Twisted Holography in B-model

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Chapter 1 contains material from String-Math 2022 conference proceedings [1].

Chapter 2 and appendix A corresponds to the paper [2], written in collaboration with Davide Gaiotto.

Chapter 3 corresponds to the paper [3], written in collaboration with Davide Gaiotto.

Chapter 4 and appendix B corresponds to the paper [4], written in collaboration with Davide Gaiotto, Justin Kulp, Brian Williams, Jingxiang Wu, Matthew Yu.

Abstract

Supersymmetric quantum field theories contain protected subsectors which can be obtained by the procedure known as twisting. The idea of twisted holography is to study holographic duals of such twists. The main example of twisted holography in this thesis is the duality between the chiral algebra subsector of $\mathcal{N} = 4$ super Yang-Mills and the B-model topological string theory on the complex manifold $SL(2, \mathbb{C})$. In this thesis, we study two aspects of the duality: the correspondence between determinant operators in the chiral algebra and "Giant Graviton" branes in the dual geometry, and the extension to non-conformal vacua of the chiral algebra.

The second BPS subsector studied in this thesis is the holomorphic twist of 4d $\mathcal{N} = 1$ super Yang-Mills. The holomorphic twist is defined as the cohomology of one supercharge and captures the quarter-BPS operators that are counted by the supersymmetric index. The twisted theory is endowed with extra structures and symmetries which are a 4d analogue of a 2d chiral algebra. We observe that the differential in the holomorphic twist receives loop corrections which make the theory topological and can be interpreted as a sign of confinement of the original theory. Finally, we present a holographic realization of the holomorphic theory in the B-model topological string theory.

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Chapter 1

Introduction and summary

1.1 Twisted holography

Twisting supersymmetric quantum field theories is the procedure of passing to the cohomology of a fermionic supercharge $\mathcal{Q}$. The operation produces a consistent subsector of the SQFT which is easier to study and may allow for exact computations. In the twisted theory several simplifications occur:

- Twisting restricts to protected (BPS) operators. Certain quantities, eg. correlation functions, might become independent of various parameters like the coupling constant.
- Dependence on certain spacetime coordinates drops out in cohomology. The fermionic supercharges satisfy anticommutation relations of the schematic form

$$\{Q, \tilde{Q}\} \sim P\,. \tag{1.1}$$

  The momenta that appear in the image of the twisting supercharge $\{\mathcal{Q}, \bullet\} \sim P$, are by definition exact in cohomology. Since momenta generate spacetime translations, dependence on the corresponding spacetime coordinates drops out in the twisted theory. There might be many twists available depending on the spacetime dimension and the number of supersymmetry (see eg. [5, 6]). For example, if all of the spacetime momenta become exact we have *the topological twist*, and if half – *the holomorphic twist*.

- Twisted theories are endowed with extra mathematical structures and symmetries, which become visible at the level of cohomology [7, 4, 8, 9, 10, 11].

For the supersymmetric theories that have a holographic dual, a natural question is whether there exists an analogous twisting procedure on the gravity side. It was proposed in [12] that the dual operation corresponds to turning on a non-zero value for a bosonic ghost field[1] of the supergravity theory. The twisted type IIB supergravity was conjectured to be equivalent to the BCOV theory. Remarkably, twisted supergravity can be quantized to all orders in perturbation theory, despite the theory being non-renormalizable [13, 14].

Applying the twisting procedures to both sides of a holographic duality should therefore produce an easier and more tractable duality, where many simplifications occur. In addition, objects that appear after twisting are typically more well-defined mathematically (eg. vertex algebras, BCOV theory). One could hence hope for a more mathematical formulation of holography, at least at the twisted level.

The main example[2] of twisted holography in this thesis is the duality between a 2d chiral algebra $\mathcal{A}_N$ and the topological B-model on $SL(2,\mathbb{C})$ proposed in [31]. The chiral algebra is a subsector of $\mathcal{N}=4$ SYM obtained by the "$Q+S$" twist of [32].[3] One can arrive at the dual geometry $SL(2,\mathbb{C})$ by considering a stack of D1-branes in the topological B-model on $\mathbb{C}^3$, whose worldvolume theory is precisely the chiral algebra $\mathcal{A}_N$ (in analogy to the original derivation by Maldacena [34]). The stack of branes deforms the complex structure:

$$\mathbb{C}^3 \setminus \mathbb{C} \longrightarrow SL(2,\mathbb{C})\,. \tag{1.2}$$

B-model on $SL(2,\mathbb{C})$ can also be seen as a twist of type IIB string theory on $\mathrm{AdS}_5 \times S^5$, where $SL(2,\mathbb{C}) \cong \mathrm{AdS}_3 \times S^3 \subset \mathrm{AdS}_5 \times S^5$. Therefore, the above duality can be regarded as a subsector of the standard $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence.

At the twisted level, many aspects of the duality simplify:

- The dependence on the 't Hooft coupling $g^2_{\mathrm{YM}}N$ drops out. As a consequence, the computations on the chiral algebra side are essentially free theory computations

[1] The role of the bosonic ghost field in the BV formalism is to gauge a local supersymmetry. Coupling a supersymmetric field theory (worldvolume theory of a brane) to the background where a bosonic ghost Ψ is non-zero is equivalent to adding the corresponding supercharge Q_Ψ to the BRST charge of the supersymmetric field theory.

[2] Other examples of twisted supergravity/holography include [15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30].

[3] For an earlier work on the holographic dual of the chiral algebra subsector of $\mathcal{N}=4$ SYM see [33].

(when considering BRST closed operators). The only parameter is the rank of the gauge group N, which gets mapped to the string coupling $g_s \sim 1/N$. Therefore, the large N computations on the gauge theory side can be readily matched with perturbative topological string theory expansion.

- The closed string field theory of the topological B-model is the Kodaira-Spencer (BCOV) theory [35, 36]. This theory has been shown to admit a unique quantization in perturbation theory [37, 13].
- D-branes in the topological B-model are holomorphic submanifolds of the target Calabi-Yau. In particular, D1-branes that will be considered in this thesis are holomorphic curves in $SL(2,\mathbb{C})$.

In [31], the holographic dictionary between single trace operators of the chiral algebra $\mathcal{A}_N$ and modifications of boundary conditions of the Kodaira-Spencer theory on $SL(2,\mathbb{C})$ was proposed. In this thesis, we study two generalizations:

1. The correspondence between determinant operators and "Giant Graviton" D1-branes in Chapter 2.
2. The duality between the chiral algebra $\mathcal{A}_N$ in non-conformal vacua and topological B-model on asymptotically $SL(2,\mathbb{C})$ geometries in Chapter 3.

In Chapter 4, we further extend the study of twisted theories and their holographic duals to the holomorphic twist of 4d $\mathcal{N}=1$ gauge theories and the topological B-model in non-commutative spacetime. Focusing on the holomorphic twist of pure $\mathcal{N}=1$ SYM, we examine its properties such as holomorphic confinement and realize it within the topological B-model.

1.2 Determinants and Giant Gravitons

As part of the standard AdS/CFT correspondence, an insertion of a determinant operator in $\mathcal{N}=4$ SYM is dual to a Giant Graviton D3-brane, which asymptotically wraps an $\mathbb{R}_+ \times S^3$ inside $\mathrm{AdS}_5 \times S^5$. In twisted holography, determinant operators in the chiral algebra $\mathcal{A}_N$ are dual to D1-branes that asymptotically wrap $\mathbb{C}^* \cong \mathbb{R}_+ \times S^1$ inside $SL(2,\mathbb{C}) \cong \mathrm{AdS}_3 \times S^3$.

When considering insertions of multiple determinants, there might be different brane configurations (eg. connected or disconnected) with the same boundary behavior. Correlation functions of multiple determinants have large N saddles [38]. Using a spectral curve construction, to each large N saddle we can associate a holomorphic curve in $SL(2,\mathbb{C})$, which we conjecture is the support of the dual D1-brane. The conjecture is tested by various holographic computations summarized in the next sections.

1.2.1 Correlation functions of determinants

The chiral algebra $\mathcal{A}_N$ is a gauged $\beta\gamma$ system of symplectic bosons X, Y valued in the adjoint representation of $U(N)$. It is convenient to define the linear combination

$$Z(u;z) \equiv X(z) + uY(z). \tag{1.3}$$

Then the symplectic boson OPE is[4]

$$Z(u;z)^a_b\, Z(v;w)^c_d \sim \delta^a_d \delta^c_b \frac{1}{N}\frac{u-v}{z-w}\,. \tag{1.4}$$

We are interested in correlation functions of determinant operators:

$$\mathcal{D}(m;u;z) \equiv \det\left(m + Z(u;z)\right), \quad m \in \mathbb{C}\,. \tag{1.5}$$

An insertion of a determinant of this form corresponds to a Giant Graviton brane wrapping a 1-dimensional complex curve, which approaches the boundary of AdS_3 at a point z along the line $b = au - m + \mathcal{O}(a^{-1})$, where $ad - bc = 1$ are the coordinates of $SL(2,\mathbb{C})$.[5] When considering insertions of multiple determinants $\mathcal{D}(m_i;u_i;z_i)$ there might be many brane configurations with the same asymptotics controlled by parameters m_i, u_i, z_i.

A method of computing correlations functions of determinant operators in the large N limit was presented in [38]. Following their prescription (also implemented in [39, 40, 41]), we fermionize the determinants:

$$\left\langle \prod_i^k \mathcal{D}(m_i;u_i;z_i) \right\rangle = \int [\mathrm{d}\psi \mathrm{d}\bar{\psi}] \left\langle \prod_i^k e^{\bar{\psi}_i(m_i + Z(u_i;z_i))\psi^i} \right\rangle , \tag{1.6}$$

[4] We include the N^{-1} factor in the OPE and also put N in front of single-trace operators. With this choice of conventions, the ribbon diagrams of genus g contribute at order N^{2-2g} in the 't Hooft expansion.

[5] The boundary behavior of a D1-brane dual to a determinant can be derived by adding a probe D1'-brane transverse to the stack of N D1-branes and finding its image in the backreacted geometry.

where $\psi^i, \bar{\psi}_i$ are auxiliary (anti)fundamental fermions and $\mathrm{d}\psi \mathrm{d}\bar{\psi} \equiv \prod_i^k \mathrm{d}\psi^i \mathrm{d}\bar{\psi}_i$. The expectation value on the right hand side stands for the chiral algebra path integral. Since the action for the symplectic bosons is free: $N \int \mathrm{Tr}\, X \bar{\partial} Y$, we can easily integrate them out:

$$\int [\mathrm{d}\psi \mathrm{d}\bar{\psi}]\, e^{-\frac{1}{2N} \sum_{i \neq j} \frac{u_i - u_j}{z_i - z_j} \bar{\psi}_i \psi^j \bar{\psi}_j \psi^i + \sum_i m_i \bar{\psi}_i \psi^i} \,. \tag{1.7}$$

To deal with the above integral, we perform the Hubbard-Stratonovich transformation by introducing auxiliary bosonic variables ρ^i_j for $i \neq j$ (and set $\rho^i_i \equiv m_i$):

$$\frac{1}{Z_\rho} \int [\mathrm{d}\psi \mathrm{d}\bar{\psi}][\mathrm{d}\rho]\, e^{\frac{N}{2} \sum_{i \neq j} \frac{z_i - z_j}{u_i - u_j} \rho^i_j \rho^j_i + \sum_{i,j} \rho^i_j \bar{\psi}_i \psi^i} \,, \tag{1.8}$$

where

$$Z_\rho = \int [\mathrm{d}\rho]\, e^{\frac{N}{2} \sum_{i \neq j} \frac{z_i - z_j}{u_i - u_j} \rho^i_j \rho^j_i} \tag{1.9}$$

and the integral $\int [\mathrm{d}\rho]$ involves only the off-diagonal components of ρ (and an appropriate contour).

After integrating out the fermions, the ρ integral takes a form suitable for a large N saddle-point approximation:

$$\frac{1}{Z_\rho} \int [\mathrm{d}\rho]\, e^{N S[\rho]} \,, \tag{1.10}$$

with the action

$$S[\rho] = \frac{1}{2} \sum_{i \neq j} \frac{z_i - z_j}{u_i - u_j} \rho^i_j \rho^j_i + \log \det \rho \,. \tag{1.11}$$

The saddle point equations are

$$\frac{z_i - z_j}{u_i - u_j} \rho^i_j + (\rho^{-1})^i_j = 0, \quad i \neq j \,. \tag{1.12}$$

They can be rewritten in a simple form as a matrix equation:

$$[\zeta, \rho] + [\mu, \rho^{-1}] = 0 \,, \tag{1.13}$$

where

- ρ is a $k \times k$ matrix, whose diagonal elements are fixed to be $\rho^i_i = m_i$ and the off-diagonal components are the variables we are solving for,
- ζ is a diagonal $k \times k$ matrix, whose diagonal elements are the positions z_i of determinants on the boundary of AdS_3,
- μ is a diagonal $k \times k$ matrix, whose diagonal elements u_i control the linear combinations of the symplectic bosons X, Y employed in determinants.

1.2.2 Spectral curve construction

To each saddle ρ, we can associate a holomorphic curve in $SL(2, \mathbb{C})$ defined as a spectral curve of a system of commuting matrices:

$$\begin{aligned} B(a) &= a\mu - \rho \\ C(a) &= a\zeta + \rho^{-1} \\ D(a) &= a\zeta\mu + \rho^{-1}\mu - \zeta\rho\,. \end{aligned} \tag{1.14}$$

The spectral curve is defined as a set of points (a, b, c, d), where $a \in \mathbb{C}$ and b, c, d are simultaneous eigenvalues of $B(a), C(a), D(a)$. The matrices are defined in such a way that:

- they commute when ρ satisfies the saddle point equations (1.13). Therefore, for each saddle ρ, they can be simultaneously diagonalized,
- they satisfy
$$aD(a) - B(a)C(a) = 1\,, \tag{1.15}$$
which constraints the simultaneous eigenvalues to lay inside the locus $ad - bc = 1$. As a result, the spectral curve is a holomorphic curve inside $SL(2, \mathbb{C})$,
- the boundary behavior of the spectral curve matches the expected boundary behavior of a D1-brane dual to k insertions of determinants $\mathcal{D}(m_i; u_i; z_i)$ in the boundary theory. Explicitly, when $a \to \infty$, the eigenvalues of matrices $B(a)$ and $C(a)$ approach
$$\begin{aligned} b_i &= au_i - m_i + \dots \\ c_i &= az_i + p_i + \dots\,, \end{aligned} \tag{1.16}$$
where $p_i \equiv [\rho^{-1}]^i_i$. This means that the spectral curve reaches the holographic boundary at k points z_i, with an asymptotic behavior controlled by u_i and m_i as expected above.

1.2.3 Holographic checks

The conjecture is tested by various holographic computations in Sections 2.3 and 2.4.

Determinant correlation functions with a single trace

First, we compare the large N expectation value of a single-trace operator in the presence of multiple determinants with a holographic Witten diagram computation.

A similar calculation to the one in Section 1.2.1 yields

$$\left\langle N \operatorname{Tr} Z(u;z)^n \prod_i^k \mathcal{D}(u_i;z_i;m_i) \right\rangle = \frac{1}{Z_\rho} \int [\mathrm{d}\rho] e^{NS[\rho]} \left[-N \operatorname{Tr}_{k\times k} \left(-\rho^{-1} \frac{\mu - u}{\zeta - z} \right)^n \right] . \tag{1.17}$$

The ρ integral is controlled by the same large N saddles (1.13). At a given saddle $\rho = \rho^*$, it is equal to evaluating $e^{NS[\rho^*]}$ times

$$-N \operatorname{Tr}_{k\times k} \left(-\rho^{-1} \frac{\mu - u}{\zeta - z} \right)^n \Bigg|_{\rho=\rho^*} . \tag{1.18}$$

The holographic computation corresponds to a closed string propagating from a boundary insertion $N \operatorname{Tr} Z(u;z)^n$ and interacting with a Giant Graviton brane. The BCOV calculation accords to integrating a bulk-to-boundary propagator sourced by $N \operatorname{Tr} Z(u;z)^n$ along the brane:

$$\int_{\mathcal{S}} \partial^{-1} \alpha_n(u;z) . \tag{1.19}$$

In Section 2.4 we show that (1.18) can be rewritten in the above form as an integral over the spectral curve $\mathcal{S} = S_{\rho^*}$.

Matching actions

An insertion of a determinant operator in the boundary can be engineered in the bulk by placing a probe D1'-brane transverse to the stack of N D1-branes. The auxiliary fermions $\psi^i, \bar{\psi}_i$, $i = 1, \ldots, N$ introduced in (1.6) can be interpreted as open strings between the probe brane and the stack [38]. The auxiliary variables ρ^{-1} behave as fermion bilinears.

It would be very interesting to match the ρ action (1.11) with the action of open strings on the dual Giant Graviton D1-brane. Instead we match the derivatives of the actions with respect to the parameters m_i, u_i, z_i. In particular, the derivative of the saddle action (1.11) with respect to m_i equals

$$p_i \equiv \frac{\partial S}{\partial m_i} = [\rho^{-1}]^i_i \,. \tag{1.20}$$

The worldvolume theory of a D1-brane is a $u(1)$-gauged $\beta\gamma$ system,[6] where the two fields, β and γ, correspond to fluctuations in the two transverse directions, which we can take to be b and c. The action of the worldvolume theory of a brane $\mathcal{S}$ parametrized by a is[7]

$$\int_{\mathcal{S}} \beta\bar{\partial}\gamma \wedge \frac{\mathrm{d}a}{a} \,. \tag{1.21}$$

Close to the brane we can expand the two fields into modes:

$$\beta(a) = \sum_n \beta_n a^n \,, \qquad \gamma(a) = \sum_n \gamma_n a^n \,. \tag{1.22}$$

The asymptotic behaviour of a brane supported at the spectral curve $\mathcal{S} = \mathcal{S}_\rho$ was found to be (1.16). Near one of the asymptotic boundaries one can then identify the zero-modes as:

$$\beta_0 = -m_i \,, \qquad \gamma_0 = p_i \,. \tag{1.23}$$

These zero-modes are conjugate with respect to the action (1.21).

The above considerations give a match of variables p_i defined as conjugate to m_i. The derivatives of the actions with respect to u_i and z_i can be matched in a similar but more cumbersome way.

[6] Not to be confused with the chiral algebra $\mathcal{A}_N$ which is a $\mathrm{u}(N)$-gauged $\beta\gamma$ system, originally appearing as a worldvolume theory of a stack of N D1-branes.

[7] The denominator comes from contracting with the $SL(2,\mathbb{C})$ volume form, which can be written as $\Omega = \frac{\mathrm{d}a\mathrm{d}b\mathrm{d}c}{a}$ in the $a \neq 0$ patch.

Modifications of determinants

Excitations of Giant Graviton D1-branes correspond to determinant modifications [42, 43, 44, 45] such as[8]

$$\det X = \frac{1}{N!}\varepsilon\varepsilon(X, X, \ldots X, X) \mapsto \frac{1}{N!}\varepsilon\varepsilon(X, X, \ldots, X, Y^2)\,. \tag{1.24}$$

One can create BRST-closed modifications of determinants by acting with the modes of the global symmetry algebra of the chiral algebra $\mathcal{A}_N$ in the infinite N limit. This algebra was identified on both sides of the duality in [31]. On the chiral algebra side, the global symmetry algebra is defined by certain modes of the BRST-closed single-trace operators:[9]

$$a^{(n)}_{p,q} \sim \oint \mathrm{d}z\, z^{p+\frac{n}{2}} \oint \mathrm{d}u\, u^{q+\frac{n}{2}}\ \mathrm{Tr}\, Z(u;z)^n, \quad |p| \leq \frac{n}{2} - 1,\ |q| \leq \frac{n}{2}\,. \tag{1.25}$$

For example, the lowest modes generate an su(2) subalgebra:

$$a^{(2)}_{0,-1} \sim \oint \mathrm{d}z\ \mathrm{Tr}\, X^2(z), \quad a^{(2)}_{0,0} \sim \oint \mathrm{d}z\ \mathrm{Tr}\, XY(z), \quad a^{(2)}_{0,1} \sim \oint \mathrm{d}z\ \mathrm{Tr}\, Y^2(z)\,. \tag{1.26}$$

The global symmetry algebra, when acting on determinants, produces BRST-closed modifications, eg.:

$$\left[a^{(2)}_{0,1}, \det X(0)\right] \sim \varepsilon\varepsilon(X, \ldots, X, Y)\,. \tag{1.27}$$

In principle, different modes can produce BRST-equivalent operators. The BRST-inequivalent modifications can be identified by studying the following matrix of two-point functions:

$$\left\langle \left[a^{(n)}_{-p,-q}, \det Y(\infty)\right]\, \left[a^{(m)}_{p,q}, \det X(0)\right] \right\rangle \Big|_{N\to\infty}\,. \tag{1.28}$$

[8]We use a schematic notation:

$$\varepsilon\varepsilon(Z_1, \ldots, Z_N) \equiv \varepsilon_{i_1\ldots i_N}\varepsilon^{j_1\ldots j_N}(Z_1)^{i_1}_{j_1}\ldots(Z_N)^{i_N}_{j_N}\,.$$

[9]There are also three other types of generators of the global symmetry algebra (see [31]) but we focus on this one.

The two types of inequivalent determinant modifications were found to be:

$$[a^{(n)}_{p,p-1}, \det X(0)] \sim n\,\varepsilon\varepsilon(X,\dots,Y^{1-2p}) \qquad , \, p \leq \frac{1}{2} \tag{1.29}$$

$$\begin{aligned}[a^{(n)}_{p,p+1}, \det X(0)] \sim\ & n\,\varepsilon\varepsilon(X,\dots,Y^{-1-2p}\partial X) \\ & + n\,\varepsilon\varepsilon(X,\dots,\partial^2 Y^{-3-2p}) \qquad , \, p < -\frac{1}{2}\,. \end{aligned} \tag{1.30}$$

On the holographic side, (the bosonic part of) the global symmetry algebra acts by holomorphic divergence-free vector fields on $SL(2,\mathbb{C})$. The modes of dual global symmetry algebra create excitations of the dual Giant Graviton brane.

The brane dual to the insertions of $\det X(0)$ and $\det Y(\infty)$ is

$$g = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix} \subset SL(2,\mathbb{C})\,. \tag{1.31}$$

Likewise, we find two types of brane excitations

$$\delta g = \begin{pmatrix} a & \delta b \\ 0 & \frac{1}{a} \end{pmatrix}, \quad \delta b \sim n a^{1-2p} \tag{1.32}$$

$$\delta g = \begin{pmatrix} a & 0 \\ \delta c & \frac{1}{a} \end{pmatrix}, \quad \delta c \sim n a^{-1-2p}\,, \tag{1.33}$$

which can be matched with the two types of determinant modifications (1.29) and (1.30).

1.3 Non-conformal vacua

In Chapter 3, as the next step in studying twisted holography, we propose that the duality can be extended to non-conformal vacua of the chiral algebra $\mathcal{A}_N$. The holographic dual geometries are deformations of $SL(2,\mathbb{C})$, originating from the backreaction of a stack of non-coincident D1-branes in the topological B-model on $\mathbb{C}^3$.

This is a twisted analog of the duality between the Coulomb branch of $\mathcal{N}=4$ SYM and the (asymptotically $\mathrm{AdS}_5 \times S^5$) multi-center supergravity solutions obtained from a near-horizon limit of a stack of non-coincident D3-branes [46, 47, 48, 49, 50].

Non-conformal (translation-invariant) vacua of a chiral algebra are a novel observable and we conjecture that they correspond to the notion of *the associated variety* of the chiral algebra.[10] In case of chiral algebras of $\mathcal{N} = 2$ SCFTs, we expect the 2d vacua to descent from the Higgs branch vacua of the 4d parent theory.[11] The conjecture is therefore in agreement with [54] which identified the Higgs branch of an $\mathcal{N} = 2$ SCFT with the associated variety of its chiral algebra subsector.

1.3.1 Higgs branch conjecture

The Coulomb branch of $\mathcal{N} = 4$ SYM is simply $\mathbb{R}^{6N}/S_N$, parametrized by the commuting vevs of the six adjoint scalars $\vec{\Phi}$. In an $\mathcal{N} = 2$ language, it decomposes into the $\mathcal{N} = 2$ Coulomb branch and the $\mathcal{N} = 2$ Higgs branch, $\mathbb{C}^N/S_N$ and $\mathbb{C}^{2N}/S_N$, respectively. We expect the translation-invariant vacua of the chiral algebra $\mathcal{A}_N$ to descend from the Higgs branch vacua.

More generally, the Higgs branch of $\mathcal{N} = 2$ SCFTs has been conjectured to correspond to the notion of the associated variety of its chiral algebra subsector [54].[12]

Motivated by the above considerations, we conjecture that the space of translation-invariant vacua of a 2d chiral algebra corresponds to the associated variety of the chiral algebra (see [58] for the definition). The conjecture can be verified in the case of gauged $\beta\gamma$ systems, which arise from Lagrangian $\mathcal{N} = 2$ SCFTs.

1.3.2 Coulomb branch geometries

The translation-invariant vacua of the chiral algebra $\mathcal{A}_N$ can be parametrized by eigenvalues (x_i, y_i) of X and Y of multiplicity $N_i = \alpha_i N$, $i = 1, \ldots, n$.

The dual geometries arise from separating the stack of N D1-branes in $\mathbb{C}^3$. The eigenvalues x_i and y_i are identified with the positions $x = x_i$, $y = y_i$ of N_i branes, where (x, y, z) are the coordinates in $\mathbb{C}^3$.

The backreaction of the stack of non-coincident branes can be described using 2^n patches of $\mathbb{C}^3$, where for each $i = 1, \ldots, n$ either $x - x_i$ or $y - y_i$ is non-zero. The coordinates x and y remain holomorphic and coincide in all patches. The coordinate z gets deformed to a new holomorphic coordinate z_I in a patch I. The holomorphic coordinate transformation

[10] Chiral algebras are referred to as vertex algebras in the math literature.

[11] In a setting without conformal symmetry, the chiral algebra subsector can be defined using a version of the Ω-deformation [51, 52, 53]. We expect, but not prove, that this procedure is compatible with the Higgs branch vevs.

[12] See also [55] and the reviews [56, 57].

on the intersection of two patches I and I' which differ by the i-th choice only ($y \neq y_i$ in I and $x \neq x_i$ in I') is

$$z_I = z_{I'} + \frac{\alpha_i}{(x - x_i)(y - y_i)} \,. \tag{1.34}$$

Let us consider two "extremal" patches: I_0 where $y - y_i$ are all non-zero and I_∞ where $x - x_i$ are all non-zero. The coordinate transformation on the intersection takes the form

$$z_0 - z_\infty = \sum_{i=1}^{n} \frac{\alpha_i}{(x - x_i)(y - y_i)} = \sum_{k \geq 0} \sum_{l \geq 0} \frac{1}{x^{k+1} y^{l+1}} \sum_{i=1}^{n} \alpha_i x_i^k y_i^l \,. \tag{1.35}$$

The first term of the sum ($k = l = 0$) describes the standard $SL(2, \mathbb{C})$ geometry obtained by a backreaction of coincident branes:[13]

$$z_0 - z_\infty = \frac{1}{xy} \,. \tag{1.36}$$

The subsequent terms $\frac{1}{x^{k+1} y^{l+1}}$ are deformations of the $SL(2, \mathbb{C})$ geometry. The coefficients can be identified with the vevs of single-trace operators in the vacuum parametrized by (x_i, y_i):

$$\left\langle \, \mathrm{Tr} X^k Y^l \, \right\rangle = \sum_{i=1}^{n} N_i x_i^k y_i^l \,. \tag{1.37}$$

1.4 Holomorphic twist

In the final Chapter 4, we move on to study 4d holomorphic theories and their holographic duals. $\mathcal{N} = 1$ theories are very interesting physically since they display phenomena such as confinement, chiral symmetry breaking and many dualities. Their holomorphic twist is defined by restricting to the cohomology of one out of the four supercharges and hence captures the 1/4-BPS subsector of the theory. Our primary motivation for studying this subject is that the twisted theory enjoys an infinite dimensional symmetry algebra, that includes a 4d analog of the Virasoro algebra. We leverage this symmetry to show that the holomorphic twist of pure supersymmetric Yang–Mills is actually a topological theory – a phenomenon we dub *holomorphic confinement*.

[13] In coordinates $w_0 = z_0 x$, $w_\infty = z_\infty y$, we get the familiar $SL(2, \mathbb{C})$ relation: $w_0 y - w_\infty x = 1$.

1.4.1 Symmetries

4d $\mathcal{N}=1$ theories have four supercharges:

$$Q_{\dot\alpha}, \tilde{Q}_\alpha\,, \qquad \alpha, \dot\alpha \in \{1,2\}\,. \tag{1.38}$$

We can define complex coordinates on $\mathbb{C}^2$:

$$z_\alpha \equiv x_{\dot 1 \alpha}\,, \quad \bar{z}_\alpha \equiv x_{\dot 2 \alpha}\,. \tag{1.39}$$

To twist, we take the cohomology with respect to $Q \equiv Q_{\dot 2}$. With this choice of twisting supercharge and complex coordinates, the antiholomorphic derivatives are Q-exact:

$$\{Q, \tilde{Q}_\alpha\} \sim \partial_{\bar{z}^\alpha}\,. \tag{1.40}$$

When discussing the holomorphic twist, it is convenient to employ a reduced superspace formalism, i.e. introduce odd spacetime coordinates $\tilde\theta^\alpha \equiv \mathrm{d}\bar{z}^\alpha$ and combine the descendants of $\tilde{Q}_\alpha$ into one superfield:

$$\mathcal{O} \equiv e^{\mathrm{d}\bar{z}^\alpha \tilde{Q}_\alpha} \mathcal{O}^{(0)} = \mathcal{O}^{(0)} + \mathcal{O}^{(1)} + \mathcal{O}^{(2)} \in \Omega^{0,\bullet}(\mathbb{C}^2, \mathfrak{g})\,. \tag{1.41}$$

Then, one can show that iff the 0-th component $\mathcal{O}^{(0)}$ is in Q-cohomology, the corresponding superfield satisfies the following equation:

$$(Q + \bar\partial)\mathcal{O} = 0\,. \tag{1.42}$$

We will call such superfields *semi-chiral*.

For each semi-chiral superfield we can define the non-negative modes of the infinite dimensional symmetry algebra as

$$\hat{\mathcal{O}}_{m,n} \equiv \oint_{S^3} \mathrm{d}^2 z\, z_1^n z_2^m \mathcal{O}(z,\bar{z})\,, \qquad n, m \geq 0\,. \tag{1.43}$$

The infinite dimensional symmetry algebra includes a subsector generated by the *stress tensor superfield* S_α (or the derivative of the stress tensor $\partial_\alpha S^\alpha$ in case the $U(1)_R$ symmetry is broken), which is a 4d analogue of the Virasoro algebra.

The non-negative modes can be combined into the so called λ-bracket:

$$\{\mathcal{O}_{1\lambda} \mathcal{O}_2\} \equiv \oint_{S^3} \mathrm{d}^2 z\, e^{\lambda z} \mathcal{O}_1(z,\bar{z}) \mathcal{O}_2(0,0)\,, \tag{1.44}$$

which computes the coefficients of the OPE between operators in Q-cohomology with the descendants of the operators in Q-cohomology.

1.4.2 Holomorphic BF theory

In this thesis, we focus on the example of the holomorphic twist of 4d $\mathcal{N}=1$ SYM with $SU(N)$ gauge group. The holomorphic twist is equivalent to the holomorphic BF theory [59, 6, 10]:

$$\int_{\mathbb{C}^2} \mathrm{d}^2 z \, \mathrm{Tr} \, b\left(\bar{\partial} c - \frac{1}{2}[c, c]\right), \tag{1.45}$$

where

$$b = b^{(0)} + b^{(1)} + b^{(2)} \in \Omega^{0,\bullet}(\mathbb{C}^2, \mathfrak{g}^\vee), \quad c = c^{(0)} + c^{(1)} + c^{(2)} \in \Omega^{0,\bullet}(\mathbb{C}^2, \mathfrak{g})[1]. \tag{1.46}$$

The field content of pure $\mathcal{N}=1$ SYM is the vector multiplet, which can be compared with the field content of the holomorphic BF theory. In particular we can match:

- The holomorphic part of field strength F_{++} with the component $b^{(0)}$.
- The gauginos $\tilde{\lambda}_\alpha$ with $\partial_{z^\alpha} c^{(0)}$.

The BRST/BV cohomology of the holomorphic BF theory is isomorphic to the Q-cohomology of the original $\mathcal{N}=1$ SYM. In particular, loop corrections to the action of the supercharge Q map to loop correction to the BRST charge, which we compute in Section 4.3.2. An important corollary of the following one-loop computation:

$$Q_1(\mathrm{Tr}\, b^2) \sim \partial_\alpha \, \mathrm{Tr}\, b \partial^\alpha c = \partial_\alpha S^\alpha \tag{1.47}$$

is that the derivative of the stress tensor of the holomorphic BF theory becomes exact at one-loop. The semi-chiral superfield $\partial_\alpha S^\alpha$ generates the action of holomorphic vector fields on $\mathbb{C}^2$ that preserve the symplectic form $\mathrm{d}z^1 \wedge \mathrm{d}z^2$, which include translations and rotations. Therefore, the holomorphic BF theory becomes topological at one-loop, which can be interpreted as evidence of confinement of the original theory.

In Section 4.5, we compute the infinite N cohomology at tree-level, which is generated by three towers of single-traces:

$$A_n = \mathrm{Tr}\, b^n \tag{1.48}$$

$$B_{n,\alpha} = \mathrm{Tr}\, b^n \partial_\alpha c \tag{1.49}$$

$$C_n = \mathrm{Tr}\, b^n \partial_\alpha c \partial^\alpha c \tag{1.50}$$

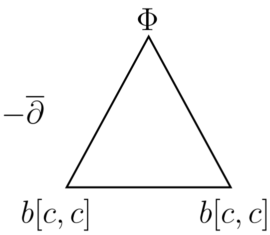
$Q_1(\Phi) \;\sim\; -\overline{\partial}$ 

Figure 1.1: One-loop correction to the action of Q.

and their derivatives.

At one loop, the differential Q_1 relates some of these towers:

$$Q_1(A_n) \sim \partial_\alpha B^\alpha_{n-1} \tag{1.51}$$
$$Q_1(B_{n,\alpha}) \sim \partial_\alpha C_{n-1}\,. \tag{1.52}$$

The one loop cohomology consists only of the C_n tower and no derivatives, which is consistent with our result that the theory becomes topological.

Finally, we realize the holomorphic BF theory within the B-model. The holomorphic BF theory is the worldvolume theory of a stack of N D3-branes wrapping $\mathbb{C}^2 \subset \mathbb{C}^3$. The c field corresponds the the ghost along the branes and the b field controls the transverse fluctuations.

The stack of branes sources a Poisson bivector field:

$$\eta = N\frac{1}{z_3}\partial_{z_1} \wedge \partial_{z_2}\,.$$

This field does not deform the complex structure but instead introduces non-commutativity in the spacetime.

This leads us to conjecture that the holomorphic BF theory is holographically dual to the BCOV theory on $\mathbb{C}^3 \setminus \mathbb{C}^2$ in the presence of Poisson bivector η.

Chapter 2

Giant gravitons in twisted holography

2.1 Introduction and conclusions

The simplest example of Twisted Holography [60, 31] is the conjectured equivalence of

- The large N 't Hooft expansion for a 2d gauged $\beta\gamma$ system valued in the adjoint representation of $\mathfrak{u}(N)$, aka the $\mathcal{A}_N$ chiral algebra.
- The B-model topological string theory (aka BCOV theory) [36, 35] on an $SL(2,\mathbb{C})$ background with appropriate holographic boundary conditions and coupling N^{-1}.

The proposal of [31] formulates a holographic dictionary for single-trace local operators whose size remains finite in the large N limit. It includes the match of a large global symmetry algebra which exists on the two sides of the duality at the leading order in N.

In this Chapter we expand the dictionary to include (sub)determinant operators, whose size scales linearly in N. We will match the insertions of such operators to the presence of a B-model D1-brane reaching the boundary of $SL(2,\mathbb{C})$ with specific boundary conditions.

Correlation functions of (sub)determinant operators admit a rich collection of large N saddles [38]. We will attach to each saddle a *spectral curve* in $SL(2,\mathbb{C})$ and identify it with the worldsheet of a dual D1-brane. The explicit identification will allow us to verify a variety of quantitative holographic predictions.

2.1.1 Relation to physical holography

The Twisted Holography setup is expected to capture a protected subsector of the standard example of holography: the duality between $\mathcal{N}=4\,U(N)$ gauge theory and Type IIB string theory on an $\mathrm{AdS}_5 \times S^5$ background [34, 61].

This is manifest on the gauge theory side: the $\mathcal{A}_N$ chiral algebra and its two-sphere correlation functions coincide with the protected chiral subsector of $\mathcal{N}=4$ gauge theory introduced by [32].[1] The operators in $\mathcal{A}_N$ represent specific position-dependent linear combinations of protected operators in the four-dimensional gauge theory. The correlation functions do not depend on the gauge coupling, but have a non-trivial large N genus expansion.

The relation between the B-model topological string theory on $SL(2,\mathbb{C})$ and Type IIB string theory on $\mathrm{AdS}_5 \times S^5$ is less explicit. It is expected to be closely analogous to the original construction relating the B-model to IIB string theory on a self-dual graviphoton background [35, 64, 65]. Both should be special cases of a general construction in twisted supergravity [12].

The derivation in [31] bypassed the twisted supergravity analysis. Instead, it employed the near-horizon limit of a stack of N D1-branes in the B-model, in analogy to the original derivation of holography from the near-horizon limit of a stack of N D3-branes in IIB string theory [34].[2]

The holographic interpretation of (sub)determinant BPS operators in $\mathcal{N}=4$ SYM is well understood [66, 67]. The insertion of a (sub)determinant operator is described holographically as the presence of a "Giant Graviton" D3-brane in the bulk, reaching the boundary in a specific manner at the insertion point.

This entry of the holographic dictionary can be derived by adding a probe D3′-brane to the standard near-horizon limit [38]. The D3′-brane is taken to be fully transverse to the stack of N D3-branes. The D3−D3′ open strings give 0-dimensional fermions coupled to the world-volume theory of the N D3-branes. They can be integrated out to give the (sub)determinant operator insertion. The image of the D3′-brane in the near-horizon geometry wraps an S^3 inside the transverse S^5 and reaches the boundary of AdS_5 at the point where the (sub)determinant operator is inserted in the dual gauge theory.

The derivation can be mimicked in twisted holography. The starting point is a probe D1'-brane transverse to the stack of N D1-branes. The D1'-brane engineers the (sub)determinant

[1] Torus correlation functions can also be embedded in the gauge theory [62, 63]. It would be interesting to study their role in Twisted Holography.

[2] Recall that Dn-branes in the B-model arise from D$(n+2)$-branes in IIB string theory [65].

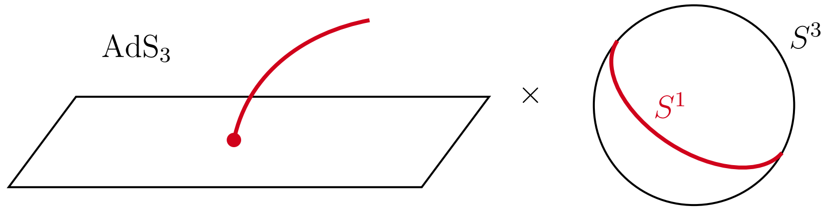


Figure 2.1: A schematic depiction of a Giant Graviton $D1$-brane wrapping $\mathbb{C}^*$ in Euclidean $\mathrm{AdS}_3 \times S^3 \approx SL(2,\mathbb{C})$.

operator in the world-volume $\mathcal{A}_N$ chiral algebra. The image of the D1'-brane in the near-horizon geometry wraps a $\mathbb{C}^*$ submanifold in $SL(2,\mathbb{C})$ and reaches the holographic boundary at the point where the (sub)determinant operator is inserted.

In both situations, the holographic dictionary only prescribes that a D-brane should reach the boundary at the insertion point with a certain asymptotic shape. The shape of the D-brane in the bulk will be determined dynamically. If multiple (sub)determinant insertions are present at the boundary, there may be semiclassical saddles where the same bulk D-brane connects all the insertion points, as well as saddles where multiple disconnected D-branes do the job.

The calculation of correlation functions of multiple determinants in $\mathcal{A}_N$ takes the form of a tree-level calculation in the physical gauge theory [38], with specific choices of determinant insertions. In the physical theory, the tree-level answer for generic determinant insertions has large N saddles which can be qualitatively matched to the D-brane saddles. A quantitative comparison, though, requires a full control of the 't Hooft coupling dependence of the answer and is currently out of reach.

As the 't Hooft coupling dependence drops out in the chiral algebra subsector, we have instead an exact answer which can be directly compared with a holographic calculation.[3] We identify the holographic dual D-brane as the spectral curve of an auxiliary set of commuting Higgs fields built from the data of the chiral algebra saddle.

A full lift of our results to the physical holographic duality would require an extra step, which we do not attempt: a match between the holomorphic B-model saddles and explicit supersymmetric D3-brane configurations in $\mathrm{AdS}_5 \times S^5$, perhaps along the lines of [70].

[3]The 't Hooft coupling drops out in a variety of protected subsectors [68, 69], but these are topological in nature, so the answers only depend on discrete data. The chiral algebra subsector allows for a non-trivial (holomorphic) dependence on the positions of the operator insertions, which must be matched by the dual D-brane geometry.

2.1.2 Saddle equations and spectral curves

The saddle equations [38] for a correlation function of k subdeterminants in $\mathcal{A}_N$ take a simple form:

$$[\zeta, \rho] + [\mu, \rho^{-1}] = 0\,, \tag{2.1}$$

where

- ρ is a $k \times k$ matrix whose diagonal elements m_i control the "length" of the i-th subdeterminant operator and whose off-diagonal elements are the variables we are solving for,
- ζ is a diagonal matrix whose entries z_i are the positions of the subdeterminant operators on the two-sphere,
- μ is a diagonal matrix whose entries u_i control the specific linear combination of the adjoint scalars employed in the subdeterminant operator.

Any solution ρ of the saddle point equations allows us to build a family of commuting $k \times k$ matrices

$$\begin{aligned} B(a) &= a\,\mu - \rho \\ C(a) &= a\,\zeta + \rho^{-1} \\ D(a) &= a\,\zeta\,\mu + \rho^{-1}\mu - \zeta\,\rho, \end{aligned} \tag{2.2}$$

which satisfy

$$aD(a) - B(a)C(a) = 1. \tag{2.3}$$

In turn, this defines a spectral curve $\mathcal{S}_\rho$ in $SL(2,\mathbb{C})$: the collection of points (a,b,c,d) such that b,c,d are simultaneous eigenvalues of $B(a)$, $C(a)$, $D(a)$. Here we identify $SL(2,\mathbb{C})$ with the locus of $ad - bc = 1$ in $\mathbb{C}^4$:

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.4}$$

Following the holographic dictionary of [31], the spectral curve reaches the holographic boundary at k points z_i, with an asymptotic shape controlled by u_i and m_i just as expected for the support of a holographic dual D1-brane.

We thus conjecture that a saddle ρ is dual to a B-model D1-brane $\mathcal{B}_\rho$ supported on $\mathcal{S}_\rho$ in $SL(2,\mathbb{C})$. The first test of this conjecture is a comparison of the saddle action $S[\rho]$

on the two sides of the duality. More precisely, we successfully compare the observables $p_i \equiv \frac{\partial S}{\partial m_i}$ conjugate to m_i, as well as other derivatives of the saddle action.

Next, we compare the leading expectation values of single-trace operators in the presence of the collection of (sub)determinant operators. The expectation values are computed from the eigenvalues of a family of auxiliary matrices

$$\frac{\mu - u}{\zeta - z}\rho^{-1}. \tag{2.5}$$

We recast the chiral algebra calculation as an integral over $\mathcal{S}_\rho$ of certain 1-forms which we identify with the bulk-to-boundary B-model propagators, completing the holographic match.

The spectral curve can be connected, but can also consist of multiple components. We analyze the genus 1 part of the large N expansion around a saddle with two disconnected components and compare it with a genus 1 calculation in the B-model. We find a non-trivial match, which allows us to probe the Chan-Paton bundle on the holographic branes. The spectral curve comes equipped naturally with a line bundle $\mathcal{L}_\rho$ given by the common eigenline of $B(a)$, $C(a)$, $D(a)$ corresponding to the b,c,d eigenvalues. It appears to control the CP bundle for $\mathcal{B}_\rho$.

2.1.3 Determinant modifications and open string boundary conditions

The holographic dictionary for operator insertions is best understood after the application of a state-operator map. In particular, the determinant operators create a state in the gauge theory which is dual to a state in the string theory where the Giant Graviton D-brane has a certain shape and the D-brane world-volume theory is in its ground state.

It is also possible to consider operators dual to a Giant Graviton D-brane in an excited state. In the physical theory, they are described by a certain class of modifications of a determinant operator [43, 44, 42, 45].

We formulate such a dictionary for twisted holography and test it with explicit calculations. The spectrum of open strings on the B-model brane is well-understood: the worldvolume theory is a $\mathfrak{u}(1)$-gauged $\beta\gamma$ system valued in the normal directions to the brane. The main challenge for us is to produce BRST-closed modifications of a determinant operator in $\mathcal{A}_N$ and to identify BRST-closed modifications which differ by BRST-exact ones.

Our strategy is to employ the generators of the global symmetry algebra from [31] to create the modifications/fluctuations on the two sides of the duality. Different generators can produce the same open string fluctuations, with specific relative coefficients. On the chiral algebra side the corresponding modifications of determinant operators should satisfy the same relations, up to BRST-exact operators. We test our proposal both by inserting a single modification in the correlation function of multiple determinants and by inserting two modifications in a two-point function.

2.1.4 Structure of the Chapter

In Section 2.2 we review the relevant aspects of the twisted holography construction and of the construction of determinant operators. In Section 2.3 we derive the large N saddles of a correlation function of (sub)determinant operators and the corresponding spectral curves. We formulate and test the conjecture that the D-branes dual to a given saddle are supported on the spectral curve, with a natural choice of the Chan-Paton bundle. In Section 2.4 we compute the large N expectation value of a single-trace operator in the presence of multiple (sub)determinants. We express the answer in terms of the spectral curves and match it with a B-model calculation. We also match the action of global symmetry generators on the two sides of the correspondence. In Section 2.5 we do a detailed test of our dictionary relating determinant modifications and open string fluctuations on the Giant Graviton. In Section 2.6 we review a protected subsector of the chiral algebra which resembles the Dijkgraaf-Vafa setup.

2.1.5 Future directions

The most striking feature of the twisted holography setup is the emergence of geometry from what are essentially free field calculations in the chiral algebra. This reminds us of the protracted efforts to derive a holographic dual description of free $\mathcal{N} = 4$ SYM [71, 72, 73, 74, 75, 76, 77] and of the Gaussian matrix model [78, 79] which we encounter in Section 2.6.

Single-trace operators only probe small fluctuations of the geometry. Determinant operators can create D-branes with non-trivial geometrical shapes. The natural next step is to study operators of size $\sim N^2$, which could produce finite modification of the geometry, as in [80]. Twisted holography could give an explicit, controllable example where multiple geometric saddles contribute to a calculation.

Another natural objective would be to prove twisted holography at all orders in the large N expansion. Deriving the world-volume theory of Giant Gravitons from the chiral algebra could be a natural first step. Many of the ingredients of the large N expansion can be expressed in terms of the spectral curve. It may be possible to recast the expansion in terms of some open string field theory, controlling all string amplitudes which include at least one boundary.

The twisted holography based on the $\mathcal{A}_N$ chiral algebra we employed in this Chapter is one member of a large family of examples. Other examples include the chiral algebras associated to $\mathcal{N} = 4$ SYM with other classical gauge groups, as well as $\mathcal{N} = 2$ gauge theories modelled on affine ADE quivers [31]. All of these examples have some IIB duals and include a variety of determinant-like operators, often dual to Giant Gravitons wrapping non-trivial cycles in the dual geometries. It may be possible to analyze the corresponding correlation functions with the tools employed in this Chapter. Correlation functions of determinants and Pfaffians in $\mathcal{N} = 4$ $SO(N)$ gauge theory would be a natural starting point. It would also be interesting to study determinant-like operators in less conventional examples, such as class $\mathcal{S}$ theories with M-theory duals [81].

