

Co-Higgs Bundles and Poisson Structures

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The relationship between co-Higgs bundles and holomorphic Poisson structures was first studied by Polishchuk, who established a one-to-one correspondence between rank 2 holomorphic co-Higgs fields Φ and Poisson structures on the total space of $\mathbb{P}(V)$. Matviichuk later extended this work by showing how to obtain co-Higgs fields from coisotropic Poisson structures σ_Φ on $\text{Tot}(V)$ and π_Φ on $\text{Tot}(\mathbb{P}(V))$ for rank greater than or equal to 3. For diagonalizable holomorphic co-Higgs fields, Matviichuk introduced the notion of strong integrability as a necessary condition for the existence of these corresponding Poisson structures, focusing his analysis on open sets $\mathcal{U} \subset \mathbb{C}$ and $\mathbb{P}^1$.

We extend these results in several directions. First, we prove that a Poisson structure on $\mathbb{P}(V)$ lifts to a quadratic Poisson structure on V if and only if $p^*\omega_X \otimes p^*\det V^*$ admits a Poisson module structure, extending Matviichuk's result to the non-Calabi-Yau case. For co-Higgs fields over $\mathcal{U} \subset \mathbb{C}^n$, we classify all cases where σ_Φ is integrable, proving that Φ must be either a function multiple of a constant matrix or have only one nonzero column. We also show that for curves of genus $g \geq 1$, all co-Higgs fields are strongly integrable and characterize their explicit forms based on genus.

Finally, we investigate the relationship between stability and Poisson geometry. We establish that Φ -invariant subbundles correspond to Poisson subvarieties of $\mathbb{P}(V)$ and prove that the eigenvariety E coincides with $\text{Zeroes}(\pi_\Phi)$. For rank 2 co-Higgs bundles over $\mathbb{P}^1$, we show the zero locus of π decomposes into two components: a 2:1 cover corresponding to the spectral curve S_Φ and fibers over the zeros of Φ . This decomposition provides geometric criteria for understanding stability. In particular, we prove that the spectral curve is irreducible and the zero locus consists only of the 2:1 cover component if and only if (V, Φ) is stable.

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Dedication

To human kindness - may it be the light that guides our discoveries and the bond that unites our understanding.

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Chapter 1

Introduction

Higgs bundles have been studied intensively, since their introduction by Hitchin [18] and further development by Simpson [31]. A Higgs bundle on a complex manifold X consists of a holomorphic vector bundle V together with a holomorphic Higgs field $\Phi : V \rightarrow V \otimes T_X^*$ taking values in the cotangent bundle T_X^* , satisfying the integrability condition $\Phi \wedge \Phi = 0$. These objects have applications across mathematics and physics, appearing in Poisson geometry, and generalized complex geometry.

Co-Higgs bundles are dual objects to Higgs bundles. A co-Higgs bundle is a pair (V, Φ) where V is a holomorphic vector bundle over X and $\Phi : V \rightarrow V \otimes T_X$ is a co-Higgs field taking values in the tangent bundle rather than the cotangent bundle. They first emerged in the work of Gualtieri [13], and were further investigated by Hitchin [17, 16], Rayan [27, 26, 28], Vicente Colmenares [34], and Boulter-Moraru [8].

The goal of this thesis is to explore the connections between co-Higgs bundles and holomorphic Poisson structures, particularly focusing on how properties of co-Higgs fields influence the properties of the corresponding Poisson geometry. In this thesis we work with quadratic Poisson structures that have coisotropic fibers. That is equivalent to requiring that the pushforward of the Poisson structure σ onto the base manifold X vanishes. Under this condition, σ induces a co-Higgs field $\Phi \in \text{Hom}(V, V \otimes T_X)$. In local coordinates we can express the Poisson structure on the total space of V as

$$\sigma = \sum_{j,k=1}^r g_{ijk} \partial_{x_i} \wedge y_j \partial_{y_k} + \sum_{i,j,k,l=1}^r h_{ijkl} y_i y_j \partial_{y_k} \wedge \partial_{y_l}$$

where $x_1, \dots, x_n$ are coordinates on a small open set $\mathcal{U} \subset X$, $y_1, \dots, y_r$ are fiberwise linear coordinates on $V|_{\mathcal{U}}$, and the functions $g_{ijk}, h_{ijkl} \in \mathcal{O}_{\mathcal{U}}$ depend only on x . Then the terms

$g_{ijk}\partial_{x_i} \wedge y_j\partial_{y_k}$ define the co-Higgs field Φ . In this thesis we mostly focus on the part of σ that induces the co-Higgs field. That is, $\sigma_\Phi = \sum_{j,k=1}^r g_{ijk}\partial_{x_i} \wedge y_j\partial_{y_k}$.

For a rank 2 vector bundle V , Polishchuk [23] established a one-to-one correspondence between co-Higgs fields on V and Poisson structures on the total space of $\mathbb{P}(V)$. Then Matviichuk [21] gave a way to obtain a co-Higgs field from quadratic coisotropic Poisson structures on $\text{Tot}(V)$ and coisotropic Poisson structures on $\text{Tot}(\mathbb{P}(V))$, when the rank of V is greater or equal to 3. However, the converse is not always possible. Therefore Matviichuk introduced the notion of strong integrability (the coefficients $s_i(\Phi)$ of the characteristic polynomial pairwise Poisson commute when viewed as functions on T_X^*) as a necessary condition for the existence of a Poisson structure corresponding to a given co-Higgs field. His work focused on diagonalizable co-Higgs bundles over an open set $\mathcal{U} \subset X$ to obtain Poisson structures on V and over $\mathcal{U} \subset \mathbb{C}$, and $\mathbb{P}^1$ to obtain Poisson structures on $\mathbb{P}(V)$. In this thesis we extend this analysis in several directions while also exploring new connections between co-Higgs bundles and Poisson structures.

We organize this thesis as follows. Chapter 2 establishes the foundational material, introducing co-Higgs bundles and Poisson structures on complex manifolds. We review the spectral correspondence for co-Higgs bundles and discuss the notion of stability, particularly over curves and surfaces.

In chapter 3, we investigate the correspondence between co-Higgs bundles and holomorphic Poisson structures, extending the results from Matviichuk [21]. We focus on three main aspects of this correspondence: The relationship between Poisson structures on the total space of V and the total space of $\mathbb{P}(V)$, the conditions under which co-Higgs fields induce integrable bivectors (Poisson structures), and the characterization of strongly integrable co-Higgs fields over curves of different genera.

In section 3.1, we start by investigating the relationship between quadratic Poisson structures on the total space of V and Poisson structures on the total space of $\mathbb{P}(V)$. While we can always obtain a Poisson structure on $\mathbb{P}(V)$ from a given quadratic Poisson structure on V , the reverse process requires more conditions. Bondal and Polishchuk [7, 23] established that when V is a r dimensional vector space, any Poisson structure on $\mathbb{P}(V)$ lifts to a quadratic Poisson structure on V . Matviichuk [21] extended this by proving that such a lift exists when the total space of V is Calabi-Yau. We further extended these results by proving that when the total space of V is not Calabi-Yau, a Poisson structure on $\mathbb{P}(V)$ lifts to a quadratic Poisson structure on V if and only if $p^*\omega_X \otimes p^*\det V^*$ admits a Poisson module structure in Theorem 3.1.8.

In section 3.2, we investigate the relationship between co-Higgs fields and quadratic Poisson structures σ on the total space of V . Matviichuk showed that it is always possible

to obtain a co-Higgs field given a quadratic Poisson structure σ on the total space of V with coisotropic fibers. However, obtaining a Poisson structure σ from a co-Higgs field requires more conditions. In the case where Φ is diagonalizable, Matviichuk gives a necessary condition called strong integrability, on the co-Higgs field Φ for σ_Φ to be integrable and showed that strong integrability implies that all diagonal entries are scalar multiples of a single vector field. We then extended this result by classifying all possible forms of co-Higgs bundles of rank r over $\mathcal{U} \subset \mathbb{C}$ that induce integrable Poisson structures. In fact, in Theorem 3.2.7, we show that these co-Higgs fields must either be a function multiple of a constant matrix or have only one nonzero column. We also extend this result further to co-Higgs fields over $\mathcal{U} \subset \mathbb{C}^2$ in Theorem 3.2.14.

In section 3.3, we investigate the relationship between traceless co-Higgs fields and Poisson structures π on the total space of $\mathbb{P}(V)$. Polishchuk established a one-to-one correspondence between rank 2 co-Higgs bundles and these Poisson structures π . Matviichuk then showed that it is always possible to obtain a unique traceless co-Higgs field from a given coisotropic Poisson structure π on $\mathbb{P}(V)$. However, going in the opposite direction requires more conditions. He proved that if Φ is a diagonalizable co-Higgs field over an open set $\mathcal{U} \subset \mathbb{C}$ then we can obtain a Poisson structure on $\mathbb{P}(V)$ if and only if Φ is strongly integrable. We generalize this result to the case of a diagonalizable traceless co-Higgs fields over a contractible open set $\mathcal{U} \subset \mathbb{C}^n$ in Theorem 3.3.4. We also provide a classification of co-Higgs fields of rank r that induce Poisson structures on $\mathbb{P}(V)$ in Theorem 3.3.6.

In Section 3.4, we study co-Higgs bundles over smooth projective curves of genus $g \geq 0$, while Matviichuk studied the case $g = 0$, we study the case $g \geq 1$. We begin by investigating the existence of nonzero traceless co-Higgs fields in Proposition 3.4.1 and Theorem 3.4.3. We then provide a classification of ruled surfaces admitting holomorphic Poisson structures, giving explicit descriptions of the co-Higgs fields inducing these structures in Theorem 3.4.4. Finally, we analyze strong integrability conditions and spectral curves in different genera in Theorem 3.4.5 and Proposition 3.4.7.

Finally, in Section 3.5, we study strongly integrable co-Higgs fields over $\mathbb{P}^1 \times \mathbb{P}^1$, giving an explicit characterization of their characteristic polynomials in Proposition 3.5.3.

Chapter 4 explores the relationship between stability of nonzero traceless co-Higgs fields and properties of their induced Poisson structure on $\mathbb{P}(V)$. Studying stability of co-Higgs bundles involves checking for Φ -invariant subbundles, particularly Φ -invariant line bundles. For rank 2 co-Higgs bundles, the absence of Φ -invariant subline bundles is equivalent to stability. While stability can be studied through the spectral curve S_Φ , in this chapter we develop a geometric perspective through the analysis of the zero locus of the induced Poisson structure π . We show that this zero locus, after removing the vertical components

corresponding to zeros of Φ , is irreducible precisely when the co-Higgs bundle is stable, establishing a connection between stability and the geometry of $\text{Zeroes}(\pi)$.

To investigate this, we focus on analyzing the relationships between the Φ -invariant subbundles of V , the eigenvariety E of Φ , and the spectral curve of Φ with the zero set of π_Φ .

In section 4.1, we study the relationship between the eigenvariety E and the zeroes of the Poisson structure π on $\mathbb{P}(V)$. The eigenvariety E of Φ , is defined as the projectivization of the set $\{(x, v) \in X \times V : x \in X, 0 \neq v \in V|_x \text{ such that } \Phi_x(v) \wedge v = 0\}$. First, we prove that E is equal to $\text{Zeroes}(\pi_\Phi)$ in Theorem 4.1.3. We then give a description of $\text{Zeroes}(\pi_\Phi)$ on $\mathbb{P}(V)$ as the disjoint union of two distinct proper subvarieties of $\mathbb{P}(V)$: The preimage $\rho^{-1}(D)$, representing points in $\mathbb{P}(V)$ over the divisor D of zeroes of Φ in X , and a subvariety $B \subset X$ in Theorem 4.1.6. Moreover, if L is a Φ -invariant subline bundle we have that the induced Poisson subvariety $\mathbb{P}(L)$ is contained in the eigenvariety E (Proposition 4.1.10).

In section 4.2, we study Φ -invariant subbundles of V and their relationship with Poisson structures. We prove that if $L \subset V$ is a Φ -invariant subbundle, then $\mathbb{P}(L)$ is a Poisson subvariety of $(\mathbb{P}(V), \pi)$ in Theorem 4.2.1. This correspondence between Φ -invariant subbundles and Poisson subvarieties provides a way to understand non-stability through the Poisson geometry.

Finally, in section 4.3, we study the connection between the Poisson subvarieties, and the zero locus of the Poisson structure π with the spectral curve S_Φ . We start by proving that the existence of a Φ -invariant subline bundle L of V is equivalent to the reducibility of the spectral curve S_Φ , providing a geometric criterion for non-stability (Theorem 4.2.5). Then for rank 2 co-Higgs bundles over $\mathbb{P}^1$, we give a geometric description of $\text{Zeroes}(\pi)$, showing it decomposes into two components: a 2:1 cover of X and fibers over the zeros of Φ . This 2:1 cover corresponds to the spectral curve S_Φ . The structure of these components provides a geometric understanding of the stability of the co-Higgs bundle. For instance, we show that when (V, Φ) is stable over $\mathbb{P}^1$, the spectral curve must be irreducible and the zero locus consists only of the 2 : 1 cover with no fiber components (Theorem 4.3.3). Therefore, the presence of vertical fiber components in the zero locus indicates the existence of Φ -invariant subbundles, indicating potential non-stability. Therefore, this geometric description provides a way to understand stability through the induced Poisson structures.

Chapter 2

Preliminaries

2.1 Co-Higgs Bundles

In this section, we introduce co-Higgs bundles over complex projective manifolds. We then discuss some known results about semistable co-Higgs bundles over curves and surfaces, focusing on the spectral correspondence and the classification of rank 2 semistable co-Higgs bundles over the projective line. Co-Higgs bundles are dual objects to Higgs bundles, while a Higgs field is a section of the endomorphism bundle tensored with the cotangent bundle, a co-Higgs field instead uses the tangent bundle. They first emerged in the work of Gualtieri in [13]. Subsequently, co-Higgs bundles were further investigated by Hitchin in [17, 16] and Rayan in [27, 26, 28]. The study of co-Higgs bundles has implications in many areas of mathematics, including complex geometry and mathematical physics, see for instance [13, 14, 20].

Definition 2.1.1. Let X be a complex manifold with tangent bundle T_X . A **co-Higgs bundle** is a pair (V, Φ) where $V \rightarrow X$ is a holomorphic vector bundle and Φ is a holomorphic section of $\text{End}(V) \otimes T_X$ satisfying the integrability condition: $\Phi \wedge \Phi = 0$ in $H^0(X, \text{End}(V) \otimes \bigwedge^2 T_X)$. Here Φ is called the co-Higgs field of V .

Note that we can think of global sections in $H^0(X, \text{End}(V) \otimes T_X)$ as elements in the vector space $H^0(X, \text{Hom}(V, V \otimes T_X))$ because

$$\text{End}(V) \otimes T_X \simeq (V^* \otimes V) \otimes T_X \simeq V^* \otimes (V \otimes T_X) \simeq \text{Hom}(V, V \otimes T_X).$$

The wedge product $\wedge$ on $H^0(\text{End}(V) \otimes T_X)$ is defined by combining two distinct operations: For the $\text{End}(V)$ component, it acts as the commutator and for the T_X component, it behaves as the standard wedge product on vector fields.

The condition $\Phi \wedge \Phi$ is always equal to zero when X is a curve, although this condition is non-trivial in higher dimensions. For instance, when X is a surface, and $\Psi, \Phi \in H^0(\text{End}(V) \otimes T_X)$, we can express Ψ and Φ in local coordinates as:

$$\Psi = \Psi_1 \partial_1 + \Psi_2 \partial_2, \quad \Phi = \Phi_1 \partial_1 + \Phi_2 \partial_2$$

where $\Psi_i, \Phi_i \in \text{End}(V)$ for $i = 1, 2$, and $\partial_i = \frac{\partial}{\partial x_i}$.

The wedge product $\Psi \wedge \Phi$ can then be computed as follows:

$$\begin{aligned} \Psi \wedge \Phi &= (\Psi_1 \partial_1 + \Psi_2 \partial_2) \wedge (\Phi_1 \partial_1 + \Phi_2 \partial_2) \\ &= [\Psi_1, \Phi_2] \partial_1 \wedge \partial_2 + [\Psi_2, \Phi_1] \partial_2 \wedge \partial_1 \\ &= ([\Psi_1, \Phi_2] - [\Psi_2, \Phi_1]) \partial_1 \wedge \partial_2 \end{aligned}$$

Here, $[-, -]$ denotes the commutator in $\text{End}(V)$, defined as $[A, B] = AB - BA$ for $A, B \in \text{End}(V)$.

Example 2.1.2. Let $V = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ be the rank 2 holomorphic vector bundle over $\mathbb{P}^1$. The co-Higgs field $\Phi \in H^0(\mathbb{P}^1, \text{End}(V) \otimes T_{\mathbb{P}^1})$ is given by

$$\Phi = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \otimes \partial_x$$

where x is a coordinate on $\mathbb{P}^1$ and ∂_x is the corresponding vector field.

A **morphism** between two co-Higgs bundles (V, Φ) and (V', Φ') on a complex manifold X is a homomorphism of vector bundles $\psi : V \rightarrow V'$ such that the following diagram commutes

$$\begin{array}{ccc} V & \xrightarrow{\psi} & V' \\ \Phi \downarrow & & \downarrow \Phi' \\ V \otimes T_X & \xrightarrow{\psi \otimes \text{Id}} & V' \otimes T_X \end{array}$$

In other words, ψ must satisfy the condition

$$\Phi' \circ \psi = (\psi \otimes \text{Id}) \circ \Phi.$$

An **isomorphism** between two co-Higgs bundles (V, Φ) and (V', Φ') on a complex manifold X is a morphism of co-Higgs bundles $\psi : V \rightarrow V'$ such that ψ is an isomorphism of vector bundles. Furthermore, (V, Φ) and (V', Φ') are isomorphic if and only if there exist an automorphism ψ of V such that $\psi \Phi \psi^{-1} = \Phi'$.

Example 2.1.3. Let $V = \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. Let us define the co-Higgs fields $\Phi, \Phi' : V \rightarrow V \otimes T_{\mathbb{P}^1}$ by the matrices

$$\Phi = \begin{pmatrix} 0 & x \\ 1 & 0 \end{pmatrix}$$

and

$$\Phi' = \begin{pmatrix} 0 & -x \\ -1 & 0 \end{pmatrix},$$

where x is the affine coordinate on $\mathbb{P}^1$. To show that (V, Φ) and (V, Φ') are isomorphic, we need to find an automorphism ψ of V such that $\psi\Phi\psi^{-1} = \Phi'$. Let $\psi : V \rightarrow V$ be the automorphism given by the matrix

$$\psi = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},$$

where i is the imaginary unit. Then we have

$$\psi\Phi\psi^{-1} = \begin{pmatrix} 0 & -x \\ -1 & 0 \end{pmatrix} = \Phi',$$

and hence (V, Φ) and (V, Φ') are isomorphic co-Higgs bundles on $\mathbb{P}^1$.

2.1.1 Stable co-Higgs bundles

Co-Higgs bundles, like Higgs bundles, can be classified based on their stability properties. Hitchin introduced the condition of stability in [18] for Higgs bundles on a curve. This condition generalizes to Higgs bundles over Kähler varieties, and consequently, to projective varieties of arbitrary dimension. We will therefore state the version of this stability condition for co-Higgs bundles.

Definition 2.1.4. Let X be a projective variety. A co-Higgs bundle (V, Φ) is called **semistable** if for every proper Φ -invariant subsheaf $W \subset V$; i.e., $\Phi(W) \subseteq W \otimes T_X$ we have

$$\mu(W) \leq \mu(V)$$

where $\mu(V) = \frac{\deg(V)}{\text{rk}(V)}$ is the slope of V . If the inequality is strict ($<$), the co-Higgs bundle is called **stable**.

We can recover the usual condition of stability for vector bundles by setting $\Phi = 0$ because any vector bundle V can be viewed as a co-Higgs bundle with zero co-Higgs field. Moreover, V is stable if and only if $(V, 0)$ is a stable co-Higgs bundle.

We can also decompose Φ uniquely into a traceless part Φ_0 and a trace part $\frac{1}{r}\text{tr}(\Phi) \cdot \text{Id}_V$ where $r = \text{rk}(V)$. Explicitly, we have $\Phi_0 = \Phi - \frac{1}{r}\text{tr}(\Phi) \cdot \text{Id}_V$. Therefore, the co-Higgs bundle (V, Φ) is stable if and only if (V, Φ_0) is stable. We may assume without loss of generality that Φ is traceless when (V, Φ) is stable. It is also important to note that stable co-Higgs bundles are simple, that is, every endomorphism V that commutes with Φ is a constant multiple of 1_V .

The following lemma tells us that (semi)stability is not affected by tensoring the co-Higgs bundle with a line bundle.

Lemma 2.1.5. *If (V, Φ) is a stable co-Higgs bundle, then $(V \otimes L, \Phi \otimes \text{Id}_L)$ is stable for any line bundle L .*

Proof. Suppose $\mathcal{F} \subset V \otimes L$ is a proper $\Phi \otimes \text{Id}_L$ -invariant subsheaf. Then $\mathcal{F} \otimes L^*$ is a proper Φ -invariant subsheaf of V . By stability of (V, Φ) we have $\mu(\mathcal{F} \otimes L^*) = \mu(\mathcal{F}) - \mu(L) < \mu(V)$. Then, $\mu(\mathcal{F}) < \mu(V \otimes L)$. Hence $(V \otimes L, \Phi \otimes \text{Id}_L)$ is stable. $\square$

2.1.2 Spectral Correspondence

The spectral correspondence for co-Higgs bundles provides a geometric interpretation of these objects. The following is a summary of section 2.2 in [21].

Given a co-Higgs field Φ on V , we can endow V with the structure of a $\text{Sym}(T_X^*)$ -module as follows: For any $\alpha \in T_X^*$, we define an endomorphism $\Phi_\alpha \in \text{End}(V)$ by:

$$\Phi_\alpha(v) = (\text{id}_V \otimes \alpha)(\Phi(v)) \quad \text{for } v \in V$$

This defines the action of elements of T_X^* on V . We extend this to an action of $\text{Sym}(T_X^*)$ on V by defining:

$$(\alpha_1 \cdots \alpha_r) \cdot v = \Phi_{\alpha_1} \circ \cdots \circ \Phi_{\alpha_r}(v)$$

for $\alpha_1, \dots, \alpha_r \in T_X^*$ and $v \in V$. The integrability condition $\Phi \wedge \Phi = 0$ ensures that this action is well-defined, i.e., it respects the symmetry relations in $\text{Sym}(T_X^*)$.

Conversely, given a vector bundle V with a $\text{Sym}(T_X^*)$ -module structure, we can construct a co-Higgs field $\Phi \in \text{Hom}(V, V \otimes T_X)$ by defining

$$\langle \Phi(v), \alpha \rangle = \alpha \cdot v$$

for any $v \in V$ and $\alpha \in T_X^*$. The properties of the $\text{Sym}(T_X^*)$ -module structure ensure Φ satisfies the integrability condition. Thus, (V, Φ) forms a co-Higgs bundle.

This correspondence extends to a relationship between locally free $\mathcal{O}_X$ -modules with $\text{Sym}(T_X^*)$ -module structure and certain vector bundles on $\mathbb{P}(T_X \oplus \mathcal{O}_X)$. The following theorem adapted for Higgs fields was proved in [6] when X is a curve, and in [30] in general.

Theorem 2.1.6. *Co-Higgs bundles on X are in one-to-one correspondence with vector bundles on $\mathbb{P}(T_X \oplus \mathcal{O}_X)$ whose support is disjoint from the divisor $D = \mathbb{P}(T_X \oplus \mathcal{O}_X) \setminus T_X$.*

This theorem provides a method for constructing co-Higgs bundles. One can embed or immerse a variety $i : \Sigma \rightarrow T_X$ such that it intersects each fiber T_x at a constant number of points (counting multiplicity), and equip Σ with a line bundle L . The vector bundle i_*L on $T_X \subset \mathbb{P}(T_X \oplus \mathcal{O}_X)$ then defines a co-Higgs bundle $(V = p_*(i_*L), \Phi)$ on X .

An alternative construction method involves spectral data $(p : \Sigma \rightarrow X, L, v)$, where p is a branched cover of smooth manifolds, L is a line bundle over Σ , and v is a vector field on Σ . The pushforward $(Tp \circ v)_*L$ defines a vector bundle on $T_X \subset \mathbb{P}(T_X \oplus \mathcal{O}_X)$, yielding a co-Higgs bundle (V, Φ) on X by Theorem 2.1.6. The co-Higgs field Φ on $V = p_*L$ can be recovered directly: for $\alpha \otimes v \in T_X^* \otimes V$, $\Phi(\alpha \otimes v) = \langle p^*\alpha, v \rangle s \in V$, where $s \in V$ is viewed as a section of L .

Spectral variety. Let (V, Φ) be a rank r co-Higgs bundle on X . At each point $x \in X$, the co-Higgs field $\Phi_x : V_x \rightarrow V_x \otimes T_{X,x}$ has $T_{X,x}$ -valued eigenvalues, which we call eigenvector fields. Let $v_1, \dots, v_r$ be these eigenvector fields counted with multiplicity. The characteristic polynomial coefficients $s_k(\Phi) \in H^0(X, S^k T_X)$ are given by the elementary symmetric functions:

$$\begin{aligned} s_1(\Phi) &= v_1 + v_2 + \dots + v_r, \\ s_2(\Phi) &= \sum_{1 \leq i < j \leq r} v_i v_j, \\ s_3(\Phi) &= \sum_{1 \leq i < j < k \leq r} v_i v_j v_k, \\ &\vdots \\ s_r(\Phi) &= v_1 v_2 \dots v_r. \end{aligned}$$

The spectral variety S_Φ of the co-Higgs field Φ is the subvariety of T_X cut out by the section:

$$\chi_\Phi(\theta) = \det(\theta - \Phi) = \theta^r - s_1(\Phi)\theta^{r-1} + \dots + (-1)^{r-1}s_{r-1}(\Phi)\theta + (-1)^r s_r(\Phi)$$

of $p^*S^rT_X$, where $p : T_X \rightarrow X$ is the natural projection, and θ is the tautological section of p^*T_X .

2.1.3 Semistable co-Higgs bundles over a curve

In this section, we present a summary of Rayan's work over curves, see [27, 26]. Note that a stable co-Higgs bundle on a curve of genus greater than one is simply a stable vector bundle with zero co-Higgs field. In the case of genus one curves (elliptic curves), the notion of a co-Higgs bundle coincides with that of a Higgs bundle, as the canonical bundle is trivial. Therefore, we only need to analyze the case $g = 0$, i.e., the projective line $\mathbb{P}^1$.

Recall that a vector bundle on the projective line can be decomposed into a direct sum of line bundles. This decomposition is unique up to the permutation of the summands as we can see in the following theorem.

Theorem 2.1.7 (Birkhoff-Grothendieck, [12]). *Every holomorphic vector bundle V on $\mathbb{P}^1$ is isomorphic to a direct sum of holomorphic line bundles. More precisely,*

$$V \cong \bigoplus_{i=1}^r \mathcal{O}_{\mathbb{P}^1}(a_i)$$

for some unique integers $a_1 \geq a_2 \geq \dots \geq a_r$, where $r = \text{rk}(V)$.

The integers $a_1, \dots, a_r$ are called the *splitting type* of V . On $\mathbb{P}^1$ the integrability condition $\Phi \wedge \Phi = 0$ is automatically satisfied. Therefore, a co-Higgs bundle is simply a pair (V, Φ) with $\Phi \in H^0(X, \text{End}(V) \otimes T_X)$.

Rayan classified semistable rank 2 co-Higgs bundles over $\mathbb{P}^1$ in [27, 26]. He starts by noticing that a semistable rank 2 co-Higgs bundle of even degree on $\mathbb{P}^1$ has underlying bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}$ or $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ and a semistable rank 2 co-Higgs bundle of odd degree on $\mathbb{P}^1$ has underlying bundle $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, up to tensoring them by an appropriate line bundle. Then, Rayan proves that for odd degree, the moduli space is a 6-dimensional subvariety of $\text{Tot}(\mathcal{O}_{\mathbb{P}^1}(2)) \times H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4))$ which is a universal elliptic curve and for even degree, he gives an explicit description for the fibers of the Hitchin map sending a co-Higgs bundle to the characteristic polynomial of its Higgs field.

Spectral curves and the Hitchin correspondence over $\mathbb{P}^1$. Now, we provide a brief overview of the spectral curve and the Hitchin correspondence for co-Higgs bundles over $\mathbb{P}^1$.

Let $\mathcal{M}_{\mathbb{P}^1}^{\text{co}}(2)$ denote the moduli space of traceless rank 2 semistable co-Higgs bundles over $\mathbb{P}^1$. The Hitchin map is defined as

$$\begin{aligned} H : \mathcal{M}_{\mathbb{P}^1}^{\text{co}}(2) &\rightarrow H^0(\mathbb{P}^1, T_{\mathbb{P}^1}^2) \\ (V, \Phi) &\mapsto \text{char } \Phi \end{aligned}$$

where $\text{char } \Phi$ denotes the characteristic polynomial of Φ . Note that $T_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(2)$, and consequently, $T_{\mathbb{P}^1}^2 \cong \mathcal{O}_{\mathbb{P}^1}(4)$. Assuming Φ is trace-free, we have $\text{char } \Phi = \eta^2(y) + \det \Phi$ where η is the tautological section of the pullback of $\mathcal{O}_{\mathbb{P}^1}(2)$ to its total space. Thus, we can identify $\text{char } \Phi$ with $\det \Phi \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4))$. This map is surjective and proper.

