

The Philosophy of Reconstructions of Quantum Theory: Axiomatization, Reformulation, and Explanation

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

The quantum reconstruction programme is a novel research program in theoretical physics aimed at deriving the key features of quantum mechanics from fundamental physical postulates. Unlike standard interpretations of quantum theory, which take the Hilbert space formalism at face value, quantum reconstructions seek to derive this formalism from axiomatic principles. Reconstructions represent a new shift in foundations of physics away from interpreting quantum theory and towards understanding its foundational origins. The reconstruction programme has been a major focus of research in physics, beginning with Hardy (2001)’s “Quantum Theory from Five Reasonable Axioms.” However, the quantum reconstruction programme has been met with very little interest in philosophy. The goal of this project is to situate the quantum reconstruction programme in a broader philosophical context, investigating themes such as scientific methodology, explanation, the applicability of mathematics to physical theories, and theory exploration and development in the philosophy of science.

I argue that reconstructions demonstrate a contemporary application of axiomatization with significant points of continuity to historical axiomatizations. I also argue that we should best understand reconstructions as provisional, practical representations of quantum theory that are conducive to theory exploration and development. Further, I contend that reconstructions function as alternative formulations of quantum theory, which is methodologically advantageous. I discuss Bokulich (2019)’s “Losing the Forest for the Ψ : Beyond the Wavefunction Hegemony” which argues that the existence of alternative formulations of quantum theory undermines our ability to literally interpret a single formulation. I argue that Bokulich (2019)’s conclusions further support the reconstructionist’s rejection of the standard interpretative project. I also argue that reconstructionists have gone beyond Bokulich (2019)’s insistence on the consideration of alternative formulations to develop a methodology that systematically constructs alternative formulations of quantum theory. Additionally, I argue that reconstructions of quantum theory are genuinely explanatory as they answer Wheeler (1971)’s “Why the quantum?” question. I contend that reconstructions are explanatory in the same spirit as Bokulich (2016)’s account of explanation in “Fiction As a Vehicle for Truth: Moving Beyond the Ontic Conception” which focuses on patterns of counterfactual dependence that correctly capture underlying dynamics. However, in order to accommodate the reconstruction case, I expand Bokulich’s account to consider theories and models as well as representations that are neither fictional nor literal interpretations. Thus, I offer an account of explanation in the reconstruction programme that is noncausal and non-interventionist, utilizing *w*-questions *a la* Woodward (2003). I conclude that reconstructions of quantum theory give us genuine insight into the structure of quantum theory via the generalized physical principles which carry physical content.

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Chapter 1

Introduction

1.1 The Quantum Reconstruction Programme

The quantum reconstruction programme is a novel research programme in the foundations of physics that seeks to elucidate the fundamental physical principles of quantum theory. The programme represents a “paradigmatic shift in the domain of the foundations of physics” (Grinbaum, 2007, 387) as it moves the focus from supplying interpretations of quantum theory toward the process of reconstructing the theory from simple principles.

Interpretations of quantum theory are primarily concerned with constructing a description about what the formalism of quantum theory tells us about the real world. Specifically, interpretations of quantum theory seek to create an ontological picture of the world based on the Hilbert space formalism, which is an articulation of quantum theory in terms of vectors in Hilbert spaces with self-adjoint operators. For example, standard interpretations of quantum mechanics tend to grant ontological status to the wavefunction, which functions as the common representation of the quantum state (Bokulich, 2020, 2). The Hilbert space formalism is the accepted formalism of quantum theory and very successfully describes the theory. However, there has not yet been a single interpretation of the theory that has reached a scientific consensus to the point where physicists consider the issue of what the formalism says about the real world is solved. Many candidate interpretations have been proposed, such as the Many-Worlds interpretation, the de Broglie-Bohm interpretation, and the hegemonic Copenhagen interpretation. Different interpretations have diverging means of attending to the two overarching problems in quantum theory, namely the measurement problem and the problem of non-locality. Further, these distinct interpretations

often come to very different and contradictory conclusions about the nature of quantum reality (Berghofer, 2024b, 5).

It is amongst this turmoil that the quantum reconstruction programme has carved out a space for itself in quantum foundations. One of the significant questions that the programme aims to answer is the “why the quantum?” question, which asks *why* quantum mechanics is so expertly described by complex Hilbert spaces.¹ The reconstruction programme is born from the idea that instead of accepting the Hilbert space formalism at the outset and devising an interpretation based on that formalism, we should aim to rederive this formalism from fundamental physical principles via axiomatization. Axiomatization is a specific methodology that reconstructionists use which includes the process of deriving a result from a set of axioms. Reconstructions are distinct attempts at rederiving the formalism of quantum theory using axiomatization as a methodology.² In this way, the formalism of quantum theory will have an unambiguous origin in terms of those physical principles. One of the primary concerns with the standard interpretative project is that the Hilbert space formalism is expressed in overtly mathematical language which does not lend itself to a clear physical interpretation. Quantum mechanics is, after all, “one of the two or three great theories of physics, and look how it’s posed!” (Fuchs and Stacey, 2016, 285). A prevailing sentiment in the reconstruction programme is that if we are unable to articulate quantum mechanics in terms of clear physical principles then the physical content of the theory would never become clear.

The reconstruction programme is a proposed solution to this problem. The idea is that we *begin* with clear physical principles and *end* with the quantum formalism. In this way, we will have clarified the connection between the mathematical language used to express quantum mechanics and the description the language purports to tell us about the physical world. Thus the reconstruction programme is “based on a plausible, almost trivial, idea: It is more straightforward to understand the physical meaning of physically meaningful principles than to understand the physical meaning of an abstract mathematical formalism” (Berghofer, 2024b, 6). The result of the quantum reconstruction programme is multiple distinct reconstructions. If one is able to derive the von Neumann postulates—the minimal mathematical and empirical results of quantum theory—from a set of axioms, it is considered a success.³ Thus there will be many different reconstructions which encode

¹I provide a more nuanced discussion of what exactly the “why the quantum?” question asks in section 5.1.

²I do sometimes refer to a reconstruction *as* an axiomatization of quantum theory to highlight reconstructions as axiomatization attempts.

³The von Neumann postulates form the foundation of the Hilbert space formalism. These postulates include the postulate that states are represented by vectors, that observable quantities are associated with

a variety of physical principles.

The first modern reconstruction was completed by Hardy (2001) with many more to follow, including Chiribella et al. (2011), Dakic and Brukner (2009), Goyal (2014), and Höhn (2017). The programme continues to gain momentum among physicists engaged in quantum foundations, leaning into the meteoric rise of quantum information theory. Quantum information theory “seeks to describe and make use of the distinctive possibilities for information processing and communication that quantum systems provide” (Timpson, 2013, 45). The theory sits at the intersection of quantum physics, computer science, and communication theory (Timpson, 2013, vii). The reconstruction programme takes seriously the idea that information may serve as a foundational way of understanding quantum mechanics.

However, in spite of the popularity of quantum reconstructions among physicists, there has been very little interest in the programme by philosophers (Berghofer, 2024b, 2). As such, there is limited philosophical literature on the programme.⁴

This dissertation aims to be one of the first major works in the philosophy of quantum reconstructions. My goal is to situate the quantum reconstruction programme in a broader philosophical context, specifically as it relates to methodology, explanation, and theory development and exploration in the philosophy of science. Further, I intend that this dissertation will serve as a defense of the goals and methodology of the programme. Quantum reconstructions have generated many successful reconstructions and as I will argue, the methodology—some of it novel methodology—is well-suited for the aims of the programme. Most importantly, I will argue that the quantum reconstruction programme is an innovative means of making progress in the foundations of physics. It is my hope that my work will encourage philosophers and physicists to appreciate the significance and value of quantum reconstructions. Better yet, I hope that this dissertation will encourage both philosophers and physicists to actively engage with quantum reconstructions.

The quantum reconstruction programme raises several important philosophical issues that I will focus on in this dissertation. The first issue is the use of axiomatic methodology in the programme, which is a central methodological feature of reconstructions. I will investigate why axiomatic methodology is used, how it is construed, and if it is methodologically beneficial. The second issue is that of alternative formulations of theories. Alternative formulations of theories are different mathematical representations of

a Hermitian operator, etc. and were posited in von Neumann’s *Mathematical Foundations of Quantum Mechanics* (1932).

⁴This is not to say that there is *no* philosophical work on the reconstruction programme, though it is minimal. For example, Adlam (2022), Koberinski and Müller (2018a), Grinbaum (2007), and Timpson (2013) all explicitly discuss the reconstruction programme, amongst others.

the same physical situation in the world. The methodology of reconstructions of quantum theory is such that alternative formulations of quantum theory are intentionally generated. I will determine if reconstructions should be considered alternative formulations of quantum theory and if having alternative formulations is methodologically advantageous. The last issue is the issue of explanation. In the reconstruction programme, physicists have the intuition that reconstructions are explanatory *in some sense*. I will explore if reconstructions are genuinely explanatory and what type of explanatory insights are revealed in the programme.

1.2 Project outline

1.2.1 Case Studies from the Quantum Reconstruction Programme

The project will proceed as follows. In Chapter 2 I motivate the quantum reconstruction programme, adding more detail to the development of the programme and why the programme arose in foundations of quantum theory. This chapter will lay out the important details from two reconstructions, Hardy (2001) and Selby et al. (2018), which act as my two case studies. I have chosen these two reconstructions because they use different physical principles to inform different sets of axioms for quantum theory. For example, Hardy (2001) and Selby et al (2018)’s reconstructions are situated in different frameworks. Hardy (2001)’s reconstruction is situated in the Generalized Probabilistic Theories framework and he offers a reconstruction based on five physically motivated axioms. These axioms involve principles like the ability to distinguish states and the continuity of state transformations. Selby et al. (2018)’s reconstruction is situated in the Process Theoretic framework. Selby et al. present a diagrammatic representation of their reconstruction, which allows abstract processes and systems to be represented graphically. The set of postulates offered in Selby et al. are framed in terms of diagrams, which they take to provide an intuitive way of capturing the structure of quantum theory. The distinctions between these two reconstructions will be made clear in Chapter 2.

In both case studies I discuss some general motivations from each and walk through the important details of the proofs. In particular, I explore how both case studies distinguish the classical from the quantum. In order to explore some of the general motivations in Hardy (2001) and Selby et al. (2018), I cover Hardy (2018) and Coecke (2012). I also articulate some of the important concepts that we will encounter throughout the dissertation. For example, the Generalized Probabilistic Theories framework is very important as many reconstructions work within it. The Generalized Probabilistic Theories framework

is also significant because of its operational formulation, which fosters an empirical connection between the framework and the laboratory. I also explore the distinction between the Generalized Probabilistic Theories framework and the Process Theoretic framework. I conclude this chapter by highlighting the important philosophical issues that arise from the case studies and the quantum reconstruction programme more generally. These include philosophical questions pertaining to axiomatic methodology, the existence of alternative formulations of theories, and the issue of explanation.

1.2.2 Methodology and the Role of Axiomatization

In Chapter 3 I focus on the use of axiomatic methodology in the reconstruction programme. I argue that reconstructions of quantum theory use axiomatic methodology in ways that are methodologically advantageous and conducive to theory exploration and development. In reconstructions, there are features of axiomatic methodology that have deep historical roots—for example, going back to Hilbert’s axiomatization of geometry—that inform how axiomatization is used in the present day. I contend that in the context of quantum reconstructions, axiomatization is best understood as facilitating provisional, practical representations of quantum theory.

Individual reconstructions serve as distinct axiomatizations of the theoretical content of quantum theory. The goal of the programme is to find a “compelling set of axioms that singles out quantum theory from among all possible theories” (Chiribella and Spekkens, 2016, 4). In this chapter I argue that reconstructions represent a modern usage of axiomatization with significant points of continuity to von Neumann’s axiomatizations in quantum mechanics. I begin in section 3.1 by identifying common axiomatic methodology shared between David Hilbert’s early axiomatizations of geometry and mechanics and a later axiomatization by Hilbert, Nordheim, and von Neumann. In section 3.2 I then analyze von Neumann’s axiomatic derivation of the trace function. Afterward I discuss Hardy (2001)’s reconstruction from five simple axioms. In particular, I contend that Hardy and von Neumann’s axiomatizations share similar methodological ordering, have a common operational framing, and insist on the empirical basis of axioms. In the reconstruction programme, interesting points of discontinuity with historical axiomatizations include the stipulation of a generalized space of theories represented by a framework such as the GPT framework and the stipulation of analytic machinery at two levels of generality (first by establishing a generalized mathematical framework and then by positing specific formulations of axioms). In light of the reconstruction programme, I argue that we should understand axiomatization attempts as being context-dependent, context which is contingent upon the goals of inquiry and the maturity of both mathematical formalism and theoretical underpinnings

within the area of inquiry. Lastly, in section 3.5 I discuss an account of axiomatization offered by Mitsch (2022). Drawing on Mitsch (2022)’s account of axiomatization, I conclude that reconstructions should best be understood as provisional, practical, representations of quantum theory that are well suited for theory development and exploration. I propose my context-dependent re-framing of axiomatization as a means of enriching Mitsch’s account.

In this chapter I also argue that the inclusion of a generalized framework is methodologically advantageous in the reconstruction programme. I argue that by establishing a generalized framework, reconstructions are better able to compare and contrast alternative formulations of quantum theory within those frameworks. This comparative platform enables us to more clearly see the distinctions between theories in the space of theories, i.e., classical, quantum, non-quantum, and extra-quantum theories. And by seeing the distinctions between theories, reconstructions function as vehicles for theory exploration.

1.2.3 Reconstructions as Alternative Formulations of Quantum Theory

Chapter 4 explores the role that alternative formulations of theories have to play in the reconstruction programme. Alternative formulations of scientific theories are important because they give us different viewpoints on features of the theory. I argue that reconstructions should be considered alternative formulations of quantum theory. In the case of reconstructions, I contend that there are methodological advantages to generating alternative formulations of quantum theory. One of the advantages is the genuine physical insights we gain when we consider these alternative formulations. In the presence of alternative formulations of a theory, we are not limited to interpreting a single formulation. What this means is that we are encouraged to explore a variety of different ways that a theory might be, which reveals distinctions between alternative formulations. I conclude that if we had not considered those alternative formulations, the distinctions between classical, quantum, and extra-quantum theories would be obfuscated.

I begin the chapter by discussing Bokulich (2019)’s “Losing Sight of the Forest for the Ψ : Beyond the Wavefunction Hegemony,” which argues that standard interpretations of quantum theory are undermined by the existence of alternative formulations of the theory. Bokulich also argues that there are benefits to having alternative formulations of theories, benefits that are not realized if alternative formulations are not explored. I argue that the reconstruction case serves as an example that reinforces Bokulich’s perspective on the advantages of exploring alternative formulations. Further, I contend that (1) Bokulich’s (2019) argument further supports the reconstructionist’s rejection of the standard interpre-

tative project. I also argue that (2) there are significant advantages to having alternative formulations of theories resultant from non-literally interpreting them. In section 4.2.2 I argue that reconstructions are an example of what I call non-literally interpreted formulations. Non-literally interpreted formulations are formulations of a theory that enable us to draw genuine insights into the physical world without literally interpreting the formalisms of the theory, i.e., directly ascribing an ontology based on the formalism. The category of a non-literally interpreted formulation allows me to establish a tripartite distinction of formulations between full formulations, non-literally interpreted formulations, and literally interpreted formulations. The distinction between the three categories is related to the attitude one has toward the role of realism and representation in scientific theories.

I contend that there are advantages to having alternative formulations of theories born from non-literally interpreting them. In reconstructions the primary advantage of non-literally interpreting a formulation are the physical insights that are illuminated via multiple alternative formulations. Further, these physical insights enable us to introduce a moderate realism in spite of a non-literal interpretation. Therefore, non-literally interpreted formulations will offer a variety of realism that is not as robust as we might expect from a full interpretation but they avoid the problems that arise from directly reading an ontology off of a mathematical formalism.

Finally, in section 4.3 I argue that (1) and (2) serve as motivation and justification of the methodology of the reconstruction programme. I conclude that the methodology of reconstructions has evolved to be perfectly suited to the aims of inquiry of the research programme. Reconstructionists have gone beyond Bokulich (2019)’s insistence that we must consider alternative formulations to develop a methodology that prioritizes the construction of alternative formulations. Specifically, I argue that reconstructionists have systematized the construction of alternative formulations of quantum theory in order to explore the space of theories. Further, reconstructionists have inverted the methodological order that is typical of the standard interpretative approach in order to elucidate the general physical principles of quantum theory. Both of the above are concerted methodological strategies that aid reconstructionists in their attempts to gain conceptual insights into the nature of quantum theory.

1.2.4 Answering the “Why the Quantum?” Question

Chapter 5 will investigate the connection between explanation and the quantum reconstruction programme. I argue that reconstructions are genuinely explanatory. Quantum reconstructions explain via their ability to give us answers to Wheeler (1971)’s “why the

quantum?” question, which asks why quantum theory has the peculiar mathematical structure that it does. I contend that reconstructions enable us to both discover the structure of theories and to illuminate the genuine physical insights that are revealed by the physical principles. Thus, there will be many answers for why quantum theory has the particular structure that it does. I conclude that the proliferation of answers is a *feature* of the reconstruction programme, not a bug. The sheer number of reconstructions has positive methodological significance for the aims of the programme, giving us complementary views on different aspects of quantum theory.

I begin in section 5.1 by exploring the type of question that is being asked in Wheeler (1971) and the way that reconstructions propose to answer it. I argue that reconstructions are genuinely explanatory in their ability to answer this “why the quantum?” question. In section 5.2 I examine Bokulich (2012)’s “Distinguishing Explanatory from Nonexplanatory Fictions,” which offers an account of explanation based on patterns of counterfactual dependencies. I develop a noncausal and non-interventionist variety of explanation for application to the reconstruction programme that utilizes Woodward (2003)’s w-questions. Section 5.3 investigates Woodward’s account of explanation, focusing on w-questions, or “what-if-things-had-been-different” questions. These questions recast explanation in terms of answering how the explanandum would have been different if the explanans had been changed. I argue that reconstructions enable us to answer w-questions by allowing us to modify the set of axioms within a reconstruction, guiding us toward a particular region in the space of theories.

I seat my account of explanation in the reconstruction programme in the context of Bokulich’s eikonic conception of explanation. The eikonic conception of explanation focuses on the importance of representations in scientific theories, allowing that representations need not be veridical in order to be genuinely explanatory (Bokulich, 2018a). I argue that reconstructions successfully track patterns of counterfactual dependence within the space of theories. When these patterns of counterfactual dependence orient us in the quantum region in the space of theories, a candidate answer for the “why the quantum?” question is provided. I assert that these patterns illuminate the distinctions between regions within the space of theories, enabling us to discover the structure of theories.

The structure of the space of theories is illuminated via the generalized physical principles which carry physical content. In section 5.3.1 I contend that these physical principles give us *genuine physical insights* in their ability to be physically interpreted. In this way, I argue that the physical insight we gain connects the structure of the space of theories to the structure of the world. I conclude by arguing that the generalized framework is the methodological feature that allows us to track these patterns of counterfactual dependence. Therefore, the generalized framework enables us to discover the distinctions

between theories and is an integral part of a reconstruction’s explanatory power.

Lastly, in section 5.4 I discuss Duwell (2020)’s account of modal understanding in the reconstruction programme. I conclude that Duwell’s account of understanding and my account of explanation in the reconstruction programme are complementary accounts. However, my perspective differs from Duwell insofar as I propose a connection to the world as it really is through the physical insights we gain from our generalized principles.

1.3 Conclusion

The chapters of this dissertation are connected in several ways. For example, in Chapter 4 I propose that reconstructions are non-literally interpreted formulations of quantum theory. The result of this designation is that reconstructions enable us to gain genuine physical insight into the world without literally interpreting the formalism. In Chapter 5, I argue that it is the discovery of genuine physical insight via the generalized physical principles that allows reconstructions to be genuinely explanatory. Thus, the physical insight that we gain from the consideration of reconstructions as non-literally interpreted formulations is integral to how reconstructions are genuinely explanatory.

The significance of the methodological features of the reconstruction programme are highlighted throughout the chapters. In Chapter 3, I explore the methodological benefits of axiomatization. These benefits include the axiomatic structure within which to modify or drop axioms, the credibility of a derivation from axiom sets, and the significance of the generalized framework for the comparison of theories. The methodological significance of the generalized framework is a common theme. Generally construed, the generalized framework is a novel and extremely important feature of the reconstruction programme, enabling us to probe the space of theories to discover distinctions between what we consider to be classical, quantum, non-quantum, and extra-quantum theories. In Chapter 4, I argue that reconstructionists have evolved their methodology via the generalized framework to develop a means of generating alternative formulations of the theory. This evolved methodology is integral to the types of questions that the programme is interested in answering, i.e., probing the boundaries of what we consider ‘quantum’ and how we might carve up the theoretical space.

Further, in Chapter 4 I highlight how methodologically advantageous it is to have multiple formulations of quantum theory based on Bokulich (2019). One may wonder why a multitude of reconstructions does not mirror the same problem that multiple interpretations of quantum theory does. Is it not the same difficulty, in terms of the sheer number of

different solutions to the same problem, as with interpretations? My answer to this is no. In Chapter 5, I contend that the generalized framework gives us the right theoretical space to track patterns of counterfactual dependence, thereby allowing us to answer the “why the quantum?” question. Being able to work with a variety of reconstructions give us different vantage points on what makes quantum theory distinctly quantum. Additionally, in Chapter 3 I assert that axiomatization gives us the right structure to change or modify axioms easily. And as I argue in Chapter 5, it is the changing or modifying of axioms that gives us candidate answers for answering the “why the quantum?” question.

The intended audience of this research include philosophers and physicists concerned with the conceptual foundations of physics. In particular, this dissertation is intended for those who are curious about other ways outside of the standard interpretative project to make progress in the foundations of quantum theory. More generally, this work is aimed at those curious about theory exploration and development in science, scientific methodology, and the application of mathematics to physical theories. This project is also intended for philosophers and physicists interested in the role of realism in the philosophy of physics. Finally, my hope is that those who are convinced that the standard interpretative project is the only pathway forward in foundations of physics will read this dissertation. Perhaps they will be convinced that the reconstruction programme has an important role to play in how we do physics.

Chapter 2

Case Studies from the Quantum Reconstruction Programme

2.1 Introduction

The quantum reconstruction programme is a research programme in foundations of physics which seeks to reconstruct quantum theory from a set of physical axioms. The basic purpose of the reconstruction programme is to achieve a better understanding of quantum theory by identifying and examining the fundamental principles of the theory. A reconstruction aims to rederive the essential mathematical and empirical foundations of quantum theory, known as the von Neumann postulates. These postulates outline the Hilbert space formalism, which describes states as vectors within a complex Hilbert space. This formalism underpins quantum mechanics, providing the framework for describing the states and evolution of quantum systems.

Methodologically, reconstructions begin by defining a generalized framework that stipulates a range of possible theories, encompassing classical, quantum, and extra-quantum theories. This landscape allows us to compare and contrast different theories, highlighting the distinctions between them. Within the general framework, specific reconstructions are constructed. Axioms are stipulated, which are based on the physical principles that researchers consider fundamental to quantum theory. Axioms are specific mathematical formulations that represent the physical principles. From these axioms, a successful reconstruction will rederive the von Neumann postulates. This process shows that the proposed physical principles, which initially are considered assumptions, along with their associated mathematical formulations, entail the von Neumann postulates.

Alongside achieving a better understanding of quantum theory, the programme is also interested in investigating where the formalism of the theory comes from and in particular *why* the complex Hilbert space so successfully describes nature. As Goyal (2022) describes, the specific goal of reconstruction is to distill “the full physical content of the formalism into physical principles and assumptions that can be expressed in natural language and that are amenable to philosophical reflection” (Goyal, 2022, 2). The hope is that articulating quantum theory in this way will enable us to understand quantum theory in terms of clear physical statements that clarify the connection between the formalism and the physical world. However, the programme has received very little attention from philosophers, in spite of significant development in physics. The reconstruction programme is distinct from standard work in philosophy of quantum theory, which often focuses on developing interpretations of quantum theory. Interpretations of quantum theory aim to construct an ontological picture of the world based on the accepted formalism of the theory.

This dissertation aims to develop a philosophical foundation for the reconstruction programme. The purpose of this chapter is to offer an overview of the two main case studies, pulling the philosophically salient features from each. These case studies serve as the foundation from which I draw my philosophical conclusions on the programme. This chapter will set the context for the research programme, specifically as it relates to the motivations and goals of reconstructions and how the programme relates to standard work in foundations of physics. I also explain important concepts such as the generalized framework, the classical bit, and the qubit. The material in this chapter will provide the necessary background for the physics examples, enabling readers to engage with the philosophical arguments presented therein.

Below, I offer an overview of the reconstructions presented in Hardy (2001) and Selby, Scandolo, and Coecke (2018). I focus on explicating the important details of the physics examples and highlight the distinctions between the two reconstructions. These distinctions are relevant as they reveal the diverse ways that researchers can reconstruct quantum theory. My reason for choosing Hardy (2001)’s reconstruction is that it was pivotal in the development of the programme as one of the first compelling reconstructions. Hardy is considered to be one of the founders of the modern reconstruction approach. In addition, I review Hardy (2018)’s “Construction Interpretation: Conceptual Roads to Quantum Gravity”. (Hardy, 2018) is relevant as it is an intentional discussion of methodology and strategy-building in quantum foundations. I then move to Selby et al (2018)’s reconstruction which is significant because it utilizes a category-theoretic mathematical framework, as opposed to the Generalized Probability Theory framework used in Hardy (2018). I also review Coecke’s “Quantum Mechanics–Take II” (Coecke, 2012). Coecke (2012) is notable because he takes it to be a continuation of von Neumann’s quantum logic, which illuminates

some of Coecke’s earlier motivations.

I begin this chapter in section 2.2 by discussing some of the motivations for the quantum reconstruction. These include the dissatisfaction with the Hilbert space formalism, the desire for clear physical statements as a foundation for quantum theory, and the search for quantum gravity. This section also clarifies the connection between the quantum reconstruction programme and the standard interpretative project. In section 2.3 I review Hardy (2001), showing how he derives quantum theory from five axioms. Section 2.3 also includes an illustration of how the classical and the quantum are often distinguished. In section 2.3.2 I examine Hardy (2018), revealing some of his deeper motivations. Section 2.4 presents work by Coecke (2012) and Selby, Scandolo, and Coecke (2018). In section 2.4.2 I walk through Selby et al. (2018)’s derivation of quantum theory from diagrammatic postulates. Throughout the chapter, I highlight how researchers select different mathematical formalisms based on their suitability for expressing the physical concepts considered to be most important. In section 2.5 I conclude with some important philosophical questions that are brought to light as a result of the reconstruction programme. These are the philosophical questions that will be answered in this dissertation.

2.2 Motivations for the quantum reconstruction programme

Reconstructions occupy a unique place in physics, holding a deep connection to foundational work in quantum theory. Recently there has been renewed interest in pursuing foundational questions in quantum theory and reconstructions are viewed as a potential way forward.

In “Why Physics Needs Quantum Foundations” Hardy and Spekkens make the case that a foundational approach is needed to make progress in physics as a whole and “should be regarded as part and parcel of the great project of theoretical physics—to gain an ever better understanding of the world around us” (Hardy and Spekkens, 2010, 4). The reconstruction programme is a part of broader work in quantum foundations. Quantum foundations seeks to determine the philosophical, mathematical, and physical underpinnings of quantum theory. Standard work in quantum foundations is focused on the range of various interpretations of quantum theory (including the Everett interpretation and de Broglie–Bohm’s theory), which aim to paint a picture of what the accepted formalism of quantum theory says about things in the world. The standard approach frames interpretations as giving solutions to both the measurement problem and the problem of nonlocality,

which are long-standing problems in quantum theory on which much ink has been spilled. However, the standard interpretative approach has failed to reveal an interpretation of quantum theory that has reached the level of scientific consensus expected from one of our fundamental physical theories. Therefore, we still lack a “comprehensive, compelling account of just what kind of physical reality is so extraordinarily elegantly encoded in the mathematical formalism of the theory” (Goyal, 2022, 2).

This lack of consensus has prompted some researchers to question what would be required in order for research in quantum foundations to cease. What would researchers in quantum foundations need to do “as a community to get in a position that we would disband?” (Fuchs and Stacey, 2016, 285). Without a shift away from the standard interpretative approach, it is possible that the debate between which interpretation of quantum theory is the correct interpretation would continue *ad nauseam*. This concern serves as a motivation for the reconstruction programme.

Hardy and Spekkens divide researchers in quantum foundations into two camps, the operationalists and the realists. Operationalists will take theory to specify “what probabilities of outcomes will be observed when a given measurement follows a given preparation” (Hardy and Spekkens, 2010, 1). The operationalist approach is generally typified by preparation, transformation, and measurement procedures of quantum systems.¹ Theoretical concepts must be definable in these operational terms. According to Hardy and Spekkens, the inhabitants of the second camp are the realists, who are interested in connecting the complex formalisms of quantum theory with the world ‘as it actually is.’ They assert that it is unclear how one is able to achieve a complete scientific world-view without some form of realism (Hardy and Spekkens, 2010, 2). In the context of quantum foundations, the realist needs an interpretation of quantum theory. However, realism is insufficient by itself because a single candidate interpretation of quantum theory has yet to be “proven sufficiently compelling to achieve a scientific consensus” (Hardy and Spekkens, 2010, 2).

One of the advantages of the operational approach is that it gives us a general mathematical language. The Generalized Probabilistic Theory framework is one framework that utilizes the operational approach and can accommodate a variety of probabilistic theories. Within this framework of probabilistic theories, quantum theory stands as a special case. The operational approach is integral to the reconstruction programme, particularly as it can free the mind “from all the baggage of classical preconceptions and previous attempts to interpret the quantum formalism” (Chiribella and Spekkens, 2016, 4). However, operationalism by itself is insufficient as it ‘bottoms out’ at the level of complex entities, which is an undesirable end (Hardy and Spekkens, 2010, 2). What is of value is reducing those

¹A more thorough discussion of the operationalist approach will be presented in section 2.3.1.

explanations to simple entities and abstract concepts, which Hardy and Spekkens take to be the end goal of the realist (Hardy and Spekkens, 2010, 2). They conclude that a shared approach is required in order to solve long-standing problems in quantum foundations (Hardy and Spekkens, 2010, 2).

Thus, the reconstruction programme represents a paradigmatic shift away from the primacy of interpretations in quantum foundations. The goal of reconstructions is to discover the fundamental physical principles that underwrite quantum theory via an axiomatic approach. To do this, reconstructions posit a set of axioms based on physical principles that successfully isolates quantum theory amongst all possible theories. In this way, the “fundamental principles that one wants to uphold can be build in from the outset” (Chiribella and Spekkens, 2016, 5). The reconstruction programme is also intimately connected to the advancement of quantum technologies and quantum information theory. The quantum information revolution revealed a quantum world that held “new possibilities for information-processing tasks—in particular, communication tasks, cryptographic tasks and computational tasks—that could not be accomplished in classical physics” (Chiribella and Spekkens, 2016, 3). The development of quantum information theory has revealed that informational concepts offer a novel perspective for understanding quantum theory.

Another motivation for the reconstruction programme is the dissatisfaction with the opaque mathematical language that is required to talk about quantum theory. Reference to, for example, complex vector spaces and tensor products does not enable us to see clear connections between the mathematics used to describe the theory and the natural world it purports to represent. There is a general sentiment amongst reconstructionists that “if we couldn’t take the structure of quantum theory and change it from this very overt mathematical speak—something that didn’t look to have much physical content at all, in a way that anyone could identify with some kind of physical principle...then the debate would go on forever and ever” (Fuchs and Stacey, 2016, 285).² The proposal then is to reduce the mathematical structure of quantum theory to clear physical statements that illuminate the connection between natural world and mathematical representation.

Physicists engaged in the contemporary reconstruction programme were not the first to be dissatisfied with the lack of a clear physical interpretation of the Hilbert space formalism. This dissatisfaction was shared by von Neumann who found the Hilbert space formulation unilluminating. This led von Neumann to develop his quantum logic, which was devised in order to capture some of the most salient physical features of quantum mechanics (Coecke, 2012). The motivation behind the development of a quantum logic

²The ‘debate’ Fuchs refers to here is the aforementioned debate regarding the correct interpretation of quantum theory.

was to investigate the physical concept of superposition more deeply, thereby revealing superposition’s physical interpretation. Essentially, von Neumann’s quantum logic would function as a reformulation of quantum theory. However, Coecke argues that quantum logic fell victim to some of the same ills that befell the Hilbert space formulation, with unclear physical interpretations and a dearth of provable quantum mechanical features (Coecke, 2012, 2).

But what kind of clear physical statements are we meant to be working towards? One of the major influences on the trajectory of the reconstruction programme is the search for crisp and clear physical principles. One oft-cited example comes from special relativity, “the speed of light is constant in all inertial frames, and the laws of physics are the same in all inertial frames” (Fuchs and Stacey, 2016, 285). Our great theories of physics—for example, general relativity—can be distilled to these types of crisp and clear physical principles. The focus on crisp and clear physical principles has an historical connection to Einstein’s 1919 principle versus constructive distinction. The distinction characterizes principle theories to consist of a set of well-confirmed empirical generalizations. According to Koberinski and Müller, principle theories explain “by showing that features of the world are deductive consequences of a small set of general principles” (Koberinski and Müller, 2018a, 3). A constructive theory, on the other hand, reduces a broad range of diverse phenomena to a single underlying mechanism (Berghofer, 2024b, 21). One of the main advantages of a principle theory is that if the quantum formalism is derived from a set of fundamental physical principles, then one “gains an explanation in the sense that quantum theory is found to be a unique consequence of easily understandable constraints. The mathematical ingredients therefore become less arbitrary, and provide a partial interpretation in terms of the role they play in satisfying the more comprehensible principles” (Koberinski and Müller, 2018a, 4). Viewing the reconstruction programme through a ‘principle theory’ lens is one way that we might gain insight into the motivations and methodology behind the programme.