2.2 The 2d chiral algebra

We will change conventions slightly compared to [31] in order to follow the standard conventions for a 't Hooft expansion, so that contributions of ribbon diagrams of genus g scale as N^{2-2g}.

The relevant chiral algebra $\mathcal{A}_N$ is defined by a system of gauged symplectic bosons valued in the adjoint representation of $U(N)$. After gauge-fixing, we have an action of the form

$$N \int \mathrm{Tr}\left(X\bar{\partial}Y + b\bar{\partial}c\right). \tag{2.6}$$

Here X, Y are a pair of bosonic spinors, aka *symplectic bosons*. The ghosts b and c are also valued in the adjoint representation of $U(N)$. The BRST charge has the schematic form

$$N \oint \mathrm{Tr}\left(c[X,Y] + \frac{1}{2}b[c,c]\right). \tag{2.7}$$

In our calculations we will never actually use the BRST charge or the ghosts. All operators we will employ are built explicitly from a basic set of BRST-closed combinations of the X and Y fields.

The chiral algebra has an $SL(2)_R$ global symmetry acting on the (X, Y) doublet of symplectic bosons.[4] Throughout the Chapter we will employ a very convenient formalism to describe elements of finite-dimensional $SL(2)$ representations: we identify a spin j representation with the space of polynomials of degree $2j$ in some auxiliary variable. The action of $SL(2)$ on the coefficients of a polynomial maps to a fractional linear transformation of the auxiliary variable:

$$p(u) \to (\gamma u + \delta)^{2j} p\left(\frac{\alpha u + \beta}{\gamma u + \delta}\right). \tag{2.8}$$

In particular, we will employ the linear combination

$$Z(u; z) \equiv X(z) + uY(z) \tag{2.9}$$

and write the symplectic boson OPE as

$$Z(u; z)Z(v; w) \sim \frac{1}{N}\frac{u - v}{z - w}. \tag{2.10}$$

In the large N limit, operators built from a finite number of fields can be expressed as regularized polynomials in single-trace operators. At leading order in N, the BRST charge will act separately on each single-trace operator in the product. We can thus focus on the BRST cohomology of single-trace operators.

A word of caution is that the large N expansion does not play well with the full 2d conformal symmetry group: most Virasoro descendants of a single-trace operator are multi-trace operators. We will thus usually focus on the $SL(2)_L$ global conformal group[5] only and discuss single-trace BRST-closed quasi-primary fields.

The single-trace operators

$$A_n(u; z) = N \,\mathrm{Tr}\, Z(u; z)^n \tag{2.11}$$

are BRST closed. They have $SU(2)_R$ spin $j = \frac{n}{2}$ and scaling dimension $\Delta = \frac{n}{2}$.

The tower of A_n operators does not exhaust the BRST cohomology of single-trace quasi-primary operators. There is a second tower of D_n operators with $SU(2)_R$ spin $j = \frac{n}{2}$ and dimension $\Delta = \frac{n}{2} + 2$. These are more complicated to write down explicitly. They can be usefully represented as specific terms in the OPE of operators in the A tower. There are also towers of BRST-closed operators B_n and C_n in ghost number ± 1, but we will not employ them.

[4]This is a subgroup of the R-symmetry of the physical 4d theory.

[5]This is a certain combination of conformal and R-symmetry transformations of the physical theory.

2.2.1 The global symmetry algebra

The chiral algebra correlation functions in the large N limit are invariant under the action of a global symmetry algebra: the subset of the Fourier modes of single-trace quasi-primaries which annihilate the vacuum at the origin and at infinity. This algebra can be matched to the algebra of holomorphic vector fields on $SL(2,\mathbb{C})$ which preserve the holomorphic three-form Ω. At least at tree level, this is a symmetry for the holographic dual B-model.[6]

The Fourier modes of $A_n(u;z)$ which annihilate the vacuum at the origin and infinity are organized in a multiplet of spin $\frac{n}{2}$ for $SL(2)_R$ and $\frac{n}{2}-1$ for the $SL(2)_L$ global conformal symmetry. We can collect them into a polynomial of degrees $(n,n-2)$ in two auxiliary variables v and w:

$$a_n(v;w) = \oint \frac{\mathrm{d}z}{2\pi i}\,(z-w)^{n-2}A_n(v;z). \tag{2.12}$$

The Fourier modes of the $D_n(v;z)$ which annihilate the vacuum at the origin and infinity are organized in a multiplet of spin $\frac{n}{2}$ for $SL(2)_R$ and $\frac{n}{2}+1$ for the $SL(2)_L$ global conformal symmetry. We can collect them into a polynomial $d_n(v;w)$ of degree n in v and $n+2$ in w.[7]

We can match these Fourier modes with the global holomorphic vector fields on $SL(2,\mathbb{C})$ which preserve the holomorphic three-form[8]

$$\Omega = \frac{\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c}{a}. \tag{2.13}$$

We can write remarkably compact expressions for the polynomial generating functions of vector fields in the same irreps of $SL(2)_L \times SL(2)_R$. We start with the polynomial $J_2(v)$ collecting the three $SL(2)_R$ generators:

$$J_2(v) = (b-va)(\partial_a + v\partial_b) + (d-vc)(\partial_c + v\partial_d) \tag{2.14}$$

and the polynomial $I_0(w)$ collecting the three $SL(2)_L$ generators:

$$I_0(w) = (c-wa)(\partial_a + w\partial_c) + (d-wb)(\partial_b + w\partial_d)\,. \tag{2.15}$$

We then match $a_n(v;w)$ to some multiple of

$$J_n(v;w) = (d-vc-wb+vwa)^{n-2}J_2(v) \tag{2.16}$$

[6]At finite N the Fourier modes of single-trace operators do not form a Lie algebra.

[7]There are also fermionic generators associated to the B_n and C_n towers. We will focus on bosonic modes here.

[8]Written here in the $a \neq 0$ patch of $SL(2,\mathbb{C})$.

and $d_n(v;w)$ to some multiple of

$$I_n(v;w) = (d - vc - wb + vwa)^n I_0(w). \tag{2.17}$$

Matching commutators constrains the relation further. In particular, we find that $a_n(v;w)$ matches $nJ_n(v;w)$.[9]

It is easy to show that these vectorfields preserve the $ad - bc = 1$ locus and thus define vectorfields on $SL(2,\mathbb{C})$. The invariance of Ω requires a bit more work. The $SL(2)_R$ variation is

$$\mathrm{d}\left(i_{J_2(v)}\Omega\right) = \mathrm{d}\left(-\frac{\mathrm{d}c}{a}(b - va)(\mathrm{d}b - v\mathrm{d}a) + (d - vc)\frac{\mathrm{d}a\mathrm{d}b}{a}\right) = \frac{\mathrm{d}a\mathrm{d}b}{a^2}(-b\mathrm{d}c + a\mathrm{d}d) = 0. \tag{2.18}$$

The desired result for $J_n(v;w)$ follows from the observation that $i_{J_2(v)}\Omega$ wedged with $\mathrm{d}(d - vc - wb + vwa)$ vanishes. The analysis for $I_n(v;w)$ is completely analogous.

There is another useful perspective. Consider the 1-forms

$$\omega_{J_2}(v) = \frac{1}{2}\left[(b - va)\,\mathrm{d}(d - vc) - (d - vc)\,\mathrm{d}(b - va)\right]. \tag{2.19}$$

It is easy to verify that

$$\mathrm{d}\omega_{J_2}(v) = \mathrm{d}(b - va)\,\mathrm{d}(d - vc) = i_{J_2(v)}\Omega. \tag{2.20}$$

Similarly, define

$$\omega_{J_n}(v;w) = \frac{1}{n}(d - vc - wb + vwa)^{n-2}\left[(b - va)\,\mathrm{d}(d - vc) - (d - vc)\,\mathrm{d}(b - va)\right]. \tag{2.21}$$

We have

$$\mathrm{d}\omega_{J_n}(v;w) = i_{J_n(v;w)}\Omega. \tag{2.22}$$

Similarly, we can define

$$\omega_{I_0}(w) = \frac{1}{2}\left[(c - wa)\,\mathrm{d}(d - wb) - (d - wb)\,\mathrm{d}(c - wa)\right] \tag{2.23}$$

and

$$\omega_{I_n}(v;w) = \frac{1}{n+2}(d - vc - wb + vwa)^n \omega_{I_0}(w). \tag{2.24}$$

These 1-forms play a useful role in describing the action of vector fields on D-branes.

[9] The simplest way to do so is to compare the leading N commutator of $\oint \frac{dz}{2\pi i} N \,\mathrm{Tr}\, X^2 Y(z)$ and $\oint \frac{dz}{2\pi i} N \,\mathrm{Tr}\, X^n(z)$ and the commutator of $b^2\partial_b - 2ba\partial_a + bd\partial_d - (ad + bc)\partial_c$ and $n(b^{n-1}\partial_a + b^{n-2}d\partial_c)$. The former gives $n \oint \frac{dz}{2\pi i} N \,\mathrm{Tr}\, X^{n+1}(z)$, the latter gives $n(n+1)(b^n\partial_a + b^{n-1}d\partial_c)$.

2.2.2 (Sub)Determinant operators

The determinant operator det $Z(u;z)$ is a BRST-closed quasi-primary operator. We will also consider the "subdeterminant" operator, which is a linear combination of quasi-primary operators

$$\mathcal{D}(m;u;z) \equiv \det\,(m + Z(u;z)) \,. \tag{2.25}$$

Different quasi-primary operators in this generating function can be obtained by expanding in powers of m.[10]

For concrete calculations, we will describe a determinant operator with the help of some auxiliary fermionic fields:[11]

$$\det\,(m + Z(u;z)) = \int \mathrm{d}\psi \mathrm{d}\bar{\psi}\, e^{\bar{\psi}(m+Z(u;z))\psi}. \tag{2.26}$$

The fermionic language is also useful to describe modifications of a determinant operator. For example, something like

$$\int \mathrm{d}\psi \mathrm{d}\bar{\psi} : e^{\bar{\psi}X\psi}\bar{\psi}O\psi : \tag{2.27}$$

defines a modification of the $\det X$ operator where one of the "X" symbols has been replaced by O.

As we will review momentarily, the $\bar{\psi}O\psi$ insertion will behave as a boundary insertion in the 't Hooft expansion. As a consequence of the large N combinatorics, these insertions will behave at the leading order in N in a manner similar to single-trace operators: we can place multiple independent insertions in the fermionic integral and the BRST charge will act separately on each at the leading order in N. It is thus possible to define a BRST cohomology of modifications of a specific determinant operator.

In practice, it is not immediately obvious how to write down BRST-closed modifications, even at the leading order in N. Our strategy will be to produce such operators by acting

[10]We can give the whole generating function a good behaviour under global conformal transformations acting on z and $SU(2)_R$ rotations acting on u by allowing m to transform in the same way as $Z(u;z)$ under fractional linear transformations.

[11]These can be interpreted as the D3-D3$'$ open string modes. It is also possible to employ a probe D3"-brane, which is only transverse to the stack of N D3-branes in the B-model directions. This would give rise to bosons and inverse determinants.

with the global symmetry generators on the (sub)determinant operators. For example, a commutator of the form

$$\oint \frac{\mathrm{d}z}{2\pi i} N \operatorname{Tr} Y^{n+1}(z) \det X(0) = \int \mathrm{d}\psi \mathrm{d}\bar{\psi} : e^{\bar{\psi} X(0) \psi} \bar{\psi} Y(0)^n \psi : + \cdots \tag{2.28}$$

gives a BRST-closed operator which at the leading order in N contains a single $X \to Y^n$ modification. This decreases the R-charge by $\frac{n+1}{2}$ and increases the scaling dimension by $\frac{n-1}{2}$. A similar construction with the component of D_{n-1} of minimal R-charge will produce a modification which decreases the R-charge by $\frac{n-1}{2}$ and increases the scaling dimension by $\frac{n+1}{2}$. The fermionic towers of single-trace operators of ghost number ± 1 would similarly give modifications of ghost number ± 1 which increase/decrease the scaling dimension/R-charge by $\frac{n}{2}$.

We conjecture that these four towers of modifications exhaust the BRST-cohomology of modifications of $\det X$. This is a non-trivial statement, which is justified by an index calculation [82] and will match the collection of open string fluctuations available on the holographic dual side.

2.3 Correlation functions of determinants

We will now compute a sphere correlation function of subdeterminant operators

$$\langle \mathcal{D}(m_1; u_1; z_1) \cdots \mathcal{D}(m_k; u_k; z_k) \rangle . \tag{2.29}$$

using the techniques of [38]. We will assume the u_i to be distinct for convenience.

Using the fermionic presentation, this can be written as

$$\left\langle \prod_i \mathcal{D}(m_i; u_i; z_i) \right\rangle = \int \prod_i [\mathrm{d}\psi^i \mathrm{d}\bar{\psi}_i] \left\langle \prod_i e^{\bar{\psi}_i (m_i + Z(u_i; z_i)) \psi^i} \right\rangle \tag{2.30}$$

and evaluated to

$$\left\langle \prod_i \mathcal{D}(m_i; u_i; z_i) \right\rangle = \int [\mathrm{d}\psi \mathrm{d}\bar{\psi}] \, e^{-\frac{1}{2N} \sum_{i \neq j} \frac{u_i - u_j}{z_i - z_j} \bar{\psi}_i \psi^j \bar{\psi}_j \psi^i + \sum_i m_i \bar{\psi}_i \psi^i} . \tag{2.31}$$

We do an Hubbard–Stratonovich transformation, by introducing auxiliary variables ρ^i_j for $i \neq j$. We also define $\rho^i_i = m_i$. We thus write

$$\left\langle \prod_i \mathcal{D}(m_i; u_i; z_i) \right\rangle = \frac{1}{Z_\rho} \int [\mathrm{d}\psi \mathrm{d}\bar{\psi}] [\mathrm{d}\rho] \, e^{\frac{N}{2} \sum_{i \neq j} \frac{z_i - z_j}{u_i - u_j} \rho^i_j \rho^j_i + \sum_{i,j} \rho^i_j \bar{\psi}_i \psi^j} , \tag{2.32}$$

where

$$Z_\rho \equiv \int [\mathrm{d}\rho]\, e^{\frac{N}{2}\sum_{i\neq j}\frac{z_i-z_j}{u_i-u_j}\rho^i_j\rho^j_i}. \tag{2.33}$$

The $\int[\mathrm{d}\rho]$ integral only involves the $k(k-1)$ off-diagonal components of ρ and employs whatever contour makes the Gaussian integral converge.

We can now integrate the fermions away to get the final answer

$$\left\langle \prod_i \mathcal{D}(m_i;u_i;z_i)\right\rangle = \frac{1}{Z_\rho}\int [\mathrm{d}\rho]\, e^{\frac{N}{2}\sum_{i\neq j}\frac{z_i-z_j}{u_i-u_j}\rho^i_j\rho^j_i}\left[\det\rho\right]^N. \tag{2.34}$$

This form is suitable for a large N analysis. We can do a saddle-point evaluation of the ρ integral, with action

$$S[\rho] = \frac{1}{2}\sum_{i\neq j}\frac{z_i-z_j}{u_i-u_j}\rho^i_j\rho^j_i + \log\det\rho. \tag{2.35}$$

The classical saddles satisfy the equations

$$\frac{z_i-z_j}{u_i-u_j}\rho^i_j + [\rho^{-1}]^i_j = 0, \qquad\qquad i\neq j, \tag{2.36}$$

i.e. in matrix form

$$[\zeta,\rho]+[\mu,\rho^{-1}] = 0, \tag{2.37}$$

where we introduced diagonal $k\times k$ matrices ζ and μ with entries z_i and u_i.

From now on, whenever ρ appears outside of an integral it refers to some specific solution of (2.37). Recall that $\rho^i_i = m_i$. It is also useful to define

$$p_i \equiv \frac{\partial S[\rho]}{\partial m_i} = \left[\rho^{-1}\right]^i_i, \tag{2.38}$$

which helps to probe the dependence of $S[\rho]$ on the choice of saddle.

2.3.1 The Spectral Curve

As anticipated in the introduction, the saddle point equations (2.37) guarantee the commutativity of a one-parameter family of $k\times k$ matrices

$$\begin{aligned} B(a) &= a\,\mu - \rho \\ C(a) &= a\,\zeta + \rho^{-1} \\ D(a) &= a\,\zeta\,\mu + \rho^{-1}\mu - \zeta\rho, \end{aligned} \tag{2.39}$$

which satisfy

$$aD(a) - B(a)C(a) = 1. \tag{2.40}$$

This allows us to associate to any saddle ρ a spectral curve $\mathcal{S}_\rho$ in $SL(2,\mathbb{C})$, with coordinates a,b,c,d constrained by

$$ad - bc = 1. \tag{2.41}$$

The spectral curve consists of points (a, b, c, d) such that b,c,d are simultaneous eigenvalues of $B(a)$, $C(a)$, $D(a)$.

The spectral curve is non-compact: as $a \to \infty$ we find k branches where b, c and d grow linearly with a. More precisely, we can expand

$$\begin{aligned} \frac{b}{a} &= u_i - \frac{m_i}{a} + \cdots \\ \frac{c}{a} &= z_i + \frac{p_i}{a} + \cdots . \end{aligned} \tag{2.42}$$

We would like to compare these boundary conditions with the holographic boundary conditions expected for a dual D1-brane.

Recall that the holographic boundary conditions of [31] control the behaviour of the B-model fields as (a, b, c, d) go to infinity with fixed ratios. Single-trace operator insertions at some point z control the behaviour of the fields at fixed $\frac{c}{a} = z$. Analogously, we expect the insertion of (sub)determinant operators to correspond to the presence of a D1-brane which asymptotically lies in the $\frac{c}{a} = z$ locus.

2.3.2 The asymptotic shape of Giant Gravitons

The near-horizon analysis of [31] starts with a stack of N D1-branes in $\mathbb{C}^3$. If we use coordinates (x, y, z) on $\mathbb{C}^3$, they sit at $x = y = 0$. We can take the probe D1$'$-brane to wrap the transverse line defined $m_0 + x + u_0 y = 0$ and $z = z_0$ to engineer a determinant operator $\mathcal{D}(m_0; u_0; z_0)$.[12]

The backreaction of the D1-branes deforms the complex structure of the ambient manifold. The x, y coordinates remain holomorphic. After a rescaling by a factor of $N^{\frac{1}{2}}$ they will be identified with b and $-a$ in $SL(2,\mathbb{C})$. The z coordinate is not holomorphic anymore,

[12] As a check, notice that if we displace the D1-branes to $x = X$ and $y = Y$, the combination $m_0 + X + u_0 Y$ will have a zero eigenvalue whenever the D1$'$-brane intersects one of the D1-branes and the action of D1$-$D1$'$ strings correspondingly vanishes.

but (xz, yz) are deformed and rescaled by a factor of $N^{\frac{1}{2}}$ to give holomorphic coordinates d and $-c$ in $SL(2,\mathbb{C})$.

The asymptotic shape of the probe D1′-brane for large (a, b, c, d) is thus predicted to be

$$\begin{aligned} \frac{b}{a} &= u_0 - \frac{m_i}{a} + \cdots \\ \frac{c}{a} &= z_0 + \cdots . \end{aligned} \tag{2.43}$$

We conclude that a D1-brane $\mathcal{B}_\rho$ supported on the spectral curve $\mathcal{S}_\rho$ has the correct shape at large (a, b, c, d) to satisfy the holographic boundary condition for the insertion of the k determinant operators $\mathcal{D}(m_1; u_1; z_1) \cdots \mathcal{D}(m_k; u_k; z_k)$.

2.3.3 The asymptotic phase space and the holographic action

The world-volume theory of a single D1-brane is just a $\beta\gamma$ system where the two fields are valued in the normal bundle to the worldsheet. The combination $\beta\bar{\partial}\gamma$ is a $(0,1)$ form valued in the second exterior power of the normal bundle, which can be contracted with Ω to give a $(1,1)$ form which can be integrated over the curve to give the action of the $\beta\gamma$ system.

Locally, we can parameterize the curve by the a coordinate. We parameterize the normal fluctuations as fluctuations β of b and γ of c. Using $\Omega = \frac{\mathrm{d}a\,\mathrm{d}b\,\mathrm{d}c}{a}$ we get a standard action

$$\int_{\mathcal{S}} \beta\bar{\partial}\gamma \wedge \frac{\mathrm{d}a}{a}. \tag{2.44}$$

We find the classical phase space by expanding β and γ into powers of a:

$$\begin{aligned} \beta(a) &= \sum_n \beta_n a^n \\ \gamma(a) &= \sum_n \gamma_n a^n. \end{aligned} \tag{2.45}$$

The Poisson brackets are

$$\{\beta_n, \gamma_m\} = \delta_{n,-m} \qquad\qquad \{\beta_n, \beta_m\} = \{\gamma_n, \gamma_m\} = 0. \tag{2.46}$$

As $a \to \infty$, the modes with positive n are non-normalizable and are fixed by the holographic boundary conditions. The mode β_0 is also fixed, as it represents a change of m_i.

The conjugate mode γ_0 and the normalizable modes with negative n are left free by the boundary conditions.

Given a solution of the classical equations of motion, the value of γ_0 will be the derivative of the classical action with respect to the fixed value of its conjugate variable β_0. Similarly, the modes with negative n give the derivative of the classical action with respect to a change in the fixed values of the modes with positive n.[13]

Applying this reasoning to each asymptotic end of the spectral curve, we learn that $\gamma_0 = p_i$ at the i-th end represents the derivative of the brane action with respect to m_i. But p_i also equals the derivative of the saddle action with respect to m_i. We thus get a full match at the leading order in N between the action for a saddle ρ of the (sub)determinant correlation function and the action of the holographic dual B-model brane $\mathcal{B}_\rho$, up to a function of z_i and u_i only.

It is not too hard to match the z_i and u_i dependence of the actions as well, though it is a bit more cumbersome. For example, the z_i derivative of the saddle action is

$$\sum_{j\neq i} \frac{1}{u_i - u_j} \rho^i_j \rho^j_i. \tag{2.47}$$

We need to compare this with the a^{-1} term in the large a expansion of $-b$. This is straightforward: the large a expansion of b is given by a systematic diagonalization of the matrix $-\mu + a^{-1}\rho$, which is just the non-degenerate perturbation theory in quantum mechanics. The u_i play the role of unperturbed energies and the ρ^i_j are the matrix elements of the perturbation. The expression above is the familiar second-order correction to the eigenvalues.

We can match the u_i derivatives in the same manner:

$$-\sum_{j\neq i} \frac{z_i - z_j}{(u_i - u_j)^2} \rho^i_j \rho^j_i = -\sum_{j\neq i} \frac{1}{z_i - z_j} [\rho^{-1}]^i_j [\rho^{-1}]^j_i \tag{2.48}$$

matches with the a^{-1} term in the large a expansion of c from the systematic diagonalization of $\zeta + a^{-1}\rho$.

[13] This is perfectly analogous to the statement that the classical mechanics action $S[x_i, x_f]$ evaluated on a solution with fixed initial and final positions x_i, x_f determines the initial and final momenta as $p_f = \frac{\partial S}{\partial x_f}$ and $p_i = -\frac{\partial S}{\partial x_i}$.

2.3.4 D-branes and spectral curves

Observe that D-branes wrapping a spectral curve are a rather common occurrence in the B-model. A natural way to produce a complicated D-brane is to start with multiple copies of a simpler D-brane and turn on matrix-valued vevs for the transverse deformations of the stack. Essentially, the world-volume theory on a stack of k coincident D1-branes is a $\mathfrak{u}(k)$-gauged $\beta\gamma$ system valued in the adjoint representation. A classical solution gives commuting holomorphic vevs to β and γ.

Probing the system with another B-brane shows that the deformed stack of branes behaves as a single brane wrapping a spectral curve with transverse position given by the eigenvalues of β and γ. The Chan-Paton bundle of the resulting D-brane is controlled by a natural line bundle on the spectral curve: the line bundle whose fiber consists of the common eigenvectors of β and γ with these eigenvalues.

In the case at hand, it is natural to identify $B(a)$ and $C(a)$ with the matrix-valued vevs of fields $\beta(a)$ and $\gamma(a)$ describing the transverse positions of a stack of k D1-branes which originally sit at $b = c = 0$.[14] We will verify at the end of the Section that the appropriate Chan-Paton bundle on $\mathcal{B}_\rho$ is indeed controlled by the eigenline bundle $\mathcal{L}_\rho$.

2.3.5 Factorization and global symmetry

The saddle solutions come automatically in $(\mathbb{C}^*)^k$ families, as we can conjugate ρ by a diagonal invertible matrix to get a new solution. The spectral curve $\mathcal{S}_\rho$ is the same for all saddles in these continuous families. If we track back the origin of this symmetry, we see that it originates from a symmetry under rescaling of $\bar{\psi}_i$ and ψ^i in opposite directions. We can interpret this as the remnant of the $U(1)$ gauge symmetry on the bulk of the probe D1′ worldsheet, which becomes a $U(1)$ global symmetry at the boundary points according to a standard holographic dictionary.

We can consider saddle solutions such that ρ is block-diagonal, with blocks ρ_a of size k_a. Then the ρ_a satisfy the saddle equations for the corresponding collection of k_a determinant operators. The spectral curve decomposes correspondingly to the collection of $\mathcal{S}_{\rho_a}$, which are typically disjoint. This describes a configuration of disconnected D1-branes.

[14] An alternative approach with equivalent results would be to describe a D1-brane as the result of tachyon condensation on a stack of space-filling D5-branes and anti-D5-branes, i.e. as a complex of sheaves $\mathbb{C}^k \to \mathbb{C}^{2k} \to \mathbb{C}^k$. One can build an appropriate differential out of commuting linear operators $b - B(a)$ and $c - C(a)$.

Conversely, "connected" holographic saddles associated to a single smooth semiclassical D1-brane correspond to "irreducible" solutions for which ρ is not block diagonal.

Recall that the chiral algebra correlation functions are covariant under both $SL(2)_L$ global conformal transformations and $SL(2)_R$ complexified R-symmetry transformations. Our definition of the spectral curve, though, seems to treat the a coordinate in a different way than the b, c and d coordinates. We will now show that the spectral curve transforms correctly under the action of $SL(2)_L \times SL(2)_R$ on $SL(2,\mathbb{C})$.

The $Z(u;z)$ field transforms non-trivially under the action of $SL(2)_L \times SL(2)_R$:

$$Z(u;z) = \frac{\gamma u+\delta}{\gamma' z+\delta'} Z\left(\frac{\alpha u+\beta}{\gamma u+\delta}; \frac{\alpha' z+\beta'}{\gamma' z+\delta'}\right). \tag{2.49}$$

We will thus also transform m_i in the same manner so that the subdeterminants transform multiplicatively:

$$\mathcal{D}(m;u;z) = \left(\frac{\gamma u+\delta}{\gamma' z+\delta'}\right)^N \mathcal{D}\left(\frac{\gamma' z+\delta'}{\gamma u+\delta} m; \frac{\alpha u+\beta}{\gamma u+\delta}; \frac{\alpha' z+\beta'}{\gamma' z+\delta'}\right). \tag{2.50}$$

Following through the derivation, the auxiliary ρ variables transform as

$$\rho \to \rho \frac{\gamma' \zeta+\delta'}{\gamma \mu+\delta}. \tag{2.51}$$

Notice that there is a bit of latitude in the definition of the action of $SL(2)_L \times SL(2)_R$, as we can combine the above formula with conjugation by any diagonal matrix.

Translations $\mu \to \mu+\beta$ or $\zeta \to \zeta+\beta'$ clearly act on the spectral curve as shifts $b \to b+\beta a$, $d \to d+\beta c$ or $c \to c+\beta' a$, $d \to d+\beta' b$. We thus only need to consider the action of inversions $\mu \to -\mu^{-1}$ or $\zeta \to -\zeta^{-1}$, as inversions and translations generate $SL(2)_L \times SL(2)_R$. We can show the details for $\mu \to -\mu^{-1}$, as the analysis for $\zeta \to -\zeta^{-1}$ is essentially identical.

Observe that an eigenvector s of $B(a)$, etc. is annihilated by

$$\begin{aligned}
&a\,\mu - \rho - b \\
&a\,\zeta + \rho^{-1} - c \\
&a\,\zeta\,\mu + \rho^{-1}\mu - \zeta\rho - d.
\end{aligned} \tag{2.52}$$

Then μs is annihilated by

$$\begin{aligned}
&b\,(-\mu^{-1}) - \rho\mu^{-1} + a \\
&a\mu^{-1}\,\zeta + \rho^{-1}\mu^{-1} - c\mu^{-1} \\
&a\,\zeta + \rho^{-1} - \zeta\rho\mu^{-1} - d\mu^{-1},
\end{aligned} \tag{2.53}$$

which is the same as being annihilated by

$$\begin{aligned} & b\,(-\mu^{-1}) - (\rho\mu^{-1}) + a \\ & b\,\zeta + (\rho\mu^{-1})^{-1} - d \\ & b\,\zeta(-\mu^{-1}) + (\rho\mu^{-1})^{-1}(-\mu^{-1}) - \zeta(\rho\mu^{-1}) + c. \end{aligned} \tag{2.54}$$

We thus find that $-a$,d,$-c$ are simultaneous eigenvalues of

$$\begin{aligned} & b\,(-\mu^{-1}) - (\rho\mu^{-1}) \\ & b\,\zeta + (\rho\mu^{-1})^{-1} \\ & b\,\zeta(-\mu^{-1}) + (\rho\mu^{-1})^{-1}(-\mu^{-1}) - \zeta(\rho\mu^{-1}), \end{aligned} \tag{2.55}$$

i.e. the spectral curve transforms under inversion $\mu \to -\mu^{-1}$, $\rho \to \rho\mu^{-1}$ as: $a \to b, b \to -a, c \to d, d \to -c$ as desired.

2.3.6 Off-diagonal fluctuations

Consider now a reducible saddle solution with two blocks ρ and ρ'. We may ask under which conditions on ρ and ρ' the reducible saddle admits infinitesimal block off-diagonal deformations.

Block off-diagonal fluctuations η of the saddle solutions[15] satisfy a complicated-looking matrix equation:

$$\zeta\eta - \eta\zeta' - \mu\rho^{-1}\eta(\rho')^{-1} + \rho^{-1}\eta(\rho')^{-1}\mu' = 0\,. \tag{2.56}$$

We will now show that one can find a zeromode η for every intersection point of $\mathcal{S}_\rho$ and $\mathcal{S}_{\rho'}$. The zeromode is built from the right eigenvector s for $B(a)$, $C(a)$, $D(a)$ and the corresponding left eigenvector s' for $B'(a)$, $C'(a)$, $D'(a)$. Let us take η to be proportional to $\rho s \otimes s' + s \otimes s'\rho'$. Then, one can rewrite the left-hand side of (2.56) in the following way[16]

$$(c\mu - d)s \otimes s' + \zeta s \otimes s'(a\mu' - b) - (a\mu - b)s \otimes s' - s \otimes s'(a\mu' - b)\zeta' \tag{2.57}$$

$$-\mu s \otimes s'(c - a\zeta') - \mu(c - a\zeta)s \otimes s' + s \otimes s'(d - b\zeta') + (c - a\zeta)s \otimes s'\mu'\,, \tag{2.58}$$

where b, c, d are the eigenvalues for s as well as s'. After cancellations, we can arrange the remaining terms as

$$s \otimes s'\big[b\zeta' + (c - a\zeta')\mu'\big] - \big[c\mu + \zeta(b - a\mu)\big]\, s \otimes s'\,. \tag{2.59}$$

[15]We assume the saddle is of the form $\left(\begin{smallmatrix} \rho & \eta \\ 0 & \rho' \end{smallmatrix}\right)$.

[16]We thank Adrián López for pointing out a mistake here in the previous version.

Both terms are equal to $ds \otimes s'$ so they cancel.

We would like to employ this observation to compare the Chan-Paton bundle of the brane B_ρ dual to some saddle ρ with the canonical line bundle $\mathcal{L}_\rho$ present on any spectral curve, whose fiber is the common eigenline to $B(a)$, $C(a)$, $D(a)$.

In order to "see" the Chan-Paton line bundle of a brane $\mathcal{B}_\rho$ we need to consider open strings stretched between $\mathcal{B}_\rho$ and some other brane. We may employ another $\mathcal{B}_{\rho'}$ for this purpose, though B-model open strings will only appear if $\mathcal{B}_\rho$ and $\mathcal{B}_{\rho'}$ intersect at some point. As we deform $\mathcal{B}_{\rho'}$ and move the intersection point, the open strings will transform as a section of the Chan-Paton line bundle of $\mathcal{B}_\rho$, and viceversa.

These open string modes are boundary local operators in the topological string worldsheet and generate the infinitesimal deformation of the two intersecting branes into a single smooth curve. It is thus natural to identify them with the solutions η of (2.56) associated to the intersection point, which indeed transform as sections of $\mathcal{L}_\rho$ as we vary the intersection point.[17]

This calculation also offers a subleading check of the holographic duality. In the presence of an off-diagonal zeromode, the semiclassical contribution of the block-diagonal saddle to the correlation function will diverge. This divergence arises from the exponentiation of an annulus diagram. On the holographic dual side, this diagram should correspond to the propagation of a closed string from $\mathcal{B}_\rho$ to $\mathcal{B}_{\rho'}$. It is easy to verify with a local calculation that this diagram in the B-model diverges in the correct manner when the two D-brane worldsheets intersect each other.

2.3.7 The inverse problem

Notice that our identification between semiclassical saddles goes only in one direction: we have built $\mathcal{B}_\rho$ from ρ, but we have not yet demonstrated that every B-model brane $\mathcal{B}$ which satisfies the correct boundary conditions can be produced from some ρ.[18]

Often, spectral curve constructions can be inverted by a pushforward operation. Here we could try to foliate $SL(2,\mathbb{C})$ by surfaces of constant a and pushforward the sheaf defining

[17] The reader may be confused by the fact that the open string modes are fermionic, whereas the deformation modes are bosonic. This is actually natural: the deformation involves a descent relation, so the deformation parameters multiply the first descendant of the fermionic operators, which is bosonic.

[18] Considering the embedding of the chiral algebra calculation in the $\mathcal{N}=4$ SYM physical theory, it is not actually obvious that this should be the case: the physical brane configurations involve a supersymmetric D3-brane in $\mathrm{AdS}_5 \times S^5$ and a B-model brane $\mathcal{B}$ in $SL(2,\mathbb{C})$ could potentially fail to lift to such a D3-brane.

$\mathcal{B}$ along these surfaces. Concretely, that means that for generic a we would consider the intersection points of the support $\mathcal{S}$ of $\mathcal{B}$ with the constant a surface and add up the fibers of the CP bundle $\mathcal{L}$ at these points. At branch points where two intersection points collide one needs to work a bit harder, but the basic idea is to produce a sheaf on the a plane whose local sections in any open set U are the same as the collection of local sections of $\mathcal{L}$ in the preimage of U under the projection to the a plane.

Because of the boundary conditions on $\mathcal{B}$, the intersection points cannot move to infinity as we vary a. The number of intersection points will thus be generically fixed. Let us denote it by k. The direct sum of the fibers at intersection points will thus give a rank k bundle. Commuting $k \times k$ matrices $B(a)$ and $C(a)$ could be defined as representing the multiplication action by b and c on the sections of $\mathcal{L}$. A possible subtlety here is that the surfaces for generic a are $\mathbb{C}^2$ parameterized by b and c, but the fiber $\mathbb{C} \times \mathbb{C}^*$ at $a = 0$ restricts $bc = 1$ and leaves d unconstrained. We expect that to simply impose the constraint $B(0)C(0) = 0$.

We can trivialize the vector bundle at finite a and write $B(a)$, $C(a)$ globally on the a plane as polynomial matrices. We know that the eigenvalues of $B(a)$, $C(a)$ grow at infinity as $u_i a$ and $z_i a$. Unfortunately, this is not enough to immediately conclude that $B(a)$, $C(a)$ themselves should grow linearly at infinity. If it did, we would be done, as $B(a)$, $C(a)$ would have to take the form (2.39).

If we work with a single matrix $B(a)$, ignoring $C(a)$, we can easily produce counter-examples. The simplest one would be to write a matrix $B(a)$ with eigenvectors $(1, b, b^2, \dots, b^{k-1})$ for an eigenvalue b. Such a matrix would have all elements equal to 0, 1 or to the coefficients of the characteristic polynomial of $B(a)$. It would be of degree k in a. In general, we can look for matrices such that the (i, j) entry has degree up to $n_i - n_j + 1$ for some collection of integers n_i. We can produce some examples of this sort, but we do not have a general understanding of the situation. We leave it to future work.

Instead, we will apply the inverse map to the special example of curves of genus 0. Parameterize the curve by a global coordinate t and denote as t_i the location of the points which are mapped to infinity in $SL(2,\mathbb{C})$. We can parameterize

$$
\begin{aligned}
a &= a_\infty + \sum_i \frac{a_i}{t-t_i} \\
b &= b_\infty + \sum_i \frac{u_i a_i}{t-t_i} \\
c &= c_\infty + \sum_i \frac{z_i a_i}{t-t_i} \\
d &= d_\infty + \sum_i \frac{u_i z_i a_i}{t-t_i}\,.
\end{aligned} \tag{2.60}
$$

We can compute

$$
ad-bc = a_\infty d_\infty - b_\infty c_\infty + \sum_i \frac{a_i}{t-t_i}\left[u_i z_i a_\infty - z_i b_\infty - u_i c_\infty + d_\infty + \sum_{j\neq i} a_j \frac{(u_i-u_j)(z_i-z_j)}{t_i-t_j}\right]. \tag{2.61}
$$

Therefore, we have constraints $a_\infty d_\infty - b_\infty c_\infty = 1$ and

$$
u_i z_i a_\infty - z_i b_\infty - u_i c_\infty + d_\infty + \sum_{j\neq i} a_j \frac{(u_i-u_j)(z_i-z_j)}{t_i-t_j} = 0\,. \tag{2.62}
$$

We can generically solve these k linear equations for the a_i in terms of a_∞, b_∞, c_∞, d_∞ and of the t_i.

Notice that we can act with fractional linear redefinitions of t. These will act on the t_i in the obvious way, but will also shift a_∞, etc. by multiples of the a_i. Using the solution for the a_i, one gets an intricate action on a_∞, b_∞, c_∞, d_∞. At this point we thus have a k-dimensional space of curves which is the quotient of a $(k+3)$-dimensional space by $SL(2,\mathbb{C})$. The point $(a_\infty, b_\infty, c_\infty, d_\infty)$ is the image of $t=\infty$, so under the reparameterization of t the point $(a_\infty, b_\infty, c_\infty, d_\infty)$ will move along the curve. We cannot use this $SL(2,\mathbb{C})$ to eliminate the a_∞, b_∞, c_∞, d_∞ parameters. We can use it instead to fix the values of 3 of the t_i.

We also find

$$
\begin{aligned}
m_i &= u_i a_\infty - b_\infty + \sum_{j\neq i} \frac{u_i-u_j}{t_i-t_j} a_j \\
p_i &= c_\infty - z_i a_\infty - \sum_{j\neq i} \frac{z_i-z_j}{t_i-t_j} a_j.
\end{aligned} \tag{2.63}
$$

Solving for the a_∞, b_∞, c_∞, d_∞ and t_i as a function of m_i seems hard. Instead, we can content ourselves with parameterizing the solutions by these parameters.

Next, we will find an associated saddle ρ. We use the following trick. Consider the k-dimensional vector s with components $\frac{1}{t-t_i}$ as a function on the curve.[19] We will now find a ρ such that this is the common eigenvector of $B(a(t))$ and $C(a(t))$. Indeed, consider the vector $(a(t)\mu - b(t))s$. As $a(t)u_i - b(t)$ is regular at $t = t_i$, the i-th entry of the vector has a simple pole at each t_j, including t_i. As $t \to \infty$, the vector goes to 0. That means we can express uniquely each entry of $(a(t)\mu - b(t))s$ as a linear combination of $\frac{1}{t-t_i}$. In other words,

$$(a(t)\mu - b(t))s = \rho s, \tag{2.64}$$

where $\rho^i_i = m_i$ and ρ^i_j is computed as a residue, i.e.

$$\left[u_i a_\infty - b_\infty + \sum_{j\neq i} \frac{u_i - u_j}{t - t_j} a_j\right] \frac{1}{t - t_i} \tag{2.65}$$

has residue at $t = t_j$:

$$\rho^i_j = -\frac{u_i - u_j}{t_i - t_j} a_j. \tag{2.66}$$

Similarly, we find $(\rho^{-1})^i_i = p_i$ and

$$(\rho^{-1})^i_j = \frac{z_i - z_j}{t_i - t_j} a_j. \tag{2.67}$$

We have thus found a natural "genus 0" k-dimensional family of saddles ρ parametrized by the t_i and a_∞, b_∞, c_∞, d_∞ with $a_\infty d_\infty - b_\infty c_\infty = 1$ modulo the $SL(2,\mathbb{C})$ action. It would be nice to know if this exhausts all saddles, or if higher genus saddles may exist.

2.4 Correlation functions of determinants and traces

In this Section we consider correlation functions which include both a collection of (sub)determinant operators and a collection of single-trace operators. In the large N expansion, these correlation functions will be controlled by the same large N saddles as we found for the

[19] This s has a zero at $t = \infty$. The construction would work equally well if we rescale the components to $\frac{t-t_0}{t-t_i}$, placing the zero at some other point t_0. Even better, we should think about s as a non-zero section of the spin bundle $K^{\frac{1}{2}}$ on the genus 0 curve.

correlation function of subdeterminant operators: the single-trace insertions can at most produce some polynomial in N, which cannot compete with the exponentials associated to the determinants.

We would like to show how the expansion around each large N saddle takes the form of a standard 't Hooft expansion. We will begin by a small exercise in normal ordering which will facilitate the analysis. Consider the combination of all (sub)determinants which will appear in the correlation function, express it in a fermionic presentation and normal-order it with respect to the Z contractions:

$$\prod_i \mathcal{D}(m_i;u_i;z_i) = \int \prod_j [\mathrm{d}\psi^j \mathrm{d}\bar{\psi}_j] : e^{-\frac{1}{2N}\sum_{i\neq j}\frac{u_i-u_j}{z_i-z_j}\bar{\psi}_i\psi^j\bar{\psi}_j\psi^i+\sum_i \bar{\psi}_i(m_i+Z(u_i;z_i))\psi^i} :_Z . \tag{2.68}$$

This means that when we evaluate correlation function containing the expression on the right, we do not do Wick contractions between the Z's in the exponent.

We can now apply the Hubbard–Stratonovich transformation

$$\prod_i \mathcal{D}(m_i;u_i;z_i) = Z_\rho^{-1}\int [\mathrm{d}\psi \mathrm{d}\bar{\psi}][\mathrm{d}\rho] : e^{\frac{N}{2}\sum_{i\neq j}\frac{z_i-z_j}{u_i-u_j}\rho^i_j\rho^j_i+\sum_{i,j}\rho^i_j\bar{\psi}_i\psi^j+\sum_i\bar{\psi}_i Z(u_i;z_i)\psi^i} :_Z . \tag{2.69}$$

The normal order operation allowed us to introduce the auxiliary ρ variables before integrating away the Z fields. We can then redefine $\rho \to \rho + \eta$ where η is dynamical and ρ is a background value which we can take to be any solution of the saddle equations (2.37), to insure the vanishing of the η tadpole at the leading order in N.[20]

We are now ready for a diagrammatic analysis of the Feynman diagrams which arise from the Z, ψ, $\bar{\psi}$ and η Wick contractions. We can use a standard ribbon graph picture: the Z propagators are represented by ribbons with two colour lines, the fermion propagators by ribbons with one colour and one flavour line, the η propagators by ribbons with two flavour lines. The only constraint is that no Z propagator can connect two $\bar{\psi}Z\psi$ vertices. Standard large N 't Hooft combinatorics apply to these ribbon graphs, so that the power of N is controlled by the topology of the surface which we obtain from the ribbon graph by filling in the closed colour lines: each connected component contributes N^{2-2g-b}, where g is the genus and b the number of closed flavour lines, which behave as boundaries of the surface. Determinant modifications can be readily included in the analysis. They behave as extra "boundary" vertices for the ribbon graphs.

Our main conjecture is that the large N expansion of correlators around a saddle ρ is holographically dual to a B-model calculation in the presence of a D-brane $\mathcal{B}_\rho$, order by order in the 't Hooft expansion. We will do various tests of this conjecture.

[20]At higher order in the N^{-1} expansion it may be useful to adjust ρ by subleading corrections.

2.4.1 Determinants and a single-trace

As our main example, consider the correlation function

$$\langle \mathcal{D}(m_1; u_1; z_1) \cdots \mathcal{D}(m_k; u_k; z_k) N \operatorname{Tr} Z(u; z)^n \rangle. \tag{2.70}$$

For simplicity, we assume that $z \neq z_i$ and $u \neq u_i$.

For illustrative purposes, we will evaluate it in a standard way [38], without the intermediate normal order step. We start from

$$\left\langle \prod_i \mathcal{D}(m_i; u_i; z_i) N \operatorname{Tr} Z(u; z)^n \right\rangle = \left\langle \int \prod_i [\mathrm{d}\psi^i \mathrm{d}\bar{\psi}_i] e^{\bar{\psi}_i (m_i + Z(u_i; z_i))\psi^i} N \operatorname{Tr} Z(u; z)^n \right\rangle, \tag{2.71}$$

evaluate the Z contractions

$$\int [\mathrm{d}\psi \mathrm{d}\bar{\psi}]\, e^{-\frac{1}{2N}\sum_{i\neq j} \frac{u_i - u_j}{z_i - z_j} \bar{\psi}_i \psi^j \bar{\psi}_j \psi^i + \sum_i m_i \bar{\psi}_i \psi^i} N \operatorname{Tr}\left(-\frac{1}{N} \sum_i \psi^i \frac{u_i - u}{z_i - z} \bar{\psi}_i \right)^n, \tag{2.72}$$

and reorganize the trace

$$-\int [\mathrm{d}\psi \mathrm{d}\bar{\psi}]\, e^{-\frac{1}{2N}\sum_{i\neq j} \frac{u_i - u_j}{z_i - z_j} \bar{\psi}_i \psi^j \bar{\psi}_j \psi^i + \sum_i m_i \bar{\psi}_i \psi^i} N \operatorname{Tr}_{k\times k}\left(-\frac{1}{N} \bar{\psi}_j \psi^i \frac{u_i - u}{z_i - z} \right)^n. \tag{2.73}$$

Here the quantity in parenthesis is treated as a $k \times k$ matrix with indices i and j.

We introduce the auxiliary ρ variables[21]

$$-Z_\rho^{-1} \int [\mathrm{d}\psi \mathrm{d}\bar{\psi}][\mathrm{d}\rho]\, e^{\frac{N}{2}\sum_{i\neq j} \frac{z_i - z_j}{u_i - u_j} \rho^i_j \rho^j_i + \sum_{i,j} \rho^i_j \bar{\psi}_i \psi^j} N \operatorname{Tr}_{k\times k}\left(-\frac{1}{N} \bar{\psi}_j \psi^i \frac{u_i - u}{z_i - z} \right)^n. \tag{2.74}$$

As we execute the fermion Gaussian integral, the maximal power of N will arise from contractions between each consecutive $\bar{\psi}_j \psi^i$ pair, giving an extra factor of N for each. We thus obtain the leading N answer

$$-Z_\rho^{-1} \int [\mathrm{d}\rho]\, e^{\frac{N}{2}\sum_{i\neq j} \frac{z_i - z_j}{u_i - u_j} \rho^i_j \rho^j_i} [\det \rho]^N N \operatorname{Tr}_{k\times k}\left(-\rho^{-1} \frac{\mu - u}{\zeta - z} \right)^n. \tag{2.75}$$

Evaluating this on some saddle ρ, the correlation function equals the "bare" correlation function of (sub)determinants $\langle \mathcal{D}(m_1; u_1; z_1) \cdots \mathcal{D}(m_k; u_k; z_k) \rangle$ times

$$-N \operatorname{Tr}_{k\times k}\left(-\rho^{-1} \frac{\mu - u}{\zeta - z} \right)^n \tag{2.76}$$

[21] As before, we integrate over the $k(k-1)$ off-diagonal components of ρ and $\rho^i_i = m_i$.

evaluated on the saddle.

