Tubings of Rooted Trees

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Chapter 3 includes material from a paper [2] coauthored by Paul-Hermann Balduf, Kurusch Ebrahimi-Fard, Lukas Nabergall, Nicholas Olson-Harris, and Karen Yeats. The material from [2] in Chapter 3 is my contribution to [2] and does not overlap with other students' theses. Chapter 4 is comprised solely of material written by me under the supervision of Karen Yeats. The first section of Chapter 5 primarily covers the work of Igor Mencattini and Alexandre Quesney in their paper [27]. The second section of Chapter 5 is original joint work between Kimia Shaban and me under the supervision of Karen Yeats.

Abstract

A tubing of a rooted tree is a broad term for a way to split up the tree into induced connected subtrees. They are useful for computing series expansion coefficients. This thesis discusses two different definitions of tubings, one which helps us understand Dyson-Schwinger equations, and the other which helps us understand the Magnus expansion. Chord diagrams are combinatorial objects that relate points on a circle. We can explicitly map rooted connected chord diagrams to tubings of rooted trees by a bijection, and we explore further combinatorial properties arising from this map. Furthermore, this thesis describes how re-rooting a tubed tree will change the chord diagram. We present an algorithm for finding the new chord diagram by switching some chords around. Finally, a different notion of tubings of rooted trees is introduced, which was originally developed by Mencattini and Quesney [27]. They defined two sub-types of tubings: vertical and horizontal which are used to find coefficients in the Magnus expansion. These two types of tubings have an interesting relationship when the forests are viewed as plane posets.

Acknowledgments

First, I want to thank my supervisor, Karen Yeats for her tremendous supervising. Her help and guidance has been invaluable to me, and I could not have done this without her. She is the best supervisor I could have asked for, and I will always be grateful for everything she has done for me and our research group.

I thank Kurusch Ebrahimi-Fard for hosting me at NTNU in Trondheim, where we worked on project that I really liked and eventually expanded on, which is covered in Chapter 5. I enjoyed working with him and he did an excellent job making my trip seamless, comfortable and productive.; I really felt welcome in Trondheim.

Additionally, I want to thank Melissa Cambridge for facilitating my time in the department, and for generally keeping everything running smoothly.

I thank Kimia Shaban for working on the Magnus expansion project with me, for taking impeccable notes, and for being a friend. Additionally, I want to thank Jette Gutzeit for helping me with my figures and for being a wonderful friend. I also thank Paul-Hermann Balduf for always troubleshooting my latex issues and providing detailed notes on our research projects. I want to thank Paul, Kurusch, Nick Olson-Harris, Lukas Nabergall, and Karen for doing a great job on the Dyson-Schwinger paper.

Finally, I want to thank my friends Mahtab, Mohammad, Ian, and Josephine, along with Kimia and Jette for always being there through the good times and bad, and I will cherish the time we spent together. Additionally, I am grateful to those friends and colleagues that I have not mentioned, but who have left a positive impact on me. And finally, I am not sure I could have made it here without the steadfast support of my partner, Shane, whom I am deeply grateful for. I give my thanks to my family who have supported me through these years.

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Chapter 1

Introduction

This thesis primarily deals with tubings of plane rooted trees and chord diagrams, and how we can use them to calculate coefficients in quantum field theory.

Rooted trees are combinatorial structures that come up frequently in various disciplines; they are particularly useful in combinatorics and computer science, (see Chapters I.2 and I.5 of [12] and references therein.) Rooted trees efficiently display hierarchical structures of sets, and consequently are useful in computer science and data organization. Additionally the hierarchical structure makes it easy to systematically count the number of distinct trees [34], which is highly advantageous in enumeration. Rooted trees are closely related to Catalan numbers; for example, the number of different binary trees with n vertices is given by the n^{th} Catalan number. Rooted trees may be plane or nonplane, and the plane structure introduces new enumerative properties and applications. In the perspective of this thesis, rooted trees are motivated by quantum field theory, where they represent the hierarchical structure of insertions of divergent Feynman diagrams into other divergent Feynman diagrams. Tubings are a way to decompose a tree into connected subtrees, which yield interesting combinatorial and geometric structures. Binary tubings, sometimes called pipings [14], are a type of tubing that have been used to solve series expansions of Dyson Schwinger equations [2, 33]. Described in this thesis is a very nice map between binary tubings of rooted trees and connected chord diagrams. Magnus tubings are another type of tubing which depend on the partial orderings of vertices in the tree [27]. There are many other types of tubings with interesting properties, such as those used to define the graph associahedron [14, 5].

Chord diagrams are combinatorial structures made of pairings of points on a circle, and have applications throughout mathematics and other sciences, such as biology. Combinatorially, they allow us to study properties of permutations and partitioning [28, 30, 31], and they can be used to model problems in areas such as graph theory [15, 16, 17] and bioinformatics [22]. A particularly interesting application of chord diagrams is studying RNA secondary structure, which involves base-pairings within an RNA molecule [20, 18]. Frequently chord diagrams are drawn on a circle where chords are edges connecting points on the circle, however, rooted chord diagrams take one end of a chord to be the root, and are thus displayed linearly. Chord diagrams may be connected or disconnected, and in the rooted sense, may be decomposable. This thesis deals with rooted connected chord diagrams. For Dyson-Schwinger equations, chord diagrams are useful in series expansions [26, 7, 6, 19, 8].

The work of this thesis is motivated primarily by the combinatorics of quantum field theory. Quantum field theory is an area of theoretical physics that combines quantum mechanics and relativity to describe the behaviour of subatomic particles. Dyson-Schwinger equations are the quantum analogs of the equations of motion. They are differential equations used in quantum field theory containing information about the behavior and interactions of quantum fields. In a perturbative approach to quantum field theory, we look for series expansion solutions to Dyson-Schwinger equations. These series expansions are usually indexed by Feynman diagrams [11], but this is not fully combinatorial because each Feynman diagram contributes a complicated integral to the sum. Yeats developed a program to give series expansions of Dyson-Schwinger equations using chord diagrams [26, 19, 7, 6], where each diagram's contribution is combinatorial. This was improved upon in [2] to a tubing expansion; Chapter 3 gives the explicit connection between tubings and chord diagram expansions of Dyson-Schwinger equations.

Another important expansion in quantum field theory is the Magnus expansion, which is discussed in more detail in Chapter 5. Other ways combinatorics comes up in quantum field theory include renormalization Hopf algebras and the graph theory of Feynman periods (see [37] and the references therein). Combinatorics also relates to vector models and other topics (see [36] and references therein).

The thesis is roughly divided into two parts, binary tubings and Magnus tubings, and the outline is as follows. First, some of the key terms and concepts needed for this research, such as rooted trees, chord diagrams, and posets, will be introduced in Chapter 2. Chapter 3 and Chapter 4 relate to a bijection between tubings of rooted trees and connected chord diagrams Theorem 3.11. Chapter 4 makes use of the map from Chapter 3 but focuses on how re-rooting trees (Definition 4.1) affects chord diagrams through the map defined. Finally, Chapter 5 introduces a different type of tubings of plane rooted forests, which we call Magnus tubings (Definition 5.4), based on the work of Mencattini and Quesney [27]. Two types of Magnus tubings are used to compute coefficients in the Magnus Expansion

(Theorem 5.9 and Theorem 5.13.) Building off of their work, we were use the perspective of plane posets to study a duality between these two types (Definition 5.15.) Finally, Chapter 6 will conclude the research and discuss open problems and future work.

The bijection of Chapter 3 and Chapter 4 opens up new questions about how manipulating a tubed rooted tree affects its chord diagram or vice-versa. Re-rooting trees is one way to study this. Tree re-rooting is an operation where, given a rooted tree, we choose a new root, changing nothing else about the graph. While it's certainly possible to re-root a tree and find the chord diagram from the bijection, we are interested in how we can find the chord diagram without knowing what the new tree looks like. Solely by knowing what the tree and chord diagram look like before re-rooting, we have found an algorithm for finding the new chord diagram, and the new tree is not necessary in this algorithm at all. It turns out that we need to put markers (in this thesis we color them) on the vertices of the tree so that we know what to do with each chord in the chord diagram. When the chord diagrams are colored like the tree, then we can just use these colors and very easily do the re-rooting process through the chord diagrams. In solving this problem, the main difficulty lies in what this coloring will be. It turns out it is slightly convoluted to determine the coloring of the chord diagram, and also includes changing the markers on the ends of some of the chords. This will be described in more detail in Chapter 4.

In [27], Mencattini and Quesney introduced two very interesting ways to create tubings on plane rooted forests, which they used to compute coefficients in the Magnus expansion, a useful equation in quantum field theory. We expanded on what Mencattini and Quesney did by viewing these tubed forests as plane posets. We created a transformation for these plane posets and found a special class of forests such that after the transformation, they are still plane rooted forests. From there, we can apply the tubings introduced by [27] to these tubed forests before and after the transformation. Furthermore, there is some possible future work with further abstraction of the transformation of forests into a poset point of view.

Chapter 2

Combinatorial Background

The following sections introduce important definitions and concepts. Rooted trees and tubings of rooted trees are essential to the ideas presented in this thesis. Rooted trees have many applications in enumeration, algebraic combinatorics, computer science, and physics. We can study rooted trees further in the perspective of tubings. These tubings are related to chord diagrams, outlined in Chapter 3. Finally, posets are introduced to present how rooted trees can be viewed as posets.

2.1 Rooted Trees

Rooted trees play a central role in this thesis, thus it is important to understand some basic properties and conventions concerning rooted trees.

Definition 2.1 (Rooted tree)**.** Recursively, a *rooted tree* is a vertex r, called the root, and a possibly empty multiset of rooted trees whose roots are the children of r. A *plane rooted tree* is a vertex r, also called the root, and a possibly empty ordered list of plane rooted trees whose roots are the children of r.

We can also think of a rooted tree as a graph with the root as a distinguished vertex, and so we will use graph theoretic language when convenient. In this thesis, rooted trees will be drawn with the roots at the top. Additionally, all trees in this thesis are plane rooted trees, so unless otherwise stated, trees will refer to plane rooted trees and forests will refer to plane forests of rooted trees.

Definition 2.2 (Plane forest)**.** A plane forest is an ordered set of plane rooted trees.

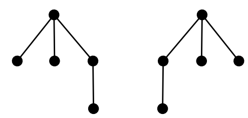


Figure 2.1: An example of two rooted trees that are graph theoretically equal, but *not* the same plane rooted tree.

Definition 2.3 (Tree traversal)**.** Given a plane tree T, a tree traversal is a specified process of visiting every vertex of T.

The tree traversal technique used in this thesis is the pre-order traversal:

Definition 2.4 (Pre-order traversal)**.** The pre-order tree traversal always begins at the root. For a given vertex i with n children, let T_k be the subtree rooted its k^{th} child (as given in its plane structure). Visit the vertex i, then traverse every subtree T_k starting at T_1 and finishing at T_n.

One can visualize pre-order traversal by beginning at the root, walking along the edges of the graph until all vertices have been visited, where the leftmost edge is the first edge. When roots are oriented at the top, then pre-order traversal takes a counter-clockwise walk around the graph. The following image is an example of pre-order traversal.

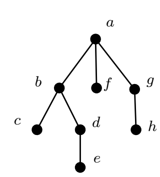


Figure 2.2: By following pre-order traversal, the resulting order on the tree is a, b, c, d, e, f, g, h

Definition 2.5 (Traversal order)**.** We call the order on the vertices resulting from the pre-order traversal the traversal order of the tree.

Note that there are many ways to traverse a tree, and thus this term is not suitable in the broader sense; however, in the scope of this thesis, this is the only type of tree traversal used, thus we call this order the traversal order.

2.2 Tubings

This thesis discusses two different notions of tubings. In this section, the notion of tubings only refers to binary tubings. Chapter 5 introduces a different definition of tubings.

Definition 2.6 (Tube)**.** Let t be a rooted tree or a plane rooted tree. A *tube* of t is a set of vertices of t that induces a connected subgraph of t.

A connected subgraph of a rooted tree t is a tree, hence is a rooted subtree.

Furthermore, in this thesis, a tube and its induced subtree will be interpreted to be synonymous. So for a tube t, the induced subtree will also be referred to as t.

In a very broad sense, we have the following definition of a tubing.

Definition 2.7 (Tubing)**.** A *tubing* of a forest is a set of tubes.

While this thesis focuses on two types of tubings, there are many more types of tubings [5, 14, 27]. Generally, we say that a tube is maximal if it contains every vertex of of the forest, and minimal if it contains only one vertex.

Definition 2.8 (Binary tubing)**.** Let T be a rooted tree or a plane rooted tree. A *binary tubing* τ of T is a set of tubes such that

1. τ contains the tube of all vertices of T, and
2. each tube t of τ either is a single vertex or there are two other tubes in τ which partition t.

(Definition 2.3 in [2].)

We can also treat our tubings recursively. Let T be a rooted tree or a plane rooted tree. A (binary) tubing τ of T can be obtained recursively as follows:

1. τ contains the outer tube consisting of all vertices of T.
2. Pick $e = uv \in E(T)$, with u closer to the root than v. Partition the vertices $V(T) = \{A|B\}$ such that B is the set of vertices of the subtree rooted at v. Said another way, B induces a connected rooted subtree with v as the root and contains all of v's descendants and A induces a connected subgraph that contains the root of T. This subgraph is itself a rooted tree and has u as a leaf. Set A and B to be tubes of τ.

3. Proceed likewise to construct tubings of the trees induced by A and B.

Every tubing of T can be constructed as above. Expanding on this in another way, to find all the tubings of a tree, for every edge in T, we “break” the edge and put the lower vertex and all of its descendants into one tube and the rest of the tree in another tube. Then for each tube and each edge in the tube, do the same until all of the tubes contain only one vertex. In other words, a tubing defines a binary tree structure on the edges of the original tree based on which stage in the hierarchy of the tubes we “break” the edge.

To improve readability, we will sometimes not draw the outermost tube, which encircles the full tree, nor the innermost tubes around each individual vertex.

