ALL UNITARY OPERATORS ARE TRICYCLES, AND MOST ARE BICYCLES

LAURENT W. MARCOUX[1], MATJAŽ OMLADIČ[2], HEYDAR RADJAVI, AND YUANHANG ZHANG[3]

Abstract. Let $\mathcal{H}$ be a complex, separable Hilbert space (of finite or infinite dimension), and let $\mathcal{U}(\mathcal{H})$ denote the group of unitary operators on $\mathcal{H}$. In the finite-dimensional setting, every unitary operator of determinant one can be expressed as the product of two operators, each unitarily equivalent to the $n \times n$ unitary cycle. In the infinite-dimensional setting, we prove that every unitary operator U is a product of three operators, each unitarily equivalent to the bilateral shift, and if the spectrum of U has non-zero Lebesgue measure, then U is a product of two operators, each unitarily equivalent to the bilateral shift.

We dedicate this paper to the memory of our dear friend and colleague, Ahmed Sourour.

1. Introduction

There is a vast and ever-growing literature dealing with the factorisation and characterisation of certain classes of bounded, linear, Hilbert space operators as a product, sum or linear combination of simpler, more tractable classes of operators [3, 6, 10, 12, 13, 15, 20, 25].

In the article [8], the authors prove that if $\mathbb{F}$ is a field of cardinality at least three and $n \geq 2$, then the semigroup $\mathcal{S}$ generated by the similarity orbit of a fixed invertible, non-scalar matrix $A \in \mathbb{M}_n(\mathbb{F})$ is a normal subgroup of the group $\mathrm{GL}_n(\mathbb{F})$ of invertible matrices if and only if the determinant of A is a root of unity in $\mathbb{F}$, in which case $\mathcal{S}$ is isomorphic to the semi-direct product of the special linear group $\mathrm{SL}_n(\mathbb{F})$ with the cyclic subgroup of $\mathbb{F}$ generated by $\det A$. The same paper also studied the semigroup generated by the similarity orbit of a fixed singular matrix in $\mathbb{M}_n(\mathbb{F})$, proving amongst other things that if $A \in \mathbb{M}_n(\mathbb{F})$ has rank equal to $n-1$, then the semigroup generated by A consists of all matrices of rank at most $n-1$. Later, in [19], the authors investigated the semigroup generated by the unitary orbit of a fixed, singular complex matrix. In 2003, Fong and Sourour [7] proved that if T is an invertible operator acting on an infinite-dimensional, separable Hilbert space $\mathcal{H}$ such that T is not a scalar modulo the ideal of compact operators, then the semigroup generated by the similarity orbit of T is the group of all invertible operators, while if U is a unitary operator which is not a scalar modulo the compact operators, then the semigroup generated by the unitary orbit of U consists of all unitary operators on $\mathcal{H}$. One cannot expect there to be a universal bound (i.e. independent of the choice of T or of U) on the number of terms required in the factorisation, as is clear by considering operators T or U

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which are close to the identity operator. More recently, in [18], the authors investigated the semigroup generated by the unitary orbit of a fixed invertible, complex matrix A, proving in particular that if U is a non-scalar unitary matrix of determinant 1, then the semigroup generated by the unitary orbit of U is the set $\mathrm{SL}_n(\mathbb{C})$ of all unitary matrices of determinant 1.

In this paper, we shall examine a particular instance of the problem related to factorisations of unitary operators as products of operators each belonging to the unitary orbit of a fixed unitary operator. More specifically, suppose that $\mathcal{H}$ is a complex, separable, finite- or infinite-dimensional Hilbert space and that $\mathcal{B}(\mathcal{H})$ denotes the algebra of bounded linear operators acting on $\mathcal{H}$. We denote by $\mathcal{K}(\mathcal{H})$ the ideal of compact operators in $\mathcal{B}(\mathcal{H})$. Given $1 \leq n \leq \infty$, we define the **n-fold ampliation** of $\mathcal{H}$ as $\mathcal{H}^{(n)} = \oplus_{j=1}^{n} \mathcal{H}$, and similarly, given $T \in \mathcal{B}(\mathcal{H})$ and n as above, we define the **n-fold ampliation** of T to be $T^{(n)} = \oplus_{j=1}^{n} T$, acting on $\mathcal{H}^{(n)}$. Denote by $\mathcal{U}(\mathcal{H})$ the set of **unitary operators** on $\mathcal{H}$. Given $T \in \mathcal{B}(\mathcal{H})$, we define the **unitary orbit** of T to be the set $\mathcal{U}(T) := \{V^*TV : V \in \mathcal{U}(\mathcal{H})\}$. We write $R \simeq T$ to mean that R is **unitarily equivalent** to T, which is to say that $R \in \mathcal{U}(T)$, and note that $\simeq$ is an equivalence relation. Suppose next that $T \in \mathcal{B}(\mathcal{H})$ is not a compact perturbation of a scalar operator, and that every $V \in \mathcal{U}(\mathcal{H})$ may be decomposed as the product of a fixed, finite number of elements from $\mathcal{U}(T)$. What can then be said about T?

Certain related results have existed for a while now and may be considered classical. For example, it is well-known that every unitary operator is a product of at most four symmetries (i.e. self-adjoint involutions) [11]. Since any symmetry in $\mathcal{B}(\mathcal{H})$ is a product of two symmetries each unitarily equivalent to the "balanced" symmetry $H := I \oplus -I$, where I denotes the identity operator acting on an infinite-dimensional and infinite-codimensional subspace of $\mathcal{H}$, each unitary operator can be written as a product of at most eight operators, each unitarily equivalent to H. (As an interesting side note, the characterisations of the set of products of two and of four symmetries is well-known, while the description of the set of products of three symmetries seems to be exceedingly complicated [13].)

In the work below, we shall demonstrate that if $2 \leq n \in \mathbb{N}$ and $V \in \mathcal{U}(\mathbb{C}^n)$ is a fixed unitary matrix, and if every unitary matrix $W \in \mathcal{U}(\mathbb{C}^n)$ of determinant 1 can be written as the product of two matrices unitarily equivalent to V, then V must be a multiple of the $n \times n$ unitary cycle

$$Q := \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

Conversely, every unitary matrix of determinant one can be expressed as the product of two matrices unitarily equivalent to Q.

In the infinite-dimensional setting, we prove the following. First, fix an orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ for $\mathcal{H}$, and consider the **bilateral shift** operator $B \in \mathcal{B}(\mathcal{H})$ which satisfies $Be_n = e_{n+1}$ for all $n \in \mathbb{Z}$. We prove that if $V \in \mathcal{U}(\mathcal{H})$ is fixed, and if every unitary operator $U \in \mathcal{U}(\mathcal{H})$ can be expressed as a product of two operators unitarily equivalent to V, then there exists $n \in \mathbb{N} \cup \{\infty\}$ such that V is unitarily equivalent to the direct sum of n copies of the bilateral shift B.

It is known [11] that every unitary operator is indeed the product of two operators, each unitarily equivalent to $B^{(\infty)}$. Thinking of the bilateral shift B as an infinite-dimensional analogue of a "**cycle**", we show that every unitary operator is the product of three operators each unitarily equivalent to B (we shall say that every unitary operator is a "**tricycle**"), and any unitary operator $U \in \mathcal{U}(\mathcal{H})$ whose spectrum has positive Lebesgue measure is a product of two operators, each unitarily equivalent to B (we refer to such a product as a "**bicycle**"). We do not know at this time whether or not every unitary operator is a bicycle, but we do have a number of examples of bicycles whose spectrum has Lebesgue measure zero.

We remind the reader that if $T \in \mathcal{B}(\mathcal{H})$, then the *essential spectrum* of T is defined as $\sigma_e(T) := \sigma(\pi(T))$, where $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the canonical quotient map of $\mathcal{B}(\mathcal{H})$ onto the Calkin algebra. If the range of T is closed and both $\mathrm{nul}(T)$ and $\mathrm{nul}(T^*)$ are finite, then T is said to be **Fredholm**, and the **Fredholm index** of T is defined to be $\mathrm{ind}(T) := \mathrm{nul}(T) - \mathrm{nul}(T^*)$. It is well-known that T is Fredholm if and only if $\pi(T)$ is invertible in the Calkin algebra.