If we use the normal order trick, the diagrammatic interpretation of this answer is very straightforward. The leading ribbon graph is a disk diagram with the topology of a wheel, with the n spokes being Z propagators. Each Z propagator gives one factor of $\frac{\mu-u}{\zeta-z}$ and each fermionic propagator gives a $-\rho^{-1}$ factor. The closed flavour line gives rise to the $\mathrm{Tr}_{k\times k}$. The overall factor of N appears because this is a disk diagram.

2.4.2 Geometric interpretation

The disk one-point function (2.76) can be computed in terms of the eigenvalues of the $k \times k$ matrix

$$R(u;z) \equiv -\frac{1}{\zeta - z}\rho^{-1}(\mu - u). \tag{2.77}$$

We will now express this answer in the language of the spectral curve. We can write

$$R(u;z) = -\frac{1}{\zeta - z}\left(D(a) - u\,C(a) - \zeta B(a) + \zeta u a\right) \tag{2.78}$$

for every a. Consider now the special values of a where the spectral curve intersects the surface $\Delta(u;z)$ defined by $d - uc - zb + uza = 0$. There are k such values: the solutions of the equation

$$\det\left[D(a) - uC(a) - zB(a) + uza\right] = 0\,. \tag{2.79}$$

The corresponding null eigenvectors are annihilated by $D(a) - uC(a) - zB(a) + uza$. That means

$$R(u;z) = b - ua \tag{2.80}$$

when acting on these eigenvectors. This means that the eigenvectors of $B(a)$, etc. at the intersection points are also eigenvectors of $R(u;z)$ with eigenvalues $b - ua$.

As long as these k eigenvectors are distinct, we can thus write

$$\mathrm{Tr}_{k\times k}\, R(u;z)^n = \int_{S_\rho} (b - ua)^n\, \delta_{\Delta(u;z)}\,, \tag{2.81}$$

where $\delta_{\Delta(u;z)}$ is a delta function supported on $\Delta(u;z)$. Indeed, the integral on the right localizes to the intersection points and gives the sum of the corresponding eigenvalues of $R(u;z)^n$. We expect this to happen for generic values of u and z. By continuity, this relation must be true even when the eigenvectors are not distinct.

2.4.3 A holographic match

Holographically, this disk 1-point function corresponds to a process where a closed string propagates from the boundary insertion corresponding to $N\,\mathrm{Tr}\,Z(u;z)^n$ to the B_ρ brane dual to the saddle. This would evaluate to[22]

$$\int_{S_\rho} \partial^{-1}\alpha_{n;u;z}, \tag{2.82}$$

where $\alpha_{n;u;z}$ is the Kodaira-Spencer field sourced by the boundary insertion.

This matches the chiral algebra calculation if we can identify

$$\alpha_{n;u;z} = \partial\left[(b-ua)^n\delta_{\Delta(u;z)}\right]. \tag{2.83}$$

This is indeed the case. The equation defining $\Delta(u;z)$ can also be written as

$$\frac{1}{a}\left(1+(b-ua)(c-za)\right)=0. \tag{2.84}$$

As we approach the boundary, the distribution is approximately supported on the two loci $\frac{c}{a}=z$ and $\frac{b}{a}=u$. We thus write

$$\alpha_{n;u;z} \sim \partial\left[(b-ua)^n\delta_{\frac{c}{a}=z}+(za-c)^{-n}\delta_{\frac{b}{a}=u}\right]. \tag{2.85}$$

The first term has a power law growth towards the boundary and is supported at the point z of the holographic boundary. It has precisely the correct quantum numbers to represent the insertion of $N\,\mathrm{Tr}\,Z(u;z)^n$ at the boundary. In particular, it is a polynomial of degree n in u, as expected.

The second term has a power law decay and represents the boundary behaviour of the field sourced by the $N\,\mathrm{Tr}\,Z(u;z)^n$ insertion. It would enter, say, in the calculation of a two-point function of $N\,\mathrm{Tr}\,Z(u;z)^n$ and another operator $N\,\mathrm{Tr}\,Z(u';z')^n$ at position $z'=\frac{c}{a}$.

We have thus matched the large N expectation value of $A_n(u;z)$ in a saddle ρ for a correlator of (sub)determinants to a B-model calculation of the same quantity in presence of the D1-brane $\mathcal{B}_\rho$.

[22] See appendix F of [31] for a review of the coupling of KS fields to D1-branes.

2.4.4 From single-trace operators to modifications

Next, consider the insertion of the global symmetry generator

$$a_n(v;w) = \oint \frac{\mathrm{d}z}{2\pi i} (z-w)^{n-2} A_n(v;z). \tag{2.86}$$

As we vary z along a closed loop, the codimension 2 loci $\Delta(v;z)$ sweep a codimension 1 locus $E(v)$, which intersects the spectral curve along some collection of loops. On $\Delta(v;z)$ we have $z = \frac{d-vc}{b-va}$. At the leading order in N in the saddle ρ, the insertion of $a_n(v;w)$ will thus produce a contour integral on these loops on $\mathcal{S}_\rho$:

$$\oint \frac{1}{2\pi i} \mathrm{d}\frac{d-vc}{b-va} \left(\frac{d-vc}{b-va} - w\right)^{n-2} (b-va)^n = n \oint \frac{1}{2\pi i} \omega_{J_n}(v;w). \tag{2.87}$$

This is another remarkable test of the holographic correspondence. Indeed, the action of a symmetry generator in the B-model is implemented by cutting $SL(2,\mathbb{C})$ along a codimension 1 locus E and gluing it back under the action of the corresponding holomorphic vector field V. Equivalently, it corresponds to turning on a vev for the Kodaira-Spencer field α which is supported on E proportional to $i_V\Omega$:

$$\alpha = i_V \Omega \delta_E. \tag{2.88}$$

Recall that a D1-brane couples by $\partial^{-1}\alpha$. If $i_V\Omega = d\omega_V$, we can take

$$\partial^{-1}\alpha = \omega_V \delta_E. \tag{2.89}$$

We have thus verified that the action of $a_n(v;w)$ on the chiral algebra side matches the action of $nJ_n(v;w)$ on the B-model side: the effect of the $J_n(v;w)$ vector fields on a D-brane supported on the spectral curve is precisely (2.87)!

Somewhat implicitly, this check also verifies the statement that the action of the global symmetry algebra onto a determinant operator produces modifications which are BRST-equivalent if the corresponding vector fields produce the same non-normalizable shape deformation in the dual D-brane. Indeed, it shows that the action of the symmetry generators is completely captured by the action of the corresponding vector fields onto the spectral curve.

In the next Section we will do a more direct test of this statement.

2.5 Two-point functions of modified determinants

In this Section we will focus on correlation functions in the presence of two determinants: $\det Y(\infty)$ and $\det X(0)$. We will employ the global symmetry generators to build a modification of both determinants and thus compute a matrix of (large N) two-point functions of modified determinants. This will show explicitly which modifications are BRST-equivalent at leading order and match them to the action of vector fields on the B-model D1-brane.

This correlation function has a single saddle, corresponding to the D-brane with support in the Cartan of $SL(2,\mathbb{C})$:[23]

$$g = \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a} \end{pmatrix}. \tag{2.90}$$

A global symmetry generator $J_n(v;w)$, given by (2.16), deforms g by

$$\delta g = \epsilon(a^{-1} + vwa)^{n-2} \begin{pmatrix} -va & -v^2 a \\ \frac{1}{a} & \frac{v}{a} \end{pmatrix}. \tag{2.91}$$

We can combine this with the reparametrization $a \to a + \epsilon va(a^{-1} + vwa)^{n-2}$ to eliminate the variation of a and d. We are left with variations

$$\delta b(a) = -\epsilon v^2 a(a^{-1} + vwa)^{n-2}, \qquad \delta c(a) = \epsilon a^{-1}(a^{-1} + vwa)^{n-2}. \tag{2.92}$$

The coefficient of $v^{k+1}w^{k-1}$ in $J_n(v;w)$ gives a variation

$$\delta b(a) = -\epsilon \binom{n-2}{k-1} a^{2k+1-n}, \tag{2.93}$$

which is a mode corresponding to a modification of $\det X(0)$ if $2k+1 \geq n$ and to a modification of $\det Y(\infty)$ otherwise. The coefficient of $v^k w^k$ in $J_n(v;w)$ gives a variation

$$\delta c(a) = \epsilon \binom{n-2}{k} a^{2k+1-n}, \tag{2.94}$$

which is a mode corresponding to a modification of $\det X(0)$ if $2k+1 > n$ and to a modification of $\det Y(\infty)$ otherwise. All the other coefficients give vanishing variations. Notice that the modes $\delta b = a^s$ and $\delta c = a^{-s}$ for all s are precisely dual under the inner product in the $\beta\gamma$ worldvolume theory of the D-brane.

We thus predict that the action of $a_n(v;w)$ on $\det X(0)$ will only produce very specific modifications at large N, up to BRST-exact operators:

[23] In appendix A.1 we discuss saddles with a small number of determinants.

- The coefficients of $v^{k+1}w^{k-1}$ for $2k+1 \geq n$ should give $-n\binom{n-2}{k-1}$ times a modification which only depends on $2k+1-n$.

- The coefficients of $v^k w^k$ for $2k+1 > n$ should give $n\binom{n-2}{k}$ times a modification which only depends on $2k+1-n$.

- All other coefficients should give BRST-exact or subleading modifications.

The same statements apply to $\det Y(\infty)$, with the opposite constraints on $2k+1-n$, producing modifications which are precisely dual under the inner product.

In the rest of the Section we will often employ individual generators rather than generating functions of them. We can write

$$a_p^{(n)}(v) \equiv \sum_{q=-\frac{n}{2}}^{\frac{n}{2}} a_{p,q}^{(n)} v^{\frac{n}{2}-q} \equiv \oint \frac{\mathrm{d}z}{2\pi i z}\, z^{\frac{n}{2}+p} A_n(v;z) \tag{2.95}$$

i.e.

$$a_{p,q}^{(n)} = \oint \frac{\mathrm{d}z}{2\pi i z} z^{p+\frac{n}{2}} \oint \frac{\mathrm{d}v}{2\pi i v} v^{q-\frac{n}{2}} A_n(v;z) \tag{2.96}$$

and then predict that the action of $a_{p,q}^{(n)}$ on $\det X(0)$ will only produce the following non-trivial modifications at large N:

- $a_{p,p-1}^{(n)}$ for $2p-1 \leq 0$ should give $(-1)^{p-\frac{n}{2}}n$ times a modification which only depends on p.

- $a_{p,p+1}^{(n)}$ for $2p+1 < 0$ should give $(-1)^{p-1-\frac{n}{2}}n$ times a modification which only depends on p.

- All other coefficients should give BRST-exact modifications.

The same statements apply to $\det Y(\infty)$, with the opposite constraints on p, producing modifications which are precisely dual under the inner product.

We will test these statements by computing correlation functions

$$\langle [a_{-p,-q}^{(m)}, \det Y(\infty)][a_{p,q}^{(n)}, \det X(0)]\rangle \tag{2.97}$$

in the large N limit. We expect to find

$$\frac{\langle [a^{(m)}_{-p,-q}, \det Y(\infty)][a^{(n)}_{p,q}, \det X(0)]\rangle}{\langle \det Y(\infty) \det X(0)\rangle}\Big|_{N\to\infty} = (-1)^{\frac{m+n}{2}+1} nmN \left(\delta_{q,p-1} - \delta_{q,p+1}\right). \tag{2.98}$$

Notice that n and m are either both even or both odd in order for the answer to be non-zero. The relative sign in the parenthesis is due to the relative sign between inner products $\langle 0|\beta_n\gamma_{-n}|0\rangle$ and $\langle 0|\gamma_n\beta_{-n}|0\rangle$ in the $\beta\gamma$ system on the D1-brane.

2.5.1 Chiral algebra computation

The correlation functions (2.97) of modified determinants are computed by contour integrals from

$$\langle \det Y(\infty) A_m(v'; z') A_n(v; z) \det X(0)\rangle. \tag{2.99}$$

The general strategy to calculate correlation functions of two determinants and two single-trace operators is to employ the formula (2.69), which in this case takes the form

$$\det Y(\infty) \det X(0) = \frac{1}{Z_\rho}\int [\mathrm{d}\psi \mathrm{d}\bar{\psi}][\mathrm{d}\rho] : e^{N\rho^1_2\rho^2_1 + \rho^1_2\bar{\psi}_1\psi_2 + \rho^2_1\bar{\psi}_2\psi_1 + \bar{\psi}_1 Y(\infty)\psi_1 + \bar{\psi}_2 X(0)\psi_2} :_Z . \tag{2.100}$$

We insert it to the correlation function[24]

$$\langle \det Y(\infty) A_m(v'; z') A_n(v; z) \det X(0)\rangle \tag{2.101}$$

$$= \frac{1}{Z_\rho}\int [\mathrm{d}\psi \mathrm{d}\bar{\psi}][\mathrm{d}\rho] e^{N\rho^1_2\rho^2_1} \left\langle : e^{\rho^1_2\bar{\psi}_1\psi_2 + \rho^2_1\bar{\psi}_2\psi_1 + \bar{\psi}_1 Y(\infty)\psi_1 + \bar{\psi}_2 X(0)\psi_2} :_Z A_m(v'; z') A_n(v; z)\right\rangle . \tag{2.102}$$

We can study terms at the highest order in N using diagrammatics explained in Section 2.4. We need at least one contraction between each of the single-trace operators and the determinant operators in order for the contour integrals in (2.97) to not vanish. In the large N limit, the dominant ribbon graphs are of order N^1 and have the topology of a disk with one fermion boundary.

All such possible diagrams are shown in figure 2.2. The two single-trace operators are connected by a certain number of "inner" Z propagators:

$$\overbrace{Z^i_j(v'; z') Z^k_l}(v; z) = \delta^i_l \delta^k_j \frac{1}{N}\frac{v' - v}{z' - z}. \tag{2.103}$$

[24] The normal ordered product $: :_Z$ introduced in Section 2.4, means we do not contact symplectic bosons within the exponent.

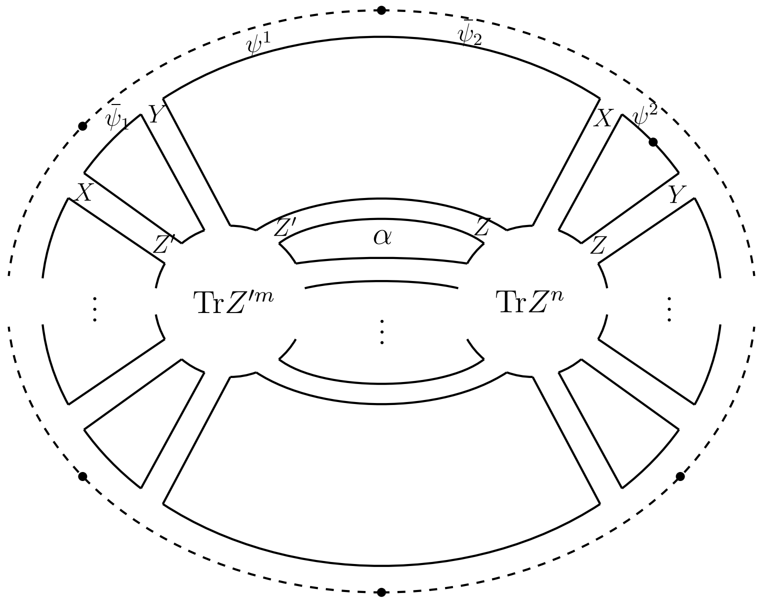


Figure 2.2: The class of ribbon diagrams which contribute to the calculation in the main text at the order of N^1.

The remaining fields are contracted with fermion vertices, $\bar\psi_1 Y(\infty)\psi_1$ or $\bar\psi_2 X(0)\psi_2$, along the unique closed flavour loop. As we go along the flavour loop, we encounter first the sequence of fermionic vertices contracted with the first single-trace operator $A_m(v';z')$ and then the sequence of fermionic vertices contracted with the second single-trace operator $A_n(v;z)$. The contractions are:

$$\overbracket{X^i_j(0)Z}^k_l(v;z) = \delta^i_l\delta^k_j \frac{1}{N}\frac{v}{z}, \qquad \overbracket{Y^i_j(\infty)Z}^k_l(v;z) = \delta^i_l\delta^k_j \frac{1}{N}. \tag{2.104}$$

The $A_n(v;z)$ operator can be "inserted" in such a diagram in n different ways, depending on which of the Z fields is connected to the "first" propagator in the diagram. The same is true for $A_m(v';z')$. The resulting nm combinatorial factor already appears in the answer (2.98). The sum over diagrams will have to reproduce the remaining $\delta_{q,p-1} - \delta_{q,p+1}$ and appropriate overall sign.

Multiple diagrams with the same topology are distinguished by two things:

- The number of inner Z propagators between single-trace operators $A_m(v';z')$ and $A_n(v;z)$, which we denote α.

- Which of the two flavours of fermions run in the two propagators in between the two sequences of fermionic vertices.

The fermion propagator is completely off-diagonal, so the two flavours of fermions have to alternate along the boundary. Let us denote by ε the difference between the numbers of contractions $\overbracket{X(0)Z}(v';z')$ and $\overbracket{Y(\infty)Z}(v';z')$. When $m-\alpha$ and $n-\alpha$ are both even there are two possible diagrams, both with $\varepsilon=0$. They give equal contributions. When $m-\alpha$ and $n-\alpha$ are odd there are two distinct diagrams, with $\varepsilon=1$ and $\varepsilon=-1$.

A diagram characterized by (α,ε) gets the contribution from all propagators:

$$(-1)^{\frac{m+n}{2}-\alpha+1}\left(\frac{v'-v}{z'-z}\right)^{\alpha}\left(\frac{v'}{z'}\right)^{\frac{m-\alpha-\varepsilon}{2}}\left(\frac{v}{z}\right)^{\frac{n-\alpha+\varepsilon}{2}}, \tag{2.105}$$

where the sign comes from fermion contractions. When inserted into the contour integrals it produces two binomial factors:

$$\oint_{\infty}\frac{\mathrm{d}z'}{2\pi iz'}z'^{\frac{m}{2}-p}\oint_{\infty}\frac{\mathrm{d}v'}{2\pi iv'}v'^{-q-\frac{m}{2}}\oint_{0}\frac{\mathrm{d}z}{2\pi iz}z^{\frac{n}{2}+p}\oint_{0}\frac{\mathrm{d}v}{2\pi iv}v^{q-\frac{n}{2}}\left(\frac{v'-v}{z'-z}\right)^{\alpha}\left(\frac{v'}{z'}\right)^{\frac{m-\alpha-\varepsilon}{2}}\left(\frac{v}{z}\right)^{\frac{n-\alpha+\varepsilon}{2}} \tag{2.106}$$

$$=(-1)^{\frac{\alpha-\varepsilon}{2}-q}\binom{\alpha}{\frac{\alpha+\varepsilon}{2}+q}\binom{\frac{\alpha+\varepsilon}{2}-p-1}{\alpha-1}\equiv s(\alpha,\varepsilon). \tag{2.107}$$

The final answer is the sum of three contributions, each of which is a sum over the possible values of α. Schematically,

$$(-1)^{\frac{m+n}{2}+1}\left(2\sum_{m-\alpha\text{ even}}(-1)^{\alpha}s(\alpha,0)+\sum_{m-\alpha\text{ odd}}(-1)^{\alpha}s(\alpha,-1)+\sum_{m-\alpha\text{ odd}}(-1)^{\alpha}s(\alpha,1)\right). \tag{2.108}$$

It turns out that the combination in parenthesis is non-vanishing only in two cases:[25]

$$=\begin{cases}+1 & p-q=1, p\leq\frac{1}{2}\\ -1 & q-p=1, p<-\frac{1}{2}.\end{cases} \tag{2.109}$$

[25]The computation is relegated to the appendix A.2. There is actually a third non-vanishing case, with $p=q=0$: the $a^{(n)}_{0,0}$ generators do not create a modification of the determinant, but they give back a multiple of the determinant itself.

We arrive at the expected

$$\frac{\langle [a^{(m)}_{-p,-q}, \det Y(\infty)][a^{(n)}_{p,q}, \det X(0)]\rangle}{\langle \det Y(\infty) \det X(0)\rangle}\Big|_{N\to\infty} = (-1)^{\frac{m+n}{2}+1} nmN \left(\delta_{q,p-1} - \delta_{q,p+1}\right). \qquad (2.110)$$

Since the modifications by $a^{(n)}_{p,q}$ do not depend on n up to a number, we can compute them using the lowest possible n:

$$[a^{(-2p+2)}_{p,p-1}, \det X(0)] = \oint_0 \frac{\mathrm{d}z}{2\pi i} \oint_0 \frac{\mathrm{d}v}{2\pi i v} v^{2p-2} N \operatorname{Tr} Z(v;z)^{-2p+2} \det X(0)$$

which gives a substitution $X \to (2-2p)Y^{1-2p}$. The second type of modification is

$$[a^{(-2p+2)}_{p,p+1}, \det X(0)] = \oint_0 \frac{\mathrm{d}z}{2\pi i} \oint_0 \frac{\mathrm{d}v}{2\pi i v} v^{2p} N \operatorname{Tr} Z(v;z)^{-2p+2} \det X(0). \qquad (2.111)$$

There are two terms which contribute at large N, schematically:

$$\varepsilon\varepsilon(X,\dots,X, Y^{-2p-2}\partial X) + \varepsilon\varepsilon(X,\dots,X,\partial^2 Y^{-2p-3}). \qquad (2.112)$$

2.6 The matrix model configuration

This Section is somewhat orthogonal to the rest of the Chapter. It describes a topological subsector where the chiral algebra correlation functions reduce to correlation functions in a Gaussian matrix model. This subsector may be related the Dijkgraaf-Vafa setup [83, 84, 85].

The definition of the topological subsector is akin to the manner the chiral algebra itself is embedded into the physical theory. Recall that the chiral algebra is embedded in the physical theory by realizing X and Y as position-dependent linear combinations of scalar fields of the physical theory. Concretely, one has four scalars $\phi_{\alpha\dot\alpha}$ and takes combinations $X = \phi_{1\dot 1} + \bar z \phi_{1\dot 2}$ and $Y = \phi_{2\dot 1} + \bar z\phi_{2\dot 2}$.

We can imitate this construction to define a subsector of the chiral algebra generated by a field

$$M(z) \equiv X(z) + zY(z) = Z(u=z;z). \qquad (2.113)$$

Correlation functions of $M(z_i)$ are independent of the positions z_i. Wick contractions of M with itself are identical to these of a Gaussian matrix model.

One way to understand the origin of this simplification is that the correlation functions of operators built from $M(z)$ preserve some extra superconformal symmetry, i.e. the fermionic linear combinations of symmetry generators:

$$\oint \frac{\mathrm{d}z}{2\pi i} \operatorname{Tr} bZ(z;z) \tag{2.114}$$

and

$$\oint \frac{\mathrm{d}z}{2\pi i} \operatorname{Tr} \partial c Z(z;z)\,. \tag{2.115}$$

In general, we can use the $SL(2)_R$ symmetry to rotate at least three u_i to coincide with some z_i. As a result, two- and three-point functions can always be taken to lie in the protected subsector. This can be computationally useful.

2.6.1 Protected determinants

As soon as we set $z_i = u_i$, the saddle equations reduce to

$$\rho + \rho^{-1} = \lambda \tag{2.116}$$

for some diagonal matrix λ. For irreducible solutions, λ has to be a multiple of the identity.

In an irreducible solution, ρ will have k_+ eigenvalues equal to some r and k_- eigenvalues equal to r^{-1}, with $r + r^{-1} = \lambda$. Before imposing the $m_i = \rho_i^i$ constraint, we have a $2k_+k_-$-dimensional space of solutions. Imposing $m_i = \rho_i^i$ enforces $k_+ r + k_- r^{-1} = \operatorname{Tr} m$ and removes $k-1$ degrees of freedom. If we also quotient by the identification $\rho \to \lambda\rho\lambda^{-1}$ for diagonal λ, we get a $2(k_+-1)(k_--1)$-dimensional space of solutions. For the $k = 1, 2, 3$ cases the space has expected dimension 0.

As we adjust the spectral problem to the simplified setting, we find

$$\begin{aligned} B(a) &= a\,\zeta - \rho \\ C(a) &= a\,\zeta - \rho + \lambda \\ D(a) &= a\,\zeta^2 - \rho\zeta - \zeta\rho + \lambda\zeta \end{aligned} \tag{2.117}$$

and thus an irreducible spectral curve is supported on $c - b = \lambda$.

As we compute correlation functions of operators with $u_i = z_i$, the Wick contractions between different operators are the same as in a Gaussian matrix model. We still do not have contractions between scalar fields in the same operator, which means that the chiral algebra correlation functions reduce to matrix model correlation functions of *normal ordered* operators.

There is a simple integral relation between a normal-ordered determinant and a standard determinant in the matrix model. If we start from the fermionized form

$$: \det(\lambda + M) := \int \mathrm{d}\psi \mathrm{d}\bar{\psi} : e^{\bar{\psi}(\lambda+M)\psi} : \tag{2.118}$$

then we can remove the normal ordering by writing

$$\det(\lambda + M) = \int \mathrm{d}\psi \mathrm{d}\bar{\psi} : e^{\bar{\psi}(\lambda+M)\psi} : e^{-\frac{1}{2N}(\bar{\psi}\psi)^2} \tag{2.119}$$

and then

$$\det(\lambda + M) = \sqrt{-\frac{N}{2\pi}} \int \mathrm{d}\psi \mathrm{d}\bar{\psi} \mathrm{d}m : e^{\bar{\psi}(m+M)\psi} : e^{\frac{N}{2}(m-\lambda)^2}. \tag{2.120}$$

At the level of saddle equations, this means that instead of fixing m_i we fix $m_i + p_i = \lambda_i$.

The saddle equations are unchanged, but now the diagonal components of ρ are free and λ is fixed. We only get irreducible solutions if all the λ_i happen to coincide. Otherwise, the only saddles are completely diagonal and the k branes S_{m_i} are disconnected from each other.

Chapter 3

Twisted holography without conformal symmetry

3.1 Introduction and conclusions

Any $\mathcal{N} = 2$ superconformal field theory in 4d contains a 2d chiral algebra[1] subsector, protected by a linear combination "$Q+S$" of super-charges and super-conformal charges of the ambient theory [32]. Many such SCFTs are also equipped with a non-trivial moduli space of super-Poincaré-invariant vacua. The space of vacua may include various "branches", distinguished by the type of BPS operators which acquire vevs. The Higgs branch of vacua can be characterized as leaving unbroken the $U(1)_r$ R-symmetry of the SCFT. According to a conjecture of [54], one can recover the Higgs branch of the space of 4d vacua as the *associated variety* of the 2d chiral algebra[2], an algebraic tool introduced originally to study the characters of a vertex algebra [58].

This conjecture is surprising: Higgs branch vevs break the 4d superconformal group to the super-Poincaré group and in particular break the "$Q + S$" symmetry protecting the chiral algebra subsector. The tension can be resolved by considering an alternative definition of the chiral algebra subsector based on a B-type Ω-deformation of the physical

[1]We use the terms 2d chiral algebra and vertex algebra interchangeably throughout the thesis. Conceptually, a 2d chiral algebra encodes the properties of holomorphic local operators supported on a two-dimensional locus in some quantum field theory. Vertex algebras provide a mathematical formalization of 2d chiral algebra, essentially via Kaluza-Klein reduction on a circle.

[2]See also [55], and the reviews [56, 57].

theory [51, 52, 53]: the B-type Ω-deformation setup only requires unbroken $U(1)_r$ R-symmetry[3] and thus appears compatible with Higgs branch vevs.[4] This leads to the obvious question: how does a Higgs branch vev affect the chiral algebra correlation functions computed in the Ω-deformed theory?

In order to answer this question, we introduce the notion of translation-invariant vacuum for a 2d chiral algebra. A simple analysis, presented in Section 3.2 of this Chapter, leads to a transparent physical interpretation of the associated variety: it coincides with the moduli space of translation-invariant vacua of the 2d chiral algebra. The conjecture of [54] thus identifies the Higgs branch vacua of the 4d $\mathcal{N}=2$ SCFT with the vacua of the corresponding 2d chiral algebra. In the Ω-deformation setup, the identification maps the expectation values of chiral algebra operators to the expectation values of the corresponding operators in the 4d theory.

Correlation functions of a 2d chiral algebra in a non-trivial vacuum are a novel observable, which may be of mathematical interest.[5] In the rest of the Chapter, we restrict our attention to $\mathcal{N}=2$ SCFTs which have a type IIB holographic dual [34, 87]. Most of our calculations will be done in the chiral algebra of $\mathcal{N}=4$ $U(N)$ gauge theory, but can be extended to other examples with some extra work.

One of the simplest variations of Maldacena duality [34] involves precisely correlation functions of $SU(N)$ $\mathcal{N}=4$ SYM computed in non-trivial super-Poincaré-invariant vacua, i.e. the Coulomb branch,[6] in flat space [46, 47, 48, 49, 50], where the six adjoint scalar fields $\vec{\Phi}$ receive diagonal vevs, with eigenvalues $\vec{y}_i$ of multiplicities $N_i \equiv \alpha_i N$, $i=1,\dots,n$.

The dual IIB supergravity solutions are obtained from a near-horizon limit of multi-center half-BPS D3-brane solutions:

$$\mathrm{d}s^2 = H(\vec{y})^{-\frac{1}{2}}\mathrm{d}x^2 + H(\vec{y})^{\frac{1}{2}}\mathrm{d}\vec{y}^{\,2}\,, \tag{3.1}$$

where x are the four directions parallel to the D3-branes and $\vec{y}$ the six directions transverse

---

[3] This should be contrasted to the Ω-deformation defined in [86], which is instead applicable to any $\mathcal{N}=2$ SQFT.

[4] We leave a proof of this fact to future work. We can find support for this assumption in the main example in this Chapter: the chiral algebra of $U(N)$ $\mathcal{N}=4$ SYM. If we identify the theory as the low-energy worldvolume theory on D3-branes, the B-type Ω-deformation is induced by a string theory Ω-background, which reduces the D3-branes to certain B-branes in the B-model topological string. These B-branes have a moduli space of vacua parameterized by the Higgs branch of the physical theory.

[5] It should be also be possible to insert non-trivial modules for the chiral algebra at points in the plane and define conformal blocks in a non-trivial vacuum.

[6] The $\mathcal{N}=4$ Coulomb branch is $(\mathbb{R}^6)^N/S_N$. It includes the $\mathcal{N}=2$ Coulomb and Higgs branches, $\mathbb{C}^N/S_N$ and $(\mathbb{C}^2)^N/S_N$, parametrized respectively by vevs of two and four of the six scalars $\vec{\Phi}$.

to the D3-branes. The solutions involve a harmonic function $H(\vec{y})$ in $\mathbb{R}^6$:

$$H(\vec{y}) = L^4 \sum_{i=1}^{n} \frac{\alpha_i}{|\vec{y}-\vec{y}_i|^4}\,, \tag{3.2}$$

with $\vec{y}_i$ being the transverse positions of the D3-brane stacks, each with a fraction α_i of the total number N of D3-branes.[7]

The $\mathbb{R}^4$ holographic boundary[8] lies at $\vec{y} \to \infty$. The asymptotic deviation of $H(\vec{y})$ from the $\mathrm{AdS}_5 \times S^5$ reference value $L^4|\vec{y}|^{-4}$ encodes the vacuum expectation values of local operators, according to the standard holographic dictionary.

The main objective of this Chapter is to study the analogous setting for the twisted holography correspondence of [31], which is expected to capture a protected subsector of Maldacena's duality. This correspondence relates the 2d chiral algebra subsector of 4d $\mathcal{N}=4$ SYM [32] to the B-model topological string theory [35, 36] on appropriate (complex) 3-dimensional Calabi-Yau geometries [60] endowed with a holographic boundary.

The Calabi-Yau geometry dual to the 2d chiral algebra in the standard conformally-invariant vacuum is the deformed conifold $SL(2,\mathbb{C})$. We follow closely the derivation of [31] and present in Section 3.3 a family of candidate 3d Calabi-Yau geometries dual to the correlation functions of the 2d chiral algebra in a non-trivial vacuum.

As a test of our proposal, we study correlation functions involving determinant operators in the chiral algebra. In the full physical theory, the insertion of such determinant operators in the boundary theory is dual to "Giant Graviton" D3-branes approaching a point of the holographic boundary of $\mathrm{AdS}_5 \times S^5$. In twisted holography, the D-branes wrap 1-dimensional complex curves in the Calabi-Yau geometry. In either case, the large N saddles of correlation functions of multiple determinants should be dual to semi-classical D-brane configurations in the bulk [38, 88, 41].

In the standard conformally-invariant vacuum, a spectral curve construction maps the large N saddles of chiral algebra correlation functions to explicit complex curves in $SL(2,\mathbb{C})$. In Section 3.4 we will use a similar but somewhat more intricate construction to extend the match to the saddles which appear for non-trivial vacua.

[7] The solution for D3-branes in flat space has an extra "1" constant term in H, which drops out in the near-horizon limit.

[8] As these vacua are not conformally invariant, correlation functions do not naturally extend to the four-sphere.

3.2 Translation-invariant vacua and associated varieties

In this Section we introduce the notion of translation-invariant vacua for a 2d chiral algebra and propose it coincides with the associated variety of the chiral algebra.

A translation-invariant vacuum $\mathcal{V}$ is a collection of translation-invariant correlation functions on the plane which satisfy OPE and have the cluster property: the correlation functions factorize in the limit where a subset of the local operators is far from the rest. In particular, correlation functions in the vacuum $\mathcal{V}$ should have a finite limit

$$\left\langle O_0(z) \prod_a O_a(z_a) \right\rangle_{\mathcal{V}} \to \langle O_0 \rangle_{\mathcal{V}} \left\langle \prod_a O_a(z_a) \right\rangle_{\mathcal{V}} , \qquad z \to \infty \,, \tag{3.3}$$

when one operator is brought to infinity.

We can use a standard strategy to determine the dependence of such correlation function on the position z of one operator: the OPE relations determine its poles as a function of correlation functions of fewer operators and factorization controls the behaviour at infinity. Every correlation function can be reconstructed recursively in this manner, given the one-point functions $\langle O_a \rangle_{\mathcal{V}}$.

The full OPE relations contain more information than the singular parts. The non-singular part of the OPE places constraints on the one-point functions $\langle O_a \rangle_{\mathcal{V}}$. For example, consider a two-point function

$$\langle O_a(z) O_b(0) \rangle_{\mathcal{V}} = \langle O_a \rangle_{\mathcal{V}} \langle O_b \rangle_{\mathcal{V}} + \sum_{n \geq 0} z^{-n-1} \langle [O_{a;n} O_b] \rangle_{\mathcal{V}} \,, \tag{3.4}$$

reconstructed from the behaviour at large z and the singular part of the OPE.[9] If we compare this to the full OPE expansion,

$$\langle O_a(z) O_b(0) \rangle_{\mathcal{V}} = \sum_{n \in \mathbb{Z}} z^{-n-1} \langle [O_{a;n} O_b] \rangle_{\mathcal{V}} \,, \tag{3.5}$$

we deduce the one-point functions of all operators which appear in the non-singular part

[9] We use the mathematical conventions here, so that $n = 0, 1, \ldots$ correspond to the singular part and $n = -1, -2, \ldots$ correspond to the finite part of the OPE.

of the OPE:[10]

$$\begin{aligned} &\langle [O_{a;n} O_b] \rangle_{\mathcal{V}} = 0\,, && n \le -2 \\ &\langle [O_{a;-1} O_b] \rangle_{\mathcal{V}} = \langle O_a \rangle_{\mathcal{V}} \langle O_b \rangle_{\mathcal{V}}\,. \end{aligned} \tag{3.6}$$

The space of operators of the form $[O_{a;n} O_b]$, $n \le -2$, forms a very nice subspace $C_2(V)$ of the vertex algebra V.[11] The quotient $R_V = V/C_2(V)$ equipped with the (commutative) product $O_a \cdot O_b \equiv [O_{a;-1} O_b] \; (\mathrm{mod}\, C_2(V))$ is called the Zhu's C_2-algebra of the VOA [89].

The relations (3.6) are equivalent to the statement that the 1-point functions $\langle O_a \rangle_{\mathcal{V}}$ define an algebra map from R_V to the complex numbers. Essentially by definition,[12] this is the same as a point in the associated variety of the VOA, which is defined as the maximal spectrum of the C_2-algebra [58]:

$$\mathcal{X}_V = \mathrm{mSpec}\, R_V\,. \tag{3.7}$$

Conversely, any algebra map from R_V to the complex numbers gives us a collection of 1-point functions, with the property that the 2-point functions derived from those via Ward identities satisfy cluster decomposition and are compatible with the full OPE.

We expect this property to be sufficient to guarantee that all n-point-functions also satisfy cluster decomposition and are compatible with the full OPE expansion. It would be nice to prove this fact. With that assumption, we find that the space of vacua for V coincides with $\mathcal{X}_V$.

The C_2-algebra is also equipped with a Poisson bracket $\{O_a, O_b\} \equiv [O_{a;0} O_b]$. The corresponding Hamiltonian flows $\{O_a, \cdot\}$ have a natural physical interpretation: they describe the infinitesimal deformation of the vacuum induced by integrating O_a on a very large circle.[13] If $O \equiv J$ is a dimension 1 current, the zero-mode of J defines a symmetry of V and the Poisson bracket $\{J, \cdot\}$ is the action of the same symmetry on R_V. Equivalently, the image of J in R_V is the moment map for the symmetry associated to J.

[10] Notice that $[O_{a;n} O_b]$ for negative n is just the regularization of a $\partial^{n-1} O_a\, O_b$ composite operator. The relations below just tell us that the regularization does not affect the factorization of vevs.

[11] If we take O_b to be the identity, we find that derivatives of any operator belong to $C_2(V)$.

[12] The maximal spectrum $\mathrm{mSpec}\, R$ of a commutative $\mathbb{C}$-algebra R is defined as the set of its maximal ideals. Quotiening R by a maximal ideal results in the field $\mathbb{C}$. Conversely, kernel of any algebra map from R to $\mathbb{C}$ is a maximal ideal.

[13] The insertion of such contour integral indeed preserves translation symmetry and the cluster property: the contour can be translated and can also be deformed to separately encircle two collections of well-separated local operators.

3.2.1 Gauged $\beta\gamma$ systems

As an example, we consider gauged $\beta\gamma$ systems, which arise as chiral algebras of $\mathcal{N}=2$ Lagrangian gauge theories. We identify their associated varieties and translation-invariant vacua as well as match them with the Higgs branches of the corresponding 4d theories.

First, consider a particularly simple example of associated variety, which occurs for $\beta\gamma$ system of free symplectic bosons Sb, with OPE

$$Y(z)X(0) \sim \frac{1}{z} \,. \tag{3.8}$$

It is easy to see that any operator which contains derivatives of the elementary fields belongs to $C_2(\mathrm{Sb})$. Then R_{Sb} consists of polynomials in two variables $x=[X]$ and $y=[Y]$ (mod $C_2(\mathrm{Sb})$) and the associated variety is $\mathcal{X}_{\mathrm{Sb}} = \mathbb{C}^2$, with the standard Poisson bracket $\{y,x\}=1$. Correspondingly, the translation-invariant vacua of Sb are labelled by the vevs

$$x = \langle X(0)\rangle_{x,y} \,, \qquad\qquad y = \langle Y(0)\rangle_{x,y} \,. \tag{3.9}$$

The associated 4d $\mathcal{N}=2$ theory is a single free hypermultiplet, whose Higgs branch is indeed $\mathbb{C}^2$. The generalization to multiple copies of the symplectic boson chiral algebra is straightforward.

There exists a universal prescription for the associated variety of a gauged chiral algebra in terms of the associated variety of the original chiral algebra. Consider a 2d chiral algebra V equipped with a Kac-Moody symmetry G at level $-2h$, with h being the dual Coxeter number of G. Such a Kac-Moody symmetry can be gauged to produce a new 2d chiral algebra, which we denote $V/\!\!/G$. Concretely, one adds a bc ghost system for G and takes cohomology with respect to the standard BRST charge. The associated variety for $V/\!\!/G$ is known to coincide with the complex symplectic quotient[14] $\mathcal{X}_V/\!\!/G$ of the associated variety of V [90, 91].

The proof in the last reference essentially shows that the operation of taking Zhu's C_2-algebra R_V commutes with BRST reduction, in the sense that $R_{V/\!\!/G}$ is obtained from R_V by adding the ghost representatives $[b]$ and $[c]$ and defining a BRST charge as the Poisson bracket with the representative of the BRST current $\{J_{\mathrm{BRST}},\cdot\}$. The BRST reduction of R_V is a derived description of the complex symplectic quotient of the corresponding associated variety: it makes the moment maps BRST-exact and imposes G-invariance.

[14]Recall that the complex symplectic quotient $\mathcal{X}/\!\!/G$ of a variety $\mathcal{X}$ with Hamiltonian G-action is defined by quotiening the vanishing locus for the moment maps by (the complexification of) G. The moment maps are the Hamiltonians for the G-action.

More physically, this theorem tells us that a vacuum for $V /\!\!/ G$ is the same as a BRST-invariant vacuum for the combination of V and the ghosts. The BRST variation of the vacuum is computed as an integral of the BRST current on a large circle, which is the same as the Poisson bracket with the current representative in R_V. This statement is fully compatible with the conjectural relation between associated varieties and Higgs branches. Indeed, consider a 4d $\mathcal{N} = 2$ SCFT with global symmetry G. As long as the $U(1)_r$ symmetry does not become anomalous, gauging G (in an $\mathcal{N} = 2$ sense) results in a new 4d $\mathcal{N} = 2$ SCFT. The Higgs branches of the two theories are related by a complex symplectic quotient and the 2d chiral algebras of the two theories are related by gauging as well. Incidentally, this reasoning also proves the conjecture for all Lagrangian theories, which give rise to gauged $\beta\gamma$ systems [32, 54].

Chiral algebra of $\mathcal{N} = 4$ SYM

In the case of $\mathcal{N} = 4$ SYM, the $\beta\gamma$ system is a pair symplectic bosons X, Y in the adjoint representation of $U(N)$ [32]. The OPE is

$$Z^a_b(u; z) Z^c_d(v; w) \sim \frac{1}{N} \frac{u - v}{z - w} \delta^a_d \delta^c_b \,, \tag{3.10}$$

where $Z(u; z)$ is the linear combination

$$Z(u; z) \equiv X(z) + uY(z) \,. \tag{3.11}$$

The purpose of the auxiliary variable u is to write expressions which are covariant under the $SL(2)_R$ symmetry[15] rotating X and Y into each other, which acts as fractional linear transformations on u.

The moment map is $[X, Y]$ and we can select a vacuum where the X and Y receive vevs that are any commuting diagonal matrices, with eigenvalues (x_i, y_i) appearing with multiplicity N_i (by definition, $\sum_i N_i = N$). The vev eigenvalues represent the positions of stacks of N_i D-branes in the transverse $\mathbb{C}^2$.

The simplest set of single-trace BRST-closed local operators, the *A-tower*, consist of the individual terms of the expansion of

$$A_n(u; z) = N \operatorname{Tr} Z(u; z)^n \tag{3.12}$$

in powers of u, i.e.

$$A_{n;s}(z) = N \operatorname{STr} X^{n-s} Y^s(z) \,, \tag{3.13}$$

[15] This is a subgroup of the R-symmetry of the 4d theory.

where $\mathrm{STr}\, X^{n-s}Y^s$ is the symmetrized trace. These form an irreducible representation of $SL(2)_R$ of dimension $(n+1)$.

In a vacuum parametrized by eigenvalues (x_i, y_i) of X and Y of multiplicity N_i, they receive vacuum expectation values

$$\langle \mathrm{STr} X^n Y^m \rangle = \sum_i N_i x_i^n y_i^m \,, \tag{3.14}$$

which we will give a holographic interpretation in the next Section.

3.3 Coulomb branch geometry

In this Section, we generalize the analysis of Section 4 of [31] and compute the backreaction of a collection of parallel *non-coincident* D-branes in the B-model/BCOV theory. We identify the backreacted geometry as dual to the chiral algebra of $\mathcal{N} = 4$ SYM in the non-trivial vacua studied above.

Consider B-model branes wrapping parallel $\mathbb{C}$'s in $\mathbb{C}^3$. We use coordinates $(x, y, z) \in \mathbb{C}^3$ so that $N_i = \alpha_i N$ branes wrap $\mathbb{C}$ defined by equations $x = x_i$, $y = y_i$. The back-reaction is described by a Beltrami differential[16]

$$\beta = \sum_{i=1}^{n} \alpha_i \frac{(\bar{x} - \bar{x}_i)\mathrm{d}\bar{y} - (\bar{y} - \bar{y}_i)\mathrm{d}\bar{x}}{(|x - x_i|^2 + |y - y_i|^2)^2} \partial_z \,, \tag{3.15}$$

which deforms the complex structure of $\mathbb{C}^3 \setminus \bigcup_i \{x = x_i, y = y_i\}$.

We can also give a Čech description of this Beltrami differential. The description involves 2^n patches of $\mathbb{C}^3$, in which, for each $i = 1, \ldots, n$, either $x - x_i$ or $y - y_i$ is non-zero. We denote a patch by an index I and denote as I_x the collection of i's for which $x - x_i$ is non-zero and as I_y the collection of i's for which $y - y_i$ is non-zero. We can trivialize the Beltrami differential in each patch by a gauge transformation, giving us a new holomorphic local coordinate z_I.[17] The coordinates x and y are holomorphic and coincide in all patches.

[16] For convenience, we set the topological string coupling to N^{-1}.

[17] In a patch I, β can be written as $\beta = \bar{\partial}\gamma_I$, where

$$\gamma_I = \sum_{i \in I_x} \left(\frac{\bar{y} - \bar{y}_i}{x - x_i} \right) \frac{\alpha_i}{|x - x_i|^2 + |y - y_i|^2} - \sum_{i \in I_y} \left(\frac{\bar{x} - \bar{x}_i}{y - y_i} \right) \frac{\alpha_i}{|x - x_i|^2 + |y - y_i|^2} \,. \tag{3.16}$$

The new holomorphic local coordinate is then $z_I = z - \gamma_I$.

On the intersection of two patches I and I' which differ by the ith choice only, where I_y and I'_x include i, we have a coordinate transformation

$$z_I = z_{I'} + \frac{\alpha_i}{(x-x_i)(y-y_i)}\,. \tag{3.17}$$

More generally, $z_I - z_{I'}$ is a sum of terms, with positive sign for each i included both in I_y and I'_x or with negative sign for each i included both in I_x and I'_y.

We can equivalently use coordinates

$$w_I = z_I \left[\prod_{i\in I_x}(x-x_i)\right]\left[\prod_{i\in I_y}(y-y_i)\right], \tag{3.18}$$

which extend to globally defined functions on the whole geometry and satisfy relations

$$w_I(x-x_i) - w_{I'}(y-y_i) = \alpha_i \left[\prod_{j\in I_x}(x-x_j)\right]\left[\prod_{j\in I'_y}(y-y_j)\right], \tag{3.19}$$

whenever I and I' differ at the ith entry only, where I_y and I'_x include i.

The holographic boundary is at $x, y \to \infty$. The coordinate $z \sim z_I$ parameterizes the boundary and is identified with the holomorphic coordinate on the 2d chiral algebra plane.[18] Holographic boundary conditions were formulated in a holomorphic language in [31].