For a generic section $\rho \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4))$ and $(V, \Phi) \in H^{-1}(\rho)$, the spectral curve X_ρ associated to $\text{char } \Phi = \rho$ is a smooth subvariety of $\text{Tot}(\mathcal{O}_{\mathbb{P}^1}(2))$ defined by

$$\eta^2(y) + \det \Phi = 0$$

The genericity of ρ ensures that X_ρ is irreducible. Moreover, X_ρ is a double cover of $\mathbb{P}^1$ under the projection $\pi : X_\rho \rightarrow \mathbb{P}^1$. For $p \in \mathbb{P}^1$, we have

$$\pi^{-1}(p) = \{b \in \text{Tot}(\mathcal{O}_{\mathbb{P}^1}(2)) : \text{char } \Phi(p)(b) = 0\}$$

Thus, $\pi^{-1}(p)$ corresponds to the eigenvalues of $\Phi(p)$. The ramification of X_ρ occurs at the four points in $\mathbb{P}^1$ where ρ vanishes. By the Riemann-Hurwitz formula, X_ρ has genus one and is therefore an elliptic curve. For generic $\rho \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4))$, the fiber $H^{-1}(\rho)$ is isomorphic to $\text{Pic}(X_\rho)$, which is also an elliptic curve. This isomorphism constitutes the Hitchin correspondence, which we describe as follows: Given a line bundle $L \rightarrow X_\rho$, the pushforward $\pi_* L$ yields a rank 2 bundle V over $\mathbb{P}^1$. The push forward of the multiplication map $- \otimes \eta : L \rightarrow L \otimes \pi^* \mathcal{O}_{\mathbb{P}^1}(2)$ provides a Higgs field for V with $\text{char } \Phi = \rho$. The irreducibility of X_ρ ensures the stability of (V, Φ) .

Conversely, for $(V, \Phi) \in H^{-1}(\rho)$, we can construct a line bundle over X_ρ as follows: Consider $\pi^* V$, a rank 2 bundle over X_ρ , and $\pi^* \Phi : \pi^* V \rightarrow \pi^* V \otimes \mathcal{O}_{\mathbb{P}^1}(2)$. The eigenline bundle L_e is defined as:

$$L_e = \ker(\pi^* \Phi - \eta \text{Id} : \pi^* V \rightarrow \pi^*(V \otimes \mathcal{O}_{\mathbb{P}^1}(2))) \otimes R$$

where R is the ramification divisor of X_ρ . This construction yields the desired element in $\text{Pic}(X_\rho)$, completing the Hitchin correspondence.

2.1.4 Semistable co-Higgs bundles over surfaces

In this subsection we present a brief summary of Rayan's and Vicente's results for surfaces in [27, 26, 34].

The following theorem suggest that the existence of non-trivial semistable co-Higgs bundles of rank 2 can be found at the non-positive end of the Kodaira spectrum.

Theorem 2.1.8 ([26]). *Let X be a surface of general type or be birational to a K3 surface. Then, if (V, Φ) is a stable, traceless rank 2 co-Higgs bundle on X with $c_1(V) = 0$, we must have that $\Phi = 0$.*

Co-Higgs bundles over Hirzebruch surfaces. The following is a brief summary of the results obtained in [34].

We define the n -th Hirzebruch surface, $\mathbb{F}_n$, as

$$\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)) \xrightarrow{\pi} \mathbb{P}^1$$

where $n \geq 0$. By the Birkhoff-Grothendieck theorem, all vector bundles over $\mathbb{P}^1$ are decomposable, justifying this choice for the definition of $\mathbb{F}_n$.

Let C_0 denote the negative section of $\pi : \mathbb{F}_n \rightarrow \mathbb{P}^1$ with $C_0^2 = -n$, and F a general fiber of π . The tangent bundle $T_{\mathbb{F}_n}$ of $\mathbb{F}_n$ is the rank 2 decomposable bundle $T_{\mathbb{F}_n} = \mathcal{O}(2F) \oplus \mathcal{O}(2C_0 + nF)$. For line bundles over $\mathbb{P}^1 \times \mathbb{P}^1$, we use the notation $\mathcal{O}(b, a) := \text{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(b) \otimes \text{pr}_2^* \mathcal{O}_{\mathbb{P}^1}(a)$ where pr_i denotes the projection onto the i -th copy of $\mathbb{P}^1$. Note that by the Künneth formula we have $H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a, b)) = H^0(\mathbb{P}^1, \mathcal{O}(a)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(b))$. Therefore any $\Phi \in H^0(\text{End}_0 V \otimes T_{\mathbb{F}_n})$ can be written as $\Phi = \Phi_1 + \Phi_2$ with $\Phi_1 \in H^0(\text{End}_0 V(2F))$ and $\Phi_2 \in H^0(\text{End}_0 V(2C_0 + nF))$. Locally, we can express Φ as

$$\Phi = \Phi_1 + \Phi_2 = \begin{pmatrix} A_1 & B_1 \\ C_1 & -A_1 \end{pmatrix} + \begin{pmatrix} A_2 & B_2 \\ C_2 & -A_2 \end{pmatrix}$$

where A_i, B_i, C_i are complex valued functions for $i = 1, 2$.

The integrability condition $\Phi \wedge \Phi = 0$ is equivalent to

$$B_1 C_2 = C_1 B_2$$

$$A_1 B_2 = B_1 A_2$$

$$C_1 A_2 = A_1 C_2$$

It is important to note that for points in $\mathbb{F}_n$ where either Φ_1 or Φ_2 has distinct eigenvalues, Φ_1 and Φ_2 must share the same eigenspaces as the following lemma suggest.

Lemma 2.1.9. *Let M_1 be an $n \times n$ complex matrix with distinct eigenvalues. If M_2 is a matrix such that $[M_1, M_2] = 0$, then M_1 and M_2 have the same eigenspaces.*

Proof. Let λ be an eigenvalue of M_1 and $v \neq 0$ be a vector in the corresponding eigenspace. Then $0 = 0v = [M_1, M_2]v = M_1M_2v - M_2M_1v = M_1(M_2v) - \lambda(M_2v)$. Therefore, $M_1(M_2v) = \lambda(M_2v)$, which implies that M_2v is also in the eigenspace of M_1 corresponding to λ . Consequently, M_2v must be a scalar multiple of v , i.e., $M_2v = \lambda'v$ for some $\lambda' \in \mathbb{C}$. This shows that every eigenvector of M_1 is also an eigenvector of M_2 , and since M_1 has distinct eigenvalues, its eigenvectors span $\mathbb{C}^n$. Thus, M_1 and M_2 share the same eigenspaces. $\square$

Vicente also proved in [34] that the moduli space $\mathcal{M}^{\text{co}}(-F, 0)$ of rank 2 stable co-Higgs bundles over $\mathbb{P}^1 \times \mathbb{P}^1$ with first Chern class $-F$ and second Chern class 0 is a 6-dimensional smooth variety isomorphic to the moduli space $\mathcal{M}_{\mathbb{P}^1}^{\text{co}}(-1)$ of rank 2 stable co-Higgs bundles of degree -1 over $\mathbb{P}^1$.

Spectral Surfaces over $\mathbb{P}^1 \times \mathbb{P}^1$. Now, we provide a brief description of the spectral surface over $\mathbb{P}^1 \times \mathbb{P}^1$ given by Vicente in [34].

The Hitchin map for the moduli space of semistable rank 2 co-Higgs bundles over $\mathbb{P}^1 \times \mathbb{P}^1$ is defined as

$$\begin{aligned} H : \mathcal{M}^{\text{co}}(2) &\rightarrow H^0(S^2(T_{\mathbb{P}^1 \times \mathbb{P}^1})) \\ (V, \Phi) &\mapsto \text{char } \Phi \end{aligned}$$

where $\text{char } \Phi = \eta^2(y) + \det \Phi$, with η being the tautological section of the pullback of $T_{\mathbb{P}^1 \times \mathbb{P}^1}$ to its total space. Explicitly, for $(V, \Phi) \in \mathcal{M}^{\text{co}}(2)$, we have:

$$H(V, \Phi) = (\det \Phi_1, -2A_1A_2 - 2B_1C_2, \det \Phi_2) \in H^0(\mathcal{O}(4, 0) \oplus \mathcal{O}(2, 2) \oplus \mathcal{O}(0, 4))$$

Note that the Hitchin map is not surjective, as $\rho_{1,2}^2 = 4\rho_1\rho_2$ for $H(V, \Phi) = (\rho_1, \rho_{1,2}, \rho_2)$.

Definition 2.1.10. For $(V, \Phi) \in \mathcal{M}^{\text{co}}(2)$, the spectral surface S_ρ associated to $\rho = \text{char } \Phi$ is defined by

$$S_\rho = \{y \in \text{Tot}(T_{\mathbb{P}^1 \times \mathbb{P}^1}) : \text{char } \Phi(y) = \eta^2(y) + \det \Phi(\theta(y)) = 0\}$$

where $\theta : T_{\mathbb{P}^1 \times \mathbb{P}^1} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. We endow S_ρ with the projection to $\mathbb{P}^1 \times \mathbb{P}^1$ inherited from the natural projection of $\text{Tot}(T_{\mathbb{P}^1 \times \mathbb{P}^1})$.

For $y = (y_1, y_2) \in \text{Tot}(T_{\mathbb{P}^1 \times \mathbb{P}^1})$ and $\rho = (\rho_1, \rho_{1,2}, \rho_2)$, the equations defining S_ρ are:

$$\begin{aligned}\eta_1^2(y_1) + \rho_1 &= 0 \\ \eta_2^2(y_2) + \rho_2 &= 0 \\ 2\eta_1(y_1)\eta_2(y_2) + \rho_{1,2} &= 0\end{aligned}$$

Here, η_1 and η_2 are the tautological sections of the pullbacks of $\mathcal{O}(2, 0)$ and $\mathcal{O}(0, 2)$ to their respective total spaces.

Proposition 2.1.11. *The first two equations of S_ρ define a reducible surface in $\text{Tot}(T_{\mathbb{P}^1 \times \mathbb{P}^1})$, and the third equation cuts out a two-dimensional subvariety of this surface.*

Proof. Multiply the first equation by $\eta_2^2(y_2)$ and use the second equation along with $\rho_{1,2}^2 = 4\rho_1\rho_2$ to obtain:

$$\begin{aligned}0 &= (\eta_1^2(y_1) + \rho_1)\eta_2^2(y_2) \\ &= \eta_1^2(y_1)\eta_2^2(y_2) - \rho_1\rho_2 \\ &= (\eta_1(y_1)\eta_2(y_2) + \rho_{1,2}/2)(\eta_1(y_1)\eta_2(y_2) - \rho_{1,2}/2)\end{aligned}$$

The third equation of S_ρ appears as one of these factors. □

Theorem 2.1.12. *For generic ρ , the spectral surface S_ρ is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$.*

Proof. Let λ_i^1 and λ_i^2 be the eigenvalues of Φ_1 and Φ_2 respectively at an unramified point $p \in \mathbb{P}^1 \times \mathbb{P}^1$. The integrability of Φ implies that Φ_1 and Φ_2 share eigenspaces, so we may assume the eigenspace of λ_1^j equals that of λ_2^j for $j = 1, 2$.

We claim that $(\lambda_1^i, \lambda_2^j) \in S_\rho$ if and only if $i = j$. This is equivalent to showing that the points of S_ρ above unramified points of $\mathbb{P}^1 \times \mathbb{P}^1$ are pairs of eigenvalues of Φ_1 and Φ_2 sharing the same eigenspace. Since Φ_1 and Φ_2 commute, λ_1^j and λ_2^j sharing eigenspaces is equivalent to $\lambda_1^j\lambda_2^j$ being an eigenvalue of $\Phi_1\Phi_2$. Thus, it must satisfy

$$\eta_1^2(\lambda_1^j)\eta_2^2(\lambda_2^j) - \text{tr}(\Phi_1\Phi_2)\eta_1(\lambda_1^j)\eta_2(\lambda_2^j) + \det(\Phi_1\Phi_2) = 0$$

This reduces to $2\eta_1(\lambda_1^j)\eta_2(\lambda_2^j) - \rho_{1,2} = 0$ which is precisely the third equation of S_ρ . Therefore, the points in S_ρ above $p \in \mathbb{P}^1 \times \mathbb{P}^1$ are $(\lambda_1^1, \lambda_2^1)$ and $(\lambda_1^2, \lambda_2^2)$, confirming that S_ρ is a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$. □

2.2 Poisson Structures on Complex Manifolds

Poisson geometry is the study of manifolds equipped with Poisson brackets and has connections with classical and quantum mechanics, dynamics, integrable systems, and gauge theory, among others [1, 33, 35]. In this section, we introduce the notion of Poisson structures on complex manifolds. We begin by defining Poisson brackets and Poisson manifolds, and provide examples in the context of complex geometry. We then discuss the correspondence between Poisson brackets and bivectors, and introduce the concept of a Poisson map and a Poisson submanifold. For more details, we refer the reader to [9, 23, 24].

2.2.1 Holomorphic Poisson structures

We begin by introducing holomorphic Poisson structures on a complex manifold or algebraic variety X .

Definition 2.2.1. Let X be a complex manifold, and $\mathcal{O}_X$ its sheaf of holomorphic functions. A **holomorphic Poisson structure** on X is a $\mathbb{C}$ -bilinear operation

$$\{-, -\} : \mathcal{O}_X \times \mathcal{O}_X \rightarrow \mathcal{O}_X$$

satisfying the following axioms for all $f, g, h \in \mathcal{O}_X$

1. Skew-symmetry: $\{f, g\} = -\{g, f\}$
2. Leibniz rule: $\{f, gh\} = \{f, g\}h + g\{f, h\}$
3. Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$

A complex manifold X equipped with a holomorphic Poisson structure is called a **(holomorphic) Poisson manifold**.

Example 2.2.2. Let $X = \mathbb{C}^{2n}$ with complex coordinates $(z_1, \dots, z_n, w_1, \dots, w_n)$. Define a Poisson bracket on X by

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial z_i} \frac{\partial g}{\partial w_i} - \frac{\partial f}{\partial w_i} \frac{\partial g}{\partial z_i} \right).$$

Then $(X, \{-, -\})$ is a Poisson manifold.

The **Hamiltonian vector field** associated to a function $f \in \mathcal{O}_X$ is the vector field X_f defined by

$$X_f(g) = \{f, g\}$$

for all $g \in \mathcal{O}_X$.

A holomorphic Poisson structure can also be characterized as a bivector. To describe this correspondence, we need to first introduce the Nijenhuis - Schouten bracket [22, 29].

Definition 2.2.3. The **Nijenhuis - Schouten bracket** is a bilinear operation

$$[-, -] : \wedge^k T_X \times \wedge^m T_X \longrightarrow \wedge^{k+m-1} T_X$$

defined by

$$[u_1 \wedge \dots \wedge u_k, v_1 \wedge \dots \wedge v_m] = \sum_{i=1}^k \sum_{j=1}^m (-1)^{i+j} [u_i, v_j] u_1 \wedge \dots \widehat{u}_i \wedge \dots \wedge u_k \wedge v_1 \wedge \dots \widehat{v}_j \wedge \dots \wedge v_m$$

for $u_i, v_j \in T_X$, and $[f, u] = -[u, f] = \iota_{df}(u)$, for $f \in \mathcal{O}_X, u \in \wedge T_X$.

Remark 2.2.4. If $k = 1 = m$ then the Nijenhuis - Schouten bracket coincides with the Lie bracket of vector fields. In the particular case of two vector fields u_1 and u_2 , we have

$$[u_1 \wedge u_2, u_1 \wedge u_2] = -2[u_1, u_2] \wedge u_2 \wedge u_1.$$

Proposition 2.2.5. A holomorphic Poisson structure on X is equivalent to a global holomorphic bivector field

$$\sigma \in \Gamma(X, \wedge^2 T_X)$$

satisfying $[\sigma, \sigma] = 0$, where $[-, -]$ denotes the Nijenhuis - Schouten bracket.

Proof. Given a Poisson bracket $\{-, -\}$, we define σ by the relation $\{g, h\} = \sigma(dg \wedge dh)$ for all $g, h \in \mathcal{O}_X$. Here, $\sigma(dg \wedge dh)$ should be understood as the evaluation of the bivector σ on the two-form $dg \wedge dh$.

Conversely, given σ , we define $\{g, h\} = \sigma(dg \wedge dh)$. This definition automatically satisfies the skew-symmetry and Leibniz rule properties of a Poisson bracket. The condition $[\sigma, \sigma] = 0$ is equivalent to the Jacobi identity for $\{-, -\}$. This equivalence can be verified by a direct calculation in local coordinates. $\square$

In general, if we write $\sigma = \sum_{i < j} \sigma^{ij} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ in local coordinates $(x_1, \dots, x_n)$, then

$$\{f, g\} = \sum_{i < j} \sigma^{ij} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_i} \right)$$

Some authors also denote $\sigma(dg \wedge dh)$ by $\langle \sigma, dg \wedge dh \rangle$. We will use both notations interchangeably for convenience when needed for computations.

Definition 2.2.6. The **zero set** of a Poisson structure σ on X is the set of points $p \in X$ where $\sigma(p) = 0$. In local coordinates, this is equivalent to all

$$\sigma^{ij}(p) = 0.$$

Another way to define a Poisson bracket is by specifying a skew-symmetric bundle map $\sigma^\sharp : T_X^* \rightarrow T_X$. Indeed, the relationship between the bundle map $\sigma^\sharp$ and the bivector σ is given by $\sigma^\sharp(\alpha) = \iota_\alpha(\sigma) = \langle \sigma, \alpha \rangle$, $\alpha \in T_X^*$, where ι_α denotes the interior product with α .

Example 2.2.7. Consider the complex manifold $X = \mathbb{C}^2$ with coordinates (z, w) . Define a skew-symmetric bundle map $\sigma^\sharp : T_X^* \rightarrow T_X$ by $\sigma^\sharp(dz) = \frac{\partial}{\partial w}$, and $\sigma^\sharp(dw) = -\frac{\partial}{\partial z}$. This corresponds to the Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial z} \frac{\partial g}{\partial w} - \frac{\partial f}{\partial w} \frac{\partial g}{\partial z}.$$

A **symplectic form** on X is a closed non-degenerate 2-form $\omega \in \Gamma(X, \wedge^2 T_X^*)$. A complex manifold equipped with a symplectic form is called a **symplectic manifold**. The definition of a Poisson bracket as a bundle map helps us to connect Poisson geometry with symplectic geometry in the following way: A bivector field σ is called **non-degenerate** if the bundle map $\omega^b : T_X^* \rightarrow T_X$ is an isomorphism. There is a bijective correspondence between Poisson structures π and non-degenerate closed 2-forms ω .

Proposition 2.2.8. *Let X be an even-dimensional complex manifold and $\omega \in \Gamma(X, \wedge^2 T_X^*)$ a non-degenerate 2-form. Then there exists a non-degenerate bivector σ on X such that*

$$\sigma^\sharp = (\omega^b)^{-1}.$$

where $\omega^b : T_X^* \rightarrow T_X$ is defined by $\omega^b(v) = \iota_v \omega$, $v \in T_X$. Moreover, σ is Poisson if and only if ω is closed.

Example 2.2.9. Let $X = \mathbb{C}^2$ with coordinates (z, w) . Define a skew-symmetric bundle map $\sigma^\sharp : T_X^* \rightarrow T_X$ by $\sigma^\sharp(dz) = \frac{\partial}{\partial w}$, and $\sigma^\sharp(dw) = -\frac{\partial}{\partial z}$. This corresponds to the Poisson bracket

$$\{f, g\} = \frac{\partial f}{\partial z} \frac{\partial g}{\partial w} - \frac{\partial f}{\partial w} \frac{\partial g}{\partial z}.$$

Moreover, $\sigma^\sharp$ is an isomorphism, so this Poisson structure is symplectic. The corresponding symplectic form is $\omega = dz \wedge dw$.

The next proposition gives a characterization of the definition of the zero set of π in terms of the bundle map.

Proposition 2.2.10. *A point $x \in X$ is a zero of the Poisson structure σ if and only if $\sigma^\sharp : T_{X,x}^* \rightarrow T_{X,x}$ is the zero map.*

Proof. Let $x \in X$ such that $\sigma_x = 0$, where σ_x is the value of the bivector field at the point x and let $\alpha \in T_x^*X$ be any co-vector at x . By definition of $\sigma^\sharp$ we have $\sigma_x^\sharp(\alpha) = \iota_\alpha \sigma_x$ and since $\sigma_x = 0$ we get $\sigma_x^\sharp(\alpha) = 0$. $\square$

The **rank** of a Poisson structure σ at a point $x \in X$ is the rank of the linear map $\sigma_x^\sharp : T_x^*X \rightarrow T_xX$. The Poisson structure is said to have **constant rank** if its rank is the same at every point.

Definition 2.2.11. A point $x \in X$ where $\sigma_x^\sharp = 0$ is called a **singular point** of the Poisson structure.

Theorem 2.2.12 (Weinstein's Splitting Theorem). *Let (X, σ) be a Poisson manifold and $x \in X$. There exist local coordinates $(p_1, \dots, p_k, q_1, \dots, q_k, y_1, \dots, y_l)$ near x such that*

$$\sigma = \sum_{i=1}^k \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q_i} + \sum_{i,j=1}^l f_{ij}(y) \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial y_j}$$

where $f_{ij}(y)$ are functions vanishing at $y = 0$.

Definition 2.2.13. A map $\varphi : (X, \{-, -\}_X) \rightarrow (Y, \{-, -\}_Y)$ between Poisson manifolds is called a **Poisson map** if

$$\{\varphi^* f, \varphi^* g\}_X = \varphi^* \{f, g\}_Y$$

for all $f, g \in \mathcal{O}_Y$.

Example 2.2.14. Let $X = \mathbb{C}^2$ with coordinates (z_1, z_2) and $Y = \mathbb{C}$ with coordinate w . Define the Poisson structures on X and Y by $\{z_1, z_2\}_X = z_1 z_2$, and $\{w, w\}_Y = 0$. Consider the map $\phi(z_1, z_2) = z_1$. For any $f, g \in \mathcal{O}_{\mathbb{C}}$, we have $f \circ \phi = f(z_1)$, $g \circ \phi = g(z_1)$, and $\{f \circ \phi, g \circ \phi\}_X = \{f(z_1), g(z_1)\}_X = \left(\frac{\partial f}{\partial z_1} \frac{\partial g}{\partial z_2} - \frac{\partial f}{\partial z_2} \frac{\partial g}{\partial z_1} \right) \{z_1, z_2\}_X$. Given that $f(z_1)$ and $g(z_1)$ do not depend on z_2 , we have

$$\{f(z_1), g(z_1)\}_X = \left(\frac{\partial f}{\partial z_1} \cdot 0 - 0 \cdot \frac{\partial g}{\partial z_1} \right) \{z_1, z_2\}_X = 0.$$

Therefore, since $\{f, g\}_Y = 0$, the projection $\phi(z_1, z_2) = z_1$ is a Poisson map from X to Y .

2.2.2 Poisson Subvarieties

In this subsection, we begin by defining Poisson subvarieties and then we provide equivalent conditions for a submanifold to be a Poisson subvariety.

Definition 2.2.15. Let (X, σ) be a Poisson manifold. A subvariety Y of X is called **Poisson** if the inclusion map $i : Y \hookrightarrow X$ is a Poisson morphism, where Y is equipped with the induced Poisson structure. In other words, the Poisson structure $\{-, -\}_Y$ of Y satisfies

$$\{f|_Y, g|_Y\}_Y = (\{f, g\}_X)|_Y$$

for all functions $f, g \in \mathcal{O}_X$.

The following lemma provides equivalent conditions for a submanifold to be a Poisson subvariety.

Lemma 2.2.16. *Let $Y \subset X$ be a submanifold of X with ideal sheaf*

$$I_Y = \{f \in \mathcal{O}_X : f|_Y = 0\}.$$

The following are equivalent:

1. Y is a Poisson submanifold
2. I_Y is a Poisson ideal. That is, $\{\mathcal{O}_X, I_Y\} \subset I_Y$.
3. $\sigma^\#(T_X^*|_Y) \subseteq T_Y$

Proof. First, assume that Y is a Poisson submanifold of X . We need to show that I_Y is a Poisson ideal, i.e., $\{\mathcal{O}_X, I_Y\} \subset I_Y$. Let $f \in I_Y$ and $g \in \mathcal{O}_X$. Since Y is a Poisson submanifold, we have that $\{g, f\}|_Y = \{g|_Y, f|_Y\}_Y$, where $\{\cdot, \cdot\}_Y$ is the induced Poisson bracket on Y . Since $f \in I_Y$, we have $f|_Y = 0$. Therefore, $\{g, f\}|_Y = \{g|_Y, 0\}_Y = 0$, which implies that $\{g, f\} \in I_Y$. Thus, $\{\mathcal{O}_X, I_Y\} \subset I_Y$.

Now, let us assume that I_Y is a Poisson ideal. We need to show that $\sigma^\#(T^*X|_Y) \subseteq T_Y$. Let $\alpha \in T^*X|_Y$. Since α is a one-form restricted to Y , around any point on Y , we can locally express it as $\alpha = df$ for some $f \in \mathcal{O}_X$.

Now we have $\sigma^\#(df) = \iota_{df}(\sigma) = \sigma(df) = \langle df, \sigma \rangle$. Let $g \in I_Y$. Then by definition of the Poisson bracket and the ideal I_Y we have $0 = \{f, g\}|_Y = \sigma(df \wedge dg)|_Y = \sigma^\#(df)(dg)|_Y = \iota_{df}\sigma(dg|_Y)$. Therefore, the vector field $\iota_{df}(\sigma) = \sigma^\#(df)$ is tangent to Y . Finally, to show that condition (3) implies condition (1) we define $\sigma_Y^\# : T_Y^* \rightarrow T_Y$ by $\sigma_Y^\# = \sigma^\#(\beta)$ for any $\beta \in T_Y^*$. This is well-defined since $\sigma^\#(T^*X|_Y) \subseteq T_Y$. Since $\sigma^\#$ is skew-symmetric and satisfies the Jacobi identity, it follows that $\sigma_Y^\#$ also does. Therefore, $\sigma_Y^\#$ defines a Poisson structure σ_Y on Y . $\square$

An important property of Poisson structures is that their zero sets are themselves Poisson subvarieties. This is established in the following proposition.

Proposition 2.2.17. *Let (X, σ) be a complex Poisson manifold. The zero set $\text{Zeroes}(\sigma)$ of σ is then a Poisson subvariety of X .*

Proof. To prove that $\text{Zeroes}(\sigma)$ is a Poisson subvariety, we show that its ideal sheaf $I_{\text{Zeroes}(\sigma)}$ is a Poisson ideal, that is, $\{\mathcal{O}_X, I_{\text{Zeroes}(\sigma)}\} \subset I_{\text{Zeroes}(\sigma)}$. Let $f \in I_{\text{Zeroes}(\sigma)}$ and $g \in \mathcal{O}_X$ be arbitrary. Therefore, it suffices to show that $\{f, g\} \in I_{\text{Zeroes}(\sigma)}$, which is equivalent to proving that $\{f, g\}|_{\text{Zeroes}(\sigma)} = 0$. Let $x \in \text{Zeroes}(\sigma)$ so that $\sigma_x = 0$. By definition of the Poisson bracket, we have $\{f, g\}(x) = \sigma_x(df \wedge dg) = 0$, implying that $\{f, g\}|_{\text{Zeroes}(\sigma)} = 0$. Consequently, $\{f, g\} \in I_{\text{Zeroes}(\sigma)}$. $\square$

To illustrate the above proposition, we provide the following example.

Example 2.2.18. Let $X = \mathbb{C}^2$ with coordinates (z, w) . Consider the holomorphic Poisson structure on X given by $\sigma = (z^2 + w^2) \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial w}$. This defines a Poisson bracket on holomorphic functions $f, g \in \mathcal{O}_X$ as follows:

$$\{f, g\} = (z^2 + w^2) \left(\frac{\partial f}{\partial z} \frac{\partial g}{\partial w} - \frac{\partial f}{\partial w} \frac{\partial g}{\partial z} \right).$$

The set of zeroes of σ is $\text{Zeroes}(\sigma) = \{(z, w) \in \mathbb{C}^2 : z^2 + w^2 = 0\}$. The ideal $I_{\text{Zeroes}(\sigma)}$ is generated by the function $h(z, w) = z^2 + w^2$. Let $f \in I_{\text{Zeroes}(\sigma)}$ and $g \in \mathcal{O}_X$ be arbitrary. Then $f = h \cdot k$ for some holomorphic function k . We need to show that $\{f, g\} \in I_{\text{Zeroes}(\sigma)}$.

$$\begin{aligned} \{f, g\} &= \{h \cdot k, g\} \\ &= h \cdot \{k, g\} + k \cdot \{h, g\} \\ &= h \cdot \{k, g\} + k \cdot (z^2 + w^2) \left(2z \frac{\partial g}{\partial w} - 2w \frac{\partial g}{\partial z} \right) \\ &= h \cdot \left(\{k, g\} + k \cdot \left(2z \frac{\partial g}{\partial w} - 2w \frac{\partial g}{\partial z} \right) \right). \end{aligned}$$

The result is again in the ideal $I_{\text{Zeroes}(\sigma)}$ as it is a multiple of h . Then $\text{Zeroes}(\sigma)$ is indeed a Poisson subvariety of $\mathbb{C}^2$.

2.2.3 Quadratic Poisson structures

In the context of holomorphic vector bundles, we introduce a special class of holomorphic Poisson structures known as quadratic Poisson structures.

Definition 2.2.19. A **quadratic Poisson structure** on the total space of a holomorphic vector bundle V over a complex manifold X is a holomorphic Poisson structure σ on the total space of V that is invariant under the fiberwise scaling action of $\mathbb{C}^*$.

This structure typically exhibits terms of quadratic order in the coordinates of the fibers. To be precise, let $(x_1, \dots, x_n)$ be local coordinates on an open subset $\mathcal{U}$ of X and $(y_1, \dots, y_r)$ be linear coordinates on the fibers of the vector bundle projection $p : V \rightarrow X$. The quadratic Poisson bivector field σ on V can be expressed as

$$\sigma = \sum_{i,j=1}^n f_{ij} \partial_{x_i} \wedge \partial_{x_j} + \sum_{i=1}^n \sum_{j,k=1}^r g_{ijk} \partial_{x_i} \wedge y_j \partial_{y_k} + \sum_{i,j,k,l=1}^r h_{ijkl} y_i y_j \partial_{y_k} \wedge \partial_{y_l}, \quad (2.1)$$

where $f_{ij}, g_{ijk}, h_{ijkl}$ are holomorphic functions on $\mathcal{U}$. The presence of terms $y_i y_j \partial_{y_k} \wedge \partial_{y_l}$ characterizes the quadratic nature of σ .