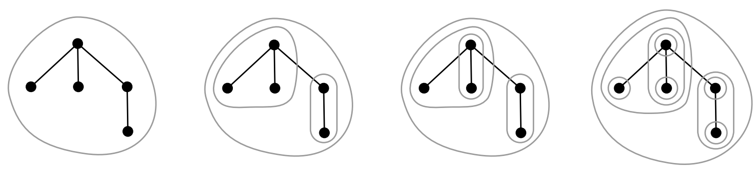


Figure 2.3: An intuitive way to build a tubing, where each tube is progressively partitioned into two tubes, shown from left to right. The left shows only the outermost tube and the right shows the entire tubing.

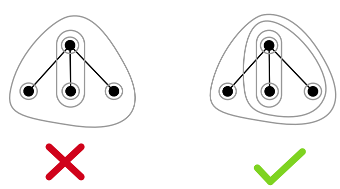


Figure 2.4: The left tubing is not a binary tubing because the outermost tube is partitioned into three tubes. The right tubing shows the correction of the left tubing by adding a new tube around some of the vertices.

Lemma 2.9. Let T be a rooted tree and τ a tubing of T. Then if t is any tube of τ, the tree induced by the vertices of t has a tubing given by t and those tubes of τ which are properly contained in t.

Proof. Immediate from the definition. □

Remark 2.10.

In view of the last lemma, a tubing is either

(i) the unique tubing of $T = \bullet$, or

(ii) determined by (τ', τ'') where τ' is a tubing of some proper rooted subtree t' and τ'' is a tubing of $t'' = T \setminus t'$, where t'' contains the root of T.

(Remark 2.7 in [2].)

Generally given a tubing τ of the tree $T \neq \bullet$ we will write t' and t'' for the two rooted trees induced by the tubes that partition the outermost tube of τ with the root of T being in t''. The sub-tubings τ' and τ'' denote the tubings of the subtrees t' and t'', respectively, as in Remark 2.10.

2.3 Chord Diagrams

Next, we introduce chord diagrams, which play a key role in the first two chapters of this thesis. Chapter 3 will discuss a map between chord diagrams of rooted trees, which is not only combinatorially interesting, but has direct applications to series expansions in quantum field theory.

Definition 2.11 (Rooted chord diagram)**.** A *rooted chord diagram* is a set of ordered pairs of the form (a_i, b_i) such that $\{a_1, b_1, a_2, b_2, \dots, a_n, b_n\} = \{1, 2, \dots, 2n\}$ and $a_i < b_i$ for each $1 \leq i \leq n$. The pair (a_i, b_i) is the i^{th} chord in the chord diagram. The chord such that $b_i = 2n$ is the *root chord* and the chord such that $a_i = 1$ is the *bud chord*.

Definition 2.12 (Right end and left end of a chord)**.** For a chord $c_i = (a_i, b_i)$, the *left end* of c_i is a_i and the *right end* is b_i.

Note in [2] and other literature [26, 7, 19, 6], the root chord is defined to be the leftmost chord, $a_i = 1$. The reason we choose the rightmost chord to be the root is in Chapter 3, we will see that the root chord corresponds to the root of a rooted tree, in the presented bijection. The term bud chord is introduced in this thesis and will be discussed more in Chapter 4.

Remark 2.13. If we consider the set $\{1, 2, \dots, 2n\}$ up to cyclic permutation, then the equivalence classes are unrooted chord diagrams.

Remark 2.14. In the graphical sense, an unrooted chord diagram is a 1-regular graph on a cyclically ordered vertex set. Here, chords are the edges of the graph. A rooted chord diagram is a 1-regular graph on a linearly ordered vertex set and the ordered pair (a_i, b_i) is a chord where a_i is its first vertex and b_i is its second vertex.

We will only consider rooted chord diagrams, so from now on, chord diagram indicates rooted chord diagram.

Definition 2.15 (Decomposable, Indecomposable)**.** A chord diagram is *decomposable* if its vertices can be partitioned into two sets, S, T, such that $s < t$ for all $s \in S$ and $t \in T$, and that there are no chords (a_i, b_i) with $a_i \in S$ and $b_i \in T$. In other words, all chords are either in set S or set T. A chord diagram is *indecomposable* if it is not decomposable.

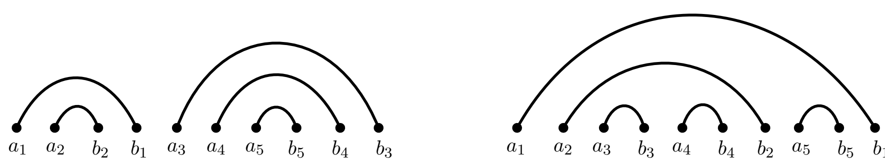


Figure 2.5: The chord diagram on the left is a decomposable chord diagram and the one on the right is an indecomposable non-crossing chord diagram

Intuitively, a chord diagram is decomposable if it is a concatenation of smaller nonempty chord diagrams.

Definition 2.16 (Crossing, Non-crossing)**.** We say a pair of chords (a_i, b_i) and (a_j, b_j) cross if $a_i < a_j < b_i < b_j$ or $a_j < a_i < b_j < b_i$. A chord diagram is *non-crossing* if it contains no pair of crossing chords.

Definition 2.17 (Intersection graph)**.** The *intersection graph* of a chord diagram is a graph such that for each chord c_i in a chord diagram, there is a vertex v_i in the intersection graph. Two vertices v_1 and v_2 are adjacent if the chords c_1 and c_2 cross in the chord diagram.

Definition 2.18 (Connected chord diagram)**.** A chord diagram is *connected* if its intersection graph is connected. A *connected component* of a chord diagram is a subset of chords S such that the chord diagram S is connected.

Every connected chord diagram is inherently indecomposable.

Note that connectivity in chord diagrams is not defined in the graph theoretical sense on the chord diagram as a 1-regular graph, but rather on the intersection graph, see Figure 2.6.

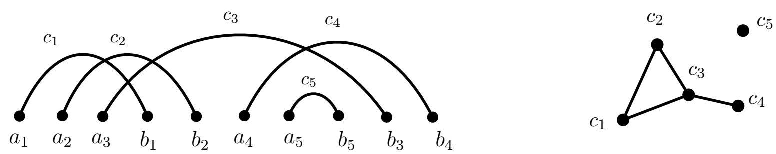


Figure 2.6: An indecomposable, non-connected chord diagram and its intersection graph.

In this thesis, we are primarily concerned with connected chord diagrams, such as in the example below.

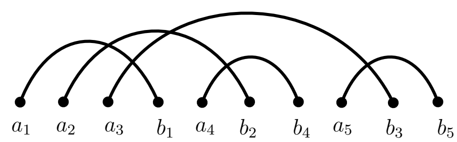


Figure 2.7: A connected chord diagram.

2.4 Posets

Partially ordered sets, or posets, are fundamental structures in combinatorics that are used to relate certain elements within a set. They are intrinsically related to rooted trees because of the hierarchical structure of them concerning the root and its descendants. In fact, as we will see in the coming chapters, forests of rooted trees can always be interpreted as posets, and thus we can broaden our perspective of rooted forests in this way.

Definition 2.19 (Partially ordered set (poset))**.** A set is a *partially ordered set* $(P, \leqslant)$ if there exists a relation, denoted $\leqslant$, such that for $a, b, c \in P$:

(i) transitive: if $a \leqslant b$ and $b \leqslant c$ then $a \leqslant c$.

(ii) reflexive: $a \leq a$

(iii) anti-symmetric: if $a \leqslant b$ and $b \leqslant a$, then $a = b$

For $a, b \in P$, if $a \leqslant b$ or $b \leqslant a$ then a and b are comparable. Otherwise, they are incomparable.

We use $a < b$ to mean $a \leq b$ and $a \neq b$.

Definition 2.20 (Antichain)**.** In a poset $(P, \leqslant)$, an *antichain* is a subset of elements of P such that all of its elements are incomparable.

A useful tool to visualize posets is a Hasse diagram. First we define the following relation.

Definition 2.21 (Covering relation)**.** For a poset $(P, \leqslant)$ and elements $a, b \in P$, we say that b *covers* a if and only if $a < b$ and there is no $c \in P$ such that $a < c < b$.

Given a poset $(P, \leqslant)$ and elements $a, b \in P$. A Hasse diagram can be drawn such that elements in P are vertices as described below.

Definition 2.22 (Hasse diagram)**.** Let $(P, \leqslant)$ be a poset and let a, b be elements of P. A *Hasse diagram* is a directed graph such that for vertices a, b, there is an edge ab such that $a \to b$ if and only if b covers a in $(P, \leqslant)$. Conventionally, edges are not drawn with arrows, but are oriented upwards.

In a Hasse diagram, maximal elements are displayed at the top and minimal elements at the bottom. In a Hasse diagram, it should be clear to see the ordering on elements, and that some elements may be incomparable. See the example below of a Hasse diagram.

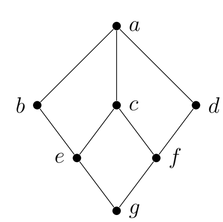


Figure 2.8: A Hasse diagram. We have orders $g < e < b < a$, $g < f < d < a$, $e < c$ and $f < c$. In this example, b, c, d are incomparable and e, f are incomparable. a is the unique maximal element, and g is the unique minimal element

Hasse diagrams are helpful in many types of posets, but particularly for us because we can easily view rooted trees as Hasse diagrams, where the root is the maximal element.

Definition 2.23 (Predecessor, Successor)**.** In an ordering, if $b < a$, then b is a *predecessor* of a and a is a *successor* of b. If a covers b then b is an *immediate predecessor* of a and a is an *immediate successor* of b

An immediate successor of a is the lowest ordered element c such that $a < c$ and the immediate predecessor of a is the highest ordered element b such that $b < a$.

When viewing posets as forests, we will sometimes refer to a predecessor as a descendent and a successor as an ancestor.

Definition 2.24 (Upset, Downset)**.** An *upset* of a poset $(P, <)$ is a subset S of P such that for every element $a \in S$, and all elements $b \in P$ such that $a < b$, b is also in S. That is, for every element in S, all of its successors are also in S. Similarly, a *downset* of $(P, <)$ is a subset S of P such that for every element $a \in S$, and all elements $b \in P$ such that $b < a$, b is also in S. That is, for every element in S, all of its predecessors are also in S.

Note. Upset may also be called upper set, or higher set. Likewise, downset may be called lower set.

Note that the empty set $\emptyset$ is both an upset and a downset.

Definition 2.25 (Ancestry partial order)**.** The *ancestry partial order*, $\prec$, of a tree is the partial order on vertices of a tree where $u \prec v$ if u is a descendent of v.

The root is maximal in ancestry order. We may view the rooted forests as Hasse diagrams. Note that this order is called level partial order in 4.1 of [27]. The name ancestry order was chosen because rooted trees have a hereditary structure, where all vertices are descendants of the root, and a vertex may have children or siblings for example.

Definition 2.26 (Generation)**.** Let a, b be two vertices in a forest. Then a and b are in the same *generation* if they have the same number of ancestors between themselves and the roots of their respective trees.

This notion is sometimes called level, and often Hasse diagrams will reflect this in the way they are drawn. In Figure 2.8, $b, c,$ and d are in the same generation and e and f are in the same generation.

The following definitions will be useful in Chapter 5.

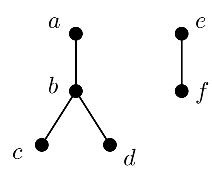
Definition 2.27 (Double poset)**.** A *double poset* is a triple $(P, \leq_1, \leq_2)$, where P is a finite set and $\leq_1$ and $\leq_2$ are two partial orders on P. (Definition 2.1 in [25])

Definition 2.28 (Plane poset)**.** A *plane poset* is a double poset $(P, \leq_h, \leq_r)$ such that for all $x, y \in P$ with $x \neq y$, x and y are comparable for $\leq_h$ if and only if x and y are not comparable for $\leq_r$. (Definition 1.2 in [13])

Definition 2.29 (Rightward order, $<_r$)**.** The *rightward order*, $<_r$, of a forest is partial ordering of vertices such that $u <_r v$ if $u < v$ in traversal order and u and v are $\prec$-incomparable.

The name rightward order was chosen because in plane forests, maximal elements appear on the right of the forest when viewed as a Hasse diagram with the roots at the top. We draw the plane structure as a left-to-right structure in the diagram. See Figure 2.9 for an example.

The practice of interpreting plane forests as plane posets will be explored in detail in Chapter 5. Below is an example of a plane poset with the two orders $<_r$ and $\prec$ (Definition 2.29 and Definition 2.25). One may observe that all pairs of vertices are comparable in exactly one of the two orders.



$c \prec b \prec a$
$d \prec b \prec a$
$e \prec f$
$a <_r e$
$a <_r f$
$b <_r e$
$b <_r f$
$c <_r d <_r e$
$c <_r d <_r f$

Figure 2.9: Example of $\prec$ (2.25) and $<_r$ (2.29) on a forest F. Here, b is a $\prec$-predecessor of a and f is a $<_r$-successor of a, for example.

Chapter 3

Bijection Between Tubed Rooted Trees and Connected Chord Diagram

We now define a bijection between tubings of rooted trees and chord diagrams. This is not only interesting on a purely mathematical level, but also has applications to Dyson-Schwinger equations, as discussed in [2]. The next section on re-rooting trees will make use of this bijection.

Definition 3.1 (Rooted tree insertion place)**.** Given a plane rooted tree T, a rooted tree insertion place (RTIP) is a place where a new subtree can be added. Specifically, at each vertex there is an insertion place before any of the children of the vertex, between any two consecutive children, and after the last child, and all insertion places are of this form.

We place an order on the RTIPs as follows. Let r be the root of T and let $t_1, t_2, \ldots, t_k$ be the list of subtrees (whose roots are the children of r in that order). Define the order on the insertion places, x_i, of T in this order:

- the insertion place before t_1.
- all insertion places of t_1 in this same order, applied recursively on t_1.
- the insertion place between t_1 and t_2.
- all insertion places of t_2 in this same order recursively on t_2.