The holomorphic formulation of the boundary conditions makes use of an "internal" $\mathbb{CP}^1$ defined by the ratio x/y as x and y are sent to ∞. In our chosen gauge, the intricacies of the deformed geometry appear asymptotically in a neighbourhood of the poles of the internal $\mathbb{CP}^1$: directions with generic x/y belong to the intersection of all patches I. Near the North and South poles of $\mathbb{CP}^1$, we can assume we are sitting respectively in either of the two "extremal" patches, I_0 where $y - y_i$ are all non-zero or I_∞ where $x - x_i$ are all non-zero. The coordinate transformation between the two patches is

$$z_0 - z_\infty = \sum_{i=1}^{n} \frac{\alpha_i}{(x-x_i)(y-y_i)} = \sum_{k\geq 0}\sum_{l\geq 0} \frac{1}{x^{k+1}y^{l+1}} \sum_{i=1}^{n} \alpha_i x_i^k y_i^l\,. \tag{3.20}$$

[18] As we mentioned in a previous footnote, correlation functions in this non-conformal setup are only defined on a plane. We do not add a point at infinity to extend the holographic boundary to $\mathbb{CP}^1$, as was done in [31].

Each individual $\frac{1}{x^{k+1}y^{l+1}}$ term after the first describes a deformation of the standard $SL(2,\mathbb{C})$ geometry[19]

$$z_0 - z_\infty = \frac{1}{xy} \tag{3.21}$$

obtained by backreation of a coincident stack of branes at $x = y = 0$.

The deformation decays as we approach the boundary and represents holographically the vevs

$$\langle \mathrm{STr}\, X^k Y^l \rangle = \sum_{i=1}^n N_i x_i^k y_i^l \tag{3.22}$$

of the corresponding chiral algebra local operators in a vacuum where X and Y go to commuting expectation values at ∞, with eigenvalues (x_i, y_i) of multiplicity N_i.

This confirms our identification of the geometry as a description of the chiral algebra correlation functions in a non-trivial vacuum.

3.3.1 Restoring $SL(2)_R$ invariance

The original $\mathbb{C}^3$ geometry has an $SL(2)_R$ symmetry rotating the x and y coordinates, which was broken by our choice of trivializations: the coordinates we use come from the trivializations

$$\frac{\bar{x}d\bar{y} - \bar{y}d\bar{x}}{(|x|^2+|y|^2)^2} = \bar{\partial}\left(\frac{\bar{y}}{x(|x|^2+|y|^2)}\right) = \bar{\partial}\left(-\frac{\bar{x}}{y(|x|^2+|y|^2)}\right) , \tag{3.23}$$

which privilege respectively the x or y coordinates.

A more general trivialization would be

$$\frac{\bar{x}d\bar{y} - \bar{y}d\bar{x}}{(|x|^2+|y|^2)^2} = \bar{\partial}\left(\frac{\bar{y} - v\bar{x}}{(x+vy)(|x|^2+|y|^2)}\right) , \tag{3.24}$$

which interpolates between the two as we vary the parameter v.

Such a trivialization gives us another family $z_{v^{-1}}$ of local coordinates on the deformed geometry (in a patch avoiding the $x + vy = x_i + vy_i$ loci):

$$z_{v^{-1}} = z_0 - \sum_{i=1}^n \frac{\alpha_i}{(x - x_i + v(y-y_i))(y-y_i)} = z_\infty + \sum_{i=1}^n \frac{\alpha_i v}{(x-x_i)(x-x_i+v(y-y_i))} . \tag{3.25}$$

[19] Using coordinates $w_0 = z_0 y$, $w_\infty = z_\infty x$, we get the familiar $SL(2,\mathbb{C})$ relation: $xw_0 - yw_\infty = 1$.

Different members of the family are related as

$$z_{u^{-1}} - z_{v^{-1}} = \sum_{i=1}^{n} \frac{\alpha_i(u-v)}{(x-x_i+u(y-y_i))(x-x_i+v(y-y_i))}\,. \tag{3.26}$$

These coordinates will be useful below.

3.4 Determinant correlation functions in a Coulomb background

Recall, the chiral algebra subsector [32] of $\mathcal{N}=4$ SYM is a gauged system of symplectic bosons X, Y in the adjoint representation of $U(N)$.

We will now compute correlation functions of determinant operators in a translation-invariant vacuum $\mathcal{V}$ of the 2d chiral algebra, where X and Y vevs are diagonal matrices with eigenvalues (x_i, y_i) of multiplicity N_i.

The determinant operators we study are

$$\mathcal{D}(m;u;z) \equiv \det\left(m + Z(u;z)\right) = \int \mathrm{d}\psi \mathrm{d}\bar{\psi}\, e^{m\bar{\psi}\psi + \bar{\psi}Z(u;z)\psi}\,, \tag{3.27}$$

which can be expressed in terms of auxiliary (anti)fundamental fermions ψ and $\bar{\psi}$.

Holographically, the insertion of such a determinant represents the presence of a "Giant Graviton" D-brane wrapping a 1-dimensional complex curve which approaches the boundary at a point z, along the line $x + uy + m = 0$.

In order to study correlation functions of multiple determinants, we follow the treatment of [38], also implemented in [92, 39].

We can use fermionization and normal ordering to express a correlator of determinants as

$$\prod_{a=1}^{k} \mathcal{D}(m_a;u_a;z_a) = \int \mathrm{d}\psi \mathrm{d}\bar{\psi} : e^{\sum_a \left[m_a \bar{\psi}_a \psi^a + \bar{\psi}_a Z(u_a;z_a)\psi^a\right]} : e^{-N^{-1}\sum_{a<b} \frac{u_a-u_b}{z_a-z_b} \bar{\psi}_a \psi^b \bar{\psi}_b \psi^a}\,. \tag{3.28}$$

Then, we apply the Hubbard–Stratonovich transformation i.e. introduce an auxiliary bosonic $k \times k$ matrix ρ, with $\rho^a_a = m_a$:

$$\prod_{a=1}^{k} \mathcal{D}(m_a;u_a;z_a) = Z_\rho^{-1} \int \mathrm{d}\psi \mathrm{d}\bar{\psi}\, \mathrm{d}'\rho : e^{\sum_{a,b} \rho^a_b \bar{\psi}_a \psi^b + \sum_a \bar{\psi}_a Z(u_a;z_a)\psi^a} : e^{N \sum_{a<b} \frac{z_a-z_b}{u_a-u_b} \rho^a_b \rho^b_a}\,, \tag{3.29}$$

where $\mathrm{d}'\rho$ is the integration measure for the off-diagonal components of ρ and Z_ρ is a normalization factor. We evaluate the correlation function of chiral algebra fields in the vacuum $\mathcal{V}$:

$$\Big\langle : e^{\bar\psi_a Z(u_a;z_a)\psi^a} : \Big\rangle_{\mathcal{V}} = e^{\bar\psi_a\psi^a \sum_{i=1}^n N_i(x_i+u_a y_i)} \tag{3.30}$$

The correlation function then becomes

$$\Big\langle \prod_{a=1}^k \mathcal{D}(m_a;u_a;z_a)\Big\rangle_{\mathcal{V}} = Z_\rho^{-1}\int \mathrm{d}'\rho\, e^{N\sum_{a<b}\frac{z_a-z_b}{u_a-u_b}\rho^a_b\rho^b_a}\prod_{i=1}^n\left[\det_{a,b}\rho^a_b+(x_i+u_a y_i)\delta^a_b\right]^{N_i}. \tag{3.31}$$

Introducing diagonal $k\times k$ matrices ζ, μ with entries z_a and u_a respectively, the large N saddle equations become

$$[\zeta,\rho]+\left[\mu,\sum_{i=1}^n\alpha_i\frac{1}{\rho+x_i+\mu y_i}\right]=0 \tag{3.32}$$

or

$$[\zeta,\rho]=\sum_{i=1}^n\alpha_i\frac{1}{\rho+x_i+\mu y_i}[\mu,\rho]\frac{1}{\rho+x_i+\mu y_i}\,. \tag{3.33}$$

3.4.1 The spectral curve

In this Section, we will map large N saddles of determinant correlation functions studied above to complex curves in the dual geometry (3.20) using a spectral curve construction.

For each saddle ρ satisfying the equations (3.32)-(3.33), we can define $k\times k$ matrices, functions of a spectral parameter y:

$$\begin{aligned} X(y) &\equiv -\mu y-\rho \\ Z_0(y) &\equiv \zeta-\sum_{i=1}^n\frac{\alpha_i}{y-y_i}\frac{1}{\rho+x_i+\mu y_i}\,. \end{aligned} \tag{3.34}$$

The definition is such that

$$[X(y),Z_0(y)]=[\zeta,\rho]+\left[\mu y+\rho,\sum_{i=1}^n\frac{\alpha_i}{y-y_i}\frac{1}{\rho+x_i+\mu y_i}\right]=0\,. \tag{3.35}$$

We can look at simultaneous eigenvectors of $X(y)$ and $Z_0(y)$ as a function of y, with eigenvalues $x(y)$ and $z_0(y)$, away from $y = y_i$. This defines a holomorphic *spectral curve* $\mathcal{S}_\rho$ in the I_0 patch of the expected dual geometry.

We can also consider the matrix

$$Z_\infty(y) = Z_0(y) - \sum_{i=1}^{n} \frac{\alpha_i}{y - y_i} \frac{1}{X(y) - x_i} = \zeta - \sum_{i=1}^{n} \frac{\alpha_i}{y - y_i} \left[\frac{1}{\rho + x_i + \mu y_i} - \frac{1}{\rho + x_i + \mu y} \right] . \tag{3.36}$$

We can also write it as

$$Z_\infty(y) = \zeta + \sum_{i=1}^{n} \alpha_i \frac{1}{\rho + x_i + \mu y_i} \mu \frac{1}{X(y) - x_i} . \tag{3.37}$$

This matrix is regular at $y = y_i$ for $i = 1, \ldots, n$ and well-defined away from $x(y) = x_i$. It commutes with $X(y)$. We can look at simultaneous eigenvectors of $X(y)$ and $Z_\infty(y)$ as a function of y, with eigenvalues $x(y)$ and $z_\infty(y)$.

If $y \neq y_i$, the matrix also commutes with $Z_0(y)$ and

$$z_\infty(y) = z_0(y) - \sum_{i=1}^{n} \frac{\alpha_i}{y - y_i} \frac{1}{x(y) - x_i} . \tag{3.38}$$

That means $x(y)$ and $z_\infty(y)$ extend the definition of the spectral curve $\mathcal{S}_\rho$ to the I_∞ patch of the expected dual geometry (3.20).

More generally, the collection of matrices

$$Z_I(y) = Z_0(y) - \sum_{i \in I_x} \frac{\alpha_i}{y - y_i} \frac{1}{X(y) - x_i} , \tag{3.39}$$

commute with $X(y)$ and their eigenvectors $z_I(y)$ satisfy (3.17) and therefore extend the spectral curve to all the patches of the expected dual geometry. The spectral curve $\mathcal{S}_\rho$ is thus a curve in the full geometry.

We conjecture that the spectral curve $\mathcal{S}_\rho$ is the support of a B-model D-brane which is dual to the gauge theory saddle ρ. As a basic test, it approaches the boundary at k locations, which at the leading order are at $x = -u_a y$, $z_0 = z_a$, $a = 1, \ldots, k$ in the I_0 patch. At sub-leading order, we find $x(y) + u_a y + \rho_a^a = 0$. As $\rho_a^a = m_a$, this is the desired boundary condition of a brane dual to an insertion of a determinant $\mathcal{D}(u_a; z_a; m_a)$.

We can also compute the sub-leading behaviour of $z_0(y)$:

$$z_0(y) \sim z_a - y^{-1} \sum_{i=1}^{n} \alpha_i \left[\frac{1}{\rho + x_i + \mu y_i} \right]_a^a . \tag{3.40}$$

Following the holographic dictionary, we expect the coefficients on the right hand side:

$$p^a \equiv \sum_{i=1}^{n} \alpha_i \left[\frac{1}{\rho + x_i + \mu y_i} \right]_a^a \tag{3.41}$$

to be the conjugate momentum to m_a on the B-model side, i.e. the derivative of the D-brane action with respect to m_a. At the same time, we recognize $p^a = \frac{\partial S}{\partial m_a}$ for the semiclassical action S at the gauge theory saddle. This ensures that the action for this B-model D-brane matches the semiclassical action for the gauge theory saddle, up to m-independent terms.

3.4.2 Restoring $SL(2)_R$ invariance

Our formalism in this Section is not manifestly covariant under $SL(2)_R$.

Let us first consider how the spectral curve transforms under inversion $\mu \to -\mu^{-1}$, $\rho \to \rho\mu^{-1}$. When acting on an eigenvector V of $X(y)$ and $Z_\infty(y)$ with eigenvalues x and z_∞ we have

$$xV = -\mu y V - \rho V \tag{3.42}$$

$$z_\infty V = \zeta V + \sum_{i=1}^{n} \alpha_i \frac{1}{\rho + x_i + \mu y_i} \mu \frac{1}{x - x_i} V , \tag{3.43}$$

which can be rewritten as

$$y\mu V = (-x\mu^{-1} - \rho\mu^{-1})\mu V \tag{3.44}$$

$$z_\infty \mu V = \left(\zeta + \sum_{i=1}^{n} \frac{\alpha_i}{x - x_i} \frac{1}{\rho\mu^{-1} + y_i + x_i\mu^{-1}} \right) \mu V . \tag{3.45}$$

That means μV is a simultaneous eigenvector for matrices:

$$Y(x) \equiv -x\mu^{-1} - \rho\mu^{-1} \tag{3.46}$$

$$Z_\infty(x) \equiv \zeta + \sum_{i=1}^{n} \frac{\alpha_i}{x - x_i} \frac{1}{\rho\mu^{-1} + y_i + x_i\mu^{-1}} , \tag{3.47}$$

with eigenvalues $y(x)$ and $z_\infty(x)$. These are built just as $X(y)$ and $Z_0(y)$, with $\mu \to -\mu^{-1}$, $\rho \to \rho\mu^{-1}$. Therefore, the spectral curve transforms under inversion as $y \to -x$, $x \to y$, $z_0 \to z_\infty$.

More generally, we can define

$$Z_{v^{-1}}(y) \equiv Z_0(y) - \sum_{i=1}^{n} \frac{\alpha_i}{(y-y_i)} \frac{1}{X(y) - x_i + v(y-y_i)}\,, \tag{3.48}$$

which can be rewritten as

$$Z_{v^{-1}}(y) = \zeta + \sum_{i=1}^{n} \alpha_i \frac{1}{\rho + x_i + \mu y_i}(\mu - v)\frac{1}{X(y) - x_i + v(y-y_i)}\,. \tag{3.49}$$

When acting on an eigenvector V of $X(y)$ and $Z_{v^{-1}}(y)$ with eigenvalues x and $z_{v^{-1}}$ we have

$$xV = -\mu y V - \rho V \tag{3.50}$$

$$z_{v^{-1}}V = \zeta + \sum_{i=1}^{n} \alpha_i \frac{1}{\rho + x_i + \mu y_i}(\mu - v)\frac{1}{x - x_i + v(y-y_i)}\,, \tag{3.51}$$

which can be rewritten as

$$y(\mu - v)V = \big(-(x+vy)(\mu-v)^{-1} - \rho(\mu-v)^{-1}\big)(\mu-v)V \tag{3.52}$$

$$z_{v^{-1}}(\mu - v)V = \zeta(\mu-v)V \tag{3.53}$$

$$+\sum_{i=1}^{n} \frac{\alpha_i}{(x+vy)-(x_i+vy_i)} \frac{1}{\rho(\mu-v)^{-1} + y_i + (x_i+vy_i)(\mu-v)^{-1}}(\mu-v)V\,. \tag{3.54}$$

Therefore, $(\mu - v)V$ is a simultaneous eigenvector for matrices parametrized by $x + vy$:

$$Y(x+vy) \equiv -(x+vy)(\mu-v)^{-1} - \rho(\mu-v)^{-1}$$

$$Z_{v^{-1}}(x+vy) \equiv \zeta + \sum_{i=1}^{n} \frac{\alpha_i}{(x+vy)-(x_i+vy_i)} \frac{1}{\rho(\mu-v)^{-1} + y_i + (x_i+vy_i)(\mu-v)^{-1}}\,,$$

with eigenvalues $y(x+vy)$ and $z_{v^{-1}}(x+vy)$. These are built just as $X(y)$ and $Z_{v^{-1}}(y)$, with $\mu \to -(\mu-v)^{-1}$, $\rho \to \rho(\mu-v)^{-1}$, $y \to -(x+vy)$, $x \to y$.

This shows that the construction of the spectral curve $\mathcal{S}_\rho$ is $SL(2)_R$-covariant.

3.4.3 Single-trace insertion and bulk-to-boundary propagator

A single-trace insertion in the correlation function of determinants can also be evaluated in the large N limit [38]. Without loss of generality, we can look at the insertion of $\mathrm{Tr}\, Y^n(w)$, or better a generating function

$$N\mathrm{Tr}\log\left(\hat{y}-Y(w)\right)=N\mathrm{Tr}\log\hat{y}-\sum_{n=1}^{\infty}\frac{1}{n\hat{y}^n}N\mathrm{Tr}Y(w)^n\,. \tag{3.55}$$

The main effect of vacuum vevs on Feynman diagrams is that Y fields can either be contracted with a propagator or replaced by their vev. The dominant Feynman diagrams have the topology of a disk, with the auxiliary fermions running along the boundary.

After some combinatorial manipulations analogous to these in Chapter 2, the correlation function is the sum of the classical vev $N\mathrm{Tr}\log\left(\hat{y}-Y_\infty\right)$ and the leading quantum correction

$$-N\mathrm{Tr}_{k\times k}\log\left(1-\frac{1}{w-\zeta}\sum_{i=1}^{n}\frac{1}{\hat{y}-y_i}\frac{\alpha_i}{\rho+x_i+\mu y_i}\right)\,. \tag{3.56}$$

This can be written in the I_0 patch as

$$-N\mathrm{Tr}_{k\times k}\log\frac{w-Z_0(\hat{y})}{w-\zeta}=N\sum_{a=1}^{k}\log(w-z_a)-N\log\det\left(w-Z_0(\hat{y})\right)\,. \tag{3.57}$$

The second term on the right hand side can be computed from a contour integral

$$\oint\log(\hat{y}-y')\partial_{y'}\log\det\left(w-Z_0(y')\right)=\sum_{y'_*(w)}\log(\hat{y}-y'_*(w)) \tag{3.58}$$

and evaluated as a sum over the intersection points y'_* of the spectral curve and the surface $z_0=w$.

We can now compare this with a B-model calculation of the leading Witten diagram: the integral of a boundary-to-bulk propagator on the conjectural D-brane world-volume[20]

$$\int_{\mathcal{S}_\rho}\partial^{-1}\alpha_w\,. \tag{3.59}$$

[20]See appendix F of [31] for a review of the coupling of KS fields to D1-branes.

Analogously to the analysis in Chapter 2, we expect to have a representative for the bulk-to-boundary propagator sourced by $\operatorname{Tr} Y^k(w)$ that is supported on the complex surface $z_0 = w$:

$$\partial^{-1}\alpha_{k;w} = y^k \delta_{z_0=w} \,. \tag{3.60}$$

Thus the bulk-to-boundary propagator for the insertion (3.55) is

$$\partial^{-1}\alpha_w[\hat{y}] = \log(\hat{y} - y)\delta_{z_0=w} \,. \tag{3.61}$$

It is straightforward to integrate $\partial^{-1}\alpha_w[\hat{y}]$ on the spectral curve: in the I_0 patch, it only receives contributions from the intersections between the spectral curve and $z_0 = w$ and reproduces (3.58).

In order to explore other patches we can rewrite (3.60) as

$$\partial^{-1}\alpha_{k;w} = y^k \delta_{\Delta(w)} \,, \tag{3.62}$$

where $\Delta(w)$ is a complex surface defined in the I_0 patch by

$$(z_0 - w)\prod_{i=1}^{n}(y - y_i) = 0 \,. \tag{3.63}$$

In the I_∞ patch it is given by

$$\left(z_\infty - w + \sum_{j=1}^{n} \frac{\alpha_j}{(x - x_j)(y - y_j)}\right)\prod_{i=1}^{n}(y - y_i) = 0 \,. \tag{3.64}$$

This surface reaches the boundary in the neighborhoods of $y = y_i$ for all $z_\infty \neq w$, eg.

$$y \sim y_1 + \frac{\alpha_1}{x(z_\infty - w)} + \frac{\alpha_1 x_1}{x^2(z_\infty - w)} - \frac{\alpha_1}{x^2(z_\infty - w)^2}\sum_{i>1}\frac{\alpha_i}{(y_1 - y_i)} + \dots \tag{3.65}$$

These regions of the surface contribute terms to the propagator which decay with the power law near the boundary and represent the vevs of other local operators in the presence of the insertion at $z = w$, starting with the vev of the identity operator:

$$\partial^{-1}\alpha_{k;w} \sim \sum_{i=1}^{n} y_i^k \delta_{\frac{y}{x}=0} + \dots \,. \tag{3.66}$$

Chapter 4

Semi-chiral operators in 4d $\mathcal{N} = 1$ gauge theories

4.1 Introduction

Four-dimensional $\mathcal{N} = 1$ supersymmetric quantum field theories have been the subject of intense investigation (see [93] for a review, and [94, 95] for a review of some modern aspects of the subject). They display phenomena of great physical interest, such as confinement and chiral symmetry breaking, which have a rich interplay with quantities that can be exactly computed or strongly constrained via supersymmetry [93].

An important tool is the study of F-terms and chiral operators, which are protected by two of the four supercharges. These constrain the space of supersymmetric vacua as well as important classes of deformations of the theories [96, 97, 98, 99, 100]. They are also the first quantities one would typically match across dualities such as Seiberg duality and its generalizations [100, 101, 102, 103].

Another important protected quantity which is available for $\mathcal{N} = 1$ superconformal field theories is the superconformal index [104, 105, 106, 107, 108, 109]. The superconformal index is a power series whose coefficients are Witten indices of the space of local operators with a given set of charges. It is a robust quantity which can be computed semi-classically, disregarding interactions.

Essentially by definition, the coefficients of the superconformal index also count, with signs, the cohomology of the spaces of local operators with respect to one of the four supercharges, denoted simply as Q in the following. The cohomology of Q thus provides a natural "categorification" of the data in the superconformal index [110, 111].

Concrete calculations in these references demonstrated surprising simplifications which can be rigorously proven and extended with the help of the *holomorphic twist* of the $\mathcal{N} = 1$ SCFTs [112]. This is a procedure which maps the full physical theory to a simplified theory whose gauge-invariant local operators are the Q-cohomology of the original theory.[1]

The notion of Q-cohomology of the space of local operators appears to be well-defined even for theories which are not superconformal. It should describe local operators which are protected by a single supercharge. In analogy with the chiral ring, we dub these operators the *semi-chiral ring* of the theory. In the main text of the Chapter, we will discuss why the term "ring" is reasonable here. Again, one can explore this structure by the *holomorphic twist* of the theory, which is also well-defined in the absence of conformal symmetry.

The twisting procedure has recently been developed in a mathematically rigorous perturbative setting [113, 6], allowing one to simplify the twisted SQFTs to holomorphic quantum field theories defined on $\mathbb{C}^2$. The simplified theories are endowed with extra symmetries and structures which are a four-dimensional analogue of a two-dimensional chiral algebra [114, 115, 9, 116, 11]. The implications of these structures on the structure of the Q-cohomology and the associated constraints on the original physical theory are mostly unexplored.

It is important to observe that the Q-cohomology is much less "robust" than the Witten index. In particular, interactions are expected to modify the naive free field theory answers. In the literature, Q-cohomology calculations have mostly been done at tree level, invoking or conjecturing non-renormalization theorems to justify disregarding perturbative or non-perturbative corrections. For example, this approach has been rather successful for planar $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [111]. However, the presence of quantum corrections to the chiral ring, such as the generalized Konishi anomaly [117], suggests the existence of quantum corrections to the Q-cohomology of general $\mathcal{N} = 1$ gauge theories, which we will make explicit in this Chapter.

The main purpose of this work is to explore the effect of interactions on the semi-chiral ring of pure $\mathcal{N} = 1$ gauge theory. We will work with the assumption that the space of local operators can be studied systematically in the far UV where the theory is asymptotically free. Concretely, that means we will start from the semi-chiral ring of the free theory and study how interactions deform the action of Q order-by-order in perturbation theory,

[1] Essentially, one adds Q to the original BRST differential of the physical theory. This enlarges the gauge symmetry of the problem and allows further simplifications to be carried on systematically.

gradually reducing the size of the cohomology by pairing up operators which were in cohomology at the previous loop order.[2]

Our main assumption will be that non-perturbative corrections may at most further reduce the size of the perturbative semi-chiral ring. Instanton effects violate the anomalous R-symmetry selection rules which constrain the action of Q in perturbation theory, allowing for further cancellation between perturbative cohomology classes.[3]

Crucially, all calculations can be done in the twisted theory.[4] Classically, this reproduces and generalizes the simplifications which were found in direct calculations in $\mathcal{N}=4$ SYM [110, 111]. Quantum mechanically, this greatly reduces the complexity of the relevant Feynman diagrams. The one-loop corrections are semi-chiral analogues of the generalized Konishi anomaly equations [119, 120, 117]. Our first observation is that the perturbative corrections cannot stop at 1-loop, as the resulting differential is not nilpotent. Indeed, for pure $SU(2)$ gauge theory we can even "bootstrap" the necessary two-loop corrections, essentially bypassing an actual calculation of the Feynman diagrams.

Companion work [7, 116] outline the general definition and computation of the Feynman diagrams which contribute to the differential and to other algebraic structures in twisted gauge theories with a general matter content and interactions.

In order to explore the structure of the cohomology, we conduct a systematic calculation of perturbative cohomology classes in pure $SU(2)$ gauge theory. We are able to include in the calculation operators with up to fourteen derivatives. The cohomology at one loop is already sufficiently sparse to forbid further perturbative corrections.

Remarkably, the quantum numbers of the non-zero perturbative cohomology representatives appear compatible with the hypothesis that instanton effects will ultimately lift all operators in the semi-chiral ring which are not in the chiral ring. Recall that the chiral ring of the theory has a single generator, the gaugino bilinear C_0.[5] It is expected to satisfy the relation $C_0^2 \sim \Lambda^6$ (in the chiral ring of $SU(2)$) and thus acquire a vev which distinguishes the two gapped vacua of the theory.[6]

[2] This procedure is somewhat analogous to a spectral sequence in mathematics and is formalized by the notion of homotopy transfer. See appendix B.4.

[3] See [118] for a review of semi-classical instanton calculations. It would be nice to do similar calculations in twisted theories.

[4] This statement can typically be promoted to a theorem in perturbation theory, but may require some caution non-perturbatively. In principle, the choice of path integration contour/solution of Ward identities for the simplified theories might not be unique, leading to the higher-dimensional analogue of conformal blocks. We will not explore this issue in this Chapter, but it would be interesting to do so.

[5] The glueball field is usally denoted S in the SUSY literature.

[6] More generally, we expect $C_0^h \sim \Lambda^{2h}$ for a gauge group with dual Coxeter element h.

If the hypothesis is correct, the semi-chiral ring computed in the UV will fully match the semi-chiral ring of the theory in the far IR, which is trivial up to the order parameter distinguishing the two vacua. It would be interesting to test further the hypothesis that the semi-chiral ring in the UV and IR should coincide for general twisted theories, perhaps by looking at gauge theories which enjoy Seiberg-like dualities. See [121] for an earlier attempt to work outside the chiral ring.

We also present systematic calculations in $SU(3)$ gauge theory up to five derivatives as well as derive some abstract results valid for any gauge group. Our main analytic result is a surprising observation: one-loop corrections to the differential in pure $\mathcal{N}=1$ gauge theory make the theory topological rather than holomorphic. We are sorely tempted to take this result as a UV manifestation of the IR confinement of the physical theory: a gapped theory is indeed topological at large distances.

Concretely, this result places strong constraints on the possible IR behaviour of the theory: the holomorphic twist of the IR degrees of freedom must also be secretly topological. Further investigation (and more examples) of such *holomorphic confinement* constraints is warranted.

Another analytic result we present is a full calculation of the semi-chiral ring in the planar approximation for $SU(N)$. The structure of the calculation leads us to propose a holographic dual description for the large N twisted theory, involving the B-model on a certain backreacted version of $\mathbb{C}^3$.

Many of our methods extend to more general gauge theories. We anticipate some of these generalizations by a uniform discussion of the large N cohomology and potential twisted holographic duals for a variety of theories, including SQCD, $\mathcal{N}=4$ SYM and quivers associated to Calabi-Yau cones [112].

4.1.1 Structure of the Chapter

Section 4.2 discusses in detail the expected properties of holomorphic twists of $\mathcal{N}=1$ supersymmetric QFTs. We review a useful superspace technology in Section 4.2.2, discuss infinite symmetry enhancement in Section 4.2.3, and the twisted version of the stress tensor multiplet in Section 4.2.4.

Section 4.3 discusses the holomorphic twist of pure $\mathcal{N}=1$ gauge theory and the presentation as a 4d bc system. In Section 4.3.1 we discuss the space of local operators in the free theory and at tree level. In Section 4.3.2 we discuss loop corrections to the BRST differential acting on local operators.

In Section 4.4 we extend the discussion to general Lagrangian gauge theories. In Section 4.5 we perform cohomology calculations in the planar approximation. We discuss a general strategy, apply it in full to pure gauge theory, and implement it schematically for several other examples.

In Section 4.6 we use the large N analysis to motivate a holographic proposal for twisted pure gauge theory and discuss twisted versions of standard holographic dualities for certain quiver gauge theories.

Section 4.7 contains some auxiliary computations of characters of spaces of local operators. Section 4.8 contains the results of our concrete calculations of local operator cohomology for pure $SU(2)$ and $SU(3)$ guage theories.

We conclude with several appendices. Appendix B.1 reviews some properties of λ-brackets. Appendix B.2 discusses the calculation of these brackets at tree level and appendix B.3 discusses the calculation of brackets at one loop. Appendix B.4 reviews the notion of homotopy transfer and the homotopy perturbation lemma. Appendix B.5 reviews an origin of L_∞ algebras in QFT. Finally, appendix B.6 contains a careful comparison of different perspectives on twisted supersymmetric QFTs.

4.2 Symmetries and kinematics

Our starting point is the four-dimensional $\mathcal{N} = 1$ supersymmetry algebra in Euclidean signature. Using the isomorphism $\mathrm{Spin}(4) \cong SU(2) \times SU(2)$, we have an isomorphism between the vector representation of Spin(4) and (the product of) the spinor representations of the two $SU(2)$'s. Thus we denote our supercharges with the two-component spinors[7] $Q_{\dot{\alpha}}$ and $\bar{Q}_\alpha$, subject to anticommutation relations:

$$\{Q_{\dot{\alpha}}, \bar{Q}_\beta\} = P_{\dot{\alpha}\beta}\,, \tag{4.1}$$

$$\{Q_{\dot{\alpha}}, Q_{\dot{\beta}}\} = \{\bar{Q}_\alpha, \bar{Q}_\beta\} = 0\,, \tag{4.2}$$

where $P_{\dot{\alpha}\beta}$ are the translation generators. Rotation generators in the two $SU(2)$ subgroups will be denoted $M_{\dot{\alpha}\dot{\beta}}$ and $\bar{M}_{\alpha\beta}$ respectively.

Without loss of generality, we choose $Q \equiv Q_{\dot{-}}$ as a twisting supercharge.[8] This choice equips Euclidean spacetime with a complex structure such that $z^\alpha \equiv x^{\dot{+}\alpha}$ are holomorphic

[7] We have switched the standard two-component spinor notation for dotted/undotted indices as dotted indices will soon drop out [122].

[8] Different choices of Q will simply define a different complex structure on $\mathbb{C}^2$. The moduli space of twisting supercharges forms the nilpotence variety of the supersymmetry algebra [123, 6] (see also 2.1.1 of [8] for an introduction).

coordinates and $\bar{z}^\alpha \equiv x^{\dot{-}\alpha}$ are anti-holomorphic.[9] We immediately see that the anti-holomorphic translations are Q-exact:

$$\{Q, \bar{Q}_\alpha\} = \partial_{\dot{-}\alpha} = \partial_{\bar{z}^\alpha} \,. \tag{4.3}$$

As a result, the twisted theory is holomorphic in a cohomological sense [124, 125].

The choice of twisting supercharge breaks the four-dimensional rotation group to the $SU(2)$ subgroup generated by the $\bar{M}_{\alpha\beta}$ in the twisted theory, which acts trivially on dotted spinors. These are holomorphic symplectic rotations of the z^α, preserving the holomorphic volume form $\mathrm{d}^2 z = \mathrm{d}z^1 \wedge \mathrm{d}z^2$.

In the presence of an unbroken $U(1)_R$ symmetry, we can combine $M_{\dot{+}\dot{-}}$ with the R-symmetry generator to define an extra twisted rotation generator[10] $M_R \equiv M_{\dot{+}\dot{-}} - R$, which extends the $SU(2)$ above to the entire $U(2)$ group of holomorphic rotations of the z^α. The gauge theories we consider in this Chapter have a $U(1)_R$ symmetry which is anomalous at one loop. The $U(2)$ rotation symmetry formally survives as a symmetry in perturbative calculations, but is broken down to $SU(2)$ by familiar instanton effects.

For cohomology computations, it is convenient to define a $\mathbb{Z}$-grading on the twisted theory such that Q increases the cohomological degree by 1. We can combine the Cartan generator $M_{\dot{+}\dot{-}}$ with the ghost number symmetry of the physical theory to play the role of the cohomological degree $C \equiv \mathrm{gh} - M_{\dot{+}\dot{-}}$. This choice is a bit unusual, but is very convenient in the absence of $U(1)_R$ symmetries.

In perturbation theory, interactions will lift some part of the cohomology but will not add new cohomology classes. Thus we expect the cohomology to still be supported in non-positive cohomological degree for $\mathcal{N} = 1$ SQFTs expanded perturbatively around the free point. It would be interesting to know if this property holds non-perturbatively as well, or in generic $\mathcal{N} = 1$ SQFTs.

4.2.1 Semi-chiral operators and semi-short superconformal multiplets

In a superconformal theory the Q-cohomology is somewhat constrained. Operators can be grouped into superconformal multiplets, generated from primary operators which are

[9] Complex conjugation acts as $(z_\alpha)^* = \epsilon_{\alpha\beta}\bar{z}^\beta$.

[10] In conventions where $[M_{\dot{+}\dot{-}}, Q_{\dot{-}}] = -Q_{\dot{-}}$ and $[R, Q_{\dot{-}}] = -Q_{\dot{-}}$. Charges and gradings are summarized in table 4.7.1.

annihilated by superconformal and special conformal generators. Each primary operator transforms in some representation of Spin(4), has some charge R under $U(1)_R$ and some scaling dimension Δ. Other "descendant" operators are obtained by acting on primaries with supercharges and derivatives.

For generic Δ, supercharges and derivatives act freely. In particular, an operator in such a supermultiplet is Q-closed only if it is Q-exact. However, when Δ achieves some minimum values (aka saturates BPS bounds), some descendant operators are missing and the multiplets become short. An obvious example is the identity operator, which has no descendants. Obviously, it is a non-trivial Q-cohomology class. Generally, short multiplets for $\mathcal{N} = 1$ superconformal symmetry are classified, see e.g. [126], with separate shortening conditions for Q's and $\bar{Q}$'s. While we don't perform the computations explicitly here, one can work out exactly the Q-cohomology of such multiplets.

It is instructive to look at some simple subalgebras of the superconformal algebra. For example, consider the $\mathfrak{su}(1|1)$ algebra generated by our supercharge $Q_{\dot{-}}$ and its Hermitian conjugate $S_{\dot{+}}$. Their anticommutator is

$$\{Q_{\dot{-}}, S_{\dot{+}}\} = \Delta + \frac{3}{2}R - M_{\dot{+}\dot{-}}\,. \tag{4.4}$$

Generic multiplets consist of two operators, which cancel in Q-cohomology. Short multiplets appear when $\Delta = M_{\dot{+}\dot{-}} - \frac{3}{2}R$ and consist of a single Q-cohomology class.

The $\mathfrak{su}(2|1)$ algebra generated by $Q_{\dot{\alpha}}$ and $S_{\dot{\beta}}$ is also very useful. It closes on

$$\{Q_{\dot{\alpha}}, S_{\dot{\beta}}\} = (\Delta + \frac{3}{2}R)\epsilon_{\dot{\alpha}\dot{\beta}} - M_{\dot{\alpha}\dot{\beta}}\,. \tag{4.5}$$

Generic multiplets consist of quadruplets of $\mathfrak{su}(2)$ representations: a primary of spin j, two Q-descendants with spins $j \pm \frac{1}{2}$, and a Q^2 descendant of spin j. For $j = 0$, the Q-descendants have spin $\frac{1}{2}$ only.

For general j, the only short multiplet ("A_1") is missing the $j - \frac{1}{2}$ Q-descendants and the Q^2 descendants. It is easy to see that there is a single Q-cohomology class, the $Q_{\dot{+}}$ descendant of the primary operator with the most positive $M_{\dot{\alpha}\dot{\beta}}$ charge. It has cohomological degree $-2j - 1$. Being a $Q_{\dot{+}}$ descendant, it is not in the chiral ring.

For $j = 0$, there are two possible short multiplets. One ("A_2") is missing the Q^2 descendant. The $Q_{\dot{+}}$ descendant gives again a Q-cohomology class, of cohomological degree -1, which is not in the chiral ring.

The second type of $j = 0$ short multiplet ("B_1") consists of the primary only, which is thus a Q-cohomology class of cohomological degree 0 and also a chiral ring element.

In conclusion, Q-cohomology elements in superconformal theories are always annihilated by the $M_{\dot{+}\dot{+}}$ rotation generator and have non-positive cohomological degree. The cohomology of degree 0 consists of the familiar *chiral ring* operators.

4.2.2 Chiral superspace and Dolbeault forms

Four-dimensional $\mathcal{N} = 1$ SQFTs are often formulated in the language of superspace [122, 127]. Spacetime is promoted to a supermanifold with extra odd coordinates $\theta^{\dot{\alpha}}$ and $\bar{\theta}^{\alpha}$, and supersymmetric multiplets of operators are collected into "superfields"[11] which depend on the superspace coordinates.

In our context, it is natural to employ a "chiral" superspace which only employs $\bar{\theta}^{\alpha}$ odd coordinates, with the property that $\bar{Q}_{\alpha} = -\partial_{\bar{\theta}^{\alpha}}$ when acting on superfields.[12] The identification $\bar{\theta}^{\alpha} \leftrightarrow \mathrm{d}\bar{z}^{\alpha}$ allows us to present superfields as Dolbeault forms of type $(0, \bullet)$. The Dolbeault operator can be written as

$$\bar{\partial} \equiv \bar{\theta}^{\alpha} \partial_{\bar{z}^{\alpha}} \,. \tag{4.6}$$

Given such a superfield/form $\mathcal{O}$, we denote its n-th form component as $\mathcal{O}^{(n)}$. The whole superfield can be reconstructed by a repeated application of $\bar{\theta}^{\alpha}\bar{Q}_{\alpha}$ on the $\mathcal{O}^{(0)}$ component:

$$\mathcal{O}[\bar{\theta}] = e^{\bar{\theta}^{\alpha}\bar{Q}_{\alpha}} \mathcal{O}^{(0)} = \mathcal{O}^{(0)} + \mathcal{O}^{(1)} + \mathcal{O}^{(2)} \,. \tag{4.7}$$

The combination $Q + \bar{\partial}$ coincides with the superspace derivative $D_{\dot{-}}$ in the physical theory and acts naturally on superfields:

$$\{Q + \bar{\partial}, \bar{Q}_{\alpha}\} = 0 \,. \tag{4.8}$$

As a result, we have the descent relation:

$$(Q + \bar{\partial})\mathcal{O}[\bar{\theta}] = (Q\mathcal{O})[\bar{\theta}] \,, \tag{4.9}$$

or, in components,

$$Q\mathcal{O}^{(k)} + \bar{\partial}\mathcal{O}^{(k-1)} = (Q\mathcal{O})^{(k)} \,. \tag{4.10}$$

[11] We notice the unfortunate convention of using the term "superfield" to denote operators on superspace, even when they are not elementary fields.

[12] In full superspace, such a definition would be combined with $D_{\dot{\alpha}} = \partial_{\theta^{\dot{\alpha}}}$. Given a standard $\mathcal{N} = 1$ superfield we can Taylor expand in $\theta^{\dot{\alpha}}$ to restrict to the chiral superspace.

We call a superfield $\mathcal{O}$ *semi-chiral*[13] if it satisfies

$$D_{\dot{-}}\mathcal{O} = (Q + \bar{\partial})\mathcal{O} = 0\,. \tag{4.11}$$

From the descent relations, we see that a superfield $\mathcal{O}$ is semi-chiral if and only if its 0-form component $\mathcal{O}^{(0)}$ is Q-closed. For semi-chiral superfields, the descent relations become:

$$Q\mathcal{O}^{(1)} + \bar{\partial}\mathcal{O}^{(0)} = 0\,, \qquad Q\mathcal{O}^{(2)} + \bar{\partial}\mathcal{O}^{(1)} = 0\,. \tag{4.12}$$

Analogously, if $\mathcal{O}^{(0)}$ is Q-exact, i.e. $\mathcal{O}^{(0)} = Q\mathcal{U}^{(0)}$ then

$$\mathcal{O} = (Q + \bar{\partial})\mathcal{U}\,. \tag{4.13}$$

In short, we can identify the cohomology of Q with the space of semi-chiral superfields modulo the image of $(Q+\bar{\partial})$, simply by a translation in the superspace directions. We will often do so implicitly.

Finally, we can comment on the OPE of semi-chiral superfields. As anti-holomorphic translations are Q-exact, non-holomorphic terms in the OPE must be Q-exact as well. Restricting to the Q-cohomology, the OPE will thus be meromorphic. Hartog's theorem forbids the existence of meromorphic functions on $\mathbb{C}^2$ singular at one point. As a consequence, the OPE must be non-singular when restricted to the Q-cohomology. This justifies the claim that semi-chiral operators can be organized in a *semi-chiral ring*. Next, we will discuss more refined algebraic structures hidden in the OPE of semi-chiral superfields.

4.2.3 Infinite dimensional symmetries

The descent relations satisfied by the components of a semi-chiral superfield can be interpreted as conservation laws which hold in the twisted theory. Each semi-chiral superfield $\mathcal{O}$ actually gives rise to infinitely many conservation laws: given any $\bar{\partial}$-closed $(2,\bullet)$ form ρ the combination $\rho\,\mathcal{O}$ is also conserved.

This is completely analogous to what happens for holomorphic operators in 2d CFTs [9, 10]. Recall that a 2d CFT always includes a holomorphic stress tensor $T(z)$, and may include other holomorphic operators. Every holomorphic local operator in 2d is automatically a conserved current, being invariant under $\bar{\partial}$. Moreover, every holomorphic local operator remains a conserved current when multiplied by a holomorphic function,

[13] In analogy to *(anti)chiral* superfields, which satisfy $D_{\dot{+}}\mathcal{O} = D_{\dot{-}}\mathcal{O} = 0$.

giving rise to various infinite symmetry enhancements. For example, the infinite Virasoro generators arise from the conservation of the holomorphic stress tensor $T(z)$.

More generally, every holomorphic operator $\mathcal{O}(z)$ in a 2d CFT gives rise to a tower of symmetry generators $\hat{\mathcal{O}}_n$, $n \in \mathbb{Z}$, acting on the space of local operators near the origin by[14]

$$(\hat{\mathcal{O}}_n A)(0) = \oint_{S^1} \frac{\mathrm{d}z}{2\pi i} z^n \mathcal{O}(z) A(0)\,. \tag{4.14}$$

For non-negative n, these modes capture the singular part of the OPE of $\mathcal{O}(z)$ with other operators. In the mathematical literature, the non-negative modes are sometimes collected in a generating function called the *λ-bracket* [128]:

$$\{\mathcal{O}\,{}_\lambda\, A\}(0) = \oint_{S^1} \frac{\mathrm{d}z}{2\pi i} e^{\lambda z} \mathcal{O}(z) A(0)\,. \tag{4.15}$$

Conversely, the negative modes $\hat{\mathcal{O}}_{-n-1}$ define the regularized products $(\partial^n \mathcal{O}\, A)$.

The contour-integral definition of the negative modes involves the singular forms $z^{-n-1}\frac{\mathrm{d}z}{2\pi i}$. In order to set up an analogy to problems in higher dimensions, it is useful to identify these singular forms as derivatives of the 2d Bochner-Martinelli kernel $\omega_{\mathrm{BM}} = \frac{1}{2\pi i}\frac{1}{z}$, i.e. the Green's function for the $\bar\partial$ operator.

Finally, the collection of holomorphic 1-forms $\frac{\mathrm{d}z}{2\pi i} z^n$ which appear in the definition of the modes spans the Dolbeault cohomology $H^{1,0}(\mathbb{C}^\times) \cong \mathbb{C}[z^{\pm 1}]\,\mathrm{d}z$. It is also useful to give a Serre-dual description as the basis of Laurent monomials in $H^{0,0}(\mathbb{C}^\times) \cong \mathbb{C}[z^{\pm 1}]$.[15] This is just the statement that the modes $\hat{\mathcal{O}}_n$ are the coefficients of a Laurent expansion of $\mathcal{O}(z)$ around $z = 0$.

We are now ready to generalize these notions to n complex dimensional holomorphic theories. The symmetries

$$\hat{\mathcal{O}}_\rho = \oint_{S^{2n-1}} \rho\, \mathcal{O}\,, \tag{4.16}$$

of the space of local operators associated to the holomorphic superfield $\mathcal{O}$ are now labelled by forms ρ living in $H^{n,\bullet}(\mathbb{C}^n - 0)$ [11, 9]. Indeed, such an operator varies by Q-exact amounts if we deform the S^{2n-1} integration contour and if we vary ρ by a $\bar\partial$-exact form.

[14] We are using a mathematical indexing convention here, omitting a shift by the conformal dimension of $\mathcal{O}(z)$ which is standard in the physics literature. With these conventions, $\hat{\mathcal{O}}_{-1}$ represents a regularized multiplication by $\mathcal{O}(0)$.

[15] The duality pairing is given by integration on S^1, with the explicit dual of the Laurent monomial z^n given by the one-form $z^{-n-1}\frac{\mathrm{d}z}{2\pi i}$.

A standard calculation reveals that the Dolbeault cohomology $H^{n,\bullet}(\mathbb{C}^n - 0)$ is concentrated entirely in degrees 0 and $n-1$. The degree $n-1$ part is dual, under integration on S^{2n-1}, to polynomials $\mathbb{C}[z_1, \ldots, z_n]$. It consists of the n-dimensional Bochner-Martinelli kernel ω_{BM} and its holomorphic derivatives.[16] It is analogous to the negative modes in 2d and captures the regularized products $(\partial_1^{k_1} \cdots \partial_n^{k_n} \mathcal{O}\, A)$. We can denote the corresponding modes as $\hat{\mathcal{O}}_{-k_1-1,\cdots,-k_n-1}$.

The degree 0 part consists of polynomial holomorphic top forms $\mathbb{C}[z_1, \ldots, z_n]\mathrm{d}^n z$.[17] The integral (4.16) thus employs the $(n-1)$-th descendant in $\mathcal{O}$. These modes are analogous to the non-negative modes in 2d and can also be combined into a λ-bracket [129]:

$$\{\mathcal{O}_1\, {}_\lambda \mathcal{O}_2\} = \oint_{S^{2n-1}} e^{\lambda \cdot z} \mathrm{d}^n z\, \mathcal{O}_1(z)\mathcal{O}_2(0)\,, \tag{4.18}$$

which generalizes the secondary products in TFTs [130].[18] In analogy with the vertex algebra situation, we will say that fields $\mathcal{O}_1$ and $\mathcal{O}_2$ have *regular* OPE if such λ-brackets vanish [131] (see also [129, 8, 132]). See appendix B.1 for some algebraic properties of this bracket.

The λ-bracket and the regularized product in n-dimensional holomorphic theories satisfy associativity axioms which generalize the axioms of a vertex operator algebra. However, these axioms are further complicated in this derived setting: the operations can be defined on local operators rather than on their Q-cohomology, so that extra higher brackets appear in the associativity relations, much as L_∞ or A_∞ algebras appear in two-dimensional TFTs [8].