A submanifold N of a holomorphic Poisson manifold (M, σ) is called **coisotropic** if $\sigma^\sharp(TN^\circ) \subset TN$, where $T_x N^\circ = \{\xi \in T_x^* M : \xi|_{T_x N} = 0\}$ for all $x \in N$. In this thesis, we are interested in quadratic Poisson structures on holomorphic vectors bundles V (and their projectivizations $\mathbb{P}(V)$) with respect to which the fibres of the vector bundle map $p : V \rightarrow X$ are coisotropic. In terms of local coordinates, these are quadratic Poisson structures σ for which $f_{ij} = 0$ for all $i, j = 1, \dots, n$ in (2.1):

Proposition 2.2.20. *Let $p : V \rightarrow X$ be a vector bundle, and let σ be a quadratic Poisson structure on V . Then, the fibers of p are coisotropic if and only if $f_{ij} = 0$ for all $i, j = 1, \dots, n$ in the local expression (2.1) of σ .*

Proof. Let N be a fiber of p and $x \in N$. In terms of the local coordinates of (2.1), we have $T_x N = \text{Span}_{\mathbb{C}}\{\partial_{y_1}|_x, \dots, \partial_{y_r}|_x\}$ and $T_x N^\circ = \text{Span}_{\mathbb{C}}\{dx_1|_x, \dots, dx_n|_x\}$. Note that for any $\xi = \sum_{i=1}^n a_i dx_i + \sum_{j=1}^r b_j dy_j \in T_x^* V$, we have $\xi|_{T_x N} = 0$ if and only if $\xi = \sum_{i=1}^n a_i dx_i$. Therefore, $\sigma^\#(T_x N^\circ) \subset T_x N$ if and only if $f_{ij} = 0$ for all $i, j = 1, \dots, n$. $\square$

In the sequel, we therefore work with holomorphic Poisson structures on V that are locally given by

$$\sigma = \sum_{i=1}^n \sum_{j,k=1}^r g_{ijk} \partial_{x_i} \wedge y_j \partial_{y_k} + \sum_{i,j,k,l=1}^r h_{ijkl} y_i y_j \partial_{y_k} \wedge \partial_{y_l}, \quad (2.2)$$

where g_{ijk}, h_{ijkl} are holomorphic functions on $\mathcal{U}$.

2.2.4 Poisson Modules

In this subsection, we define Poisson modules and explore their properties.

Definition 2.2.21. A **Poisson connection** on a vector bundle V over a Poisson manifold (X, σ) is a map $\nabla : V \rightarrow V \otimes T_X$ that satisfies the Leibniz rule:

$$\nabla(fs) = -s \otimes \sigma^\#(df) + f \nabla(s)$$

for $f \in \mathcal{O}_X$ and $s \in \Gamma(V)$, where $\sigma^\# : T_X^* \rightarrow T_X$ is the canonical bundle map induced by the Poisson structure.

This definition allows us to define a bracket operation between functions and sections of a vector bundle V :

$$\{f, s\} = \nabla_{df} s, \quad f \in \mathcal{O}_X, s \in \Gamma(V).$$

With this bracket operation, we can introduce the concept of flatness for Poisson connections.

Definition 2.2.22. A Poisson connection is called **flat** if one has

$$\{\{f, g\}, s\} = \{f, \{g, s\}\} - \{g, \{f, s\}\}, \quad f, g \in \mathcal{O}_X, s \in \Gamma(V).$$

A vector bundle V with a flat Poisson connection ∇ is called a **Poisson module**.

Example 2.2.23. The canonical line bundle ω_X of a Poisson manifold (X, σ) carries the canonical Poisson module structure defined by $\{f, \mu\} = \text{Lie}_{\sigma^\sharp(df)} \mu$, $f \in \mathcal{O}_X$, $\mu \in \omega_X$, where Lie denotes the Lie derivative.

An important property of Poisson modules is that they can be combined to form new Poisson modules.

Proposition 2.2.24. *Let V_1 and V_2 be two Poisson modules with connections $\nabla^1 : V_1 \rightarrow V_1 \otimes T_X$ and $\nabla^2 : V_2 \rightarrow V_2 \otimes T_X$, respectively. The tensor product $V_1 \otimes V_2$ is then a Poisson module with connection given by*

$$\nabla(s \otimes s') = \nabla^1(s) \otimes s' + s \otimes \nabla^2(s'),$$

where $s \in \Gamma(V_1)$, and $s' \in \Gamma(V_2)$.

Example 2.2.25. Let L be a line bundle over a complex manifold X and V be a rank r vector bundle over X . We have the following:

- i) If $L^{\otimes r}$ admits a Poisson module structure, then L also admits a Poisson module structure.
- ii) A Poisson module structure on V uniquely defines a Poisson module structure on V^* .

Proposition 2.2.26 ([23]). *Let $p : L \rightarrow X$ be a line bundle over a Poisson manifold (X, σ) . Then, there is a one-to-one correspondence between the Poisson module structures on L^* with respect to σ and $\mathbb{C}^*$ -invariant Poisson structures on the total space of L rendering the projection map p Poisson.*

Chapter 3

Poisson Structures and Co-Higgs bundles

In this chapter, we study the correspondence between co-Higgs fields and holomorphic Poisson structures. Specifically, given a holomorphic vector bundle V on a complex manifold X , any holomorphic Poisson structure on $\mathbb{P}(V)$ induces a co-Higgs field on V . The problem is to determine when the converse is true, namely, when do co-Higgs fields on V lead to holomorphic Poisson structures on $\text{Tot}(V)$ and $\text{Tot}(\mathbb{P}(V))$? This was first studied by Polishchuk in [23] in the case when V has rank 2, where it is proved that there is a one-to-one correspondence between holomorphic Poisson structures on $\mathbb{P}(V)$ and co-Higgs fields on V . However, Matviichuk showed in [21] that the converse is not always possible when the rank of V is greater or equal to 3. Matviichuk nonetheless introduced the notion of strong integrability as a necessary condition for the existence of a Poisson structure corresponding to a given diagonalizable co-Higgs field [21].

In section 3.1 we study the relationship between quadratic Poisson structures on the total space of V and Poisson structures π on the total space of $\mathbb{P}(V)$. In fact, if we have a quadratic Poisson structure σ on $\text{Tot}(V)$ then by pushing forward σ along the projection map we obtain a Poisson structure π on $\text{Tot}(\mathbb{P}(V))$. A natural question to ask is if it is possible to obtain a quadratic Poisson structure σ on $\text{Tot}(V)$ from a given Poisson structure π on $\text{Tot}(\mathbb{P}(V))$. In this section we address this question.

Matviichuk proved that for diagonalizable co-Higgs fields, strong integrability forces all diagonal entries to be scalar multiples of a single vector field. In section 3.2, we extend this result by giving a classification of rank r co-Higgs bundles over $U \subset \mathbb{C}$ that induce Poisson structures (integrable bivectors). We then generalize this classification to co-Higgs

fields over $U \subset \mathbb{C}^2$.

Matviichuk's work focused on how to obtain Poisson structures π on the total space of $\mathbb{P}(V)$ from diagonalizable traceless co-Higgs bundles over an open set $\mathcal{U} \subset \mathbb{C}$. In section 3.3, we generalize this result to higher-dimensional base spaces. Concretely, we extend Matviichuk's theorem (Corollary 3.4.7 in [21]) to diagonalizable traceless co-Higgs fields on vector bundles over contractible open sets $\mathcal{U} \subset \mathbb{C}^n$, providing a better understanding of the relationship between co-Higgs fields and Poisson structures in higher dimensions. Moreover, we describe explicitly the form of the co-Higgs fields that induce a Poisson structure on the total space of $\mathbb{P}(V)$.

We are also interested in studying the existence and properties of holomorphic Poisson structures associated with co-Higgs fields on curves of different genera. In section 3.4 we start by studying the existence of nonzero traceless co-Higgs fields over curves of genus $g \geq 0$ and then we analyze strong integrability conditions and spectral curves in different genera. Finally, in section 3.5 we give a characterization of the characteristic polynomial of strongly integrable co-Higgs fields over the 0-th Hirzebruch surface $\mathbb{P}^1 \times \mathbb{P}^1$.

In this chapter, we denote by σ a Poisson structure on the total space of V and by π a Poisson structure on the total space of $\mathbb{P}(V)$. For a smooth complex manifold X , let $(x_1, \dots, x_n)$ be local coordinates. We denote by $\partial_{x_i} = \frac{\partial}{\partial x_i}$, $i = 1, \dots, n$ the local basis of the tangent sheaf T_X over $\mathcal{O}_X$, and by $\xi_i = dx_i$, $i = 1, \dots, n$ the local basis of the cotangent sheaf T_X^* over $\mathcal{O}_X$.

3.1 Poisson structures on V and $\mathbb{P}(V)$

In this subsection, we study the relationship between quadratic Poisson structures on the total space of a holomorphic vector bundle V and Poisson structures on the total space of its projectivization $\mathbb{P}(V)$.

Let V be a rank r holomorphic vector bundle on the complex manifold X . Recall that a quadratic Poisson structure σ on V can be written locally as

$$\sigma = \sum_{i,j=1}^n f_{ij} \partial_{x_i} \wedge \partial_{x_j} + \sum_{i=1}^n \sum_{j,k=1}^r g_{ijk} \partial_{x_i} \wedge y_j \partial_{y_k} + \sum_{i,j,k,l=1}^r h_{ijkl} y_i y_j \partial_{y_k} \wedge \partial_{y_l},$$

where $x_1, \dots, x_n$ are coordinates on a small open set $\mathcal{U} \subset X$, $y_1, \dots, y_r$ are fiberwise linear coordinates on $V|_{\mathcal{U}}$, and the functions $f_{ij}, g_{ijk}, h_{ijkl} \in \mathcal{O}_{\mathcal{U}}$ depend only on x .

If we want to obtain a Poisson structure π on $\mathbb{P}(V)$ from a given quadratic Poisson structure σ on V , we just need to pushforward σ along the projection $V \setminus 0 \rightarrow \mathbb{P}(V)$.

Example 3.1.1. Let $V = \mathbb{C}^3$ be a vector space (so that V is a rank-3 vector bundle over a point). Consider a quadratic Poisson structure σ on V given by

$$\sigma = \tilde{h}_{01}\partial_{y_0} \wedge \partial_{y_1} + \tilde{h}_{20}\partial_{y_2} \wedge \partial_{y_0} + \tilde{h}_{12}\partial_{y_1} \wedge \partial_{y_2},$$

where y_0, y_1, y_2 are linear coordinates on V and $\tilde{h}_{ij}$ are quadratic homogeneous polynomials in y_0, y_1, y_2 . Then its projectivization on $\mathbb{P}^2 = \mathbb{P}(V)$ is given by the holomorphic bivector

$$\pi = h\partial_{u_1} \wedge \partial_{u_2},$$

where we are working in the affine chart where $y_0 \neq 0$ so that $u_1 = y_1/y_0, u_2 = y_2/y_0$, and

$$h(u_1, u_2) = \frac{y_0\tilde{h}_{12} + y_1\tilde{h}_{20} + y_2\tilde{h}_{01}}{y_0^3}$$

may be a polynomial of degree three in u_1, u_2 . Note that while the original Poisson structure involves quadratic polynomials $\tilde{h}_{ij}$, the projectivized structure may increase the degree. For instance, if $\sigma = 3y_1y_2\partial_{y_0} \wedge \partial_{y_1}$, then $\pi = 3u_1u_2^2\partial_{u_1} \wedge \partial_{u_2}$.

A natural question to ask is whether or not one can reverse the process. To be precise, does any holomorphic Poisson structure on $\mathbb{P}(V)$ lift to a quadratic Poisson structure on V ? In the case where X is a point so that V is just an r -dimensional vector space over $\mathbb{C}$ and $\mathbb{P}(V) = \mathbb{P}^{r-1}$, the answer is yes:

Theorem 3.1.2 ([7, 23]). *Let V be an r -dimensional complex vector space. Then any Poisson structure π on $\mathbb{P}(V)$ admits a lift to a quadratic Poisson structure σ on V .*

Example 3.1.3. Let $V = \mathbb{C}^{n+1}$ and let π be a Poisson structure on $\mathbb{P}^n$ given by

$$\pi = \sum_{i < j} \frac{f_{ij}}{y_0^3} \partial_{u_i} \wedge \partial_{u_j},$$

where $u_i = y_i/y_0$ are affine coordinates on the chart where $y_0 \neq 0$, and f_{ij} are homogeneous polynomials of degree 3. Then, there exists a canonical lift of π to a $\mathbb{C}^*$ -invariant Poisson structure σ on $\mathbb{C}^{n+1}$ given by

$$\sigma = \sum_{i < j} \frac{1}{y_0} \left(f_{ij} - \frac{y_i}{n+1} \sum_{k=1}^n \frac{\partial f_{kj}}{\partial y_k} + \frac{y_j}{n+1} \sum_{k=1}^n \frac{\partial f_{ki}}{\partial y_k} \right) \partial_{y_i} \wedge \partial_{y_j}$$

where $y_0, y_1, \dots, y_n$ are linear coordinates on $\mathbb{C}^{n+1}$ [23, Theorem 12.2].

In the general case, one needs to impose further conditions on V to ensure that any holomorphic Poisson structure π on $\mathbb{P}(V)$ lifts to a quadratic Poisson structure σ on V . Matviichuk proves in Theorem 3.1.4 that we can obtain a quadratic Poisson structure on V if the total space of the vector bundle V is Calabi-Yau, that is, $\omega_{\text{Tot}(V)} \simeq \mathcal{O}_{\text{Tot}(V)}$.

Theorem 3.1.4 ([21], Theorem 3.1.5). *Let V be a vector bundle over X whose total space is Calabi-Yau, that is, $\omega_{\text{Tot}(V)} \cong \mathcal{O}_{\text{Tot}(V)}$. Then, any holomorphic Poisson structure π on $\mathbb{P}(V)$ lifts to a quadratic Poisson structure σ on V . If X is compact, such a lift is unique.*

Remark 3.1.5. In Theorem 3.1.4, Matviichuk actually proves that the lift obtained is a unimodular quadratic Poisson structure. We refer the reader interested in unimodular holomorphic Poisson structures to sections 2.1 and 3.3 of Matviichuk's thesis [21].

Matviichuk also considers the case where the total space of V is not Calabi-Yau when $X = \mathbb{P}^1$, which occurs if and only if $\deg V \neq -2$. He obtains the following result:

Theorem 3.1.6 ([21], Corollary 4.0.7). *Let V be a holomorphic vector bundle on $\mathbb{P}^1$ with $\deg(V) \neq -2$ and denote by p the induced projection $\mathbb{P}(V) \rightarrow \mathbb{P}^1$. Then, a Poisson structure π on $\mathbb{P}(V)$ lifts to a quadratic Poisson structure on V if and only if the line bundle $p^*\mathcal{O}_{\mathbb{P}^1}(1)$ admits a Poisson module structure with respect to π .*

We can generalize Theorem 3.1.6 to the case where V is a holomorphic vector bundle on any complex manifold X and the total space of V is not Calabi-Yau. Before stating our theorem, we need the following result that describes the anticanonical bundle of V .

Proposition 3.1.7. *Let V be a rank r holomorphic vector bundle on the complex manifold X and denote by p the induced projection $\mathbb{P}(V) \rightarrow X$. Then,*

$$\omega_{\mathbb{P}(V)}^{-1} = p^* \det(V) \otimes p^* \omega_X^{-1} \otimes \mathcal{O}_{\mathbb{P}(V)}(r).$$

Proof. Let us consider the following short exact sequences:

$$0 \rightarrow T_{\mathbb{P}(V)/X} \rightarrow T_{\mathbb{P}(V)} \rightarrow p^*T_X \rightarrow 0, \quad (3.1)$$

$$0 \rightarrow \mathcal{O}_{\mathbb{P}(V)} \rightarrow p^*V \otimes \mathcal{O}_{\mathbb{P}(V)}(1) \rightarrow T_{\mathbb{P}(V)/X} \rightarrow 0, \quad (3.2)$$

where equation (3.2) is just the relative Euler sequence. From (3.1), we have

$$\begin{aligned} \omega_{\mathbb{P}(V)}^{-1} &:= \det T_{\mathbb{P}(V)} = \det T_{\mathbb{P}(V)/X} \otimes p^*(\det T_X) \\ &= \det T_{\mathbb{P}(V)/X} \otimes p^* \omega_X^{-1}. \end{aligned}$$

And from the exact sequence (3.2), we have that

$$\det T_{\mathbb{P}(V)/X} = \det(p^*V \otimes \mathcal{O}_{\mathbb{P}(V)}(1)) = p^* \det(V) \otimes \mathcal{O}_{\mathbb{P}(V)}(r).$$

Therefore, $\omega_{\mathbb{P}(V)}^{-1} = p^* \det(V) \otimes \mathcal{O}_{\mathbb{P}(V)}(r) \otimes p^* \omega_X^{-1}$. $\square$

We can now prove the main result of this section:

Theorem 3.1.8. *Let V be a holomorphic vector bundle on the complex manifold X and p be the induced projection $\mathbb{P}(V) \rightarrow X$. If the total space of V is not Calabi - Yau, then a Poisson structure π on $\mathbb{P}(V)$ lifts to a quadratic Poisson structure σ on V if and only if the bundle $p^* \omega_X \otimes p^* \det V^*$ admits a Poisson module structure with respect to π .*

Proof. Let π be a Poisson structure on $\mathbb{P}(V)$. Suppose that π lifts to a quadratic Poisson structure σ on V . This means, in particular, that σ vanishes along the zero section of V and therefore lifts to the blow-up $\tilde{V}$ of V along its zero section. Recall that $\tilde{V}$ is isomorphic to the total space of $\mathcal{O}_{\mathbb{P}(V)}(-1)$. A direct computation in local coordinates then shows that this isomorphism takes the Poisson structure on $\tilde{V}$ to a $\mathbb{C}^*$ -invariant Poisson structure on $\mathcal{O}_{\mathbb{P}(V)}(-1)$ that projects onto π . Conversely, suppose that $\mathcal{O}_{\mathbb{P}(V)}(-1)$ admits a $\mathbb{C}^*$ -invariant Poisson structure that projects onto π . Note that the complement of the zero section in $\mathcal{O}_{\mathbb{P}(V)}(-1)$ is isomorphic to the complement of the zero section in V , and this isomorphism takes the $\mathbb{C}^*$ -invariant Poisson structure on $\mathcal{O}_{\mathbb{P}(V)}(-1)$ to a quadratic Poisson structure on V that is a lift of π away from the zero sections. Since any quadratic Poisson structure on the complement of the zero section extends uniquely over the zero section itself, this proves that a Poisson structure π on $\mathbb{P}(V)$ admits a unique lift to a quadratic Poisson structure σ on V if and only if the total space of $\mathcal{O}_{\mathbb{P}(V)}(-1)$ admits a $\mathbb{C}^*$ -invariant Poisson structure projecting onto π . The proof of the theorem therefore boils down to showing that the total space of $\mathcal{O}_{\mathbb{P}(V)}(-1)$ admits a $\mathbb{C}^*$ -invariant Poisson structure projecting onto π if and only if the bundle $p^* \omega_X \otimes p^* \det V^*$ admits a Poisson module structure with respect to π .

By Proposition 2.2.26, the total space of $\mathcal{O}_{\mathbb{P}(V)}(-1)$ admits a $\mathbb{C}^*$ -invariant Poisson structure projecting onto π if and only if $\mathcal{O}_{\mathbb{P}(V)}(1)$ admits a Poisson module structure with respect to π . And referring to section 2.2.4, $\mathcal{O}_{\mathbb{P}(V)}(1)$ admits a Poisson module structure with respect to π if and only if $\mathcal{O}_{\mathbb{P}(V)}(1)^{\otimes r}$ does; moreover, the anticanonical bundle $\omega_{\mathbb{P}(V)}^{-1}$ has a canonical Poisson module structure with respect to π . By Proposition 3.1.7, we have that

$$\mathcal{O}_{\mathbb{P}(V)}(1)^{\otimes r} = \mathcal{O}_{\mathbb{P}(V)}(r) \simeq p^* \omega_X \otimes p^* \det V^* \otimes \omega_{\mathbb{P}(V)}^{-1}.$$

Thus, $\mathcal{O}_{\mathbb{P}(V)}(1)^{\otimes r}$ admits a Poisson module structure with respect to π if and only if $p^* \omega_X \otimes p^* \det V^*$ does so. Therefore, $\mathcal{O}_{\mathbb{P}(V)}(-1)$ admits a $\mathbb{C}^*$ -invariant Poisson structure projecting

onto π if and only if $p^*\omega_X \otimes p^*\det V^*$ admits a Poisson module structure with respect to π . $\square$

Remark 3.1.9. Note that when $X = \mathbb{P}^1$, we have $\omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$ and $\det V = \mathcal{O}_{\mathbb{P}^1}(m)$ for some integer m . Consequently, $p^*\omega_{\mathbb{P}^1} \otimes p^*\det V^* = p^*\mathcal{O}_{\mathbb{P}^1}(-(m+2))$ and the statement of Theorem 3.1.8 becomes “a Poisson structure π on $\mathbb{P}(V)$ lifts to a quadratic Poisson structure σ on V if and only if the bundle $p^*\mathcal{O}_{\mathbb{P}^1}(-(m+2))$ admits a Poisson module structure with respect to π .” However, $p^*\mathcal{O}_{\mathbb{P}^1}(-(m+2))$ admits a Poisson module structure with respect to π if and only if $p^*\mathcal{O}_{\mathbb{P}^1}(1)$, and we thus recover the statement of Theorem 3.1.6.

3.2 Co-Higgs bundles and Poissons structures on V

In this subsection, we discuss the relationship between quadratic Poisson structures on the total space of a holomorphic vector bundle V on X and co-Higgs fields $\Phi \in H^0(X, \text{End } V \otimes T_X)$ on V . Starting with a quadratic Poisson structure σ on V with coisotropic fibers, we always obtain a co-Higgs field on Φ on V . However, not every co-Higgs field Φ on V gives rise to a quadratic Poisson structure σ on V . Nonetheless, there are necessary and sufficient conditions that ensure in some cases that σ is integrable, which we show below. We begin by describing how to obtain a co-Higgs field from a quadratic Poisson structure.

Let $p : V \rightarrow X$ be a holomorphic vector bundle of rank r on the n -dimensional complex manifold X . Let σ be a quadratic Poisson structure on the total space of V with respect to which the fibres of p are coisotropic. In local coordinates, we saw in section 2.2.3 that

$$\sigma = \sum_{i=1}^n \sum_{j,k=1}^r g_{ijk} \partial_{x_i} \wedge y_j \partial_{y_k} + \sum_{i,j,k,l=1}^r h_{ijkl} y_i y_j \partial_{y_k} \wedge \partial_{y_l}, \quad (3.3)$$

where $x_1, \dots, x_n$ are coordinates on a small open set $\mathcal{U} \subset X$, $y_1, \dots, y_r$ are fiberwise linear coordinates on $V|_{\mathcal{U}}$, and the functions $g_{ijk}, h_{ijkl} \in \mathcal{O}_{\mathcal{U}}$ depend only on the variables $x_1, \dots, x_n$. Under this assumption, the terms $g_{ijk} \partial_{x_i} \wedge y_j \partial_{y_k}$ in (3.3) define a co-Higgs field Φ_{σ} on V as follows.

Recall that $\mathbb{C}^*$ -invariant vector fields $\sum_{j,k} a_{jk} y_j \partial_{y_k}$ on V tangent to the fibres of p are in one-to-one correspondence with sections $\phi = (a_{jk})^T$ of $\text{End } V$. We set

$$\Phi = \sum_{i=1}^n \sum_{j,k} g_{ijk}(x) \frac{\partial}{\partial x_i} \otimes y_j \frac{\partial}{\partial y_k}.$$

Note that

$$\langle \Phi, dx_i \rangle = \sum_{j,k} g_{ijk}(x) y_j \frac{\partial}{\partial y_k}$$

for all $i = 1, \dots, n$, which corresponds to the section of $\text{End } V$ whose matrix representation with respect to the charts giving the fibrewise linear coordinates $y_1, \dots, y_r$ is $(g_{ijk}(x))^T$.

This can also be done in a coordinate-free way:

Lemma 3.2.1 ([21], Lemma 3.2.1). *Let $p : V \rightarrow X$ be a holomorphic vector bundle and σ be a quadratic Poisson structure on the total space of V with respect to which the fibers of p are coisotropic. Then, σ induces a co-Higgs field $\Phi \in H^0(X, \text{End } V \otimes T_X)$ on X , given by the formula*

$$\langle \Phi, \alpha \rangle = \sigma^\sharp(p^* \alpha), \quad \text{for any } \alpha \in T_X^*.$$

Remark 3.2.2. Here, $\sigma^\sharp : T_V^* \rightarrow T_V$ is the bundle map induced by the Poisson structure σ , and $p^* \alpha \in T_V^*$ is the pullback to $\text{Tot}(V)$ of the 1-form α on X .

Conversely, one can obtain a quadratic holomorphic bivector σ on V given a co-Higgs field $\Phi \in H^0(X, \text{End } V \otimes T_X)$. In local coordinates, if we write

$$\Phi = \sum_{i,j,k} \phi_{ijk}(x) \frac{\partial}{\partial x_i} \otimes y_j \frac{\partial}{\partial y_k},$$

then

$$\sigma_\Phi = \sum_{i,j,k} \phi_{ijk}(x) \frac{\partial}{\partial x_i} \wedge y_j \frac{\partial}{\partial y_k}$$

is a holomorphic bivector on $\text{Tot}(V)$.

Remark 3.2.3. For a line bundle L on a manifold X , a co-Higgs field on L is simply a vector field $v \in H^0(X, T_X)$ because $\text{End } L \simeq \mathcal{O}_X$ in this case. And each vector field v lifts uniquely to a quadratic Poisson structure σ on L via the formula $\sigma = \tilde{v} \wedge \text{Eul}$, where $\tilde{v}$ is any choice of a $\mathbb{C}^*$ -invariant lift of v to L and Eul is the Euler vector field on L . Conversely, given a quadratic Poisson structure σ on L , we can obtain the corresponding co-Higgs field v by pushing forward σ along the projection map $L \rightarrow X$.

Example 3.2.4. Consider the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$. Let $v = t \frac{\partial}{\partial t}$ be a vector field on the the affine coordinate chart $\mathcal{U} = \{[1 : t] \mid t \in \mathbb{C}\} \subset \mathbb{P}^1$. We have coordinates (t, y) on the total space of $\mathcal{O}(1)|_{\mathcal{U}}$, where t is the base coordinate and y is the fiber coordinate. A $\mathbb{C}^*$ -invariant lift of v is $\tilde{v} = t \frac{\partial}{\partial t} - y \frac{\partial}{\partial y}$ and the Euler vector field is $\text{Eul} = y \frac{\partial}{\partial y}$. Therefore,

the quadratic Poisson structure on $\mathcal{O}(1)|_{\mathcal{U}}$ corresponding to the vector field $t\frac{\partial}{\partial t}$ on $\mathcal{U} \subset \mathbb{P}^1$ is given by

$$\sigma = \tilde{v} \wedge \text{Eul} = \left(t\frac{\partial}{\partial t} - y\frac{\partial}{\partial y} \right) \wedge y\frac{\partial}{\partial y} = ty\frac{\partial}{\partial t} \wedge \frac{\partial}{\partial y}.$$

Conversely, given this quadratic Poisson structure σ on $\mathcal{O}(1)|_{\mathcal{U}}$, we can recover the original vector field v as follows. Let $p : \mathcal{O}(1)|_{\mathcal{U}} \rightarrow \mathcal{U}$ be the bundle projection. For any $\alpha \in T^*\mathcal{U}$, we have $v = p_*(\sigma^\#(p^*\alpha))$. Taking $\alpha = dt$, we get $\sigma^\#(p^*dx) = \sigma^\#(dt) = \iota_{dt}(ty\frac{\partial}{\partial t} \wedge \frac{\partial}{\partial y}) = -ty\frac{\partial}{\partial y}$. Therefore, $v = p_*(-ty\frac{\partial}{\partial y}) = -tp_*(y\frac{\partial}{\partial y}) = t\frac{\partial}{\partial t}$.

A natural question to ask is, given a co-Higgs field $\Phi \in H^0(X, \text{End } V \otimes T_X)$, when does the corresponding quadratic bivector σ_Φ becomes a Poisson structure on $\text{Tot}(V)$? In other words, under what conditions is σ_Φ integrable, that is, when do we have $[\sigma_\Phi, \sigma_\Phi] = 0$? When the co-Higgs field Φ is diagonalizable, Matviichuk gives a necessary condition on Φ , namely, strong integrability, for σ_Φ to be integrable.

To define strong integrability, we first introduce the k -th trace of a co-Higgs field. For $\Phi \in H^0(X, \text{End}(V) \otimes T_X)$ on a rank r vector bundle V , let

$$\tau_k(\Phi) = \text{Tr}(\Phi^k) \in H^0(X, S^k T_X), \quad k = 1, 2, \dots, r.$$

These traces can be interpreted as elements of $\text{Hom}(S^k T_X^*, \mathcal{O}_X)$ through the composition:

$$S^k T_X^* \longrightarrow \text{End}(V) \xrightarrow{\text{Trace}} \mathcal{O}_X$$

$$\omega_1 \otimes \dots \otimes \omega_k \longmapsto \langle \Phi, \omega_1 \rangle \dots \langle \Phi, \omega_k \rangle \longmapsto \text{Tr}(\langle \Phi, \omega_1 \rangle \dots \langle \Phi, \omega_k \rangle).$$

Definition 3.2.5 ([21]). A co-Higgs field Φ is called **strongly integrable** if the trace functions $\tau_k(\Phi)$ pairwise Poisson commute as functions on the symplectic manifold T^*X .

