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journal homepage: www.elsevier.com/locate/ffaOn the number of irreducible factors with a given multiplicity in function fields [☆]Sourabhashis Das^{a,*}, Ertan Elma^b, Wentang Kuo^a, Yu-Ru Liu^a^a Department of Pure Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, N2L 3G1, Canada^b Department of Mathematics and Computer Science, University of Lethbridge, 4401 University Drive, Lethbridge, Alberta, T1K 3M4, Canada

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ABSTRACT

Let $k \geq 1$ be a natural number and $f \in \mathbb{F}_q[t]$ be a monic polynomial. Let $\omega_k(f)$ denote the number of distinct monic irreducible factors of f with multiplicity k . We obtain asymptotic estimates for the first and the second moments of $\omega_k(f)$ with $k \geq 1$. Moreover, we prove that the function $\omega_1(f)$ has normal order $\log(\deg(f))$ and also satisfies the Erdős-Kac Theorem. Finally, we prove that the functions $\omega_k(f)$ with $k \geq 2$ do not have normal order.

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1. Introduction

In 1940, Erdős and Kac, [4], proved the remarkable result that the number of distinct prime factors, $\omega(n)$, of a natural number n obeys the Gaussian law in the sense that

$$\lim_{x \rightarrow \infty} \frac{1}{x} \left| \left\{ 3 \leq n \leq x : a \leq \frac{\omega(n) - \log \log n}{\sqrt{\log \log n}} \leq b \right\} \right| = \Phi(b) - \Phi(a) \quad (1)$$

where

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^t e^{-\frac{u^2}{2}} du. \quad (2)$$

Various approaches to the Erdős-Kac theorem have been pursued, for example, see [1], [5], [6], [7], [8], [9].

In [3], the second and the fourth authors considered a refinement of the function $\omega(n)$ in the following way. Let $\omega_k(n)$ be the number of distinct prime factors of n with multiplicity $k \geq 1$. In [3], the first and the second moments of $\omega_k(n)$ are obtained. Moreover, they showed that $\omega_1(n)$ has normal order¹ $\log \log n$, that $\omega_1(n)$ satisfies the Gaussian law given in (1), and that the function $\omega_k(n)$ with $k \geq 2$ does not have normal order $F(n)$ for any nondecreasing nonnegative function $F(n)$.

In this work, we consider the refined version of the function field analogue of $\omega(n)$. Let $\mathbb{F}_q[t]$ be the ring of polynomials in the variable t with coefficients from the finite field $\mathbb{F}_q$ with q elements. Let M be the set of monic polynomials in $\mathbb{F}_q[t]$ with degree $\deg(f) \geq 1$ and P be the set of monic irreducible polynomials in $\mathbb{F}_q[t]$ which is the analogue of prime numbers in the ring $\mathbb{Z}$. For $f \in \mathbb{F}_q[t]$, define $|f| := q^{\deg(f)}$.

For a monic polynomial $f \in M$ and a monic irreducible $l \in P$, let $\nu_l(f)$ be the multiplicity of l in the unique factorization of f , that is, $\nu_l(f)$ is the largest nonnegative integer such that $l^{\nu_l(f)} \mid f$ but $l^{\nu_l(f)+1} \nmid f$. Let

$$\omega(f) := \sum_{l \mid f} 1$$

which counts the number of distinct monic irreducible factors $l \in P$ of $f \in M$, and

$$\omega_k(f) := \sum_{\substack{l \mid f \\ \nu_l(f)=k}} 1, \quad (k \geq 1)$$

¹ Let $f, F : \mathbb{N} \rightarrow \mathbb{R}_{\geq 0}$ be two functions such that F is non-decreasing. Then $f(n)$ is said to have *normal order* $F(n)$ if for any $\epsilon > 0$, the number of $n \leq x$ that do not satisfy the inequality

$$(1 - \epsilon)F(n) \leq f(n) \leq (1 + \epsilon)F(n)$$

is $o(x)$ as $x \rightarrow \infty$.

which counts the number of distinct monic irreducible factors of f with a given multiplicity $k \geq 1$. Note that the set of functions $\omega_k(f)$ with $k \geq 1$ is a refinement of $\omega(f)$ in the sense that

$$\omega(f) = \sum_{k \geq 1} \omega_k(f)$$

for all $f \in M$.

For a natural number $n \geq 1$, let M_n be the set of monic polynomials in $\mathbb{F}_q[t]$ with degree n . By the work of A. Knopfmacher and J. Knopfmacher, [10, Theorems 1 and 4 or the comment on Page 111], the mean value of $\omega(f)$ over all polynomials in M_n is given by

$$\sum_{f \in M_n} \omega(f) = q^n(\log n) + A_1 q^n + O\left(\frac{q^n}{n}\right) \quad (3)$$

with

$$A_1 = \gamma + c_1, \quad (4)$$

where γ is the Euler-Mascheroni constant, and $c_1 = \sum_{r=1}^{\infty} (\pi_q(r) - q^r/r) q^{-r}$ with $\pi_q(r)$ denoting the number of monic irreducible polynomials $l \in P$ with degree $\deg(l) = r$.

First we obtain refinements of the result in (3) by estimating the first moment of the functions $\omega_k(\cdot)$ with $k \geq 1$.

Theorem 1.1. *For an integer $m \geq 2$, let*

$$L(m) := \sum_{l \in P} \frac{1}{|l|^m}. \quad (5)$$

We have

$$\sum_{f \in M_n} \omega_1(f) = q^n(\log n) + q^n(A_1 - L(2)) + O\left(\frac{q^n}{n}\right)$$

where A_1 is defined in (4). Moreover for $k \geq 2$, as $n \rightarrow \infty$, we have

$$\sum_{f \in M_n} \omega_k(f) = (L(k) - L(k+1))q^n + O\left(n \cdot q^{\frac{n}{k}}\right).$$

For the second moments of $\omega_k(\cdot)$, we obtain the following result.

Theorem 1.2. *Let A_1 and $L(m)$ be defined as in (4) and (5), respectively. As $n \rightarrow \infty$, we have*

$$\sum_{f \in M_n} \omega_1^2(f) = q^n (\log n)^2 + c_2 q^n (\log n) + c_3 q^n + O\left(\frac{q^n (\log n)}{n}\right),$$

where

$$c_2 := 1 + 2A_1 - 2L(2)$$

and

$$c_3 := (A_1 - 2L(2))(A_1 + 1) - \frac{\pi^2}{6} + (L(2))^2 + 2L(3) - L(4). \quad (6)$$

Moreover, for $k \geq 2$, we have

$$\sum_{f \in M_n} \omega_k^2(f) = c'_k q^n + O\left(n \cdot q^{\frac{n}{k}}\right),$$

where

$$c'_k := (L(k) - L(k+1))(L(k) - L(k+1) + 1) - L(2k) + 2L(2k+1) - L(2k+2).$$

Note that the error terms for ω_k with $k \geq 2$ as in Theorem 1.1 and Theorem 1.2 improves upon their respective error terms in the integer case (see [3, Theorem 1.1 and Theorem 1.2]) by at least a factor of $\frac{(k-1)^2}{k(3k-1)}$ in the exponent.