[16] The Bochner-Martinelli kernel ω_{BM} is the $(0, n-1)$-form defined to be the fundamental class of S^{2n-1} when wedged with $\mathrm{d}^n z$ (some conventions define ω_{BM} to be this $(n, n-1)$-form instead). So, for any continuously differentiable function f on the closure of a domain D,

$$f(0) = \oint_{\partial D} f(z)\, \omega_{\mathrm{BM}}\, \mathrm{d}^n z - \int_D \bar{\partial} f(z)\, \omega_{\mathrm{BM}}\, \mathrm{d}^n z\,. \tag{4.17}$$

When f is holomorphic, the second term disappears. When f is cohomologically holomorphic, the second term is Q-exact, being proportional to the first descendant of f.

[17] The dual $(0, n-1)$ forms are best described by Čech representatives of the form $\mathbb{C}[z_1^{-1}, \ldots, z_n^{-1}]\frac{1}{z_1 \cdots z_n}$ at the intersection of the n patches in $\mathbb{C}^n$ where one coordinate is non-zero.

[18] In topologically twisted theories, the modes would instead be characterized by the de Rham cohomology $H^\bullet(\mathbb{R}^d - 0)$. This is supported entirely in degrees 0 and $d-1$, each with dimension 1. In this case, the "mode" extracted by the "degree 0 class" is just the $(d-1)$th descendant $\mathcal{O}^{(d-1)}$ itself. Instead of a λ-bracket, we just have a single secondary product [130]. Alternatively, we can think of the λ-bracket as a generating function for the many secondary products in holomorphically twisted theories [129].

4.2.4 Twisting the stress tensor supermultiplet

As a warm-up, consider a theory with 4d $\mathcal{N}=1$ supersymmetry equipped with a conserved flavour current. The conserved current is part of a real *linear* supermultiplet $\mathcal{J}[\theta,\bar\theta]$ such that

$$D^2\mathcal{J}=0\,,\qquad\qquad \bar D^2\mathcal{J}=0\,. \tag{4.19}$$

We immediately see that $J[\bar\theta]\equiv D_{\dot+}\mathcal{J}|_{\theta=0}$ is a semi-chiral superfield, which plays a role analogous to that of a Kac-Moody current in 2d CFT.

The zero mode $\hat J_{0,0}$ implements global flavour rotations, but the remaining positive modes $\hat J_{n,m}$ give an action of a much larger symmetry algebra: holomorphic position-dependent flavour rotations. The negative modes give extra Grassmann-odd symmetry generators which essentially multiply an operator by derivatives of J [11, 133, 10].

A similar analysis, but slightly more involved, is also possible for the stress tensor multiplet. The most general "$\mathcal{S}$"-supermultiplet was derived in [134]. It includes a superfield $S_{\dot\alpha\beta}$. The component

$$S_\alpha\equiv D_{\dot+}S_{\dot+\alpha}|_{\theta=0} \tag{4.20}$$

satisfies $(Q+\bar\partial)S_\alpha=\bar Y_{\dot+\alpha}$, where $\bar Y_{\dot+\alpha}$ is a component of a closed form $\bar Y_\mu \mathrm{d}x^\mu$. In particular,

$$\partial_{\dot+\alpha}\bar Y^\alpha_{\dot+}=0\,. \tag{4.21}$$

Therefore, $\partial_{z^\alpha}S^\alpha$ is semi-chiral. The operator $\bar Y_\mu$ vanishes for theories endowed with a conserved R-symmetry current, in which case S_α itself is semi-chiral.

The one-form component $S^{(1)}_\alpha$ which enters the definition of $\widehat{(\partial_\alpha S^\alpha)}_{n,m}$ or $\hat S^\alpha_{n,m}$ includes the holomorphic part of the physical stress tensor. The $\widehat{(\partial_\alpha S^\alpha)}_{n,m}$ positive modes generate Hamiltonian holomorphic symplectomorphisms of spacetime:

$$\int_{S^3} f(z)\partial_\alpha S^\alpha(z)\mathcal{O}(w)=-\int_{S^3}(\partial_\alpha f(z))S^\alpha(z)\mathcal{O}(w)\,, \tag{4.22}$$

which are diffeomorphisms generated by the Hamiltonian vectorfield $\partial_\alpha f$. In the case S_α itself is semi-chiral, we expect the positive modes $\hat S^\alpha_{n,m}$ to generate holomorphic diffeomorphisms of spacetime. The negative modes give extra Grassmann-odd symmetry generators which essentially multiply an operator by derivatives of $\partial_\alpha S^\alpha$ or S_α respectively [135, 9, 133].

If a quantum field theory is classically scale invariant but has a non-zero beta-function at one-loop, we will thus find that $(Q+\bar\partial)S_\alpha$ vanishes classically but not at one-loop, while

$(Q + \bar{\partial})\partial_\alpha S^\alpha$ vanishes exactly. This is a simple example of 1-loop correction to the action of Q.

In pure gauge theory, we will find a startling result: $\partial_\alpha S^\alpha$ *is* $(Q + \bar{\partial})$*-exact at one-loop* – so, the theory actually becomes topological at first-order in perturbation theory. We dub this phenomenon *holomorphic confinement*, as we expect it to remain true at all orders in perturbation theory and even non-perturbatively. If this expectation holds, it implies remarkable constraints on the IR behaviour of the theory: the low energy effective field theory must also become topological upon holomorphic twist.

4.2.5 A simple example: the chiral multiplet

The free (anti)chiral multiplet is described by an (anti)chiral superfield Φ, i.e. a superfield annihilated by both superspace derivatives $D_\pm$, which also satisfies an equation of motion $\bar{D}^2\Phi = 0$.

Clearly

$$\gamma \equiv \Phi \tag{4.23}$$

itself is a semi-chiral superfield.[19] The complex conjugate $\bar{\Phi}$ also contains a semi-chiral superfield:

$$\beta \equiv D_{\dot{+}}\bar{\Phi}\,. \tag{4.24}$$

Together they form a four-dimensional analogue of the two-dimensional chiral $\beta\gamma$ system. The $U(1)$ current superfield $|\Phi|^2$ gives us $J = \beta\gamma$. A slightly more involved calculation produces the holomorphic stress tensor

$$S_\alpha = \beta\partial_\alpha\gamma - \Delta\,\partial_\alpha(\beta\gamma)\,. \tag{4.25}$$

Much as in a 2d $\beta\gamma$ system, the value of the coefficient Δ in the second term determines the holomorphic scaling dimension of β and γ and is proportional to the R-charge of Φ. The dependence drops out for theories with no conserved R-symmetry:

$$\partial_\alpha S^\alpha = \partial_\alpha\beta\partial^\alpha\gamma\,. \tag{4.26}$$

Brackets can be readily calculated using the Green's function pairing β and γ, which is the Bochner-Martinelli kernel. E.g. β generates holomorphic shifts of γ, etc.

[19] We do not even need to set $\theta = 0$, as Φ is independent of θ.

A small variation of this system is a chiral multiplet with a superpotential, with an equation of motion $\bar{D}^2\Phi = \partial_\Phi \mathcal{W}(\Phi)$. Now β is not semi-chiral. Instead, we have

$$(Q+\bar{\partial})\beta = \partial_\gamma \mathcal{W}(\gamma)\,. \tag{4.27}$$

Notice that

$$(Q+\bar{\partial})\beta\partial_\alpha\gamma = \partial_\alpha \left[\mathcal{W} - \Delta\,\gamma\partial_\gamma\mathcal{W}\right]\,, \tag{4.28}$$

so $S_\alpha = \beta\partial_\alpha\gamma$ is not a semi-chiral superfield unless $\mathcal{W}$ is quasi-homogeneous with weight 1, but $\partial_\alpha S^\alpha$ always is. This example is also sufficiently rich to admit 1-loop corrections to some of the brackets [116].

4.3 Twisted gauge theory as a holomorphic BF theory

The holomorphic twist of four-dimensional $\mathcal{N}=1$ pure gauge theory has been identified in the BV formalism with a holomorphic version of the BF theory [59, 6, 10]. The identification is non-trivial and involves simplifications which modify the field content *without changing the Q-cohomology*. The twisted theory exists (classically) on any complex surface Y, but at the quantum level there is an anomaly which requires that Y be Calabi–Yau.[20] In the language of the previous Section 4.2, the holomorphic stress tensor S_α is not closed at 1-loop, so only complex symplectomorphisms are symmetries of the theory. In this Chapter, we will focus on the properties of local operators, so we can take $Y=\mathbb{C}^2$ with its standard holomorphic volume form $\mathrm{d}^2z \equiv \mathrm{d}z^1\wedge\mathrm{d}z^2$.

The elementary fields are collected in a bosonic superfield[21] b and a fermionic superfield c:

$$b = b^{(0)} + b^{(1)} + b^{(2)} \in \Omega^{0,\bullet}(\mathbb{C}^2,\mathfrak{g}^\vee), \quad c = c^{(0)} + c^{(1)} + c^{(2)} \in \Omega^{0,\bullet}(\mathbb{C}^2,\mathfrak{g})[1]\,, \tag{4.29}$$

valued in the adjoint representation of the gauge group.[22]

[20] This is a consequence of the Grothendieck–Riemann–Roch theorem. On curved space there is an anomaly to the one-loop quantization of holomorphic BF theory which is proportional to the characteristic class $c_1(T_Y)$ times the quadratic Casimir of the gauge group G in the adjoint representation. For semi-simple groups, the only way this anomaly vanishes is if $c_1(T_Y)=0$.

[21] If we do not explicitly fix a holomorphic volume form it is more natural to view $b^{(i)}$ as a $(2,i)$ Dolbeault form. In this Chapter we will always fix the volume form d^2z and view $b^{(i)}$ as a $(0,i)$ form.

[22] Strictly speaking c is valued in the adjoint representation and b in the coadjoint representation. We will use a nondegenerate invariant pairing on the Lie algebra to identify the two representations.

The $c^{(0)}$ field is a ghost and has cohomological degree 1. The descendant $c^{(1)}$ has cohomological degree 0 and can be identified with the anti-holomorphic part of the gauge connection. Accordingly, the 2-form part of the combination $\bar\partial c - \frac{1}{2}[c,c]$ is the anti-holomorphic field strength.

For matrix Lie groups G, the action of holomorphic BF theory is written as

$$\int \mathrm{d}^2 z\ \mathrm{Tr}\, b\left(\bar\partial c - \frac{1}{2}[c,c]\right). \tag{4.30}$$

Expanding out the action, we find the BF term:

$$\int \mathrm{d}^2 z\ \mathrm{Tr}\, b^{(0)}\left(\bar\partial c^{(1)} - \frac{1}{2}[c^{(1)},c^{(1)}]\right). \tag{4.31}$$

The cohomological degree of $b^{(0)}$ is -2. The holomorphic part of the connection is absent in this description.

Field redefinitions notwithstanding, we can compare the field content of the BF theory to the Q-cohomology of the space of fields of a free $\mathcal{N}=1$ pure gauge theory:

- the $(2,0)$ part of the field strength $F_{\dot+\dot+}$ corresponds to $\mathrm{d}^2 z\, b^{(0)}$,
- the two gaugino components $\bar\lambda_\alpha$, $\alpha = 1,2$ which satisfy the Dirac equation $\epsilon_{\alpha\beta}\partial^\alpha\bar\lambda^\beta = \partial_\alpha\bar\lambda^\alpha = 0$ are identified with the two holomorphic derivatives $\partial_{z^\alpha}c^{(0)}$. In fact, $\bar\lambda_\alpha$ is the leading component of an anti-chiral superfield $\overline{\mathcal{W}}_\alpha$, which can be identified with the superfield $\partial_{z^\alpha}c$.

The basic action above actually corresponds to a self-dual limit of the twisted theory. The full twisted action also includes the twisted version of the usual kinetic F-term $\int d^2\bar\theta\,\tau\,\overline{\mathcal{W}}_\alpha\,\overline{\mathcal{W}}^\alpha$:

$$\int \mathrm{d}^2 z\,\tau\,\mathrm{Tr}\,\partial_\alpha c\,\partial^\alpha c\,. \tag{4.32}$$

This is a total derivative in the BF theory, though, so it only affects instanton calculations. It is also possible to make the coupling $\tau(z)$ position dependent and holomorphic. Mathematically, this corresponds to the statement that the space of translation invariant quantizations of holomorphic BF theory on $\mathbb{C}^2$ is a torsor for the abelian group $H^4(\mathfrak{g}[[z_1,z_2]])$.

4.3.1 Local operators

In the BV/BRST formalism, the $c^{(0)}$ field plays the role of a ghost for the gauge symmetry of the BF theory and thus $c^{(0)}$ itself is forbidden from appearing in local operators: only its derivatives are allowed. Indeed, Faddeev–Popov ghosts are only suitable to gauge-fix the non-zero modes of the gauge group. The correct way to deal with constant gauge transformations is to only consider local operators which are strictly invariant under them.[23] This is consistent with the form of semi-chiral operators in the physical free gauge theory, which involves the gauginos $\bar{\lambda}_\alpha \sim \partial_\alpha c^{(0)}$ and their derivatives, up to the equation of motion $\partial_\alpha \bar{\lambda}^\alpha = 0$.

Local operators in the free theory

The Q-cohomology of the space of local operators in the free limit of the holomorphic BF theory consists of (graded symmetric) polynomials in $b^{(0)}$ and its holomorphic derivatives and holomorphic derivatives of $c^{(0)}$ which are invariant under constant gauge transformations. Thus, as a graded vector space, the space of local operators is the G-invariant set

$$\mathcal{V} \equiv \mathbb{C}\left[\partial_{z^1}^n \, \partial_{z^2}^m \, c^{(0)}, \partial_{z^1}^k \, \partial_{z^2}^l \, b^{(0)}, \, n+m>0, \, k+l \geq 0\right]^G . \tag{4.33}$$

This is also consistent with the field content of semi-chiral operators in the free physical theory. Operators in the free physical theory that are built from the holomorphic field strength $F_{\dot{+}\dot{+}}$ are mapped to $b^{(0)}$, and the two modes of the gaugino $\bar{\lambda}_\alpha$ modulo the equation of motion are mapped to $\partial_\alpha c^{(0)}$.

Notice that the kinetic term in (4.30) pairs $b^{(0)}$ and $c^{(1)}$ as well as $b^{(1)}$ and $c^{(0)}$. As a consequence $b^{(0)}$ and $c^{(0)}$ have a regular OPE and the above composite operators have no renormalization ambiguities in the free theory. In the BV formalism, the quantum corrections to Q in the free theory involve the *BV Laplacian* Δ_{BV}, which roughly implements a single Wick contraction on the fields in the operator.[24] The absence of $b^{(1)}$ and $c^{(1)}$ in the above cohomology representatives thus also means that quantum corrections[25] do not change the free Q-cohomology $\mathcal{V}$.

The operators in the Q-cohomology for the free BF theory can be promoted to semi-chiral superfields which are polynomials in b and its holomorphic derivatives as well as

[23] The rule is different when one studies interactions, which can include the field $c^{(0)}$. Furthermore, interactions are defined up to total derivatives, so they are identified as the cohomology of the combination $Q + \partial$.

[24] To avoid UV divergences, this operator must be suitably regularized.

[25] We emphasize that we are studying the free theory and quantum corrections thereof in this subsection.

holomorphic derivatives of c. The resulting semi-chiral operators also have no renormalization ambiguities.

Local operators in the classical interacting theory

Classically, the BRST transformations are modified in the interacting theory to

$$Qc = -\bar{\partial}c + \frac{1}{2}[c,c]\,, \qquad Qb = -\bar{\partial}b + [c,b]\,. \tag{4.34}$$

Observe that these BRST transformations act sensibly on the space of G-invariant local operators which do not contain c's but only their derivatives: the BRST transformation of any operator equals an infinitesimal G rotation with parameter c plus corrections proportional to derivatives of c. For example,

$$\begin{aligned}(Q+\bar{\partial})\partial_\alpha c &= [c,\partial_\alpha c]\,,\\ (Q+\bar{\partial})\partial_\alpha\partial_\beta c &= [c,\partial_\alpha\partial_\beta c] + [\partial_\alpha c,\partial_\beta c]\,,\\ (Q+\bar{\partial})\partial_\alpha b &= [c,\partial_\alpha b] + [\partial_\alpha c, b]\,.\end{aligned} \tag{4.35}$$

Thus we can drop the terms containing c when Q acts on G-invariant operators.

Here we encounter the simplest example of *homotopy transfer*, or a spectral sequence. The cohomology of the local operators with respect to the full interacting (tree level) differential

$$Q = Q_{\text{free}} + Q_{\text{int}} \tag{4.36}$$

can be computed in stages by first computing the $Q_{\text{free}} = -\bar{\partial}$ cohomology, which we denoted $\mathcal{V}$, defined in (4.33), and then computing the cohomology of an appropriate version of Q_{int} acting on $\mathcal{V}$.[26]

Omitting the $^{(0)}$ superscripts for legibility, the restricted action of Q_{int} is just

$$Q_{\text{int}}c = \frac{1}{2}[c,c]\,, \qquad Q_{\text{int}}b = [c,b]\,. \tag{4.37}$$

The situation is particularly simple here because Q_{int} does not change the overall form degree or number of anti-holomorphic derivatives in an operator. In particular, it maps $\mathcal{V}$ back to itself.

[26]Homotopy transfer is a method used to recognize the full cohomology as the cohomology of a particular differential acting on $\mathcal{V}$. Said differently, there is a spectral sequence whose first page is the free cohomology $\mathcal{V}$. The differential computing the next page is given by the restricted action of Q_{int} on $\mathcal{V}$. In general, there may be further terms in the spectral sequence, but for the example at hand the spectral sequence collapses at this stage.

This action matches naturally the interacting SUSY transformations in the twisted theory, as formulated in past work on the categorification of the superconformal index [110, 111, 136]. In these works, the role of $\mathcal{V}$ is played by the space of G-invariant words in a collection of BPS letters and their holomorphic covariant derivatives, which precisely matches the semi-chiral superfields in the twisted theory and their holomorphic derivatives, with c omitted. The leading correction to the action of the supercharge comes from the presence of the holomorphic connection in the covariant derivatives, so that $Q_{\dot{-}}$ maps $D_{z^\alpha} \to [\bar{\lambda}_\alpha, \cdot]$. This reproduces the corrections proportional to $\partial_\alpha c$ in (4.35).

The holomorphic twist explains this simplification and extends it to the quantum theory, allowing one to compute the further loop corrections to the cohomology which will be discussed momentarily.

Lie algebra cohomology

A concise way to describe the action of Q_{int} on $\mathcal{V}$ is in terms of relative Lie algebra cohomology. The Lie algebra cohomology of a Lie algebra $\tilde{\mathfrak{g}}$ is computed by the Chevalley–Eilenberg complex $C^\bullet(\tilde{\mathfrak{g}})$ which is the space of graded polynomials on the vector space $\tilde{\mathfrak{g}}[1]$ equipped with the Chevalley–Eilenberg differential. If G is a group whose Lie algebra $\mathfrak{g}$ is a Lie subalgebra of $\tilde{\mathfrak{g}}$, then the relative Lie algebra cohomology of $\tilde{\mathfrak{g}}$ relative to $\mathfrak{g}$ is the cohomology of a subcomplex $C^\bullet(\tilde{\mathfrak{g}}|\mathfrak{g})$ of $C^\bullet(\tilde{\mathfrak{g}})$ which as a graded vector space is G-invariants of the graded symmetric algebra on the graded vector space $(\tilde{\mathfrak{g}}/\mathfrak{g})[1]$. The differential is the usual Chevalley–Eilenberg differential. There is a similar construction for relative Lie algebra cohomology with values in a module M which we denote by $C^\bullet(\tilde{\mathfrak{g}}|\mathfrak{g}; M)$ [137]. For appearances of relative Lie algebra cohomology in the context of supersymmetric gauge theory see [111, 136].

In the example of holomorphic BF theory, the tree-level cohomology of local operators is equivalent to the Lie algebra cohomology of $\tilde{\mathfrak{g}} = \mathfrak{g}[[z_1, z_2]]$ relative to the subalgebra $\mathfrak{g}$ (consisting of constant polynomials) with coefficients in the module $M = \mathrm{Sym}(\mathfrak{g}[[z_1, z_2]]^*) = \mathbb{C}\left[\partial_{z^1}^k \partial_{z^2}^l b,\, k, l \in \mathbb{Z}_{\geq 0}\right]$. Following the notation above, we denote this by

$$C^\bullet\left(\mathfrak{g}[[z_1, z_2]] \mid \mathfrak{g};\, \mathrm{Sym}(\mathfrak{g}[[z_1, z_2]]^*)\right). \tag{4.38}$$

The generators of $\mathfrak{g}[[z_1, z_2]]$ play the role of derivatives $\partial_1^n \partial_2^m c^{(0)}$ of the ghost field, while subalgebra $\mathfrak{g} \subset \mathfrak{g}[[z_1, z_2]]$ represents the Lie algebra of constant gauge transformations, which in the notation above was given by the ghost $c^{(0)}$ and is removed from the calculation.

Unfortunately, the computation of most Lie algebra cohomologies is notoriously difficult. Remarkable simplifications occur in the large N limit for $U(N)$ gauge theories. In

Section 4.5 we will discuss the large N limit and in Section 4.8 some explicit calculations in $SU(2)$ and $SU(3)$ gauge theories.

We can present a collection of cohomology classes which exist for all G. Consider any G-invariant polynomial $A_a \equiv P_a(b)$. As it does not contain derivatives, this is trivially Q_{int}-closed. As it does not contain c fields, it cannot be Q_{int}-exact. In $SU(N)$ gauge theory, such operators can be written as polynomials in

$$A_n = \operatorname{Tr} b^n\,, \qquad\qquad n \geq 1\,. \tag{4.39}$$

As Q_{int} commutes with derivatives ∂_α and satisfies a Leibniz rule, polynomials in the A_a and their derivatives will also be Q_{int}-closed and give a large collection of cohomology classes.

We can do better. We can introduce an auxiliary fermionic variable ε and combine the elementary fields as $C(\varepsilon) \equiv c + \varepsilon b$. The BRST transformations become

$$Q_{\text{int}} C(\varepsilon) = \frac{1}{2}[C(\varepsilon), C(\varepsilon)]\,. \tag{4.40}$$

First of all, this allows us to recast the problem as a Lie algebra cohomology problem:

$$C^\bullet\left(\mathfrak{g}[[z_1, z_2, \varepsilon]] \mid \mathfrak{g}\right). \tag{4.41}$$

Second, as the commutator is local in ε, Q_{int} must commute with all vectorfields in z_α and ε. This includes vectorfields

$$\begin{aligned} \xi &= \partial_\varepsilon \quad : \, c \to b \\ \tilde{\xi}_\alpha &= \varepsilon\partial_\alpha \, : \, b \to \partial_\alpha c\,, \end{aligned} \tag{4.42}$$

which anticommute to translations ∂_α.

We can define

$$\begin{aligned} B_{a-1,\alpha} &\equiv \tilde{\xi}_\alpha A_a \\ C_{a-2} &\equiv \tilde{\xi}_1 \tilde{\xi}_2 A_a\,. \end{aligned} \tag{4.43}$$

These are manifestly Q_{int}-closed. As $\xi A_a = 0$, we have

$$\begin{aligned} \xi B_{a-1,\alpha} &= \partial_\alpha A_a \\ \xi C_{a-2} &= 2\partial_\alpha B^\alpha_{a-1}\,. \end{aligned} \tag{4.44}$$

The first relation implies that $B_{a-1,\alpha}$ cannot be Q_{int}-exact, as $\partial_\alpha A_a$ is not. In order to draw the same conclusion for C_{a-2}, we need to verify that $\partial_\alpha B^\alpha_{a-1}$ is not Q_{int}-exact. That follows from the relation

$$\partial_\alpha B^\alpha_{a-1} = \tilde{\xi}^\alpha \partial_\alpha A_a\,. \tag{4.45}$$

We thus obtained three collections of tree-level cohomology classes. For $SU(N)$ gauge group we can denote them as symmetrized traces:

$$\begin{aligned} A_n &= \operatorname{Tr} b^n, && n \geq 2 \\ B_{n,\alpha} &= \operatorname{STr} b^n \partial_\alpha c, && n \geq 1,\ \alpha = 1,2 \\ C_n &= \operatorname{STr} b^n \partial_\alpha c\, \partial^\alpha c, && n \geq 0\,. \end{aligned} \tag{4.46}$$

The first element in the B tower is the stress tensor superfield:

$$S_\alpha = \operatorname{Tr} b \partial_\alpha c\,. \tag{4.47}$$

This is perfectly analogous to the stess tensor in a two-dimensional bc system. The pure gauge theory is scale-invariant and has a $U(1)_R$ R-symmetry at tree level, and indeed S_α is a semi-chiral superfield at tree-level, as anticipated in Section 4.2.4.

The auxiliary symmetries we introduced are also the zero-modes of certain currents. The symmetry ξ is generated by $A_2 = \operatorname{Tr} b^2$. The symmetries $\tilde{\xi}_\alpha$ are generated by the primitive $\operatorname{Tr} c\partial_\alpha c$ of C_0. Higher modes of these currents, together with those of the ghost number current $\operatorname{Tr} bc$ and the stress tensor $\operatorname{Tr} b\partial_\alpha c$, give the action of more general vector-fields in z_α and ϵ.

We list our conventions for the Lie algebra of $SU(N)$ and its Chevalley–Eilenberg complex that we will use in the remainder of the Chapter. We will use capital letters $A, B, \ldots$ for the adjoint gauge indices. The Killing form is taken to be δ^{AB} so that adjoint indices can be raised and lowered freely:

$$\operatorname{Tr} t^A t^B = -\frac{1}{2}\delta^{AB}, \quad \operatorname{Tr} t^A [t^B, t^C] = -\frac{1}{2} f^{ABC}\,, \tag{4.48}$$

where the trace is taken in the defining representation and f^{ABC} is the structure constant $[t^A, t^B] = f^{ABC} t^C$ such that

$$\sum_{A,B} f^{ABC} f^{ABD} = N\delta^{CD}\,. \tag{4.49}$$

In particular, for $SU(2)$, we take the generators to be $t^A = -\frac{i}{2}\sigma^A$ and $f^{ABC} = \epsilon^{ABC}$.

In this notation, the tree level differential (the Chevalley–Eilenberg differential) $Qb = [c,b]$, $Qc = \frac{1}{2}[c,c]$ reads

$$Qb^A = \epsilon^{ABC} c^B b^C, \quad Qc^A = \frac{1}{2}\epsilon^{ABC} c^B c^C. \tag{4.50}$$

We will also use the relation

$$b^2 = b_A b^A = -2\,\mathrm{Tr}\, b^2\,. \tag{4.51}$$

4.3.2 Loop corrections

The classical answer for the cohomology of local operators in the interacting theory must receive perturbative quantum corrections. As an example, consider the holomorphic stress tensor. Famously, the pure gauge theory is *not* scale invariant quantum-mechanically. Both scale-invariance and the R-symmetry are broken at one-loop. The latter anomaly is one-loop exact and is computed by a famous triangle diagram and is proportional to $\mathrm{Tr}\, F \wedge F$. It is supersymmetrized to the Konishi anomaly equation [119, 120, 117], which involves the chiral superfield $\mathrm{Tr}\, \overline{\mathcal{W}}_\alpha \overline{\mathcal{W}}^\alpha$. (See [138] for a discussion of the Konishi anomaly in terms of the stress tensor supermultiplets from [134].)

These anomalies must have a counterpart in the holomorphic theory. A direct translation of [138] gives e.g.

$$(Q + \bar{\partial}) S_\alpha \sim \partial_\alpha \,\mathrm{Tr}\left(\partial_\beta c\, \partial^\beta c\right)\,. \tag{4.52}$$

We will momentarily recover this by a direct calculation of a one-loop triangle diagram in the holomorphic BF theory.

As we discussed in the previous Section, this relation controls the quantum-mechanical failure of the theory to be invariant under general holomorphic coordinate transformations: an infinitesimal coordinate transformation generated by a holomorphic vector field v^α shifts the action by an amount proportional to

$$\int \mathrm{d}^2 z\, v^\alpha\, \partial_\alpha \,\mathrm{Tr}\left(\partial_\beta c\, \partial^\beta c\right)\,. \tag{4.53}$$

In turns, this shifts the holomorphic gauge coupling τ by a multiple of $\partial_\alpha v^\alpha$. This is the holomorphic generalization of the scale anomaly of the physical theory.

If v^α is divergence-free with respect to the standard holomorphic volume form on $\mathbb{C}^2$, then this anomaly vanishes as the density above is a total derivative. Thus complex symplectomorphisms of spacetime remain a symmetry of the quantum theory.

A reader may be surprised by the statement that the action of a supercharge may receive quantum corrections. Indeed, statements of the opposite flavour are sometimes made in the literature.[27]

The origin of this phenomenon is, of course, operator renormalization. As we regulate a composite operator we introduce extra terms in the action of symmetries, which may then survive as we send the regulator to 0. The origin of these corrections is manifest in the fact that they do not act "a letter at the time" on the operator, but they simultaneously change multiple fields.[28]

Loop corrections and higher brackets

More formally, the full quantum corrected differential in the BV formalism is defined on the space of operators:

$$\bar{\partial} + \hbar\Delta_{\mathrm{BV}} + \{\mathcal{I}, \cdot\}_{\mathrm{BV}}\,, \tag{4.55}$$

where the BV Laplacian Δ_{BV} accounts for the quantum corrections of the free theory and the last term encodes the effect of the interactions. A proper definition of the differential requires a careful renormalization, addressing both UV and IR divergences.

Our general expectation/conjecture is that the cohomology of the full quantum differential is computed as the cohomology of a certain differential $\mathbf{Q}$ acting on the cohomology $\mathcal{V}$ of the underlying free and classical theory, at least in perturbation theory. The differential $\mathbf{Q}$ should be interpreted as a quantum-corrected version of the classical interacting differential Q_{int}.

Operationally, the action of $\mathbf{Q}$ on some operator $\mathcal{O}$ is computed in two steps. First regularize the combination of the local operator and the exponential of the integrated

[27] A more shocking statement is actually true: even the action of translations receives quantum corrections in QFTs. Consider a current J^μ which is conserved classically but broken at 1-loop. Then $P_\mu J^\mu \equiv \partial_\mu J^\mu$ vanishes classically but not quantum-mechanically.

[28] The action of Q still satisfies the Leibniz rule, but only for the loop corrected product $O \circ O' \equiv \widehat{O}_{-1,-1}O'$. One may try to get rid of the quantum corrections by using the loop-corrected product to multiply elementary letters in a composite operator, but such an effort is thwarted by the fact that the regularized product is not associative.

For example, we could try to define the stress tensor as $\mathrm{Tr}(b \circ \partial_\alpha c)$. This would actually coincide with $\mathrm{Tr}\, b\partial_\alpha c$. The Leibniz rule for Q_{int} would produce

$$\mathrm{Tr}([c,b] \circ \partial_\alpha c) + \mathrm{Tr}(b \circ [c, \partial_\alpha c])\,, \tag{4.54}$$

but that fails to vanish due to a 1-loop correction to the regularized product and reproduces the expected anomaly.

interaction Lagrangian. Then act with Q_{free} and rewrite the answer as the regularization of the combination of some new local operator $\mathbf{Q}\mathcal{O}$, and the exponential of the integrated interaction Lagrangian. The resulting $\mathbf{Q}$ operator will be scheme-dependent. In order for the answer to be scheme-independent, we expect that the $\mathbf{Q}$ obtained in different schemes should be related by quasi-isomorphisms (see [116]).

The result is a perturbative expansion for $\mathbf{Q}$

$$\mathbf{Q} = Q_0 + Q_1 + \cdots\,, \tag{4.56}$$

where Q_n is computed by certain n-loop Feynman diagrams in the holomorphic theory.[29]

In order to develop some intuition, we can show that the tree-level differential Q_0 is the expected BRST operator Q_{int} arising from the classical interaction. Denote the interacting part of the action (4.30) as

$$\int_{\mathbb{R}^4} \mathcal{I}^{(2)}(z,\bar{z})\mathrm{d}^2 z\,. \tag{4.57}$$

i.e. $\mathcal{I} = -\frac{1}{2}\operatorname{Tr} b[c,c]$. This integrated operator is naively BRST invariant, as its variation is a total derivative. In the presence of a local operator $\mathcal{O} \in \mathcal{V}$ placed at the origin, though, renormalization may lead to a BRST anomaly. If we simply turn off the interaction in a small ball of radius ϵ surrounding the operator, the BRST anomaly will be

$$Q_0\mathcal{O} = Q_{\text{free}}\left[\int_{|z|>\epsilon} \mathcal{I}(z,\bar{z})\mathrm{d}^2 z\; \mathcal{O}(0,0)\right] = \int_{|z|=\epsilon} \mathcal{I}(z,\bar{z})\mathrm{d}^2 z\,\mathcal{O}(0,0)\,. \tag{4.58}$$

Here we have used the fact that $\mathcal{O},\mathcal{I} \in \mathcal{V}$ are both semi-chiral and thus are annihilated by $Q_{\text{free}} = -\bar{\partial}$. Integration by parts gives an integral on a small 3-sphere surrounding the origin.

By definition (4.16), this is the zero-mode of $\mathcal{I}$ acting on $\mathcal{O}$ in the free theory, which we can also write as

$$Q_0\mathcal{O} = \{\mathcal{I},\mathcal{O}\}\,, \tag{4.59}$$

where the bracket is the $\lambda = 0$ specialization of the $\lambda-$bracket (4.18). When seen as an element of the Q_{free} cohomology, it is independent of ϵ. The integral greatly simplifies in the $\epsilon \to 0$ limit. Indeed:

- In the absence of Wick contractions, the integral goes to 0 as ϵ^3.

[29] One may wonder how to take the cohomology of an operator defined by a formal power series. A natural strategy is to observe that $Q_0^2 = 0$ and use homotopy transfer to map $Q_1 + \cdots$ to the Q_0 cohomology. One then takes the cohomology of the new leading term, etc. See appendix B.4 for details.

- Each Wick contraction gives a propagator $P(z,\bar{z})$ which is a $(0,1)$ form as explained below. So for the dimensional reason, we can have at most one Wick contraction, involving the single descendant in $\mathcal{I}^{(1)}$. Denoting the remaining collection of uncontracted fields as $F(z,\bar{z})$, we have

$$\int_{|z|=\epsilon} \mathrm{d}^2 z\, F(z,\bar{z}) P(z,\bar{z}) \to F(0,0) + O(\epsilon) \tag{4.60}$$

  as $\epsilon \to 0$. In other words, the fields in $\mathcal{I}$ which survive the Wick contractions can be placed at the origin. As a consequence, removing the regulator projects $Q_0\mathcal{O}$ back to $\mathcal{V}$.

The general form of Q_n is that of a multi-linear "higher bracket" with $n+2$ slots, see appendix B.1, with all but one of the slots taken by the interaction:

$$Q_n\mathcal{O} = \frac{1}{(n+1)!}\{\mathcal{I},\mathcal{I},\cdots,\mathcal{I},\mathcal{O}\}\ . \tag{4.61}$$

One can argue that these higher brackets satisfy certain quadratic axioms and that the resulting $\mathbf{Q}$ is nilpotent order-by-order in perturbation theory thanks to these axioms and an anomaly-cancellation Maurer-Cartan condition on the interaction Lagrangian:

$$\frac{1}{2}\{\mathcal{I},\mathcal{I}\} + \frac{1}{6}\{\mathcal{I},\mathcal{I},\mathcal{I}\} + \cdots = 0\,. \tag{4.62}$$

See [116] for more details. The basic procedure for evaluating the higher brackets can be described in the following steps:

1. Take the basic integral
$$\int_{\mathbb{R}^{4n}} \prod_{i=1}^{n} \mathcal{I}(z_i,\bar{z}_i)\mathrm{d}^2 z_i\, \mathcal{O}(0,0)\,. \tag{4.63}$$

2. Do a maximal number of Wick contractions using regularized propagators.

3. Act with Q_{free} and integrate the resulting $\bar{\partial}$ by parts to act on the product of propagators.

4. Set to 0 all descendants and project all surviving fields to the origin, by setting $\bar{z}_i = 0$ and Taylor-expanding in z_i to get an element of $\mathcal{V}$.

5. Do the resulting Feynman integrals, which turn out to be finite and independent of the regulator.

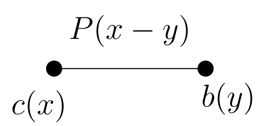
The main feature of the Feynman diagram calculations is that they involve superspace propagators which coincide with the Bochner-Martinelli kernel: one-forms $P(x, \bar{x}, \mathrm{d}\bar{x})$ of Dolbeault type $(0,1)$ (and Grassman odd):

$$P(x, \bar{x}, \mathrm{d}\bar{x}) \equiv \frac{1}{4\pi^2} \frac{\bar{x}^2 \mathrm{d}\bar{x}^1 - \bar{x}^1 \mathrm{d}\bar{x}^2}{|x|^4}\,, \tag{4.64}$$

where $|x|^2 = x^1\,\bar{x}^1 + x^2\,\bar{x}^2$. The propagator is the Green's function for $\bar{\partial} = \mathrm{d}\bar{x}^1 \frac{\partial}{\partial \bar{x}^1} + \mathrm{d}\bar{x}^2 \frac{\partial}{\partial \bar{x}^2}$, giving the relation

$$\bar{\partial} P(x, \bar{x}, \mathrm{d}\bar{x}) = \mathrm{d}\bar{x}^1 \mathrm{d}\bar{x}^2 \delta^4(x)\,. \tag{4.65}$$

This puts an immediate upper bound on the number of propagators allowed at any loop order: Q_n involves an integral over $2n+2$ anti-holomorphic variables and thus $2n+1$ propagators. In what follows we will avoid writing the dependence on the antiholomorphic coordinate and write propagators as $P(x)$. Each propagator removes a b and a c superfield:



(4.66)

and each interaction $\mathcal{I}(x) = -\frac{1}{2}\operatorname{Tr} b[c,c](x)$, adds two c superfields and one b superfield. The net number of c fields thus increases by 1, while the net number of b fields decreases by n. A simple scaling argument also shows that Q_n will increase the number of each type of derivative by n.

Notice that the action of the differential is well-defined and can in principle be computed on all polynomials in the elementary fields and their holomorphic derivatives. For a gauge theory with compact gauge group, we then restrict the action of the general differential to operators which involve derivatives of c but not c itself and are G-invariant.

Tree level

The only tree-level diagram has a single internal line, contracting a single field in $\mathcal{O}(x, \bar{x})$ with a field in $\mathcal{I}(y, \bar{y})$. In particular, it satisfies the Leibniz rule: other uncontracted fields in $\mathcal{O}(x, \bar{x})$ go along for the ride. It also commutes with ∂_α.

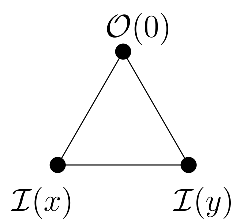
If we take $\mathcal{O} = b$, under the action of the tree level Q_0 we have

$$\begin{aligned} Q_0\, b(x) = \{\mathcal{I}, b\}(x) &= \int_{|y-x|>\epsilon} \mathrm{d}^2 y\;\; \bar{\partial}_y P(y-x)\, [c(y), b(y)] \\ &= [c(x), b(x)] - \bar{\partial}_x \int_{|y-x|>\epsilon} \mathrm{d}^2 y\, P(y-x)\, [c(y), b(y)]\,, \end{aligned} \tag{4.67}$$

where we performed a single Wick contraction between b in $\mathcal{O}$ and c in $\mathcal{I}$ in two ways. We then integrated by parts the free differential to get to the second line. The result is just $[c(x), b(x)]$ up to a term exact in the free cohomology $\mathcal{V}$. Similarly, we find $Q_0\, c = \{\mathcal{I}, c\} = \frac{1}{2}[c, c]$, reproducing the action of Q_{int} in eq. (4.37). It is easy to see that $Q_0\mathcal{O}$ coincides with $Q_{\text{int}}\mathcal{O}$ in general, as they both satisfy the Leibniz rule and commute with derivatives.

One loop and beyond

At one-loop, $Q_1\mathcal{O} = \frac{1}{2}\{\mathcal{I}, \mathcal{I}, \mathcal{O}\}$ only receives contributions from triangle diagrams. The resulting Wick contractions remove two fields in $\mathcal{O}$ and two fields in each of the two interaction vertices, as shown below:



.

Without explicitly evaluating the integral, we see the resulting Q_1 acts on *pairs* of fields within an operator. It also has to respect the gauge symmetry and the perturbative spacetime symmetry $U(2)$. For example, it will transform one b field into a c field and add two holomorphic derivatives, with spacetime indices contracted. For more details involving computing this bracket for specific operators $\mathcal{O}$ see appendix B.3. An important computation (B.19) involves acting with Q_1 on $\mathrm{Tr}\, b^2$, i.e.

$$Q_1\, \mathrm{Tr}\, b^2 \propto \partial_\alpha \left(\mathrm{Tr}\, b\partial^\alpha c\right)\,. \tag{4.68}$$

This is $\partial_\alpha S^\alpha \equiv \partial_\alpha \left(\mathrm{Tr}\, b\partial^\alpha c\right)$, i.e. the derivative of the stress tensor. Even though $\partial_\alpha S^\alpha$ belongs to a nontrivial cohomology class at tree level, it becomes Q-exact at one loop! As a consequence of the filtration discussed above and more explicitly in (4.71), higher order corrections all act trivially on $\operatorname{Tr} b^2$, so (4.68) does not receive further perturbative corrections.

As we discussed before and review in detail in the appendices B.2.1 and B.3, $\partial_\alpha S^\alpha$ generates Hamiltonian symplectomorphisms. The exactness of $\partial_\alpha S^\alpha$ implies that the image of all Q-closed operators under Hamiltonian symplectomorphisms must be Q-exact as well.

As a consequence, the holomorphic derivatives of all local operators vanish in cohomology. Effectively, the holomorphic theory becomes topological at one loop. We dub this peculiar phenomenon *Holomorphic Confinement*, as it is compatible with the expectation that the physical gauge theory should become gapped, i.e. topological, in the IR. Conversely, whatever low energy effective description applies to the physical theory, it should become topological upon holomorphic twist.

Although the general triangle diagram is straightforward to evaluate, and we do so in [7], it turns out that the general structure we just described together with the constraint $\{Q_0, Q_1\} = 0$ we derive momentarily, is sufficient to "bootstrap" the detailed form of Q_1 up to two unknown coefficients, which can be determined by direct integration.

Experimentally, the analogous statement seems to apply to all Q_n's. The crucial requirement is that the relation

$$\mathbf{Q}^2 \mathcal{O} = 0, \quad \forall\, \mathcal{O} \in \mathcal{V} \tag{4.69}$$

with $\mathbf{Q}$ as in (4.56), should hold order-by-order in perturbation theory. The first non-trivial equation is $\{Q_0, Q_1\} = 0$. At higher order, one has

$$\sum_{n_1+n_2=n} Q_{n_1} Q_{n_2} \mathcal{O} = 0, \quad \forall\, \mathcal{O} \in \mathcal{V},\, \forall\, n \geq 0\,. \tag{4.70}$$

Denote the space of the monomials by $V(N_b, N_c, J_1, J_2)$ where b or its derivatives appear N_b times, c or its derivatives appear N_c times, and the total number of derivatives with respect to z_i is J_i. We can make the following observation based on the structure of Feynman diagrams:

- The action of Q_n is linear.
- Q_n annihilates monomials of length shorter than $n+1$, since there are not enough fields in the monomial to perform the Wick contractions needed:

$$Q_n V(N_b, N_c, J_1, J_2) = 0, \quad \text{if } N_b + N_c < n+1\,. \tag{4.71}$$

- Q_n takes a monomial of length $n+1$ and outputs monomial of length 2. More precisely, it takes a monomial with $(n+1)$ b fields and outputs a monomial of length two that contains a single b field and a single c field. It also takes a monomial with n b fields and one c field to a monomial with two c fields:

$$Q_n : V(n+1,0,J_1,J_2) \to V(1,1,J_1+n,J_2+n), \tag{4.72}$$

$$Q_n : V(n,1,J_1,J_2) \to V(0,2,J_1+n,J_2+n)\,. \tag{4.73}$$

- Q_n acts as derivation if the length is longer than $n+1$.

Another experimental observation is that $Q_1^2 \neq 0$, which implies $Q_2 \neq 0$. In [7] we indeed compute the only Feynman diagram which contributes to the two-loop differential and find it non-vanishing.

Using the contents of the third bullet above, one can define an auxiliary classical symmetry $U(1)_b$ under which b has charge 1 and c has charge 0, commuting with derivatives. Then, the tree level Q_0 preserves the symmetry, while higher Q_n reduce the $U(1)_b$ charge.

The existence of this filtration makes the action of $\mathbf{Q}$ well-defined on any given monomial in the fields. It also helps ensure that one can define a good cohomology problem in perturbation theory via homotopy transfer, as a spectral sequence.

4.4 Adding matter

In this Section, we discuss briefly how the holomorphic theory is modified for gauge theories with matter. As explained in Section 4.2.5, chiral multiplets transforming in some representation R of the gauge group give rise to bosonic superfields γ transforming in R as well as fermionic superfields β transforming in $\bar{R}$; after complexification we identify this with the dual representation R^*.

4.4.1 Twisted SQCD

The action for the matter field takes the form of a "gauged $\beta\gamma$ system"[30]

$$\int \mathrm{d}^2 z\, \beta \left(\bar{\partial}\gamma - [c,\gamma]\right) + W(\gamma)\,, \tag{4.74}$$

[30] Analogous gauged $\beta\gamma$ systems occur in 2d, e.g. in the chiral algebra subsector of 4d $\mathcal{N}=2$ gauge theories [32].

where we included a superpotential $W(\gamma)$, see [9]. The BRST transformation of c is unchanged from (4.34). The remaining fields transform at tree-level as

$$Q_0 b = -\bar{\partial} b + [c, b] + \gamma\beta\,, \qquad Q_0\gamma = -\bar{\partial}\gamma + [c, \gamma]\,, \qquad Q_0\beta = -\bar{\partial}\beta + [c, \beta] + \partial_\gamma W\,. \quad (4.75)$$

Local operators are built as before, including β, γ and their derivatives. From the point of view of the free theory, the bc and $\beta\gamma$ systems are essentially identical, so the computation of the free brackets involves literally the same Feynman diagrams. The main difference is in the form of the interactions.

An immediate consequence of the modified BRST transformations is that G-invariant polynomials in b are not Q-closed anymore. On the other hand, chiral operators of the physical theory map to G-invariant polynomials in γ, modulo polynomials multiple of $\partial_\gamma W$.

In the absence of superpotentials, β's and γ's appear on the same footing in the twisted theory, potentially leading to interesting symmetry enlargements which persist beyond tree level.

For example, a natural extension of supersymmetric Yang-Mills theory for $SU(N)$ is supersymmetric QCD (SQCD). At the level of the twist this gives rise to matter fields $(\gamma^i, \tilde{\beta}^a)$ transforming in the fundamental (respectively anti-fundamental) representation and $(\tilde{\gamma}_a, \beta_i)$ anti-fundamental (respectively fundamental) representation with no superpotential. In particular, the partners $\tilde{\beta}^a$ of the anti-fundamental chiral multiplets and the fundamental chiral multiplets γ^i have the same gauge quantum numbers, extending the naive $SU(N_f)_L \times SU(N_f)_R \times U(1)_B$ flavour symmetry to $SU(N_f|N_f)$.[31]

Concretely, we can build a holomorphic stress tensor of the form

$$S_\alpha = \mathrm{Tr}\, b\partial_\alpha c + c_1 \tilde{\Gamma}^A \partial\, \Gamma_A + c_2 \partial\, \tilde{\Gamma}^A \Gamma_A \qquad (4.76)$$

for appropriate constants c_1, c_2. The $\mathrm{Tr}\, b^2$ operator is not in tree-level cohomology anymore, and neither the stress tensor or its derivatives are exact anymore, at least in perturbation theory.