Another important motivation behind the reconstruction programme is the search for quantum gravity. The expectation for a theory of quantum gravity is that it will reduce “to quantum theory in one limit and to general relativity in another” (Hardy and Spekkens, 2010, 2), likely modifying both quantum theory and general relativity in the pursuit of their unification. Hardy and Spekkens assert that it is likely that a new mathematical framework driven by the conceptual insights generated by foundational work will be necessary for the development of quantum gravity (Hardy and Spekkens, 2010, 4). In particular, Hardy argues that it is plausible that the pathway to the discovery of quantum gravity will be analogous to Einstein’s pathway to Relativistic Gravity (Hardy, 2018). The hope is that the reconstruction programme will reveal the necessary conceptual and mathematical insights

to illuminate the way to a theory of quantum gravity.

2.3 Hardy

2.3.1 Reconstructing quantum theory from five reasonable axioms

Lucien Hardy provides an axiomatic reconstruction of quantum theory in “Quantum Theory from Five Reasonable Axioms” (Hardy, 2001). The motivation behind Hardy’s approach is to “recover the basic structure of quantum theory along with the most general type of quantum evolution possible” (Hardy, 2001, 1). Hardy’s reconstruction falls under the Generalized Probability Theory framework in light of the notion that quantum theory is a novel type of probability theory (Hardy, 2001, 1).

In the reconstruction programme, the purpose behind a framework is to generate a space of alternative theories or *foil theories* to quantum theory. Foil theories serve as contrasts in order to highlight the distinctive characteristics that we are looking for (Chiribella and Spekkens, 2016, 5). Within the generalized framework, foils are those points in the landscape that are nonquantum, thereby specifying “a way the world might have been had it not been quantum” (Chiribella and Spekkens, 2016, 5). These points of contrast highlight the differences between quantum, non-quantum, and extra-quantum theories within the space of theories. This is imperative because in order to isolate the fundamental principles of quantum theory, we must discover which physical principles orient us within the quantum region in the space of theories.

The framework of generalized probabilistic theories

The framework of Generalized Probability Theories allows us to describe any theory which can be articulated in an operational description. The type of experimental situation considered in the Generalized Operational Theory framework includes preparation, transformation, and measurement devices. A preparation device prepares physical systems in some state, the state of which can be changed via a number of knobs (Hardy, 2001, 3). After the system is released via the pressing of a button it passes through a second device, a transformation device. This device has the ability to transform the state of the system using knobs that can, for example, control fields which affect the system (Hardy, 2001, 3). Finally, the system passes through a measurement device. Knobs on the measurement device can be

adjusted to determine what measurement is made. Due to the framework’s generality, “any physical experiment, quantum, classical or other, can be viewed as an experiment of the type described here” (Hardy, 2001, 3). For example, though classical probability theory and quantum theory are distinct, a common operational articulation of both within the GPT framework gives us a shared language with which to speak. This softens the problem of incommensurability in such cases where we are interested in articulating the relations between disparate theories (Duwell, 2020, 10). The problem of incommensurability was proposed by Kuhn in *The Structure of Scientific Revolutions*. The problem refers to the idea that scientific paradigms are so different from one another that they cannot be directly compared via some common standard (Oberheim and Hoyningen-Huene, 2024). The operational articulation is important because it represents physical systems that can be interacted with, by, for example, setting knobs or pressing buttons. This is significant because it gives us a clearer idea about the physical conclusions we might draw from our investigations.

The basic setting that Hardy considers is one involving preparation, transformation, and measurement devices. A state is associated with each preparation, defined as below:

The state associated with a particular preparation is defined to be (that thing represented by) any mathematical object that can be used to determine the probability associated with the outcomes of any measurement that may be performed on a system prepared by the given preparation. (2001, p. 2)

Thus, the state can be represented by a list of probabilities resulting from all of the possible measurements that could be made. This collection of all possible states is called the state space. However, a smaller set of probabilities “pertaining to a set of carefully chosen measurements may be sufficient to determine the state” (Hardy, 2001, 2). This is true for both classical probability theory and quantum theory. To further characterize the type of system we are considering, Hardy specifies integers K and N , K representing the number of degrees of freedom (the minimum number of probability measurements required to determine the state) and N representing the maximum number of states that are capable of being distinguished from one another in a single measurement (Hardy, 2001, 2). According to Hardy, classical probability theory has $K = N$ whereas quantum theory has $K = N^2$. This is an example of how a generalized framework is helpful for the comparison of different theories.

Hardy’s five axioms include probabilities, simplicity, subspaces, composite systems, and continuity, privileging in particular the notion of continuity. The five axioms are:

1. Probabilities. Relative frequencies are given for given measurements on an ensemble of systems via a specific preparation procedure.
2. Simplicity. Given N where $N = 1, 2, \dots$, the minimum value is taken that is consistent with the axioms.
3. Subspaces. A system whose state belongs to an M dimensional subspace behaves like a system of dimension M .
4. Composite systems. A composite system comprising A, B satisfies $N = N_A N_B$ and $K = K_A K_B$.
5. Continuity. There exists a continuous reversible transformation between pure states in a system.

Hardy’s proof that quantum theory can be derived from these five axioms begins by showing how the state can be described by a real vector p “whose entries are probabilities and that the probability associated with an arbitrary measurement is given by a linear function, $r \cdot p$, of this vector (the vector r is associated with the measurement)” (Hardy, 2001, 2). It is then shown that $K = N^r$ where r is a positive integer. As per the simplicity axiom, $r = 2$. This is because $r = 1$ is ruled out by the continuity axiom. Quantum theory is recovered for a two dimensional Hilbert space when $N = 2$, $K = 4$ are considered (Hardy, 2001, 4). Quantum theory for general N is constructed in light of the subspace axiom. Further, the most general evolution of the state is obtained, showing that “the state of a composite system can be represented by a positive operator on the tensor product of the Hilbert spaces of the subsystems” (Hardy, 2001, 2).

The classical bit, qubit, and Bloch ball

In order to better understand Hardy’s reconstruction, I will explain the different ways we can represent a classical and quantum object. Classical probability theory and quantum theory result in different types of state spaces. The state of a classical bit is either 1 or 0. For example, we can think of this as a coin toss landing on either ‘heads or tails’. Classically, a state “corresponds to a probability distribution over mutually distinguishable alternatives (such as probability p for “heads” and $1 - p$ for “tails” for a coin), and there is always a unique way to decompose that state into its alternatives” (Koberinski and Müller, 2018a, 7). In other words, in a classical setting the system will have two distinct and distinguishable possibilities.

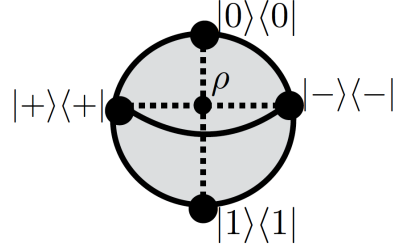


Figure 2.1: The state space of a qubit, or Bloch ball (Koberinski and Müller, 2018a, 6).

A quantum bit, or a qubit, will also have two distinct states. As illustrated in 2.1, a qubit is a representation of the quantum state where states can be $|0\rangle$ and $|1\rangle$, with $|0\rangle$ and $|1\rangle$ as vectors. If we consider a particle, states might correspond to ‘spin up’ and ‘spin down’ after a measurement. However, unlike the classical bit, a qubit can also be in a state of superposition, which is a linear combination of $|0\rangle$ and $|1\rangle$, represented as $|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$.

The Bloch sphere is a three dimensional representation of the qubit. Every point that lies on the boundary of the sphere is a pure state and orthogonal states are on antipodal points of the sphere (Müller, 2021, 18). A signature of nonclassicality is that the decomposition of mixed states into pure states is not unique. So for example in 2.1, the mixed state $\rho = \frac{1}{2}1$ will have $\rho = \frac{1}{2} |0\rangle \langle 0| + \frac{1}{2} |1\rangle \langle 1| = \frac{1}{2} |+\rangle \langle +| + \frac{1}{2} |-\rangle \langle -|$ where $|+/-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + / - |1\rangle)$ (Koberinski and Müller, 2018a, 6). The Bloch ball representation is important as it gives us an intuitive impression of how classical probability theory and quantum theory are distinct. The representation is also important as it gives us a visualization of the physical consequences of the quantum state space, e.g., that continuous reversibility can be thought of as rotations of the Bloch ball (Hardy, 2001, 22).

Proof of the derivation of quantum theory from five reasonable axioms

Based on the above axioms, Hardy shows that axioms 1–4 imply probability theory and the addition of axiom 5 implies quantum probability theory (Hardy, 2001, 15).³ He argues

³Hardy claims that classical probability theory can be derived by removing axiom 5, however the question of how and under what conditions classicality is achieved within the broader space of theories requires some nuance. Hardy uses the continuity axiom to exclude theories that are non-quantum, including classical probability theory (Hardy, 2001, 15). This axiom effectively eliminates alternatives to the Bloch

that the only solution to satisfy all five axioms is with the special case of $N = 2$, which represents quantum theory. More specifically, the $N = 2$ case is illustrated by the Bloch sphere, which is a three dimensional representation of the quantum state. Essentially, Hardy rules out alternatives to the Bloch sphere, showing that deviations from $N = 2$ fail to reproduce it. The $K = N$ case—the classical case—is ruled out as “the only pure vectors are the basis vectors themselves which corresponds to classical probability theory” (Hardy, 2001, 20). This means that $K = 1$ produces a discrete set of vectors, a finite number of pure states.

In contrast, the continuity axiom necessitates an infinite set of pure states. Continuity requires that since “reversible transformations always transform pure states to pure states it follows from this axiom that we can transform any pure state to any other pure state along a continuous trajectory through the pure states” (Hardy, 2001, 15). Continuous reversibility corresponds to unitary evolution, which can be thought of as rotations of the Bloch sphere (Hardy, 2001, 22). And all points on the surface of the Bloch sphere correspond to pure states (Hardy, 2001, 9). Hence, since a ball can be continuously rotated there will be infinitely many pure states. This is problematic for classical systems of finite dimension N because there are only N pure states for such systems, precluding a continuous trajectory through the pure states (Hardy, 2001, 15). Hardy isolates the $K = N^2$ as the special case which reproduces quantum theory, ruling out the $K = N$ case which represents classical probability theory. Therefore, Hardy successfully reproduces quantum theory on the basis of physical principles articulated as axioms within the GPT framework. “Quantum Theory from Five Reasonable Axioms” (2001) represented the beginning of the modern reconstructive effort in quantum foundations.

2.3.2 Construction Interpretation

In “The Construction Interpretation: Conceptual Roads to Quantum Gravity”, Hardy offers a methodology that is aimed at “obtaining an interpretation through dedicating ourselves to a conceptually driven construction...of a theory of Quantum Gravity” (Hardy, 2018, 7). A theory of quantum gravity is “perhaps the holy grail of modern physics (Hardy and Spekkens, 2010, 3). This ‘interpretation’ is born out of Hardy’s search for an underlying theory of quantum gravity that will unite general relativity and quantum theory. Specifically, Hardy argues that drawing an analogy between Einstein’s pathway to relativistic gravity and the pathway to quantum gravity will direct us toward a ‘conceptually-driven’

sphere. However, removing the continuity assumption could result in either classical or quantum theories unless additional constraints on the transformation properties of states are imposed.

construction of a theory of quantum gravity (Hardy, 2018, 1). The argument is based on the plausibility that the search for quantum gravity will find success in a similar manner to that of Einstein’s discovery of the curvature of spacetime.

To do so, Hardy discusses Newton’s law of universal gravitation, which was problematic as it implied action at a distance without a mediating substance (Hardy, 2018, 5). After many attempts at mechanical explanations, it was not until much later that a solution was offered by Einstein in the form of general relativity (also referred to as relativistic gravity), which “limits to Newtonian Gravity on the one hand and to special relativistic field theories (such as Maxwell’s electromagnetism) on the other” (Hardy, 2018, 5):

Newtonian Gravity $\longleftarrow$ Relativistic Gravity $\longrightarrow$ Special Relativistic Field Theories

However, Hardy argues that it was not a discovery that fit neatly in between Newtonian gravity and special relativistic field theories. The curvature of spacetime was a notion very alien to the Newtonian picture which takes space and time to be separate metrics. Rather, both Newtonian gravity and special relativistic field theories underwent revision in order for the full impact of general relativity to be expressed. These revisions resulted in corrections to Newtonian gravity at speeds close to the speed of light and to special relativistic field theories when gravitationally induced curvature is large (Hardy, 2018, 5). The introduction of these limits allowed for general relativity to nest itself between the two disparate theories, thereby functioning as a unifying underlying theory.

The problem of action at a distance without mediating substances was also resolved by the Newton–Cartan formalism. The Newton–Cartan formalism separated space and time into distinct metrics, thereby denoting space and time as the “stuff” that mediated gravitational force between distant masses (Hardy, 2018, 6). In light of the new theory of general relativity, we can now interpret the Newton–Cartan formalism. The Newton–Cartan formalism agrees with the predictions of Newton’s gravitational theory, which assumes the $c \rightarrow \infty$ limit of GR. But the Newton–Cartan theory was formulated *after* General Relativity. So it was not that Einstein ‘discovered’ the curvature of spacetime by using the Newton–Cartan formalism as a stepping stone, nor by using any of the proposed mechanical models (Hardy, 2018, 6). Hardy argues that it was a ‘conceptual approach’ that led Einstein to general relativity. It is this pathway that Hardy argues is analogous to the search for quantum gravity. He likens the plethora of interpretations of quantum theory to the mechanical models of gravitation, arguing that interpretations of quantum theory are “at best, an attempt to guess the analogue of the Newton–Cartan formalism without first attempting the solve the deeper problem of Quantum Gravity” (Hardy, 2018, 6). It only

became obvious how to formulate Newton–Cartan theory after GR had been formulated. And it was conceptual innovation that was required in order to formulate general relativity. Hardy sees the pathway to quantum gravity as analogous to the Newton–Cartan formalism and general relativity case.

Thus, Hardy anticipates that quantum gravity will modify our current conceptions of general relativity and quantum theory in a way similar to how general relativity modified Newtonian gravity and special relativistic field theories (Hardy, 2018, 6). In such a case, “if Quantum Gravity requires a radical departure from existing theories, the most interpretations of Quantum Theory can aspire to is to be the correct limiting version of Quantum Gravity just as the Newton–Cartan formalism is the correct limit of General Relativity” (Hardy, 2018, 7). Thus, Hardy is arguing that generating more interpretations of quantum theory is unlikely to be illuminating in the same way that interpretations of the Newton–Cartan formalism were not illuminating. The Newton–Cartan formalism is not equivalent to GR as it lacked the conceptual innovation of the spacetime metric. Interpreting the Newton–Cartan formalism did not reveal the curvature of spacetime, a conceptual approach to physics did (Hardy, 2018, 6). If quantum gravity requires a modification to both quantum theory and general relativity, then focusing on interpreting the accepted formalism of quantum theory is focusing on the wrong thing. Rather, Hardy claims that focusing on understanding the conceptual foundations of quantum theory is more likely to solve the foundational issue of how to devise a theory of quantum gravity.

General Relativity $\longleftarrow$ Quantum Gravity $\longrightarrow$ Special Relativistic Quantum
Field Theories

For Hardy, the purpose of the paper is to “nudge researchers working on the foundations of Quantum Theory along a similar kind of path but, this time, guided by attempting to solve the problem of Quantum Gravity employing the conceptual tools developed in foundations” (Hardy, 2018, 6). This different methodology is a rejection of the standard interpretative project and a proposed strategy to discover quantum gravity. And it is through the reconstruction programme that we might gain the necessary conceptual insights in order to solve the problem of quantum gravity. It is possible that the reconstruction programme will illuminate a modification to quantum theory and this modification may reveal a new mathematical framework that will lead us to a deeper, more fundamental theory (Hardy and Spekkens, 2010, 4). Ultimately, Hardy maintains that “there is a danger in finding an interpretation of Quantum Theory when that theory will, likely, be replaced by a deeper theory of Quantum Gravity” (Hardy, 2018, 44). And perhaps, like Einstein’s discovery of the curvature of spacetime, quantum gravity will be born from concerted conceptual work in quantum foundations.

2.4 Coecke

2.4.1 Quantum mechanics, take II

In “The Logic of Quantum Mechanics—Take II” (2012), Bob Coecke offers a category-theoretic reconstruction of quantum mechanics. The approach he puts forth is a diagrammatic process logic which takes the notion of compositionality to be a fundamental feature of quantum mechanics. To be precise, the notion of compositionality has to do with how we describe “multiple quantum systems given that we know how to describe the individual quantum systems” (Coecke, 2012, 2), i.e., the interactions between separate systems. It is the interactions between separate systems that Coecke takes to truly characterize quantum behaviour (Coecke, 2011, 2). “Logic—Take II” is an attempt at a second logic of quantum mechanics after Birkhoff and von Neumann (Birkhoff and von Neumann, 1936). Coecke’s revival of quantum logic carries with it the same spirit as the first quantum logic, though he privileges the notion of composition rather than superposition.

Coecke (2012) has the very ambitious goal of discovering a broad quantitative logic that will serve as a foundational logic in a variety of domains including quantum protocols, linguistics, Bayesian inference, and natural language processing (Coecke, 2012, 4). For Coecke, part of the appeal of a broad quantitative logic is its potential heuristic usefulness. One of Coecke’s main goals is to represent processes and Coecke’s graphical logic is well adapted to information theoretic applications, particularly in computation. Coecke (2012) is a precursor to the diagrammatic language, inspired by category theory, that is developed in Selby, Scandolo, and Coecke (2018). Coecke (2012) illuminates the potential for category theory as a different mathematical framework within which to articulate quantum theory instead of the popular GPT framework.

2.4.2 Reconstructing quantum theory from diagrammatic postulates

In “Reconstructing quantum theory from diagrammatic postulates” (2018) Selby, Scandolo, and Coecke provide a diagrammatic reconstruction of quantum theory under the Category-Theoretic framework. The Category-Theoretic framework is centred on the operational notion of composition, for example on the composing of “two systems in parallel (this *and* that) and composing two physical processes in a sequence (do this *and then* do that)” (Chiribella and Spekkens, 2016, 7). The focus on composition is partially motivated by

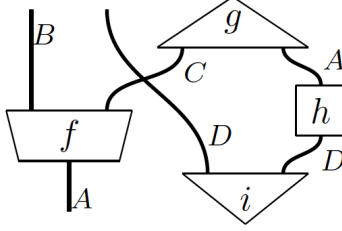


Figure 2.2: A diagrammatic representation of processes. A collection of *systems* is denoted by labelled wires and a collection of *processes* is denoted by labelled boxes. Labelled boxes are connected via input wires at the bottom and output wires at the top (Selby et al., 2018, 5).

the sentiment that composition is more primitive than the notion of probability (Chiribella and Spekkens, 2016, 7).

The postulates or axioms of this reconstruction can be stated in diagrammatic, category-theoretic, and process-theoretic terms. In their reconstruction, Selby et al. begin by accepting the process theory framework which informally means that “nature is fundamentally about processes and how they compose” (Selby et al., 2018, 4). Process theories are represented via diagrams, which Selby et al. suggest offer a more intuitive visualization of concepts in quantum information and computation (Selby et al., 2018, 4). The diagrammatic representation of process theories consist of a collection of systems which are denoted by labeled wires, and a collection of processes, which are denoted by labelled boxes with input and output wires (Selby et al., 2018, 5). They assert that the process-theoretic framework will allow “one to describe essentially any conceivable physical theory” (Selby et al., 2018, 14), mirroring the sentiment expressed regarding the GPT framework. The process theoretic framework allows Selby et al. to state key postulates in diagrammatic terms which are then translated to both category-theoretic and process-theoretic terms. They claim that the process-theoretic description establishes the fact that “the conceptual bare-bones of quantum theory concerns the matter in which systems and processes compose” (Selby et al., 2018, 1). This is because quantum theory can be reconstructed from simple process-theoretic terms. As it relates specifically to processes, Selby et al. suggest that an operational understanding of processes can be given wherein “one can think of the processes as corresponding to a single use of a piece of lab equipment in a single run of an experiment” (Selby et al., 2018, 6). This point is relevant as it conveys how Selby et al.’s reconstruction is connected to the general operational framework.

Selby et al. understand reconstructions to have developed in several distinct phases. The first phase included von Neumann’s quantum logic and was focused on mathematical axiomatization inspired by Hilbert’s work in 20th century physics and mathematics (Selby et al., 2018, 3). Selby et al. describe this approach as having a high degree of mathematical sophistication, corresponding to “conceptually motivated mathematical structures, e.g., lattices or generalised probability spaces” (Selby et al., 2018, 3). The second phase, born out of Hardy (2001)’s reconstruction, was a concerted shift away from mathematical axiomatizations toward a more conceptual grounding based on principles. For example, the kinds of principles used in Hardy (2001) are abstract principles which are based on clear physical concepts. However, Selby et al. argue that “the postulate-based approaches lacked the mathematical clarity of the earlier attempts” (Selby et al., 2018, 3). This phase also spurred on *categorical quantum mechanics*, which Selby et al. take to have drastic effects on reconstructions by changing the paradigm from states to processes. The third wave of reconstruction attempts is typified by the mathematical ‘spirit’ of the first phase with the concern for concepts that typify the second (Selby et al., 2018, 4). Selby et al. understand their reconstruction to be a part of the third wave.

Proof of the derivation of quantum theory from diagrammatic postulates

Selby et al. first construct a full process theoretic description of quantum theory, which includes classical and quantum systems and their interactions, thus establishing a space of possible theories that includes classical, quantum, and hybrid classical–quantum systems (Selby et al., 2018, 16). This is because even though a process logic underlies quantum theory, the processes are ‘classically controlled’ (Selby et al., 2018, 15). The classical control of processes is trying to capture the idea that our interactions with the world will be via classical inputs and outputs. From this space of theories, they then restrict to the fully quantum systems (*only* quantum) via two more postulates, rederiving the von Neumann postulates (Selby et al., 2018, 16). The axioms are:

1. The theory has a classical interface.
2. The theory has cups and caps.
3. The theory has a sharp dagger.
4. All processes admit symmetric purifications.

A classical interface describes how we interact with, control, and learn about the theory (Selby et al., 2018, 4). Selby et al. claim that the classical interface postulate provides

a more ‘operational layer’ to their theory, insofar as it describes the world via classical inputs and outputs (Selby et al., 2018, 14). A process theory must have classical systems with classical processes between them.⁴ Cups and caps are cup-shaped and cap-shaped wires for each system (Selby et al., 2018, 5).⁵ In a process theory, cups and caps mean that inputs can be ‘bent’ into outputs and vice versa (Selby et al., 2018, 10). This also means that inputs can be connected to inputs and outputs can be connected to outputs, meaning that the theory has maximal correlations (Selby et al., 2018, 10). Maximal correlations are important for the representation of entangled systems.⁶ Cups and caps formalize the idea that a process theory is blind with respect to inputs and outputs (Selby et al., 2018, 10). Process theories typically require that outputs are wired to inputs (Selby et al., 2018, 6), so cups and caps ‘relax’ this requirement.

Sharp daggers represent time reversal, meaning that the dagger of a state can be interpreted as a test for that state (Selby et al., 2018, 11). Conceptually, the dagger represents a reflection of a process. Informally, “the time reverse of a process can be thought of as ‘playing the video of that process in reverse’, in particular, for a state this corresponds to a ‘test’ for that state” (Selby et al., 2018, 4). The *sharpness* of a dagger is added in order to fully characterize the Hermitian adjoint of quantum theory.⁷ The Hermitian adjoint is an operator $O^\dagger = O$ where the eigenvalues are real. Intuitively, the ‘realness’ of eigenvalues allows us to represent physical quantities in quantum mechanics (Timpson, 2013, 245).

The symmetric purification postulate is what Selby et al. call their novel postulate (Selby et al., 2018, 1). Symmetric purification is a ‘time symmetric’ version of standard purification. In contrast, the standard purification postulate is satisfied by quantum theory but not by classical probability theory, so it functions to isolate quantum theory (Selby et al., 2018, 5). Symmetric purification states that “every process arises from a pure process by ‘discarding’ a system to both the future and the past” (Selby et al., 2018, 14). We can think operationally of discarding as a “way to describe a scenario where we have

⁴Classical systems will have a number of features, including a natural number dimension n , the composition of systems as given by the product of their dimensions $n \otimes m = nm$, that states correspond to $1 \times n$ matrices, etc. (Selby et al., 2018, 6).

⁵In Coecke (2012), he notes that the dagger or adjoint could be understood, from the perspective of a Birkhoff–von Neumann quantum logic, as “the analog to an orthocomplementation on the lattice structure” (Coecke, 2012, 8).

⁶Entangled quantum systems have a combined state that cannot be separated into a product of the individual states of the systems. Entanglement creates a ‘link’ between, for example, two particles such that the state of one particle is dependent on the state of the other, in spite of being spatially separated (Timpson, 2013, 252).

⁷The Hermitian adjoint is the conjugate transpose of a matrix or operator. The sharp dagger represents a unary algebraic operator, operating on a matrix or operator and returning its Hermitian adjoint.

no knowledge about, control over, or interaction with the future of a system” (Selby et al., 2018, 14). Intuitively, discarding means that there is a way that we are able to ‘ignore’ or ‘throw away’ a system (Selby et al., 2018, 7).

Unlike standard purification, symmetric purification holds for both classical and quantum theory. Due to this, Selby et al. reconstruct classical, quantum, and hybrid systems all together, then single out the fully quantum systems (Selby et al., 2018, 5). They are able to single out the fully quantum systems either through the existence of *pure* cups or the lack of non-trivial leaks. Pure cups allow us to represent the connection between two pure processes, more specifically representing the existence of entangled states (Selby et al., 2018, 5). Pure processes can be thought of as states without any mixed components. Non-trivial leaks can be understood as a “generalised no-broadcasting theorem, or, that information gain always causes a disturbance of a quantum system” (Selby et al., 2018, 29). Non-trivial leaks capture the idea that information gain causes disturbance of quantum systems, which Selby et al. take to be an unsurprising feature of nature. If one holds the view that measurements should be described by interactions between systems, then it is unsurprising that this interaction influences the system being measured (Selby et al., 2018, 29).

Selby et al. prove that these two postulates are equivalent in any process theory with cups and caps (Selby et al., 2018, 5). They also prove that the “standard purification postulates is implied by the conjunction of the symmetric purification postulates and the existence of a pure cup” (Selby et al., 2018, 5). From this they conclude that standard purification can be parsed into two pieces, one that applies to both classical *and* quantum theory, and the other that only applies to quantum theory. Thus, the uniquely quantum piece of the postulate is clearly identified. Selby et al. conclude that it is “the existence of correlations mediated by *pure* states (and/or effects) that is the distinguishing feature of quantum theory” (Selby et al., 2018, 28). The collection of the above axioms allows Selby et al. to reproduce quantum theory within the space of theories, connecting a process theory to the Hilbert space model (Selby et al., 2018, 5).

One point to highlight is how the Generalized Probability Theory framework and the Process-Theoretic (PT) framework are distinct. As described above, Hardy (2001)’s reconstruction is in the GPT framework, specifically under the sub-umbrella of the Convex Operational theory framework (Chiribella and Spekkens, 2016, 8). The Operational-Probabilistic Theories (OPTs) framework is a hybrid framework, based on both the notion of composition from the category-theoretic framework as well as probabilistic structure from an operational description (Chiribella and Spekkens, 2016, 8). The GPT framework encompasses theories that can be described via experimental probabilities “while remaining noncommittal about whether it is the framework of convex operational theories or the more

general framework of operational–probabilistic theories” (Chiribella and Spekkens, 2016, 8). Thus both convex operational theories and OPTs are instances of the GPT framework.

Selby et al. prove that a diagrammatic presentation of GPTs can be subsumed under the process theoretic framework (Selby et al., 2018, 1). They assert that the probabilistic layer of the GPT framework can be represented by the classical interface in process theoretic language (Selby et al., 2018, 30).⁸ A distinct feature of the process–theoretic description is its inclusion of C^* –algebras, which allows Selby et al. to describe quantum and classical systems (Selby et al., 2018, 31). General C^* –algebras do not allow for the property of transitivity, i.e., reversible transformations between any pair of pure states (Selby et al., 2018, 31). This is an important distinction between the process theoretic framework and the GPT framework.⁹ The result of this distinction is that the majority of other reconstructions will rule out the full C^* –algebraic theory as most reconstructions take reversible transformations to be an important postulate (Selby et al., 2018, 31).

Selby et al.’s reconstruction is more computationally–minded, which is reflected in the type of axioms chosen. This is because the diagrammatic representation lends itself well to representing quantum computational processes. But the reader need not understand the minutiae of the postulates in order to benefit from the above examples. What is significant for my purposes is that the reader can see the distinctions between Hardy (2001) and Selby et al. (2018). In particular, it is important to note how the GPT and process–theoretic framework are distinct and how the types of axioms within reconstructions can be so diverse. But the key insight I want readers to grasp is that different mathematical frameworks are chosen because they are better suited for expressing the physical concepts researchers deem most important.

As I have described above, the GPT framework is based on probabilities whereas the process theory framework is based on processes. The process theory framework is category–theoretic. Hardy (2001)’s reconstruction is Convex Operational, which falls under the GPT framework. These different frameworks give us diverse ways of articulating and representing the space of theories. Different frameworks will give us access to different mathematical tools. For example, a category–theoretic representation gives Selby et al. access to cups

⁸More precisely, Selby et al. reconstruct something *similar* to the GPT framework as they fail to obtain that the state space is closed (Selby et al., 2018, 30). In an operational setting, closure is a “very natural assumption as, up to any finite error, a state space should be indistinguishable from its closure” (Selby et al., 2018, 30). My understanding is that this is not problematic for Selby et al. as a complete recovery of the GPT framework is not a primary goal. Further, Selby et al. maintain that the process–theoretic formulation of quantum theory is the most appropriate formulation, *not* the GPT framework.

⁹The inclusion of a C^* –algebra is controversial as C^* –algebras load in assumptions integral to quantum theory rather than deriving them (Koberinski and Müller, 2018a, 2).

and caps (Selby et al., 2018, 5) and a C^* -algebra. Selby et al. have a diagrammatic representation whereas Hardy maintains a strictly mathematical representation. Therefore, the distinct frameworks give us different ways of representing our chosen axioms. Selby et al.’s axioms all refer to diagrammatic processes whereas Hardy’s axioms are more clearly based on generalizations of physical ideas, e.g., continuity. As Hardy claims, his reconstruction is mathematically straightforward as ‘essentially linear algebra’ (Hardy, 2001, 3). Selby et al.’s axioms require a greater background in quantum mechanics and computation in order to be understood. However, both reconstructions successfully derive the minimal empirical and mathematical results required by quantum theory, i.e., the von Neumann postulates. This is a key feature of the reconstruction programme.

2.5 Conclusion

I have introduced Hardy (2001) and Selby et al. (2018) because they are excellent examples of the diverse approaches to reconstructing quantum theory via axiomatization. There are several philosophical issues that are raised by thinking about these examples from physics and the reconstruction programme in its entirety. Below I touch on the research questions that I will address in chapters 3–5.

Axiomatization is one of the main methodological features of the reconstruction programme and the use of axiomatic methodology raises some interesting philosophical questions. For example, what is the purpose of using axiomatization as a part of the methodology? This is philosophically interesting if one is interested in what types of methodologies are effective in physics. Is there any continuity of methodology in the way that reconstructionist’s use axiomatization to that of, for example, the way that Hilbert used axiomatization? These questions are relevant to reconstructionists if they wish to know if axiomatic methodology will help or hinder them in their efforts to clarify the conceptual foundations of quantum mechanics. How best should we understand axiomatization attempts in the context of the quantum reconstruction programme? Further, what kind of representations of quantum theory does axiomatization facilitate?

Another issue raised is the existence of alternative formulations of a theory. As we have seen, the reconstructions by Hardy (2001) and Selby et al (2018) can be considered alternative formulations of quantum theory. Both reconstructions have different sets of axioms within different frameworks that nonetheless allow the successful derivation of the von Neumann postulates. Thus, we have different ways of representing the same physical theory. But are these different representations alternative formulations of quantum theory?

This raises the philosophical question of what the value is in devising a scientific methodology that generates these alternative formulations. To what end do we generate alternative formulations of a theory? Does that not result in a sort of underdetermination, where we have multiple successful formulations with different ontological conclusions? In such a case, it is not obvious how the programme can be a guide to understanding quantum theory. However, the existence of alternative formalisms is to be expected in the reconstruction programme, which is designed to generate alternative formulations of quantum theory. Are there methodological advantages to considering alternative formulations within the context of reconstructions? But the existence of alternative formulations of quantum theory does not by itself help us answer the questions we are interested in answering. It does not, for example, tell us *what to do* with all of the different formulations. What is the value of a multitude of alternative formulations of quantum theory? What type of insights do we get from the consideration of alternative formulations?

The final philosophical issue covered is that of explanation. If science seeks to explain, then what does explanation look like in the reconstruction programme? From a philosophical perspective, are reconstructions providing explanations for why quantum theory is the way it is? In other words, does the isolation of physical principles that are distinctly quantum principles explain the differences between classical, quantum, and extra-quantum theories? Presumably, reconstructionists believe that by identifying which physical principles are distinctly quantum principles, they are explaining *why quantum theory is the way it is*. Are they successful in providing an explanation in the above sense? And if there is explanation in the reconstruction programme, what kind of explanation is it? What type of explanatory insights are born from the quantum reconstruction programme?