Moreover, the trace functions $\tau_k(\Phi)$ and the characteristic polynomial coefficients $s_k(\Phi)$ of the spectral variety of Φ are related by the Newton's identities:

$$\tau_k(\Phi) = (-1)^{k-1} k s_k(\Phi) + \sum_{i=1}^{k-1} (-1)^{k-i+1} s_{k-i}(\Phi) \tau_i(\Phi), \quad 1 \leq k \leq r.$$

Therefore, the strong integrability condition is equivalent to the pairwise commutativity of the coefficients $s_k(\Phi)$ of the characteristic polynomial of Φ , where $s_k(\Phi)$ are the elementary symmetric functions in the eigenvalues of Φ .

Having defined the strong integrability condition, we are now ready to state the following result from Matviichuk [21].

Lemma 3.2.6 ([21], Lemma 3.4.2 and 4.0.4). *Let $\Phi \in H^0(X, \text{End}(V) \otimes T_X)$ be a co-Higgs field. Let $\mathcal{U} \subset X$ be an analytic open set on which Φ can be diagonalized:*

$$\Phi = \begin{pmatrix} v_1 & 0 & \dots & 0 \\ 0 & v_2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & v_r \end{pmatrix}, v_i \in H^0(\mathcal{U}, T_X), i = 1, 2, \dots, r.$$

We have the following:

- (a) *The bivector σ_Φ is an integrable quadratic Poisson structure on $V|_{\mathcal{U}}$ if and only if Φ is strongly integrable on $\mathcal{U}$.*
- (b) *If $\mathcal{U} \subset \mathbb{C}$ is a simply connected, $V = \mathcal{O}_{\mathcal{U}}^r$, and Φ is strongly integrable, then there exists a single holomorphic vector field v on $\mathcal{U}$ and complex constants $\lambda_1, \lambda_2, \dots, \lambda_r$ such that each diagonal entry of Φ can be expressed as $v_i = \lambda_i v$ so that $\Phi = v \text{diag}(\lambda_1, \dots, \lambda_r)$.*

The following theorem extends part (b) of Lemma 3.2.6 by classifying all possible forms of co-Higgs bundles of arbitrary rank that induce integrable quadratic Poisson structures.

Theorem 3.2.7. *Let (V, Φ) be a rank r co-Higgs bundle over an analytic open set $\mathcal{U} \subset \mathbb{C}$, where Φ is locally given by*

$$\Phi = (a_{ij}(x))^T$$

with $a_{ij}(x) \in H^0(\mathbb{C}, \mathcal{O}(\mathcal{U}))$. Then, Φ induces an integrable quadratic bivector

$$\sigma_\Phi = \sum_{i,j=1}^r a_{ij}(x) \partial_x \wedge y_i \partial_j$$

on the total space of V if and only if Φ is of one of the following two forms.

1. *An $r \times r$ matrix with only one nonzero row:*

$$\Phi = \begin{pmatrix} 0 & \dots & a_{1j}(x) & \dots & 0 \\ 0 & \dots & \vdots & \dots & 0 \\ \vdots & \vdots & a_{ij}(x) & \vdots & \vdots \\ 0 & \dots & \vdots & \dots & 0 \\ 0 & \dots & a_{r-1j}(x) & \dots & 0 \\ 0 & \dots & a_{rj}(x) & \dots & 0 \end{pmatrix}^T$$

with $a_{ij}(x) \in H^0(\mathbb{C}, \mathcal{O}(\mathcal{U}))$, $i = 1, \dots, r$.

2. A function multiple of a constant $r \times r$ matrix:

$$\Phi = a(x)A$$

with $a(x) \in H^0(\mathbb{C}, \mathcal{O}(\mathcal{U}))$ and A a constant $r \times r$ matrix.

Proof. The induced quadratic bivector σ_Φ is given by

$$\sigma_\Phi = \sum_{i,j=1}^r a_{ij} \partial_x \wedge y_i \partial_j.$$

For σ_Φ to be integrable, we need $[\sigma_\Phi, \sigma_\Phi] = 0$. A direct computation gives

$$\begin{aligned} [\sigma_\Phi, \sigma_\Phi] &= \sum_{\substack{i,j,k,l=1 \\ j \neq l}}^r [a_{ij} \partial_x, a_{kl} \partial_x] \wedge y_i \partial_j \wedge y_k \partial_l \\ &= \sum_{\substack{i,j,k,l=1 \\ j \neq l}}^r (a_{ij}(x) a'_{kl}(x) - a_{kl}(x) a'_{ij}(x)) \partial_x \wedge y_i \partial_j \wedge y_k \partial_l. \end{aligned}$$

Note that the summand is symmetric in the ij and kl . That is,

$$(a_{ij}(x) a'_{kl}(x) - a_{kl}(x) a'_{ij}(x)) \partial_x \wedge y_i \partial_j \wedge y_k \partial_l = (a_{kl}(x) a'_{ij}(x) - a_{ij}(x) a'_{kl}(x)) \partial_x \wedge y_k \partial_l \wedge y_i \partial_j$$

for all i, j, k, l . Therefore, $[\sigma, \sigma] = 0$ if and only if

$$a_{ij}(x) a'_{kl}(x) - a_{kl}(x) a'_{ij}(x) = 0 \tag{3.4}$$

for all i, j, k, l with $j \neq l$.

Let us first assume that only the j_0 -th column of Φ^T is nonzero so that $a_{ij}(x) = 0$ for all $j \neq j_0$ but $a_{ij_0}(x) \neq 0$ for at least one $1 \leq i \leq r$. Referring to equation (3.4), there are no constraints on the coefficients $a_{ij_0}(x)$ appearing in the j_0 -th column. Hence,

$$\Phi = \begin{pmatrix} 0 & \cdots & a_{1j_0}(x) & \cdots & 0 \\ 0 & \cdots & \vdots & \cdots & 0 \\ \vdots & \vdots & a_{ij_0}(x) & \vdots & \vdots \\ 0 & \cdots & \vdots & \cdots & 0 \\ 0 & \cdots & a_{r-1j_0}(x) & \cdots & 0 \\ 0 & \cdots & a_{rj_0}(x) & \cdots & 0 \end{pmatrix}^T$$

with $a_{ij_0}(x) \in H^0(\mathbb{C}, \mathcal{O}(\mathcal{U}))$, $i = 1, \dots, r$, proving 1.

Let us now assume that the at least two columns of Φ are not zero, say, the j_0 -th and j_1 -th columns with $j_0 < j_1$. Without loss of generality, we can assume that $a_{1j_0}(x) \neq 0$. Equation (3.4) then tells us that

$$a_{1j_0}(x)a'_{ij}(x) - a_{ij}(x)a'_{1j_0}(x) = 0$$

for all $i = 1, \dots, r$ and $j \neq j_0$, implying that

$$a_{ij}(x) = c_{ij}a_{1j_0}(x)$$

for some $c_{ij} \in \mathbb{C}$ for all $i = 1, \dots, r$ and $j \neq j_0$. If all the other elements in the j_0 -th column are zero, that is, $a_{ij_0}(x) = 0$ for all $i = 2, \dots, r$, then we obtain that

$$\Phi^T = a_{1j_0}(x)B$$

with B a constant $r \times r$ matrix. Otherwise, assume that at least one element of the j_0 -th column other than $a_{1j_0}(x)$ is nonzero. Assume this element is $a_{i_0j_0}(x)$ for some $2 \leq i_0 \leq r$. Then, by equation (3.4), we have

$$a_{i_0j_0}(x)a'_{ij}(x) - a_{ij}(x)a'_{i_0j_0}(x) = 0$$

for all $i = 1, \dots, r$ and $j \neq j_0$, implying that

$$a_{ij}(x) = d_{ij}a_{i_0j_0}(x)$$

for some $d_{ij} \in \mathbb{C}$ for all $i = 1, \dots, r$ and $j \neq j_0$. Since we assumed that the j_1 -th column is nonzero, we have that $a_{i_1j_1}(x) \neq 0$ for some $1 \leq i_1 \leq r$, which means that

$$a_{i_1j_1}(x) = c_{i_1j_1}a_{1j_0}(x) = d_{i_1j_1}a_{i_0j_0}(x)$$

with $c_{i_1j_1}, d_{i_1j_1} \neq 0$. In other words, $a_{i_0j_0}(x) = c_{i_0}a_{1j_0}(x)$ for some $c_{i_0} \in \mathbb{C}$. Since this is true for any nonzero element of the first column other than $a_{1j_0}(x)$, we again obtain that

$$\Phi^T = a_{1j_0}(x)B$$

with B a constant $r \times r$ matrix. Putting all this cases together, we obtain

$$\Phi = a(x)A$$

with $a(x) \in \Gamma(\mathcal{O}(U))$ and A a constant $r \times r$ matrix if at least two columns of Φ are nonzero, proving 2. $\square$

Remark 3.2.8. Note that if we restrict Theorem 3.2.7 to the case where Φ is diagonalizable, we obtain the same result as Matviichuk in Lemma 3.2.6.

Remark 3.2.9. It follows from Theorem 3.2.7 that if (V, Φ) is a rank r co-Higgs bundle over an analytic open subset $\mathcal{U} \subset \mathbb{C}$ with constant Higgs field $\Phi = (a_{ij})^T$, $a_{ij} \in \mathbb{C}$, then the induced quadratic Poisson structure σ_Φ is always integrable, that is, $[\sigma_\Phi, \sigma_\Phi] = 0$.

Theorem 3.2.10. *Let (V, Φ) be a rank r co-Higgs bundle over an analytic open set $\mathcal{U} \subset \mathbb{C}$. If the induced quadratic Poisson structure σ_Φ is integrable, then Φ is strongly integrable.*

Proof. Suppose that σ_Φ is integrable. If Φ is locally given by $\Phi = (a_{ij}(x))^T$ with $a_{ij}(x) \in H^0(\mathbb{C}, \mathcal{O}(\mathcal{U}))$, then it follows from Theorem 3.2.7 that either Φ has only one nonzero row, say, the j -th row $(a_{1j}(x), \dots, a_{rj}(x))$, or else $\Phi = a(x)A$ for some $a(x) \in H^0(\mathbb{C}, \mathcal{O}(\mathcal{U}))$ and constant $r \times r$ matrix A . In the first case,

$$\text{char}(\Phi) = \theta^r - \left(\sum_{i=1}^r a_{ij} \right) \theta^{r-1}$$

so that $s_0(\Phi) = 1$, $s_1(\Phi) = (\sum_{i=1}^r a_{ij})\xi$, and $s_k(\Phi) = 0$ for all $1 < k \leq r$. Consequently, $\{s_i(\Phi), s_j(\Phi)\} = 0$, proving that Φ is strongly integrable. In the second case, we have

$$\tau_k(\Phi) = \text{Tr}(\Phi^k) = a(x)^k \text{Tr}(A^k) = c_k a(x)^k$$

with $c_k = \text{Tr}(A^k) \in \mathbb{C}$. Therefore,

$$\{\tau_i(\Phi), \tau_j(\Phi)\} = c_i c_j \{a(x)^i \xi^i, a(x)^j \xi^j\} = 0,$$

implying that Φ is strongly integrable. □

Remark 3.2.11. Theorem 3.2.10 tells us that strong integrability of Φ is a necessary condition for the integrability of σ_Φ when X is 1-dimensional. The converse is, however, not true in general as illustrated by the following example.

Example 3.2.12. Let $\mathcal{U}$ be an open subset in $\mathbb{C}$ containing 0 and $V = \mathcal{O}_\mathcal{U} \oplus \mathcal{O}_\mathcal{U}$ be the trivial rank 2 holomorphic vector bundle on $\mathcal{U}$. Define a co-Higgs field Φ on V as

$$\Phi = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Then Φ is strongly integrable because $s_1(\Phi) = 2\xi$ and $s_2(\Phi) = 2\xi^2$, which Poisson commute. Now, let y_1 and y_2 be fiberwise linear coordinates on the first and second summands of V respectively. The bivector σ_Φ on V corresponding to Φ has the form

$$\sigma_\Phi = \partial_x \wedge y_1 \partial_{y_1} + \partial_x \wedge y_2 \partial_{y_2} + x \partial_x \wedge y_2 \partial_{y_1}.$$

Calculating the Schouten bracket $[\sigma_\Phi, \sigma_\Phi]$, we get

$$[\sigma_\Phi, \sigma_\Phi] = -2\partial_x \wedge y_2\partial_{y_1} \wedge y_2\partial_{y_2} \neq 0,$$

implying that σ_Φ is not integrable.

Remark 3.2.13. Theorem 3.2.7 can be generalized to $\mathcal{U} \subset \mathbb{C}^n$, but the equations are more complicated to solve. Indeed, for $\sigma_\Phi = \sum_{i=1}^n \sum_{j,k=1}^r g_{ijk} \partial_{x_i} \wedge y_j \partial_{y_k}$, we have

$$[\sigma_\Phi, \sigma_\Phi] = \sum_{i,l=1}^n \sum_{j,k,m,p=1}^r [g_{ijk} \partial_{x_i}, g_{lmp} \partial_{x_l}] \wedge y_j \partial_k \wedge y_m \partial_p + [y_j \partial_k, y_m \partial_p] \wedge g_{ijk} \partial_{x_i} \wedge g_{lmp} \partial_{x_l}.$$

Therefore, the condition $[\sigma_\Phi, \sigma_\Phi] = 0$ is equal to $[g_{ijk} \partial_{x_i}, g_{lmp} \partial_{x_l}] = 0$ and $[y_j \partial_k, y_m \partial_p] = 0$, for $1 \leq j, k, m, p \leq r$ and $1 \leq i, l \leq n$.

The following theorem generalizes Theorem 3.2.7 for the case when $\mathcal{U} \subset \mathbb{C}^2$.

Theorem 3.2.14. *Let (V, Φ) be a rank r co-Higgs bundle over an analytic open set $\mathcal{U} \subset \mathbb{C}^2$, where $\Phi = \Phi_1 + \Phi_2$ with Φ_1 and Φ_2 locally given by*

$$\Phi_1 = (a_{ij}(x_1, x_2))^T, \quad \Phi_2 = (b_{ij}(x_1, x_2))^T$$

for some $a_{ij}(x_1, x_2), b_{ij}(x_1, x_2) \in H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$. Then, Φ induces an integrable quadratic bivector

$$\sigma_\Phi = \sum_{i,j=1}^r a_{ij}(x_1, x_2) \partial_{x_1} \wedge y_i \partial_j + \sum_{i,j=1}^r b_{ij}(x_1, x_2) \partial_{x_2} \wedge y_i \partial_j$$

on the total space of V if and only if Φ is of one of the following forms:

1. *A pair of $r \times r$ matrices each with only one nonzero row:*

$$\Phi_1 = \begin{pmatrix} 0 & \cdots & a_{1j}(x_1, x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & a_{rj}(x_1, x_2) & \cdots & 0 \end{pmatrix}^T,$$

$$\Phi_2 = \begin{pmatrix} 0 & \cdots & b_{1k}(x_1, x_2) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & b_{rk}(x_1, x_2) & \cdots & 0 \end{pmatrix}^T,$$

where $a_{ij}, b_{ik} \in H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$ and j, k are fixed rows (with possibly $j=k$) that satisfy:

- If $j \neq k$, there are no additional constraints on the functions a_{ij} and b_{ik} .
- If $j = k$, the functions must satisfy

$$a_{ij} \frac{\partial b_{mj}}{\partial x_1} - b_{mj} \frac{\partial a_{ij}}{\partial x_1} = b_{ij} \frac{\partial a_{mj}}{\partial x_2} - a_{mj} \frac{\partial b_{ij}}{\partial x_2}$$

for all i, m .

2. Function multiples of constant matrices:

$$\Phi_1 = f(x_1, x_2)A, \quad \Phi_2 = g(x_1, x_2)B$$

with $f, g \in H^0(\mathcal{U}, \mathcal{O}_{\mathcal{U}})$ and A, B constant $r \times r$ matrices satisfying

$$f \frac{\partial g}{\partial x_1}(A_{ij}B_{kl}) - g \frac{\partial f}{\partial x_1}(B_{kl}A_{ij}) = g \frac{\partial f}{\partial x_2}(B_{ij}A_{kl}) - f \frac{\partial g}{\partial x_2}(A_{kl}B_{ij}),$$

where A_{ij} and B_{ij} are the entries of the constant matrices A and B , respectively.

Proof. The induced quadratic bivector σ_{Φ} needs to satisfy $[\sigma_{\Phi}, \sigma_{\Phi}] = 0$. A direct computation gives

$$\begin{aligned} [\sigma_{\Phi}, \sigma_{\Phi}] &= \sum_{i,j,k,l=1}^r [a_{ij}\partial_{x_1}, a_{kl}\partial_{x_1}] \wedge y_i\partial_j \wedge y_k\partial_l \\ &+ \sum_{i,j,k,l=1}^r [b_{ij}\partial_{x_2}, b_{kl}\partial_{x_2}] \wedge y_i\partial_j \wedge y_k\partial_l \\ &+ \sum_{i,j,k,l=1}^r [a_{ij}\partial_{x_1}, b_{kl}\partial_{x_2}] \wedge y_i\partial_j \wedge y_k\partial_l \\ &+ \sum_{i,j,k,l=1}^r [b_{ij}\partial_{x_2}, a_{kl}\partial_{x_1}] \wedge y_i\partial_j \wedge y_k\partial_l. \end{aligned}$$

Therefore, we have the following equations:

$$a_{ij} \frac{\partial a_{kl}}{\partial x_1} - a_{kl} \frac{\partial a_{ij}}{\partial x_1} = 0, \tag{3.5}$$

$$b_{ij} \frac{\partial b_{kl}}{\partial x_2} - b_{kl} \frac{\partial b_{ij}}{\partial x_2} = 0, \tag{3.6}$$

$$a_{ij} \frac{\partial b_{kl}}{\partial x_1} - b_{kl} \frac{\partial a_{ij}}{\partial x_1} = b_{ij} \frac{\partial a_{kl}}{\partial x_2} - a_{kl} \frac{\partial b_{ij}}{\partial x_2}. \quad (3.7)$$

For case (1), if Φ_1 and Φ_2 each have only one nonzero row, then equations (3.5) and (3.6) are automatically satisfied. Equation (3.7) is also satisfied when $j \neq k$ because then both sides are zero, and when $j = k = j_0$ we have that $a_{ij_0} \frac{\partial b_{kj_0}}{\partial x_1} - b_{kj_0} \frac{\partial a_{ij_0}}{\partial x_1} = b_{ij_0} \frac{\partial a_{kj_0}}{\partial x_2} - a_{kj_0} \frac{\partial b_{ij_0}}{\partial x_2}$.

For case (2), if $\Phi_1 = f(x_1, x_2)A$ and $\Phi_2 = g(x_1, x_2)B$, then $a_{ij}(x_1, x_2) = f(x_1, x_2)A_{ij}$ and $b_{ij}(x_1, x_2) = g(x_1, x_2)B_{ij}$, where A_{ij} and B_{ij} are the entries of the constant matrices A and B respectively. Substituting this into equation (3.5), we obtain

$$\begin{aligned} a_{ij} \frac{\partial a_{kl}}{\partial x_1} - a_{kl} \frac{\partial a_{ij}}{\partial x_1} &= (f(x_1, x_2)A_{ij}) \left(\frac{\partial f}{\partial x_1} A_{kl} \right) - (f(x_1, x_2)A_{kl}) \left(\frac{\partial f}{\partial x_1} A_{ij} \right) \\ &= f \frac{\partial f}{\partial x_1} (A_{ij}A_{kl} - A_{kl}A_{ij}) \\ &= f \frac{\partial f}{\partial x_1} [A, A] \\ &= 0. \end{aligned}$$

Similarly, we have that equation (3.6) is satisfied for $\Phi_2 = g(x_1, x_2)B$.

Conversely, if more than one column is nonzero in either Φ_1 or Φ_2 , then following similar arguments as in the one-dimensional case in Theorem 3.2.10 and using equations (3.5), (3.6), and (3.7), we can show that Φ_1 and Φ_2 must be function multiples of constant matrices that commute. $\square$

3.3 Co-Higgs bundles and Poisson structures on $\mathbb{P}(V)$

In this section, we describe the correspondence between quadratic Poisson structures on the total space of $\mathbb{P}(V)$ and co-Higgs fields on V . In the case where the rank of V is two, Polishchuk established a one-to-one correspondence between co-Higgs fields and quadratic Poisson structures in [23]. We begin by describing this correspondence when the rank of V is two, and then proceed to examine it for the case where the rank is greater or equal to three, where more conditions are needed.

Let V be a rank 2 vector bundle over a manifold X , and let $\rho: \mathbb{P}(V) \rightarrow X$. The tangent bundle over $\mathbb{P}(V)$ fits into the following short exact sequence

$$0 \rightarrow T_F \rightarrow T_{\mathbb{P}(V)} \rightarrow \rho^* T_X \rightarrow 0, \quad (3.8)$$

where T_F is the tangent bundle along the fibers. From this, we have

$$T_F \otimes \rho^* T_X \subset \wedge^2 T_{\mathbb{P}(V)}. \quad (3.9)$$

Furthermore, $\rho_*(T_F \otimes \rho^* T_X) = \rho_* T_F \otimes T_X$ by the projection formula. Then

$$\begin{aligned} \rho_* T_F \otimes T_X &= \rho_*(\text{Aut}(\mathbb{P}(V))) \otimes T_X = \rho_*(\text{Aut}(V)/\mathbb{C}^*) \otimes T_X = \rho_*(\text{End}(V)/\mathbb{C}^*) \otimes T_X \\ &= \text{End}_0(V) \otimes T_X. \end{aligned}$$

Therefore, a traceless co-Higgs field on V can be obtained from a Poisson structure on the projective bundle $\mathbb{P}(V)$.

An explicit formula to obtain a Poisson structure π on $\mathbb{P}(V)$ from a traceless co-Higgs field Φ on a rank 2 vector bundle $V \rightarrow X$ is the following:

Let

$$\Phi = \sum_{i=1}^n (f_{i11} \partial_{x_i} \otimes y_1 \partial_{y_1} + f_{i12} \partial_{x_i} \otimes y_1 \partial_{y_2} + f_{i21} \partial_{x_i} \otimes y_2 \partial_{y_1} - f_{i11} \partial_{x_i} \otimes y_2 \partial_{y_2}),$$

where $f_{ijk} \in \mathcal{O}_{\mathcal{U}}$, $x_1, \dots, x_n$ are coordinates on $\mathcal{U} \subset X$, and y_1, y_2 are fiberwise linear coordinates on $V|_{\mathcal{U}}$. The Poisson lift of Φ to $\mathbb{P}(V|_{\mathcal{U}})$ is then

$$\pi = \sum_{i=1}^n (2f_{i11} \partial_{x_i} \wedge \hat{y}_1 \partial_{\hat{y}_1} + f_{i12} \partial_{x_i} \wedge \hat{y}_1^2 \partial_{\hat{y}_1} + f_{i21} \partial_{x_i} \wedge \partial_{\hat{y}_1}), \quad (3.10)$$

where $\hat{y}_1$ is the affine coordinate y_1/y_2 .

Example 3.3.1 (Poisson structure of a rank 2 co-Higgs bundle over $\mathbb{P}^1$). Let

$$V = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2)$$

be a rank 2 vector bundle over $\mathbb{P}^1$ with $a_1, a_2 \in \mathbb{Z}$. Since $T_{\mathbb{P}^1} = \mathcal{O}(2)$, a traceless co-Higgs field Φ on V is then a holomorphic section of $\text{End}_0 V \otimes \mathcal{O}(2)$. We can therefore write the co-Higgs field Φ as

$$\Phi = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & -f_{11} \end{pmatrix},$$

where $f_{11} \in H^0(\mathbb{P}^1, \mathcal{O}(2))$, $f_{12} \in H^0(\mathbb{P}^1, \mathcal{O}(a_2 - a_1 + 2))$, and $f_{21} \in H^0(\mathbb{P}^1, \mathcal{O}(a_1 - a_2 + 2))$. Let y_1, y_2 be fiberwise linear coordinates on $V|_U$. Then, Φ can be written in the form

$$\Phi = f_{11} \partial_x \otimes y_1 \partial_{y_1} + f_{21} \partial_x \otimes y_1 \partial_{y_2} + f_{12} \partial_x \otimes y_2 \partial_{y_1} - f_{11} \partial_x \otimes y_2 \partial_{y_2},$$

where $U = \{[1 : x = x_2/x_1] \mid x \in \mathbb{C}\} \subset \mathbb{P}^1$. Therefore, by (3.10), the Poisson structure of Φ over $\mathbb{P}(V|_U)$ is given by

$$\begin{aligned}\pi &= 2f_{11}\partial_x \wedge \hat{y}_1\partial_{\hat{y}_1} + f_{21}\partial_x \wedge \hat{y}_1^2\partial_{\hat{y}_1} + f_{12}\partial_x \wedge \partial_{\hat{y}_1} \\ &= (2f_{11}\hat{y}_1 + f_{21}\hat{y}_1^2 + f_{12})\partial_x \wedge \partial_{\hat{y}_1},\end{aligned}$$

where $\hat{y}_1 = y_1/y_2$ is the affine coordinate on the fibers of $\mathbb{P}(V|_U)$.

More generally, if $p : V \rightarrow X$ is a holomorphic vector bundle on X of any rank, then any Poisson structure on $\mathbb{P}(V)$ with coisotropic fibers gives rise to a co-Higgs field. In fact, Matviichuk used the one-to-one correspondence between vector fields on $\mathbb{P}(V)$ tangent to the fibers of p and sections of $\text{End}_0(V)$, the traceless endomorphisms of V , to show this:

Lemma 3.3.2 ([21], Lemma 3.2.3). *Let V be a rank r holomorphic vector bundle on X , $\rho : \mathbb{P}(V) \rightarrow X$ be its projectivization, and π be a Poisson structure on $\mathbb{P}(V)$ with coisotropic fibers. Then, π induces a unique traceless co-Higgs field $\Phi_0 \in H^0(X, \text{End}_0(V) \otimes T_X)$, given by the formula*

$$\langle \Phi_0, \alpha \rangle = \pi^\sharp(\rho^*\alpha), \quad \text{for any } \alpha \in T_X^*.$$

In local coordinates, if we write $\langle \Phi_0, dx_i \rangle = \sum_{j,k} \Phi_{ijk} y_j \frac{\partial}{\partial y_k}$ then we have

$$\Phi_0 = \sum_{i,j,k} \Phi_{ijk}(x) \frac{\partial}{\partial x_i} \otimes y_j \frac{\partial}{\partial y_k}.$$

Conversely, one can obtain a quadratic holomorphic bivector π_{Φ_0} on $\mathbb{P}(V)$ from a traceless co-Higgs field $\Phi_0 \in H^0(X, \text{End}_0 V \otimes T_X)$. In local coordinates, if we write

$$\Phi_0 = \sum_{i,j,k} \phi_{ijk}(x) \frac{\partial}{\partial x_i} \otimes y_j \frac{\partial}{\partial y_k},$$

then

$$\pi_{\Phi_0} = \sum_{i,j,k} \phi_{ijk}(x) \frac{\partial}{\partial x_i} \wedge \hat{y}_j \frac{\partial}{\partial \hat{y}_k}$$

is a holomorphic bivector on $\mathbb{P}(V)$, where $\hat{y}_j = y_j/y_r$.

Now to obtain an integrable bivector π_{Φ_0} , that is, a Poisson structure on $\mathbb{P}(V)$ from a diagonalizable co-Higgs bundle, we also need the condition of strong integrability as the following theorem suggests.

Theorem 3.3.3 ([21], Corollary 3.4.7). *Let Φ be a diagonalizable traceless co-Higgs field on a holomorphic vector bundle over an open set $\mathcal{U} \subset \mathbb{C}$. Then, the quadratic bivector π_Φ is integrable if and only if Φ is strongly integrable.*

The following is a generalization of Theorem 3.3.3 to a diagonalizable traceless co-Higgs field on a holomorphic vector bundle over a contractible open set $\mathcal{U} \subset \mathbb{C}^n$.

Theorem 3.3.4. *Let Φ be a diagonalizable traceless co-Higgs field on a holomorphic vector bundle V over a contractible open set $\mathcal{U} \subset \mathbb{C}^n$. Then, π_Φ is a quadratic Poisson structure on $\mathbb{P}(V)$ if and only if Φ is strongly integrable.*

Proof. Suppose that Φ is strongly integrable. The induced bivector σ_Φ is then a quadratic Poisson structure on V by Lemma 3.2.6 b). Hence, the projectivization π_Φ of σ_Φ gives a quadratic Poisson structure on $\mathbb{P}(V)$.

Now assume that Φ induces a quadratic Poisson structure π_Φ on $\mathbb{P}(V)$. Since every vector bundle over a contractible space is trivial, the canonical bundle ω_V of V is trivial. Therefore, by Theorem 3.1.4, π_Φ admits a further lift to a quadratic Poisson structure σ_Φ on V . Lemma 3.2.6 then implies that Φ is strongly integrable. $\square$

Theorem 3.3.5. *If Φ is a traceless co-Higgs field on a vector bundle over an open set $\mathcal{U} \subset \mathbb{C}$, then π_Φ is integrable if σ_Φ is integrable.*

Proof. Assume that σ_Φ is integrable, then it induces a Poisson structure $\tilde{\pi}$ on $\mathbb{P}(V)$ by pushing forward along the projection $p : V \setminus 0 \rightarrow \mathbb{P}(V)$. Moreover, this induced Poisson structure $\tilde{\pi}$ must coincide with π_Φ since they both come from the same co-Higgs field Φ . Therefore, π_Φ is integrable. $\square$

Corollary 3.3.6. *Let (V, Φ) be a rank $r + 1$ traceless co-Higgs bundle over an analytic open set $\mathcal{U} \subset \mathbb{C}$, where Φ is locally given by*

$$\Phi = (a_{ij}(x))^T$$

with $a_{ij}(x) \in H^0(\mathbb{C}, \mathcal{O}(\mathcal{U}))$. Then, Φ induces an integrable Poisson structure

$$\pi_\Phi = \sum_{i,j,k} a_{ijk}(x) \partial_x \wedge \hat{y}_j \partial_{\hat{y}_k}$$

on $\mathbb{P}(V)$ if Φ is of one of the following two forms.