Let $g, G : M \rightarrow \mathbb{R}_{\geq 0}$ be two functions. We say $G(f)$ is non-decreasing if $G(f) \geq G(h)$ for all f, h with $\deg(f) \geq \deg(h)$. Then $g(f)$ is said to have normal order $G(f)$ for a non-decreasing function $G(f)$ if for any $\epsilon > 0$, the number of polynomials f with degree n that do not satisfy the inequality

$$(1 - \epsilon)G(f) \leq g(f) \leq (1 + \epsilon)G(f)$$

is $o(q^n)$ as $n \rightarrow \infty$.

The first and the second moment estimates in Theorem 1.1 and Theorem 1.2 allow us to obtain the normal order of $\omega_1(\cdot)$.

Theorem 1.3. *Let c_3 be defined as in (6). As $n \rightarrow \infty$, we have*

$$\sum_{f \in M_n} (\omega_1(f) - \log n)^2 = q^n \log n + c_3 q^n + O\left(\frac{q^n \log n}{n}\right).$$

Let $\epsilon' \in (0, 1/2)$. Then the number of monic polynomials f of degree n such that

$$\frac{|\omega_1(f) - \log n|}{\sqrt{\log n}} \geq (\log n)^{\epsilon'}$$

is $o(q^n)$ as $n \rightarrow \infty$ and thus $\omega_1(f)$ has normal order $\log(\deg(f))$.

However, unlike the function $\omega_1(\cdot)$, we prove that the functions $\omega_k(\cdot)$ with $k \geq 2$ do not have normal order.

Theorem 1.4. *Let $k \geq 2$ be a fixed integer. Then the function $\omega_k(f)$ does not have normal order $G(f)$ for any non-decreasing function $G : M \rightarrow \mathbb{R}_{\geq 0}$.*

In [14], Zhang proved a function field analogue of the Erdős-Kac Theorem that

$$\lim_{n \rightarrow \infty} \frac{\left| \left\{ f \mid f \in M_m, m \leq n, \frac{\omega(f) - \log m}{\sqrt{\log m}} \leq a \right\} \right|}{\left| \{f \mid f \in M_m, m \leq n\} \right|} = \Phi(a), \quad (7)$$

where $|\{\cdot\}|$ denotes the cardinality of the corresponding set. A generalization of this result can be found in the work of the fourth author [12] and in the work of Lalín and Zhang [11].

In our final main result, we prove that the function $\omega_1(\cdot)$ obeys the same Gaussian distribution as in the Erdős-Kac Theorem.

Theorem 1.5. *Let $a \in \mathbb{R}$. Then*

$$\lim_{n \rightarrow \infty} \frac{1}{q^n} \left| \left\{ f \in M_n \mid \frac{\omega_1(f) - \log n}{\sqrt{\log n}} \leq a \right\} \right| = \Phi(a),$$

where $\Phi(a)$ is defined in (2).

For a monic polynomial $f \in M$, let $\Omega(f)$ be the number of monic irreducible factors of f . Then $\Omega(f)$ satisfies the Erdős-Kac Theorem (see [12, Remark, Page 605]). Let $\Omega_k(f)$ be the number of monic irreducible factors of f with a given multiplicity $k \geq 1$. Note that $\Omega_k(f) = k \cdot \omega_k(f)$ and $\Omega(f) = \sum_{k \geq 1} \Omega_k(f)$ for all $f \in M$. We can deduce similar results for $\Omega_k(f)$ as our results above. In particular, we can prove that $\Omega_1(f)$ has normal order $\log(\deg(f))$ and also satisfies the Erdős-Kac Theorem. We can also prove that the functions $\Omega_k(f)$ with $k \geq 2$ do not have normal order $G(f)$ for any non-decreasing function $G : M \rightarrow \mathbb{R}_{\geq 0}$.

2. Lemmata

In this section, we prove some lemmata that are needed in the proof of Theorem 1.1 and Theorem 1.2. We start by recalling the analogue of the prime number theorem for polynomials. We recall that $\pi_q(n)$ denotes the number of monic irreducible polynomials $l \in P$ with degree $\deg(l) = n$.

Lemma 2.1. [13, Theorem 2.2] (The prime number theorem for polynomials) For $n \geq 1$, we have

$$\pi_q(n) = \frac{q^n}{n} + O\left(\frac{q^{\frac{n}{2}}}{n}\right).$$

Lemma 2.2. For $R > 0$, let $F(z)$ be a meromorphic function on $|z| \leq R$ with at most finitely many poles in the open disk $|z| < R$ and analytic on the circle $|z| = R$. Suppose that $F(z)$ is analytic at $z = 0$ with the power series representation

$$F(z) = \sum_{n=0}^{\infty} c_n z^n$$

in the open disk $|z| < r$ for some $r > 0$. As $N \rightarrow \infty$, we have

$$c_N = - \sum' \operatorname{Res} \left(\frac{F(z)}{z^{N+1}} \right) + O \left(R^{-N} \max_{|z|=R} |F(z)| \right)$$

where the implied constant is absolute and the sum calculates the residues of $F(z)/z^{N+1}$ at the poles of $F(z)/z^{N+1}$ in $|z| < R$, except the one that comes from a possible pole at $z = 0$.

Proof. Applying the residue theorem to the integral $\frac{1}{2\pi i} \int_{|z|=R} \frac{F(z)}{z^{N+1}} dz$ and using the power series of $F(z)$ around $z = 0$, we obtain the desired result. $\square$

Lemma 2.3. (Mertens' theorem for $\mathbb{F}_q[x]$) Let A_1 be defined as in (4). As $n \rightarrow \infty$, we have

$$\sum_{\substack{l \in P \\ \deg(l) \leq n}} \frac{1}{|l|} = \log n + A_1 + O \left(\frac{1}{n} \right).$$

Proof. By Lemma 2.1, we have

$$\pi_q(m) = \frac{q^m}{m} + r_m$$

where $r_m \ll \frac{q^{m/2}}{m}$ as $m \rightarrow \infty$. Thus

$$\sum_{\substack{l \in P \\ \deg(l) \leq n}} \frac{1}{|l|} = \sum_{m=1}^n q^{-m} \pi_q(m) = \sum_{m=1}^n \frac{1}{m} + \sum_{m=1}^n r_m q^{-m}.$$

By [2, Theorem 1], we have

$$\sum_{m=1}^n \frac{1}{m} = \log n + \gamma + O \left(\frac{1}{n} \right). \quad (8)$$