These philosophical issues will be explored in chapters 3–5. What we will see is that the methodology of reconstructions is well-suited to the aims of the research programme. Further, reconstructions of quantum theory hold the potential to illuminate pathways forward in the foundations of physics.

Chapter 3

Methodology and the Role of Axiomatization

¹Quantum theory is one of the most successful scientific theories to date, underpinning nearly all of our best theories of physics, excepting gravity. It is a theory that is overwhelmingly experimentally accurate, has led to a variety of technologies, and has driven advances in mathematics and other areas of science. However, a singular interpretation of quantum theory which describes what the theory says about the nature of reality is greatly debated, with little consensus amongst physicists and philosophers as to what the answer may be. The standard interpretational approaches to quantum mechanics (exemplified by the Everett interpretation or de-Broglie-Bohm’s theory) typically supply an interpretation based on the accepted mathematical formalism for quantum theory. Interpretation tends to be the main focus of standard work in quantum foundations, focusing on providing solutions to the measurement problem and analyzing the problem of nonlocality. In spite of the insights discovered from the standard interpretational approach,² a unified picture of what the formalism of quantum theory says about what exists in the world has yet to emerge.

The reconstruction programme refers to “both a mathematical and a conceptual paradigm that allows one to derive the usual formulation of quantum theory from a set of primitive assumptions” (Selby et al., 2018, 1). The aim of reconstructions is to reformulate quantum theory from a base set of physical principles which are used to motivate a set of

¹This chapter has been published previously in (Oddan, 2024).

²Including no-go theorems, the Hilbert space formalism, and new concepts such as Bohr’s *complementarity*, to name a few (Goyal, 2022, 15).

mathematically-formulated axioms; this reformulation is done in order to derive the key postulates of quantum theory (the von Neumann postulates). The overarching goal of the reconstruction programme is to “find a compelling set of axioms that singles out quantum theory from among all possible theories” (Chiribella and Spekkens, 2016, 4). One common ambition is to subvert the importance of interpretation and instead focus on developing an abstract generalization of the fundamental features of quantum mechanics, seeking to “remove the interpretative bottleneck” (Goyal, 2022, 18) through the derivation of the mathematical formalism. The hope is that reformulation from primitive assumptions will reveal the core physical structures that underlie the mathematical formalism.

I argue that the reconstruction programme represents a modern usage of axiomatization with significant points of continuity to historical uses of axiomatization. The historical context of the method reveals how axiomatization developed, particularly as it was applied to the physical sciences. We see the axiomatic method in David Hilbert’s formalization of mathematics and its application to a young quantum mechanics in 1928 by Hilbert, Nordheim, and von Neumann entitled *Über die Grundlagen der Quantenmechanik*. I will show that there are important continuous features of axiomatization in Hilbert, Nordheim, and von Neumann and von Neumann’s axiomatizations which are also present in reconstructions of quantum theory. Here I explore Lucien Hardy (2001)’s “Quantum Theory from Five Reasonable Axioms”, a reconstruction which was pivotal in the development of the research programme. Further, I argue that we should understand axiomatization attempts as being context-dependent, context which is relative to the goals of inquiry and the maturity of the mathematical formalism and theoretical concepts within an area of knowledge. Using Mitsch (2022) as a starting point, I show that not only should we understand axiomatizations as context-dependent, but we can also have a more robust understanding of axiomatizations as axiomatic completions. I take the context-dependency of axiomatization as a means of enriching Mitsch’s account.

I begin in section 3.1 by identifying strands of development of the axiomatic method that are continuous with the use of axiomatization in reconstructions, starting with David Hilbert’s axiomatization of geometry and his axiomatization of mechanics. I then discuss a paper by Hilbert, Nordheim, and von Neumann, which was an earlier axiomatization of a young quantum mechanics before moving on in section 3.2 to von Neumann’s axiomatic derivation of the trace function. Throughout this historical overview I highlight how the maturity of the area of inquiry influenced the ease with which an axiomatization could proceed. Section 3.3 gives a sketch of the general methodology of reconstructions of quantum theory, then focuses on a discussion of Hardy (2001)’s ‘Five Axioms’. Here I argue that there are several points of continuity between von Neumann’s axiomatization and Hardy’s reconstructions, including similar methodological ordering, a similarly

operational framing, and the insistence on the empirical basis of axioms. I conclude that these points of continuity situate Hardy’s reconstruction as an historical successor to von Neumann’s axiomatization and thus to Hilbert’s axiomatic project. In section 3.4 I show how reconstructions use features rooted in earlier axiomatizations but manifest differently in a matured quantum theory. I highlight these methodological features, which include the stipulation of a framework as a generalized space of theories and stipulation of analytic machinery at two levels of generality (once at the framework level and again at the level of axioms). Lastly in section 3.5 I discuss Mitsch’s account of axiomatization as axiomatic completion attempts, arguing that this is a good way to understand the value of axiomatization in the case of reconstructions. I apply this understanding to the reconstruction program, showing that we should conceptualize reconstructions as axiomatic completions. Thus I show that reading axiomatizations as context-dependent is complementary to the notion of axiomatic completions.

3.1 A Brief History of Axiomatization

David Hilbert is widely considered to be the father of the axiomatic programme which called for the formalization of mathematics in an axiomatic form. This had resounding effects on foundations of mathematics and any scientific endeavour that used mathematics as a basis (Zach, 2019). Hilbert’s interest in applying axiomatization went well beyond foundations of mathematics, extending his attention to physics. This extension led to Hilbert’s Sixth Problem, a ‘programmatic call’ for the axiomatization of physical theories. Hilbert took inspiration from the success of axiomatization in the foundations of geometry, hoping to apply the method to ‘those physical sciences’ where mathematics plays a major role.

Hilbert’s axiomatization of geometry was an attempt to derive key propositions in Euclidean geometry (such as Desargue’s theorems) from physical axioms. The question was *which* physical assumptions (for example, the parallel postulate) would result in the derivation of those key propositions. Hilbert succeeded in this endeavour by “translating geometric statements into a formalism whose necessary assumptions had already been elucidated (the arithmetic equations); with the formalism’s structure already clear, the axiom candidates for Euclidean geometry could be translated into the formalism’s language and their relationship precisely characterized” (Mitsch, 2022, 5). Hence, in light of these physical assumptions Hilbert could identify a formalism into which the theory in question could be translated, the arithmetic equations. It could then be determined which axioms were necessary for the derivation of the key propositions by relying on the structure of the

identified formalism.

However, the application of axiomatization to physical sciences which were closely connected to experimentation and empirical information challenged the plausibility of the expansion of the use of axiomatization beyond mathematics. Hilbert had seen that axiomatization had the benefit of clarifying starting assumptions and concepts, and the axiomatization of geometry exemplified the power of the methodology. So Hilbert attempted to axiomatize mechanics in 1905 (Corry, 1997, 131). Though the basic facts of the discipline (e.g., vector addition) were well established, the arrangements of those basic concepts (e.g., nuanced readings of time, space, mass, microscopic mechanical descriptions, etc.) were still open to revision.³ Hilbert’s axiomatization of mechanics, however, “was not followed by an analysis of the independence of the axioms, based on the construction of partial models, such as Hilbert had carried out for geometry” (Corry, 1997, 187). Mechanics, compared to geometry, had to consider a range of observed phenomena which made the axiomatization much more difficult. Hilbert himself maintained that there was yet much work to do in the axiomatic treatment of mechanics (Corry, 1997, 131).

Just as axiomatization encountered roadblocks in mechanics, the problems faced in the axiomatization of quantum mechanics were magnified tenfold. Although Hilbert called for the full mathematization of physics, quantum mechanics was very much in its infancy when Hilbert presented the Göttingen lectures in 1905 (Lacki, 2000, 280). By 1925 there were several formalisms vying to be *the* formalism of the discipline, including Heisenberg’s matrix mechanics, Dirac’s q -numbers, Schrödinger’s wave mechanics, and the operator calculus of Born–Wiener (Lacki, 2000, 281). The existence of multiple formulations posed significant problems for what we could reasonably conclude about the physical world—in light of multiple mathematical descriptions, which could we physically interpret? Likewise, was it possible that a more fundamental and undiscovered formulation underpinned the variant formulations? This latter question is what Hilbert, Nordheim, and von Neumann explore in 1928.

3.1.1 Hilbert, Nordheim, and von Neumann

Hilbert et al.’s 1928 paper was an attempt to find an underlying basis between the disparate formulations of quantum theory using axiomatization. This paper is significant because it is an explicit description of how the authors see axiomatization proceeding in the very

³In his discussion of mechanics, Hilbert discusses several differing approaches to its foundations, including Hertz’s perspective that force was explained by rigid connections between bodies and Boltzmann’s presentation that focused on the central forces between “any two mass points” (Corry, 1997, 142).

‘young’ quantum theory. Hilbert, Nordheim, and von Neumann state that “this way is thus that of an axiomatization, as it has been carried out, for example, with geometry” (Hilbert et al., 1928, 2). Rédei understands Hilbert, Nordheim, and von Neumann to characterize axiomatization to include (Rédei, 2005, 47):

1. Physical axioms
2. Analytic machinery (i.e., formalism)
3. Physical interpretation

The basis of physical axioms is meant to be empirical, gained by past experiences and trends (Hilbert et al., 1928, 2). These physical axioms will be “formulated for certain physical quantities and relations among them” (Rédei and Stöltzner, 2006, 3). Thus the physical axioms must be related in certain ways—where certain ‘physical *demands*’ must be met—as indicated by our past experiences. The analytic machinery is a formalism constructed to reflect those relations. Thus the analytic machinery is a “mathematical structure containing quantities that have the same relation among themselves as the relation between the physical quantities” (Rédei, 2005, 47). The analytic machinery is then given a physical interpretation based on those physical demands (Hilbert et al., 1928, 2). In other words, our physical interpretation is ‘read off’ of the formalism in whatever way satisfies those physical demands given by the physical axioms. This is the ‘suggested’ method of axiomatization put forth by Hilbert et al. (Hilbert et al., 1928, 3) which I will label ‘optimal axiomatization’. Hilbert, Nordheim, and von Neumann assert that this method is inspired by the axiomatization of geometry:

Through the axioms the relations between the elements of geometry, point, line, plane, are characterized, and then it is shown that these relations are exactly satisfied by an analytic apparatus, namely the linear equations. Through it one can again recover geometric propositions from the properties of the linear equations. (Hilbert et al., 1928, 2)

Hilbert, Nordheim, and von Neumann maintain that *optimally* the three steps listed above follow in a ‘natural’ temporal order, where the physical axioms are robust enough to completely articulate the analytic machinery necessary to describe them (Rédei, 2005, 48). In the context of quantum mechanics, the analytic machinery Hilbert, Nordheim, and von Neumann refer to is the general probability relations of the theory (Hilbert et al., 1928,

1). However, the authors assert that “the above suggested procedure of axiomatization is not typically followed in physics now, but rather is the way to the erection of a new theory” (Hilbert et al., 1928, 3). Thus axiomatization as it is used in the practice of science and particularly in the development of new theories will differ from the ‘optimal’ method above. Instead, Hilbert, Nordheim, and von Neumann claim that the analytic machinery is typically already formulated and the physical axioms are then determined based on the analytic machinery (Rédei, 2005, 48). Thus, we gain insight into what the physical axioms might be based on an established formalism. Hilbert, Nordheim, and von Neumann maintain that in the practice of physics the physical axioms are determined via an interpretation of the formalism, rather than the physical interpretation being born out of the physical axioms (Rédei and Stöltzner, 2006, 4). I call this different characterization of axiomatization ‘practical’ axiomatization. Indeed, Lacki notes that Hilbert, Nordheim, and von Neumann maintained “the difficulty, even impossibility, of carrying out the axiomatization process in its most ambitious form, namely that going from the physical requirements to the finding of the formalism” (Lacki, 2000, 297). This was the last of Hilbert’s work in physics, though the axiomatic program was carried forward notably by John von Neumann.

From the work by Hilbert, Nordheim, and von Neumann, what we observe is that axiomatization as developed by Hilbert for geometry had to become more flexible (specifically as it relates to the order in which analytic machinery and physical axioms are posited) in light of the method’s application to the physical sciences. It was the application of axiomatization to mechanics that highlighted the problem of the strictness of axiomatization, particularly in light of physical theories with greater demands on empirical information. Hilbert et al. (1928)’s paper was the realization and attempted execution of the ‘loosening off’ of axiomatization in order to contend with the formalism of an immature quantum mechanics. Rédei and Stöltzner maintain that axiomatization became more flexible (‘soft’), less well-defined, and more intuitive (Rédei and Stöltzner, 2006, 3). This increase in flexibility widened the scope of what axiomatization could be applied to and the order in which axioms, physical interpretation, and analytic machinery could be posited.

The Hilbert, Nordheim, and von Neumann (1928) paper is significant for my purposes because it illustrates how the way an axiomatization proceeds would change depending on the area of inquiry being axiomatized. Geometry offered an area of inquiry that could more easily be axiomatized as its experimental foundations were generally accepted and the theory could be “turned into a *pure mathematical science*” (Hilbert, 1898-9 *Mechanik* Göttingen Lecture). However, mechanics lacked a clear arrangement of basic concepts which proved difficult for a similarly precise axiomatic treatment that was enjoyed by geometry. The axiomatic treatment of quantum mechanics posed additional challenges due to its multiple formulations. The lack of a single well-established mathematical language

hindered the articulation of key theorems and concepts. John von Neumann aimed to resolve this by utilizing axiomatization to ascertain the correct formulation of quantum mechanics.

3.2 Von Neumann on Axiomatization

Von Neumann published two important papers in 1927, the *Mathematische Begründung der Quantenmechanik* (von Neumann, 1927a) and *Wahrscheinlichkeitstheoretischer Aufbau der Quantenmechanik* (von Neumann, 1927b). In these works von Neumann presented quantum mechanics in a mathematically rigorous manner that solidified the use of the abstract Hilbert space formalism (Lacki, 2000, 300). It is in *Aufbau* (von Neumann, 1927b) that the trace function is derived, which von Neumann had determined was a key proposition in any formulation of quantum mechanics (Mitsch, 2022, 14). The goal was now to identify which basic assumptions would result in the derivation of the trace function.

In the introduction of this proof, von Neumann highlights the problem “of the determination of the probability for obtaining a determinate value of a given physical quantity” (Lacki, 2000, 301). In quantum mechanics, due to the impossibility of certain simultaneous measurements⁴ a measurement will not necessarily result in a definite value being attributed to a physical quantity (Lacki, 2000, 301). Von Neumann’s aim was to define the *statistics* of physical quantities over an ensemble of identical systems, which would yield an expectation value.⁵ A trace function is the sum of the diagonal elements of a matrix (sum of the complex eigenvalues), essentially determining measured probabilities while maintaining basis invariance. Hence a trace is the same regardless of coordinates. In the proof, von Neumann begins by stipulating two physical axioms regarding the expectation values of physical quantities in a statistical ensemble (an ensemble being a collective of systems that are characterized by the operator U ⁶):

A. Expectation value assignments are linear (where ‘ a ’, ‘ b ’ represent a given physical quantity:

$$Exp(\alpha a + \beta b + \dots) = \alpha Exp(a) + \beta Exp(b) \dots \quad (3.1)$$

⁴This is a result of the uncertainty principle, which rules out simultaneous measurements of incompatible properties or conjugated quantities (pairs of observables whose operators do not commute).

⁵An expectation value is the average value predicted from a collection of measurements.

⁶ U here is some statistical operator that is both positive (real, non-negative valued) and linear (Rédei and Stöltzner, 2006, 5).

B. Expectation value assignments are positive:

$$Exp(a) \geq 0 \quad (3.2)$$

Rédei and Stöltzner argue that both of these axioms are physical axioms in the sense that they are based in our experience and observations in the laboratory (empirical) (Rédei and Stöltzner, 2006, 3). They also note that the physical quantities a and b are left unspecified while the two axioms represent the expectation values of the observed statistical behaviour of an ensemble of systems.⁷ It is here that von Neumann draws insight from the ‘analytic machinery,’ i.e., formalism, which is the set of all selfadjoint operators on a Hilbert space (Hermitian operators representing physical quantities) (Rédei and Stöltzner, 2006, 5). In other words, von Neumann assumes that the physical quantities were represented by a type of operator that would only result in real eigenvalues. He then stipulates:

C. If operators $S, T, \dots$ represent physical quantities $a, b, \dots$ then the operator $\alpha S + \beta T + \dots$ represents the physical quantity $\alpha a + \beta b, \dots$

D. If operator S represents physical quantity a then the operator $f(S)$ represents the physical quantity $f(a)$.

These two requirements fill out the physical interpretation, linking the physical axioms to the analytic machinery. Von Neumann clearly stipulates that the physical quantities are represented by all linear operators on a Hilbert space, thus the properties of the physical quantities are postulated by considering the ‘structural’ properties of the Hilbert space operators (Rédei and Stöltzner, 2006, 5). Hence C and D represent and capture “the central focus in quantum mechanics on the functional relationship among experimentally measurable quantities” (Mitsch, 2022, 16).

The collection of $A + B + C + D$ resulted in the derived

$$Exp(a) = Tr(US) \quad (3.3)$$

with a representing a physical quantity with its associated operator S (Rédei and Stöltzner, 2006, 5). This is the trace formula that is “the heart of the whole story” (Rédei and Stöltzner, 2006, 5).

Von Neumann successfully derived the trace function and in doing so isolated the basic assumptions necessary for its derivation. This derivation was based on the determination of

⁷By unspecified I take Rédei and Stöltzner to be designating a general scope in terms of what those physical quantities could be, even if they can be specified at the time of measurement. This detail is not expanded on in Rédei and Stöltzner.

physical requirements which then enabled him to isolate the central proposition of quantum mechanics. As we can see from the consideration of *Aufbau* and the joint Hilbert, Nordheim, and von Neumann paper, von Neumann was driven by the ‘spirit of axiomatization’ imparted to him by his work with David Hilbert, and the trace function case illustrates his employment of the method. In the trace function case, von Neumann employed axiomatic methodology in order to identify the unique formalism for quantum mechanics. Von Neumann was working in an immature area of inquiry, both in the development of its mathematical formulation as well as its theoretical underpinnings. The derivation of the trace function was a step toward providing a solid mathematical foundation on which to build a unified representation of quantum mechanics.

Thus far, I have traced axiomatization from Hilbert’s axiomatization of geometry to von Neumann’s continuation of the method in the derivation of the trace function. I have highlighted how the maturity of an area of inquiry, in both its mathematical formalism as well as its theoretical underpinnings, influences the way in which an axiomatization could proceed. An area of inquiry with a mature mathematical formalism (such as geometry) and theoretical underpinnings could be axiomatized in a more straight-forward way wherein formal notions such as completeness, uniqueness, and independence, for example, could be surveyed. However, axiomatization of an immature area of inquiry requires more leniency in how an axiomatization is ordered. In particular, the maturity of an area of inquiry influences whether physical axioms are stipulated before or after looking at the analytic machinery. In von Neumann’s trace function case we see how the axiomatic project continued as he searches for an underlying formalism for quantum mechanics. Von Neumann’s main goal was to discover a unique formalism for quantum mechanics, one that unambiguously derived the key proposition of the theory.⁸ This derivation was made possible by his discovery of physical axioms which would give rise to the key proposition.

Next I introduce the reconstruction programme in the foundations of quantum theory. I demonstrate the continuity in axiomatic methodology from von Neumann’s trace function case to Hardy’s “Five Axioms,” particularly concerning methodological ordering, an operational reading, and the empirical basis of axioms. I conclude that these points of continuity support the interpretation that the reconstruction programme is a continuation of Hilbert’s axiomatic project.

⁸While this is true at this stage, it should be noted that von Neumann later abandons the Hilbert space formalism he was so integral in developing. He was aware of significant conceptual problems—including an infinite trace—in the derivation of the trace function (Rédei, 1996, 495). Von Neumann moved to develop his type II operator algebra (now called a von Neumann algebra) in an attempt to solve the conceptual problems in the Hilbert space formalism (Rédei, 1996, 495).

3.3 Methodology of Reconstructions of Quantum Theory

This section details how axiomatization is used in Hardy (2001)’s reconstruction, highlighting significant similarities and differences to von Neumann’s axiomatization. What we will see is the continuation of the historical story of axiomatization from von Neumann, but used in a very modern context.

Methodologically, reconstructions begin by establishing a framework which describes a landscape of possible theories including classical, quantum, and extra-quantum theories. This framework is used to stipulate foil theories which serve as nonquantum contrasts (Chiribella and Spekkens, 2016, 5). These nonquantum foils enable us to pinpoint the uniquely quantum case, and it is the relations between theories within the landscape that are studied. These foils function as a means of answering Wheeler (1971)’s “Why the quantum?” question, which asks both why our world is described by quantum theory and why it is not described by something else. Thus if we can envision a landscape of sufficient variety, then we are tackling both the question “why this?” and “why not *that*?” (Chiribella and Spekkens, 2016, 4). A framework establishes a broad theory space via a generalized mathematical formalism which best represents the core principles of an area of inquiry. For example, the Generalized Probability Theory (GPT) framework describes any theory that can be articulated in terms of experimental probabilities, and the generalized formalism represents the core principles in terms of preparation, transformation, and measurement procedures.

Under the general framework, individual axiomatizations pick out what researchers take to be the fundamental physical principles of quantum theory. An axiom is a particular mathematical representation within the broader framework, serving as a mathematical description of a physical principle. Axioms function as specific constraints on the framework in order to isolate what is uniquely quantum in the space of theories. Different axiomatizations will highlight different physical principles that researchers take to be fundamental to quantum theory. It is these physical principles which will be given divergent mathematical formulations which are not mathematically translatable. From these axioms, a successful reconstruction will derive the von Neumann postulates which are fundamental to quantum theory, i.e., the Hilbert space formalism in terms of linear operators in a Hilbert space. This process demonstrates that the postulated physical principles, initially considered as assumptions at the axiom level, along with their corresponding mathematical expressions, are compatible with the von Neumann postulates.

3.3.1 Hardy’s ‘Five Axioms’

Hardy (2001)’s seminal “Quantum Theory from Five Reasonable Axioms” kick-started the reconstruction programme and is an excellent example of how axiomatization is active in theoretical physics today. Hardy aims to “recover the basic structure of quantum theory along with the most general type of quantum evolution possible” (Hardy, 2001, 1). He seats his reconstruction in the Generalized Probability Theory framework based on his intuition that quantum theory is a new type of probability theory. Hardy’s five posited axioms are based first on ‘states’ typified as integers K and N . K represents the number of degrees of freedom (the minimum number of probability measurements required to determine the state) and N represents the maximum number of states that are capable of being distinguished from one another in a single measurement (Hardy, 2001, 2). The five axioms are:

1. Probabilities. Relative frequencies are given for given measurements on an ensemble of systems via a specific preparation procedure.
2. Simplicity. Given N where $N = 1, 2, \dots$, the minimum value is taken that is consistent with the axioms.
3. Subspaces. A system whose state belongs to an M subspace behaves like a system of dimension M .
4. Composite systems. A composite system comprised of A , B satisfies $N = N_A N_B$ and $K = K_A K_B$.
5. Continuity. There exists a continuous reversible transformation between pure states in a system.

All of the axioms are physically motivated and abstractly formulated. Hardy maintains that these axioms are less obscure than the axioms of traditional approaches (e.g., complex Hilbert spaces, Hermitean operators, etc.) and that these fundamental axioms can be articulated most clearly in the context of probability theory. The axioms are meant to be less obscure because they are based on clear, generalized, empirical physical principles. In contrast, the traditional formulation of quantum theory reveals very little about what physical insights we might derive from the formalism. Hardy discovers that it is the continuity axiom that is the feature that picks out the quantum case. By omitting the continuity axiom Hardy claims to obtain classical probability theory ($K = N$) instead of quantum

theory ($K = N^2$), with the aid of the simplicity axiom (Hardy, 2001, 2).⁹ Thus, Hardy’s axiomatization begins with the acceptance of the formalism of probability theory situated within the GPT framework, postulates physical principles which are then encoded mathematically as axioms, and ends with the derivation of the von Neumann postulates. Below, I identify the methodological order that Hardy follows in his axiomatization, comparing it to earlier applications of axiomatization:

1. Analytic machinery (probability theory, specifically Generalized Probability Theory)
2. Identification of physical principles (e.g., subspaces, composite systems)
3. Analytic machinery, physical principles formulated as mathematical axioms
4. Derivation (e.g., von Neumann postulates)

Hardy situates his reconstruction in the Generalized Probability Theory framework. The physical principles are stipulated as axioms via a mathematical formulation within the framework. The ordering of Hardy’s reconstruction aligns with ‘practical’ axiomatization from Hilbert, Nordheim, and von Neumann insofar as some analytic machinery is accepted (GPT) before the physical axioms are articulated.

We can note that the ‘physical interpretation’ step noted in Hilbert, Nordheim, and von Neumann’s view is not present in Hardy’s reconstruction. Hardy explicitly states that he is “principally interested in deriving the structure of quantum theory rather than solving the interpretational problems” (Hardy, 2001, 10). The focus on the structure of quantum theory is to be expected of the reconstruction programme, which does not aim to provide a full, conventional interpretation of quantum theory. However, there is a subtle point to be made as it relates to ‘interpretation’ in both Hilbert, Nordheim, and von Neumann and in Hardy’s reconstruction. Hilbert et al.’s aim of physically interpreting the analytic apparatus (which is identified via empirical information) is to “fully formulate the physical demands so that the analytic apparatus is clearly defined” (Hilbert et al., 1928, 2). This motivation conveys that what we really want are statements about real, physical things. I take ‘physical interpretation’ here to be closer to ‘those real physical objects the analytic

⁹Although Hardy claims to obtain classical probability theory with the removal of axiom 5, the issue of how and when we are situated in a classical region in the space of theories is a subtle one. Hardy uses the continuity axiom to rule out various theories that do not correspond to quantum theory, including classical probability theory (Hardy, 2001, 15). It is sufficient to rule out alternatives to the Bloch sphere. However, omitting continuity could result in either classical or quantum theories in the absence of assumptions about the transformation properties of states. I am thankful to a reviewer for this insight.

apparatus is meant to represent’ rather than interpretations of quantum theory in a modern sense.¹⁰ In its modern sense, interpretation refers to an attempt to provide a full ontic account of the connection between theory and world (for example, the Everettian interpretation). For Hilbert, Nordheim, and von Neumann, we have a more minimal understanding of interpretation concerned mostly with how the analytic machinery connects to the physical facts of the domain of inquiry within an axiomatization. Von Neumann’s trace function case is likewise not solving for interpretation in the modern sense, and better aligns with Hilbert, Nordheim, and von Neumann’s ideas about the mathematics–world connection. The trace function, as a necessary proposition in any formulation of quantum mechanics, is a piece of the analytic apparatus that is designated as physically representative. The way the physical principles are ‘true’ in von Neumann is similar to Hardy. Although the ‘physical interpretation’ step is absent from Hardy’s reconstruction, there is a ‘thin’ physical interpretation given via the identification of the axioms. In reconstructions, the physical intuitions that inform the axioms are taken as ‘true’ in some sense, as researchers choose the physical features they take to be central to quantum mechanics (in contrast to classical mechanics or other foil theories). The physical principles are empirically well–verified and understood to be unlikely to be revised. This is because they are born from clear empirical generalizations that are meant to be physically transparent. Hence, the basis of the physical principles are taken to be accurate representations of the world as true generalizations of phenomena.¹¹

A point of continuity between von Neumann and Hardy’s ‘Five Axioms’ is a shared operationalist framing. In the trace function case von Neumann restricts his representation of requirements C and D to only those quantities that are experimentally measurable and based on experience and intuition. We can see that this bears a similarity to Hardy’s reading of his axiomatization as operational. This is not to say that von Neumann’s axioms are articulated in an operationalist fashion, but rather that Hardy’s operationalism and von Neumann’s ‘intuitive, empirical basis’ are similar in spirit. In both accounts, the basis of the axioms must be empirical. In Hardy’s case, the GPT framework is operational, describing only that which is experimentally measurable. What this means is that Hardy’s axioms, which are mathematically formulated based on an overarching physical principle, will be necessarily connected to empirical information. Hence, the formalism used to express operational axioms will necessarily have a thin physical interpretation.

As I have shown, there are points of continuity between von Neumann’s historical

¹⁰Mitsch avoids the conflation of ‘physical interpretation’ by instead referring to ‘physical facts’ in Hilbert, Nordheim, and von Neumann’s work. Mitsch maintains that ‘physical facts’ is a more apt terminology as it relates to Hilbert’s account (Mitsch, 2022, 4).

¹¹I expand more on this point in section 5.3.1.

axiomatization of the trace function and Hardy’s modern quantum reconstruction. In particular, both axiomatizations are ordered similarly according to the ‘practical’ ordering, have a similar operationalist flair, and base axioms on empirical observations. Therefore, Hardy’s reconstruction should be considered an historical successor to axiomatization as it was used by von Neumann.

3.4 Diverging Methodology in the Reconstruction Programme

Thus far I have tracked the evolution of the method of axiomatization, beginning with Hilbert, Nordheim, and von Neumann and continuing with von Neumann. Reconstructions can be regarded as extensions of this method. However, what is perhaps more interesting are the ways in which certain features rooted in aspects of earlier applications of axiomatization diverge in the reconstruction programme. To this end, I will now demonstrate that axiomatizations are tailored to the different mathematical and theoretical context that the reconstruction programme finds itself in.

3.4.1 The Space of Theories

One important feature in von Neumann’s work and reconstructions is the interest in capturing a general characterization of quantum theory. For von Neumann this feature arises from his interest in discovering a generalized formalism for quantum mechanics in his trace function example. The trace function was a fundamental feature of any formalism in quantum mechanics. Von Neumann was looking for the widest formal generalization of quantum theory in order to provide a unified mathematical representation of the theory, and this interest in generalization informed his goal to derive the unique formalism for quantum mechanics.¹² In reconstructions, the interest in generalization arises out of the stipulation

¹²It is possible to read von Neumann’s later shift from the Hilbert space formalism (HSF) to type II operator algebras as similar to the interest in different formulations in the reconstruction case. As Rédei notes, von Neumann abandons the idea that the HSF is the exclusive framework for quantum mechanics (Rédei, 1996, 495). Von Neumann looks to different mathematical formulations to solve conceptual problems in the HSF. It does appear that there is a potential similarity with the methodology of the reconstruction programme. However, the discussion in this paper is confined to von Neumann’s work within the HSF in the period up to 1932. Von Neumann began investigating the Type II framework after this period. It would be worth investigating this possible similarity with von Neumann’s later work in mind. I am thankful to a reviewer for this thought-provoking insight.

of a broad landscape of theories. Within this generalized theoretical space, reconstructionists aim to discover the physical principles that isolate the quantum region in the space of theories. I proceed now to emphasize how this important feature is represented in the methodology of reconstructions.

The methodology of reconstructions establishes a framework that stipulates an expansive landscape of theories, allowing for the exploration of possible theories that are *not* quantum theories but especially those close in theory space to quantum theory. This framework is devised to highlight the distinctions between these regions in theory space. Individual reconstructions will pick out the fundamental features of quantum theory differently which means that a different set of theories within the fixed landscape will be highlighted. This also results in divergent physical insights insofar as the physical principles which inform axioms will confer different physical interpretations. For example, Hardy’s ‘Five Axioms’ is formulated within the Convex Operational Theories framework¹³ as one axiomatization. Other axiomatizations under the Convex Operational Theories framework include Barnum et al. (2007), Barnum and Wilce (2011), Barnum et al. (2012), and Barrett (2007). I suggest that establishing a broad space of theories is methodologically important as this framework is more likely to include crucial features that might have been overlooked in a narrower set of theories. The methodology of reconstructions provides the necessary structure to do this, both in the stipulation of the broad framework and the use of axiomatization as a constraint.

Axiomatic methodology in reconstructions is even more significant in the field of quantum foundations since one of the main advantages of the method is conceptual clarification. Here I take ‘conceptual clarification’ to be the simplification of the chosen fundamental physical principles with an accompanying mathematical formulation. Fundamental physical principles are intended to assume as little as possible while being firmly based on empirical information. Axiomatization functions well here because it allows us to ‘toggle off’ an axiom which will affect the space of theories. Toggling is akin to testing an electrical panel. If we switch off a breaker we are able to see the effect on the lighting in the room. In the context of reconstructions, this is methodologically advantageous because it is a means of clarifying which physical concepts and formalisms are ones we consider to be foundational. According to Rédei, Hilbert also supported the dropping of single axioms in order to study alternative theories and ‘deepen the foundations’ (Rédei, 2005, 4).¹⁴

¹³The Convex Operational Theories framework is a further specification of the Generalized Probability Theory framework (Chiribella and Spekkens, 2016, 8). Hence, Hardy’s ‘Five Axioms’ is within the GPT framework, stipulated specifically as a convex operational variety.

¹⁴ Stöltzner (2002) gives a much more detailed account of the different ways Hilbert deepens the foundations. There may be echoes of these different ways of deepening foundations in different implementations

Both the methodology of reconstructions and axiomatization provide the structure (a framework *and* axioms) to generate alternative theories while placing fundamental principles centrally. As Chiribella and Spekkens note:

Given an axiomatic derivation of quantum theory, it suffices to modify a single axiom in order to get a consistent alternative. Furthermore, this approach can be used to avoid an important pitfall of more ad hoc approaches to developing alternatives to quantum theory, namely, that the latter may inadvertently violate fundamental principles that one would prefer not to abandon (Chiribella and Spekkens, 2016, 5).