1. An $r \times r$ matrix with only one nonzero row:

$$\Phi = \begin{pmatrix} 0 & \cdots & a_{1j}(x) & \cdots & 0 \\ 0 & \cdots & \vdots & \cdots & 0 \\ \vdots & \vdots & a_{ij}(x) & \vdots & \vdots \\ 0 & \cdots & \vdots & \cdots & 0 \\ 0 & \cdots & a_{r-1,j}(x) & \cdots & 0 \\ 0 & \cdots & a_{r,j}(x) & \cdots & 0 \end{pmatrix}^T$$

with $a_{ij}(x) \in H^0(\mathbb{C}, \mathcal{O}(\mathcal{U}))$, $i = 1, \dots, r$.

2. A function multiple of a constant $r \times r$ matrix:

$$\Phi = a(x)A$$

with $a(x) \in H^0(\mathbb{C}, \mathcal{O}(\mathcal{U}))$ and A a constant $r \times r$ matrix.

Proof. Follows from Theorems 3.3.5 and 3.2.7. □

3.4 Co-Higgs bundles and strong integrability on curves of genus $g \geq 0$

Bartocci and Macri classified complex projective surfaces admitting holomorphic Poisson structures in [5]. This classification contained, in particular, some ruled surfaces. Recall that a ruled surface is the projectivization $\mathbb{P}(V)$ of a holomorphic rank 2 vector bundle V over a curve. In fact, Hirzebruch surfaces correspond to ruled surfaces over $\mathbb{P}^1$. Furthermore, as we saw in section 3.3, when V has rank 2, any holomorphic Poisson structure on $\mathbb{P}(V)$ comes from a co-Higgs field on V . In this section, we consider co-Higgs bundles (V, Φ) of any rank over a smooth projective curve X of genus $g \geq 0$ and investigate conditions under which Φ may induce an integrable quadratic Poisson structure π_Φ on $\mathbb{P}(V)$.

3.4.1 Existence of nonzero traceless co-Higgs fields

In order to produce nonzero quadratic Poisson structures, we need nonzero traceless co-Higgs fields. We therefore begin by determining what types of holomorphic vector bundles admit nonzero traceless co-Higgs fields $\Phi \neq 0$ on curves of genus $g \geq 0$.

Proposition 3.4.1. *Let V be a rank r holomorphic vector bundle on a curve X of genus $g \geq 0$.*

1. *If $g = 0$, then V always admits a nonzero traceless Higgs field Φ .*
2. *If $g = 1$, then V admits a nonzero traceless Higgs field Φ if and only if V is unstable.*
3. *If $g \geq 2$ and V admits a nonzero traceless Higgs field Φ , then V is unstable.*

Proof. If $g = 0$, we can express V as a direct sum of line bundles $V = \bigoplus_{i=1}^r \mathcal{O}(a_i)$ with $a_1, \dots, a_r \in \mathbb{Z}$. Then,

$$\text{End}(V) \otimes T_X \cong (V^* \otimes V) \otimes \mathcal{O}(2) \cong \bigoplus_{i,j=1}^r \mathcal{O}(a_j - a_i + 2).$$

Sections of $\text{End}(V) \otimes T_X$ can therefore be written in matrix form with ij -th entry a section of $\mathcal{O}(a_j - a_i + 2)$. In particular, the diagonal entries are all sections of $\mathcal{O}(2)$. Hence, for any nonzero degree 2 homogeneous polynomial of $p \in H^0(\mathbb{P}^1, \mathcal{O}(2))$, we can construct a nonzero traceless co-Higgs field Φ by setting

$$\Phi = \begin{pmatrix} p & & & & \\ & p & & & \\ & & \ddots & & \\ & & & p & \\ & & & & -(r-1)p \end{pmatrix}.$$

Now if $g = 1$, then $T_X = \mathcal{O}$ so that $H^0(X, \text{End } V \otimes T_X) = H^0(X, \text{End } V)$. Assume that V is unstable. We consider two cases: In the first case, we assume that V is decomposable so that $V = V_1 \oplus V_2$ with V_1 of rank r_1 and V_2 of rank r_2 . In this case, define Φ as

$$\Phi = \text{Id}_{V_1} \oplus \left(-\frac{r_1}{r_2} \right) \text{Id}_{V_2}.$$

Thus Φ is a nonzero traceless co-Higgs field of V . In other words, V always admits nonzero traceless co-Higgs fields.

In the second case, we assume that V is indecomposable of rank r and degree d . If $(r, d) = 1$, then V is stable by [32, Corollary 4.19], contradicting our assumption. Therefore, $(r, d) = h > 1$. Set $r' = r/h$ and $d' = d/h$. Then, by [3, Lemma 26], there exist a line

bundle L of degree 0 such that $V \simeq E_A(r, d) \otimes L$, where $E_A(r, d)$ is a fixed indecomposable bundle of rank r and degree d . Moreover, by [3, Lemma 24], $E_A(r, d) \simeq E_A(r', d') \otimes F_h$, where F_h is the unique indecomposable rank h bundle of degree 0 such that $h^0(X, F_h) \neq 0$ (see [3, Theorem 5]). Therefore,

$$V = E_A(r', d') \otimes F_h \otimes L.$$

We can now compute

$$\begin{aligned} \text{End } V &= (E_A(r', d') \otimes F_h \otimes L)^* \otimes (E_A(r', d') \otimes F_h \otimes L) \\ &= \text{End}(E_A(r', d')) \otimes \text{End } F_h \\ &= (\mathcal{O} \oplus \text{End}_0(E_A(r', d'))) \otimes \text{End } F_h \\ &= \text{End } F_h \oplus (\text{End}_0(E_A(r', d')) \otimes \text{End } F_h). \end{aligned}$$

This implies that

$$h^0(X, \text{End } V) \geq h^0(X, \text{End } F_h).$$

However, $F_h^* \simeq F_h$ by [3, Corollary 1] so that $\text{End } F_h \simeq F_h \otimes F_h$. Moreover, $h^0(X, F_h \otimes F_h) = \min\{h, h\}$ by [3, Lemma 17(i)], implying that

$$h^0(X, \text{End } F_h) = h > 1.$$

Thus, $h^0(X, \text{End } V) > 1$ and so $h^0(X, \text{End}_0 V) \neq 0$. This means that V admits a nonzero traceless co-Higgs field.

Conversely, we now prove that if $g = 1$ and V is stable, then any traceless co-Higgs field on V must be zero. By [8, Proposition 2.2], since V is stable and $g = 1$, any co-Higgs field Φ on V takes the form $\Phi = c \cdot \text{Id}_V$ for some $c \in \mathbb{C}$, where Id_V is the identity endomorphism on V . However, Φ is assumed to be traceless. Therefore, $c = 0$ and consequently $\Phi = 0$.

For $g \geq 2$, we prove the contrapositive and assume that V is stable. Since $g \geq 2$, we have $\deg(T_X) = 2 - 2g < 0$. Referring to [8, 26], when V is stable and $\deg(T_X) < 0$, the only co-Higgs field on V is the zero map. $\square$

Remark 3.4.2. When $g \geq 2$, many unstable bundles do not admit nonzero traceless Higgs fields. We first note that since $\deg T_X < 0$ in this case, $h^0(X, T_X) = 0$ and so

$$H^0(X, \text{End}_0 V \otimes T_X) = H^0(X, \text{End } V \otimes T_X).$$

Suppose, for example, that $V = L_1 \oplus \cdots \oplus L_r$ with each L_i a line bundle on X . We can assume, without loss of generality, that $L_1 = \mathcal{O}$ and $\deg L_i \geq \deg L_j$ for all $i, j = 1, \dots, r$. Then,

$$\text{End } V \otimes T_X = \bigoplus_{i,j=1}^r L_i^* \otimes L_j \otimes T_X$$

and

$$h^0(X, \text{End } V \otimes T_X) = \sum_{i,j=1}^r h^0(X, L_i^* \otimes L_j \otimes T_X).$$

For all $i \geq j$,

$$\deg L_i^* \otimes L_j \otimes T_X = \deg L_j - \deg L_i + \deg T_X \leq \deg T_X < 0$$

so that $h^0(X, L_i^* \otimes L_j \otimes T_X) = 0$. Hence,

$$h^0(X, \text{End } V \otimes T_X) = \sum_{i < j} h^0(X, L_i^* \otimes L_j \otimes T_X).$$

Nonetheless, if in addition, $\deg L_i - \deg L_j > \deg T_X$ when $i < j$, then we have

$$h^0(X, L_i^* \otimes L_j \otimes T_X) = 0$$

for all i, j , implying $h^0(X, \text{End}_0 V \otimes T_X) = 0$. However, if there are some $i_1, i_2 \in \{1, \dots, r\}$ such that $i_1 > i_2$ and $\deg L_{i_2} - \deg L_{i_1} + \deg T_X \geq 2g - 1$, then $h^0(X, L_{i_1}^* \otimes L_{i_2} \otimes T_X) \neq 0$, implying that $h^0(X, \text{End}_0 V \otimes T_X) \neq 0$.

We can obtain a more precise description for which holomorphic vector bundles admit nonzero traceless co-Higgs fields for direct sums of line bundles:

Theorem 3.4.3. *Let X be a smooth projective curve of genus $g \geq 0$ and $V = L_1 \oplus \dots \oplus L_r$ be a decomposable vector bundle on X with $\deg(L_i) \leq \deg(L_j)$ for $i \leq j$. Then, V admits a nonzero traceless co-Higgs field in the following cases:*

1. *If and only if $0 \leq \deg(L_j) - \deg(L_i) \leq 2$ for some $i < j$ when $g = 0$.*
2. *If and only if either $\deg(L_i) > \deg(L_j)$ for some $i < j$, or $L_i \cong L_j$ for some $i \neq j$ when $g = 1$.*
3. *If $\deg(L_j) - \deg(L_i) \geq 4g - 3 > 0$ for some $i < j$ when $g \geq 2$. In particular, all co-Higgs fields on V are strictly upper triangular as $r \times r$ matrices with respect to the decomposition $L_1 \oplus \dots \oplus L_r$.*

Proof. For any $\Phi \in H^0(X, \text{End}(V) \otimes T_X)$, we have $\Phi = (\phi_{ij})$ with

$$\phi_{ij} \in H^0(X, \text{Hom}(L_i, L_j \otimes T_X))$$

for all $i, j = 1, \dots, r$. Therefore, $\Phi \neq 0$ if and only if $\phi_{i_0 j_0} \neq 0$ for some $1 \leq i_0, j_0 \leq r$ if and only if $h^0(X, \text{Hom}(L_{i_0}, L_{j_0} \otimes T_X)) \neq 0$. We therefore need to analyze

$$H^0(X, \text{Hom}(L_i, L_j \otimes T_X)) \cong H^0(X, L_i^* \otimes L_j \otimes T_X)$$

for each genus.

Case $g = 0$. In this case, we have that $T_X \cong \mathcal{O}(2)$. Therefore, $H^0(X, L_i^* \otimes L_j \otimes T_X) \cong H^0(X, \mathcal{O}(\deg(L_j) - \deg(L_i) + 2))$. This is nonzero if and only if $\deg(L_j) - \deg(L_i) + 2 \geq 0$. Since $\deg(L_i) \geq \deg(L_j)$ for $i < j$, this is equivalent to $0 \leq \deg(L_j) - \deg(L_i) \leq 2$. We can then construct a nonzero Φ with $\phi_{ij} \neq 0$ and with diagonal entries $\phi_{ii} \in H^0(\mathcal{O}(2))$. Moreover, we can write Φ as

$$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1r} \\ 0 & \phi_{22} & \cdots & \phi_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_{rr} \end{pmatrix},$$

where $\phi_{ij} \in H^0(\mathcal{O}(\deg(L_i) - \deg(L_j) + 2))$ is nonzero for some $i < j$ with $0 \leq \deg(L_j) - \deg(L_i) \leq 2$, $\phi_{ii} \in H^0(\mathcal{O}(2))$, and $\sum_i \phi_{ii} = 0$.

Case $g = 1$. In this case, $T_X \cong \mathcal{O}_X$. Therefore, $H^0(X, L_i^* \otimes L_j \otimes T_X) \cong H^0(X, L_i^* \otimes L_j)$. This is nonzero if and only if $\deg(L_i) > \deg(L_j)$ or $L_i \cong L_j$. If $\deg(L_i) > \deg(L_j)$ for some $i < j$, then $H^0(L_j^* \otimes L_i) \neq 0$, allowing a nonzero upper-triangular Φ . If $L_i \cong L_j$ for some $i \neq j$, then $H^0(L_i^* \otimes L_j) \cong H^0(L_j^* \otimes L_i) \cong \mathbb{C}$, allowing nonzero (i, j) and (j, i) entries in Φ . If neither condition holds, then $L_1 \cong \cdots \cong L_r$ and only scalar multiples of the identity are possible, which are not trace-free co-Higgs fields. Therefore, we have that Φ can be either

$$\Phi = \begin{pmatrix} 0 & \phi_{12} & \cdots & \phi_{1r} \\ 0 & 0 & \cdots & \phi_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $\phi_{ij} \in \mathbb{C}$ is nonzero for some $i < j$, or

$$\Phi = \begin{pmatrix} \phi_{11} & \cdots & \phi_{1j} & \cdots & \phi_{1r} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \phi_{i1} & \cdots & \phi_{ij} & \cdots & \phi_{ir} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \phi_{r1} & \cdots & \phi_{rj} & \cdots & \phi_{rr} \end{pmatrix},$$

where $\phi_{ij}, \phi_{ji} \in \mathbb{C}$ are nonzero for some $i \neq j$ and $\phi_{kk} \in H^0(\mathcal{O}_X)$ for all k .

Case $g \geq 2$. Here, $\deg(T_X) = 2 - 2g < 0$. To guarantee that $h^0(X, L_i^* \otimes L_j \otimes T_X) \neq 0$, we need $\deg(L_j) - \deg(L_i) + \deg(T_X) \geq 2g - 1$, which is equivalent to $\deg(L_i) - \deg(L_j) \leq 3 - 4g < 0$, in which case, there exists a nonzero $\phi_{ij} \in H^0(L_i^* \otimes L_j \otimes T_X)$ for some $i < j$. Conversely, if no such i, j exist, then we may have $H^0(X, L_i^* \otimes L_j \otimes T_X) = 0$ for all i, j , so Φ must be zero. Thus, in this case, we can write Φ locally as

$$\Phi = \begin{pmatrix} 0 & \phi_{12} & \cdots & \phi_{1r} \\ 0 & 0 & \cdots & \phi_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

where $\phi_{ij} \in H^0(L_i^* \otimes L_j \otimes T_X)$ is nonzero for some $i < j$. □

3.4.2 Holomorphic Poisson structures on ruled surfaces

We now give a classification of ruled surfaces admitting holomorphic Poisson structures and give an explicit description of the co-Higgs field inducing these Poisson structures when $g = 0, 1$.

Theorem 3.4.4. *Let X be a smooth projective curve of genus $g \geq 0$. The classification of rank 2 holomorphic vector bundles V on X admitting a nonzero traceless co-Higgs field Φ is the following:*

1. *If $g = 0$, then all rank 2 holomorphic vector bundle V on X admits nonzero traceless co-Higgs fields, and these correspond to holomorphic sections of $\text{End}_0 V \otimes \mathcal{O}(2)$.*
2. *If $g = 1$, then V admits a nonzero traceless co-Higgs field if and only if V is unstable. More specifically,*
 - (a) *If V is decomposable, then $V = L_1 \oplus L_2$ for some line bundles L_1 and L_2 on X with $\deg L_1 \geq \deg L_2$. And the traceless co-Higgs field on V are given by*

$$\Phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

with $a \in \mathbb{C}$, $b \in H^0(X, L_2^ \otimes L_1)$, and $c \in \mathbb{C}$, with $c = 0$ if $\deg L_1 > \deg L_2$.*

- (b) If V is indecomposable, then $V = F_2 \otimes L$ for some line bundle L on X , where F_2 is the unique non trivial extension

$$0 \rightarrow \mathcal{O} \xrightarrow{i_2} F_2 \xrightarrow{p_2} \mathcal{O} \rightarrow 0.$$

In this case, any traceless co-Higgs field on V is nilpotent and of the form

$$\Phi = c(p_2 \circ i_2)$$

for some $c \in \mathbb{C}$.

3. If $g \geq 2$ and V admits a nonzero traceless co-Higgs field, then V is unstable. However, not all unstable bundles admit nonzero traceless co-Higgs fields. Specifically, V admits a nonzero traceless co-Higgs field in the following cases:

- (a) V is decomposable and $\deg V \geq 3g - 2$ or $\deg V = 2g - 2$.
- (b) V is decomposable with $2g - 2 < \deg V \leq 3g - 3$ and $-K_X - \Lambda^2 V$ is effective.
- (c) V is indecomposable and $\deg V = 2g - 2$.

Finally, non-trivial Poisson structures on $\mathbb{P}(V)$ are in one-to-one correspondence with nonzero traceless co-Higgs field on V .

Proof. The proof follows from Proposition 3.4.1 and [5, Corollary 2.2]. The correspondence between nonzero traceless co-Higgs fields on V and non-trivial Poisson structures on $\mathbb{P}(V)$ follows from the work of Polishchuk [23] and is further elaborated in the classification of Poisson surfaces by Bartocci and Macri [5]. $\square$

3.4.3 Spectral curves and strong integrability

In sections 3.2 and 3.3, we saw that a necessary condition for co-Higgs field Φ on a holomorphic vector bundle V to induce an integrable Poisson structure on either V or $\mathbb{P}(V)$ is strong integrability. In this section, we discuss strong integrability for co-Higgs fields on curves of genus $g \geq 0$.

The case $g = 0$ was first investigated by Matviichuk in [21]. We start this section by providing a brief summary of his results. When $g = 0$, we have that $T_X = \mathcal{O}(2)$, and the characteristic polynomial of Φ has the form

$$\chi_\Phi = \theta^r + s_1(\Phi)\theta^{r-1} + \cdots + s_{r-1}(\Phi)\theta + s_r(\Phi)$$

with $s_k(\Phi) \in H^0(\mathbb{P}^1, S^k T_X) = H^0(\mathbb{P}^1, \mathcal{O}(2k))$. Note that $s_k(\Phi)$ is a homogeneous polynomial of degree $2k$ for all $k = 1, \dots, r$. Referring to Lemma 3.2.6, Φ is strongly integrable if and only if the eigenvalue vector fields $v_k \in H^0(\mathcal{U}, T_{\mathbb{P}^1})$ pairwise Lie commute on any open set $\mathcal{U} \subset \mathbb{P}^1$ where Φ can be diagonalized. Specifically, for a strongly integrable co-Higgs field Φ of rank r on $\mathbb{P}^1$, the characteristic polynomial must have the form

$$\chi_\Phi(\theta) = \theta^{r-km} \prod_{i=1}^m (\theta^k - \lambda_i \rho),$$

for some $1 \leq k \leq r$, $0 \leq m \leq r$ with $km \leq r$, $\lambda_i \in \mathbb{C}$, and $\rho \in H^0(\mathbb{P}^1, S^k T_{\mathbb{P}^1})$ [21, Proposition 4.0.1]. Furthermore, Matviichuk proves that for rank $r > 2$, if the spectral curve is smooth and connected, then the co-Higgs field Φ does not admit a Poisson structure on $\mathbb{P}(V)$ [21, Proposition 4.3.1].

We however also saw in sections 3.2 and 3.3 that strong integrability is not sufficient for the integrability of a quadratic Poisson structure. Given that all holomorphic vector on $\mathbb{P}^1$ are decomposable, we can nonetheless use Theorem 3.2.7 to classify all quadratic Poisson structures on vector or projective bundles on $\mathbb{P}^1$ thus improving the results of Matviichuk stated above.

Theorem 3.4.5. *Let $V = \mathcal{O}(m_1) \oplus \dots \oplus \mathcal{O}(m_r)$ be a rank r holomorphic vector bundle on $\mathbb{P}^1$ and let Φ be a co-Higgs field on V . The induced quadratic bivector σ_Φ on the total space of V is then integrable if and only if Φ is of one of the following two forms.*

1. An $r \times r$ matrix with only one nonzero row:

$$\Phi = \begin{pmatrix} 0 & \cdots & a_{1j}(z) & \cdots & 0 \\ 0 & \cdots & \vdots & \cdots & 0 \\ \vdots & \vdots & a_{ij}(z) & \vdots & \vdots \\ 0 & \cdots & \vdots & \cdots & 0 \\ 0 & \cdots & a_{r-1,j}(z) & \cdots & 0 \\ 0 & \cdots & a_{rj}(z) & \cdots & 0 \end{pmatrix}^T$$

where $a_{ij}(z) \in H^0(\mathbb{P}^1, \mathcal{O}(m_i - m_j + 2))$ for all $i = 1, \dots, r$ and fixed j . In this case, the spectral curve of Φ is reducible and is given by

$$S_\Phi : \theta^{r-1}(\theta - \sum_{i=1}^r a_{ij_0}(z)) = 0.$$

2. A function multiple of a constant matrix:

$$\Phi = a(z)A$$

where $a(z) \in H^0(\mathbb{P}^1, \mathcal{O}(2))$ and A is a constant $r \times r$ matrix. In this case, the spectral curve of Φ is given by

$$S_\Phi : \prod_{i=1}^r (\theta - a(z)\alpha_i) = 0,$$

where α_i are the eigenvalues of A .

Proof. Let $U_0 = \{[1 : z]\}$ and $U_1 = \{[w : 1]\}$ be an open cover of $\mathbb{P}^1$ with the transition function $w = 1/z$. Since V is locally trivial, in each chart we can express $\Phi = (a_{ij}(z))$ as an $r \times r$ matrix. The induced Poisson structure σ_Φ in local coordinates is given by

$$\sigma_\Phi = \sum_{i,j=1}^r a_{ij}(z) \partial_z \wedge y_i \partial_j$$

where $a_{ij}(z)$ are the entries of Φ in the local trivialization. Therefore, by Theorem 3.2.7, σ_Φ is integrable if and only if Φ takes the following two forms: Φ has only one nonzero row, say, the j -th row $(a_{1j}(z), \dots, a_{rj}(z))$, or $\Phi = a(z)A$ for some $a(z) \in H^0(\mathbb{C}, \mathcal{O}(\mathcal{U}))$ and a constant $r \times r$ matrix A . In the first case,

$$\text{char}(\Phi) = \theta^{r-1}(\theta - \sum_{i=1}^r a_{ij_0}(z)).$$

Thus, the spectral curve S_Φ is given by $\theta^{r-1}(\theta - \sum_{i=1}^r a_{ij_0}(z)) = 0$ and it is reducible.

In the second case,

$$\begin{aligned} \text{char}(\Phi) &= \det(\theta - \Phi) \\ &= \det(\theta - a(z)A) \\ &= \prod_{i=1}^r (\theta - a(z)\alpha_i) \end{aligned}$$

Therefore, the spectral curve S_Φ is given by $\prod_{i=1}^r (\theta - a(z)\alpha_i) = 0$. □

Remark 3.4.6. Since any integrable quadratic bivector on the total space of V induces an integrable quadratic bivector on $\mathbb{P}(V)$, the integrable quadratic bivectors σ_Φ of Theorem 3.4.5 produce integrable quadratic bivectors π_Φ for the projectivizations $\mathbb{P}(V)$.

We end this section by investigating the case where $g \geq 1$.

Proposition 3.4.7. *Let (V, Φ) be a rank r co-Higgs bundle on a smooth projective curve X of genus $g \geq 1$. Then, Φ is strongly integrable. In particular, the characteristic polynomial of Φ has the form*

$$\chi_\Phi = \theta^r + a_1\theta^{r-1} + \cdots + a_{r-1}\theta + a_r$$

for some $a_1, \dots, a_r \in \mathbb{C}$, which are all zero when $g \geq 2$.

Proof. For $g = 1$, we have $T_C \cong \mathcal{O}_C$. Therefore, $s_k(\Phi) \in H^0(C, \mathcal{O}_C) \cong \mathbb{C}$ for all k , and the characteristic polynomial of Φ takes the form $\chi_\Phi = \theta^r + a_1\theta^{r-1} + \cdots + a_{r-1}\theta + a_r$ where $a_i \in \mathbb{C}$. Since $s_k(\Phi)$ are constants, their Poisson brackets vanish $\{s_i(\Phi), s_j(\Phi)\} = 0$ for all i, j .

Now, for $g \geq 2$, we have $\deg(T_X) = 2 - 2g < 0$. This implies that $H^0(X, T_X) = 0$. Then $s_k(\Phi) = 0 \in H^0(X, T_X)$ for all k and the strongly integrability condition $\{s_i(\Phi), s_j(\Phi)\} = 0$ for all i, j is trivially satisfied. Moreover, the characteristic polynomial of Φ is $\chi_\Phi = \theta^r$. Therefore, $a_i = 0$ for $i = 1, \dots, r$.

Therefore, all co-Higgs bundles on a smooth projective curve of genus $g \geq 1$ are strongly integrable. $\square$

Remark 3.4.8. Note that in the case where $g \geq 2$, the characteristic polynomial of Φ is

$$\chi_\Phi = \theta^r.$$

Hence, all the eigenvalues of Φ are zero and Φ is nilpotent. Moreover, its spectral curve is

$$S_\Phi : \theta^r = 0,$$

which is non-reduced.

Whereas for $g = 1$, the characteristic polynomial of Φ takes the form

$$\chi_\Phi = \theta^r + a_1\theta^{r-1} + \cdots + a_{r-1}\theta + a_r$$

for some $a_i \in \mathbb{C}$. This implies that the eigenvalues of Φ are constant over $\mathbb{C}$. Thus, we can factor the characteristic polynomial as

$$\chi_\Phi = (\theta - \alpha_1)^{m_1}(\theta - \alpha_2)^{m_2} \cdots (\theta - \alpha_s)^{m_s},$$

where $\alpha_1, \dots, \alpha_s \in \mathbb{C}$ are the distinct eigenvalues of Φ whose multiplicities are $m_1, \dots, m_s$, respectively, so that $m_1 + \cdots + m_s = r$. The spectral curve of Φ is then

$$S_\Phi : (\theta - \alpha_1)^{m_1}(\theta - \alpha_2)^{m_2} \cdots (\theta - \alpha_s)^{m_s} = 0,$$

which is reducible. If the eigenvalues α_i all have multiplicity $m_i = 1$, then S_Φ is smooth with irreducible components $\theta = \alpha_1, \dots, \theta = \alpha_r$. Otherwise, the components with $m_i > 1$ are non-reduced so that S_Φ is not smooth.

3.5 Strong integrability condition for rank 3 co-Higgs bundles over $\mathbb{P}^1 \times \mathbb{P}^1$

In this section, we give a characterization of the characteristic polynomial of a strongly integrable co-Higgs field over $\mathbb{P}^1 \times \mathbb{P}^1$.

Let $\mathcal{M}^{\text{co}}(3)$ be the moduli space of traceless rank 3 semistable co-Higgs bundles over $\mathbb{P}^1 \times \mathbb{P}^1$. The Hitchin map is defined as follows:

$$\begin{aligned} H : \mathcal{M}^{\text{co}}(3) &\rightarrow H^0(S^3(T_{\mathbb{P}^1 \times \mathbb{P}^1})) \\ (V, \Phi) &\mapsto \text{char}(\Phi) \end{aligned}$$

where $S^3(T_{\mathbb{P}^1 \times \mathbb{P}^1}) \cong \mathcal{O}(6, 0) \oplus \mathcal{O}(4, 2) \oplus \mathcal{O}(2, 4) \oplus \mathcal{O}(0, 6)$.

Let (x_1, x_2) be local coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$, and $e_1 = \frac{\partial}{\partial x_1}$, $e_2 = \frac{\partial}{\partial x_2}$ be the corresponding basis elements for the tangent bundle $T_{\mathbb{P}^1 \times \mathbb{P}^1}$. Then on an open set $\mathcal{U}$ we can write

$$\Phi = \Phi_1 \otimes e_1 + \Phi_2 \otimes e_2.$$

with $\Phi_1 = (a_{ij})$, $\Phi_2 = (b_{ij})$ where $a_{ij} \in H^0(\mathcal{U}, \mathcal{O}(2, 0))$, $b_{ij} \in H^0(\mathcal{U}, \mathcal{O}(0, 2))$ and $\text{Tr}(\Phi_1) = 0$, $\text{Tr}(\Phi_2) = 0$.

Since Φ is traceless the characteristic polynomial is given by $\eta^3 - \frac{1}{2} \text{tr}(\Phi^2)\eta - \det(\Phi)$ where η is the tautological section of the pullback of $T_{\mathbb{P}^1 \times \mathbb{P}^1}$ to its own total space. The spectral surface S_ρ associated to $\rho = \text{char}(\Phi)$ is given by those points $y \in \text{Tot}(T_{\mathbb{P}^1 \times \mathbb{P}^1})$ such that

$$\text{char}(\Phi(y)) = \eta^3(y) - \frac{1}{2} \text{tr}(\Phi^2)\eta(y) - \det \Phi(\theta(y)) = 0$$

where $\theta : T_{\mathbb{P}^1 \times \mathbb{P}^1} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$.

