

Modal Interaction in Electrostatic MEMS Mirrors

by

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Abstract

The impetus of this work is to introduce nonlinear modal interactions as novel actuation mechanism for electrostatic MEMS-based scanning micromirrors. *Modal interactions* refer to the engagement of two or more modes of vibration in a system, creating a bridge to channel vibration energy from a directly excited mode to one or more of the coupled modes. In chapter two, this report carries out a comprehensive literature review of the different types of mode coupling in nonlinear resonators. First, internal resonance in general nonlinear oscillators are addressed. Second, we limit our focus to mode coupling in electrostatic MEMS.

As an initial test-bed, we examine in chapter three the modulation equations governing a system of two nonlinearly coupled 1-DOF oscillators involved in a 2:1 parametric modal interaction. Simulations show that as the excitation frequency varies in the vicinity of the directly excited higher-frequency oscillator, the amplitude of its motions saturate. Meanwhile, the amplitude of the lower-frequency oscillator undergoes large motions under the influence of a parametric ‘energy pump’.

The fourth chapter reports on nonlinear modal interaction in a MEMS made of an electrostatically actuated curved-beam. We characterize the first few in-plane and out-of-plane bending modes of the beam. Thermal noise excitation is utilized to extract the out-of-plane natural frequencies, whereas the in-plane natural frequencies are captured using pulse excitation. Then, the frequency response of the MEMS in the neighbor of the first symmetric and second symmetric in-plane modes. Characterization results discloses a 2:1 ratio between the second symmetric and the first anti-symmetric in-plane modes. We show that this anti-symmetric mode can be effectively excited via the energy channel between it and the second symmetric mode when the latter is driven directly by external electrostatic forcing.

In the fifth chapter, we establish bending-torsional equations of motion for a symmetric electrostatic MEMS actuator that can capture the 2:1 modal interaction between its in-plane bending and out-of-plane rotational motions. Our approach demonstrates that incorporating the linear slopes into the cross-sectional shear strains efficiently originates quadratic couplings between the bending and torsional motions whose existence depends on non-vanishing first moments of

area of the microbeam's cross-section. According to imperfections in microdevice fabrication, we assumed a minuscule offset in positions between the centroid of the as-fabricated and as-designed cross-sections of the microbeams. Energy approach is exploited to derive the equations of motion (EoM). The static response of the MEMS actuator together with its tuned eigenmodes are examined in this chapter.

Chapter six reports the frequency- and voltage-displacement behaviors of the mirror addressing the 2:1 and 3:1 flexural-torsional internal resonance experimentally and numerically. The numerical simulation results indicate that the in-plane motion, which is the directly excited mode, saturates upon the initiation of a 2:1 energy pathway between the bending and torsional motions. Through suitable tuning of the AC frequency, the amplitude of the in-plane motion is minimized, while the amplitude of the torsional motion, an indirectly excited mode, is maximized. The numerical simulation results demonstrate that the actuator's torsional motion, when subjected to a 1:2:1 electro-flexural-torsional modal interactions, is triggered by applying a maximum voltage of 10 V, resulting in about 15 degrees rotational angle. Further, prolific frequency combs are generated as a result of secondary Hopf bifurcations along the large-amplitude response branches, capturing quasi-periodicity in the MEMS dynamics. The experimental results demonstrate the mirror's dynamics exhibiting 3:1 flexural-torsional modal interaction that provides an efficient out-of-plane rotation drive through in-plane excitation.

The present study is a platform for the implementation of a novel actuation mechanism of MEMS scanning micromirrors using parametric modal interaction. Conclusion remarks and propose future work with the are presented seven chapter.

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Chapter 1

Introduction

1.1 MEMS Resonators

Special thanks to recent advancements in micro-fabrication technology, a universal movement has emerged for designing small-scale engineering devices, such as sensors and actuators. Among different groups of microstructures, micro-electro-mechanical systems (MEMS) have found a significant contribution in the development of sensing and actuating tasks. There are many applications of such systems in which the MEMS is set to be acting like a resonator to provide a desired task. It should be noted that the term resonator clarifies that dynamic characteristics of MEMS are exploited to implement actuating, sensing, or manipulating acts.

1.2 Electrostatic MEMS Actuators

Recent advancements in micro-technology have led Micro-Electro-Mechanical systems to become the pioneer candidate in designing micro-scale actuators. However, the implementation of their working principle depends on the structures dimensions, fabrication technology, application, and the maximum consumed power. Accordingly, the working principle governing

MEMS actuators is generally schemed using electrostatic [1–8] piezoelectric [9, 10], electromagnetic [11–14] and electrothermal [15–20] actuations. It is worth noting that fabrication of the MEMS operating based on electrostatic excitation is much easier comparing to that of the other three families. Moreover, electrostatic excitation results in fast dynamic response, low energy loss, and improved signal-to-noise ratio.

The trace of capacitive MEMS actuators are found in many industrial applications. For instance, micromotors producing either translational or rotational motions are known as one of the main components in manipulating micromechanical systems. They can be fabricated in-plane, on x-y stages for example, to control the driving forces for microrelays [21–23], micromirrors [24, 25] and microgrippers [26]. In such applications, the actuation mechanism for the micromotor is mostly devised based on developing an electrostatic field between a stationary and a moving electrode.

Electrostatic actuation with a moderately low operating voltage effectively contributes to the design of microshutters [27]. Furthermore, electrostatic MEMS capacitors having curved electrodes are employed elaborately to manipulate the light transmission in microshutters [28].

Electrostatic MEMS actuators have also proven successful in medical and biomedical applications of scanning micromirrors [29–34]. For such applications, it is common to design the MEMS based on one of these four groups' configuration; linear comb-actuator, vertical comb-actuator, rotary comb-actuator, and parallel plate-based electrostatic actuators.

In addition, it has been shown in the literature that the idea of constructing electrical field between two parallel electrodes highly deserves to be used for microrelays [35]. In such devices, preparing a design based on electrostatic actuation allows us to reduce their size and weight, and holding electrical current substantially.

1.3 MEMS Scanning Micromirrors

Scanning mirrors gained market share in the past decade finding applications in Light Detection and Ranging (LiDAR) systems [36], optical coherence tomography [37], and smart glasses [38].

Smart glasses rely on scanning mirrors to raster an incoming laser beam onto a display surface [39, 40] or directly on the retina [41]. The mirrors must provide the scanning range and speed necessary for such modern projection systems. This is usually achieved using resonant mirrors, especially for the fast axis of the raster scan.

State-of-the-art scanning mirrors rely on electrostatic and piezoelectric actuation. Piezoelectric materials deform under applied voltage due to asymmetric accumulation of charges. This deformation is used to actuate scanning mirrors. However, piezoelectrically actuated mirrors suffer from creep and hysteresis and require more sophisticated control systems [42]. Moreover, piezoelectric materials have high dielectric constants (relative permittivities) and, therefore, create significant parasitic capacitance [43]. This introduces additional control problems and drastically increases power consumption, a critical issue in battery-operated systems such as smart glasses.

Electrostatic mirrors are simpler and more robust to fabricate. They are driven by a voltage difference between two opposing surfaces and utilize the resultant attraction to move the mirror. This is usually achieved using staggered or angled comb fingers [44, 45] but is limited in terms of scanning range and operating frequencies due to limits on the forces generated by their combfingers as well as their large footprint and mass. The size of the combfinger actuators also creates significant damping forces and incurs large parasitic capacitance. Additionally, the nonuniform overlap of the staggered fingers during actuation [46] leads to a loss of control authority at larger angles. Therefore, they require complicated control electronics.

Additionally, electrostatic mirrors exhibit lower power consumption and simpler control electronics, except for staggered and angled comb finger actuators. However, their range of motion is usually limited by the captive gap. For instance, smaller capacitive gaps for parallel-plate mirrors suffer a very limited field of view (rotation range), while larger capacitive gaps scale voltage requirements beyond reason. Moreover, electrostatic parallel plate actuators suffer from the pull-in phenomenon, thereby limiting the motion range to about one-third of the available capacitive gap [47].

Electrothermal actuation can also be used to rotate the scanning mirrors by utilizing the ther-

mal expansion due to Joule heating. However, this method suffers from even higher power consumption. It also has a slow response, making it unsuitable for high-speed scanning mirrors. Electrostatic MEMS scanning micromirrors, however, have proved their outstanding role in modern technology within a wide range of applications such as, microscopy, micropositioning, micromachining, imprint lithography, and precision measurements. Furthermore, electrostatic scanning mirrors serve a function for commercialized devices including projectors, bioimaging technologies [48–50], televisions [51, 52], tunable lasers [53], and hard disk storage. Medical and biomedical areas of research are another applications where the employment of scanning micromirrors have resulted in invaluable achievements, making the life easier. For instance, we can mention submicron biomedical systems including microscopy, optical scanning confocal, microscopy, laparoscopy, medical endoscopy, and optical coherence tomography (OCT) [29]. It should be noted that in those applications, scanning micromirrors are utilized to satisfy the following tasks; scanning the objective area, manipulating the light sources for surgery, and obtaining the bioimaging from the surgical locations. With this in mind, it is crucially important to so design and model such devices that their characteristics to be compatible with the desired operating conditions.

MEMS technology is known as a well-established process in fabricating scanning micromirrors, providing adequate motion characteristics that is needed to satisfy the above-stated applications. It is worth noting that there are still limitations and considerations in the fabrication process of such devices that depend on the operating conditions relating to each application [29]. These processes results in various kinds of mechanisms to actuate the mirrors.

Totally, there are four commonly used actuating principles to manipulate the motion of MEMS-based scanning micromirrors; electrostatic, piezoelectric, electrothermal, and electromagnetic excitation schemes. Among these four actuating mechanism, electrostatic approach provides the mirrors with fast speed dynamics, large scanning area, and low consumed power. However, there are two drawbacks for this method of excitation introduced in the literature from which these systems are still suffering from, pull-in phenomenon and high driving voltage.

It has been shown in the literature that electrostatic actuation itself can be classified into four

main sub-domains; linear comb drive, vertical comb drive; rotary comb drive, and parallel plate capacitor-based actuator [37, 54–56]. Among these, the idea of creating electrical field between two parallel electrodes placed in front of each other making a transversal capacitor leads MEMS systems to experience strong nonlinearity in both static and dynamic regimes. Actually, the electrostatic force developed between two transversally-movable parallel electrodes is nonlinearly dependent on the amount of the initial gap distance between them, producing a strong nonlinear displacement-dependent force. The presence of this force in MEMS dynamics, which is known as the primary source of nonlinearity, causes various nonlinear phenomena such as, jump, multi-valued response, hysteresis, saturation, modal interaction, and bifurcation to be experienced in dynamic behavior of such systems.

Based on this study, a novel actuating mechanism can be proposed that efficiently improves the performance of scanning micromirrors by using internal resonance characteristics and saturation phenomena. To address this problem, it is required to study the probability of internal resonance in nonlinear resonators first, and second, the conditions under which modal interaction can result in saturation phenomenon should be examined. Therefore, the next chapter focuses on the literature survey of internal resonance and saturation phenomenon happening in both general nonlinear resonators and electrostatic MEMS systems.

1.4 Research Motivation

The impetus of this research is to benefit from autoparametric modal interaction characteristics to improve the performance of scanning micromirrors. In general, there are four different types of actuation for MEMS-based scanning/projection micromirrors, among which, electrostatic actuation potentially provides the mirror with larger stroke comparing to electromagnetic and piezoelectric actuation mechanisms, however, it is still less than electrothermal scheme. It is worth noting that electrothermal actuation approach requires more complex fabrication technology in comparison to electrostatic-based actuation scheme, therefore, it deserves to propose an efficient electrostatic-based actuation mechanism resulting in large displacement, in the same

order of what which is obtained under electrothermal mechanism.

The scanning mirrors utilized in smart glasses create a raster scan using two consecutive mirrors, a fast mirror and a slow one. The slow mirror scans the incoming laser across the vertical axis at a rate equal to the frame rate of the display, while the fast mirror scans the laser beam horizontally. The fast mirror must be fast enough to catch up with the vertical resolution. Thus, its speed must be half the number of vertical lines (vertical display resolution) multiplied by the frame rate. However, it cannot be faster than the ability of the laser modulator to modulate the color of the laser beam. Therefore, the fast mirror frequency (F_f) is limited by:

$$\frac{1}{2}N_v F_r < F_f < \frac{R}{2N_H} \quad (1.1)$$

where N_v and N_H are the vertical and horizontal display resolutions, respectively, F_r is the frame rate, and R is the laser color modulation rate. Figure 1.1 shows the basics of raster scanning displays using a fast and a slow-scanning mirror. However, the limitations of electrostatic and piezoelectric actuators impose challenges in achieving the required specifications. Two solutions based on electrostatic actuation have been proposed thus far to overcome such challenges:

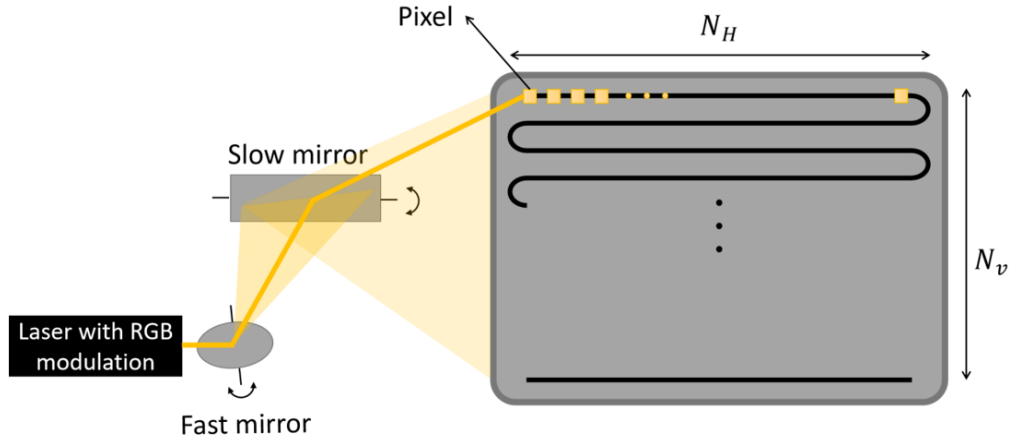


Figure 1.1: Raster scanning using two scanning mirrors

- 1) The use of staggered or angled comb fingers to rotate micromirrors out-of-plane has proven

successful in the past [44, 45]. However, it remains inefficient in terms of the maximum scan angle, and the massive structure of the staggered comb fingers limits the operating frequency. The larger footprint of the actuator also generates significant damping via drag forces. Additionally, the nature of the staggered comb fingers requires complicated control electronics because of their huge parasitic capacitance and the non-uniform overlap of the staggered fingers during actuation [57]. This leads to a loss of control authority at larger angles. Therefore, the use of staggered or angled comb fingers poses an engineering challenge and requires innovative solutions. A schematic of staggered comb fingers is shown in Figure 1.2.

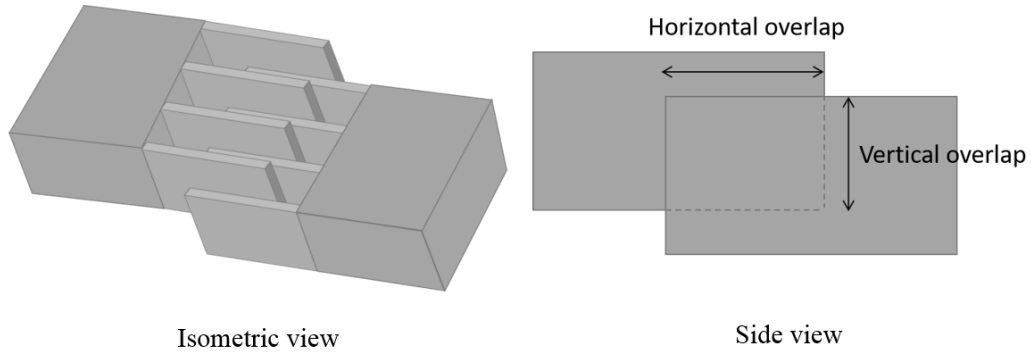


Figure 1.2: Staggered comb fingers

2) The use of an in-plane comb finger drive to create a rotational motion via parametric excitation has been proposed by STMicroelectronics [58]. Autoparametric excitation due to the modal interaction of the 2:1 type has been previously implemented to indirectly excite the required mode of actuation in two coupled mass-spring-damper large scale systems [59–63].

This study proposes the employment of parametric modal interaction between in-plane and out-of-plane vibration modes of scanning mirrors to produce large rotation stroke, allowing for a large field of view by removing the actuation electrode from the path of the mirror plate. This can be obtained through designing in-plane interdigitated comb-finger actuators or in-plane microbeam-based actuator that will reduce both the actuator footprint and the parasitic capacitance. With this in mind, two-to-one parametric internal resonance can be effectively utilized to drive the mirror with sufficiently large out-of-plane rotation, employed in the absence of direct

out-of-plane excitation. In order to create parametric modal interaction applicable for MEMS scanning micromirrors, a necessary condition needs to be satisfied by design, and that is, there must be a commensurate ratio between the desired in-plane and out-of-plane vibration modes. This allows the vibration energy to exchange between two motion-directions while the in-plane motion is directly excited using electrostatic actuation. Figure ?? displays the out-of-plane rotation of a back-etched mirror under in-plane direct excitation.

Chapter 2

Literature Review

This chapter focuses on introducing different types of internal resonance happening on the dynamics of both general resonators and MEMS-based oscillators. For each case, the impressive results that are the consequences of this phenomenon are addressed. The literature survey begins with the section that concerns on internal resonance occurring in nonlinear mechanical resonators. The second part concentrates on modal interaction taking place on nonlinear dynamics of electrostatic MEMS. This section is divided into two subsections, first addresses the occurrence of internal resonance happening in one-plane, meaning that there is an energy channel between two vibration modes that both are related to the same plane, and second introduces some valuable researches that have studied the dynamic characteristics of a MEMS device experiencing internal resonance between two motions happening on two perpendicular directions, namely, autoparametric modal interaction.

2.1 Internal Resonance in Nonlinear Resonators

Nonlinear oscillators behave differently from what is experienced about linear resonators. In general, the dynamic response of a nonlinear oscillator is a combination of linear and nonlinear response characteristics. Elaborately speaking, it should be noted that corresponding to each nonlinear resonator, there is a linear system that is obtained through eliminating the impact of all nonlinear phenomenon included in the system dynamics. Therefore, it can be stated that each nonlinear resonator is introduced as a linear system which is affected by some nonlinearities. The addition of nonlinearity to a linear system leads to the emergence of some interesting phenomena in both static and dynamic responses of the system such as multi-valued response [64, 65], jump [66, 67], bifurcation [68–70], secondary resonance [71–75], parametric resonance [76–79], internal resonance/modal interaction [80–85].

Among these nonlinear phenomena, modal interaction has been the focused of attention of many researchers working on nonlinear resonators. An interesting motion characteristic appearing in this nonlinear event is that the system response is contributed by at least two vibration modes and energy exchange occurs between the engaged modes. There is a necessary condition for a nonlinear resonator to experience modal interaction, and that is, there should be a commensurable ratio between two of the system's natural frequencies. For instance, if the second natural frequency of a resonator is twice its first natural frequency, the system is potent to experience two-to-one internal resonance, and its dynamic response is constructed by the two vibration modes while the excitation frequency is set to be varying in the vicinity of one of these two natural frequencies. Similarly, if the ratio between two vibration modes is equal to three, the oscillator has the potential to include another order of internal resonance, defined as three-to-one modal interaction.

There are many valuable research works addressing the occurrence of different types of internal resonance, happening in nonlinear oscillators [61, 65, 80, 86–91]. The dynamic of a planar elastic pendulum undergoing two-to-one internal resonance was studied by A. H. Nayfeh [64]. He derived the governing motion equations using energy method. He examined the origination of each nonlinear term contributing the resonator dynamics that can cause the system to expe-

rience internal resonance between its radial and rotational motions. Moreover, he addressed the dynamics of a nonlinear energy sink (NES) using a two degree-of-freedom lumped mass model. It was indicated that the efficacy of the oscillation absorber is substantially improved while the autoparametric resonance of the NES is actuated. The term autoparametric indicates that mode coupling occurs between two motions that are happening along two perpendicular directions. This kind of internal resonance is going to be addressed in detail, in next sections.

Nonlinear normal modes (NNM) of a shallow arch beam subjected to flexible boundary conditions were extracted by Yi et al. [92]. They obtained the relationships between the linear natural frequencies of the undamped structure that satisfy two-to-one internal resonance condition. For both transversal and rotational stiffness, they examined the required conditions for the activation of two-to-one internally resonant NNMs.

Mechanical resonators constructed by compound structures are highly probable to undergo internal resonances. Bajaj et al. [93] studied one-to-one and two-to-one mode engagements in a single degree-of-freedom resonator which is coupled to a series of nonlinear pendulums. They utilized the method of averaging to extract the governing modulation equations. Their findings reveals that the system's frequency-response behavior forms an M-shape curve including different types of bifurcations such as, Hopf and sub-critical pitchfork bifurcations, while autoparametric resonance is activated.

The beneficial use of internal resonance in nonlinear energy harvesters is apparent in the literature as well. Xiong et al. [94] presented a novel technique to tune the commensurable ratio between the natural frequencies of an energy harvester through adding an auxiliary component to the main oscillator, resulting in a two degree-of-freedom resonator. Comparing to the linear generation of their harvester, they proved both experimentally and theoretically that the internally coupled nonlinear resonator remarkably widens the resonance bandwidth of the energy harvester. Yang et al. [95] conducted experimental and theoretical studies on two-to-one modal interaction in a low frequency energy harvester. Their results indicated that internal resonance characteristics included in nonlinear magnetic force causes the efficiency of energy harvesting multiplies that of uncoupled-mode resonant.

Mode coupling phenomenon has been also employed on mass sensing application. Du et al. [96] proposed a mass sensing mechanism working based on three-to-one internal resonance in a compound structure made of two ductile cantilevers connected at their roots. Their outputs indicated that not only the response amplitude of the short cantilever grows noticeably, but also the resonance frequency shift enhances compared to the case of single degree-of-freedom resonator while the coupled system is excited near its super-harmonic resonance.

2.2 Internal Resonance in Electrostatic MEMS

In general, electrostatic MEMS systems are composed of one stationary and one movable electrode, forming a capacitive-based device based on parallel plate assumption. There are two different actuating principle in this devices, 1) interdigitated comb fingers, where the movable electrode is allowed to experience large displacement, and 2) transverse electrostatic force where the movable electrode cannot be displaced too much due to lack of sufficient space between the two electrodes. In both cases, the electrostatic force is defined as displacement-dependent force developed between the fixed and movable electrodes. In the former case, the electrostatic force is linearly depended on the displacement of movable electrode; however, in the last case, the force is nonlinearly expressed in terms of displacement field, and it is introduced as the main source of nonlinearity in electrostatic MEMS sensors and actuators, softening the mechanical stiffness of the flexible structure.

There is another source of nonlinearity in MEMS devices that comes from large deformation, known as geometrical nonlinearity. Actually, while the constitutive particles of flexible electrodes undergo large displacement due to resonant or large voltage amplitude, larger amount of strain energy is stored in structures compared to when the particles experience infinitesimal displacements. The difference between the stored strain energy for the two cases of infinitesimal and relatively large deformations is compensated by geometrical nonlinearity arising from nonlinear strain-displacement relationships. Depending on resonator's boundary conditions, static and dynamic responses of MEMS systems are affected by different types of geometrical non-

linearities. For fully-clamped boundary conditions, mid-plane stretching effect is included in MEMS dynamics [97], however, for clamped-free boundary conditions, the geometrical nonlinearity is generated due to inextensibility of the flexible component, as observed in cantilever beams [97].

Considering these two sources of nonlinearity, it can be concluded that MEMS oscillators are prone to experience interesting nonlinear phenomena expected to happen about nonlinear resonators. In the following, we aim to explain different occasions under which a MEMS device can experience a specific type of internal resonance.

2.2.1 One-Plane Internal Resonance

Electrostatic MEMS systems operate in-plane or out-of-plane depending on design and fabrication process. In this section, we aim to concentrate on the case in which mode engagement occurs between two vibration modes related to one-direction motion, either in-plane or out-of-plane. It is worthwhile to note that in this case, both the electrostatic force and the vibration of the movable electrode happen in same plane, and there is a commensurate ratio between the vibration modes laying in the same plane.

Many researchers have investigated the steady-state dynamic response of MEMS systems addressing this type of internal resonance. As observed in the literature, the systems are involved in complex frequency-response characteristics while they are actuated near their internal resonance regions.

One-to-one internal resonance characteristics were analyzed by B. K. Hammad et al. [98] to propose a MEMS-based tunable filter. In their work, the flexible electrode of the MEMS device is made of an H-shape structure containing two clamped-clamped beams that are weakly connected through a slender coupling beam. They analytically studied the dynamic response of the filter using distributed-parameter model and multiple-time scales method. They benefited from one-to-one modal interaction between in-phase and out-of-phase vibration modes that both happening out-of-plane, to generate rectangular-shape frequency-response curves that are qualified

for bandpass filters. Because their MEMS device is made of two identical resonators weakly coupled using a mechanical spring (third beam), they also reported on mode localization phenomenon and the relation between its intensity and exchange energy between the two engaged modes.

Z. Wang et al. [99] examined three-to-one internal resonance characteristics of a microbeam under electrostatic actuation and flexible supports. In their work, they have reported on the occurrence of modal interaction between symmetric and anti-symmetric modes. They have illustrated that for the case in which one end is assumed to be flexible and the other end is rigidly clamped (breaking the symmetry of the device), modal interaction happens and it causes the vibration energy to exchange between the first two modes. However, there is no modal interaction between the symmetric and anti-symmetric modes while the two supports are considered to be rigidly clamped. Furthermore, they have reported on the emergence of Hopf bifurcation in the frequency-response behavior of the MEMS resonator while the two modes are coupled. Moreover, they investigated the influence of tensile axial load on the system response, and found that reducing the spring stiffness or DC voltage amplitude enhances the strength of modal interaction. Also, they indicated that comparing to the case of zero axial load, tensile axial load causes the internal resonance characteristics to become more sensitive to the system parameters.

Two-to-one internal resonance of an electrically actuated arch microbeam was studied by Hajjaj et al. [100]. They conducted both experimental and analytical investigations to capture modal interaction between the first two symmetric modes of the arch resonator. They employed electrothermal tuning mechanism to match the two vibration modes, meeting two-to-one commensurable ratio. They observed that under large AC voltage amplitude, simultaneous excitation of the two engaged modes are driven together with the activation of internal resonance. They demonstrated that the MEMS system can experience rich dynamics including quasi-periodic and chaotic motions while modal interaction occurs. Dynamic behavior of a series of electrostatically coupled microbeams forming a microbeam array, operating in nonlinear regime, was comprehensively assessed by S.

Gutschmidt et al. [101]. In their study, the MEMS device is composed of multiple clamped-

clamped beams that are coupled through interior electrostatic forcing. In addition, a nonlinear electrodynamic damping parameter was included in their model for both preventing the response amplitude to tend to infinity and resulting in better agreement between the numerical and experimental outputs; however, this type of damping was included as a lumped energy dissipation source in their model. It should be mentioned that they used local mode shapes to discretize the motion's equations to a set of nonlinear ODEs, and then, extracted the global natural frequencies of the linearized compound MEMS under the bias electrostatic force. In their work, both one-to-one and three-to-one internal resonances were examined using multiple-time scales method.

Ruzziconi et al. [102] reported on the emergence of two-to-one modal interaction between the third and fifth modes of a clamped-clamped MEMS resonator. In their work, in order to excite both symmetric and anti-symmetric modes while characterizing the MEMS device, the movable electrode is covered by a half-length electrode. They actuated the resonator using a combination of both DC and AC voltages provided that the AC frequency is set to be varying near the third mode (exciting from below), and examined each part of the measured frequency-response curves meticulously to address the participation of each mode in forming the MEMS resonant response. In order to show each mode contribution in constructing the system response beyond the activation of internal resonance, they took Fast Fourier Transformation (FFT) of the measured data, displaying two main peaks corresponding to the two resonant modes. They also performed comprehensive numerical simulations to validate the frequency-response curves obtained from experiment.

Hajjaj et al. [103] experimentally studied the occurrence of multiple internal resonances in arch MEMS resonators. As the tuning mechanism, they passed electrical current through the arch resonator to commensurate the natural frequencies, providing the required condition to meet two-to-one, three-to-one and four-to-one internal resonances. They observed an improvement about the frequency stabilization while the arch is driven in modal interaction regime. One application for the occurrence of internal resonance is stabilizing resonant frequencies. This phenomenon was observed by X. Wang et al. [104] while investigating the dynamics of a MEMS device, which is made of two electrostatically coupled double-ended tuning fork (DETF) resonators. In their

study, they prepared a condition under which three-to-one internal resonance is met between the fundamental natural frequencies of the two resonators. In order to show the concept of frequency stabilization, they directly actuated the low-frequency resonator through applying DC and AC voltages in such a manner that the AC frequency was assumed to be varying in the neighbor of this resonator's natural frequency, and demonstrated that the resonance frequency becomes stable as the coupling electrostatic voltages decreases. They used electronic experimental set-up including Lock-in-Amplifier (LIA) to monitor the frequency-response behavior of the MEMS system.

Alcheikh [105] conducted both experimental and analytical investigations on characterizing a MEMS device made of V-shape structures to address linear modal interaction. They used electrothermal tuning mechanism to adjust the MEMS natural frequencies. They proved that mode veering can occur between different vibration modes, for specific values of the thermal voltage. Moreover, they found that their analytical results can predict the occurrence of mode crossing as well, and this is because that geometrical uncertainties were not included in their mathematical model, providing symmetry. However, mode crossing cannot occur in reality because there is always a level of geometrical imperfection coming from fabrication process that results in asymmetry.

Nonlinear dynamic response of a MEMS resonator addressing three-to-one internal resonance were analyzed by D. A. Czaplewski et al. [106] using bifurcation diagrams. They showed that the steady state behavior of the system cannot be sustained at the onset of internal resonance. However, for higher actuation level, the system oscillations is stable and sustainable but the resonator amplitude is found to be modulated.

Houri et al. [107] reported on the emergence of supercritical Hopf bifurcation in a clamped-clamped MEMS resonator for the case in which there is three-to-one commensurable ratio between the first and third resonant modes. In their work, the microresonator was fabricated out-of-plane and the system oscillations were measured optically using Laser Doppler Vibrometer (LDV). They showed that the system response experiences period doubling bifurcation after the onset of mode engagement. In order to validate their experimental outputs, they employed the

method of multiple-time scales to analytically capture the MEMS dynamics.

Asadi et al. [108] experimentally investigated one-to-two and two-to-one internal resonances arising from strong nonlinearity in an asymmetric microbeam resonator. They demonstrated mode coupling between the second and third flexural modes of the resonator, resulting in high energy transfer to the vibration mode which is internally excited. Furthermore, they proved that quadratic nonlinearity coming from mid-plane stretching plays a significant role in the energy exchange process. They also modeled the system dynamics mathematically and solved it analytically using harmonic balance method (HBM).

Xia et al. [109] proposed a novel mass sensing mechanism working based on three-to-one internal resonance characteristics that is capable to trigger the first time warning and the second time warning as the mass perturbation is incremented. In their work, the MEMS device is composed of two weakly coupled duffing-type resonators, forming an H-shape MEMS-based mass sensor. They showed that by changing the eigenfrequency of the high frequency mode, both the operating range of the first time detection and the threshold of the second time detection are tuned. Moreover, they put an argument that the linearity of the sensing mechanism can be enhanced by changing the sign of the coupling term between the low-frequency resonator and high-frequency resonator.

2.2.2 Parametric Modal Interaction

In this section, the occurrence of mode coupling between two perpendicular motions is discussed. This type of mode engagement is known as parametric internal resonance. As some examples addressing such situations, we can mention the following cases; existence of modal interaction between two flexural modes that each one is related to one specific plane; internal resonance between one torsional mode and one flexural mode; mode involvement between one flexural mode and one longitudinal vibration mode, etc. Generally, while there is a commensurate ratio between two natural frequencies corresponding to two different motion directions, the resonator is potent to experience parametric internal resonance. Under this condition, the vibration energy of the mode which is directly excited by an external source can be parametrically transferred to the motion occurring in other plane. With this in mind, saturation might take place force-amplitude behavior of resonators. In recent years, this phenomenon has attracted the interest of many researchers working on dynamic analysis of MEMS-based devices. In the following, some of these works are explained.

Nayfeh et al. [61] examined the complex dynamics of a two degree-of-freedom L-shape structure both experimentally and analytically. In order to investigate the potential two-to-one parametric internal resonance, they excited the primary resonance of the higher mode together with the sub-harmonic resonance of the lower mode (excitation from above). Their results indicated that for specific values of the excitation frequency, the system undergoes periodic, quasi-periodic and chaotic regimes. They also demonstrated that keeping the excitation frequency constant, the oscillation amplitude is saturated after the forcing amplitude reaches a threshold.

Parametric mode coupling and amplification in a torsional micro-electro-mechanical resonator was experimentally examined by R. Baskaran et al. [110]. In their study, they benefited from interdigitated out-of-plane comb fingers to excite the MEMS device near the internal resonance which is composed of one flexural and one torsional resonant mode. Their findings reveal that the emergence of parametric internal resonance between the two mechanical modes is mainly because of the time-varying stiffness generated by displacement-dependent electrostatic force.

Saturation phenomenon occurring in nonlinear dynamic response of an H-shape MEMS gy-

roscope was addressed by Sarrafan et al. [111]. They introduced a planar microresonator provided by two-to-one commensurable ratio between two desired flexural modes that is made by design. In their design, the two motions happen in two different in-plane directions so that while high-frequency mode, which is X-motion, is excited directly through the drive electrode, the low-frequency mode, which is Y-motion, is run auto-parametrically. They examined the force-response behavior of the MEMS and observed that while the AC voltage reaches a threshold, the amplitude of X-motion is saturated and the vibration energy is transferred to Y-direction. This phenomenon activates the Y-motion with remarkable amplitude using indirect excitation. They addressed the same phenomenon on the dynamics of a MEMS gyroscope made of compound structures, benefiting from the traditional pendulum-spring vibration mechanism [112].

Gadhvi et al. [113] introduced a robust mechanism to measure angular velocity using two-to-one autoparametric resonance characteristics. To this aim, they performed their experimental tests on a commercialized in-plane sensor fabricated based on MEMS Interdigitated Design for Inertial Sensors (MIDIS) technology, which was encapsulated by the technology of Teledyne DALSA Semiconductor Inc. In their experiment, the MEMS sensor is placed on a rate table which is under a constant angular velocity. Then, electrostatic voltage is applied across the MEMS capacitor through the drive electrodes, and in following to that, the motion-induced current is measured through the sense electrode. They observed that the resonance bandwidth enhances as the DC voltage increases up to a critical value, and then, it starts shrinking as the DC voltage grows beyond that threshold.

Zhang et al. [114] experimentally investigated the two-to-one modal interaction between the fundamental extensional and flexural modes of a microresonator made of two cantilevers carrying a pair of shuttle mass at their tips. They also verified the dynamic response of the system using a two degree-of-freedom oscillator model containing quadratic nonlinearities. A standard silicon on insulator (SOI) process consisting two side-by-side cantilever beams with bulk mass on their tips was performed to fabricate the resonator. They schemed an electrical circuit including electrostatic actuation and differential piezoresistive sensing methods to measure the motion characteristics of the MEMS device. Their findings prove that the resonance frequency of the longitudinal mode can be tuned due to its internal coupling with the flexural mode, and this tun-

ing process causes hysteresis in frequency-response behavior of the system which is observed through forward and backward sweep.

Xia et al. [115] proposed a smart and novel sensing mechanism that is implemented based on an auto-parametric internal resonance system (AIRS). They demonstrated that for a T-shape microresonator, where there is a two-to-one commensurate ratio between its fundamental X- and Y-motion modes, the peak of the M-shape frequency response curve becomes flattened by adding extra damping to the cantilever beam. They showed that the slope of the flat tip of the frequency-response curve is independent to both the excitation amplitude and the excitation frequency. However, this slope changes while the clamped-clamped beam is externally disturbed by added mass. Regarding this, they proposed a mass sensing mechanism using AIRS that related the slope of the flat tip of the frequency-response curve to the amount of the added mass deposited on the fully-clamped beam.

Daqaq et al. [46] examined the dynamic behavior of a micromirror scanner under two-to-one internal resonance. They showed that the mirror undergoes complicated dynamics including aperiodic response while modal interaction is activated. This causes the mirror to become out of focus, and consequently distorts images.

Chen et al. [116] used nonlinear energy transfer to propose a novel technique maintaining the vibration energy of resonators even after the external excitation supply is off. Actually, they showed that using mode engagement, they can overcome the energy dissipation caused by viscous damping, and keep the system's oscillation stable without inducing cyclic energy from external excitation source. They utilized one-to-one modal involvement occurring between bending and torsional modes, and to model it analytically, they considered two coupled damped resonators including a traditional duffing oscillator with a linear one. Their results indicated that increasing the actuation voltage causes the resonant frequency shifts to the right, and eventually, it is stabilized as the electrostatic voltage reaches a specific value.

Mustafa et al. [117] investigated the nonlinear dynamics of a vibration-absorbing device made of a column-pendulum structure through performing both analytical and experimental studies. In their work, the coupled structure is composed of a flexible cantilever having a bulk mass

at its end. The pendulum is vertically connected to the tip mass through a string, making a compound resonator. To address various bifurcations within the complex nonlinearities embedded in the system dynamics, they captured the frequency-amplitude of the steady-state response of the system. As a part of their study, they focused on two-to-one internal resonance between the first resonant mode of the column and the linearized natural frequency of the pendulum. To inspect this, they set the excitation frequency to be varying twice the fundamental natural frequency of the beam column, whereas the beam resonant mode itself is nearly two times the pendulum natural frequency. This refers to the condition of excitation from above, meaning, simultaneous actuation of the one-second and one-fourth sub-harmonic resonances of the beam and pendulum, respectively. Their results proved that as the forcing amplitude grows, the column motion is saturated and vibration motion transmits to the pendulum.

The dynamics of a MEMS-based gyroscope is investigated by Noori et al. [118]. They proposed a novel technique to measure angular velocity using two-to-one internal resonance characteristics happening between transversal and rotational motions. Their MEMS device is composed of a clamped-clamped microbeam with a cantilever at its middle, forming a T-shape structure. They demonstrated that while the excitation level hits a certain value, the low-frequency mode is auto-parametrically actuated due to the occurrence of saturation phenomenon. Furthermore, their outputs confirm that the MEMS dynamic response is not remarkably affected by the operating pressure while the resonator is set to be working under internal resonance; however, this is not the case as traditional linear oscillations. Motion saturation in parametric modal interaction has been examined by the same group, concerning on H-shape MEMS-based structures [111, 119]. To observe energy transfer between to in-plane motions, they prepared their design in such a way that the occurrence of two-to-one internal resonance is potentially embedded in the MEMS dynamics. To do this, they were inspired by macroscale spring-pendulum system that it is capable to switch between the spring and pendulum resonant modes, while mode coupling exists. To validate their experimental results, they developed a lumped-mass model to describe the system dynamics.

Nabholz et al. [120] reported on mode coupling occurring between the drive mode and parasitic mode of a MEMS gyroscope. To analyze the system dynamics analytically, they constructed

a two degree-of-freedom system including two coupled resonators with cubic nonlinearities. They also monitored the system response undergoing three-to-one internal resonance experimentally. As their main finding, they concluded that the model can predict the measurement with high accuracy if the steady-state response of the MEMS is augmented by the contribution of the transient response.

A nonlinear energy harvester working based on internal resonance was proposed by Yang et al. [81]. In this research work, the authors examined the dynamics of an energy harvester whose resonance bandwidth becomes efficiently broadband due to the occurrence of three-to-one parametric modal interaction. They proposed an in-plane device composed of two cantilevers having magnets at their tips. This causes the two resonators become magnetically coupled. They employed a two degree-of-freedom lumped mass system to analytically model the dynamics of the energy harvester, and used the method of multiple-time scales to extract the solution. Their results indicated that the combination of internal resonance and magnet nonlinearity results in the formation of M-shape frequency-response behavior that substantially improves the efficacy of the energy harvester.

A high-frequency narrow-band filter made of a T-shape structure was proposed by R. Wang et al. [121]. Their device was composed of a fully-clamped beam and a cantilever with a tip mass attached to its end. Also, the cantilever was coated by a piezoelectric layer, and this is for harvesting electrical power as the structure undergoes large deformation. To improve the vibration amplitude, edge distribution in the bandpass frequency, and the suppression ability outside the passband frequency domain the structure was actuated near its auto-parametric modal interaction. This harnessed energy serves a function in supplying power for a MEMS switch that is integrated to the T-shape resonator.

One area where auto-parametric motion is specialized to is the dynamic analysis of gyroscopes. MEMS gyroscopes are designed to measure both angular and translational motions. Hamed et al. [122] investigated simultaneous resonances in a MEMS gyroscope operating in XY plane, measuring translational displacements. They modeled the MEMS using a two-degree-of-freedom system, considering one degree-of-freedom for each resonator's motion, oscillating

along one specific axis. They reported on one-to-one auto-parametric resonance happening between the drive mode and the sense mode. In their study, the benefited from the method of averaging to extract the frequency-response equation addressing auto-parametric modal coupling. Their main target was to suppress the large-amplitude vibration caused by modal interaction using an active control method. Their results indicated that saturation phenomenon occurs for larger values of the linear damping coefficient and specific contribution of nonlinearity.

Microelectromechanical systems equipped with comb-finger actuation mechanisms are also potent to experience auto-parametric resonance. Czaplewski et al. [91] conducted an experimental study to introduce a novel nonlinear response of a MEMS resonator, operating in a vigorous mode-coupling region. They addressed the occurrence of cyclic-fold on an invariant circle (CFIC) bifurcation in the MEMS dynamics through examining one-to-three internal resonance between the fundamental bending and torsional modes. They showed that the frequency at which cyclic-fold bifurcation occurs saturates while the device is driven with moderately large voltage amplitude and the drive frequency is swept near one-third the torsional natural frequency.

Conclusively, it can be mentioned that researchers working on internal resonance in MEMS dynamics have been mainly interested in characterizing the response of such systems, studying the probability of different types of bifurcations; stable and unstable dynamic responses; and multivaluedness that cause the systems' dynamics to become complex. As observed in the literature, there are a few works focusing on the application of internal resonance to improve the actuating or sensing mechanisms. Therefore, it is worth to conduct an investigation addressing a novel application of modal interaction, employing indirect excitation in conjunction with saturation phenomenon to improve the performance of MEMS actuators. Moreover, it should be noted that since the dynamic response generated by mode coupling is strongly complex, it can potentially provide us with the employment of various actuating and sensing mechanisms.

Chapter 3

Parametric Modal Interaction: Analytical Study

In this chapter, the dynamics of two nonlinearly couple resonators are examined analytically. To investigate mode coupling and saturation phenomenon, their frequency and force responses of the system were evaluated numerically. It is observed that when the system is excited near twice the higher mode natural frequency, autoparametric resonance occurs. As a result, the response amplitude of the lower mode experiences a sudden jump up to a nontrivial value when the force amplitude exceeds a threshold. It is also shown that further increasing the forcing amplitude, the higher frequency resonator oscillates with a periodic orbit whose amplitude remains constant while the lower frequency resonator vibrates with larger amplitudes in the phenomenon known as saturation.

3.1 Interactions in Two Degrees-of-Freedom

In this section, an initial investigation of two degrees-of-freedom structures previously studied in the literature is carried out to gain a deeper understanding of autoparametric internal resonance and its conditions.

3.1.1 L-shaped structure

Nayfeh and Ibrahim [64] studied an L-shaped structure composed of two thin cantilever beams in a vertical plane, figure 2.1.7 in their work. The clamped end of the horizontal beam is connected to a shaker, providing base excitation. A bulk mass M_1 is attached to the tip of the horizontal beam. The root of the vertical beam is located in M_1 and a shuttle mass M_2 is placed along that beam at a distance D from M_1 . The position of the shuttle mass on the vertical beam is adjustable, making it possible to tune the natural frequency of the second global (system) mode f_2 to be twice the first natural frequency of the first f_1 .

Nayfeh and Ibrahim [64] examined experimentally the nonlinear dynamic response of this structure in the vicinity of primary resonance of the second global mode. They derived the system equations of motion from a Lagrangian considering all possible quadratic nonlinearities as,

$$\ddot{v}_1 + \omega_1^2 v_1 + \epsilon[2\mu_1 \dot{v}_1 + \delta_1 v_1^2 + \delta_2 v_1 v_2 + \delta_3 v_2^2 + \delta_4 \dot{v}_1^2 + \delta_5 \dot{v}_1 \dot{v}_2 + \delta_6 \dot{v}_2^2 + \delta_7 v_1 \ddot{v}_1 + \delta_8 v_1 \ddot{v}_2 + \delta_9 v_2 \ddot{v}_1 + \delta_{10} v_2 \ddot{v}_2] = G \cos(\Omega t) \quad (3.1)$$

$$\ddot{v}_2 + \omega_2^2 v_2 + \epsilon[2\mu_2 \dot{v}_2 + \alpha_1 v_1^2 + \alpha_2 v_1 v_2 + \alpha_3 v_2^2 + \alpha_4 \dot{v}_1^2 + \alpha_5 \dot{v}_1 \dot{v}_2 + \alpha_6 \dot{v}_2^2 + \alpha_7 v_1 \ddot{v}_1 + \alpha_8 v_1 \ddot{v}_2 + \alpha_9 v_2 \ddot{v}_1 + \alpha_{10} v_2 \ddot{v}_2] = 0 \quad (3.2)$$

here, v_1 and v_2 are the generalized coordinates corresponding to the first transversal bending mode of the horizontal and vertical cantilevers, respectively. Because the two resonators are not coupled linearly, the natural frequency of the horizontal cantilever corresponds to the natural frequency of the first global mode ($\omega_1 = 2\pi f_1$) and the natural frequency of the vertical cantilever corresponds to the natural frequency of the second global ($\omega_2 = 2\pi f_2$). The terms marked by the coefficients δ_i and α_i ($i=1,2,3$) describe the contributions of geometric nonlinearities, and the terms marked by the coefficients δ_i and α_i ($i \geq 4$) describe the contributions of inertial nonlinearities. Base excitation appears as an external excitation in the horizontal beam equation of motion. The vertical cantilever beam is autoparametrically excited through the nonlinear coupling terms. We will call the case where interactions are triggered by driving the lower

frequency mode a 1:2 interaction.

3.1.2 Coupled column-pendulum resonator

Mustafa and Ertas [123] proposed a nonlinear energy sink (NES) made of a cantilever beam carrying a tip mass M and a pendulum with a point mass m , figure 1 in their paper. With the aim of reducing the vibrations of the beam, the pendulum was designed such that its natural frequency was one-half of the first natural frequency of the cantilever, $\omega_1 \approx \frac{1}{2}\omega_2$, meeting the condition for a 2:1 internal resonance.