Since it will be used repeatedly throughout the remainder of the paper, we also pause to recall the Weyl-von Neumann-Berg Theorem. The following formulation is from [4, Corollary II.4.2]

1.1. Theorem. [The Weyl-von Neumann-Berg Theorem] *Let $N \in \mathcal{B}(\mathcal{H})$ be a normal operator. Given $\varepsilon > 0$, there exist a diagonalisable normal operator D and a compact operator K satisfying $\|K\| < \varepsilon$ such that $N = D + K$. Furthermore, D can be chosen so that $\sigma(D) = \sigma(N)$, including the multiplicities of the isolated eigenvalues.*

A standard consequence of this result is the following:

1.2. Theorem. [4, Theorem II.4.4] *Suppose that M and N are normal operators on a separable Hilbert space $\mathcal{H}$. Then M and N are approximately unitarily equivalent (i.e. $\overline{\mathcal{U}(N)} = \overline{\mathcal{U}(M)}$) if and only if*

(a) *$\sigma_e(M) = \sigma_e(N)$, and*
(b) *$\mathrm{nul}(M - \lambda I) = \mathrm{nul}(N - \lambda I)$ for all $\lambda \in \mathbb{C} \setminus \sigma_e(M)$.*

2. An apéritif: the finite-dimensional case

2.1. In the finite-dimensional setting, the determinant plays a central role in the question of factorisation. Having said this, a wonderful result of Sourour's [22] states that if $\mathbb{F}$ is a field, $2 \leq n \in \mathbb{N}$, $T \in \mathbb{M}_n(\mathbb{F})$ is a non-scalar invertible matrix, and $\alpha_j, \beta_j \in \mathbb{F}$, $1 \leq j \leq n$ satisfy $\prod_{j=1}^n \alpha_j \beta_j = \det T$, then there exist matrices $A, B \in \mathbb{M}_n(\mathbb{F})$ with $\sigma(A) = \{\alpha_j\}_{j=1}^n$ and $\sigma(B) = \{\beta_j\}_{j=1}^n$ (counted according to algebraic multiplicity) such that $T = AB$.

If one were to try to factor an arbitrary unitary matrix U as a product of a finite number of operators, each equivalent to a fixed operator $T \in \mathcal{B}(\mathbb{C}^n)$, then one would require $\det T$ to be a root of $\det U$ for each unitary matrix, which is impossible. For this reason, we concentrate on the most interesting case, namely the problem of expressing every unitary matrix U of determinant one as such a product. We denote the special unitary group in $\mathcal{B}(\mathbb{C}^n)$ by $\mathrm{SU}_n(\mathbb{C})$. We continue to use Q to denote the unitary cycle defined in Section 1. The next result establishes the importance of this operator for our problem. As always, we set $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

2.2. Proposition. *Let $n \geq 2$, $V \in \mathcal{U}(\mathbb{C}^n)$, and suppose that $W \in \mathrm{SU}_n(\mathbb{C})$ implies that there exist $V_1, V_2 \in \mathcal{U}(V)$ such that $W = V_1 V_2$.*

Then there exists $\theta \in \mathbb{T}$ satisfying $\theta^{2n} = 1$ such that $V \simeq \theta Q$, where Q is the $n \times n$ unitary cycle.

Proof. Let $\omega = e^{\frac{2\pi i}{n}}$ and note that $\omega^k I_n \in \mathrm{SU}_n(\mathbb{C})$ for all $0 \leq k \leq n$. Thus, for each $0 \leq k \leq n$, $\omega^k I_n = V_{1,k} V_{2,k}$ for some $V_{1,k}, V_{2,k} \in \mathcal{U}(V)$ and so

$$\omega^k \sigma(V) = \omega^k \sigma(V_{1,k}^*) = \sigma(V_{2,k}) = \sigma(V).$$

From this it readily follows that for some $\theta \in \mathbb{T}$,

$$\sigma(V) = \{\theta, \omega\theta, \omega^2\theta, \ldots, \omega^{n-1}\theta\},$$

whence

$$V \simeq \theta Q.$$

Taking $k = 0$ shows that

$$1 = \det(I_n) = \det(V_{1,0}) \det(V_{2,0}) = \det(V)^2,$$

so that $\det(V) \in \{-1, 1\}$. But $\det(V) = \theta^n(1 \cdot \omega \cdot \omega^2 \cdots \omega^{n-1}) = (-1)^{n-1}\theta^n$. Hence $\theta^n \in \{-1, 1\}$, and so $\theta^{2n} = 1$.

□

The above result motivates the following definition:

2.3. Definition. *Let $n \geq 2$, and denote by Q the $n \times n$ unitary cycle in $\mathcal{B}(\mathbb{C}^n)$. A **bicycle** is an operator of the form $Q_1 Q_2$, where each of Q_1 and Q_2 lie in the unitary orbit of Q.*

We next wish to establish a converse to Proposition 2.2.

2.4. Proposition. *Let $n \geq 2$ and $U \in \mathcal{U}(\mathbb{C}^n)$ be a diagonal unitary matrix with determinant 1. There exists a diagonal unitary V such that*

$$U = VQV^*Q^*,$$

where Q is the $n \times n$ unitary cycle.

Proof. Let $U = \mathrm{diag}(z_1, \cdots, z_n)$. Define $V := \mathrm{diag}(z_1, z_1 z_2, \cdots, \prod_{i=1}^n z_i = 1)$. Recalling the definition of the $n \times n$ unitary cycle $Q \in \mathbb{M}_n(\mathbb{C})$ from Section 1, we find that an easy calculation yields

$$QV^*Q^* := \mathrm{diag}\{1, \overline{z_1}, \overline{z_1 z_2}, \cdots, \overline{z_1 z_2 \cdots z_{n-1}}\},$$

and thus $U = VQV^*Q^*$.

□

2.5. Proposition. *Let $n \geq 2$, $U \in \mathrm{SU}_n(\mathbb{C})$. Then U is a bicycle.*

Proof. Clearly, we only need to consider the case where U is diagonal. Since the unitary cycle Q is unitarily equivalent to its adjoint, the decomposition $U = VQV^*Q^*$ from Proposition 2.4 can be rewritten as

$$U = (VQV^*)(WQW^*)$$

with some unitary operator W.

□

3. The significance of the bilateral shift

3.1. We now turn our attention to the infinite-dimensional setting. **From now on, we will always denote $\mathcal{H}$ as an infinite-dimensional, separable, complex Hilbert space**, unless otherwise specified. We begin by justifying our interest in the bilateral shift operator, which will play the role that the unitary cycle Q played in the finite-dimensional setting. In anticipation of this, we define the following.

3.2. Definition. *Let $\mathcal{H}$ be an infinite-dimensional, separable, complex Hilbert space with orthonormal basis $\{e_k\}_{k\in\mathbb{Z}}$, and denote by B the bilateral shift operator satisfying $Be_k = e_{k+1}$ for all $k \in \mathbb{Z}$. For $n \geq 1$, we define an n-**cycle** to be an operator of the form $B_1B_2\cdots B_n$, where each B_j is unitarily equivalent to B. We use the terms **bicycle** and **tricycle** to refer to 2-cycles and 3-cycles respectively.*

3.3. Remark. We note that an n-cycle V is automatically an $(n+1)$-cycle. To see this, it suffices to prove the unintuitive fact that B itself is a bicycle. One of the key properties of the bilateral shift which will prove invaluable below is that it is, in the parlance of [14], J_n**-stable** for all $n \geq 2$. That is, by [14, Proposition 3.28], given $n \geq 2$,

$$B \simeq J_n(B) := \begin{bmatrix} 0 & I & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \ddots & \cdots & \vdots & \vdots \\ \vdots & & & \ddots & \cdots & & \vdots \\ \vdots & & & & \ddots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & I \\ B & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \in \mathcal{B}(\mathcal{H}^{(n)}).$$

We may likewise define $J_n(T)$ for any $T \in \mathcal{B}(\mathcal{H})$. By [14, Lemma 2.4], if $X \simeq Y$, then $J_n(X) \simeq J_n(Y)$. Now $B \simeq B^*$, from which we deduce that $B \simeq J_n(B) \simeq J_n(B^*)$.

