves (90).

Now, since C is not local, it follows that $C \not\subseteq \eta(e)$; so by (90) and without loss of generality, we may assume that there exists a vertex

$$x' \in (B(u) \cup \eta(u)) \setminus \eta(e) \subseteq (B(u) \cup \eta(u)) \setminus (B(v) \cup \eta(v))$$

such that $\{x', y'\} \subseteq C$. Similarly, since C is not local, it follows that $C \not\subseteq B(u) \cup \eta(u)$, and so by (90), there exists a vertex $y' \in (B(v) \cup \eta(v)) \setminus (B(u) \cup \eta(v))$. But then $\{x', y'\}$ is a 2-subset of C which is not local in η, a contradiction (see Figure 10.2). This completes the proof of Theorem 10.1. ■

The mere statement of our main result in this section relies on a number of further definitions, which we will provide below.

For the following four sets of definitions, let (G, a, T, η) be a club.

- **Rungs.** For an edge $e \in E(T)$ with ends u, v, by an *$\eta(e)$-rung*, we mean a path P in $\eta(e) \subseteq \eta(T)$ for which either $|V(P)| = 1$ and $P \subseteq \eta(e,u) \cap \eta(e,v)$, or P has an end in $\eta(e,u) \setminus \eta(e,v)$, an end in $\eta(e,v) \setminus \eta(e,u)$, and $P^* \subseteq \eta^\circ(e)$. Equivalently, a path P in $\eta(e)$ is an $\eta(e)$-rung if P has an end in $\eta(e,u)$, an end in $\eta(e,v)$, and $|P \cap \eta(e,u)| = |P \cap \eta(e,v)| = 1$.
  It follows from (S5) that every vertex in $\eta(e) \setminus \eta^\circ(e)$ is contained in an $\eta(e)$-rung. In particular, if $\eta(e,u) \subseteq \eta(e,v)$, then we have $\eta(e,u) = \eta(e,v)$ (for otherwise each vertex in $\eta(e,v) \setminus \eta(e,u)$ fails to satisfy both bullet conditions of (S5)). Similarly, if $\eta(e,v) \subseteq \eta(e,u)$, then we have $\eta(e,u) = \eta(e,v)$. An $\eta(e)$-rung is said to be *long* if it is of non-zero length.

- **Tame structures.** Let $e \in E(T)$. We write $\tilde{\eta}(e)$ for the set of all vertices in $\eta(e)$ that are not in any $\eta(e)$-rung (so $\tilde{\eta}(e) \subseteq \eta^\circ(e)$.) We say that η is *tame* if $\eta(v) = \varnothing$ for every $v \in V(T)$, and $\tilde{\eta}(e) = \varnothing$ for every $e \in E(T)$. In other words, η is tame if and only if every vertex in $\eta(T)$ is in an $\eta(e)$-rung for some $e \in E(T)$.

- **Substantial structures.** We say that η is *substantial* if for every $e \in E(T)$, there exists a long $\eta(e)$-rung in G. Equivalently, η is substantial if for every edge $e \in E(T)$ with ends u, v, we have $\eta(e,u) \neq \eta(e,v)$, and so $\eta(e,u) \setminus \eta(e,v), \eta(e,v) \setminus \eta(e,u) \neq \varnothing$. For a (T,a)-strip-structure η' in G, we write $\eta \leq \eta'$ to mean that $\eta(o) \subseteq \eta'(o)$ for every $o \in V(T) \cup E(T) \cup (E(T) \times V(T))$. One may observe (using (S4) and the smoothness of T, in particular) that if η is substantial and $\eta \leq \eta'$, then η' is also substantial.

- **Rich structures.** We say η is *rich* if a is trapped in $\eta^+(T)$, and for every edge $e = lv \in E(T)$ where l is a leaf, we have $|\eta(e,l)| = 1$. It follows that if η is rich, then T has exactly $|N_G(a)|$ leaves. Moreover, for each edge $e = lv \in E(T)$ where l is a leaf of T, since T is smooth, it follows that v has degree at least three in T, which in turn implies that $\eta(e,v) \cap \eta(e,l) = \varnothing$ (for otherwise we have $\eta(e,l) \subseteq \eta(e,v)$, and the single neighbor of a in $\eta(e,l)$ violates the assumption that a is trapped in $\eta^+(T)$).

A *pyramid* is a graph Σ consisting of a vertex a, a triangle $\{b_1, b_2, b_3\}$ disjoint from a and three paths P_1, P_2, P_3 in Σ at most one of which has length exactly one, such that for each $i \in [3]$, the ends of P_i are a to b_i, and for all distinct $i, j \in [3]$, the sets $V(P_i) \setminus \{a\}$ and $V(P_j \setminus \{a\})$ are disjoint, and b_ib_j is the only edge of G with an end in $V(P_i) \setminus \{a\}$ and an

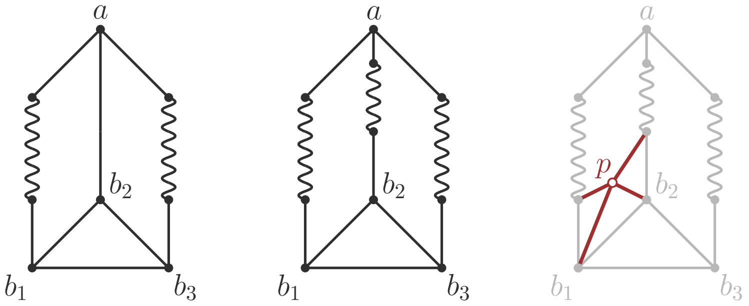

Figure 10.3: From left to right: A short pyramid, a long pyramid, and a jewel (at b_3). Squiggly lines represent paths of arbitrary (possibly zero) length.

end in $V(P_j) \setminus \{a\}$. We say that a is the *apex* of Σ, the triangle $b_1b_2b_3$ is the *base* of Σ, and P_1, P_2, P_3 are the *paths* of Σ. The pyramid Σ is said to be *long* if each of its paths has length at least two. For a graph G, by a *(long) pyramid in G* we mean an induced subgraph of G which is a (long) pyramid. See Figure 10.3.

Let G be a graph and let Σ be a pyramid in G with apex a, triangle $b_1b_2b_3$, and paths P_1, P_2, P_3. Let $i \in [3]$ and let $p \in V(G) \setminus V(\Sigma)$. We say p is a *jewel for* Σ *at* b_i if p is anticomplete to P_i (in particular, p is anticomplete to a), and for every $j \in [3] \setminus \{i\}$, we have $N_{P_j}(p) = N_{P_j}[b_j]$ (see Figure 10.3). By a *jewel for* Σ we mean a jewel for Σ at one of b_1, b_2 or b_3. For the most part, we will be tacking pyramids and jewels that are "canonical" relative to a given strip structure. The precise definitions are as follows. Let (G, a, T, η) be a club.

- **Seagulls and claws.** By a *seagull in T* we mean a triple (v, e_1, e_2) where $v \in V(T)$ and e_1, e_2 are two distinct edges of T incident with v. By a *claw in T* we mean a 4-tuple (v, e_1, e_2, e_3) where $v \in V(T)$ and e_1, e_2, e_3 are three distinct edges of T incident with v.

- **Canonical pyramids.** Let (v, e_1, e_2, e_3) be a claw in T. By an *η-pyramid at* (v, e_1, e_2, e_3), we mean a pyramid Σ in $\eta^+(T)$ with apex a, base $b_1b_2b_3$ and paths P_1, P_2, P_3, such that the following hold for each $i \in [3]$ (see also Figure 10.4).
  - $b_i \in \eta(e_i, v)$.
  - There is a leaf l_i of T with the following property. Let Λ_i be the unique path in T from v to l_i. Then we have:
    - $e_i \in E(\Lambda_i)$; and

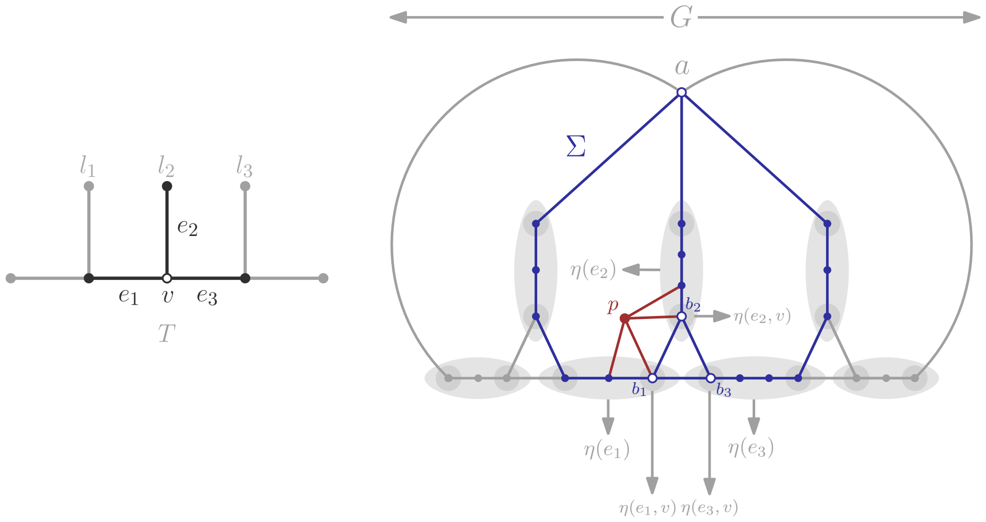

Figure 10.4: Left: A smooth tree T, a branch vertex v of T and a claw (v, e_1, e_2, e_3) in T. Right: An η-pyramid Σ, which is an η-pyramid at v, or more specifically, an η-pyramid at (v, e_1, e_2, e_3). The vertex p is η-jewel, a η-jewel at v, and a η-jewel at (v, e_1, e_2).