  $\vdots$

- the insertion place after t_k

Additionally, we say that for RTIP x_i, its vertex is the vertex where a tree is joined to when inserted at x_i.

In other words, during a pre-order traversal on T, each vertex is given at least one insertion place. If a vertex v_i is a leaf, it only has one insertion place. If v_i has k children, then it has $k+1$ insertion places (between each of the children and to the left and right of them.)

To think of this intuitively, this order on insertion places can be visualized by drawing the tree and walking along the edges counter clockwise starting to the left of the root, where every time a vertex is visited an insertion place is added. See Figure 3.1.

When we say a tree has insertion place x_i, we mean that x_i is the i^{th} insertion place on the tree.

Lemma 3.2. Let T be a rooted tree with n vertices. Then the number of insertion places of T is $2n-1$.

Proof. Let T be a rooted tree with n vertices and let P_n be the number of insertion places of T. We claim that $P_n = 2n-1$ by induction. Base case $n=1$. There is only one insertion place on a single vertex. So $P_1 = 1$. Now, suppose for a plane rooted tree with n vertices that $P_n = 2n-1$. Suppose a new vertex, v_j, is added at an insertion place, x, on an existing vertex v_i. Note that every plane rooted tree on $n+1$ vertices can be constructed by adding a new vertex in this way to a plane rooted tree on n vertices. Then the new vertex v_j has one insertion place; v_i loses the x insertion place (replaced by the edge v_0v_1) and thereby gains two new insertion places on either side of the edge v_0v_1. Therefore, $P_{n+1} = P_n - 1 + 2 + 1 = P_n + 2$ for $n \geq 1$. Thus $P_{n+1} = 2n - 1 + 2 = 2(n+1) - 1$. □

Next we define similar insertion places for chord diagrams. The way chord diagrams are constructed is by inserting one chord diagram into another at an insertion place. The chord diagram insertion places can be visualized as the space between ends of chords in a chord diagram, see Figure 3.1. If there are n chords in a given chord diagram, then there are $2n$ chord ends, so the number of places between them is $2n-1$. The following is a formal definition of chord diagram insertion places for indecomposable chord diagrams.

Definition 3.3 (Chord diagram insertion place)**.** Let C be a chord diagram of size n. Let y_i be the pair $(i, i+1)$. Then the *chord diagram insertion places (CDIPs)* for C are given by y_i for $i \in \{1, 2, \ldots, 2n-1\}$. We order the chord diagram insertion places by their index.

Lemma 3.4. Let C be a chord diagram of size n. Then the number of insertion places of C is $2n-1$.

Proof. Let Q_n be the number of CDIPs on a chord diagram with n chords. We claim that $Q_n = 2n - 1$. This follows directly from the definition of chord diagram insertion places. There are exactly $2n - 1$ pairs $\{i, i + 1\}$ for $i \in \{1, 2, \ldots, 2n - 1\}$. □

Below shows the chord diagram insertion places and rooted tree insertion places.

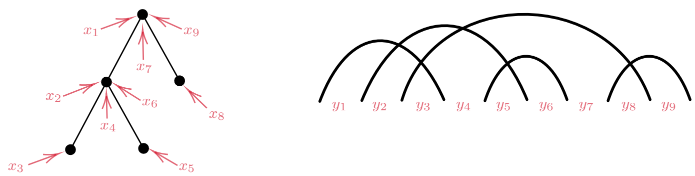


Figure 3.1: Left: rooted tree with RTIPs (Definition 3.1) x_i indicated (Definition 3.3). Right: The chord diagram with CDIPs y_i indicated.

Lemma 3.5. Let T'' a rooted tree with insertion places $x''_1, x''_2, \ldots, x''_k$ and let T' be a tree with insertion places $x'_1, x'_2, \ldots, x'_l$. Let T be the rooted tree constructed by inserting T' into T'' at T'''s IP x''_i.

1. The new tree T has $k + l - 1$ insertion places.
2. The indexing of the insertion places of T is as follows:
   - Every insertion place $x''_j \in \{x''_1, x''_2, \ldots, x''_{i-1}\}$ in T'' becomes x_j in T. In other words, there is no change from T'' to T for all insertion places before x_i.
   - The insertion place x''_i in T'' gets replaced by new insertion places because this is where T' is being inserted. The new insertion places are determined as follows. Every insertion place $x'_j \in \{x'_1, x'_2, \ldots, x'_l\}$ in T' becomes x_{j+i-1} in T.
   - Every insertion place $x''_j \in \{x''_{i+1}, x''_{i+2}, \ldots, x''_k\}$ in T'' becomes x_{j+l-1} in T.

   Therefore the new indices are as follows:

$$
\begin{array}{ccc}
x''_1 \mapsto x_1 & x''_i \mapsto x_i & x''_{i+1} \mapsto x_{i+l} \\
x''_2 \mapsto x_2 & \mapsto x_{i+1} & x''_{i+2} \mapsto x_{i+l+1} \\
\vdots & \vdots & \vdots \\
x''_{i-1} \mapsto x_{i-1} & \mapsto x_{i+l-1} & x''_k \mapsto x_{k+l-1}
\end{array}
$$

Proof.

1. T'' has k insertion places and T' has l insertion places. Because IP x_i is being replaced by the l insertion places, it must not be counted. Therefore T has $k+l-1$ insertion places.

2. - The insertion places in T'' before the IP where T' is inserted are $\{x''_1, x''_2, \ldots, x''_{i-1}\}$. They are not affected by the insertion of T' at x_i, so they remain the same in T.
   - IP x''_i gets replaced by IPs of T'. So we are adding l IPs after x_{i-1}, so T''s j^{th} insertion place becomes the $(j+i-1)^{th}$ IP of T.
   - This follows from the previous two points. That the last IP of T' is x_{i+l-1} so the rest of the IP's must be $x_{i+1}, \ldots, x_{k+l-1}$. Because l IPs are added before the j^{th} IP of T'' in the insertion, but one IP (x_i) is removed, then x''_j of T'' becomes x_{j+l-1} in T.

□

See the following example of how this indexing works.

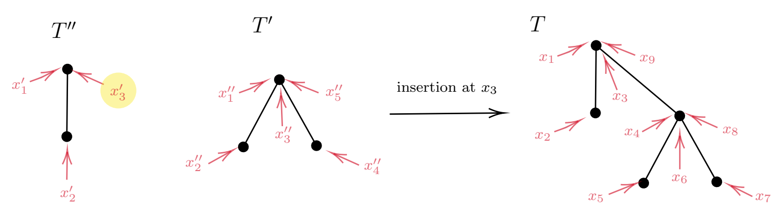


Figure 3.2: Here, $k = 3$ and $l = 5$, so T'' has 3 insertion places and T' has 5 insertion places. T is obtained by inserting T' into T'' at insertion place x_3, so $i = 3$. In the tree T, x_1, x_2, x_3 correspond to x'_1, x'_2, x'_3 in T' $x_4, \ldots, x_8$ correspond to $x''_1, \ldots, x''_5$ in T'', and x_9 is the new insertion place arising from the insertion.

The primary goal of this section is to define a bijection between tubings of rooted trees and connected chord diagrams. We have seen that when the same size, they have the same number of insertion places, which may give us insight of the connection between these two. Additionally, chord diagrams can be made by inserting some smaller chord diagram into another one [32], as we've seen with rooted trees, and specifically tubings of rooted trees. This insertion process will be defined in Definition 3.8. This bijection gives a connection between tubing expansions of Dyson-Schwinger equations and chord diagram expansions of Dyson-Schwinger equations.

We have seen that chord diagrams with n chords and rooted trees with n vertices have the same number of insertion places. Now we introduce a map from RTIPs to CDIPs.

Definition 3.6. Given a plane rooted tree t with n vertices and a chord diagram C with n chords, let x_i be the ith RTIP of t and let y_i be the ith CDIP of C under the orders given in Definition 3.1 and Definition 3.3. Then define the map f via $f(x_i) = y_i$.

Lemma 3.7. The number of RTIPs on a plane rooted tree t with n vertices is the same as the number of CDIPs on a chord diagram C with n chords and therefore f is a bijection between the RTIPs of t and the CDIPs of C.

Proof. Let P_n be the number of RTIPs on a plane rooted tree with n vertices. Let Q_n be the number of CDIPs on a chord diagram with n chords. By Lemma 3.4 and Lemma 3.2, $P_n = Q_n$ and hence f is a bijection. □

In a general sense, one can insert a chord diagram into another at any insertion place. Definition 3.8 describes the particular method we will use, but it is not the only way a chord diagram can be inserted. Below we have a corollary pertaining to the insertion of a chord diagram into another yielding an indecomposable chord diagram.

Now we are ready to define the main bijection.

Definition 3.8. Let τ be a tubing of a plane rooted tree t with n vertices. Then, define the chord diagram $\phi(\tau)$ as follows.

Base case: if $t = \bullet$ then τ is the unique tubing of $\bullet$ and $\phi(\tau)$ is the unique chord diagram with one chord.

Recursive step: Let t'', t', τ'', and τ' be as in Remark 2.10. Let C'' and C' be $\phi(\tau'')$ and $\phi(\tau')$ respectively. Let x be the insertion place where t' is inserted into t''. Let y be the CDIP given by $f(x)$ (relative to C''). Then define $C = \phi(\tau)$ as follows: Insert C' into y such that the resulting chord diagram is a disconnected indecomposable chord diagram. Let q' be the bud chord of C'. Take the left end of q' and move it before all chords of C''. The resulting chord diagram is C. Graphically, the procedure is shown in Figure 3.3.

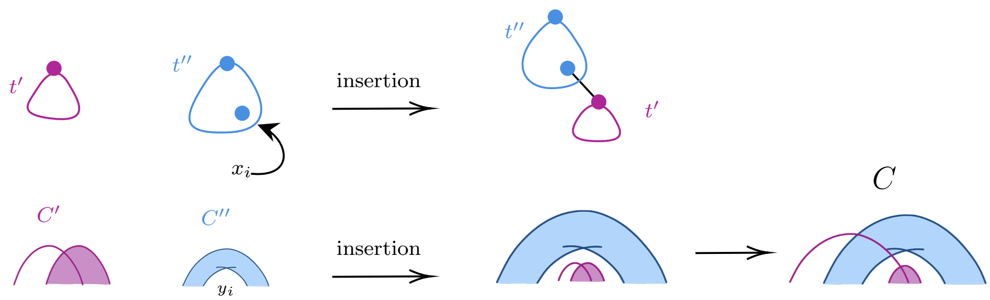


Figure 3.3: Recursive step of the insertion of a chord diagram into another.

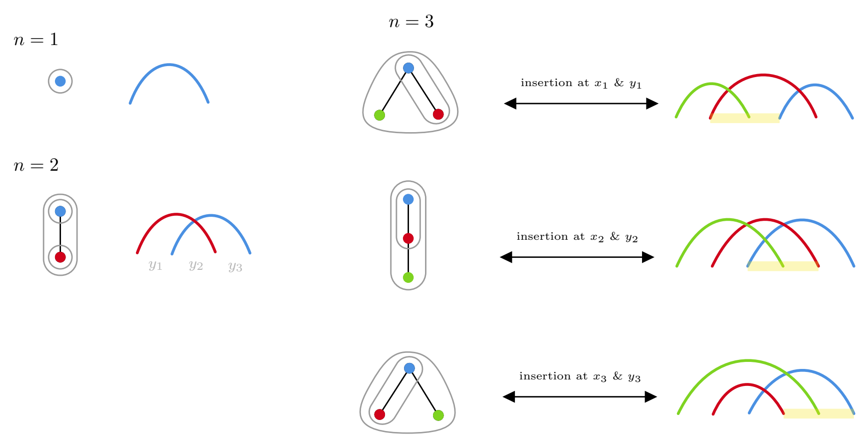


Figure 3.4: On the left, we have the first two rooted trees and chord diagrams where $n = 1$ on top and $n - 2$ in the middle. On the bottom left, the RTIPs and CDIPs are shown. On the right, we have the insertion of a singleton into each of the three insertion places on the $n = 2$ rooted tree/chord diagram.

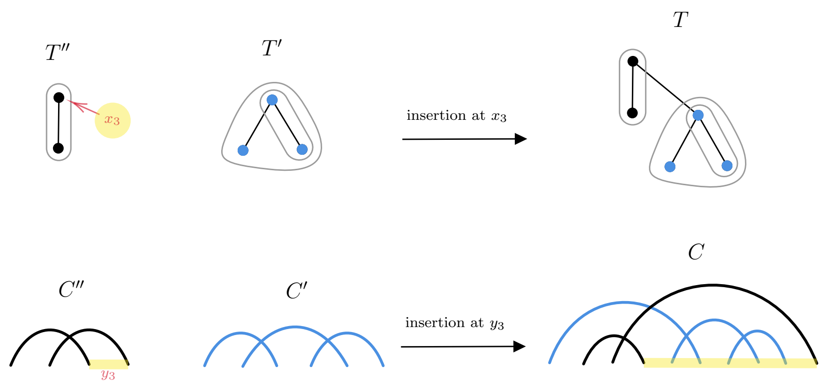


Figure 3.5: An example of the insertion of C' into the third insertion place of C''.

Wilf and Nijenhuis described this chord diagram insertion in [32] used this insertion to enumerate chord diagrams, and Stein proved in [35] that every connected chord diagram is generated exactly once in this way. It is called root-share decomposition in [26].

Corollary 3.9. By Lemma 3.7, and for chord diagrams C'' and C', let C be the indecomposable chord diagram obtained by inserting C' into C'' at the i^{th} insertion place of C'', defined in Definition 3.8. Then the indices of C are the same as described in Lemma 3.5.