Also, we have

$$\sum_{m=1}^n r_m q^{-m} = c_1 - \sum_{m=n+1}^{\infty} r_m q^{-m} = c_1 + O\left(\sum_{m=n+1}^{\infty} \frac{q^{-m/2}}{m}\right) = c_1 + O\left(\frac{q^{-n/2}}{n}\right).$$

Hence the desired result follows. $\square$

Lemma 2.4. Let $g : P \rightarrow \mathbb{C}$ be a function such that $|g(l)| \leq 1$. Let $k \in \mathbb{N}$. Let $s = \sigma + it$, $\sigma, t \in \mathbb{R}$. Define

$$B_{g,k}(s) := \sum_{l \in P} \frac{1}{|l|^{ks}(|l|^s - g(l))}.$$

Then $B_{g,k}(s)$ is absolutely convergent for $\sigma > \frac{1}{k+1}$. Moreover, we have

$$B_{g,k}(s) = \sum_{f \in M} \frac{b_{g,k}(f)}{|f|^s},$$

where

$$b_{g,k}(f) = \begin{cases} g(l)^{\alpha-(k+1)} & \text{if } f = l^\alpha \text{ for some } l \in P \text{ and } \alpha \geq k+1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. For $\sigma > \frac{1}{k+1}$, we have

$$\sum_{l \in P} \frac{1}{|l|^{ks}(|l|^s - g(l))} \ll \sum_{l \in P} \frac{1}{|l|^{(k+1)\sigma}} \ll \sum_{n \geq 1} \frac{q^n}{q^{n(k+1)\sigma}} \ll 1,$$

and thus the series $B_{g,k}(s)$ is absolutely convergent for $\sigma > \frac{1}{k+1}$. Since $|g(l)| \leq 1$ for all $l \in P$ and $\left|\frac{g(l)}{|l|^s}\right| < 1$ for $\sigma > \frac{1}{k+1}$, we have

$$B_{g,k}(s) = \sum_{l \in P} \frac{1}{|l|^{(k+1)s}} \left(1 + \frac{g(l)}{|l|^s} + \left(\frac{g(l)}{|l|^s}\right)^2 + \dots\right) = \sum_{f \in M} \frac{b_{g,k}(f)}{|f|^s}$$

which completes the proof. $\square$

Proposition 2.1. Let $g : P \rightarrow \mathbb{C}$ be a function such that $|g(l)| \leq 1$. Let $k \in \mathbb{N}$. Define

$$a_{g,k}(f) := \sum_{\substack{l \in P, \ l|f \\ \nu_l(f) \geq k+1}} (1 + g(l) + (g(l))^2 + \dots + (g(l))^{\nu_l(f)-(k+1)}),$$

with the convention that the empty sum is taken to be zero. Define

$$C_{g,k} := \sum_{l \in P} \frac{1}{|l|^k(|l| - g(l))}.$$

Then, we have

$$\sum_{f \in M_n} a_{g,k}(f) = C_{g,k} q^n + O\left(n \cdot q^{\frac{n}{k+1}}\right),$$

where the implied constant is absolute.

Proof. Let $s = \sigma + it$, $\sigma, t \in \mathbb{R}$. We define

$$A_{g,k}(s) := \sum_{f \in M} \frac{a_{g,k}(f)}{|f|^s} = \sum_{n \geq 1} \left(\sum_{f \in M_n} a_{g,k}(f) \right) q^{-ns}.$$

Note that, we will show later in this proof that $A_{g,k}(s)$ is absolutely convergent for $\sigma > 1$.

The zeta function $\zeta_q(s)$ in $\mathbb{F}_q[t]$ is defined by

$$\zeta_q(s) = \sum_{f \in M} \frac{1}{|f|^s} = \frac{1}{1 - q^{1-s}} \quad (\sigma > 1).$$

Thus, we have

$$\zeta_q(s) B_{g,k}(s) = \sum_{f \in M} \frac{1}{|f|^s} \cdot \sum_{f \in M} \frac{b_{g,k}(f)}{|f|^s} = \sum_{f \in M} \frac{\sum_{d|f} b_{g,k}(d)}{|f|^s}.$$

Notice that

$$\sum_{d|f} b_{g,k}(d) = \sum_{\substack{l \in P, l|f \\ \nu_l(f) \geq k+1}} \left(\sum_{j=k+1}^{\nu_l(f)} (g(l))^{j-(k+1)} \right) = a_{g,k}(f).$$

Since $\zeta_q(s)$ and $B_{g,k}(s)$ converge absolutely for $\sigma > 1$ and $\sigma > \frac{1}{k+1}$ respectively, we conclude that

$$\sum_{f \in M} \frac{a_{g,k}(f)}{|f|^s} = A_{g,k}(s) = \zeta_q(s) B_{g,k}(s)$$

converges absolutely in the region $\sigma > 1$. Additionally, $A_{g,k}(s)$ admits a meromorphic continuation up to $\sigma > \frac{1}{k+1}$ with simple poles at $1 + i \frac{2\pi k}{\log q}$ where $k \in \mathbb{Z}$. Making the change of variable $u = q^{-s}$, we introduce $\mathcal{B}_{g,k}(u)$ ($= B_{g,k}(s)$) and $\mathcal{A}_{g,k}(u)$ ($= A_{g,k}(s)$) as

$$\mathcal{B}_{g,k}(u) = \sum_{l \in P} \frac{u^{(k+1)\deg(l)}}{1 - g(l)u^{\deg(l)}} \quad \text{and} \quad \mathcal{A}_{g,k}(u) = \sum_{f \in M} a_{g,k}(f) u^{\deg(f)}.$$

Notice that $\mathcal{A}_{g,k}(u)$ admits a meromorphic continuation to $|u| < q^{-\frac{1}{k+1}}$ with only a simple pole at q^{-1} . Moreover as $\zeta_q(s) = \mathcal{Z}_q(u) = \frac{1}{1-qu}$, we have

$$\mathcal{A}_{g,k}(u) = \mathcal{Z}_q(u) \mathcal{B}_{g,k}(u) = \frac{\mathcal{B}_{g,k}(u)}{1-qu}.$$