The conceptual clarification comes out of the derivation of quantum theory from a physically perspicuous set of axioms and those axioms are not likely to be subject to revision. By placing those physical principles centrally, we are more easily able to track the principles and their interactions with other posited features. Axiomatization as it was used first by von Neumann and then by the reconstructionists has the advantage of flexibility insofar as the adding, dropping, or modifying of a single axiom enables one to devise a large number of alternative theories. The construction of foil theories within the framework also allows us to prove the independence of a set of axioms: “if one axiom is independent from another, then one should be able to devise a foil theory that satisfies the former but violates the latter” (Chiribella and Spekkens, 2016, 5). Axiomatization is particularly advantageous in the context of theory development, exploration, and ‘deepening foundations’ because it opens up the space of theories we are willing to consider.

3.4.2 Axiomatic Starting Points

One difference between Hardy’s reconstruction and von Neumann’s axiomatizations is the starting point of both researchers. Specifically, how mature the mathematical formalism and theoretical concepts is in both uses of axiomatization influences how each axiomatization proceeds. In the case of the trace function, the insight that was desired at the outset of the axiomatization was insight into the *formalism*. Von Neumann was looking to

of the reconstruction project. For example, Stöltzner notes a type of deepening in the sense that deepening mathematical foundations might yield concepts that are physically more fundamental (Stöltzner, 2002, 257). This is similar in spirit to how Hardy uses a minimal Hilbert space formalism in order to highlight continuity as an important physical axiom of quantum theory. The potential connection between Stöltzner’s types of deepening and the reconstruction project will be explored in future work. I am grateful to a reviewer for this valuable insight.

define the statistics of physical quantities over an ensemble of identical systems. The trace function was a key piece of new mathematical machinery developed via axiomatization, accomplished in part by leaning on trusted physical axioms (the linearity and positive value of expectation values). Von Neumann’s starting point included intuitive physical axioms and a minimal formalism of probability theory used to articulate those axioms. This result was pivotal because the discipline was still trying to find its mathematical and conceptual footing (Lacki, 2000, 281).

Von Neumann’s use of axiomatization in the above cases diverges from Hardy’s ‘Five Axioms’, given that Hardy has a mature formalism to work with (the Hilbert space formalism). This mature formalism acts not only as a solid mathematical foundation from which to begin his reconstruction but also as a constraint as the end point. That is, Hardy employs an operational description in order to mathematically define the physical principles which inform his axioms, and the derivation of the von Neumann postulates indicates that the reconstruction is successful. Hardy also takes the GPT framework to be fixed, even if axioms can be formulated in different ways. However, a different framework could be chosen, for example one that is device-independent (Popescu and Rohrlich, 1998) or category-theoretic (Selby et al., 2018). A change in framework results in a different generalized formalism and different allowable formulations of axioms. Thus, the insight that is desired in Hardy’s reconstruction (and in reconstructions in general) is predominantly *physical* insight, accomplished in part by leaning on a mature formalism. Hardy’s starting point includes intuitive physical postulates, a minimal, mature formalism, and the specific results we need to reproduce at the end of the reconstruction. The end point is a proof that the von Neumann postulates are recovered, which also functions as a proof that the formulations of Hardy’s axioms are well-formed. In this case, when an axiom is well-formed the mathematical formulation of the physical principle will have the right mathematical ingredients to derive the VNP.

What is highlighted is how the goals of inquiry shift relative to the maturity of the area of inquiry. Goyal (2022) offers a helpful distinction between the *developmental* and *reflective* phase of a theory. In the developmental phase, the workability of a theory is more important than, for example, mathematical rigour. The result is that mathematical formalism often lacks clear physical connection (Goyal, 2022, 1). The reflective phase establishes the connection between mathematical formalism and its physical content, occurring after a theory’s formalism has satisfactorily captured “some basic regularities in nature’s workings” (Goyal, 2022, 1). This distinction allows us to highlight the difference in goals between von Neumann’s derivation of the trace function and the overarching goal of the reconstruction programme. Von Neumann works in the developmental phase of quantum theory, whereas reconstructionists work in the reflective phase. Both von Neumann and the

reconstructionists are searching for the proper formulation of quantum mechanics. While each share a similar goal, the maturity of the area of inquiry being axiomatized is quite different. For reconstructions, the more mature area of inquiry allows them to integrate the framework level in the methodology, which allows them to situate quantum mechanics in a generalized theory space. Due to the maturity of the formalism, reconstructionists know of the von Neumann postulates which function as an indicator of success. Von Neumann is working to establish a unique mathematical language, while the reconstructionists focus on discovering a set of axioms that picks out the quantum case in the landscape. They have a similar goal, but the field they are working in has different levels of development.

The above discussion leads us to a unique feature of reconstructions that distinguishes the programme from von Neumann's axiomatizations. Von Neumann's trace function example occurs in the context of probability theory. His focus was how to determine the probability of obtaining a determinate value for a physical quantity. However, the analytic machinery that is used in the axiomatization is stipulated once, which is at the level of axioms (the linearity and positivity of expectation value assignments). This is where the reconstruction programme diverges, as analytic machinery is stipulated at two points. Once at the framework level and once in the specific formulations of axioms. In Hardy's case, Generalized Probability Theory is stipulated as the broad, overarching framework that describes the space of theories, which includes a generalized mathematical formalism. The specific formulations of the axioms are the second instance where analytic machinery is stipulated, which in Hardy's case include, for example, the physical principle of composition represented mathematically as subsystems A and B satisfying $N = N_A N_B$ and $K = K_A K_B$. The key difference is the stipulation of a framework. Here two instances of analytic machinery are stipulated at different 'levels' of generality, a broad overarching framework and the specific formulations of axioms. Although von Neumann's derivation of the trace function occurs in the context of probability theory, a specific framework is not *stipulated*, particularly a framework that has the same degree of generality as the framework in reconstructions. It is partly the stipulation of a broad framework in reconstructions that is methodologically advantageous as it enables us to probe the boundaries of what is uniquely quantum. It might seem imprudent to stipulate such a generalized space of theories if what we are trying to accomplish is the identification of a set of axioms that isolates quantum theory. However, *because* we already know the formalism that typifies quantum theory (the von Neumann postulates) we can use it as a constraint within that space of theories. Thus, concerns about the scope of the space of theories are offset by the known formalism. In von Neumann's trace function and hidden variables cases, axiomatization was used to arrive at a broader formalism for quantum mechanics in the hopes of establishing a formalism on which to build.

3.4.3 Inference tracking

There are many advantages to axiomatization as a methodology, whether Hilbert's more formal notion of axiomatization or later von Neumann's more flexible version. Ultimately, axiomatization as a methodology is understood to lend epistemic credibility as a semi-formal means of parsing out fundamental and easily understandable constraints (axioms stipulated from physical postulates) that give rise to some solution or collection of solutions. Axiomatization provides a more directed means of inference tracking, particularly in the physical sciences. Part of the trustworthiness of axiomatization is being able to find mistakes within a derivation from a set of axioms. This derivation facilitates the finding of mistakes in our reasoning. If the stipulated axioms are resistant to revision, like Hilbert stressed, then it is more likely that the resulting derivations and conclusions of those derivations will preserve that resistance to revision as well.

The significance of how an axiomatization is ordered has been highlighted throughout the historical story. The key distinction between 'optimal' axiomatization and 'practical' axiomatization is whether the physical postulates help us to identify the analytic machinery, or if the analytic machinery gives us insights into what the physical postulates are. The optimal version begins with the stipulation of physical postulates which are then given a mathematical formulation. The practical version is the inverse. It was important for Hilbert, Nordheim, and von Neumann to separate formalism and physical interpretation because the mathematics was resistant to revision while physical interpretation was not. Hilbert, Nordheim, and von Neumann, and particularly Hilbert, wanted to remove any specifics of meaning in order to de-empiricize a theory, thus transforming it into a pure mathematical exercise¹⁵. However, I argue that in the context of reconstructions the ordering of an axiomatization, though it was significant for Hilbert, Nordheim, and von Neumann, is important *not* due to *when* analytic machinery and physical content is ordered but rather *that* the connection is stipulated at all. And so, the designation of optimal versus practical ordering is not the important insight we get from following such a procedure.

Rather, the insight we get by distinguishing between formalism and physical content is pinpointing where our mathematics applies to the physical world. What is significant is not that we stipulate mathematical formalism before physical content or the inverse. It is the specification that some mathematical feature x is representative of some physical insight y . The insight desired, however, is relative to the goals one has in performing

¹⁵To de-empiricise a theory was to disentangle empirical content from the analytic machinery. This was successful when a formalism was identified whose structure represented the important relations of the theory without empirical content.

an axiomatization. If we are, e.g., looking for physical insight, then as we see in reconstructions it becomes important to seek out the formalism–physical content connection. In other axiomatizations this connection may be less important such as those axiomatizations where understanding of the mathematical formalism is being sought. Von Neumann uses physical insights in his trace function case in order to search for the right mathematics. Axiomatization provides the methodological structure to facilitate the bridging of our mathematics to what it purports to represent. It is important that it is the physical axioms in reconstructions that are taken to be unlikely to be revised, rather than mathematical formalism, which is a feature that is distinctive in how axiomatization is used.¹⁶ Though the physical principle is unlikely to be revised, this is not necessarily the case for how that physical principle is mathematically formulated as an axiom. How those physical principles are formulated as axioms is dependent on the choice of researchers. The methodology of reconstructions also cements the importance of physical intuitions that guide researchers in the choice of formalism, not only at the level of axioms but at the framework level. Axiomatization provides the semi-formal structure to aid researchers in specifying the connection between physical ideas and their mathematical representations.

Next I show how a liberalized axiomatic method solidifies the role axiomatization can play in reconstructions, relying in part on Mitsch (2022)’s recent work on ‘Hilbert-Style Axiomatic Completion’.

3.5 Reconstructions as Axiomatic Completions

In “Hilbert–Style Completion: On von Neumann and Hidden Variables in Quantum Mechanics” (2022) Mitsch offers a nuanced interpretation of Hilbert’s axiomatic method. Mitsch’s elucidation of the role that axiomatization plays in the development of theories is directly applicable to the reconstruction programme. I apply this understanding to demonstrate that Mitsch’s conception of axiomatization is the best means of understanding the value of the methodology. I also argue that a context–dependent understanding of axiomatization complements Mitsch’s. In particular, Mitsch’s description of axiomatization as a provisional, practical,¹⁷ meta-mathematical procedure concerned with orienting and

¹⁶This relates to how reconstructions are also distinct from the standard interpretational project in quantum mechanics: the standard interpretational approach accepts the formalism of quantum mechanics and instead aims to provide a physical interpretational of that formalism.

¹⁷Mitsch is not referring to ‘practical’ in the sense of the ordering of analytic machinery and physical concepts that I have outlined prior from Hilbert, Nordheim, and von Neumann. Rather, ‘practical’ refers more to the notion that axiomatization contributes to scientific progress (Mitsch, 2022, 3).

ordering an area of inquiry aligns with my conception of the methodology. With this understanding in mind, what we see is that reconstructions of quantum theory successfully use axiomatization in a way that has and will continue to clarify and develop quantum theory.

Mitsch argues that Hilbert’s axiomatic method was both provisional and practical, rather than a strict formalization “in the service of radical epistemological or metaphysical goals” (Mitsch, 2022, 2).¹⁸ According to Mitsch, the goal of an axiomatization is a meta-mathematical one, wherein the relationship between mathematics and reality is addressed “insofar as an axiomatization will identify necessary physical assumptions based on the theorems central to an area of inquiry” (Mitsch, 2022, 6). In contrast, the mathematician determines what propositions result from a given set of axioms.

This meta-mathematical project proceeds, in the Hilbertian tradition (Mitsch, 2022, 6):

1. The identification of central theorems and concepts of an area of knowledge.
2. The identification of a formalism whose structure reflects the above area of knowledge.
3. The determination of the necessity of the candidate axioms based on the formalism.

Mitsch calls the relations between candidate axioms the “*uniqueness question*: are the other axioms sufficient for deciding the structure of the formalism w.r.t. the candidate axiom?” (Mitsch, 2022, 6). Uniqueness questions determine if and when the collection of axioms is representative for the theory while admitting no other realizations (Mitsch, 2022, 20). An axiomatic completion is achieved when all uniqueness questions are answered, in which case the axiomatic structure of the theory of the area of knowledge is completely determined (Mitsch, 2022, 6). However, Mitsch maintains that axiomatic completions, even if they are successful, are provisional—they “generate provisional representations of reality insofar as axiomatic completions rely on fallible steps” (Mitsch, 2022, 7). This is because the central theorems and concepts are liable to change within the theory of an area of knowledge. Axiomatic completions are also practical as “they are a tool meant to generate helpful representations of a field of knowledge” (Mitsch, 2022, 7) while simultaneously orienting and ordering a theory (Mitsch, 2022, 8). A theory is *oriented* when the independence of propositions is surveyed and an axiomatization orients an area of inquiry by directing

¹⁸Mitsch disagrees with Lacki that Hilbert was interested in axiomatization strictly in terms of logical clarification and rational reconstruction (Mitsch, 2022, 3). Rather, Mitsch interprets Hilbert as having a more pragmatic and liberal idea of what axiomatization could achieve.

our attention to certain physical, epistemological, or mathematical considerations (Mitsch, 2022, 29). A theory is *ordered* when the lack of contradictions between propositions is guaranteed. An axiomatic completion of a theory is successful when it is able to both orient and order an area of knowledge alongside its mathematical investigation (Mitsch, 2022, 8).

Mitsch’s description of axiomatization is a nuanced reading of axiomatization as a liberalized methodology that is pragmatically useful, in contrast with the notion of axiomatization as a strict formalization. He concludes that “von Neumann effectively summarized and clarified where we *had been*—in physics as well as in mathematics—in an effort to identify where we *could go*” (Mitsch, 2022, 30). I concur with this understanding of the methodological role that axiomatization has to play in the development and exploration of theories. For my purposes, I am primarily interested in Mitsch’s characterization of axiomatization as a practical, provisional, meta-mathematical methodology which aims to order and orient an area of knowledge. This is precisely how we should understand the role that axiomatization plays in the reconstruction programme. Individual reconstructions should be taken as provisional, alternative formulations of quantum mechanics. The methodology of reconstructions should be understood as a predominantly meta-mathematical process which enriches both our mathematical and physical conceptions in foundations of quantum theory. The meta-mathematical project first identifies central theorems and concepts, then identifies a formalism whose structure reflects those theorems and concepts, and lastly determines the necessity of candidate axioms based on the formalism. It is here where we understand an axiomatization to be provisional, as the central theorems and concepts may change. Below we see how the meta-mathematical project plays out in the methodology of reconstructions:

1. The identification of central theorems and concepts of an area of knowledge (e.g., the von Neumann postulates, specific axioms like composite systems)
2. The identification of a formalism whose structure reflects the above area of knowledge (e.g., the GPT framework *a la* Hardy)
3. The determination of the necessity of candidate axioms based on the formalism (e.g., the continuity axiom in ‘Five Axioms’)

The central postulates to be derived in the reconstruction programme are the von Neumann postulates, the minimal theoretical and empirical result that must be recovered in any axiomatization of quantum theory. Specific axioms may differ depending on what researchers take to be fundamental to quantum theory. For example, both Hardy in ‘Five Axioms’

and Selby et al. (2018) include the physical concept of composition (composite *systems* in Hardy and the composition *of processes* in Selby et al.) as a central concept of quantum theory. The identification of an appropriate formalism is by framework choice, such as an operational formulation in Hardy’s case or a process theory formulation in Selby et al.¹⁹ In reconstructions, formalism stipulated at two points adds another level of specification to the analytic machinery. Lastly, we determine the necessity of candidate axioms in light of the axiomatic structure of the formalism.

This final step requires a more careful treatment in the reconstruction programme. Mitsch describes this final step as determining if the collection of axioms is representative for the theory while admitting no other realizations. An axiomatic completion is achieved if the axiomatic structure of the theory has been completely determined, i.e., if the necessity of candidate axioms is determined. Uniqueness questions help us decide if other axioms are needed. Von Neumann’s goal is to show that transformation theory is a unique representation of quantum theory that did not allow for hidden variables. Hence, transformation theory is unique insofar as the collection of axioms was sufficient to describe quantum theory with no other realizations. The result is a single formulation of quantum theory which is a general characterization of the theory. In the reconstruction case, successful reconstructions will rederive the von Neumann postulates from specified sets of axioms and in doing so will highlight a collection of axioms necessary for its derivation. These sets of axioms will not be unique in the sense that there will be no other realizations. There will be many different sets of axioms which isolate the quantum space of theories within a framework. Thus, reconstructions will not be unique in the sense that von Neumann meant.

Have we achieved an axiomatic completion in the context of reconstructions? The answer is not obvious. Reconstructions fail to be unique in the sense that there is a single realization that represents quantum theory. However, reconstructions *do* succeed in determining axiomatic structures which describe quantum theory when they isolate the quantum region in the space of theories. Reconstructions are able to reproduce the von Neumann postulates by considering which axioms are necessary to situate us in a quantum space of theories. If we are able to isolate the quantum region within the space of theories, then other candidate axioms are not required. What is unique in the reconstruction programme is the region in theory space that is quantum theory. Hence, I maintain that reconstructions successfully determine the axiomatic structure of quantum theory by isolating the quantum region in the space of theories. There will however be many realizations, in which case there will be no unique reconstruction which represents the theory. However

¹⁹A process theory formalism is meant to capture the intuition that “the conceptual bare-bones of quantum theory concerns the manner in which systems and processes compose” (Selby et al., 2018, 1).

the lack of a unique reconstruction makes sense in the context of the programme. The goal of reconstructionists is to discover a general characterization of quantum theory in a contrastive theoretical space. This contrastive space enables researchers to highlight the differences between the classical, the quantum, and other possible theories. The results of the programme are different candidates for the distinct features of quantum theory. Thus, the intention is not to determine a single formulation for quantum theory, but to highlight the distinctions between the classical, quantum, extra-quantum, and non-quantum in the space of theories. Therefore I argue that reconstructions successfully determine axiomatic structures for quantum theory, even if there are multiple realizations. The methodology is designed to give us the theoretical space (the framework) in order to investigate what the core physical principles of quantum theory might be.

But one could ask why, if von Neumann was successful in achieving an axiomatic completion of quantum mechanics in 1932, we should consider the reconstruction programme to be examples of another attempt at axiomatic completions of quantum theory. Why, in other words, is the project ongoing if von Neumann already succeeded? This is where the provisional nature of an axiomatic completion comes to the fore. If we understand axiomatization attempts as being context-dependent (relative to the goals and maturity of the area of inquiry), then it is unproblematic that the search for axiomatic completions continues. In von Neumann's hidden variables case, the goal was to show that transformation theory is a unique representation of quantum mechanics (*excluding* hidden variables). In the reconstruction programme, the goal is to identify a set of axioms which isolates the quantum case in the space of theories. The goals of inquiry differ between both cases. As I have argued above, the maturity of the mathematical formalism for quantum mechanics is quite different in the present day, as are the central theorems and concepts. The context of the area of knowledge has shifted, which is why axiomatic completion attempts continue. And this contextual reading is complementary to the provisional nature of axiomatic completions. Therefore, I argue that Mitsch's account of axiomatic completions is further enriched by framing axiomatizations as context dependent, relative to the goals and maturity of the area of inquiry. As we can see, both the mathematical and meta-mathematical project will be relative to the maturity of the area of inquiry. An immature area of inquiry will result in a greater degree of difficulty in identifying central theorems and concepts as well as an accompanying formalism whose structure reflects the relations between them. The central theorems, concepts and formalism are liable to change, and this will be relative to how mature the theory is. Thus, steps (1) and (2) in the meta-mathematical project will be influenced, which inevitably influences step (3). In the reconstruction case, because quantum theory is reasonably mature, reconstructionists have a clearer idea of what steps 1-3 should look like, even if they are still investigating the arrangement of candidate axioms.

However, as long as the provisional nature of axiomatic completions is maintained, axiomatization functions as a means of structuring how we probe the mathematical, conceptual, and meta-mathematical structure of theories.

According to Mitsch, the meta-mathematical goal of an axiomatization is to address the connection between mathematics and the reality it purports to represent. This goal is achieved when the necessary physical assumptions are identified, based on the central theorems of an area of inquiry. The mathematical goal concerns the derivation of the propositions that result from a set of axioms. In the reconstruction case, the goal of rederiving the von Neumann postulates (the central concepts) is the meta-mathematical project. As I have described, reconstructionists aim to determine which physical assumptions are necessary to pick out the quantum case. As per Mitsch’s distinction, this is a meta-mathematical procedure. Further, the provisional nature of an axiomatic completion is complementary to reconstructions. Though the chosen axioms stipulated at the outset are taken to be fundamental and *resistant* to revision, both the collection of those axioms and their mathematical formulation can be altered. Reconstructions are best understood as provisional representations of quantum theory, while having a decidedly pragmatic flair. Recall that an axiomatization is pragmatic insofar as it generates different representations of a field of knowledge. Here Mitsch refers to Hilbert’s consideration of both Boltzmann and Hertz’s axiomatizations of Lagrangian mechanics. According to Hilbert, both representations contributed by unearthing a deeper layer of mechanics (Mitsch, 2022, 8). However, separate representations²⁰ will be evaluated via their ability to order and orient an area of inquiry (Mitsch, 2022, 8). An axiomatization is also pragmatic insofar as it orients an area of inquiry by surveying the independence of propositions. In the reconstruction programme we see this in the construction of foil theories where we might test the independence of axioms. The methodology of reconstructions is well-suited to contend with questions of independence, aligning with Mitsch’s reading of axiomatization.

As I have emphasized, the reconstruction programme results in multiple axiomatizations of quantum theory, based on different physical principles which are encoded as axioms. Though the common theoretical and empirical result (the von Neumann postulates) must be derived, separate reconstructions will act as different representations of quantum theory. Since researchers will carve up the essence of quantum theory in divergent ways, reconstructions will be distinct in their physical axioms, the formalism used within the axiomatization, and the associated physical implications of the axiomatization.²¹ The gen-

²⁰Which might even be *mutually inconsistent*, which is an unproblematic end in the interest of scraping out the mathematical core of a theory (Mitsch, 2022, 8).

²¹For example, the physical implications of time-reversibility (Selby et al., 2018, 4) is difficult to comprehend.

eration of multiple representations of an area of inquiry is a result of axiomatization, a result that is useful in the exploration and development of theories. Although the reconstruction programme is relatively new, it has already enriched mathematics, physics, and philosophy. The focus on operational formulations has become an effective methodological tool, while also spurring research on how to understand such an approach (Adlam, 2022). Recent work offers some potential general interpretative insights that can be drawn from several operational reconstructions including different notions of space, measurement, and time (Goyal, 2022, 35). More practically, the reconstruction programme is tied intimately with the advancement of quantum technologies and quantum information. For example, device-independent cryptography such as Barrett, Hardy, and Kent’s “No signaling and quantum key distribution” (Barrett et al., 2005) arose out of foundational work on information-processing (Chiribella and Spekkens, 2016, 5). Though there is much work to be done in this area, there is great promise in the reconstruction programme.

I have shown, using Mitsch’s articulation of axiomatization, that we should understand reconstructions of quantum theory as provisional, alternative formulations of quantum mechanics. They are best understood as a primarily meta-mathematical process which has benefited mathematics, physics, and philosophy, and is likely to continue to enrich foundational work in quantum theory.

3.6 Conclusion

Reconstructions of quantum theory employ a modern version of axiomatization with important points of continuity to axiomatization as it was practiced by von Neumann. Hardy’s ‘Five Axioms’ shares a number of core features of axiomatization with the form of axiomatization that was used by von Neumann. These core features include a shared operational approach, similarly empirical axioms, and the same methodological ordering between analytic machinery and physical axioms.

However, though there are points of continuity between these cases, I have demonstrated that the ways in which historical features of axiomatization appear in reconstructions are methodologically distinct. Specifically, I show that the inclusion of a generalized framework is methodologically advantageous, primarily in its capacity to provide a comparative platform of alternative formulations of quantum mechanics within the framework. Further, the stipulation of analytic machinery at two distinct levels of generality allows for greater specification of the mathematical tools used in a reconstruction. This specification is significant because it allows us to both constrain the space of theories and to clearly identify the success condition of deriving the von Neumann postulates. These features

are methodologically advantageous in the context of the reconstruction programme, which enables reconstructions to be vehicles for theory exploration. In light of historical uses of axiomatization and its modern usage in reconstructions, I showed that axiomatization should be construed as context-dependent, where the relevant context is contingent on the goals of inquiry and the maturity of the mathematical formalism and theoretical concepts in an area of knowledge. A mature formalism and set of central theorems and concepts allow reconstructionists to integrate a generalized framework in the methodology, which aids researchers in identifying the quantum case in the space of theories.

Using Mitsch (2022)’s understanding of axiomatizations as axiomatic completions, I show that reconstructions are provisional, practical representations of quantum theory that order and orient the discipline. This characterization clarifies the role that axiomatization plays in the exploration and development of theories. The context-dependency of axiomatization further enriches Mitsch (2022)’s account of axiomatic completions, as both the mathematical and meta-mathematical project will be relative to the maturity and goals of the area of inquiry.

Thus, just as von Neumann took seriously the task of “spreading the spirit of axiomatization” (Lacki, 2000, 281), so too do reconstructionists of quantum theory.

Chapter 4

Reconstructions as Alternative Formulations of Quantum Theory

Entertaining a range of ways to apply and conceptualize a murky theory, we entertain as well a range of ways forward—that is, a range of ways to develop potential successor theories. And that stands to be a superior strategy for scientific progress than narrow adherence to a single interpretation we’ve declared the winner in order to sustain a stringent sense of understanding that might stand at cross-purposes with working physics. (Ruetsche 2024, p. 13)

Reconstructions of quantum theory aim to derive the formalism of quantum theory from a small set of clear physical principles. The underlying motivation of the programme is to gain a better understanding of the theory by identifying and assessing the foundational principles at its core. To do this, reconstructions stipulate a set of axioms informed by key physical principles in order to derive the von Neumann postulates, which are the minimal empirical and mathematical results required for quantum theory. The reconstruction programme is primarily focused on where the peculiar mathematics of quantum theory—complex Hilbert spaces, for example—comes from. Reconstructionists reject the standard interpretative approach wherein one accepts a mathematical formalism and gives that formalism a literal interpretation. The primary reason for this rejection is the lack of an agreed-upon interpretation of the theory that solves both the measurement problem and the problem of non-locality (Koberinski and Müller, 2018b, 1). In particular, reconstructionists take issue with the Hilbert space formalism as overtly mathematical and abstract, lacking the kind of physical *je ne sais quoi* we might expect from one of our great physical theories (Fuchs and Stacey, 2016, 285).

Conversely, the standard interpretative project in quantum theory typically develops ontological conclusions based on an accepted mathematical formalism. The goal is to reveal the way the physical world really is based on the formalism of the theory. But reconstructionists do not want to start by assuming the Hilbert space formalism is correct and literally interpret that formalism. Why interpret a formalism we think is not the most appropriate language in which to express the theory? In response, the reconstruction programme developed a methodology that begins with physical principles and ends with the derived formalism, inverting the order of the standard interpretative project. The result of this research programme is multiple successful reconstructions, i.e., alternative formulations that reproduce the key features of quantum theory by encoding different sets of physical principles. These reconstructions are complementary views about what is special about quantum theory.

Several important questions arise from the reconstruction programme. What is the value in having multiple formulations of quantum theory? What is the benefit of rederiving formalism we already have (von Neumann postulates)? What sort of things should we consider reconstructions to be? In this chapter, I will lay out the philosophical reasons for why having alternative formulations of quantum theory is valuable. To do so, I utilize Bokulich (2019)’s “Losing Sight of the Forest for the Ψ ,” where she argues that alternative formulations of a theory undercuts the legitimacy of interpreting a single formulation of that theory. Bokulich argues that there are significant advantages to having alternative formulations of theories, advantages that are otherwise not seen if alternative formulations are not explored. I maintain that the reconstruction case serves as an example that demonstrates and supports Bokulich’s view on the benefits of considering alternative formulations. Further, I argue that Bokulich (2019) gives us a different reason to reject the starting point of accepting the Hilbert space formalism: once you see that we can have multiple formulations of the same theory with the same empirical consequences but different physical consequences, we are not justified in literally interpreting one formalism. Reconstructionists however do not offer this as a reason for rejecting the Hilbert space formalism. Therefore, I argue that (1) the argument in Bokulich (2019) that standard interpretations of quantum mechanics are undermined by the existence of alternative formulations is an additional argument in support of reconstructionist’s initial rejection of the standard interpretative project. Further, I argue that (2) there are advantages to having alternative formulations of theories born from non-literally interpreting them. In the reconstruction case, these benefits come primarily in the form of the physical insights we get from multiple alternative formulations. And it is these physical insights that allow us to introduce a form of realism without literally interpreting the formalism of reconstructions.

I argue that reconstructions are examples of what I call non-literally interpreted for-

mulations. Non-literally interpreted formulations do not ascribe an ontology based on an accepted formalism but still give us genuine insights into the physical world. This new category results in a tripartite distinction of formulations between full formulations, non-literally interpreted formulations, and literally interpreted formulations or full interpretations. I argue that the distinction predominantly hinges upon the attitude one takes toward the role of realism and representation. Non-literally interpreted formulations offer a variety of realism that is not as robust as the realism we expect out of a full interpretation. However, non-literally interpreted formulations do not read an ontology off of a formalism, the act of which is problematic for a variety of reasons (Bokulich 2019). Finally, I argue that both (1) and (2) serve as further motivation for the methodology of the reconstruction programme. I contend that reconstructionists go beyond Bokulich (2019)’s insistence on the consideration of alternative formulations to develop a methodology that systematically constructs alternative formulations. I conclude that reconstructionists are evolving their methodology via the new mechanism of systematically constructing alternative formulations and by inverting the methodological order of the standard interpretative approach.

The chapter will proceed as follows. I begin in section 4.1 by discussing Bokulich (2019) which explores multiple cases of alternative formulations of quantum theory that reveal important insights not seen in standard formulations. Bokulich (2019) argues that the existence of alternative formulations means we are not justified in drawing conclusions about the underlying ontology based solely on one particular formalism. Thus, she concludes that *if* we want to read ontology off of mathematical structure then we must consider alternative formulations of a theory. She supports her argument via Lagrangian Quantum Hydrodynamics, which she argues is a full alternative formulation of quantum theory. Further, Bokulich argues that alternative formulations of quantum theory undermine the legitimacy of the standard interpretative project, which must choose a single formulation to interpret. In section 4.2 I illustrate the reconstruction case, highlighting parallels between Bokulich’s cases and reconstructions that involve comparable notions of equivalence. I argue that reconstructions are distinct from the standard interpretative project and are geared toward the discovery of new formulations of quantum theory. Using Bokulich’s account of alternative formulations, I argue that reconstructions of quantum theory are full alternative formulations of quantum theory. Understanding reconstructions as alternative formulations is important because it shows that there is more than one way to formulate quantum theory. This implies that we are not justified in interpreting any single formulation, which is a common approach in the standard interpretative project.¹ I conclude

¹In “The miracles argument meets quantum mechanics: Toward a locavore philosophy of physics” (2024) Ruetsche offers a similar view about the usefulness of retaining different interpretations of quantum

that (1) the reconstructionist’s rejection of the standard interpretation project is further supported by Bokulich (2019)’s argument for the consideration of alternative formulations.

Bokulich highlights several advantages of exploring alternative formulations of theories. I explicate how the reconstruction approach benefits from the key advantages that Bokulich describes when we consider alternative formulations. I argue that reconstructionists gain the same benefits from considering alternative formulations by using a new method of systematically constructing alternative formulations. More specifically to the reconstruction case, I argue that (2) these benefits are a result of non-literally interpreting those alternative formulations. In this way, I claim that reconstructions signify a shift in quantum foundations away from thinking about realism in terms of standard interpretations. As Bokulich argues, one potential way of injecting realism into quantum theory is to move away from supplying interpretations and to instead move toward inferential realism, an attitude of ‘what true things can we learn?’ (Bokulich, 2020, 38). To this end, in section 4.2.2 I develop the notion of a non-literally interpreted formulation as a middle ground between a full formulation and a full interpretation. A non-literally interpreted formulation allows us to gain genuine physical insight into the way the world is amongst the myriad of alternative reconstructions without literally interpreting the formalism. The non-literally interpreted formulation represents a shift away from thinking of theories as literal construals and toward this new attitude of ‘what true things can we learn?’ And this new attitude is an effective means of gaining a better understanding of quantum theory writ large.

In 4.2.3 I explore some ways that the reconstruction case diverges from Bokulich’s examples. This includes the use of the axiomatic method in the reconstruction case and the general strategy that the quantum will resemble the classical in Bokulich’s case. I argue that these are methodological and motivational differences that have no bearing on the role that alternative formulations play in the reconstruction programme. Section 4.3 discusses how reconstructions have shifted the focus from the standard interpretative project toward the development of alternative formulations of quantum theory. I further consider how this shift has produced genuine insights into foundations of quantum theory, primarily in the form of physical insights awarded by inference. I argue that arguments

mechanics. Ruetsche argues that it is a “mistake to afflict upon our best theories a single, uniform interpretation meant to apply in all circumstances” (Ruetsche, 2024, 1). Instead, the ‘Locavore’ approach is proposed, which values the pursuit of local projects of ‘making sense,’ i.e., valuing the context-sensitivity of ongoing projects in physics that “*fosters that theory’s capacity to function as physics*” (Ruetsche, 2024, 10). Similar to my view, Ruetsche argues for the methodological advantages—for example, the fruitfulness of the consideration of different interpretations—that arise from moving away from a “unique global univocal account” (Ruetsche, 2024, 11) of the story about the world we should take quantum mechanics to be telling us.

(1) and (2) collectively motivate the methodology of reconstructions. Here I contend that reconstructionists have evolved their methodology to be aimed toward generating alternative formulations of quantum theory. I conclude that reconstructions of quantum theory are a promising pathway forward in exploring and solving foundational problems.