The base of the beam moves vertically and excites its transverse bending mode parametrically. The pendulum is excited autoparametrically via nonlinear coupling terms between the beam and pendulum. A reduced-order model of the system motion can be found as follows [64],

$$\ddot{v}_1 + \omega_1^2 v_1 + \epsilon[2\mu_1 \dot{v}_1 + \delta_1 v_1^2 + \delta_2 v_1 v_2 + \delta_3 v_2^2 + \delta_4 \dot{v}_1^2 + \delta_5 \dot{v}_1 \dot{v}_2 + \delta_6 \dot{v}_2^2 + \delta_7 v_1 \ddot{v}_1 + \delta_8 v_1 \ddot{v}_2 + \delta_9 v_2 \ddot{v}_1 + \delta_{10} v_2 \ddot{v}_2] = 0 \quad (3.3)$$

$$\ddot{v}_2 + \omega_2^2 v_2 + \epsilon[2\mu_2 \dot{v}_2 + \alpha_1 v_1^2 + \alpha_2 v_1 v_2 + \alpha_3 v_2^2 + \alpha_4 \dot{v}_1^2 + \alpha_5 \dot{v}_1 \dot{v}_2 + \alpha_6 \dot{v}_2^2 + \alpha_7 v_1 \ddot{v}_1 + \alpha_8 v_1 \ddot{v}_2 + \alpha_9 v_2 \ddot{v}_1 + \alpha_{10} v_2 \ddot{v}_2 + G \cos(\Omega t) v_2] = 0 \quad (3.4)$$

where v_1 is the rotation angle describing the position of the pendulum with respect to the vertical and v_2 is a generalized coordinate describing the vibrations of the first transverse bending mode of the beam. In this case, base excitation appears as parametric excitation of the higher frequency resonator (beam). The pendulum is designed such that its natural frequency satisfies the condition for principle parametric resonance with the beam, namely,

$$\omega_2 \approx 2\omega_1 + \epsilon\sigma_1 \quad (3.5)$$

where σ_1 is a detuning parameter defining the nearness of the two natural frequencies.

This system of nonlinear ordinary differential equations, Eqs. (3.3) and (3.4), was solved analytically by Nayfeh and Ibrahim [64] using the method of multiple scales, for the case of an

excitation frequency varying near twice the higher (beam) natural frequency

$$\Omega \approx 2\omega_2 + \epsilon\sigma_2 \quad (3.6)$$

where σ_2 is a detuning parameter defining the nearness of the excitation frequency and the higher natural frequency. They obtained a first-order approximation of the solution in the form

$$\begin{aligned} v_1(t) &= p_1 \cos \frac{\Omega}{4}t + q_1 \sin \frac{\Omega}{4}t \\ v_2(t) &= p_2 \cos \frac{\Omega}{2}t + q_2 \sin \frac{\Omega}{2}t \end{aligned} \quad (3.7)$$

where p_1 and q_1 , and p_2 and q_2 are slowly varying coefficients. It is seen from Eq. (3.7) that the period of the low frequency resonator is twice that of the high frequency resonator and the period of the high frequency resonator is twice that of the excitation. The modulation equations, describing the slow evolution of the system, were defined as

$$\begin{aligned} p_1' &= -\mu_1 p_1 - \nu_1 q_1 - \Lambda_1(p_2 q_1 - p_1 q_2) \\ q_1' &= -\mu_1 q_1 + \nu_1 p_1 - \Lambda_1(p_1 p_2 + q_1 q_2) \\ p_2' &= -\mu_2 p_2 - \nu_2 q_2 + 2\Lambda_2 p_1 q_1 - g q_2 \\ q_2' &= -\mu_2 q_2 + \nu_2 p_2 - \Lambda_2(p_1^2 - q_1^2) - g p_2 \end{aligned} \quad (3.8)$$

where Λ_1 and Λ_2 are the nondimensional effective nonlinearities and μ_1 and μ_2 are the nondimensional viscous damping coefficients corresponding to the first and second resonators, respectively, and g is the nondimensional amplitude of the parametric force. The detuning parameters ν_1 and ν_2 are defined as

$$\nu_1 = \frac{1}{4}\sigma_2 + \frac{1}{2}\sigma_1, \quad \nu_2 = \frac{1}{2}\sigma_2 \quad (3.9)$$

Fixed points of the modulation equations, Eqs. (3.8), correspond to steady-state periodic motions of the system. To find the fixed points, we set the right hand-sides of Eqs. (3.8) equal to zero leading to a set of four coupled nonlinear algebraic equations. The stability and bifurcations of

the fixed points is determined by evaluating the Jacobian of that algebraic system at the fixed point and examining their eigenvalues.

3.2 Numerical Simulations

In this section, the steady-state response of the column-pendulum system is investigated by inspecting the equilibrium points of the modulation equations. As a baseline, we recreated the frequency-response curves reported by Nayfeh and Ibrahim [64] for the system parameter values of:

$$\Lambda_1 = 1, \quad \Lambda_2 = 0.5, \quad \mu_1 = 0.3, \quad \mu_2 = 0.2, \quad \sigma_1 = 0.1, \quad g = 0.7$$

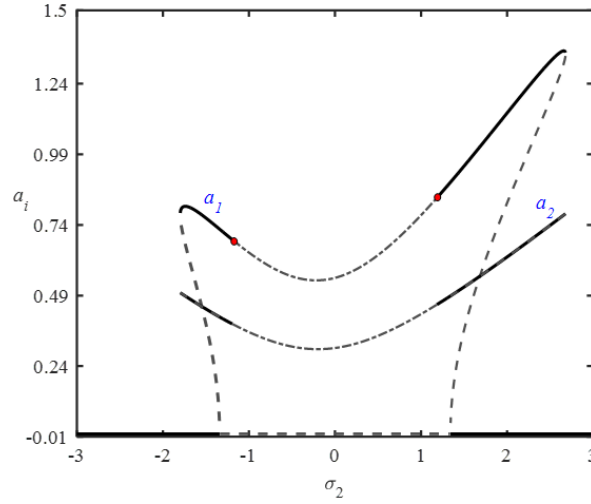


Figure 3.1: The frequency-response curves of the column-pendulum system under parametric excitation.

The simulation results shown in figure 3.1 are in close agreement with those report by Nayfeh and Ibrahim [64]. The amplitude of the orbits were obtained from the Cartesian representation of Eqs. (3.9) as

$$a_i = \sqrt{p_i^2 + q_i^2}$$

The trivial response of the system ($a_1 = a_2 = 0$) is stable everywhere except in a region around $\sigma_2 = 0$, marked in dash lines. In that region, the trivial solution is unstable and the response of both resonators jumps to resonant branches. Interestingly, the nondimensional motions of the secondary resonator (pendulum) are larger than those of the primary resonator (beam). We note that the frequency-response curves are not symmetric with respect to $\sigma_2 = 0$, with orbits on the right of the resonant branch being larger than those on the left, since the effective nonlinearities of the two resonators are unequal and a detuning exists between their natural frequencies ($\sigma_1 \neq 0$).

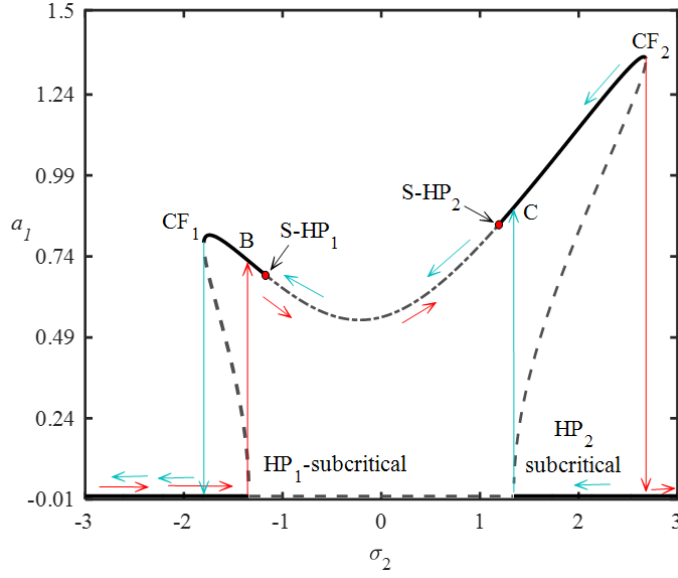


Figure 3.2: The frequency-response curve of the first resonator (pendulum).

We show in figure 3.2 a detailed study of the frequency-response curve of the lower frequency resonator. Faraway from the nontrivial region, the pendulum is stationary. In a frequency up-sweep (marked by red arrows in the figure), the system encounters a subcritical primary Hopf bifurcation (HP_1) at $\sigma_2 \approx -1.344$ where the trivial solution loses stability, and the response jumps up to points B along a branch of stable nontrivial solutions. Continuing the sweep, a pair of complex conjugate eigenvalues of the fixed point enters the right-hand side of the complex plane crossing the imaginary axis at $\sigma_2 \approx -1.194$ where a secondary Hopf bifurcation ($S-HP_1$) occurs. Beyond that point, the fixed points of the modulation equations are unstable and the

system response jumps to a branch of quasiperiodic orbits. The branch of unstable periodic orbits shown in figure 3.2 regains stability in a reverse secondary Hopf bifurcation (S-HP₂) at $\sigma_2 \approx 1.183$. Continuing the frequency up-sweep, beyond this point, the stable orbits along the resonant branch increase in size until they encounter a branch of unstable orbits emerging from the right-end of the nontrivial region at HP₂ in cyclic-fold bifurcation (CF₁) at $\sigma_2 \approx 2.678$. Beyond that point, trivial solutions prevail.

In a frequency down-sweep starting from that trivial solution, a reverse subcritical primary Hopf bifurcation (HP₂) occurs at $\sigma_2 \approx 1.344$ with the solution jumping up to point *C* along the resonant branch. Continuing with the down-sweep, the response encounters the secondary Hopf bifurcations S-HP₂ and S-HP₁ and the intervening quasiperiodicity then arrives at a branch of stable periodic orbits. This branch is terminated in a cyclic-fold bifurcation (CF₁) at $\sigma_2 \approx -1.799$ where it meets a branch of unstable solutions that emerges from the left-end of the nontrivial region at HP₁. Beyond that point, trivial solutions prevail.

The parameter values examined above assume relatively high damping uncharacteristic in electrostatic MEMS where our interest lies in fabrication of high Q-factor MEMS resonators to alleviate the need for large amplitude actuation voltages. Therefore, we will investigate the following system parameter values in the rest of this section:

$$\Lambda_1 = 1, \quad \Lambda_2 = 0.5, \quad \mu_1 = \mu_2 = 0.02, \quad \sigma_1 = 0.2, \quad g = 0.7 \quad (3.10)$$

Those values reduce the system damping by one order-of-magnitude and double the detuning between the natural frequencies of the resonators to account for microfabrication uncertainties.

Increasing the detuning between the natural frequencies σ_1 results in a larger nontrivial region, where the trivial solution is unstable as shown figure 3.3, extending between HP₁ at $\sigma_2 = -1.4$ and HP₂ at $\sigma_2 = 1.4$. It also results in a reduction in the size of the orbits along the unstable stretch of the resonant branch which approaches a minimum size at $\sigma_2 = -0.38$. The secondary Hopf bifurcations S-HP₁ and S-HP₂, demarcating the ends of this stretch, move out and away from $\sigma_2 = 0$ to locations almost equal to those of HP₁ and HP₂, thereby precluding the possibility of realizing periodic orbits directly from rest.

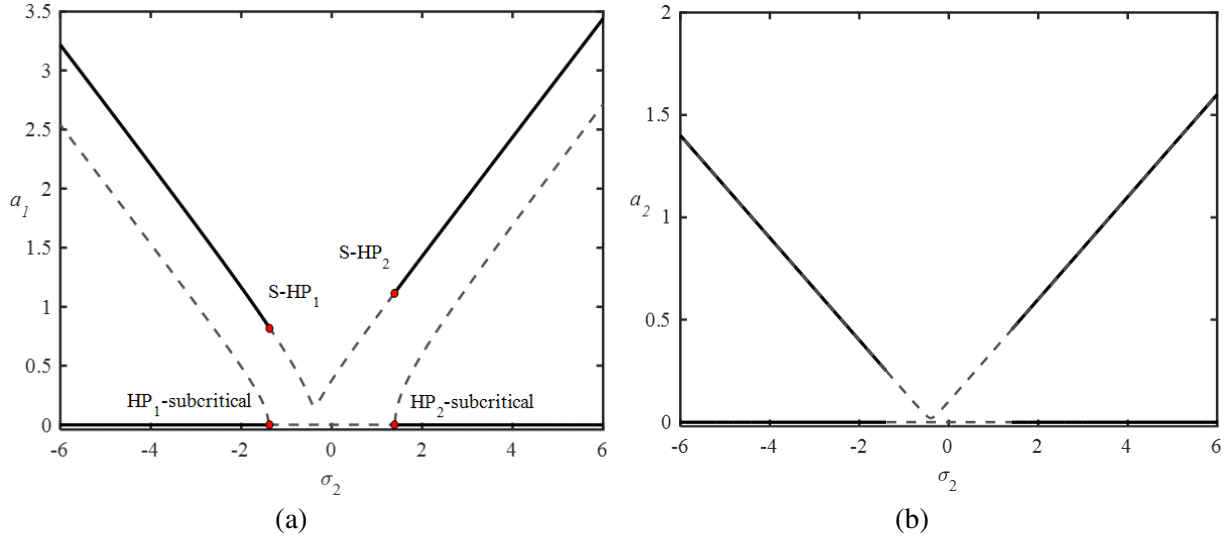


Figure 3.3: The frequency-response curves of (a) the first and (b) the second resonators under lower damping and larger modal detuning σ_1 .

Decreasing the damping level in the system, μ_1 and μ_2 , results in larger amplitude resonant branches that extend over longer frequency ranges as the cyclic-fold bifurcations CF_1 and CF_2 move outside our region of interest $-6 \leq \sigma_2 \leq 6$. As a result, multivaluedness prevails outside the nontrivial region with three possible steady-state responses: a stable large-amplitude periodic orbit, an unstable smaller-amplitude periodic orbit (marked in dashed lines), and the trivial stable solution. The response lands on either of the stable solutions (marked in solid lines) depending on the initial conditions. In practice, as we move along the large amplitude branches towards larger detuning values $|\sigma_2|$, the response will either jump down to the trivial branch, due to external disturbances, or unmodeled nonlinear damping mechanisms will become significant, thereby invalidating this model.

The evolution of those stable and unstable periodic orbits are depicted in in figure 3.4 as functions of the detuning parameter σ_2 . Stable orbits are shown in blue, unstable orbits are shown in black lines, and the trivial stable solution is shown as a dotted black line.

The force-displacement curve of the nonlinear system is presented in figure 3.5 for a case where the forcing frequency is detuned to the lower side ($\sigma_2 = -0.1$) of the high frequency res-

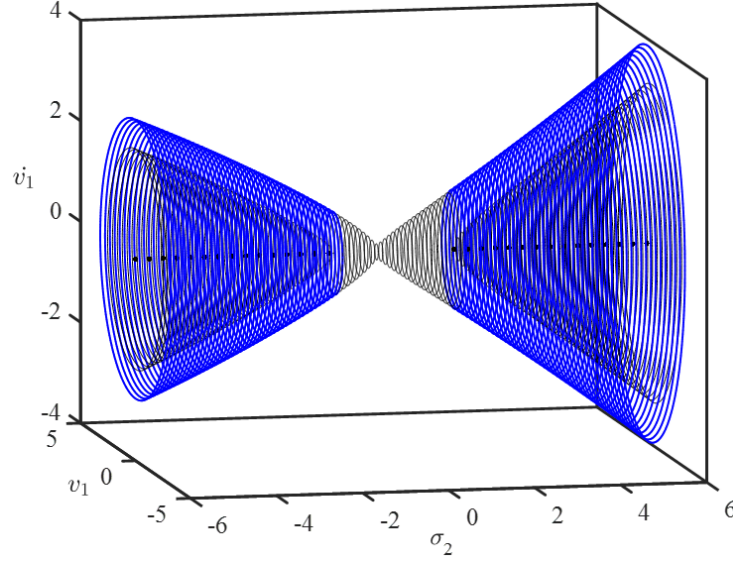


Figure 3.4: Stable orbits (blue lines), unstable orbits (black lines), and the trivial solution (black dotted line) as functions of the detuning parameter σ_2 .

onator's natural frequency. The trivial solution is stable until it encounters a subcritical primary Hopf bifurcation at $g = 0.053$. Beyond that point, the trivial solution loses stability and coexists with a branch of unstable periodic orbits (both shown in dashed lines). At the bifurcation point, the first resonator's amplitude grows continuously while the second resonator's amplitude jumps up discontinuously to a large-amplitude unstable branch. It is observed that at this forcing frequency, the system response is either trivial or unstable. The corresponding force-response curve for the case of $\sigma_2 = 0.1$ is shown in figure 3.6. Three types of bifurcations occur within a small range of the forcing amplitude. Initially, the trivial solution is stable for small forcing amplitudes. It loses stability through a subcritical primary Hopf (HP) bifurcation at $g = 0.053$ where the trivial solution becomes unstable. Beyond the bifurcation, the response jumps up to a small branch of stable (nontrivial) orbits (solid line). decreasing the forcing amplitude, this loses stability through a cyclic-fold (CF) bifurcation at $g = 0.028$ where it meets another branch of unstable orbits emerging from HP. Increasing the forcing amplitude along the stable branch, it loses stability through a secondary Hopf bifurcation (S-HP) at $g = 0.057$. Although in this

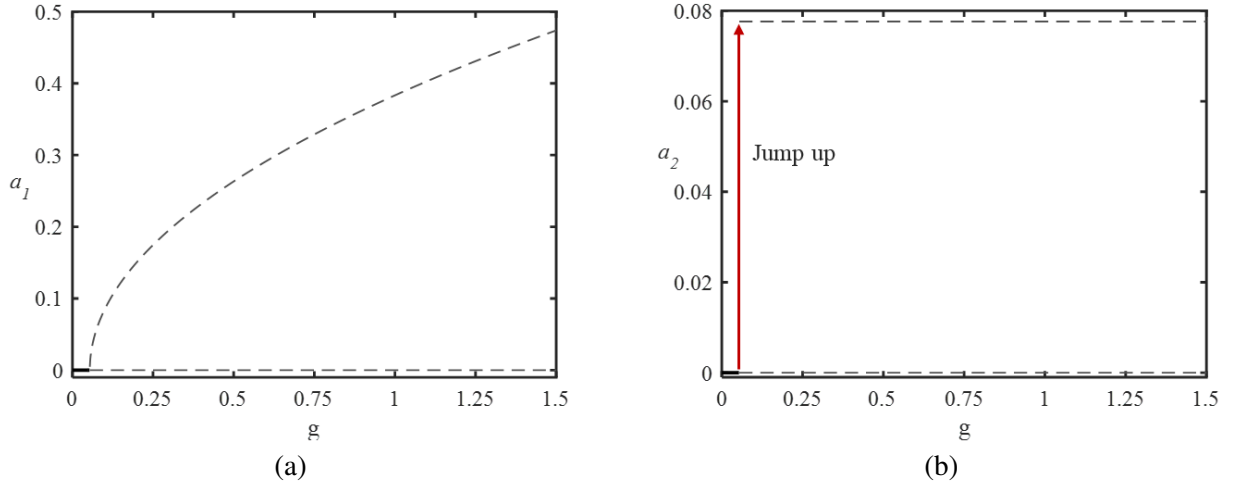


Figure 3.5: Amplitudes of the (a) first and (b) second resonators as functions of the excitation amplitude for the case of $\sigma_1 = 0.2$ and $\sigma_2 = -0.1$.

case, periodic orbits can be obtained directly from rest, this is only possible for a small range of the forcing amplitude $0.053 \leq g \leq 0.057$. Increasing the detuning between the two natural frequencies to $\sigma_1 = 1$ and the detuning between the excitation frequency and the higher of the two natural frequencies to $\sigma_2 = 2$, while keeping the other parameters as listed in Eq. (3.10), results in a qualitatively similar force response with the same number and type of solution branches separated by the same bifurcations as shown in figure 3.7. Larger detuning allows the trivial solution to retain stability for much larger excitation amplitudes with the subcritical primary Hopf bifurcation, where the trivial solution loses stability, occurring at $g = 1$. On the other hand, the resonant response (nontrivial solutions) appear over an extended range of excitation amplitudes. Because resonant response appears at a much larger forcing amplitude, it is also much larger than those realized in the previous case. However, the stable part of the resonant branch continues to co-exist with the stable trivial solution, as in the previous case, with the secondary Hopf bifurcation appearing at $g \approx 1$. As a result, the steady-state response of the system can not settle on the stable periodic orbits starting from rest. One way to overcome this drawback is to excite the system from rest with a forcing amplitude beyond the Hopf bifurcation ($g > 1$) then decrease the excitation amplitude gradually to a value lower than the secondary Hopf bifurcation ($g < 1$)

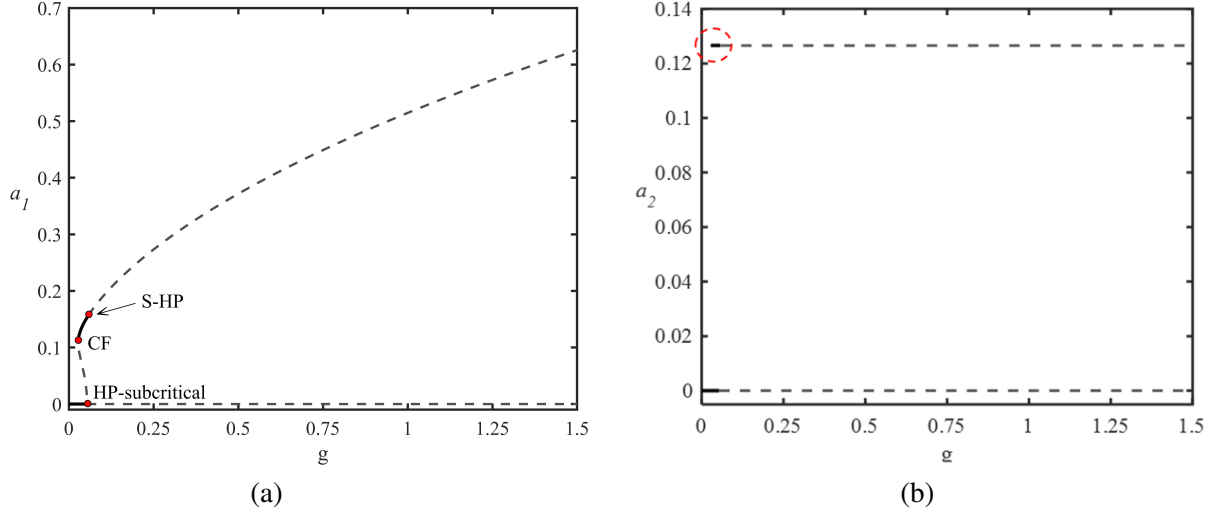


Figure 3.6: Amplitudes of the (a) first and (b) second resonators as functions of the excitation amplitude for the case of $\sigma_1 = 0.2$ and $\sigma_2 = 0.1$.

in order to realize larger stable periodic motions. We also note that the amplitude of the higher frequency resonator saturate while that of the lower frequency resonator continues to increase with forcing amplitude across the stable and unstable stretches of the resonator branch.

For the sake of completeness, we also examine a case of large detuning between the two natural frequencies $\sigma_1 = 1$ and lower forcing $g = 0.5$ while keeping the other parameters as defined in Eq. (3.10). The combination of low damping and lower forcing results in an intermediate sized frequency response, figure 3.8, larger than that seen in figure 3.2 and smaller than that seen in figure 3.3. The lower forcing amplitude results in a smaller nontrivial region, where the trivial solution is unstable, extending from a subcritical primary Hopf at $\sigma_2 = -1$ to a reverse subcritical primary Hopf at $\sigma_2 = 1$. A lower forcing amplitude g_1 combined with a larger detuning between the natural frequencies σ_1 result in the separation of the resonant branch into two resonant branches. The right (higher frequency) resonant branch continues to originate from primary Hopf bifurcation. Likewise, the coexisting smaller unstable branch continues to originate from the reverse primary Hopf bifurcation. Larger detuning (σ_1) also continues to locate the secondary Hopf along the resonant branch very close to the location of the primary Hopf

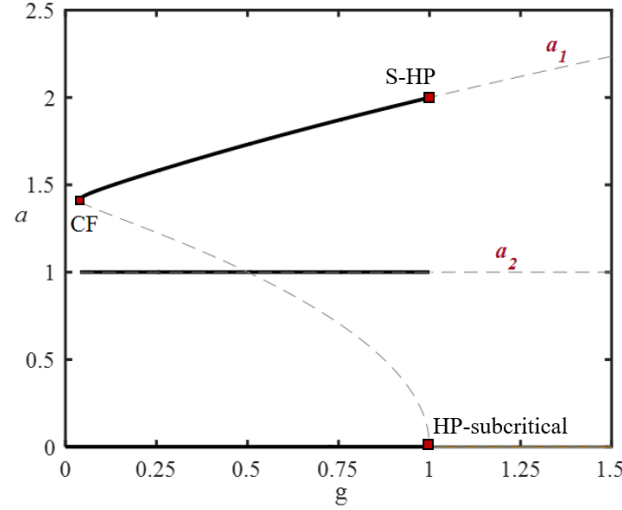


Figure 3.7: The force-response curves of the two resonators for the case of $\sigma_1 = 1$ and $\sigma_2 = 2$.

along the trivial branch with the net effect that the stable periodic orbits co-exist with unstable periodic orbits and a stable trivial solution. On the other hand, the left side of the resonant branch turns into a set of two isolated branches of solutions, a large branch of stable orbits and a smaller branch of unstable orbits, that meet at a cyclic-fold bifurcation and co-exist with the stable trivial solution.

3.3 Conclusions

Operation of scanning micromirrors requires the realization of steady-state (stable) periodic motions with a large angular stroke preferably at low actuation voltage. Our results above show that this would be hard to realize via internal resonance for cases where the detuning between the two natural frequencies σ_1 is large. While we found stable periodic orbits in those cases, they were typically not directly realizable from rest due to their co-existence with a stable trivial solution. While it is possible to reach those stable periodic orbits by initiating actuation from rest with a forcing amplitude and frequency in the nontrivial region then sweeping the frequency or force slowly to reach those inaccessible stable orbit, this calls for an elaborate actuation protocol

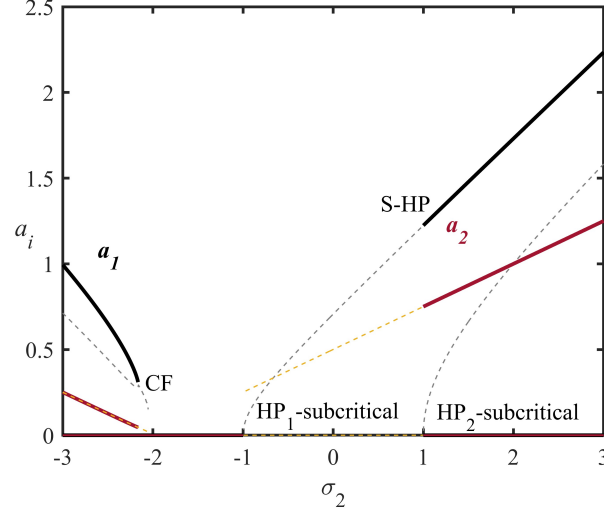


Figure 3.8: The frequency-response curves of the lower frequency a_1 (black lines) and higher frequency a_2 (red and yellow lines) resonators for the case of $\sigma_1 = 1$ and $g = 0.5$.

which is an undesirable system characteristic.

Although uncertainties in micro-fabrication make control of the detuning between the natural frequency hard, it is nonetheless a desirable system characteristic since it will allow for a larger nontrivial region where stable periodic orbits are directly accessible from rest. This can be achieved by deliberately creating a positive detuning between the two natural frequencies $\sigma_1 > 0$ large enough to withstand fabrication uncertainties then actively tuning the higher frequency after fabrication using, for example Joule heating to reduce σ_1 .

Our results here seem to suggest that higher Q-factor systems are desirable since they result in larger orbits for the same actuation level. This conclusion, however, needs further investigation since this chapter did not thoroughly study the dynamics along the unstable branch between the two secondary Hopf bifurcations. Lower damping is likely to result in complex dynamics in that region. But it shouldn't impact our conclusions about the stable periodic orbits.

Chapter 4

Modal Interaction in Curved Microbeams

In this chapter, we examine modal interaction in electrostatic curved microbeams. First, the natural frequencies and Q-factors of the microbeams' first few modes are found experimentally. It is shown that two-to-one ratio exists between the first anti-symmetric and the first symmetric modes and between the second symmetric and the first anti-symmetric modes. Since modal interaction between a symmetric mode and an anti-symmetric mode has not been studied in the literature, we focus on the second case.

4.1 MEMS Device

The MEMS device is composed of a clamped-clamped initially-curved beam fabricated in the PiezoMUMPs process out of a single crystal silicon-on-insulator (SOI) wafer. The crystal silicon of the structural layer is boron doped to increase its conductivity. The beam (flexible electrode) is actuated via a stationary electrode placed on its concave side. Figure 4.1(a) shows a schematic of the beam and its actuation scheme. The undeformed location of the neutral axis (NA) is dubbed $w_o(x)$, motions away from that position are dubbed $w(x; t)$, h_0 is the initial rise of the beam at mid-point, g_0 is the initial distance between the baseline (x-axis) and the stationary electrode.

The bias, amplitude, and frequency of the voltage across the actuation capacitor are dubbed V_{DC} , V_{AC} , and Ω , respectively.

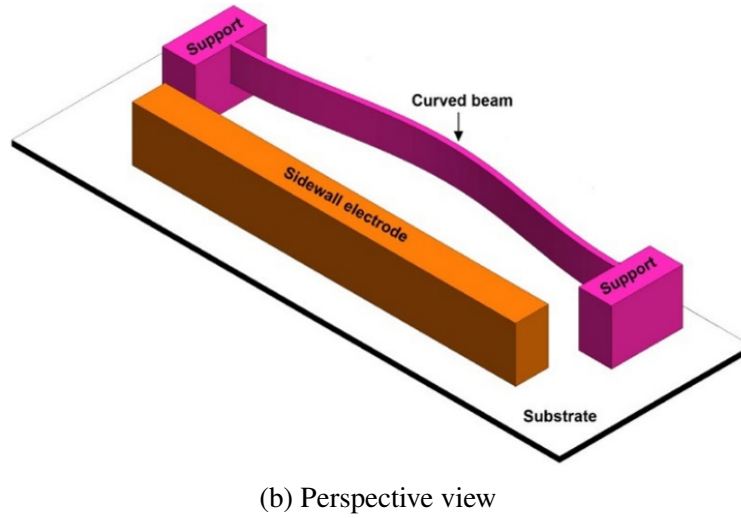
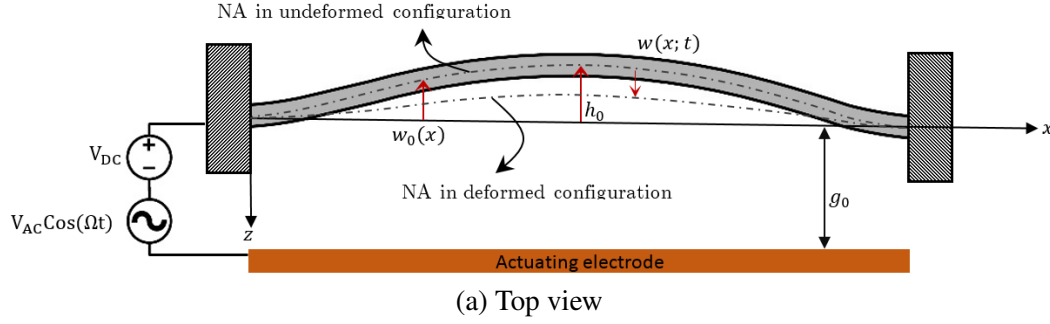


Figure 4.1: Schematics of (a) top and (b) perspective views of the clamped-clamped electrostatically actuated curved beam [124].

A perspective view of the MEMS device is illustrated in figure 4.1(b) [124]. Figures 4.2(a) and 4.2(b) depict 10X and 20X microscopic top views of the arch microbeam, respectively. The small dumbbell on the counter electrode, figure 4.2(b), marks the beam's mid-point. A profilometer image of the curved microbeam is shown in figure 4.2(c) under 10X optical magnification providing a color map that describes the height across the device as measured by the profilometer.

The dimensions of the device as measured by the profilometer are listed in Table 4.1.

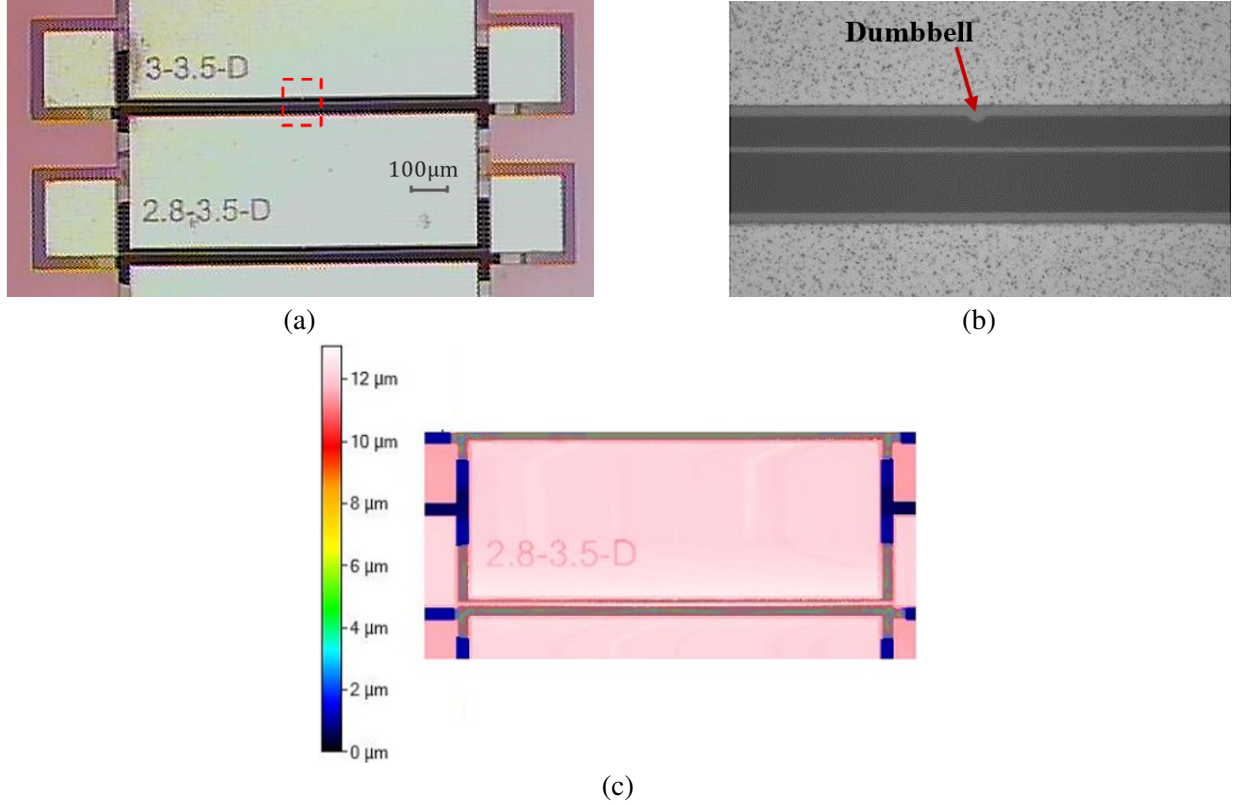


Figure 4.2: (a) Optical microscope image (10X) of two curved beams and their side electrodes, (b) zoomed image (20X) of the area marked by red dashed line in (a) showing the dumbbell, and (c) profilometer image of the curved-beam and its actuation electrode.

4.2 Device Characterization

In this section, the natural frequencies and corresponding modal damping (Q-factor) of the first few transverse (in-plane and out-of-plane) bending ration modes of the microbeam are characterized. In macro structures, impact hammers are used in experimental modal analysis to apply an impulse excitation at a point along a degree-of-freedom. Accelerometers and strain gauges are

Table 4.1: Geometrical properties of the curved-beam

Parameter	Value [μm]
Beam length	1000
Beam thickness	3.5
Beam width	10
Initial rise at mid-point	3.55
Initial gap distance	10.51

attached at various points to measure the response. Where needed, a mass cancellation process is carried out to account for the added mass of the sensors and evaluate the unshifted natural frequencies of the structure.

However, for micro/nano-scale structures, the system motion is measured either optically, using a Laser Doppler Vibrometer (LDV), or electrically, using Lock-in-Amplifier directly or based on the method of harmonic detection of resonance (HDR) [125]. In this study, the curved microbeam is characterized using optical measurement. The out-of-plane velocity of a point on the microstructure is measured using a laser beam focused as that point. This output signal represent the unloaded structural motions.

A discrete conceptualization of the continuous microbeam transverse vibrations is shown in figure 4.3. The beam is modeled as a set of n concentrated rigid masses m_i . Each pair of masses is connected through a transverse linear springs. Viscous damping occurs due to the interaction between the masses and surrounding air as well as supports. Point forces act on F_i on each mass representing the projection of the external force onto each mass. The laser beam is focused on one mass m_d along the transverse direction to measure its velocity. Under this condition, the three following cases concerning the nature of the external excitation are considered and the frequency content of the system's response is established and corresponding to each case.

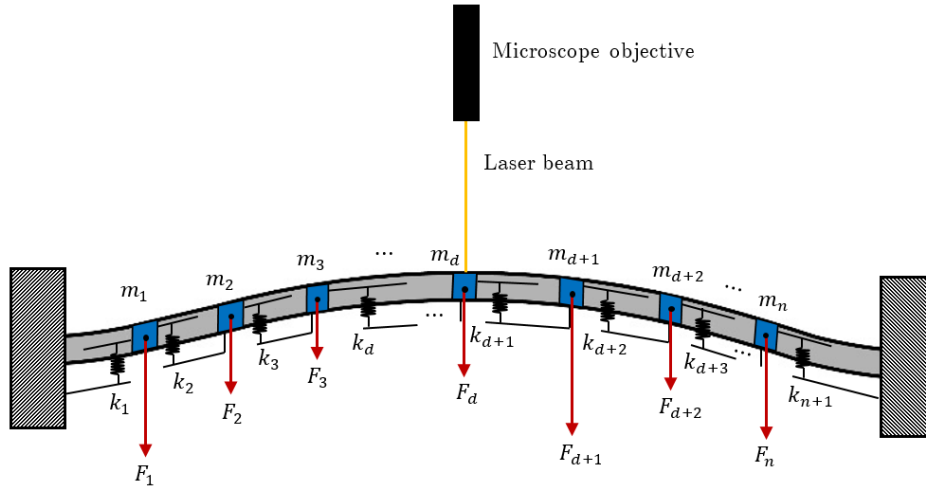


Figure 4.3: Multi degrees-of-freedom model of an out-of-plane clamped-clamped curved-beam structure.

4.2.1 Thermal noise excitation

We investigate the frequency content of the system's response under thermal noise excitation. No potential difference is applied between the movable and stationary electrodes. Different classes of noise processes exist in ambient environment such as, external disturbances, electromagnetic radiation noise, and thermomechanical noise. External disturbances and electromagnetic radiation arising from ambient vibrations and power lines, respectively, have a low-frequency content, extending from sub-hertz to hundreds of hertz. Typical MEMS have natural frequencies in the kilo hertz range and are, thus, dynamically isolated from those oscillations. Since the MEMS, in this experimental paradigm, are unactuated, they are also not exposed to electrical noise in the actuation signal. On the other hand, thermomechanical noise is uniformly distributed across the frequency spectrum. It is also proportional to ambient temperature. At room temperature (25°C), its power is enough to excite the lower frequency resonances of high-Q MEMS, in the kilo to mega hertz range, where stiffness is low.

Most MEMS have low Q-factors in the air due to the nonlinear squeeze-film damping effect, As a result, their resonant response under thermomechanical noise is hard to measure with a low

signal-to-noise ration (SNR). Placing MEMS under vacuum increases their Q-factors and, therefore, the SNR of resonant response to thermomechanical noise. The motions of the discretized microbeam under thermomechanical noise forces F^{ns} can be described by the following set of equations of motion:

$$\begin{aligned}
m_1 \ddot{x}_1 + c_1 \dot{x}_1 + L_1(x_1, x_2, \dots, x_d, \dots, x_n) &= f_1^{ns}(t) \\
\vdots & \\
m_d \ddot{x}_d + c_d \dot{x}_d + L_d(x_1, x_2, \dots, x_d, \dots, x_n) &= f_d^{ns}(t) \\
\vdots & \\
m_n \ddot{x}_n + c_n \dot{x}_n + L_n(x_1, x_2, \dots, x_d, \dots, x_n) &= f_n^{ns}(t)
\end{aligned} \tag{4.1}$$

where x_i ($i=1,2,\dots,n$) is the displacement of the i^{th} mass, L_i is a linear operator describing the linear stiffness of the mass, c_i s is the linear viscous damping coefficient of the mass, and $f^{ns}(t)$ is the projection of the thermomechanical noise force applied to the mass.

In experiment, the system response is measured over a time horizon T . Thermomechanical noise forces can be written in terms of the frequency grid points

$$\omega_k = \frac{2(k-1)\pi}{T}, \quad k = 1, 2, \dots, N$$

of the resulting FFT as follows:

$$f_n^{ns}(t) = \sum_{k=1}^N F_{nk}^{ns} e^{i\omega_k t} + cc \tag{4.2}$$

Substituting Eq. (4.2) into the system equations of motion, Eqs. (4.1), gives:

$$\begin{aligned}
m_1 \ddot{x}_1 + c_1 \dot{x}_1 + L_1(x_1, \dots, x_d, \dots, x_n) &= F_{11}^{ns} e^{i\omega_1 t} + \dots + F_{1N}^{ns} e^{i\omega_N t} + cc \\
\vdots \\
m_d \ddot{x}_d + c_d \dot{x}_d + L_d(x_1, \dots, x_d, \dots, x_n) &= F_{d1}^{ns} e^{i\omega_1 t} + \dots + F_{dN}^{ns} e^{i\omega_N t} + cc \\
\vdots \\
m_n \ddot{x}_n + c_n \dot{x}_n + L_n(x_1, \dots, x_d, \dots, x_n) &= F_{n1}^{ns} e^{i\omega_1 t} + \dots + F_{nN}^{ns} e^{i\omega_N t} + cc
\end{aligned} \tag{4.3}$$

Because the system (4.3) is linear, its solution can be obtained using superposition. Hence, the steady-state response at the measurement point under thermal noise excitation can be written in the following form:

$$x_d(t) = \sum_{k=1}^N X_{dk} e^{i\omega_k t} + cc \tag{4.4}$$

where X_{dk} is the magnitude of the FFT at ω_k . Due to linear coupling L_i , the measured response at a given frequency, for example X_{d1} , is a function of the magnitude of the noise force at that frequency over all points F_{i1}^{ns} . It can, therefore, be concluded that the frequency content of the MEMS response to thermal noise excitation is non-zero at all frequency grid points.

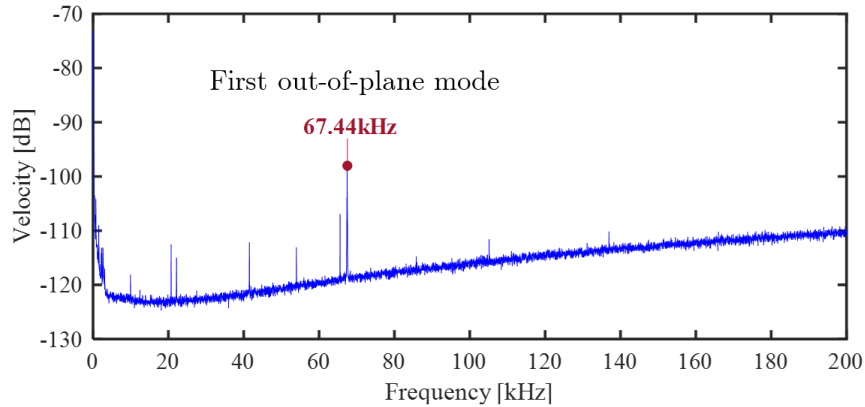


Figure 4.4: The FFT of the microbeam's mid-point velocity under thermal noise excitation.

The FFT of the velocity at the beam mid-point ($x_d = 0.5$) under thermomechanical noise

excitation is shown in figure 4.4. The chip is placed flat so that the top surface of the beam is directly exposed to the laser spot allow the vibrometer to measure out-of-plane motions. The test was performed in a vacuum chamber at a pressure of 5.4 mTorr to enhance the Q factor of the system. A peak is observed at $f_1^o = 67.44 \text{ kHz}$ corresponding to the natural frequency of the first symmetric out-of-plane bending mode. Because the mid-point is a node of the first anti-symmetric mode, the FFT contains no information about it.

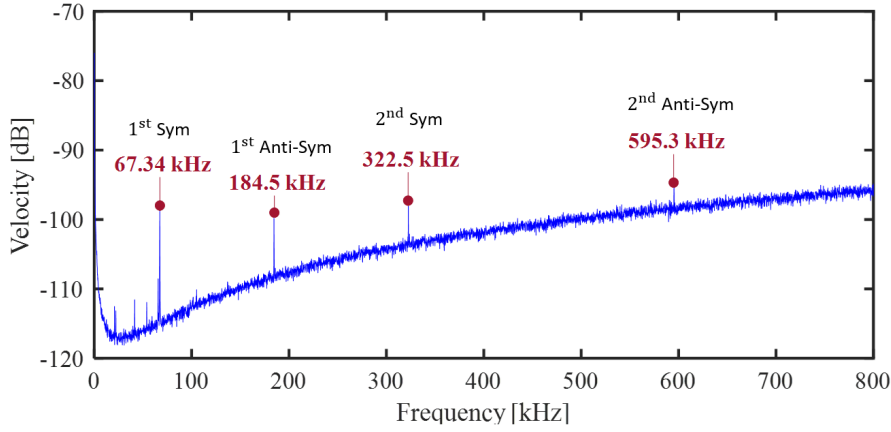


Figure 4.5: The FFT of the microbeam's quarter-span velocity under thermal noise excitation.

With that in mind, the laser spot was focused at the quarter-span point to capture the natural frequencies of the first few out-of-plane bending modes, figure 4.5. In addition to the first symmetric mode, we found the natural frequencies of the first anti-symmetric mode at $f = 184.5 \text{ kHz}$, the second symmetric mode at $f = 322.5 \text{ kHz}$, and the second anti-symmetric mode at $f = 595.3 \text{ kHz}$.