Let $\eta = \eta_1 + \eta_2$ where η_1 and η_2 are the tautological sections of the pullback of $\mathcal{O}(2, 0)$, and $\mathcal{O}(0, 2)$, respectively, to their own total spaces. Then the characteristic equation of Φ becomes

$$\text{char}(\Phi(y)) = (\eta_1 + \eta_2)^3(y) - \frac{1}{2} \text{tr}(\Phi^2)(\eta_1 + \eta_2)(y) - \det \Phi(\theta(y)) = 0$$

Note that $\det(\Phi) = \frac{1}{3}(\Phi^3) = \frac{1}{3}(\Phi \otimes \Phi \otimes \Phi) = \frac{1}{3} \text{tr}(\Phi_1^3 e_1^3 + (3\Phi_1^2 \Phi_2) e_1^2 \otimes e_2 + (3\Phi_1 \Phi_2^2) e_2^2 \otimes e_1 + \Phi_2^3 e_2^3)$ and $\frac{1}{2} \text{tr}(\Phi^2) = \frac{1}{2}(\Phi_1^2 e_1^2 + (2\Phi_1 \Phi_2) e_1 \otimes e_2 + \Phi_2^2 e_2^2)$.

If $H(V, \Phi) = (\rho_1, \rho_{1,2}, \rho_{2,1}, \rho_2)$ we have that the equations of the spectral surface are:

$$\text{Coefficient of } e_1^3 (\rho_1) : \eta_1^3 - \frac{1}{2} \text{tr}(\Phi_1^2) \eta_1 - \frac{1}{3} \text{tr}(\Phi_1^3) = 0 \quad (3.11)$$

$$\text{Coefficient of } e_1^2 e_2 \text{ } (\rho_{1,2}) : 3\eta_1^2 \eta_2 - \frac{1}{2} \text{tr}(\Phi_1^2) \eta_2 - \text{tr}(\Phi_1 \Phi_2) \eta_1 - \text{tr}(\Phi_1^2 \Phi_2) = 0 \quad (3.12)$$

$$\text{Coefficient of } e_1 e_2^2 \text{ } (\rho_{2,1}) : 3\eta_1 \eta_2^2 - \frac{1}{2} \text{tr}(\Phi_2^2) \eta_1 - \text{tr}(\Phi_1 \Phi_2) \eta_2 - \text{tr}(\Phi_1 \Phi_2^2) = 0 \quad (3.13)$$

$$\text{Coefficient of } e_2^3 \text{ } (\rho_2) : \eta_2^3 - \frac{1}{2} \text{tr}(\Phi_2^2) \eta_2 - \frac{1}{3} \text{tr}(\Phi_2^3) = 0 \quad (3.14)$$

Example 3.5.1. Let V be a trivial rank 3 vector bundle over $\mathbb{P}^1 \times \mathbb{P}^1$. We define the co-Higgs field Φ as follows:

$$\Phi = \Phi_1 \otimes e_1 + \Phi_2 \otimes e_2 = \begin{pmatrix} 2\omega_1 & 0 & 0 \\ 0 & -\omega_1 & 0 \\ 0 & 0 & -\omega_1 \end{pmatrix} \otimes e_1 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \omega_2 & 0 \\ 0 & 0 & -\omega_2 \end{pmatrix} \otimes e_2.$$

where $\omega_1 \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2, 0))$ and $\omega_2 \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(0, 2))$ are nonzero sections. In this case we have that $\text{tr}(\Phi_1^2) = 6\omega_1^2$, $\text{tr}(\Phi_2^2) = 2\omega_2^2$, $\text{tr}(\Phi_1^3) = 6\omega_1^3$, $\text{tr}(\Phi_2^3) = 0$, $\text{tr}(\Phi_1 \Phi_2) = 0$, $\text{tr}(\Phi_1^2 \Phi_2) = 0$, and $\text{tr}(\Phi_1 \Phi_2^2) = 0$. Therefore, the equations of the spectral surface are given by:

$$\eta_1^3 - 3\omega_1^2 \eta_1 - 2\omega_1^3 = 0 \quad (3.15)$$

$$3\eta_1^2 \eta_2 = 0 \quad (3.16)$$

$$3\eta_1 \eta_2^2 - \omega_2^2 \eta_1 = 0 \quad (3.17)$$

$$\eta_2^3 - \omega_2^2 \eta_2 = 0 \quad (3.18)$$

From equation (3.16) we have two cases: In the first case if $\eta_2 \neq 0$, then $\eta_1 = 0$ and thus we have $\eta_2^3 - \omega_2^2 \eta_2 = 0$. This factors as $\eta_2(\eta_2^2 - \omega_2^2) = 0$, giving us a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ in the η_2 direction, plus the zero section $\eta_1 = 0$. In the second case we have that if $\eta_1 \neq 0$, then $\eta_2 = 0$. Therefore we have $\eta_1^3 - 3\omega_1^2 \eta_1 - 2\omega_1^3 = 0$. This is a triple cover of $\mathbb{P}^1 \times \mathbb{P}^1$ in the η_1 direction.

Proposition 3.5.2. Let Φ be a co-Higgs bundle of rank 3 on $\mathbb{P}^1 \times \mathbb{P}^1$. Then if Φ is strongly integrable its characteristic polynomial has the form

$$\begin{aligned} \chi_\Phi(\theta) = & \theta^3 - (f_1 e_1^2 + f_{1,2} e_1 \otimes e_2 + f_{2,1} e_2 \otimes e_1 + f_2 e_2^2) \theta \\ & - (g_1 e_1^3 + g_{1,2} e_1^2 \otimes e_2 + g_{2,1} e_1 \otimes e_2^2 + g_2 e_2^3) \end{aligned}$$

where $f_1, f_2 \in H^0(\mathbb{P}^1, \mathcal{O}(4))$, $f_{1,2}, f_{2,1} \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2, 2))$, $g_1 \in H^0(\mathbb{P}^1, \mathcal{O}(6))$, $g_2 \in H^0(\mathbb{P}^1, \mathcal{O}(6))$, $g_{1,2} \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(4, 2))$, $g_{2,1} \in H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2, 4))$, and there exist $\alpha_1, \alpha_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \epsilon_1, \epsilon_2 \in \mathbb{C}^*$ such that

$$\begin{aligned} f_1 &= \alpha_1 \omega_1^2, & f_2 &= \alpha_2 \omega_2^2, & f_{1,2} &= \gamma_1 \omega_1 \omega_2, & f_{2,1} &= \gamma_2 \omega_1 \omega_2 \\ g_1 &= \delta_1 \omega_1^3, & g_2 &= \delta_2 \omega_2^3, & g_{1,2} &= \epsilon_1 \omega_1^2 \omega_2, & g_{2,1} &= \epsilon_2 \omega_1 \omega_2^2 \end{aligned}$$

Proof. The characteristic polynomial of Φ is given by $\det(\theta - \Phi) = \theta^3 - s_1(\Phi)\theta^2 + s_2(\Phi)\theta - s_3(\Phi)$, and since Φ is traceless we have that $s_1(\Phi) = \text{tr}(\Phi) = 0$. Let (x_1, x_2) be affine coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\xi_1 = dx_1, \xi_2 = dx_2$ the corresponding cotangent coordinates. We can express $s_2(\Phi)$ and $s_3(\Phi)$ as:

$$\begin{aligned} s_2(\Phi) &= f_1(x_1)\xi_1^2 + f_{1,2}(x_1, x_2)\xi_1\xi_2 + f_2(x_2)\xi_2^2 \\ s_3(\Phi) &= g_1(x_1)\xi_1^3 + g_{1,2}(x_1, x_2)\xi_1^2\xi_2 + g_{2,1}(x_1, x_2)\xi_1\xi_2^2 + g_2(x_2)\xi_2^3 \end{aligned}$$

where f_1, f_2 are polynomials of degree 4, $f_{1,2}$ is a polynomial of bi-degree $(2, 2)$, g_1, g_2 are polynomials of degree 6, $g_{1,2}$ is a polynomial of bi-degree $(4, 2)$, and $g_{2,1}$ is a polynomial of bi-degree $(2, 4)$. The Poisson bracket on $T^*(\mathbb{P}^1 \times \mathbb{P}^1)$ is given by

$$\{F, G\} = \frac{\partial F}{\partial x_1} \frac{\partial G}{\partial \xi_1} + \frac{\partial F}{\partial x_2} \frac{\partial G}{\partial \xi_2} - \frac{\partial F}{\partial \xi_1} \frac{\partial G}{\partial x_1} - \frac{\partial F}{\partial \xi_2} \frac{\partial G}{\partial x_2}$$

Now since Φ is strongly integrable we have that $\{s_2(\Phi), s_3(\Phi)\} = 0$. Therefore, we have the following partial differential equations:

$$\text{Terms with } \xi_1^4 : 3f_1'(x_1)g_1(x_1) - 2f_1(x_1)g_1'(x_1) = 0$$

$$\begin{aligned} \text{Terms with } \xi_1^3\xi_2 : 2f_1'(x_1)g_{1,2}(x_1, x_2) + 3\frac{\partial f_{1,2}}{\partial x_1}(x_1, x_2)g_1(x_1) + \frac{\partial f_{1,2}}{\partial x_2}(x_1, x_2)g_{1,2} \\ - 2f_1(x_1)\frac{\partial g_{1,2}}{\partial x_1}(x_1, x_2) - f_{1,2}(x_1, x_2)g_1'(x_1) - f_{1,2}\frac{\partial g_{1,2}}{\partial x_2}(x_1, x_2) = 0 \end{aligned}$$

$$\begin{aligned} \text{Terms with } \xi_1^2\xi_2^2 : f_1'(x_1)g_{2,1}(x_1, x_2) + 2\frac{\partial f_{1,2}}{\partial x_1}(x_1, x_2)g_{1,2}(x_1, x_2) + 2\frac{\partial f_{1,2}}{\partial x_2}(x_1, x_2)g_{2,1}(x_1, x_2) \\ + f_2'g_{1,2} - 2f_1(x_1)\frac{\partial g_{2,1}}{\partial x_1}(x_1, x_2) - f_{1,2}(x_1, x_2)\frac{\partial g_{1,2}}{\partial x_1}(x_1, x_2) \\ - 2f_2(x_2)\frac{\partial g_{1,2}}{\partial x_2}(x_1, x_2) - f_{1,2}\frac{\partial g_{2,1}}{\partial x_2}(x_1, x_2) = 0 \end{aligned}$$

$$\begin{aligned} \text{Terms with } \xi_1\xi_2^3 : 3\frac{\partial f_{1,2}}{\partial x_2}(x_1, x_2)g_2(x_2) + 2f_2'(x_2)g_{2,1}(x_1, x_2) + g_{2,1}\frac{\partial f_{1,2}}{\partial x_1}(x_1, x_2) \\ - f_{1,2}(x_1, x_2)\frac{\partial g_{2,1}}{\partial x_1}(x_1, x_2) - 2f_2(x_2)\frac{\partial g_{2,1}}{\partial x_2}(x_1, x_2) - f_{1,2}g_2' = 0 \end{aligned}$$

$$\text{Terms with } \xi_2^4 : 3f_2'(x_2)g_2(x_2) - 2f_2(x_2)g_2'(x_2) = 0$$

From the first partial differential equation we have $3f_1'(x_1)g_1(x_1) - 2f_1(x_1)g_1'(x_1) = 0$. This implies that $f_1(x_1) = \alpha_1\omega_1(x_1)^2, g_1(x_1) = \alpha_1^{3/2}\omega_1(x_1)^3$ for some polynomial ω_1 of degree

2 and some $\alpha_1 \in \mathbb{C}^*$. Indeed, dividing both sides by $f_1(x_1)g_1(x_1)$, assuming neither is zero, we have $\frac{3f_1'(x_1)}{f_1(x_1)} - \frac{2g_1'(x_1)}{g_1(x_1)} = 0$. Therefore we have $f_1(x_1)^3 = Kg_1(x_1)^2$ where K is a positive constant. Recall that $f_1(x_1)$ is a polynomial of degree 4 and $g_1(x_1)$ is a polynomial of degree 6. For the equation $f_1(x_1)^3 = Kg_1(x_1)^2$ to hold, there must exist a polynomial $\omega_1(x_1)$ of degree 2 such that $f_1(x_1) = \alpha_1\omega_1(x_1)^2$, and $g_1(x_1) = \alpha_1^{3/2}\omega_1(x_1)^3 = \delta_1\omega_1(x_1)^3$, where $\alpha_1 \in \mathbb{C}^*$. Similarly, from the last partial differential equation we have that there exists a degree 2 polynomial $\omega_2(x_2)$ such that $f_2(x_2) = \alpha_2\omega_2(x_2)^2$, and $g_2(x_2) = \delta_2\omega_2(x_2)^3$. We can also see that $f_{1,2} = \gamma_1\omega_1(x_1)\omega_2(x_2)$, $f_{2,1} = \gamma_2\omega_1(x_1)\omega_2(x_2)$, $g_{1,2} = \epsilon_1\omega_1^2(x_1)\omega_2(x_2)$, and $g_{2,1} = \epsilon_2\omega_1(x_1)\omega_2^2(x_2)$ are solutions to the other partial differential equations. $\square$

Proposition 3.5.3. *Let Φ be a co-Higgs bundle of rank 3 on $\mathbb{P}^1 \times \mathbb{P}^1$. Then if Φ is strongly integrable its characteristic polynomial has the form*

$$\chi_\Phi(\theta) = (\theta - \lambda_1\omega_1e_1 - \mu_1\omega_2e_2)(\theta - \lambda_2\omega_1e_1 - \mu_2\omega_2e_2)(\theta - \lambda_3\omega_1e_1 - \mu_3\omega_2e_2)$$

where $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3 \in \mathbb{C}^*$ and $\omega_1, \omega_2 \in H^0(\mathbb{P}^1, \mathcal{O}(2))$.

Proof. Expanding $\chi_\Phi(\theta)$ we have

$$\begin{aligned} \chi_\Phi(\theta) = & \theta^3 - (\lambda_1 + \lambda_2 + \lambda_3)\omega_1e_1\theta^2 - (\mu_1 + \mu_2 + \mu_3)\omega_2e_2\theta^2 \\ & + [(\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\omega_1^2e_1^2 \\ & + (\lambda_1\mu_2 + \lambda_1\mu_3 + \lambda_2\mu_1 + \lambda_2\mu_3 + \lambda_3\mu_1 + \lambda_3\mu_2)\omega_1\omega_2(e_1 \otimes e_2 + e_2 \otimes e_1) \\ & + (\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3)\omega_2^2e_2^2]\theta \\ & - [\lambda_1\lambda_2\lambda_3\omega_1^3e_1^3 \\ & + (\lambda_1\lambda_2\mu_3 + \lambda_1\lambda_3\mu_2 + \lambda_2\lambda_3\mu_1)\omega_1^2\omega_2e_1^2 \otimes e_2 \\ & + (\lambda_1\mu_2\mu_3 + \lambda_2\mu_1\mu_3 + \lambda_3\mu_1\mu_2)\omega_1\omega_2^2e_1 \otimes e_2^2 \\ & + \mu_1\mu_2\mu_3\omega_2^3e_2^3] \end{aligned}$$

Now, matching the corresponding coefficients we get

$$\begin{aligned} \rho_1 &= (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3)\omega_1^2 = \alpha_1\omega_1^2 \\ \rho_2 &= (\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3)\omega_2^2 = \alpha_2\omega_2^2 \\ \rho_{1,2} + \rho_{2,1} &= (\lambda_1\mu_2 + \lambda_1\mu_3 + \lambda_2\mu_1 + \lambda_2\mu_3 + \lambda_3\mu_1 + \lambda_3\mu_2)\omega_1\omega_2 = (\gamma_1 + \gamma_2)\omega_1\omega_2 \\ \sigma_1 &= \lambda_1\lambda_2\lambda_3\omega_1^3 = \delta_1\omega_1^3 \\ \sigma_2 &= \mu_1\mu_2\mu_3\omega_2^3 = \delta_2\omega_2^3 \\ \sigma_{1,2} &= (\lambda_1\lambda_2\mu_3 + \lambda_1\lambda_3\mu_2 + \lambda_2\lambda_3\mu_1)\omega_1^2\omega_2 = \epsilon_1\omega_1^2\omega_2 \\ \sigma_{2,1} &= (\lambda_1\mu_2\mu_3 + \lambda_2\mu_1\mu_3 + \lambda_3\mu_1\mu_2)\omega_1\omega_2^2 = \epsilon_2\omega_1\omega_2^2 \end{aligned}$$

Therefore we arrive to the same expression of the characteristic polynomial of Φ in Proposition [3.5.2](#). $\square$

Chapter 4

Stable co-Higgs bundles and Poisson structures

In this chapter, we investigate how the geometry properties of a co-Higgs bundle (V, Φ) influence those of the induced quadratic Poisson structure π_Φ on $\mathbb{P}(V)$. In order to do this, we define the eigenvariety E of the co-Higgs field Φ as the projectivization of the set

$$\{(x, v) \in X \times V \mid x \in X, 0 \neq v \in V|_x \text{ such that } \Phi_x(v) \wedge v = 0 \in \wedge^2 V|_x \otimes T|_{X,x}\},$$

which was first introduced by Matviichuk in [21], and prove that it is equal to the set of zeroes of π_Φ in section 4.1. We then study the connection between Φ -invariant subbundles of V and Poisson subvarieties of $\mathbb{P}(V)$ as well as between the eigenvariety and the spectral variety of the co-Higgs field Φ in section 4.2. We end the chapter by analyzing holomorphic Poisson structures on Hizebruch surfaces in section 4.3, in particular, classifying the ones coming from (semi)stable co-Higgs pairs.

4.1 Eigenvarieties and zeroes of quadratic bivectors

Let (V, Φ) be a co-Higgs bundle on a complex manifold X and π_Φ be the induced quadratic holomorphic bivector on $\mathbb{P}(V)$. In this section, we do not assume that π_Φ is an integrable Poisson structure on $\mathbb{P}(V)$. We study instead the connection between the eigenvariety of Φ (which we define below) and the set of zeroes of its corresponding quadratic holomorphic bivector π_Φ . Understanding how these two objects are related is important in determining what the properties of the co-Higgs field Φ tell us about the zeroes of the bivector π_Φ .

We start by defining the eigenvariety of the co-Higgs field Φ . This definition was first introduced by Matviichuk in [21].

Definition 4.1.1. Let V be a vector bundle over a complex manifold X equipped with a co-Higgs field $\Phi \in H^0(X, \text{End}(V) \otimes T_X)$. The **eigenvariety** $E \subset \mathbb{P}(V)$ of the co-Higgs field Φ is the projectivization of the set given by

$$\{(x, v), x \in X, 0 \neq v \in V_x \text{ such that } \Phi_x(v) \wedge v = 0 \in \wedge^2 V_x \otimes T_{X,x}\} \subset V.$$

Note that if v is an eigenvector of $\Phi_x : V_x \rightarrow V_x \otimes T_{X,x}$, $x \in X$, then there exists $\lambda \in \mathbb{C}$ such that $\Phi_x(v) = \lambda v \otimes t$ for some $t \in T_{X,x}$. Therefore,

$$\Phi_x(v) \wedge v = (\lambda v \otimes t) \wedge v = \lambda(v \wedge v) \otimes t = 0,$$

since $v \wedge v = 0$ in $\wedge^2 V_x$. Conversely, suppose that $0 \neq v \in V_x$ is such that $\Phi_x(v) \wedge v = 0$ in $\wedge^2 V_x \otimes T_{X,x}$. This means that $\Phi_x(v)$ and v are linearly dependent in $V_x \otimes T_{X,x}$. Since $v \neq 0$, there must exist $\lambda \in \mathbb{C}$ and $t \in T_{X,x}$ such that $\Phi_x(v) = \lambda v \otimes t$. But this precisely means that v is an eigenvector of Φ_x with eigenvalue λ . Therefore, the eigenvariety E consists of all points $(x, [v]) \in \mathbb{P}(V)$ where $x \in X$ and $[v]$ is a line in V_x such that v is an eigenvector of the co-Higgs field Φ at x . In other words, the eigenvariety of Φ is also defined as

$$E = \{(x, [v]) \in \mathbb{P}(V) \mid \Phi_x(v) = \lambda v \text{ for some } \lambda \in \mathbb{C}\} \subset \mathbb{P}(V).$$

Example 4.1.2. Let $X = \mathbb{C}$ with coordinate x , and let $V = \mathcal{O} \oplus \mathcal{O}$ be the trivial rank 2 vector bundle over X . We define a traceless co-Higgs field Φ on V by

$$\Phi = \begin{pmatrix} x & 1 \\ 1 & -x \end{pmatrix} \otimes \partial_x.$$

Given that X is 1-dimensional, the co-Higgs field Φ vacuously satisfies the integrability condition $\Phi \wedge \Phi = 0$. Let us then compute the eigenvariety E of Φ . The eigenvalues of Φ are given by the characteristic equation

$$\det(\Phi - \lambda I) = \lambda^2 - x^2 - 1 = 0.$$

We thus have $\lambda = \pm\sqrt{x^2 + 1}$, and the eigenvectors are of the form (v_1, v_2) for two holomorphic functions satisfying

$$(x \pm \sqrt{x^2 + 1})v_1 + v_2 = 0.$$

The eigenvariety of Φ is then the subvariety of $\mathbb{P}(V) = \mathbb{P}^1 \times \mathbb{P}^1$ given by

$$E = \left\{ \left(x, \left[1 : - \left(x + \sqrt{x^2 + 1} \right) \right] \right) \mid x \in \mathbb{C} \right\} \cup \left\{ \left(x, \left[1 : - \left(x - \sqrt{x^2 + 1} \right) \right] \right) \mid x \in \mathbb{C} \right\}.$$

Matviichuk in Lemma 3.3.4 [21] proved that for a diagonalizable traceless co-Higgs field Φ on a rank r vector bundle V over an analytic open set $\mathcal{U} \subset \mathbb{C}^n$, the eigenvariety coincides with the zero set of π on $\mathbb{P}(V)$. The following theorem shows that the eigenvariety E of Φ is isomorphic to the zero set of the quadratic bivector π_Φ on $\mathbb{P}(V)$ obtained from Φ even when Φ is not diagonalizable.

Theorem 4.1.3. *Let (V, Φ) be a rank r zero trace co-Higgs bundle over an n -dimensional complex manifold X , and π_Φ a quadratic bivector on $\mathbb{P}(V)$ with coisotropic fibers induced by the co-Higgs field Φ on V . Then the eigenvariety $E \subset \mathbb{P}(V)$ of Φ is equal to the zero set of π_Φ .*

Proof. To simplify the notation in the proof, we set $\pi = \pi_\Phi$. We first verify that $\pi((x, [v])) = 0$ for all $(x, [v]) \in E$, implying that the zero set of π is contained in the eigenvariety E of Φ . Let $(x, [v])$ be a point in E so that $x \in X$ and $[v]$ is the subspace of V_x spanned by the nonzero vector $v \in V_x$. Since $(x, [v]) \in E$, we have $\Phi_x(v) = \lambda(x)v$ for some scalar $\lambda(x) \in \mathbb{C}$. Choose a local coordinate chart $(\mathcal{U}, (x_1, \dots, x_n))$ on X containing x and a local trivialization of V over $\mathcal{U}$. In this trivialization, we can express Φ as a matrix-valued vector field:

$$\Phi = \sum_{i=1}^n A_i(x) \otimes \partial_{x_i} = (a_{ijk}(x))^T \otimes \partial_{x_i},$$

where $a_{ijk}(x) = a_{ijk}(x_1, \dots, x_n)$ are holomorphic functions on $\mathcal{U}$ for all $j, k = 0, \dots, r-1$. Let $v \in \mathbb{C}^r \setminus 0$. The condition $\Phi_x(v) = \lambda(x)v$ can be written in local coordinates as:

$$\sum_{j=0}^{r-1} a_{ijk}(x) v_j = \lambda(x) v_k$$

for each $k = 0, \dots, r-1$, where $r-1$. Using projective coordinates $\hat{y}_k = y_k/y_0$ on $\mathbb{P}(V)$, this equation becomes:

$$\sum_{j=0}^{r-1} a_{ijk} \hat{y}_j = \lambda \hat{y}_k.$$

When $k = 0$, we have

$$\lambda = \sum_{j=0}^{r-1} a_{ij0} \hat{y}_j \tag{4.1}$$

since $\hat{y}_0 = 1$, and when $1 \leq k \leq r-1$, we have

$$\sum_{j=0}^{r-1} a_{ijk} \hat{y}_j = \lambda \hat{y}_k. \tag{4.2}$$

Now, consider the vector field $X = \sum_{j,k=0}^{r-1} a_{ijk}(x) y_j \frac{\partial}{\partial y_k}$ on $V|_{\mathcal{U}}$. Let

$$\varphi_0 : U_0 \rightarrow \mathbb{C}^{r-1}, [y_0 : \cdots : y_{r-1}] \mapsto (\hat{y}_1, \dots, \hat{y}_{r-1}),$$

be the standard affine chart on $\mathbb{P}(V|_{\mathcal{U}})$, where $U_0 = \{[y_0 : y_1 : \cdots : y_{r-1}] \in \mathbb{P}(V|_{\mathcal{U}}) \mid y_0 \neq 0\}$. Then,

$$\begin{aligned} Y = \varphi_{0,*}(X) &= \varphi_{0,*} \left(\sum_{j,k=0}^{r-1} a_{ijk}(x) y_j \frac{\partial}{\partial y_k} \right) \\ &= \sum_{j=0}^{r-1} a_{ij0}(x) \varphi_{0,*} \left(y_j \frac{\partial}{\partial y_0} \right) + \sum_{j=0,k=1}^{r-1} a_{ijk}(x) \varphi_{0,*} \left(y_j \frac{\partial}{\partial y_k} \right). \end{aligned}$$

Using the fact that

$$\varphi_{0,*} \left(y_j \frac{\partial}{\partial y_k} \right) = \begin{cases} \hat{y}_j \frac{\partial}{\partial \hat{y}_k}, & \text{if } k \neq 0, \\ -\hat{y}_j \sum_{l=1}^{r-1} \left(\hat{y}_l \frac{\partial}{\partial \hat{y}_l} \right), & \text{if } k = 0, \end{cases}$$

we have

$$\begin{aligned} Y &= - \sum_{j=0}^{r-1} \sum_{l=1}^{r-1} a_{ij0}(x) \hat{y}_j \hat{y}_l \frac{\partial}{\partial \hat{y}_l} + \sum_{j=0}^{r-1} \sum_{k=1}^{r-1} a_{ijk}(x) \hat{y}_j \frac{\partial}{\partial \hat{y}_k} \\ &= - \sum_{k=1}^{r-1} \left(\sum_{j=0}^{r-1} a_{ij0}(x) \hat{y}_j \right) \hat{y}_k \frac{\partial}{\partial \hat{y}_k} + \sum_{k=1}^{r-1} \left(\sum_{j=0}^{r-1} a_{ijk}(x) \hat{y}_j \right) \frac{\partial}{\partial \hat{y}_k} \end{aligned}$$

Substituting the equations (4.1) and (4.2), we find that $Y = 0$ at the point $(x, [v])$. By the correspondence between vector fields on $\mathbb{P}(V)$ tangent to the fibers and trace-free endomorphisms of V , the vanishing of Y at $(x, [v])$ implies that $\pi^\#(\rho^*\alpha) = 0$ for all $\alpha \in T_x^*X$, where $\rho : \mathbb{P}(V) \rightarrow X$ is the projection. To show that $\pi^\#$ is zero for all $\alpha \in T_{(x,[v])}^*\mathbb{P}(V)$, we use the decomposition $T_{(x,[v])}\mathbb{P}(V) = T_F \oplus H$, where T_F is the tangent space to the fiber and H is the horizontal space. The Poisson structure π respects this decomposition because its fibers are coisotropic. Since $\pi^\#(\rho^*\alpha) = 0$ for all $\alpha \in T_x^*X$, and π vanishes on $T_F \times T_F$ (due to the coisotropic condition), we conclude that $\pi^\# : T_{(x,[v])}^*\mathbb{P}(V) \rightarrow T_{(x,[v])}\mathbb{P}(V)$ is the zero map.

Conversely, suppose $\pi((x, [v])) = 0$ for some $(x, [v]) \in \mathbb{P}(V)$. This means that $\pi^\#(\rho^*\alpha) = 0$ for all $\alpha \in T_x^*X$. By the correspondence between vector fields on $\mathbb{P}(V)$ tangent to the fibers and zero trace endomorphisms of V , this implies that the vector field Y vanishes

at $(x, [v])$. From the calculation of Y above, we can see that this is only possible if $\Phi_x(v) = \lambda(x)v$ for some scalar $\lambda(x)$, which means $(x, [v]) \in E$. Now we can cover X with coordinate charts and apply the same reasoning on each chart. The holomorphicity of Φ ensures that these local results glue together to give a global correspondence between E and the zero set of π . Therefore, we have shown that the eigenvariety E is precisely the zero set of the Poisson structure π over all of X . $\square$

Remark 4.1.4. If Φ is the zero co-Higgs field, then π_Φ is the zero Poisson structure on $\mathbb{P}(V)$ and its zero set (and thus, eigenvariety) is all of $\mathbb{P}(V)$. We are therefore interested in the case where $\Phi \neq 0$.

Example 4.1.5. Consider the co-Higgs bundle we defined in example 4.1.2, where $X = \mathbb{C}$ with coordinate x , and $(V = \mathcal{O} \oplus \mathcal{O}, \Phi)$ with Φ given by

$$\Phi = \begin{pmatrix} x & 1 \\ 1 & -x \end{pmatrix} \otimes \partial_x.$$

The corresponding quadratic bivector is then

$$\pi_\Phi = 2x\partial_x \wedge \hat{y}_1\partial_{\hat{y}_1} + \partial_x \wedge \hat{y}_1^2\partial_{\hat{y}_1} + \partial_x \wedge \partial_{\hat{y}_1} = (2x\hat{y}_1 + \hat{y}_1^2 + 1)\partial_x \wedge \partial_{\hat{y}_1},$$

where $\hat{y}_1 = y_1/y_2$ is the affine coordinate on the fibers of $\mathbb{P}(V)$. Note that π_Φ is integrable since $\mathbb{P}(V)$ is two-dimensional, giving us a quadratic Poisson structure on $\mathbb{P}(V)$, and it vanishes when $2x\hat{y}_1 + \hat{y}_1^2 + 1 = 0$. Solving this equation for $\hat{y}_1$, we obtain two components $\hat{y}_1 = -x \pm \sqrt{x^2 + 1}$. The component $\hat{y}_1 = -(x + \sqrt{x^2 + 1})$ corresponds to

$$\left\{ \left(x, \left[1 : - \left(x + \sqrt{x^2 + 1} \right) \right] \right) \mid x \in \mathbb{C} \right\}$$

in homogeneous coordinates, and the second component $\hat{y}_1 = -(x - \sqrt{x^2 + 1})$ is given by

$$\left\{ \left(x, \left[1 : - \left(x - \sqrt{x^2 + 1} \right) \right] \right) \mid x \in \mathbb{C} \right\}.$$

These are precisely the two components of E we computed earlier.