4.1 Bokulich’s “Losing Sight of the Forest for the Ψ ”

Bokulich (2019) argues that alternative formalisms in quantum mechanics offer an important lesson for the connection between the formalism of a theory and the ontology we are justified in attributing to a theory. Bokulich claims that the distinction between the representational vehicle, i.e., the mathematics, and what is taken to be represented in the world is a distinction that is often overlooked, particularly in quantum mechanics (Bokulich, 2020, 2). For example, in quantum mechanics Ψ is often taken to represent the wavefunction both as a mathematical representation and as the ‘thing in the world.’ Bokulich argues that this is problematic because it both serves to ‘reify’ the formalism as well as potentially importing properties of the representational vehicle that are *not* properties of the thing in the world (Bokulich, 2020, 2). This problem reveals itself particularly in ‘wavefunction realism,’ “which takes the wavefunction to be a concrete physical object, and takes the $3n$ -dimensional configuration space on which the wavefunction is defined, to be the space that we (despite appearances to the contrary) actually live in” (Bokulich, 2020, 2). Bokulich maintains that the realist project is often understood in quantum theory as trying to read off a *literal* depiction of the world from the mathematical formalism. However, Bokulich argues that the wavefunction is not something that can provide a literal depiction as the wavefunction is an estimation of a quantum state based on a collection of various measurements from an ensemble of identically prepared systems (Bokulich, 2020, 8). She contends that we must be doubly cautious about the ontological implications of a formalism when the “entity or state in question is unobservable and only indirectly accessible” (Bokulich, 2020, 8).

What Bokulich is describing is typical of how standard interpretations of quantum mechanics proceed insofar as they posit an ontology via some particular interpretation of the formalism. Bokulich claims that it is problematic to assume the correctness of the formalism outright, maintaining that the formalism of quantum mechanics is not necessarily mapped onto the world. She argues that the consideration of alternative formalisms is *particularly* important for the realist who tries to read off an ontology directly from the mathematics (Bokulich, 2020, 2). To serve as a contrast, Bokulich explores an alternative formalism to the standard formalism of quantum mechanics, Lagrangian quantum hydro-

dynamics (LQH), that does not use the wavefunction Ψ but rather represents the quantum state as a “displacement function of a continuum of interacting ‘particles’ following space-time trajectories, $q_i(a, t)$ ” (Bokulich, 2020, 2). The result of this different representation of the quantum state is an alternative formalism for quantum mechanics *without* the wavefunction. According to Bokulich, an ‘interpretation’ provides some kind of answer to the question of what is physically real (Bokulich, 2020, 3). Bokulich takes Lagrangian quantum hydrodynamics to be a robust alternative formalism that is equivalent to standard formulations (Bokulich, 2020, 3). Bokulich’s primary goal is to argue that the consideration of alternative formalisms “is a critical prerequisite to any such realist project that attempts to read the ontological implications of a theory off the formalism” (Bokulich, 2020, 3).

Continuum mechanics serves as the overarching framework under which classical and quantum hydrodynamic formulations are developed (Bokulich, 2020, 11). The continuum framework is based on understanding fluid flow as a continuum rather than an atomistic understanding, even though the continuum assumption is “strictly false as a depiction of real fluids” (Bokulich, 2020, 11). Bokulich considers the continuum framework to be a ‘credentialed fiction’, which serves as a powerful mathematical framework for representing idealized fluid movement. A ‘credentialed fiction’ is a fictional representation that is genuinely explanatory via its ability to correctly capture relevant patterns of inference (Bokulich, 2016).² Bokulich argues that the continuum framework “provides insights that would be difficult (if not impossible) to achieve on an atomistic approach” (Bokulich, 2020, 11).

Classical hydrodynamics, described by both the Eulerian and Lagrangian versions, provides a robust foundation for the description of fluids. The Lagrangian version in particular reveals a relabeling symmetry where no particular label of a particle is preferred over another. This relabeling symmetry corresponds to a conservation law.³ The development of quantum hydrodynamics follows a similar progression to that of classical hydrodynamics. Holland (2005) developed the Lagrangian version of quantum hydrodynamics, showing that we get the quantum version of the relabeling symmetry wherein the quantum state is described by a displacement amplitude. This relabeling symmetry similarly corresponds via Noether’s first theorem to a conservation law (Bokulich, 2020, 28). Thus, we get the quantum versions of the conservation laws seen in the classical case (Bokulich, 2020, 28). Both the quantum version of the relabeling symmetry and the corresponding conservation law are ‘new insights’ revealed to us through the alternative formalism. Bokulich notes that the Eulerian and Lagrangian formulations of hydrodynamics are equivalent insofar

²I discuss Bokulich’s ‘credentialed fictions’ and how we might understand them in Chapter 5.

³The conservation law of potential vorticity, which is the “idea that a rotating column of fluid, despite changing shape, will conserve volume and angular momentum” (Bokulich, 2020, 15).

as they can be mathematically transformed into one another, although they are not mere coordinate transformations of each other (Bokulich, 2020, 29). The formulations are two different ways of describing the same physical system. The choice of formulation is based on “calculational tractability or the physical insight it provides” (Bokulich, 2020, 30). Hence, the usage will be context dependent.

Lagrangian quantum hydrodynamics was developed in close *formal analogy* to classical hydrodynamics. Just as classical hydrodynamics is described by both Eulerian and Lagrangian versions, the Lagrangian quantum version was developed in 2005, mirroring the Eulerian quantum version (1927-1932) (Bokulich, 2020, 21). The Eulerian and Lagrangian formulations give us different conceptions of fluid flow. In the Eulerian formulation, fluid properties are ascribed to particular points in space, meaning that the flow of a fluid is understood as the changing of properties defined at various spatial locations over a period of time, using a fixed inertial coordinate system (Bokulich, 2020, 12). In the Lagrangian picture, a fluid is represented as a ‘dense set of fluid particles’ or *parcel* with its own properties, following a classical trajectory and maintaining its identity through fluid evolution (Bokulich, 2020, 12). These individual fluid parcels—which are ‘pieces of the continuum’—are individually labeled. Hence the Lagrangian formulation introduces new variables (particle labels) into the formalism not present in the Eulerian formulation (Bokulich, 2020, 22). In this case, we track particles and describe the variation in properties along the trajectory of each fluid particle. These are different ways of conceiving of the nature of fluids, which are based on different physical intuitions.

Bokulich argues that the existence of the LQH undercuts the primacy of the wavefunction formulation of quantum mechanics. She contends that much like the Eulerian and Lagrangian formulations of fluid flow paint different pictures of the same physical system, so too do the LQH and wavefunction formulations of quantum mechanics. Different physical notions are present in both the Eulerian versus Lagrangian and LQH and wavefunction cases. In the Eulerian versus Lagrangian case, fluid flow is understood differently. In the LQH versus wavefunction case, the quantum state is understood differently. LQH understands the quantum state as a displacement function of a continuum whereas in the wavefunction formulation, the quantum state is in a $3n$ -dimensional configuration space. Bokulich argues that contrasting the LQH and the wavefunction formulation of quantum mechanics supports similar conclusions to the Eulerian versus Lagrangian classical case. The LQH and Ψ formulation utilize different mathematical formalisms, are intertransformable, and share the “common mathematical data of the initial conditions” (Bokulich, 2020, 33), but paint different ontological pictures. She concludes that the hegemony of the wavefunction is thus undermined and that the consideration of alternative formalisms is necessary, particularly for any realist project aiming to read an ontology off of the math-

ematics. Thus, EQH and LQH are full formulations that are mathematically translatable and are “taken to be—in some not yet philosophically precise sense—equivalent” (Bokulich, 2020, 29).

Bokulich concludes that the consideration of different mathematical formulations can give rise to significant results—e.g., the quantum version of conservation laws—that would have been obscured in another representation. The elision of formalism and ontology leads us to reify mathematical formalism and assume that the properties of a representation are properties of the world. The benefits of *separating* formalism and ontology are several. Therefore, Bokulich claims that it behooves us to consider other formulations for four main reasons (Bokulich, 2020, 3).

The first and primary reason to consider alternative formulations is to halt the aforementioned elision between the formalism and things in the world. The existence of alternative formulations “undercuts a facile identification of the formalism with the world, bringing to the fore the substantive questions about what sorts of ontological inferences can legitimately be made” (Bokulich, 2020, 3). We can see this lesson most clearly in the comparison between the LQH and Ψ case. Wavefunction realism asserts Ψ as the representation of the quantum state as well as an entity in the world. However, as Bokulich has argued, LQH offers a different formulation of quantum theory that does not contain the wavefunction, and rather considers the quantum state to be represented by a displacement function following spacetime trajectories (Bokulich, 2020, 2). These two different representations are empirically equivalent (yielding the same predictions), intertransformable, but paint different ontological pictures insofar as the LQH formulation will not contain the wavefunction. It is the existence of LQH that undercuts the legitimacy of drawing the ontological conclusion that the wavefunction is an existent entity.

The second reason is that certain phenomena may be better represented in some formalisms rather than others. The fertility of the hydrodynamic description is one such example. The classical Lagrangian formulation gives rise to the relabeling symmetry wherein “potential energy terms in the Lagrangian are unaffected by particle-label variations $\delta \mathbf{a}(\mathbf{x}, t)$ that leave the density and entropy unchanged” (Bokulich, 2020, 14). This symmetry corresponds via Noether’s theorem to a conservation law which “is scientifically one of the most important results in geophysical fluid dynamics” (Bokulich, 2020, 14). Most important is the quantum version of the Lagrangian formulation which likewise results in the relabeling symmetry (gauge freedom) which corresponds to a conservation law. There is also a pragmatic point to be made about LQH versus Ψ regarding the quantum evolution. In the wavefunction case, the wavefunction is required to derive trajectories. However, in the LQH case the quantum evolution can be derived from trajectories alone (Bokulich, 2020, 33). Thus there will be certain formulations that may be preferred for

ease of calculation or other pragmatic concerns (testability, simplicity, etc.).⁴

The third reason to consider alternate formalisms is that certain formalisms may be better suited to suggesting further lines of inquiry than others as it relates to theory development (Bokulich, 2020, 3). This point is well-made by the hydrodynamic representation. For example, Holland constructs a full (Eulerian and Lagrangian) hydrodynamic ‘fluid’ model of the electromagnetic field which can be extended to produce Lagrangian trajectory formulation of any *general* field theory, a feat which “thus provides researchers with new conceptual and computational resources for exploring field theories” (Bokulich, 2020, 25). And although the significance of the conservation theorems from LQH are unclear in relativistic quantum mechanics, it is argued that “the conservation law associated with the particle relabeling symmetry is what allows one to define global simultaneity manifolds, restoring absolute simultaneity, and allowing a foliation into a $3+1$ space and time” (Bokulich, 2020, 28). These are but a few examples of why Holland asserts the LQH is very fertile and why Bokulich understands this to be a result of the consideration of alternative formulations.

Bokulich’s final reason is that alternative formalisms “can challenge common presuppositions about what features are supposedly demanded by the theory (e.g., the presumption that the quantum state must live in a $3n$ -dimensional configuration space)” (Bokulich, 2020, 4). Bokulich specifically addresses ‘configuration-space realism’ which is the presumption that since the quantum state in the Ψ formulation ‘lives’ in a $3n$ -dimensional configuration space, “we too must live in this high-dimensional abstract space, and the three-dimensional physical space of our experience is in fact an illusion” (Bokulich, 2020, 34). However as Bokulich concludes, the LQH formulation shows that the quantum state of a many-body system (i.e., where n is the number of particles in the universe) can be represented in ordinary three-dimensional physical space (Bokulich, 2020, 35). Thus the abstract $3n$ -dimensional configuration space that is required in the Ψ formulation is *not* required in LQH, which challenges the presupposition that a $3n$ -dimensional space is a necessary feature of quantum mechanics and of our world. Bokulich also maintains that nonlocality is captured in the LQH formulation, an essential feature of any formulation of quantum mechanics (Bokulich, 2020, 36). We will see that these four benefits also are realized by the reconstruction programme, which will be discussed in section 4.2.

⁴Bokulich explicitly makes this point, referring to Feynman’s path integral formulation of quantum mechanics (Bokulich, 2020, 33).

4.1.1 Formulation versus Interpretation

There is an important distinction to be made between a formulation and an interpretation, a distinction which is likewise relevant for the reconstruction programme. Bokulich considers LQH to be a ‘full formulation or representation’ or ‘full alternative formalism’, *not* an interpretation (Bokulich, 2020, 3). By this I take Bokulich to define a formulation as a mathematical description that does not intend to posit the mathematical representation as literally describing things in the world. EQH and LQH are different formulations that describe the same physical system but are not intended to supply literal descriptions. This is due in part to the continuum assumption which is ‘strictly false.’ As a result, the continuum assumption results in both EQH and LQH being best understood as idealizations. Bokulich refers us to Takabayasi in order to emphasize the distinction between formulation and interpretation wherein “the hydrodynamic analogy...though sometimes useful for the analysis of the Schrödinger equation and rather appropriate to make one visualize the presence of internal force, does not necessarily prove the *reality* of the hydrodynamic picture” (Takabayasi, 1952, 150). Although the hydrodynamic analogy is a fictional representation of quantum mechanics, the formulation grants us insights into certain features of quantum mechanics. Thus as I understand Bokulich, a formulation stipulates a mathematical formalism where we might have some empirical information or correspondence but the representation is not intended to describe what is happening in the world.

In contrast to a formulation, Bokulich understands an interpretation of quantum mechanics “to answer the question of what quantum mechanics (on any formalism) tells us is physically real” (Bokulich, 2020, 3). An interpretation of quantum mechanics also seeks to solve the measurement problem, while reading mathematical formalism as a literal depiction of the world (Bokulich, 2020, 5). One way that Bokulich carves out the distinction between formulation and interpretation is by contrasting LQH and the de Broglie–Bohm (dBB) interpretation of quantum mechanics. DBB is an interpretation of quantum mechanics that attributes ontological status to the wavefunction, with a dual ontology of trajectories *and* wavefunctions or pilot–waves (Bokulich, 2020, 33). Thus, Ψ is taken to be an objectively real field. Bohmian mechanics aims to solve the measurement problem by “underpinning the indeterministic and statistical story told by standard quantum mechanics with a completely deterministic, causal story of individual quantum systems and processes [while] the LQH formulation, on its own, has no interpretive ambitions” (Bokulich, 2020, 27). The formalism in the dBB interpretation is taken to be a literal depiction of the world. Typically, researchers would not seek to interpret a formalism they believe to be ‘strictly false’, or one that is heavily idealized.⁵

⁵Although researchers would not typically interpret a heavily idealized formalism, this does not preclude

Later, I introduce a third category that is intermediate between a full formulation and an interpretation, a non-literally interpreted formulation. This third category will have greater ambitions than a full formulation insofar as a non-literally interpreted formulation will contain some genuine physical content which tells us something about the real world. However, a non-literally interpreted formulation will not ascribe an ontology based on a mathematical representation to the extent that an interpretation does. I will argue that reconstructions are an example of this third category.

4.1.2 Cases of equivalence

Equivalence plays an important role in Bokulich (2019) because the presence of equivalent formulations undermines our ability to literally interpret a single formulation. For example, LQH and the Ψ formulations of quantum theory reveal “the importance of recognizing the legitimacy of both” representations (Bokulich, 2020, 10), which functions as a criticism of the standard interpretative project. As we will see in section 4.2, the reconstruction programme allows us to draw similar conclusions.

As Bokulich claims, EQH and LQH are equivalent in a sense that is not yet philosophically precise. She does assert that they are intertranslatable, empirically equivalent, and describe the same physical system. Equivalence is important because Bokulich wants to establish that we have two equivalent formulations of the same physical system that nonetheless give rise to different ontological pictures. EQH and LQH are empirically equivalent insofar as they yield the same predictions, but Bokulich is aiming for a more robust equivalence than mere empirical equivalence, which is where intertranslatability becomes important. The intertranslatability of EQH and LQH means that the two formulations are physically equivalent, though they do not imply the same ontologies.⁶ Due to this, there is a further implication that EQH and LQH are theoretically equivalent (Bokulich 2019, 29).

Bokulich draws the same conclusions for the LQH versus wave function case, as she takes both formulations to be empirically equivalent and intertransformable, while describing the same physical system. Collectively, this undermines wavefunction realism and ‘shifts the burden of proof’ of what ontological conclusions can be made directly from a formalism (Bokulich, 2020, 34). Although Bokulich does not explicitly explain what is

them from using specific idealization methods, e.g., Feynman diagrams.

⁶Bokulich does not go into detail about *how* exactly intertranslatability results in physical equivalence, though presumably having the same mathematical formulation implies that the physical consequences will likewise be the same. However, we can still have different ontologies, i.e., different features of the formulations that we take to *exist*.

meant by intertransformability in the LQH versus Ψ case—other than that formulations are able to be mathematically transformed into one another—we can infer that intertransformability refers in part to the wavefunction as a derived quantity. Bokulich asserts that the wavefunction is able to be derived *from* the displacement function, in which case LQH could accommodate Ψ if desired (Bokulich, 2020, 32). In this way, one could transform LQH into a formulation with wavefunctions. It will be helpful to provide some details about how LQH and the Ψ formulations result in different theoretical descriptions.

In general, wavefunction realism will take the fundamental quantum state to be represented by the wavefunction, which is an existent entity. For example, the de Broglie–Bohm interpretation has a dual ontology of trajectories and wavefunctions (or pilot waves) (Bokulich, 2020, 33). DBB also asserts the existence of a corpuscle (of mass m) which follows one of the trajectories obtained from the wavefunction. On the other hand, LQH conceives of the quantum state as a displacement function of a continuum of interacting ‘particles’ following spacetime trajectories, dispensing both the corpuscle and wavefunction (Bokulich, 2020, 26). In the dBB interpretation the motion of a corpuscle is determined by the external quantum potential, however in LQH the trajectories are determined by interactions between fluid parcels (Bokulich, 2020, 26). The Schrödinger equation is recast in the form of Newton’s second law (Bokulich, 2020, 23):

$$m \frac{\partial^2 q_i(a)}{\partial t^2} = - \frac{\partial}{\partial q_i} (V(x) + V_Q(x))|_{x=q(a,t)} \quad (4.1)$$

The trajectories described by the displacement function can be calculated directly by the recast Schrödinger equation, providing “a self-contained formulation of quantum mechanics, without reference to the wavefunction” (Bokulich, 2020, 32). In other words, the quantum evolution (equation 4.1) is dependent on the congruence of trajectories alone, whereas the Ψ formulation must obtain trajectories via the wavefunction. In the dBB interpretation, the time-dependent Schrödinger equation must first be solved in order to obtain the wavefunction which is then used to derive trajectories (Bokulich, 2020, 26).

The most significant difference that Bokulich highlights between LQH versus Ψ formulation is that the formulations result in several differences in theoretical descriptions. In the LQH formulation the wavefunction is absent—though it can be derived—and replaced by a displacement function, whereas the wavefunction in the Ψ formulation is taken to be a real object. This difference lies in what each approach considers to be the quantum state, leading to distinct mathematical representations, i.e., the wavefunction versus $q_i(a, t)$. In the case of the wavefunction, an ontological claim is made regarding that entity. In the dBB interpretation, both trajectories and the wavefunction are existent entities. However,

LQH only requires trajectories but does not posit that those trajectories exist in the world. These two formulations also have different descriptions of physical space, as the wavefunction formulation necessitates a $3n$ -dimensional space whereas LQH can be represented in ordinary physical 3-space (Bokulich, 2020, 35). Further, the Ψ formulation considers $3n$ -dimensional configuration space to be the ‘space we actually live in’ (Bokulich, 2020, 5). The last distinction that is highlighted is how the quantum evolution is derived. In LQH, the quantum evolution (equation 4.1) is derived from trajectories alone while in the dBB interpretation, the wavefunction must first be derived which then allows us to derive trajectories. This distinction I take to be more of a pragmatic point but it also serves to highlight how the LQH formulation is a ‘simpler’ derivation in the sense that fewer ‘starting assumptions’—e.g., a solved Schrödinger equation, the derived wavefunction, and then trajectories in the dBB—are required to derive the quantum evolution. Further, the dynamical equations are significant to establish that theories have the same predictive consequences and are therefore theoretically equivalent.

Due to the theoretical equivalence of the two representations, Bokulich concludes that the primacy of the Ψ formulation is undermined. Neither LQH nor the Ψ formulation should be read as literal depictions, in which case the standard interpretative project is problematic. But where does that leave us, particularly if we have realist predilections? Formulations of theories without any import on the physical world will be no consolation to the realist. To this end, Bokulich explores three different philosophical approaches toward LQH in the spirit of “realist explorations of quantum theory” (Bokulich, 2020, 35).

4.1.3 Interpretations of LQH

One of Bokulich (2019)’s main conclusions is that it behooves researchers to stop reading ontologies off of mathematical formalisms, particularly in those cases where alternative formulations are present. However, the realist will require some sort of interpretation of a formalism if their project is to be realist in any robust sense. According to Bokulich, the first option is the ‘Many Interacting Worlds (MIW)’ interpretation which regards *all* trajectories in the quantum ensemble to be ‘valid and real’ while assuming a finite number of worlds (Bokulich, 2020, 36). However, the MIW interpretation must assume that the number of worlds is uncountably infinite in order to be equivalent to standard quantum dynamics. The second option is to understand LQH as admitting both a wavefunction *and* a trajectory concept of state which gives rise to a new variety of wave-particle duality (Bokulich, 2020, 37). Whether or not such a duality should be understood as a “feature of our mathematical representations, or...as revealing an ontological duality inherent in the world remains unclear” (Bokulich, 2020, 38). The final option and the one that Bokulich

takes to be the most promising is inferential realism, which I will explain in the next paragraph. Bokulich concludes that LQH is permissive insofar as the formulation admits three different interpretations, which is problematic for the standard interpretative approach. A single formulation that gives rise to three different interpretations undermines the ease with which we can claim that an interpretation represents the objective world.

For Bokulich the presence of alternative formulations that give different ontological pictures shifts the discussion away from whether or not a representation is a ‘literally true depiction’ of the world and rather toward a different question: what true physical insights and correct inferences do different representations allow us to draw about the world (Bokulich, 2020, 16)? This is a question that points Bokulich toward what she calls ‘inferential realism’, which she asserts is the best means of exploring the potential realist implications of alternative formulations like LQH. Inferential realism emphasizes “the importance of a plurality of representations” (Bokulich, 2020, 38) and understands science as aiming at ‘fruitful metaphors, analogies and models’, not as literal representations (Bokulich, 2020, 39).⁷ The plurality of representations that result from inferential realism “are aimed at and purport to tell us about the world, but their connection is often looser involving pragmatic elements; hence, a greater caution is required in drawing ontological conclusions” (Bokulich, 2020, 39). Much like the Eulerian versus Lagrangian example, the use of different representations will be context dependent but still committed to realism insofar as those different representations yield genuine knowledge of the world (Bokulich, 2020, 39). Bokulich argues that the hydrodynamic analogy and the formalism of continuum mechanics offer substantial explanatory power and fertility, in spite of being a non-literal representation of the world (Bokulich, 2020, 39). The ‘credentials’ come not from the interpretation of the formalism as a literal representation but from the correct inferences it licenses, i.e., the new insights derived (Bokulich, 2020, 40). Further, different representations will better suited to different contexts and domains. In section 4.3 I discuss how reconstructions are an example of the shift in attitude that we see in Bokulich’s proposed inferential realism. I argue that by shifting away from the standard interpretation project, reconstructionists have gained genuine insights into the nature of quantum theory.

In the next section, I cover the reconstruction case, showing that we should consider reconstructions to be alternative formulations of quantum mechanics. In doing so, I contend that (1) Bokulich’s argument for the value of alternative formulations supports reconstructionist’s rejection of the standard interpretative approach. I then argue that (2) there are benefits in the reconstruction case to non-literally interpreting alternative formulations. I

⁷How this process works, i.e., how metaphors, analogies, and models can give us genuine insights into the real world, is a result of Bokulich’s “Fiction as a Vehicle for Truth” (Bokulich, 2016). I explore this work and its relation to the reconstruction programme in Chapter 5.

then propose the new category of a non-literally interpreted formulation, which I argue facilitates a moderate form of realism.

4.2 The Reconstruction Case

In reconstructions of quantum theory, a framework is established that articulates a generalized mathematical formalism describing a space of theories. Under the general framework, individual axiomatizations are devised which pick out what researchers take to be the fundamental physical principles of quantum theory. These physical principles are then given mathematical formulations in order to serve as axioms. From those axioms, a successful reconstruction will derive the von Neumann postulates, which are the fundamental postulates of quantum theory, i.e., the Hilbert space formalism in terms of linear operators in a Hilbert space. The derivation of the von Neumann postulates isolates the quantum case in the space of theories. In this way, the physical principles which are postulated as *guesses* at the axiom level and their mathematical formulations are shown to be compatible with the von Neumann postulates by a successful derivation. As a result of this methodology, each axiomatization within the reconstruction programme will serve as a separate attempt at correctly capturing the fundamental principles of quantum theory. Since researchers will choose different principles which encode different physical content, separate axiomatizations function as different formulations. As I establish in Chapter 2, we can consider both case study reconstructions to be full, alternative formalisms that represent quantum theory.

4.2.1 Equivalence in reconstructions

For Bokulich, the equivalence of alternative formulations which nonetheless have different physical consequences undermines the legitimacy of affixing an ontology to an accepted formalism. But how should we understand equivalence in the reconstruction case? In reconstructions, different axiomatizations will not be intertranslatable as they will be based on different physical principles which are not mathematically transformable. Alternative axiomatizations are theoretically equivalent insofar as each set of axioms entails the von Neumann postulates as a mathematical consequence. However, alternative axiomatizations may not be physically equivalent because the chosen axioms encode different physical principles. This presents a case where alternative descriptions of the same physical state of affairs are theoretically equivalent yet physically distinct. The standard formulation of quantum mechanics versus the LQH formulation is an example of such a case. Though

the Ψ and LQH formulation are theoretically equivalent, they nonetheless give different physical descriptions for the same situation in the world. In the reconstruction case, alternative reconstructions are physically distinct insofar as they offer different explanations to the ‘why the quantum’ question, as discussed in Chapter 5.

For example, the mathematical representation that is used in Hardy’s ‘Five Axioms’ is ‘essentially linear algebra’ while Selby et al.’s axiomatization is diagrammatic. The fundamental mathematical element of the representation in Selby et al.’s reconstruction is the process, which is denoted by “labelled boxes with input wires (at the bottom) and output wires (at the top)” (Selby et al., 2018, 5). The fundamental physical element of the representation for Hardy is the state, which is defined to be any mathematical object associated with a particular preparation “that can be used to determine the probability associated with the outcomes of any measurement that may be performed on a system prepared by the given preparation” (Hardy, 2001, 2). It is represented by a positive (Hermitian) operator $\hat{p}$ satisfying $0 \leq \text{tr}(\hat{p}) \leq 1$ (Hardy, 2001, 6). As Hardy’s reconstruction is operational, the above description is unsurprising. We can see however that the mathematical representations used in these two reconstructions will be quite different. One such example is that Selby et al. load in composition at the outset via a C^* algebra, while composition is derived in Hardy as the tensor product of the Hilbert space of the subsystems (Hardy, 2001, 16). Selby et al. also do not “obtain that the state space is closed” (Selby et al., 2018, 30). Thus there are several formal features that will not be translatable between the two reconstructions. They will also give different descriptions of physical space. For example, ‘Diagrammatic Postulates’ includes time-reversibility, which they say can be intuitively thought of as “playing the video of that process in reverse” (Selby et al., 2018, 4) represented by a sharp dagger. How this would be interpreted physically is unclear, a problem Hardy gets to avoid in ‘Five Axioms’ as he does not assert time-reversibility as a postulate. As it relates to ontologies and descriptions of physical space, the reconstruction cases will be rather nuanced as reconstructions do not posit ontologies in the way that the Ψ formulation does.

In both the EQH versus LQH and LQH versus Ψ cases, we have examples where different formulations of the same physical system are empirically equivalent and yet lead to different ontological pictures. In LQH versus Ψ , this undermines wavefunction realism. In the reconstruction case, reconstructions which rederive the von Neumann postulates are considered to be empirically equivalent but paint different ontological pictures via different physical principles. Thus, the reconstruction case likewise undermines those projects where ontology is read off of mathematical formalism, which functions as a criticism of the standard interpretative project. Therefore, I conclude that the reconstructionist’s initial rejection of the standard interpretative project is further supported by Bokulich (2019)’s

argument that the standard interpretative project is undermined by the existence of alternative formulations of quantum theory. And as Bokulich argues, there are benefits to separating mathematical formalism and ontology. Below, I explore how the reconstruction programme benefits from exploring alternative formulations. Thus, I maintain that the reconstruction case exemplifies and supports Bokulich’s view on the benefits of distinguishing mathematical formalism from ontology and considering alternative formulations.

First, exploring alternative formulations of a theory is beneficial because “it undercuts a facile identification of the formalism with the world, bringing to the fore the substantive questions about what sorts of ontological inferences can legitimately be made” (Bokulich, 2020, 3). In other words, exploring alternative formulations challenges the simple idea that formalism represents reality. When this simple idea is challenged, the question of what sorts of ontological inferences we can make is highlighted. As I have argued, reconstructions should be considered alternative formulations of quantum theory with different mathematical representations. The presence of a multitude of axiomatizations in the reconstruction programme will undercut the ease and credibility of directly reading an ontology off of a formalism. However, this is unproblematic as providing an interpretation is not the primary goal of the programme. Rather, the intention is to look more closely at the mathematical structure and foundational physical principles of quantum theory. Reconstructions benefit from not equating formalism with ontology, and their methodology is well-suited to exploring the mathematical structure of quantum theory.

The second benefit of exploring alternative formulations is that “some phenomena or questions are more readily treated in one formalism rather than another” (Bokulich, 2020, 3). An excellent example of this benefit in reconstructions is in Selby et al. (2018), whose central physical concept is composition. The intuition that composition acts as the primary feature of quantum mechanics motivated the interest in a category-theoretic framework, which Coecke in particular takes to be a ‘natural arena’ for representing processes (Coecke, 2011, 10). Selby et al. take *categorical quantum mechanics* to change the paradigm from focusing on states to focusing on processes while also having “a full and faithful diagrammatic representation (a.k.a. *quantum picturalism*)” (Selby et al., 2018, 4). It is the diagrammatic representation that is meant to provide a more intuitive picture of quantum information and computation. Category theory’s abstract nature and mathematical flexibility help capture the physical intuition that Selby et al. see as fundamental to quantum mechanics. Thus, the representation chosen is a pictorial process formulation represented via category theory, which is more appropriate for Selby et al.’s aims than standard representations of quantum theory.

Hardy shares a similar interest in exploring different mathematical formalisms in his ‘Construction Interpretation’ (Hardy, 2018). Recall that one of Hardy’s grounding mo-

tivations as it relates to quantum foundations is to discover a pathway toward quantum gravity, which he believes may be found in an analogous manner to Einstein’s discovery of General Relativity (Hardy, 2018). The intention of the Construction Interpretation is to find a theory of Quantum Gravity that has as limits General Relativity and Quantum Field Theory, recovering at the very least the parts of the theory that have been empirically verified (Hardy, 2018, 20). This is not to say that the conceptual and mathematical formalisms must be consistent across all three theories. Indeed, Hardy maintains that Quantum Gravity could have drastically different conceptual and mathematical formalisms, mirroring how different Newtonian Mechanics is from General Relativity (Hardy, 2018, 20). There is a likelihood that several candidate theories may arise, in which case Hardy asserts that appeals to values such as “simplicity, elegance, or conceptual coherence” (Hardy, 2018, 20) may help distinguish the different theories, though it will be experiment that is the final arbiter. However, it is likely that a new mathematical formalism will be required in order to accommodate General Relativity and Quantum Field Theory.

The hope is that by exploring a variety of formulations of quantum theory, we will discover something that will lead us to a theory of Quantum Gravity. That ‘something’ will likely be formal, i.e., a new mathematical tool that will accommodate both Quantum Gravity and General Relativity. Indeed, Hardy and Spekkens maintain that “if we have a mathematical framework that is rich enough to contain a theory of quantum gravity (in much the same way that the mathematics of Hilbert space is sufficient for quantum theory and the mathematics of tensor calculus is sufficient for general relativity) then we could expect that a few suitably chosen postulates would narrow us down to the right theory” (Hardy and Spekkens, 2010, 4). This is based on the idea that new formalisms will push us toward future lines of theory development. Hardy likens looking for interpretations to looking for the mechanical models of gravitation, which were not successful (Hardy, 2018, 6). Rather, Hardy asserts that it is a conceptual approach that is needed. However, there is still a quiet desire to find the ‘right’ axiomatization to describe quantum mechanics via this method. Optimally, formulating quantum gravity will involve selecting fundamental principles from both quantum theory and general relativity. In such a case, quantum theory should be recoverable from quantum gravity in a suitable limit.

The third benefit of exploring alternative formulations is that “some formalisms may be more fertile in suggesting future lines of theory development” (Bokulich, 2020, 3). As I have stated, Selby et al. argue that the category-theoretic foundation of their process theory captures the core aspects of quantum mechanics more effectively than the standard formulation, which centres on states. They also believe that this formulation will more naturally lend itself to quantum computing (particularly due to its pictorial representation) and quantum information theory, which is a major motivation. There is a hope

that discovering a generalized process theory will lead to key developments in quantum foundations as well as other areas of inquiry including natural language processing. The translatability of Selby et al.’s postulates between diagrammatic, category–theoretic, and process–theoretic formulations also allows for a good degree of applicability, which may prove fertile in terms of theory development (Selby et al., 2018, 4).