4.2.2 Pulse excitation

The linear response of the curved beam was also investigated under pulse excitation to better characterize the in-plane response actuated by the side electrode. A function generator was used to apply a voltage pulse $V(t)$ to the side electrode while the beam was grounded. To examine the resulting force, the expression for electrostatic force per unit length is expanded around the rest

position ($w(x) = 0$) using a Taylor series expansion, as follows:

$$\begin{aligned} F_{elec} &= \frac{\epsilon b V(t)^2}{2(g_0 - w)^2} \\ &= \alpha V^2 \left(1 + \frac{2}{g_0} w + \frac{3}{g_0^2} w^2 + \frac{4}{g_0^3} w^3 + \frac{5}{g_0^4} w^4 + \dots \right) \end{aligned} \quad (4.5)$$

where the coefficient $\alpha = \frac{\epsilon b}{2g_0^2}$ represents the electromechanical coupling. Assuming small motions ($w \ll 1$), we drop the nonlinear terms and linearize the electrostatic force as follows:

$$\begin{aligned} F_{elec} &= \alpha V^2 \left(1 + \frac{2}{g_0} w \right) \\ &= \alpha V^2 + \frac{2\alpha V^2}{g_0} w \\ &= F_\delta(t) + P_\delta(t) \end{aligned} \quad (4.6)$$

where $F_\delta(t)$ and $P_\delta(t)$ are external and parametric impulse forces, respectively. The parametric excitation term contributes to the tuning of the linear stiffness. Since the duty cycle, ratio of pulse width to period, is small, we neglect its contribution to linear stiffness and reduce the set of ODEs describing the discretized system dynamics to the form

$$\begin{aligned} m_1 \ddot{x}_1 + c_1 \dot{x}_1 + L_1(x_1, \dots, x_d, \dots, x_n) &= F_\delta(t) \\ \vdots \\ m_d \ddot{x}_d + c_d \dot{x}_d + L_d(x_1, \dots, x_d, \dots, x_n) &= F_\delta(t) \\ \vdots \\ m_n \ddot{x}_n + c_n \dot{x}_n + L_n(x_1, \dots, x_d, \dots, x_n) &= F_\delta(t) \end{aligned} \quad (4.7)$$

The external pulse excitation can be expanded in terms of its frequency content as

$$F_\delta(t) = \sum_{k=1}^N f_k^p e^{i\omega_k t} + cc \quad (4.8)$$

Inserting Eq. (4.8) into Eq. (4.7), one can extract the frequency spectrum of the target degree-of-freedom $x_d(t)$ by solving the linear system of ODEs simultaneously

$$\begin{aligned}
m_1 \ddot{x}_1 + c_1 \dot{x}_1 + L_1(x_1, \dots, x_d, \dots, x_n) &= f_1 e^{i\omega_1 t} + \dots + f_N e^{i\omega_N t} + cc \\
\vdots \\
m_d \ddot{x}_d + c_d \dot{x}_d + L_d(x_1, \dots, x_d, \dots, x_n) &= f_1 e^{i\omega_1 t} + \dots + f_N e^{i\omega_N t} + cc \\
\vdots \\
m_n \ddot{x}_n + c_n \dot{x}_n + L_n(x_1, \dots, x_d, \dots, x_n) &= f_1 e^{i\omega_1 t} + \dots + f_N e^{i\omega_N t} + cc
\end{aligned} \tag{4.9}$$

expressed in the form of Eq. (4.4). While pulse excitation and thermal noise excitation are wideband, the forcing level varies with frequency in the former case and is uniform in the latter case.

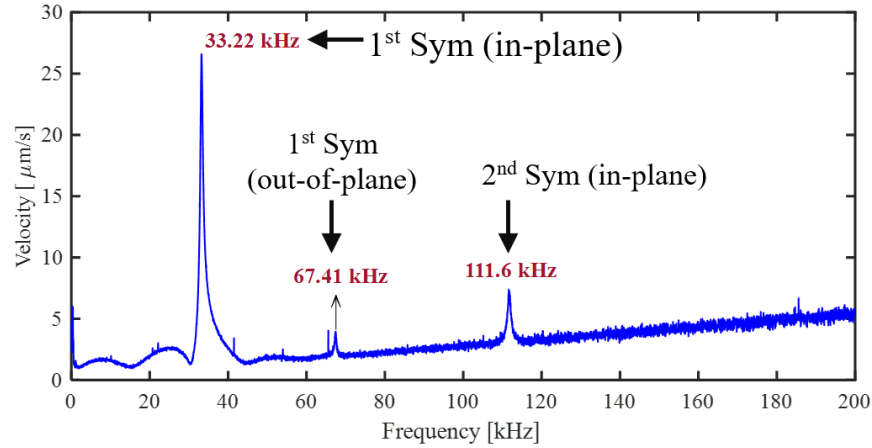


Figure 4.6: The FFT plot of the mid-point velocity under a pulse excitation with a frequency of $f = 15$ Hz, a duty cycle of 0.1%, and an amplitude of 15V.

The FFT of the curved-beam mid-point velocity under pulse excitation is shown in figure 4.6. In this case, the laser beam was focused on the side wall of the beam to measure in-plane motions [xx]. The pulse frequency was set to $f = 15$ Hz, the duty cycle to 0.1%, and its amplitude to 15V. We observed the natural frequencies of the first three modes, namely the first in-plane symmetric mode at $f_1^i = 33.22$ kHz, the first out-of-plane symmetric mode at $f_1^o = 67.41$ kHz, and the third

in-plane symmetric mode at $f = 111.6$ kHz. We note that those results are in close agreement with the results obtained from thermal noise excitation for the first out-of-plane symmetric mode.

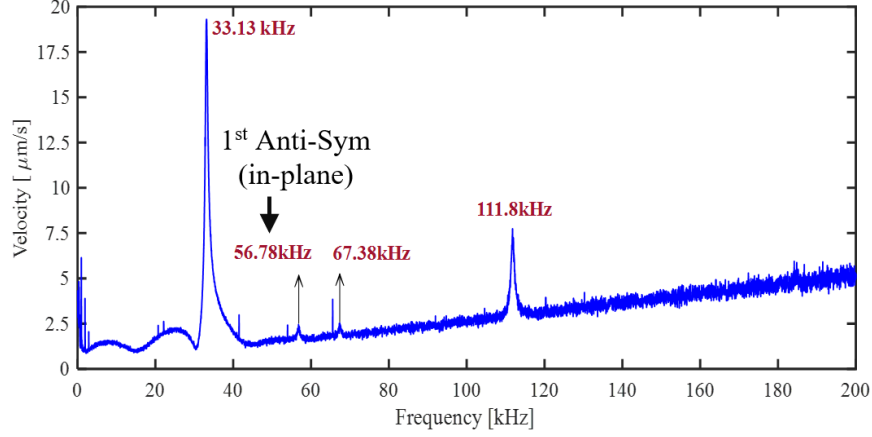


Figure 4.7: The FFT plot of the quarter-span point velocity under a pulse excitation with a frequency of $f = 15$ Hz, a duty cycle of 0.1%, and an amplitude of 15V.

The FFT of the quarter-span point velocity is shown in figure 4.7. In addition to the three modes observed at mid-point, a peak appears at $f = 56.78$ kHz corresponding to the natural frequency of the first anti-symmetric in-plane mode. We note that the magnitude of this peak is much smaller than those of the symmetric modes since the projection of the pulse force delivered by the symmetric electrode on the anti-symmetric mode is very small.

The frequency resolution of the FFT resulting from pulse excitation is a function of the pulse period T , which determines the maximum observation time. While a lower pulse frequency, provides a longer observation time and better frequency resolution, a higher pulse frequency provides more power at the cost of inferior frequency resolution. This makes higher frequency pulse trains more effective at observing higher frequency and, therefore, stiffer modes.

We used a pulse train with a higher frequency (1 kHz), the same duty cycle of 0.1%, and a higher amplitude of (37.5 V) to investigate higher frequency in-plane modes. The time-history of the mid-point velocity under this excitation is shown in figure 4.8. The frequency response of the system, figure 4.9, shows the natural frequency of the third symmetric in-plane mode at

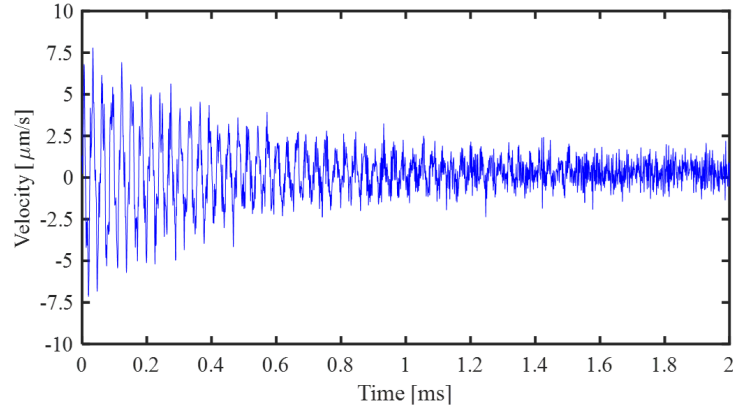


Figure 4.8: The time history of the mid-point velocity under a voltage pulse train with a frequency of 1 kHz, a duty cycle of 0.1%, and an amplitude of 37.5 V.

$f = 262.4 \text{ kHz}$. The first and second in-plane symmetric modes were also observed. We note that the accuracy of their natural frequencies reported here is lower than that reported earlier.

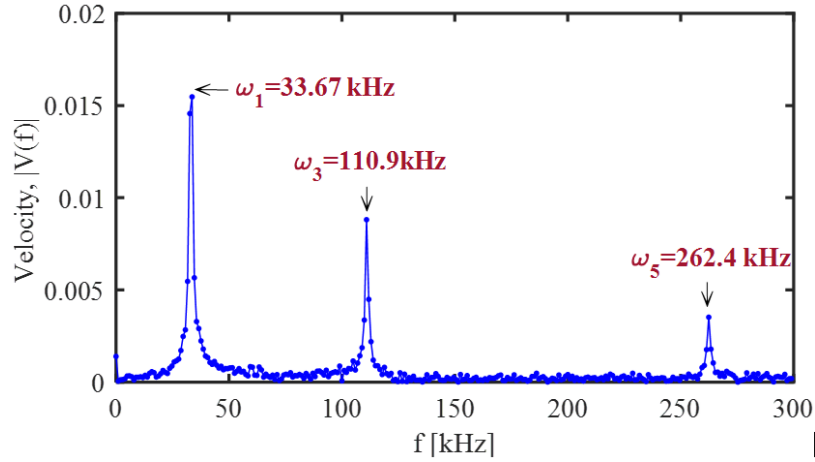


Figure 4.9: FFT plot of the MEMS response at the beam's middle-point under the pulse excitation of 1kHz frequency and %0.1 duty cycle, and the voltage amplitude of 37.5V.

4.2.3 Frequency sweeps

Thermal noise and pulse excitations are appropriate methods for modal analysis over a wideband frequency spectrum. Where we are more interested in detailed analysis of the frequency response within a narrow interval, frequency sweeps can also be used. This section introduces the use of frequency sweeps in linear modal analysis of electrostatic MEMS.

The electrostatic force resulting from a biased voltage waveform

$$V(t) = V_{\text{DC}} + V_{\text{AC}} \cos(\Omega t)$$

is proportional to the voltage squared:

$$\begin{aligned} F_{\text{elec}} &\propto \alpha V(t)^2 \\ &= \alpha \left(V_{\text{DC}}^2 + \frac{V_{\text{AC}}^2}{2} \right) + 2\alpha V_{\text{DC}} V_{\text{AC}} \cos(\Omega t) + \frac{1}{2} \alpha V_{\text{AC}}^2 \cos(2\Omega t) \\ &= F_s + F_1 e^{i\Omega t} + F_2 e^{i2\Omega t} + cc \end{aligned} \quad (4.10)$$

It contains a static component:

$$F_s = \alpha \left(V_{\text{DC}}^2 + \frac{V_{\text{AC}}^2}{2} \right)$$

a component at the first harmonic of the voltage signal,

$$F_1 = 2\alpha V_{\text{DC}} V_{\text{AC}}$$

and a component at the second harmonic of the signal

$$F_2 = \frac{1}{2} \alpha V_{\text{AC}}^2$$

In our implementation of the frequency sweep technique, the bias and amplitude of the signal were held fixed while the signal frequency was increased (swept-up) at a constant slew rate $\dot{\Omega}$.

Using Eq. (4.10) in Eq. (4.5), the displacement-dependent electrostatic force can be ex-

pressed in terms of external excitation, stiffness modification, and parametric excitation as follows:

$$\begin{aligned}
F_{elec} &= (F_s + F_1 e^{i\Omega t} + F_2 e^{i2\Omega t} + cc) \left(1 + \frac{2}{g_0} w + \frac{3}{g_0^2} w^2 + \frac{4}{g_0^3} w^3 + \dots \right) \\
&= \underbrace{F_s + F_1 e^{i\Omega t} + F_2 e^{i2\Omega t}}_{\text{External excitation}} + \underbrace{F_s \left(\frac{2}{g_0} w + \frac{3}{g_0^2} w^2 + \frac{4}{g_0^3} w^3 + \dots \right)}_{\text{Stiffness modification}} \\
&\quad + \underbrace{F_1 e^{i\Omega t} \left(\frac{2}{g_0} w + \frac{3}{g_0^2} w^2 + \frac{4}{g_0^3} w^3 + \dots \right)}_{\text{First harmonic parametric excitation}} + \underbrace{F_2 e^{i2\Omega t} \left(\frac{2}{g_0} w + \frac{3}{g_0^2} w^2 + \frac{4}{g_0^3} w^3 + \dots \right)}_{\text{Second harmonic parametric excitation}} \quad (4.11)
\end{aligned}$$

We note that for parametric excitation to result in a resonant response, its frequency should be near twice the natural frequency of a mode or an integer m sub-multiple of that $\frac{2\omega_n}{m}$ and its amplitude must be above the activation level for that parametric resonance. In modal analysis, we limit our interest to small enough excitation to avoid undue influence of nonlinearities including the activation of parametric resonances. Therefore, the set of equations of motion describing the discrete dynamic system under electrostatic excitation can be written in the following form:

$$\begin{aligned}
m_1 \ddot{x}_1 + c_1 \dot{x}_1 + L_1(x_1, \dots, x_d, \dots, x_n) &= F_s + F_1 e^{i\Omega t} + F_2 e^{i2\Omega t} \\
\vdots \\
m_d \ddot{x}_d + c_d \dot{x}_d + L_d(x_1, \dots, x_d, \dots, x_n) &= F_s + F_1 e^{i\Omega t} + F_2 e^{i2\Omega t} \quad (4.12) \\
\vdots \\
m_n \ddot{x}_n + c_n \dot{x}_n + L_n(x_1, \dots, x_d, \dots, x_n) &= F_s + F_1 e^{i\Omega t} + F_2 e^{i2\Omega t}
\end{aligned}$$

where the linear modification of stiffness ($F_s \frac{2}{g_0} w$) have been absorbed into the set of linear operators L_i . Based on linear vibration theory, the displacement of the target point is obtained as:

$$x_d(t) = X_d^s + X_{d1} e^{i\Omega t} + X_{d2} e^{i2\Omega t} + cc \quad (4.13)$$

The displacement contains a static component X_d^s that represents the response to F_s , a first har-

monic component X_{d1} that represents response to in reply F_1 , and a second harmonic component X_{d2} that represents the response to F_2 . The velocity, on the other hand, is composed only of two harmonic components:

$$\dot{x}_d(t) = \Omega \dot{X}_{d1} e^{i\Omega t} + 2\Omega \dot{X}_{d2} e^{i2\Omega t} + cc \quad (4.14)$$

In interpreting the response to electrostatic forcing, the multi-frequency nature of the excitation illustrated above should be kept in mind.

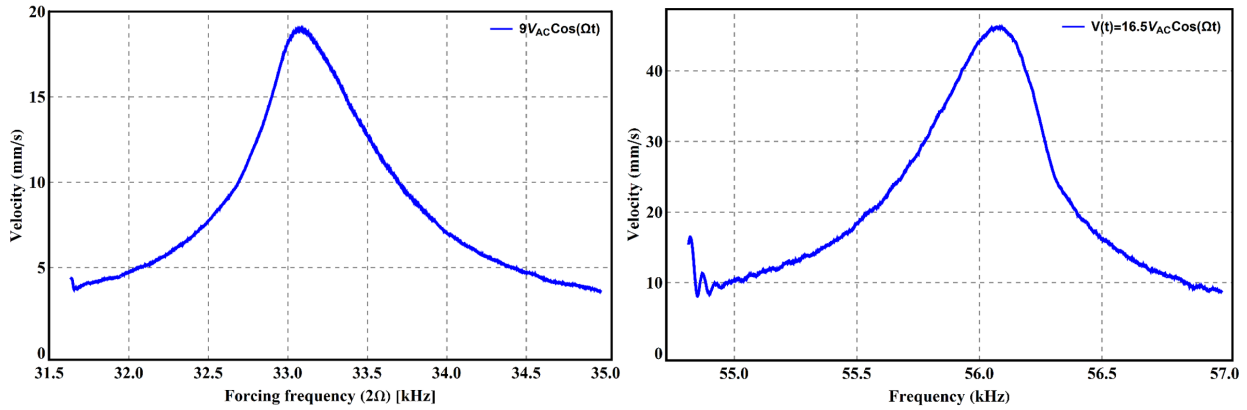


Figure 4.10: The frequency-response curves of the mid-point velocity RMS in the vicinity of the (a) first in-plane symmetric mode and (b) the second in-plane symmetric mode.

The frequency-response curves of the MEMS mid-point velocity in the vicinity of the first and second in-plane symmetric modes are shown in figures 4.10(a) and (b), respectively. The microbeam was excited in air with a harmonic waveform. As a result, the forcing term in both cases was exclusively F_2 . The amplitude and slew rate were set to $V_{AC} = 9$ V and $\dot{\Omega} = 20$ kHz/s for the first symmetric mode and to $V_{AC} = 16.5$ V and $\dot{\Omega} = 22$ kHz/s for the second symmetric mode. The curves represent the root mean-square (RMS) of the velocity over a moving time-window extending over an integer number of the forcing periods [126]. The first and second symmetric in-plane bending modes appear to slightly softening and hardening, respectively. Inspecting the half-power bandwidth of the frequency-response curves, the Q-factor of the first and second symmetric modes are obtained as 53.67 and 130.543, respectively.

4.3 Modal Interaction between Symmetric and Anti-Symmetric Modes

We compared the experimental results reported above with a finite element model (FEM) of the curved-beam's built in COMSOL software. Table 4.2 compares the measured and simulated natural frequencies. It can be seen that two-to-one ratio exist between the second symmetric and first anti-symmetric modes, and therefore, the MEMS has the potential for internal resonance and energy exchange between those modes.

Table 4.2: The natural frequencies of the curved beam structure [kHz]

		1 st Symm.	1 st Anti-Sy.	2 nd Symm.	2 nd Anti-Sy.	3 rd Symm.
Experiment	In-plane	33.13	56.78	111.8	-	265
	Out-of-plane	67.38	184.5	322.5	595.3	-
FEM	In-plane	33.782	55.06	112.09	178.47	267.22
	Out-of-plane	67.889	187.1	366.38	605.07	-

On the other hand, the symmetry of microbeam's actuation and boundary conditions make it hard to excite the anti-symmetric modes directly as was illustrated in subsection 4.2.2. In this study, we target the potential interaction identified above as a possible route to excite the first anti-symmetric mode.

In the experiment, the laser spot was focused on a node of second symmetric mode to eliminate its contributions in the observed motions and to capture the motions of the first anti-symmetric mode only. The microbeam was actuated in air by a harmonic voltage waveform resulting in electrostatic forcing composed only of the F_2 component. The forcing frequency was swept-up in the vicinity of the second symmetric mode resonance

$$2f = \frac{\Omega}{\pi} = \frac{\omega_3 + \sigma}{2\pi}$$

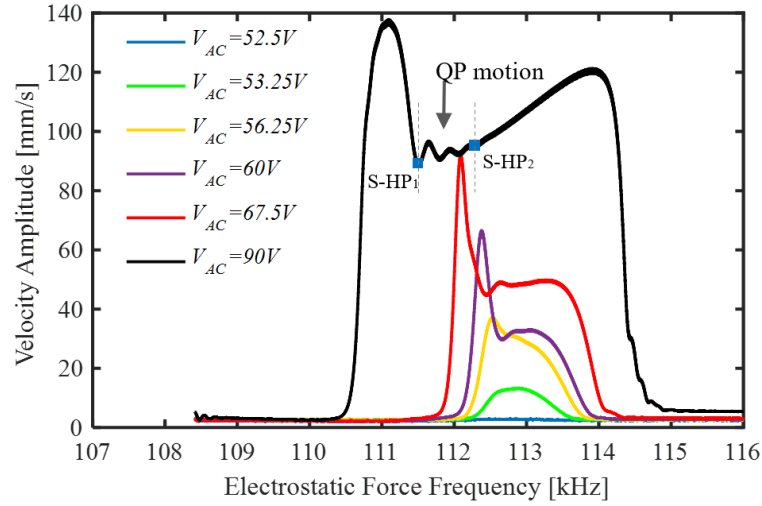


Figure 4.11: The frequency-response curves of the velocity RMS at a node of the second symmetric mode in the vicinity of its primary resonance for six values of the excitation amplitude V_{AC} and no bias ($V_{DC} = 0$).

For low voltage amplitudes V_{AC} , no motions were observed. As the forcing level increased, we crossed the activation level of the interaction and nontrivial oscillations were observed. The activation level was found to lie in the range $52.5 \text{ V} < V_{AC} < 53.55 \text{ V}$. The frequency-response curves corresponding to the lower and upper bounds of this range are shown as blue and green lines in figure 4.11. At the upper end of the range, nontrivial motions were observed over a narrow frequency range where only a trivial response was available at the lower end.

Further increase in the forcing level, resulted in wider nontrivial regions and larger, more complex, nontrivial oscillations. At higher actuation levels ($V_{AC} \geq 60 \text{ V}$), the frequency-response curves take the M-shape characteristic of modal interactions, figure 4.11. At the highest excitation level of $V_{AC} = 90 \text{ V}$, the trivial response loses stability via a primary Hopf bifurcation at $2f = 110.4 \text{ kHz}$. Beyond this bifurcation, nontrivial stable response emerges and grows swiftly up to a maximum within a narrow frequency range. Increasing the excitation frequency, the velocity drops gradually to a minimum where it encounters a secondary Hopf bifurcation, marked S-HP₁, at $2f = 111.5 \text{ kHz}$. At this point, the nontrivial response enters a quasi-periodic regime that continues until a reverse secondary Hopf bifurcation, marked S-HP₂, occurs at

$2f = 112.3 \text{ kHz}$. As the nontrivial response returns to one-period periodicity, its size grows with the excitation frequency towards a second maxima, corresponding to the right side of the M-shaped curve, before dropping to the trivial solution.

We compare in figure 4.12 the frequency spectrum of the velocity at a node of the second symmetric mode for two excitation frequencies along the frequency-response curve of $V_{AC} = 90 \text{ V}$, shown above, one outside and another inside the nontrivial region. Outside the nontrivial region, 4.12(a), a larger peak appears at the forcing frequency $2f$ due to the finite size of the laser spot capturing residual motions of the second symmetric mode around its node and a smaller peak appears at half of that frequency f due to residual bias voltage that result in a small F_1 component in the electrostatic force. Both peaks, however, are two orders-of-magnitude smaller than the peak seen inside the nontrivial region at f , figure 4.12(b), as parametric resonance channels energy from the higher forcing frequency $2f$ to the resonant frequency of the lower first anti-symmetric mode.

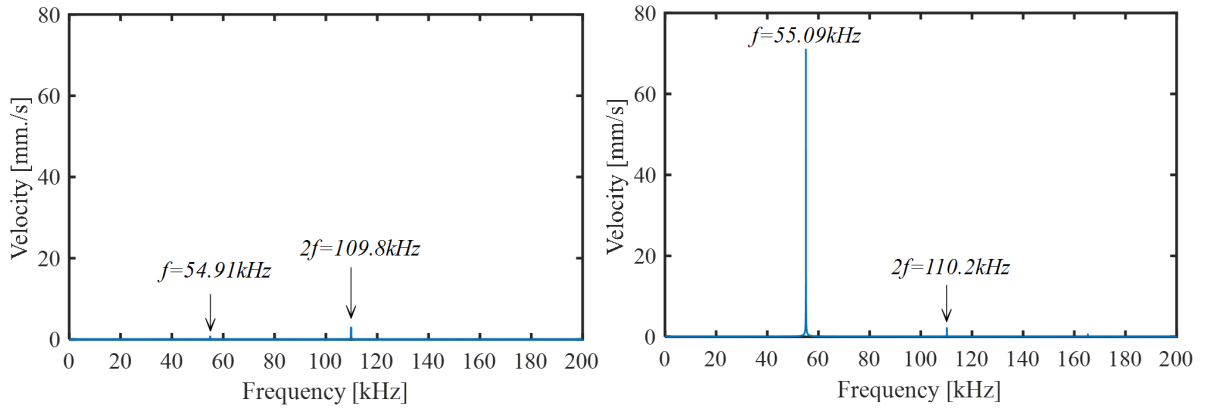


Figure 4.12: The FFT of the velocity at a node of the second symmetric mode for (a) an excitation frequency outside ($2f = 110 \text{ kHz}$), and (b) another inside ($2f = 111 \text{ kHz}$) the nontrivial region and a voltage amplitude of $V_{AC} = 90 \text{ V}$

The time-histories of the velocity at the node of the second symmetric mode, corresponding to the frequency-domain results described above, are shown in figure 4.13. The contrast between the time-envelop of the velocity outside the nontrivial region (red lines) and inside it (blue lines) illustrate a jump in the response size once the excitation frequency falls within the nontrivial

region and channel energy into the first anti-symmetric mode.

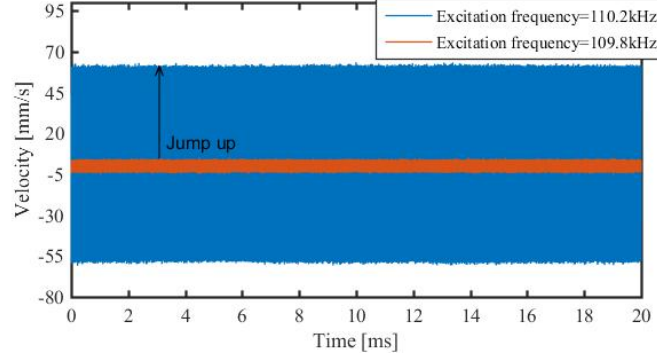


Figure 4.13: The time-history of the velocity at a node of the second symmetric mode for (a) an excitation frequency outside ($2f = 110$ kHz), and (b) another inside ($2f = 111$ kHz) the nontrivial region and a voltage amplitude of $V_{AC} = 90$ V

To investigate the amplitude and phase relationships of the anti-symmetric mode, we used a lock-in amplifier to drive the microbeam and measure the optical head's output voltage. Another microbeam specimen was utilized for this experiment where the activation level of the two-to-one modal interaction was found to lie in the range $15.75 \text{ V} < V_{AC} < 16.5 \text{ V}$. The frequency-response curves of the amplitude and phase of the velocity component at half the excitation frequency f , corresponding to first anti-symmetric mode, measured for a node of the second symmetric mode are shown in figure 4.14. Increasing the voltage amplitude widens the nontrivial region within the frequency spectrum and increases the size of the nontrivial responses. The frequency-amplitude curves take the characteristic M-shape as the forcing level increases ($V_{AC} > 21 \text{ V}$).

While the frequency-amplitude curves realized at resonance f are similar to the frequency-response curves of the total velocity, figure 4.11, for lower forcing levels ($V_{AC} \leq 21 \text{ V}$), there is a marked difference between the response curves at larger forcing levels. Namely, variations in the velocity component at the resonant frequency f are minimal as the excitation frequency varies while large variations are seen in the total velocity indicating that the larger values realized close to the maxima of the M-shaped curves are due to higher harmonics.

The phase-frequency curves for lower forcing levels ($V_{AC} \leq 21 \text{ V}$), encounter a single sharp

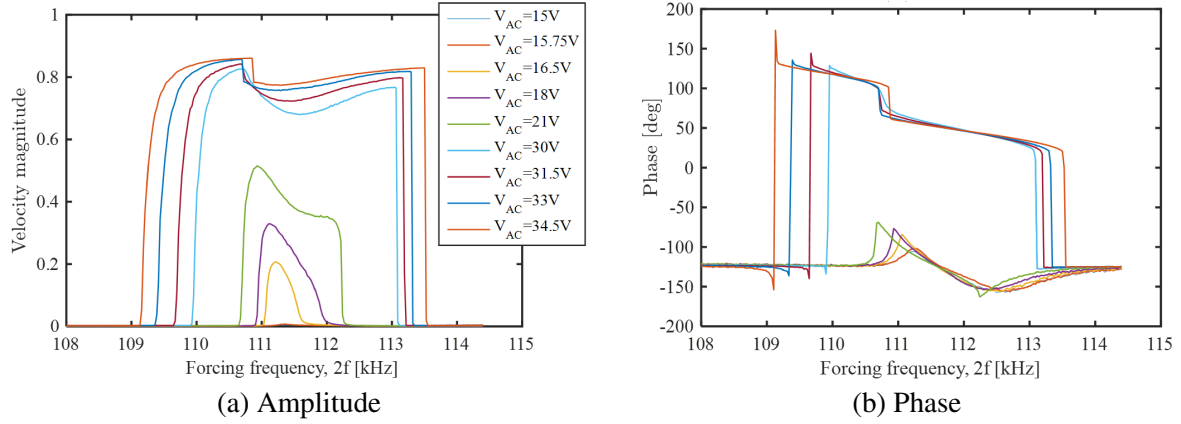


Figure 4.14: The frequency-response curves of the (a) amplitude and (b) phase of the velocity component at half the excitation frequency f for a node of the second symmetric mode as the excitation frequency $2f$ is varied in the vicinity of its primary resonance for nine values of the signal amplitude V_{AC} and no bias ($V_{DC} = 0$).

peak during forward sweeps of the excitation frequency at the primary Hopf bifurcation. On the other hand, the phase-frequency curves for higher forcing levels ($V_{AC} > 21$ V), encounter two jumps during forward sweeps, the first at the secondary Hopf bifurcation and the second upon the jump down from the nontrivial response to the trivial response.

The same experimental setup was also used to investigate the amplitude and phase relationships of the directly excited second symmetric mode. In this case, the lock-in amplifier was used to measure the output voltage of the optical head while the laser spot was focused on the microbeam mid-point. It was set to lock onto half the excitation frequency f , the excitation frequency $2f$, and twice the excitation frequency $4f$, thereby measuring only symmetric mode's amplitude and phase at those frequencies. The amplitudes are reported in figures 4.15(a), (c) and (e), respectively, in terms of the output voltage RMS. Comparing among the amplitudes of the modal response at f , $2f$, and $4f$, it can be seen that the component at the forcing frequency is dominant. In contrast, the dominant component of the of the anti-symmetric modes appears at half the forcing frequency f illustrating the difference between the directly and indirectly excited modes in a two-to-one internal resonance. Further, we note that the symmetric mode acquires a component at f only while the internal resonance is active. This component sees a sudden

increase in amplitude at the secondary Hopf bifurcation, $f = 110.44$ kHz and a gradual drop in phase beyond this point as it appears to absorb energy channeled back from the anti-symmetric mode.

Finally, it should be noted that this type of modal interaction is classified in the group of one-plane internal resonances, where the two engaged modes belong to the same DOF. Our experiments proved that the energy channel between a symmetric mode and an anti-symmetric mode can be activating where a commensurate ratio exists between their natural frequencies. This indirect actuation method may prove an efficient way to actuate anti-symmetric modes in symmetric MEMS.

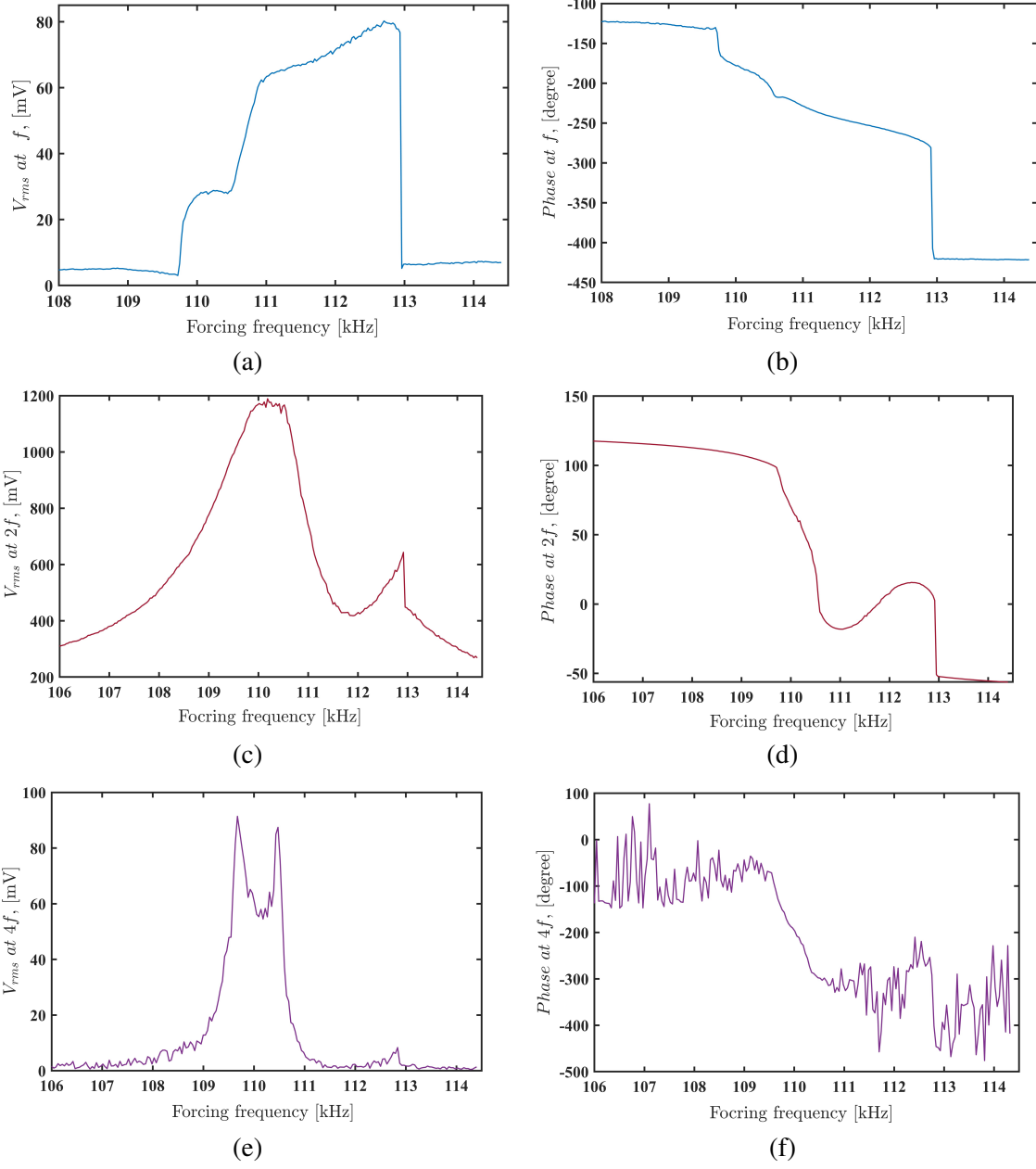


Figure 4.15: The frequency-response curves of the (a), (c), and (e) amplitude and (b), (d), and (f) phase of the mid-point velocity components at f (a and b), $2f$ (c and d), and $4f$ (e and f) as the excitation frequency $2f$ is varied in the vicinity of primary resonance of the second symmetric mode for a signal amplitude of $V_{AC} = 30$ V and no bias ($V_{DC} = 0$).

Chapter 5

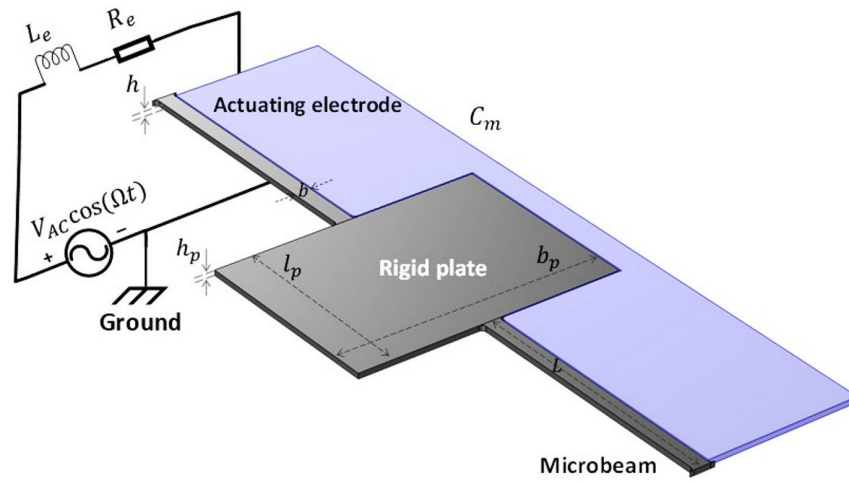
Electrostatic MEMS Scanning Mirror: Mathematical Modeling

We establish the bending-torsion equations of motion for an electrostatic micromirror with a view to investigating modal interactions between its in-plane bending and out-of-plane rotational motions. The mirror is composed of two suspension microbeams supporting a rectangular plate. Among them, they form a clamped-clamped compound structure, figure 5.1. In the following, we exploit Hamilton's principle to derive the equations of motion (EoM) governing that interaction.

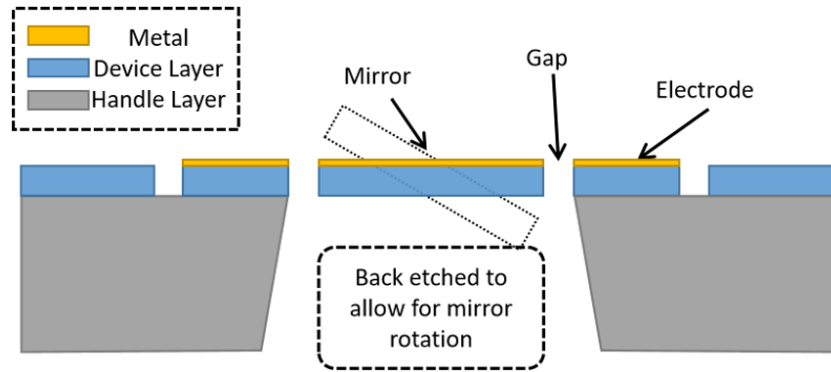
5.1 Imperfections in MEMS

Micro-scale fabrication inevitably introduces uncertainties into the geometric and material properties of microstructures. Depending on the fabrication process and the parameter in question, these uncertainties can reach up to 5–10% of the as-designed value. Those imperfections can manifest in various ways:

- 1) deviations in feature dimensions and shapes due to limitations in lithography and etching techniques [127].



(a)



(b)

Figure 5.1: (a) Schematic of the electrostatic MEMS mirror and its actuation circuit. (b) Structure of parallel plate actuated mirror with back etching to allow for a large range of mirror rotation.

- 2) conformal deposition resulting in flat surfaces and straight lines in the layout appearing as stepped-up or -down structural layers.
- 3) surface roughness [128], which can affect layer thinness, specially for thin layers.
- 4) misalignment of masks in photolithography, potentially distorting and misaligning device features, layers, or components [129, 130].
- 5) residual stresses encountered during fabrication, which affects structural stiffness and stability [131].
- 6) material impurities, inclusions, or flaws, which can compromise the structural integrity and electrical characteristics of the device [132].
- 7) thin film defects, such as porosity, pinholes, cracks, or non-uniformity, which can impact mechanical, electrical, or optical properties [133].
- 8) under-etching and over-etching which changes the released lengths and may undermine the rigidity of the device anchors [134].
- 9) inadequate adhesion between different layers, risking delamination or detachment [135].

The resulting disparities between the as-design dimensions and as-fabricated dimensions can introduce a measure of asymmetry into symmetrically designed devices as well be discussed further below.

We study microbeams with ideally (as-designed) rectangular cross-sections. Following the convention in the literature [97, 136–139], we position a Lagrangian coordinate system $OXYZ$ at the centroid of its left-end with the x -axis extending along the neutral axis (NA), figure 5.2. For homogeneous microbeams, this axis passes through the geometric centers of the cross-sections. Due to micro-scale fabrication imperfections, the as-fabricated cross-sections will be perturbed away from the as-designed cross-sections, for example as shown by the blue and red cross-sections in figure 5.2. As a result, deviations will exist between the positions of the ideal C_d and as-fabricated C_f centroids. The green curve, perturbed non-deterministically around the x -axis (black line), depicts the locus of positions of C_f . In this study, we will investigate the impact of this misalignment on the vibrations of microbeams.

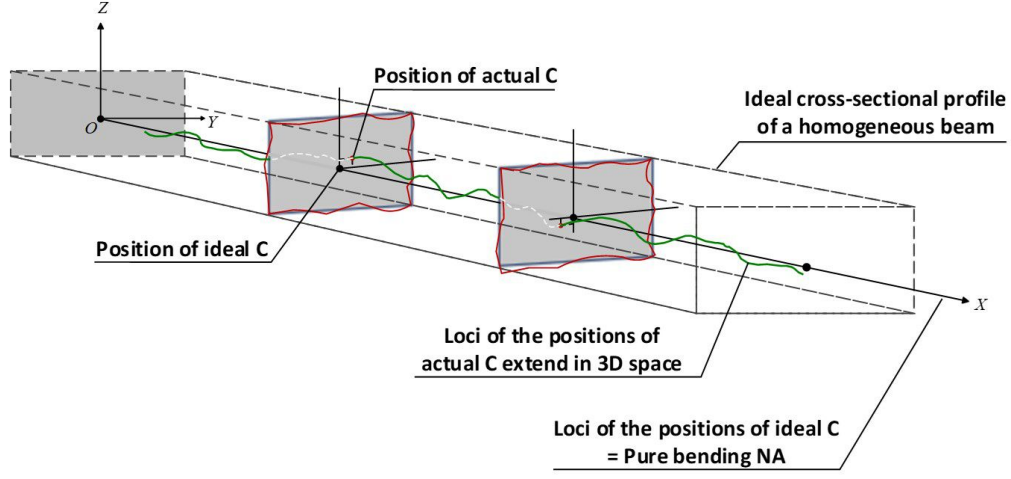


Figure 5.2: Schematic of the microbeam. The as-designed beam profile is shown in dashed black lines. The green curve represents the locus of the as-fabricated centroids.

5.2 Cross-sectional imperfection

Microbeams may experience initial upward or downward curvature depending on the fabrication process. Numerous studies [103, 140–142] have investigated the impact of initial curvature, an imperfection, on their dynamics. However, the influence of cross-sectional asymmetry on the dynamics of microbeams has rarely been considered. One objective of this study is to investigate the impact of this asymmetry on the dynamics of microbeams.

It is a standard practice in deriving the microbeam equations of motion to ignore section imperfections in calculating the cross-sectional first and second moments of area. This assumption, however, may eliminate inherent coupling between the bending and torsional motions of the microbeams.

The cross-section of a microbeam at a position x along its axis is shown in figure 5.3. The figure shows the deviations of the as-fabricated section edges away from those of the as-designed section and their respective centroids C_d and $C_f(x)$. The coordinates of $C_f(x)$, are denoted (y_o, z_o) . We assume that the perturbations of $C_f(x)$ away from C_d are stochastic along the beam

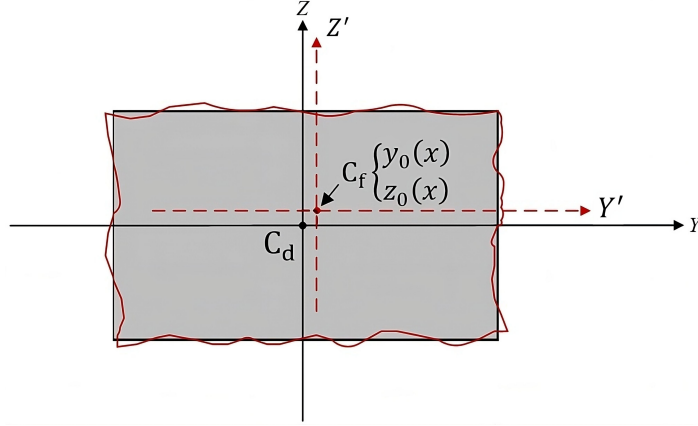


Figure 5.3: Variations between the as-designed, gray rectangle, and as-fabricated, red outline curve, beam cross-sections.

axis and calculate the first moments of area of the as-fabricated cross-section with as:

$$\begin{aligned} Q_{yy}(x) &= \iint_{A_f} z dA = A_f(x) z_o(x) \neq 0 \\ Q_{zz}(x) &= \iint_{A_f} y dA = A_f(x) y_o(x) \neq 0 \end{aligned} \quad (5.1)$$

where A_f is the as-fabricated section area. As per the micro-fabrication process handbook [143], there is a 5% tolerance in design feature sizes, which encompasses imperfections resulting in asymmetry of symmetrically designed sections. Hence, the terms $y_o(x)$ and $z_o(x)$ in Eq. (5.1) are assumed to vary be 2-3% of the microbeam's thickness and width, respectively, while perturbations of the section area are assumed negligible $A_f = A_d$. In the following, we formulate the equations of motion of the proposed electrostatic MEMS mirror.

5.3 Mathematical model

5.3.1 Mirror design

The electrostatic MEMS mirror, figure 5.4, is composed of a rigid plate supported by two microbeams. The beams are identical with a length of L and a thickness of h . They are clamped at one end and connected to the plate at the other end. The plate's length and thickness are denoted by l_p and h_p . The beams and plate are fabricated in the same structural layer with a common width b . The mirror is made from single crystal silicon $\langle 100 \rangle$ with Young's modulus of $E = 128 \text{ GPa}$, Poisson's ratio of $\nu = 0.22$, a modulus of rigidity of $G = 52 \text{ GPa}$, and density of $\rho = 2230 \text{ kg/m}^3$. It is equipped with a single planar U-shaped actuation electrode, spanning its entire length of the beams and plate. The initial capacitive gap is denoted g_0 . The as-designed dimensions of the mirror are listed in 5.1.

Table 5.1: The geometry of the electrostatic MEMS.

Parameters	Value (μm)
Microbeam length, L	510
Microbeam width, h	31
Microbeam thickness, b	10
Plate length, l_p	350
Plate width, h_p	505
Plate thickness, b	10
Initial gap, g_0	2.5

The mirror is excited in-plane by applying a voltage waveform with an amplitude V_{AC} and a frequency Ω

$$V(t) = V_{AC} \cos(\Omega t)$$

between the side electrode and the movable mirror structure. The MEMS baseline capacitance C_o at rest is made of the parallel-plate capacitor between one of the microbeams C_b and the side

electrode and half of the rigid plate and the side electrode C_p and expressed as:

$$\begin{aligned} C_o &= 2C_b + 2C_p \\ &= \frac{2\epsilon Lb}{(g_o - w)} + \frac{\epsilon l_p b}{(g_o - w)} \end{aligned} \quad (5.2)$$

where ϵ is the permittivity of the medium.