- $P_i = \Gamma_i \cup \{a\}$, where Γ_i is a path in $G[\bigcup_{e \in E(\Lambda_i)} \eta(e)]$ such that $R_i = \Gamma_i \cap \eta(e_i)$ is a *long* $\eta(e_i)$-rung and $\Gamma_i \cap \eta(e)$ is an $\eta(e)$-rung for each $e \in E(\Lambda_i) \setminus \{e_i\}$.

In particular, assuming u_i to be the ends of e_i distinct from v and c_i to be the unique vertex in $N_{R_i}(b_i) = N_{P_i}(b_i)$ for each $i \in [3]$, we have $b_i \in \eta(e_i, v) \setminus \eta(e_i, u_i)$ and $c_i \in \eta(e_i) \setminus \eta(e_i, v)$.

For a branch vertex $v \in V(T)$, by an *η-pyramid at v* we mean an η-pyramid at (v, e_1, e_2, e_3) for some claw (v, e_1, e_2, e_3) in T. Also, by an *η-pyramid* we mean an η-pyramid at v for some branch vertex $v \in V(T)$. It follows that every η-pyramid is a long pyramid, and if η is substantial, then there is a η-pyramid at every claw (v, e_1, e_2, e_3) in T.

- **Canonical jewels.** Let (v, e_1, e_2) be a seagull in T. A vertex $p \in G \setminus \eta^+(T)$ is said to be a *η-jewel at* (v, e_1, e_2) if for some edge $e_3 \in E(T) \setminus \{e_1, e_2\}$ incident with v, there exists an η-pyramid Σ at (v, e_1, e_2, e_3) with base $b_1b_2b_3$ where $b_i \in \eta(e_i, v)$ for each $i \in [3]$, such that p is a jewel for Σ at b_3 (see Figure 10.4). In particular, for each $i \in \{1, 2\}$, the vertex p is adjacent to b_i and its unique neighbor c_i in P_i^*. Since Σ is an η-pyramid at (v, e_1, e_2, e_3), it follows that, assuming u_i to be the end of e_i distinct from v, the vertex p has a neighbor in $\eta(e_i, v) \setminus \eta(e_i, u_i)$, namely b_i, and a neighbor in $\eta(e_i) \setminus \eta(e_i, v)$, namely c_i.

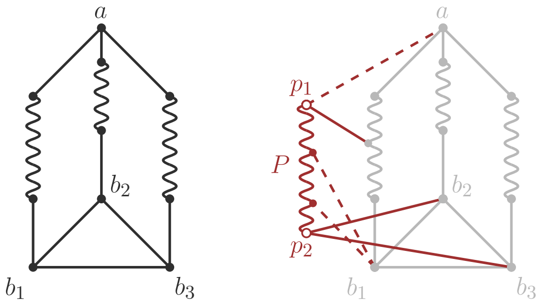

Figure 10.5: A pyramid (left) and a corner path at b_1 (right). Squiggly lines represent paths of arbitrary (possibly zero) length and dashed lines represent possible edges.

For a vertex $v \in V(T)$, by a *η-jewel at v* we mean a η-jewel at (v, e_1, e_2) for some seagull (v, e_1, e_2) in T. By a *η-jewel* we mean a η-jewel at v for some branch vertex $v \in V(T)$. We denote by JW_η the set of all η-jewels. It follows that $\mathsf{JW}_\eta \subseteq G \setminus \eta^+(T)$.

We are now in a position to state the main result of this section:

Theorem 10.2. *Let (G, a, T, η) be a club where G is (theta, prism)-free and η is tame, substantial, and rich. Then there is a substantial and rich (T, a)-strip-structure ζ in G such that $\eta \leq \zeta$, and for every vertex $w \in G \setminus \zeta^+(T)$, either w is anticomplete to $\zeta^+(T)$ in G or $w \in \mathsf{JW}_\zeta$.*

The proof of Theorem 10.2 makes critical use of Theorem 10.3 below. In rough terms, for a (theta, prism)-free graph G and a pyramid Σ in G whose apex is trapped in Σ, Theorem 10.3 characterizes the adjacency between Σ and any path in $G \setminus \Sigma$. For the exact statement, we need to give a few definitions. Let G be a graph and let Σ be a pyramid in G with apex a, base $b_1b_2b_3$ and paths P_1, P_2, P_3. A set $X \subseteq V(\Sigma)$ is said to be *local (in Σ)* if either $X \subseteq V(P_i)$ for some $i \in [3]$ or $X \subseteq \{b_1, b_2, b_3\}$. Let P be a path in $G \setminus \Sigma$ with (not necessarily distinct) ends p_1, p_2. For $i \in [3]$, we say P is a *corner path for Σ at b_i* if p_1 has at least one neighbor in $V(P_i) \setminus \{b_i\}$, p_2 is complete to $\{b_1, b_2, b_3\} \setminus \{b_i\}$, and except for the edges imposed by the latter conditions, there is no edge with an end in $V(P)$ and an end in $V(\Sigma) \setminus \{b_i\}$ (see Figure 10.5). By a *corner path for Σ* we mean a corner path for Σ at one of b_1, b_2 or b_3.

Observe that, if G is a graph, Σ is a pyramid in G and P is a path in $G \setminus \Sigma$ such that $N_\Sigma(P)$ is local in Σ, then there is no path in P which is a corner path for Σ, nor is there a jewel for Σ in $V(P)$. The following shows that if G is (theta, prism)-free, then the converse is also true:

Theorem 10.3. *Let G be a (theta, prism)-free graph, let H be an induced subgraph of G, let $a \in V(H)$ be trapped in H and let P be a path in $G \setminus H$. Let Σ be a pyramid in H with apex a, base $b_1b_2b_3$, and paths P_1, P_2, P_3. Then $N_\Sigma(P)$ is local in Σ if and only if there is no path in P which is a corner path for Σ, and $V(P)$ contains no jewel for Σ.*

Proof. We only need to prove the "if" implication. Suppose for a contradiction that there exists a path P in $G \setminus H$ for which $N_\Sigma(P)$ is not local in Σ, there is no path in P which is a corner path for Σ, and there is no jewel for Σ in $V(P)$. We choose such a path P with $|V(P)|$ minimum. It follows that $N_\Sigma(X)$ is local in Σ for every path X in G with $V(X) \subsetneq V(P)$. Also, since a is trapped in H, it follows that Σ is a long pyramid, and $V(P) \subseteq V(G) \setminus V(H)$ is anticomplete to $N_\Sigma[a]$.

First, assume that $|V(P)| = 1$, say $V(P) = \{p\}$. We deduce:

(91) *There exists $i \in [3]$ for which p is anticomplete to P_i.*

Suppose for a contradiction that p has a neighbor in each of P_1, P_2, P_3. Since $N_\Sigma(p)$ is not local in Σ and p is not a corner path for Σ, we may assume without loss of generality that p has a neighbor in P_1^* and a neighbor in P_2^*. For each $i \in [3]$, traversing P_i from a to b_i, let x_i be the first neighbor of p in P_i. Since a is trapped, it follows that $x_1 \in P_1^*$, $x_2 \in P_2^*$ and $x_3 \in P_3 \setminus N_\Sigma[a]$ (in particular, $x_3 = b_3$ is possible). But then G contains a theta with ends a, p and paths $(a\text{-}P_i\text{-}x_i\text{-}p : i \in [3])$, a contradiction. This proves (91).