The fourth and final benefit of considering alternative formulations is that “alternative formalisms can challenge common presuppositions about what features are supposedly demanded by the theory” (Bokulich, 2020, 4). This lesson can also be seen in the reconstruction programme as different axiomatizations will highlight the physical principles researchers take to be fundamental to quantum theory. These physical principles will be given divergent mathematical formulations which are not mathematically translatable. This will result in a wide variety of alternative formalisms based on divergent physical principles. The features demanded by each axiomatization will differ radically. For example, ‘Diagrammatic Postulates’ requires features such as cups and caps (Selby et al., 2018, 10), symmetric purification (Selby et al., 2018, 14)), and the necessity of a sharp dagger structure in order to correspond to the Hilbert space model. Hardy’s ‘Five Axioms’ will not require any of the above features, instead relying on subspaces, continuity, etc. As I have discussed, there are several features in Hardy’s ‘Five Axioms’ that will not be present in ‘Diagrammatic Postulates’ (e.g., states) while the inverse will also be true (e.g., processes).

Therefore, I argue that the reconstruction case is an example that supports Bokulich’s view on the benefits of separating formalism and ontology as well as considering alternative formulations of theories. The consideration of alternative formulations has paved the way for new insights into the nature of quantum theory, including, for example, distinct interpretative insights on the nature of space, time, and measurement (Goyal, 2022, 35). More specifically to the reconstruction case, there are advantages to having alternative formulations that are a result of non–literally interpreting them. To this end, I argue that reconstructions are examples of non–literally interpreted formulations. Below I introduce the category of non–literally interpreted formulations which results in a tripartite distinction of formulations.

4.2.2 A tripartite distinction of formulations

Recall that according to Bokulich’s account the the EQH versus LQH formulations are both fictional hydrodynamic formulations based on the ‘strictly false’ continuum assumption in hydrodynamics. Both representations are fictional descriptions that are nonetheless fruitful representations. However, the reconstruction case diverges from the EQH versus

LQH case because reconstructions are not fictional descriptions. Instead, I consider reconstructions to be ‘non-literally interpreted formulations,’ a category which falls in between a full formulation at the one end and a full interpretation at the other. A non-literally interpreted formulation does not ascribe an ontology based on a formalism but it *does* give us genuine insights into the physical world. The introduction of this third category results in a tripartite distinction between a full formulation, a non-literally interpreted formulation, and a literally interpreted formulation or full interpretation. The distinction has much to do with the attitude one takes toward the role of realism and its connection to representation. The distinction also hinges upon the degree to which we literally read ontology off of a formalism.

Full alternative formulations will be empirically equivalent in a minimal sense, i.e., will agree with the predictions of the theory. For example, a full formulation like LQH captures the empirical content of quantum theory. If one is an instrumentalist, which is the view that theories are merely instruments for capturing prediction (Chakravartty, 2017), then full formulations will be sufficient. We can use the formalism for its predictive consequences without interpreting the formalism as representing the world. Other than being a reliable tool for prediction, the instrumentalist need not require full formulations to say anything of the world as it really is. Further, multiple formulations of a theory are unproblematic if they accurately capture predictive consequences.

However, a full formulation will be unsatisfactory for the realist. The full-blooded realist wants to determine the furniture of the world by literally interpreting the formalism and ascribing an ontology based on that formalism. Capturing mere prediction will be insufficient. But the full-blooded realist will require that a single formulation of a theory must be *chosen* amongst alternatives and literally interpreted, e.g., the Ψ formulation. *Not* picking a single formulation to interpret would result in multiple potentially contradictory pictures of reality and the standard interpretative project is driven to tell a *single* story about the ontology of the world. Because an ontology is read directly off of a formalism, the full-blooded realist is required to pick which formalism to interpret. But as we have seen in Bokulich (2019), we cannot trust a single formulation and interpret it because we will have different formulations that are empirically equivalent but have different physical consequences.

So is it possible to non-literally interpret a formulation while saying more about the furniture of the world? I propose that the answer to this is yes. Say, for example, that we have a full formulation as in Bokulich (2019) but we want to inject some realism into our notion of scientific theories. However, we do not want to literally interpret the formalism as representing the world as we would in a literally-interpreted formulation. This middle point between a full formulation and a literally-interpreted formulation or full interpretation is

what I call a *non-literally interpreted formulation*.

A non-literally interpreted formulation is a formulation that contains some physical content that corresponds to the world as it really is. In the reconstruction case, I argue that it is the physical principles that correspond to the world as it really is. Physicists take the physical principles that serve as axioms to be true in the sense that they give us information on the state of the world. Let us take the continuity axiom as an example. As I argue in Chapter 5, Hardy (2001)’s principle of continuity can be interpreted as representing either the physical implementation of a computational process or as physical time evolutions. Both of these interpretations are physically meaningful. They, for example, place restrictions on the types of physical transformations that systems in the world can undergo. In this way we can understand physical principles as true generalizations of phenomena which have a physical interpretation. Thus, the physical principles that act as axioms provide insights into the state of the world because they represent real constraints on physical processes. Additionally, the physical principles chosen by researchers are empirically well-confirmed within the theory. So too are the von Neumann postulates which are the minimal empirical and mathematical results derived from a successful reconstruction.

A non-literally interpreted formulation differs from a full formulation in its ability to describe what is happening in the world. The hydrodynamic analogy, though strictly false, still gives us insights we would have otherwise not gained in another formulation. We cannot however understand the continuum framework to be a literal representation from which we affix ontologies based on the representation. But a non-literally interpreted formulation will not go so far as a full interpretation wherein an ontology is read directly off of a mathematical formalism, as in the dBB interpretation. However, we do interpret parts of the representation, i.e., the physical principles as having real physical content. It is this real physical content that enables reconstructions to be non-literally interpreted formulations. But reconstructions will not be idealizations in the same sense that EQH and LQH are, as ‘credentialed fictions’. This is due in part to the operational approach. Recall that the operational framework captures “all conceivable situations in physics that can be cast in operational language, in particular typical laboratory situations where there is a preparation procedure, followed by a measurement (and possibly transformation procedures in between both)” (Koberinski and Müller, 2018b, 2). It is from the operational basis that we gain a direct connection to experiment and empirical correspondence. However, my claim is that reconstructions go further than mere empirical correspondence when we interpret our physical principles as true generalizations of phenomena. This is a subtle point as it relates to the reconstruction programme, which does not have full interpretive ambitions but will have a thin physical interpretation. To be more precise, the reconstruction programme stands in opposition to the standard interpretative project wherein an inter-

pretation is directly read off of a formalism (such as wavefunction realism), however that does not preclude reconstructions from carrying a very minimal physical interpretation, i.e., the interpretation of physical principles which gives us physical insight into the way the world is.

Hence, a non-literally interpreted formulation opens the door to a moderate kind of realism. In reconstructions, this realism takes the form of the physical insights that we get from multiple formulations via the physical principles that inform our axioms. This physical content will not elicit a fully fleshed out description of the ontology that is expected from a full interpretation. But this is unproblematic given that one of the motivations behind the programme is to subvert the standard interpretative project. The key distinction between a non-literally interpreted formulation and a literally interpreted formulation is the choosing of a single formulation that we then literally interpret. Importantly, in the reconstruction case the physical insight that is gained is awarded not by literally interpreting the formalism but is awarded by *inference*. This hooks up nicely to Bokulich's inferential realism. Recall that Bokulich articulates inferential realism in terms of 'what true things can we learn?' Science aims at and purports to tell us about the world, but the representations we use are more loosely connected to the world as it really is. The connection between representation and the world is often pragmatic and requires us to be careful about the ontological conclusions we draw. In reconstructions, physical insight is gained via inference insofar as reconstructions invert the order followed by standard interpretations. Reconstructions choose instead to begin with physical principles, encode those principles as axioms and then rederive the quantum formalism. Reconstructions have the characteristic feature that physical principles inform the mathematically formulated axioms. But the connection between the representation, e.g., the mathematical formulation of the continuity principle, and the world, e.g., as a physical time evolution, is looser and can be physically interpreted in different ways. That does not, however, preclude us from learning true things about the world. The result of the programme is multiple reconstructions with complementary views about what is special about quantum theory. Therefore, non-literally interpreted formulations allow us to say something as realists. However, the presence of multiple successful reconstructions will require us to be the kind of realist who is amenable to a pluralistic view of reconstructions.

4.2.3 Dissimilarities between Bokulich's examples and the reconstruction case

While the reconstruction case has many similarities to the LQH case, there are also some dissimilarities between them worth mentioning. However, these dissimilarities are either

methodological or motivational and do not affect my conclusions regarding the role of alternative formulations in the reconstruction programme.

One dissimilarity between Bokulich’s examples and reconstructions is that reconstructions do not employ the strategy that the quantum will resemble the classical as in the LQH case. Reconstructions are looking for different formulations of quantum theory using axiomatization as a core strategy. However, in the LQH case, the assumed resemblance to classicality is part of the justification for developing a *quantum* version of Lagrangian hydrodynamics. This is a difference in motivations. The development of LQH was motivated by the idea that because there were Eulerian and Lagrangian formulations of classical hydrodynamics, there must be Eulerian and Lagrangian formulations of quantum hydrodynamics. In other words, the quantum case would be analogous to the classical case under the framework of continuum mechanics. The assumption that the quantum would resemble the classical resulted in the new insights discovered in the LQH formulation. This motivation is not necessarily shared by the reconstruction programme, though in principle individual reconstructions could use formal analogy as a strategy at their disposal.

Another obvious difference between the reconstruction programme and Bokulich’s cases is that the axiomatic method is integral to the methodology of reconstructions whereas axiomatization is not used in the EQH versus LQH and LQH versus Ψ cases. As I argue in Chapter 3, the axiomatic method plays a central role in the reconstruction programme and is methodologically advantageous in many ways: the axiomatic method facilitates inference tracking, provides a semi-formal structure, and confirms the physical features and mathematical formulations of those physical features. The semi-formal structure of the axiomatic method also adds a degree of reliability because it facilitates the tracking of inferences. Another methodological distinction is the stipulation of a generalized framework that describes a landscape of theories. Within the framework, separate axiomatizations are devised which function as different formulations. Thus in the reconstruction programme the mathematical representations are stipulated at two ‘levels’, an overarching framework and the separate formulations nested under that framework. This differs from Bokulich’s examples as the varying formulations are centred at a ‘lower’ level in theory development, a level which is analogous to the separate axiomatizations level. Due to the presence of both a framework and axiomatization level in the reconstruction programme, the mathematical formulation is stipulated at two points, which also means that physical intuition can be integrated at both levels. Both the use of axiomatization and the designation of a landscape of theories act as helpful structure in reconstructions.

One important feature to note is how different physical intuitions motivate the set up of the different formulations in the hydrodynamic example. The Eulerian and Lagrangian formulations understand fluid flow in distinct ways (as properties at spatial locations which

change over time or as a set of ‘parcels’ with classical trajectory, respectively) which guides the choice of mathematical formulation. Recall that in the reconstruction programme, physical intuition can influence the choice of mathematical framework, as in Selby et al.’s case where category theory is chosen to represent the physical intuition of composition. Physical intuitions also influence the choice of axioms, for example, where Hardy chooses the axiom of continuity as being central to quantum theory. In the reconstruction programme, physical intuition has the opportunity to affect the choice of mathematics at two different places because mathematical formalism is stipulated at two different points. As we have seen in both Bokulich’s cases and reconstructions, the choice of mathematical representation (hydrodynamic versus wavefunction, or process versus state) influences the theoretical description that results from those representations. The choice of mathematical representation is highly important.

In sum, though there are a few distinctions between the reconstruction case and Bokulich’s cases, I conclude that they are distinctions which have no import on my argument for the integral role of alternative formulations in the reconstruction programme. In the next section I will argue that the drive for alternative formulations is similar between Bokulich and the reconstruction programme, with a specific focus on mathematical formulation rather than interpretation. I will also establish that this drive naturally lends itself to subverting standard interpretation, a value that is shared between reconstructionists and Bokulich. However, I argue that reconstructionists go beyond Bokulich by evolving their methodology to systematically generate alternative formulations.

4.3 The Drive for Alternative Formulations

One of the main motivators behind the exploration of alternative formulations in the reconstruction programme is the dissatisfaction with the Hilbert space formalism. In particular, reconstructionists oppose the acceptance of the abstract Hilbert space formalism at the outset and rather aim to derive a minimal version of the Hilbert space formalism as a consequence of the key physical principles. In this way, the mathematical formalism is derived as a consequence of the clear physical principles rather than being “merely heaped over and above the mathematical formalism of quantum mechanics” (Grinbaum, 2007, 388). The standard interpretative project accepts the Hilbert space formalism and reads an ontology off of that formalism, a process that the reconstruction programme takes to be problematic. This relates to how Hardy and Spekkens describe the ‘realists’, who want to connect formalism with the world ‘as it really is.’ Realists aim to discover the deeper reality underlying the equations of quantum mechanics, which will account for *why* the formalism is the

way it is (Hardy and Spekkens, 2010, 1). Hardy and Spekkens compare the realist camp with the operationalists, who typify quantum formalism by preparation, transformation, and measurement procedures. This can be construed as an instrumentalist approach where the theory “merely specifies what probabilities of outcomes will be observed when a given measurement follows a given preparation” (Hardy and Spekkens, 2010, 1). They conclude that what is needed is a shared approach as operationalism ‘bottoms out’ at complex entities and realists have yet to provide an interpretation “sufficiently compelling to achieve a scientific consensus” (Hardy and Spekkens, 2010, 2). Hardy and Spekkens want to push beyond an instrumental view to gain genuine physical insights. The goal is some variety of realism, though not by way of literally interpreting the Hilbert space formalism. The reconstruction programme is an attempt at this shared approach.

As I have shown throughout the chapter, Bokulich makes a strong case for the consideration of alternative formulations and against the standard approach to interpreting quantum mechanics in her exploration of LQH. One of Bokulich’s claims is that we can gain new insights via the consideration of alternative formulations that are obscured in standard formulations, and this claim is justified by the LQH case. The four lessons above serve to bolster the justifications for the consideration of alternative formulations. However, the crux of the 2019 paper is that an ontology should not be ascribed to a mathematical formulation as a literal representation of the world. This is the result of the elision between mathematical representation and things in the world, which is problematic both because it reifies the mathematics and also encourages us to postulate properties of the representational vehicle as properties of reality (Bokulich, 2020, 2). The antidote to these problems is the consideration of alternative formulations. It is the standard approach to interpreting quantum mechanics, i.e., the literal construal of a formalism, that is problematic for Bokulich.

As it relates to reconstructions, the hope of many physicists is that by considering alternative reconstructions we might discover insights into the nature of quantum theory via the derivation of the formalism from key physical principles. At the core of much of this—particularly as it relates to foundations of quantum theory—is the search for quantum gravity. This is one of the main motivations behind the reconstruction programme, and relates particularly to the consideration of alternative formulations: it is believed by some that quantum gravity will require a new mathematical framework that has as limits both general relativity and quantum theory as “in the past, when two less fundamental theories have been unified into a deeper, more fundamental theory, the unification has typically required an entirely new mathematical framework, motivated by conceptual insights from the two component theories” (Hardy and Spekkens, 2010, 4).

The methodology of the reconstruction programme is designed to generate these con-

ceptual insights for quantum theory. Rather than continuing to supply interpretations based on an accepted formalism, the new methodology offered by the reconstruction programme acts as a means of generating new formulations of quantum theory. To do so, reconstructionists invert the order of operations: start with physical principles, turn those principles into axioms, and then rederive the formalism. Multiple formulations of quantum theory give us insights into the fundamental principles of the theory. These insights are made possible because reconstructionists reject the outright assumption of the correctness of the Hilbert space formalism. By rederiving the formalism from physical principles, we confirm that those physical principles are fundamental to quantum theory.

As I have argued in (1), Bokulich gives reconstructionists another argument in favour of rejecting the standard interpretative project. In the presence of alternative formulations, our ability to literally interpret a single formulation is undermined. In Bokulich's cases, physicists are already looking at different formulations of quantum theory in their explorations of EQH and LQH. But as I contend, reconstructionists go beyond Bokulich to propose ways to systematically construct alternative formulations. In light of my argument (2), reconstructionists are getting the same benefits of considering alternative formulations that Bokulich describes via the new mechanism of generating alternative formulations. Further, the sort of thing we should consider reconstructions to be are non-literally interpreted formulations. This category of formulation gives rise to a moderate realism wherein we gain genuine physical insight without reading an ontology directly off of a mysterious formalism. Non-literally interpreted formulations acts as a middle ground between a full formulation which does not purport to give us constraints on the way the world really is and an interpreted formulation which aims to tell us a story about the world based on a literal construal of the accepted mathematical formalism.

Both the argument (1) in support of reconstructionist's rejection of the standard interpretative project and the argument (2) that there are advantages to non-literally interpreting formulations (2) serve as further motivation for the methodology of the programme. I contend that reconstructionists are evolving their methodology by systematizing the development of alternative formulations and by inverting the order of the standard interpretative project. Thus, I conclude that reconstructions represent a shift toward thinking about realism as an attitude of 'what true things can we learn?' *a la* Bokulich (2019). This attitudinal shift is geared toward the generation of new formulations of quantum theory.

Chapter 5

Answering the “Why the Quantum?” Question

Reconstructions of quantum theory are a novel research program in theoretical physics which aims to uncover the unique physical features of quantum theory via axiomatization. The programme subverts the importance of interpretation and instead focuses on developing an abstract generalization of the fundamental features of quantum mechanics, seeking to, as Goyal describes, “remove the interpretative bottleneck” (Goyal, 2022, 18) through the derivation of the quantum formalism. However, how we might philosophically understand reconstructions is a new area of research, as the programme itself is relatively young.

Reconstructions occur within a framework which designates a space of theories. For example, the Generalized Probability Theories (GPT) framework is an operational framework which is articulated in terms of preparation, measurement, and transformation procedures that could be done in a laboratory. The GPT framework describes a broad collection of theories including classical, quantum, non-quantum, and super-quantum theories. Separate reconstructions are based on different physical principles which researchers understand to be fundamental to quantum theory. Those generalized¹ physical principles are given mathematical representations in order to serve as axioms within the reconstruction. From these

¹Abstractness and generality are two distinct though interrelated concepts. Abstractness is a “measure of the degree to which physical interpretation is specified” (Fraser and Koberinski, 2024, 4). A more abstract concept, for example, will have less physically contentful descriptions. However, generality is a measure of the scope of applicability of a framework, model, or theory (Fraser and Koberinski, 2024, 4). Thus, the more general a concept, framework, etc. is, the more phenomena will be included.

axioms, the minimal mathematical and empirical results (the von Neumann postulates) will be derived if the reconstruction is successful, thereby reproducing quantum theory.

One of the primary motivations of the research programme is to answer Wheeler (1971)’s “Why the quantum?” question which asks why our world is described by quantum theory and not by some other theory. The hope is that by isolating specific physical principles within a reconstruction, we will be able to determine which principles are fundamental to quantum theory. If we determine which principles are fundamental to the theory, then we will have an explanation for why quantum theory is the way it is. Physicists are contending with the quantum question and proposing answers that they take to be explanatory, such as the principle of continuity (Hardy 2001) or tomographic locality (Koberinski and Müller 2018). They might think, for example, that the principle of continuity *explains* why quantum theory is the way it is as without continuity the von Neumann postulates cannot be derived.

Intuitively, reconstructionists feel like the answers that are prompted by the vague “why the quantum?” question are explanatory in some sense. But this intuition does not stop at the level of learning more about different ways of *formulating* quantum theory. There is a sentiment amongst some physicists that answering the “why the quantum?” question does not only give us knowledge of quantum *theory*, it also gives us knowledge of the *world*. But it is not obvious in what sense reconstructions are explanatory or how precisely we might understand the “why the quantum?” question. Further, it is not clear if we are learning about quantum theory as a representation or if we are *also* learning about the kind of world that is described by quantum theory.

Hence, there is work to be done by philosophers to investigate if the above intuitions can be justified and made precise. For example, what makes reconstructions explanatory, and what kind of explanation is being given? What precisely is the explanandum? The “why the quantum?” question, for example, is vaguely phrased and is often used to convey different meanings. What precise question is even being asked? Further, what is the explanans? What is *doing* the explaining and what kind of explanation is being supplied? Do we gain knowledge of the world from these explanations, or do we merely learn more about how quantum theory can be represented?

My purpose in this chapter is to determine whether there are activities prompted by these ‘quantum questions’ that provide legitimate explanations by philosophical standards. This is an exercise in applied philosophy, an activity where we look at real life situations—in this case, reconstructions of quantum theory—and aim to apply existing philosophical accounts to those situations. These philosophical accounts may be accounts of explanation, of understanding, or perhaps of ontology. However, if those existing philosophical accounts

are applied to the real life case study and are found wanting, new philosophical accounts must be developed. I will answer the above questions in this chapter.

I argue that reconstructions of quantum theory do provide answers to Wheeler (1971)’s “Why the quantum?” question and are therefore genuinely explanatory. However, this question can be interpreted in a variety of ways. In section 5.1 I offer a more precise version of the “why the quantum?” question, arguing that the question is really two questions in tandem. I claim that reconstructions answer the “why the quantum?” question because they satisfy Bokulich (2012)’s account of explanation. Bokulich offers an account of explanation that focuses on patterns of counterfactual dependence which correctly capture real underlying dynamics. These patterns are explanatory insofar as they allow us to see the difference in the explanandum if factors in the explanans are changed in a variety of ways (Bokulich, 2012, 730). I offer an account of explanation in the programme that is non-causal, non-interventionist, and utilizes “what-if-things-had-been-different” questions *à la* Woodward (2003). I provide this account of explanation while endorsing Bokulich’s eikonic² conception of explanation, which focuses on representations as being integral to how science explains. Under the eikonic conception, representations need not be veridical representations in order to be genuinely explanatory (Bokulich, 2018a). I show that the reconstruction programme successfully tracks patterns of counterfactual dependence within the space of theories articulated by a generalized framework. In reconstructions, these patterns of counterfactual dependence arise out of the derivation of the von Neumann postulates from a set of axioms, which situates a reconstruction in the space of quantum theories. I contend that when a reconstruction is determined to be in the space of quantum theories, a candidate answer for the quantum question is provided. Further, I argue that as axioms are changed, different worlds within the landscape of theories are picked out, highlighting the classical versus quantum distinction by isolating specific axioms that situate us within the quantum space of theories. I contend that patterns of counterfactual dependence reveal the relationships between regions in the theoretical landscape and these patterns enable us to discover the modal structure of theories.

However, my account of explanation departs from Bokulich’s in a couple of ways. I shall argue that Bokulich’s account of explanation remains too restrictive when applied to the case of reconstructions as not all successful explanations involve *models* in the ways she requires. In particular, I shall argue that theories can play the role in scientific explanations that Bokulich reserves for models. Further, Bokulich’s account is attending specifically to how fictionalist representations can be explanatory. However, reconstructions of quantum theory are not fictional representations. While my account of explanation differs from Bokulich’s account in these respects, it remains in the same spirit as Bokulich. Bokulich

²‘Eikonic’ is Bokulich’s stipulative term for the conception, not a common term.

emphasizes that her account defends an eikonic conception of explanation which allows for flexibility in how and what we represent. Reconstructions are an example of a case in which we need to be more flexible in how and what we represent.

Not only do we discover the modal structure of theories, but we also discover knowledge about the world via the physical principles. I argue that we gain physical insight into the structure of quantum theory via the generalized physical principles on which reconstructions are based. This physical insight bridges the gap between the structure of theories and the structure of the world. My claim is that the structure of theories represented by counterfactual dependencies reflects the structure of the world via the physical principles that are fundamental to quantum theory. This is because the physical principles can be interpreted as having physical content. This physical insight means that reconstructions are examples of what I call non-literally interpreted formulations, as I argued in Chapter 4. A non-literally interpreted formulation is a formulation of a theory that gives us some physical constraint on the way the world is without aspiring to be a literal interpretation. In contrast, a literal interpretation aims to tell a story about the way the world is based on the mathematical formalism of a theory. The formalism is understood to be a direct representation of the physical world. I argue that reconstructions are non-literally interpreted formulations because they are non-literal representations that nonetheless give us some physical constraints on the way the world must be. And these physical insights are examples of genuine explanation in the reconstruction programme. Successful explanation is the result of tracking correct inferences to counterfactual dependencies, inferences which can capture relations among regions in the landscape of theories without the representation being given a literal interpretation.

Furthermore, I argue that the methodological machinery that provides the necessary theoretical space for us to track these patterns of counterfactual dependence is the generalized framework of classical, quantum, and extra-quantum theories. This generalized framework enables us to probe what is distinctly quantum via non-quantum foil theories which serve as points of contrast. I contend that the generalized framework facilitates this contrastive exercise as an example of contrastive explanation. Thus, I argue that the generalized framework is integral to revealing the patterns of counterfactual dependence in reconstructions and is a significant part of their explanatory power. It is via the contrastive exercise that we are better able to articulate the distinction between classical, quantum, and extra-quantum theories, distinctions which sit at the heart of the explanandum.

In “Understanding quantum phenomena and quantum theories,” Duwell (2020) offers a modal account of understanding that can be applied to the reconstruction programme. This modal account focuses on understanding representations of theoretical content and phenomena. The primary distinction between Duwell (2020)’s account and my own is

that I claim that we learn about both quantum theory as a representation and we gain knowledge about conditions in the world that make quantum theory true. Duwell’s account does not require that the world is represented as it really is. In contrast, I suggest that reconstructions tell us something about the world as well as telling us something about the representations of quantum theory.

The chapter is organized as follows. I begin in section 5.1 by describing Wheeler’s “Why the quantum?” question, the answering of which is at the heart of the explanatory project for the programme. In section 5.2 I discuss Alisa Bokulich’s views on scientific explanation. I use both her eikonic conception of explanation and her account of explanation as the foundation for my account of explanation in quantum reconstructions. I claim that the reconstruction case is sufficiently similar to Bokulich (2018a)’s case of quantum dots. Next I explore a disagreement between Woodward (2003) and Bokulich regarding the acceptability of a noncausal, non-interventionist account of scientific explanation. The explanations supplied by reconstructions are best captured by a non-interventionist and noncausal account because the types of interventions that occur in reconstructions are outside of the scope of what we typically understand interventions to be, e.g., physical interventions on a system. Interventions in the reconstruction case include the changing of axioms within a reconstruction which highlights different areas in the space of theories. This is best understood through a modal lens as we change what is possible and not possible in the space of theories. And these types of interventions cannot be causal in the usual sense, e.g., as physical events causing other physical events. Further, a noncausal account is better suited to the reconstruction programme as the space of theories is a non-veridical representation. I argue that in a later paper, Woodward (2018) opens the door for the sort of explanation that Bokulich (2018a) and I are interested in.

In section 5.3 I illustrate how counterfactual dependencies play out in the reconstruction case. I discuss how we are able to gain physical insight into the structure of quantum theory from the generalized physical principles. I then entertain some objections to my account including a difference in the sort of counterfactuals presented in the reconstruction case. I show that the difference in cases do not undercut the claim that reconstructions provide a scientific explanation, particularly in the context of the eikonic conception of explanation. I conclude that reconstructions successfully provide an explanation of the “why the quantum?” question in spite of these differences. Finally, in section 5.4 I discuss Duwell (2020)’s modal account of understanding in the reconstruction programme, I explore some of the similarities amongst Duwell, Bokulich, and my accounts alongside some dissimilarities. Ultimately, I maintain that Duwell’s account of understanding is complementary to my account of explanation in reconstructions. However, my account is distinguished from Duwell’s in that I posit a connection to the world as it really is via the

physical interpretation of our fundamental principles.

5.1 The Quantum Question

Wheeler’s “why the quantum?” question is born from an interest in why quantum theory is the way it is and not some other way. The question is often broken down into two distinct questions: “why this?” and “why not that?” (Chiribella and Spekkens, 2016). It is these two questions that the reconstruction programme primarily aims to answer. However, the “why the quantum?” question by itself is not overly clear. Does the question ask us, for example, why we choose quantum theory over some other theory? Or does it perhaps ask us why quantum theory has some particular set of qualities?

Put more precisely, I claim that the “why this?” branch of the quantum question encourages us to investigate the justification of the quantum formalism. In other words, the question asks *why quantum theory is so perfectly described by the Hilbert space formalism*. Interest in the justification of the quantum formalism is a departure from the standard interpretative project where the formalism of quantum theory is not the subject of scrutiny. Instead, the goal of standard interpretative work is to determine what the accepted formalism of quantum theory says about reality. Though the formalism of quantum theory is generally agreed upon amongst physicists, the connection between the mathematical formulation of the theory and what that formulation tells us about the physical world remains unanswered. There is a sentiment that affixing a physical interpretation to a mysterious formalism cannot be satisfactory if, as Grinbaum asserts, “it is merely heaped over and above the mathematical formalism of quantum mechanics, instead of coming all the way along with the formalism as it rises in a derivation of the theory” (Grinbaum, 2007, 388). In the reconstruction programme, one solution to this is to place the physically significant concepts of quantum theory centrally at the outset of the derivation rather than attempting to see where those physical concepts might fit onto an already accepted formalism. Probing the foundations of the quantum formalism in the right way means we should figure out what physical principles will give rise to the formalism of quantum theory. Hence the foundation of the quantum formalism should be born from the physical principles that typify quantum theory. Ultimately, answering the “why this?” question will point us toward specific formulations of quantum theory that single out quantum theory from among all possible theories.

I also claim that the precise version of the “why not that?” branch of the quantum question is ones that asks *what specific features of quantum theory distinguish it from other*

theories. An answer to the “why this?” question will optimally reveal what abstract physical concept(s) are quantum, which will solidify the distinction between classical, quantum, and extra-quantum theories. Exploring the myriad of physical concepts researchers believe to be fundamental to quantum theory requires points of contrast in order to highlight which collections of axioms reveal quantum features. Non-quantum foil theories enable us to highlight what is distinctively quantum via contrast. Thus, answering the “why not that?” question depends on having alternative theories to facilitate this contrastive exercise. In the reconstruction programme, the methodological machinery that does this is the stipulation of the generalized framework of classical, quantum, and extra-quantum theories. Within the generalized framework we are able to investigate how the proposed physical principles—encoded as mathematical axioms—relate to one another, including the independence of axioms and the completeness of a set of axioms (Chiribella and Spekkens, 2016, 5).

The reconstruction programme is oriented around answering these two questions and the central ambition is to acquire a deeper understanding of quantum theory. In this chapter I argue that these questions are answered in the reconstruction programme and that the answers yield genuine explanations. Next, I explore Bokulich (2012)’s account of scientific explanation, which focuses on patterns of counterfactual dependence as explanatory.

5.2 Bokulich on Scientific Explanation

Bokulich is interested in establishing both a *conception* and *account* of scientific explanation. According to Bokulich, a conception of scientific explanation is a “view about what explanations are” (Bokulich, 2018a, 793). This is contrasted with an account of explanation, which is an assertion about how explanations work (Bokulich, 2018a, 793). The conception of explanation put forth by Bokulich is the “eikonic” (from the Greek *eikon*, meaning representation) conception of explanation, a representational conception of scientific explanation which holds that “not only are the *explanantia* representations of the relevant processes (entities, structures, etc.), but so too are the *explananda* representations of the phenomena to be explained” (Bokulich, 2016, 269).³ The conception is born out of the claim that scientists study phenomena via some simplified representation that is contextualized to the specific area of inquiry, research program, or explanatory project (Bokulich, 2018a, 801). The eikonic conception is contrasted with the dominant view of

³*Explananda* describes the phenomena to be explained while an *explanantia* describes the proposed explanation (Woodward and Ross, 2021).

scientific explanation, which is the ontic conception. The ontic conception describes explanations as the “concrete causes or mechanisms in the world themselves” (Bokulich, 2018a, 793). The eikonic conception permits multiple representations of the same phenomena in light of the notion that certain representations are better suited to a particular purpose than others (Bokulich, 2016, 269). The multitude of representations allows for phenomena to be represented “at one of many different possible levels of abstraction or generality” (Bokulich, 2016, 269).

Beyond the eikonic conception, Bokulich is committed to two types of explanatory pluralism. Bokulich holds the view that ‘type I’ explanatory pluralism is one wherein scientists use different types of explanations with respect to different phenomena (Bokulich, 2018b, 5). ‘Type II’ explanatory pluralism holds that “there can be more than one scientifically acceptable explanation of a single, given phenomenon” (Bokulich, 2018b, 5). It is the type II variety Bokulich maintains is more controversial, as it both presupposes type I, but also asserts that the different types of explanation we use can correctly be applied to the same phenomenon (Bokulich, 2018b, 6). Bokulich argues that type II explanatory pluralism is a legitimate extension of explanatory pluralism, i.e., that there are cases where a single given phenomenon is given multiple acceptable explanations.

Another feature of Bokulich’s view of scientific explanation is the commitment to model-based explanation. This feature is based on the view that “many explanations in science proceed by way of an idealized model, in terms of what I have called model-based explanation” (Bokulich, 2018b, 6). Bokulich’s account of model-based explanation draws on James Woodward’s view which articulates explanations as providing patterns of counterfactual dependence between the explanans and the explanandum (Bokulich, 2012, 730). Bokulich’s approach allows for a variety of types of model explanations, including causal, mechanistic, structural, etc., depending on the origin of the counterfactual dependence. According to Bokulich, a model explanation must satisfy three conditions (Bokulich, 2018b, 6). I have labelled these conditions for ease of reference later:

1. **Idealization condition:** The explanans in question must reference a scientific model which contains a degree of abstraction, idealization, or fictionalization.
2. **Counterfactual condition:** The model explains the explanandum by conveying how the model correctly captures the patterns of counterfactual dependence of the target system, which allows us to answer a range of ‘what-if-things-had-been-different’ questions.
3. **Justificatory condition:** The model must include a justificatory step which specifies the domain of applicability of the model and “where and to what extent the

model can be trusted as an adequate representation of the target for the purpose(s) in question” (Bokulich, 2012, 730).