Another important example is that of $\mathcal{N} = 4$ SYM. In that example, the holomorphic twist consists of a holomorphic bc system together with three $\beta\gamma$ systems each transforming

[31]For $SU(2)$ gauge group, there is an extension of the flavor symmetries to the strange Lie supergroup $P(2N_f - 1)$ which contains $SU(2N_f)$ as its even part. It would be interesting to compare these symmetry enhancements across dualities such as Seiberg duality. They would provide a non-trivial test of the conjecture that the simplification of twisted gauge theories to gauged $\beta\gamma$ systems holds beyond perturbation theory.

in the adjoint representation. The superpotential is $\mathrm{Tr}[\gamma^1, \gamma^2]\gamma^3$. Following [111] we can collect all the fields together into a single adjoint superfield

$$C(\varepsilon_1, \varepsilon_2, \varepsilon_3) \equiv c + \varepsilon_i \gamma^i - \frac{1}{2}\epsilon^{ijk}\varepsilon_i \varepsilon_j \beta_k - \varepsilon_1 \varepsilon_2 \varepsilon_3 b\,, \tag{4.77}$$

which transforms at tree-level as

$$Q_0 C(\varepsilon_i) = \frac{1}{2}[C(\varepsilon_i), C(\varepsilon_i)]\,. \tag{4.78}$$

This makes manifest an enhanced group of symmetries, consisting of holomorphic vector-fields in the $\mathbb{C}^{2|3}$ parameterized by z_α and ε_i which preserve the volume element $\mathrm{d}^2 z \mathrm{d}^3 \varepsilon$.

The tree-level BRST cohomology can thus be presented in terms of relative Lie algebra cohomology as

$$C^\bullet\left(\mathfrak{gl}_N \otimes \mathbb{C}[[z_1, z_2, \varepsilon_i]] \mid \mathfrak{gl}_N\right). \tag{4.79}$$

In the next Section, this will allow us to make contact with the large N analysis of [111] to provide an explicit description of the large N BRST cohomology.

4.4.2 Twisted theories from twisted branes

In the context of holography, another very rich class of examples include certain quiver gauge theories with superpotentials, which arise from D3 branes located at the tip of a three-dimensional Calabi-Yau cone X. The holomorphic twist of such quivers gives, somewhat obviously, a gauged $\beta\gamma$ system based on the same quiver with superpotential.

Less obviously, but rather naturally, the twisted theory takes the form of an open string field theory for the corresponding D-branes in the B-twisted Calabi-Yau sigma model with target X [112]. This follows from the relation between twists of D-brane world-volume theories and twisted supergravity [12, 37, 31], which here should be a B-model on $\mathbb{C}^2 \times X$. From this perspective, the description of the twisted worldvolume theory on $\mathbb{C}^2$ becomes very explicit. If B denotes the algebra of functions on a non-singular Calabi–Yau manifold X, then the complex of fields of the worldvolume theory on a stack of N twisted D3 branes located at a point $p \in X$ is

$$\Omega^{0,\bullet}(\mathbb{C}^2) \otimes \mathrm{Ext}_B(\mathcal{O}_p^{\oplus N}, \mathcal{O}_p^{\oplus N})[1]\,, \tag{4.80}$$

where $\mathcal{O}_p$ is the skyscraper sheaf at p. Moreover, in this situation the Ext-algebra $\mathrm{Ext}_B(\mathcal{O}_p, \mathcal{O}_p)$ is equipped with a cyclic structure which determines the BV structure on this complex of fields.

When X is a Calabi–Yau cone then one should replace the algebra of functions by a non-commutative resolution B. In terms of the quiver determining the Calabi–Yau cone this is the non-commutative Jacobi algebra associated to the superpotential. In any case, the resulting complex of fields of the holomorphic twist of the worldvolume theory probing the singular point p of X is completely analogous to the smooth case; it can be written as

$$\Omega^{0,\bullet}(\mathbb{C}^2) \otimes \mathfrak{gl}_N[A][1] \tag{4.81}$$

where we have introduced the Koszul dual algebra $A = \mathrm{Ext}_B(\mathcal{O}_p, \mathcal{O}_p)$. Notice that A will, in general, admit a model as an A_∞-algebra.

We will unpack this description. The $N \times N$ matrix valued fields C_i of the holomorphic twist of the four-dimensional gauge theory can be put in correspondence with the boundary local operators a^i for the B-branes wrapping $\mathbb{C}^2$ at the tip of the cone X, and the tree-level differential takes the form

$$QC_i = f_i^{jk} C_j C_k + f_i^{jkt} C_j C_k C_t + \cdots \tag{4.82}$$

where f are the structure constants for the A_∞ algebra A of boundary local operators in the B-twisted sigma model.[32]

A single D3 brane usually decomposes into several fractional D-branes, each of which gives rise to a separate gauge group. Elements in a certain subalgebra A_0 of A will be dual to the c ghosts for the quiver gauge fields, others will be dual to the b fields and the rest to the β and γ fields from the chiral multiplets.

In a compact form, we can write

$$C = \sum_i C_i a^i \tag{4.84}$$

and the differential as a Maurer-Cartan equation for deformations of the B-branes:

$$QC = \frac{1}{2}(C, C) + \frac{1}{3}(C, C, C) + \cdots \tag{4.85}$$

where the parentheses indicate the operations of the A_∞ algebra.

[32]Their transformations should follow from the open string field theory action

$$\int \mathrm{d}^2 z\, \eta^{ij} C_i \bar{\partial} C_j + \eta^{ijk} C_i C_j C_k + \cdots . \tag{4.83}$$

where $\eta^{\cdots}$ are disk correlation functions in the B-twisted sigma model.

Figure 4.1: Quiver diagram for $\mathcal{N} = 4$ SYM.

This is the standard string theory dictionary: the boundary couplings C_i for the world-volume theory of the string can be promoted to fields on the D-brane world-volume. In the BV formalism, the Maurer–Cartan equation simultaneously encodes gauge invariance of a deformed boundary and determines the BRST differential for the world-volume theory. This is a version of the well-known relation between boundary beta functions and world-volume equations of motion.[33]

The tree-level BRST cohomology can be expressed in terms of Lie algebra cohomology as

$$C^\bullet \left(\mathfrak{gl}_N \left[A\right] \left[\left[z_1, z_2\right]\right] \mid \mathfrak{gl}_N \left[A_0\right]\right). \tag{4.86}$$

where $\mathfrak{gl}_N \left[A_0\right]$ is the gauge theory Lie algebra, which is a direct sum of $\mathfrak{gl}_{k_a N}$ subalgebras. This form will allow us to give a very streamlined analysis of the large N cohomology, following [141, 112]. This "categorifies" well-known simplifications in the computation of superconformal indices at large N [104, 141].

We provide a few examples. Perhaps surprisingly, we can also associate the holomorphic twist of pure gauge theory to a brane system. Indeed, the tree-level cohomology can be recovered from $A = \mathbb{C}[\varepsilon]$, which would formally arise from a point-like brane in $Y = \mathbb{C}$.

$\mathcal{N} = 4$ SYM and $\mathbb{C}^3$

The example of $\mathcal{N} = 4$ supersymmetric Yang–Mills theory arises from probing the tip of $X = \mathbb{C}^3$ (the origin, say) and corresponds to the quiver with one vertex and three edges together with the superpotential $W = xyz - xzy$, shown in figure 4.1. In the holomorphic twist the resulting quiver gauge theory is a bc system valued in $\mathfrak{gl}(N)$ together with three

[33]It is also completely analogous to the standard presentation of open string field theory—the field theory encoding the world-volume theory of D-branes [139]. See also [140] and many references therein. String field theory is usually not a local field theory, but it is for the B-model [35], essentially due to the absence of worldsheet instantons.

adjoint-valued $\beta\gamma$ systems that we denote by β_i, γ^i for $i = 1, 2, 3$. The superpotential is $W = \operatorname{Tr} \gamma^1[\gamma^2, \gamma^3]$ and from (4.75) we find the action of the tree level BRST charge

$$\begin{aligned} Q_0 c &= \frac{1}{2}[c, c] \\ Q_0 b &= [c, b] + \sum_{i=1}^{3} [\beta_i, \gamma^i] \\ Q_0 \gamma^i &= [c, \gamma^i] \\ Q_0 \beta_i &= [c, \beta_i] + \frac{1}{2} \sum_{j,k=1}^{3} \epsilon_{ijk} [\gamma^j, \gamma^k] \,. \end{aligned} \tag{4.87}$$

The algebra A is defined as follows. Each path gives rise to a Grassman generator, i.e.

$$\varepsilon_i \varepsilon_j = -\varepsilon_j \varepsilon_i \,. \tag{4.88}$$

The node gives rise to an identity e

$$ee = e, \quad e\varepsilon_i = \varepsilon_i e = \varepsilon_i \,, \tag{4.89}$$

which recovers (4.77) and (4.78). This agrees with the description of the holomorphic twist of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory given in (4.77), which we cast as the bc system valued in the super Lie algebra $\mathfrak{gl}(N)[\varepsilon_1, \varepsilon_2, \varepsilon_3]$. Therefore, in this example, we have $A = \mathbb{C}[\varepsilon_1, \varepsilon_2, \varepsilon_3]$ and $A_0 = \mathbb{C}$.

The conifold

Next, let's consider a stack of N D3 branes probing the Calabi Yau cone over the Einstein manifold $T^{1,1}$. This is a quiver with two nodes and four edges $x_1, x_2 \colon 0 \to 1$, $y_1, y_2 \colon 1 \to 0$, shown in Figure 4.2. This quiver is equipped with the Klebanov–Witten superpotential

$$W = x_1 y_1 x_2 y_2 - x_1 y_2 x_2 y_1, \tag{4.90}$$

see [142]. The field content of the holomorphically twisted gauge theory is

- a bc system valued in $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 = \mathfrak{gl}(N) \oplus \mathfrak{gl}(N)$,
- a pair of $\beta\gamma$ systems where

$$\gamma_i \in \operatorname{Hom}(\mathbb{C}^N, \mathbb{C}^N), \quad i = 1, 2 \tag{4.91}$$

  transform in the fundamental of $\mathfrak{g}_1$ and the antifundamental of $\mathfrak{g}_2$,

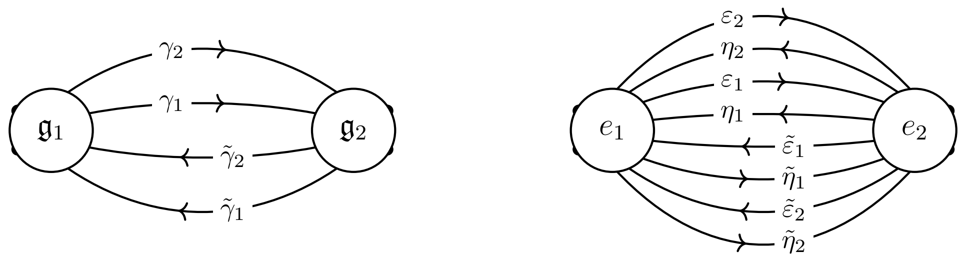


Figure 4.2: Conifold quiver.

- another pair of $\beta\gamma$ systems where

$$\tilde{\gamma}_i \in \mathrm{Hom}(\mathbb{C}^N, \mathbb{C}^N), \quad i = 1, 2 \tag{4.92}$$

transform in the antifundamental of $\mathfrak{g}_1$ and the fundamental of $\mathfrak{g}_2$.

The superpotential becomes

$$W = \mathrm{Tr}\left(\gamma_1\tilde{\gamma}_1\gamma_2\tilde{\gamma}_2 - \gamma_1\tilde{\gamma}_2\gamma_2\tilde{\gamma}_1\right). \tag{4.93}$$

From (4.75) we find the action of the tree level BRST charge

$$\begin{aligned}
&Q_0 c_1 = \frac{1}{2}[c_1, c_1], && Q_0 c_2 = \frac{1}{2}[c_2, c_2]\\
&Q_0\gamma_i = c_1\gamma_i - \gamma_i c_2 && Q_0\tilde{\gamma}_i = -\tilde{\gamma}_i c_1 + c_2\tilde{\gamma}_i\\
&Q_0 b_1 = [c_1, b_1] + \frac{1}{2}\sum_{i=1}^{2}\left(\gamma_i\beta_i - \tilde{\beta}_i\tilde{\gamma}_i\right) && Q_0 b_2 = [c_2, b_2] - \frac{1}{2}\sum_{i=1}^{2}\left(\beta_i\gamma_i - \tilde{\gamma}_i\tilde{\beta}_i\right)\\
&Q_0\beta_1 = \beta_1 c_1 + c_2\beta_1 + \tilde{\gamma}_1\gamma_2\tilde{\gamma}_2 - \tilde{\gamma}_2\gamma_2\tilde{\gamma}_1\\
&Q_0\beta_2 = \beta_2 c_1 + c_2\beta_2 + \tilde{\gamma}_2\gamma_1\tilde{\gamma}_1 - \tilde{\gamma}_1\gamma_1\tilde{\gamma}_2\\
&Q_0\tilde{\beta}_1 = c_1\tilde{\beta}_1 + \tilde{\beta}_1 c_2 + \gamma_2\tilde{\gamma}_2\gamma_1 - \gamma_1\tilde{\gamma}_2\gamma_2\\
&Q_0\tilde{\beta}_2 = c_1\tilde{\beta}_2 + \tilde{\beta}_2 c_2 + \gamma_1\tilde{\gamma}_1\gamma_2 - \gamma_2\tilde{\gamma}_1\gamma_1\,.
\end{aligned} \tag{4.94}$$

This BRST differential encodes the A_∞ algebra A which abstractly is the Koszul dual of the non-commutative Jacobi ring associated to the quiver 4.2 with potential. Each node defines an generator e_v with $v = 1, 2$, which are idempotent and pairwise orthogonal, i.e.

$$e_1 e_1 = e_1, \quad e_2 e_2 = e_2 \quad e_1 e_2 = 0\,. \tag{4.95}$$

It has natural multiplication with the paths ϵ_i and $\tilde{\epsilon}_i$

$$\epsilon_{x\to y} e_y = \epsilon_{x\to y}, \quad e_x \epsilon_{x\to y} = \epsilon_{x\to y} \,. \tag{4.96}$$

In particular, it is easy to see $e \equiv \sum e_v = e_1 + e_2$ is a multiplicative identity

$$e\epsilon_{v_1\to v_2} = \epsilon_{v_1\to v_2} e = \epsilon_{v_1\to v_2} \,. \tag{4.97}$$

If a path and a node don't compose, their multiplication vanishes, e.g. $e_1\eta_1 = \eta_1 e_2 = 0$. We also require

$$\epsilon_i \tilde{\epsilon}_j = \tilde{\epsilon}_i \epsilon_j = \epsilon_i \epsilon_j = \tilde{\epsilon}_i \tilde{\epsilon}_j = 0 \tag{4.98}$$

Furthermore there are nontrivial 3-nary A_∞ brackets[34] $(-,-,-)\colon A^{\times 3} \to A$, dual to the relations determined by the non-commutative derivatives of the superpotential.

$$\begin{aligned}
\eta_1 &\equiv (\tilde{\epsilon}_1, \epsilon_2, \tilde{\epsilon}_2) = -(\tilde{\epsilon}_2, \epsilon_2, \tilde{\epsilon}_1)\\
\eta_2 &\equiv (\tilde{\epsilon}_2, \epsilon_1, \tilde{\epsilon}_1) = -(\tilde{\epsilon}_1, \epsilon_1, \tilde{\epsilon}_2)\\
\tilde{\eta}_1 &\equiv (\epsilon_2, \tilde{\epsilon_2}, \epsilon_1) = -(\epsilon_1, \tilde{\epsilon_2}, \epsilon_2)\\
\tilde{\eta}_2 &\equiv (\epsilon_1, \tilde{\epsilon_1}, \epsilon_2) = -(\epsilon_2, \tilde{\epsilon_1}, \epsilon_1) \,.
\end{aligned} \tag{4.99}$$

All of the 2-nary multiplications between ϵ_i, $\tilde{\epsilon}_i$, η_i and $\tilde{\eta}_i$ vanish except

$$\tilde{\eta}_i \tilde{\epsilon}_i = \epsilon_i \eta_i, \quad \eta_i \epsilon_i = \tilde{\epsilon}_i \tilde{\eta}_i, \quad \forall i \,. \tag{4.100}$$

η_i and $\tilde{\eta}_i$ multiply with the node e_i naturally according to the path, e.g. $\eta_1 e_1 = e_2 \eta_1 = \eta_1$. All other 3-nary brackets vanish.

Then if we define the superfield to be

$$C = c_1 e_1 + c_2 e_2 + \gamma_i \epsilon_i + \tilde{\gamma}_i \tilde{\epsilon}_i + \beta_1 \eta_1 + \beta_2 \eta_2 + \tilde{\beta}_1 \tilde{\eta}_1 + \tilde{\beta}_2 \tilde{\eta}_2 - b_1 \epsilon_1 \eta_1 - b_2 \eta_2 \epsilon_2 \,.$$

We find (4.94) is simply

$$Q_0 C = [C, C] + (C, C, C) \,, \tag{4.101}$$

where with a slight abuse of notation, $[x, y]$ denotes the appropriate action according to the representations. For example, if both x and y are in adjoint representation, $[-,-]$ is $\frac{1}{2}$ times the bracket $[x, y]$ for adjoint action; if x is in adjoint representation and y is in the fundamental, then the $[-,-]$ is simply the matrix multiplication. Note that e_v, η_i and $\tilde{\eta}_i$ commute with all the fields and ϵ_i, $\tilde{\epsilon}_i$ anticommute with the c_i, β_i and $\tilde{\beta}_i$.

Lastly, in this example A_0 is determined by the number of nodes, i.e. $A_0 = \mathbb{C} \oplus \mathbb{C}$.

[34] 2-nary bracket is simply denoted as the multiplication.

4.4.3 $\mathcal{N}=2$ gauge theories

For $\mathcal{N}=2$ gauge theories, the "algebra" presentation of the tree-level cohomology problem involves an auxiliary Lie algebra which appeared before in [143], built from the data of a Lie algebra $\mathfrak{g}$ and a symplectic representation R of $\mathfrak{g}$. The bosonic part of the algebra is $T^*\mathfrak{g}$, with generators t^a and $\tilde{t}_a$ and non-trivial commutators $[t^a, t^b] = f^{ab}_c t^c$ and $[t^a, \tilde{t}_b] = f^{ac}_b \tilde{t}_c$.[35] The fermionic part of the algebra is ΠR, with generators ρ^i and commutators $[t^a, r^i] = f^{ai}_j r^j$ and $\{r^i, r^j\} = f^{ija}\tilde{t}_a$, where we raised an index in the structure constants using the symplectic form.

Then

$$C^\bullet\bigg(\ (T^*\mathfrak{g} \oplus \Pi R) \otimes \mathbb{C}[[z_1, z_2, \varepsilon]] \mid \mathfrak{g}\ \bigg), \tag{4.102}$$

reproduces the tree-level cohomology of the twisted $\mathcal{N}=2$ gauge theory with gauge group G and matter representation R. The chirals in the matter hypermultiplets give γ_i and β_i fields packaged in $\Pi R[[\epsilon]]$, while the adjoint chirals in the vectormultiplets are associated to the $\tilde{t}_a$ generators.

4.4.4 Quivers from orbifold projections

Affine ADE $\mathcal{N}=2$ quivers with $SU(d_a N)$ gauge groups arise from N D3 branes in probing the singular point of $\mathbb{C}^2/\Gamma \times \mathbb{C}$, where Γ is the corresponding finite subgroup of $SU(2)$. More general $\mathcal{N}=1$ quivers arise from cones of the form $\mathbb{C}^3/\Gamma$ for a discrete subgroup Γ of $SU(3)$[144, 145].

In all of these examples, the matter content of the theory is obtained by "orbifold projection" of the fields in an $U(|\Gamma|N)$ $\mathcal{N}=4$ gauge theory. The idea is to embed Γ into the gauge group as a permutation group of $|\Gamma|$ copies of $\mathbb{C}^N$ as well as in the $SU(3)$ flavour group in the obvious way. One then projects onto Γ-invariant fields.

The permutation action of Γ decomposes $\mathbb{C}^{N|\Gamma|}$ into Nd_a copies of each representation R_a of dimension d_a. Then the gauge fields are projected to $U(d_a N)$ gauge fields, while fields which transform in the (anti)fundamental representation (3) of $SU(3)$ form $d_a N \times d_b N$ matrices for every copy of R_a into $(3) \otimes R_b$ (or viceversa).

[35] Here we are simplifying the discussion and only working with the totalized $\mathbb{Z}/2$ grading. Like twists of $\mathcal{N}=1$ theories (without superpotential) there exists a mixed ghost/parity grading by the integers where the bosonic part is $T^*[2]\mathfrak{g}$, for example.

We can implement the orbifold projection directly in the twisted theory. We start from the algebra $\mathfrak{gl}_{|\Gamma|} \otimes \mathbb{C}[[\varepsilon_1, \varepsilon_2, \varepsilon_3]]$ with Γ acting as permutations on $\mathbb{C}^{|\Gamma|}$ and as a subgroup of $SU(3)$ on ε_i. The correct algebra A is then the Γ-invariant part of $\mathfrak{gl}_{|\Gamma|} \otimes \mathbb{C}[[\varepsilon_1, \varepsilon_2, \varepsilon_3]]$.

4.5 The large N cohomology

We have argued in Section 4.3.1 and Section 4.4 that the BRST complex of local operators at tree level for a variety of twisted $SU(N)$ theories can be expressed as a certain cochain complex

$$C^\bullet \left(\mathfrak{gl}_N \left[A \otimes \mathbb{C}[[z_1, z_2]]\right] | \mathfrak{gl}_N \left[A_0\right]\right) . \tag{4.103}$$

For pure gauge theory, $A = \mathbb{C}[[\epsilon]]$ for an odd variable ϵ. For $\mathcal{N} = 4$ supersymmetric Yang–Mills theory, $A = \mathbb{C}[[\epsilon_i]]$ for three odd variables ϵ_i. For quivers associated to D-branes at Calabi-Yau singularities, A is the algebra of boundary local operators in the associated B-model D-branes.

If we work at large N, these cohomology problems can be solved in a rather uniform way. For $\mathcal{N} = 4$ SYM, a conjectural solution was proposed in [111], in a form which can be easily generalized. As in the previous Section, denote as

$$C(z, \varepsilon) = c(z) + \cdots \tag{4.104}$$

the polynomial in ε_i which collects all the fields in the problem and depends holomorphically on z_α. We view this as a linear element in the complex (4.38). The differential (called the Chevalley–Eilenberg differential) acting on $C(z, \varepsilon)$ is dual to the Lie bracket on $\mathfrak{gl}_N[[z_\alpha, \varepsilon_i]]$ and hence can be schematically written as

$$QC(z, \varepsilon) = \frac{1}{2}[C(z, \varepsilon), C(z, \varepsilon)] \,. \tag{4.105}$$

We formally extend the space of fields to include not only functions in the variables z_α, ε_i, but also differential forms in these variables. This amounts to replacing $\mathbb{C}[[z_\alpha, \varepsilon_i]]$, which we view as functions on the formal super disk $\hat{D}^{2|\mathcal{N}-1}$, with de Rham forms on the formal super disk. There is also a formal de Rham differential defined by

$$\mathrm{d}C(z, \varepsilon) = \mathrm{d}z_\alpha \frac{\partial C}{\partial z_\alpha} + \mathrm{d}\varepsilon_i \frac{\partial C}{\partial \varepsilon_i} \,. \tag{4.106}$$

We extend the de Rham differential as a derivation and declare that it commutes with the original differential as in

$$Q\mathrm{d}C(z, \varepsilon) = [C(z, \varepsilon), \mathrm{d}C(z, \varepsilon)] \,. \tag{4.107}$$

Notice that the right hand side contains no $c(z)$-term. So, all coefficients in the expansion of the expression

$$\mathrm{Tr}\,[\mathrm{d}C(z,\varepsilon)]^n \tag{4.108}$$

to powers in the variables z, ε, $\mathrm{d}z$, and $\mathrm{d}\varepsilon$ are valid Q-closed operators.

Some linear combinations of these operators actually vanish, simply because

$$\mathrm{d}\,\mathrm{Tr}\,[\mathrm{d}C(z,\varepsilon)]^n = 0\,. \tag{4.109}$$

The conjecture of [111] is that the coefficients of the expansion of these operators in ε_i, $\mathrm{d}\varepsilon_i$ and $\mathrm{d}z^\alpha$, modulo the above relation, generate the single-trace cohomology at large N. We will discuss below the possible loop corrections to this statement.

If we repeat the analysis exactly in the same way with a single ϵ variable, we can formulate a similar conjecture for pure gauge theory. Expanding $C(z,\varepsilon) = c(z) + \varepsilon b(z)$, we have

$$\mathrm{d}C(z,\varepsilon) = \mathrm{d}\varepsilon\, b(z) + \mathrm{d}z_\alpha \partial_\alpha c(z) + \varepsilon \mathrm{d}z_\alpha \partial_\alpha b(z) \tag{4.110}$$

and thus

$$\begin{aligned}\mathrm{Tr}[\mathrm{d}C(z,\varepsilon)]^n = {} & \mathrm{d}\varepsilon^n\,\mathrm{Tr}\,b^n + \mathrm{d}z_\alpha \mathrm{d}\varepsilon^{n-1}\mathrm{STr}\,b^{n-1}\partial_\alpha c + \mathrm{d}z^1\mathrm{d}z^2\mathrm{d}\varepsilon^{n-2}\mathrm{STr}\,b^{n-2}\partial_\alpha c\,\partial^\alpha c \\ & + \varepsilon \mathrm{d}z_\alpha \mathrm{d}\varepsilon^{n-1}\mathrm{STr}\,b^{n-1}\partial_\alpha b + \varepsilon \mathrm{d}z^1\mathrm{d}z^2 d\varepsilon^{n-2}\mathrm{STr}\,b^{n-2}\partial_\alpha b \partial_\alpha c\,. \end{aligned} \tag{4.111}$$

For general n, the first three terms are our familiar A_n, $B_{n-1,\alpha}$ and C_{n-2} towers:

$$\begin{aligned}\mathrm{Tr}[\mathrm{d}C(z,\varepsilon)]^n = {} & \mathrm{d}\varepsilon^n A_n + \mathrm{d}z_\alpha \mathrm{d}\varepsilon^{n-1} B_{n-1,\alpha} + \mathrm{d}z^1\mathrm{d}z^2\mathrm{d}\varepsilon^{n-2} C_{n-2} \\ & + \varepsilon \mathrm{d}z_\alpha \mathrm{d}\varepsilon^{n-1} n^{-1}\partial_\alpha A_n + \varepsilon \mathrm{d}z^1\mathrm{d}z^2 d\varepsilon^{n-2}(n-1)^{-1}\partial_\alpha B_{n-1,\alpha}\,. \end{aligned} \tag{4.112}$$

The two last terms do not give us new fields, as they can be rewritten as derivatives of other representatives. These are precisely the relations from (4.109).

For small n the statements have to be adjusted in a minor way and possibly corrected to account for the difference between $U(N)$ and $SU(N)$.

4.5.1 Cohomology of the tree-Level differential

We now provide a proof of these expectations and a generalization for all A, following [112, 31].

Our method uses a theorem of Loday, Quillen, and Tysgan (LQT) which relates the (non-relative) Lie algebra cohomology to the *cyclic homology* of $\mathbb{C}[[z_\alpha, \varepsilon_i]]$ [146, 147].

Applied to our situation, the Hochschild–Serre spectral sequence implies a general relationship between the relative Lie algebra cohomology and the absolute cohomology which takes the form of an isomorphism

$$H^\bullet(\mathfrak{gl}_N(A')|\mathfrak{gl}_N(A_0)) \otimes H^\bullet(\mathfrak{gl}_N(A_0)) \simeq H^\bullet(\mathfrak{gl}_N(A')). \tag{4.113}$$

Here A' is a general algebra, and A_0 is a subalgebra, but we will employ the result for $A' = A \otimes \mathbb{C}[[z_1, z_2]]$ with $A_0 \subset A$. We assume that $\mathfrak{gl}_N(A_0)$ is a reductive subalgebra in order for the above isomorphism to hold; this is true in all examples we consider. Thus, to obtain relative Lie algebra cohomology we will simply remove the contribution coming from $H^\bullet(\mathfrak{gl}_N(A_0))$. With this in mind, we follow the Loday, Quillen, Tsygan prescription for computing the absolute cohomology, and after this we will extract the relative answer.

The theorem of Loday, Quillen, and Tsygan can be explained in an intuitive way. We study single-trace operators

$$\operatorname{Tr} C_{i_1} \cdots C_{i_n} . \tag{4.114}$$

The action of the differential closes on single-trace operators

$$Q \operatorname{Tr} C_{i_1} \cdots C_{i_n} = \sum_{a,k} (-1)^{\cdots} f_{i_a}^{j_1, \cdots, j_k} \operatorname{Tr} C_{i_1} \cdots C_{i_{a-1}} C_{j_1} \cdots C_{j_k} C_{i_{a+1}} \cdots C_{j_n} . \tag{4.115}$$

Here the sign $(-1)^{\cdots}$ accounts for various Koszul signs. This differential defines a degree-shifted version of the dual of cyclic homology of A', denoted as $HC_\bullet(A')^\vee[-1]$.

Including multi-trace operators, we arrive to the theorem of Loday, Quillen, and Tsygan [146, 147] relating the large N Lie algebra cohomology of $\mathfrak{gl}_N(A')$ to the cyclic homology of the algebra

$$H^\bullet\left(\mathfrak{gl}_\infty(A')\right) \simeq \operatorname{Sym}\left(HC_\bullet(A')^\vee[-1]\right).$$

Let a_m denote the elements in the A_∞ algebra $A' = A \otimes \mathbb{C}[[z_1, z_2]]$ with A_∞ operations $(-, \dots, -)$. The cyclic homology itself can be described by linear combinations of formal symbols $[a_1 \cdots a_{i_n}]$ defined up to cyclic rotations (with appropriate Koszul signs) dual to the single-trace operators, placed in a degree shifted by $n-1$, with differential

$$\begin{aligned} Q[a_1 \cdots a_n] = &\sum_{u<v} (-1)^{\cdots} [a_1 \cdots a_{u-1} (a_u, \cdots, a_v) a_{v+1} \cdots a_n] + \\ &+ \sum_{u>v} (-1)^{\cdots} [a_{v+1} \cdots a_n (a_u, \cdots, a_v) a_1 \cdots a_{u-1}] , \end{aligned} \tag{4.116}$$

where the sum is over all pairs of u and v.

Before we continue, we should remind ourselves of the 2d TFT interpretation of cyclic homology. Intuitively, each boundary condition in the TFT gives a "boundary state", i.e. a state for the 2d TFT compactified on the circle. If we produce a complicated boundary state by taking N copies of an elementary boundary condition and deforming it by some couplings C_i, we should be able to expand the boundary state in powers of the couplings.

A more precise statement is that the coefficient of $\mathrm{Tr}\, C_{i_1} \cdots C_{i_n}$ in the expansion is mapped to an element in the space $Z_{S^1}[S^1]$ of "S^1-equivariant" states for the 2d TFT compactified on a circle. The images of the planar Q are the coefficients of the BRST variation of the boundary state, so the map from single-trace operators to $Z_{S^1}[S^1]$ commutes with the BRST differential.

In a string field theory language, the map tells us how the closed string fields couple to local operators in the brane world-volume.

The cyclicity of the trace turns out to be an important complication in the computation of cyclic homology. A very similar differential acting on words which are not cyclically symmetric defines the notion of Hochschild homology $HH_\bullet(A')$ of A' (valued in A'), which is usually more computable.

For example, we can think about a commutative algebra A' as defining the affine scheme $X = \mathrm{Spec}(A')$. Then, a theorem of Hochschild, Kostant, and Rosenberg [148] asserts that the Hochschild homology of A' can be identified with the algebra of de Rham forms on X.[36]

Crucially, the cyclic cohomology can be computed from $HH_\bullet(A')$ via a spectral sequence involving the *Connes B operator*, which is an algebraic analogue to the de Rham differential [149].

The first page of the spectral sequence is

$$HH_\bullet(A') \otimes \mathbb{C}[u^{-1}].$$

Here u is a parameter of ghost number $+2$ and $HH_\bullet(A')$ is the ordinary Hochschild homology of the algebra A'. The differential at this page in the spectral sequence is uB where B is the Connes B-operator acting on the Hochschild homology. The variable u corresponds to the generator of $H^\bullet(BS^1)$ and its presence encodes the fact that the cyclic homology is an S^1-equivariant version of ordinary Hochschild homology.[37]

[36] The grading is opposite to the one usually used for de Rham forms.

[37] The Hochschild homology $HH_\bullet(A')$ also has a 2d TFT interpretation, involving TFT states without S^1 equivariance and boundaries decorated by a boundary local operator. The difference between equivariant and non-equivariant states in a string theory language concerns the inclusion or exclusion of the $c_0 - \bar{c}_0$ mode of the bulk ghosts associated to circle rotations.

Order-by-order in u, at order u^0 we find a copy $HH_\bullet(A')$ modulo the image of B. At the next order u^{-1}, we have a copy in $HH_\bullet(A')u^{-1}$ of the kernel of B modulo the image of B, etcetera. In other words, we get a single copy of $HH_\bullet(A')/(B\,HH_\bullet(A'))$ and infinitely many copies of the cohomology of B.

For our algebra $A' = \mathbb{C}[[z_1, z_2, \varepsilon]]$, the Connes B operator is explicit to describe. We apply the Hochschild, Kostant, Rosenberg theorem to express the Hochschild homology of A' as

$$HH_\bullet(A') \simeq \mathbb{C}[[z_\alpha, \mathrm{d}z_\alpha, \varepsilon, \mathrm{d}\varepsilon]].$$

Here the cohomology degree of z_α is zero, $\mathrm{d}z_\alpha$ is degree -1, ε is degree $+1$, and $\mathrm{d}\varepsilon$ is degree zero. With this identification, the Connes B operator is simply the de Rham differential

$$B = \mathrm{d}z_\alpha \partial_{z_\alpha} + \mathrm{d}\varepsilon \partial_\varepsilon.$$

The cohomology of B is very simple and consists of $\mathbb{C}$ only. It gives us a summand of $\mathbb{C}[u^{-1}]$ which is actually the difference between the relative, and the absolute Lie algebra cohomologies.

The part we are interested in is the quotient of $\mathbb{C}[[z_\alpha, \mathrm{d}z_\alpha, \varepsilon, \mathrm{d}\varepsilon]]$ by the image of the de Rham differential. At this point, we have already essentially reproduced our conjectures above: the cohomology is generated by the coefficients in $\mathrm{Tr}\,[\mathrm{d}C(z,\varepsilon)]^n$ modulo the relation following from $\mathrm{d}\,\mathrm{Tr}\,[\mathrm{d}C(z,\varepsilon)]^n = 0$.

More concretely, we compute the second page of the spectral sequence converging to $HC_\bullet(A')$ by using an auxiliary spectral sequence that splits the differential B into two pieces. A similar computation with a single complex coordinate and two odd variables was done in [31]. The first differential in this auxiliary spectral sequence is the piece of uB which is the differential acting on the ε-coordinate, and does not affect the holomorphic z_α-variables.[38] The cohomology of $HH_\bullet(A')[u^{-1}]$ with respect to $u\mathrm{d}\varepsilon\partial_\varepsilon$ is isomorphic to

$$\varepsilon\mathbb{C}[[z_\alpha, \mathrm{d}z_\alpha, \mathrm{d}\varepsilon]] \oplus \mathbb{C}[[z_\alpha, \mathrm{d}z_\alpha]][u^{-1}]. \tag{4.117}$$

The remaining differential is simply $u\mathrm{d}z_\alpha\partial_{z_\alpha}$, which is basically the holomorphic de Rham differential on $\mathbb{C}^2$. The next page of the spectral sequence is the cohomology of (4.117) with respect to this differential. This cohomology is

$$\varepsilon\mathbb{C}[[z_\alpha, \mathrm{d}\varepsilon]] \oplus (\oplus_\alpha \varepsilon \mathrm{d}z_\alpha \mathbb{C}[[z_\alpha, \mathrm{d}\varepsilon]]) \oplus \mathbb{C}[[z_\alpha]] \oplus \mathrm{d}z_1\mathrm{d}z_2\mathbb{C}[[z_\alpha]][1] \oplus u^{-1}\mathbb{C}[[u^{-1}]]. \tag{4.118}$$

[38] This term in the differential can be thought of as the Koszul differential resolving a point inside of $\mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[\mathrm{d}\varepsilon])$.

For the summand in degree -1 have made the identification of the quotient space of one-forms on $\mathbb{C}^2$ modulo exact one-forms with two-forms on $\mathbb{C}^2$ using the Poincaré lemma. There are no further terms in the spectral sequence.

Removing the summand $\mathbb{C} \subset \mathbb{C}[[z_\alpha]]$ and $u^{-1}\mathbb{C}[u^{-1}]$ corresponds to taking the large N limit of the relative, rather than the absolute, Lie algebra cohomology. Thus, in summary we see that the relative large N cohomology is $\mathrm{Sym}^\bullet(V^\vee[-1])$ where V is the vector space

$$\varepsilon\mathbb{C}[[z_\alpha, \mathrm{d}\varepsilon]] \oplus (\oplus_\alpha \varepsilon \mathrm{d}z_\alpha \mathbb{C}[[z_\alpha, \mathrm{d}\varepsilon]]) \oplus \mathbb{C}[[z_\alpha]]/\mathbb{C} \oplus \mathrm{d}z_1\mathrm{d}z_2\mathbb{C}[[z_\alpha]][1] \tag{4.119}$$

Here, $\mathbb{C}[[z_\alpha]]/\mathbb{C}$ is power series modulo constant functions. We interpret each of these summands as tree-level single trace operators present in the large N limit:

- Consider the term $\varepsilon\mathbb{C}[[z_\alpha, \mathrm{d}\varepsilon]]$. This corresponds to single trace operators which have ghost number zero. Tracing through (4.111), which compares with de Rham cohomology, this term is generated by the primary tower of operators

  $$A_n = \mathrm{Tr}\{b^n\}, \quad n \geq 0$$

  and their z_α-derivatives.

- Next consider the term $\varepsilon \mathrm{d}z_\alpha \mathbb{C}[[z_\alpha, \mathrm{d}\varepsilon]]$. This corresponds to single trace operators which have ghost number $+1$, and by comparing with (4.111), is generated by the primary tower of operators

  $$B_{n,\alpha} = \mathrm{Tr}\, b^n \partial_\alpha c, \quad n \geq 0,\ \alpha = 1,2$$

  and their z_α-derivatives.

- Consider the term $\mathbb{C}[[z_\alpha]]/\mathbb{C}$. This corresponds to single trace operators which have ghost number $+1$. This term is generated by holomorphic descendants of the local operator $\mathrm{Tr}\, c$ where at least one holomorphic derivative appears.

- Consider the term $\mathrm{d}z_1\mathrm{d}z_2\mathbb{C}[[z_\alpha, \mathrm{d}\varepsilon]][1]$. This corresponds to single trace operators which are of ghost number $+2$ and is generated by the primary tower of operators

  $$C_n = \mathrm{Tr}\, b^n \partial_\alpha c\, \partial^\alpha c, \quad n \geq 0$$

  and their z_α-derivatives.[39]

[39]Notice that the relative order of the fields in the trace is immaterial, as we can change the relative order by adding Q-exact operators with a single insertion of $\partial_\alpha b$:

$$Q\partial_\alpha b = [\partial_\alpha c, b] + \cdots \tag{4.120}$$

Summarizing, we find the following single-trace primaries in the tree-level cohomology:

$$\begin{aligned} A_n &= \operatorname{Tr} b^n, && n \geq 1 \\ B_{n,\alpha} &= \operatorname{Tr} b^n \partial_\alpha c, && n \geq 1,\ \alpha = 1,2 \\ C_n &= \operatorname{Tr} b^n \partial_\alpha c\, \partial^\alpha c, && n \geq 0\,. \end{aligned} \tag{4.121}$$

By the computation above, these operators together with their derivatives and the derivatives of $\operatorname{Tr} c$ form a basis of the large N single-trace tree-level cohomology.

The general structure of this calculation should generalize to all algebras of the form $A' = A \otimes \mathbb{C}[[z_\alpha]]$. Then

$$HH_\bullet(A') \simeq \mathbb{C}[[z_\alpha, \mathrm{d}z_\alpha]] \otimes HH_\bullet(A).$$

The Connes B operator for A' still splits as

$$B = \mathrm{d}z_\alpha \partial_{z_\alpha} \otimes \mathrm{id} + \mathrm{id} \otimes B_A$$

where B_A is the Connes operator for the algebra A.

The copies of the B cohomology at non-trivial powers of u^{-1} generalizes the $u^{-1}\mathbb{C}[u^{-1}]$ factor above. It has the correct quantum numbers to represent the difference between the absolute and relative cohomology. At order u^0 we can apply a similar approach as above by employing a spectral sequence which first computes the B_A cohomology. The first page in this spectral sequence computing the large N relative Lie algebra cohomology is thus the u^0 part of the cyclic homology of A tensored with $\mathbb{C}[[z_\alpha, \mathrm{d}z_\alpha]]$.

If A is commutative, we can give a more intuitive explanation of this by employing superfields C valued in $\mathbb{C}^2 \times \operatorname{Spec}(A)$. Then we could formally define dC using the Connes B as a differential d and build Q-closed operators $\operatorname{Tr}(dC)^n$ as before.

4.5.2 Cohomology of the loop-corrected differential

Starting with the above results about the tree level operators at large N, we can now discuss the effect of the one-loop correction to the differential.

The first important observation is that the differential now mixes operators with different numbers of traces: a single-trace operator can be mapped to a single trace operator with an extra factor of N, if the differential acts on neighbouring letters in the trace, or to a product of two traces if it acts on other pairs of letters.

Conversely, the 1-loop differential acting on the product of two traces could act on each individually or merge them into a single trace.[40]

In order to have a standard large N combinatorics, it is useful to incorporate a factor of N^{-1} in the propagators of the fields and a factor of N in the interaction. One can normalize single-trace operators with an overall power of N. Then the planar loop differential maps single-trace to single-trace with no extra factors of N. The leading $1-2$ process is suppressed by a power of N^2, but the $2-1$ process is of order 1.

In the pure gauge theory, we can schematically discuss the action of the one-loop differential on single-trace operators.

The action on A_n is particularly simple:

$$Q_1 A_n \sim n \mathrm{Tr}\, b^{n-2} \partial_\alpha b \partial^\alpha c\,. \tag{4.122}$$

The relative order of the symbols in the trace can be changed by adding tree-level exact terms such as $Q_0 \mathrm{Tr}\, b^{k_1} \partial_\alpha b\, b^{k_2} \partial^\alpha b$. Then, by using the fact that the action of Q_0 on the individual fields gives commutators, and canceling with the operator in (4.122), we can thus rewrite the Q_1 action on the tree-level cohomology in terms of $B_{n,\alpha}$:

$$Q_1 A_n = \frac{2n}{n-1} \partial_\alpha B^\alpha_{n-1}\,. \tag{4.123}$$

In cohomology, therefore, the one-loop differential eliminates the A_n tower for $n > 1$ and divergences of the $B_{n,\alpha}$ tower. The action on $B_{n,\alpha}$ is a bit more complicated. Barring magical cancellation, it must produce a multiple of $\partial_\alpha C_n$ up to tree-level exact terms. On the other hand, the action of Q_1 on C_n must vanish as there are no single-trace operators with three c's in the tree-level cohomology.

Summarizing, in the planar limit we see that the one-loop differential eliminates almost every single-trace operator. In the $SU(N)$ theory, only the C_n survive, even though the derivatives of C_n are trivial in one-loop cohomology. These remain even in the $SU(N)$ theory.[41]

Stripping off the singleton, i.e. working with an $SU(N)$ gauge theory, we thus find that the one-loop corrections produce a *topological* theory with single-trace observables associ-

[40] This "problem" could be avoided if we define multi-trace operators via regularized products, effectively adding to the operator extra terms with fewer traces. This would introduce a different, possibly worse problem: the regularized product is not associative.

[41] We do not expect the differential mapping two traces to a single trace to change this conclusion, as acting on $C_n C_m$ it would produce single-trace operators with too many c's.

ated to C_n. The quantum numbers of the operators prevent further perturbative corrections to this answer. Indeed, as they have all even ghost number, even non-perturbative corrections are impossible.

4.6 Hints of twisted holography

We will now discuss a holographic realization of holomorphic BF theory in the B-model topological string, leading to conjectural examples of twisted holography for the twists of pure gauge theory and SQCD with a fixed number of flavours. At tree level, this setup can be treated in parallel to other twisted holography examples based on the B-model, e.g. the cases of the 2d chiral algebra in $\mathcal{N}=4$ supersymmetric Yang–Mills theory [31] or the holomorphic twist of $\mathcal{N}=4$ supersymetric Yang–Mills theory [12], and other theories associated to D3 branes at singularites as we touched upon in Section 4.4.

For the B-model with flat target $\mathbb{C}^{d_1+d_2}$ one can consider branes wrapping d_1 of the directions. As we recalled in Section 4.4, the topological B-model presents the tree-level local operators in the world-volume theory on N branes in terms of the Lie algebra cohomology of $\mathfrak{gl}_N[[z_\alpha, \varepsilon_i]]$, where $\alpha = 1, \dots, d_1$ label the bosonic holomorphic coordinates and $i = 1, \dots, d_2$ label the fermionic coordinates. We can describe it, as before, in terms of the composite superfield $C(z, \varepsilon)$.

In the case at hand we are considering $d_1 = 2, d_2 = 1$, so the total dimension is 3. This is a familiar situation, where the two-dimensional B-twisted sigma-model computes top forms on the moduli space of complex structures of a Riemann surface, which are integrated to give topological string amplitudes. The closed string sector of the B-model topological string on a Calabi–Yau threefold admits an elegant description in terms of Kodaira–Spencer theory [35, 37, 13].

Theories associated to D3 branes at singularities, on the other hand, have $d_1 = 2, d_2 = 3$ and involve a more exotic "supercritical" topological string theory, defined in dimension greater than 3. Loop amplitudes in such a theory vanish unless insertions with sufficiently negative ghost number are present on the worldsheet. The result can still be expressed in the language of Kodaira–Spencer theory [37].

Generally speaking, the bulk B-model fields are divergence-free holomorphic poly-vectorfields on $\mathbb{C}^{d_1+d_2}$, but in the neighbourhood of the brane they can be roughly thought of as divergence-free holomorphic poly-vectorfields on $\mathbb{C}^{d_1|d_2}$, essentially by identifying ∂_{ε_i} as bosonic coordinates $\tilde{z}^i$ and ε_i as $\partial_{\tilde{z}^i}$ in the poly-vectorfield. They can thus be paired up naturally with the forms $\mathrm{Tr}(\mathrm{d}C)^n$ on $\mathbb{C}^{d_1|d_2}$ to describe the coupling of the brane to the bulk fields.

This setup can be employed to study the world-volume theory of the branes, the BF theory, in two related ways:

- The bulk closed string fields must admit a non-anomalous coupling to the world-volume theory living on the brane. Anomaly cancellation can be used to constrain the differential and brackets of the worldvolume theory, for any number N of branes.[42]

- Open-closed duality predicts that the open-closed string theory in the presence of a stack of N branes should be equivalent to the closed string theory in a backreacted geometry. The standard dictionary of open-closed duality relates the planar loop expansion of the field theory to classical calculations in the bulk closed string theory, with a precise relation between the number of loops and the power of the backreaction on the two sides. Quantum effects in the bulk govern the N^{-1} expansion in the field theory.

On general grounds, the backreaction of the D-branes is a d_1-vector valued in $(0, d_2 - 1)$ forms:

$$N\omega_{\mathrm{BM}} \prod_{i=\alpha}^{d_1} \partial_{z_\alpha} \tag{4.124}$$

where ω_{BM} is now the Green's function for $\bar{\partial}$ in the transverse directions.

The effects of this backreaction can be drastically different depending on the value of d_2 and are typically non-geometric in nature. It is not necessarily obvious how to define a decoupling limit to derive a standard holographic correspondence from the open-closed duality statement. The anomaly cancellation results, on the other hand, can be safely used to constrain the properties of the D-brane worldvolume theory.