It is a standard result that B is unitarily equivalent to the operator $M_z \in \mathcal{B}(L^2(\mathbb{T}, \mu))$, where $M_z f(z) = zf(z)$, $\mathbb{T}$ is the unit circle in the complex plane, and μ is Lebesgue measure on $\mathbb{T}$. Let $H = M_x$ be the operator of multiplication by $f(x) = x$ on $L^2([0,1], m)$, where m denotes Lebesgue measure on $\mathbb{R}$ (restricted to $[0,1]$). Then via the functional calculus, $B \simeq e^{2\pi \mathrm{i}H} \simeq \begin{bmatrix} e^{2\pi\frac{\mathrm{i}H}{2}} & 0 \\ 0 & e^{2\pi\frac{-\mathrm{i}H}{2}} \end{bmatrix}$. Note that

$$B \simeq J_2(B^*) = \begin{bmatrix} 0 & I \\ B^* & 0 \end{bmatrix} \simeq J_2(B) = \begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix}.$$

Hence,

$$B \simeq \begin{bmatrix} e^{2\pi\frac{\mathrm{i}H}{2}} & 0 \\ 0 & e^{2\pi\frac{-\mathrm{i}H}{2}} \end{bmatrix} = \left(\begin{bmatrix} 0 & I \\ B^* & 0 \end{bmatrix}\right)\left(\begin{bmatrix} B & 0 \\ 0 & e^{2\pi\frac{\mathrm{i}H}{2}} \end{bmatrix}\begin{bmatrix} 0 & I \\ B & 0 \end{bmatrix}\begin{bmatrix} B^* & 0 \\ 0 & e^{2\pi\frac{-\mathrm{i}H}{2}} \end{bmatrix}\right)$$

is a bicycle.

3.4. Let X be a metric space, and suppose that μ, ν are Borel measures on X. We write $\mu \ll \nu$ to indicate that μ is absolutely continuous with respect to ν. If μ and ν are mutually absolutely continuous (i.e. if they are *equivalent*), we write $[\mu] = [\nu]$.

The following notions are from [17]. Let G be a locally compact topological group, and denote by $M(G)$ the set of all probability measures on G. For $a \in G$, set $R_a(x) = xa$, $L_a(x) = ax$, $x \in G$.

For $\mu \in M(G)$, $R_a(\mu), L_a(\mu)$ are the right- and left-translates of μ, and are defined as follows. Given a Borel subset E of G,

$$R_a(\mu)(E) := \mu(R_a^{-1}(E)) = \mu(Ea^{-1}),$$

and

$$L_a(\mu)(E) := \mu(L_a^{-1}(E)) = \mu(a^{-1}E).$$

Define $RA(\mu) = \{a \in G; R_a(\mu) \ll \mu\}$, $LA(\mu) = \{a \in G; L_a(\mu) \ll \mu\}$.

Let E be a subset of G. A measure $\mu \in M(G)$ is said to be right E-quasi-invariant (resp. left E-quasi-invariant) if $E \subset RA(\mu)$ (resp. $E \subset LA(\mu)$). In the event that $E = G$, we say μ is right quasi-invariant (resp. left quasi-invariant).

3.5. Theorem.[17, Corollary 6] *Let G be a locally compact topological group and $\mu \in M(G)$ be right (or left) quasi-invariant, Then μ is equivalent to the Haar measure of G.*

If μ is a regular Borel measure on $\mathbb{C}$, and if μ has compact support $\Omega \subseteq \mathbb{C}$, then we define the normal operator N_μ via $N_\mu f(z) = zf(z)$, $z \in \Omega$, $f \in L^2(\Omega, \mu)$.

3.6. Theorem.[2, Theorem IX 10.1] *Let $\mathcal{H}$ be a separable Hilbert space.*

(a) *If $N \in B(\mathcal{H})$ is a normal operator, then there is a sequence (possibly finite) of measures $\{\mu_n\}$ on $\mathbb{C}$ (each having compact support) such that $\mu_{n+1} \ll \mu_n$ for all n and*

$$N \simeq N_{\mu_1} \oplus N_{\mu_2} \oplus \cdots .$$

(b) *If N and $\{\mu_n\}$ are as in (a) and $M \simeq N_{\nu_1} \oplus N_{\nu_2} \oplus \cdots$., where $\nu_{n+1} \ll \nu_n$ for all n, then $N \simeq M$ if and only if $[\mu_n] = [\nu_n]$ for all n.*

Equipped with the above two theorems, we can now present the result alluded to in the introduction of the paper.

3.7. Theorem. *Let $V \in \mathcal{U}(\mathcal{H})$, and suppose that for every $W \in \mathcal{U}(\mathcal{H})$ there exist $V_1, V_2 \in \mathcal{U}(V)$ such that $W = V_1V_2$. Then there exists $n \in \mathbb{N} \cup \{\infty\}$, such that $V \simeq B^{(n)}$.*

Proof. First note that I being of the form V_1V_2 implies that V is unitarily equivalent to V^*. Now note that for every z on the unit circle $\mathbb{T}$, the operator $\overline{z}I$ is of the form V_1V_2, which implies that $zV \simeq V^*$. We conclude that V is unitarily equivalent to zV, for any $z \in \mathbb{T}$.

By Theorem 3.6(a), there is a sequence (possibly finite) of measures $\{\mu_n\}$ on $\mathbb{T}$ such that $\mu_{n+1} \ll \mu_n$ for all n and

$$V \simeq N_{\mu_1} \oplus N_{\mu_2} \oplus \cdots .$$

Hence, for any $z \in \mathbb{T}$,

$$zV \simeq N_{\mu_{z,1}} \oplus N_{\mu_{z,2}} \oplus \cdots ,$$

where $\mu_{z,n}$ is the translate of μ_n under L_z. It is easy to see that for each $z \in \mathbb{T}$, $\mu_{z,n+1} \ll \mu_{z,n}$ for all n. Since $V \simeq zV$ for all $z \in \mathbb{T}$, by Theorem 3.6(b), we have that

$$[\mu_{z,n}] = [\mu_n], \quad \text{for all } n \text{ and for all } z \in \mathbb{T}.$$

By Theorem 3.5, μ_n is equivalent to the Haar measure of $\mathbb{T}$, which is Lebesgue measure m. Hence, V is unitarily equivalent to the bilateral shift of some (possibly infinite) multiplicity.

□

3.8. We are seeking a unitary operator V such that every unitary operator W can be factored as the product of two operators unitarily equivalent to V. By Theorem 3.7, the only available candidates for V – up to unitary equivalence – are operators of the form $B^{(n)}$, where $n \in \mathbb{N} \cup \{\infty\}$.

The fact that every unitary operator is a product of two operators unitarily equivalent to $B^{(\infty)}$ is due to Halmos and Kakutani [11, Theorem 1]. For the sake of completeness, we include a quick sketch of the proof here.

3.9. Theorem. [Halmos and Kakutani]. *Let $U \in \mathcal{B}(\mathcal{H})$ be a unitary operator. There exist two operators V_1 and V_2, each unitarily equivalent to $B^{(\infty)}$, such that $U = V_1V_2$.*

Proof. It is a standard result (see, e.g., [11, Theorem 1]) that given any normal operator N acting on an infinite-dimensional, separable Hilbert space, we can find a sequence $(\mathcal{H}_n)_{n\in\mathbb{Z}}$ of pairwise orthogonal reducing subspaces for N, all infinite-dimensional, with respect to which we may write $N = \oplus_{n\in\mathbb{Z}} N_n$. Indeed, it was shown in [9, Lemma 3] that if $H = H^* \in \mathcal{B}(\mathcal{H})$, then there exists an infinite-dimensional subspace $\mathcal{M}_1$ of $\mathcal{H}$ which reduces H, such that $\mathcal{M}_1^\perp$ is also infinite-dimensional. The proof there extends *mutatis mutandis* to normal operators, and so give N, we may find a reducing subspace $\mathcal{M}_1$ for N of infinite dimension and co-dimension. As then pointed out in the proof of Theorem 1 in [11], by applying this argument to $N|_{\mathcal{M}_1^\perp}$, one obtains an infinite-dimensional subspace $\mathcal{M}_2$ of $\mathcal{M}_1^\perp$ which reduces $N|_{\mathcal{M}_1^\perp}$ (and which therefore reduces N), such that $\mathcal{M}_1^\perp \ominus \mathcal{M}_2$ is infinite-dimensional. A simple induction process then produces the desired sequence of reducing subspaces for N.

Applying this argument to U and producing the appropriate sequence $(\mathcal{H}_n)_{n\in\mathbb{Z}}$, we obtain the decomposition $U = \oplus_{n\in\mathbb{Z}} U_n$. It is also not hard to see that if $W \in \mathcal{U}(\mathcal{H})$ is a unitary operator satisfying $W\mathcal{H}_n = \mathcal{H}_{n+1}$ for all $n \in \mathbb{Z}$, then W is unitarily equivalent to $B^{(\infty)}$. Letting $Y := WU$, we see that Y is again unitary and satisfies $Y\mathcal{H}_n = \mathcal{H}_{n+1}$ for all $n \in \mathbb{Z}$, so that $Y \simeq B^{(\infty)}$ and $U = W^*Y$. But $B \simeq B^*$, implying that $W^* \simeq W \simeq B^{(\infty)}$, completing the proof.