Woodward’s notion of counterfactual dependence, which allows us to answer w-questions (or ‘what-if-things-had-been-different’ questions), is central to Bokulich’s overall view of scientific explanation. However, there are places where Bokulich disagrees with Woodward (2003), primarily in Woodward’s causal construal of explanation and the necessity and definition of interventions. Below, I discuss the resolution of these disagreements between Bokulich and Woodward, where a path toward a noncausal, non-interventionist account of explanation is established. This resolution is important for my purposes as I claim that reconstructions provide a similar type of explanation.

Woodward on Explanation

Woodward (2003)’s *Making Things Happen* offers an interventionist account of causal explanation, which requires that causal claims correctly describe patterns of counterfactual dependence. For Woodward, counterfactual dependence is analyzed in terms of ‘what-if-things-had-been-different’ questions, which means that explanation “must enable us to see what sort of difference it would have made for the explanandum if the factors cited in the explanans had been different in various possible ways” (Woodward, 2003, 11). ‘What-if-things-had-been-different’ questions are different questions that are posed to determine what the explanandum variable depends upon. In other words, successful explanation reveals to us how changes to variables in the explanans results in changes in the explanandum. Woodward calls these changes ‘interventions,’ which are understood as possible physical manipulations that change the system being intervened on (Woodward, 2018, 122). Hence this variety of causal explanation must allow us to track cause and effect relationships when we intervene on some system. Successful explanation is the result of conveying this cause and effect ‘what-if-things-had-been-different’ information.

Although Bokulich draws on Woodward (2003)’s suggestion that explanation is understood in terms of tracking patterns of counterfactual dependencies, Bokulich rejects that counterfactual dependencies must be construed via an interventionist lens. It is the interventionist requirement that restricts Woodward (2003)’s account to a causal account (Bokulich, 2012, 730). Bokulich generalizes the Woodwardian approach to encompass both causal and noncausal accounts of explanation (Bokulich, 2018b, 8). A noncausal explanation is one where the “explanatory model is decoupled from the different possible kinds of causal mechanisms that could realize the explanandum phenomenon, such that the explanans is not a representation (even an idealized one) of any causal process or mechanism”

(Bokulich, 2018b, 14). What this means is that a noncausal explanation will not aim to represent causal processes or mechanisms in its explanatory model. Similarly, interventions need not be understood as physical manipulations, which broadens what we conceive difference-making factors to be.

Later work by Woodward (2018) explores possible forms of noncausal scientific explanation. These forms of noncausal explanation result from loosening or changing the requirements of his 2003 Woodwardian framework, including the interventionist requirement. One possible form of a noncausal explanation is one where *w*-questions are answered but *not* by answering questions about what happens under interventions. Rather, in these noncausal cases *w*-questions are answered by substituting “claims about what would happen under different sorts of changes...e.g., changes that correspond to a purely mathematical or conceptual variations not having an interpretation in terms of a possible physical interventions” (Woodward, 2018, 124). Woodward offers the example of whether stable planetary orbits are possible for dimension $n \geq 4$, given assumptions about gravitational potential in an n -dimensional space, Newton’s laws of motion, and a conception of what the stability of planetary orbits consists of (Woodward, 2018, 124). We learn that stable planetary orbits are not possible for spaces of dimension $n \geq 4$. We also learn whether or not stable orbits are possible for different values of n . In this case an ‘intervention’ would consist of changing the dimensionality of space. But viewing an intervention in this way is “outside the bounds of useful or perhaps even intelligible application of the intervention concept” (Woodward, 2018, 124). Woodward maintains that patterns of dependence are tracked between the dimensionality of space and the possibility of stable orbits. Thus, the example is explanatory even if the intervention requirement is dropped (Woodward, 2018, 125). Further, if explanations without interventions can track dependency relations—which reveal that changes in the explanans depend upon changes to variables in the explanandum—then they can be construed as noncausal (Woodward, 2018, 126). *W*-questions will focus on variations in factors in the explanans and how those factors make a difference in whether or not the explanandum holds (Woodward, 2018, 134). Other types of noncausal explanation include mathematical explanation (Lange 2013) and minimal model explanations (Batterman and Rice 2014). Woodward concludes that non-interventionist, noncausal explanations are possible in those cases where there is empirical dependence or conceptual or mathematical relations between explanans and explanandum (Woodward, 2018, 145).

Like Bokulich, I borrow the notion of *w*-questions from Woodward as a means of describing what explanations consist of. And like Bokulich, I focus on a more permissive view of *w*-questions, which does not require interventions. Rather, *w*-questions may be thought of as “citing noncausal difference-making factors which have explanatory import” (Woodward, 2018, 144). I also take on board Bokulich’s eikonic conception of scientific

explanation, which focuses on how scientific explanations are largely about *representations*. The eikonic conception is oriented toward viewing scientific explanation in light of scientific practice to clarify what is really explained when scientists *do* science. From the eikonic conception we also emphasize the value and significance of multiple representations that are contextualized to an area of inquiry and can be conveyed at different levels of abstraction. This is directly applicable to the reconstruction programme. In Chapter 4 I argue that each reconstruction should be understood as a distinct formulation or representation of quantum theory. These representations will encode different physical principles—which are generalized concepts—as integral to quantum theory. Different reconstructions will highlight various features that answer the “why the quantum” question. Thus, there will be multiple candidate explanans for the “why the quantum” question. Not only are separate reconstructions distinct representations of quantum theory, but they also do not have to be veridical representations in order to be explanatory. As we will see below, Bokulich’s quantum dot example cites classical periodic orbits as a source of explanation, in spite of the fact that classical periodic orbits are *fictional* in a quantum context. Being fictional, they cannot be causal, as fictional representations are not the kinds of things that can participate in causal relationships through physical interventions.

Bokulich characterizes her account of explanation as a *model-based* account. But as I will expand on later, reconstructions are not models.⁴ Models are an idealized representation of a target system and must reliably reproduce certain features at the basis of our inquiry. Typically, models represent *phenomena*. But reconstructions are representations of a *theory* based on generalized physical principles. I maintain that we can still take lessons from Bokulich in spite of this difference. In fact, I contend that we can generalize Bokulich’s account to accommodate theories as well as models as facilitators of explanation. What is most significant for my purposes is the proposal that tracking patterns of counterfactual dependence yields genuine explanation. As Bokulich stipulates, the model must capture the significant patterns of counterfactual dependence of the target system. Capturing these patterns of counterfactual dependence means that the model must be able to “reproduce” the important features of the explanandum phenomenon. The model must also be able to show us how the target system would change in response to changing the values of parameters within the model. Bokulich’s account of model explanation below is more narrowly focused on distinguishing explanatory from nonexplanatory *fictions*. However, her more general account of scientific explanation, specifically how patterns of counterfactual dependence are explanatory, supports my argument that reconstructions of

⁴There is some nuance to be added to this assertion. It is possible that reconstructions could be models in the Tarskian sense of instantiations of theory or possibly in Giere’s less formal sense of models. This connection will be pursued in future work. I am grateful to a committee member for this comment.

quantum theory provide genuine explanation. As I show in the reconstruction case, these patterns of counterfactual dependence are captured within the generalized framework, in response to the changing of axioms within a reconstruction.

Electron trajectories in quantum dots

Bokulich offers the example of classical periodic orbits in quantum dots to argue that we can have a fictional representation that is nonetheless genuinely explanatory. Importantly, classical periodic orbits are regarded as fictional from the perspective of our current best theory, which is a quantum theory. Quantum dots are devices that contain electrons within a two-dimensional region (Bokulich, 2012, 727). These are quantum-mechanical systems with quantized energy levels which “behave as a single quantum phase coherent-unit for which interference effects are important” (Bokulich, 2012, 727). When coupled weakly to external leads, it is possible for electrons to tunnel into the dot which is typically blocked by the Coulomb repulsion of the electron present in the dot. The result is that no current will flow. However, changing the voltage will compensate for the Coulomb repulsion resulting in peaks in conductance with unusual features (Bokulich, 2012, 727). What we expect in a classical counterpart to a quantum dot is that the motion of an electron will be chaotic. In this case, the wave function that represents the dot exhibits a certain randomness. Based on this randomness a statistical theory portrays these wave functions as random and uncorrelated with each other (Bokulich, 2012, 727).

However, recent experiments showed large unexpected correlations between conductance peaks and “a consensus started to emerge that these puzzling features of the Coulomb-blockade conductance peaks in a quantum dot were explained by the properties of the particular classical periodic orbits in the dot” (Bokulich, 2012, 728). A classical periodic orbit is a description of the quantum dot which obeys classical mechanics. If we assume that electrons in the quantum dot obey classical mechanics, there will be a small number of periodic orbits in which the electron will retrace its path. These unstable orbits “dominate the quantum dynamics” (Bokulich, 2012, 728). Hence, integrating classical periodic electron orbits provided an explanation for the unexpected correlations between conductance peaks. However, “despite the striking success of this classical-orbit explanation of the many details of the conductance properties of quantum dots, a troubling feature of this explanation is there are no such classical periodic electron orbits in the quantum dot—they are fictions!” (Bokulich, 2012, 728). There is no classical periodic orbit in a quantum system because the system is described by quantum mechanics, not by classical mechanics. Bokulich argues that this is an example of an explanatory fiction (Bokulich, 2012, 728).

Using the quantum dot case, Bokulich considers the Idealization condition (1) to be

satisfied insofar as the conductance properties of quantum dots are explained via classical periodic orbits, orbits which are a fictional representation of the behaviour of electrons (Bokulich, 2012, 730). The Counterfactual condition (2) requires that the pattern of counterfactual dependence of the explanandum phenomenon is captured in the semiclassical model. In the quantum dot case, the patterns of counterfactual dependence are captured in an equation that shows how the classical periodic orbits modulate the heights of the conductance peaks (Bokulich, 2012, 730). The unexpected features of the quantum dot conductance peaks are shown to be dependent on certain features of the classical periodic orbits model. Further, Bokulich maintains that the semiclassical model allows physicists to answer a variety of what-if-things-had-been-different questions (Bokulich, 2012, 731). For example, physicists can now understand how the interference around each periodic orbit oscillates—which accounts for the peaks in conductance—as system parameters vary, including variations in the magnetic field or in the number of electrons in the dot (Bokulich, 2012, 731). Ultimately, Bokulich contends that “the semiclassical model is one of the best ‘windows’ that they [physicists] have into the quantum dynamics of the dot” (Bokulich, 2012, 731). And semiclassical models are not merely phenomenological models useful only for making predictions. Rather, physicists understand semiclassical models to be genuinely explanatory, giving us insight into the structure of the wave function (Bokulich, 2012, 731). The formal structure of the classical periodic orbits mirrors the structure of the wavefunction of the quantum dot. The semiclassical models supply answers to ‘what-if-things-had-been-different’ questions, hence the Counterfactual condition (2) is satisfied.

Thus, we have a case where a fictional model is nonetheless able to support inferences that track the counterfactual dependencies that we see in the world. Bokulich contends that models need not be literal representations of the target system in order to facilitate inferences that track counterfactual dependencies, counterfactual dependencies which are genuinely explanatory. To serve as a point of contrast, Bokulich offers an example of a nonexplanatory scientific representation.

Ptolemaic epicycles

Bokulich’s example of a nonexplanatory representation is the epicycles of Ptolemaic astronomy. Ptolemaic astronomy is a geocentric model with the sun, moon, stars, etc., orbiting the earth in perfect circular orbits. Epicycles, which are smaller circular orbits centred on a point on the main circular orbit around earth, were introduced in order to account for the retrograde motion of the planets. The inclusion of epicycles successfully predicted the retrograde motion of the planets, which had the appearance of a planet moving backward

across the sky. Although Ptolemaic astronomy allows us to make certain precise predictions about the locations of planets, Bokulich maintains that we do not want to consider Ptolemaic epicycles as an explanatory fiction (Bokulich, 2012, 732). On her view, the Ptolemaic epicycle example is best understood as a merely phenomenological model and prediction does not qualify as explanation (Bokulich, 2012, 732). This is because Ptolemaic epicycles do not capture the real underlying dynamics of the planets and are not relevant in explaining the retrograde motion of planets relative to the current context of science (Bokulich, 2012, 734).

What then will be the crucial difference in explanatory and nonexplanatory representations? According to Bokulich, it is not only the patterns of counterfactual dependence which enable us to answer what-if-things-had-been-different questions. The scope of what-if-things-had-been-different questions will typically be more limited in a phenomenological model, but Bokulich does not believe scope alone can cement the distinction (Bokulich, 2012, 733). Rather, what this suggests is that “the difference between explanatory and nonexplanatory models is determined by something like a contextual relevance relation set by the current state of scientific knowledge” (Bokulich, 2012, 733). A contextual relevance relation will determine which items (entities, properties, processes, etc.) we include in our explanatory store relative to the current state of scientific knowledge. The crucial distinction between explanatory and nonexplanatory representations, then, must be that explanatory representations are only those that adequately reflect the relevant features of the world. Adequacy here is not merely empirical adequacy, however. Bokulich argues that “what is to count as an adequate fictional representation is something that has to be negotiated by the relevant scientific community and will depend on the details of the particular science, the nature of the target system, and the purposes for which the scientists are deploying the model” (Bokulich, 2012, 734). Hence, the determination of the adequacy of a fictional representation is a result of the third condition in Bokulich’s model explanation account, i.e., the justificatory step.

Applied to the two cases, Bokulich argues that “given the relevance relation set by contemporary science, epicycles are irrelevant to the explanation of retrograde motion” (Bokulich, 2012, 735). The ‘actual structure’ of the solar system that is relevant for these explanatory purposes is the heliocentric model of the universe, rather than the geocentric Ptolemaic model. However, Bokulich maintains that classical periodic orbits are explanatory fictions because their fictional representation captures real features of the dynamics of quantum dots (Bokulich, 2012, 735). The quantum dynamics that are captured include how the modulation of peaks is determined by the period of the classical orbit, that the frequency of oscillations is proportional to the area covered by the orbit, and that the peak distribution is determined by the Lyapunov exponent of the classical orbit (Bokulich, 2012,

729). By using the fictional electron orbits—the use of which is constrained by the rules of semiclassical mechanics—we are able to gain genuine physical insight into the true nature of electron dynamics, hence the Justificatory condition (3) is satisfied (Bokulich, 2012, 735). Bokulich concludes that the fictional classical periodic orbits are explanatory “not because they are a literally true description of the electron’s behavior—but because they are a fictional representation that is able to generate genuine knowledge of the true underlying quantum dynamics” (Bokulich, 2012, 735). This account integrates what Bokulich calls a moderate realism, insofar as what distinguishes fictions as genuinely explanatory is that they generate real physical insight (Bokulich, 2012, 736).

On the matter of interventions, the quantum dot case does include interventions but not exactly in the Woodwardian (2003) sense. For example, the number of electrons can be changed by controlling gate voltage, which is a physical intervention (Bokulich, 2012, 727). However, this intervention is on the quantum system, i.e., the explanandum. The intervention does not occur within the semiclassical model, i.e., the explanans because it is not possible to intervene on non-existent fictions. So even though we have interventions in the quantum dot case, they are not interventions on the explanans, which makes Woodward (2003)’s account of interventions inapplicable. However, if the intervention requirement is dropped as per Woodward (2018), Woodward’s account becomes applicable to this case. We can also have noncausal explanation, as in the quantum dot case above, which is allowable in Woodward (2003, 2018). As Bokulich maintains, classical periodic orbits cannot enter into causal relationships as they do not exist in quantum systems (Bokulich, 2018b, 7). Explanation arises from the discovery that features in the quantum dot conductance peaks depend upon certain features in the classical periodic model. The semiclassical model has the relevant formal structure (periodic orbits) that functions as an explanatory representation by capturing patterns of counterfactual dependence in quantum dots.

Next, I apply Bokulich’s account of explanation to the reconstruction programme, arguing that reconstructions satisfy the requirement of tracking counterfactual dependencies and are thus genuinely explanatory.

5.3 Reconstructions

For my purposes, the most important aspect of Bokulich’s account of explanation is the notion of patterns of counterfactual dependence. I argue that reconstructions of quantum theory satisfy Bokulich’s main requirement of correctly tracking counterfactual dependencies. Recall that in the reconstruction case, the explanatory project is one that seeks to answer the “why the quantum?” question, which asks both “why this?” and “why not

that?” Thus in the context of reconstructions, an example of a ‘what-if-things-had-been-different question’ would be “what region will be picked out if I removed axiom 3?” or “if I remove axiom 7, will a quantum region in the space of theories be picked out?” I contend that reconstructions provide an answer to these types of questions when a reconstruction has successfully reproduced the von Neumann postulates, thereby orienting us in the quantum space of theories.

In order to illustrate my account more precisely, recall Hardy (2001)’s “Quantum Theory from Five Reasonable Axioms.” Hardy chooses five axioms he takes to typify quantum theory including:

1. Probabilities
2. Simplicity
3. Subspaces
4. Composite systems
5. Continuity

His reconstruction is in the Generalized Probability Theory (GPT) Framework, which is articulated in terms of preparation, measurement, and transformation procedures. Hardy successfully derives the von Neumann postulates, thereby situating this reconstruction within the space of quantum theories in the superstructure of Generalized Probability Theories. Since derivation of the von Neumann postulates is what constitutes being within the space of quantum theories, this is one way of answering the “why the quantum” question. In other words, quantum theory is the way it is because it is based on the above physical principles. As Hardy also shows, it is the axiom of continuity that is the uniquely quantum axiom.⁵ Removal of axiom five results in the von Neumann postulates no longer being reproduced.

The collection of the above axioms without continuity fails to situate us in a quantum world. This is another answer to a “w-question”: ‘what type of world is picked out when the continuity axiom is dropped?’ The removal of the continuity axiom orients us instead to either a non-quantum, quantum, or classical world which does not answer our “why the

⁵Not every reconstruction will specify one axiom that is ‘uniquely’ quantum, i.e., the axiom that determines if the region of space is classical, quantum, or non-quantum. Some reconstructions are content to identify a set of axioms that jointly entail the VNP, while others further specify which axiom, if removed, renders the reconstruction as non-quantum.

quantum” question. It does however answer the “why not that” question. If we remove the continuity axiom then we do not reproduce the VNP in the GPT framework. Thus, the removal of continuity from the collection of physical principles in the axiom set fails to orient us in a quantum space in the space of theories. Within the superstructure of the GPT, these are but two examples of the patterns of counterfactual dependencies within the reconstruction programme. It reveals that changes to the set of axioms, i.e., removing continuity affects where we are oriented in the space of theories, i.e., not being within the the quantum region. My claim is that this means the continuity axiom is physically revealing.

I invoke counterfactuals as a means of describing change in the theoretical space. Patterns of counterfactual dependence are a way to describe what happens when we change axioms within a reconstruction, which will change the region that is highlighted in the space of theories. In the reconstruction case, counterfactuals refer to the different worlds picked out by different sets of axioms. The patterns of counterfactual dependence allow us to explore the properties of theories within the theory space. For example, we can look at a specific reconstruction to see if the theory space picked out by the reconstruction supports entanglement.⁶ In doing so we can clarify how much or how little properties such as entanglement depend on certain axioms. We gain information from these types of inferences. Suppose we have reconstruction R_1 , which contains axioms a, b, c, d . We want to track the property of entanglement. We discover that the axiom set does not produce entanglement, so we add axiom e . Adding axiom e is a ‘what-if-things-had-been’ different question: ‘what if we added axiom e ’? With the addition of axiom e , we can now produce entanglement. We have thus learned that entanglement depends on axiom e . In this way, we are able to track the changes in the space of theories as a result of applying or removing axioms in the axiom set.

5.3.1 Physical principles, physical content

In the quantum dot case, we gain genuine physical insight via the fictional semiclassical model. According to Bokulich, the key distinction between true explanations, like those for quantum dots, and non-explanations, such as those involving epicycles, lies in offering genuine physical insights. This raises the natural question of whether something that can reasonably be called genuine physical insights exists in the reconstruction case. My claim is that the abstract physical principles have physical content which expresses a physical

⁶I thank Philip Goyal for the example of entanglement in conversations at *The Quantum Reconstruction Program and Beyond* conference.

constraint on the way the world is. Recall that because physical principles are used to give physical interpretations of axioms, the axioms are not purely mathematical. What this means is that the space of theories is not just a space of mathematical possibilities. Mathematics is used to formulate the axioms but those axioms are physically interpreted.

Let us take the continuity axiom as an example. Continuity requires that there exists a continuous reversible transformation on a system between any two pure states of that system (Hardy, 2001, 2). In “Quantum theory as a principle theory: insights from an information–theoretic reconstruction” Koberinski and Müller note that continuous reversibility is well–motivated in quantum theory via experience in the laboratory. Koberinski and Müller isolate continuous reversibility as a potential physical principle that is a *genuinely physical* kernel of quantum reality (Koberinski and Müller, 2018a, 11). They claim that continuous reversible transformations can be interpreted as either a computational process or as physical time evolutions and are integral in highlighting the quantum region in the space of theories (Koberinski and Müller, 2018a, 12). As we recall from Chapter 2, transformations are any operation that we can apply to a particle, including switching on a magnetic field or implementing a gate in computation. A qubit is a representation of the quantum state space which can be visualized as a Bloch ball (a ball in three dimensions). Rotations of the three dimensional ball correspond to unitary transformations, which map an initial state ρ_i to a final state $\rho_f = U\rho_i U^\dagger$ (Koberinski and Müller, 2018a, 7). Continuity is a result of the fact that a three–dimensional ball can be continuously rotated, as the qubit has infinitely many pure states.

Interpreting continuity as representing either the physical implementation of a computational process or time evolution is physically meaningful. These are two ways we can understand an abstract principle to represent the physical world. Therefore, my claim is that we have isolated a feature of quantum theory that reveals a physical constraint on the structure of quantum theory. If the theory is to be quantum theory rather than a classical or non–quantum theory, it must have these types of transformations. And these types of transformations represent physical transformations that systems in the world can undergo. By isolating a feature that is central to deriving quantum theory we provide an answer to the “why the quantum” question. And describing abstract physical principles as a key part of the representation enables us to, for example, translate the continuity axiom into what physical content it might have.

5.3.2 Knowledge of theory versus knowledge of the world

At this juncture, I must draw an important distinction between counterfactuals in terms of theories and the world as described by those theories. The generalized framework describes

a space of theories. The counterfactuals—which are essentially if–then statements—within that space show us all of the different ways that the space of theories could be. When we change axioms, i.e., when we change the principles we take to be true, we change what is possible and not possible in the space of theories. Probing the space of theories enables us to provide answers to the question of ‘what if the theory had been different?’ The result is that we learn about quantum theory as a representation and the relationships between different theories within the space of theories. Within the space of theories we will learn the structure of quantum theory when a reconstruction successfully derives the VNP. Thus, we discover some of the structure of the space of theories within the representation.

But the quantum question does not just ask us why quantum theory is the way it is as a *representation*. A representation does not necessarily tell us anything about the world. The quantum question is an inquiry into what is special about the quantum world held up against all the other ways the *world* might have been. In other words, we want to learn about the conditions in the world that make quantum theory true alongside the way representations represent. This is particularly important if we have realist predilections. Reconstructions illuminate the structure of quantum theory within the space of theories. But it is a further claim to assert that the structure of quantum theory in the space of theories reflects the structure of the world.

Put plainly, there is a distinction between our knowledge of theories and our knowledge of the actual world. In the reconstruction case, we can envision the space of theories as a modal space of theories. The space of theories in reconstructions is a case in which possible theories represent the actual world or possible worlds which are not actual because the laws of physics are different. What I am *not* advocating for is a multiverse account where all possible worlds are actual. The patterns of counterfactual dependence that we track in the space of theories give us modal information about the space of theories when we change axioms within a reconstruction. As we change out axioms, we secure modal knowledge about the theory space, i.e., modal knowledge of theories. The theories that are ruled out in the space of theories are those that are empirically inadequate for describing quantum theory, e.g., classical theories. But here is where more nuance is required. Axioms are part of the theory. However, I have argued above with the continuity example that axioms of quantum theory *also* represent constraints on physical processes in the *actual* world. Hence, what I am arguing for is that the modal knowledge we gain about the space of theories teaches us something about the kind of world that is described by quantum theory.⁷

⁷The debate between modal fictionalism and modal realism is not relevant for my purposes as it is not the main debate to which I am contributing. The debate is not significant here because the metaphysical question is not relevant to the philosophy of science debate, which is centred on epistemology.

That ‘something’ that we learn about the actual world is the physical principles that are postulated at the beginning of the reconstruction. While there is immense value in engaging with the representations of quantum theory, I claim that reconstructionists learn knowledge about the physical world via the physical principles they discover are integral to the theory. For example, the physical principles may describe the possible ways in which a quantum system can change its state over time, i.e., *evolve* and ways in which it is not possible for a quantum system to change its state over time. Within the space of theories, we learn which physical principles hold in our world and which do not, i.e., which physical principles are empirically adequate for describing quantum theory. The physical insights, which have physical content, enable us to navigate the space of theories. As we can recall in Bokulich’s quantum dot case, classical periodic orbits provide genuine physical insight into the world because they track the real world quantum dynamics. In the reconstruction case, the counterfactual dependencies track the physical content in the axioms which represent possible physical processes in the world. In this way, we get some kind of knowledge of the world.

The continuity axiom, for example, represents the physical implementation of a computational process or time evolution. The types of transformations in the continuity example represent physical transformations that systems in the world can undergo. The continuity example illustrates how we learn about the modal space of theories, for example, by highlighting differences between quantum and classical theories. However, because we physically interpret the principle of continuity as representing some physical process in the world, we learn that the theory that has continuity as an axiom is empirically adequate in our world.

Further, if the physical principles have physical content then this means that modifying, adding, or dropping a physical principle in an axiom set will reveal physical distinctions in the space of theories. Removing continuous reversibility and its physical interpretation results in a failure to reproduce the von Neumann postulates. This shows us both that quantum theory requires continuous reversibility but also that some non-quantum regions do not have this physical feature. Since the physical principles have physical content, we can track how changing these physical principles in the axiom set results in changes in the space of theories, meaning we see the changes in how to carve up the space of theories. In this way, reconstructions give us information on the structure of the space of theories. As Bokulich maintains, representations must capture real entities, processes, or structures in the world in order to be explanatory (Bokulich, 2012, 734). My claim is that the physical content in the principles enables us to learn about the structure of the *world*.

Admittedly, the sense in which we shift from securing modal knowledge about the theory space to the assertion that modal knowledge teaches us about our actual world is

qualified. It is clear that we are learning *something* about the world but it is not clear as to what exactly we learn. The continuity axiom is a case in point. The continuity axiom does not give us an explicit description of features in the world. Furthermore, in Chapter 4 I have argued that reconstructions should be considered non-literal representations, meaning that we do not take the mathematical formalism to directly describe ontological features of the world. The non-literal nature of the representation makes it challenging to provide a neat physical description of the aspects of the world being represented.⁸ But the central focus of the reconstruction programme is to load in physical content at the outset and the continuity axiom is meant to encode a fundamental physical principle. Reconstructionists hope that they are learning something true about which physical principles hold in our world. The point of starting with physical principles is to determine what is physically possible in the world. It is with a charitable interpretation of the physical principles as representing physical processes in the world, we gain knowledge about the world as it really is.

5.3.3 Comparing the quantum dot and reconstruction case

The quantum dot case and the reconstruction case will provide different kinds of explanations. I am focusing on a representation of theories and a framework which encompasses a broad space of theories. Bokulich’s quantum dot case is focused on one model for a particular type of phenomenon. Both cases provide noncausal explanations and neither have interventions in the strict Woodwardian (2003) sense. In reconstructions, what we might consider ‘interventions’ would occur when we change, remove, or add an axiom in the axiom set. The effect we see is which region in the space of theories is highlighted. This sense of intervention is different than Woodward (2003) which requires a physical manipulation of some kind. It is also different than Bokulich’s quantum dot case, where there are physical manipulations on the explanandum but there are no interventions on the explanans. Changing an axiom in the axiom set is certainly not a physical intervention. In the reconstruction case we may be outside of the bounds of what we might usefully consider an intervention. However, this is unproblematic as Woodward (2018) drops the intervention requirement, maintaining that interventions are not necessary for scientific explanation.

The difference in how interventions are understood in the quantum dot and reconstruction case result in different types of counterfactual questions. Counterfactuals in the

⁸In contrast to, say, Bohmian mechanics. As I expand on in section 5.3.3, Bohmian mechanics takes the formalism of the theory quite literally, granting ontological status to the wavefunction and trajectories of quantum particles. This is an example of a full interpretation.

quantum dot case capture the dependencies between features of the semiclassical models and features in quantum dot conductance. Counterfactuals in reconstructions refer to which region in the space of theories is highlighted, which is dependent on the posited axioms. In this way, the reconstruction case requires us to be more permissive on what counts as a *w*-question and what counts as an explanatory answer. *W*-questions will not be articulated in terms of interventions but rather in terms of difference-making changes, e.g., the removal of continuity. Explanatory answers will show us whether or not the VNP are derivable, indicating whether or not we are in the quantum region. And being in the quantum region will give us an answer for the ‘why the quantum’ question. But we still get explanation out of both. The physical constraints we learn are such because of the physical principles that are fundamental to the theory as without certain physical principles, we do not get quantum theory. This is an example of a counterfactual.

An objection could be raised that the different types of counterfactual dependencies in the quantum dot and reconstruction cases undercut the explanation in the reconstruction case. In the quantum dot case, counterfactuals latch on to specific properties in the target system (such as the number of electrons) and how the explanandum changes in response. However, in the case of reconstructions counterfactuals refer not to different properties possessed within a target system but instead refer to different worlds picked out by distinct sets of axioms. In the QDot case, changes to variables in the model are explained by classical periodic orbits. The semiclassical model correctly captures the quantum dynamics and the patterns of counterfactual dependence. The quantum dot case is at a much lower level of generality as it is dealing with specific properties, entities etc. at the model level.

This points to a difference between Bokulich’s account and my own. Bokulich’s account refers to models where reconstructions refer to theories. Although the reconstruction case is about theories rather than models, I argue that we nonetheless obtain explanations. This is a generalization of Bokulich’s account of explanation, which is restricted to models. Models are meant to represent small classes of phenomena and must reproduce important features of the target system being represented (Frigg and Hartmann, 2020). Theories capture the essential generalizations of classes of phenomena without specifying phenomenon (Frigg and Hartmann, 2020). Recall that the generalized framework of a reconstruction designates a classical, quantum, and extra-quantum space of theories. Via a set of axioms, a reconstruction picks out a particular world in the space of theories that could be classical, quantum, or extra-quantum. A successful reconstruction will stay within the space of the quantum theories, represented by the derivation of the von Neumann postulates. In the reconstruction case, counterfactuals will not be about the different properties that a target system could possess but about the different worlds that are picked out by a reconstruction. This distinction points to the different level of generality between Bokulich’s quantum dot

case and the reconstruction programme. The semiclassical model provides the representation in the quantum dot example, whereas the representation in the reconstruction case is a framework which describes a space of *theories*. The patterns of counterfactual dependence will give us information on how to navigate the space of theories and the framework provides the representation. What we see is that reconstructions are situated at a higher level of generality in a framework–theory–model schema. This higher level accounts for the difference in what counterfactual dependency looks like in the reconstruction versus quantum dot case. The GPT framework correctly represents these relations between regions in the space of theories. Thus, the framework generates genuine knowledge of the true underlying relations between regions in the space of theories. This difference in the level of generality is where the reconstruction case and the quantum dot case are dissimilar. However, this dissimilarity does not undercut the explanation. The fictional semiclassical model provides an explanation of the unexpected correlations we see in the quantum dot case. In the reconstruction case, changing axioms within a reconstruction highlights a region in the space of theories which explains how those regions are distinct. The level of generality does not detract from the explanation. Thus, I argue that theories can fulfill the role in scientific explanations that Bokulich assigns to models.

The legitimacy of the shift from models to theories in an account of explanation is supported further by the eikonic conception. Recall that the eikonic conception is a permissive view on the types of representations we might have in the practice of science. A plurality of representations is not a weakness but a strength, enabling us to learn more about a phenomenon or entity than we would with a single representation (Bokulich, 2018a, 8). In the reconstruction case, representations will include a framework and abstract physical principles. This representation is appropriate for the goals of inquiry as we try to figure out the abstract structure of quantum theory and which physical principles are fundamental to the theory. This type of investigation requires a broad representation which includes spaces of theories. The eikonic conception allows us a wide array of representations. Because of this, we are liberated from adhering to a singular, definitive representation. What we need is access to a range of representations that will give us insights into different aspects of quantum theory, i.e., alternative successful reconstructions. In that, the reconstruction programme has already succeeded.

However, as Bokulich has argued, it is not just that we have captured counterfactual dependencies which reveals to us whether or not we have an adequate representation of the target system. In the quantum dot case, the semiclassical model representation captures real features of the underlying quantum *dynamics*, but it is the contextual relevance relation that distinguishes explanatory from nonexplanatory models. This justification step is satisfied in order to justify the explanatory use of a fictional description to capture real

dynamics. Bokulich contends that some fictional models can be explanatory, as in the quantum dot versus Ptolemaic epicycles example. In brief, Bokulich maintains that a good fictional representation will be a representation of the true mechanism and will capture “the relevant structural features or patterns of counterfactual dependence—then it can (when used with due diligence) stand in for the true mechanism in the scientist’s reasoning and explanations, even though the ontology of the fictional mechanism is wrong” (Bokulich, 2016, 269). Scientists may choose to use a fictional representation because there are cases where the fictional representation better suits an explanatory project and the sorts of insights that are desired (Bokulich, 2016, 270).