The origin of the global coordinate system, $OXYZ$ is positioned at the centroid of the left end's cross-section, figure 5.4(a), with the X -axis passing through the as-designed cross-sectional centroids. Its unit vectors are denoted $\mathbf{i}_a$, $\mathbf{i}_b$, and $\mathbf{i}_c$. The $ABCD$ plane depicts the microbeam's cross-section at $x = s$ from the origin O . The undeformed beam element extends out of this plane in the positive direction of the X -axis with a length dx . The coordinate system, $oxyz$, with the unit vectors $\mathbf{i}_x$, $\mathbf{i}_y$, and $\mathbf{i}_z$ and origin o along the X -axis in the undeformed configuration, is locally attached to the $ABCD$ plane. The two coordinate systems are originally aligned, therefore

$$\mathbf{i}_a = \mathbf{i}_x, \quad \mathbf{i}_b = \mathbf{i}_y, \quad \mathbf{i}_c = \mathbf{i}_z$$

The translational displacement field of point o is defined as

$$\mathbf{D}(x; t) = \begin{bmatrix} u(x; t) \\ v(x; t) \\ w(x; t) \end{bmatrix}$$

where u , v , and w are the x -, y -, and z -displacement components translating a point P along the NA to P^* and a point Q , at a distance dx along the NA, to Q^* . The Euler angles describing the rigid-body rotations of the microbeam's section under the displacement field are denoted $\psi(x; t)$, $\theta(x; t)$, and $\phi(x; t)$. These rotations occur around the z -axis, followed by the y' -axis, and finally, the x'' -axis, figure 5.4(b). The local coordinate system of the cross-section in the deformed configuration is denoted $\zeta\eta\xi$ with unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$. It is worth pointing out that for homogeneous beams with cross-sections having two axes of symmetry, the bending NA

passes through torsional center [97].

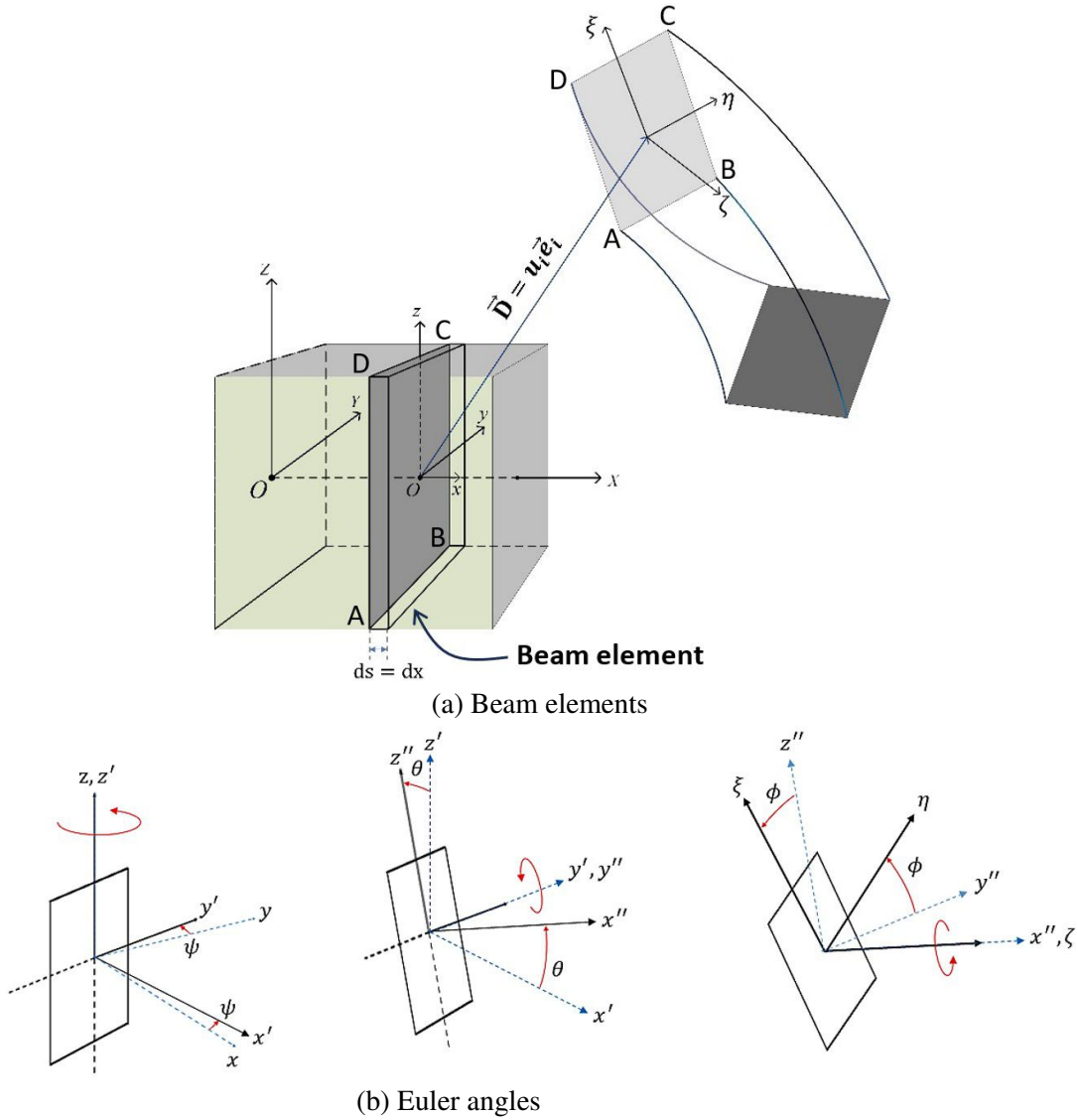


Figure 5.4: (a) A schematic of the microbeam coordinate systems, the undeformed, and deformed beam elements. (b) The successive Euler angles describing the orientation of the cross-section in the beam element's deformed configuration.

5.3.2 Equations of motion

We are interested in studying the coupling between in-plane bending, w , and torsional rotation ϕ of the mirror, both of which exhibit symmetry with respect to the mid-span. Therefore, we focus on half of the structure and ignore the out-of-plane transverse displacement, thus setting $v \approx 0$ and $\psi \approx 0$. The concept of a local displacement field is utilized to formulate the structural kinematics [97]. Accordingly, the motion of the beam element, figure 5.4(a), is decomposed into rigid body motion and deformation. First, the center of the cross-section displaced from its rest position to final position through the vector $\mathbf{D}$ and the orientation of xyz is aligned with $\zeta\eta\xi$ through the three successive Euler angles ψ , θ , and ϕ . Second, the deformation of the beam element is taken into account with respect to the $\zeta\eta\xi$, clamping the face of the beam element where the $\zeta\eta\xi$ coordinate system is attached. It is worth noting that this approach captures the nonlinear curvature of the NA in the deformed configuration [97]. However, it ignores the slope of the NA due to the assumption of clamping for the face of the beam element placed on the $\eta\xi$ -plane.

In the following, we incorporate the NA slope into the strain potential energy via defining the local displacement field in the $oxyz$ coordinate system. The position vector of a point P on the $\eta\xi$ -plane can be expressed as

$$\mathbf{r}_P = s \mathbf{i}_a + u(s; t) \mathbf{i}_x + w(s; t) \mathbf{i}_z + y \mathbf{i}_2 + z \mathbf{i}_3 \quad (5.3)$$

The transformation matrix $[\mathbf{T}]$ from the undeformed element orientation to the deformed elements orientation:

$$\begin{bmatrix} \mathbf{i}_1 \\ \mathbf{i}_2 \\ \mathbf{i}_3 \end{bmatrix} = [\mathbf{T}] \begin{bmatrix} \mathbf{i}_x \\ \mathbf{i}_y \\ \mathbf{i}_z \end{bmatrix}$$

is given by,

$$[\mathbf{T}] = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ \sin \phi \sin \theta & \cos \phi & \sin \phi \cos \theta \\ \cos \phi \sin \theta & -\sin \phi & \cos \phi \cos \theta \end{bmatrix} \quad (5.4)$$

The position vector can, therefore, be written as

$$\begin{aligned} \mathbf{r}_P &= (s + u)\mathbf{i}_x + w\mathbf{i}_z + y(T_{21}\mathbf{i}_x + T_{22}\mathbf{i}_y + T_{23}\mathbf{i}_z) + z(T_{31}\mathbf{i}_x + T_{32}\mathbf{i}_y + T_{33}\mathbf{i}_z) \\ &= (s + u + T_{21}y + T_{31}z)\mathbf{i}_x + (T_{22}y + T_{32}z)\mathbf{i}_y + (w + T_{23}y + T_{33}z)\mathbf{i}_z \end{aligned} \quad (5.5)$$

The angle, θ can be found in terms of the displacement of the displacement field as [xx]

$$\begin{aligned} \theta &= \tan^{-1} \left(\frac{-w'}{1 + u'} \right) \\ &\approx -w' + u' w' + \frac{1}{3}w'^3 + O(w'^4) \end{aligned} \quad (5.6)$$

where the prime stands for the derivative with respect to x and $\mathcal{O}(w'^4)$ denotes terms of order w'^4 or higher. The angular velocity of the $ABCD$ plane can be expressed as follows:

$$\begin{aligned} \boldsymbol{\omega} &= \dot{\theta} \mathbf{i}_{y''} + \dot{\phi} \mathbf{i}_1 \\ &= \dot{\phi} \mathbf{i}_1 + \dot{\theta} \cos \phi \mathbf{i}_2 - \dot{\theta} \sin \phi \mathbf{i}_3 \\ &= \omega_1 \mathbf{i}_1 + \omega_2 \mathbf{i}_2 + \omega_3 \mathbf{i}_3 \end{aligned} \quad (5.7)$$

where the overdot stands for the derivative with respect to time t . The velocity of particle P is found by taking the derivative of the position vector, Eq. (5.3), with respect to time.

$$\begin{aligned} \dot{\mathbf{r}}_P &= (\dot{u} + T_{11}(z\omega_2 - y\omega_3) - z\omega_1 T_{21} + y\omega_1 T_{31}) \mathbf{i}_x \\ &\quad + (-z\omega_1 T_{22} + y\omega_1 T_{32}) \mathbf{i}_y \\ &\quad + (\dot{w} + T_{13}(z\omega_2 - y\omega_3) - z\omega_1 T_{23} + y\omega_1 T_{33}) \mathbf{i}_z \end{aligned} \quad (5.8)$$

Integrating of the kinetic energy of a mass element ρdV over the microbeam volume, we obtain

its total kinetic energy as

$$T_b = \frac{1}{2} \int_V \rho (\dot{\mathbf{r}}_P \cdot \dot{\mathbf{r}}_P) dV \quad (5.9)$$

The position of the half-plate center of mass can be written in terms of the microbeam tip displacement as:

$$\begin{aligned} \mathbf{r}_c &= L \mathbf{i}_x + u(L; t) \mathbf{i}_x + w(L; t) \mathbf{i}_z + \frac{l_p}{4} \mathbf{i}_1(L; t) \\ &= \left(L + u + \frac{l_p}{4} T_{11} \right) \big|_L \mathbf{i}_x + \left(\frac{l_p}{4} T_{12} \right) \big|_L \mathbf{i}_y + \left(w + \frac{l_p}{4} T_{13} \right) \big|_L \mathbf{i}_z \end{aligned} \quad (5.10)$$

The translational kinetic energy of the half-plate is written as,

$$\begin{aligned} T_{Pl}^t &= \frac{1}{2} \frac{M_p}{2} (\dot{\mathbf{r}}_c \cdot \dot{\mathbf{r}}_c) \\ &= \frac{M_p}{4} \left[\left(\dot{u} + \frac{l_p}{4} \dot{T}_{11} \right)^2 + \left(\frac{l_p}{4} \dot{T}_{12} \right)^2 + \left(\dot{w} + \frac{l_p}{4} \dot{T}_{13} \right)^2 \right]_L \end{aligned} \quad (5.11)$$

where M_p is the mass of the mirror plate. Moreover, the kinetic energy of the half-plate rigid body rotation can be written in the following form.

$$T_{Pl}^r = \frac{1}{2} I_{\zeta\zeta} \omega_1^2(L; t) + \frac{1}{2} I_{\eta\eta} \omega_2^2(L; t) + \frac{1}{2} I_{\xi\xi} \omega_3^2(L; t) \quad (5.12)$$

where I_{ii} are the mass moments of inertia of the half-plate, with respect to a coordinate system parallel to $\zeta\eta\xi$ with its origin at the half plate's center of mass. Hence, the total kinetic energy of a suspension microbeam and the half-plate is given by the summation of Eqs. (5.9), (5.11) and (5.12)

$$T = T_b + T_{Pl}^t + T_{Pl}^r \quad (5.13)$$

The curvatures of the deformed beam element can be defined as follows [97].

$$\begin{aligned}\rho_1(x; t) &= \mathbf{i}'_2 \cdot \mathbf{i}_3 = \phi' \\ \rho_2(x; t) &= \mathbf{i}'_3 \cdot \mathbf{i}_1 = \theta' \cos \phi \\ \rho_3(x; t) &= \mathbf{i}'_1 \cdot \mathbf{i}_2 = -\theta' \sin \phi\end{aligned}\tag{5.14}$$

The length of a deformed element extending out of the plane $A^*B^*C^*D^*$, figure 5.5(a), can be written as

$$ds^* = (1 + e) dx$$

where e is the axial strain given as:

$$e(x; t) = \sqrt{(1 + u')^2 + w'^2} - 1\tag{5.15}$$

We define a local displacement field that captures the deformation ascribed by the displacement field $\mathbf{D}$ to the element ds with respect to the origin of the $\zeta\eta\xi$ coordinate system as follows

$$\begin{aligned}U_x(x, y, z; t) &= u_1^0(x; t) + z \sin \theta_2(x; t) - y \sin \theta_3(x; t) \\ U_y(x, y, z; t) &= u_2^0(x; t) - z \sin \theta_1(x; t) \\ U_z(x, y, z; t) &= u_3^0(x; t) - z (1 - \cos \theta_2(x; t)) + y \sin \theta_1(x; t)\end{aligned}\tag{5.16}$$

where the displacements u_1^0 , u_2^0 , and u_3^0 take place along and the angles θ_1 , θ_2 , and θ_3 describe rotations around the ζ , η , and ξ -axes, respectively. We can write the linearized curvatures in terms of the local angles as:

$$\begin{aligned}\rho_1 &= \theta'_1 \\ \rho_2 &= \theta'_2 \\ \rho_3 &\approx 0\end{aligned}\tag{5.17}$$

Figure 5.5(a) depicts the variations in the directions $\mathbf{i}_1$ and $\mathbf{i}_3$ as the NA extends from point

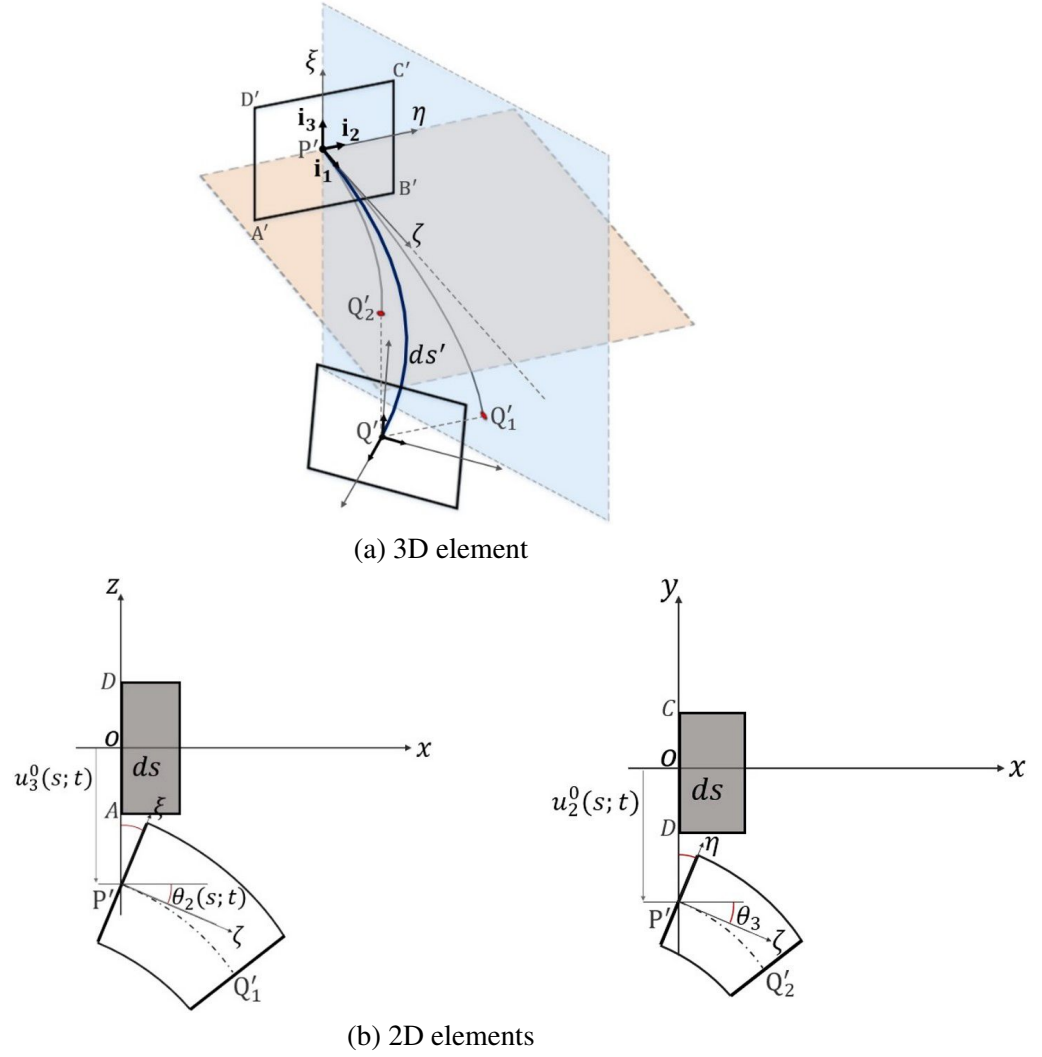


Figure 5.5: (a) The deformed beam element ds^* . (b) Projections of the undeformed ds and deformed ds^* beam elements onto the $\zeta\xi$ - and $\zeta\eta$ -planes where $\theta_2 = u_3^{0'}$ and $\theta_3 = u_2^{0'}$.

P^* to Q^* . The projection of point Q onto the $\zeta\xi$ -plane is dubbed Q_1^* and the projection of Q onto the $\zeta\eta$ -plane is dubbed Q_2^* describing the curvatures ρ_2 and ρ_3 , respectively. The slopes of the tangents to ds^* at P^* can be set to zero with respect the $\zeta\eta\xi$ coordinate system. Integrating the first two of Eqs. (5.17) over dx yields non-zero values for the rotation of the face of the beam element representing the absolute slopes of the curve ds^* at P^* as

$$\begin{aligned}\theta_1 &= \phi \\ \theta_2 &= \theta \\ \theta_3 &= 0\end{aligned}\tag{5.18}$$

Using Eq. (5.6), we can further reduce the expression for the rotation around the η -axis to

$$\theta_2 \approx -w'$$

and using Eq. (5.18), we can also write:

$$\rho_2 \approx -w''$$

Finally, using figure 5.5(b), we obtain

$$u_1^{0'} = e, \quad u_2^{0'} = \theta_3, \quad u_3^{0'} = \theta_2\tag{5.19}$$

To interpret these slopes, we rigidly transfer $\overline{P^*Q_1^*}$ and $\overline{P^*Q_2^*}$ onto the xz - and xy -plane. The incorporation of the absolute slopes θ_2 and θ_3 , into the cross-sectional shear strain, ε_{xy} , never explored before, generates coupling between the transverse and torsional motions of the beam. We note that $u_2^{0'}$ and $u_3^{0'}$ were assumed to be zero in [97].

We calculate the Green-Lagrange strain tensor using the nonlinear strain-displacement rela-

relationship for ε_{xy} as:

$$\begin{aligned}
\varepsilon_{xy} &= \frac{1}{2} \left(\frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} + \frac{\partial U_x}{\partial x} \frac{\partial U_x}{\partial y} + \frac{\partial U_y}{\partial x} \frac{\partial U_y}{\partial y} + \frac{\partial U_z}{\partial x} \frac{\partial U_z}{\partial y} \right) \\
&= -\frac{1}{2} z \rho_1 + \frac{1}{2} \left[(u_1^{0'} + z \theta_2' \cos \theta_2 - y \theta_3' \cos \theta_3) (-\sin \theta_3) + (u_3^{0'} - z \theta_2' \sin \theta_2 + y \theta_1') \theta_1 \right] \\
&= -\frac{1}{2} z \rho_1 + \frac{1}{2} (-w' - z w'' w' + y \phi') \phi \\
&= -\frac{1}{2} z \phi' + \frac{1}{2} (-w' - z w'' w' + y \phi') \phi + H.O.T
\end{aligned} \tag{5.20}$$

and the linear relationships for the other components as:

$$\begin{aligned}
\varepsilon_{xx} &= \frac{\partial U_x}{\partial x} = e - y \rho_3 + z \rho_2 \\
\varepsilon_{xz} &= \frac{1}{2} \left(\frac{\partial U_x}{\partial z} + \frac{\partial U_z}{\partial x} \right) = \frac{1}{2} y \rho_1 \\
\varepsilon_{yy} &= \varepsilon_{zz} = \varepsilon_{yz} = 0
\end{aligned} \tag{5.21}$$

The variations of the total strain potential energy of the microbeams is expressed as follows.

$$\delta U_s = \int (\sigma_{xx} \delta \varepsilon_{xx} + 2\sigma_{xy} \delta \varepsilon_{xy} + 2\sigma_{xz} \delta \varepsilon_{xz}) dV \tag{5.22}$$

The MEMS capacitance is the summation of the capacitance formed between the suspension microbeams and the stationary side electrode, C_b , and the capacitance between the middle plate and the side electrode, C_p , Figure 5.6. Assuming parallel-plate capacitance, the differential capacitance, $dC_b(x; t)$ between an element of the microbeam side dA and the actuating electrode can be written as

$$dC_b = \frac{\epsilon dA}{d(\zeta)} \tag{5.23}$$

where ζ is an axis parallel to the beam edge with an origin at the NA, $dA = d\zeta dx$, and $d(\zeta)$ is

the distance between dA and the actuation electrode expressed as follows

$$d(\zeta) = \left(g_o + \frac{h}{2}\right) - (w + r \cos \chi) \quad (5.24)$$

where $r = \sqrt{\frac{h^2}{4} + \zeta^2}$ and $\chi = \tan^{-1}\left(\frac{2\zeta}{h}\right) + \phi$. As the mirror rotates in the positive ϕ direction, its top corner protrudes above the side electrode at:

$$\gamma^+ = \frac{1}{2} \left(\frac{b - h \sin \phi}{\cos \phi} \right)$$

and we ignore points above that level in evaluating the capacitance. Similarly, when the mirror rotates in the negative ϕ direction, its bottom corner protrudes below the side electrode at:

$$\gamma^- = -\frac{1}{2} \left(\frac{b + h \sin \phi}{\cos \phi} \right)$$

and we ignore points below that level in evaluating the capacitance.

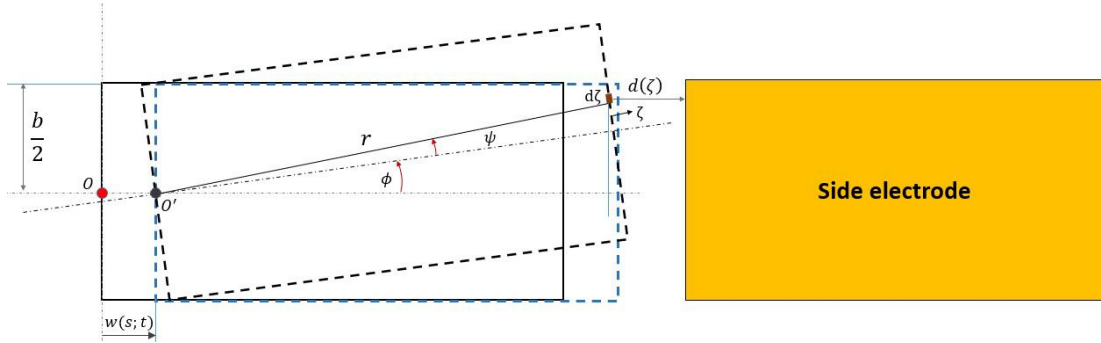


Figure 5.6: A section through the electrostatic micromirror and its side electrode.

The capacitance per unit length of the microbeam can be written as follows.

$$C_b(t) = \begin{cases} \int_{-\frac{b}{2}}^{\gamma^+} \frac{\epsilon}{d(\zeta)} d\zeta & 0 \leq \phi < 2 \tan^{-1}\left(\frac{b}{h}\right) \\ \int_{\gamma^-}^{\frac{b}{2}} \frac{\epsilon}{d(\zeta)} d\zeta & -2 \tan^{-1}\left(\frac{b}{h}\right) < \phi \leq 0 \end{cases} \quad (5.25)$$

We can evaluate the electrostatic force and torque per unit length of the microbeam from the electrical potential energy stored in the capacitor

$$U_b^e = \frac{1}{2} C_b V^2$$

as follows.

$$\begin{aligned} F_{bz}^e(w, \phi) &= \frac{\partial U_b^e}{\partial w} \\ &= \begin{cases} \int_{-\frac{b}{2}}^{\gamma^+} \frac{\epsilon V^2}{2d^2(\zeta)} d\zeta, & 0 \leq \phi < 2 \tan^{-1}(\frac{b}{h}) \\ \int_{\gamma^-}^{\frac{b}{2}} \frac{\epsilon V^2}{2d^2(\zeta)} d\zeta, & -2 \tan^{-1}(\frac{b}{h}) < \phi \leq 0 \end{cases} \end{aligned} \quad (5.26)$$

$$\begin{aligned} \tau_{bx}^e(w, \phi) &= \frac{\partial U_b^e}{\partial \phi} \\ &= \begin{cases} - \int_{-\frac{b}{2}}^{\gamma^+} \frac{\epsilon V^2 r \sin \chi}{2d^2(\zeta)} d\zeta, & 0 \leq \phi < 2 \tan^{-1}(\frac{b}{h}) \\ - \int_{\gamma^-}^{\frac{b}{2}} \frac{\epsilon V^2 r \sin \chi}{2d^2(\zeta)} d\zeta, & -2 \tan^{-1}(\frac{b}{h}) < \phi \leq 0 \end{cases} \end{aligned} \quad (5.27)$$

where the Euler angle ϕ is assumed to occur round the NA during and reduce to the beam torsion angle. Although there exists an orientation difference between the x - and ζ -axes, it is ignored for the purposes of deriving the electrostatic torque.

The electrostatic force and torque applied to the rigid plate can be obtained similarly as

$$F_{pz}^e(w(L; t), \phi(L; t)) = \frac{\epsilon l_p V^2}{4} \begin{cases} \int_{-\frac{b}{2}}^{\gamma^+} \frac{1}{d^2(\zeta)} d\zeta, & 0 \leq \phi < 2 \tan^{-1}(\frac{b}{h_p}) \\ \int_{\gamma^-}^{\frac{b}{2}} \frac{1}{d^2(\zeta)} d\zeta, & -2 \tan^{-1}(\frac{b}{h_p}) < \phi \leq 0 \end{cases} \quad (5.28)$$

$$\tau_{px}^e(w(L; t), \phi(L; t)) = \frac{\epsilon l_p V^2}{4} \begin{cases} - \int_{-\frac{b}{2}}^{\gamma^+} \frac{r \sin \chi_p}{d^2(\zeta)} d\zeta, & 0 \leq \phi < 2 \tan^{-1}(\frac{b}{h_p}) \\ - \int_{\gamma^-}^{\frac{b}{2}} \frac{r \sin \chi_p}{d^2(\zeta)} d\zeta, & -2 \tan^{-1}(\frac{b}{h_p}) < \phi \leq 0 \end{cases} \quad (5.29)$$

where $\chi_p = \tan^{-1} \left(\frac{2\zeta}{h_p} \right) + \phi$. Their conservative work can expressed as

$$\delta W_{ext} = \int_0^L (F_{bz}^e(w, \phi) \delta w + \tau_{bx}^e(w, \phi) \delta \phi) dx + F_{pz}^e(L; t) \delta w + \tau_{px}^e(L; t) \delta \phi \quad (5.30)$$

The non-conservative work of viscous damping of in-plane bending c_w and torsion c_ϕ are expressed as

$$\delta W_{ext}^{nc} = \int_0^L (c_w \dot{w} \delta w + c_\phi \dot{\phi} \delta \phi) dx + c_w \dot{w}(L; t) \delta w + c_\phi \dot{\phi}(L; t) \delta \phi \quad (5.31)$$

Finally, the equations of motion of the electrostatic scanning mirror are obtained employing the extended Hamilton's principle to minimize the system's Lagrangian between two arbitrary instants, t_1 and t_2 . Using Eqs. (5.13), (5.22), (5.30), and (5.31), we write:

$$\int_{t_1}^{t_2} (\delta T - \delta U_s + \delta W_{ext} + W_{ext}^{nc}) dt = 0 \quad (5.32)$$

Using variational calculus and truncating the nonlinear stiffness and inertia, arising from nonlinear curvature, up to third order $\mathcal{O}(w^3)$ and $\mathcal{O}(\phi^3)$, yields the following equations of motion:

$$\rho A \ddot{u} + \rho I_{yy} (w' \ddot{w}'' + w'' \ddot{w}') - EA(u'' + w'' w') - EI_{yy} (w'''' w' + w'' w''') = 0 \quad (5.33)$$

$$\begin{aligned} \rho A \ddot{w} + c_w \dot{w} + \rho I_{yy} (\phi^2 \ddot{w}'' + 2\phi \dot{\phi} \ddot{w}'' + 2\phi' \phi \ddot{w}' + 2\phi \dot{\phi}' \ddot{w}' + 2\phi' \dot{\phi} \ddot{w}' + w' \ddot{u}'' + 2u' \ddot{w}'' \\ + 2u'' \ddot{w}' + w'' \ddot{u}' + 2\dot{u}'' \ddot{w}' + 2\dot{u}' \ddot{w}'' + 2w'^2 \ddot{w}'' - \ddot{w}'' + 4w' w'' \ddot{w}' + 2w'' \dot{w}'^2 \\ + 4w' \dot{w}' \ddot{w}'') + \rho I_{zz} \left(-\phi^2 \ddot{w}'' - 2\phi \dot{\phi} \ddot{w}'' - 2\phi' \phi \ddot{w}' - 2\phi \dot{w}' \dot{\phi}' - 2\phi' \dot{\phi} \ddot{w}' \right) \\ + EA(-u' w'' - u'' w' - \frac{3}{2} w'^2 w'') + EI_{yy} (-w'''' \phi^2 - 4w''' \phi' \phi - 2w'' \phi'' \phi \\ - u'''' w' - 2u' w'''' - 2w'^2 w'''' + w'''' - 2w'' \phi'^2 - 2w''^3 - 3u''' w'' - 4u'' w''' \\ - 8w' w''' w'') + EI_{zz} (w'''' \phi^2 + 4w''' \phi' \phi + 2w'' \phi'' \phi + 2w'' \phi'^2) = F_{bz}^e(w, \phi) \end{aligned} \quad (5.34)$$

$$\rho J_o \ddot{\phi} + c_\phi \dot{\phi} - \frac{1}{2} G J_o \phi'' + \rho I_{yy} \phi \dot{w}'^2 - \rho I_{zz} \phi \dot{w}'^2 - EI_{yy} w''^2 \phi + EI_{zz} w''^2 \phi = \tau_{bx}^e(w, \phi) \quad (5.35)$$

subject to the boundary conditions:

$$\begin{aligned} u(0; t) &= 0, \quad u(L; t) = 0, \\ w(0; t) &= 0, \quad w'(0; t) = 0, \quad w'(L; t) = 0 \\ \phi(0; t) &= 0 \end{aligned} \tag{5.36}$$

$$-M_p \ddot{w}(L; t) + EI_{yy} w'''(L; t) + F_{pz}^e(w, \phi) = 0 \tag{5.37}$$

$$J_{op} \ddot{\phi}(L; t) + \frac{1}{2} G J_o \phi'(L; t) + \tau_{px}^e(w, \phi) = 0 \tag{5.38}$$

where $J_o = I_{yy} + I_{zz}$ and $J_{op} = \frac{1}{24} M_p (h_p^2 + b^2)$. The bending and torsion have been ordered at $w = \mathcal{O}(\varepsilon)$ and $\phi = \mathcal{O}(\varepsilon)$ while axial displacement has been ordered at a higher order of $u = \mathcal{O}(\varepsilon^2)$ [97] and terms of order $\mathcal{O}(\varepsilon^4)$ have been dropped.

The microbeam's longitudinal inertia is negligible within the frequency range of interest, and we can write: $\rho A \ddot{u} \approx 0$ and rewrite Eq. (5.33) as:

$$\begin{aligned} \rho I_{yy} (w' \ddot{w}') - EA \left(u' + \frac{1}{2} w'^2 \right)' - EI_{yy} (w' w''')' &= 0 \\ \frac{\partial}{\partial x} \left[\rho I_{yy} (w' \ddot{w}') - EA \left(u' + \frac{1}{2} w'^2 \right) - EI_{yy} (w' w''') \right] &= 0 \end{aligned}$$

Integrating this equation of the axial position x , yields:

$$\rho I_{yy} (w' \ddot{w}') - EA \left(u' + \frac{1}{2} w'^2 \right) - EI_{yy} (w' w''') = C(t) \tag{5.39}$$

Integrating again over the beam length, and using the boundary conditions on u in Eqs. (5.36), yields

$$C(t) = \frac{\rho I_{yy}}{L} \int_0^L w' \ddot{w}' dx - \frac{EA}{2L} \int_0^L w'^2 dx - \frac{EI_{yy}}{L} \int_0^L w' w''' dx \tag{5.40}$$

Finally, using this equation to substitute for the constant $C(t)$ in Eq. (5.39) results in the follow-

ing expression:

$$u' = \frac{1}{EA} \left[-\frac{\rho I_{yy}}{L} \int_0^L w' \ddot{w} dx + \frac{EA}{2L} \int_0^L w'^2 dx + \frac{EI_{yy}}{L} \int_0^L w' w''' dx - \frac{EA}{2} w'^2 - EI_{yy} w' w''' + \rho I_{yy} w' \ddot{w} \right] \quad (5.41)$$

Higher order spatial derivatives of u can be obtained from Eq. (5.35). Substituting with u and its spatial derivatives in Eqs. (5.34) and (5.35), we obtain:

$$\begin{aligned} \rho A \ddot{w} + c_w \dot{w} + EI_{yy} w'''' - \frac{EA}{2L} w'' \int_0^L w'^2 dx + \rho I_{yy} (\phi^2 \ddot{w}'' + 2\dot{\phi} \dot{w}'' + 2\phi' \phi \ddot{w}') \\ + 2\phi \dot{\phi}' \dot{w}' + 2\phi' \dot{\phi} \dot{w}' - \ddot{w}'') + \rho I_{zz} (-\phi^2 \ddot{w}'' - 2\dot{\phi} \dot{w}'' - 2\phi' \phi \ddot{w}') \\ - 2\phi \dot{\phi}' \dot{w}' - 2\phi' \dot{\phi} \dot{w}') + EI_{yy} w' (w' w'''' + 2w'' w''') - \frac{EI_{yy}}{L} w'' \int_0^L w' w''' dx \end{aligned} \quad (5.42)$$

$$\begin{aligned} + EI_{yy} (-w'''' \phi^2 - 4w''' \phi' \phi - 2w'' (\phi'' \phi - w' w''') - 2w'' \phi'^2 + w'''^3) \\ - \frac{EI_{yy}}{L} w'''' \int_0^L w'^2 dx + EI_{zz} (w'''' \phi^2 + 4w''' \phi' \phi + 2w'' \phi'' \phi + 2w'' \phi'^2) = F_{bz}^e \\ \rho J_o \ddot{\phi} + c_\phi \dot{\phi} - \frac{1}{2} G J_o \phi'' + \rho (I_{yy} - I_{zz}) \phi \dot{w}'^2 + E (I_{zz} - I_{yy}) \phi w''^2 \\ - \frac{1}{2} G Q_{yy} (w'' \phi + w' \phi') = \tau_{bx}^e \end{aligned} \quad (5.43)$$

subject to the boundary conditions of Eqs. (5.36)-(5.38). The nonlinear inertia and stiffness terms marked in blue and gray, respectively, in Eqs. (5.42) and (5.43) arise from nonlinear curvature, Eq. (5.6). These terms are proportional to the gap-to-length ratio of the microbeams which is negligible in our case $(\frac{g_0}{L})^2 = 0.000024$. We will, therefore, ignore them for the rest of this study. The term highlighted in purple in Eq (5.36) defines the mid-plane stretching induced in the microbeam, stemming from the rigidity of the support and plate on either side of the beam.

Ignoring the small blue and gray highlighted terms, the equations of motion reduce to:

$$\rho A \ddot{w} + c_w \dot{w} + EI_{yy} w'''' - \frac{EA}{2L} w'' \int_0^L w'^2 dx = F_{bz}^e(w, \phi) \quad (5.44)$$

$$\rho J_o \ddot{\phi} + c_\phi \dot{\phi} - \frac{1}{2} G J_o \phi'' - \underbrace{\frac{1}{2} G Q_{yy} (w'' \phi + w' \phi')}_{\text{Quadratic stiffness}} = \tau_{bx}^e(w, \phi) \quad (5.45)$$

The quadratic stiffness terms that couples in-plane bending and torsional motions, highlighted in brown in Eq. (5.45), arise from the potential strain energy component ε_{xy} . This can be seen by substituting with the expression for ε_{xy} , Eq. (5.20), into the potential energy, Eq. (5.22), to get:

$$\begin{aligned} \delta U_s &= \int_{\Omega} 2G \varepsilon_{xy} \delta \varepsilon_{xy} dV \\ &= \frac{1}{2} G \int (-z \phi' - w' \phi - z w'' w' \phi + y \phi' \phi) (-z \delta \phi' - \phi \delta w' - w' \delta \phi \\ &\quad - z w' \phi \delta w'' - z w'' \phi \delta w' - z w'' w' \delta \phi + y \phi \delta \phi' + y \phi' \delta \phi) dV \\ &= \frac{1}{2} G \int_{\Omega} (z w' \phi \delta \phi' + \dots) dV \end{aligned}$$

The quadratic term, therefore, arises from the interaction between the torsional strain (highlighted in blue) with the slope term $u_3^{0'}$ (highlighted in brown). This is a previously unexplored approach to establish quadratic coupling between the transverse and torsional motions. The existence of this term is contingent on a non-zero first moment of area Q_{yy} as can be seen by reformulating the last integral obtained above

$$\frac{1}{2} G \int (z w' \phi \delta \phi') dA dx = \frac{1}{2} G \left(\int z dA \right) \left(\int_0^L w' \phi \delta \phi' dx \right)$$

where the term colored in blue is Q_{yy} as per Eqs. (5.1). As discussed in section 5.2, Q_{yy} does not vanish for symmetrically designed beams because of microfabrication imperfections. We utilize integration by parts to highlight the extraction of the quadratic coupling term via Hamilton

Principle

$$\begin{aligned} \frac{1}{2}GQ_{yy} \int_0^L w' \phi \delta \phi' dx &= \frac{1}{2}GQ_{yy} (w' \phi \delta \phi) \Big|_0^L - \frac{1}{2}GQ_{yy} \int_0^L (w'' \phi + w' \phi') \delta \phi dx \\ &= \frac{1}{2}GQ_{yy} (w' \phi \delta \phi) \Big|_0^L - \frac{1}{2}GQ_{yy} \int_0^L \underbrace{(w'' \phi + w' \phi')}_{\text{Quadratic stiffness}} \delta \phi dx \end{aligned}$$

The first two terms on the left hand-side vanish identically as per the boundary conditions, Eqs. (5.36). The remaining two terms are those of quadratic coupling between bending and torsion. The first, curvature dependent, term $(w'' \phi)$ dominates the second, slope dependent, term $(w' \phi')$.

5.4 Static and eigenfrequency analysis

The linear undamped in-plane bending $\varphi_i(x)$ and torsion $\psi_i(x)$ eigenfunctions were found analytically by dropping the forcing, damping, and nonlinear terms from Eqs. (5.37), (5.38), (5.44), and (5.45) and solving the resulting linear equations of motion and their corresponding boundary conditions by separation of variables. Figures 5.7a and 5.7b illustrate the first five bending and torsion eigenfunctions, respectively.

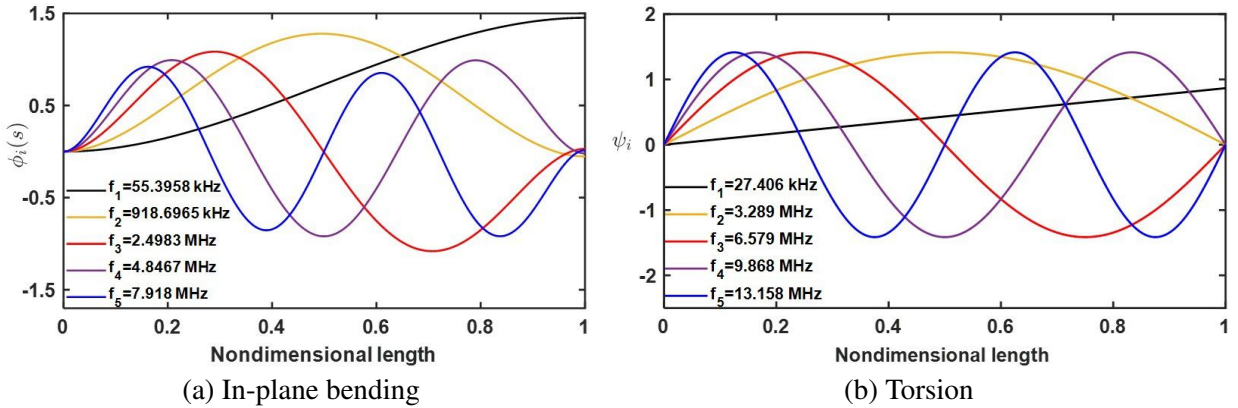
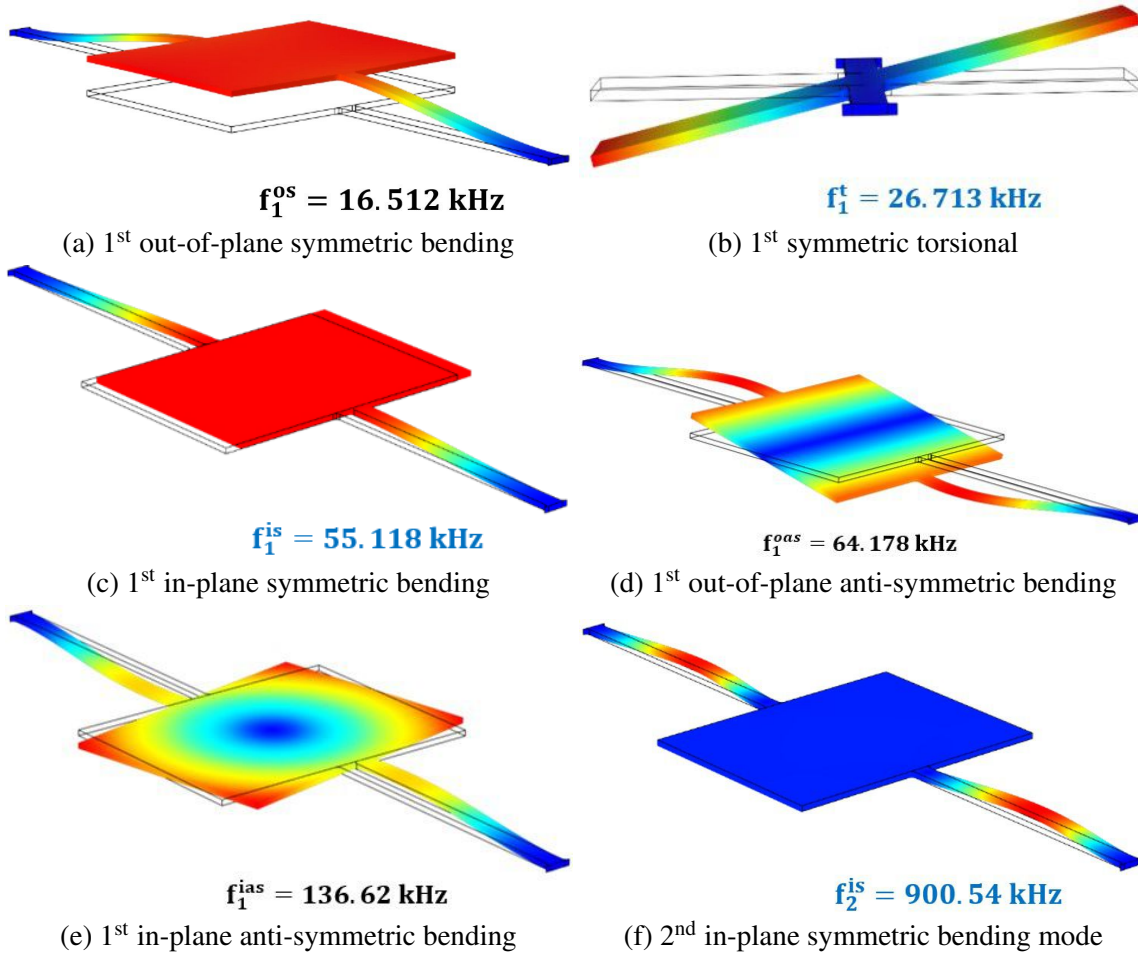


Figure 5.7: Analytically obtained eigenfunctions of in-plane bending and torsion of the half-mirror under no load.

We also carried out the eigenvalue analysis in COMSOL for the full mirror under no load and compared it to the analytical results. The global eigenfunctions of the mirror obtained from FEM analysis are shown in figure 5.9. The modes are denoted “t” for torsional modes, “os” for out-of-plane symmetric bending modes, “is” for in-plane symmetric bending modes, “oas” for out-of-plane anti-symmetric bending modes, and “ias” for in-plane anti-symmetric bending modes. The results indicate the presence of a near 2:1 ratio between the first in-plane symmetric bending and the first torsional modes with the analytical solution reporting a ratio of: $\frac{f_1^{is}}{f_1^t} = \frac{55.3958}{27.406} = 2.021$ and the FEM simulation reporting a ratio of: $\frac{f_1^{is}}{f_1^t} = \frac{55.118}{26.713} = 2.063$.