□

We refer to a unitary operator $W \in \mathcal{B}(\mathcal{H})$ as a **permutation unitary operator** if there exists a countable orthonormal basis $\{e_\lambda : \lambda \in \Lambda\}$ for $\mathcal{H}$ such that $We_\lambda = e_{\varrho(\lambda)}$ for some permutation ϱ of Λ. Obviously $B^{(\infty)}$ is a permutation unitary operator. As an immediate consequence of Theorem 3.9, we obtain the following.

3.10. Corollary. *Every unitary is a product of two permutation unitaries.*

Returning to the problem of factoring unitary operators as products of operators unitarily equivalent to $B^{(n)}$, we next turn our attention to the case where the multiplicity of the bilateral shift is equal to 1. As we shall see, this case is far more challenging.

We begin by establishing some notation. Let $\Omega := \mathcal{U}(B)$ be the unitary orbit of the bilateral shift (of multiplicity one). For $n \geq 1$, we set

$$\Omega_n := \{B_1 B_2 \cdots B_n : B_j \in \Omega, 1 \leq j \leq n\}.$$

It is worth noting that Ω_n is invariant under unitary conjugation for each $n \geq 1$, and since $B \simeq B^*$, each Ω_n is self-adjoint. Furthermore, since $B \simeq \theta B$ for all $\theta \in \mathbb{T}$, we also have that $\theta\,\Omega_n = \Omega_n$ for all $\theta \in \mathbb{T}$, $n \geq 1$.

3.11. Proposition. *Every diagonalisable unitary operator U acting on a separable, infinite-dimensional Hilbert space $\mathcal{H}$ belongs to Ω_2.*

Proof. The proof is analogous to that given for Proposition 2.4.

First we choose an orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$ which diagonalises $U = \operatorname{diag}(u_n)_{n\in\mathbb{Z}}$. Let $B_1 \in \mathcal{U}(B)$ be the operator $B_1 e_n = e_{n+1}$ for all $n \in \mathbb{Z}$. With respect to the same orthonormal basis, define

$$V = \operatorname{diag}(v_n)_{n\in\mathbb{Z}},$$

where

$$v_n = \begin{cases} u_0 u_1 \cdots u_n & \text{if } n \geq 0 \\ 1 & \text{if } n = -1 \\ \overline{u_{n+1} u_{n+2} \cdots u_{-1}} & \text{if } n \leq -2. \end{cases}$$

A routine computation shows that $W := B_1 V^* B_1^* = \operatorname{diag}(w_n)_{n\in\mathbb{Z}}$, where

$$w_n = \begin{cases} 1 & \text{if } n = 0 \\ \overline{u_0 u_1 \cdots u_{n-1}} & \text{if } n \geq 1 \\ u_n u_{n+1} \cdots u_{-1} & \text{if } n < 0. \end{cases}$$

A second routine computation then yields

$$U = VW = (V B_1 V^*) B_1^*.$$

Since $V B_1 V^*, B_1^* \in \mathcal{U}(B)$, we are done.

□

3.12. Corollary. *Every unitary operator in $\mathcal{B}(\mathcal{H})$ lies in the norm-closure of Ω_2. In fact,*

$$\overline{\Omega_2} = \mathcal{U}(\mathcal{H}) = \overline{\mathcal{U}(B)} \cdot \overline{\mathcal{U}(B)}.$$

Proof. That $\overline{\Omega_2} = \mathcal{U}(\mathcal{H})$ is an immediate consequence of the Weyl-von Neumann-Berg Theorem, which asserts that every normal operator is a norm-limit of diagonalisable normal operators with the same spectrum. Thus if the original normal operator is unitary, so can we choose the approximating diagonalisable normal operator.

Moreover, the Weyl-von Neumann-Berg Theorem also proves that $B^{(\infty)} \in \overline{\mathcal{U}(B)}$. Applying Theorem 3.9, we obtain the second equality.

□

3.13. Remarks.

(a) Voiculescu [24] has shown that if $N \in \mathcal{B}(\mathcal{H})$ is normal and $\varepsilon > 0$, then we can find a diagonalisable normal operator D with $\sigma(D) = \sigma(N)$ such that $|||N - D||| < \varepsilon$, where $||| \cdot |||$ represents the Hilbert-Schmidt norm. If N is unitary, then D too must be unitary, yielding a strengthening of (the first half of) the conclusion of Corollary 3.12.

(b) Murphy [16] has shown that every unitary operator U is a product of two diagonalisable unitary operators. By combining this with Proposition 3.11, we conclude that $\mathcal{U}(\mathcal{H}) = \Omega_4$. That is, every unitary operator is a product of four operators, each unitarily equivalent to the bilateral shift B, or in other words, every unitary operator is a 4-cycle. As indicated by the title, we shall show that every unitary is in fact the product of three such operators, each unitarily equivalent to B, and that when the Lebesgue measure of the spectrum of the unitary U is positive, then U lies in Ω_2.

4. Most unitary operators are bicycles

4.1. Although we have not managed to show that every unitary operator is a bicycle, we shall demonstrate that every unitary operator whose spectrum has positive Lebesgue measure is. Furthermore, we have several sufficient conditions for belonging to Ω_2 which are satisfied by unitary operators V for which $\sigma(V)$ has zero Lebesgue measure. In this sense, "most" unitary operators are bicycles.

The next result is crucial to the remainder of this section. Suppose that we are given $n \in \mathbb{N}$ and an operator $T = \oplus_{j=1}^{n} T_j \in \mathcal{B}(\mathcal{H})$, where each T_j acts on an infinite-dimensional, separable subspace of $\mathcal{H}$. We can associate to this decomposition of T a notion of "operator determinant" $\Delta(T) = T_1 T_2 \cdots T_n$. Of course, the same operator T can admit many such direct sum decompositions relative to different decompositions of the space, and in fact we should be associating to the *unitary orbit* of T a *set* of "operator determinants". We wish to exploit an interesting property of such operator determinants.

4.2. Theorem. *Let $n \geq 2$ be an integer, and suppose that $V, V_j \in \mathcal{B}(\mathcal{H})$ are unitary, $1 \leq j \leq n$. Suppose furthermore that V is unitarily equivalent to $\oplus_{j=1}^{n} V_j$ and that for some permutation ϱ of the set $\{1, 2, \ldots, n\}$,*

$$\prod_{j=1}^{n} V_{\varrho(j)} \quad \text{is a bicycle.}$$

Then V is a bicycle.

Proof. Since we are only asking that V be unitarily equivalent to $\oplus_{j=1}^{n} V_j$, we are free to order these summands in any way we choose. In light of this, by relabelling the factors if necessary, we may assume without loss of generality that $V = \oplus_{j=1}^{n} V_j$, and it furthermore suffices to consider the case where

$$V_n V_{n-1} V_{n-2} \cdots V_2 V_1$$

is a bicycle. If $X_j \in \mathcal{U}(\mathcal{H})$ are unitary, $j = 1, \cdots, n$ and $B_1, B_2 \in \mathcal{U}(B)$, we may define the unitary operator

$$U := \begin{bmatrix} & & & X_1 \\ & & X_2 & \\ & \iddots & & \\ X_n & & & \end{bmatrix},$$

and set (recalling the definition of $J_n(B_j)$, $j = 1, 2$ from Remark 3.3):

$$T := J_n(B_1)UJ_n(B_2)U^* = \begin{bmatrix} X_2X_1^* & & & \\ & \ddots & & \\ & & X_nX_{n-1}^* & \\ & & & B_1X_1B_2X_n^* \end{bmatrix}.$$

Our strategy is to show that we may choose suitable X_j, $j = 1, \cdots, n$ and $B_1, B_2 \simeq B$ such that $T = V$. Note that if we can do this, then using the fact that $J_n(B_1) \simeq B \simeq J_n(B_2)$, we conclude that T is a bicycle.

Since $\prod_{j=1}^n V_{n-j+1}$ is a bicycle, there exist B_1, B_2 such that $B_1, B_2 \simeq B$ and $\prod_{j=1}^n V_{n-j+1} = B_1B_2$. Set $X_1 = I$, $X_2 = V_1, X_3 = V_2V_1, \cdots, X_n = \prod_{j=1}^{n-1} V_{n-j}$. Then it is straightforward to check that

$$X_{j+1}X_j^* = V_j,\ 1 \leq j \leq n-1,$$

and that

$$B_1X_1B_2X_n^* = B_1B_2(\prod_{j=1}^{n-1} V_{n-j})^* = (\prod_{j=1}^{n} V_{n-j+1})(\prod_{j=1}^{n-1} V_{n-j})^* = V_n.$$

Thus $V = T$, and hence V is a bicycle.