Remark 3.10. The description provided in Definition 3.8 is a more intuitive way to describe the map ϕ. The following is a more formal definition of the recursive step of the map. With $t'', t', \tau'', \tau', C''$, and C' as above, define $C = \phi(\tau)$ as follows:

Let $\{1, \ldots, 2n\}$ be the vertices of C'' with CDIP $y = \{i, i+1\}$. Similarly let $\{1, \ldots, 2m\}$ be the vertices of C' with bud chord $(1, k)$. The vertices of C are $\{1, \ldots, (2n + 2m)\}$. The chords of C are defined from the chords of C'' and C' as follows:

(i) For each chord $\{a_j, b_j\}$ from C':

- the bud chord $\{1, k\}$ becomes $\{1, k + i\}$
- all other chords $\{a_j, b_j\}$ become $\{a_j + i, b_j + i\}$.

(ii) For each chord $\{a_j, b_j\}$ from C'':

- if $a_j \leq i$ and $b_j \leq i$, then $\{a_j, b_j\}$ becomes $\{a_j + 1, b_j + 1\}$
- if $a_j \leq i$ and $b_j > i$, then $\{a_j, b_j\}$ becomes $\{a_j + 1, b_j + 2m\}$

- if $a_j > i$ and $b_j \leq i$, then $\{a_j, b_j\}$ becomes $\{a_j + 2m, b_j + 2m\}$

(or to say it another way, if a_j (or b_j) $\leq i$, then a_j (or b_j) becomes $a_j + 1$ (or $b_j + 1$), and if a_j (or b_j) $> i$, then a_j (or b_j) becomes $a_j + 2m$ (or $b_j + 2m$)).

Write $\tau' \oplus_x \tau''$ for the tubing obtained by inserting τ' into τ'' at insertion place x and write $C' \oplus_y C''$ for chord diagram obtained by inserting C' into C'' at insertion place y in the above manner.

Theorem 3.11. ϕ is a bijection from the set of tubed rooted plane trees with n vertices to the set of rooted connected chord diagrams with n chords. (Theorem 5.15 in [2].)

Proof. To prove this, we construct the inverse map, $\mu : C \mapsto \tau$.

Let C be a rooted connected chord diagram. If C is just one chord, then define $\mu(C)$ to be the unique tubing of $\bullet$. Otherwise, let r be the bud chord. The removal of the bud chord results in either a connected chord diagram, or a disconnected, indecomposable chord diagram. Note that this indecomposable diagram will be a sequence of nested connected components. Recall the definition of disconnected chord diagrams from Definition 2.17. We take r together with all but the outermost component of $C \backslash r$ and call it C', and let the rest of the chord diagram, namely the outer component, be $C'' = C \backslash C'$, which yields connected chord diagrams C'' and C'.

The last thing we need is the insertion place where C' is inserted into C'' to give the chord diagram C. To specify this, let η be the smallest index in C greater than 1 of a chord end in C'. In C, the insertion place to the left of this chord is $y_{\eta-1}$. So, looking at insertion places of C'', we see that $C = C' \oplus_{\eta-2,\eta-1} C''$ and so $y_{\eta-2}$ is the insertion place in C'' we are looking for. This is because when C' is inserted into C'', the bud chord of C' is shifted all the way to the left of C'', as described in Definition 3.8; and by Lemma 3.5, the CDIPs to the left of C' in C are increased by 1 after insertion, thus in the inverse, we subtract 1. Recall from Definition 3.6 and Lemma 3.7 that there exists a map f^{-1} that takes a CDIP and gives an RTIP. Recursively let $\tau'' = \mu(C'')$ and $\tau' = \mu(C')$ and $x = f^{-1}(y)$ as an RTIP of t''. Now define t to be the insertion of t' into t'' at x and define τ to be the tubing of t given by the tubes of τ'' and τ' along with one tube containing all of t. Then t and τ give $\mu(C)$.

Directly from these definitions, we see that μ and ϕ are mutual inverses, and the number of vertices in t is equal to the number of chords in C. $\square$

Remark 3.12. Let's look at the physical context in an example. In a piece of Yukawa theory studied in [4], the Dyson-Schwinger equation is

$$G(x,L) = 1 + xG(x, \frac{\partial}{\partial\rho})^{-1}(e^{L\rho} - 1)F(\rho)\Big|_{\rho=0}$$

where $F(\rho) = \sum_{i\geq 0} c_i \rho^{i-1}$.

This is a special case of (1) from [2]. The tubing solution of this equation is

$$G(x,L) = 1 + \sum_t x^{|t|} \sum_\tau \prod_{\substack{v\in V(t)\\ v\neq rt(t)}} c_{b(v,\tau)-1} \sum_{i=1}^{b(\tau)} c_{b(\tau)-i} \frac{L^i}{i!}$$

where the outer sum is over plane rooted trees and the middle sum is over tubings of t. This is the corresponding special case of the solution discussed in [2] on page 9. Earlier work [26] gave a chord diagram solution to the same Dyson-Schwinger equation, which has the form

$$G(x,L) = 1 + \sum_C x^{|C|} c_0^{|C|-k-1} c_{d_k - d_{k-1}} c_{d_{k-1}-d_{k-2}} \cdots c_{d_2 - d_1} \sum_{i=1}^{d_1} c_{d_1 - i} \frac{L^i}{i!}$$

where the outer sum is over rooted connected chord diagrams C and the terminal chords of C are at indices $d_1 < \cdots < d_k$. The definition of terminal chord and the order in which they are indexed can be found in [26] and [2]. The bijection described in this chapter provides the link between these two expansions. The verification of the various parameters can be found in Section 5.3 of [2].

Chapter 4

Re-rooting Tubed Rooted Trees

In this chapter, the previous bijection between chord diagrams and tubed rooted trees will be used to study how the chord diagram corresponding to a tubed rooted tree changes when trees undergo a process called re-rooting. We aim to be able to find the chord diagram of a re-rooted tree without finding the re-rooted tree first, in other words, to go directly from one chord diagram to another. Part of the motivation of studying this is that tubings don't rely on the rooted structure, but the bijection does, and this relationship is interesting. In the broader sense, this doesn't have immediate applications, but it is useful in exploring ways in which chord diagrams relate to the rooted tree, and understanding the bijection from Chapter 3 more thoroughly. It is an interesting interpretation of re-rooting trees, which is a common tree operation in mathematics.

4.1 Preliminaries

Broadly speaking, re-rooting a rooted tree is the process of taking any of the tree's other vertices to be the root, while preserving the graphical structure of the tree. The following is a specific process of re-rooting a tubed rooted tree, which will be simply be called re-rooting, given in the following definition.

Definition 4.1 (Re-rooting)**.** Let T be a rooted tree with $n > 1$ vertices and root r_B. Let r_P be the leftmost child of r_B, which is the second vertex in traversal order (r_B being the first). *Re-rooting* the tree is a map, ϕ that takes r_P to be the new root, while preserving the graph structure of the tree. The adjacency of the vertices of T remains the same, but it becomes a different rooted tree T'. Additionally, the planar structure is preserved, where the order of a vertex's children is preserved before and after re-rooting, and when a parent becomes a child the cylic order around the vertex is preserved. With tubed rooted trees, the tubes remain the same on the tree during the re-rooting, meaning for a tube t_i of T, there is a tube t'_i in $\phi(T)$ which contains all the same vertices in t_i.

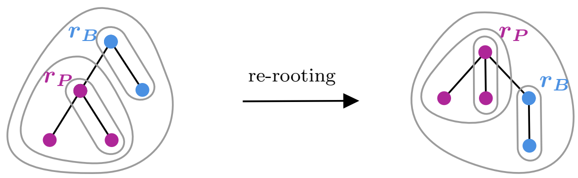


Figure 4.1: An example of re-rooting a tubed rooted tree.

For the rest of this chapter, for some feature κ of a tubed rooted tree, such as vertex or tube, we use $c(\kappa)$ to be the corresponding feature of its chord diagram. For example, for the vertex v in a tree, $c(v)$ is the chord corresponding to v. κ could also indicate a tube or the tree itself. Note however for insertion places, the notation used in Definition 3.1 and Definition 3.3 is x_i and y_i for RTIP and CDIP respectively. Although $c(x_i) = y_i$, in this chapter we will continue to use the notation x_i and y_i for insertion places of trees and chord diagrams.

Definition 4.2 (Rooted path)**.** A tree is a *rooted path* if it is a path in the graph theoretic sense and any of its vertices is the root. Note that the root does not need to be one of the terminal vertices.

Lemma 4.3. Let r_B be the root of a tree T and let r_P be the root of the tree T' after re-rooting T. Suppose r_B has q children and r_P has s children, which implies r_B has $(q+1)$ insertion places in T, and r_P has $(s+1)$ insertion places in T. Then:

1. r_B has q insertion places in T'
2. r_P has $s+2$ insertion places in T'

We use the letters B and P because they are drawn in blue and purple respectively in the figures.

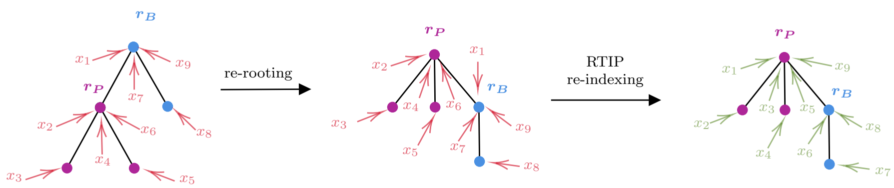


Figure 4.2: In this example, r_B has $q = 2$ children and 3 insertion places, x_1, x_7, x_9, in T and r_P has $s = 2$ children and 3 insertion places, x_2, x_4, x_6 in T. After re-rooting, in T', r_B has two insertion places and r_P has 4 insertion places.

Proof.

1. In T, r_B has q children, but r_P is one of them. Thus, after re-rooting, r_P is no longer a child of r_B. So r_B has $q - 1$ children in T', and thus has q insertion places.

2. In T, r_B is not a child of r_P but after re-rooting, r_B is a child of r_P. Additionally, r_P does not lose any children in the re-rooting. Thus, in T', r_P has $s + 1$ children and therefore has $s + 2$ insertion places.

□

The first and last insertion places essentially merge during the re-rooting. This can be thought of as the first one rotating clockwise and merging with the first one in its path, which is the last insertion place of T. See Figure 4.3. Additionally, the insertion place that r_P gains is the last insertion place of the tree T', which can be visualized as splitting what was the first insertion place of r_P in two.

Lemma 4.4. Let T be a tree with n vertices, and let x_i be an insertion place of T and let x'_i be the same insertion place relative to its vertex after re-rooting. Then, during every re-rooting, $x'_i = x_{i-1}$ for $2 \leq i \leq 2n - 1$. In other words, if a tube of a tubing is located on the i^{th} insertion place, then after re-rooting, it will be in the $(i-1)^{\text{th}}$ insertion place (for $2 \leq i \leq 2n - 1$.)

Proof. A tree T with $n > 1$ vertices has root r_B and after re-rooting, the new tree T' has root r_P. Let $x_1, \ldots, x_m$ be the insertion places of the tree T. Then x_1 is the first insertion place on the root r_B before r_P and x_2 is the first insertion place on r_P. Let T' be the tree after re-rooting with insertion places $x'_1, \ldots, x'_m$. Because r_P is the root after re-rooting, x_2 becomes the first insertion place of the tree T', so $x'_1 = x_2$. After the re-rooting, x_1 disappears because r_B loses an insertion place.

Because the order of the insertion places doesn't change in the re-rooting, the loss of x_1 causes every insertion place index to decrease by 1.

□

Furthermore, because the i^{th} RTIP is exactly the i^{th} CDIP for every i among all insertion places, thus the result is analogously true for CDIPs.

See the below figure for a visualization.

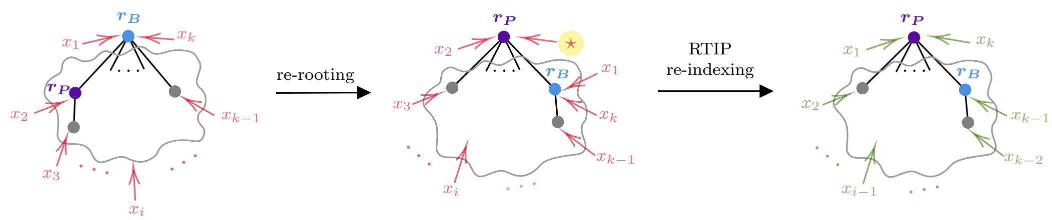


Figure 4.3: This shows the IP shifts on re-rooting a general tree. The last insertion place after re-rooting, denoted by $\star$ is the new IP gained by r_P. One can see that the insertion place marked x_1 combines with x_k after re-rooting.

Definition 4.5 (Bud of a tube)**.** The *bud* of a tube is a vertex defined recursively as follows:

1. If the tube is a single vertex, then that vertex is the bud
2. Suppose the tube t is comprised of tube t' being inserted into tube t''. Then the bud of t is the bud of t'.

Intuitively, the bud can be understood to be the last vertex inserted of each tube, which is the bud of the last inserted tube. The following is an example of a tubed rooted tree and its buds.

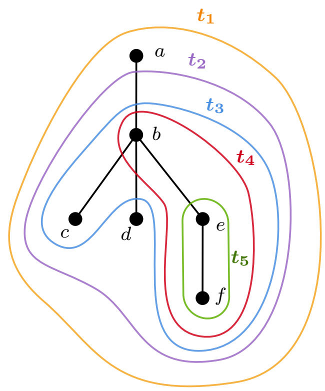


Figure 4.4: An example of a tubed rooted tree with tubes marked $t_1, \ldots, t_5$ (only marking tubes of size > 1 because otherwise, the bud is trivial). The buds, b_i, of each tube t_i are $b_1 = d, b_2 = d, b_3 = c, b_4 = f, b_5 = f$

Lemma 4.6.

1. The root of a tube, r_i, always corresponds to the chord whose right end is the farthest right chord of all the chords in that tube.

2. The bud of a tube, b_i, always corresponds to the chord whose left end is the farthest left chord of all the chords in that tube.

Proof. We use Theorem 3.11.

By induction. When the tree is a single vertex, then that vertex is both the root and bud, and the chord diagram is a single chord, thus its right end is the farthest right in the chord diagram and its left end is the farthest left in the chord diagram. Now, assume the statement is true for every tree and chord diagram with $k < n$ vertices/chords. Let t some tree with n vertices such that t is obtained by the tube t' being inserted into t''.