To solve for the mirror static equilibrium $w_s(x)$ and $\phi_s(x)$, we set the time derivatives and

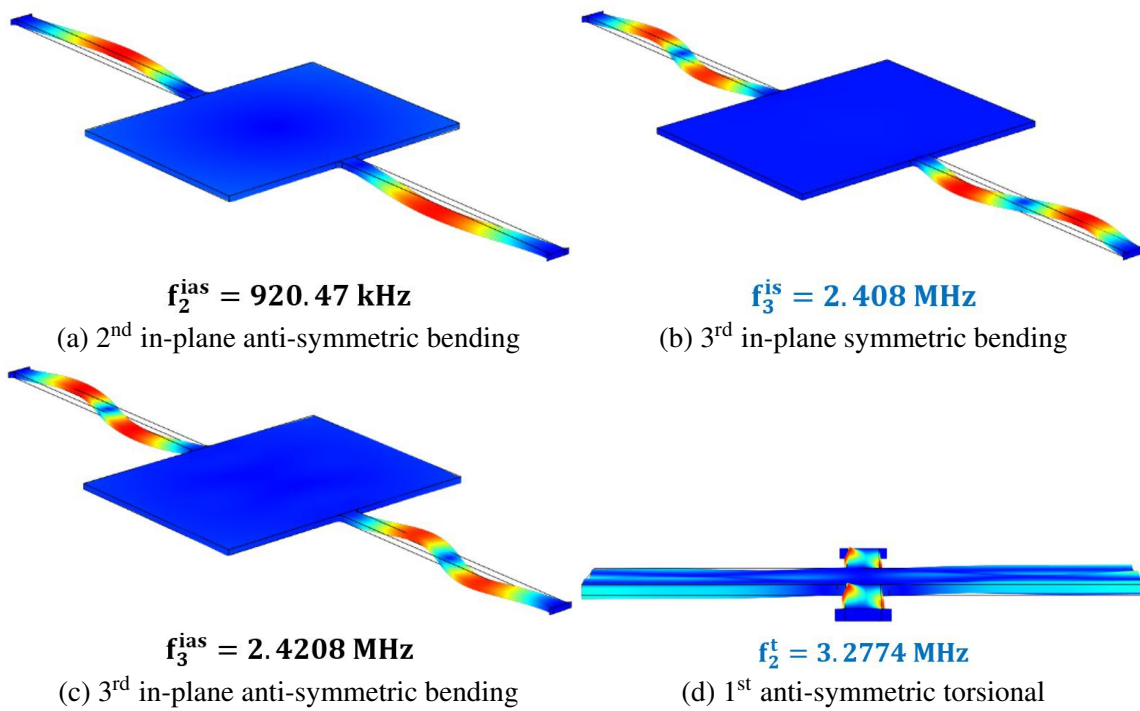


Figure 5.9: Global modes of the mirror.

time-varying terms the Eqs. (5.37), (5.38), (5.44), and (5.45) equal to zero. Inspecting Eqs. (5.38) and (5.45), we find that torsion possess a trivial solution $\phi_s(x) = 0$.

To obtain the static in-plane displacement as a function of the DC voltage V_{DC} , we write it as a linear combination of N eigenfunctions $\varphi_i(x)$:

$$w_s(x) = \sum_{i=1}^N q_{si} \varphi_i(x) \quad (5.46)$$

where q_{si} are modal coefficients attributing each mode's contribution to static equilibrium. Substituting Eq. (5.46) and setting $\phi_s(x) = 0$ in Eqs. (5.37) and (5.44), multiplying those equations by $\varphi_n(x)$, integrating over the beam span, applying integration by parts to the w_d'''' term, and using the boundary condition, Eq. (5.37) to substitute for the result, yields a set of nonlinear algebraic equations in terms of the are modal coefficients q_{si} .

We solved the resulting reduced-order model (ROM) for the five lowest modes ($N = 5$) of the half-mirror numerically in Matlab for voltage in the range of $V_{DC} = [0, 200]$ V. For validation, we compare the results to those obtained from FEM analysis conducted in COMSOL in figure 5.10. The voltage-mirror displacement curves for the ROM (solid line) and FEM (red dots) are in close agreement. The ROM found that static pull-in occurs at $V_P = 195.2$ V where the branch of stable static equilibria meets another branch of unstable static equilibria (dashed line). The pull-in voltage obtained from the FEM simulation was lower at $V_P = 192$ V suggesting that the ROM underestimates the electrostatic force.

To derive the eigenvalue problem of the mirror around an arbitrary equilibrium $w_s(x)$ and $\phi_s(x)$, we substitute a small vibrations expansion of the state variables

$$\begin{aligned} w(x; t) &= w_s(x) + w_d(x; t) \\ \phi(x; t) &= \phi_s(x) + \phi_d(x; t) \end{aligned} \quad (5.47)$$

into the equations of motion, Eqs. (5.44), and (5.45), and boundary conditions, Eqs. (5.37) and (5.38), drop the terms balanced by the static solution, linearize the result by dropping high-order terms, and linearize the electrostatic force and torque in terms of w_d and ϕ_d to obtain the

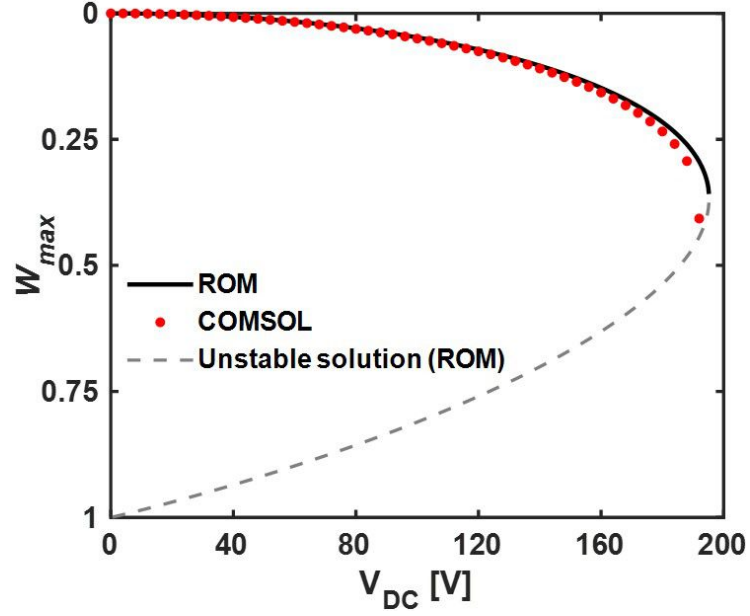


Figure 5.10: The voltage-mirror displacement $w_s(L)$ curve.

following equations:

$$\ddot{w}_d + \alpha_1 w_d'''' - 2\alpha_3 w_s'' \int_0^1 (w_s' w_d') dx - \alpha_3 w_d'' \int_0^1 w_s'^2 dx = \frac{2\alpha_e b V_{DC}^2 w_d}{(g_0 - w_s)^3} \quad (5.48)$$

$$\ddot{\phi}_d - \beta_1 \phi_d'' - \beta_3 w_s'' \phi_d = \frac{b^2 - 3h(g_0 - w_s)}{2(g_0 - w_s)^3} \beta_e b V_{DC}^2 \phi_d \quad (5.49)$$

subject to the boundary conditions.

$$w_d(0; t) = 0, \quad w_d'(0; t) = 0, \quad w_d'(L; t) = 0, \quad \phi_d(0; t) = 0 \quad (5.50)$$

$$-\ddot{w}_d + \alpha_2 w_d''' + 2\alpha_{ep} b V_{DC}^2 \frac{w_d}{(g_0 - w_s)^3} = 0 \quad ; x = L \quad (5.51)$$

$$\ddot{\phi}_d + \beta_2 \phi_d' - \frac{b^2 - 3h_p(g_0 - w_s)}{(g_0 - w_s)^3} \beta_{ep} b V_{DC}^2 \phi_d = 0 \quad ; x = L \quad (5.52)$$

where, the parameters appearing in those equations are defined as follows.

$$\begin{aligned}
\alpha_1 &= \frac{EI_{yy}}{\rho A} \quad , \quad \alpha_2 = \frac{EI_{yy}}{M_p} \quad , \quad \alpha_3 = \frac{E}{2\rho L} \quad , \quad \alpha_4 = \frac{EI_{zz}}{\rho A} \\
\alpha_e &= \frac{\epsilon_0}{2\rho A} \quad , \quad \alpha_{ep} = \frac{\epsilon_0 l_p}{4M_p} \\
\beta_1 &= \frac{G}{2\rho} \quad , \quad \beta_2 = \frac{GJ_o}{2J_{op}} \quad , \quad \beta_3 = \frac{GQ_{yy}}{2\rho J_o} \\
\beta_e &= \frac{\epsilon_0}{6\rho J_o} \quad , \quad \beta_{ep} = \frac{\epsilon_0 l_p}{24J_{op}}
\end{aligned} \tag{5.53}$$

To separate the variables in the loaded eigenvalue problem at the equilibrium $w_s(x)$ and $\phi_s(x)$, we write the small motions state variables in the harmonic form:

$$\begin{aligned}
w_d(x; t) &= \Psi(x)e^{i\omega t} \\
\phi_d(x; t) &= \Phi(x)e^{i\omega t}
\end{aligned} \tag{5.54}$$

where the loaded eigenfunctions $\Psi(x)$ and $\Phi(x)$ are, in turn, described in terms of N and M , respectively, unloaded eigenfunctions as:

$$\begin{aligned}
\Psi(x) &= \sum_{i=1}^N A_i \varphi_i(x) \\
\Phi(x) &= \sum_{i=1}^M P_i \psi_i(x)
\end{aligned} \tag{5.55}$$

Substituting the harmonic forms of Eqs. (5.54), into Eqs. (5.48), (5.49), (5.51), and (5.52), multiplying equations (5.48) and (5.51) by $\varphi_n(x)$, multiplying equations (5.49) and (5.52) by $\psi_n(x)$, integrating over the beam span and employing the orthogonality properties of the eigenfunctions, applying integration by parts to the w_d'''' and ϕ_d'' terms, and using the boundary conditions, Eqs.

(5.51) and (5.52) to substitute for the results, yields

$$\begin{aligned}\sum_{i=1}^N A_i K_{ni} - \omega^2 \sum_{i=1}^N A_i \delta_{ni} &= 0 \\ \sum_{i=1}^N P_i K_{ni}^t - \omega^2 \sum_{i=1}^N P_i \delta_{ni} &= 0\end{aligned}\tag{5.56}$$

where, δ_{ni} is the Kronecker delta function and K_{ni} and K_{ni}^t denote the components of the linear stiffness matrices for the bending and torsional modes, respectively, defined as follows:

$$\begin{aligned}K_{ni} = & - \int_0^L \varphi'_n \varphi_i''' dx - \alpha_3 \left(\int_0^L \varphi_n w_s'' dx \int_0^L 2w'_s \varphi'_i ds + \int_0^L \varphi_n \varphi_i'' dx \int_0^L w_s'^2 dx \right) \\ & - 2\alpha_e b V_{DC}^2 \int_0^L \frac{\varphi_n \varphi_i}{(g_0 - w_s)^3} dx - \frac{2\alpha_{ep} b V_{DC}^2}{\alpha_4} \frac{\varphi_n(L) \varphi_i(L)}{(g_0 - w_s(L))^3}\end{aligned}\tag{5.57}$$

$$\begin{aligned}K_{ni}^t = & \beta_1 \int_0^L \psi'_n \psi'_i dx - \beta_3 \int_0^L \psi_n \psi_i w_s'' dx - \beta_e b V_{DC}^2 \int_0^L \psi_n \psi_i \frac{b^2 - 3h(g_0 - w_s)}{2(g_0 - w_s)^3} dx \\ & - \frac{\beta_1}{\beta_2} \beta_{ep} b V_{DC}^2 \frac{b^2 - 3h_p(g_0 - w_s(L))}{(g_0 - w_s(L))^3} \psi_n(L) \psi_i(L)\end{aligned}\tag{5.58}$$

Repeating this process of all N and M eigenfunctions, respectively, we write the resulting systems of equations in matrix form as

$$\begin{aligned}([K_b] - \omega^2[I])\mathbf{A} &= 0 \\ ([K_t] - \omega^2[I])\mathbf{P} &= 0\end{aligned}\tag{5.59}$$

Given the mirror dimensions listed in Table 1, the algebraic system of Eqs. (5.59) was solved numerically for the first $N = 5$ and $M = 5$ eigenvalues and eigenfunctions of the DC-loaded mirror. Figures 5.11(a) and (b) compares the linear natural frequencies of the first in-plane bending and torsional modes as functions of bias voltage V_{DC} obtained from this solution with FEM simulations results from COMSOL. The two set of results are in excellent agreement confirming convergence of the ROM. Consistent with the static displacement results they also show that the ROM underestimates the electrostatic force. As voltage increases, the first in-plane bending

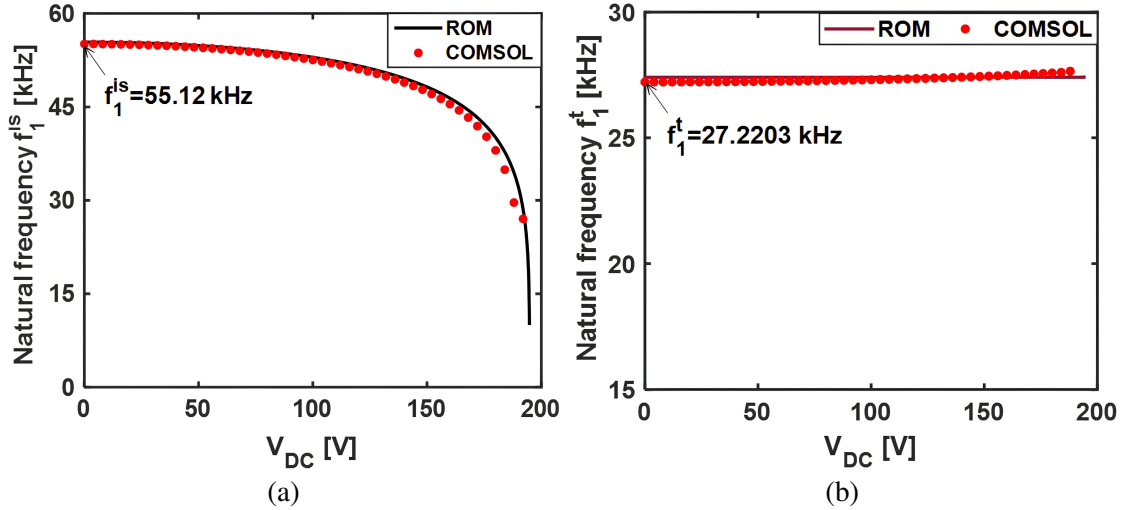


Figure 5.11: The natural frequency-voltage relationships of the electrostatic micromirror for (a) the first in-plane symmetric bending mode and (b) first torsional symmetric mode obtained from ROM (solid lines) and FEM simulations (red dots).

mode softens, whereas the first torsional mode stiffens slightly. Pull-in occurs due to instability of in-plane bending under higher electrostatic fields as the natural frequency of the first in-plane mode f_1^{is} approaches and crosses zero at same values reported above for static pull-in, namely $V_P = 195.2$ V in the ROM and $V_P = 192$ V in the FEM simulation.

In figure 5.12, we plot twice the natural frequency of the first symmetric torsional $2f_1^t$ to the natural frequency of the first symmetric in-plane bending f_1^{is} . The two modes have commensurate ratio of 2:1 at a voltage of $V_{DC} = 52$ V as per the ROM model and 44 V as per the FEM simulation.

We compare in figure 5.13 the first symmetric in-plane bending mode shape and its spatial derivatives under a bias voltage of $V_{DC} = 88$ V as the number of bending modes retained in the modal expansion, Eq. (5.56), is increased from $N = 1$ to 9. A noticeable change is observed in the spatial variation of the third-order $\Psi'''(x)$ and fourth-order $\Psi''''(x)$ derivatives of the mode shape, figures 5.13(d) and 5.13(e) but not in the mode shape itself or its lower derivatives. No changes are observed in the mode or any of its derivatives at lower bias voltages ($V_{RMS} < 40$

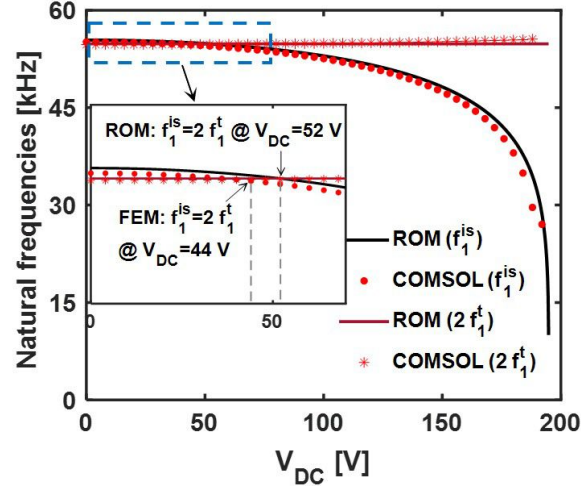


Figure 5.12: Variation in twice the natural frequency of the first symmetric torsional mode $2f_1^t$ and the natural frequency of the first in-plane symmetric bending mode with voltage show a 2:1 commensurate ratio.

V) as the number of modes is increased from $N = 1$ to 9. These results suggest that as the bias voltage increases further and approaches pull-in voltage V_p , significant differences will appear between the unloaded and loaded mode shapes and their derivatives.

5.5 Dynamic analysis

To reduce the voltage required to realize primary resonance of the first in-plane bending mode, we use resonant excitation [144] and match the natural frequency of the mode and twice the electrical resonance of an RLC circuit made by introducing an inductor L_e in series with and an auxiliary capacitor C_a in-parallel with the MEMS, $\Omega \approx \omega_b^1$. The charge flow $Q(t)$ through this circuit can be described as:

$$L_e \ddot{Q} + R_e \dot{Q} + \frac{1}{C_t} Q = V_{AC} \cos(\Omega t) \quad (5.60)$$

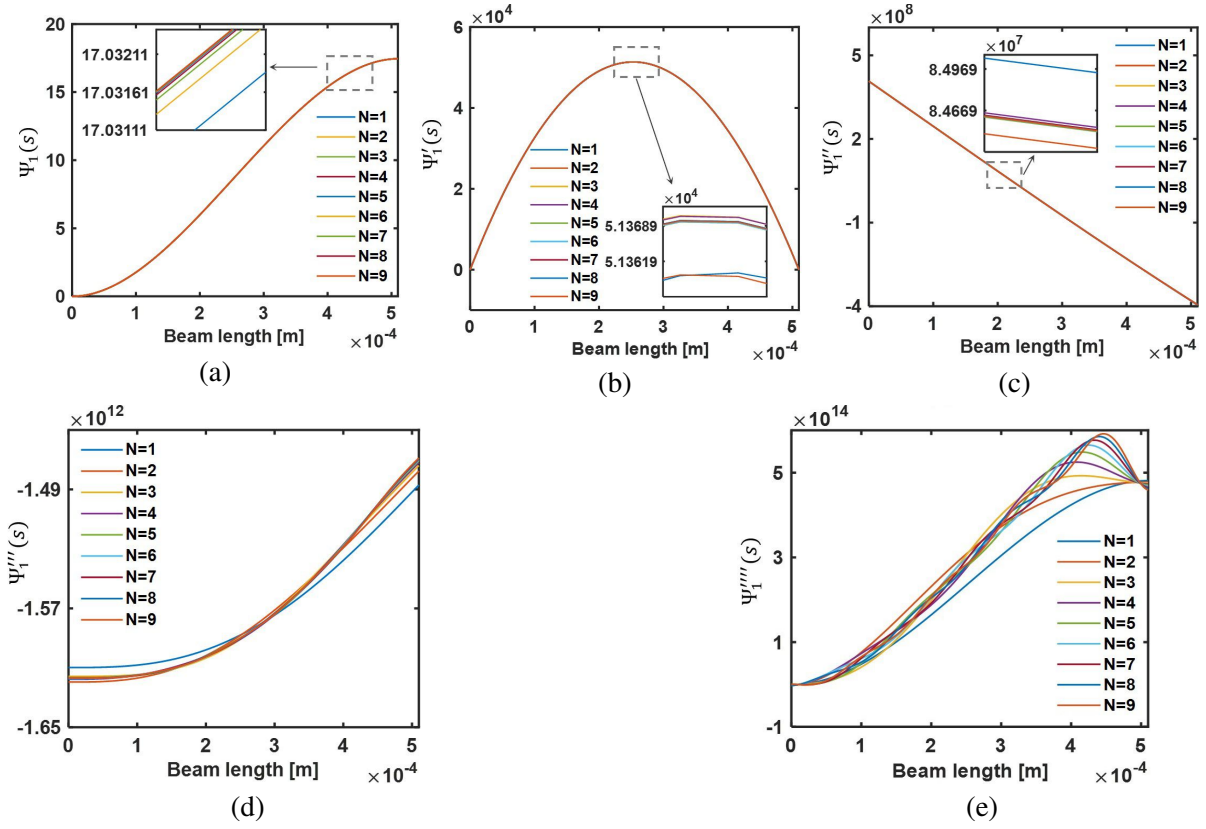


Figure 5.13: (a) The first in-plane bending mode shape $\Psi(x)$ and its (b) first $\Psi'(x)$, (c) second $\Psi''(x)$, (d) third $\Psi'''(x)$, and (e) fourth $\Psi''''(x)$ derivatives under a bias voltage of $V_{DC} = 88$ V.

where, R_e denotes parasitic resistance in the circuit, C_t represents the total capacitance of the half-structure:

$$C_t(t) = C_{bt}(t) + C_p(t) + C_a \quad (5.61)$$

and a microbeam capacitance C_{bt} is evaluated from Eq. (5.25) as:

$$C_{bt}(t) = \begin{cases} \int_0^L \int_{-\frac{b}{2}}^{\gamma^+} \frac{\epsilon}{d(\zeta)} d\zeta dx & 0 \leq \phi < 2 \tan^{-1} \left(\frac{b}{h} \right) \\ \int_0^L \int_{\gamma^-}^{\frac{b}{2}} \frac{\epsilon}{d(\zeta)} d\zeta dx & -2 \tan^{-1} \left(\frac{b}{h} \right) < \phi \leq 0 \end{cases} \quad (5.62)$$

Using the micromirror dimensions, Table 1, the capacitance of the beam C_{bt} and plate C_p at rest are found to be in the pico-Farad range which is much smaller than the total capacitance ($C_t \gg C_{bt}$) and $C_t \gg C_p$) required to achieve resonant matching within realistic inductance L_e values. Further, those capacitance vary as the mirror moves which makes it unrealistic to sue them in frequency tuning. Therefor, we set:

$$C_t(t) \approx C_a$$

and therefore obtain the electrical resonance as

$$\omega_e \approx \frac{1}{\sqrt{L_e C_0}}$$

Using auxiliary capacitance and inductance of $C_a = 1$ nF and $L_e = 35$ mH which set the electrical resonance equal to half the natural frequency of the first in-plane bending mode.

The electrical resonator equation, Eq. (5.60), is nonlinearly coupled with mirror moitons via C_t . The equations of motion describe flexural-torsional of the mirror, Eqs (5.44) and (5.45), are

solved via Galerkin expansion. Towards that end, we assuming the modal expansion form

$$\begin{aligned} w(x; t) &= \sum_{i=1}^3 q_i(t) \Psi_i(x) \\ \phi(x; t) &= \sum_{i=1}^3 p_i(t) \Phi_i(x) \end{aligned} \quad (5.63)$$

and use it to discretize the equations of motion. Only the first three modes are considered in each direction since the natural frequencies of the higher-order modes in each case are much larger than the first three, figure 5.7, and their modal residuals are, therefore, expected to be negligible in the frequency range of interest. Further, based on this analysis in section 5.4, we set the number of unloaded modes used to construct the loaded modes $\Psi_i(x)$ and $\Phi_i(x)$ to $N = M = 5$ for low RMS voltage values ($V_{RMS} \leq 40$ V) and to $N = M = 9$ for higher RMS voltage values.

Substituting with Eqs. (5.63) into Eqs. (5.37), (5.38), (5.44), and (5.45), applying the Galerkin method, and using integration by parts to satisfy the boundary conditions, Eqs. (5.37) and (5.38), yields a set of six second-order nonlinear ODEs as follows:

$$\begin{aligned} \sum_{i=1}^N \ddot{q}_i \delta_{ni} + c_w \sum_{i=1}^N K_{ni}^{(1)} \dot{q}_i + \alpha_1 \sum_{i,j,k=1}^N q_i q_j q_k K_{nijk}^{(1)} + 4\alpha_1 \sum_{i,j,k=1}^N q_i q_j q_k K_{nijk}^{(2)} \\ + \alpha_1 \sum_{i,j,k=1}^N q_i q_j q_k K_{nijk}^{(3)} - \frac{\alpha_1}{L} \sum_{i,j,k=1}^N q_i q_j q_k K_{nijk}^{(4)} \\ - \alpha_3 \sum_{i,j,k=1}^N q_i q_j q_k K_{nijk}^{ms} - \alpha_1 \sum_{i=1}^N K_{ni}^{(2)} q_i = F_n(t) \end{aligned} \quad (5.64)$$

$$\sum_{i=1}^N \ddot{p}_i \delta_{ni} + c_\phi \sum_{i=1}^N K_{ni}^{(1)} \dot{p}_i + \beta_1 \sum_{i=1}^N K_{ni}^{(2)} p_i - \beta_3 \sum_{i,j=1}^N p_i q_j K_{nij}^q = F_n^t(t) \quad (5.65)$$

where, the components of the linear, quadratic, and cubic stiffness matrices along with the elec-

trostatic force and torque for the n^{th} bending and torsional modes are defined as below:

$$\begin{aligned}
K_{ni}^{(1)} &= \int_0^L \Psi_n \Psi_i dx, \quad K_{ni}^{l(2)} = \int_0^L \Psi'_n \Psi_i''' dx, \quad K_{nijk}^{(1)} = \int_0^L \Psi_n \Psi'_i \Psi'_j \Psi_k''' dx \\
K_{nijk}^{(2)} &= \int_0^L \Psi_n \Psi'_i \Psi_j'' \Psi_k''' dx, \quad K_{nijk}^{(3)} = \int_0^L \Psi_n \Psi_i'' \Psi_j'' \Psi_k'' dx \\
K_{nijk}^{(4)} &= \left(\int_0^L \Psi_n \Psi_i''' dx \right) \left(\int_0^L \Psi'_j \Psi'_k ds \right) + \left(\int_0^L \Psi_n \Psi_i'' dx \right) \left(\int_0^L \Psi'_j \Psi_k''' dx \right) \\
K_{nijk}^{ms} &= \left(\int_0^L \Psi_n \Psi_i'' dx \right) \left(\int_0^L \Psi'_j \Psi'_k ds \right) \\
Kt_{ni}^{(1)} &= \int_0^L \Phi_n \Phi_i dx, \quad Kt_{ni}^{(2)} = \int_0^L \Phi'_n \Phi'_i ds, \quad Kt_{nij}^q = \int_0^L \Phi_n \Phi_i \Psi_j'' dx
\end{aligned}$$

$$\begin{aligned}
F_n(t) &= \alpha_e \left(\frac{Q}{C_t} \right)^2 \begin{cases} \int_0^L \Psi_n(x) \int_{-\frac{b}{2}}^{\gamma^+} \frac{1}{d^2(\zeta)} d\zeta dx & 0 \leq \phi < 2\tan^{-1}\left(\frac{b}{h}\right) \\ \int_0^L \Psi_n(x) \int_{\gamma^-}^{\frac{b}{2}} \frac{1}{d^2(\zeta)} d\zeta dx & -2\tan^{-1}\left(\frac{b}{h}\right) < \phi \leq 0 \end{cases} \\
&+ \alpha_1 \frac{\alpha_{ep}}{\alpha_2} \left(\frac{Q}{C_t} \right)^2 \Psi_n(L) \begin{cases} \int_{-\frac{b}{2}}^{\gamma^+} \frac{1}{d^2(\zeta)} d\zeta & 0 \leq \phi < 2\tan^{-1}\left(\frac{b}{h_p}\right) \\ \int_{\gamma^-}^{\frac{b}{2}} \frac{1}{d^2(\zeta)} d\zeta & -2\tan^{-1}\left(\frac{b}{h_p}\right) < \phi \leq 0 \end{cases} \\
F_n^t(t) &= -3\beta_e \left(\frac{Q}{C_t} \right)^2 \begin{cases} \int_0^L \Phi_n(x) \int_{-\frac{b}{2}}^{\gamma^+} \frac{r \sin \chi}{d^2(\zeta)} d\zeta dx & 0 \leq \phi < 2\tan^{-1}\left(\frac{b}{h}\right) \\ \int_0^L \Phi_n(x) \int_{\gamma^-}^{\frac{b}{2}} \frac{r \sin \chi}{d^2(\zeta)} d\zeta dx & -2\tan^{-1}\left(\frac{b}{h}\right) < \phi \leq 0 \end{cases} \\
&- \beta_1 \frac{6\beta_{ep}}{\beta_2} \left(\frac{Q}{C_t} \right)^2 \Phi_n(L) \begin{cases} \int_{-\frac{b}{2}}^{\gamma^+} \frac{r \sin \chi}{d^2(\zeta)} d\zeta & 0 \leq \phi < 2\tan^{-1}\left(\frac{b}{h_p}\right) \\ \int_{\gamma^-}^{\frac{b}{2}} \frac{r \sin \chi}{d^2(\zeta)} d\zeta & -2\tan^{-1}\left(\frac{b}{h_p}\right) < \phi \leq 0 \end{cases}
\end{aligned}$$

Those equations coupled with Eq. (5.60) form a system of equations that describes the electromechanical dynamics of the mirror. They were then converted into $2n$ state-space form and numerically integrating using Runge-Kutta method. A combination of the shooting and arc-length continuation schemes were scripted in Matlab to capture the periodic orbits of the system. The stability of the periodic solutions was assessed via analyzing the eigenvalues of the system's monodromy matrix. In the following, we study the efficacy of the proposed actuation technique

via examining the frequency and force response behaviors of the mirror.

5.6 Conclusions

In cases where there is a 2:1 commensurate ratio between the natural frequencies of a micromirror's bending and torsional modes, the torsional mode can be indirectly excited through initiation of a 2:1 autoparametric modal interaction. We demonstrated that the channel necessary for the exchange of energy between the bending and torsional motions is established through the presence of the quadratic coupling terms, originating from a non-zero first moment of area, a consequence of cross-sectional asymmetry in homogeneous microbeams. Those terms identically vanish were geometric imperfections to be ignored. For the first time, we unravel the fact that the quadratic stiffness term, essential for the triggering of the 2:1 autoparametric modal interaction between the in-plane bending and torsional motions, arise from the existence of those non-zero first-moments of area in the support microbeams due to imperfections. This finding highlights the importance of fabrication imperfections frequently encountered in micro-scale devices.

We employed a local displacement field to expand the strain potential energy of the deformed structure, encompassing nonlinear curvatures, and the previously unexplored impact of linearizing the slope of the neutral axis in its deformed configuration. Our approach demonstrates that incorporating the linear slope into the cross-sectional shear strains efficiently accounts for quadratic coupling between the bending and torsional motions whose existence depends on non-vanishing cross-sectional first moments of area. We account for imperfections in the microdevice fabrication, by introducing small perturbations in the 'as-fabricated' centroid of the microbeam cross-section away from its as-design position.

Chapter 6

Dynamic Response: Simulation and Experimental Results

We demonstrate the impact of geometric imperfections on the inception of the 2:1 modal interaction between a bending and a torsional mode in symmetrically designed electrostatic MEMS.

6.1 Bending-torsional modal interaction

6.1.1 Simulation results

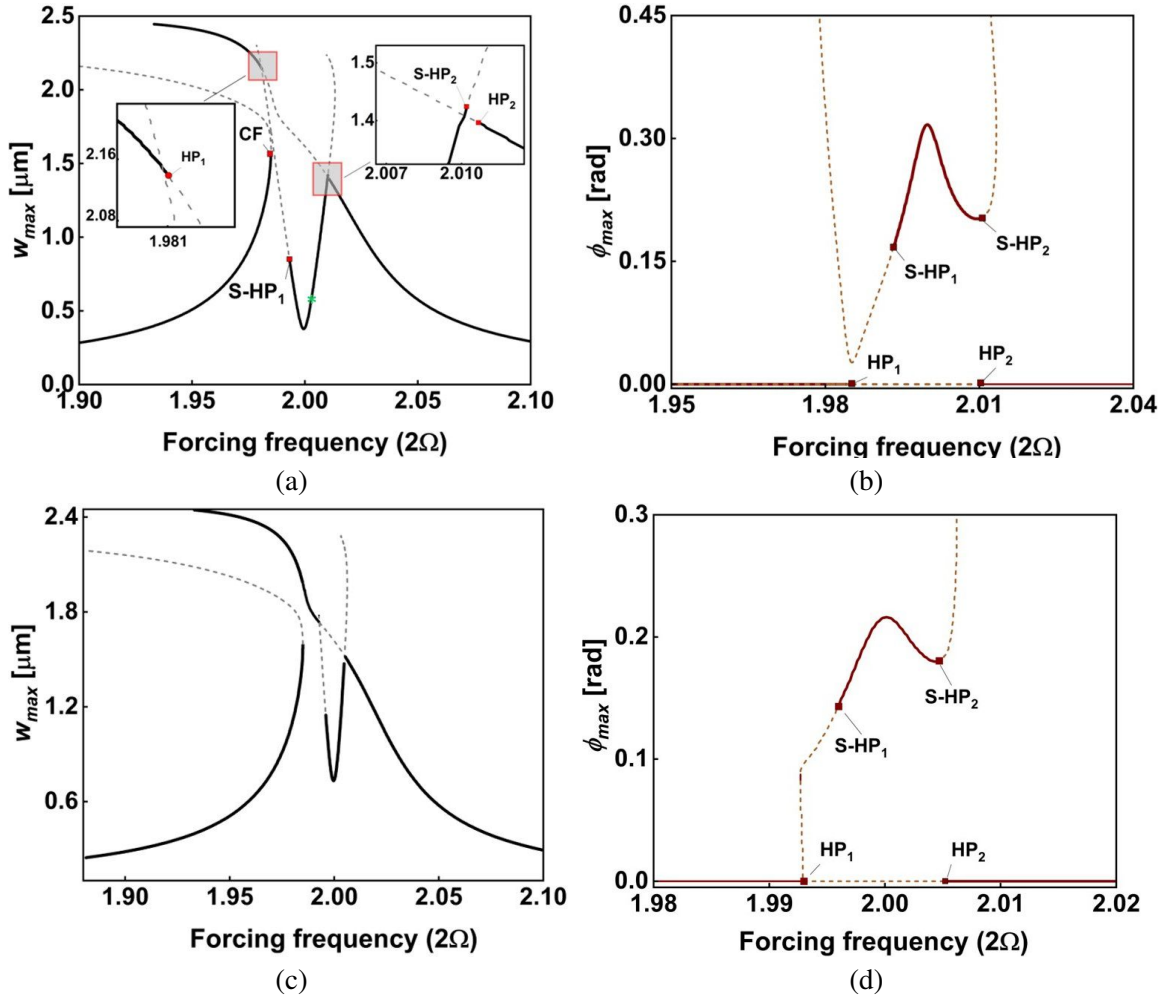
The frequency-response curves of the electrostatic micromirror in-plane displacement and rotation angle under an unbiased voltage amplitude of $V_{AC} = 88 \text{ V}$ for three values of z_0 are shown in figure 6.1. The bending and torsional viscous damping parameters are set to be $c_w = 0.042$ and $c_\phi = 0.003$, respectively. To excite primary resonance of the first in-plane bending mode, the frequency of the forcing component F_2 was swept in the vicinity of its natural frequency ($2\Omega = \omega_1$).

Figures 6.1(a) and (b) with z_0 set at 3.5% of the beam's thickness, $z_0 = 0.035 h$. Figure 6.1a illustrates that the amplitude of the bending motion enhances along a small-amplitude branch

until it reaches a cyclic-fold (CF) bifurcation at $2\Omega = 1.985$ as the AC frequency grows from lower frequencies. However, the torsional motion extends along the trivial branch. By backward sweeping the AC frequency at this turning point, the stability of the MEMS is lost, and the bending motion transitions to a large-amplitude branch, whereas the torsional motion remains trivial. Setting the forcing frequency at $2\Omega = 2.10$, and performing a backward sweep of the parameter, the in-plane motion's amplitude gradually increases as the forcing frequency approaches the natural frequency of the bending mode. However, the torsional motion remains trivial. This trend continues until reaching $2\Omega = 2.0107$. Further decreasing the forcing frequency, the response undergoes a primary Hopf (HP_2) bifurcation, leading to instability in the system's response. Continuing backward along this branch, the unstable response observes a reverse primary Hopf (HP_1) bifurcation at $2\Omega = 1.9811$, retrieving stability on the large-amplitude branch. The periodic orbits of the bending motion continue to expand along this branch until dynamic pull-in occurs. Up to this point, the MEMS's frequency-response behavior is not contributed by modal interaction between the bending and the torsional modes. To observe the formation of an energy channel between the two motions, we set the force frequency at $2\Omega = 2.003$ and perform a single run using the long-time integration method, for zero initial conditions. With a sufficiently large AC voltage amplitude ($V_{AC} = 88$ V), which exceeds the activation threshold for a 2:1 modal interaction between the in-plane bending and torsional modes, the steady-state dynamic response of the MEMS is attracted to a stable periodic orbit that coexists with the previously discussed large-amplitude unstable branch. This solution, marked by a green star in figure 6.1a, is contributed by a non-trivial torsional motion, initiated through an energy exchange between this motion and the in-plane bending mode. Taking this stable response as a starting point, we continued the MEMS dynamics along the 2:1 IR-branch (branch on which the bending and the torsional motions interact) via backward sweeping the AC frequency. The amplitude of the bending motion decreases to a minimum, while the amplitude of the torsional motion increases to a maximum, occurring at $2\Omega = 1.9994$. Further decreasing the AC frequency, the MEMS surpasses its extrema, with a notable increase in the amplitude of the bending motion and a corresponding decrease in the amplitude of the torsional motion and it continues until the system's dynamics experience a secondary Hopf ($S-HP_1$) at $2\Omega = 1.9933$. At this point, the MEMS response transitions to a quasi-periodic

regime, leading to the formation of local frequency combs centered around the forcing frequency and its higher harmonics. This will be discussed in the following. To capture the right-half of the 2:1 IR-branch, we return back to the starting point and continue the MEMS dynamics via forward sweeping the AC frequency. Likewise, the amplitude of the periodic orbits for bending motion expands, while the torsional motion's amplitude diminishes when transitioning to higher forcing frequencies. The maximum rotation angle of the plate happens at the frequency at which the in-plane bending motion is at its lowest level. In essence, the in-plane transverse and torsional motions exhibit contrasting behaviors, signifying that as one amplitude increases, the other decreases, while the MEMS operates within the 2:1 modal interaction regime. Figures 6.1c and 6.1d depict the frequency-response behaviors of the MEMS with a reduced misalignment in the position of the cross-sectional centroid, $z_0 = 0.025 h$. The results indicate that the overall dynamics remain consistent with the previous case, albeit with the distinction that the large amplitude unstable branch and the frequency range where modal interaction occurs are shortened. Further decreasing the value of z_0 to $0.01h$ results in the disappearance of the M-shape frequency-response curve for the bending motion, figure 6.1e. This leads to the conclusion that increased symmetry in the beam cross-section corresponds to a higher activation level for the 2:1 modal interaction between the flexural and torsional motions. Consequently, in symmetrically designed electrostatic MEMS, with extremely small offsets in the position of the cross-sectional centroid, the microstructure undergoes pull-in before the triggering of torsional motion through a 2:1 internal resonance between the torsional and in-plane bending modes.

Figure 6.2 illustrates the electrostatic MEMS dynamic responses for three different values of the transverse viscous damping coefficient and AC voltage amplitude, 1) $c_w = 0.042$, $V_{AC} = 88 V$, 2) $c_w = 0.032$, $V_{AC} = 77 V$, and 3) $c_w = 0.022$, $V_{AC} = 65 V$, for $z_0 = 0.035 h$. Here, the torsional viscous damping parameter remains as previous, $c_\phi = 0.003$. As seen, decreasing the damping parameter from 0.042 to 0.032 advances the activation level of the internal resonance, reducing the AC voltage amplitude by 10 V. This prevents the occurrence of cyclic-fold bifurcation and the emergence of a large-amplitude unstable branch following it. Moreover, the merging of stable periodic solutions from the left non-IR-branch with the first secondary Hopf-bifurcation (S-HP₁) located on the 2:1 IR-branch leads to the onset of period-doubling bi-



furcation, PD. Here, the MEMS's response along the 2:1 IR-branch until reaching the forcing frequency at which primary Hopf bifurcation (HP) occurs is entirely characterized by stable periodic orbits. Nevertheless, with a further reduction in the damping parameter to 0.022 and a corresponding decrease in the waveform amplitude to 65 V, the system displays dynamics similar to those depicted in figure 6.2a. Comparing figures 6.2a and 6.2e reveals that decreasing the damping parameter and AC voltage amplitude avoids the MEMS from undergoing cyclic-fold bifurcation; expands the large-amplitude unstable branch situated between the two primary Hopf bifurcations, HP_1 and HP_2 ; leads the left stable branch of the M-shape frequency-response curve

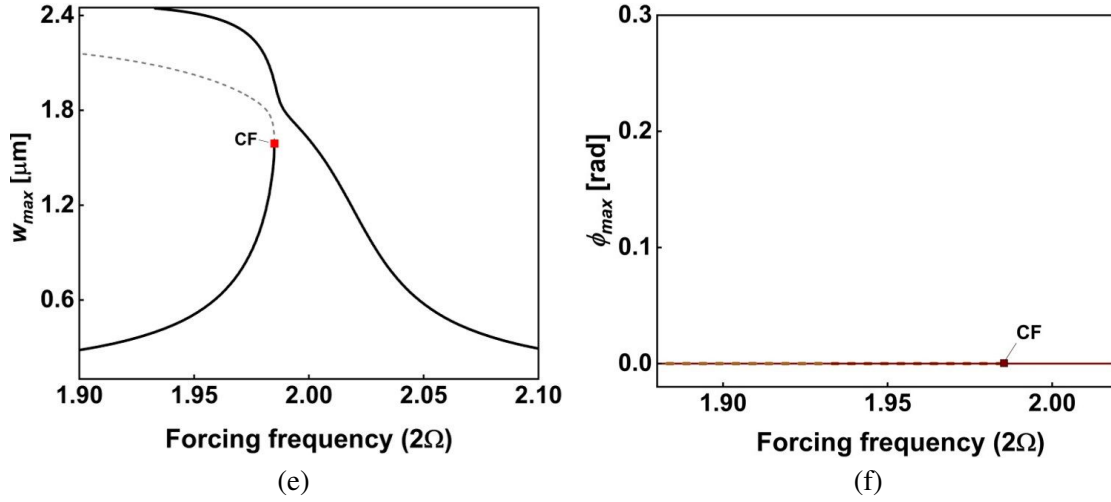
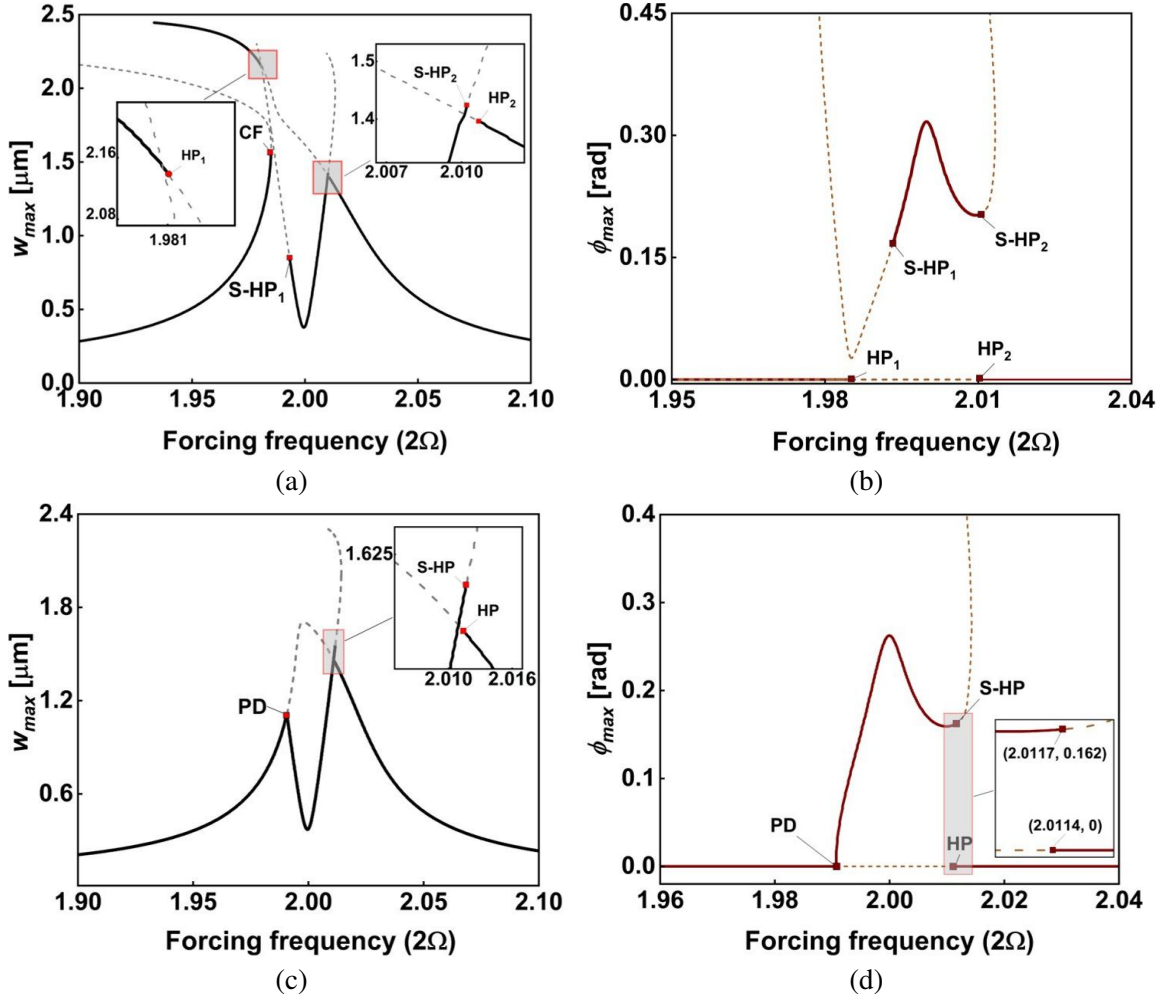


Figure 6.1: The frequency-response curves of the electrostatic micromirror (a), (c), and (e) in-plane displacement and (b), (d), and (f) rotation angle under voltage waveform with an amplitude of $V_{AC} = 88 \text{ V}$ in the vicinity of primary resonance of the first in-plane bending mode, for three values of the cross-sectional imperfection (a) and (b) $z_0 = 0.035 h$, (c) and (d) $z_0 = 0.025 h$, and (e) and (f) $z_0 = 0.01 h$, respectively.

to bifurcate at $2\Omega = 1.9938$ under a period-doubling bifurcation, PD; and confines the quasi-periodic regime to a frequency range determined by a secondary Hopf bifurcation, S-HP, and the right peak of the M-shaped curve where a primary Hopf bifurcation (HP₂) occurs on the right large-amplitude solution branch. Referring to figures 6.2d and 6.2f, it is evident that with lower values of the linear damping parameter and AC voltage load, the maximum rotation angle of the plate decreases. However, it allows the plate to undergo either exclusively non-trivial stable rotations or primarily non-trivial stable rotations accompanied by quasi-periodicity, confined within a narrow frequency range along the 2:1 IR-branch. However, this scenario differs in figure 6.2b, where a relatively broad frequency range (between HP₁ and S-HP₁) exists, in which the system's motions are determined to be quasi-periodic. Therefore, it is concluded that to enhance stability in the MEMS operating under the 2:1 modal interaction, we need to set a balance between the damping parameter and the applied voltage amplitude.