□

As $I = BB^*$ and $B^* \simeq B$, I is clearly a bicycle. Thus, by Theorem 4.2, we have the following corollary.

4.3. Corollary. *Let $U \in \mathcal{B}(\mathcal{H})$ be a unitary operator. Then $U \oplus U^*$ is a bicycle.*

4.4. Our main goal is to prove that every unitary operator U whose spectrum has positive Lebesgue measure is a bicycle. We shall accomplish this in a number of steps, each of which involves describing a class of unitary operators admitting an "operator determinant" which we can show is a bicycle. From there, we apply Theorem 4.2 to deduce that the members of that class are themselves bicycles.

We briefly recall the following version of the Spectral Theorem.

4.5. Theorem.[2, IX, Theorem 2.2] *If N is a normal operator, there is a unique spectral measure E on the Borel subsets of $\sigma(N)$ such that*

(1) *$N = \int z dE(z)$;*
(2) *if G is a non-empty relatively open subset of $\sigma(N)$, $E(G) \neq 0$; and*
(3) *for $A \in \mathcal{B}(\mathcal{H})$, we have that $AN = NA$ and $AN^* = N^*A$ if and only if $AE(\Delta) = E(\Delta)A$ for every Δ.*

The following characterisation of the non-essential spectrum of a normal operator is also from Conway [2].

4.6. Proposition.[2, XI Proposition 4.2] *Let $N \in \mathcal{B}(\mathcal{H})$ be a normal operator. Then*

$$\sigma(N) \setminus \sigma_e(N) = \{\lambda \in \sigma(N) : \lambda \text{ is an isolated point of } \sigma(N), \text{ and } \dim\ker(\lambda I - N) < \infty\}.$$

4.7. Remark. Let $N \in \mathcal{B}(\mathcal{H})$ be a normal operator. From the above proposition, we have that

(a) each cluster point of $\sigma(N)$ belongs to $\sigma_e(N)$;
(b) $\sigma_e(N) = \sigma(N)$ in the case that $\sigma(N)$ is connected; and
(c) $\sigma(N) \setminus \sigma_e(N)$ is a countable set.

4.8. Theorem. *Let $V \in \mathcal{B}(\mathcal{H})$ be a unitary operator and suppose that*

(a) *$\sigma(V)$ is a perfect subset of $\mathbb{T}$ (i.e. $\sigma(V)$ has no isolated points), and*
(b) *$\sigma(V)$ is symmetric with respect to some line passing through the origin.*

Then V is a bicycle.

Proof. We first observe that it suffices to prove the result after replacing condition (b) by condition

(b′) $\sigma(V)$ is symmetric with respect to the x-axis.

Indeed, suppose that (a) and (b) hold. There exists $\theta \in \mathbb{T}$ such that $\sigma(\theta V)$ is symmetric with respect to the x-axis, while $\sigma(\theta V)$ clearly remains a perfect set. If the Theorem holds for (a) and (b′), then θV is a bicycle. But Ω_2 is invariant under multiplication by elements of $\mathbb{T}$, and so V must itself be a bicycle.

Suppose therefore that (a) and (b′) hold. Our next goal is to prove that under the above hypotheses, there exist unitary operators $V_1, V_2, V_3, V_4 \in \mathcal{B}(\mathcal{H})$ such that

(i) $\sigma_e(V_j) = \sigma(V_j)$, $j = 1, 2, 3, 4$;
(ii) $V \simeq V_1 \oplus V_2 \oplus V_3 \oplus V_4$; and
(iii) $V_4V_3V_2V_1$ is a compact perturbation of I.

If we can do this, then since $V_4V_3V_2V_1$ is normal and a compact perturbation of the identity, it is diagonalisable. By Proposition 3.11, it is a bicycle, and then by Theorem 4.2, so is V.

Since $\sigma(V)$ is a perfect subset of $\mathbb{T}$ and is symmetric with respect to the x-axis, there exist $\theta_0 \in (0, \pi)$ such that $\sigma(V) \cap \Upsilon_j \neq \varnothing$, where

$$\begin{aligned}
\Upsilon_1 &= \{e^{i\theta} : \theta \in (0, \theta_0)\} \\
\Upsilon_2 &= \{e^{i\theta} : \theta \in (\theta_0, \pi)\} \\
\Upsilon_3 &= \{e^{i\theta} : \theta \in (\pi, 2\pi - \theta_0)\} \\
\Upsilon_4 &= \{e^{i\theta} : \theta \in (2\pi - \theta_0, 2\pi)\}.
\end{aligned}$$

We next set

$$\begin{aligned}
\Delta_1 &= \sigma(V) \cap \{e^{i\theta} : \theta \in [0, \theta_0)\} \\
\Delta_2 &= \sigma(V) \cap \{e^{i\theta} : \theta \in [\theta_0, \pi)\} \\
\Delta_3 &= \sigma(V) \cap \{e^{i\theta} : \theta \in [\pi, 2\pi - \theta_0)\} \\
\Delta_4 &= \sigma(V) \cap \{e^{i\theta} : \theta \in [2\pi - \theta_0, 2\pi)\}.
\end{aligned}$$

By the Spectral Theorem 4.5, there exists a unique measure E such that

$$V = \int_{\mathbb{T}} \lambda dE_\lambda.$$

Define

$$W_j = \int_{\mathbb{T}} \chi_{\Delta_j}(\lambda)\lambda dE_\lambda = \int_{\Delta_j} \lambda dE_\lambda,\ j = 1, 2, 3, 4,$$

where χ_E denote the characteristic function of E.

Then $\sigma(W_j) = \overline{\Delta_j}$, $j = 1, 2, 3, 4$. Clearly

$$V = W_1 \oplus W_2 \oplus W_3 \oplus W_4.$$

Let $\{\theta_n\}_{n=1}^\infty$ be a dense subset of $\sigma(V) \cap \Upsilon_1$, $\{\eta_n\}_{n=1}^\infty$ be a dense subset of $\sigma(V) \cap \Upsilon_2$. Since $\sigma(V)$ is a perfect subset of $\mathbb{T}$ and is symmetric with respect to the x-axis, one concludes that $\{2\pi - \eta_n\}_{n=1}^\infty$ is dense in $\sigma(V) \cap \Upsilon_3$, and $\{2\pi - \theta_n\}_{n=1}^\infty$ is dense in $\sigma(V) \cap \Upsilon_4$. Fix an orthonormal basis $\mathcal{E} := \{e_n\}_{n=1}^\infty$ of $\mathcal{H}$. With respect to $\mathcal{E}$, define

$$\begin{aligned}
D_1 &= \operatorname{diag}\{e^{i\theta_1}, \cdots, e^{i\theta_n}, \cdots\} \\
D_2 &= \operatorname{diag}\{e^{i\eta_1}, \cdots, e^{i\eta_n}, \cdots\} \\
D_3 &= \operatorname{diag}\{e^{i(2\pi-\eta_1)}, \cdots, e^{i(2\pi-\eta_n)}, \cdots\} \\
D_4 &= \operatorname{diag}\{e^{i(2\pi-\theta_1)}, \cdots, e^{i(2\pi-\theta_n)}, \cdots\}.
\end{aligned}$$

It is routine to check that

$$D_4 D_3 D_2 D_1 = I.$$

and that

$$\sigma(D_j) = \overline{\Delta_j},\ j = 1, 2.$$

Since $\overline{\Delta_j}$ has no isolated points, it follows from Remark 4.7 that

$$\sigma_e(D_j) = \sigma(D_j) = \overline{\Delta_j} = \sigma(W_j) = \sigma_e(W_j),\ j = 1, 2, 3, 4.$$

By the Weyl–von Neumann–Berg Theorem, there exist unitary operators U_j and compact operators K_j such that

$$U_j^* W_j U_j = V_j := D_j + K_j,\ j = 1, 2, 3, 4.$$

Define

$$U = U_1 \oplus U_2 \oplus U_3 \oplus U_4.$$

Then

$$V \simeq U^* V U = V_1 \oplus V_2 \oplus V_3 \oplus V_4,$$

and there exists a compact operator $K \in \mathcal{B}(\mathcal{H})$ such that

$$V_4 V_3 V_2 V_1 = D_4 D_3 D_2 D_1 + K = I + K.$$

Note that $I + K$ is unitary and a compact perturbation of the identity, hence it is diagonalisable. By Proposition 3.11, $V_4 V_3 V_2 V_1 = I + K$ is a bicycle. Finally, by Theorem 4.2, V is a bicycle.