Let (V, Φ) be a rank r co-Higgs bundles on X and denote by $\rho : \mathbb{P}(V) \rightarrow X$ the natural projection. Suppose that Φ is not the zero co-Higgs field. The following theorem describes the zeroes of the induced quadratic bivector π_Φ on $\mathbb{P}(V)$ as the disjoint union of two distinct proper subvarieties of $\mathbb{P}(V)$. These subvarieties consist of the preimage $\rho^{-1}(D)$, representing points in $\mathbb{P}(V)$ over the divisor D of zeroes of Φ in X , and a subvariety $B \subset X$.

Theorem 4.1.6. *Let V be a rank r holomorphic vector bundle on X together with a nonzero co-Higgs field $\Phi \in H^0(X, \text{End}_0(V) \otimes T_X)$. Furthermore, let $\rho : \mathbb{P}(V) \rightarrow X$ the natural projection and π_Φ be the induced quadratic bivector on $\mathbb{P}(V)$. Then,*

$$\text{Zeroes}(\pi_\Phi) = B + \rho^{-1}(D),$$

where D is the divisor of zeroes of Φ in X and B is a subvariety of $\mathbb{P}(V)$ that projects onto X . In fact, if $r = 2$ or if Φ_x has distinct eigenvalues for all $x \in X$, then B is a finite cover of X , which is an $r : 1$ cover in the latter case.

Proof. Let $x \in X$ be a point and consider the fibre $\rho^{-1}(x) = \mathbb{P}(V_x)$ of ρ over x . Then π_Φ vanishes at a point $[v] \in \mathbb{P}(V_x)$ if and only if $v \in V_x$ is an eigenvector of $\Phi_x \in \text{End}_0(V_x)$ by Theorem 4.1.3. If $x \in D$, then $\Phi_x = 0$ on V_x , so every vector in V_x is an eigenvector of Φ_x . In other words, π_Φ vanishes identically on $\mathbb{P}(V_x)$. The locus of such fibres is precisely $\rho^{-1}(D)$. Moreover, since we are assuming that Φ is not the zero co-Higgs field, $\rho^{-1}(D)$ is a proper subvariety of the eigenvariety E of Φ . The subvariety B of $\mathbb{P}(V)$ is thus obtained by removing from E the irreducible components of E corresponding to the fibres in $\rho^{-1}(D)$. Finally, to see why B projects onto X , we note that if $x \notin D$, then Φ_x is nonzero and $B \cap \rho^{-1}(x) = B \cap \mathbb{P}(V_x)$ is the union of the projectivizations of the eigenspaces of Φ_x , which is non-empty.

Let us now prove that if $r = 2$ or Φ is diagonalizable, then B is a finite cover of X , which is of order r in the latter case. It is enough to check this for points that are not in $\rho^{-1}(D)$. Suppose that $x \notin D$ so that Φ_x is nonzero and traceless. If $r = 2$, then Φ_x has eigenvalues $\pm \lambda_x$. If $\lambda_x = 0$, then Φ_x is the nilpotent matrix

$$N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In particular, 0 is the only eigenvalue of Φ_x and its eigenspace E_x^0 is 1-dimensional. But if $\lambda_x \neq 0$, then Φ_x has two distinct eigenvalues λ_x and $-\lambda_x$ with distinct 1-dimensional eigenspaces $E_x^{\lambda_x}$ and $E_x^{-\lambda_x}$ in V_x , respectively. Since these eigenspaces are 1-dimensional, they correspond to points $\mathbb{P}(E_x^0)$, $\mathbb{P}(E_x^{\lambda_x})$, and $\mathbb{P}(E_x^{-\lambda_x})$ in $\mathbb{P}(V_x)$ that trace out the subvariety B as x varies. Note that if $\Phi_x = N$ for all x , then B is a section of $\mathbb{P}(V)$. Otherwise, B is a $2 : 1$ cover of X branched over the points where $\Phi_x = N$. The situation is similar if $r > 2$ and Φ_x has distinct eigenvalues for all $x \in X$ because then Φ_x has r distinct eigenvalues $\lambda_1, \dots, \lambda_r$ with distinct 1-dimensional eigenspaces $E_x^{\lambda_1}, \dots, E_x^{\lambda_r}$ in V_x , respectively. Since these eigenspaces are 1-dimensional, they correspond to distinct points $\mathbb{P}(E_x^{\lambda_1}), \dots, \mathbb{P}(E_x^{\lambda_r})$

in $\mathbb{P}(V_x)$, which trace out the subvariety B as x varies that is an $r : 1$ cover of X . Combining these two cases, we see that

$$\text{Zeroes}(\pi_\Phi) = B + \rho^{-1}(D)$$

with B a finite cover of X . □

Remark 4.1.7. Before looking at some examples, we make a few remarks.

- (i) Note that when Φ has no zeroes then, $B = \text{Zeroes}(\pi_\Phi) = E$, where E is the eigenvariety of Φ .
- (ii) If V has rank 2, then E is a divisor in $\mathbb{P}(V)$.
- (iii) If the rank of V is ≥ 3 and Φ is not diagonalizable, then the component B of E can be quite complicated. Take for example the trivial rank 3 holomorphic vector bundle $V = \mathcal{O}^{\oplus 3}$ on $X = \mathbb{C}$ together with the constant co-Higgs field

$$\Phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In this case, $D = 0$ since Φ is nowhere-vanishing so that $B = E$ by (i). Moreover, 0 is the only eigenvalue of Φ and its eigenspace is the 2-dimensional subspace

$$E_x^0 = W := \text{span}_{\mathbb{C}}\{(1, 0, 0), (0, 0, 1)\} \subset V_x$$

for every $x \in X$. Consequently, $B = X \times \mathbb{P}(W) = X \times \mathbb{P}^1$, which is not a finite cover of X .

Example 4.1.8. Consider the co-Higgs bundle we defined in examples 4.1.2 and 4.1.5, where $X = \mathbb{C}$ with coordinate x , and $(V = \mathcal{O} \oplus \mathcal{O}, \Phi)$ with Φ given by

$$\Phi = \begin{pmatrix} x & 1 \\ 1 & -x \end{pmatrix} \otimes \partial_x$$

with corresponding Poisson structure π_Φ on $\mathbb{P}(V) = X \times \mathbb{P}^1$. In this case, Φ has no zeroes so that $D = 0$ and

$$\text{Zeroes}(\pi_\Phi) = B = E,$$

where E is the eigenvariety of Φ , which we calculated to be

$$E = \left\{ \left(x, \left[1 : - \left(x + \sqrt{x^2 + 1} \right) \right] \right) \mid x \in \mathbb{C} \right\} \cup \left\{ \left(x, \left[1 : - \left(x - \sqrt{x^2 + 1} \right) \right] \right) \mid x \in \mathbb{C} \right\}.$$

Note that since V has rank 2, the zero set of π_Φ is a divisor in $\mathbb{P}(V)$.

Example 4.1.9. Let $(V = \mathcal{O} \oplus \mathcal{O}, \Phi)$ be a rank 2 co-Higgs bundle over $X = \mathbb{C}$ with Φ is given by the matrix

$$\Phi = \begin{pmatrix} a(x) & 0 \\ 0 & -a(x) \end{pmatrix} \otimes \partial_x$$

for some polynomial $a(x)$ of degree m , where x is the complex coordinate in $\mathbb{C}$. Then, Φ vanishes at the m zeroes $p_1, \dots, p_m$ of $a(x)$ so that $D = \{p_1, \dots, p_m\}$. Note that we are not assuming that the p_i 's distinct so that some of the zeroes may be repeated. Moreover, at every point $x \in \mathbb{P}^1$, the eigenvalues are $\pm a(x)$ with corresponding eigenvectors $v_+ = (1, 0)$ and $v_- = (0, 1)$, respectively, implying that $B = X \times [1 : 0] + X \times [0 : 1]$ in this case. The zero set in $\mathbb{P}(V) = X \times \mathbb{P}^1$ of corresponding bivector π_Φ is then

$$\text{Zeroes}(\pi_\Phi) = (X \times [1 : 0] + X \times [0 : 1]) + \sum_{i=1}^m p_i \times \mathbb{P}^1.$$

Note that since V has rank 2 and X is a curve, the zero set of π_Φ is a divisor in $\mathbb{P}(V)$.

We now prove that Φ -invariant subline bundles correspond to components of the eigenvariety E , showing that the existence of such subbundles implies the reducibility of E .

Proposition 4.1.10. *Let (V, Φ) be a co-Higgs bundle on X with nonzero Higgs field Φ , and let E be its eigenvariety. If L is a Φ -invariant subline bundle of V , then $\mathbb{P}(L) \subset E$. This means, in particular, that the eigenvariety E of Φ is reducible.*

Proof. Let $(x, [v]) \in \mathbb{P}(L)$. Then, $(x, v) \in L_x$ with $v \neq 0$. Note that all the elements of L_x are of the form $(x, \lambda v)$ for some $\lambda \in \mathbb{C}$ since L is a line bundle and $v \neq 0$. Moreover, $\Phi(L) \subset L \otimes T_X$ by Φ -invariant of L . Hence, $\Phi_x(v) \in L_x \otimes T_{X,x}$ so that $\Phi_x(v) = \lambda v \otimes t$ for some $\lambda \in \mathbb{C}$ and $t \in T_{X,x}$. In other words, $(x, [v]) \in E$. $\square$

Corollary 4.1.11. *If the eigenvariety of Φ is irreducible, then Φ has no zeroes and V does not have Φ -invariant subline bundles. If V has rank 2, this also means that (V, Φ) is stable.*

4.2 Φ -invariant subbundles, Poisson subvarieties and stability

Let (V, Φ) be a co-Higgs bundle on a complex manifold X and suppose that the induced quadratic holomorphic bivector π_Φ is integrable giving us a holomorphic Poisson structure

on $\mathbb{P}(V)$. In this section, we are interested in studying the conditions under which certain submanifolds of $\mathbb{P}(V)$ become Poisson subvarieties with respect to the Poisson structure induced by the co-Higgs field. Specifically, we prove that if $L \subset V$ is a Φ -invariant subbundle, then $\mathbb{P}(L)$ is a Poisson subvariety of $(\mathbb{P}(V), \pi_\Phi)$.

4.2.1 Relationship with Poisson subvarieties

We saw in section 4.1, Proposition 4.1.10, that if L is a Φ -invariant subline bundle of V , then $\mathbb{P}(L)$ is a component of the zero set of π_Φ . This means, in particular, that $\mathbb{P}(L)$ is a Poisson subvariety of $\mathbb{P}(V)$ with induced Poisson structure the zero Poisson structure. A natural question to ask is whether the projectivization of any Φ -invariant subbundle of V is also a Poisson subvariety. As stated in the following theorem, the answer is yes.

Theorem 4.2.1. *Let (V, Φ) be a zero trace co-Higgs bundle over a complex manifold X that induces a quadratic Poisson structure π_Φ on $\mathbb{P}(V)$. If L is a Φ -invariant subbundle of V , then $\mathbb{P}(L)$ is a Poisson subvariety of $\mathbb{P}(V)$ with the induced Poisson structure. In particular, if L is a line bundle, then the restriction of π_Φ to $\mathbb{P}(L)$ is the zero Poisson structure.*

Proof. Let us show that $\pi^\sharp(T^*\mathbb{P}(V)|_{\mathbb{P}(L)}) \subseteq T\mathbb{P}(L)$, which implies that $\mathbb{P}(L)$ is a Poisson subvariety by Lemma 2.2.16. Let $\mathcal{U} \subset X$ be an open set over which both V and L are trivial. Let $(x_1, \dots, x_n)$ be local coordinates on $\mathcal{U}$ and $(y_1, \dots, y_r)$ be fiber coordinates on $V|_{\mathcal{U}}$, chosen such that $L|_{\mathcal{U}}$ is given by the equations $y_1 = \dots = y_{r-k} = 0$, where $k = \text{rk}(L)$.

The co-Higgs field Φ can be written locally as

$$\Phi = \sum_{i=1}^n \sum_{j,l=1}^r g_{ijl} \partial_{x_i} \otimes y_j \partial_{y_l}$$

where g_{ijl} are holomorphic functions on $\mathcal{U}$. Since L is Φ -invariant, for any section s of L , $\Phi(s)$ must be a section of $L \otimes T_X$. This means that if we apply Φ to any vector of the form $(0, \dots, 0, v_{r-k+1}, \dots, v_r)$, the result must have zeros in the first $r - k$ components. This implies

$$g_{ijl} = 0 \quad \text{for all } i = 1, \dots, n, \quad r - k + 1 \leq j \leq r, \quad 1 \leq l \leq r - k.$$

On $\mathbb{P}(V)|_{\mathcal{U}}$, we can use projective coordinates $\hat{y}_j = y_j/y_r$ for $j = 1, \dots, r - 1$. In these coordinates, $\mathbb{P}(L)$ is defined by $\hat{y}_1 = \dots = \hat{y}_{r-k} = 0$. Moreover, the Poisson structure π_Φ

on $\mathbb{P}(V)$ takes the form

$$\pi = \sum_{i=1}^n \sum_{j,l=1}^{r-1} (g_{ijl} \hat{y}_j - g_{ijr} \hat{y}_j \hat{y}_l) \partial_{x_i} \wedge \partial_{\hat{y}_l}.$$

Let $p \in \mathbb{P}(L)$ and $\alpha \in T_p^* \mathbb{P}(V)$. Thus, we can write $\alpha = \sum_{i=1}^n \alpha_i dx_i + \sum_{j=1}^{r-1} \beta_j d\hat{y}_j$ and

$$\pi_{\Phi}^{\sharp}(\alpha) = \sum_{i=1}^n \sum_{j,l=1}^{r-1} (g_{ijl} \hat{y}_j - g_{ijr} \hat{y}_j \hat{y}_l) (\alpha_i \partial_{\hat{y}_l} - \beta_l \partial_{x_i}).$$

At points of $\mathbb{P}(L)$ where $\hat{y}_1 = \dots = \hat{y}_{r-k} = 0$, and since some of the coefficients g_{ijl} vanish because L is Φ -invariant, we have that

$$\pi_{\Phi}^{\sharp}(\alpha) = \sum_{i=1}^n \sum_{j,l=r-k+1}^{r-1} (g_{ijl} \hat{y}_j - g_{ijr} \hat{y}_j \hat{y}_l) (\alpha_i \partial_{\hat{y}_l} - \beta_l \partial_{x_i})$$

This shows that $\pi_{\Phi}^{\sharp}(\alpha)$ is a linear combination of ∂_{x_i} and $\partial_{\hat{y}_l}$ for $l \geq r - k + 1$, which span the tangent space $T_p \mathbb{P}(L)$. Hence, $\pi_{\Phi}^{\sharp}(\alpha) \in T_p \mathbb{P}(L)$. Since this holds for all $p \in \mathbb{P}(L)$, a similar computation can be made on any trivializing open set. We then conclude that $\pi_{\Phi}^{\sharp}(T^* \mathbb{P}(V)|_{\mathbb{P}(L)}) \subseteq T \mathbb{P}(L)$ globally, and thus $\mathbb{P}(L)$ is a Poisson subvariety. Finally, referring to Proposition 4.1.10, we see that if L is a line bundle, then π_{Φ} induces the zero Poisson structure on $\mathbb{P}(L)$. $\square$

Example 4.2.2. Let (V, Φ) be a rank 2 co-Higgs bundle over $\mathbb{P}^1$ where $V = \mathcal{O}(1) \oplus \mathcal{O}(-1)$ and Φ is given locally by

$$\Phi = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \otimes \partial_x,$$

where x is the affine coordinate on $\mathbb{P}^1$. Let $L_1 = \mathcal{O}(1) \subset V$. Then, L_1 is Φ -invariant since

$$\Phi \left(\begin{pmatrix} s \\ 0 \end{pmatrix} \right) = \begin{pmatrix} xs \\ 0 \end{pmatrix} \otimes \partial_x$$

for any section s of $\mathcal{O}(1)$.

The induced Poisson structure π_{Φ} on $\mathbb{P}(V)$ is given by $\pi_{\Phi} = (-2x\hat{y})\partial_x \wedge \partial_{\hat{y}}$ in the affine coordinate $\hat{y} = y_2/y_1$ (and is integrable since $\mathbb{P}(V)$ is a surface). Let us verify that $\mathbb{P}(L_1)$ is a Poisson subvariety of $\mathbb{P}(V)$. Along $\mathbb{P}(L_1)$, we have $\hat{y} = 0$ so that

$$\begin{aligned} \pi_{\Phi}^{\sharp}(dx) &= (-2x\hat{y})\partial_{\hat{y}} = 0 \\ \pi_{\Phi}^{\sharp}(d\hat{y}) &= (2x\hat{y})\partial_x = 0 \end{aligned}$$

showing that $\pi_\Phi^\sharp(T_{\mathbb{P}(V)}^*|_{\mathbb{P}(L_1)}) \subseteq T\mathbb{P}(L_1)$. Similarly, $L_2 = \mathcal{O}(-1) \subset V$ is also Φ -invariant and $\mathbb{P}(L_2)$ is a Poisson subvariety of $\mathbb{P}(V)$. Note that it is not surprising that π_Φ restricts to the zero Poisson structure on $\mathbb{P}(L_1)$ and $\mathbb{P}(L_2)$ since L_1 and L_2 are line bundles, as stated in Theorem 4.2.1.

Remark 4.2.3. Although π_Φ restricts to the zero Poisson structure on $\mathbb{P}(L)$ when L is a Φ -invariant subline bundle of V , the same is not necessarily true if L is a Φ -invariant bundle of higher rank, as illustrated by the next example.

Example 4.2.4. Let $V = \mathcal{O}^{\oplus 3}$ on $X = \mathbb{C}$ and consider the constant co-Higgs field

$$\Phi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Referring to Corollary 3.3.6, since Φ is constant, it induces an integrable quadratic Poisson structure π_Φ on $\mathbb{P}(V) = X \times \mathbb{P}^2$. Let x be the complex coordinate in X . In terms of the affine coordinates $\hat{y}_2 = y_2/y_1$ and $\hat{y}_3 = y_3/y_1$ on the $\mathbb{P}^2$ fibres of $\mathbb{P}(V)$, the Poisson structure is given by

$$\pi_\Phi = (1 - \hat{y}_2^2)\partial_x \wedge \partial_{\hat{y}_2} - \hat{y}_2\hat{y}_3\partial_x \wedge \partial_{\hat{y}_3}.$$

Let $L = \mathcal{O} \oplus \mathcal{O}$ correspond to the first two factors of V . We then see that L is a Φ -invariant rank 2 subbundle of V , implying that $\mathbb{P}(L)$ is a Poisson subvariety of $\mathbb{P}(V)$ with Poisson structure the restriction of π_Φ to $\mathbb{P}(L)$. Since $\mathbb{P}(L)$ is given by $\hat{y}_3 = 0$, the restriction of π_Φ to $\mathbb{P}(L)$ is

$$\pi_\Phi|_{\mathbb{P}(L)} = (1 - \hat{y}_2^2)\partial_x \wedge \partial_{\hat{y}_2},$$

which is not the zero Poisson structure on $\mathbb{P}(L)$.

4.2.2 Relationship with stability and spectral varieties

To study the stability of a co-Higgs bundle (V, Φ) on a complex manifold X , one needs to examine the Φ -invariant subsheaves of V (if any). In fact, if X is a curve, then it is enough to only consider Φ -invariant subbundles of V . On the other hand, we proved in section 4.2.1 that Φ -invariant subbundles of V correspond to Poisson subvarieties of $(\mathbb{P}(V), \pi_\Phi)$ when the quadratic bivector π_Φ is integrable. And, in particular, if L is Φ -invariant subline bundle of V , then $\mathbb{P}(L)$ is component of the eigenvariety E of Φ . This means that if V admits Φ -invariant subline bundles, then its eigenvariety is reducible. Conversely, Corollary 4.1.11 tells us that if the eigenvariety of Φ is irreducible, the V does not admit Φ -invariant

subline bundles. There is a comparable relationship between the existence of a Φ -invariant subline bundle of V and the reducibility of the spectral variety S_Φ of Φ , which provides a geometric criterion for the potential unstability of (V, Φ) . We refer the reader, for example, to [4] and the references therein. Nonetheless, for the convenience of the reader, we state and prove the result for rank r co-Higgs bundles on curves.

Theorem 4.2.5. *Let X be a curve and (V, Φ) be a rank r co-Higgs bundle on X such that $\Phi \not\equiv 0$ and S_Φ is reduced. Then, there exists a Φ -invariant subline bundle L of V if and only if the spectral curve S_Φ of Φ is reducible.*

Proof. Assume that L is a Φ -invariant subline bundle of V , and let's consider the spectral curve S_L associated to the co-Higgs field $\Phi|_L : L \rightarrow L \otimes T_X$. The Φ -invariance of L ensures that $\Phi|_L$ is well-defined and the eigenvectors corresponding to $\Phi|_L$ lie within L . Since S_Φ is reduced, the eigenvalues of the restriction of Φ to V_x are distinct for generic $x \in X$. Therefore, $S_L \subsetneq S_\Phi$.

Note that $\Phi|_{L|_{L_x}} \in \text{Hom}(L_x, L_x \otimes (T_X)_x) = \text{Hom}(\mathbb{C}, \mathbb{C})$. This homomorphism, being a complex linear map, can be represented by multiplication by a complex number (its eigenvalue). If S_L were a point, it would imply that $\Phi|_L$ acts as scalar multiplication by a single complex number across all of L . However, since $\Phi|_L$ is derived from the non-trivial co-Higgs field Φ on V , its action varies non-trivially across L , resulting in a spectral curve S_L that is more than just a single point. Since both S_Φ and S_L have dimension 1, then S_L is a component of S_Φ , implying that S_Φ is reducible.

Now assume that the spectral curve S_Φ of Φ is reducible, then it can be expressed as the union of two proper closed subsets, say S_1 and S_2 . The spectral curve S_Φ is given by the equation $\det(\Phi(x) - \eta \text{Id}) = 0 \subset \text{Tot}(T_X)$ where η is the tautological section of the pullback of T_X to its own total space. Consider $(x, w) \in T_X$ where $w \in (T_X)_x$. Then $\eta(x, w) \in (\pi^* T_X)_{(x, w)} = (T_X)_{\pi(x, w)} = (T_X)_x$ and since $w \in (T_X)_x$ we conclude that $\eta(x, w) = w$. That is w is an eigenvalue of Φ because if $(x, w) \in \det(\Phi(x) - \eta(x, w) \text{Id}) = 0$ then $\det(\Phi(x)) = w \cdot \text{Id}$.

Let us now take $(x, w) \in S_\Phi$ and without loss of generality, assume that $(x, w) \in S_1$. Since S_Φ is reduced let's define the line bundle L_1 as the eigenspace of dimension 1 generated by w . That is,

$$L_1 := \ker(\Phi(x) - \eta(p^{-1}(x)) \cdot \text{Id}_E)|_{S_1},$$

where $p : T_X|_{S_1} \rightarrow X$. Since every eigenspace is Φ -invariant we have that L_1 is Φ -invariant. In the case where $\Phi \neq 0$, but $\Phi(x) = 0$ then we choose a neighbourhood $U \subset X$ around zero and write $\Phi(x) = p(x)\Phi_0$, where $p(x)$ is a homogeneous polynomial of degree one and

Φ_0 is a nowhere vanishing matrix. Then we apply the argument above to Φ_0 to construct an invariant subline bundle L_1 of V . $\square$

The following example illustrates the correspondence between reducible spectral curves and the Φ -invariant line subbundles for a rank 2 co-Higgs bundle.

Example 4.2.6. Consider the rank 2 holomorphic vector bundle $V = \mathcal{O}(1) \oplus \mathcal{O}(-1)$ on $\mathbb{P}^1$ and the co-Higgs field Φ on V given locally by

$$\Phi = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} \otimes \partial_x,$$

where x is an affine coordinate on $\mathbb{P}^1$. The spectral curve $S_\Phi \subset \text{Tot}(T_{\mathbb{P}^1})$ is given by

$$\det(\eta - \Phi) = (\eta + x)(\eta - x) = 0,$$

where η is the tautological section of $p^*T_{\mathbb{P}^1}$. Note that S_Φ is reducible since it splits into two components. Indeed, $S_\Phi = S_1 \cup S_2$ with $S_1 := \{\eta = -x\}$ and $S_2 := \{\eta = x\}$. Moreover, at each point $x \in \mathbb{P}^1$, the fiber of S_Φ over x consists of two distinct points, $\eta = x(x)$ and $\eta = -x(x)$, which correspond to the eigenvalues of $\Phi_x : V_x \rightarrow V_x \otimes T_{\mathbb{P}^1, x}$. The associated eigenspaces then form two Φ -invariant subline subbundles of V , namely,

$$L_1 = \text{span}_{\mathcal{O}} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} \cong \mathcal{O}(1) \text{ and } L_2 = \text{span}_{\mathcal{O}} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \cong \mathcal{O}(-1).$$

Remark 4.2.7. The spectral variety $S_\Phi \subset \text{Tot}(T_X)$ of Φ is also closely related to its eigenvariety $E \subset \mathbb{P}(V)$. In fact, Matviichuk proves that there is a canonical morphism $\text{Eig} : E \rightarrow S_\Phi$ sending eigenvectors to their eigenvalues, which is an isomorphism over an open set $\mathcal{U} \subset X$ if Φ_x has distinct eigenvalues for all $x \in \mathcal{U}$ [21, Lemma 3.3.2]. This result, together with Theorem 4.1.6 gives us:

Corollary 4.2.8. *Let (V, Φ) be a rank r co-Higgs bundle on X whose co-Higgs field Φ is such that Φ_x has distinct eigenvalues for all $x \in X$. Then, $\text{Zeroes}(\pi_\Phi) = E$ is isomorphic to S_Φ .*

4.3 Holomorphic Poisson structures on Hirzebruch surfaces

We end this chapter by giving an explicit analysis of the the geometric properties of holomorphic Poisson structures on Hirzebruch surfaces. Recall that Hirzebruch surfaces correspond to projectivizations of rank 2 holomorphic bundles on $\mathbb{P}^1$. Therefore, they are all of

the form $\mathbb{P}(V)$ with

$$V = \mathcal{O}(m_1) \oplus \mathcal{O}(m_2)$$

for some integers m_1, m_2 . Moreover, we have seen all holomorphic Poisson structures on a Hirzebruch surface $\mathbb{P}(V)$ are induced from co-Higgs fields on the underlying bundle V (see Theorem 3.4.4). We describe in detail the zero sets of there Poisson structures and classify all the ones coming from (semi)stable co-Higgs pairs (V, Φ) .

4.3.1 Eigenvarieties

Let us consider rank 2 holomorphic vector bundles over the projective line $\mathbb{P}^1$. Such bundles can be written as $V \cong \mathcal{O}(m_1) \oplus \mathcal{O}(m_2)$ for some integers m_1, m_2 with $m_1 \geq m_2$. Since $\text{End}_0(V) \cong \text{End}_0(V \otimes L)$ we can always assume that any rank 2 vector bundle V over $\mathbb{P}^1$ is isomorphic, up to tensoring by a line bundle, to $\mathcal{O} \oplus \mathcal{O}(-d)$ for some non-negative integer d . We will now analyze the zero set of the Poisson structure on $\mathbb{P}(V)$ induced by a co-Higgs field, for different values of d .

Consider the rank 2 holomorphic vector bundle $V = \mathcal{O} \oplus \mathcal{O}(-d)$, $d \geq 0$, over $\mathbb{P}^1$, and let $\Phi \in H^0(\mathbb{P}^1, \text{End}_0(V) \otimes T_{\mathbb{P}^1})$ be a zero trace co-Higgs field on V . If Φ is the zero co-Higgs field, then $\text{Zeroes}(\pi_\Phi) = \mathbb{P}(V)$. Otherwise, note that

$$\text{End}_0(V) \cong \text{Hom}(\mathcal{O}, \mathcal{O}) \oplus \text{Hom}(\mathcal{O}(-d), \mathcal{O}) \oplus \text{Hom}(\mathcal{O}, \mathcal{O}(-d)) \cong \mathcal{O} \oplus \mathcal{O}(d) \oplus \mathcal{O}(-d).$$

And since $T_{\mathbb{P}^1} \cong \mathcal{O}(2)$, we have $\text{End}_0(V) \otimes T_{\mathbb{P}^1} \cong \mathcal{O}(2) \oplus \mathcal{O}(2+d) \oplus \mathcal{O}(2-d)$. Therefore, sections of $\text{End}_0(V) \otimes T_{\mathbb{P}^1}$ can be represented by traceless 2×2 matrices

$$\Phi(x) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

with $a \in H^0(\mathbb{P}^1, \mathcal{O}(2))$, $b \in H^0(\mathbb{P}^1, \mathcal{O}(2+d))$, and $c \in H^0(\mathbb{P}^1, \mathcal{O}(2-d))$. Furthermore, $2+d > 0$ and $2-d \leq 0$ if and only if $0 \leq d \leq 2$. Suppose that $[x_1 : x_2]$ are homogeneous coordinates on $\mathbb{P}^1$. Then, a and b are homogeneous polynomials in x_1 and x_2 of degrees 2 and $2+d$, respectively. Whereas, $c = 0$ if $d > 2$, or c a homogeneous polynomials in x_1 and x_2 of degree $2-d$ if $0 \leq d \leq 2$.