1. Suppose t'' has k vertices. The root, r'', of t'' is also the root of t. Additionally, $c(r'')$ is the rightmost chord of $c(t'')$ by the induction hypothesis. Then because $c(t')$ is inserted into a CDIP, and all CDIPs are to the left of the right end of the root chord, thus the root chord $c(r'')$ is the rightmost chord of $c(t)$.

2. Suppose t' has k vertices. The bud, b of t is the bud of t', by definition. By the induction hypothesis, the bud of t corresponds to leftmost chord of $c(t')$. When $c(t')$ is inserted into $c(t'')$ to form $c(t)$, the left end of the bud of $c(t')$ is pulled all the way to the left, and thus the bud chord is the leftmost chord in $c(t)$.

□

Observe that for tree a T and tube $t_i = T$, the bud of t_i corresponds to the bud chord as defined in Definition 2.11, which is the leftmost chord of the entire chord diagram.

Definition 4.7 (Root change)**.** If a tube t_i has a different root before and after re-rooting, then we say it has a *root change*.

Definition 4.8 (Bud change)**.** If a tube t_i has a different bud before and after re-rooting, then we say it has a *bud change*.

This leads us to the following lemma, which is important in the partitioning the vertices discussed in the next section.

Note that at least one tube exists that contains both r_B and r_P because the maximal tube does.

Definition 4.9 (t_*)**.** For a tree T which $n > 1$ vertices, let t_* be the minimal tube containing both r_B and r_P.

Lemma 4.10. Let T be a tree with at least 2 vertices. In the re-rooting of T, the only subtubes of T which have a root change are t_* and all tubes containing t_*. All tubes not contained in t_* and tubes inserted into t_* have the same root in T and $\phi(T)$.

Proof. Let T be the tree being re-rooted. Let t_* be as defined in Definition 4.9. Because t_* contains both r_B and r_P, then it has size > 1, so it is made up of two disjoint subtubes. Furthermore, one of these subtubes contains r_B and the other contains r_P because otherwise t_* would not be minimal. Let t_B and t_P be these disjoint subtubes containing r_B and r_P, respectively. In T, t_B is the root, and in $\phi(T)$, t_B is the root. Thus, r_B is the root of t_* and r_P is the root of $\phi(t_*)$, so t_* has a root change. Furthermore, tubes containing t_* have a root change because their roots are the root of t_*, which has a root change. Now we consider the tubes not containing t_*, broken down in two cases: proper subtubes of t_* and tubes not contained in t_*.

Case 1: All subtubes of t_* are contained in either t_B or t_P. The root of t_B is r_B in both T and $\phi(T)$ and likewise for t_P and r_P. Thus the roots of all subtubes of t_B and t_P remain the same in T and $\phi(T)$ because each subtube is inserted into disjoint subtube that contains the root t_B or t_P.

Case 2: All tubes not contained in t_*, which don't contain t_*, are contained in some tube inserted into a vertex of t_*. Let t_i be some such tube contained in a tube inserted into t_* at vertex $u \in t_*$ and let t_i have root r_i. Then for any vertex $v \in t_i$, there is a unique path from v to t_P that must go through r_i and u (where $r_i \preceq u$). Because the tube inserted into t_* is inserted at u before and after re-rooting, then after re-rooting, $r_i \preceq u$, and the path from v to t_B also goes through r_i and u. Thus r_i is the root of t_i after re-rooting.

□

This leads to the following corollary.

Corollary 4.11. For a tree T, every tube that does not contain t_* is the same rooted tree before and after re-rooting.

Proof. Because every tube that is not T or t_* has the same root before and after re-rooting, there is no way that the tree may change if the root is the same. □

Corollary 4.12. Let T be a tree with at least 2 vertices. Every time T is re-rooted, there is exactly one tube, namely t_*, in which a bud change occurs.

Proof. This follows from Lemma 4.10 and Corollary 4.11. Because no tubes other than tubes containing t_* change, the only bud changing tube must contain t_*. Now, t_* must have a bud change because it is comprised of two disjoint subtubes: t_P and t_B. By definition, the bud of T is in t_P. But after re-rooting, t_P contains the new root, and therefore the bud of t_B must be the bud of t_* after re-rooting. Furthermore if $t_i \supset t_*$, then t_i is obtained by inserting some tube t' into t_* or into some tube containing t_*. Therefore, the bud of t_i is in t' and because the insertion of t' also happens after re-rooting with t' unchanged, the bud of t_i remains unchanged after re-rooting. □

Definition 4.13 (Active tube)**.** The unique tube with the bud change, denoted t_* is called the *maximal active tube*, and all subtubes of it are called *active tubes*.

Definition 4.14 (Inactive tube)**.** All tubes that are inserted into t_*, and not contained in t_* are called *inactive tubes*.

Note that tubes properly containing t_* are neither active nor inactive. When we say an *inactive insertion*, we mean that a tube is inserted into a vertex in t_*. A tubing such that the maximal tube is strictly larger than t_*, has n inactive insertions if n tubes were inserted into it or a previous inactive insertion. Additionally, vertices within active tubes are called active vertices. See the example below.

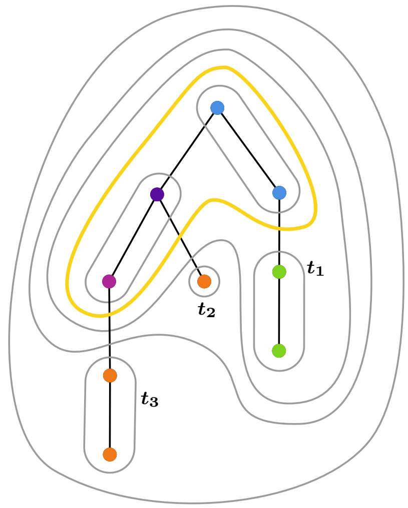


Figure 4.5: This tubed rooted tree has 3 inactive insertions, marked t_1, t_2, and t_3. All vertices in these tubes are inactive vertices, and the vertices in t_* are active vertices.

4.2 Marking Tubed Rooted Trees

The method for re-rooting the chord diagram corresponding to a tubed rooted tree relies on partitioning the vertices of the tree into four markings. In all the figures in this chapter, we color these four as blue (B), purple (P), orange (L), and green (R). The L and R are named so because in rooted paths, the orange and green vertices are on the left and right, respectively. For every vertex v_i in the rooted tree, we mark the chord corresponding to that vertex, $c(v_i)$ the same. However, for some of the chords, we will mark just the left end of the chord differently, which will be explained in more detail at the end of this section.

Let t_* (which we will color yellow in the figures) be the maximal active tube, that is, the tube with the bud change. We know that t_* is split up into two subtubes; call the two disjoint maximal subtubes t_B and t_P. Then the root, r_B, of the entire tree is also the root of the tube t_B, and the new root after the re-rooting, r_P, is the root of the tube t_P. Additionally, let b be the bud of t_B and b_P be the bud of t_P. We color all vertices in t_B and t_P blue and purple, respectively. We will give r_P a darker color of purple to indicate its importance in the re-rooting, but this is not strictly necessary for the rules we define. In the chord diagram, chords $c(t_P)$ and $c(t_B)$ are colored the same as the vertices.

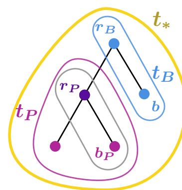


Figure 4.6: coloring the vertices of the active tubes. The maximal active tube, t_*, is shown in yellow.

Determining the coloring of vertices of inactive tubes is straightforward if the tree is a path. In the path case, we partition the rest of the vertices as follows. Let the tube inserted into vertices of P (here, the bud of t_P) be marked L (colored orange), and all other tubes inserted into L vertices also be marked L. Similarly, let the tube inserted into B-vertices (i.e. the bud b) be marked R (colored green), and all other tubes inserted into vertices of R also be marked R. Visually, we see that all vertices of L are on the left side of the tree, and all vertices of R are on the right side of the tree.

Figure 4.7: Example of a path tree partitioned with markings B,P, L, and R and its corresponding chord diagram.

Now, if the tree is *not* a path, we have the exact same marks B, P, L, R for vertices, but determining how to color each vertex is not as simple. We use the same rule for the active vertices as described earlier, but the inactive tubes have new rules. For inactive vertices, we introduce a method for determining which marking they get, using a special insertion place, x_μ, which we will call the split place. The purpose of x_μ is to determine whether an inserted tube is L or R marked.

Let T'' be a tree with insertion places $x_1, \ldots, x_\mu, \ldots x_{n+1}$. If a tube is inserted into T'' into any IP x_j such that $1 \leq j \leq \mu$, then it is marked L. Otherwise it is marked R. In other words, a tube t_i inserted at x_μ or any IP before x_μ (in pre-order traversal), then t_i is marked L.

We first will define the split place y_μ of a chord diagram, as it is straightforward and easy to see.

Definition 4.15 (Split place)**.** The *split place*, y_μ of a chord diagram is the CDIP immediately to the left of the right end of the root r_P of $c(t_P)$. For chord diagram $c(t'')$ with spit place $y''_{\mu''}$, if the inactive insertion $c(t')$ is inserted into or to the left of y_μ, then $c(t')$ will be colored orange, and if $c(t')$ is inserted to the right of y_μ, then $c(t')$ will be colored green.

Visually, we can see y_μ as the rightmost insertion place under r_P (dark purple).

The following lemma explains how we get the exact value of the index μ of the split place of a chord diagram.

Lemma 4.16. Suppose a chord diagram $c(t'')$ is composed only of active chords, meaning its corresponding tubed rooted tree t is the maximal active tube, t_*. Suppose the tube t_P has m insertion places. Then the split place, y_μ, of $c(t'')$ has index $\mu = m + 1$.

Otherwise suppose a chord diagram $c(t'')$ has some inactive insertions. Let μ'' be the index of the split place $y''_{\mu''}$ in the chord diagram $c(t'')$. Suppose a chord diagram $c(t')$ with k insertion places is inserted into $c(t'')$ to form $c(t)$ with split place y_μ

1. If $c(t')$ is inserted at or to the left of $y''_{\mu''}$ (and therefore colored orange), then the index of the split place of $c(t)$ is $\mu = \mu'' + k + 1$

2. If $c(t')$ is inserted to the right of $y''_{\mu''}$ (and therefore colored green), then the index of the split place of $c(t)$ is $\mu = \mu'' + 1$.

Proof. The first part of the lemma concerns the case where all the tubes are active tubes, meaning all vertices are either in t_P or in t_B. Now, $c(t_P)$ has m insertion places, thus the rightmost one, y''_m is the insertion place immediately to the left of the right end of $c(r_P)$. Because $c(t'')$ is comprised of $c(t_P)$ being inserted into the first insertion place of $c(t_B)$, we have that $\mu = m + 1$ because the bud chord of $c(t_B)$ adds one insertion place to the left of $c(t_P)$.

The second part of the lemma concerns the case where there are inactive tubes in the chord diagram $c(t'')$.

1. Insertion of $c(t')$ into IP y''_i where $i \leq \mu''$. This means there are k IPs added to the left of $y''_{\mu''}$ in addition to one extra one because the insertion splits the y''_i of $c(t'')$ into two insertion places. So the rightmost IP under r_P has index $\mu'' + k + 1$.

2. Insertion of $c(t')$ into IP y''_j where $\mu'' \leq j$. This means that the number of insertion places that $c(t')$ adds to the chord diagram doesn't affect $y''_{\mu''}$ except for the bud of $c(t')$ whose left end is added to the left of $y''_{\mu''}$. Thus the rightmost IP under r_P has index $\mu'' + 1$.

□

The following is an example of how this works.

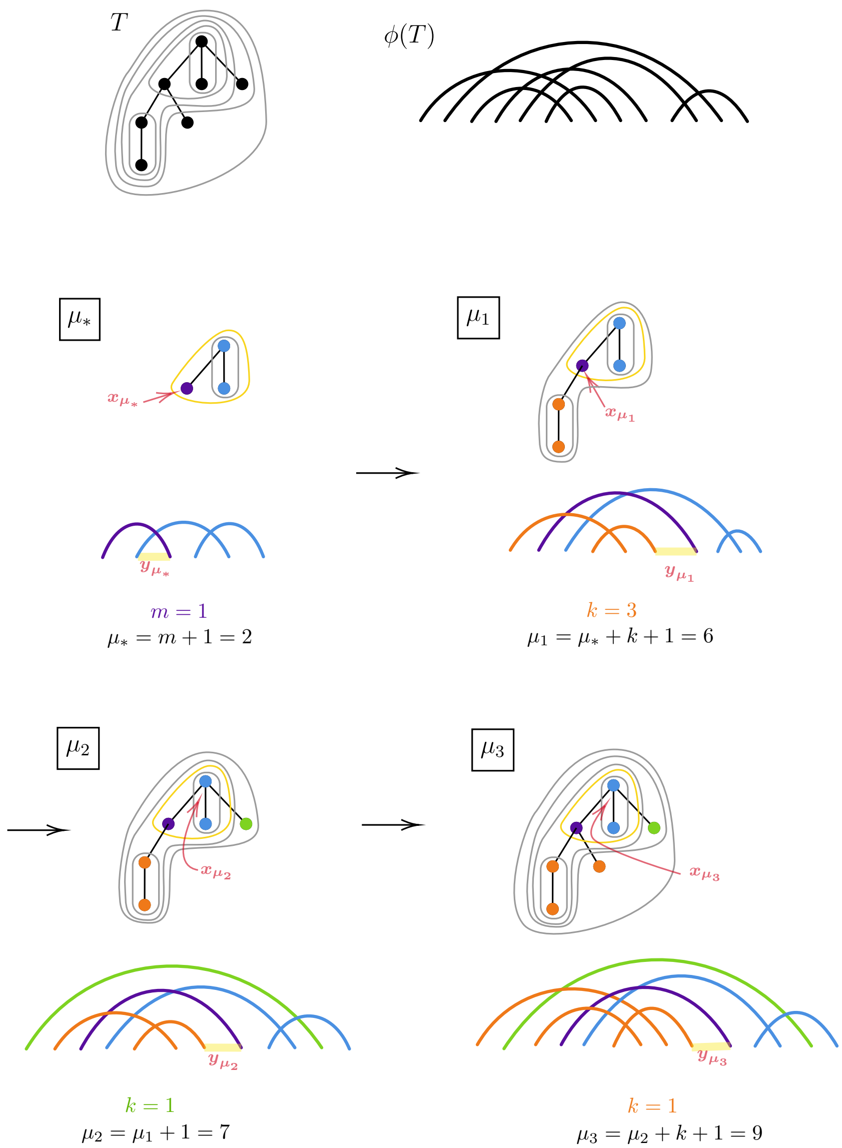


Figure 4.8: How to recursively build the chord diagram with the correct coloring using the index μ. One can see that y_μ in the chord diagram corresponds to x_μ in the rooted tree, and that the coloring of the vertices agrees with the coloring of the chords. The first insertion is colored orange because it is inserted at x_{μ_*}, the second is colored green because it is inserted after x_{μ_1} and the third is colored orange because is inserted before x_{μ_2}. Finally, we end up with x_{μ_3}, which tells us how to color future insertions.