4.6.1 Holomorphic BF theory from the B-Model

The world-volume theory of a stack of N D-branes wrapping $\mathbb{C}^2$ in $\mathbb{C}^3$ is precisely holomorphic BF theory with gauge group $U(N)$.[43] The lowest component of the b field describes

[42]In lower-dimensional examples (topological defects of complex dimension one) the constraints can be discussed elegantly in the language of Koszul duality [150, 151, 8, 152]. It is expected that the notion of Koszul duality can be extended to higher dimensional defects and a similar language will apply here.

[43]Holomorphic BF theory is the dimensional reduction of holomorphic Chern-Simons theory, which is the open string field theory of the B-model topological string on a space-filling brane [153].

transverse fluctuations of the branes while the lowest component of the c field is the ghost for the world-volume gauge fields.

We will now show more explicitly that the operators $A_n, B_{n,\alpha}, C_n$ (4.46) in BF theory can be naturally coupled to the closed string fields of the B-model in the neighbourhood of the brane.

The closed string fields of the Kodaira–Spencer theory are given by

$$\beta \in \mathrm{P.V.}^{0,\bullet}(\mathbb{C}^3)[2], \quad \mu \in \mathrm{P.V.}^{1,\bullet}(\mathbb{C}^3)[1], \quad \pi \in \mathrm{P.V.}^{2,\bullet}(\mathbb{C}^3). \tag{4.125}$$

These are Dolbeault $(0,\bullet)$ forms with coefficients in the zero, first, and second exterior power of the holomorphic tangent bundle of the target. The ghost number of a field is determined by both the polyvector degree and Dolbeault degree. If $\alpha^{(i)}$ denotes the $(0,i)$ Dolbeault component of a field, then $\beta^{(i)}$ has ghost number $i-2$, $\mu^{(i)}$ has ghost number $i-1$, and $\pi^{(i)}$ has ghost number i. The fields μ, π satisfy the constraint that they are divergence-free, meaning that $\partial_\Omega \mu = 0$ and $\partial_\Omega \pi = 0$. In local coordinates $\mu = \mu_\alpha \partial_{z_\alpha}$, the first constraint is simply $\partial_{z_\alpha}\mu_\alpha = 0$ and similarly for the bivector field.

Let us take coordinates (z_1, z_2, w) on $\mathbb{C}^3$ with the complex codimension brane wrapping the first two coordinate planes. The most basic first-order coupling of closed string fields to a stack of N D-branes of complex codimension one is

$$N \int_{\mathbb{C}^2} \partial^{-1}\beta \,. \tag{4.126}$$

Notice that only the $(0,2)$ Dolbeault component appears in the coupling expression above. Here, we view β as a Dolbeault type $(3,2)$ form on $\mathbb{C}^3$ via the holomorphic volume form; $\partial^{-1}\beta$ is a non-local expression which schematically denotes a $(2,2)$ form β' with the property that $\partial\beta' = \beta$. Fluctuations in the transverse direction, whose coordinate is w, are controlled by the eigenvalues of the lowest component of the open string b-field via:

$$\mathrm{Tr}\,\beta^{(0)}(b) = \sum_{n=0}^{\infty} \frac{1}{n!}\,\mathrm{Tr}\, b^n \partial_w^n \beta|_{w=0} \,. \tag{4.127}$$

Therefore, the basic coupling leads to a tower of single-trace couplings

$$\int_{\mathbb{C}^2} N \frac{1}{n!}\,\mathrm{Tr}\, b^n\, \partial_w^{n-1}\beta|_{w=0}\, \mathrm{d}z_1 \mathrm{d}z_2 \tag{4.128}$$

involving the $A_n = \mathrm{Tr}\, b^n$ local operators in the tree-level cohomology of the 4d holomorphic BF theory.

Notice that in our B-model inspired analysis of the large N tree-level cohomology, A_n was identified as the coefficient of $\mathrm{d}\varepsilon^n$ in $\mathrm{Tr}(\mathrm{d}C)^n$. The above coupling is consistent with an identification $\mathrm{d}\varepsilon \to \partial_w$.

There are analogous couplings between the $B_{n,\alpha} = \mathrm{Tr}\, b^n \partial_\alpha c$ and $C_n = \mathrm{Tr}\, b^n \partial_\alpha c \partial^\alpha c$ operators involving the transverse derivatives of the components $\mu_\alpha \partial_{z_\alpha}$ of the field μ and the field π given schematically by

$$\int_{\mathbb{C}^2} \frac{1}{(n-1)!} B_{n,\alpha}\, \partial_w^{n-1} \mu_\alpha|_{w=0}\, \mathrm{d}z_1 \mathrm{d}z_2 \tag{4.129}$$

$$\int_{\mathbb{C}^2} \frac{1}{(n-1)!} C_n\, \partial_w^{n-1} \pi|_{w=0}\, \mathrm{d}z_1 \mathrm{d}z_2\,. \tag{4.130}$$

All the couplings can be expressed concisely in terms of $\mathrm{Tr}(\mathrm{d}C)^n$. Observe that the replacement $\mathrm{d}\varepsilon \to \partial_w$, $\varepsilon \to \mathrm{d}w$ maps the differential $B = \mathrm{d}\varepsilon \partial_\varepsilon + \mathrm{d}z^\alpha \partial_\alpha$ to the ∂ operator in $\mathbb{C}^3$. Thus $\mathrm{Tr}(\mathrm{d}C)^n$ becomes a ∂-closed form in $\mathbb{C}^3$, which can be naturally paired with the closed string fields.

These linear couplings can give rise to an anomaly which is bilinear in the bulk field, involving the bracket of two local operators on the brane. We expect it to be cancelled by a tree-level bulk Feynman diagram involving a cubic bulk vertex. This is the general principle of Koszul duality: the algebraic structures in the brane world-volume theory should be constrained or even determined by anomaly cancellation in terms of bulk Feynman diagrams. It should represent a target space manifestation of the relations between open- and closed- string field theory.

The brane backreaction[44] is captured by the following closed string field which is a bivector:[45]

$$\pi_{br} = N \frac{1}{w} \partial_{z_1} \wedge \partial_{z_2}. \tag{4.131}$$

At leading order, the effect of π_{br} on the classical closed string theory is to modify the differential acting on polyvector fields to

$$\bar{\partial} \to \bar{\partial} + \{\pi_{br}, -\}_S\,, \tag{4.132}$$

[44]The backreaction can be obtained by solving the equation of motion of the Kodaira-Spencer theory with a D-brane source term $N\delta_{w=0}$.

[45]This field has charge -1 under the $U(1)_b$ rotation symmetry in the plane transverse to the branes, so that a perturbative expansion in this back-reaction breaks $U(1)_b$ just in the same way as the quantum corrections in the 4d holomorphic BF theory.

where $\{-,-\}_S$ is the Schouten bracket.[46]

This backreaction also produces a BRST anomaly in the couplings to the brane, which we expect to be cancelled by the planar part of the 1-loop BRST transformations of A_n, $B_{n,\alpha}$ and C_n. Generally speaking, turning on a non-trivial closed string field which is a bivector has the effect of making spacetime noncommutative.

4.6.2 Large N SQCD

In this Section we discuss some preliminary observations about the holomorphic twist of SQCD, see Section 4.4.1. We briefly recall that the twist of SQCD can be described as a higher dimensional gauged $\beta\gamma$ system, with β, γ and $\tilde{\beta}$, $\tilde{\gamma}$ fields arising respectively from (anti)fundamental chiral multiplets.

Recal also that the β fields from the fundamental chiral multiplets have the same gauge quantum numbers as the $\tilde{\gamma}$ fields from the anti-fundamental chiral multiplets, and vice versa. As a result, the $U(N_f)_L \times U(N_f)_R$ classical global symmetry of physical SQCD is enhanced to $U(N_f|N_f)$.

Remarkably, we can engineer this holomorphic theory in the B-model as well, by adding $(N_f|N_f)$ space-filling branes, which support an $U(N_f|N_f)$ holomorphic Chern-Simons theory. This opens up the possibility of a twisted holography analysis.

In the large N limit, we expect the (anti-)fundamental matter to contribute mesons $\tilde{\Gamma}_A b^n \Gamma^B$ to the BRST cohomology of local operators, together with their derivatives.

There does not seem to be space for any loop corrections at the leading order in N.

It thus appears that the holomorphic twist of SQCD, at the leading order in N, enjoys a symmetry by the super Lie algebra $\mathfrak{u}(N_f|N_f)[z_1, z_2, \partial_w]$, deformed by the effect of the

$$\eta = \frac{1}{w}\partial_{z_1} \wedge \partial_{z_2} \tag{4.134}$$

background.

At low energy, for $N_f \ll N$, SQCD is expected to have a runaway behaviour due to an instanton-generated superpotential with no minimum [154, 155]. It would be interesting

[46] In local coordinates, for $\beta, \gamma \in PV^{\bullet,\bullet}(\mathbb{C}^3) = C^\infty(\mathbb{C}^3)[\varepsilon_i \equiv \partial_i, \mathrm{d}\bar{z}^i]$, it is computed by

$$\{\beta, \gamma\}_S = (\partial_{\varepsilon_i}\beta)(\partial_{z^i}\gamma) + (-1)^{|\beta|}(\partial_{z^i}\beta)(\partial_{\varepsilon_i}\gamma)\,. \tag{4.133}$$

to study this phenomenon in the holomorphic twist and holographically. In practice, it should mean that the operator "1" is Q-exact after instanton corrections are included.

4.7 Indices and characters for local operators

Supersymmetric indices offer a good way to gain some intuition about the cohomology calculations which characterize BPS operators. Depending on the specific calculation we are interested in, we can flavour the indices with fugacities for various actual or spurious symmetries. Crucially, the indices can be computed in the free theory and are unaffected by the interacting differentials.

Index calculations are not useful to study the full non-perturbative pure gauge theory case, which only has an $SU(2)$ rotational symmetry commuting with the super charge: the collection of protected local operators in the free theory with a given $SU(2)$ charge is infinite-dimensional and thus cannot be sensibly counted.[47]

In a perturbative setup, we can grade operators by the full $U(2)$ symmetry, as explained at the beginning of Section 4.2. Both the b field and the holomorphic derivatives have positive charge under the diagonal generator M_R and no fields of the BF theory have negative charge (see table 4.7.1). The counting problem is thus well-defined.

Introducing traditional fugacities p, q for the $U(2)$ Cartan generators, the contribution to the index of a b field of gauge fugacity z and its derivatives is

$$\frac{1}{\prod_{n,m\geq 0}(1-p^{m+1}q^{n+1}z)}, \tag{4.135}$$

which we can regard as the plethystic exponential of the single particle index $\frac{pqz}{(1-p)(1-q)}$. Here pqz is the fugacity for b, which is a boson, and each derivative adds a factor of p or q. The contribution of a c field, keeping only derivatives, is

$$\prod_{n,m\geq 0|(n,m)\neq(0,0)}(1-p^m q^n z)\,. \tag{4.136}$$

[47]We do expect the cohomology for each cohomological degree to be typically finite-dimensional, so that a weighed character of the cohomology is well-defined. For example, in the pure free gauge theory the only letters of non-negative cohomological degree are $\partial_\alpha c$, and each can only appear once in an operator. As long as the interactions can be described as turning on a differential on the free cohomology, the result will be finite-dimensional. An exception would be theories with an infinite-dimensional chiral ring which cannot be decomposed into finite-dimensional sectors of given charge under some flavour symmetry.

As we build the full index, the contributions of b and c fields with the same gauge charge almost cancel out, leaving the products

$$\prod_{m\geq 1}(1-p^m z)\prod_{n\geq 1}(1-q^n z) \tag{4.137}$$

for each generator of the gauge group G.

We can manipulate the product of q-dependent factors over the generators of G with the help of the Kac-Weyl denominator formula:

$$\prod_{m\geq 1}(1-q^m)^r \prod_{\alpha}\prod_{m\geq 1}(1-q^m z_\alpha) = \sum_{\lambda\in\Lambda_G^+}(-1)^{\epsilon(\lambda)} q^{(\lambda,\lambda+2\rho)/(2h_G)}\chi_\lambda(z)\,. \tag{4.138}$$

Here $(-1)^{\epsilon(\lambda)}$ is a sign which will be immaterial in the following and the sum on the right hand side runs over the positive roots Λ_G^+ of the group. The product on the left hand side runs over roots α for G.

We can expand the product of p-dependent factors in the same manner and combine the two sums. The projection over G invariants maps $\chi_\lambda(z)\chi_\mu(z)\to\delta_{\lambda,\mu}$ as the product of two irreps of G contains a (single) G-invariant element if and only if the two irreps coincide.

We thus get

$$I^G_{\rm pert}[p,q] = \sum_{\lambda\in\Lambda_G^+}(pq)^{(\lambda,\lambda+2\rho)/(2h_G)}\,. \tag{4.139}$$

For example,

$$I^{SU(2)}_{\rm pert}[p,q] = \sum_{\ell\geq 0}(pq)^{\frac{\ell(\ell+1)}{2}}\,. \tag{4.140}$$

Notice that the index is a function of pq only: it does not receive contributions from operators with non-trivial $SU(2)$ spin. This is compatible with the exactness of $\partial_\alpha S^\alpha$, discussed in Section 4.3.2, which implies that all operators with non-trivial $SU(2)$ spin are Q-exact at 1-loop.

4.7.1 Large N index

If we include the fugacity x for the spurious symmetry $U(1)_b$ giving charge 1 to b, as well as the usual p, q rotation fugacities employed in calculations of the superconformal index, the single-letter index $f(x;p,q)$ is given by

$$1-f(x;p,q) = \frac{1-x}{(1-p)(1-q)}\,. \tag{4.141}$$

The fugacity x should be set to pq when we introduce loop corrections. Non-perturbative corrections would further constrain $p^N q^N = 1$ as the $U(1)_R$ symmetry is broken to a discrete subgroup by anomalies.

The large N index for an $U(N)$ theory is computed by a standard formula [104]:

$$I^{\infty}_{\text{pert}}[x;p,q] = \prod_{n=1}^{\infty} \frac{1}{1-f(x^n;p^n,q^n)} = \prod_{n=1}^{\infty} \frac{(1-p^n)(1-q^n)}{1-x^n}\,. \tag{4.142}$$

This index should count polynomials in single-trace operators. The single-trace index is the Plethystic logarithm:

$$I_{\text{single}}[x;p,q] = \frac{x}{1-x} - \frac{p}{1-p} - \frac{q}{1-q}\,. \tag{4.143}$$

We expect all single trace operators in the tree-level cohomology to be organized into towers of derivatives of a "primary" field, with the exception of $\partial_\alpha \mathrm{Tr}\, c$, as $\mathrm{Tr}\, c$ is disallowed. Excluding the tower of derivatives of $\mathrm{Tr}\, c$, the index for the remaining single-trace primaries is

$$1 + (1-p)(1-q)\left(-1 + I_{\text{single}}[x;p,q]\right) = \frac{x - px - qx + pq}{1-x}\,. \tag{4.144}$$

This agrees with our large N cohomology calculations in Section 4.5: we see the contribution of the primaries

$$\begin{aligned} A_n &= \mathrm{Tr}\, b^n, & & n \geq 1 \\ B_{n,\alpha} &= \mathrm{Tr}\, b^n \partial_\alpha c, & & n \geq 1,\ \alpha = 1,2 \\ C_n &= \mathrm{Tr}\, b^n \partial_\alpha c\, \partial^\alpha c, & & n \geq 0\,. \end{aligned} \tag{4.145}$$

This confirms the result that these operators together with their derivatives, and the derivatives of $\mathrm{Tr}\, c$, form a basis of the large N single-trace tree-level cohomology.

In the case of gauge group $SU(N)$ and infinite N, the derivatives of $\mathrm{Tr}\, c$ vanish and the cohomology is generated by primaries

$$A_n,\ n \geq 2, \qquad B_{n,\alpha},\ n \geq 1, \qquad C_n,\ n \geq 0 \tag{4.146}$$

and their derivatives.

The 1-loop corrected cohomology for $U(N)$ appears to only include the C_n operators and no derivatives, as well as derivatives of $\mathrm{Tr}\, c$. This reproduces the single-particle index in (4.143) where $x = pq$, without any cancellation, i.e. each term in the single-trace index corresponds to one single-trace cohomology class.

	R	$M_{+\dot{-}}$	M_R	C_R	C
Q	-1	-1	0	1	1
b	0	2	2	0	-2
c	-1	-1	0	1	1
∂_α	0	1	1	0	-1

Table 4.1: We collect here the charges of elementary fields under various generators used in the main text. The twisted rotation generator was defined as $M_R = M_{+\dot{-}} - R$ and the "perturbative" and "non-perturbative" cohomological degrees as $C_R = \mathrm{gh} - R$ and $C = C_R - M_R = \mathrm{gh} - M_{+\dot{-}}$, respectively.

4.8 Numerical analysis for $N = 2, 3$

In this Section, we study the cohomology of local operators for the holomorphic twist of pure $\mathcal{N} = 1$ SYM for $SU(N)$ gauge group with $N = 2, 3$, formulated as holomorphic BF theory as described in Section 4.3. The space of local operators can be organized by the charge $M_R \equiv M - R$, with finite-dimensional eigenspaces. We use symbolic manipulation software to compute the cohomology up to the following values for the charge:

$$\begin{aligned} M_R &= 14 \quad \text{for} \quad N = 2\,, \\ M_R &= 8 \quad \text{for} \quad N = 3\,. \end{aligned} \tag{4.147}$$

Up to the tree-level stage, our calculation is the pure gauge theory analog of [111, 136], which computed numerically the low energy cohomology of semi-chiral operators in $\mathcal{N} = 4$ SYM for $SU(N)$ with $N = 2, 3, 4$. The main bottleneck in numerical calculations of cohomology of local operators appears to be the large number of gauge invariant operators and of monomials which appear in individual gauge invariant operators, especially for $N > 2$.

The computational strategy is straightforward:

1. We pick a specific basis for the adjoint representation of the gauge group and work with the explicit components for all the fields. In this way, the finite N trace relations are implemented automatically.[48]

2. We produce a basis for the space of gauge invariant operators

$$\mathcal{V}_\bullet(n_1, n_2) = \oplus_{|b|} \mathcal{V}_{|b|}(n_1, n_2)$$

[48] This choice is also the main limitation of our method, as the number of possible monomials in the fields grows dramatically as M_R or N increase.

built out of the operator b and its derivatives as well as derivatives of c. Operators in $\mathcal{V}_{|b|}(n_1, n_2)$ have charges n_1, n_2 under the Cartan generators of $U(2)$, which add up to M_R, have charge $|b|$ under the auxiliary $U(1)_b$ which commutes with the tree-level differential Q_0 and the cohomological grading (which we have left implicit in the expression above). Recall that $|b|$ simply counts the number of times b appears in a local operator; all other fields are weighted trivially under $U(1)_b$. The perturbative differential commutes with $U(2)$, so that we get a collection of complexes $\mathcal{V}_\bullet(n_1, n_2)$ indexed by the n_i.

3. We compute the action of

$$Q_0 \colon \mathcal{V}^i_{|b|}(n_1, n_2) \to \mathcal{V}^{i+1}_{|b|}(n_1, n_2) \tag{4.148}$$

on the basis and find a basis for the representatives of the Q_0 cohomology

$$H^i(\mathcal{V}_{|b|}(n_1, n_2), Q_0).$$

4. We compute the action of the one-loop differential Q_1 on the representatives of the Q_0 cohomology:

$$Q_1 \colon H^i(\mathcal{V}_{|b|}(n_1, n_2), Q_0) \to H^{i+1}(\mathcal{V}_{|b|-1}(n_1, n_2), Q_0)\,. \tag{4.149}$$

We then compute the representatives of the Q_1 cohomology

$$H^\bullet(H^\bullet(\mathcal{V}_\bullet(n_1, n_2), Q_0), Q_1).$$

This procedure gives the one-loop approximation to the perturbative cohomology. In principle, we should continue and compute the two-loop differential

$$Q_2 \;:\; H^\bullet(H^\bullet(\mathcal{V}_{|b|}(n_1, n_2), Q_0), Q_1) \to H^\bullet(H^\bullet(\mathcal{V}_{|b|-2}(n_1, n_2), Q_0), Q_1) \tag{4.150}$$

as in a spectral sequence. Up to the values of M_R we consider, though, we find that the 1-loop cohomology is sufficiently sparse to forbid non-zero higher loop differentials. Our answer

$$H^\bullet(\mathcal{V}_\bullet(n_1, n_2), Q_{\mathrm{pert}}) \equiv \bigoplus_{|b|} H^{\bullet + n_1 + n_2}(H^\bullet(\mathcal{V}_{|b|}(n_1, n_2), Q_0), Q_1) \tag{4.151}$$

is thus exact to all orders in perturbation theory.

A potential strategy for non-perturbative calculations would be to start from $H^\bullet(\mathcal{V}_\bullet(n_1, n_2), Q_{\mathrm{pert}})$ and compute a 1-instanton correction to the differential. Such a correction must lower

$n_1 + n_2$ by multiples of $2N$, the anomaly introduced by a 1-instanton effect, while increasing cohomological degree by 1:

$$Q_{\text{1-inst}} \,:\, H^i(\mathcal{V}_\bullet(n_1, n_2), Q_{\text{pert}}) \to H^{i+1}(\mathcal{V}_\bullet(n_1 - N, n_2 - N), Q_{\text{pert}})\,. \tag{4.152}$$

It would be the beginning of another spectral sequence-like calculation incorporating n-instanton effects.

For $N = 2$, we will find multiple pairs of perturbative cohomology classes which could in principle be lifted by instanton effects. Indeed, our calculations are compatible with the conjecture that the whole perturbative cohomology should be lifted by instanton effects, with the exception of the identity operator and of the gaugino condensate $C_0 = \partial_\alpha c \partial^\alpha c$, with no non-trivial operator with the charge of C_0^2. This is somewhat remarkable, as the perturbative cohomology is otherwise rather sparse.

Such a result would fully agree with the space of local operators in the far infrared: the IR theory has two gapped vacua distinguished by the vev of C_0, which is expected to square to a multiple of the identity operator, proportional to the appropriate power of the strong coupling scale. The far infrared effective theory in each gapped vacuum is not expected to support any other non-trivial local operators. We will comment on this point further at the end of the Section.

For $N = 3$ one may similarly hope for instanton effects to cancel all perturbative cohomology classes except C_0 and C_0^2, as the theory is expected to have three gapped vacua in the IR. Unfortunately, we cannot push our calculation to the order $p^5 q^5$, which would be needed to find a perturbative cohomology class whose 1-instanton image could cancel the other perturbative cohomology class we find below.

4.8.1 Cohomology of the tree-level differential

By the numerical analysis outlined above, we find that all cohomology classes at tree level (up to the studied values in equation (4.147)) are of the "same type" as in the infinite N case, i.e. their representatives are polynomials in single trace primaries $A_n, B_{n,\alpha}, C_n$ given in (4.46) and their derivatives. Therefore, there is a surjective map from infinite N tree-level cohomology into $SU(2)$ and $SU(3)$ cohomology, up to the level we checked:[49]

$$H^i_{SU(\infty)}(\mathcal{V}_{|b|}(n_1, n_2), Q_0) \twoheadrightarrow H^i_{SU(N)}(\mathcal{V}_{|b|}(n_1, n_2), Q_0)\,, \qquad N = 2, 3\,. \tag{4.153}$$

[49] The map is not injective: e.g. due to trace relations operators C_n vanish for odd n in $SU(2)$.

M_R	$N=2$	$N=3$	$N=\infty$
2	pqu^2		
3	$-pux-qux+p^2qu^2+pq^2u^2$		
4	$x^2-p^2ux-2pqux-q^2ux$ $+p^3qu^2+p^2q^2u^2+pq^3u^2$	$+pqu^2x+p^2q^2u^4$	
5	$p^4qu^2+p^3q^2u^2-p^3ux+p^2q^3u^2$ $-2p^2qux+pq^4u^2-2pq^2ux+px^2$ $-q^3ux+qx^2$	$+p^3q^2u^4+p^2q^3u^4-p^2qu^3x$ $+p^2qu^2x-pq^2u^3x+pq^2u^2x$ $-pux^2-qux^2$	
6	$p^5qu^2+p^4q^2u^4+p^4q^2u^2-p^4ux$ $+p^3q^3u^4+p^3q^3u^2-p^3qu^3x$ $-2p^3qux+p^2q^4u^4+p^2q^4u^2$ $-p^2q^2u^3x-2p^2q^2ux+p^2x^2$ $+pq^5u^2-pq^3u^3x-2pq^3ux$ $+pqu^2x^2+pqx^2-q^4ux+q^2x^2$	$+p^4q^2u^4+p^3q^3u^4-p^3qu^3x$ $+p^3qu^2x+p^2q^4u^4-3p^2q^2u^3x$ $+p^2q^2u^2x-p^2ux^2-pq^3u^3x$ $+pq^3u^2x+pqu^2x^2-2pqux^2$ $-q^2ux^2+x^3$	$+p^3q^3u^6$ $+p^2q^2u^4x$ $+pqu^2x^2$

Table 4.2: Tree-level characters counting cohomology classes with charges $p^{n_1}q^{n_2}u^{C_R}x^b$. Each column should be added to the previous one to get the full answer.

In the analogous setting for $\mathcal{N}=4$ SYM with $SU(2)$ gauge group, this statement fails at $M_R \simeq 8$, due to the appearance of a "non-multitrace" cohomology representative which is not a polynomial in infinite N single-trace cohomology representatives [136, 156, 157]. Here we do not find any non-multitrace cohomology classes of the tree level differential up to $M_R = 14$. It is possible that such states will appear at higher quantum numbers. We do not have a clear expectation either way.

The tree-level characters for $N = 2, 3$ and $N = \infty$ are listed in table 4.2.

4.8.2 Cohomology of the loop-corrected differential

As predicted, the derivatives of cohomology classes are exact at 1-loop for finite and infinite N. More surprisingly, we find several cohomology classes at $N = 2$, which are not polynomials in the infinite N single-trace cohomology of the 1-loop differential. These are the "non-multitrace" cohomology classes with respect to the 1-loop differential.

The 1-loop cohomology classes for $N = 2, 3$ compared to $N = \infty$ are shown in table 4.3.

M_R	$N=2$	$N=3$	$N=\infty$
2	C_0	C_0	C_0
4	-	C_1, C_0^2	C_1, C_0^2
6	C_2	?	C_2, C_0C_1, C_0^3
8	-	?	$C_3, C_2C_0, C_1^2, C_1C_0^2, C_0^3$
10	$C_4, B_{1,2}\partial_2 C_0\partial_1^2 C_0$	?	$C_4, C_3C_0, C_2C_1, C_2C_0^2, C_1^2C_0, C_1C_0^3, C_0^5$
12	$B_{1,2}\partial_1\partial_2 B_{1,2}\partial_1^2 C_0$	?	$C_5, C_4C_0, C_3C_1, C_3C_0^2, C_2^2, C_2C_1C_0, C_2C_0^3,$ $C_1^3, C_1^2C_0^2, C_1C_0^3, C_0^6$
14	$C_6, B_{3,2}\partial_2 C_0\partial_1^2 C_0$	?	$C_6, C_5C_0, C_4C_1, C_4C_0^2, C_3C_2, C_3C_1C_0, C_3C_0^3,$ $C_2^2C_0, C_2C_1^2, C_2C_1C_0^2, C_2C_0^4, C_1^3C_0, C_1^2C_0^3, C_1C_0^5, C_0^7$

Table 4.3: Cohomology representatives at 1-loop. For $N=2$ we find cohomology classes which are not of the form of the cohomology classes at infinite N.

We include the characters counting 1-loop cohomology classes of various charges:

$$\begin{aligned}\chi^{SU(2)}_{\text{1-loop}}(u,p,q) =&1 + pqu^2 + p^3q^3u^2 + p^5q^5u^2 - p^5q^5u^5 + p^6q^6u^4 \\ &+ p^7q^7u^2 - p^7q^7u^5 + \mathcal{O}(p^8q^8)\,,\end{aligned} \tag{4.154}$$

$$\chi^{SU(3)}_{\text{1-loop}}(u,p,q) =1 + pqu^2 + p^2q^2u^4 + p^2q^2u^2 + \mathcal{O}(p^3q^3)\,, \tag{4.155}$$

$$\chi^{\infty}_{\text{1-loop}}(u,p,q) =\frac{1}{\prod_{n=1}^{\infty}(1-p^nq^nu^2)} = \chi^{SU(3)}_{\text{1-loop}}(u,p,q) + \mathcal{O}(p^3q^3)\,. \tag{4.156}$$

If we replace the fugacity u for C_R with a fugacity v for the nonperturbative cohomological degree C (see table 4.7.1), the $SU(2)$ character becomes

$$\begin{aligned}\chi^{SU(2)}_{\text{1-loop}}(v,p,q) =&1 + pq + \left(p^3q^3v^{-4} - p^5q^5v^{-5}\right) + \left(p^5q^5v^{-8} - p^7q^7v^{-9}\right) + p^6q^6v^{-8} \\ &+ p^7q^7v^{-12} + \mathcal{O}(p^8q^8)\,.\end{aligned} \tag{4.157}$$

We grouped together pairs of terms which could be lifted by 1-instanton effects.

It seems plausible that the term $p^7q^7v^{-12}$ coming from the cohomology class C_6 could be paired up with a term $-p^9q^9v^{-13}$ coming from a potential cohomology class $B_{5,2}\partial_2 C_0\partial_1^2 C_0$ at charge $M_R = 18$, which is above the values studied by us. More generally one can conjecture the contributions from cohomology classes C_{2n} for $n>0$:

$$p^{2n+1}q^{2n+1}v^{-4n} \tag{4.158}$$

could be lifted by the contributions from "non-multitrace" cohomology classes

$$B_{2n-1,2}\partial_2 C_0\partial_1^2 C_0$$

which take the form

$$-p^{2n+3}q^{2n+3}v^{-4n-1}\,. \tag{4.159}$$

The term $p^6q^6v^{-8}$ could potentially be cancelled by a contribution $-p^8q^8v^{-9}$ from a new "non-multitrace" cohomology class at charge $M_R = 18$, which is above the values we studied numerically.

4.8.3 Comparison to compactification on a two-torus

We should elaborate on the possibility of matching operators between the UV and far IR theory. This discussion will be somewhat orthogonal to the rest of the Section.

In two-dimensional topological twists, it was found that the space of local operators computed in the UV could be fully matched to the space of local operators in the far IR, but only including the effect of gapped particles and solitons [158]. The two-dimensional $(2,2)$ supersymmetric $\mathbb{C}P^1$ sigma model is closely related to pure four-dimensional $\mathcal{N}=1$ gauge theory. Indeed, it is expected to arise from a supersymmetric compactification on a two-torus of the four-dimensional theory [159, 160]. Solitons in the two-dimensional theory are related to domain walls in the four-dimensional theory.

The supersymmetric torus compactification is compatible with the holomorphic twist: we can consider the twisted theory on $\mathbb{C}^* \times \mathbb{C}^*$ and reduce it to a topological theory on $\mathbb{R}^2$. Local operators in the two-dimensional theory, though, could arise either from local operators in the four-dimensional theory or from holomorphic surface defects wrapping the internal two-torus. In particular, the local operators in the twisted $\mathbb{C}P^1$ sigma model which create solitons are likely to arise in such manner.

A better option, for our purpose, is to consider the twisted theory on the product of two holomorphic cigars and reduce it to the same topological theory on $(\mathbb{R}^+)^2$, with specific boundary conditions. Then four-dimensional local operators map to "corner" local operators at the junction of the two boundaries. It would be nice to characterize the resulting boundary conditions in the twisted $\mathbb{C}P^1$ sigma model and verify that the corner local operators computed via the web formalism of [158] match the 1, C_0 local operators expected in four dimensions.

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APPENDICES

Appendix A

Appendices for Chapter 2

A.1 Examples

A.1.1 $k = 1$

The saddle equations are trivial. We have $\rho = m_1$ and thus $p_1 = m_1^{-1}$. This is reasonable: a one-point function of $\mathcal{D}(m_1; u_1; z_1)$ can only pick the m_1^N term, so the saddle action is $\log m_1$.

The spectral curve is given by $b = u_1 a - m_1$, $c = z_1 a + m_1^{-1}$, $d = u_1 z_1 a + u_1 m_1^{-1} - z_1 m_1$, i.e.

$$g = \begin{pmatrix} 1 & u_1 \\ z_1 & u_1 z_1 \end{pmatrix} a + \begin{pmatrix} 0 & -m_1 \\ m_1^{-1} & (u_1 m_1^{-1} - z_1 m_1) \end{pmatrix}. \tag{A.1}$$

If we identify $t = a$, this is a standard genus 0 spectral curve with one puncture at $t = \infty$.

A.1.2 $k = 2$

For an irreducible solution, the saddle equations require

$$-\rho_2^1 \rho_1^2 = \frac{u_1 - u_2}{z_1 - z_2} - m_1 m_2 \tag{A.2}$$

and we have

$$p_1 = -\frac{z_1 - z_2}{u_1 - u_2} m_2, \qquad p_2 = -\frac{z_1 - z_2}{u_1 - u_2} m_1. \tag{A.3}$$

The two-point function $\langle \mathcal{D}(m_1; u_1; z_1)\mathcal{D}(m_2; u_2; z_2)\rangle$ has a tree-level saddle action

$$\log \frac{z_1 - z_2}{u_1 - u_2} - 1 - m_1 m_2 \frac{z_1 - z_2}{u_1 - u_2}. \tag{A.4}$$

The spectral curve is of genus 0. We can place the punctures at $t = 0$ and $t = \infty$ and write without loss of generality:

$$g = \begin{pmatrix} 1 & u_1 \\ z_1 & u_1 z_1 \end{pmatrix} t + \begin{pmatrix} 1 & u_2 \\ z_2 & u_2 z_2 \end{pmatrix} \frac{a_2}{t} + \begin{pmatrix} a_\infty & b_\infty \\ c_\infty & d_\infty \end{pmatrix}. \tag{A.5}$$

The constraint $\det g = 1$ fixes all but two parameters.

As we placed a puncture at $t = \infty$, we should replace the basis $\frac{1}{t-t_i}$ of functions vanishing at $t = \infty$ which we employed in the general case in section 2.3.7 with some functions that vanish at some other point t_0 instead of $t = \infty$. If we use $(t - t_0)$ and $(t - t_0)/t$ as entries of the eigenvector, we arrive to the expected form of ρ, with $\rho_1^2 = u_1 - u_2$, and can determine the two remaining parameters in terms of m_1 and m_2. We get

$$\begin{aligned} a_\infty &= -\frac{m_1 - m_2}{u_1 - u_2} \\ b_\infty &= -\frac{m_1 u_2 - m_2 u_1}{u_1 - u_2} \\ c_\infty &= -\frac{m_1 z_1 - m_2 z_2}{u_1 - u_2} \\ a_\infty &= -\frac{m_1 u_2 z_1 - m_2 u_1 z_2}{u_1 - u_2} \\ a_2 &= -\frac{\rho_2^1 \rho_1^2}{(u_1 - u_2)^2}. \end{aligned} \tag{A.6}$$

A.1.3 $k = 3$

The $k = 3$ saddle equations have generically two irreducible solutions. The solutions involve a square root and are cumbersome to write down directly. Instead, we can realize them in terms of genus 0 spectral curves. We can write

$$g = \begin{pmatrix} 1 & u_1 \\ z_1 & u_1 z_1 \end{pmatrix} a_1 t + \begin{pmatrix} 1 & u_2 \\ z_2 & u_2 z_2 \end{pmatrix} \frac{a_2}{t} + \begin{pmatrix} 1 & u_3 \\ z_3 & u_3 z_3 \end{pmatrix} \frac{a_3}{t-1} + \begin{pmatrix} a_\infty & b_\infty \\ c_\infty & d_\infty \end{pmatrix}. \tag{A.7}$$

The constraint $\det g = 1$ allows one to express, say, a_2, a_3, c_∞ and d_∞ in terms of a_1, a_∞, d_∞.

We can build ρ by using the eigenvector with components $(t-t_0)$, $(t-t_0)/t$, $(t-t_0)/(t-1)$. This expresses the masses m_i in terms of a_1, a_∞, d_∞. The relations are linear in a_∞, d_∞, but impose a quadratic equation for a_1. We thus find a nice parameterization of the saddles ρ in terms of a_1 and two of the m_i, together with a quadratic relation to impose the value of the third m_i.

A.2 Binomial sum

After change of variables $x = \frac{\alpha+\varepsilon}{2} + q$ in (2.107), the three sums in (2.108) take the form:

$$S(\varepsilon) = \sum_{x=\lceil \max(q+\varepsilon/2, 2q+\varepsilon) \rceil}^{q-p+\varepsilon} (-1)^x \binom{x-q-p-1}{2x-2q-\varepsilon-1} \binom{2x-2q-\varepsilon}{x}, \tag{A.8}$$

where $\varepsilon \in \{0, \pm 1\}$ and the total sum is equal

$$(-1)^{\frac{m+n}{2}+1}(2S(0) + S(-1) + S(1)). \tag{A.9}$$

We will use the binomial identity:

$$S(B, C) \equiv \sum_{x=0}^{B} (-1)^x \binom{x+C-1}{2x-B+C-1} \binom{2x-B+C}{x} = \begin{cases} 1, & B = 0 \\ 2(-1)^B, & B > 0. \end{cases} \tag{A.10}$$

There are 9 ranges for variables $p-q$ and $p+q$ that we have to consider:

1. $p - q \geq 2$: all three sums are zero.
2. $p - q = 1, p + q \leq 0$:
$$S(0) = 0, \qquad S(-1) = 0, \qquad S(1) = 1. \tag{A.11}$$
3. $p - q = 0, p + q \leq -1$:
$$S(0) = 1, \qquad S(-1) = 0, \qquad S(1) = -2. \tag{A.12}$$
4. $p - q = 0, p + q = 0$:
$$S(0) = 1, \qquad S(-1) = 0, \qquad S(1) = -1. \tag{A.13}$$

5. $p-q=-1, p+q \leq -1$:

$$S(0)=-2, \qquad S(-1)=1, \qquad S(1)=2. \tag{A.14}$$

6. $p-q=-1, p+q=0$:

$$S(0)=-1, \qquad S(-1)=1, \qquad S(1)=1. \tag{A.15}$$

7. $p-q \leq -2, p+q \leq -1$:

$$S(0)=2(-1)^{q-p}, \qquad S(-1)=-2(-1)^{q-p}, \qquad S(1)=-2(-1)^{q-p}. \tag{A.16}$$

8. $p-q \leq -2, p+q=0$:

$$S(0)=(-1)^{2q}, \qquad S(-1)=-(-1)^{2q}, \qquad S(1)=-(-1)^{2q}. \tag{A.17}$$

9. $p+q \geq 1$: all sums are zero again.

The total sum is non-zero only for:

$$2S(0)+S(-1)+S(1)=\begin{cases} +1, & p-q=1, p \leq \frac{1}{2} \\ -1, & p-q=-1, p<\frac{1}{2} \\ +1, & p=q=0, \end{cases} \tag{A.18}$$

where the last case does not correspond to a modification of a determinant.

Appendix B

Appendices for Chapter 4

B.1 Properties of λ-brackets

The λ-brackets describe a multilinear operation on the space of local operators modulo total derivatives. They control how the BRST differential is affected by position dependent interactions with non-zero holomorphic momenta, denoted λ.[1] A general λ-bracket $\{\bullet\, {}_{\lambda_1} \cdots\, {}_{\lambda_{n-1}} \bullet\}_n$ satisfies the properties

$$\{\ldots {}_{\lambda_{i-1}} (\partial + \lambda_i) O\, {}_{\lambda_i} \ldots\}_n = 0\,, \tag{B.1}$$

$$\partial\{\ldots {}_{\lambda_{n-1}} O\}_n = \{\ldots {}_{\lambda_{n-1}} (\partial - \sum_{i=1}^{n-1} \lambda_i) O\}_n\,. \tag{B.2}$$

Notice that the last entry of the brackets behaves in a slightly different manner from the others. The brackets are graded symmetric under permutations of the remaining arguments together with the λ parameters. They are graded symmetric under permutation of all arguments if we define $\lambda_n \equiv -\sum_{i=1}^{n-1} \lambda_i - \partial$, with the latter symbol acting outside the bracket.

The λ-brackets should be understood as generating series for a tower of brackets:

$$\{\bullet\, {}_{\lambda_1} \cdots\, {}_{\lambda_{n-1}} \bullet\}_n \equiv \sum_{k_*} \left[\prod_i \frac{\lambda_i^{k_i}}{k_i!}\right] \{\bullet\,, \ldots\,, \bullet\}_{n;k_*}\,. \tag{B.3}$$

[1]Properties of these brackets are discussed and derived in [7, 116]. Mathematically, they should describe part of the information contained in a holomorphic factorization algebra [161, 124]. We also thank Ahsan Khan for sharing unpublished notes on λ-brackets at an early stage of this work.

The brackets used in the BRST charges at the loop level and Maurer Cartan equations are the λ-bracket evaluated at $\lambda_i = 0$. The MC equations for holomorphic position-dependent couplings involve the full brackets.

A particularly simple situation is one where all brackets with more than two entries vanish. Then the 2-ary λ bracket $\{\bullet\, {}_\lambda \bullet\}$ satisfies axioms such as:

- $\{\partial A\, {}_\lambda\, B\} = -\lambda\{A\, {}_\lambda\, B\}$.
- $\partial\{A\, {}_\lambda\, B\} = -\lambda\{A\, {}_\lambda\, B\} + \{A\, {}_\lambda\, \partial B\}$. This implies $\{A\, {}_\lambda\, \partial B\} = (\partial + \lambda)\{A\, {}_\lambda\, B\}$.
- $\{A\, {}_\lambda\, B\} = (-1)^{|A||B|}\{B\, {}_{-\lambda-\partial}\, A\}$.
- $\{A\, {}_{\lambda_1}\{B\, {}_{\lambda_2}\, C\}\} - (-1)^{(|A|+1)(|B|+1)}\{B\, {}_{\lambda_2}\{A\, {}_{\lambda_1}\, C\}\} + (-1)^{|A|}\{\{A\, {}_{\lambda_1}\, B\}\, {}_{\lambda_1+\lambda_2} C\} = 0$

where $|A|$ and $|B|$ denotes the fermion parity of the operators.

In the case of one complex dimension with vanishing higher brackets, such a λ-bracket[2] generate the familiar Lie conformal algebra[3] [128], which encodes the singular part of the OPE, i.e. the λ-bracket is a generating function for the non-negative VOA operations. The information contained in the negative VOA operations is captured by a *regularized product* which satisfy compatibility axioms with the λ-bracket. These axioms guarantee that together the λ-bracket and regularized product encode the chiral algebra.

B.2 Bracket on the tree level

In this section, we give more details on the computations of the tree level bracket, i.e. λ-bracket of two arguments. Tree level brackets are simply given by performing a single Wick contraction of free fields.

$$\{b^A, c^B\} = \delta^{AB}\,, \tag{B.4}$$

where capital letters denote adjoint gauge indices. Our Lie algebra conventions are collected at the end of section 4.3.1. Therefore, given two operators $\mathcal{O}_1$ and $\mathcal{O}_2$, its tree level bracket is simply

$$\{\mathcal{O}_1, \mathcal{O}_2\} = \delta^{AB}\frac{\delta\mathcal{O}_1}{\delta c^A}\frac{\delta\mathcal{O}_2}{\delta b^B} + (-1)^{|\mathcal{O}_1|}\delta^{AB}\frac{\delta\mathcal{O}_1}{\delta b^A}\frac{\delta\mathcal{O}_2}{\delta c^B}\,, \tag{B.5}$$

[2] λ bracket in one complex dimension satisfies the same set of properties as above except a few sign differences.

[3] There is also a great set of lecture notes by Victor G. Kac available at https://web.archive.org/web/20220320093700/https://w3.impa.br/~heluani/files/lect.pdf.

where $|\mathcal{O}|$ denotes the fermion degree. Care with signs is needed when taking functional derivative with respect to Grassman variables. Our convention is

$$\frac{\delta}{\delta c^A}\left(c^B c^C\right) = \delta^{AB} c^C - \delta^{AC} c^B \,. \tag{B.6}$$

In the rest of the section, we will demonstrate the computation with more explicit examples.

B.2.1 Bracket with stress tensor: Hamiltonian symplectomorphisms

The stress tensor for the bc system, i.e. holomorphic twist of the pure gauge theory, is $2\,\mathrm{Tr}\, b\,\partial_\alpha c$. We are interested in vector fields on $\mathbb{C}^2$ that preserves the symplectic form $\omega = \mathrm{d}z_1 \wedge \mathrm{d}z_2$. Since all closed forms are exact on $\mathbb{C}^2$, all symplectic vector fields are actually Hamiltonian

$$X_f = \frac{\partial f}{\partial z_1}\frac{\partial}{\partial z_2} - \frac{\partial f}{\partial z_2}\frac{\partial}{\partial z_1} \,. \tag{B.7}$$

Given any two functions f, g on $\mathbb{C}^2$, the corresponding vector fields have the commutation relation

$$[X_f, X_g] = X_{\{f,g\}} \,, \tag{B.8}$$

where the Poisson bracket is defined to be

$$\{f, g\} = \frac{\partial f}{\partial z_1}\frac{\partial g}{\partial z_2} - \frac{\partial g}{\partial z_1}\frac{\partial f}{\partial z_2} \,. \tag{B.9}$$

Let's choose the basis $f_{m,n} = z_1^{m+1} z_2^{n+1}$ for any $m \in \mathbb{Z}_{\geq -1}$ and $n \in \mathbb{Z}_{\geq -1}$, which has the commutation relation

$$\{f_{m,n}, f_{m',n'}\} = \left[(m+1)\,(n'+1) - (m'+1)\,(n+1)\right] f_{m+m',n+n'} \,. \tag{B.10}$$

Define the mode generator

$$g_{m,n} = z_1^{m+1} z_2^{n+1} \{\partial_\alpha S^\alpha, -\} \,, \tag{B.11}$$

with $\partial_\alpha S^\alpha = 2\,\mathrm{Tr}\,\partial_\alpha b \partial^\alpha c = -\partial_\alpha b_A \partial^\alpha c^A$. A straightforward computation gives

$$[g_{m,n}, g_{m',n'}] = \left((m+1)(n'+1) - (m'+1)(n+1)\right) g_{m+m',n+n'} \,, \quad m, n \geq -1. \tag{B.12}$$

These are precisely the commutation relations (B.10) of the positive modes of the Hamiltonian vector field $\mathrm{Ham}(\mathbb{C}^2)$ on $\mathbb{C}^2$. Acting on a test field, we have

$$g_{m,n}\mathcal{O} = -X_{f_{m,n}}\mathcal{O}\,, \tag{B.13}$$

which, in particular, includes $SU(2)$ rotations and translations

$$\begin{aligned} g_{-1,0}\mathcal{O} = \partial_1\mathcal{O}, \quad g_{0,-1}\mathcal{O} = -\partial_2\mathcal{O} \\ -\frac{1}{2}(g_{1,-1} + g_{-1,1})\mathcal{O} = \Big(x_1\partial_2 - x_2\partial_1\Big)\mathcal{O}\,. \end{aligned} \tag{B.14}$$

B.2.2 Brackets for large N tree level cohomology

In this section, we calculate the tree level λ-brackets between the three towers of operators introduced in section 4.3.1 and section 4.5. We will work out the brackets explicitly in for $U(2)$. To reduce clutter, we normalize $A_n = \frac{(2i)^n}{n}\operatorname{Tr} b^n$ and further define $B_{n,\alpha}$ and C_n through the relation

$$A_n[b + \eta^\alpha\partial_\alpha c] = A_n + \eta^\alpha B_{n-1,\alpha} + \eta^1\eta^2 C_{n-2}\,. \tag{B.15}$$

A straightforward computation of Wick contraction yields[4]

$$
\begin{aligned}
\{A_{n\,\lambda}\, A_m\} &= 0 \\
\{A_{n\,\lambda}\, B_{m\alpha}\} &= 2\Big((n-1)\partial_\alpha + (m+n-1)\lambda_\alpha\Big) A_{n+m-1} \\
\{A_{n\,\lambda}\, C_m\} &= \frac{2(m+1)}{n+m-1}\Big((n-1)\partial^\alpha + (m+1)\lambda^\alpha\Big) B_{n+m-1,\alpha} \\
\{B_{n,\alpha\,\lambda}\, A_m\} &= (-2n)\left(\partial_\alpha + \frac{n+m-1}{n}\lambda_\alpha\right) A_{n+m-1} \\
\{B_{n\alpha\,\lambda}\, B_{m\beta}\} &= (-2n)\left(\frac{m}{n+m-1}\left(\partial_\alpha + \frac{n+m-1+\delta_{\alpha\beta}}{n}\lambda_\alpha\right) B_{n+m-1,\beta}\right. \\
&\qquad\qquad \left. + \frac{n-1}{n+m-1}\left(\partial_\beta + \frac{n+m-1+\delta_{\alpha\beta}}{n-1+\delta_{\alpha\beta}}\lambda_\beta\right) B_{n+m-1,\alpha}\right) \\
\{B_{n,\alpha\,\lambda}\, C_m\} &= (-2n)\frac{m+1}{n+m}\left(\partial_\alpha + \frac{n+m}{n}\lambda_\alpha\right) C_{n+m-1} \\
\{C_{n\,\lambda}\, A_m\} &= -2n\frac{n+1}{n+m-1}\left(\partial^\alpha + \frac{m+n-1}{n}\lambda^\alpha\right) B_{n+m-1,\alpha} \\
\{C_{n\,\lambda}\, B_{m\alpha}\} &= 2n\frac{n+1}{n+m}\left(\partial_\alpha + \frac{n+m}{n}\lambda_\alpha\right) C_{n+m-1} \\
\{C_{n\,\lambda}\, C_m\} &= 0
\end{aligned}
\tag{B.16}
$$

up to Q_0 exact terms.