Since $\mathcal{A}_{g,k}(u)$ is analytic at $u = 0$, we use the power series of $\mathcal{A}_{g,k}(u)$ around $u = 0$ (in a small disk $|u| < r < q^{-1}$) in Lemma 2.2 to complete the proof. In order to do that, we write $\mathcal{A}_{g,k}(u)$ as

$$\mathcal{A}_{g,k}(u) = \sum_{n=0}^{\infty} \left(\sum_{f \in M_n} a_{g,k}(f) \right) u^n$$

and for any $\epsilon \in (0, 1/4]$, we choose $R = q^{-\left(\frac{1}{k+1} + \epsilon\right)} > q^{-1}$ to satisfy the hypothesis of the lemma for the function $\mathcal{A}_{g,k}(u) = \frac{\mathcal{B}_{g,k}(u)}{1-qu}$. Thus applying the lemma, we obtain

$$\sum_{f \in M_n} a_{g,k}(f) = -\text{Res}_{u=q^{-1}} \frac{\mathcal{B}_{g,k}(u)}{u^{n+1}(1-qu)} + O\left(R^{-n} \max_{|u|=R} \left| \frac{\mathcal{B}_{g,k}(u)}{1-qu} \right| \right), \quad (9)$$

where the implied constant is absolute. Note that $\mathcal{B}_{g,k}(q^{-1}) = C_{g,k}$ and thus

$$-\text{Res}_{u=q^{-1}} \frac{\mathcal{B}_{g,k}(u)}{u^{n+1}(1-qu)} = C_{g,k} q^n. \quad (10)$$

Moreover, on $|u| = R = q^{-\left(\frac{1}{k+1} + \epsilon\right)}$, we have $\frac{1}{1-qu} \ll 1$ and $\mathcal{B}_{g,k}(u) \ll \frac{1}{q^{(k+1)\epsilon-1}}$. Thus, the error term in (9) is bounded by

$$\ll q^{\frac{n}{k+1}} \frac{q^{n\epsilon}}{q^{(k+1)\epsilon-1}}.$$

We choose ϵ such that $q^{(k+1)\epsilon} = \frac{n+1}{n}$. Then $\epsilon = \frac{1}{(k+1)\log q} \log\left(\frac{n+1}{n}\right) \leq \frac{1}{4}$ for $k \geq 1$, $q \geq 2$, $n \geq 3$, and we have

$$\begin{aligned} \frac{q^{n\epsilon}}{q^{(k+1)\epsilon-1}} &= q^{n \frac{1}{(k+1)\log q} \log\left(\frac{n+1}{n}\right)} \frac{1}{\frac{n+1}{n} - 1} \\ &= n \exp\left(\frac{1}{k+1} n \log\left(1 + \frac{1}{n}\right)\right) \\ &\ll n, \end{aligned}$$

where the implied constant is absolute. Thus the error term in (9) is bounded by $nq^{\frac{n}{k+1}}$, and this together with (10) completes the proof. $\square$

Lemma 2.5. *As $n \rightarrow \infty$, we have*

$$\sum_{m=1}^{n-1} \frac{1}{m(n-m)} = \frac{2 \log n}{n} + \frac{2\gamma}{n} + O\left(\frac{1}{n^2}\right).$$

Proof. Note that, using symmetry and (8), we have

$$\begin{aligned} \sum_{m=1}^{n-1} \frac{1}{m(n-m)} &= \frac{1}{n} \sum_{m=1}^{n-1} \left(\frac{1}{n-m} + \frac{1}{m} \right) \\ &= \frac{2}{n} \sum_{m=1}^{n-1} \frac{1}{m} \\ &= \frac{2 \log(n-1)}{n} + \frac{2\gamma}{n} + O\left(\frac{1}{n(n-1)}\right). \end{aligned}$$

We complete the proof by noticing that as $n \rightarrow \infty$,

$$\log(n-1) = \log n + \log\left(1 - \frac{1}{n}\right) = \log n + O\left(\frac{1}{n}\right). \quad \square$$

Lemma 2.6. *As $n \rightarrow \infty$, we have*

$$\sum_{m=1}^{n-1} \frac{1}{m^2(n-m)} = \frac{\pi^2}{6n} + \frac{2 \log n}{n^2} + O\left(\frac{1}{n^2}\right).$$

Proof. Notice that

$$\sum_{m=1}^{n-1} \frac{1}{m^2(n-m)} = \frac{1}{n} \left(\sum_{m=1}^{n-1} \frac{1}{m(n-m)} + \sum_{m=1}^{n-1} \frac{1}{m^2} \right).$$

Using

$$\sum_{m=1}^{n-1} \frac{1}{m^2} = \sum_{m=1}^{\infty} \frac{1}{m^2} - \sum_{m=n}^{\infty} \frac{1}{m^2} = \frac{\pi^2}{6} + O\left(\int_n^{\infty} \frac{dt}{t^2}\right) = \frac{\pi^2}{6} + O\left(\frac{1}{n}\right)$$

together with Lemma 2.5 completes the proof. $\square$

Lemma 2.7. *As $n \rightarrow \infty$, we have*

$$\sum_{m=1}^{n-1} \frac{q^{-m/2}}{m(n-m)} = \frac{\log\left(1 + \frac{1}{q^{1/2}-1}\right)}{n} + O\left(\frac{1}{n^2}\right).$$

Proof. Notice that

$$\begin{aligned}\sum_{m=1}^{n-1} \frac{q^{-m/2}}{m(n-m)} &= \frac{1}{n} \left(\sum_{m=1}^{n-1} \frac{q^{-m/2}}{n-m} + \sum_{m=1}^{n-1} \frac{q^{-m/2}}{m} \right) \\ &= \frac{1}{n} \sum_{m=1}^{n-1} \frac{q^{-m/2}}{n-m} + \frac{1}{n} \sum_{m=1}^{\infty} \frac{q^{-m/2}}{m} + O\left(\frac{q^{-n/2}}{n^2}\right).\end{aligned}$$

For the second sum on the right hand side above, by using

$$-\log(1-x) = \sum_{m=1}^{\infty} \frac{x^m}{m} \quad (|x| < 1)$$

with $x = q^{-1/2} < 1$, we obtain

$$\frac{1}{n} \sum_{m=1}^{\infty} \frac{q^{-m/2}}{m} = \frac{-\log(1 - q^{-1/2})}{n} = \frac{\log\left(1 + \frac{1}{q^{1/2}-1}\right)}{n}.$$

Thus we complete the proof by showing

$$\sum_{m=1}^{n-1} \frac{q^{-m/2}}{n-m} = O\left(\frac{1}{n}\right)$$

We write

$$\sum_{m=1}^{n-1} \frac{q^{-m/2}}{n-m} = \sum_{m=1}^{n-1} \frac{q^{-(n-m)/2}}{m} = q^{-n/2} H_n(q^{1/2}),$$

where

$$H_n(x) = \sum_{m=1}^{n-1} \frac{x^m}{m}.$$

We then notice that

$$\left(1 - \frac{1}{x}\right) H_n(x) = \sum_{m=1}^{n-1} \frac{x^m}{m} - \sum_{m=0}^{n-2} \frac{x^m}{m+1} = \frac{x^{n-1}}{n-1} - 1 + \sum_{m=1}^{n-2} \frac{x^m}{m(m+1)}$$

which gives

$$x^{-n} H_n(x) = \frac{1}{(n-1)(x-1)} - \frac{x^{-(n-1)}}{x-1} + \frac{x}{x-1} \sum_{m=1}^{n-2} \frac{x^{-(n-m)}}{m(m+1)}.$$