The value of μ for the split place y_μ in a chord diagram gives us the split place x_μ in the tubed rooted tree. Thus, we use the value given in the above lemma for coloring the vertices of tubed rooted trees.

Another observation is that when the insertion is colored orange, in the new tree, x_μ remains the same relative to the vertex it is attached to, even though the index μ changes. For example, in Figure 4.8, x_{μ_2} and x_{μ_3} are in the same place relative to the vertices. Furthermore, when the insertion is colored green, x_μ shifts to a higher place in the pre-order traversal, the degree of which depends on the size of the insertion. In Figure 4.8, x_{μ_2} is the next insertion place after x_{μ_1} because $k = 1$ in the green insertion.

Finally, we will adjust the colors on the ends of some of the chords, which will be important in distinguishing between different types of chords during the re-rooting. Recall the bud of the tube t_* is marked B (colored blue).

Changing ends of chords

In the chord diagram, color just the left end of $c(b)$ orange. Additionally, for each set of orange chords corresponding to an inserted tube, mark the left end of its bud chord R. For the example in Figure 4.7, the updated chord diagram is shown below.

Figure 4.9: Example from the path rooted path in Figure 4.7, where the left end of some chords have new colorings.

An example for changing the marking of the ends of chords corresponding to a non-path tree is shown below.

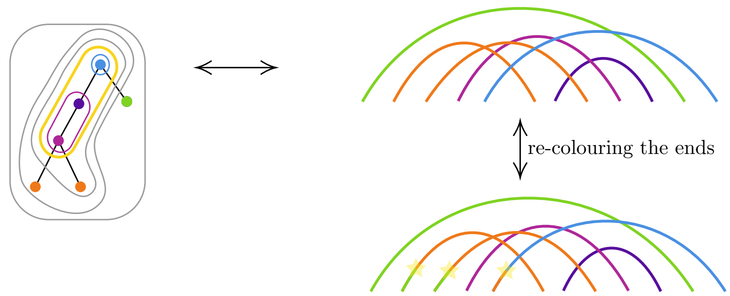


Figure 4.10: We show how to change the color of ends of certain chords.

The non-path example from Figure 4.8 is shown below with the updated chord ends.

Figure 4.11: Updated chord end marking from Figure 4.8

Definition 4.17 (Block)**.** In a chord diagram, for a chord marking w, a consecutive *block* of ends W is a collection of ends of chords $a_i, \ldots, a_j$ such that every end is marked w. The chords whose ends are in a consecutive block are not necessarily all colored w, but the ends in the block must be.

For example, in Figure 4.11, the three leftmost chord ends are a consecutive block of ends because they are each colored green, despite the second and third chord being orange.

4.3 Re-rooting Tubed Rooted Trees

The goal of this chapter is to learn how re-rooting affects the chord diagrams. More specifically, given a tubed rooted tree, after identifying its coloring, we can find the chord diagram corresponding to the next tree after re-rooting directly from the initial chord diagram.

First, the algorithm for going from initial chord diagram to post-re-rooting chord diagram is described below.

Definition 4.18 (Algorithm for re-rooting chord diagrams)**.** *We perform the re-rooting on the chord diagrams by first identifying which steps will be taken and then performing them simultaneously.*

1. The right end of the chord $c(r_P)$ is moved to all the way to the right, so that it is the rightmost chord.

2. For each maximum consecutive block of $c(L_e)$ chord ends, swap the block with the block's immediate leftward neighbor.

We call this map ρ. Thus for a chord diagram $c(T)$, the algorithm produces $\rho(c(T))$.

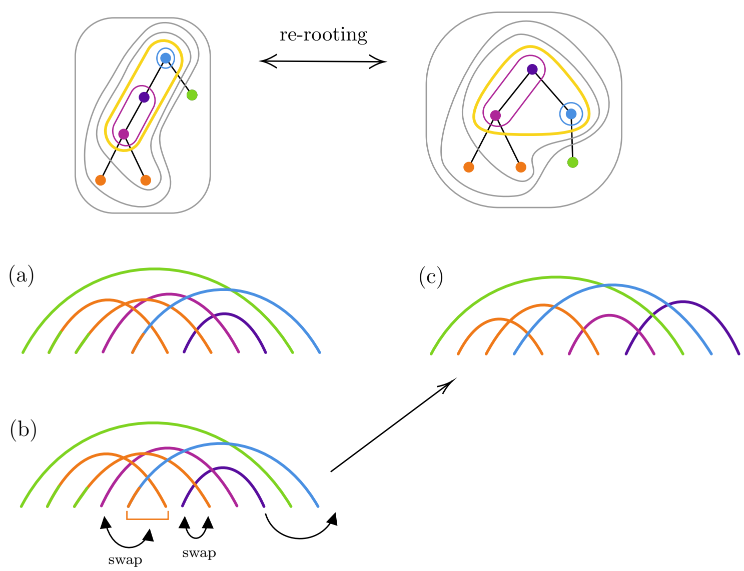


Figure 4.12: An example of how re-rooting works. (a) is the chord diagram of the tree on the left, with the full coloring. (b) shows the steps needed to be taken, where the block of orange chord ends is indicated. (c) shows the chord diagram after performing the algorithm, which is in fact the correct chord diagram corresponding to the tree on the right.

We use the following lemma to prove that this algorithm works.

Lemma 4.19. In the chord diagram $c(t_*)$, the following two properties hold:

1. The immediate leftward neighbor of $c(b)$ is the b_P (the bud of $c(t_P)$.) Furthermore, the bud of $c(t_P)$ is the only chord that is leftward of $c(b)$.

2. The bud chord of $c(t_B)$, $c(b)$, is the only chord of $c(t_B)$ that lies between the first and last ends of $c(t_P)$. All other ends of $c(t_B)$ lie to the right of $c(t_P)$ ends.

Proof. This follows from the construction of a chord diagram from the rooted tree. Recall that for a tube t_i, the bud of t_i corresponds to the leftmost chord in the chord diagram $c(t_i)$. So by construction of the chord diagram $c(t_*)$, $c(t_P)$ is inserted into the first CDIP of $c(t_B)$, meaning that the only chord to the left of this CDIP is the bud $c(b)$ of $c(t_B)$. Then by the construction, the bud of $c(t_P)$ is pulled to the left of the bud of $c(b)$. Thus the immediate and only leftward neighbor of $c(b)$ is the bud of $c(t_P)$. Because $c(t_P)$ is inserted in the first insertion place, all chords of $c(t_B)$ other than the bud $c(b)$ lie to the right of $c(t_P)$.

□

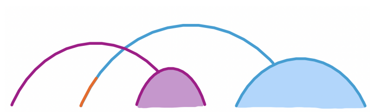


Figure 4.13: Base case of a chord diagram where the tubed tree has no inactive tubes, as in the maximal tube is t_*.

Theorem 4.20. For a rooted tree T and its re-rooted tree $\phi(T)$, the algorithm defined in Definition 4.18 takes $c(T)$ to $c(\phi(T))$.

Proof. By induction on the number of inactive insertions into t_*.

Base case: Suppose the tree T is t_*, meaning there are no inactive tubes. First note that in T, t_P is inserted into the first IP of t_B and in $\phi(T)$, t_B is inserted into the last IP of t_P. The first item in the algorithm is moving the right end of $c(r_P)$ all the way to the right of the chord diagram. This means that r_P becomes the new root and all of $c(t_B)$, except the bud b is in the last IP of $c(t_P)$, by Lemma 4.19. Because the left end of $c(b)$ is colored orange, the second item of the algorithm is swapping $c(b)$ with $c(b_P)$. By Lemma 4.19, $c(b)$ has only one leftward neighbor, $c(b_P)$, and therefore b becomes the new bud of the chord diagram. The result is the chord diagram $c(t_B)$ being inserted into the last IP of $c(t_P)$, and the bud being pulled to the left, which is exactly the chord diagram construction of inserting t_B into the last IP of t_P. Thus $\rho(c(t_*))$ is $c(\phi(t_*))$.

Induction hypothesis: Suppose for all trees t with fewer than n inactive insertions, we have that $\rho(c(t))$ is the same as $c(\phi(t))$. Suppose t'' is a tubed tree with $k < n$ inactive insertions. Let T be the tree obtained by inserting a tubed tree t' into t'' at insertion place x_i. Recall that x_μ determines whether t' is marked L or R. First, some things to note are as follows.

1. t' does not have a root-change, so t' and $c(t')$ do not change during re-rooting.
2. The index of the insertion place of t' in t must decrease by 1 after re-rooting T. In other words, t' is inserted into $\phi(t'')$ at $x_i - 1$.
3. Let b' be the bud of t'. Then b' is the bud of T.

Now, we have two cases: t' is inserted at or before x_μ, in which case t' is L-marked, or after x_μ, in which case it is R-marked. Let $c_L(t')$ be the chords ends of $c(t')$ that are L marked (all of them except the bud, which is R-marked).

Case 1: t' is L-marked (orange)

- Subcase 1: t' is inserted within or immediately to the left of a L-block of chords, called A. Here, $c_L(t')$ together with A forms a maximum consecutive block of L ends. By the induction hypothesis, the chord c_* which is immediately to the left of A in $c(t'')$ swaps with all chords in A by the map ρ. Thus, when $c(t')$ is within A or is the immediate leftward neighbor of A, it automatically gets included in this swap that occurs in t''. This swap satisfies condition 2. Thus, in $c(T)$, c_* swaps with the maximum consecutive block of L ends to form $\rho(c(T))$. This is the same as $c(\phi(T))$ because of condition 2.
- Subcase 2: $c(t')$ is inserted immediately to the right of A. Again, $c_L(t')$ together with A form a maximum consecutive block of L ends. By the induction hypothesis, ρ swaps A with the chord c_*. If c_* were to *only* swap with A, then the index of the insertion place of $c(t')$ would not decrease at all, so the only way its IP can decrease by one is if c_* also swaps with $c_L(t')$. Thus if c_* swaps with the maximum consecutive block of L ends, which includes $c_L(t')$, then $\rho(c(T))$ is the same as $c(\phi(T))$, which satisfies condition 2.

- Subcase 3: t' has no L-neighbors. From the induction hypothesis, in the chord diagram, locally at y_i, nothing happens in t'', because there are no L blocks. So when $c(t')$ is inserted, the only way for condition 2 can be satisfied is if $c_L(t')$ swaps with its immediate leftward neighbor, which is equivalent to inserting $c(t')$ into the IP y_{i-1}, which is $c(\phi(T))$.

Case 2: t' is marked R (green)

By definition of μ in the chord diagram, $c(t')$ is inserted into $c(t'')$ rightward of all chords in $c(t_P)$ and leftward of the right end of t_B's bud, $c(r_B)$. By the induction hypothesis, in $\rho(c(t''))$, the chord $c(r_P)$ jumps over all chords in $c(t_B)$, so if $c(t')$ is inserted somewhere between $c(r_P)$ and $c(r_B)$, then $c(r_P)$ also jumps over $c(t')$. Thus inserting $c(t')$ into y_{i-1} of $c(\phi(t''))$ is the same as performing ρ on T, therefore $\rho(c(T))$ is the same as $c(\phi(T))$.

□

Chapter 5

Magnus Expansion Tubings

5.1 Background

Introduced by Wilhelm Magnus in 1954 [23], the Magnus expansion is a tool in applied mathematics and mathematical physics that is used to solve first-order homogeneous linear differential equations with time-dependent coefficients. It is particularly useful in quantum mechanics as it provides highly precise approximations of the time evolution operator, which essentially provides a way to map an initial state of the system to its state at any later time. Combinatorics fits into this story in that forests can used to break the Magnus expansion down cleanly and systematically to get a better conceptual understanding of the series. The Magnus expansion can be seen as a special case of Lie series, hence can solve differential equations from Lie theory. Further, there is a post-Lie case, which is also studied in [27, 10, 9, 1]. Thus we can relate algebraic mathematics and combinatorics to the Magnus expansion.

This chapter concerns a different definition of tubings applied to computing the Magnus expansion coefficients, inspired by Mencattini and Quesney from their paper [27] about post-Lie algebras and the Magnus expansion. Mencattini and Quesney introduced two different types of nested tubings of forests and developed a method for using these tubings in Magnus expansion coefficient computations. We have expanded on the work of Mencattini and Quesney by focusing on the ordered properties of these forests, as well as revised some of their definitions to better fit this new perspective.

In order to understand these tubings, we must define two orderings on the forest which align with the definition of plane posets, and thus we can explore the plane poset properties of forests and study these tubings. We define a special type of forest, which we call leftward forests, that allow a duality between the two types of tubings using the plane poset property. This work opens the door to extending these nested tubings beyond forests to posets.

In [27], the forests are drawn with the roots at the bottom. For consistency with the rest of this thesis, the roots are drawn at the top. Additionally, regarding the ancestry partial order defined in Chapter 2, drawing the roots at the top allows the forests to be more naturally viewed as Hasse diagrams for the ancestry partial order, defined in the next section. When we change the convention from [27] so that the roots are at the top, the traversal order and ancestry order are preserved.