□

4.9. Corollary. *Let $V \in \mathcal{B}(\mathcal{H})$ be a unitary operator and suppose that $\sigma(V)$ is a non-degenerate arc of $\mathbb{T}$. Then V is a bicycle. In particular, if $\sigma(V) = \mathbb{T}$, then V is a bicycle.*
Proof. If $\sigma(V)$ is a non-degenerate arc (possibly $\mathbb{T}$ itself), then it is clearly symmetric with respect to some line passing through the origin, and Theorem 4.8 applies.

□

As pointed out in Remark 3.3, B itself is a bicycle. Of course, for any unitary $U \in \mathcal{U}(\mathcal{H})$, $\sigma(B \oplus U) = \mathbb{T}$, and so by Corollary 4.9, $B \oplus U$ is a bicycle. In particular, $B^n \simeq B^{(n)}$ is a bicycle for $n \in \mathbb{N} \cup \{\infty\}$.

The following theorem is a special case of a result of Schechter [21, Theorem 2.1]. To avoid any possible confusion, we remark that for $n \geq 1$, we denote the n-fold tensor product of a Hilbert space $\mathcal{H}$ with itself by $\mathcal{H}^{\otimes n}$. Similarly, for $k \geq 1$, $I^{\otimes k}$ denotes the k-fold tensor product of the identity operator with itself.

4.10. Theorem. *Let $n \geq 2$ be a positive integer and let $\mathcal{M} = \mathcal{H}^{\otimes n}$. Let $A_j \in \mathcal{B}(\mathcal{H})$, and let $T_j \in \mathcal{B}(\mathcal{M})$ defined by*

$$T_j = I^{\otimes(j-1)} \otimes A_j \otimes I^{\otimes(n-j)}, \quad 1 \leq j \leq n.$$

Then the spectrum of $T_1 \cdots T_n$ consists of those $\mu \in \mathbb{C}$ such that $\mu = \prod_{j=1}^n \lambda_j$, where $\lambda_j \in \sigma(A_j), 1 \leq j \leq n$. That is,

$$\sigma(T_1 \cdots T_n) = \sigma(T_1) \cdots \sigma(T_n).$$

In Corollary 4.9, we showed that if V is a unitary operator whose spectrum is a non-degenerate arc in $\mathbb{T}$, then $V \in \Omega_2$. We now show that it suffices that $\sigma(V)$ *contain* a non-degenerate arc.

4.11. Theorem. *Let $V \in \mathcal{B}(\mathcal{H})$ be a unitary operator and assume that $\sigma(V)$ contains a non-degenerate arc. Then V is a bicycle.*
Proof. Note that for $\lambda \in \mathbb{T}$, V is bicycle if and only if λV is a bicycle. Thus, by scaling V by an appropriate $\lambda \in \mathbb{T}$, we may assume without loss of generality that $1 \in \text{int}(\sigma(V))$, where $\text{int}(\sigma(V))$ denotes the interior of $\sigma(V)$ in the relative topology it inherits as a subset of $\mathbb{T}$. Then there exists $\theta_0 \in (0, \frac{1}{10}\pi)$ such that $\{e^{i\theta} : 0 \leq \theta \leq 2\theta_0\} \subset \sigma(V)$. Let $N = \lfloor \frac{4\pi}{\theta_0} \rfloor$. Then

$$(\sum_{j=1}^{N} \frac{j}{N})\theta_0 = \frac{N(N+1)}{2N}\theta_0 \geq \frac{1}{2}(\frac{4\pi}{\theta_0})\theta_0 \geq 2\pi. \tag{1}$$

Let

$$\Gamma_j = \{e^{i\theta} : \frac{j-1}{N}\theta_0 \leq \theta < \frac{j}{N}\theta_0\}, \ j = 1, 2, \cdots, N.$$

Then as we argued in Theorem 4.8, by the Spectral Theorem, there exists a unique projection-valued measure E such that

$$V = \int_{\mathbb{T}} \lambda dE_\lambda.$$

Define

$$V_j = \int_{\Gamma_j} \lambda dE_\lambda, \ j = 1, \cdots, N.$$

Then $\sigma(V_j) = \overline{\Gamma_j}$ is connected, and hence by Remark 4.7 $\sigma_e(V_j) = \overline{\Gamma_j} = \sigma(V_j)$, $j = 1, 2, \cdots, N$. As equation (1) holds, it is easy to check that

$$\sigma(V_1)\cdots\sigma(V_N) = \mathbb{T}. \tag{2}$$

Define

$$V_0 = \int_{\mathbb{T}\setminus(\cup_{j=1}^N \Gamma_j)} \lambda dE_\lambda,$$

from which it clearly follows that

$$V = V_0 \oplus V_1 \oplus \cdots \oplus \cdots \oplus V_N.$$

Set $\mathcal{M} = \mathcal{H}^{\otimes(N+1)} \oplus \mathcal{H}$,

$$R_0 = T_0 \oplus V_0 := [(e^{i(2\theta_0)}I) \otimes I^{\otimes N}] \oplus V_0; \text{ and}$$
$$R_j = T_j \oplus I := [I \otimes I^{\otimes(j-1)} \otimes V_j \otimes I^{\otimes(N-j)}] \oplus V_j,\ j \le i \le N.$$

Then $R_j \in \mathcal{B}(\mathcal{M})$, $j = 0, 1, \cdots, N$. Recall the fact that if A, B are invertible operators, then $\sigma(AB) = \sigma(BA)$. By this fact, Theorem 4.10 and equation (2),

$$\begin{aligned}\sigma((\prod_{j=0}^N R_{N-j})) = \sigma(\prod_{j=0}^N R_j) &\supset [\prod_{j=0}^N \sigma(T_j)] \\ &= (e^{i(2\theta_0)}) \prod_{j=1}^N \sigma(T_j) = (e^{i(2\theta_0)}) \prod_{j=1}^N \sigma(V_j) \\ &= e^{i(2\theta_0)}\mathbb{T} \\ &= \mathbb{T}.\end{aligned}$$

Therefore, by Remark 4.7,

$$\sigma_e((\prod_{j=0}^N R_{N-j})) = \sigma((\prod_{j=0}^N R_{N-j})) = \mathbb{T}.$$

Since $e^{i2\theta_0} \in \sigma_e(V_0)$ and $\sigma_e(V_j) = \sigma(V_j) = \sigma(R_j) = \sigma_e(R_j)$, $j = 1, \cdots, N$, by the Weyl–von Neumann-Berg Theorem, there exists $K_j \in \mathcal{K}(\mathcal{M})$, $j = 0, 1, \cdots, N$ such that

$$V_j \simeq R_j + K_j,\ 0 \le j \le N.$$

Thus

$$V \simeq \oplus_{j=0}^N (R_j + K_j).$$

Note that

$$\prod_{j=0}^N (R_{N-j} + K_{N-j}) - \prod_{j=0}^N R_{N-j} \in \mathcal{K}(\mathcal{M}),$$

and therefore

$$\sigma_e(\prod_{j=0}^N (R_{N-j} + K_{N-j})) = \sigma_e((\prod_{j=0}^N R_{N-j})) = \mathbb{T}.$$

We conclude that

$$\sigma(\prod_{j=0}^N (R_{N-j} + K_{N-j})) = \mathbb{T}.$$

By Corollary 4.9, $\prod_{j=0}^{N}(R_{N-j}+K_{N-j})$ is a bicycle. Finally, by Theorem 4.2, V is a bicycle. □

The following is the main result of this section.

4.12. Theorem. *Let $V \in \mathcal{B}(\mathcal{H})$ be a unitary operator and assume that $\sigma(V)$ has positive Lebesgue measure. Then V is a bicycle.*