A similar analysis is performed for the case where the overall offset between the centroid of



the actual cross-section and that of the ideal one is set to be 2.5% of the microbeam's thickness, $z_0 = 0.025 h$. Figure 6.3 shows that this minuscule imperfection in position of the cross-section centroid, preventing the first moment of area of the microbeam's cross-section from vanishing (non-zero coefficients for the necessary quadratic nonlinearity coupling the bending and torsional motions), is sufficient to instigate the creation of an energy channel bridging the in-plane bending and torsional motions. The overall dynamics of the system is replicated for the same parameters' values, with the difference that the frequency range of the 2:1 IR-branch becomes more restricted for enhanced symmetry in the microbeam's cross-section.

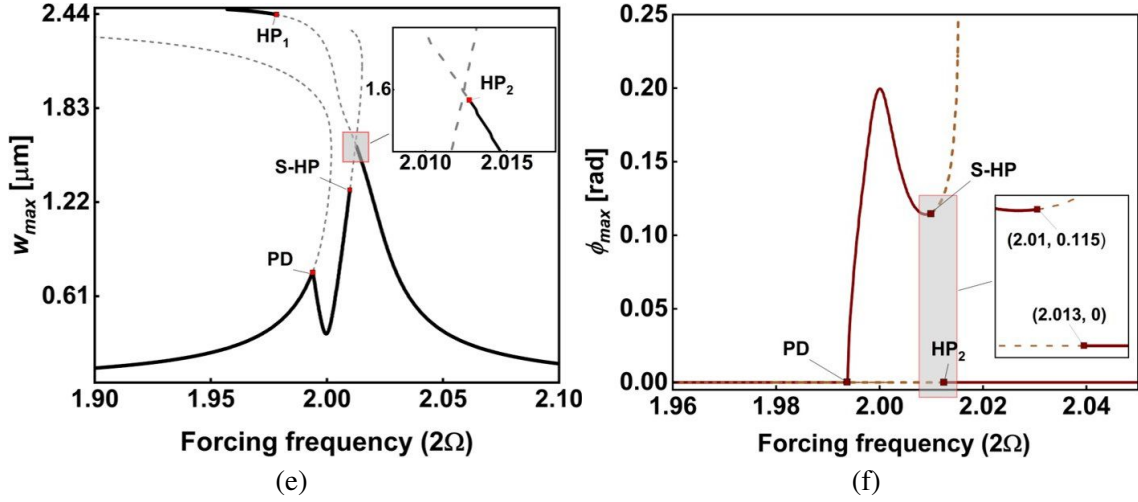
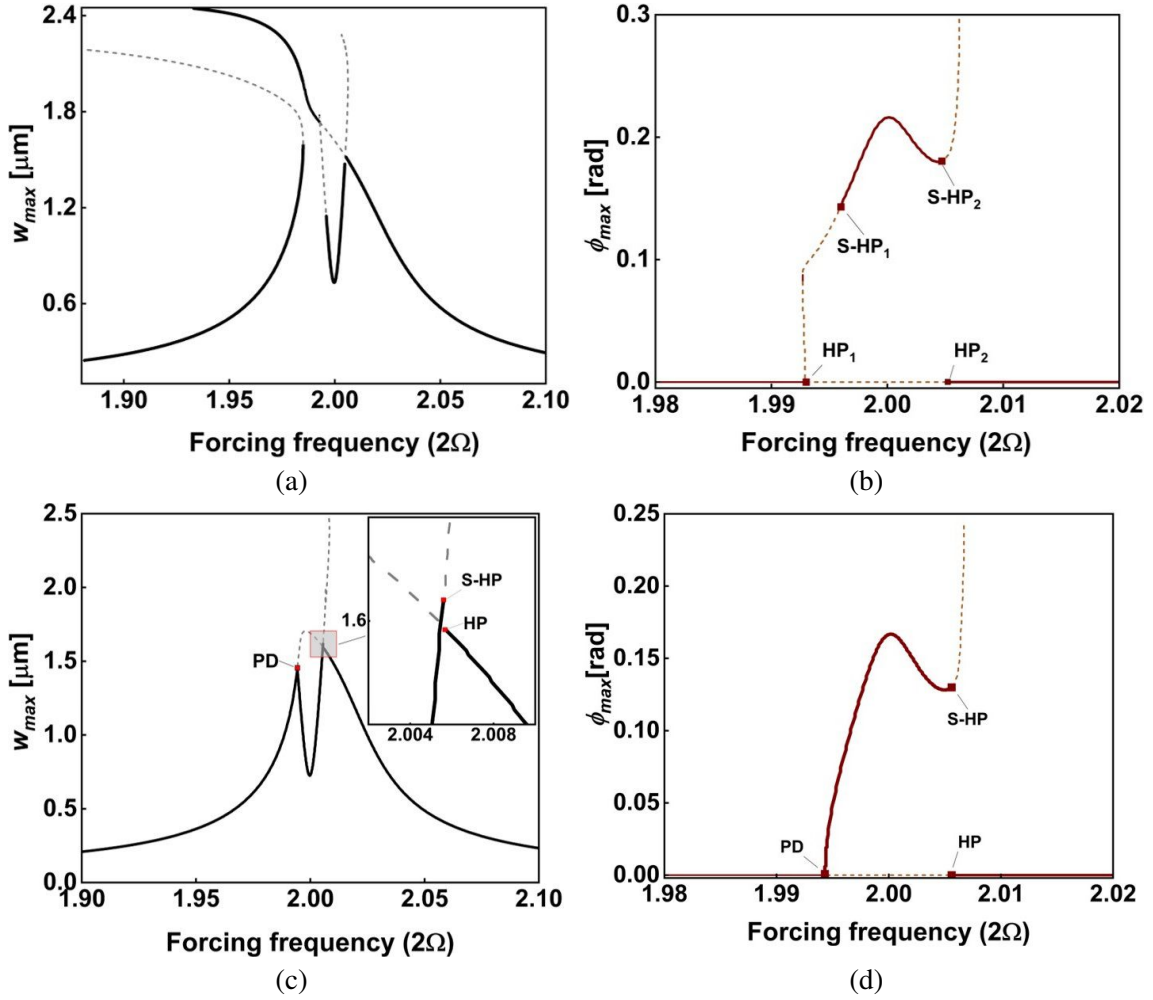


Figure 6.2: The mirror's frequency-displacement behaviors for different values of the transverse viscous damping, for $z_0 = 0.035 h$ and $c_\phi = 0.003$. The figure illustrates the frequency-response curves for the in-plane displacement of the microbeam's tip, for: (a) $c_w = 0.042$ and $V_{AC} = 88 V$, (c) $c_w = 0.032$ and $V_{AC} = 77 V$, and (e) $c_w = 0.022$ and $V_{AC} = 65 V$. The amplitudes of the rotational motion at the tip of the microbeam versus the forcing frequency are shown in (b), (d) and (f) corresponding to the parameter sets defined in (a), (c), and (e).

The steady-state dynamics of the MEMS is examined via freezing the AC voltage frequency at $\Omega = 1.0002$, for the following parameters: $z_0 = 0.035 h$, $c_w = 0.042, 0.032$, and $c_\phi = 0.003$, and are illustrated in figure 6.4. This allows us to capture the voltage-displacement behavior for the bending and torsional motions of the electrostatic actuator. Increasing the amplitude of the AC waveform from zero to $V_{AC} = 52.15 V$, figure 6.4a, the in-plane bending motion gradually grows while the torsional motion remains trivial. Increasing AC voltage load further leads to the branching of the dynamic response under a primary Hopf bifurcation, resulting in the doubling of the period of the actuator's in-plane response. The 2:1 modal interaction between the in-plane bending and torsional motions initiates at this point. Amplifying the voltage beyond the bifurcation point, the amplitude of the stable bending motion reaches a nearly saturated state; however, the amplitude of the torsional motion starts to increase. The findings indicate that applying an AC voltage of 70 V induces in-plane motion with an amplitude less than $0.5 \mu m$ and torsional motion with 0.216 radian. This analysis demonstrates the efficiency of the proposed



actuation mechanism using indirect excitation, channeling the vibration energy from the bending mode to the torsional mode. Figure 6.4b displays the voltage-displacement curve for $c_w = 0.032$. It shows that reducing the damping parameter advances the onset of modal interaction, while maintaining the overall dynamics of the system unchanged.

Figure 6.5 explores the influence of the cross-sectional imperfection parameter, z_0 , which introduces non-zero first moments of area for the suspension microbeams, subsequently, giving rise to quadratic nonlinear terms between the in-plane bending and torsional motions, on the MEMS voltage-displacement behavior. The AC frequency and the damping parameters are set

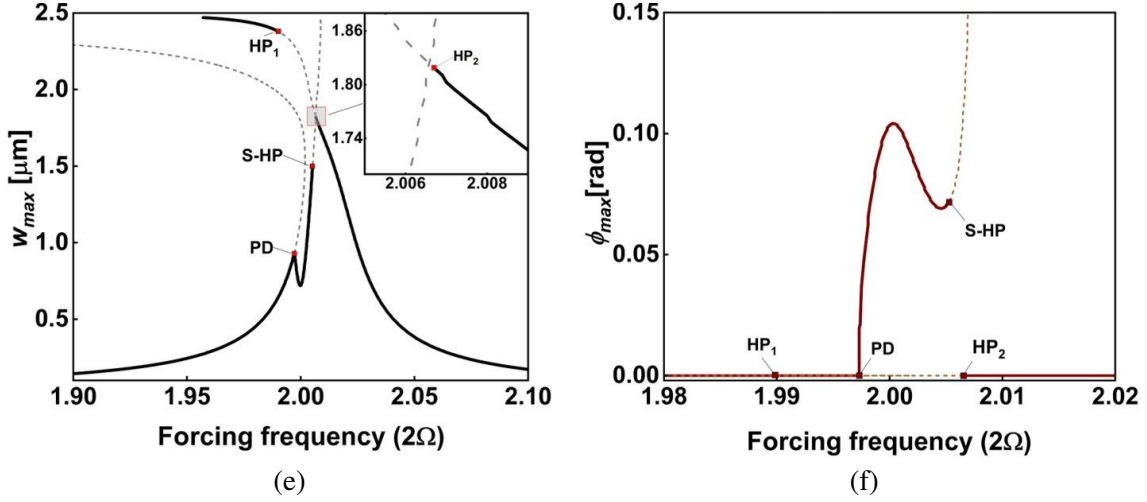


Figure 6.3: The influence of the transverse viscous damping on the frequency-displacement behavior of the MEMS, for $z_0 = 0.025 h$ and $c_\phi = 0.003$. The figure illustrates the frequency-response curves for the in-plane displacement of the microbeam's tip, for: (a) $c_w = 0.042$ and $V_{AC} = 88 V$, (c) $c_w = 0.032$ and $V_{AC} = 77 V$, and (e) $c_w = 0.022$ and $V_{AC} = 65 V$. Correspondingly, the amplitudes of the rotational motion at the tip of the microbeam versus the forcing frequency are shown in (b), (d) and (f).

to be as previous. The results reveal that increased symmetry, reduced imperfection, in the microbeam cross-section retards the development of energy exchange between the flexural and torsional motions of the microsuspensions. Reducing the z-component of the overall offset of the cross-sectional centroid from 3.5% to 2.5% of the microbeam's thickness delays the inception of modal interaction by about 20 V, figure 6.5a. The dashed lines in the figure represent the unstable periodic solutions where the two motions are not engaged. Those solutions correspond to the trivial unstable branches for the torsional motion as depicted in figure 6.5b.

In the following, we aim to demonstrate that the voltage value required for the activation of the 2:1 flexural-torsional modal interaction in the proposed electrostatic MEMS can be reduced substantially via exploiting electro-mechanical resonance syntonization. To this, the inductance of the electrical circuit is tuned so that the electrical resonant frequency of the RLC circuit is set to match with half of the natural frequency of the first in-plane bending mode of the mechanical structure. This amplifies the voltage amplitude across the MEMS capacitance while the system

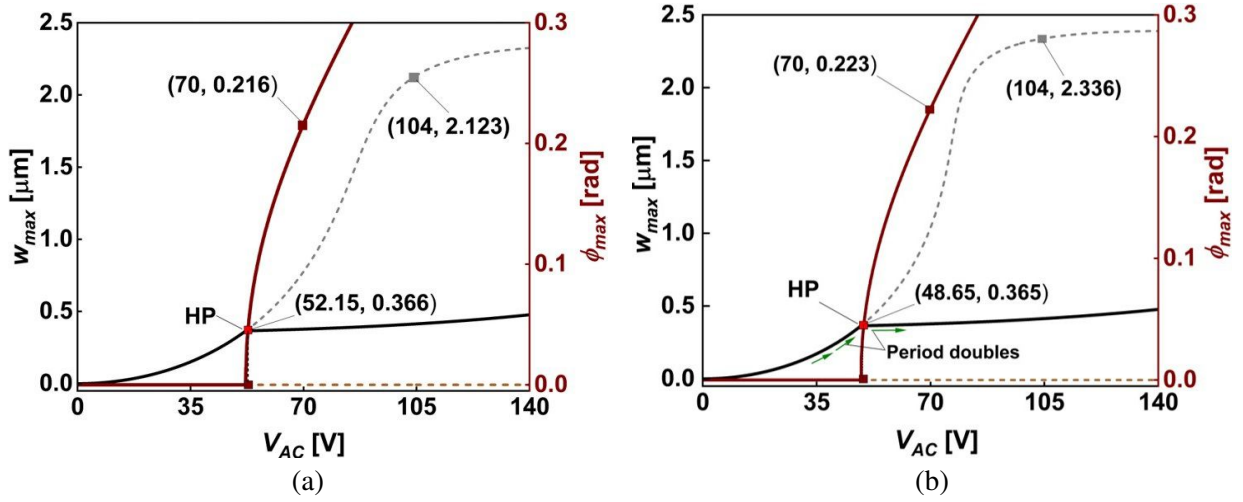


Figure 6.4: The influence of the transverse viscous damping coefficient on the voltage-displacement behavior of the MEMS. The overall offset in position of the centroid of the beam's cross-section is assumed to be $z_0 = 0.035 h$, for $\Omega = 1.0002$ and $c_\phi = 0.003$. The left and right vertical axes refer to the black and brown curves, respectively, expressing the amplitudes of the plate's in-plane transverse and rotational displacements at the tip of the microbeam. The in-plane damping parameter is (a) $c_w = 0.042$, and (b) $c_w = 0.032$.

operates under unbiased waveforms. For unbiased voltage signals, AC frequency should be set to half of the natural frequency of a mechanical mode to directly excite this mode, $\Omega \approx \frac{1}{2}\omega_b^1$. And because the electrical resonant frequency is also set at half of the natural frequency of the mechanical mode of interest, $\omega_e \approx \frac{1}{2}\omega_b^1$, the electrical charge circulating within the RLC circuit resonates due to the nearness of the supplied AC voltage's frequency to the electrical resonance, $\Omega \approx \omega_e$. The electrical charge resonance results in an amplification of voltage observed across the MEMS capacitance, thereby intensifying the electrostatic force applied to the microstructure. It is worth noting that the electrostatic force and torque are at $2\Omega \approx \omega_b^1$. So, the electrostatic field amplification leads to an amplification in the system's response (bending motion) happening at 2Ω . Up to this point, the MEMS has undergone a 1:2 electro-mechanical modal interaction, where the electrical mode is directly excited using an unbiased voltage signal, and the mechanical (flexural) motion is indirectly excited through an energy channel formed by the quadratic non-linearity associated with electrical charge present in the dynamics of the mechanical side. This

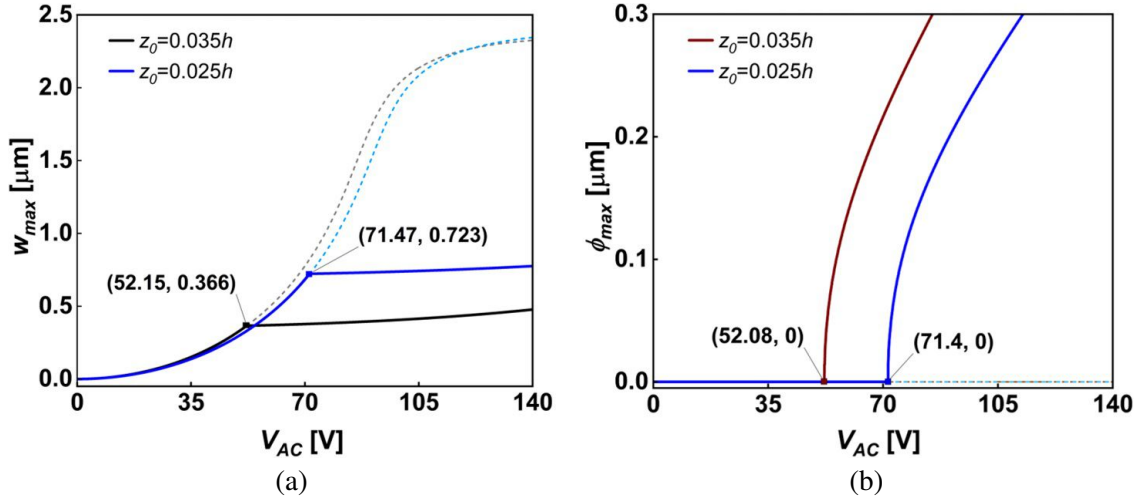


Figure 6.5: The influence of the overall offset in position of the centroid of the microbeam's cross-section on the voltage-displacement of the MEMS. The system's parameters are as follows: $c_w = 0.042$, $c_\phi = 0.003$, and $\Omega = 1.0002$. (a) illustrates the amplitude of the in-plane transverse displacement at the microbeam's tip, for $z_0 = 0.035 h$ (black curve) and $z_0 = 0.025 h$ (blue curve). (b) depicts the amplitude of the plate's rotational motion, for $z_0 = 0.035 h$ (brown curve) and $z_0 = 0.025 h$ (blue curve).

channel supplies an auto-parametric excitation for the mechanical motion, occurring at twice the AC frequency. On the other hand, the bending motion resonates because the electrostatic force frequency is located near the natural frequency of the first in-plane flexural mode. This implies that there are two factors contributing to the amplification of the mechanical in-plane bending motion: one is due to resonance in the applied electrostatic force intensity, and the other is due to the proximity of the electrostatic force frequency to the mechanical mode's natural frequency. This technique enables us to effectively bridge the in-plane flexural motion of the microbeams to their torsional motion via supplying lower voltages. Conclusively, the proposed actuation technique for electrostatic MEMS actuators features 1:2:1 electro-flexural-torsional (one electrical mode incorporated with two mechanical modes) modal interactions upon activating the energy channel between the bending and torsional motions of the movable electrode. The steady-state dynamics of the MEMS actuator in the presence of electro-mechanical resonance syntonization is illustrated in figure 6.6. Here, the system's parameters are as follows: $z_0 = 0.035 h$, $c_w = 0.042$,

$c_\phi = 0.003$, $V_{AC} = 10$ V. The bending motion is characterized by a butterfly-shaped frequency-displacement curve, in which the left and right peaks of the curve are formed by the electrical and mechanical bending modes, respectively. The V-shaped stable branch features the flexural-torsional modal interaction. Sweeping the AC frequency from below half of the bending mode's natural frequency, the in-plane motion grows without the presence of torsional motion. This trend persists until the MEMS dynamics meet a cyclic-fold bifurcation (CF_1) at $2\Omega = 1.8809$, leading to instability in the response. Turning the sweep direction at this point, the system's dynamics develop along an unstable large-amplitude periodic orbit for the bending motion, whereas the torsional mode still remains trivial. Continuing this solution branch leads the system to experience the second cyclic-fold bifurcation (CF_2) happening at $2\Omega = 1.7877$, forming the left peak of the M-shaped curve. Turning the sweep direction from backward to forward enables the capture of the large-amplitude periodic response for the in-plane transverse motion. Moving along this branch, the amplitude of the bending motion decreases gradually via increasing the AC frequency until the forcing frequency reaches $2\Omega = 1.9918$. At this point, the dynamics of the electro-flexural system undergo a secondary Hopf bifurcation, denoted by $S-HP_1$, provided that the periodic motions become unstable. Increasing the AC frequency further propels the MEMS into a quasi-periodic regime that is incorporated with only the electrical and flexural modes. However, the frequency interval associated with this behavior is extremely narrow so that slightly growing the sweeping parameter causes the system's response to exhibit branching under a period-doubling bifurcation (PD_1) at $2\Omega = 1.9935$. It is worth noting that there is no stable periodic orbit co-existing with the quasi-periodic response within this frequency range, between $S-HP_1$ and PD_1 . The initiation of torsional motion along the unstable non-trivial branch is observed in figure 6.6b when the forcing frequency surpasses PD_1 . It should be noted that the solutions obtained along the V-shaped branch for the bending (shown in figure 6.6a), and the reversed V-shaped branch for the torsional and electrical motions (shown in figures 6.6b and 6.6c), are outcomes of a 1:2:1 electro-flexural-torsional modal interactions happening in the dynamics of the electrostatic MEMS actuator. Continuing along this branch, the amplitude of the in-plane transverse motion starts to decrease, while correspondingly, the torsional motion grows rapidly until the steady-state response undergoes the second secondary Hopf bifurcation, as de-

noted by S-HP₂ at $2\Omega = 1.9936$, recovering stability in the response. Further increasing the parameter, the bending motion amplitude drops to a minimum, while the plate's rotation angle reaches its maximum at $2\Omega = 1.9984$. Beyond this frequency, the torsional motion initiates a reduction once again until the stability of the MEMS dynamics is lost under the third secondary Hopf bifurcation, S-HP₃, at $2\Omega = 2.0051$. Continuing the parameter sweep beyond this point, the dynamic response transitions to a quasi-periodic regime involving all three modes, electrical, flexural, and torsional. This situation resembles the behavior observed at the beginning of constructing this solution branch. This behavior extends until the unstable non-trivial solutions to the torsional motion merge its unstable trivial solutions under a reversed period-doubling (PD₂) bifurcation at $2\Omega = 2.0059$. Further increasing the AC frequency, the torsional motion remains vanishing; however, the unstable electrical and bending periodic motions retrieve their stability along the large-amplitude branch under the fourth secondary Hopf (S-HP₄) bifurcation occurring at $2\Omega = 2.0381$.

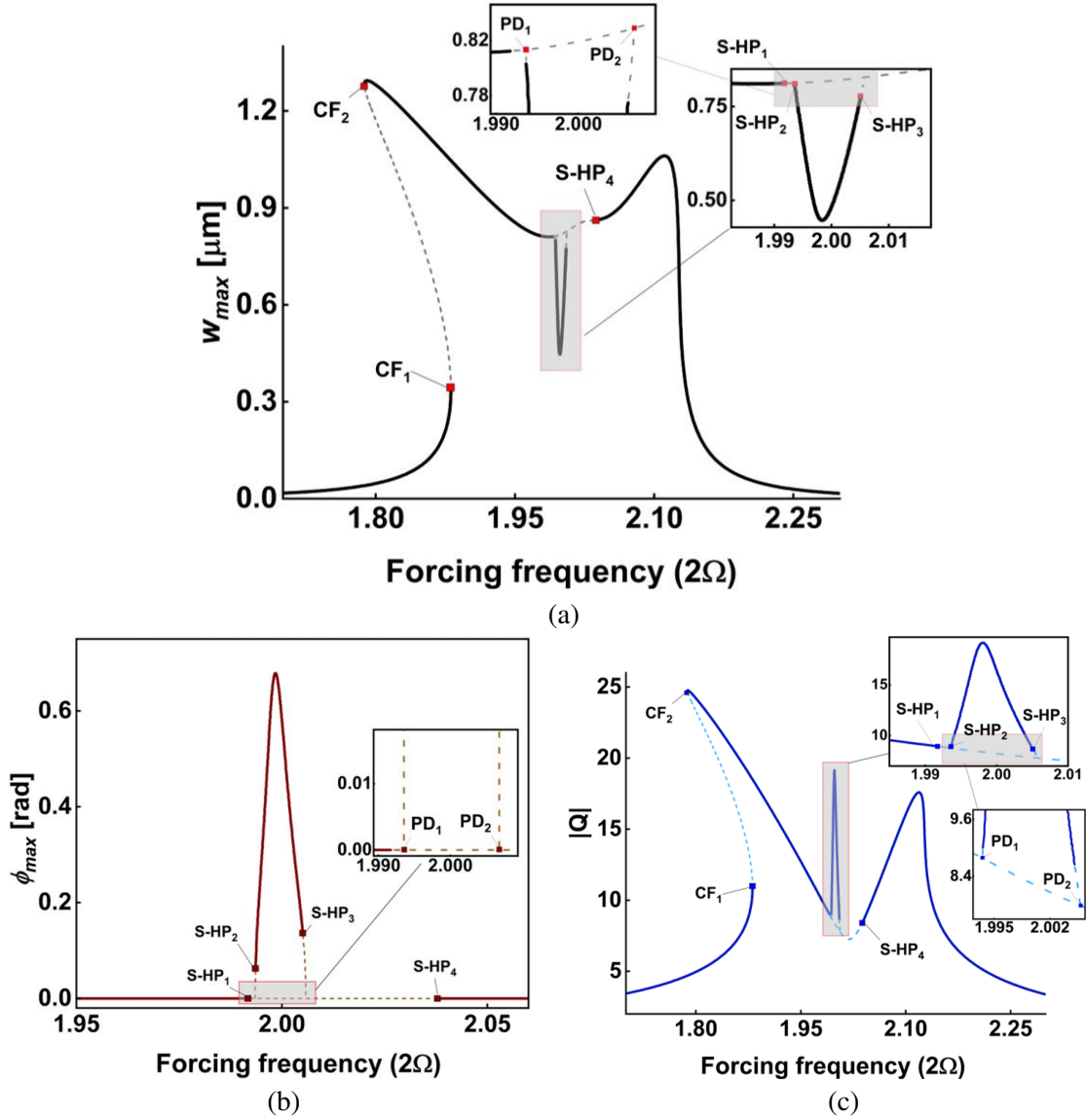
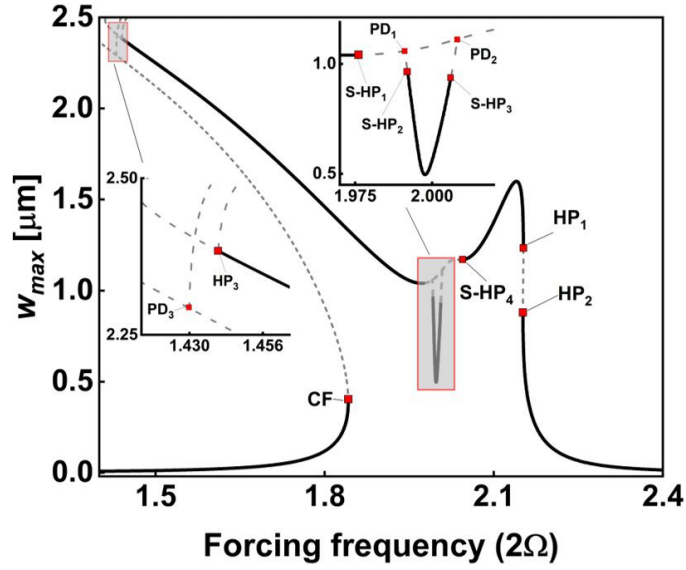
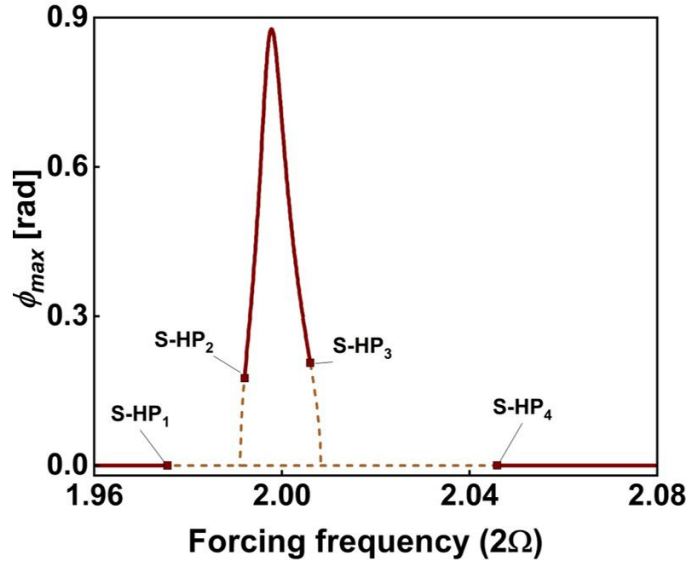


Figure 6.6: The steady-state dynamics of the MEMS in the presence of electro-mechanical resonance syntonization, for $z_0 = 0.035 h$, $c_w = 0.042$, $c_\phi = 0.003$, $V_{AC} = 10 V$. Here, the electrical resonant frequency is located at half of the natural frequency of the first in-plane bending mode, $\omega_e = \frac{1}{2}\omega_b^1$, introducing a 1:2 modal interaction between the electrical and mechanical/bending modes of the MEMS. (a), (b), and (c) depict the frequency-response curves for the plate's transverse and rotational motions and the electrical current, respectively.

Driving the electrostatic MEMS using an unbiased AC waveform with amplitude of $V_{AC} = 15\text{ V}$ keeping the cross-sectional imperfection and damping parameters identical to the previous case, provides the system with richer dynamics, figure 6.7. The overall steady-state behavior of the actuator is similar to what was presented in figure 6.6; however, the main distinctions are as the following: 1) the second cyclic-fold bifurcation forming the left peak of the M-shaped frequency-response curve for the bending motion (shown in figure 6.6) disappears in figure 6.7, and instead, the large-amplitude unstable solution branch continues until dynamic pull-in occurs. In addition, the large-amplitude stable motions near pull-in undergo a primary Hopf bifurcation (HP_3) at $2\Omega = 1.4403$. 2) Traversing the stable large-amplitude solution branch and extending the forcing frequency beyond the right peak of the M-shaped curve lead to the consecutive occurrence of two primary Hopf bifurcations, HP_1 and HP_2 , where the former introduces instability, while the later restores stability in the system dynamics. Upon comparing figures 6.6 and 6.7, a noteworthy finding is that, despite a 5 V increase in AC voltage leads to a roughly $0.3\text{ }\mu\text{m}$ growth in the large-amplitude transverse motion, the amplitude of the local minima for the in-plane bending motion, observed within the frequency range activating electro-flexural-torsional modal interactions, remains nearly constant. This implies saturation in the in-plane motion and enhancement in the rotation of the microbeam's tip.



(a)



(b)

Figure 6.7: The steady-state dynamics of the MEMS in the presence of electro-mechanical resonance syntonization, for $z_0 = 0.035 h$, $c_w = 0.042$, $c_\phi = 0.003$, $V_{AC} = 15 V$. The electrical resonant frequency is adjusted at half of the shifted natural frequency of the first in-plane bending mode, $\omega_e = \frac{1}{2}\omega_b^1$. (a) illustrate the frequency-response curve for the in-plane bending motion of the microbeam's tip, and (b) depicts the frequency-rotational displacement curve for the rigid plate.

Figure 6.8 illustrates the voltage-response behaviors of the electrostatic MEMS while the electrical resonant frequency is syntonized with half of the natural frequency of the first in-plane bending mode, for $z_0 = 0.035 h$ and $z_0 = 0.025 h$. The AC frequency is freezed at $\Omega = 1.0002$, and the damping parameters are set to be $c_w = 0.042$ and $c_\phi = 0.003$. Figure 6.8a depicts that the periodic orbits associated with the transverse motion extends in amplitude as the applied AC voltage signal intensifies. For the overall offset of 3.5% of the microbeam's thickness in the z -component of the cross-sectional centroid, the MEMS's dynamics loses its stability at $V_{AC} = 3.53 V$ through secondary Hopf bifurcation, as a result of the 1:2:1 electro-flexural-torsional modal interactions (black curve in figure 6.8a). This triggers the plate's rotational motion accompanied by a period-doubling bifurcation in the structure's in-plane transverse motion. However, for $z_0 = 0.025 h$, growth in the in-plane bending motion, without the involvement of the torsional mode, extends until the AC voltage amplitude reaches 8.1 V. Up to this point, the MEMS operates under 1:2 internal resonance between its electrical and bending modes. Further increasing the waveform amplitude propels the system into instability through a secondary Hopf bifurcation, S-HP₁, at this AC voltage. It is important to note that this bifurcation is contributed solely by the electrical and the bending modes of the MEMS. Keep forward sweeping the parameter up to $V_{AC} = 8.58 V$ results in growth and decrease in the amplitudes of the in-plane bending motion and electrical current, respectively. Raising the voltage amplitude beyond 8.58 V results in the appearance of a co-existing unstable solution branch, identified by a doubling of the period in the in-plane bending motion. The period-doubling bifurcation is the result of the torsional mode engagement. The MEMS actuator operates under the 1:2:1 electro-flexural-torsional modal interactions while moving along this unstable branch. Further increasing the voltage amplitude allows the system to retrieve its stability through a reversed secondary Hopf bifurcation, S-HP₂, at $V_{AC} = 8.65 V$. In contrast to the scenario where the electrical mode is not engaged, the system's torsional mode is triggered at voltages one-tenth of the amplitude required to initiate the 2:1 modal interaction between the structure's in-plane bending and torsional motions. As seen, we can indirectly drive the plate's rotational motion through forming an energy channel that transfers the vibration energy from the in-plane bending motion to the torsional motion supplying 3.53 V in the presence of electro-mechanical resonance syntonization. This highlights the efficiency of

the proposed actuation mechanism that eliminates the need for electrodes underneath the plate to induce out-of-plane rotations. Similar to the previous results presented in figure 6.5, the smaller the imperfection parameter, the higher the voltage required to initiate the modal interaction. Moreover, moving along the stable non-trivial branch for torsional motion is accompanied by a gradual increase in the in-plane bending displacement with a small slope. Here, the in-plane motion almost saturates. However, it is noteworthy that the microstructure does not travel more than one-third of the in-plane capacitive gap when the plate rotates out-of-plane with an angle of about 0.3 radian, achieved by applying 15 V for $z_0 = 0.025 h$, Figure 6.8b. The electrical current flowing through the RLC circuit, depicted in figure 6.8c, includes information about the in-plane bending and torsional motions. Therefore, it can be effectively utilized to capture the 1:2:1 electro-flexural-torsional modal interactions via implementing a motion-induced current measurement method.

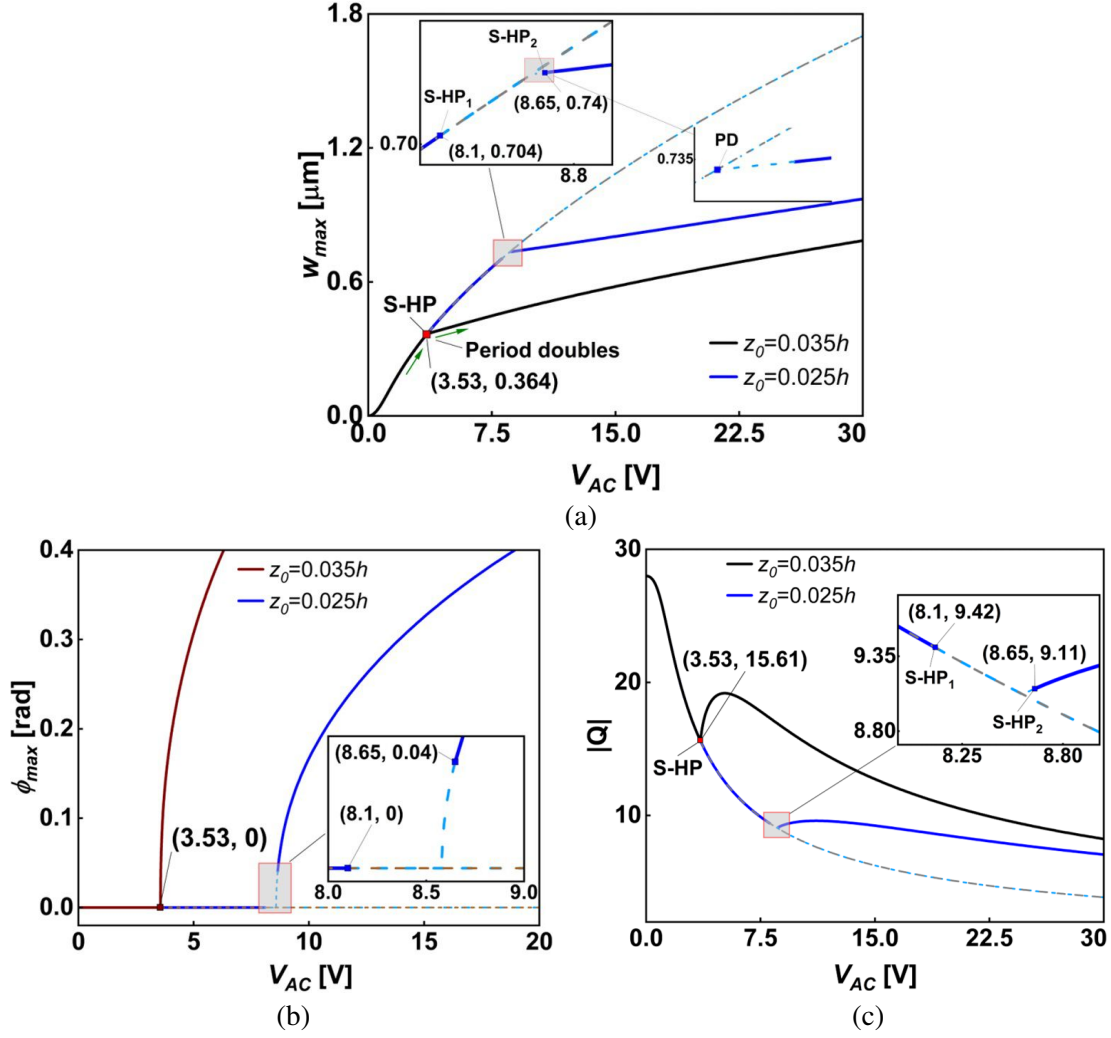


Figure 6.8: The voltage-displacement behaviors of the electrostatic MEMS in the presence of electro-mechanical resonance syntonization, for two different values of the cross-sectional imperfection parameter, $z_0 = 0.035 h$ and $z_0 = 0.025 h$. The AC frequency is freezed at $\Omega = 1.0002$, and the viscous damping parameters are assumed to be $c_w = 0.042$ and $c_\phi = 0.003$. (a), (b), and (c) display the voltage-response curves for the plate's in-plane transverse and rotational displacements and the electrical current circulating through the RLC circuit, respectively.

The time evolution for the in-plane bending and torsional motions of the movable electrode is displayed in figure 6.9, for $z_0 = 0.035 h$, $c_w = 0.042$, $c_\phi = 0.003$, and $V_{AC} = 88 V$. These responses correspond to the frequency-response curves presented in Figures 6.1a and 6.1b. Here, the AC frequency is set to linearly vary near half of the first bending mode's natural frequency with a sweep rate of 1×10^{-7} (non-dimensional frequency per non-dimensional time). It is important to note that, in order to avoid significant reduction in the numerical integration tolerances, we scale up the time variable via non-dimensionalizing it with respect to the first torsional natural frequency of the structure. However, the MEMS's time histories are plotted over the dimensional time. In the figure, the profiles of the time evolutions reflect the frequency-displacement curves presented in figures 6.1a and 6.1b. The MEMS motion begins with periodic in-plane bending oscillations and the amplitude of the steady-state response gradually grows as the AC frequency is swept forward, and correspondingly, the torsional mode of the movable electrode remains vanishing. This behavior trends until reaching the instant $t = 0.8 s$. Beyond this point, the structure's torsional motion is triggered, and the MEMS's dynamics route to a quasi-periodic regime, as indicated by beating responses in its time envelopes, and once again transition to periodicity at $t = 0.9 s$, as the AC frequency is increased further. Those windows are marked by dashed yellow rectangles in Figures 6.9a and 6.9c, and their zoomed views are illustrated in Figures 6.9b and 6.9d, for the bending and torsional motions, respectively. The frequency content of the system's responses at three different regimes, points A, B, and C, are calculated by taking Fast Fourier Transformation (FFT) of the MEMS's time-domain response. Figures 13e and 13d confirm the in-plane transverse motion contains only a single harmonic at the forcing frequency, and the plate's out-of-plane rotation angle is trivial, while the MEMS is set to operate at point A (before the initiation of the 2:1 modal interaction). Nevertheless, at point C, the torsional motion is non-trivial, owing to the 2:1 flexural-torsional modal engagement, and oscillates with a frequency that equals half of the forcing frequency, $\frac{1}{2}\omega_f = \Omega$, and simultaneously, the bending motion maintains the harmonic at the forcing frequency and its amplitude becomes minimized. This is shown in Figures 6.9g and 6.9h. Special thanks to their rich frequency spectra, the FFTs of the MEMS's bending and torsional vibrations at point B are individually depicted in figures 6.10a and 6.11a, respectively.

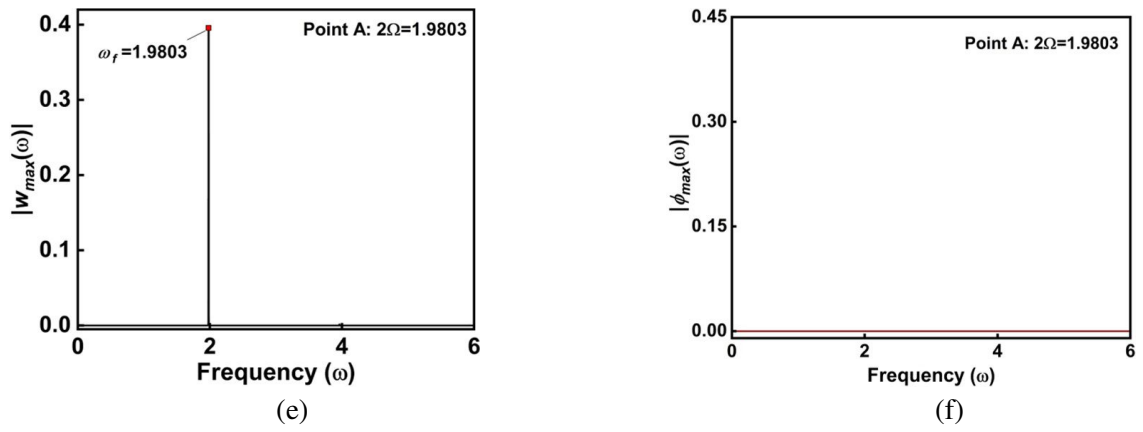
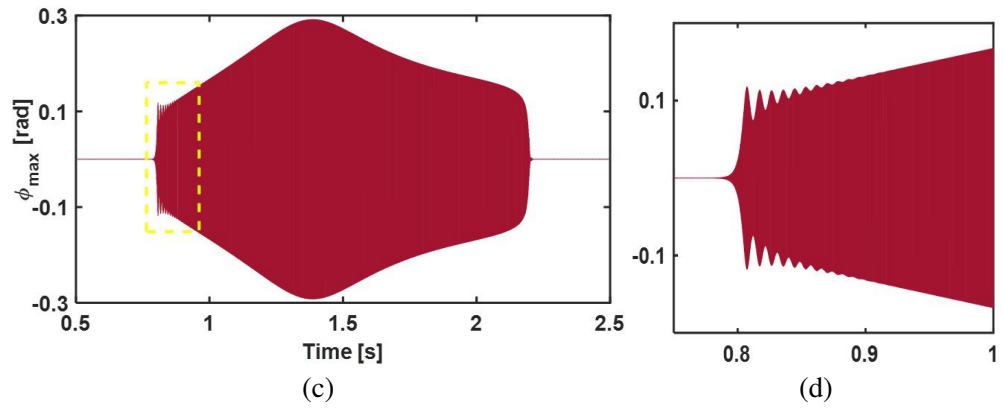
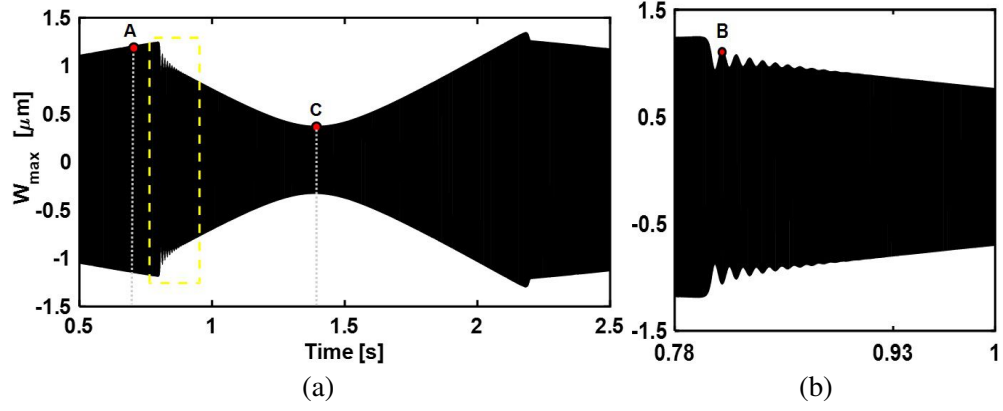
The frequency-domain representation in figure 6.10a depicts the dynamic response of the suspension microbeam with a mid-span plate oscillating at the AC frequency ($\Omega = 0.9960$) corresponding to point B in the response shown in figure 6.9b. The FFT amplitude is plotted in dB scale versus non-dimensional frequency. The in-plane displacement of the microbeam's tip includes a harmonic at the forcing frequency, as a fundamental harmonic, together with extra five harmonics appearing at integer multiples of the forcing frequency within the frequency interval from 0 to 13. Each peak is symmetrically surrounded by equally spaced teeth generating localized frequency combs. These combs emerge due to the occurrence of a secondary Hopf bifurcation incorporated by the bending and torsional motions. For instance, inspecting the FFT plot in the vicinity of the first harmonic, we observe that there are 23 teeth generated on either side of the centered peak, figure 6.10b. Those small peaks are equidistant with a frequency spacing of $\delta\omega = \frac{\omega_f}{449} = 0.0044$. Contrasting figures 6.10b and 6.10c, the density of the produced frequency combs decreases as we move from the first harmonic to the second one. The count of teeth appearing on either side of the second harmonic is nine. This pattern remains nearly identical for the combs formed around the third harmonic. However, as we shift to the fourth harmonic, the density of the combs increases once more, reaching 26 equidistant small peaks distributed on either side of the centered peak. On the contrary, the frequency combs generated around the fifth harmonic extend over a narrower frequency span compared to those around the fourth harmonic. Magnifying the frequency-domain response near the sixth harmonic reveals the formation of a frequency comb having 13 effective teeth on either side of the centered peak. This examination indicates that the proposed electrostatic MEMS actuator is also well-suited for producing localized frequency combs repeating at integer multiples of the forcing frequency. The frequency content of the torsional motion, corresponding to point B in the response illustrated in figure 6.9d, demonstrate denser frequency combs localized around half of the forcing frequency ($\frac{1}{2}\omega_f$) and its odd integer multiples up to 11 times the forcing frequency, figure 6.11a. Zooming the FFT diagram in the vicinity of the fundamental peak reveals the generation of a local frequency comb with 32 teeth on either side of the centered frequency. Unlike the bending motion, the teeth density remains almost unchanged as moving to the frequency combs localized around the higher-order harmonics. It is important to mention that the frequency spacing is equal to that

of the in-plane bending motion. Moreover, the results demonstrate the capability to construct a compound readout signal in the frequency domain via combining the frequency spectra of the two resonators, namely the bending and torsional motions. This allows for the generation of a more extensive frequency comb using quasi-periodicity that spans a continuous frequency range.