By (91) and without loss of generality, we may assume that p is anticomplete to P_3. Since $N_\Sigma(p)$ is not local in Σ, it follows that p has a neighbor in P_1 and a neighbor in P_2, and there exists $j \in \{1, 2\}$ for which p has a neighbor in P_j^*. For each $j \in \{1, 2\}$, traversing P_j from a to b_j, let x_j and y_j be the first and the last neighbor of p in P_j, respectively. Then we have $x_j \in P_j^* \setminus N_{P_j}(a)$ for some $j \in \{1, 2\}$. Indeed, this happens for all $j \in \{1, 2\}$:

(92) *We have $x_1 \in P_1^* \setminus N_{P_1}(a)$ and $x_2 \in P_2^* \setminus N_{P_2}(a)$.*

Suppose not. Since $N_\Sigma(p)$ is not local in Σ and p is anticomplete to $N_\Sigma(a)$, we may assume without loss of generality that p has a neighbor in P_1^* and $x_2 = y_2 = b_2$. But now G contains a theta with ends a, b_2 and paths $a\text{-}P_1\text{-}x_1\text{-}p\text{-}b_2$, $a\text{-}P_2\text{-}b_2$ and $a\text{-}P_3\text{-}b_3\text{-}b_2$, a contradiction. This proves (92).

(93) *$N_{P_j}(p)$ is a 2-clique for every $j \in \{1, 2\}$.*

Suppose not. Then we may assume without loss of generality that either $x_1 = y_1$ or x_1 and y_1 are distinct and non-adjacent. By (92), for each $j \in \{1, 2\}$, we have $x_j \in P_j^* \setminus N_{P_j}(a)$.

It follows that if $x_1 = y_1$, then G contains a theta with ends a, x_1 and paths a-P_1-x_1, a-P_2-x_2-p-x_1 and a-P_3-b_3-b_1-P_1-x_1, a contradiction. Therefore, x_1 and y_1 are distinct and non-adjacent. But then G contains another theta, this time with ends a, p and paths a-P_1-x_1-p, a-P_2-x_2-p and a-P_3-b_3-b_1-P_1-y_1-p, again a contradiction. This proves (93).

The case $|V(P)| = 1$ is almost concluded. By (93), for each $j \in \{1,2\}$, we have $N_{P_j}(p) = \{x_j, y_j\}$ where x_j is adjacent to y_j. It follows that if $y_j \in P_j^*$ for some $(j, j') \in \{(1,2),(2,1)\}$, then G contains a prism with triangles $\{x_j, y_j, p\}$ and $\{b_1, b_2, b_3\}$ and paths x_j-P_j-a-P_3-b_3, y_j-P_j-b_j and p-$y_{j'}$-$P_{j'}$-$b_{j'}$, a contradiction. Hence, we have $y_j = b_j$ for every $j \in \{1,2\}$. But now p is a jewel corner for Σ at b_3, a contradiction.

From now on, assume that $|V(P)| > 1$. Let p_1 and p_2 be the distinct ends of P. For every $i \in [3]$, let $P'_i = P_i \setminus N_{P_i}[a]$. By the minimality of $V(P)$ of and since $N_\Sigma(P)$ is not local in Σ, we may assume without loss of generality that

- $N_\Sigma(p_1) \subseteq P'_1$ and p_1 has a neighbor in $P'_1 \setminus \{b_1\}$; and
- p_2 has a neighbor in P'_2, and either $N_\Sigma(p_2) \subseteq P'_2$ or $N_\Sigma(p_2) \subseteq \{b_1, b_2, b_3\}$.

It follows from the choice of P that P^* and $V(\Sigma) \setminus \{b_1\}$ are anticomplete in G. For each $i \in \{1,2\}$, traversing P_i from a to b_i, let x_i and y_i be the first and the last neighbor of p_i in P_i, respectively. So we have $x_1 \in P'_1 \setminus \{b_1\}$, $y_1 \in P'_1$ and $x_2, y_2 \in P'_2$. The latter can be made more informative:

(94) *We have* $x_2 \in P'_2 \setminus \{b_2\}$.

Suppose not. Then we have $x_2 = y_2 = b_2$, and so $b_2 \in N_\Sigma(p_2) \subseteq \{b_1, b_2, b_3\}$. Since P is not a corner path for Σ at b_1, it follows that p_2 is not adjacent to b_3. But now G contains a theta with ends a, b_2 and paths a-P_1-x_1-p_1-P-p_2-b_2, a-P_2-b_2 and a-P_3-b_3-b_2, a contradiction. This proves (94).

From (94) and the minimality of P, it follows that P^* and $V(\Sigma)$ are anticomplete in G, and for every $i \in \{1,2\}$, we have $N_\Sigma(p_i) = N_{P'_i}(p_i)$, $x_i \in P'_i \setminus \{b_i\}$ and $y_i \in P'_i$.

(95) *For every* $i \in \{1,2\}$*, the vertices* x_i *and* y_i *are distinct and adjacent.*

Suppose not. Then we may assume without loss of generality that either $x_1 = y_1$ or x_1 and y_1 are distinct and non-adjacent. In the former case, G contains a theta with ends a, x_1 and paths a-P_1-x_1, a-P_2-x_2-p_2-P-p_1-x_1 and a-P_3-b_3-b_1-P_1-x_1, and in the latter case, G contains a theta with ends a, p_1 and paths a-P_1-x_1-p_1, a-P_2-x_2-p_2-P-p_1 and a-P_3-b_3-b_1-P_1-y_1-p_1, both of which are contradictions. This proves (95).

To finish the proof, note that by (95), $N_{P_i}(p) = \{x_i, y_i\}$ is a 2-clique for every $i \in \{1,2\}$. But now G contains a prism with triangles $\{p_1, x_1, y_1\}$ and $\{p_2, x_2, y_2\}$ and paths P, x_1-P_1-a-P_2-x_2 and y_1-P_1-b_1-b_2-P_2-y_2, a contradiction. This completes the proof of Theorem 10.3. ■

We are ready to prove Theorem 10.2, which we restate:

Theorem 10.2. *Let (G, a, T, η) be a club where G is (theta, prism)-free and η is tame, substantial, and rich. Then there is a substantial and rich (T, a)-strip-structure ζ in G such that $\eta \leq \zeta$, and for every vertex $w \in G \setminus \zeta^+(T)$, either w is anticomplete to $\zeta^+(T)$ in G or $w \in \mathsf{JW}_\zeta$.*

Proof. Let η be a tame, substantial, and rich (T, a)-strip-structure in G such that $\eta(T)$ is maximal with respect to inclusion. Let $M = G \setminus (\eta^+(T) \cup \mathsf{JW}_\eta)$.

(96) *Let P be a path in M with ends p_1 and p_2 such that there exist $x_1 \in N_{\eta(T)}(p_1)$ and $x_2 \in N_{\eta(T)}(p_2)$ for which $\{x_1, x_2\}$ is not local in η, and such that $|V(P)| \geq 1$ is minimum subject to this property. Then there exists $(j_1, j_2) \in \{(1,2),(2,1)\}$ and $f = v_1v_2 \in E(T)$ such that $x_{j_1} \in B(v_{j_1}) \setminus \eta(f)$ and $x_{j_2} \in (B(v_{j_2}) \cup \eta(f)) \setminus B(v_{j_1})$.*

Suppose not. For each $i \in \{1,2\}$, let $e_i \in E(T)$ be such that $x_i \in \eta(e_i)$ (hence $e_1 \neq e_2$) and let s_i be an end of e_i such that there exists a path Λ_0 (possibly of length zero) from s_1 to s_2 in $T \setminus \{e_1, e_2\}$. We claim that there is a vertex $v \in \Lambda_0$ such that $B(v) \cap \{x_1, x_2\} = \varnothing$. First, suppose that $s_1 \neq s_2$. Let v_1 be the unique neighbor of s_1 in Λ_0. Then we have $x_1 \notin B(v_1)$ and $x_2 \notin B(s_1)$. Also, since $f = s_1v_1$ does not satisfy (96), we have either $x_1 \notin B(s_1)$ or $x_2 \notin B(v_1)$. But then either $v = s_1$ or $v = v_1$ satisfies the claim. Thus, we may assume that $v = s_1 = s_2$. Since neither e_1 nor e_2 satisfies (96), we have $x_1 \notin B(s_1)$ and $x_2 \notin B(s_2)$. In other words, we have $B(v) \cap \{x_1, x_2\} = \varnothing$, and the claim follows. Henceforth, let v be as promised by the above claim. For each $i \in \{1,2\}$, let u_i be the end of e_i distinct from s_i (hence $u_1 \neq u_2$). Let $\Lambda = u_1$-s_1-Λ_0-s_2-u_2 and let u'_1, u'_2 be the neighbors of v in Λ such that Λ traverses u_1, u'_1, v, u'_2, u_2 in this order (so possibly $u_1 = u'_1$ or $u_2 = u'_2$). For each $i \in \{1,2\}$, let $e'_i = u'_iv$. Since T is smooth, one may choose a vertex $u'_3 \in N_T(v) \setminus V(\Lambda)$. Let $e'_3 = u'_3v$. For each $i \in [3]$, let T_i be the component of $T \setminus (N_T(v) \setminus \{u'_i\})$ containing v (so $e'_i \in E(T_i)$). Since $B(v) \cap \{x_1, x_2\} = \varnothing$ and since η is tame and substantial, it follows that there is an η-pyramid Σ at (v, e'_1, e'_2, e'_3) with apex a, base $b_1b_2b_3$ and paths P_1, P_2, P_3 such that we have:

- $b_i \in \eta(e'_i, v)$ and $P^*_i \subseteq \eta(T_i) \setminus B(v)$ for each $i \in [3]$; and

- $x_i \in P_i^*$ for each $i \in \{1,2\}$. In particular, $N_\Sigma(P)$ is not local in Σ and P is not a corner path for Σ.

On the other hand, since $V(P) \subseteq M \subseteq V(G) \setminus (\eta^+(T) \cup \mathsf{JW}_\eta)$ and since Σ is an η-pyramid, it follows that $V(P)$ contains no jewel for Σ. Moreover, a is trapped in $\eta^+(T)$ because η is rich. Therefore, we can apply Theorem 10.3 to G, $H = \eta^+(T)$, a, P and Σ to deduce that P contains a corner path for Σ. Note also that by the second bullet above, for every vertex $x \in V(\Sigma) \setminus \{a\}$, either $\{x, x_1\}$ or $\{x, x_2\}$ is not local in η. From this, the minimality of $|V(P)|$ and the fact that η is rich, it follows that P^* and $V(\Sigma)$ are anticomplete in G. But now P itself is a corner path for Σ, a contradiction with the second bullet above. This proves (96).

(97) *Let P be a path in M with ends p_1 and p_2 such that there exist $x_1 \in N_{\eta(T)}(p_1)$ and $x_2 \in N_{\eta(T)}(p_2)$ for which $\{x_1, x_2\}$ is not local in η, and such that $|V(P)| \geq 1$ is minimum subject to this property. Let $(j_1, j_2) \in \{(1,2),(2,1)\}$ and $f = v_1v_2 \in E(T)$ be as guaranteed by* (96) *applied to P, x_1 and x_2. Then we have $N_{\eta(T)}(P^*) \subseteq \eta(f, v_{j_1})$ and $N_{\eta(T)}(\{p_1, p_2\}) \subseteq \eta(f) \cup B(v_1) \cup B(v_2)$.*

Suppose not. Without loss of generality, we may assume that $j_1 = 1$ and $j_2 = 2$. By the minimality of $|V(P)|$, we have $N_{\eta(T)}(P^*) \subseteq \eta(f, v_1)$. So one of p_1 and p_2, say the former, has a neighbor in $x'_1 \in \eta(T) \setminus (\eta(f) \cup B(v_1) \cup B(v_2))$. Let T_1 be the component of $T \setminus \{v_2\}$ containing v_1 and let T_2 be the component of $T \setminus \{v_1\}$ containing v_2. It follows that there exists $(i,j) \in \{(1,2),(2,1)\}$ such that $x'_1 \in \eta(T_j) \setminus B(v_j)$. But now we claim that P, x'_1 and x_i violate (96). This is immediate if $|V(P)| = 1$. If $|V(P)| > 1$, then by the minimality of $|V(P)|$, we have $i = 2$, from which the claim follows directly. This proves (97).

(98) *Let P be a path in M with ends p_1 and p_2 such that there exist $x_1 \in N_{\eta(T)}(p_1)$ and $x_2 \in N_{\eta(T)}(p_2)$ for which $\{x_1, x_2\}$ is not local in η, and such that $|V(P)| \geq 1$ is minimum subject to this property. Suppose that there exist $(k_1, k_2) \in \{(1,2),(2,1)\}$, $f = v_1v_2 \in E(T)$ and $e_1 \in E(T) \setminus \{f\}$ incident with v_{k_1}, such that p_{k_1} has a neighbor in $\eta(e_1, v_{k_1})$ and p_{k_2} has a neighbor in $(B(v_{k_2}) \cup \eta(f)) \setminus B(v_{k_1})$. Then p_{k_1} is complete to $B(v_{k_1}) \setminus (\eta(e_1, v_{k_1}) \cup \eta(f))$.*

Due to symmetry, we may assume that $k_1 = 1$ and $k_2 = 2$. Let $e_3 \in E(T) \setminus \{e_1, f\}$ be incident with v_1 and let $b_3 \in \eta(e_3, v_1)$ be arbitrary. We need to show that p_1 is adjacent to b_3. Suppose for a contradiction that p_1 and b_3 are non-adjacent. Let $b_1 \in \eta(e_1, v_1)$ be adjacent to p_1 and let $x \in (B(v_2) \cup \eta(f)) \setminus B(v_1)$ be adjacent to p_2. Let T_2 be the component of $T \setminus (N_T(v_1) \setminus \{v_2\})$ containing v_1 (so $f \in E(T_2)$). Also, for each $i \in \{1,3\}$, let u_i be the end of e_i distinct from v_1 and let T_i be the component of $T \setminus (N_T(v_1) \setminus \{u_i\})$ containing

v_1 (so $e_i \in E(T_i)$). By (96) and (97), there exists an edge $f' = v_1'v_2' \in E(T)$ such that $N_{\eta(T)}(\{p_1, p_2\}) \subseteq \eta(f') \cup B(v_1') \cup B(v_2')$. This, along with the minimality of $|V(P)|$, implies that p_1 is anticomplete to $(\eta(T_1) \cup \eta(T_3)) \setminus B(v_1)$, the sets $V(P) \setminus \{p_1\}$ and $\eta(T_1) \cup \eta(T_3)$ are anticomplete, and the sets $V(P) \setminus \{p_2\}$ and $\eta(T_2) \setminus B(v_1)$ are anticomplete, as well. Since p_2 is adjacent to $x \in (B(v_2) \cup \eta(f)) \setminus B(v_1)$ and since η is tame, it follows that there exists a path P_2 in G from a to p_2 with $P_2^* \subseteq \eta(T_2) \setminus B(v_1)$. Also, for each $i \in \{1, 3\}$, there exists a path P_i in G from a to b_i with $P_i^* \subseteq \eta(T_i) \setminus B(v_1)$. Since η is rich, it follows that $V(P)$ and $N_G[a]$ are anticomplete, and in particular P_1 has length at least two. But now G contains a theta with ends a and b_1 and paths P_1, a-P_2-p_2-P-p_1-b_1 and a-P_3-b_3-b_1, a contradiction. This proves (98).

The following is immediate from (98) and the assumption that T is smooth.

(99) *Let P be a path in M with ends p_1 and p_2 such that there exists $x_1 \in N_{\eta(T)}(p_1)$ and $x_2 \in N_{\eta(T)}(p_2)$ for which $\{x_1, x_2\}$ is not local in η, and such that $|V(P)| \geq 1$ is minimum subject to this property. Suppose that there exist $(k_1, k_2) \in \{(1, 2), (2, 1)\}$ and $f = v_1v_2 \in E(T)$ such that $x_{k_1} \in B(v_{k_1}) \setminus (\eta(f))$ and $x_{k_2} \in (B(v_{k_2}) \cup \eta(f)) \setminus B(v_{k_1})$. Then p_{k_1} is complete to $B(v_{k_1}) \setminus \eta(f)$.*

We employ all previous statements to show that:

(100) *Let D be a component of M. Then $N_{\eta(T)}(D)$ is local in η.*

Suppose not. By Theorem 10.1, there exist $x_1', x_2' \in N_{\eta(T)}(D)$ such that $\{x_1', x_2'\}$ is not local in η. For each $i \in \{1, 2\}$, let p_i' be a neighbor of x_i' in D. Since D is connected, there exists a path P in $D \subseteq M$ from p_1' to p_2'. In other words, there exists a path P' in M with ends p_1', p_2' as well as vertices $x_1' \in N_{\eta(T)}(p_1')$ and $x_2' \in N_{\eta(T)}(p_2')$ such that $\{x_1', x_2'\}$ is not local in η. Therefore, we may choose a path P in M with ends p_1 and p_2 such that there exist $x_1 \in N_{\eta(T)}(p_1)$ and $x_2 \in N_{\eta(T)}(p_2)$ for which $\{x_1, x_2\}$ is not local in η, and such that $|V(P)| \geq 1$ is minimum subject to this property. As a result, we can apply (96) to P, x_1 and x_2. Let $(j_1, j_2) \in \{(1, 2), (2, 1)\}$ and $f = v_1v_2 \in E(T)$ be as in (96). We may assume without loss of generality that $j_1 = 1$ and $j_2 = 2$; in particular, v_1 is a branch vertex of T. By (97), we have $N_{\eta(T)}(P^*) \subseteq \eta(f, v_1)$ and $N_{\eta(T)}(\{p_1, p_2\}) \subseteq \eta(f) \cup B(v_1) \cup B(v_2)$. From (99) applied to $(k_1, k_2) = (1, 2)$, it follows that p_1 is complete to $B(v_1) \setminus \eta(f)$. Also, from (99) applied to $(k_1, k_2) = (2, 1)$, it follows that either p_2 is complete to $B(v_2) \setminus \eta(f)$ and $B(v_2) \setminus \eta(f) \neq \varnothing$, or p_2 is anticomplete to $B(v_2) \setminus \eta(f)$. Note that if $|V(P)| > 1$, then by the minimality of $|V(P)|$, we have $N_{\eta(T)}(p_1) \subseteq B(v_1)$ and $N_{\eta(T)}(p_2) \subseteq (B(v_2) \cup \eta(f)) \setminus B(v_1)$. Let us define

$$\eta' : V(T) \cup E(T) \cup (E(T) \times V(T)) \to 2^{G \setminus \{a\}}$$

as follows:

- Let $\eta'(o) = \eta(o)$ for every $o \in V(T) \cup (E(T) \setminus \{f\}) \cup ((E(T) \times V(T)) \setminus \{(f, v_1), (f, v_2)\})$.
- Let $\eta'(f) = \eta(f) \cup V(P)$ and let $\eta'(f, v_1) = \eta(f, v_1) \cup \{p_1\}$.
- Let
  - $\eta'(f, v_2) = \eta(f, v_2) \cup \{p_2\}$ if p_2 is complete to $B(v_2) \setminus \eta(f)$ and $B(v_2) \setminus \eta(f) \neq \varnothing$;
  - $\eta'(f, v_2) = \eta(f, v_2)$ if p_2 is anticomplete to $B(v_2) \setminus \eta(f)$.

From the assumption that η is tame, substantial and rich, and the fact that p_2 is adjacent to $x_2 \in B(v_2) \cup \eta(f)) \setminus B(v_1)$, it is straightforward to check that η' is also a tame, substantial and rich (T, a)-strip-structure in G. But $\eta'(T) = \eta(T) \cup P$, a contradiction with the maximality of $\eta(T)$. This proves (100).

The proof is almost concluded. Let X be the union of all the components D of M where D and $\eta^+(T)$ are anticomplete in G. Since η is rich, it follows that a is anticomplete to $M \setminus X$, as well. So for every component D of $M \setminus X$, we have $N_{\eta^+(T)}(D) = N_{\eta(T)}(D) \neq \varnothing$.

By (100), for every component D of $M \setminus X$, the set $N_{\eta(T)}(D)$ is local in η. Let $\mathcal{D}$ be the set of all components D of $M \setminus X$ for which we have $N_{\eta^+(T)}(D) \subseteq B_\eta(v)$ for some $v \in V(T)$. Breaking the ties arbitrarily and by the definition of X, we may write $\mathcal{D} = \bigcup_{v \in V(T)} \mathcal{D}_v$, where

- for all distinct $u, v \in V(T)$, we have $\mathcal{D}_u \cap \mathcal{D}_v = \varnothing$; and
- for all $v \in V(T)$ and every $D \in \mathcal{D}_v$, we have $N_{\eta^+(T)}(D) \subseteq B_\eta(v)$ and $N_{\eta^+(T)}(D) \neq \varnothing$.

For each edge $e = uv \in E(T)$, let $\mathcal{D}_e$ be the set of all components D of $M \setminus X$ for which we have $N_{\eta^+(T)}(D) \subseteq \eta(e)$ and

- either $N_{\eta(T)}(D) \cap \eta^\circ(e) \neq \varnothing$, or;
- $N_{\eta(T)}(D) \cap (\eta(e, u) \setminus \eta(e, v)) \neq \varnothing$ and $N_{\eta(T)}(D) \cap (\eta(e, v) \setminus \eta(e, v)) \neq \varnothing$.

From the definition of X, it follows that every component of $M \setminus X$ belongs to exactly one of the sets $\{\mathcal{D}_v, \mathcal{D}_e : v \in V(T), e \in E(T)\}$ (again, since η is rich, a is anticomplete to each such component).

We now define

$$\zeta : V(T) \cup E(T) \cup (E(T) \times V(T)) \to 2^{G \setminus \{a\}}$$

as follows. For all $v \in V(T)$ and $e \in E(T)$, let

$$\zeta(v) = \bigcup_{D \in \mathcal{D}_v} D;$$

$$\zeta(e) = \eta(e) \cup \left(\bigcup_{D \in \mathcal{D}_e} D \right);$$

$$\zeta(e,v) = \eta(e,v).$$

It is easily seen that ζ satisfies the conditions (S1)-(S8) from the definition of a (T,a)-strip-structure in G. In particular, since η is rich, it follows that ζ satisfies (S2), and from the definitions of X, $\mathcal{D}_v$'s and $\mathcal{D}_e$'s, it follows that ζ satisfies (S5) and (S7).

Now, observe that $\eta \leq \zeta$. Since η is substantial and rich, since $\eta \leq \zeta$, and from the definitions of X and ζ, it follows that ζ is a substantial and rich (T,a)-strip-structure with $\mathsf{JW}_\zeta = \mathsf{JW}_\eta$. Moreover, note that we have $\zeta^+(T) = \eta(T)^+ \cup (M \setminus X)$, and so

$$G \setminus (\zeta^+(T) \cup \mathsf{JW}_\zeta) = G \setminus (\zeta^+(T) \cup \mathsf{JW}_\eta) = X.$$

Hence, the sets $G \setminus (\zeta^+(T) \cup \mathsf{JW}_\zeta)$ and $\zeta^+(T)$ are anticomplete in G. This completes the proof of Theorem 10.2. ■

10.2 Jewels under the loupe

In this section, we revisit jewels for strip structures and collect a number of results concerning their properties in various settings. This will help attune Theorem 10.2 for its application in the proof of Theorem 10.9.

First, we introduce some notation. Let (G,a,T,ζ) be a club. For a vertex $v \in V(T)$, we denote by $\mathsf{JW}_{\zeta,v}$ the set of all ζ-jewels at v. It follows that $\mathsf{JW}_{\zeta,v} = \varnothing$ if v is a leaf of T. We also refine the notation $\mathsf{JW}_{\zeta,v}$ to the seagulls at v, as follows. For a vertex $v \in V(T)$ and an edge $e \in E(T)$ incident with v, we denote by $\zeta_e(v)$ the set of all components D of $\zeta(v)$ for which we have $N_{B(v)}(D) \subseteq \zeta(e,v)$, or equivalently, $N_{\zeta(T)\setminus\zeta(e,v)}(D) = \varnothing$. Given a seagull (v,e_1,e_2) in T with u_i being the end of e_i distinct from v for each $i \in \{1,2\}$, we define

$$\zeta(v,e_1,e_2) = \zeta(e_1) \cup \zeta(e_2) \cup \zeta_{e_1}(u_1) \cup \zeta_{e_2}(u_2) \cup \zeta(v).$$

The notation $\mathsf{JW}_{\zeta,(v,e_1,e_2)}$ then stands for the set of all ζ-jewels at (v,e_1,e_2).

The following describes, in (theta, prism)-free graphs, the adjacency of jewels at a given seagull:

Theorem 10.4. *Let (G,a,T,ζ) be a club where G is theta-free. Then the following hold.*

(a) *For every seagull* (v, e_1, e_2) *in* T, *we have* $N_{\zeta^+(T)}(\mathsf{JW}_{\zeta,(v,e_1,e_2)}) \subseteq \zeta(v, e_1, e_2)$. *Consequently, for every vertex* $v \in V(T)$, *we have* $N_{\zeta^+(T)}(\mathsf{JW}_{\zeta,v}) \subseteq \zeta(N_T[v])$, *and for every two distinct vertices* $v, v' \in V(T)$, *we have* $\mathsf{JW}_{\zeta,v} \cap \mathsf{JW}_{\zeta,v'} = \varnothing$.