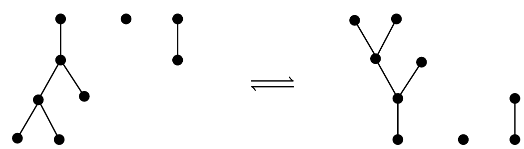


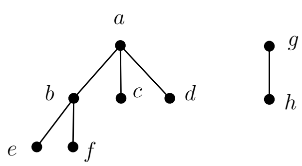
Figure 5.1: The underlying forest are the same, but in this thesis we use the convention with the roots drawn on the top (left) and [27] uses the convention that roots be at the bottom (right).

5.2 Preliminary Definitions

Recall the ancestry order $\prec$ (2.25) and rightward order $<_r$ (2.29) from the poset section of Chapter 2. We will make use of these two orders along with the traversal order (Definition 2.5) in this section. Note that the traversal order that we made use of in Chapter 3 and Chapter 4 is the same as what [27] calls the canonical order of a tree in Section 4.1. Additionally, we use the definition of rightward order $<_r$ in place of a slightly different definition from [27], which they call *horizontal order* ($<_h$) in Section 4.5 of [27]. However, this definition of rightward order is stronger in that $<_h \subseteq <_r$. Horizontal order is defined as the order of the set of roots of the forest increasing from left to right, thus, $<_r$ when restricted to the set of roots is $<_h$ in [27].

Note that the traversal order from Definition 2.5 is the common refinement of the rightward order and the ancestry order.

Recall from Chapter 2 an example of a plane forest and the partial orders on the vertices in Figure 2.9. The following is another example of a plane forest and the three orders defined.



(i) $a < b < e < f < c < d$

$g < h$

(ii) $e \prec b \prec a$

$f \prec b$

$c \prec a$

$d \prec a$

$g \prec h$

(iii) $a <_r g$

$a <_r h$

$b <_r c <_r d <_r g$

$e <_r f <_r c <_r d <_r h$

Figure 5.2: Here, we have (i) traversal orders on each tree (ii) ancestry orders (iii) rightward orders.

Definition 5.1. Root(F) is the set of roots of a forest F. (Defined in Section 4.5 of [27].)

Definition 5.2 (Horizontal forest)**.** A *horizontal forest* is a forest where all trees are trivial, i.e. single vertices.

Note that in Definition 33 of [27], horizontal forests are denoted by $\{\bullet^{\times n}\}$.

When viewed as posets, a horizontal forest is an antichain in the ancestry order ($\prec$) because none of the vertices have children. Also observe that for a horizontal forest F, $\text{Root}(F) = V(F)$.

5.3 A Different Kind of Tubings

In this section, we will introduce a new definition of tubings of trees and forests introduced by Mencattini and Quesney in [27], which we call Magnus tubings, or M-tubings for short, so as to differentiate between other notions of tubings mentioned in this thesis. Additionally, throughout this chapter, M-tubings will just be called tubings.

Let F be a forest and $v \in V(F)$, which we will view as a poset.

Definition 5.3 (b_v)**.** The set b_v is the set of children of v. In other words, v covers every element in b_v in ancestry order $\prec$. (From Section 4.5 of [27].)

Now, we define M-tubes and M-tubings of plane rooted forests.

Let F be a plane rooted forest with Root(F). We introduce a virtual vertex v_* and edges between v_* and all $v \in \text{Root}(F)$. Call this new forest F_*. Then F_* is a tree with root v_*. Note that Root(F) is b_{v_*}.

Definition 5.4 (M-tube of a forest)**.** A set of vertices of F is an *M-tube* t_i if for $t_* = (t_i \cup v_*)$, t_* is an upset of $(V(F_*), \prec)$ and for each $v \in t_*$, $(t_* \cap b_v)$ is an upset of $(b_v, <_r)$.

This definition is a revised version of definitions introduced by Mencattini and Quesney [27]. In their paper, they have a different definition for a tube of a tree and a tube of a forest (Definition 44 and Definition 45 in [27].) Introducing the vertex v_* allows us to transform a forest to a tree, in a way which automatically incorporates their definition of a tube of a forest into their definition of a tube of a tree. It's important to observe that when F is a tree, adding this virtual vertex is unnecessary, and thus in this chapter, when F is a tree, we may just use F in place of F_*. Note that when we refer to a tube t_i of a forest, t_i does not include the vertex v_*. In their paper, a tube of a forest is defined as a subset of vertices such that its intersection with the set of roots is an $<_r$-upset of the roots of F and its intersection with each tree is a possibly empty tube. The way we envelope their definition of a tube of a forest into their definition of a tube of a tree is by adding the vertex v_*. Because v_* must be in t_* (otherwise it is not an upset), we have that b_{v_*} is exactly $\text{Root}(F)$. The tube's intersection with the roots is an upset of $(b_v, <_r)$ and thus satisfies [27]'s definition of tube of a forest.

One way to conceptualize the notion of a tube is when we say upset of $(b_v, <_r)$ of a tree, essentially what that means is if a vertex v is in a tube, then all of its siblings rightward of v are also in the tube. Recall that $\emptyset$ is an upset of $(b_v, <_r)$. We can also see that for any tree, the root and the entire tree are always both tubes, and for any forest, the $<_r$-maximal root of $(\text{Root}(F), <_r)$ and entire forest is a tube.

Definition 5.5 (Nested M-Tubing)**.** Two tubes are *nested* if one of them is a subset of the other. A *nested M-tubing*, τ, of a forest F is a collection of pairwise nested, nonempty tubes of F such that τ contains at least two tubes and it contains the maximal tube (Definition 47 and 48 in [27].)

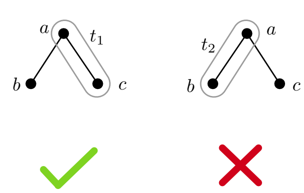


Figure 5.3: On the left: t_1 is a tube of a forest F. On the right, t_2 is an upset of $V(F)$ but it is not a tube because $b_a = \{b, c\}$ and $b <_r c$, so $(t_2 \cap b_a)$ is not an upset of b_a.

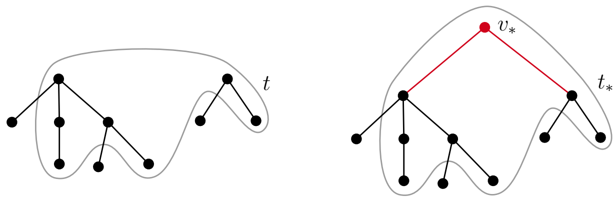


Figure 5.4: An example of how use the virtual vertex v_* to find a tube of a forest. On the left is the forest F with tube t and on the right is the forest F_* with tube t_*.

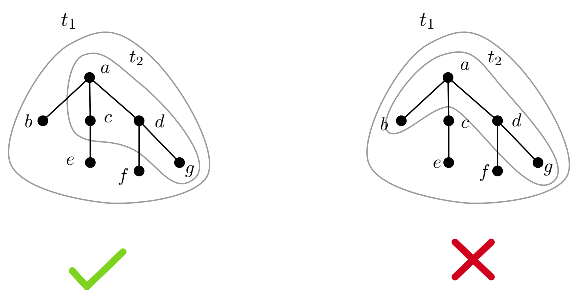


Figure 5.5: Another example of a tube and a non-tube. On the left, $t_2 \cap b_a = \{c, d\}$ which is an upset of b_a. On the right $t_2 \cap b_a = \{b, d\}$ which is not an upset of b_a.

Definition 5.6 (Boundary)**.** Let τ be a nested tubing where $\{t_1, t_2, \ldots, t_k\}$ is the set of tubes in τ such that the nesting of τ is $t_1 \subset t_2 \subset \cdots \subset t_{k-1} \subset t_k$. For a tube $t_i \in \tau$, the *boundary*, ∂t_i, of t_i is $\partial t_i = (t_i \backslash t_{i-1})$. If a tube does not contain any subtubes, then the boundary is the tube itself. (From section 4.5 of [27].)

In other words, the boundary is all the vertices in t_i that are not contained in any nested subtubes of t_i. This definition is worded differently from [27]'s definition of the boundary, which contains a typo and is not very specific. See Figure 5.6 and Figure 5.7 for examples of boundaries.

5.4 Vertical and Horizontal Nested Tubings

The following section describes two types of M-tubings of forests that are useful in determining Magnus expansion coefficients, vertical nested tubings and horizontal nested tubings. Both are used for computing coefficients, but the method using vertical nested tubings is recursive, while the method using horizontal nested tubings is not.

5.4.1 Vertical Nested Tubings

Definition 5.7 (Vertical Nested Tubing)**.** A tubing τ of a forest F is a *vertical nested tubing (VNT)* if for all $t_i \in \tau$:

(i) ∂t_i is not a horizontal forest with more than one vertex

(ii) Either all the roots of ∂t_i are children of a vertex v of a subtube of t_i or are children of v_*

(Vertical nested tubings are defined in Definition 49 in [27].)

The definition in [27] is worded slightly differently, but the definition given above is a cleaned up version that reduces ambiguity. The case where all the roots are children of v_* implies that the roots of the boundary are in $\mathrm{Root}(F)$, and so none of them are children of any vertex of any subtree of t_i.

For a forest F, let $\mathcal{V}(F)$ be the set of all of its vertical nested tubings. Note that Definition 50 in [27] calls this $Tub(F)$ instead.

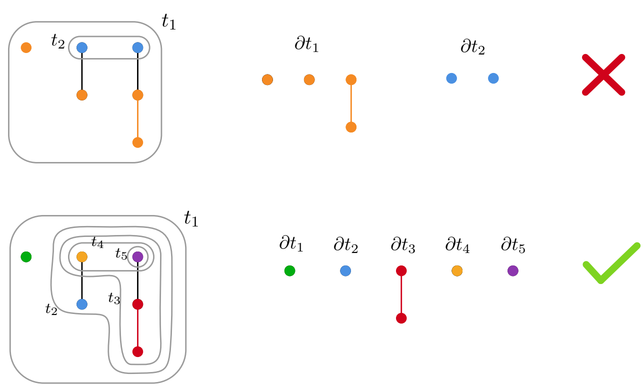


Figure 5.6: On top: a tubing of the forest on top shows a non-example of a VNT. Both conditions in Definition 5.7 fail. The first fails because ∂t_2 is a horizontal forest with 2 vertices and the second fails because the roots of ∂t_1 are not children of the same vertex of a subtube. On bottom: a tubing of the same forest satisfies all the conditions, thus is a VNT. The s are here to distinguish each boundary.

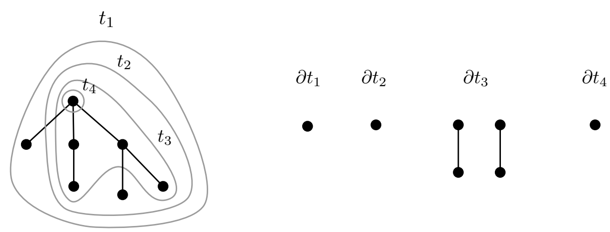


Figure 5.7: Another example of a vertical nested tubing.

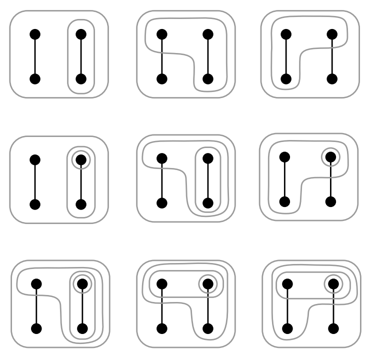


Figure 5.8: All VNTs of a forest F (Example 52 in [27].)

5.4.2 Horizontal Nested Tubings

Definition 5.8 (Horizontal Nested Tubing)**.** A tubing τ of a forest F is a *horizontal nested tubing (HNT)* if the boundary of each tube is a horizontal forest. [Definition 53 in [27].]

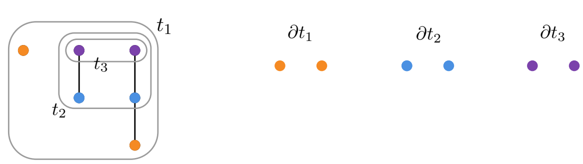


Figure 5.9: An example of a horizontal nested tubing. All the boundaries are horizontal forests of size 2.

For a forest F, let $\mathcal{H}(F)$ be the set of all of its horizontal nested tubings. In [27], this is called $hTub(F)$.

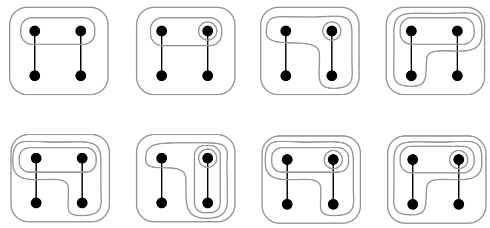


Figure 5.10: All HNTs of a forest F (Example 55 in [27].)

5.5 Magnus Expansion Coefficients

The motivation of Mencattini and Quesney [27] for defining the vertical and horizontal nested tubings was to compute coefficients of the Magnus expansion. This section will outline how they use the nested tubings to do this.

5.5.1 Coefficients from Vertical Nested Tubings

The method for finding coefficients of the Magnus expansion using vertical nested tubings is recursively defined. Recall that $\mathcal{V}(F)$ is the set of vertical nested tubings of a forest F. Let $\mathcal{F}'$ be the set of forests that does not include horizontal forests. We will denote the coefficient for a forest F as m_F.

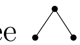
Theorem 5.9. (Proposition 58 in [27]) For the singleton tree, let $m_\bullet = 1$. For every forest $F \in \mathcal{F}'$, write the coefficient of F in the Magnus expansion to be

$$m_F = \sum_{t \in \mathcal{V}(F)} m_t$$

where m_t is the coefficient determined by the subtree induced by the tube t in a tubing. Then the formula for m_t is

$$m_t = \frac{-1}{|t|!} \prod_{t' \in t} m_{\partial t'}$$

where $|t|$ is the number of tubes in a particular tubing.