Taking $x = q^{1/2}$, we obtain

$$x^{-n} H_n(x) = O\left(\frac{1}{n}\right) + O\left(\sum_{m=1}^{n-1} \frac{1}{m^2(n-m)}\right)$$

where the implied constants are absolute and using Lemma 2.6 completes the proof. $\square$

3. Proof of Theorem 1.1

To prove Theorem 1.1, we take $g(l) = -1$ for all $l \in P$ in Proposition 2.1. Then we have

$$a_k(f) := a_{g,k}(f) = \sum_{\substack{l \in P, \, l|f \\ \nu_l(f) \geq k+1 \\ \nu_l(f) - k \text{ odd}}} 1,$$

and note that

$$\omega_1(f) = \omega(f) - a_1(f) - a_2(f). \quad (11)$$

Moreover, by the definition of $C_k := C_{g,k}$ in Proposition 2.1, we have

$$C_1 = \sum_{l \in P} \frac{1}{|l|(|l|+1)} \quad \text{and} \quad C_2 = \sum_{l \in P} \frac{1}{|l|^2(|l|+1)},$$

and

$$C_1 + C_2 = L(2).$$

Thus, by (3) and Proposition 2.1, we have

$$\begin{aligned} \sum_{f \in M_n} \omega_1(f) &= \sum_{f \in M_n} (\omega(f) - a_1(f) - a_2(f)) \\ &= q^n \left(\log n + A_1 + O\left(\frac{1}{n}\right) \right) - (C_1 + C_2) q^n + O\left(n \cdot q^{\frac{n}{2}}\right) \\ &= q^n (\log n) + q^n (A_1 - L(2)) + O\left(\frac{q^n}{n}\right). \end{aligned}$$

We now consider the cases $k \geq 2$. Since

$$\omega_k(f) = a_{k-1}(f) - a_{k+1}(f)$$

and

$$C_{k-1} - C_{k+1} = \sum_{l \in P} \left(\frac{1}{|l|^{k-1}(|l|+1)} - \frac{1}{|l|^{k+1}(|l|+1)} \right) = L(k) - L(k+1),$$

by Proposition 2.1, the result follows. $\square$

4. Proof of Theorem 1.2

To prove Theorem 1.2, we start with the first moment of ω_1 . For $l \in P$ and $f \in M$, and a positive integer k , we use $l^k || f$ to denote that $l^k | f$ and $l^{k+1} \nmid f$. We have

$$\begin{aligned} \sum_{f \in M_n} \omega_1^2(f) &= \sum_{f \in M_n} \left(\sum_{\substack{l \in P \\ l || f}} 1 \right)^2 = \sum_{f \in M_n} \sum_{\substack{l, h \in P \\ l || f, h || f}} 1 \\ &= \sum_{f \in M_n} \omega_1(f) + \sum_{f \in M_n} \sum_{\substack{l, h \in P, l \neq h \\ l || f, h || f}} 1. \end{aligned} \quad (12)$$

The first sum on the right-hand side above can be estimated by Theorem 1.1. For the second sum on the right-hand side above, we have

$$\sum_{f \in M_n} \sum_{\substack{l, h \in P, l \neq h \\ l || f, h || f}} 1 = \sum_{\substack{l, h \in P, l \neq h \\ \deg(l), \deg(h) \leq n}} \sum_{\substack{f \in M_n \\ l || f, h || f}} 1,$$

and the inner sum on the right-hand side can be written as

$$\sum_{\substack{f \in M_n \\ l || f, h || f}} 1 = \sum_{\substack{f \in M_n \\ (lh) | f}} 1 - \sum_{\substack{f \in M_n \\ (l^2 h) | f}} 1 - \sum_{\substack{f \in M_n \\ (th^2) | f}} 1 + \sum_{\substack{f \in M_n \\ (l^2 h^2) | f}} 1.$$

For $w \in M$ with $\deg(w) \leq n$, the number of multiples of w in M_n is $\frac{q^n}{|w|}$, and thus we have

$$\sum_{\substack{l, h \in P, l \neq h \\ \deg(l), \deg(h) \leq n}} \sum_{\substack{f \in M_n \\ l || f, h || f}} 1 = q^n (S_1(n) - 2S_2(n) + S_3(n)),$$

where

$$\begin{aligned} S_1(n) &= \sum_{\substack{l, h \in P, l \neq h \\ \deg(lh) \leq n}} \frac{1}{|l||h|}, \\ S_2(n) &= \sum_{\substack{l, h \in P, l \neq h \\ \deg(l^2 h) \leq n}} \frac{1}{|l|^2 |h|}, \end{aligned}$$

$$S_3(n) = \sum_{\substack{l, h \in P, l \neq h \\ \deg(l^2 h^2) \leq n}} \frac{1}{|l|^2 |h|^2}.$$

Now we show that

$$S_1(n) = \log^2 n + 2A_1 \log n + A_1^2 - \frac{\pi^2}{6} - L(2) + O\left(\frac{\log n}{n}\right). \quad (13)$$

We have

$$S_1(n) = \sum_{\substack{l, h \in P \\ \deg(ls) \leq n}} \frac{1}{|l||h|} - \sum_{\substack{l \in P \\ \deg(l) \leq n/2}} \frac{1}{|l|^2}. \quad (14)$$

For the last term above, we have

$$\sum_{\substack{l \in P \\ \deg(l) \leq n/2}} \frac{1}{|l|^2} = \sum_{l \in P} \frac{1}{|l|^2} + O\left(\sum_{m > n/2} \frac{q^{-m}}{m}\right) = L(2) + O\left(\frac{q^{-n/2}}{n}\right)$$

since $\pi_q(m) = q^m/m + O(q^{m/2}/m)$ for $m \geq 1$. Moreover, we have

$$\begin{aligned} \sum_{\substack{l \in P \\ \deg(l) \leq n-1}} \frac{1}{|l|(n - \deg(l))} &= \sum_{m=1}^{n-1} \frac{1}{m(n-m)} + O\left(\sum_{m=1}^{n-1} \frac{q^{-m/2}}{m(n-m)}\right) \\ &= \frac{2 \log n}{n} + O\left(\frac{1}{n}\right) \end{aligned}$$

by Lemma 2.5 and Lemma 2.7. Thus for the first term on the right-hand side of (14), by Lemma 2.3, we have

$$\begin{aligned} \sum_{\substack{l, h \in P \\ \deg(lh) \leq n}} \frac{1}{|l||h|} &= \sum_{\substack{l \in P \\ \deg(l) \leq n-1}} \frac{1}{|l|} \sum_{\substack{h \in P \\ \deg(h) \leq n - \deg(l)}} \frac{1}{|h|} \\ &= \sum_{\substack{l \in P \\ \deg(l) \leq n-1}} \frac{1}{|l|} \left(\log(n - \deg(l)) + A_1 + O\left(\frac{1}{n - \deg(l)}\right) \right) \\ &= \sum_{\substack{l \in P \\ \deg(l) \leq n-1}} \frac{\log(n - \deg(l))}{|l|} + A_1 \log n + A_1^2 + O\left(\frac{\log n}{n}\right). \end{aligned}$$