Proof. Since $\sigma(V)$ has positive Lebesgue measure, by Remark 4.7, so does $\sigma_e(V)$. Set $A = \{\theta \in [0, 2\pi) : e^{i\theta} \in \sigma_e(V)\}$. Then $m(A) > 0$, where again, m denotes Lebesgue measure on $\mathbb{R}$. Hence, there exist $0 < \theta_1 < \theta_2 < 2\pi$ such that the sets

$$A_0 := [0, \theta_1) \cap A,\ A_1 := [\theta_1, \theta_2) \cap A, \text{ and } A_2 := [\theta_2, 2\pi) \cap A$$

all have positive Lebesgue measure. It is clear that

$$A = A_0 \sqcup A_1 \sqcup A_2.$$

By Steinhaus' Theorem [23, Page 99], $A_1 + A_2$ contains a nonempty interval. Define $\Gamma_j = e^{iA_j} = \{e^{i\theta} : \theta \in A_j\}$, $j = 0, 1, 2$. Then $\Gamma_1\Gamma_2 = e^{A_1+A_2}$ contains an open arc. Then, arguing once again as we did in the proof of Theorem 4.8, by the Spectral Theorem, there exists a unique projection-valued measure E such that

$$V = \int_{\mathbb{T}} \lambda dE_\lambda.$$

Define

$$V_0 = \int_{\Gamma_0 \cup (\sigma(V) \setminus \sigma_e(V))} \lambda dE_\lambda,$$

and

$$V_j = \int_{\Gamma_j} \lambda dE_\lambda,\ j = 1, 2.$$

By the above constructions, $\sigma_e(V_0) \neq \varnothing$, and $\sigma_e(V_j) \supset \Gamma_j$, $j = 1, 2$. Choose θ_0 such that $e^{i\theta_0} \in \sigma_e(V_0)$. Set $\mathcal{M} = (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus \mathcal{H}$, and consider the operators

$$\begin{aligned} R_0 = T_0 \oplus V_0 &:= ((e^{i\theta_0} I) \otimes I \otimes I) \oplus V_0; \\ R_1 = T_1 \oplus V_1 &:= (I \otimes V_1 \otimes I) \oplus V_1; \text{ and} \\ R_2 = T_2 \oplus V_2 &:= (I \otimes I \otimes V_2) \oplus V_2. \end{aligned}$$

Then $R_j \in \mathcal{B}(\mathcal{M})$, $j = 0, 1, 2$. By Theorem 4.10,

$$\sigma(R_2R_1R_0) = \sigma(R_0R_1R_2) \supset \sigma(T_0T_1T_2) = e^{i(\theta_0)}\sigma(T_1)\sigma(T_2) \supset e^{i\theta_0}\Gamma_1\Gamma_2$$

contains an open arc. Therefore, by Remark 4.7, $\sigma_e(R_2R_1R_0)$ contains an open arc.

Observe next that $e^{i\theta_0} \in \sigma_e(V_0)$, and $\sigma_e(V_j) = \sigma(V_j) = \sigma(R_j) = \sigma_e(R_j)$, $j = 1, 2$. By the Weyl-von Neumann-Berg Theorem, there exists $K_j \in \mathcal{K}(\mathcal{M})$, $j = 0, 1, 2$ such that

$$V_j \simeq R_j + K_j,\ 0 \leq j \leq 2.$$

In particular,

$$V \simeq \oplus_{j=0}^{2}(R_j + K_j).$$

Note furthermore that

$$(R_2 + K_2)(R_1 + K_1)(R_0 + K_0) - R_2R_1R_0 \in \mathcal{K}(\mathcal{M}),$$

and that

$$\sigma((R_2+K_2)(R_1+K_1)(R_0+K_0)) \supset \sigma_e((R_2+K_2)(R_1+K_1)(R_0+K_0)) = \sigma_e(R_2R_1R_0).$$

Consequently, $\sigma((R_2+K_2)(R_1+K_1)(R_0+K_0))$ contains a non-degenerate arc. By Theorem 4.11, $(R_2+K_2)(R_1+K_1)(R_0+K_0)$ is a bicycle. Finally, by Theorem 4.2, V is a bicycle.

□

We finish this section with a few isolated results concerning operators in Ω_2.

4.13. Theorem. *Let $V \in \mathcal{B}(\mathcal{H})$ be a unitary operator and suppose that $\sigma_e(V)$ has only finitely many components. Then V is a bicycle.*

Proof. If $\sigma_e(V)$ has positive Lebesgue measure, then by Theorem 4.12, V is a bicycle. Otherwise, if $\sigma_e(V)$ has Lebesgue measure zero, then from the assumption that $\sigma_e(V)$ has only finitely many components, we may conclude that $\sigma_e(V)$ is a finite set. Set $\sigma_e(V) = \{\lambda_1, \cdots, \lambda_N\}$. Then there exists $K_i \in \mathcal{K}(\mathcal{H})$ such that

$$V \simeq \oplus_{j=1}^{N}(\lambda_j I + K_j).$$

Note that the V's "operator determinant"

$$W := \prod_{j=1}^{N}[(\lambda_{N-j+1}I + K_{N-j+1})]$$

is both unitary and a compact perturbation of a scalar operator. Hence W is diagonalisable. By Proposition 3.11, W is a bicycle. Thus by Theorem 4.2, V is a bicycle.

□

4.14. Our final result in this section shows that every permutation unitary is a bicycle. To do this, we shall require a combinatorial result due to Droste [5]. But first we establish some notation and terminology.

Fix a countably infinite set Λ, and let Σ_0 denote the group of all permutations of Λ. For $\varrho \in \Sigma_0$, let $[\varrho]$ denote the conjugacy class of ϱ. The set $\text{supp}(\varrho) := \{a \in \Lambda : \varrho(a) \neq a\}$ is called the **support** of ϱ.

4.15. Theorem.[5, Lemma 3.1] *Let σ, α, β be permutations of a countable set Λ such that σ has infinite support and α and β each have at least one infinite orbit. Then σ is a product of two permutations conjugate to α and β, respectively.*

4.16. Theorem. *Every permutation unitary W acting on an infinite-dimensional, separable, complex Hilbert space is a bicycle.*

Proof. We shall identify the permutation unitary operator W with the corresponding permutation of the orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$. If W has an infinite cycle (i.e. there exists a basis vector e_{n_0} whose orbit under W is infinite), then W has infinite support. Since, as a permutation of $\mathbb{Z}$, B has one infinite orbit, by Theorem 4.15, there exists $B_1, B_2 \in \mathcal{U}(B)$ such that $W = B_1B_2$.

If W has no infinite cycle, then $W \simeq \oplus_{k=1}^{\infty}W_k$, where each W_k is a finite cycle, $k \in \mathbb{N}$. Then W is unitarily equivalent to some diagonalisable unitary operator D. By Proposition 3.11, W is a bicycle.

□

5. Every unitary operator is a tricycle

5.1. We observed in Section 3 that every unitary operator is a 4-cycle. Combining this with the fact that $B \in \Omega_2$ (see Remark 3.3), we conclude that every unitary operator is an n-cycle for any $n \geq 4$. It would be interesting to know whether or not every unitary is a bicycle. In this section, we prove that every unitary operator is a tricycle.

5.2. Lemma. *Let $V \in \mathcal{B}(\mathcal{H})$ be a unitary operator. Then there exists $B_1 \simeq B$ such that*

$$\sigma(VB_1) = \mathbb{T}.$$

Proof. Choose $\alpha \in \sigma_e(V)$. Then by the Weyl–von Neumann–Berg Theorem, there exist unitary operators W_1, W_2 from $\mathcal{H} \oplus \mathcal{H}$ to $\mathcal{H}$, $\tilde{V}_1, \tilde{B}_1 \in \mathcal{U}(\mathcal{H} \oplus \mathcal{H})$ and compact operators $K_1, K_2 \in \mathcal{K}(\mathcal{H} \oplus \mathcal{H})$ such that

$$W_1^* V W_1 = \tilde{V}_1 := \begin{bmatrix} \alpha I & \\ & V \end{bmatrix} + K_1, \; W_2^* B W_2 = \tilde{B}_1 := \begin{bmatrix} B & \\ & B \end{bmatrix} + K_2.$$

Now,

$$\sigma_e(\tilde{V}_1 \tilde{B}_1) = \sigma_e(\begin{bmatrix} \alpha I & \\ & V \end{bmatrix} \begin{bmatrix} B & \\ & B \end{bmatrix}) = \sigma_e(\alpha B) \cup \sigma_e(VB) \supseteq \sigma_e(\alpha B) = \mathbb{T}.$$

But then $\mathbb{T} = \sigma_e(\tilde{V}_1 \tilde{B}_1) \subseteq \sigma(\tilde{V}_1 \tilde{B}_1) \subseteq \mathbb{T}$.

Finally, we set $B_1 = W_1 \tilde{B}_1 W_1^*$, so that

$$\sigma(VB_1) = \sigma(W_1(\tilde{V}_1 \tilde{B}_1) W_1^*) = \sigma(\tilde{V}_1 \tilde{B}_1) = \mathbb{T}.$$

□

5.3. Theorem. *Every unitary is a tricycle.*

Proof. Let $V \in \mathcal{U}(\mathcal{H})$. By Lemma 5.2, there exists $B_1 \simeq B$ such that

$$\sigma(VB_1) = \mathbb{T}.$$

By Corollary 4.9, VB_1 is a bicycle. Thus $V = (VB_1)B_1^*$ is a tricycle.