(b) *Assume further that* G *is prism-free and* ζ *is rich. Let* (v, e_1, e_2) *be a seagull in* T *and let* x *be a* ζ*-jewel at* (v, e_1, e_2). *Let* R *be a long* $\zeta(e_i)$*-rung for some* $i \in \{1, 2\}$, *let* r *be the end of* R *in* $\zeta(e_i, v)$ *and let* r' *be the unique neighbor of* r *in* R. *Then either* x *is anticomplete to* R *or* $N_R(x) = \{r, r'\}$.

Proof. We begin with a common set-up for the proofs of both 10.4(a) and 10.4(b). Fix a seagull (v, e_1, e_2) in T and a vertex $x \in \mathsf{JW}_{\zeta,(v,e_1,e_2)}$. Then v is a branch vertex of T. For each $i \in \{1, 2\}$, let u_i be the end of e_i distinct from v and let T_i be the component of $T \setminus (N_T(v) \setminus \{u_i\})$ containing v. Let T' be the component of $T \setminus \{u_1, u_2\}$ containing v. Since $x \in \mathsf{JW}_{\zeta,(v,e_1,e_2)}$ is a ζ-jewel, there exist an edge $e_3 \in E(T) \setminus \{e_1, e_2\}$ incident with v and a ζ-pyramid Σ at (v, e_1, e_2, e_3) with apex a, base $b_1b_2b_3$, and paths P_1, P_2, P_3 such that x is a jewel for Σ at b_3. In particular, $P_j \cap \zeta(e_j)$, for each $j \in [3]$, is a long $\zeta(e_j)$-rung R_j whose end in $\zeta(e_j, v)$ is b_j. Also, x is anticomplete to P_3 (and so x is not adjacent to a). For each $j \in \{1, 2\}$, assuming c_j to be the unique vertex in $N_{R_j}(b_j) = N_{P_j}(b_j)$, it follows that x is adjacent to both $b_j \in \zeta(e_j, v) \setminus \zeta(e_j, u_j)$ and $c_j \in \zeta(e_j) \setminus \zeta(e_j, v)$. Therefore, there are paths Q_j, S_j of length at least two in G from a to x such that

$$b_j \in Q_j^* \subseteq (\zeta(T') \setminus \zeta(v)) \cup (\zeta(e_j, v) \setminus \zeta(e_j, u_j))$$

and

$$c_j \in S_j^* \subseteq \zeta(T_j) \setminus (B(v) \cup \zeta(u_j) \cup \zeta(v)).$$

Now, in order to prove 10.4(a), it suffices to show that $N_{\zeta^+(T)}(x) \subseteq \zeta(v, e_1, e_2)$. Suppose for a contradiction that x has a neighbor $y \in \zeta^+(T) \setminus \zeta(v, e_1, e_2)$. Since x is not adjacent to a, it follows that $y \in \zeta(T) \setminus \zeta(v, e_1, e_2)$. Assume that $y \in \zeta(T') \setminus \zeta(v)$. Then by (S5) and (S7) from the definition of a strip structure, there is a path Q' of length at least two in G from a to x with $Q'^* \subseteq \zeta(T') \setminus \zeta(v)$. But now there is a theta in G with ends a, x and paths a-S_1-x, a-S_2-x and a-Q'-x, a contradiction. It follows that $y \in \zeta(T_1 \cup T_2) \setminus \zeta(v, e_1, e_2)$. In other words, for some $(i, i') \in \{(1, 2), (2, 1)\}$, we have $y \in \zeta(T_i) \setminus (\zeta(e_i) \cup \zeta_{e_i}(u_i) \cup \zeta(v))$. Thus, by (S5) and (S7) from the definition of a strip structure, and by the definition of $\zeta_{e_i}(u_i)$, there is a path S_i' of length at least two in G from a to x such that $S_i'^* \subseteq \zeta(T_i) \setminus (\zeta(e_i) \cup \zeta_{e_i}(u_i) \cup \zeta(v))$. But then there is a theta in G with ends a, x and paths a-Q_i-x, a-S_i'-x and a-$S_{i'}$-x, a contradiction. This proves 10.4(a).

Next, we prove 10.4(b). Assume that ζ is rich, and let i, R, r, r' be as in 10.4(b). By symmetry, we may assume that $i = 1$. Suppose x has a neighbor $y \in R$. Let $P_1' =$

$(P_1 \setminus R_1) \cup R$. Let Σ' be the pyramid with apex a, base rb_2b_3, and paths P_1', P_2 and P_3. Recall that a is trapped in $\zeta^+(T)$ because ζ is rich. Moreover, Σ' is a pyramid in $\zeta^+(T)$, x is adjacent to $y \in P_1'$ and to $b_2, c_2 \in P_2$, and x is anticomplete to P_3. It follows that $N_{\Sigma'}(x)$ is not local in Σ', and x is not a corner path for Σ'. So we can apply Theorem 10.3 to G, $H = G[\zeta^+(T)]$, a, $P = x$ and Σ' to deduce that x is a jewel for Σ' at b_3. In particular, we have $N_R(x) = N_{P_1'}(x) = \{r, r'\}$. This completes the proof of Theorem 10.4. ■

The following shows for every rich (T, a)-strip-structure in a (theta, prism)-free graphs of bounded clique number, there are only a few jewels at each vertex of T.

Theorem 10.5. *Let $t \in \mathbb{N}$ and let (G, a, T, ζ) be a club where G is (theta, prism, K_{t+1})-free and ζ is rich. Then for every vertex $v \in V(T)$, we have $|\,\mathsf{JW}_{\zeta,v}\,| < t^5$.*

Proof. If v is a leaf of T, then $\mathsf{JW}_{\zeta,v} = \varnothing$ and the result is immediate. So we may assume that v has degree at least three in T (because T is smooth). Also, by (S3) and (S4) from the definition of a strip structure, and from the assumption that G is K_{t+1}-free, it follows that $3 \leq |N_T(v)| \leq t$, and thus:

$$\binom{|N_T(v)|}{2}(t+1)^3 \leq \frac{t(t-1)(t+1)^3}{2} < \frac{t^3(t+1)^2}{2} = \frac{t^3(t^2+2t+1)}{2} < \frac{t^3(t^2+3t)}{2} \leq t^5.$$

Therefore, in order to prove $|\,\mathsf{JW}_{\zeta,v}\,| < t^5$, it is enough to show that $|\,\mathsf{JW}_{\zeta,(v,e_1,e_2)}\,| \leq (t+1)^3$ for every seagull (v, e_1, e_2) in T.

Suppose for a contradiction that $|\,\mathsf{JW}_{\zeta,(v,e_1,e_2)}\,| \geq (t+1)^3$ for some seagull (v, e_1, e_2) in T. Since G is K_{t+1}-free, it follows from Theorem 3.17 that there is a stable set $X \subseteq \mathsf{JW}_{\zeta,(v,e_1,e_2)}$ in G with $|X| = 3$. Let u_1 and u_2, respectively, be the ends of e_1 and e_2 other than v. For each $i \in \{1, 2\}$ and every $x \in X$, since x is a ζ-jewel at (v, e_1, e_2), it follows that there is a long $\zeta(e_i)$-rung R_i^x such that $Q_i^x = R_i^x \setminus \zeta(e_i, v)$ is a path in $\zeta(e_i) \setminus \zeta(e_i, v)$ from a neighbor of x to a vertex in $\zeta(e_i, u_i) \setminus \zeta(e_i, v)$; in particular, R_i^x contains a neighbor of x. Therefore, for each $i \in \{1, 2\}$, we may pick a non-empty set $\mathcal{R}_i$ of long $\zeta(e_i)$-rungs such that every vertex in X has a neighbor in at least one rung in $\mathcal{R}_i$, and subject to this property, $\mathcal{R}_i$ minimal with respect to inclusion. We deduce:

(101) *$|\mathcal{R}_i| > 1$ for some $i \in \{1, 2\}$.*

Suppose not. Then for every $i \in \{1, 2\}$, there exists a long $\zeta(e_i)$-rung S_i such that every vertex in X has a neighbor in S_i. Let s_i be the end of S_i in $\zeta(e_i, v)$ and s_i' be unique neighbor of s_i in S_i. By Theorem 10.4(b), the sets X and $\{s_1', s_2'\}$ are complete in G. But now $G[X \cup \{s_1', s_2'\}]$ is a theta with ends s_1', s_2', a contradiction. This proves (101).