Again by Lemma 2.3, we have

$$\begin{aligned}
\sum_{\substack{l \in P \\ \deg(l) \leq n-1}} \frac{\log(n - \deg(l))}{|l|} &= \log n \sum_{\substack{l \in P \\ \deg(l) \leq n-1}} \frac{1}{|l|} + \sum_{\substack{l \in P \\ \deg(l) \leq n-1}} \frac{\log\left(1 - \frac{\deg(l)}{n}\right)}{|l|} \\
&= \log^2 n + A_1 \log n + \sum_{\substack{l \in P \\ \deg(l) \leq n-1}} \frac{\log\left(1 - \frac{\deg(l)}{n}\right)}{|l|} \\
&\quad + O\left(\frac{\log n}{n}\right).
\end{aligned}$$

Since $\pi_q(m) = \frac{q^m}{m} + O\left(\frac{q^{m/2}}{m}\right)$, we have

$$\begin{aligned}
\sum_{\substack{l \in P \\ \deg(l) \leq n-1}} \frac{\log\left(1 - \frac{\deg(l)}{n}\right)}{|l|} &= \sum_{1 \leq m \leq n-1} \pi_q(m) \frac{\log\left(1 - \frac{m}{n}\right)}{q^m} \\
&= \sum_{1 \leq m \leq n-1} \frac{\log\left(1 - \frac{m}{n}\right)}{m} + O\left(\frac{1}{n}\right).
\end{aligned}$$

Either by using Riemann sums for the negative decreasing function $F(x) := \frac{\log(1-x)}{x}$ on $[0, 1]$ and the fact that $\int_0^1 F(x) dx = -\frac{\pi^2}{6}$, or by using the Taylor series of $\log\left(1 - \frac{m}{n}\right)$ and the bound $\left|\sum_{m \leq r} m^j - \frac{r^{j+1}}{j+1}\right| \leq r^j$ for all natural numbers j and r (which can be proved by induction on j), one can show that

$$\sum_{1 \leq m \leq n-1} \frac{\log\left(1 - \frac{m}{n}\right)}{m} = -\frac{\pi^2}{6} + O\left(\frac{\log n}{n}\right).$$

Combining the results above, we obtain the estimate in (13) for $S_1(n)$.

Now, we consider $S_2(n)$. We show that

$$S_2(n) = L(2) \log n + A_1 L(2) - L(3) + O\left(\frac{1}{n}\right). \quad (15)$$

We have

$$\begin{aligned}
S_2(n) &= \sum_{\substack{l, h \in P \\ \deg(l^2 h) \leq n}} \frac{1}{|l|^2 |h|} - \sum_{\substack{l \in P \\ \deg(l) \leq n/3}} \frac{1}{|l|^3} \\
&= \sum_{\substack{l, h \in P \\ \deg(l^2 h) \leq n}} \frac{1}{|l|^2 |h|} - L(3) + O\left(\frac{q^{-2n/3}}{n}\right).
\end{aligned}$$

Since by Lemma 2.7,

$$\begin{aligned}
\sum_{\substack{l, h \in P \\ \deg(l^2 h) \leq n}} \frac{1}{|l|^2 |h|} &= \sum_{\substack{h \in P \\ \deg(h) \leq n-2}} \frac{1}{|h|} \left(\sum_{\substack{l \in P \\ \deg(l) \leq \frac{n-\deg(h)}{2}}} \frac{1}{|l|^2} \right) \\
&= L(2) \sum_{\substack{h \in P \\ \deg(h) \leq n-2}} \frac{1}{|h|} + O \left(\sum_{\substack{h \in P \\ \deg(h) \leq n-2}} \frac{q^{-\frac{n-\deg(h)}{2}}}{|h|(n-\deg(h))} \right) \\
&= L(2)(\log n + A_1) + O \left(\frac{1}{n} \right) + O \left(\sum_{m=1}^{n-1} \frac{q^{-\frac{m}{2}}}{m(n-m)} - \frac{q^{-1/2}}{n-1} \right) \\
&= L(2)(\log n + A_1) + O \left(\frac{1}{n} \right),
\end{aligned}$$

we obtain the estimate in (15) for $S_2(n)$.

Now we consider $S_3(n)$ for which we show that

$$S_3(n) = L(2)^2 - L(4) + O \left(\frac{1}{n} \right). \quad (16)$$

We have

$$\begin{aligned}
S_3(n) &= \sum_{\substack{l, h \in P \\ \deg(l^2 h^2) \leq n}} \frac{1}{|l|^2 |h|^2} - \sum_{\substack{l \in P \\ \deg(l) \leq n/4}} \frac{1}{|l|^4} \\
&= \sum_{\substack{l, h \in P \\ \deg(l^2 h^2) \leq n}} \frac{1}{|l|^2 |h|^2} - L(4) + O \left(\frac{q^{-3n/4}}{n} \right).
\end{aligned}$$

Since

$$\begin{aligned}
\sum_{\substack{l, h \in P \\ \deg(l^2 h^2) \leq n}} \frac{1}{|l|^2 |h|^2} &= \sum_{\substack{l \in P \\ \deg(l) \leq \frac{n-2}{2}}} \frac{1}{|l|^2} \left(\sum_{\substack{h \in P \\ \deg(h) \leq \frac{n-2\deg(l)}{2}}} \frac{1}{|h|^2} \right) \\
&= L(2) \left(L(2) + O \left(\frac{q^{-(n-2)/2}}{n} \right) \right) + O \left(\frac{1}{n} \right) \\
&= L(2)^2 + O \left(\frac{1}{n} \right),
\end{aligned}$$

we obtain the estimate in (16) for $S_3(n)$. Hence, by Theorem 1.1, (12), (13), (15), and (16), we obtain the first assertion in Theorem 1.2 about the second moment of ω_1 .