We consider four cases, namely, $d = 0, 1, 2$, and $d > 2$.

Case $d = 0$. In this case, $\mathbb{P}(V) = \mathbb{P}^1 \times \mathbb{P}^1$. Moreover, a, b, c are homogeneous polynomials of degree 2. The zeroes of Φ then occur at points $x \in \mathbb{P}^1$ where $a(x) = b(x) = c(x) = 0$. Since a, b, c are homogeneous polynomials of degree 2, they can have up to two common zeroes. We have three cases, based on the common zeroes of a, b and c .

1. a, b, c have the same zeroes: In this case, $a(x_1, x_2) = \alpha p(x_1, x_2)$, $b(x_1, x_2) = \beta p(x_1, x_2)$, and $c(x_1, x_2) = \gamma p(x_1, x_2)$, where $p(x_1, x_2)$ is a homogeneous polynomial of degree 2 and $\alpha, \beta, \gamma \in \mathbb{C}$. We can then write $\Phi = p(x_1, x_2)A$ for some constant matrix 2×2 matrix

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}.$$

Since $p(x_1, x_2)$ is a homogeneous polynomial of degree 2, it has two zeroes, say p_1 and p_2 . We then have that $D = p_1 + p_2$. Moreover, Φ can take the following forms up to conjugation:

(a) $\Phi = p(x_1, x_2) \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ where $\lambda \neq 0$.

(b) $\Phi = p(x_1, x_2) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$

Note that if $\Phi(x) \neq 0$, then its eigenspaces correspond to the eigenspaces of the constant matrix A because Φ is then a nonzero multiple of A . In the first case, A has two distinct 1-dimensional eigenspaces L_λ and $L_{-\lambda}$ corresponding to its distinct eigenvalues λ and $-\lambda$. Suppose that these eigenspaces are spanned by the eigenvectors v_λ and $v_{-\lambda}$, respectively. Then, $B = \mathbb{P}^1 \times [v_\lambda] + \mathbb{P}^1 \times [v_{-\lambda}] \subset \mathbb{P}^1 \times \mathbb{P}^1$. Hence,

$$\text{Zeroes}(\pi_\Phi) = (\mathbb{P}^1 \times [v_\lambda] + \mathbb{P}^1 \times [v_{-\lambda}]) + (p_1 \times \mathbb{P}^1 + p_2 \times \mathbb{P}^1).$$

In the second case, A only has one eigenspace L_0 , which is 1-dimensional and corresponds to the unique eigenvalue $\lambda = 0$. If this eigenspace is spanned by the eigenvector v_0 , then $B = (\mathbb{P}^1 \times [v_0]) \subset \mathbb{P}^1 \times \mathbb{P}^1$. Hence,

$$\text{Zeroes}(\pi_\Phi) = (\mathbb{P}^1 \times [v_0]) + (p_1 \times \mathbb{P}^1 + p_2 \times \mathbb{P}^1).$$

2. a, b, c only have one common zero. We can then write

$$a = \alpha(x_1, x_2)p(x_1, x_2)$$

$$b = \beta(x_1, x_2)p(x_1, x_2)$$

$$c = \gamma(x_1, x_2)p(x_1, x_2)$$

for some degree 1 homogeneous polynomials $\alpha(x_1, x_2), \beta(x_1, x_2), \gamma(x_1, x_2)$, and $p(x_1, x_2)$ such that $\alpha(x_1, x_2), \beta(x_1, x_2)$, and $\gamma(x_1, x_2)$ have no common zeroes. Consequently,

$$\begin{aligned} \Phi &= p(x_1, x_2) \begin{pmatrix} \alpha(x_1, x_2) & \beta(x_1, x_2) \\ \gamma(x_1, x_2) & -\alpha(x_1, x_2) \end{pmatrix} \\ &= p(x_1, x_2)(Ax_1 + Bx_2) \end{aligned}$$

for some constant matrices

$$A_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & -\alpha_1 \end{pmatrix} \text{ and } A_2 = \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & -\alpha_2 \end{pmatrix}.$$

Since $p(x_1, x_2)$ is a homogeneous polynomial of degree one, it has one zero, say p_1 . We then have $D = p_1$ and

$$\text{Zeroes}(\pi_\Phi) = B + (p_1 \times \mathbb{P}^1)$$

with B a $2 : 1$ cover of $\mathbb{P}^1$. If A_1 and A_2 have a common eigenvector, which occurs if and only if $\det([A_1, A_2]) = 0$, then V has a Φ -invariant subline bundle, implying that B is reducible by Proposition 4.1.10. In other words, the curve B has two components, which are sections of $\rho : \mathbb{P}(V) \rightarrow \mathbb{P}^1$. Otherwise, B is an irreducible double cover of $\mathbb{P}^1$ branched along the zeroes of degree 2 homogeneous polynomial $\det(Ax_1 + Bx_2) = -(\alpha(x_1, x_2)^2 + \beta(x_1, x_2)\gamma(x_1, x_2))$, implying that B has genus 0.

3. a, b, c do not have a common zero. In this case, we can write:

$$\Phi = \begin{pmatrix} a(x_1, x_2) & b(x_1, x_2) \\ c(x_1, x_2) & -a(x_1, x_2) \end{pmatrix} = A^{11}x_1^2 + A^{12}x_1x_2 + A^{22}x_2^2,$$

where $a(x_1, x_2), b(x_1, x_2), c(x_1, x_2)$ are homogeneous polynomials of degree 2 without a common zero and A^{11}, A^{12}, A^{22} are constant 2×2 matrices. We then have $D = \emptyset$ and

$$\text{Zeroes}(\pi_\Phi) = B,$$

where B is a $2 : 1$ cover of $\mathbb{P}^1$. If A^{11}, A^{12}, A^{22} have a common eigenvector, then V has a Φ -invariant subline bundle so that B is reducible by proposition 4.1.10, whose two components are sections of $s_\pm : \mathbb{P}^1 \rightarrow \mathbb{P}(V)$. Otherwise, B is an irreducible double cover of $\mathbb{P}^1$ branched along the zeroes of degree 4 homogeneous polynomial $\det \Phi = -(a(x_1, x_2)^2 + b(x_1, x_2)c(x_1, x_2))$, implying that B has genus 1.

Case $d > 2$. In this case, a and b have degrees 2 and $d+2 \geq 5$, respectively, and $c = 0$. Therefore, Φ takes the form

$$\Phi = \begin{pmatrix} a(x_1, x_2) & b(x_1, x_2) \\ 0 & -a(x_1, x_2) \end{pmatrix}.$$

The zeroes of Φ occur at points $x \in \mathbb{P}^1$ where $a(x) = b(x) = 0$. And since a has degree two, a and b have at most two common zeroes so that $D = \emptyset$, p_1 or $p_1 + p_2$. We can therefore write

$$\Phi = p(x_1, x_2) \begin{pmatrix} \alpha(x_1, x_2) & \beta(x_1, x_2) \\ 0 & -\alpha(x_1, x_2) \end{pmatrix}$$

for some homogeneous polynomials $p(x_1, x_2)$, $\alpha(x_1, x_2)$ and $\beta(x_1, x_2)$ such that α and β have no common zeroes and the zeroes of p are the common zeroes of a and b (if any). Furthermore, the eigenvalues of $\Phi(x)$ are $\pm a(x_1, x_2)$ at $x = [x_1 : x_2]$ with corresponding eigenspaces $L_{a(x_1, x_2)} = \text{span}_{\mathbb{C}}\{(1, 0)\}$ and $L_{-a(x_1, x_2)} = \text{span}_{\mathbb{C}}\{(\beta(x_1, x_2), 2\alpha(x_1, x_2))\}$. In fact, the eigenspaces are the fibres of two eigenbundles $L_{\pm a}$ of V corresponding to the eigenfunctions $\pm a$. The curve B thus has two components given by $\mathbb{P}(L_a)$ and $\mathbb{P}(L_{-a})$, which are sections $s_{\pm} : \mathbb{P}^1 \rightarrow \mathbb{P}(V)$ that intersect at the zeroes of a . Hence,

$$\text{Zeroes}(\pi_{\Phi}) = (\mathbb{P}(L_a) + \mathbb{P}(L_{-a})) + \rho^{-1}(D)$$

with $D = \emptyset$, p_1 or $p_1 + p_2$ so that $\rho^{-1}(D) = \emptyset$, $p_1 \times \mathbb{P}^1$, or $(p_1 \times \mathbb{P}^1 + p_2 \times \mathbb{P}^1)$.

Case $d = 2$. In this case, $\deg(a) = 2$, $\deg(b) = 4$, and $\deg(c) = 0$. Therefore, we have two subcases to consider:

1. If c is a nonzero constant, then Φ takes the form

$$\Phi = \begin{pmatrix} a(x_1, x_2) & b(x_1, x_2) \\ c & -a(x_1, x_2) \end{pmatrix}.$$

Since $c \neq 0$ we have that $\Phi(x_1, x_2) \neq 0$, implying that $D = \emptyset$. Therefore,

$$\text{Zeroes}(\pi_{\Phi}) = B,$$

where B is a $2 : 1$ cover of $\mathbb{P}^1$ that is reducible if and only if V has a Φ -invariant subline bundle by Proposition 4.1.10, in which case B has two components that are sections of $s_{\pm} : \mathbb{P}^1 \rightarrow \mathbb{P}(V)$. Moreover, V has a Φ -invariant subline bundle if and only if its spectral variety $\lambda^2 = a^2(x_1, x_2)^2 + cb(x_1, x_2)$ is reducible by Proposition 4.2.5. Otherwise, B is an irreducible double cover of $\mathbb{P}^1$ branched along the zeroes of degree 4 homogeneous polynomial $\det \Phi = -(a(x_1, x_2)^2 + b(x_1, x_2)c)$, implying that B has genus 1.

2. If $c = 0$, then Φ takes the form

$$\Phi = \begin{pmatrix} a(x_1, x_2) & b(x_1, x_2) \\ 0 & -a(x_1, x_2) \end{pmatrix}.$$

An analysis similar to the one in the case $g > 2$ gives us

$$\text{Zeroes}(\pi_\Phi) = (\mathbb{P}(L_a) + \mathbb{P}(L_{-a})) + \rho^{-1}(D)$$

with $D = \emptyset$, p_1 or $p_1 + p_2$ so that $\rho^{-1}(D) = \emptyset$, $p_1 \times \mathbb{P}^1$, or $(p_1 \times \mathbb{P}^1 + p_2 \times \mathbb{P}^1)$.

Case $d = 1$. In this case, a , b , and c are homogeneous polynomials of degrees 2, 3 and 1, respectively. The zeroes of Φ occur at points $x \in \mathbb{P}^1$ where $a(x) = b(x) = c(x) = 0$. Since $\deg(c) = 1$, we have that $c(x)$ has exactly one zero. The polynomials $a(x), b(x), c(x)$ can thus have at most one common zero so that $D = p_1$ or $D = \emptyset$. We again have two cases:

1. If a, b, c share a common zero, say p_1 , we can write

$$\begin{aligned} a &= \alpha(x_1, x_2)p(x_1, x_2) \\ b &= \beta(x_1, x_2)p(x_1, x_2) \\ c &= \gamma p(x_1, x_2), \end{aligned}$$

where $p(x_1, x_2)$ is a homogeneous polynomial of degree 1 whose zero is p_1 , and α, β, γ are homogeneous polynomials of degrees 1, 2, and 0 respectively. Moreover,

$$\Phi = p(x_1, x_2) \begin{pmatrix} \alpha(x_1, x_2) & \beta(x_1, x_2) \\ \gamma & -\alpha(x_1, x_2) \end{pmatrix}$$

We then have $D = p_1$ and

$$\text{Zeroes}(\pi_\Phi) = B + (p_1 \times \mathbb{P}^1)$$

with B a 2 : 1 cover of $\mathbb{P}^1$ that is reducible if and only if V has a Φ -invariant subline bundle by Proposition 4.1.10, in which case B has two components that are sections of $s_\pm : \mathbb{P}^1 \rightarrow \mathbb{P}(V)$. Moreover, V has a Φ -invariant subline bundle if and only its spectral variety $\lambda^2 = a^2(x_1, x_2)^2 + cb(x_1, x_2)$ is reducible by Proposition 4.2.5. Otherwise, B is an irreducible double cover of $\mathbb{P}^1$ branched along the zeroes of degree 2 homogeneous polynomial $\det \Phi = -(\alpha(x_1, x_2)^2 + \beta(x_1, x_2)\gamma)$, implying that B has genus 0.

2. If a, b, c do not share a common zero, we have $D = \emptyset$. In this case, we can write:

$$\Phi = \begin{pmatrix} a(x_1, x_2) & b(x_1, x_2) \\ c(x_1, x_2) & -a(x_1, x_2) \end{pmatrix},$$

where $a(x_1, x_2), b(x_1, x_2), c(x_1, x_2)$ are homogeneous polynomials of degree 2, 3 and 1, respectively, without a common zero. Then,

$$\text{Zeroes}(\pi_\Phi) = B,$$

where B is a $2 : 1$ cover of $\mathbb{P}^1$ that is reducible if and only if V has a Φ -invariant subline bundle by Proposition 4.1.10, in which case B has two components that are sections of $s_\pm : \mathbb{P}^1 \rightarrow \mathbb{P}(V)$. Otherwise, B is an irreducible double cover of $\mathbb{P}^1$ branched along the zeroes of degree 4 homogeneous polynomial $\det \Phi = -(a(x_1, x_2)^2 + b(x_1, x_2)c(x_1, x_2))$, implying that B has genus 1.

4.3.2 Stability and spectral curve

Having analyzed the zero set of the Poisson structure for various values of d , we now turn our attention to the stability of these co-Higgs bundles.

We will start by giving a classification of (semi)stable traceless co-Higgs bundles (V, Φ) on $\mathbb{P}^1$ and their corresponding holomorphic Poisson structures on $\mathbb{P}(V)$. As we have seen in section 2.1.3, Rayan significantly restricts the possible options for the underlying bundle of a stable co-Higgs bundle over $\mathbb{P}^1$. He states that for $V = \mathcal{O} \oplus \mathcal{O}(-d)$, the co-Higgs bundle (V, Φ) can be stable only if $0 \leq d \leq 2$. We therefore only need to consider three cases, namely, $V = \mathcal{O} \oplus \mathcal{O}$, $V = \mathcal{O} \oplus \mathcal{O}(-1)$, and $V = \mathcal{O}(1) \oplus \mathcal{O}(-1)$, which correspond to $d = 0$, $d = 1$, and $d = 2$ respectively (up to tensoring with a suitable line bundle).

Theorem 4.3.1. *Let (V, Φ) be a rank 2 traceless co-Higgs bundle over $\mathbb{P}^1$. Then, up to tensoring V by a line bundle and conjugating Φ by an automorphism of V , the (semi)stable pairs are of the following forms:*

1. $V = \mathcal{O}(1) \oplus \mathcal{O}(-1)$ with $\Phi_0 = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$, where b is a homogeneous polynomial of degree 4 on $\mathbb{P}^1$, and the induced Poisson structure on $\mathbb{P}(V|_{\mathcal{U}})$ is $\pi_\Phi = [b\hat{y}_1^2 + 1]\partial_x \wedge \partial\hat{y}_1$.
2. $V = \mathcal{O} \oplus \mathcal{O}(-1)$ with $\Phi_0 = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$, where b is a homogeneous polynomial of degree 3 on $\mathbb{P}^1$, and the induced Poisson structure on $\mathbb{P}(V|_{\mathcal{U}})$ is $\pi_\Phi = [b\hat{y}_1^2 + 1]\partial_x \wedge \partial\hat{y}_1$.
3. $V = \mathcal{O} \oplus \mathcal{O}$ with $\Phi_0 = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$, where a, b are homogeneous polynomials of degree 2 on $\mathbb{P}^1$, and the induced Poisson structure on $\mathbb{P}(V|_{\mathcal{U}})$ is $\pi_\Phi = [b\hat{y}_1^2]\partial_x \wedge \partial\hat{y}_1$ or $[2a\hat{y}_1 + b\hat{y}_1^2]\partial_x \wedge \partial\hat{y}_1$, respectively.

Here, x is the affine coordinate on $\mathcal{U} \subset \mathbb{P}^1$. Moreover, y_1, y_2 are fiberwise linear coordinates on $V|_{\mathcal{U}}$ and $\hat{y}_1$ is the affine coordinate y_1/y_2 .

Proof. Case I: $V = \mathcal{O}(1) \oplus \mathcal{O}(-1)$. In this case, we can write $\phi_0 \in H^0(\mathbb{P}^1, \text{End}_0 V \otimes \mathcal{O}(2))$ locally as

$$\Phi_0 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

with $a \in H^0(\mathbb{P}^1, \mathcal{O}(2))$, $b \in H^0(\mathbb{P}^1, \mathcal{O}(4))$ and $c \in \mathbb{C}$. Moreover, the automorphisms of V are of the form

$$\begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$$

with $d, f \in \mathbb{C}^*$ and $e \in H^0(\mathbb{P}^1, \mathcal{O}(2))$. Note that Φ_0 is stable if and only if $\mathcal{O}(1)$ is not Φ_0 -invariant and this happens if and only if $c \neq 0$. Since we are considering Φ_0 up to conjugation by an automorphism of V and $c \neq 0$, we can conjugate by

$$\psi = \begin{pmatrix} 1 & -a/c \\ 0 & 1/c \end{pmatrix}$$

to obtain

$$\Phi'_0 = \psi \Phi_0 \psi^{-1} = \begin{pmatrix} 0 & b' \\ 1 & 0 \end{pmatrix},$$

where $b' = cb + a^2$. Therefore, the conjugacy class of ϕ_0 is completely determined by $b' \in H^0(\mathbb{P}^1, \mathcal{O}(4))$, and the stable co-Higgs fields are parameterized by $H^0(\mathbb{P}^1, \mathcal{O}(4)) \simeq \mathbb{C}^5$, the set of all homogeneous polynomials of degree 4 in the homogeneous coordinates x, y on $\mathbb{P}^1$. Then, such stable co-Higgs fields are of the form

$$\Phi_0 = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix},$$

where b is a homogeneous polynomial of degree 4 on $\mathbb{P}^1$. Therefore, the induced Poisson structure on $\mathbb{P}(V|_{\mathcal{U}})$ is $\pi_\Phi = [b\hat{y}_1^2 + 1]\partial_x \wedge \partial\hat{y}_1$.

Case II: $V = \mathcal{O} \oplus \mathcal{O}(-1)$. In this case, the elements $\phi_0 \in H^0(\mathbb{P}^1, \text{End}_0 V \otimes \mathcal{O}(2))$ are of the form

$$\Phi_0 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

with $a \in H^0(\mathbb{P}^1, \mathcal{O}(2))$, $b \in H^0(\mathbb{P}^1, \mathcal{O}(3))$ and $c \in \mathcal{O}(1)$. Moreover, the automorphisms of V are of the form

$$\begin{pmatrix} d & e \\ 0 & f \end{pmatrix}$$

with $d, f \in \mathbb{C}^*$ and $e \in H^0(\mathbb{P}^1, \mathcal{O}(1))$. Note that Φ_0 is stable if and only if $\mathcal{O}$ is not ϕ_0 -invariant and this happens if and only if $c \neq 0$. Since we are considering Φ_0 up to conjugation by an automorphism of V and $c \neq 0$ we can conjugate by

$$\psi = \begin{pmatrix} 1 & -a/c \\ 0 & 1/c \end{pmatrix}$$

to obtain

$$\Phi'_0 = \psi \Phi_0 \psi^{-1} = \begin{pmatrix} 0 & b' \\ 1 & 0 \end{pmatrix},$$

where $b' = cb + a^2$. The conjugacy class of Φ_0 is therefore completely determined by $b' \in H^0(\mathbb{P}^1, \mathcal{O}(3))$, and the stable co-Higgs fields are parameterized by $H^0(\mathbb{P}^1, \mathcal{O}(4)) \simeq \mathbb{C}^4$, the set of all homogeneous polynomials of degree 4 in the homogeneous coordinates x, y on $\mathbb{P}^1$. Then, such stable co-Higgs fields are of the form

$$\Phi_0 = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix}$$

where b is a homogeneous polynomial of degree 3 on $\mathbb{P}^1$. Therefore, the induced Poisson structure on $\mathbb{P}(V|_{\mathcal{U}})$ is $\pi_\Phi = [b\hat{y}_1^2 + 1]\partial_x \wedge \partial\hat{y}_1$.

Case III: $V = \mathcal{O} \oplus \mathcal{O}$. In this case, we have that $\text{End}(V) = \mathcal{O}^{\otimes 4}$. Therefore, $\text{End}_0 V \otimes \mathcal{O}(2) = \mathcal{O}(2)^{\otimes 4}$. Note that $\mu(V) = 0 = \mu(\mathcal{O})$. It follows that (V, Φ) is semistable and we have the following two cases up to conjugation:

$$\Phi_0 = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix},$$

where a, b are homogeneous polynomials of degree 2, with induced Poisson structure on $\mathbb{P}(V|_{\mathcal{U}})$ given by $\pi_\Phi = [b\hat{y}_1^2]\partial_x \wedge \partial\hat{y}_1$ or $[2a\hat{y}_1 + b\hat{y}_1^2]\partial_x \wedge \partial\hat{y}_1$, respectively. $\square$

Remark 4.3.2. For $V = \mathcal{O} \oplus \mathcal{O}$, a co-Higgs pair (V, Φ) is semistable but not stable if and only if there is a Φ -invariant copy of $\mathcal{O}$ in V . That is, there exist a nonzero $v \in H^0(V)$ such that $\Phi(v) = v \otimes \lambda$ where $\lambda \in H^0(\mathcal{O}(2))$. Moreover, we can write $\Phi = Ax^2 + Bxy + Cy^2$ where A, B , and C are 2×2 complex valued matrices. Then $\Phi(v) = v \otimes \lambda$ if and only if v is a common eigenvector of A, B , and C . Therefore

$$\Phi = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix},$$

where b is a homogeneous polynomial of degree 2 on $\mathbb{P}^1$, see [26, p. 42]. In this case, the Poisson structure on $\mathbb{P}(V|_{\mathcal{U}})$ is given by $\pi_\Phi = [b\hat{y}_1^2]\partial_x \wedge \partial\hat{y}_1$.

Let us now describe the zeroes of the Poisson structure that come from (semi)stable co-Higgs fields on $\mathbb{P}^1$.

Theorem 4.3.3. *Let (V, Φ) be a stable rank 2 co-Higgs bundle over $\mathbb{P}^1$ where either $V = \mathcal{O} \oplus \mathcal{O}(-1)$ or $V = \mathcal{O}(1) \oplus \mathcal{O}(-1)$. Moreover, let π_Φ be the holomorphic Poisson structure on $\mathbb{P}(V)$ induced by Φ . We then have the following:*

1. *The spectral curve S_Φ is irreducible and smooth for generic Φ .*
2. *The set $\text{Zeroes}(\pi_\Phi)$ is isomorphic to the spectral curve S_Φ .*

Proof. We are considering $V = \mathcal{O} \oplus \mathcal{O}(-d)$ for $d = 1$ or 2 . By Theorem 4.3.1, we can write Φ in the form

$$\Phi = \begin{pmatrix} 0 & b \\ 1 & 0 \end{pmatrix},$$

where b is a nonzero homogeneous polynomial of degree 3 when $d = 1$ and of degree 4 when $d = 2$. The spectral curve $S_\Phi \subset \text{Tot}(T_{\mathbb{P}^1})$ is defined by the equation

$$\det(\eta \text{Id} - \Phi) = \eta^2 - b = 0$$

where η is the tautological section of $p^*T_{\mathbb{P}^1}$ and $p : T_{\mathbb{P}^1} \rightarrow \mathbb{P}^1$ is the bundle projection.

For the case $d = 1$, the polynomial $b(x)$ has odd degree equal to 3, thus it cannot be a perfect square. Therefore, S_Φ is irreducible in this case. Whereas $b(x)$ has degree 4 when $d = 2$. Thus, $-b(x) = f^2(x)$ for some f with $\deg(f) = 2$. That is, Φ would be nilpotent at the roots of f , contradicting stability. Therefore, $b(x)$ cannot be a perfect square and S_Φ is irreducible. Moreover, the spectral curve S_Φ is smooth if and only if $b(x)$ has no repeated roots. For a generic choice of $b(x)$, this condition will be satisfied. However, there exist stable co-Higgs bundles where $b(x)$ may have repeated roots, leading to singular (but still irreducible) spectral curves.

At each point $x \in \mathbb{P}^1$, the eigenvalues $\lambda_\pm = \pm\sqrt{b(x)}$ are distinct, with corresponding eigenspaces forming two disjoint sections of $\mathbb{P}(V)$. By Corollary 4.2.8, these sections constitute $\text{Zeroes}(\pi_\Phi)$ and form a 2:1 cover of $\mathbb{P}^1$ isomorphic to S_Φ . Note that Φ has no zeroes on $\mathbb{P}^1$, as the polynomial b is always nonzero (otherwise, Φ would be nilpotent, contradicting stability). Moreover, since Φ has no zeroes, there are no vertical components in $\text{Zeroes}(\pi_\Phi)$ by Theorem 4.1.6. Thus, $\text{Zeroes}(\pi_\Phi)$ consists only of this 2:1 cover of $\mathbb{P}^1$, forming a smooth, irreducible curve in $\mathbb{P}(V)$ that is isomorphic to the spectral curve S_Φ . $\square$

Remark 4.3.4. This theorem establishes a difference between stable and unstable co-Higgs bundles. In the cases $d = 1$ and $d = 2$, Poisson structure coming from stable co-Higgs pairs have a simpler zero set, which is isomorphic to the smooth, irreducible spectral curve.

The situation is, however, different in the semistable case, where $V = \mathcal{O} \oplus \mathcal{O}$ and $\Phi_0 = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}$ with a, b are homogeneous polynomials of degree 2. This time, Φ has zeroes at the two points on $\mathbb{P}^1$ where b vanishes. Therefore, in addition to the 2:1 cover B of $\mathbb{P}^1$, the zeroes of π_Φ also include two vertical components of the form $x \times \mathbb{P}^1$, where x are the zeroes of b .

In conclusion, for stable co-Higgs bundles (V, Φ) on $\mathbb{P}^1$, the zero set of π_Φ tends to be simpler, typically consisting of a smooth, irreducible curve (isomorphic to the spectral curve S_Φ). Semistable bundles may have additional fibre components in the zero locus.

Remark 4.3.5. Theorem 4.3.3 suggests that stability is related to the irreducibility of the spectral curve and Corollary 4.2.8 shows that the zero locus of π encodes information about both the spectral curve and the zeros of Φ . These results indicate that the stability of (V, Φ) may appear in the geometry of the zero locus of π , particularly in its relation to the spectral curve.

Chapter 5

Outlook

In this chapter, we present some possible directions in which the ideas in this thesis can be taken further.

Co-Higgs bundles (V, Φ) that induce holomorphic quadratic Poisson structures on $\text{Tot}(V)$ and $\text{Tot}(\mathbb{P}(V))$. In this thesis, we classified all possible forms of co-Higgs bundles of rank r over an open set $\mathcal{U} \subset \mathbb{C}$ that induce integrable Poisson structures on V (Theorem 3.2.7) and on $\mathbb{P}(V)$ (Corollary 3.3.6). We further extended these results for rank r co-Higgs bundles over an open set $\mathcal{U} \subset \mathbb{C}^2$ (Theorem 3.2.14). Using similar techniques to the ones we used to prove Theorems 3.2.7 and 3.3.6, one should be able to generalize this even further to any open set $\mathcal{U} \subset \mathbb{C}^n$, $n > 2$, although the equations involved might be more complicated to solve, see Remark 3.2.13. Moreover, we used these local forms to study co-Higgs bundles and their corresponding quadratic bivectors on Riemann surfaces in section 3.4. A natural question is to use these techniques to study co-Higgs bundles and their corresponding quadratic holomorphic bivectors on higher-dimensional complex manifolds. We have started doing this on Hirzebruch and Hopf surfaces in collaboration with Eric Boulter and Ruxandra Moraru [2].

Explicit descriptions of the zero set of π_Φ . Let (V, Φ) be a co-Higgs bundle with induced quadratic Poisson structure π_Φ on the complex manifold X . Moreover, let $\rho : \mathbb{P}(V) \rightarrow X$ be the natural projection. While we have established that $\text{Zeroes}(\pi_\Phi)$ decomposes as $B + \rho^{-1}(D)$, where D is the divisor of zeros of Φ and B is a subvariety of $\mathbb{P}(V)$ (see Theorem 4.1.6), the subvariety B becomes more complex for rank $r \geq 3$ bundles when Φ_x does not have distinct eigenvalues for all $x \in X$. For instance, in the case of the trivial

rank 3 bundle with a nilpotent co-Higgs field, we showed that B can be a product $X \times \mathbb{P}^1$ rather than a finite cover of X . This suggests several questions worthy of investigation: What is the structure of the component B for co-Higgs fields of rank $r \geq 3$? Can we classify these structures based on the Jordan canonical form of Φ ? Can we acquire a better understanding of the relationship between B , the spectral curve S_Φ , and stability for all co-Higgs fields.

Stable holomorphic Poisson structures. Another direction of investigation is to develop a notion of stability for holomorphic Poisson structures motivated by that of stable co-Higgs bundles. This would potentially allow us to define moduli spaces of holomorphic Poisson structures. The results in this thesis demonstrate that the stability of co-Higgs bundles manifests geometrically in the structure of the zero set of their induced quadratic Poisson structures. This suggests that one might be able to characterize ‘nice’ holomorphic Poisson structures through conditions analogous to those defining stable co-Higgs bundles. Such a notion could potentially be defined in terms of Poisson subvarieties, examples of which are given by Φ -invariant subbundles (see Theorem [4.2.1](#)).

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