Figure 6.12 illustrates the time histories for the transverse displacement and electrical current of the MEMS in the presence of electro-mechanical resonance synchronization, 1:2:1 modal interactions between the electrical, flexural, and the torsional modes. Here, the system's parameters are as follows: $z_0 = 0.035 h$, $c_w = 0.042$, $c_\phi = 0.003$, and $V_{AC} = 10 V$. The temporal profiles correspond to the frequency-response curves illustrated in figure 6.6. The MEMS's motion commences with small-amplitude periodic oscillations around its shifted equilibrium position as the AC frequency grows linearly with the sweep rate same as previous. The response jumps up to the large-amplitude envelope at $t = 0.256 s$. As the time goes on, the amplitudes of the in-plane bending motion and electrical current reduce along the large-amplitude periodic orbits, and eventually, the MEMS's dynamics route to a quasi-periodic regime at $t = 0.713 s$, figures 6.12a and 6.12c. This behavior does not linger and the system retrieves periodicity as approaching $t = 0.861 s$.

The dashed yellow and red rectangles in the figures capture the responses within the windows limited by the first and fourth secondary Hopf bifurcations shown in figures 6.6a and 6.6c. These windows encompass the transient quasi-periodic response that is a result of the 2:1 bending-torsional modal interaction, and the steady-state quasi-periodic response that arises from a 1:2 electro-mechanical/bending modal interaction. The transitions from the former quasi-periodicity to the later one is more obvious in the time evolution of the electrical current. In figures 6.12b and 6.12d, the time envelopes exhibit equidistant sharp peaks distributed throughout the quasi-periodic regime, forming time-domain combs. For a certain AC frequency within this window, $\Omega = 1.0002$, time histories for the in-plane displacement of the microbeam's tip, plate's rotation angle, and the electrical current passing across the capacitance, are shown in figures 6.13a, 6.13c, and 6.13e, respectively. Correspondingly, their FFT plots are computed and depicted in figures 6.13b, 6.13d, and 6.13f. Here, the system is released from its rested configuration. It is demonstrated that the MEMS's dynamics evolve along quasi-periodic orbits in its transient response;

however, the steady-state responses settle into periodic orbits formed via energy exchange between the bending and torsional motions.



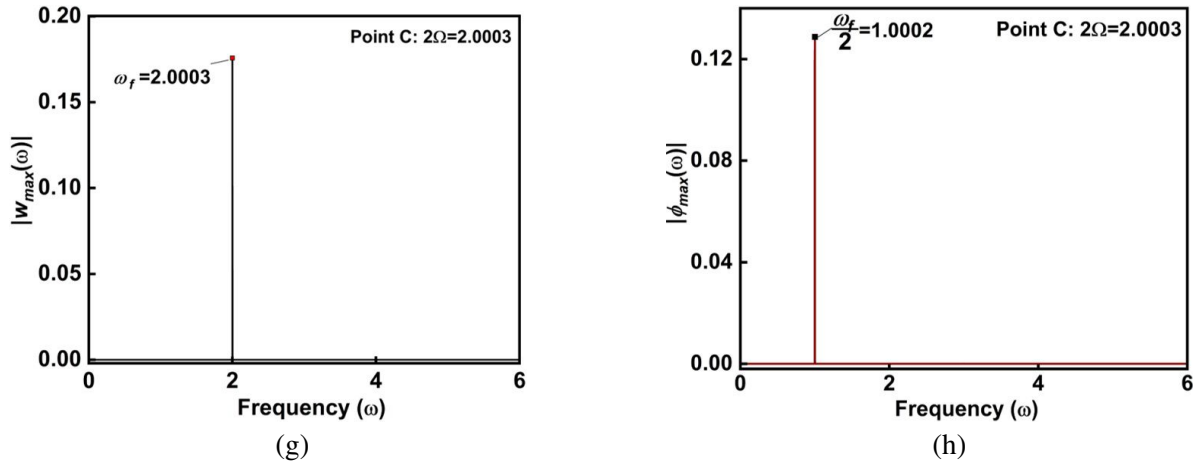
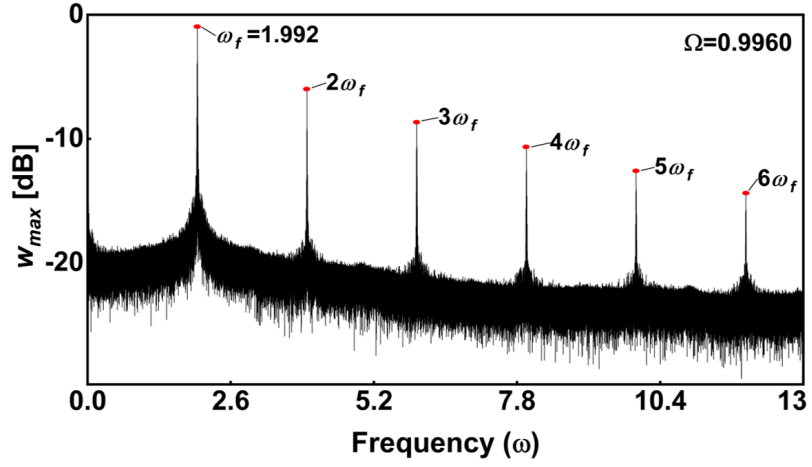
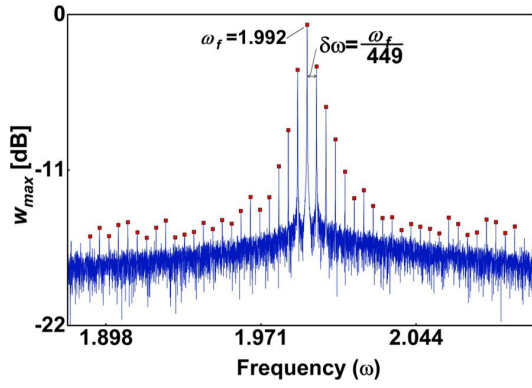


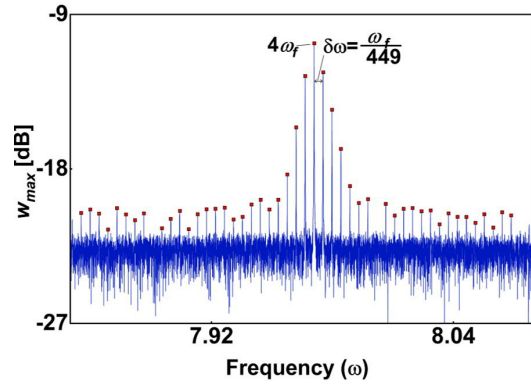
Figure 6.9: Time histories of the MEMS's motions, and FFT diagrams of the steady-state dynamic response of the system, for $z_0 = 0.035 h$, $c_w = 0.042$, $c_\phi = 0.003$, and $V_{AC} = 88 V$. (a) and (c) depict the temporal envelope for the in-plane bending and torsional motions, respectively. The dashed yellow rectangles embed the regions where the responses exhibit beating, and their zoomed views are shown in (b) and (d) for the bending and torsional displacements, respectively. FFT plots of the transverse motion of the microbeam's tip are shown in (e) and (g), while (f) and (h) illustrate the FFT plots of the plate's rotational displacement. These plots are obtained in steady-state at the forcing frequencies corresponding to points A and C, respectively.



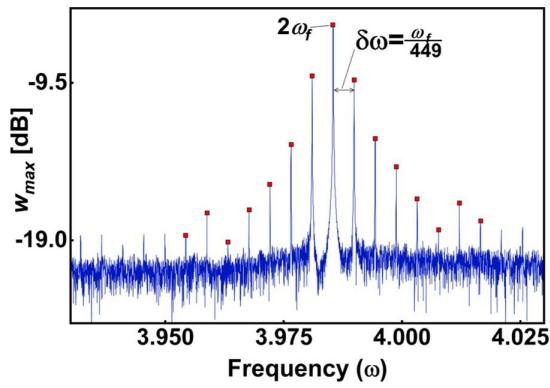
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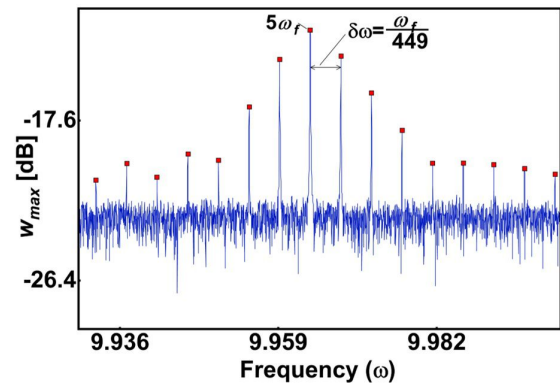
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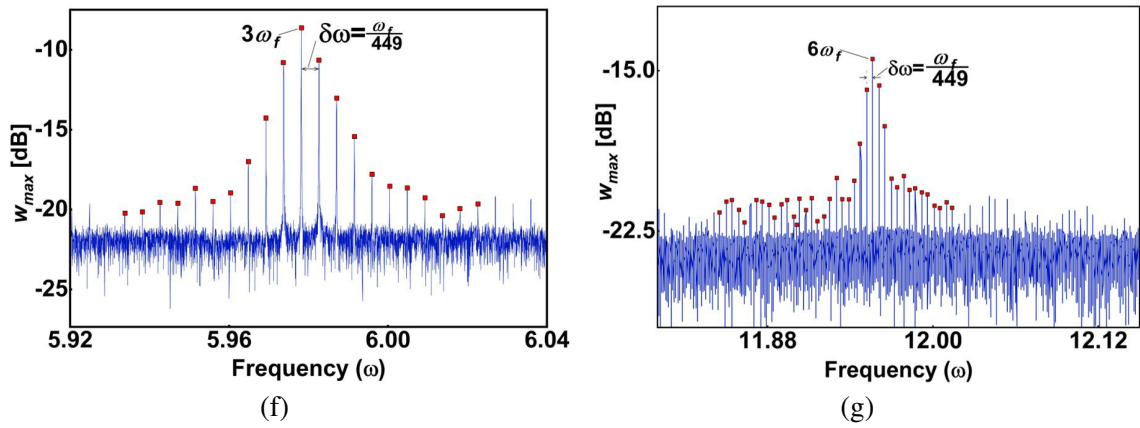
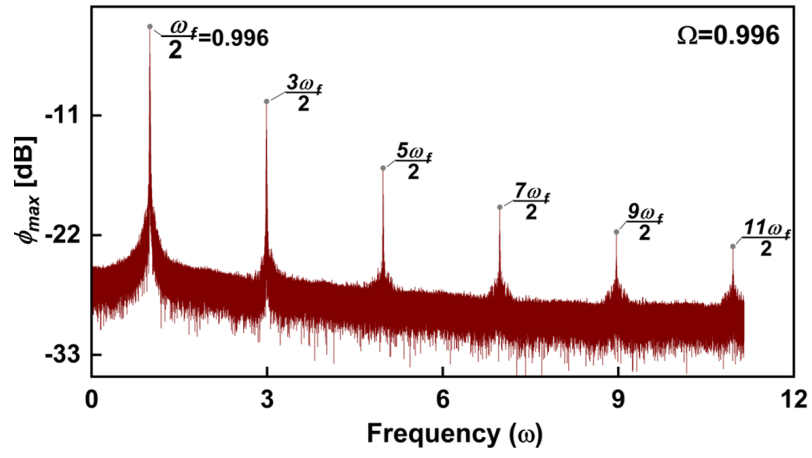
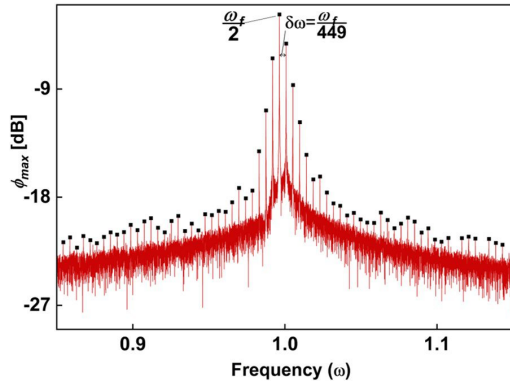


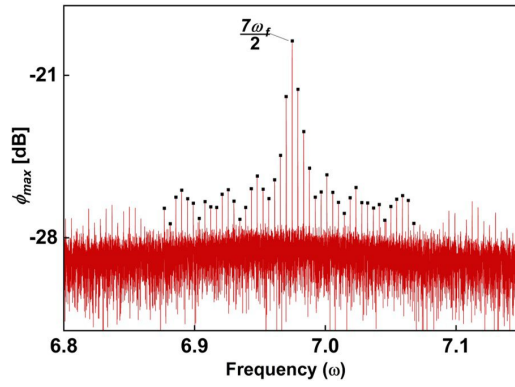
Figure 6.10: (a) The frequency content of the plate's in-plane motion for $\Omega = 0.996$ that corresponds to point B shown in figure 6.9b. The values of the damping coefficients, imperfection parameter, and the AC voltage are similar to those assumed in figure 6.9. The first harmonic is at the forcing frequency, $\omega_f = 2\Omega = 1.992$. The FFT diagram in the vicinity of each harmonic is magnified and displayed in (b), (c), (d), (e), (f), and (g), demonstrating the generation of localized frequency combs with a frequency spacing of $\delta\omega = \frac{\omega_f}{449} = 0.0044$.



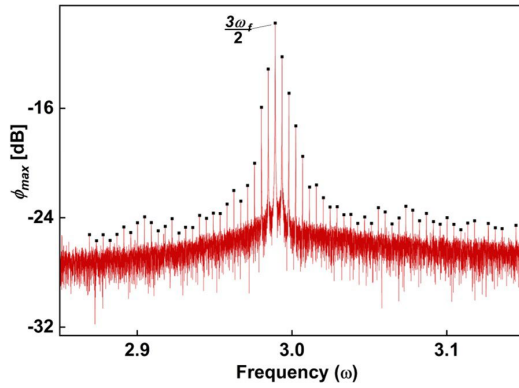
(a)



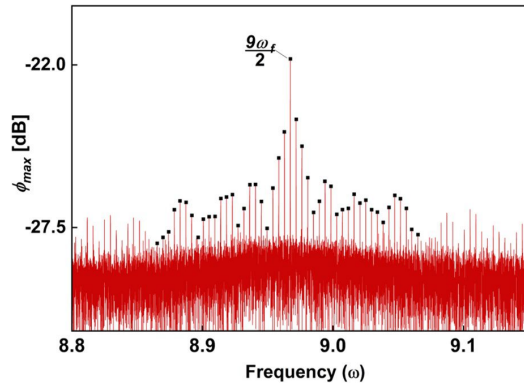
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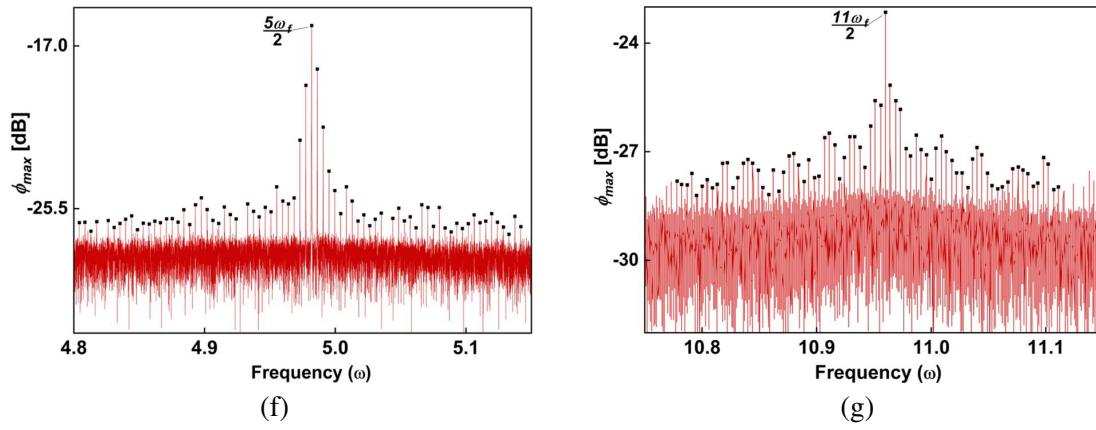


Figure 6.11: (a) The frequency content of the plate's rotational motion for $\Omega = 0.996$ that corresponds to point B shown in figure 6.9b. The system's parameters are consistent with those assumed in figure 6.9. The first harmonic is at half of the forcing frequency, $\frac{1}{2}\omega_f = \Omega = 0.966$. The FFT diagram in the vicinity of each harmonic is zoomed and shown in (b), (c), (d), (e), (f), and (g), demonstrating the formation of localized frequency combs with a frequency spacing of $\delta\omega = \frac{\omega_f}{449} = 0.0044$.

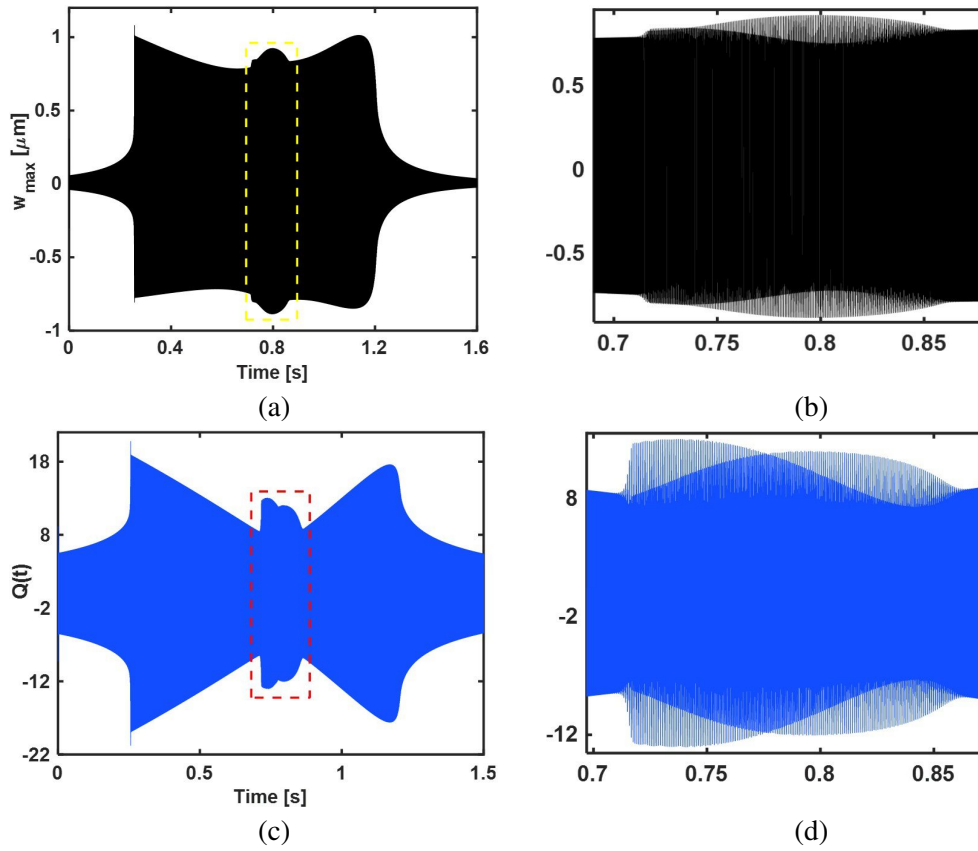


Figure 6.12: Time history of the electrostatic MEMS with electro-mechanical resonance synchronization. The simulated time envelopes correspond to the frequency-response curves presented in figures 6.6a and 6.6c. (a) and (c) depict the time histories for the plate's in-plane motion and the electrical current, respectively. The dashed rectangles indicate the regions where the MEMS's dynamics undergo quasi-periodicity, generating combs in the time-domain, and their zoomed views are illustrated in (b) and (d).

Referring to figures 6.13a and 6.13c, the transition from quasi-periodicity to periodicity in the in-plane flexural displacement is synchronized with gradual enlargements in the amplitude of the torsional motion. The 2:1 internal resonance channel leads to a reduction and saturation in the amplitude of the in-plane motion, while simultaneously driving the torsional motion to grow up to a maximum, depending on the input voltage amplitude. This phenomenon is characterized by the occurrence of period-doubling bifurcation, figures 6.13b and 6.13d. Furthermore, the electrical current's transient dynamics feature a more interesting response as compared to that of the in-plane displacement, figure 6.13e. The equidistant sharp teeth emerging in the quasi-periodic regime introduce time-domain combs. Also, the figure illustrates that the electrical current amplifies when the torsional motion initiates, contributing to the effective capacitance defined between the movable and stationary electrodes. These results indicate that the proposed electrostatic MEMS actuator, operating on the principle of 1:2:1 electro-flexural-torsional modal interactions, is well-suited for the application of periodic out-of-plane rotation drives and quasi-periodic-based time- and frequency-domain combs generated from both the transient and steady-state dynamics of the MEMS.

Tuning the AC frequency to locate the MEMS's dynamics within the frequency interval limited by the second period-doubling (PD_2) and the fourth secondary Hopf ($S-HP_4$) bifurcations, where the torsional motion lies on the unstable trivial branch, leads the actuator to operate under quasi-periodicity in its steady-state, figure 6.6. Here, the AC frequency is set at $\Omega = 1.005$. Unlike the results presented in figures 6.10 and 6.11, the frequency combs observed here arise from a 1:2 modal interaction happening between the electrical and mechanical/bending modes of the MEMS. The FFT diagram of the electrical current flowing through the RLC circuit is depicted in figure 6.14a. The result exhibits the formation of five combs within the shown non-dimensional frequency range. The first comb is centered around the AC frequency containing 12 equally spaced teeth on either side. As seen, the base profiles of the teeth for the first two combs, located at and three times the AC frequency, follow a Lorentzian-type curve. Moving to higher frequencies, the signal power drops, leading to the disappearance of this curve. The magnified views of the frequency combs are shown in figures 6.14b-6.14e. As compared to the unsynchronized electro-mechanical resonance case, the frequency spacing of the combs has increased,

$\delta\omega = \frac{\Omega}{21.62}$, with enhanced symmetry in the amplitudes of the teeth surrounding the center frequencies. Moreover, the number of teeth remains almost consistent while transitioning from one set of frequency combs to another.

It is worth noting that the stability of the frequency combs generated, as a result of secondary Hopf bifurcations in coupled resonators, can be easily compromised. This is because small disturbances in the systems' parameters, such as the forcing frequency, can lead to a transition from quasi-periodicity to a periodic regime, causing the combs to vanish. With this in mind, a robust and nonlinear control mechanism is required to maintain the system's dynamics at a desired operating condition, generating reliable frequency combs. This subject will be studied in our upcoming work.

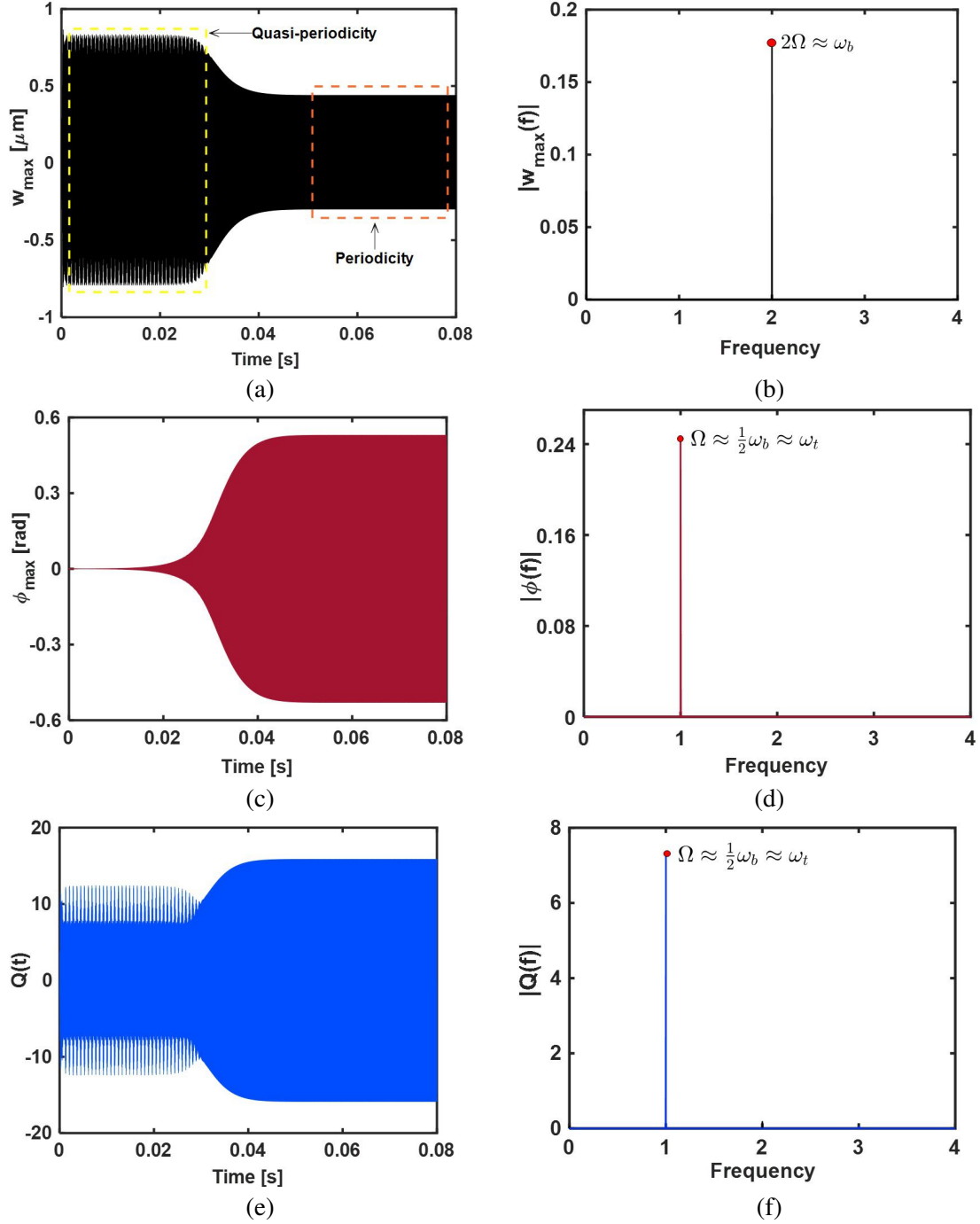


Figure 6.13: Time histories for the plate's in-plane transverse and out-of-plane rotational displacements, and the electrical current, for $\Omega = 1.0002$, are shown in (a), (c), and (e), respectively. The system's parameters are equal to those presented in the caption of figure 6.11. The FFT plots for (a), (c), and (e) are shown in (b), (d), and (f), respectively, indicating the manifestation of a period-doubling bifurcation in the MEMS's response upon the initiation of the 2:1 mode engagement between the in-plane and torsional motions.

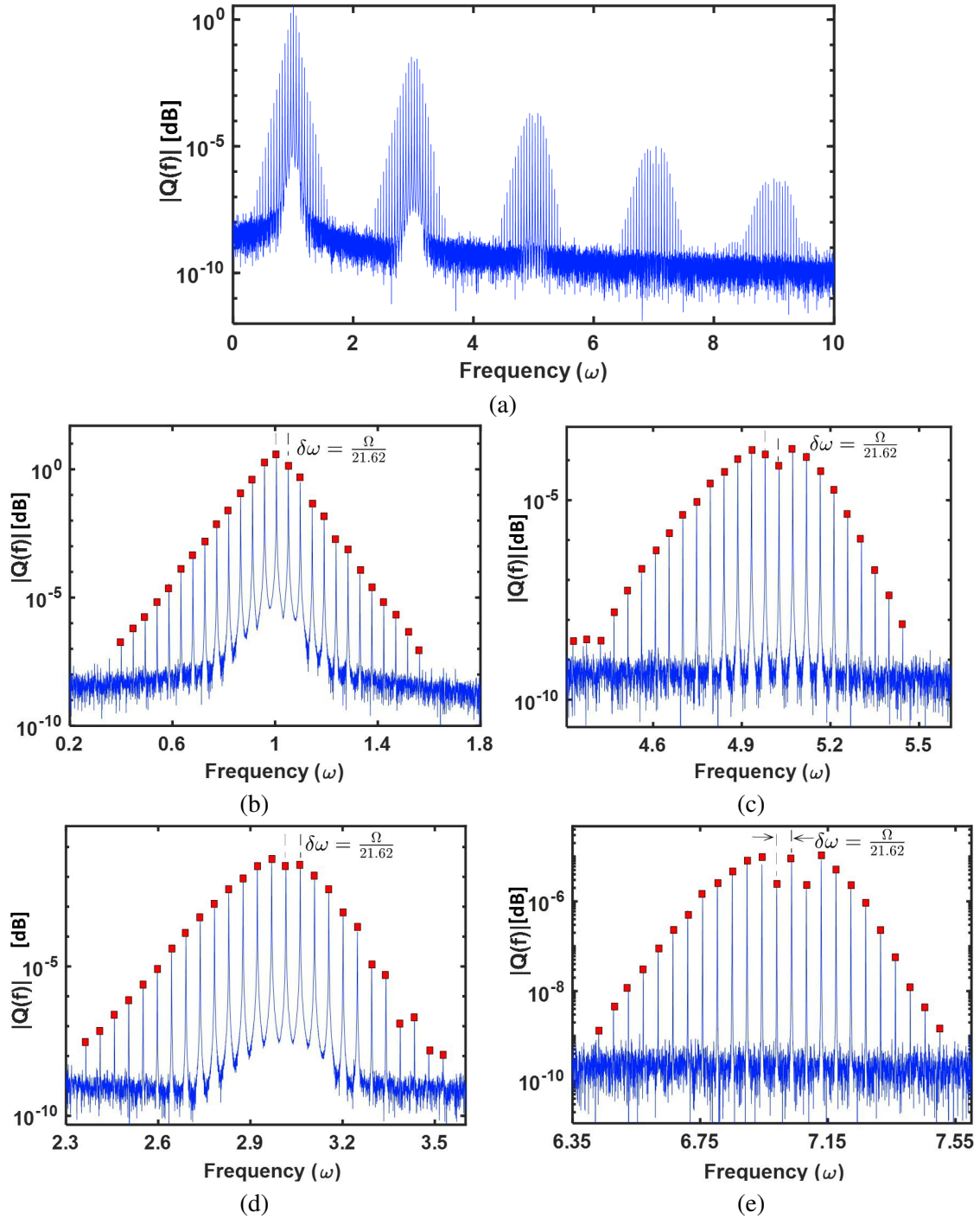


Figure 6.14: The FFT diagram of the MEMS's electrical current in the presence of electro-mechanical resonance syntonization, for $\Omega = 1.005$. The system's parameters are equal to those presented in the caption of figure 6.6. The operating frequency is located between the second period-doubling (PD_2) and the fourth secondary Hopf ($S-HP_4$) bifurcations shown in Figure 6.6a. The zoomed view for the frequency combs localized around the first four harmonics appearing at $\frac{1}{2}\omega_f$, $\frac{3}{2}\omega_f$, $\frac{5}{2}\omega_f$, and $\frac{7}{2}\omega_f$ are shown in (b), (c), (d), and (e), respectively.

6.2 Experiments

The dynamics of the MEMS mirror operating in a modal interaction regime is studied experimentally. We, first, characterize the mirror's in-plane bending and torsional eigenmodes, and second, electrostatically drive the mirror in the vicinity of its in-plane resonant mode, employing frequency sweep test, to capture its frequency- and voltage-displacement behaviors. These steps are presented in the following section.

6.2.1 MEMS characterization

Figure 6.15 depicts the microscopic image of the electrostatic MEMS mirror scanned by a white-light profilometer. figure 6.16 shows the FFT plots of the velocity of an edge and center of the plate. The mirror is driven by thermal noise excitation developed inside a vacuum with pressure of 2 mbar. This degree-of-freedom experiences motion for the first two out-of-plane bending modes and the first torsional mode of the structure. The MEMS's time-domain velocity was measured optically using an LDV, and then it was fed to the Vibrometer's computer to perform data post-processing, such as taking FFT from it. To locate the first out-of-plane mode, the laser spot was positioned at the mirror's center, point A depicted in figure 6.15b. However, the laser spot was focused on an edge of the mirror to capture the first torsional and the second out-of-plane modes of the MEMS, point B shown in figure 6.15b. This experiment characterizes the first and second out-of-plane bending modes' natural frequencies at $f_1^{ob} = 18.3438$ kHz and $f_2^{ob} = 71.1406$ kHz, respectively. Further, the first torsional mode's natural frequency is determined to be at $f_1^t = 21.0312$ kHz. Inspecting the half power bandwidth (HPB) of these three vibration modes, figure 6.17, we obtain their corresponding quality factors to be $Q_1^{ob} = 1145$, $Q_2^{ob} = 2788$, and $Q_1^t = 1035$. It is noteworthy that the other peaks emerging in the FFT diagrams do not represent mechanical motions. Here, the mirror was placed on flat surface so that the laser beam shines perpendicular to the structure's top surface.

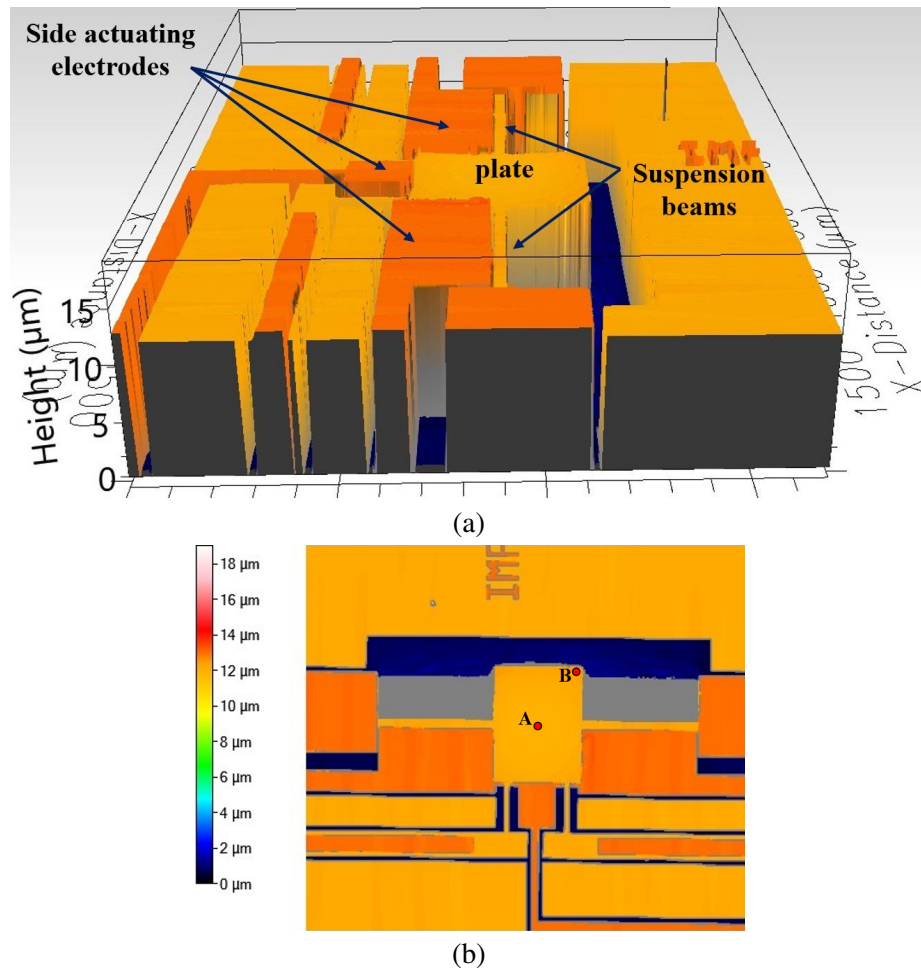


Figure 6.15: Scanned image of the mirror by a white-light profilometer using its 20X magnification lens. (a) Shows the MEMS's isometric view, and (b) depicts its top view.

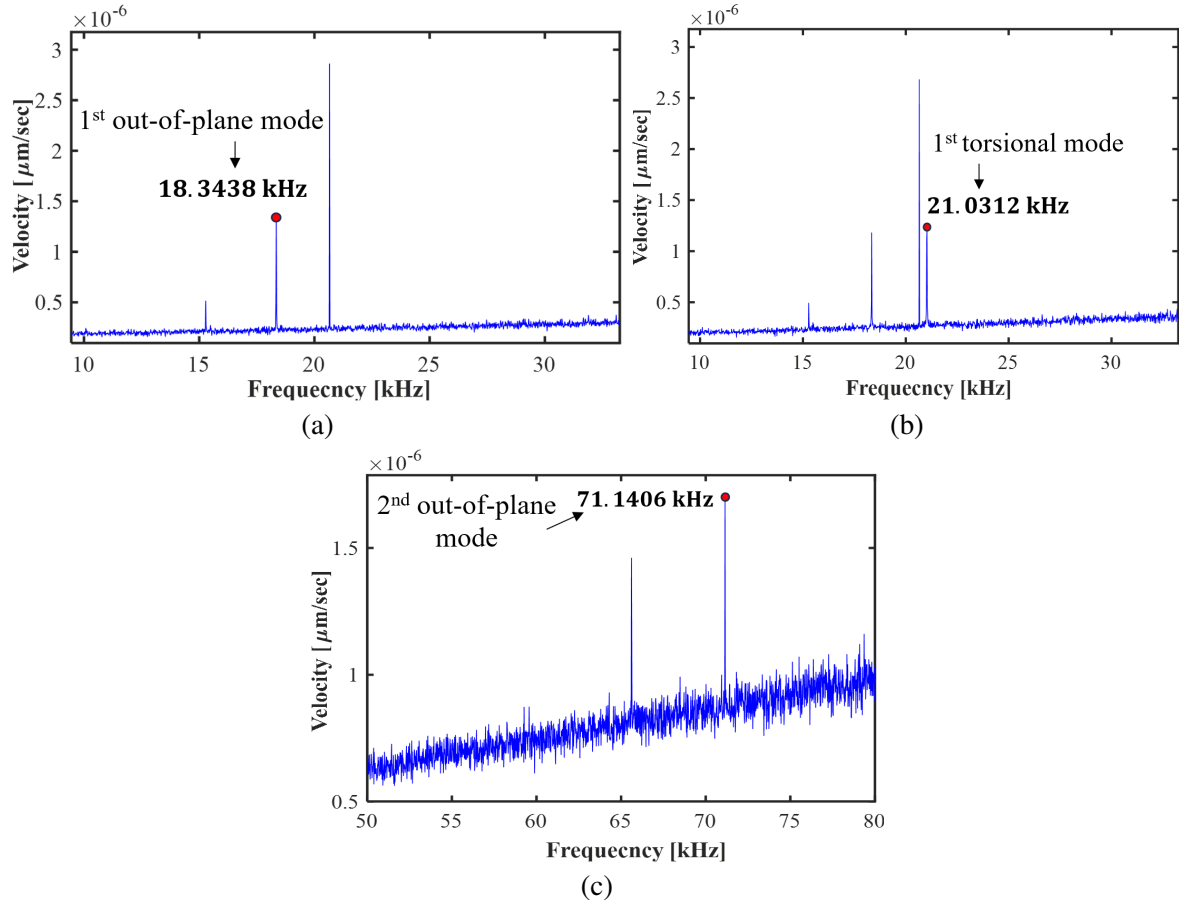
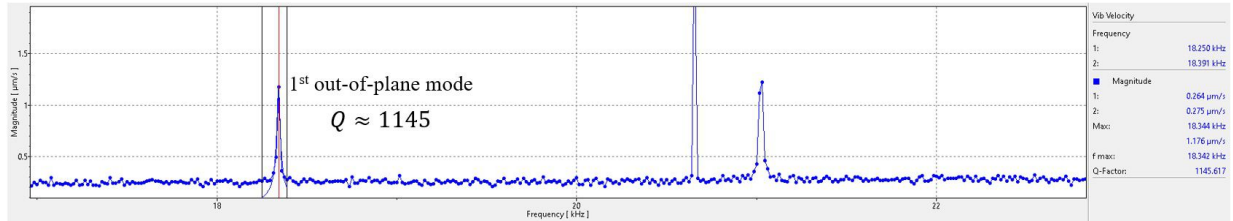
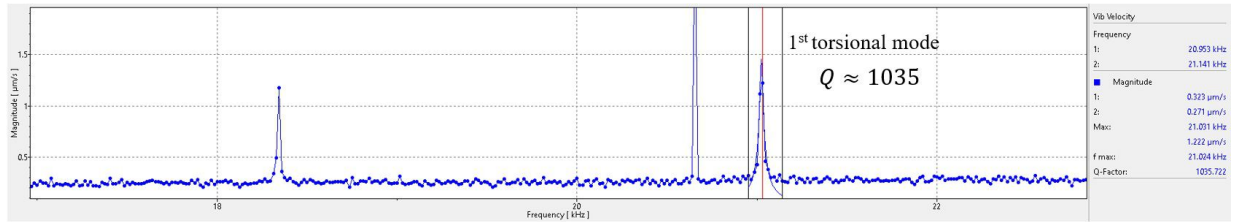


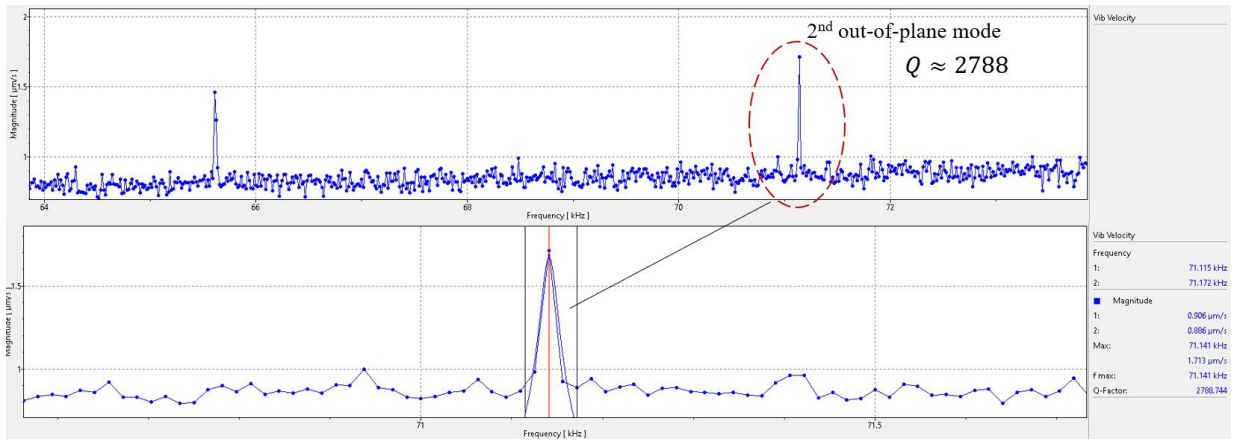
Figure 6.16: FFT plots of the mirror's velocity under thermal noise excitation. (a) 1st out-of-plane bending mode, where the laser spot was positioned at the mirror's center (point A in Figure 6.15), (b) 1st torsional/scanning mode, and (c) 2nd out-of-plane bending mode, where the laser spot was focused at an edge of the mirror (point B in Figure 6.15).



(a)



(b)



(c)

Figure 6.17: Zoomed FFT plots of the mirror's velocity under thermal noise excitation monitoring the Q-factors of the mirror's (a) 1st out-of-plane bending mode, (b) 1st torsional mode, and (c) 2nd out-of-plane bending mode.

To capture the linear frequency-response curve of the mirror in the vicinity of its out-of-plane bending and torsional modes, we exploited base excitation by placing the MEMS on a piezoelectric stage. The piezoelectric base provides rigid-body translation for the MEMS frame, and subsequently introducing a uniformly distributed force exerted to the structure, leading it to deform. The frequency of the distributed force is equal to the frequency of the waveform applied across the piezoelectric material. It is worth noting that the total displacement of any arbitrary material point of the flexible electrode (mirror structure) results from its deformation-induced motion and the rigid-body translation of its frame. By applying a biased AC voltage across the piezoelectric capacitance in the form of $V_{pie}(t) = V_{DC} + V_{AC} \cos(\Omega t)$, employing a 15X voltage amplifier, and sweeping the AC frequency in the vicinity of the first and the second out-of-plane bending modes, the mirror's frequency-velocity curves are captured, Figure 6.18a and 6.18b.