Now we consider the second moment of ω_k for $k \geq 2$. We have

$$\begin{aligned}
 \sum_{f \in M_n} \omega_k^2(f) &= \sum_{f \in M_n} \omega_k(f) + \sum_{f \in M_n} \left(\sum_{\substack{l, h \in P, l \neq h \\ l^k || f, h^k || f}} 1 \right) \\
 &= \sum_{f \in M_n} \omega_k(f) + \sum_{\substack{l, h \in P, l \neq h \\ \deg(lh) \leq n/k}} \left(\sum_{\substack{f \in M_n \\ (lh)^k | f}} 1 - \sum_{\substack{f \in M_n \\ l^{k+1} h^k | f}} 1 - \sum_{\substack{f \in M_n \\ l^k h^{k+1} | f}} 1 + \sum_{\substack{f \in M_n \\ (lh)^{k+1} | f}} 1 \right) \\
 &= \sum_{f \in M_n} \omega_k(f) + q^n \sum_{\substack{l, h \in P, l \neq h \\ \deg(lh) \leq n/k}} \left(\frac{1}{|l|^k} - \frac{1}{|l|^{k+1}} \right) \left(\frac{1}{|h|^k} - \frac{1}{|h|^{k+1}} \right). \quad (17)
 \end{aligned}$$

We have

$$\begin{aligned}
 &\sum_{\substack{l, h \in P, l \neq h \\ \deg(lh) \leq n/k}} \left(\frac{1}{|l|^k} - \frac{1}{|l|^{k+1}} \right) \left(\frac{1}{|h|^k} - \frac{1}{|h|^{k+1}} \right) \\
 &= \sum_{\substack{l, h \in P \\ \deg(lh) \leq n/k}} \left(\frac{1}{|l|^k} - \frac{1}{|l|^{k+1}} \right) \left(\frac{1}{|h|^k} - \frac{1}{|h|^{k+1}} \right) - \sum_{\substack{l \in P \\ \deg(l) \leq n/2k}} \left(\frac{1}{|l|^k} - \frac{1}{|l|^{k+1}} \right)^2
 \end{aligned}$$

and

$$\begin{aligned}
 \sum_{\substack{l \in P \\ \deg(l) \leq n/2k}} \left(\frac{1}{|l|^k} - \frac{1}{|l|^{k+1}} \right)^2 &= \sum_{l \in P} \left(\frac{1}{|l|^k} - \frac{1}{|l|^{k+1}} \right)^2 - \sum_{\substack{l \in P \\ \deg(l) > n/2k}} \left(\frac{1}{|l|^k} - \frac{1}{|l|^{k+1}} \right)^2 \\
 &= L(2k) + L(2k+2) - 2L(2k+1) + O\left(\frac{q^{-\frac{n}{2k}(2k-1)}}{n}\right).
 \end{aligned}$$

Moreover, we have

$$\begin{aligned}
 &\sum_{\substack{l, h \in P \\ \deg(lh) \leq n/k}} \left(\frac{1}{|l|^k} - \frac{1}{|l|^{k+1}} \right) \left(\frac{1}{|h|^k} - \frac{1}{|h|^{k+1}} \right) \\
 &= \sum_{\substack{l \in P \\ \deg(l) \leq (n/k)-1}} \left(\frac{1}{|l|^k} - \frac{1}{|l|^{k+1}} \right) \sum_{\substack{h \in P \\ \deg(h) \leq (n/k)-\deg(l)}} \left(\frac{1}{|h|^k} - \frac{1}{|h|^{k+1}} \right) \\
 &= \sum_{\substack{l \in P \\ \deg(l) \leq (n/k)-1}} \left(\frac{1}{|l|^k} - \frac{1}{|l|^{k+1}} \right) \left(L(k) - L(k+1) + O\left(\frac{q^{-(k-1)(\frac{n}{k}-\deg(l))}}{(n/k)-\deg(l)}\right) \right).
 \end{aligned}$$

The contribution of the error term above is bounded by

$$\begin{aligned} &\ll q^{-\frac{n(k-1)}{k}} \sum_{\substack{l \in P \\ \deg(l) \leq \lfloor n/k \rfloor - 1}} \frac{1}{|l|(\lfloor n/k \rfloor - \deg(l))} \\ &\ll \frac{\log \lfloor n/k \rfloor}{\lfloor n/k \rfloor \cdot q^{\frac{n}{k}(k-1)}} \\ &\ll \frac{\log n}{n \cdot q^{\frac{n}{k}(k-1)}}. \end{aligned}$$

Since

$$\sum_{\substack{l, h \in P \\ \deg(lh) \leq (n/k) - 1}} \left(\frac{1}{|l|^k} - \frac{1}{|l|^{k+1}} \right) = L(k) - L(k+1) + O\left(\frac{1}{n \cdot q^{\frac{n}{k}(k-1)}}\right),$$

we have

$$\begin{aligned} &\sum_{\substack{l, h \in P \\ \deg(lh) \leq n/k}} \left(\frac{1}{|l|^k} - \frac{1}{|l|^{k+1}} \right) \left(\frac{1}{|h|^k} - \frac{1}{|h|^{k+1}} \right) \\ &= (L(k) - L(k+1))^2 + O\left(\frac{\log n}{n \cdot q^{\frac{n}{k}(k-1)}}\right). \end{aligned}$$

Hence, by Theorem 1.1, (17), and the estimates above, we obtain the second assertion in Theorem 1.2. $\square$

5. Proof of Theorem 1.3

We now show how Theorem 1.3 follows from Theorem 1.1 and Theorem 1.2.

Proof. Note that

$$\sum_{f \in M_n} (\omega_1(f) - \log n)^2 = \sum_{f \in M_n} \omega_1^2(f) + \log^2 n \sum_{f \in M_n} 1 - 2 \log n \sum_{f \in M_n} \omega_1(f).$$

Applying Theorem 1.1 and Theorem 1.2 completes the first part of the proof. We thus have

$$\sum_{f \in M_n} (\omega_1(f) - \log n)^2 \ll q^n \log n. \quad (18)$$

Let E_n be the set of monic polynomials f of degree n such that

$$\frac{|\omega_1(f) - \log n|}{\sqrt{\log n}} \geq (\log n)^{\epsilon'}, \quad (0 < \epsilon' < 1/2).$$

Let $|E_n|$ be the cardinality of E_n . Thus we have

$$\sum_{f \in M_n} (\omega_1(f) - \log n)^2 \geq \sum_{f \in E_n} (\omega_1(f) - \log n)^2 \geq \sum_{f \in E_n} (\log n)^{1+2\epsilon'} = (\log n)^{1+2\epsilon'} |E_n|$$

which together with (18) yields

$$|E_n| \ll \frac{q^n}{(\log n)^{2\epsilon'}} \text{ as } n \rightarrow \infty.$$

This proves $|E_n| = o(q^n)$ as $n \rightarrow \infty$. Let $\epsilon > 0$ be any real number. Since $\epsilon' < 1/2$, there exist $n_0 \in \mathbb{N}$ such that $(\log n)^{-(\frac{1}{2}-\epsilon')} \leq \epsilon$ for all $n \geq n_0$. Hence as $n \rightarrow \infty$, the number of polynomials f of degree n satisfying

$$\frac{|\omega_1(f) - \log n|}{\log n} \geq \epsilon$$

is $o(q^n)$ which implies that $\omega_1(f)$ has normal order $\log(\deg(f))$. $\square$