By (101) and without loss of generality, we may assume that $|\mathcal{R}_1| > 1$. This, together with the minimality of $\mathcal{R}_1$, implies that there are distinct vertices $x, y \in X$ as well as distinct, long $\zeta(e_1)$-rungs $R_x, R_y \in \mathcal{R}_1$ such that x has a neighbor in R_x, y has a neighbor in R_y, x is anticomplete to R_y, and y anticomplete to R_x. Let r_x and r_y be the ends of R_x and R_y in $\zeta(e_1, v)$, respectively. Let r_x' be the unique neighbor of r_x in R_x and let r_y' be the unique neighbor of r_y in R_y. So we have $r_x', r_y' \in \zeta(e_1) \setminus \zeta(e_1, v)$. Again, by Theorem 10.4(b), we have $N_{R_x \cup R_y}(x) = \{r_x, r_x'\}$ and $N_{R_x \cup R_y}(y) = \{r_y, r_y'\}$. It follows that $r_x, r_x' \in R_x \setminus R_y$ and $r_y, r_y' \in R_y \setminus R_x$. Furthermore, r_x is anticomplete to $R_y \setminus \{r_y\}$, as otherwise there is a long $\zeta(e_1)$-rung R with $R \subseteq (R_y \setminus \{r_y\}) \cup \{r_x\}$ and $N_R(x) = \{r_x\}$, which violates the second assertion of Theorem 10.4. Similarly, we deduce that r_y is anticomplete to $R_x \setminus \{r_x\}$.

Now, let

$$G_1 = G[(B(u_1) \setminus \zeta(e_1, u_1)) \cup ((R_x \cup R_y) \setminus \{r_x, r_y\})]$$

and let

$$G_2 = G[(B(u_2) \setminus \zeta(e_2, u_2)) \cup Q_2^x \cup Q_2^y].$$

Since ζ is rich, it follows that G_1 and G_2 are connected. Consequently, there exists a path Q_1 in G_1 from r_x' to r_y', and there exists a path Q_2 from x to y with $Q_2^* \subseteq G_2$. In addition, since v is a branch vertex of T, we may choose an edge $e_3 \in E(T) \setminus \{e_1, e_2\}$ incident with v. By Theorem 10.4(a), $\{x, y\}$ is anticomplete to $\zeta(e_3, v)$. Let Q_3 be the path from r_x to r_y with $Q_3^* \subseteq \zeta(e_3, v)$ (so Q_3 has length one or two). Then there is a prism in G with triangles $\{x, r_x, r_x'\}$ and $\{y, r_y, r_y'\}$ and paths Q_1, Q_2, Q_3, a contradiction. This completes the proof of Theorem 10.5. ■

The last two results in this section examine the connectivity within $G \setminus \zeta^+(T)$ for a (T, a)-strip-structure ζ arising from Theorem 10.2.

Theorem 10.6. *Let G be a theta-free graph and let $a \in V(G)$. Let T be a smooth tree and let ζ be a (T, a)-strip-structure in G. Let $v, v' \in V(T)$ be distinct and let P be a path in $G \setminus \zeta^+(T)$ with ends x, x' such that $x \in \mathsf{JW}_{\zeta,v}$, $x' \in \mathsf{JW}_{\zeta,v'}$, and P^* is anticomplete to $\zeta^+(T)$. Then v and v' are adjacent in T.*

Proof. Suppose not. By Theorem 10.4(a), x and x' are distinct. Let Λ be the path in T from v to v'. Then Λ has length at least two, and so there are two distinct edges $f, f' \in E(\Lambda)$ such that f is incident with v and f' is incident with v'. Let u be the end of f distinct from v and

u' be the end of f' distinct from v'. Let (v, e_1, e_2) and (v', e'_1, e'_2) be two seagulls in G such that $x \in \mathsf{JW}_{\zeta,(v,e_1,e_2)}$ and $x' \in \mathsf{JW}_{\zeta,(v',e'_1,e'_2)}$. For each $i \in \{1, 2\}$, let u_i be the end of e_i distinct from v and let u'_i be the end of e'_i distinct from v'. Without loss of generality, we may assume that $u_2, u'_2 \notin \Lambda$. Let T_2 be the component of $T \setminus (N_T(v) \setminus \{u_2\})$ containing v and let T'_2 be the component of $T \setminus (N_T(v') \setminus \{u'_2\})$ containing v'. Let T' be the component of $T \setminus \{u', u'_2\}$ containing v'. Since x is a ζ-jewel at (v, e_1, e_2), it follows that x is not adjacent to a, and x has a neighbor $c \in \zeta(e_2) \setminus \zeta(e_2, v) \subseteq \zeta(T_2) \setminus (B(v) \cup \zeta(u_2) \cup \zeta(v))$. Therefore, there is a path Q of length at least two in G from a to x such that $c \in Q^* \subseteq \zeta(T_2) \setminus (B(v) \cup \zeta(u_2) \cup \zeta(v))$. Also, since x' is a ζ-jewel at (v', e'_1, e'_2), it follows that x' is not adjacent to a, and x' has a neighbor $b' \in B(v') \setminus (\zeta(f', u') \cup \zeta(e'_2, v'))$ and a neighbor $c' \in \zeta(e'_2) \setminus \zeta(e'_2, v') \subseteq \zeta(T'_2) \setminus (B(v') \cup \zeta(u'_2) \cup \zeta(v'))$. Therefore, there are paths P', Q' of length at least two in G from a to x' for which we have $b' \in P'^* \subseteq (\zeta(T') \setminus \zeta(v')) \cup (\zeta(f', v') \setminus \zeta(f', u'))$ and $c' \in Q'^* \subseteq \zeta(T'_2) \setminus (B(v') \cup \zeta(u_2) \cup \zeta(v'))$. But now there is a theta in G with ends a, x' and paths a-P'-x', a-Q'-x', and a-Q-x-P-x', a contradiction. This proves Theorem 10.6. ■

Theorem 10.7. *Let $t \in \mathbb{N}$ and let (G, a, T, ζ) be a club such that G is (theta, prism, K_{t+1})-free, ζ is rich and every vertex in $G \setminus (\zeta^+(T) \cup \mathsf{JW}_\zeta)$ is anticomplete to $\zeta^+(T)$ in G. Let $x \in V(G) \setminus (\zeta^+(T) \cup \mathsf{JW}_\zeta)$. Then there exists $S_x \subseteq V(G) \setminus (\zeta^+(T) \cup \{x\})$ with $|S_x| < 2t^5$ such that S_x separates x and $\mathsf{JW}_\zeta \setminus S_x$ in $G \setminus \zeta^+(T)$. Consequently, S_x separates x and $\zeta^+(T)$ in G.*

Proof. By Theorem 10.4(a), $(\mathsf{JW}_{\zeta,v} : v \in V(T))$ is a partition of JW_ζ into pairwise disjoint subsets. Let G' be the graph obtained from $G \setminus \zeta^+(T)$ by contracting the set $\mathsf{JW}_{\zeta,v}$ into a vertex z_v for each $v \in V(T)$ with $\mathsf{JW}_{\zeta,v} \neq \varnothing$, and then adding a new vertex z such that $N_{G'}(z) = \{z_v : v \in V(T), \mathsf{JW}_{\zeta,v} \neq \varnothing\}$. We claim that there is a set $Y \subseteq V(G') \setminus \{x, z\}$ of at most two vertices which separates x and z in G'. Suppose not. By Menger's Theorem **??**, there are three pairwise internally disjoint paths in G' from x to z. So there exists $S \subseteq T$ with $|S| = 3$ as well as three paths $(P_v : v \in S)$ in $G \setminus \zeta^+(T)$ all having x as an end and otherwise pairwise disjoint, such that for each $v \in S$, P_v has an end $y_v \in \mathsf{JW}_{\zeta,v}$ distinct from x, and we have $P^*_v \subseteq G \setminus (\zeta^+(T) \cup \mathsf{JW}_\zeta)$. For all distinct $v, v' \in S$, it follows that $P_{v,v'} = y_v$-P_v-x-$P_{v'}$-$y_{v'}$ is a path in $G \setminus \zeta^+(T)$ from $y_v \in \mathsf{JW}_{\zeta,v}$ to $y_{v'} \in \mathsf{JW}_{\zeta,v'}$ such that $P^*_{v,v'} \subseteq G \setminus (\zeta^+(T) \cup \mathsf{JW}_\zeta)$. In particular, $P^*_{v,v'}$ and $\zeta^+(T)$ are anticomplete in G. But then by Theorem 10.6, S is a 3-clique in T, a contradiction. The claim follows.