□

6. Final remarks

6.1. In trying to determine whether or not $\mathcal{U}(\mathcal{H}) = \Omega_2$, Theorem 4.12 reduces the problem to the case of unitary operators U for which $\sigma(U)$ has Lebesgue measure zero. The case where U is diagonalisable is handled by Proposition 3.11. Theorem 4.13 shows that we need only consider unitary operators whose essential spectrum has infinitely many components, and Theorem 4.8 shows we need only consider the case where $\sigma(U)$ is either not perfect or is not symmetric with respect to any line passing through the origin. Unfortunately, we have not been able to resolve these final cases.

The fact that the bilateral shift plays a central role in our decomposition led a colleague of ours, P. Guinand, to ask whether one can decompose a general unitary as a product of operators unitarily equivalent to either the unilateral forward shift S, or its adjoint S^*. Here, as always, the unilateral forward shift operator is defined by fixing an orthonormal basis $\{e_n\}_{n=1}^{\infty}$ for $\mathcal{H}$ and requiring that $Se_n = e_{n+1}$ for all $n \geq 1$. Although this is only defined up

to unitary equivalence, we are considering the unitary orbit of S and of its adjoint, and thus the particular choice of the orthonormal basis is not important. We remark, incidentally, that Fredholm index considerations clearly require the presence of both classes of operators to obtain an invertible operator. To our surprise, the answer to Guinand's question is yes.

6.2. Theorem. *Let S be the unilateral shift operator acting on the infinite-dimensional, separable, complex Hilbert space $\mathcal{H}$. There exists $S_1, S_2 \in \mathcal{U}(S)$ such that $B = S_2^* S_1$,where B denotes a bilateral shift acting on $\mathcal{H}$.*

Proof. Let $H = M_x$ be the multiplication operator on $L^2([0,1], m)$. Then

$$B \simeq e^{2\pi \mathrm{i} H} \simeq \begin{bmatrix} e^{2\pi \frac{\mathrm{i}H}{2}} & 0 \\ 0 & e^{2\pi \frac{-\mathrm{i}H}{2}} \end{bmatrix}.$$

Set $V := e^{2\pi \frac{\mathrm{i}H}{2}}$, and note that V is a unitary operator. The set $\Theta_2 := \{X_2^* X_1 : X_1, X_2 \in \mathcal{U}(S)\}$ is clearly invariant under unitary conjugation, as is $\mathcal{U}(B)$, and thus we may assume without loss of generality that

$$B = \begin{bmatrix} V & 0 \\ 0 & V^* \end{bmatrix}.$$

By Proposition 3.29 of [14], S is J_2-stable, which is to say that $S \simeq J_2(S) = \begin{bmatrix} 0 & I \\ S & 0 \end{bmatrix}$. So

$$S^* \simeq \begin{bmatrix} 0 & S^* \\ I & 0 \end{bmatrix} \simeq \begin{bmatrix} 0 & I \\ S^* & 0 \end{bmatrix}.$$

Now set $W = \begin{bmatrix} 0 & I \\ V & 0 \end{bmatrix}$, and note that W is a unitary operator. Define $S_1 = W \begin{bmatrix} 0 & I \\ S & 0 \end{bmatrix} W^*$, so that $S_1 \in \mathcal{U}(S)$, and define $S_2 = (\begin{bmatrix} 0 & I \\ S^* & 0 \end{bmatrix})^*$, so that $S_2 \in (\mathcal{U}(S^*))^* = \mathcal{U}(S)$. Note that

$$S_2^* S_1 = \begin{bmatrix} 0 & I \\ S^* & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ V & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ S & 0 \end{bmatrix} \begin{bmatrix} 0 & V^* \\ I & 0 \end{bmatrix} = \begin{bmatrix} V & 0 \\ 0 & S^* S V^* \end{bmatrix} = \begin{bmatrix} V & \\ 0 & V^* \end{bmatrix} = B.$$

□

6.3. Corollary. *Let $U \in \mathcal{U}(\mathcal{H})$.*

(a) *There exist $S_1, S_2, \ldots, S_5, S_6 \in \mathcal{U}(S)$ such that $U = S_6^* S_5 S_4^* S_3 S_2^* S_1$.*
(b) *If $U \in \Omega_2$, then there exist T_1, T_2, T_3 and $T_4 \in \mathcal{U}(S)$ such that $U = T_4^* T_3 T_2^* T_1$.*

Proof. Part (a) is an immediate consequence of Theorem 6.2 and Theorem 5.3. Part (b) follows directly from Theorem 6.2.

□

6.4. After completing the manuscript, we were informed by P. Guinand that, as an experiment, he has obtained a GEMINI-3-assisted combinatorial proof of Theorem 6.2. We outline it here. Let $\mathcal{H}$ be an infinite-dimensional, separable Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathbb{Z}}$. Enumerate the elements of $\mathbb{Z}$ as the sequence $(s_n)_{n=0}^{\infty} = (0, -1, 1, -2, 2, -3, \ldots)$. Define two maps S_1, S_2 by setting $S_1 e_{s_n} = e_{s_{n+1}}$, $n \geq 0$ and $S_2^* e_{s_{n+1}} = e_{s_n+1}$ while $S_2^* e_{s_0} = 0$. Then S_1, S_2 are each unitarily equivalent to the unilateral forward shift, while $S_2^* S_1 \simeq B$.

6.5. It is natural to ask if there are analogous factorisation results for unitary elements of the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Once again, the Fredholm index plays a role here. For this reason, we concentrate on unitary elements of the Calkin algebra which have zero Fredholm index. If $t \in \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is invertible, then we may define the index of t as $\mathrm{ind}(t) := \mathrm{nul}(T) - \mathrm{nul}(T^*)$, where $T \in \mathcal{B}(\mathcal{H})$ is any operator satisfying $\pi(T) = t$. By the Brown-Douglas-Fillmore Theorem [1], to say that a unitary element u of the Calkin algebra has index zero is equivalent to saying that $u = \pi(U)$ for some unitary operator $U \in \mathcal{B}(\mathcal{H})$.

6.6. Proposition. *Let $u \in \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ be a unitary element and suppose that* $\mathrm{ind}(u) = 0$. *Then there exist two elements $B_1, B_2 \in \mathcal{U}(B)$ such that $u = \pi(B_1)\pi(B_2)$.*

Proof. The key is to observe that by the Weyl–von Neumann–Berg Theorem, $B^{(\infty)}$ is approximately unitarily equivalent to B, and as such, given $\varepsilon > 0$, there exist $V \in \mathcal{B}(\mathcal{H})$ unitary and $K \in \mathcal{K}(\mathcal{H})$ compact with $\|K\| < \varepsilon$ such that $B^{(\infty)} = V^*BV + K$. More generally, if $Y = W^*B^{(\infty)}W$ for some unitary operator W, then $Y = W^*V^*BVW + W^*KW$, and thus $\pi(Y) = \pi(W^*V^*BVW)$, where $W^*V^*BVW \in \mathcal{U}(B)$.

Choose $U \in \mathcal{B}(\mathcal{H})$ unitary such that $u = \pi(U)$. By the Halmos-Kakutani Theorem 3.9, $U = V_1V_2$, where $V_1 \simeq B^{(\infty)} \simeq V_2$. From the previous paragraph, we see that there exist $B_1, B_2 \in \mathcal{U}(B)$ such that $\pi(B_1) = \pi(V_1)$, $\pi(B_2) = \pi(V_2)$. But then

$$u = \pi(U) = \pi(V_1)\pi(V_2) = \pi(B_1)\pi(B_2).$$

□

6.7. We finish with two questions we have not yet been able to resolve.

Question 1. Suppose that $U \in \mathcal{U}(\mathcal{H})$ and that $\sigma(U)$ has Lebesgue measure 0. Is $U \in \Omega_2$? That is, is every unitary a bicycle?

Question 2. Let $N \in \mathcal{B}(\mathcal{H})$ be a normal operator. For each $m \geq 2$, characterise the set

$$\Delta_m(N) := \{N_1N_2 \cdots N_m : N \simeq N_1 \oplus N_2 \oplus \cdots \oplus N_m\},$$

where each such N_j is assumed to be acting upon a subspace of infinite dimension and infinite co-dimension of $\mathcal{H}$. This question also makes sense for non-normal operators $T \in \mathcal{B}(\mathcal{H})$, although we cannot guarantee in that case that the sets $\Delta_m(T)$ will be non-empty.

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Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1
E-mail address: Laurent.Marcoux@uwaterloo.ca

Department of Mathematics, University of Ljubljana, Ljubljana, Slovenia
E-mail address: matjaz@omladic.net

Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1
E-mail address: hradjavi@uwaterloo.ca

School of Mathematics, Jilin University, Changchun 130012, P.R. CHINA
E-mail address: zhangyuanhang@jlu.edu.cn