6. Proof of Theorem 1.4

Proof. First assume that there exist a polynomial f_0 of degree n_0 such that $G(f_0) > 0$. Then $G(f) > 0$ for all $f \in M_n$ with $n > n_0$. For $n \in \mathbb{N}$, we define

$$\mathcal{N}_0(n) = \{f \in M_n \mid \omega_k(f) = 0\}.$$

Using $q \geq 2$, $|M_2| - \pi_q(2) = (q^2 + q)/2$, and $|M_3| - \pi_q(3) = (2q^3 + q)/3$, we obtain

$$\begin{aligned} \sum_{l \in P} \frac{1}{|l|^2} &\leq \sum_{f \in M} \frac{1}{|f|^2} - 1 - q^{-4} \left(q^2 - \frac{q^2 - q}{2} \right) - q^{-6} \left(q^3 - \frac{q^3 - q}{3} \right) \\ &\leq \zeta_q(2) - 1 - \frac{q+1}{2q^3} - \frac{2q^2+1}{3q^5} \\ &= \frac{q}{q-1} - 1 - \frac{q+1}{2q^3} - \frac{2q^2+1}{3q^5} \\ &\leq \frac{23}{32}. \end{aligned}$$

Thus, we deduce

$$|\mathcal{N}_0(n)| = q^n - \sum_{\substack{f \in M_n \\ f \notin \mathcal{N}_0(n)}} 1 \geq q^n - \sum_{l \in P} \sum_{\substack{f \in M_n \\ l^k \parallel f}} 1 \geq q^n - q^n \sum_{l \in P} \frac{1}{|l|^k} \geq \frac{9}{32} q^n.$$

Thus the number of monic polynomials f of degree n for which $G(f) > 0$ and $\omega_k(f) = 0$ is not $o(q^n)$, and for such an f ,

$$|\omega_k(f) - G(f)| > \frac{G(f)}{2} \quad (19)$$

is satisfied. This proves that $\omega_k(f)$ does not have normal order $G(f)$ when $G(f)$ is not identically 0.

Next, we suppose $G(f) = 0$ for all $f \in M$. For $n \in \mathbb{N}$, we define

$$\mathcal{N}_1(n) = \{f \in M_n \mid \omega_k(f) = 1\}.$$

Let $h \in P_1$ denote a fixed monic irreducible of degree 1. We use this h to obtain

$$|\mathcal{N}_1(n)| \geq \sum_{\substack{f \in M_n, \nu_h(f)=k \\ \nu_l(f) < k \ \forall l \in P \text{ with } l \neq h}} 1 \geq q^{n-k} \left(1 - q^{-1} - \sum_{\substack{l \in P \\ l \neq h}} \frac{1}{|l|^k} \right) \geq \frac{1}{32} q^{n-k}.$$

Thus the number of monic polynomials f of degree n for which $G(f) = 0$ and $\omega_k(f) = 1$ is not $o(q^n)$, and all such f 's again satisfy (19). This completes the proof. $\square$

7. Proof of Theorem 1.5

We begin this section by proving a second version of Erdős-Kac theorem for function fields which is a consequence of the first version given in (7).

Theorem 7.1. (*Erdős-Kac theorem for function fields - Version II*) Let $a \in \mathbb{R}$. Then

$$\lim_{n \rightarrow \infty} \frac{1}{q^n} \left| \left\{ f \in M_n \mid \frac{\omega(f) - \log n}{\sqrt{\log n}} \leq a \right\} \right| = \Phi(a)$$

where $\Phi(a)$ is defined in (2).

Proof. Let us denote the set

$$\mathcal{Z}(n) := \left\{ f \mid f \in M_m, m \leq n, \frac{\omega(f) - \log m}{\sqrt{\log m}} \leq a \right\}.$$

Therefore

$$\begin{aligned} \frac{|\mathcal{Z}(n)|}{|\{f \mid f \in M_m, m \leq n\}|} &= \frac{|\mathcal{Z}(n-1)|}{|\{f \mid f \in M_m, m \leq n-1\}|} \frac{|\{f \mid f \in M_m, m \leq n-1\}|}{|\{f \mid f \in M_m, m \leq n\}|} \\ &\quad + \frac{\left| \left\{ f \in M_n \mid \frac{\omega(f) - \log n}{\sqrt{\log n}} \leq a \right\} \right|}{q^n} \frac{q^n}{|\{f \mid f \in M_m, m \leq n\}|}. \end{aligned}$$

Taking the limits $n \rightarrow \infty$ on both sides and using (7) completes the proof. $\square$

Proof of Theorem 1.5. For $f \in M_n$ and a function $g : M \rightarrow \mathbb{R}_{\geq 0}$, we define

$$r_g(f) := \frac{g(f) - \log n}{\sqrt{\log n}}$$

and

$$D(g, n, a) := \frac{1}{q^n} |\{f \in M_n \mid r_g(f) \leq a\}|.$$

Note that $D(\omega, n, a) \leq D(\omega_1, n, a)$. Thus, by Theorem 7.1, we obtain

$$\Phi(a) \leq \liminf_{n \rightarrow \infty} D(\omega_1, n, a). \quad (20)$$

Let $\epsilon > 0$ be a real number. Let us denote the set

$$\mathcal{D}(n, \epsilon) = \left\{ f \in M_n \mid \frac{\omega(f) - \omega_1(f)}{\sqrt{\log n}} \leq \epsilon \right\}.$$

Notice that

$$D(\omega_1, n, a) \leq D(\omega, n, a + \epsilon) + \frac{1}{q^n} |M_n \setminus \mathcal{D}(n, \epsilon)|. \quad (21)$$

By (11) and the calculation in the proof of Theorem 1.1, we obtain

$$\begin{aligned} \sum_{f \in M_n \setminus \mathcal{D}(n, \epsilon)} (\omega(f) - \omega_1(f)) &\leq \sum_{f \in M_n} (a_1(f) + a_2(f)) \\ &= L(2)q^n + O\left(n \cdot q^{\frac{n}{2}}\right). \end{aligned}$$

Moreover, we have

$$\sum_{f \in M_n \setminus \mathcal{D}(n, \epsilon)} (\omega(f) - \omega_1(f)) > \epsilon \sqrt{\log n} |M_n \setminus \mathcal{D}(n, \epsilon)|$$

which together with the previous inequality yields

$$|M_n \setminus \mathcal{D}(n, \epsilon)| \ll \frac{q^n}{\epsilon \sqrt{\log n}}.$$

Thus $|M_n \setminus \mathcal{D}(n, \epsilon)| = o(q^n)$ as $n \rightarrow \infty$. Using (21) and Theorem 7.1 with $a + \epsilon$ instead of a , we deduce that

$$\limsup_{n \rightarrow \infty} D(\omega_1, n, a) \leq \Phi(a + \epsilon).$$

Since ϵ is arbitrary, combining the above inequality with (20) completes the proof. $\square$

Data availability

No data was used for the research described in the article.

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