Let Y be as in the above claim. For each $y \in Y$, if $y = z_v$ for some $v \in V(T)$, then let $A_y = \mathsf{JW}_{\zeta,v}$. Otherwise, let $A_y = \{y\}$. Let $S_x = \bigcup_{y \in Y} A_y$. Then $S_x \subseteq G \setminus (\zeta^+(T) \cup \{x\})$ separates x and $\mathsf{JW}_\zeta \setminus S_x$ in $G \setminus \zeta^+(T)$. Also, by Theorem 10.5, we have $|S_x| < 2t^5$. This completes the proof of Theorem 10.7. ■

10.3 Small cutsets

Here we complete the proof of Theorem 7.3, beginning with the following:

Theorem 10.8. *Let (G, a, T, ζ) be a club where G is theta-free and ζ is rich. Then for every $v \in V(T)$, there exists at most one edge $f \in E(T)$ such that $\zeta(f, v)$ is not a clique.*

Proof. Suppose for a contradiction that there are two distinct edges $f_1, f_2 \in E(T)$ where for each $i \in \{1, 2\}$, there exist $x_i, y_i \in \zeta(f_i, v)$ such that x_i is not adjacent to y_i. Then both f_1 and f_2 are incident with v, and so is not a leaf of T. It follows that $H = x_1\text{-}x_2\text{-}y_1\text{-}y_2\text{-}x_1$ is a hole of length four in G. Since ζ is rich, it follows that a is anticomplete to $V(H)$. Let $f_1 = u_1v$. Let l_1 be a leaf of T such that the unique path Λ_1 in T from l_1 to v contains u_1 (so $f_1 \in E(\Lambda_1)$). Let R_{x_1} be a $\zeta(f_1)$-rung containing x_1 and let R_{y_1} be a $\zeta(f_1)$-rung containing y_1. Since ζ is rich, it follows that $H_1 = G[V(R_{x_1}) \cup V(R_{y_1}) \cup B(u_1)]$ is connected, and so there is a path Q in H_1 from x_1 to y_1. In particular, Q has length at least two and $Q^* \subseteq (B(u_1) \cup \zeta(f_1)) \setminus B(v)$. But now there is a theta in G with ends x_1, y_1 and paths Q, $x_1\text{-}x_2\text{-}y_1$ and $x_1\text{-}y_2\text{-}y_1$, a contradiction. This completes the proof of Theorem 10.8. ■

Theorem 10.9. *Let $t \in \mathbb{N}$ and let (G, a, T, ζ) be a club such that G is (theta, prism, K_{t+1})-free, ζ is rich and every vertex in $G \setminus (\zeta^+(T) \cup \mathsf{JW}_\zeta)$ is anticomplete to $\zeta^+(T)$ in G. Then for every vertex $x \in G \setminus N_G[a]$, there exists a set $S_x \subseteq G \setminus \{a, x\}$ with $|S_x| < 4t^6$ such that S separates a and x in G.*

Proof. Since G is K_{t+1}-free, it follows that T has maximum degree at most t, and so $t \geq 3$. For each vertex $v \in V(T)$, we define four sets of vertices $C_v, K_v, \mathcal{M}_v, \mathcal{N}_v \subseteq V(G)$ as follows.

If v is a leaf of T, then let $C_v = B(v)$. Otherwise, let $C_v = \varnothing$. It follows that $|C_v| \leq 1$.

Let K_v be a maximal clique of G contained in $B(v)$. It follows that $|K_v| \leq t$. Moreover, if v is a leaf of T, then by the richness of ζ, we have $K_v = B(v) = C_v$ (and so $|K_v| = 1$), and if v is a branch vertex of T, then by Theorem 10.8, K_v contains all but possibly one of the sets $\zeta(f, v)$ for $f \in E(T)$.

Let

$$\mathcal{M}_v = \bigcup_{w \in N_T(v)} \mathsf{JW}_{\zeta,w}$$

and let

$$\mathcal{N}_v = \bigcup_{w \in N_T(S)} K_w.$$

Then the following is immediate from Theorems 10.4, 10.5 and 10.6:

(102) *For every $v \in V(T)$, we have*

- $\mathcal{M}_u \cup \mathcal{N}_v \subseteq G \setminus (\mathsf{JW}_{\zeta,v} \cup \{a\})$;
- $|\mathcal{M}_u \cup \mathcal{N}_v| < t(t^5 + t) < 4t^6$; *and*
- $\mathcal{M}_u \cup \mathcal{N}_v$ *separates* a *and* $\mathsf{JW}_{\zeta,v}$ *in* G.

Now, for each $x \in G \setminus N_G[a]$, we define S_x as follows.

First, assume that $x \in \zeta^+(T) \setminus N_G[a]$. Then either $x \in \zeta(e)$ for some edge $e = uv \in E(T)$, or $x \in \zeta(v)$ for some branch vertex $v \in V(T)$. In the former case, let

$$\mathcal{E}_x = \mathcal{M}_u \cup \mathcal{M}_v;$$

$$\mathcal{I}_x = \mathcal{N}_u \cup \mathcal{N}_v \cup C_u \cup C_v.$$

and in the latter case, let

$$\mathcal{E}_x = \mathcal{M}_v \cup \mathsf{JW}_{\zeta,v}$$

$$\mathcal{I}_x = \mathcal{N}_v.$$

Let $S_x = \mathcal{E}_x \cup \mathcal{I}_x$. Observe that since $x \in G \setminus N_G[a]$, we have $S_x \subseteq G \setminus \{a, x\}$. Also, we have $|\mathcal{I}_x| \leq 2t^2$, and by Theorem 10.5, we have $|\mathcal{E}_x| < 2t^6$. It follows that $|S_x| < 4t^6$. Moreover, from Theorem 10.4 and the fact that ζ is rich, it is easy to check that for every path P in G from a to x, if $V(P) \subseteq \zeta^+(T)$, then P contains a vertex from $\mathcal{I}_x$, and otherwise P contains a vertex from either $\mathcal{I}_x$ or $\mathcal{E}_x$. Therefore, S_x separates a and x in G.

Next, assume that $x \in \mathsf{JW}_\zeta$. By Theorem 10.4(a), there exists a unique vertex $v \in V(T)$ such that $x \in \mathsf{JW}_{\zeta,v}$. Let $S_x = \mathcal{M}_v \cup \mathcal{N}_v$. Then by (102), we have $S_x \subseteq G \setminus \{a, x\}$, $|S_x| < 4t^6$ and S_x separates a and x in G.

Finally, assume that $x \in V(G) \setminus (\zeta^+(T) \cup \mathsf{JW}_\zeta)$. Then we choose S_x to be as in Theorem 10.7, which implies that $S_x \subseteq G \setminus \{a, x\}$, $|X| < 2t^5 < 4t^6$ and S_x separates a and x in G. This completes the proof of Theorem 10.9. ■

We conclude by proving Theorem 7.3:

Theorem 7.3. *Let $t \in \mathbb{N}$ and let G be a (theta, prism, K_{t+1})-free graph. Let H be an induced subgraph of G that is isomorphic to the line graph of a proper subdivision of a tree which is not a path. Let $a \in V(G) \setminus V(H)$ such that a is trapped in $G[V(H) \cup \{a\}]$. Then for every vertex $x \in G \setminus N_G[a]$, there exists $S_x \subseteq V(G) \setminus \{a, x\}$ with $|S_x| < 4t^6$ such that S_x separates a and x in G.*

Proof. Since H is isomorphic to the line graph of a proper subdivision of a tree, it follows that there is a smooth tree T where H is isomorphic to the line graph of a proper subdivision of T. Since a is trapped in $G[V(H) \cup \{a\}]$, it follows that $N_G(a)$ is exactly the set of all simplicial vertices of H (all of which have degree one in H); in particular, T has exactly $|N_G(a)|$ leaves. Also, one may observe that there is a tame, substantial, and rich (T, a)-strip-structure η in G with $\eta(T) = H$ (see Figure 10.1). Therefore, we can apply Theorem 10.2 to the club (G, a, T, η), deducing that there is a substantial and rich (T, a)-strip-structure ζ in G such that every vertex in $G \setminus (\zeta^+(T) \cup \mathsf{JW}_\zeta)$ is anticomplete to $\zeta^+(T)$ in G. Hence, by Theorem 10.9 applied to the club (G, a, T, ζ), for every vertex $x \in G \setminus N_G[a]$, there exists $S_x \subseteq G \setminus \{a, x\}$ with $|S_x| < 4t^6$ such that S_x separates a and x in G. This completes the proof of Theorem 7.3. ■

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