This point relates to reconstructions insofar as reconstructionists use different representations that suit their explanatory projects. The use of the GPT framework, for example, allows us to probe the boundaries of what is quantum and what we might consider to be not-quantum. By stipulating a framework that encompasses such a broad space of theories, reconstructionists are able to get a better sense of where those distinctions may lie. And by keeping reconstructions within a framework, we are able to focus on the physical principles of quantum theory without the baggage of other information, such as specific phenomena. The representations of both the framework and physical principles fit the explanatory project of focusing on the formalism of quantum theory. The representations also suit the explanatory project of answering the quantum question.

For my purposes, one of the crucial ideas from Bokulich’s work is the notion that we can get genuine scientific explanations out of models or theories without conceiving of that model or theory as providing a literal, veridical representation of a target system. Reconstructions do not provide a fictional representation of quantum theory like the semi-classical model does in Bokulich’s case. Reconstructions provide a representation that is idealized and non-literal, but not fictional. A point of contrast would be an interpretation, which is understood to be a veridical representation of phenomenon. An example of an interpretation is de Broglie-Bohm’s (dBB) theory, which grants ontological status to the wavefunction and trajectories of quantum particles. The theory includes descriptions of concrete entities (particles described by their position) and processes (how particle positions change over time). The Bohmian explanation of the double-slit experiment describes a particle *guided by* a wave, resolving the dilemma of the wave-particle paradox. The *real* wave causes the interference we observe in the double-slit experiment. In this case, the mathematical representation is taken to be a literal, veridical representation of the world. The explanation includes specific entities and processes to describe and explain a particular anomaly we see in experiment. The Bohmian interpretation also provides a causal explanation, where the wave function *causes* the inference and where the representation is meant to stand for causal entities and processes. Contrast this with the physical princi-

ples that inform the axioms in a reconstruction. Physical principles such as the continuity axiom are abstract generalizations of the key features of quantum mechanics. Concrete entities or processes such as a specific system, particular pure states, or transformations, are not described. However, we still get physical content from these principles as in the case of continuity, i.e., as a constraint on the kinds of transformations required by the theory understood as the physical implementation of a computational process or physical time evolution. As for the mathematical representation, the space of theories will not be taken as a literal, veridical representation of the world. An ontology will not be directly read off of the formalism which represents the space of theories, e.g., the GPT framework. Reconstructions are not interpreted in the way that de Broglie–Bohm’s theory is interpreted. The Bohmian interpretation makes ontic claims about specific entities, processes in the theory. In reconstructions, the representation is at its core a generalization of physical principles or phenomena. What is explained is whether a collection of axioms reproduces the VNP. The account of explanation will be neither causal nor interventionist. We are, however, learning about physical constraints on the world based on principles which explain why quantum theory is the way it is.

Recall the concept of a non-literally interpreted formulation, which I developed in Chapter 4. My claim is that reconstructions are an example of a non-literally interpreted formulation, which is a formulation of a theory that contains some physical content without aspiring to be a literal interpretation like the Bohmian interpretation. In the reconstruction case, physical content is a result of these abstract physical principles, which in spite of their abstract nature provide physical content. This physical content provides a minimal constraint on what is happening in the world. Thus continuity acts as a physical constraint on the way that quantum theory must be. A formulation that is not interpreted sets out a mathematical formalism where we might have some empirical information or correspondence, but does not aspire to give an account of what is happening in the world.

The question of what ontological claims can be made in reconstructions is an active research topic in the programme. Although reconstructions are not interpreted in the same sense as the Bohmian example, they are still explanatory. Successful explanation is the result of models—or *theories*, as I argue—that support tracking correct inferences to counterfactual dependencies, inferences which nonetheless capture something real about the world without the model or theory itself being given a literal interpretation. In the reconstruction programme, the inferences that track counterfactual dependencies are the inferences from specified axioms that do or do not result in the von Neumann postulates. In Bokulich’s example, models need not be literal representations of target systems in order to be explanatory. In the reconstruction programme, I argue that we should draw a similar conclusion, that points in the landscape (in the space of theories) do not need

to be literal representations in order for reconstructions to be explanatory. The abstract physical concepts described by axioms will not describe concrete things or processes in the world, and yet we are still able to track real counterfactual dependencies in spite of this abstract articulation.

In this way, reconstructionists get their explanations. Reconstructionists are interested in a very particular explanatory purpose: to answer the “why the quantum” question. Answering that question encourages us to ask related questions such as “why the quantum and not the classical?” and “why the quantum and not the extra-quantum?” Hence, there will be multiple explananda.⁹ The way reconstructionists answer these questions is to designate the theoretical space which allows them to explain via contrast why axiom sets or postulates are classical, quantum, or extra-quantum. Axiomatization provides the right inferential structure to posit axioms and derive conclusions from those axioms. Alternative axiomatizations can be compared in how they track and highlight the theoretical space. The methodology in its entirety enables us to track the underlying relationships between theories in the theory space. Thus, we learn information about ‘why the quantum?’ It is the changing of axioms that explains the result when different regions in the landscape are picked out. And we can describe this process in part with counterfactuals.

In sum, I consider reconstructions to be genuinely explanatory as they explain why quantum theory is the way it is and not some other theory. They do so when they identify the key physical principles that are fundamental to the theory. And we are able to determine the structure of the space of theories via patterns of counterfactual dependence that highlight different worlds within the space of theories. These patterns enable us to learn knowledge about the world via the physical content contained in physical principles. However, my account of explanation in the reconstruction programme departs from Bokulich’s in a handful of ways. I build on Bokulich’s argument that explanations involve models, suggesting that successful explanations can also incorporate theories. Further, I contend that Bokulich’s account of explanation, which is applied to a *fictional* representation, also applies to non-fictional cases as well, as in the reconstruction programme. I conclude that these minimal differences in cases are not problematic, particularly in light of the eikonic conception.

Next, I discuss how my account of explanation in the reconstruction programme connects to Duwell (2020)’s modal account of understanding.

⁹This differs from Bokulich’s quantum dot case where the explanandum is the unexpected correlations between conductance peaks. I do not see this difference in explanandum versus explananda as problematic, particularly in light of the eikonic conception. Contextualized to the area of inquiry, the multiple explananda arising out of reconstructions is a reasonable inclusion.

5.4 Duwell’s modal view of understanding

In “Understanding quantum phenomena and quantum theories,” Duwell (2020) illustrates a modal view of *understanding* of quantum phenomena and quantum theory. Duwell argues that new work in foundations of physics—including the reconstruction programme—generates understanding without being focused on the goal of developing new interpretations. I discuss Duwell because I see us as having a shared goal. Duwell and I are both trying to establish that work in quantum foundations that is not oriented toward providing an interpretation of quantum theory is epistemically and pragmatically valuable. This is in response to a prevailing notion that research which does not foster interpretations is of limited value (Duwell, 2020, 2). I briefly discuss Duwell’s account and draw some conclusions as it relates to my proposed view of explanation. Our views are similar in several ways, including the focus on representation, the value of alternative representations, and that non-veridical representations can still tell us something about the world.

However, the key difference between Duwell’s account and mine is that I focus on the importance of the physical principles underlying the axioms as representing abstract physical content, suggesting that these principles reflect actual features of the world. In contrast, Duwell’s modal account of understanding does not require that the representations of theoretical content or phenomena tell us anything of the actual world.

5.4.1 Modal understanding

Duwell develops a modal view of understanding utilizing Le Bihan (2017)’s modal framework. Duwell begins by adopting Le Bihan’s view of phenomena where phenomena is “associated with stable, reproducible, and factual features of the observable world, whereby ‘observable’ is meant in sense of van Fraassen (van Fraassen, 1980), being detectable by unaided senses” (Duwell, 2020, 2). Modal understanding of some Phenomena P entails that (Duwell, 2020, 2):

Definition 1 (Modal Understanding of Phenomena P) *One has some modal understanding of phenomena P , $MU(P)$, if and only if one knows how to navigate some of the possibility space associated with P .*

The possibility space $S(P)$ for phenomena P is “the set of representations of some of P and their relations” (Duwell, 2020, 2). Duwell takes a permissive view of representations including mathematical models, diagrams, simulations, toy models, and so forth (Duwell,

2020, 2). A representation only needs to represent some subset of P rather than all of P . These representations, however, must adhere to adequacy conditions. Adequacy conditions may be implicit, explicit, quantitative, etc., and will vary in different scientific domains and research projects (Duwell, 2020, 2). The choice of adequacy conditions will be largely pragmatic. In order to navigate a possibility space:

Definition 2 (Navigating a Possibility Space for P) *One knows something about how to navigate $S(P)_A$ if and only if*

1. *One knows how some representations in $S(P)_A$ represent some of P , or*
2. *One knows how some representations in $S(P)_A$ relate, or*
3. *One knows something about how P in conjunction with A restrict the boundaries of $S(P)_A$.*

Duwell maintains that modal understanding comes in degrees which is “associated with the comprehensiveness of one’s ability to navigate $S(P)$ ” (Duwell, 2020, 4). First, a representation R affords better understanding than R' if R represents more of the phenomena than R' . Second, “the more representations that one knows how they represent P , the greater one’s understanding” (Duwell, 2020, 4). Third, one has more understanding if they know how different representations relate to one another. Lastly, the more one knows how the possibility space is constrained by the adequacy conditions and the phenomena, the more understanding one has (Duwell, 2020, 4).

Different adequacy conditions will correspond to different subtypes of understanding phenomena under this modal view (Duwell, 2020, 4). Duwell’s example is when one is interested in deterministic representations of quantum phenomena, indeterministic representations, or perhaps representations that are relativistically invariant. Duwell also contends that a modal understanding of phenomena does not require those representations to represent the world as it actually is “as opposed to how it might appear to us” (Duwell, 2020, 4). In this view, false theories can grant us modal understanding insofar as the adequacy conditions are met (Duwell, 2020, 4).

The above is a brief overview of Duwell’s modal understanding of *phenomena*. What is more relevant for my purposes is Duwell’s extension of his view on modal understanding to a modal understanding of *theories*. Here Duwell extends the notion of modal understanding but applies it to a theory T , decomposing understanding down to *internal* and *extensional* understanding of theories. Internal understanding of a theory is understanding “we can

have of a theory *independent* of the representational uses that we might put it to” (Duwell, 2020, 5). Extensional understanding is understanding that is dependent on the ability of a theory to represent (Duwell, 2020, 5). More specifically, internal modal understanding of a theory is defined as:

Definition 3 (Internal Modal Understanding of a Theory T) *One has some internal modal understanding of a theory T , $IMU(T)$, if and only if one knows how to navigate some of the internal possibility space associated with T .*

The internal possibility space $IS(T)$ for T is the set of representations of the theoretical content of T and their relations (Duwell, 2020, 5). An example of different representations of theoretical content are different axiomatizations of some theory T , as in the reconstruction programme. Navigating the internal possibility space for T mirrors the above rendition for phenomena:

Definition 4 (Navigating the Internal Possibility Space for T) *One knows something about how to navigate $IS(T)_A$ if and only if*

1. *One knows how some representations in $IS(T)_A$ represent some content of T , or*
2. *One knows how some representations in $IS(T)_A$ relate, or*
3. *One knows how T and A restrict boundaries of $IS(T)_A$.*

As was the case with understanding phenomena, internal understanding of a theory is a matter of degree, associated with one’s ability to navigate the possibility space for the theory (Duwell, 2020, 6). Again, better understanding is afforded when a representation represents more theoretical content. The more that one knows how the theoretical content is represented, the more understanding one has (Duwell, 2020, 6). More understanding is afforded when one knows how representations of the theory’s content relate, and one knows more when one knows how the theory and the adequacy conditions constrain the possibility space. Duwell argues further that some representational vehicles better facilitate understanding than others (Duwell, 2020, 6). Similar to the phenomena case, different subtypes of internal understanding are associated with different adequacy conditions.

The extensional modal understanding of a theory is defined as:

Definition 5 (Extensional Modal Understanding of a Theory T) *One has some extensional modal understanding of a theory T , $EMU(T)$, if and only if one knows how to navigate some of the extensional possibility space associated with T .*

The extensional possibility space $ES(T)$ for theory T is the set of *phenomena* that T can be taken to represent and their relations (Duwell, 2020, 7). The main distinction here is that internal understanding contains representations of theoretical content, whereas extensional understanding contains phenomena (Duwell, 2020, 7). One knows how to navigate the extensional possibility space when:

Definition 6 (Navigating the Extensional Possibility Space for T) *One knows something about how to navigate $ES(T)_A$ if and only if*

1. *One knows how some phenomena in $ES(T)_A$ can be represented by T , or*
2. *One knows how some phenomena in $ES(T)_A$ relate, or*
3. *One knows how T and A restrict boundaries of $ES(T)_A$.*

In sum, Duwell’s framework covers modal understanding of both phenomena and some theory T . Modal understanding of phenomena is gained when we know how some representation represents some phenomena, or when we know how those representations relate to one another. Modal understanding of some theory T can be either internal or extensional. Internal modal understanding of theory T is gained when we learn how representations represent theoretical content. Extensional modal understanding of theory T is gained when we learn how theoretical content represents phenomena. The crucial difference between internal modal understanding of a theory and extensional modal understanding is that $IS(T)$ contains representations of theoretical content, whereas $ES(T)$ contains phenomena (Duwell, 2020, 7).

How does Duwell’s framework hook up to reconstructions? Duwell’s account applies to the reconstruction programme insofar as the programme falls under the umbrella of work in quantum foundations that Duwell discusses. Specifically, Duwell shows how reconstructions are able to facilitate internal and extensional modal understanding of quantum theory and quantum phenomena. Internal modal understanding is facilitated, for example, when we learn how different theories are related within an operational framework (Duwell, 2020, 10). And knowing how quantum mechanics is distinctive in light of foil theories might “help one develop new axioms, which provide additional representations of quantum mechanical

content, which facilitates internal modal understanding of quantum mechanics” (Duwell, 2020, 12). And from this internal modal understanding we gain navigating power. Extensional modal understanding is facilitated when, for example, physical principles within a framework give us new ways of representing phenomena (Duwell, 2020, 11).

For example, Duwell argues that principles such as Information Causality and Non-trivial Communication Complexity can facilitate modal understanding and can “provide easy answers to w-questions” (Duwell, 2020, 11).¹⁰ This can help us carve up the distinction between super-quantum correlations and quantum correlations within the context of a device-independent operational framework.¹¹ Super-quantum correlations are those that are more non-local than quantum correlations, exceeding the Cirel’son bound which measures how non-local correlations are (Duwell, 2020, 10). The goal in a reconstruction is to determine which principles pick out quantum correlations within the space of no-signaling correlations, which are correlations that do not permit faster than light signaling. Duwell concludes that the principles of Information Causality and Non-trivial Communication help us in identifying the boundary between quantum and super-quantum correlations, giving us modal understanding (Duwell, 2020, 11). In my example, I argue that continuity serves a similar function in the space of theories in the GPT framework. We learn that continuity is a feature that is necessary to isolate the quantum region in the landscape of theories. I have argued that we should understand principles as having physical content despite their abstract nature, which means that we also gain physical information on the differences within the space of theories. In this way, the principles act as constraints on the way the world is, both in the physical content that is required in a theory and in the physical distinctions between theories. An explanation is yielded when we see the patterns of counterfactual dependence between physical principles and the effects on the space of theories, i.e., when we use those principles to answer w-questions.

What these types of understanding highlight is the importance of the space of theories. Much of Duwell’s discussion aligns with my own, including the emphasis on comparison and contrast between theories within the general mathematical framework (Duwell, 2020, 12). As I have argued, the generalized framework enables us to learn distinctions between theories by way of investigating the spaces near the quantum space of theories.

There are several similarities to be drawn between Duwell’s, Bokulich’s, and my account. My account utilizes Bokulich’s eikonic conception of explanation which emphasizes

¹⁰Information Causality is offered in Pawłowski et al. (2009) and Non-trivial Communication Complexity is posited in Van Dam (2013).

¹¹The device-independent operational framework is distinct because systems only consider classical inputs and outputs and their correlations (Duwell, 2020, 10).

the integral role that representations play in scientific explanation. This conception encourages that phenomena be represented by multiple alternative representations. Duwell likewise focuses on the significance of representation and how representations are related (Duwell, 2020, 3). His account argues that the more representations that one knows how they represent phenomena or theory, the greater understanding one has (Duwell, 2020, 4). Duwell also argues that some “representational vehicles may better facilitate understanding than others” (Duwell, 2020, 4). The sentiment that some questions or phenomena are better treated in some representations rather than others is also shared by Bokulich (Bokulich, 2020, 3). She extends this line of reasoning to argue for the value of the exploration of alternative formulations of theories, particularly in any project where ontological conclusions are drawn from a formalism (Bokulich, 2020, 3). Thus, we are in agreement on the importance of representations in their ability to represent of phenomena and theoretical content. We also agree that representations need not be veridical in order to be epistemically or pragmatically valuable. For Duwell, modal understanding can be facilitated even if the representation does not represent the way the world actually is, as long as adequacy conditions are met (Duwell, 2020, 3).

And as I have argued, we can have an account of explanation in the reconstruction programme even if the landscape of theories is not taken to be a literal representation. Bokulich also argues that it behooves us to search for alternative formulations (representations) of theories without the goal of interpretation in mind (2019), which I see as a criticism of the standard interpretative project. In this way Duwell, Bokulich, and myself are arguing for the value of projects in foundations of physics that are not oriented toward providing an interpretation. And like Duwell, I agree that reconstructions are an example of the value gained when we do not limit our investigations to searching for interpretations. Rather, there is much to be gained by reconceptualizing how to make progress in foundations of physics through a reconstructionist lens.

5.4.2 Differences between Duwell (2020) and my account

Although my account (and therefore Bokulich’s account) is similar in many ways to Duwell’s, there is an important distinction between the two. The main distinction between Duwell’s account and my own is the emphasis of the role of the physical principles that carry physical content, meaning that these physical principles represent features of the world. This distinction again points to the difference between understanding or having knowledge of the modal space of theories versus having knowledge about the conditions in the world that make quantum theory true.

Duwell’s concept of understanding focuses on representations and their connection to phenomena. Understanding involves knowing how representations represent, how content within some representation is related to other content, or how possible representations are constrained based on that content. Duwell’s category of internal modal understanding of theories means that we understand how to navigate the space of possible theories and their representations. His category of extensional understanding of a theory has to do with understanding phenomena and their representations. The term phenomena is defined in terms of stable, reproducible, and factual features of the observable world, where observable means that phenomena is detectable by the unaided senses. And modal understanding of phenomena “does not require that representations of phenomena adequately represent the way the world actually is as opposed to how it might appear to us” (Duwell, 2020, 4).

In Duwell’s terms, I have focused primarily on internal understanding of a theory as it relates to the reconstruction programme. Internal understanding of a theory is ‘knowledge of the theory,’ as I have put it. But as we see from extensional understanding of a theory, Duwell characterizes this as understanding *phenomena*, not understanding of the world. I have argued that the axioms which represent physical processes enable us to learn what is possible and impossible in the world. The important point is that I take the axioms to represent the world as it really is. However, as I have maintained, the non-literal nature of the representation in reconstructions makes it difficult to give a tidy physical description of the features of the world that are represented. But if we, for example, think about the time evolution interpretation of the continuity axiom, we know that time evolution is determined by the main laws of physics in our world. This is a nomic possibility, possible according to the laws of nature (Kment, 2021). In the continuity example, what we are essentially learning about is what kinds of time evolutions or quantum gates are possible or impossible in our world.

Duwell’s understanding of either theories or of phenomena, internal or extensional, does not purport to represent the world as it actually is. This is the main distinction between our two accounts. Therefore, Duwell’s account of internal understanding of theories is compatible with how I have articulated modal knowledge of patterns of counterfactual dependence in the space of theories. However, I provide an account of how the physical principles in a reconstruction can give us knowledge about the conditions in the world that make quantum theory true. Duwell’s extensional account is limited to representations of phenomena which need not represent the world.

5.5 Conclusion

I have shown that reconstructions of quantum theory can be explanatory as non-literal representations which correctly capture patterns of counterfactual dependencies. These patterns are born out of the changing of axioms and whether or not those changes pick out the quantum space of theories. The result is an account of explanation in the reconstruction programme that is noncausal and non-interventionist in a similar manner to that of Bokulich (2018a). What is explained is what features distinguish quantum theories from other theories like, for example, the principle of continuity. And knowing what features distinguish quantum theories from other theories in the space of theories explains why quantum theory has the mathematical structure that it does. Thus, I claim that reconstructions are genuinely explanatory. The generalized framework, e.g., GPT supports the tracking of these patterns of counterfactual dependencies within the landscape of theories. The patterns of counterfactual dependence reveal the structure of quantum theory. Hence, the generalized framework is necessary as a means of facilitating contrastive explanation via the space of classical, quantum, and extra-quantum theories.

Thus, reconstructions are able to provide answers to ‘why the quantum’ questions, which helps us explain the distinctions between classical, quantum, and extra-quantum theories. In this way, we learn of the structure of the space of theories. Not only are we able to explain the distinctions between theories, but we also gain genuine physical insight via the physical principles which can be interpreted as having physical content. This physical content bridges the gap between the space of theories as a representation and knowledge of the world. Therefore, I claim that the structure of the space of theories reflects the structure of the world.

And once we are provided explanations, we are aided in confronting how to understand the programme writ large as in Duwell. Both Duwell’s account and the account presented here lead to a modal view of what we hope to gain from the reconstruction programme. They also provide further support for the value in shifting away from a focus on interpretations in foundations of physics, toward other ways of making progress in quantum foundations. However, my account of explanation allows us to say something about the physical world alongside learning about quantum theory as a representation.

Chapter 6

Conclusion

Despite a century of efforts to peer through the quantum veil, we are still left in the strange predicament of being in possession of an astonishingly powerful theoretical tool that we scarcely comprehend, *viz.* a tool that we do not know how to speak about in natural language in a coherent, comprehensive, and precise manner—let alone with any logical–philosophical precision and systemacity—without resorting to theoretical jargon or leaning on mathematical crutches. (Goyal 2022, p. 38)

6.1 Toward a new approach

The quantum reconstruction programme has positioned itself to be an alternative way of making progress in quantum foundations. Much of philosophy of quantum physics has centred on the role that interpretations play in our understanding of the quantum world. However, quantum reconstructions are in the business of refining the physically–obscure features of quantum theory. Reconstructions offer a distinct perspective on *how* we might go about determining what one of our greatest theories of physics tells us about the natural world. But in spite of the programme’s positive reception within the physics community, the philosophical interest in the reconstruction programme has been minimal. Berghofer speculates that the reason for this lack of interest might be that “the existing successful reconstructions seem to put pressure on the most prominent interpretations we find in philosophy of physics” (Berghofer, 2024b, 2). Perhaps that is correct. But regardless of the particular reason *why* philosophers have paid little heed to the quantum reconstruction programme, my research serves to bridge a gap in the philosophical literature.

This dissertation has sought to establish the methodological significance of the reconstruction programme, to establish reconstructions as alternative formulations of quantum theory, and to offer an account of explanation for the programme. I have sought to lay the groundwork for establishing a philosophical foundation for quantum reconstructions. I have identified methodological tools that physicists can and are using to further the aims of the reconstruction programme. These include axiomatic methodology, the intentional generation of alternative formulations, and the utilization of the generalized framework which establishes the space of theories. This work has also argued that the unique methodological approach that reconstructions take is tailor-made to solve the specific problem that reconstructionists are contending with. More importantly, I have argued that the methodology of the programme is a valuable methodology in the context of quantum reconstructions and quantum foundations more broadly.

In Chapter 3 I have argued that reconstructions utilize axiomatic methodology with historical roots tracing back to Hilbert’s work in geometry, Hilbert, Nordheim, and von Neumann’s work in early quantum mechanics, and von Neumann’s work in foundations of quantum mechanics. I have argued that axiomatization provides helpful methodological structure that facilitates the modification of axiom sets within a reconstruction. The result of this is that we are better able to see, situated within the generalized framework, changes in the space of theories as a result of different axioms. However, I have maintained that we should understand these axiomatizations as provisional and practical representations of quantum theory, which help us gain a better understanding of the required axioms for the theory. The provisional and practical nature of reconstructions aligns with my argument in Chapter 4 which asserts that representations need not be veridical in order to be valuable representations. Provisional representations can be modified and we do not need to conceive of representations as being literally true in order for them to be effective representations.

Further, I have shown that reconstructions will be context-dependent. In particular, axiomatizations will depend on the maturity of the mathematical and theoretical concepts within an area of inquiry as well as the goals of inquiry. Reconstructionists can use the well-developed mathematical formalism to their advantage in their efforts to solidify the physical principles of quantum theory. The context-dependency of axiomatization complements Bokulich (2018)’s eikonic conception which I adopt in Chapter 5. The eikonic conception is predicated on the idea that in the actual practice of science, scientists study phenomena via simplified representations that are contextualized to the area of inquiry and explanatory project (Bokulich, 2018a, 801). The eikonic conception aligns well with the context-dependency of axiomatization attempts of quantum theory.

As I have argued throughout the dissertation, the reconstruction programme has pro-

duced genuine insight into the structure and nature of quantum theory. Goyal argues that reconstructions offer distinct interpretative insights on the nature of space, time, and measurement (Goyal, 2022, 35). Reconstructionists will not take distinct axiomatizations as literal construals of things in the world. However, as I argue in Chapter 3, though axiomatizations and the space of theories will not be literal representations, we do track real physical content via the physical principles. The ways in which those physical principles orient us in the space of theories reveals the modal structure of the space of theories. Interpreting the physical principles as having physical content enables us to connect the modal structure of the space of theories to the kind of world described by quantum theory. For example, Hardy (2001)’s inclusion of the continuity axiom orients us in the quantum space of theories, where the removal of continuity does not. This modal structure reveals differences in the space of theories. But it is the interpretation of the physical principles as having physical content that reveals constraints on the way the world really is. A methodology geared toward developing alternative formulations and away from the standard interpretative project has already been fruitful.

The idea that we should move from viewing representations as literal depictions to seeing them as tools for facilitating correct inferences and true physical insights falls under Bokulich (2018)’s eikonic conception of explanation. As I argue in Chapter 5, reconstructions are genuinely explanatory. I endorse the eikonic conception in my account of explanation in the reconstruction programme, arguing that we gather genuine physical insights into the structure of quantum theory with representations playing a major role. These insights come in the form of patterns of counterfactual dependence via the physical principles which have physical content. My account of explanation shifts the focus from representations as literal construals and toward Bokulich’s question of ‘what true things can we learn?’ As I argue, we learn physical constraints on the real world via the physical principles within a reconstruction. My account is also fully compatible with the features of inferential realism that Bokulich (2019) outlines. I argue that reconstructions are an example of this shift in how we might view realism in the context of quantum foundations, toward an attitude of ‘what true physical insights and correct inferences can we learn?’ Further, I say that reconstructions do this successfully. Non-literally interpreted formulations and a moderate realism are integrated via the correct inferences licensed by the representations within a reconstruction.

This variety of realism will not placate the full-blooded realist because it is not a full interpretation. Full interpretations typically accept a mathematical formalism and construct an ontological description of the world based on that formalism. However, my account is within the same family of Bokulich’s inferential realism. Reconstructions aim to justify *why* the formalism of quantum mechanics ‘is the way it is’ rather than just accepting

the formalism and ‘heaping over’ an interpretation. The methodology of the reconstruction programme is a means of exploring different formulations of quantum mechanics while leaving the interpretation of the formalism by the wayside. As I have argued, the shift in focus toward alternative formulations and the new insights already gained reveal that the reconstruction programme is a promising way forward.

6.2 Future Directions

This dissertation lays the groundwork for further philosophical exploration into the quantum reconstruction programme. One primary concern will be the connection between reconstructions and interpretations, some of which I have discussed in the preceding pages. This issue has received some very recent attention in the philosophical literature. Berghofer argues that reconstructions are themselves in need of interpretation (Berghofer, 2024b). He contends that reconstructions put pressure on Ψ -ontic interpretations which take the wavefunction to be an existent entity (Berghofer, 2024a). Ψ -ontic interpretations make up a good portion of the popular interpretations in philosophy, including Bohmian mechanics and the Many-Worlds interpretation (Berghofer, 2024a, 2). In such a case, quantum reconstructions would be at odds with standard interpretations. Instead, Berghofer advocates for a Ψ -doxastic interpretation, which hinges upon conceptualizing quantum theory as representing *degrees of belief*. The most influential doxastic interpretation is QBism, or Quantum Bayesianism, which applies “a personalist Bayesian account of probability to quantum probabilities (Fuchs et al., 2014, 1).

Drawing out the connection between interpretation and reconstruction is a potential way forward in the literature, though it is worth considering *what* precisely we mean when we say ‘interpretation.’ Berghofer does acknowledge this consideration, noting that ‘interpretation’ is used ambiguously and inconsistently in the literature (Berghofer, 2024b, 7). As I have discussed, interpretations typically include the broad acceptance of the mathematical formalism and focus on providing an ontology based on that formalism. As we can see, this understanding of interpretation is hard to square with reconstructions, which have in many senses contrary methodology. One future direction to pursue would be to determine what ‘interpretation’ amounts to in the context of reconstructions, building on the work presented in chapters 4 and 5.

In my view, the relation between reconstructions and interpretation is likely closer to Goyal’s “interpretive insights” (Goyal, 2022). Goyal views reconstructions as *stepping stones* toward an interpretation. This is done by reflecting on the principles of the reconstruction, “rather than attempting to decipher the inscrutable abstract mathematical

postulates in which quantum theory is standardly cast” (Goyal, 2022, 18). An example of an interpretive insight is the notion of space, as operational reconstructions “make no explicit reference to the notion of space—to the idea that objects are localized in space, or that space has a certain dimension, topology, or metric” (Goyal, 2022, 34). Thus, one interpretive insight we might draw is that quantum theory does not necessarily require a notion of space. But an interpretive insight does not a full interpretation make. These interpretive insights are rather abstract insights. They certainly do not construct an ontology based on a mathematical formalism. In this way, I see my view of the role of reconstructions as aligning closer to Goyal’s interpretive insights.

At first glance it may appear that the ethos of quantum reconstructions are anathema to the standard interpretative project. I believe it is likely that quantum reconstructions will provide opposing views to standard interpretations on the nature of the quantum world, as Berghofer has suggested. But I also believe it is likely that quantum reconstructions will deliver insights that might be integrated into work in standard interpretations. I anticipate that the project of interpreting quantum reconstructions and investigations into the role that interpretation has to play in reconstructions will capture the philosophical zeitgeist in the near future of the programme.

These insights may arise, for example, from investigating extra-quantum theories in the space of theories, those that violate known features of quantum theory as we understand it. This is a question that remains outstanding and is an active area of research in quantum foundations. For example, extra-quantum theories have super-quantum correlations that exceed the Cirel’son bound, being more non-local than quantum correlations (Duwell, 2020, 10). The Cirel’son bound is an upper limit for correlations between spatially separated quantum systems. However, the GPT framework admits theories with these super-quantum correlations. Thus the GPT framework might provide the right context to ask “given that stronger correlations than those of quantum theory are consistent with no-signalling, why aren’t quantum correlations stronger than they in fact are?” (Timpson, 2013, 184).¹ Perhaps reconstructions will provide a helpful perspective in exploring these extra-quantum theories.

A potential future direction could involve broadening my analysis of reconstructions beyond the two case studies I have examined so far. My focus on Hardy (2001) and Selby et al. (2018) was guided by the distinct physical principles underlying each of these accounts. It is possible that the conclusions I have made are more specific to the Hardy and Selby et al. examples. The physical principles in Hardy and Selby et al. can more easily

¹Interestingly, this question cannot be posed if we begin with a C^* -algebraic framework (Timpson, 2013, 184). A C^* -algebraic theory will not admit super-quantum correlations.

be interpreted as having physical content. This ease of interpretation has enabled me to develop a non—literal version of realism. However, information—theoretic reconstructions may be more naturally suited to an instrumentalist perspective rather than a realist one. To that end, a potential direction would be to expand my account to consider reconstructions that are more information theoretic, for example (Clifton et al., 2003).

One of the grounding motivations for the quantum reconstruction programme is the search for a theory of quantum gravity. To this end, it is possible that through the clarification of the boundaries of quantum theory we might discover new mathematical and physical features that will illuminate a pathway to quantum gravity, as Hardy hoped for (Hardy, 2018). Progress toward a theory of quantum gravity in the context of reconstructions is yet to be seen. But thus is the nature of physics. It progresses as it will.

6.3 Concluding remarks

Reconstructions illustrate the value of broadening our investigations beyond the typical search for interpretations. As I have shown, significant insights can be gained by reconceptualizing how to advance foundational work in physics through a reconstructionist lens. I end this project, as is appropriate, with a quote from Lucien Hardy:

The way I see it, in terms of this program of finding some postulates or axioms or whatever, is that we’re looking at some strange object, and the original mathematical axioms of quantum theory sort of give us a bunch...of strange vantage points on this strange object. Maybe we’re looking at it from odd directions. And as we get better postulates, we’re finding better ways to look at it from. So, rather than looking at it from some direction where it doesn’t make sense, perhaps we can see it face-on, and we see it from several different directions. So we’re moving towards some more reasonable set of vantage points on the given object, so then it wouldn’t disturb us of course whether these axioms are unique or not—at least they should somehow be a sensible way of looking at that object. —Hardy in (*Fuchs and Stacey 2016, p. 302.*)

Just as the quantum reconstruction programme shines the light on quantum theory from different vantage points, so too do different approaches to quantum foundations shine light on what quantum theory tells us of the natural world. In reconstructions, these different vantages take the sharp, jagged corners “off of the edge of quantum theory” (Fuchs and Stacey, 2016, 302). Research programmes outside of the usual standard interpretative

project provide new vantages on foundations of physics, smoothing out the jagged edges of how we understand physics. Perhaps these different vantages, provided by both the programme as a whole and individual reconstructions, will illuminate what quantum theory means, rather than merely knowing how quantum theory is used.

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