As discussed in section 4.6, calculations above could be reproduced by holographic consideration. We leave this as a future direction.

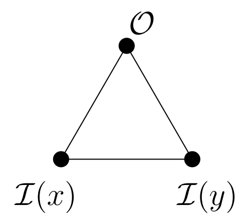
B.3 Bracket at one loop

In this section of the appendix, we explain the computation of the one loop brackets on operators, and perform some explicit examples, filling in some of the details in section 4.3.2. According to the prescription in (4.61), we need to compute the triple bracket

$$Q_1\mathcal{O} \equiv \frac{1}{2}\{\mathcal{I},\mathcal{I},\mathcal{O}\} \tag{B.17}$$

[4] We use the convention $\psi^\alpha\chi_\alpha = \psi_2\chi_1 - \psi_1\chi_2 = -\psi_\alpha\chi^\alpha$

with all λ parameters set to zero. The nontrivial contribution comes from Wick contractions removing two fields in $\mathcal{O}$ and two fields in each of the two interaction vertices, as shown below:



.

We see immediately that if $\mathcal{O}$ consist of only a single field, the triple brackets vanish identically. Q_1 acting on a pair of fields can be nonvanishing. In the example of pure gauge theory, described in 4.3, the relevant operators we are interested in are $b_A b_B$, $b_A c^B$ and their derivatives, with open adjoint gauge indices A, B. On operators of more than two fields, the action of Q_1 is simply the sum of Q_1 on all pairs of fields,

$$Q_1\Big(f_1 \dots f_{i-1} f_i f_{i+1} \dots f_{j-1} f_j f_{j+1} \dots\Big) = \sum_{i<j} (-1)^{\cdots} Q_1(f_i f_j)\Big(f_1 \dots f_{i-1} f_{i+1} \dots f_{j-1} f_{j+1} \dots\Big) \tag{B.18}$$

where the signs $(-1)^{\cdots}$ are due to the (anti)commutation relation to bring f_i and f_j to the front.

We will first compute the triple bracket by evaluating the one loop diagram given above. The one loop diagram has been calculated in [7]. As a result, the Q_1 action on $b_A b_B$ is

$$Q_1(b_A b_B) = \partial_\alpha b_A(x) \partial^\alpha c^B(x) + \partial_\alpha b_B(x) \partial^\alpha c^A(x) - 2\delta_{AB} \partial_\alpha b_C(x) \partial^\alpha c^C(x) \tag{B.19}$$

and on $b_A c^B$ is

$$Q_1(b_A c^B) = (-1)\Big(\partial_\alpha c^A(x) \partial^\alpha c^B(x) - \delta_A^B\, \partial_\alpha c_D(x) \partial^\alpha c^D(x)\Big)\,. \tag{B.20}$$

On the other hand, in the example of $SU(2)$ pure gauge theory, we can also evaluate the Q_1 action on operators by bootstapping the relation (4.69). Choose the initial ansatz

$$Q_1(b_1 c_1) = C_2 \partial_1 c_1 \partial_2 c_1 + C_1 \partial_1 c_2 \partial_2 c_2 + \dots \tag{B.21}$$

where the indices are adjoint gauge indices and $C_{1,2}$ are arbitrary coefficients. Comparing to (B.20), we find $C_2 = 0$ and $C_1 = -2$ but we will leave it unfixed below. Surprisingly, up to three loop level, we find a unique solution of (4.69) parametrized by the arbitrary coefficients $C_{1,2}$. In particular, the action of Q_1 on the most general type of operator of the form $V(1,1,n_1+m_1,n_2+m_2)$ is given by:

$$Q_1(\partial_1^{n_1}\partial_2^{n_2}b_A\partial_1^{m_1}\partial_2^{m_2}c_B) = -C_2\,\mathcal{F}(c_A,c_B) + C_1\frac{2\,\epsilon_{ADE}\,\epsilon_{BCE}}{m_1+m_2+2}\,\mathcal{G}(c_C,c_D)\,, \tag{B.22}$$

$$Q_1(\partial_1^{n_1}\partial_2^{n_2}b_A\partial_1^{m_1}\partial_2^{m_2}b_B) = -C_2\,\mathcal{H}(b_A,b_B,c_A,c_B) + C_1\frac{2\epsilon_{ADE}\,\epsilon_{BCE}}{m_1+m_2+2}\,\mathcal{K}(b_C,b_D,c_C,c_D)\,. \tag{B.23}$$

Where the explicit form of the functions are given by

$$\begin{aligned}
\mathcal{F}(c_A,c_B) &= \partial_1^{n_1+1}\partial_2^{n_2}c_A\partial_1^{m_1}\partial_2^{m_2+1}c_B - \partial_1^{n_1}\partial_2^{n_2+1}c_A\partial_1^{m_1+1}\partial_2^{m_2}c_B\,,\\
\mathcal{H}(b_A,b_B,c_A,c_B) &= \partial_1^{n_1+1}\partial_2^{n_2}c_A\partial_1^{m_1}\partial_2^{m_2+1}b_B - \partial_1^{n_1}\partial_2^{n_2+1}c_A\partial_1^{m_1+1}\partial_2^{m_2}b_B\\
&\quad - \partial_1^{n_1+1}\partial_2^{n_2}b_A\partial_1^{m_1}\partial_2^{m_2+1}c_B + \partial_1^{n_1}\partial_2^{n_2+1}b_A\partial_1^{m_1+1}\partial_2^{m_2}c_B\,,\\
\mathcal{G}(c_C,c_D) &= \sum_{k=0}^{m_1}\sum_{l=0}^{m_2}\frac{1}{k+l+1}\binom{m_1}{k}\binom{m_2}{l}\\
&\quad \left[\partial_1^{m_1-k+1}\partial_2^{m_2-l}c_C\,\partial_1^{n_1+k}\partial_2^{n_2+l+1}c_D - \partial_1^{m_1-k}\partial_2^{m_2-l+1}c_C\,\partial_1^{n_1+k+1}\partial_2^{n_2+l}c_D\right],\\
\mathcal{K}(b_C,b_D,c_C,c_D) &= \sum_{k=0}^{m_1}\sum_{l=0}^{m_2}\frac{1}{k+l+1}\binom{m_1}{k}\binom{m_2}{l}\\
&\quad \Big[\partial_1^{m_1-k+1}\partial_2^{m_2-l}c_C\,\partial_1^{n_1+k}\partial_2^{n_2+l+1}b_D - \partial_1^{m_1-k}\partial_2^{m_2-l+1}c_C\,\partial_1^{n_1+k+1}\partial_2^{n_2+l}b_D\\
&\quad - \partial_1^{m_1-k+1}\partial_2^{m_2-l}b_C\,\partial_1^{n_1+k}\partial_2^{n_2+l+1}c_D + \partial_1^{m_1-k}\partial_2^{m_2-l+1}b_C\,\partial_1^{n_1+k+1}\partial_2^{n_2+l}c_D\Big]\,.
\end{aligned}$$

One can check explicitly that the resulting Q_1 is not nilpotent. This implies the need for a two-loop correction, so that $Q_1^2 + \{Q_0, Q_2\} = 0$ from (4.70). For example,

$$Q_2\left(b_Ab^Ab_Bb^B\right) = \frac{20C_1^2}{3}\epsilon_{ABC}b^A\partial_\alpha\partial_\beta b^B\partial^\alpha\partial^\beta c^C \tag{B.24}$$

$$Q_2\left(b_Ab^A\partial_\alpha c_B\partial^\alpha c^B\right) = -\frac{10C_1^2}{9}\epsilon_{ABC}\partial_\alpha c^A\partial^\alpha\left[\partial_1^2c^B\partial_2^2c^C - \partial_1\partial_2c^B\partial_1\partial_2c^C\right]\,. \tag{B.25}$$

B.4 Homotopy transfer

Consider a super vector space X with a differential Q (an odd nilpotent linear operator). Denote the cohomology Q in X as $H(X,Q)$. Suppose that we are given a decomposition of supervector spaces

$$X = U \oplus V \oplus W \tag{B.26}$$

such that $QU \subset U$ and $W = \mathrm{im}(Q|_V)$, which implies $QW = 0$. This immediately implies that

$$H(X, Q) = H(U, \tilde{Q} = Q|_U)\,. \tag{B.27}$$

Indeed, the kernel of Q equals the kernel of $\tilde{Q}$ plus W, and the image of Q is the image of $\tilde{Q}$ plus W.

An explicit quasi-isomorphism between $(U, \tilde{Q})$ and (X, Q) can be presented in terms of a cochain *contraction* (π, ι, h) from (X, Q) to $(U, \tilde{Q})$. Let $\pi : X \to U$ be the projection, $\iota : U \to X$ be the inclusion let $h : X \to X$ be such that the following identities hold

$$\begin{aligned} \pi Q &= \tilde{Q}\pi \\ Q\iota &= \iota\tilde{Q} \\ \pi\iota &= 1_U \\ Qh + hQ &= 1_X - \iota\pi \\ \pi h &= 0 \\ h\iota &= 0 \\ h^2 &= 0\,. \end{aligned} \tag{B.28}$$

Indeed, we can define h as vanishing on the U and V summands and the inverse of Q on W. The first two relations say that π, ι are chain maps. The remaining relations encode that (π, ι, h) is a cochain contraction.

A systematic calculation of the cohomology of any cochain complex (X, Q) will effectively produce a contraction as above. Indeed, we can let $V \subset X$ be the complement of the kernel of Q. The cohomology U is then presented as any complement of $W = \mathrm{im}(Q|_V)$ in the kernel of Q.

B.4.1 Deforming a contraction

Next, consider a situation where the differential Q is a perturbation of a simpler differential Q_0, that is

$$Q = Q_0 - \delta \tag{B.29}$$

for some 'small' differential δ. Also, assume that we have a contraction for Q_0:

$$X = U_0 \oplus V_0 \oplus W_0 \tag{B.30}$$

such that $Q_0 U_0 \subset U_0$ and $\mathrm{im}(Q|_{V_0}) = W_0$. Denote by $\tilde{Q}_0$, π_0, ι_0 and h_0 the corresponding maps supplying a contraction as above.

We can look for a deformation of the contraction for Q_0 to a contraction for Q. For example, define

$$\Delta \equiv \iota_0 \pi_0 + \{h_0, Q\} = 1_X - \{h_0, \delta\}\,. \tag{B.31}$$

If we can invert Δ, we can define new maps

$$\begin{aligned}
\tilde{Q} &\equiv \pi_0 Q \Delta^{-1} \iota_0 = \tilde{Q}_0 - \pi_0 \delta \frac{1_X}{1_X - h_0 \delta} \iota_0 \\
\iota &\equiv \Delta^{-1} \iota_0 = \frac{1_X}{1_X - h_0 \delta} \iota_0 \\
\pi &\equiv \pi_0 \Delta^{-1} = \pi_0 \frac{1_X}{1_X - \delta h_0} \\
h &\equiv h_0 \Delta^{-1} = h_0 \frac{1_X}{1_X - \delta h_0}\,.
\end{aligned} \tag{B.32}$$

We claim that these maps define the desired deformed contraction. Some relations follow immediately from the expressions above:

$$\begin{aligned}
\pi \iota &= 0\,, \\
\pi h &= 0\,, \\
h \iota &= 0\,, \\
h^2 &= 0\,.
\end{aligned} \tag{B.33}$$

Using

$$\tilde{Q} = \pi_0 \Delta^{-1} Q \iota_0 = \pi Q \iota_0\,, \tag{B.34}$$

and a bit of work, we can verify another set of relations:

$$\begin{aligned}
\tilde{Q}\pi &= \pi Q \iota_0 \pi_0 \Delta^{-1} = \pi Q - \pi Q \{h_0, Q\} = \pi Q - \pi \{h_0, Q\} Q \\
&= \pi Q - \pi \iota_0 \pi_0 Q + \pi \Delta Q = \pi Q - \pi_0 Q + \pi_0 Q = \pi Q\,, \\
\iota \tilde{Q} &= \Delta^{-1} \iota_0 \pi_0 Q \iota = Q\iota - \{h_0, Q\} Q \iota = Q\iota - Q\{h_0, Q\}\iota \\
&= Q\iota - Q\iota_0 \pi_0 \iota + Q \Delta \iota = Q\iota - Q\iota_0 + Q\iota_0 = Q\iota\,.
\end{aligned} \tag{B.35}$$

The final relation takes a bit more work. First, we can compute the commutator

$$\{Q_0, h\} = \frac{1_X}{1_X - h_0 \delta} \{Q_0, h_0\} \frac{1_X}{1_X - \delta h_0} - h\{Q_0, \delta\} h$$

$$= \frac{1_X}{1_X - h_0\delta}\frac{1_X}{1_X - \delta h_0} - \iota\pi - h\delta^2 h$$
$$= 1_X + \frac{h_0\delta}{1_X - h_0\delta} + \frac{\delta h_0}{1_X - \delta h_0} - \iota\pi \,. \tag{B.36}$$

From this it follows that
$$\iota\pi + \{Q, h\} = 1_X \,, \tag{B.37}$$
as desired.

B.4.2 Perturbative cohomology

In a perturbative setting we may have a differential given as a power series in some formal parameter
$$Q = Q_0 + \hbar Q_1 + \hbar^2 Q_2 + \cdots \tag{B.38}$$
Homotopy transfer offers a way to recursively *define* a "perturbative cohomology" in such setting, as long as the Q_i have a sufficiently "triangular" form.

Namely, we can start by taking the cohomology U_0 of Q_0 and treating the rest of the sum as $-\delta$, inverting the corresponding
$$\Delta \equiv \iota_0\pi_0 + \{h_0, Q\} = 1_X + \hbar\{h_0, Q_1\} + \hbar^2\{h_0, Q_2\} + \cdots \tag{B.39}$$
perturbatively. The result is a new perturbative differential
$$\tilde{Q} = \pi_0(\hbar Q_1 + \cdots)\Delta^{-1} i_0 = \hbar Q_0^{(1)} + \hbar^2 Q_1^{(1)} + \cdots \tag{B.40}$$
which is now a formal power series starting at order $\hbar$.

We can then compute a contraction for $Q_0^{(1)}$ and repeat the procedure to get a new pertubative differential on the cohomology $U_0^{(1)}$ of $Q_0^{(1)}$, etcetera. As long as we have some kind of filtration controlling the form of the Q_i, the procedure will converge to some limiting $U_0^{(\infty)}$ which can be taken as the definition of perturbative cohomology.

B.5 Maurer-Cartan equations and quantum field theories

A student of QFT is usually familiar with the idea that any QFT can be formally/ perturbatively deformed by adding to the action a general linear combination of local operators (modulo total derivatives) multiplied by couplings:

$$\sum_i g_i \int d^D x\, \mathcal{O}^i(x)\,. \tag{B.41}$$

The parameterization of the space of deformations is scheme-dependent beyond the first order of perturbation theory. The deformation space can be described as a *formal* (i.e. defined in perturbation series) *pointed* (i.e. equipped with a base point) *super* (if we include deformations by Grassmann odd operators) *manifold*.

Another familiar notion in QFT is that of a beta function. If the base theory is scale-invariant, an infinitesimal scale transformation will act on the couplings, thus defining a formal vector-field

$$\beta = \sum_i \beta_i(g)\partial_{g_i} = \beta^j_{1;i} g_j \partial_{g_i} + \frac{1}{2}\beta^{jk}_{2;i} g_j g_k \partial_{g_i} + \cdots\,. \tag{B.42}$$

Even if the base theory is not scale invariant, an infinitesimal scale transformation will give an infinitesimal deformation of the theory. We will still get a formal vectorfield $\beta = \beta_{0;i}\partial_{g_i} + \cdots$.

If the base theory is described in a BRST formalism, we can extend the space of formal deformations to include all operators in the BRST complex. Accordingly, we have a larger collection of couplings g_i, some of which will have non-zero ghost number/cohomological degree. If we take the deformed theory with some couplings g_i and act with the BRST differential on correlation function, a possible BRST anomaly will manifest itself as an odd variation of the couplings, i.e. an odd vectorfield vector field

$$\mathcal{Q} = \sum_i \eta_i(g)\partial_{g_i} = \eta^j_{1;i} g_j \partial_{g_i} + \frac{1}{2}\eta^{jk}_{2;i} g_j g_k \partial_{g_i} + \cdots\,. \tag{B.43}$$

In other words, correlation functions will be annihilated by the action of $Q_{\mathrm{BRST}} + \mathcal{Q}$. By definition, the odd vectorfield $\mathcal{Q}$ is nilpotent. We thus have what is called a *formal pointed dg-supermanifold*.

The condition for a deformation of the theory to be non-anomalous thus takes manifestly the form of a Maurer-Cartan equation, with coefficients $\eta^{j_i\cdots j_n}_{n;i}$. The constraint $\mathcal{Q}^2 = 0$ tells us that the coefficients satisfy the axioms of an L_∞ algebra.

Notice that we did not assume that the QFT we are referring to should be topological or holomorphic. The appearance of ∞-structures is simply due to the formal deformation nature of the problem.

We can also consider position-dependent deformations of the theory. This leads to λ-brackets and their non-holomorphic analogues (see [116] for more details).

B.5.1 Homotopy transfer for odd vector-fields and L_∞ algebras

The basic homotopy transfer formula from appendix B.4 can be widely generalized. For example, consider an L_∞ algebra, presented dually as the collection of coefficients in an odd nilpotent formal vectorfield $\mathcal{Q}$.

There is a "sum over trees" construction for L_∞ algebras which is analogous to the contraction formulae above [162]. The analogy becomes sharp in the language of the formal supermanifolds. Suppose that we have a contraction for Q, the linear part of $\mathcal{Q}$. This gives us dual linear maps ι, π, h relating the g_i's and some smaller set of $\tilde{g}_a$. We can define as before

$$\Delta = \iota\pi + \{h, \mathcal{Q}\}\,, \tag{B.44}$$

and

$$\begin{aligned} \tilde{\mathcal{Q}} &\equiv \pi\mathcal{Q}\Delta^{-1}\iota\,, \\ \mathcal{I} &\equiv \Delta^{-1}\iota\,, \\ \Pi &\equiv \pi\Delta^{-1}\,, \\ \mathcal{H} &\equiv h\Delta^{-1}\,, \end{aligned} \tag{B.45}$$

which give an odd vectorfield $\tilde{\mathcal{Q}}$ on the $\tilde{g}_a$ as well as morphisms relating it to $\mathcal{Q}$.

Dually, these formulae reproduce the 'sum over tree' formulae. As we expand out the expressions for the coefficients of $\tilde{\mathcal{Q}}$, we find sums of terms involving sequences of operations which alternate the action of L_∞ brackets and of the coefficients in h.

B.6 Rigid SUSY, twists, and indices

In the bulk of the document, our goal is to understand the (perturbative) algebra of local operators in a SUSY QFT protected by the minimal amount of supersymmetry. These operators form a "semi-chiral ring," and are computed by taking the Q-cohomology of any supercharge in the $\mathcal{N} = 1$ theory (the particular supercharge is immaterial). Essentially by definition of locality, the collection of local operators in any QFT should only depend on the details of the spacetime manifold through the contribution of gauge-invariant functions of the metric and other background fields, and their derivatives at the point, which can affect operator mixing. After a holomorphic twist, we do not expect the surviving background fields, such as the Beltrami differential encoding the complex structure of spacetime, to have local invariants. Hence the semi-chiral ring should be insensitive to the spacetime manifold that the theory is placed on.

There are many other quantities of interest in SUSY QFTs which *are* sensitive to the underlying spacetime. Perhaps the best example of a BPS quantity which depends on the spacetime is the supersymmetric partition function $\mathcal{Z}_{\mathcal{M}}$. The partition function, with arbitrary background fields turned on, captures the correlation functions of local operators which couple to those background fields e.g. $\mathcal{Z}_{\mathcal{M}}[A]$ tells us about correlation functions of the current j^μ which couples to A_μ. The partition function also describes the "topological" response of the theory to non-trivial background field configurations, including non-trivial topologies themselves. In the case that the flat space theory is superconformal (and thus has a $U(1)_R$ symmetry), the supersymmetric partition function on $S^3 \times S^1$ is essentially the superconformal index, which in turn counts (with signs) the (dimensions of the Q-cohomologies of the) local operators in the semi-chiral ring. We shall review the precise correspondence between these quantities below.

In this appendix we will review and disambiguate the terminology employed here and in the supersymmetry literature. Specifically, we will:

1. Give a brief review of the literature on rigid supersymmetry and define what we mean by the word "twist." The goal here will be to disentangle the study of BPS quantities, roughly captured by passing to the Q-cohomology (and what we will call twisting), from the process of turning on a non-trivial background field. This is summarized in figure B.1.[5]

2. Describe precisely how to do computations in a twisted theory by trading it for a simpler theory which is a deformation retract of the original one, and comment on the information of the untwisted theory that is captured by the twisted theory.

3. Separate the notions of supersymmetric partition function, supersymmetric index, and superconformal index, and explain under what conditions and to what extent they are the same on $S^3 \times S^1$.

As we will explain, there are two ways to obtain the twisted/cohomological theory on a non-trivial spacetime $\mathcal{L}^{\text{tw}}_{\mathcal{M}}$.[6] Starting with the untwisted flat space theory, schematically denoted $\mathcal{L}_{\mathbb{R}^{2d_1+d_2}}$, one can:

[5]To re-iterate, our definition of "twist" only means to pass to the cohomology of a nilpotent (not necessarily scalar) operator Q. In the literature, twist can sometimes mean the entire passage from $\mathcal{L}_{\mathbb{R}^4} \to \mathcal{L}^{\text{tw}}_{\mathcal{M}}$. Unfortunately, twist can also refer to turning on background gauge fields, which we will also do (but not call twisting).

[6]Note: we write everything schematically with a Lagrangian $\mathcal{L}$ for simplicity, but a Lagrangian is not essential for the rigid SUSY based arguments below.

$$
\begin{tikzcd}
\mathcal{L}_{\mathbb{R}^{2d_1+d_2}} \arrow[r, "Q\text{-cohomology}"] \arrow[d, "\substack{\text{Couple to}\\\text{off-shell}\\\text{SUGRA}}"'] & \mathcal{L}^{\mathrm{tw}}_{\mathbb{C}^{d_1}\times\mathbb{R}^{d_2}} \arrow[d, "\substack{\text{Define}\\\text{on } \mathcal{M}}"] \\
\mathcal{L}_{\mathcal{M}} \arrow[r, "Q\text{-cohomology}"'] & \mathcal{L}^{\mathrm{tw}}_{\mathcal{M}}
\end{tikzcd}
$$

Figure B.1: Down-moving arrows always involve turning on some background fields in order to define the theory on a curved manifold: on the left, there is the extra constraint to preserve supersymmetry on $\mathcal{M}$; on the right, many local aspects of the geometry are unimportant. Right-moving arrows always indicate passing to a Q-cohomology, i.e. restricting attention to BPS quantities of the theory.

1. **Put it on $\mathcal{M}$ then twist.** Start by trying to place the flat space theory on a desired curved manifold $\mathcal{M}$. Since this generically breaks the supersymmetries of the original flat space theory, one must turn on particular background fields in order to preserve SUSY. This may not be possible at all and will generally require choices beyond the geometry of $\mathcal{M}$, such as a complex structure for a 4d $\mathcal{N}=1$ theory equipped with a $U(1)_R$ symmetry, or a complex symplectic structure for a generic 4d $\mathcal{N}=1$ theory. If the theory can be placed on $\mathcal{M}$ while preserving at least one nilpotent scalar supercharge, then one can study the Q-twisted theory on $\mathcal{M}$.

2. **Twist then put it on $\mathcal{M}$.** Alternatively, we can start by passing from the flat space theory to the twisted theory $\mathcal{L}^{\mathrm{tw}}_{\mathbb{C}^{d_1}\times\mathbb{R}^{d_2}}$, a holomorphic-topological QFT which we can try to place on different curved manifolds by using an atlas of coordinate patches related by diffeomorphisms which preserve the holomorphic-topological structure.[7] For example, the twist of a 4d $\mathcal{N}=1$ theory equipped with a $U(1)_R$ symmetry can be placed on an arbitrary complex manifold/coupled to an arbitrary Beltrami differential. In the absence of R-symmetry, it can be placed on any complex symplectic manifold.

We schematically sketch these two paths in figure B.1. In the main body of the text we take path 2.

In the next section, we briefly review the literature on rigid supersymmetry that encompasses the left-most arrow of this diagram, since it is the only one which is not obvious from

[7]In some situations, e.g. with extended supersymmetry, there may be other non-geometric background fields which can be turned on and are needed to match the supergravity setup. The S^2 partition function of 2d $(2,2)$ theories or the S^4 partition function of 4d $\mathcal{N}=2$ theories are likely examples.

the text. Then we review what it means to compute and study the twisted/cohomological theory. Finally, we discuss the particular case that $\mathcal{M} = S^3 \times \mathbb{R}$ and/or $\mathcal{M} = S^3 \times S^1$, and the relationship between the supersymmetric partition function, the supersymmetric index, and the superconformal index.

B.6.1 Rigid SUSY and curved spacetimes

Let us very briefly review the first approach, and, in particular, explain how to place a SUSY theory on a curved spacetime $\mathcal{M}$ while preserving supersymmetry. A systematic answer for this problem was first described by [163], with solutions tabulated in [164, 165]. Important follow-ups for index calculations are described in [166, 167] (see [168] for a superspace approach and [169] for a self-contained review).

As previously mentioned, if one naively tries to place a flat space theory $\mathcal{L}_{\mathbb{R}^4}$ on some manifold $\mathcal{M}$ by minimally coupling it to the metric $g_{\mu\nu}$ for $\mathcal{M}$, then the theory on $\mathcal{M}$ will not generically be supersymmetric.[8] In particular, there is only a supercharge on $\mathcal{M}$ for each covariantly constant spinor ζ

$$\nabla_\mu \zeta = 0\,. \tag{B.46}$$

Note: even when SUSY is preserved, there's no reason to expect the SUSY algebra on $\mathcal{M}$ to be the same as on $\mathbb{R}^4$ since the isometries of $\mathcal{M}$ will be different from isometries of $\mathbb{R}^4$.[9]

The *rigid supersymmetry* construction of [163] provides a systematic procedure for determining if and how one can place the flat space theory $\mathcal{L}_{\mathbb{R}^4}$ on $\mathcal{M}$, without breaking supersymmetry, for a much larger class of $\mathcal{M}$ than specified by (B.46). The idea is to couple the theory to background off-shell supergravity in a possibly non-minimal way, enriching the Killing spinor equation (B.46) with additional background field terms. Then, by tuning the values of the background fields in this supergravity multiplet, the Killing spinor equation may admit more solutions on $\mathcal{M}$. The procedure is as follows:

1. **Pick a supermultiplet $\mathcal{S}_\mu$.** Every local Lorentz-invariant SUSY QFT has a real conserved symmetric stress tensor $T_{\mu\nu}$, which lives inside a stress tensor supermultiplet alongside the conserved supercurrent $S_{\mu\alpha}$ [134] (see also section 4.2.4). The

[8] Intuitively, perturbations to the flat space metric couple to the stress tensor $T^{\mu\nu}$, and $[Q, T^{\mu\nu}] \neq 0$, so generically $\mathcal{L}_{\mathcal{M}}$ is not supersymmetric. In a similar vein, (non-flat) background connections for global symmetries will also break SUSY.

[9] However, the SUSY algebra on $\mathcal{M}$ should Inönu-Wigner contract to the SUSY algebra for $\mathbb{R}^4$, where the contraction parameter is some characteristic length scale for $\mathcal{M}$. Also see the comments around (1.1) of [163].

other component fields $\{\mathcal{X}_i\}$ in the supermultiplet are determined by the precise supermultiplet $\mathcal{S}_\mu$ we are using, but $T_{\mu\nu}$ and $S_{\mu\alpha}$ are universal.

2. **Make $g_{\mu\nu}$ a superfield.** We promote the background metric $g_{\mu\nu}$ to a (component of a) background supergravity multiplet. The multiplet contains $g_{\mu\nu}$, the gravitino $\Psi_{\mu\alpha}$, and some additional auxiliary fields $\{\mathcal{Y}_i\}$ which can source the $\{\mathcal{X}_i\}$ operators in the supercurrent multiplet $\mathcal{S}_\mu$.

3. **Couple $\mathcal{L}_{\mathbb{R}^4}$ to supergravity background.** We couple $\mathcal{L}_{\mathbb{R}^4}$ to the appropriate background off-shell supergravity multiplet (as dictated by $\mathcal{S}_\mu$).

4. **Tune the background so $\mathcal{L}_{\mathcal{M}}$ is supersymmetric.** To obtain a supersymmetric theory on $\mathcal{M}$, we must tune the values for the background off-shell supergravity multiplet appropriately. We set $g_{\mu\nu}$ to look like $\mathcal{M}$, and set the gravitinos (and any other fermionic background fields) to zero so we have a bosonic background, i.e. $\Psi = 0 = \tilde{\Psi}$. Since SUSY variations of the bosonic background fields are proportional to the gravitinos, they vanish. By solving for $\delta\Psi = 0$ and $\delta\tilde{\Psi} = 0$, we obtain the allowed values for the auxiliary bosonic background fields $\{\mathcal{Y}_i\}$ so that $\mathcal{L}_{\mathcal{M}}$ is supersymmetric.

That this last step of tuning the background off-shell supergravity fields works, and can be done at all, is non-obvious. The intuition is that one should actually think of the theory $\mathcal{L}_{\mathbb{R}^4}$ as being coupled to standard dynamical off-shell supergravity. Off-shell because we *do not* integrate out the auxiliary fields. Then, by rescaling the fields in the dynamical supergravity multiplet and taking the "rigid limit" Planck mass $M_P \to \infty$, the supergravity fluctuations decouple and we can "freeze in" the admissible background values. This also makes it clear that the resulting rigid SUSY algebra can be thought of as the subalgebra of local superdiffeomorphisms that preserve the particular background.

Some important notes about this procedure:

- Since the gravitino variation always contains a covariant derivative of ζ
$$0 = \delta_\zeta \Psi_{\mu\alpha} = \nabla_\mu \zeta_\alpha + F(g_{\mu\nu}, \mathcal{Y}_i, \zeta)\,, \tag{B.47}$$
we obtain a generalization of the Killing spinor equation (B.46), enriched by the background fields. There can be yet additional dependence of F on background connections for global symmetries.

- We can use this construction in the reverse. By solving the Killing spinor equation (B.47), we can determine which $\mathcal{M}$ admit supersymmetric backgrounds [164, 165].

- Similarly, the construction makes it clear that we do not need specific information about $\mathcal{L}_{\mathbb{R}^4}$, other than the stress tensor multiplet $\mathcal{S}_\mu$ it admits. In particular, we don't even need to have a Lagrangian description of the flat space theory.[10]

To pick a guiding concrete example, if the flat space $\mathcal{N}=1$ theory has an unbroken $U(1)_R$ symmetry, then the theory admits an $\mathcal{R}$-multiplet and couples to background fields for "new minimal supergravity" [170, 171]

$$\mathcal{R}_\mu = (j^{(R)}_\mu, S_{\alpha\mu}, \tilde{S}_{\mu\dot{\alpha}}, T_{\mu\nu}, C_{\mu\nu})\,, \tag{B.48}$$

$$\mathcal{H}_\mu = (A^{(R)}_\mu, \Psi_{\alpha\mu}, \tilde{\Psi}_{\mu\dot{\alpha}}, g_{\mu\nu}, B_{\mu\nu})\,, \tag{B.49}$$

where $C_{\mu\nu}$ is a conserved anti-symmetric two-form associated to "string charges" [134]. In Euclidean signature, left and right-handed spinors are independent, so their variations are set to zero independently $\delta_\zeta \Psi_{\alpha\mu} = 0$ and $\delta_{\tilde{\zeta}}\tilde{\Psi}_{\mu\dot{\alpha}} = 0$. Thus we simply read off the new Killing spinor equations from the gravitino variations of new minimal supergravity:

$$(\nabla_\mu - iA^{(R)}_\mu)\zeta = -\frac{i}{2}V^\nu \sigma_\mu \tilde{\sigma}_\nu \zeta\,, \tag{B.50}$$

$$(\nabla_\mu + iA^{(R)}_\mu)\tilde{\zeta} = +\frac{i}{2}V^\nu \tilde{\sigma}_\mu \sigma_\nu \tilde{\zeta}\,, \tag{B.51}$$

where $V^\mu = \frac{i}{2}\epsilon^{\mu\nu\rho\lambda}\partial_\nu B_{\rho\lambda}$ and is conserved $\nabla_\mu V^\mu = 0$ [163, 166, 167]. In Euclidean signature, A_μ and V^μ are generally allowed to be complex, but one assumes $g_{\mu\nu}$ is real.

Solutions to the above Killing spinor equations in the off-shell new minimal supergravity background (B.50-B.51) exist if and only if $\mathcal{M}$ is a Hermitian manifold, in which case the background field values $A^{(R)}_\mu$ and V^μ are (almost) fully determined from the complex structure $J^\mu{}_\nu$ and Hermitian metric on $\mathcal{M}$ (see [164] for details). For example, $V^\mu = \frac{1}{2}\nabla_\nu J^\nu{}_\mu$. When $\mathcal{M}$ is Kähler, $J^\nu{}_\mu$ is covariantly constant, so V^μ vanishes and the Killing spinor equations reduce to the familiar holomorphic twist of $\mathcal{N}=1$ theories on Kähler manifolds by an R-symmetry background as described in [172, 173, 174]. We will return to this below. Other stronger facts are obtained if more supercharges are preserved.

The analysis above can be repeated without a $U(1)_R$ symmetry for theories admitting an FZ-multiplet. The FZ-multiplet couples to the auxilliary fields of "old minimal supergravity" [175, 176]: a metric g_μ, gravitinos $\Psi_{\mu\alpha}$ and $\tilde{\Psi}_{\alpha\mu}$, a vector field b^μ, and two scalars M and $\tilde{M}$. There are also similar results for "conformal supergravity" and Lorentzian signatures [177, 178, 179, 180, 181, 182].

[10] With the caveat that background fields can enter the action non-linearly. A description of the coupling thus includes information on how the stress tensor multiplet varies as the background fields are turned on.

An important consequence of having a $U(1)_R$ symmetry in the new-minimal supergravity treatment, is that the Q which is guaranteed to survive on a Hermitian manifold $\mathcal{M}$ is a scalar under holomorphic coordinate trasformations [163, 164]. This is not the case without a $U(1)_R$ symmetry for the old-minimal supergravity treatment [165]. A closely related (but strictly stronger) fact is that with and without $U(1)_R$ symmetry, an $\mathcal{N}=1$ theory can be placed on a manifold locally isometric to $\mathcal{M}_3 \times \mathbb{R}$, where $\mathcal{M}_3$ is a maximally symmetric space, while preserving four supercharges; but only with a $U(1)_R$ symmetry they can be made time-independent (i.e. they commute with the generator of isometries along the $\mathbb{R}$ factor).

One way to understand this scalarity of Q is as follows [164]: the holonomy group of the Levi-Civita connection for a Kähler manifold $\mathcal{M}$ is $U(2)$, and at the level of Lie algebras $\mathfrak{u}(2) = \mathfrak{u}(1) \times \mathfrak{su}(2)$ where these are subalgebras of the chiral $\mathfrak{su}(2)$ symmetries of flat spacetime. By turning on a $U(1)_R$ background, one can cancel the $U(1)$ component in the $U(2)$ holonomy group, leaving a scalar supercharge ζ to the Killing spinor equations

$$(\nabla_\mu - iA_\mu^{(R)})\zeta = 0 \tag{B.52}$$

as in [172, 173, 174]. On a more general Hermitian manifold $\mathcal{M}$, the metric and complex structure are covariantly constant with respect to the Chern connection $\nabla_\mu^{(c)}$, not the Levi-Civita connection. In this general Hermitian case, the Killing spinor equations in the off-shell new minimal supergravity background (B.50-B.51) become the exceptionally simple

$$(\nabla_\mu^{(c)} - iA_\mu^{(c)})\zeta = 0 \tag{B.53}$$

when written in terms of the Chern connection (see [164] for details on the Chern-connection and gauge field $A_\mu^{(c)}$). Thus the entire story for the Kähler manifold and the holonomy of the Levi-Civita connection being cancelled by $A_\mu^{(R)}$ repeats for the $U(2)$ holonomy of the Chern connection and $A_\mu^{(c)}$.

B.6.2 Twisted theory on curved spacetime

Oppositely to the story above, we can instead take a flat space theory $\mathcal{L}_{\mathbb{R}^4}$ and *twist* it by passing to the Q-cohomology. As previously mentioned, these Q-cohomologies capture the BPS local operators comprising the semi-chiral ring. In the main body of the chapter, our journey ends here for $\mathcal{N}=1$ SYM: *we are computing the (perturbative) semi-chiral ring of local operators on flat space, and studying its structure and holomorphic descendants.*

There are now two (essentially equivalent) things we mean by the "twisted theory" $\mathcal{L}^{\mathrm{tw}}_{\mathbb{C}^2}$. On one hand, we can study the cohomological theory written in terms of the original field variables on $\mathbb{C}^2$. On the other hand, we can study an effective repackaging of these Q-cohomologies into new fields, presenting the theory in a different way. Specifically, we find a different set of field variables that correspond to Q-cohomologies and when treating both in the BV formalism provides a deformation retract of the respective chain complexes. For example. In our case, the twisted sector of $\mathcal{N}=1$ SYM can be presented in terms of the original field variables, or can be effectively described by a bc system/holomorphic BF theory.

Now, *if* we are interested in trying to place the twisted flat-space theory $\mathcal{L}^{\text{tw}}_{\mathbb{C}^2}$ with Lorentz group K on a non-trivial manifold $\mathcal{M}$, and *if* we also have a $U(1)_R$ symmetry (or other global symmetries), then we can talk about turning on background fields again to preserve the Q-cohomological/holomorphic structure on $\mathcal{M}$. In particular, it may be possible to find a homomorphism $K \hookrightarrow K \times U(1)_R$, so that the twisting supercharge Q is a scalar under the image of K: the new Lorentz group K'. In this case, there is a new stress tensor T' (whose components integrate to new Lorentz group rotations) is different from T, satisfying $T' = \{Q, S\}$.[11] In flat-space, this is a trivial redefinition of field data, but alters the way that the theory $\mathcal{L}^{\text{tw}}_{\mathbb{C}^2}$ couples to curved space. More specifically, a Q-exact stress tensor makes placing $\mathcal{L}^{\text{tw}}_{\mathbb{C}^2}$ on $\mathcal{M}$ sensible at all. Since Q is a K' scalar, the theory can be put on any $\mathcal{M}$ without breaking the Q-cohomology/holomorphic structure i.e. while preserving Q and $Q^2 = 0$.

It is important to reiterate that our definition of twist in this thesis is precisely passing to Q-cohomology, to study the BPS operators, i.e. the right-moving arrows in figure B.1. Not the act of turning on a background gauge field/twist for $U(1)_R$ symmetry, i.e. down-moving arrows in figure B.1. We stress this point because the flat-space twisted theory $\mathcal{L}^{\text{tw}}_{\mathbb{C}^2}$, with or without a $U(1)_R$ symmetry, is a theory that exists, which captures the correlation functions of the semi-chiral subsector of the full flat-space theory $\mathcal{L}_{\mathbb{R}^4}$. Turning on a background symmetry is required to put the theory on a non-trivial manifold.

Some extended remarks about twisting are in order. We have emphasized that a main step in constructing the twist of a supersymmetric theory involves taking the cohomology with respect to a nilpotent supercharge Q. More precisely, this step in the twisting procedure involves deforming the original BRST operator Q_{BRST} by the supercharge Q, see [113]. If one is only interested in what happens to the cohomology of local operators when passing to a twist, then there is a spectral sequence which does involve taking the

[11] Here there is an unfortunate historical overloading of terminology which we have tried to avoid. In the literature, the homomorphism, "new Lorentz group K'," and "new stress tensor," are called the "twisting homomorphism," "twisted Lorentz group," and "twisted stress tensor" respectively. Referring to the fact that objects are affected by a nontrivial $U(1)_R$ background. We do not use such language because a priori these "twists" do not have to be related to taking a Q-cohomology.

Q-cohomology.

There is another step in the twisting procedure that we have not discussed. To make sense of the deformed/twisted BRST operator $Q_{\mathrm{BRST}} + Q$ one uses R-symmetry to shift the ghost number grading so that this operator is homogenous. The output of twisting typically collapses the $\mathbb{Z} \times \mathbb{Z}/2$ (ghost $\times$ parity) bigrading of a supersymmetric field theory to a $\mathbb{Z}/2$ grading. In the original theory, the BRST operator has bidgree $(1, 0)$ with respect to this bigrading and Q has bidegree $(0, 1)$. One chooses a copy of $U(1)$ in the R-symmetry to lift this to a $\mathbb{Z}$ grading, which we interpret as the ghost grading in the twisted theory. Of course, if the R-symmetry does not contain a $U(1)$ factor then this step is not possible. We refer to [113, 6] for more details.

B.6.3 Partition functions, supersymmetric index, and superconformal index

In the rigid SUSY story, we emphasized the existence of a nilpotent time-independent scalar supercharge Q on the Hermitian manifold $\mathcal{M}$ when there is a $U(1)_R$ symmetry. This can be used to show, modulo anomalies, that $\mathcal{Z}_{\mathcal{M}}$ is independent of the Hermitian metric on spacetime, and is locally a holomorphic function of the complex structure moduli and line bundle moduli (in the presence of background fields) of $\mathcal{M}$ [167].

Alternatively, in the Q-cohomology story, we emphasized the existence of a new Lorentz group on flat space under which Q was a spacetime scalar, and such that a new stress tensor $T'_{\mu\nu}$ was Q-exact. Infinitesimal responses of the holomorphic theory to a curved background are computed by correlation functions of the new stress tensor $T'_{\mu\nu}$, and hence it is also clear that $\mathcal{Z}_{\mathcal{M}}$ (or really any correlation function of operators in the semi-chiral ring) only depends (locally) on the complex structure moduli since Q-exact deformations leave correlation functions unchanged.

Indeed, using a linearized analysis around flat space, one can show that the change $\Delta\mathcal{L}$ to the Lagrangian under infinitesimal variations of the complex structure and Hermitian metric lead to Q-exact deformations of the Lagrangian, except for terms proportional to the holomorphic complex structure moduli [166]. The conclusion is that $\mathcal{Z}_{\mathcal{M}}$ is locally a holomorphic function of the complex structure moduli. The fact that it is only *locally* a holomorphic function emphasizes that we may still have anomalies leading to phase ambiguities of the partition function [183].

Now let us focus on the special case of interest. The rigid SUSY analysis tells us that the only 4-manifolds admitting four supercharges must be locally isometric to $\mathcal{M}_3 \times \mathbb{R}$ where

$\mathcal{M}_3$ is a space of constant curvature [163, 164]. In particular, manifolds with topology of $S^3 \times S^1$ are an example. Without $U(1)_R$ symmetry, a theory with an FZ-multiplet can be placed on a manifold locally isometric to $S^3 \times \mathbb{R}$ while preserving 4 supercharges, but only with a $U(1)_R$ symmetry can those supercharges be made time-independent in the sense that they commute with the generator of translations along the $\mathbb{R}$ direction [163, 164]. Hence, we once again focus from hereout on theories which have a non-anomalous $U(1)_R$ symmetry.

Every complex manifold diffeomorphic to $S^3 \times S^1$ is a Hopf manifold: a quotient of $\mathbb{C}^2 - \{(0,0)\}$ by the free action of a discrete group G [184, 185]. Of primary interest are those manifolds such that $G \cong \mathbb{Z}$, in which case we are left with a *primary Hopf surface*; secondary Hopf surfaces can be obtained by finite group quotients from the primary Hopf surfaces. If the coordinates of $\mathbb{C}^2 - \{(0,0)\}$ are z_1, z_2, we obtain a primary Hopf surface by quotienting

$$(z_1, z_2) \sim (pz_1 + \lambda z_2^m, qz_2) \tag{B.54}$$

where $m \in \mathbb{N}$ and p, q, λ are complex parameters, such that $0 < |p| \leq |q| < 1$ and $(p - q^m)\lambda = 0$ [186]. We will take $\lambda = 0$ from here.

Given all this, it is completely reasonable to ask for the supersymmetric partition function $\mathcal{Z}_{\mathcal{M}}(p, q, u)$ on $\mathcal{M} = S^1 \times S^3$, where p and q label complex structure moduli of the Hopf surface, and u labels line bundle moduli for background gauge fields. By supersymmetric partition function we mean the partition function with $(-1)^F$ inserted, so that fermions have periodic boundary conditions on the S^1. Antiperiodic boundary conditions would explicitly break supersymmetry.

We define the supersymmetric index to be the (refined) Witten index of the theory on $S^3 \times \mathbb{R}$ in Hamiltonian quantization [187, 188, 105, 104, 106, 107]

$$\mathcal{I}(p, q, u) = \mathrm{Tr}_{\mathcal{H}_{S^3}} \left((-1)^F p^{j_\ell + j_r - \frac{R}{2}} q^{j_\ell - j_r - \frac{R}{2}} \zeta^{J_{\mathrm{fl}}} \right) , \tag{B.55}$$

where $2j_\ell$ and $2j_r$ are charges under the Cartan's $M_{+\dot{-}}$ and $\bar{M}_{+-}$ of the spacetime $SU(2)$s, R is $U(1)_R$ charge, and J_{fl} is some collection of $U(1)$ flavour charges associated to a corresponding set of fugacities ζ. Since S^3 is compact, this is a well-defined (signed) count of the states of the theory on S^3. Based on the counting of quantum numbers, the index is well-defined along RG flows (so long as all pertinent symmetries are preserved along the flow). Note that this definition of the index as a refined Witten index makes sense if the theory is superconformal or not. However, if the theory is superconformal, then the counting of states in the supersymmetric index can be interpreted as a counting of $1/(4\mathcal{N})$-BPS local operators by the state-operator correspondence, which we will call the superconformal index. In theories with a superconformal symmetry, this turns the four-dimensional index into a counting problem of gauge invariant local operators in the limit of vanishing coupling.

The relationship between the supersymmetric index to the superconformal index is clear. The relationship between the supersymmetric partition function and the supersymmetric index is that when $\mathcal{M} = S^3 \times S^1$, then

$$\mathcal{Z}_{\mathcal{M}}(p, q, u) = e^{-\mathcal{F}(p,q)} \mathcal{I}(p, q, u) \,, \tag{B.56}$$

where the prefactor $\mathcal{F}$ depends on the a and c central charges of the theory [189], and are interpreted as a supersymmetric Casimir energy [186, 190, 169].