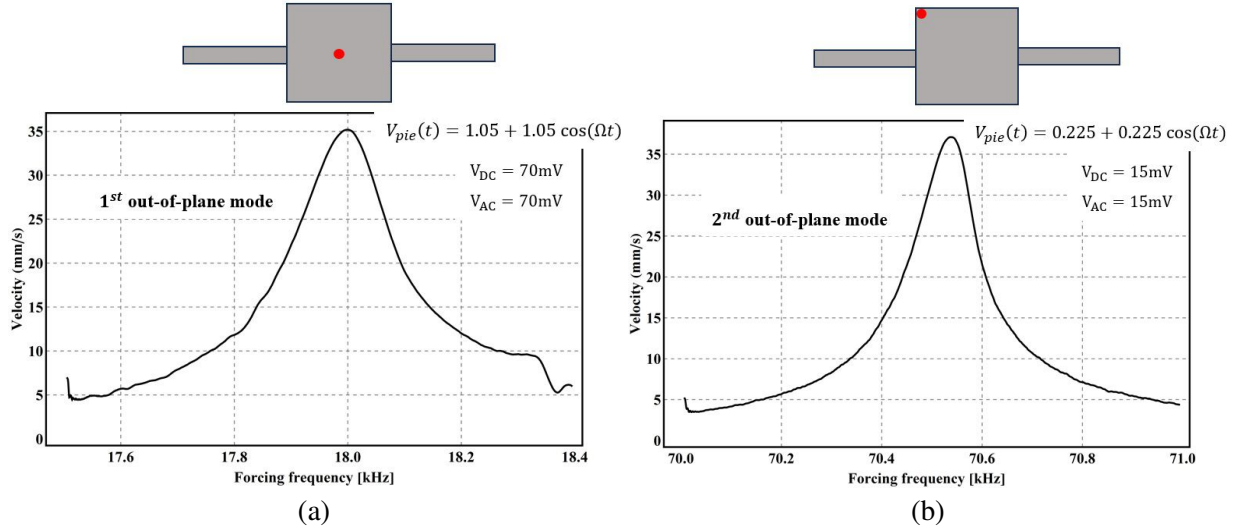


Figure 6.18: The mirror's linear frequency-velocity response curves under base excitation in air. The figure depicts the resonant peaks for, (a) the 1st out-of-plane bending mode, where the voltage amplitudes were set to be $V_{DC} = V_{AC} = 1.05 \text{ V}$, and (b) the 2nd out-of-plane bending mode, where $V_{DC} = V_{AC} = 0.225 \text{ V}$.

This experiment was carried out in an air environment, which means that the resonant peaks observed at $f_1^{ob} \approx 18 \text{ kHz}$ and $f_2^{ob} \approx 70.57 \text{ kHz}$ correspond to the damped natural frequencies of the first and second out-of-plane bending modes. It's worth noting that these frequencies are

slightly lower than those measured under thermal noise excitation in a vacuum. To capture the first torsional mode's natural frequency and Q-factor using base excitation, the laser spot was focused at two edges of the plate when the waveform frequency of the piezoelectric stage was set to vary in a frequency range from 20.7 kHz to 21.07 kHz, figure 6.19.

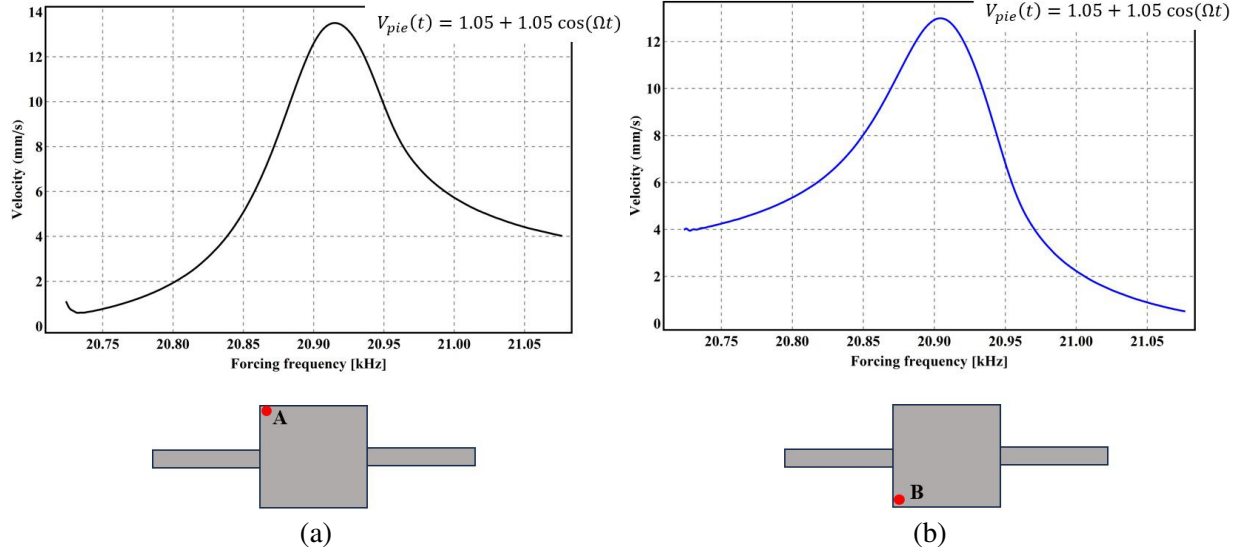
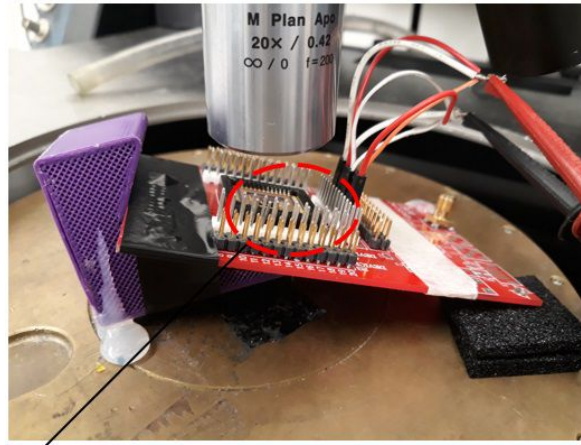


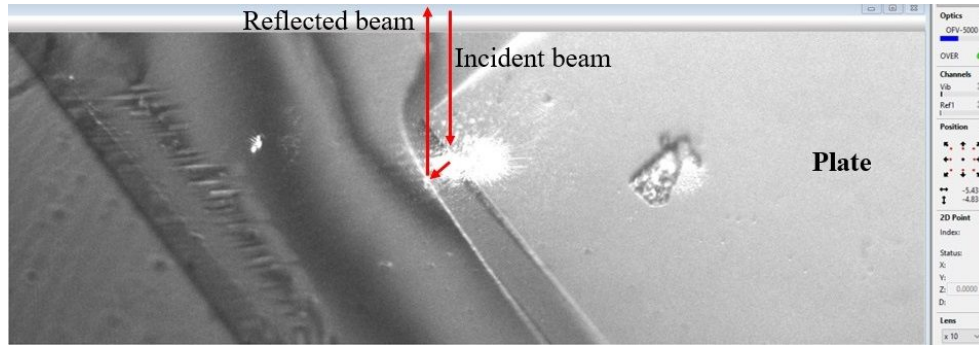
Figure 6.19: The measured linear frequency-velocity response curves of the mirror operating under base excitation in air. The figure illustrates the resonant peaks for the 1st torsional mode, corresponding to two nodal-edge points of the plate, for the voltage amplitudes of $V_{DC} = V_{AC} = 1.05$ V.

This experiment locates the torsional mode's resonant peak at $f_1^t \approx 20.92$ kHz, which is slightly damped due to air viscosity as compared to what was obtained in vacuum, figure 6.16b.

To measure the mirror's in-plane motion, we mounted the MEMS on a ramp with a tilt angle of 30 degrees, figure 6.20. This allows us to expose the structure's side surface to the incident laser beam. Since the mirror's side surface is not thoroughly perpendicular to the laser beam direction, the measured signal encompasses information related to both in-plane and out-of-plane motions.



(a)



(b)

Figure 6.20: (a) The MEMS chip placed in a PCB chip carrier mounted on a ramp with 30 degrees orientation with respect to the horizontal, (b) microscopic image depicting the incident and reflected laser beams that are guided by the plate's side wall and the silicon wafer's edge.

The FFT of the velocity of a point belonging to the side surface of the mirror is depicted in figure 6.21. Here, the MEMS operates under a pulse excitation with amplitude of 28.5 V, frequency of 8 Hz, and duty cycle of 0.1%, which is developed through applying potential difference between the side actuating electrode and the movable electrode. It is worth mentioning that the above-stated nodal point displaces not only in-plane, as expected from in-plane excitation, but also out-of-plane. This out-of-plane displacement occurs due to the electrostatic torque exerted as a result of misalignment. By positioning the laser spot on the side surface of the mirror rested on a ramp, we are able to construct a projection of the total displacement along the laser beam

direction that carries information about in-plane and out-of-plane motions.

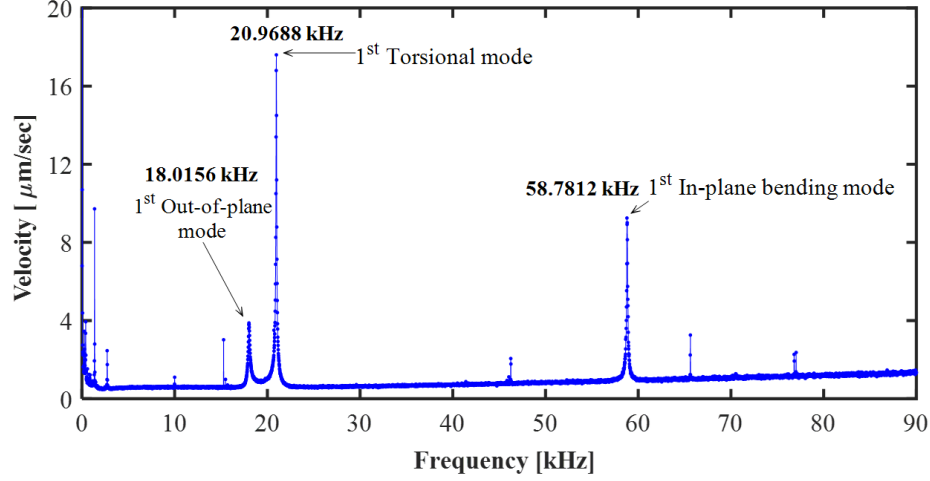


Figure 6.21: The measured FFT of the velocity of a point located on the mirror's side surface, for a pulse excitation with amplitude of 28.5 V, frequency of 8 Hz, and duty cycle of 0.1%.

The first in-plane bending and first torsional modes' natural frequencies are obtained $f_1^{ib} \approx 58.7812$ kHz and $f_1^t \approx 20.9688$ kHz. The torsional mode's natural frequency shifts noticeably to the right compared to the results obtained from base excitation. This shift indicates that the torsional mode slightly hardens in the presence of the electrostatic field, which confirms the outcomes of the tuned eigenfrequency analysis performed using our numerical and FEM simulations, figures 5.11. The ratio between the two mode's natural frequencies is $\frac{f_1^{ib}}{f_1^t} = \frac{58.7812 \text{ kHz}}{20.9688 \text{ kHz}} \approx 2.8033$. The torsional natural frequency is about 5 kHz lower than that of the as-designed, which arises from fabrication imperfection.

6.2.2 3:1 flexural-torsional modal interaction

In the PiezoMUMPs fabrication process, the structural layer's thickness is found to be about $1\mu\text{m}$ thinner than the thickness of the device layer due to back over-etching. With this vision in mind, we expect to observe a nonlinear energy pathway exploiting 3:1 modal interaction happening

between the first in-plane bending and first torsional modes. To this, we placed the mirror on flat surface again and measured its out-of-plane motion optically.

Applying a waveform that is slightly biased and sweeping its frequency near half of the natural frequency of the first in-plane bending mode result in the emergence of a time-domain velocity response shown in figure 6.22. It is important to note that the device-under-test (DUT) has on-wafer bias positive charge. Hence, we applied a waveform with a bias negative voltage to cancel the DC voltage observed that is across the MEMS capacitance. However, there is still a minor DC charge accumulated on the MEMS's electrodes even after applying a bias negative voltage. This can be observed via inspecting the FFT diagram of the mirror's velocity for a certain AC frequency, resulting in a very small-amplitude harmonic appearing at AC frequency. In this case, the main contribution of the forcing signal takes place at a harmonic that is twice the AC frequency. Here, AC frequency (f_{ac}) was swept from 29.2 kHz to 30.4 kHz within 800 ms, and the applied waveform was in the form of $V(t) = -278 \times 15 \text{ mV}_{DC} + 29.4225 V_{AC} \cos(2\pi \cdot f_{ac} t)$.

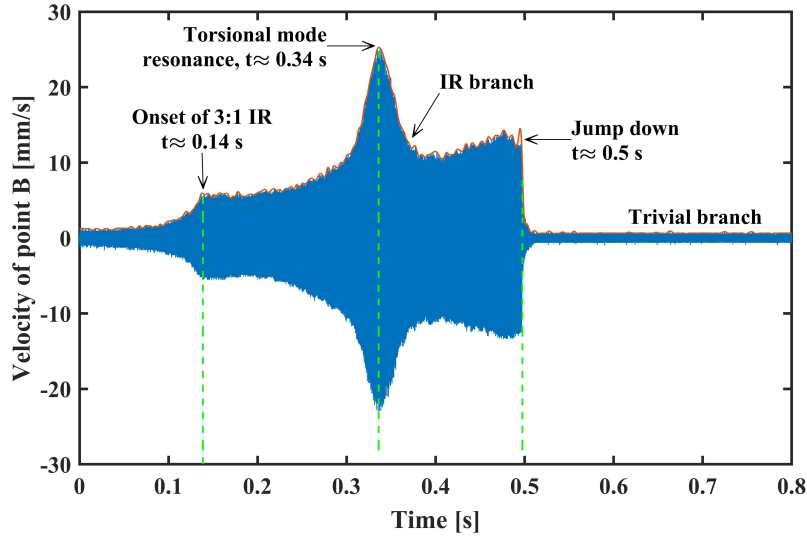


Figure 6.22: The measured time-domain velocity response of an edge of the plate, for a waveform excitation in the form of $V(t) = -278 \times 15 \text{ mV}_{DC} + 29.4225 V_{AC} \cos(2\pi \cdot f_{ac} t)$. The AC frequency was swept in the vicinity of half of the resonant frequency of the first in-plane bending mode, from 29.2 kHz to 30.4 kHz within 800 ms.

The envelope of the measured velocity response is obtained via taking RMS of its time-domain data, fitting the signal profile. This envelope is mapped in the frequency domain to depict the frequency-velocity response of point B of the mirror, figure 6.23. Sweeping up the AC frequency starting from 58.4 kHz, the MEMS's response extends along with a non-IR and almost trivial branch provided that its amplitude gradually grows as the forcing frequency approaches 58.8 kHz. Up to this point, the out-of-plane torsional motion contains a single harmonic that appears at the forcing frequency. The existence of the forcing frequency in torsional motion is due to a linear coupling between this motion and the in-plane bending mode. This coupling stems from an electrostatic torque projection that mostly arises from misalignment in the position of the flexible electrode (suspension microbeams and the plate). Further increasing the signal frequency beyond 58.8 kHz leads to an abrupt emergence of a harmonic at almost one-third the forcing frequency with small amplitude in the frequency content of point B's velocity. This is along with a sudden jump-up in the MEMS's response at forcing frequency, $f_f \approx 58.8$ kHz. However, the jump-up at this point is less observable because of the fast sweep.

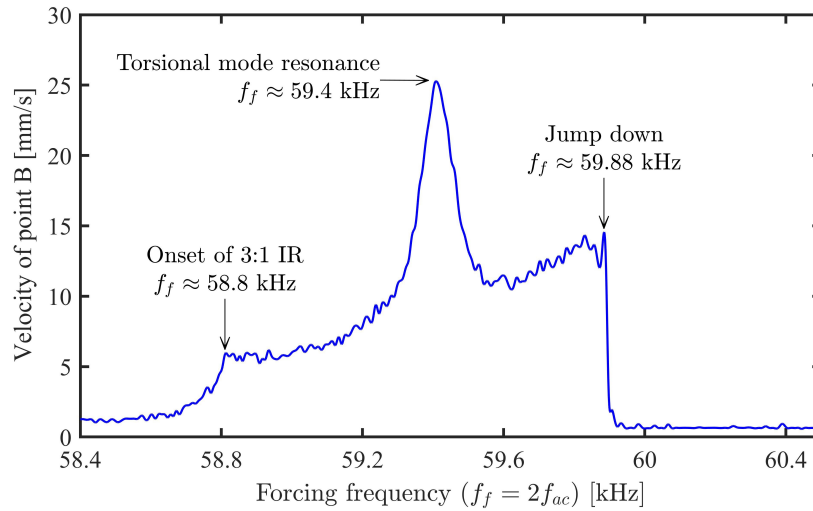


Figure 6.23: The frequency-velocity response curve corresponding to figure 6.22 via taking RMS of its measured time-domain data.

Moving forward along the IR branch, the torsional mode's contribution in constructing the mirror's out-of-plane motion enhances (amplitude of the harmonic appearing at one-third the forcing frequency grows gradually). At $f_f \approx 59.4$ kHz, the torsional motion undergoes resonance forming a peak in the velocity of point B. This resonance occurs due to the activation of a nonlinear energy exchange between the first in-plane bending mode which is excited directly, and the first torsional mode which is driven from *above* auto-parametrically. The velocity amplitude is about 25 mm/s at the resonant frequency. The MEMS's response passes the resonance point and jumps down onto a trivial branch at $f_f \approx 59.88$ kHz as the waveform frequency is swept up. In fact, the presence of input noise causes instability in the system's dynamics and propels its motion towards the trivial branch. We repeated the test for voltage amplitudes of $V_{DC} = -298 \times 15$ mV and $V_{AC} = 28.995$ V, figure 6.24.

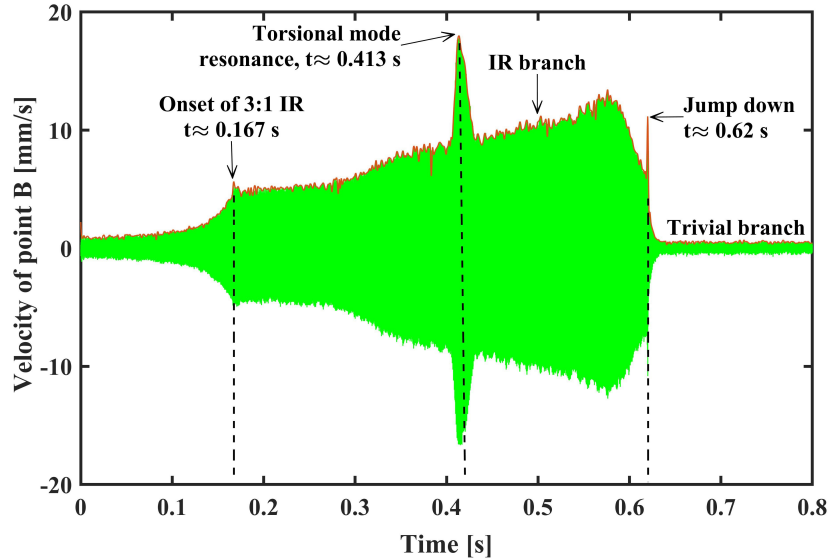


Figure 6.24: The measured time-domain velocity response of an edge of the plate, for a waveform excitation in the form of $V(t) = -298 \times 15 \text{ mV}_{DC} + 28.995 \text{ V}_{AC} \cos(2\pi \cdot f_{ac} t)$. The AC frequency was swept in the vicinity of half of the resonant frequency of the first in-plane bending mode, from 29.2 kHz to 30.4 kHz within 800 ms.

Comparing figures 6.22 and 6.24, we observe notable distinctions, particularly evident at lower AC voltage amplitudes. Those differences encompass the delayed onset of the 3:1 flexural-

torsional modal interaction, the timing of the torsional mode resonance, and the moment at which the system's response transitions to the trivial branch. Figure 6.25 shows the envelopes of the measured time-domain velocities depicted in figures 6.22 and 6.24 mapped in the frequency-domain.

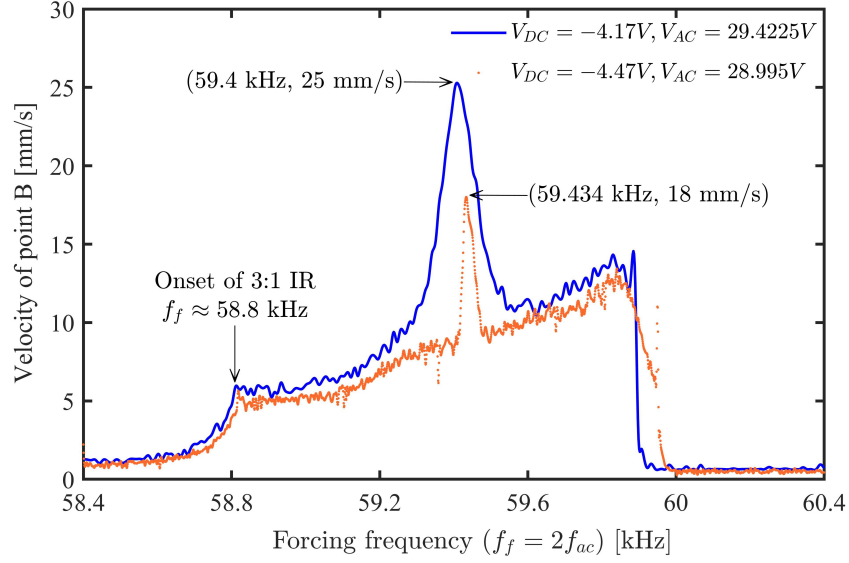


Figure 6.25: The frequency-velocity response curves of the mirror at point B. The signal frequency was set to linearly vary from 29.2 kHz to 30.4 kHz within 800 ms. The voltage amplitudes are $V_{DC} = -4.17V$ and $V_{AC} = 29.4225V$, for the blue solid curve, and $V_{DC} = -4.47V$ and $V_{AC} = 28.995V$, for the orange dotted curve.

Referring to figure 6.25, the MEMS's dynamics are potent to experience a rout-to-quasi-periodicity in the vicinity of the jump-down frequency. For the orange dotted curve, the system's response extends along the IR branch beyond approximately $f_f \approx 59.88$ kHz. In this scenario, the steady-state periodic orbits gradually shrink in size and transition to a quasi-periodic regime. This behavior becomes more evident with longer sweep times and moderate AC voltage amplitudes, as experienced for $V_{AC} = 28.995$ V. Figures 6.26a and 6.26b exhibit the time history and frequency-velocity response behaviors of point B of the mirror for a biased waveform with amplitudes of $V_{DC} = -4.17$ V and $V_{AC} = 32.2725$ V as its AC frequency was swept upward in a frequency range from 29.2 kHz to 30.4 kHz within 8 seconds.

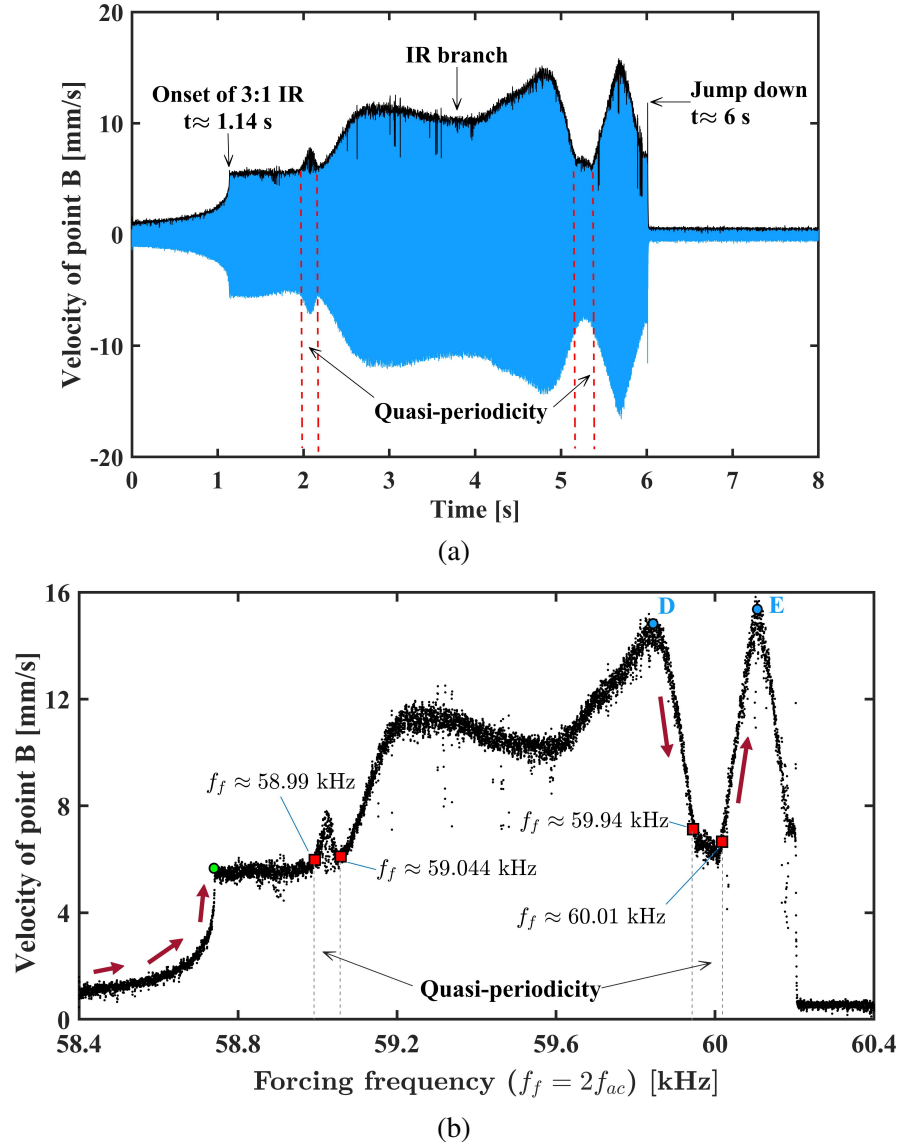


Figure 6.26: (a) The measured velocity's time-history of point B of the mirror, for $V_{DC} = -4.17V$ and $V_{AC} = 32.2725V$. The signal frequency was swept in a frequency range from 29.2 kHz to 30.4 kHz within 8 s. (b) Depicts the mirror's frequency-response curve corresponding to the measured time-domain data shown in figure 6.26a.

It is important to mention that in optical out-of-plane measurement, we capture the projection of the absolute velocity of point B along vertical direction (laser beam direction) instantaneously. Similar to previous, the mirror's rotational motion grows gradually when the signal frequency is swept upward starting from 58.4 kHz. This trend continues until the sweeping parameter reaches a point with frequency of $f_f = 58.74$ kHz, marked by a green circle, leading the motion to jump up to the IR branch. Further increasing the forcing frequency, the response's amplitude remains almost constant until reaching $f_f = 58.99$ kHz, where the MEMS's dynamics experience a secondary Hopf bifurcation, routing to a quasi-periodic regime. However, this behavior is temporary and confined to a narrow frequency bandwidth. As the forcing frequency increases further, reaching $f_f = 59.044$ kHz, the mirror regains its periodicity through a reverse secondary Hopf bifurcation. Keep increasing the waveform frequency, the torsional motion constantly grows and it experiences a maximum with amplitude of 15 mm/s at point D , $f_f = 59.84$ kHz, shown in the figure. After this point, the mirror's velocity decreases along the IR branch, and its dynamics transition to quasi-periodicity once more through the third secondary Hopf bifurcation happening at $f_f = 59.94$ kHz. This behavior takes place within a pretty narrow frequency bandwidth similar to the previous case. The experimental results show that the energy transfer from the mirror's in-plane bending mode to its torsional mode is improved again as the forcing frequency is swept up further, passing $f_f = 60.01$ kHz. Moving along the IR branch, the velocity of point B experiences the second maximum at point E , $f_f = 60.1$ kHz. Unlike the findings shown in figure 6.25, the torsional mode in this case did not exhibit resonance to create a peak along the IR branch. Our experimental investigations reveal that the absence of a finely tuned 3:1 commensurate ratio between the natural frequencies of the in-plane and torsional modes results in unstable nonlinear energy exchange between these modes. Before addressing this issue, we present the formation of frequency combs that arises from quasi-periodicity as a result of secondary Hopf bifurcation happening in the MEMS's dynamics.

The frequency spectrum of the mirror's velocity is illustrated in figure 6.27. This FFT corresponds to the system's response at a forcing frequency of $f_f = 59.0156$ kHz, lain within the first quasi-periodic zone as shown in figure 6.26a. Here, the electrostatic voltages are $V_{DC} = -4.17V$ and $V_{AC} = 32.2725V$.

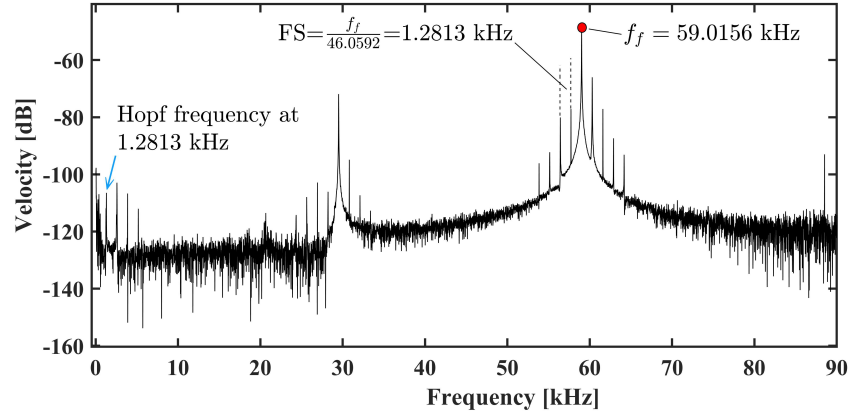


Figure 6.27: The frequency spectrum of the mirror's velocity at point B. The input voltage amplitudes are identical to those presented in the caption of figure 6.26. The forcing frequency is tuned at $f_f = 59.0156$ kHz.

There are two dominant peaks occurring at 59.0156 kHz and 29.5078 kHz that correspond to the forcing frequency and half of it, respectively. As previously mentioned, the harmonic appearing at the AC frequency is because of a small DC charge present on the electrodes. Moreover, there are four minor and equidistant peaks on either side of the forcing frequency with a frequency spacing (FS) of 1.2813 kHz. It is worth noting that the observed frequency spacing corresponds to the Hopf frequency happening at 1.2813 kHz. This forms a frequency comb that is centered at the forcing frequency with nine teeth. The presence of those teeth, harmonics with minor amplitudes, is a strong indication of having beating-type envelopes for the time-history of the system's response, demonstrating quasi-periodicity in the MEMS's dynamics. The emergence of quasi-periodic behaviors arising from secondary Hopf bifurcation was reported in our numerical simulations, performed for the 2:1 flexural-torsional modal interaction.

Returning to instability in the torsional mode's resonance, we performed several sweep tests for different values of AC voltage amplitude, figure 6.28. Here, the waveform's frequency was set to vary in a range from 29.2 kHz to 30.4 kHz, and the DC component of the electrostatic voltage was set to be $V_{DC} = -4.17V$.

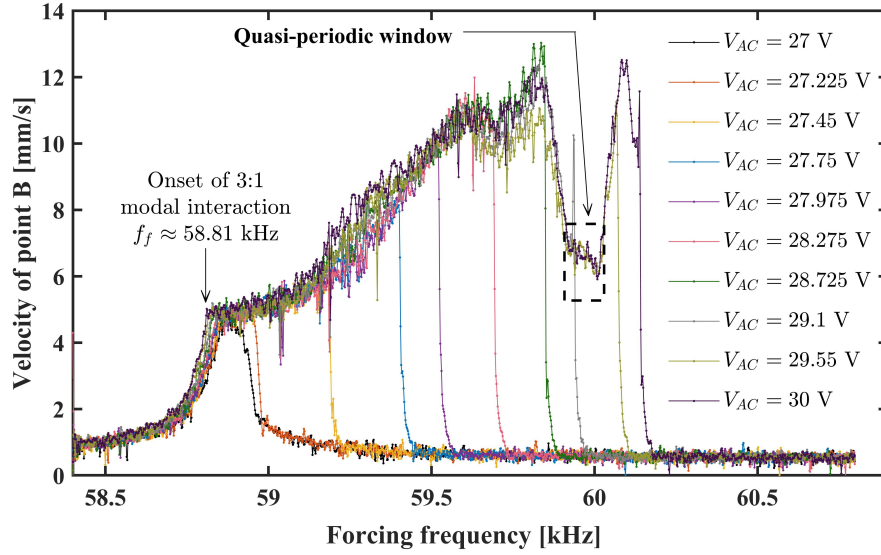


Figure 6.28: The measured frequency-response curves of the mirror at its top left edge, for various values of AC voltage amplitude. The nonlinear energy pathway is triggered at the forcing frequency of $f_f = 58.81$ kHz, creating a non-zero harmonic at about one-third the in-plane bending mode's natural frequency in the frequency spectrum of the MEMS.

Here, the mirror's velocity at its top left corner was captured for different values of AC voltage amplitude, starting from 27 V to 30 V. The voltage was swept with a fine resolution, and that is because, a small variation in the forcing amplitude results in a significant change in the amplitude of the system's response. This characteristic has been widely observed in resonators where a vibration mode is driven parametrically. In this experiment, the frequency sweep lasted for 800 ms, which explains why the jump phenomenon at $f_f \approx 58.81$ kHz, marking the initiation of the modal interaction, appears to be smooth. It's important to note that in none of these frequency sweeps does the torsional mode undergo resonance to form a peak on the IR branch. This is primarily attributed to the distorted commensurate ratio between the natural frequencies of the bending and torsional modes. Comparing the performance of the electrostatic MEMS mirror and the curved beam resonator investigated in Chapter four, we can conclude that a nonlinear energy pathway that bridges two modes oscillating in one-plane forms faster than two modes resonating in two perpendicular planes such as, bending-torsional modal interaction.

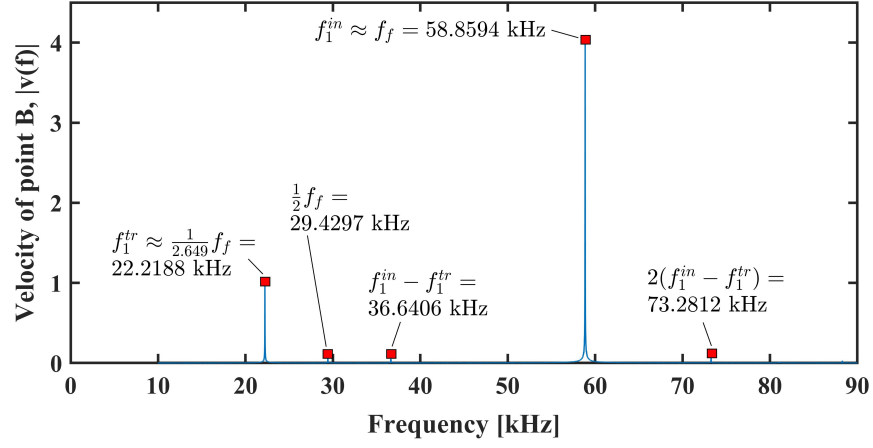
Figure 6.29 depicts the frequency spectra of the MEMS mirror that operates under AC waveform with negative and positive DC voltages. Our experiments show that during a frequency sweep, the commensurate ratio between the two modes' natural frequencies fluctuates around its value reported in the characterization section. It is important to mention that the displayed FFT plots were captured at instances when the amplitude of the motion located in the vicinity of the torsional mode became maximum during a slow frequency sweep test.

In figures 6.29a and 6.29b, we biased the AC signal with $V_{DC} = -0.351 \times 15 \text{ V}$ to cancel the positive charges that exist on the capacitance's electrodes by default. However, in figure 6.29c, we exploited the inherent positive charge ($V_{DC} = +0.351 \times 15 \text{ V}$) to actuate the mirror with an enhanced biased electrostatic force, thereby improving the commensurate ratio between the resonant frequencies of the in-plane bending and torsional modes during a sweep test. In the former case, the AC frequency was adjusted at half of the first in-plane bending mode's natural frequency, but in the later case, it is tuned at the in-plane bending mode's resonant frequency itself to directly exciting this mode.

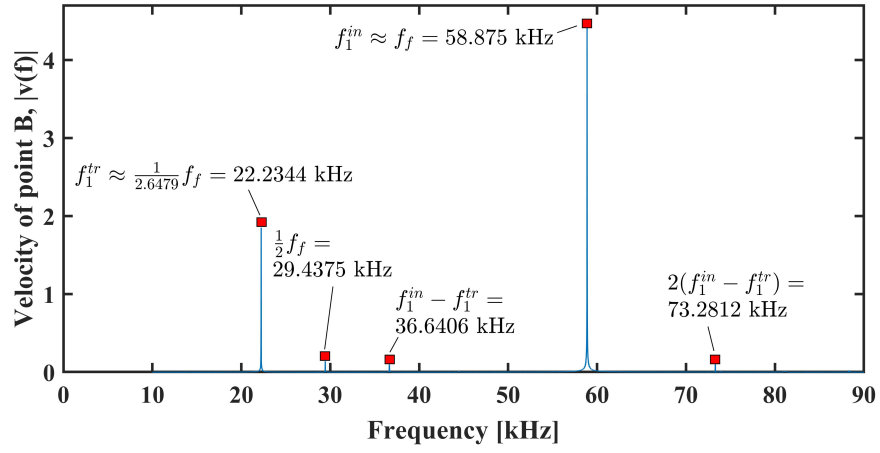
In figure 6.29a, the AC voltage amplitude was set to be $V_{AC} = 29.1 \text{ V}$. Here, the primary dominant peak appears at the forcing frequency, which is in the vicinity of the in-plane bending mode's natural frequency, $f_1^{in} \approx f_f = 58.8594 \text{ kHz}$. The secondary dominant harmonic emerges in the neighbor of the torsional mode's resonant frequency, which is approximately $\frac{1}{2.65}$ times the forcing frequency, $f_1^{tr} \approx \frac{1}{2.65} f_f = 22.2188 \text{ kHz}$. The figure illustrates the presence of a minor harmonic at the AC frequency, $\frac{1}{2} f_f = 29.4297 \text{ kHz}$, indicating that there are still DC charges remaining on the device-under-test (DUT). Moreover, the in-plane bending mode's resonant frequency is modulated at the torsional natural frequency, producing a tiny harmonic at $f_1^{in} - f_1^{tr} = 36.6406 \text{ kHz}$. Also, twice the modulated frequency appears at $2(f_1^{in} - f_1^{tr}) = 73.2812 \text{ kHz}$. The same scenario happens for $V_{AC} = 29.55 \text{ V}$ in figure 6.29b, with the distinction being that the response amplitudes at the aforementioned harmonics are enhanced compared to those depicted in figure 6.29a.

Figure 6.29c exhibits the frequency spectrum of the mirror's motion, where the frequency of the positively biased AC waveform is set at the resonant frequency of the first in-plane bending

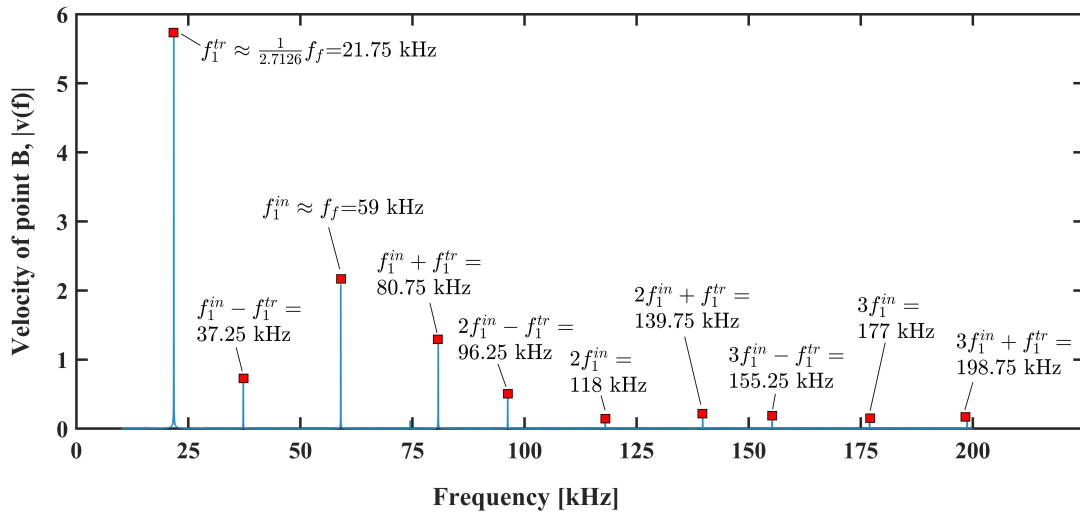
mode, $f_1^{in} \approx f_f = f_{ac} = 59$ kHz. Here, the AC voltage load is $V_{AC} = 31.05$ V. The FFT plot was recorded during a frequency sweep, starting from 58.5 kHz to 59.2 kHz. The frequency spectrum is composed of two mean harmonics that correspond to the forcing frequency and the torsional mode's resonant frequency happening at $f_1^{in} \approx f_f = 59$ kHz and $f_1^{tr} \approx \frac{1}{2.7126} f_f = 21.75$ kHz, respectively. The other contributions are built based on those two harmonics, as follows. The forcing frequency is modulated at the torsional resonant frequency that creates two equidistant peaks at either side of it, $f_1^{in} - f_1^{tr} = 37.25$ kHz and $f_1^{in} + f_1^{tr} = 80.75$ kHz. Then, the forcing frequency is modulated again at the first modulated frequency obtained in the last step, creating a harmonic at $2f_1^{in} - f_1^{tr} = 96.23$ kHz, and $f_1^{tr} = 21.75$ kHz. The sixth harmonic appearing in the FFT plot corresponds to twice the forcing frequency, $2f_f = 118$ kHz. Further, the seventh small peak is constructed from the forcing frequency where it is modulated at $f_1^{in} + f_1^{tr}$, generating a harmonic at $2f_1^{in} + f_1^{tr} = 139.75$ kHz. The same scenario occurs for the eighth and the ninth peaks. Eventually, the tenth harmonic emerging in the FFT diagram is obtained from the forcing frequency modulated at the seventh harmonic, creating a peak at $3f_1^{in} + f_1^{tr} = 198.75$ kHz. It is worth noting that these harmonics are expected to correspond to integer multiples of the dominant peak, which represents the resonant frequency of the torsional mode, $2f_1^{tr}, 3f_1^{tr}, 4f_1^{tr}, \dots$, in case of a nearly perfect 3:1 commensurate ratio between the two modes' natural frequencies. Comparing these three frequency spectra, we can conclude that the nonlinear energy exchange between the in-plane bending and torsional modes intensify when the mirror operates larger



(a)



(b)



(c)

Figure 6.29: The measured FFTs of the mirror's motion at point B. The mirror is under negative bias voltage of $V_{DC} = -0.351 \times 15V$ and AC voltage of (a) $V_{AC} = 29.1V$ and (b) $V_{AC} = 29.55V$. (c) Depicts the frequency spectrum of the MEMS's response for $V_{DC} = +0.351 \times 15V$ and $V_{AC} = 31.05V$.

Chapter 7

Conclusions

7.1 Conclusions on MEMS scanning mirrors

The impetus of this study was to introduce a novel actuation mechanism for an electrostatic MEMS scanning mirror that exploits modal coupling, between the first in-plane bending ($f_1^{in} = 58.7812$ kHz) and first torsional ($f_1^{tr} = 20.92$ kHz) modes of the mirror, to indirectly excite the torsional (scanning) mode without resorting to direct external excitation. In-plane excitation allowed us to completely remove the substrate underneath the microplate, thereby providing the mirror with a clear path to realize large rotational angles and reducing the damping it encounters without placing undue demands on the actuation voltage. Underpinning this achievement is an energy channel that exists between the in-plane and out-of-plane modes, which we exploited to induce parametric modal interaction.

Based to our experimental findings, we can make the following conclusions:

- We can realize in-plane actuated scanning mirrors by employing the 3:1 bending-torsion modal interaction to channel energy from bending to torsion.
- We can achieve large rotational angles without escalating voltage requirements (< 35 V).
- We can facilitate the introduction of control electronics in electrostatic MEMS mirrors.

Based on our analytical investigation of the 2:1 bending-torsion modal interaction, we can make the following conclusions:

- The presence of non-zero first moments of area in the as-fabricated cross-sections of symmetrical electrostatic MEMS actuators can produce the quadratic stiffness term, essential for triggering 2:1 autoparametric modal interaction between the in-plane bending and torsion.
- A first-order approximation of the absolute slope of the tangent to the deformed beam middle axis is necessary in order to bring up the quadratic coupling between bending, torsion, and nonlinear curvature of the beam neutral axis.
- A misalignment in the position of the centroid of the as-fabricated cross-sections, on the order of 3 – 5%, is adequate to initiate the 2:1 flexural-torsional modal interaction.
- Simulation results reveal saturation of the in-plane bending motion, while the amplitude of the torsional motion continues to increase with increasing electrostatic excitation.

7.2 Conclusions on underlying research

We conducted experiments to examine the dynamics of electrostatic MEMS clamped-clamped curved beams. The third in-plane bending mode of the microbeam was driven directly, whereas the second in-plane bending mode was excited parametrically (from *above*). By exploiting the 2:1 modal interaction between these two modes, we demonstrated that the first anti-symmetric bending mode can be effectively excited by energy transfer from the second symmetric mode.

In parallel to this, we analyzed numerically the modulation dynamics of a macro-scale non-linear energy sink (NES) composed of two nonlinearly coupled resonators. The higher natural frequency resonator excited parametrically, while the lower natural frequency excitation was driven autoparametrically. The modulation equations were solved numerically to capture the system's frequency- and force-response curves describing the steady-state dynamics of the two resonators. The results showed that the level activation of 2:1 modal interactions is highly dependent on the balance between damping mechanisms and the parametric energy pump.

Building upon the insights gained from these two stages, we engineered a symmetric electrostatic MEMS scanning mirror that features 2:1 ratio between the first in-plane bending and first torsional modes of the microbeams supporting the mirror plate.

7.3 Recommendations for future research

The actuation mechanism proposed in this study is based on energy transfer between two vibration modes of the flexible electrode in electrostatic MEMS mirrors. This phenomenon, however, requires a commensurate ratio between the natural frequencies of the target modes. Our experience with scanning mirrors show that a slight variation in the structural layer thickness may result in a significant deviation between the resonant frequencies of the in-plane bending and torsional modes. This mismatch narrows the operational range available for modal interaction. However, this effect was not as pronounced for the MEMS curved-beam resonator where the two interacting modes were aligned (in-plane). Further, we found that longer time was needed for the formation of an energy channel between the in-plane and torsional modes compared to the two bending modes. Taking these observations into consideration, we suggest the following to improve the mirror's performance:

- Giving that the energy pathway of the 2:1 parametric modal interaction is stronger than that of the 3:1 modal interaction, breaking the symmetry of the mirror's structure can potentially strengthen the quadratic energy channel between the in-plane and torsional modes of the MEMS mirror.
- Tuning mechanism should be added to compensate for any mistuning σ between the natural frequencies of the bending and torsional modes

$$f_1^{in} = 2f_1^{tr} + \sigma$$

that arise from fabrication imperfections. In case of positive detuning ($\sigma > 0$), thermal Joule heating can be deployed to reduce the in-plane natural frequency f_1^{in} while leaving

the torsional natural frequency unaffected. In the case of negative detuning ($\sigma < 0$), laser cooling may be a solution to shift up the in-plane natural frequency while leaving the torsional natural frequency unaffected.

- Our experimental observations indicate that the large-amplitude IR-branch encompassing the torsional motions become unstable during long-time measurements. It might be worth it to explore implementation of a nonlinear control mechanism to stabilize the steady-state dynamics of the mirror in the periodic and quasi-periodic regimes.

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