Example 5.10. The following is an example of computing coefficients of the tree .

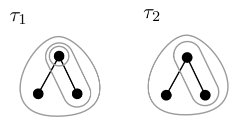


Figure 5.11: There are two VNTs on this forest, as shown here.

The calculation is:

$$m_F = m_{\tau_1} + m_{\tau_2}$$

$$m_F = \left(\left(\frac{-1}{3!}\right) \times 1 \times 1 \times 1\right) + \left(\left(\frac{-1}{2!}\right)\left(\frac{-1}{2}\right) \times 1\right) = \frac{1}{12}$$

We can easily see that the term $\frac{-1}{2}$ in m_{τ_2} comes from the unique tubing of .

Example 5.11. Using the coefficient obtained from Example 5.10, we can find the coefficients of the forest below.

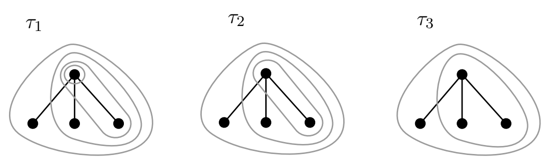


Figure 5.12: There are three VNTs on this forest, as shown here.

$$m_F = m_{\tau_1} + m_{\tau_2} + m_{\tau_3}$$

$$m_F = \left(\left(\frac{-1}{4!}\right) \times 1 \times 1 \times 1 \times 1\right) + \left(\left(\frac{-1}{3!}\right)\left(\frac{-1}{2}\right) \times 1 \times 1\right) + \left(\left(\frac{-1}{2!}\right)\left(\frac{-1}{12}\right) \times 1\right) = 0$$

5.5.2 Coefficients from Horizontal Nested Tubings

While finding the Magnus expansion coefficient from vertical nested tubings is recursive, the following is a closed form formula when using horizontal nested tubings.

The following lemma is useful for the formula for coefficients given by horizontal nested tubings, which follows easily from Definition 5.6. Recall from Definition 5.6, the tubes of τ are nested such that $t_1 \subset t_2 \subset \cdots \subset t_{k-1} \subset t_k$.

Lemma 5.12. Let τ be an M-tubing as described in Definition 5.6. For all $t_i, t_j \in \tau$, we have that $\partial t_i \cap \partial t_j = \emptyset$. Furthermore, every vertex is contained in $\partial t_1 \cup \partial t_2 \cup \cdots \cup \partial t_k$.

Proof. Take $v \in V(F)$. Because t_k is the whole forest, v is in at least one tube. Then, there exits i such that $v \in t_i$ and $v \notin t_{i-1}$, meaning t_i is the smallest tube containing v. Thus $v \in \partial t_i$ but $v \notin \partial t_j$ for $i \neq j$. For $j < i$ $v \notin t_j$ so $v \notin \partial t_j$. For $j > i$, $j - 1 \geq i$, so by the nesting property, $v \in t_i \subseteq t_{j-1} \subset t_j$. Thus $v \in t_{j-1}$ and $v \in t_j$, so v cannot be in the boundary of t_j. □

Recall that $\mathcal{H}(F)$ is the set of all horizontal nested tubings of a forest F. Then, as with vertical nested tubings, the Magnus expansion coefficient given by horizontal nested tubings is given below:

Theorem 5.13. (Theorem 64 in [27]) The coefficient in the Magnus expansion for a forest F is given by

$$m_F = \sum_{\tau \in \mathcal{H}(F)} m_\tau$$

Where the formula for m_τ is given by

$$m_\tau = \frac{(-1)^{|t|-1}}{|t|} \prod_{t_i \in \tau} \frac{1}{|\partial t_i|!}$$

where $|\partial t_i|$ is the number of vertices in the boundary ∂t_i and $|t|$ is the number of tubes in the tubing τ.

Note that in Section 4.6.2 of [27] the formula is presented differently and involves some algebra not covered in this thesis.

Example 5.14. The coefficient from the forest below is $\frac{1}{180}$.

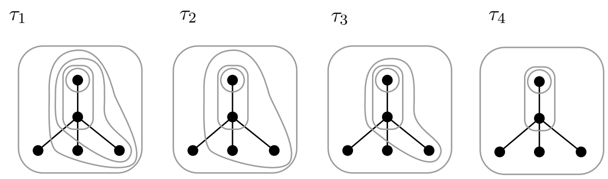


Figure 5.13: There are four HNTs on this forest, as shown here.

$$m_F = m_{\tau_1} + m_{\tau_2} + m_{\tau_3} + m_{\tau_4}$$

$$m_F = \frac{1}{5}\left(\times 1^5\right) + \frac{-1}{4}\left(\left(1^3 \times \left(\frac{1}{2!}\right)\right) + \left(1^3 \times \left(\frac{1}{2!}\right)\right)\right) + \frac{1}{3}\left(1^2 \times \frac{1}{3!}\right) = \frac{1}{180}$$

Because τ_2 and τ_3 both have the same set of boundary sizes, $(1, 2, 1, 1)$, the term $\frac{-1}{4}\left(1^3 \times \left(\frac{1}{2!}\right)\right)$ gets counted twice.

This example is given in Example 65 in [27].

5.6 Double Posets and Tubings

Given the orderings on plane rooted forests described in this chapter, we now focus on viewing these forests as plane posets (recall Definition 2.27 and Definition 2.28.) This section will describe how we transform certain forests using this plane poset definition to other plane rooted forests while preserving the nested tubings. There is potential to expand this idea to all posets, rather than just plane forests, however this thesis only covers plane forests. The following section of this chapter makes use of the tubings from [27] but is original work.

Given a forest, as viewed as a plane poset with orderings $\prec$ and $<_r$, recall that for any two vertices, u, v, that either u and v are comparable by the $\prec$ relation or the $<_r$, but not both. Thus, we may easily swap the orders $\prec$ and $<_r$ for every pair of vertices in the forest. The hope is to find duality between vertical nested tubings and horizontal nested tubings on the same graph by swapping the orders on every pair of vertices in a forest. We define a transformation φ.

Definition 5.15. Let φ be a function that takes as input $(V(F), <_r, \prec)$ and swaps the orderings on F for all vertices; that is, for any two vertices, u, v, if $u <_r v$ in F, then $u \prec v$ in $\varphi(F)$, and if $u \prec v$ in F, then $u <_r v$ in $\varphi(F)$.

Additionally, $\varphi(\varphi(F)) = F$ because if $u <_r v$ in F, then $u \prec v$ in $\varphi(F)$, then $u <_r v$ in $\varphi(\varphi(F))$ and vice versa for $u \prec v$ in F.

When we start with a forest, F, the resulting poset $\varphi(F)$ is not necessarily a forest of rooted trees or even a forest. The following example illustrates this.

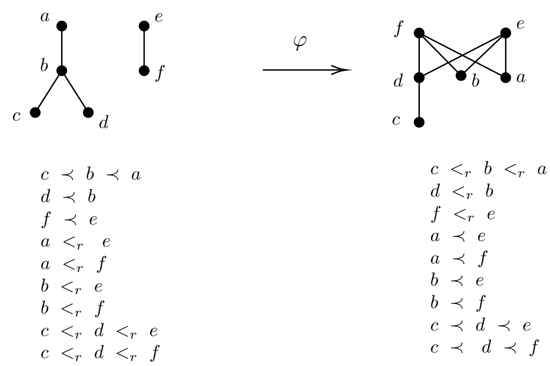


Figure 5.14: Example of the forest F shown in Figure 2.9. Here, $\varphi(F)$ is not a forest.

5.6.1 Ladders and Horizontal Forests

Definition 5.16. A *rooted ladder* is a rooted path (recall Definition 4.2) such that a terminal vertex is the root.

Because this thesis only deals with rooted trees, rooted ladders will simply be referred to as ladders, when unambiguous.

Lemma 5.17. Rooted ladders and horizontal forests map to each other in φ. For any rooted ladder tree T, $\varphi(T)$ is a horizontal forest, and for every horizontal forest F, $\varphi(F)$ is a ladder tree.

Proof. Suppose a tree T is a ladder with vertices $v_1 \prec v_2 \prec \cdots \prec v_n$. There are no two vertices v_i, v_j such that $v_i <_r v_j$. Then in the φ transformation, this ordering becomes $v_1 <_r v_2 <_r \cdots <_r v_n$, and no two vertices v_i, v_j such that $v_i \prec v_j$. By definition, this is a horizontal forest.

The argument is the same the other direction. $\square$

Lemma 5.18. Suppose F is a ladder and let $\hat{F}$ be the horizontal forest $\varphi(F)$. Then every nested tubing of F is a nested tubing of $\hat{F}$.

Proof. Let τ be a nested tubing of F. First, we know that the nested structure is preserved, so it suffices to show that that all the tubes of τ are tubes of $\hat{F}$. Let $\hat{F}_*$ be $\hat{F}$ with virtual vertex $\hat{v}_*$ (defined in Definition 5.4) and let $\hat{t}_* = (t_i \cup \hat{v}_*)$. Thus $\hat{t}_*$ is an upset of $(V(\hat{F}_*), \prec)$ because it is made of root $\hat{v}_*$ and a subset of $\hat{v}_*$'s children. Therefore, the first condition of Definition 5.4 is satisfied, that t_* is an upset of $(V(\hat{F}_*), \prec)$.

Now, the second condition of a tube states that for each $v_i \in t_*$, $(t_* \cap b_v)$ is an upset of $(b_v, <)$. Let $\hat{b}_v$ be the b_v value for vertices in $\hat{F}$. Note that for all $v \in \hat{F}$, $\hat{b}_v = \emptyset$, so we only need to verify that $(\hat{t}_* \cap \hat{b}_{v_*})$ is an upset of $(\hat{b}_{v_*}, <)$ in $\hat{F}$. Recall that $\hat{b}_{v_*} = \mathrm{Root}(\hat{F}) = V(\hat{F})$, so $(\hat{t}_* \cap \hat{b}_{v_*}) = ((t_i \cup v_*) \cap V(\hat{F})) = t_i$. Therefore, we need to verify that t_i is an upset of $V(\hat{F})$. Because t_i is a tube of F, then t_i is an upset of $(V(F), \prec)$ In the transformation φ, we have that $(V(F), \prec)$ becomes $(V(\hat{F}), <_r)$, thus t_i must be an upset of $(V(\hat{F}), <_r)$. In a horizontal forest, the traversal order is the same as $<_r$, thus t_i is an upset of $(V(\hat{F}), <)$, and the second condition is satisfied. ☐

Now, we have a parallel proof in the other direction.

Lemma 5.19. Suppose F is a horizontal forest and let $\hat{F}$ be the ladder $\varphi(F)$. Then every nested tubing of F is a nested tubing of $\hat{F}$.

Proof. Let τ be a nested tubing of F. As stated in the previous lemma, it suffices to show that all the tubes of τ are tubes of $\hat{F}$. Let t_i be a tube of τ. Because $\hat{F}$ is a tree, the first condition of Definition 5.4 is equivalent to saying that a tube t_i is an upset of $V(\hat{F}, \prec)$, which we want to prove. Let $\hat{v}_*$ be the virtual root of F_* and $t_* = (t_i \cup v_*)$. Because t_i is a tube of F, then $(t_* \cap b_{v_*}) = (t_* \cap V(F))$ is an upset of $(b_{v_*}, <) = (V(F), <)$. Thus t_i is an upset of $(V(F), <_r)$. In the transformation, $(V(F), <_r)$ becomes $(V(\hat{F}), \prec)$. Thus t_i is an upset of $(V(\hat{F}), \prec)$, as desired.

Now, for the second condition, we wish to show that $(t_i \cap b_v)$ is an upset of $(b_v, <)$. $\hat{F}$ is a ladder, so for any $v \in t_i$, either b_v is a single vertex, u, or $b_v = \emptyset$. Thus $(t_i \cap b_v) = b_v$ or $(t_i \cap b_v) = \emptyset$. The empty set is an upset of b_v and b_v is an upset of itself. The second condition is satisfied.

☐

Now we have a similar result for ladders and horizontal forests, which is more to the crux of this chapter, that draws a relation between HNTs and VNTs.

Remark 5.20. Any tube t_i of a ladder is either a single vertex or a ladder, because t_i is an upset of $(V(F), \prec)$. Moreover, any tube t_i of a horizontal forest is also a horizontal forest.

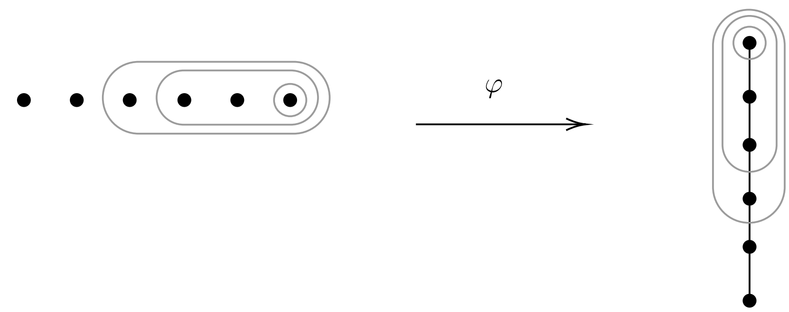


Figure 5.15: An example that shows this characterization.

Lemma 5.21. Let F be a ladder and $\hat{F}$ be the horizontal forest after the transformation φ. Then:

1. All nested tubings of F are VNTs
2. All nested tubings of $\hat{F}$ are HNTs.

Proof. 1. Let τ be any nested tubing of ladder F and let $t_i \in \tau$. From Remark 5.20 we have that ∂t_i is either a single vertex or a ladder, so condition (i) of Definition 5.7 is satisfied. Furthermore, there is only one root of ∂t_i, so condition (ii) is satisfied.

2. Let τ be any nested tubing of horizontal forest $\hat{F}$ and let $t_i \in \tau$. Then because $\hat{F}$ is a horizontal forest and from Remark 5.20, we have that every tube is also a horizontal forest, and Definition 5.8 is satisfied.

□

Lemma 5.22. For each ladder F and horizontal forest $\hat{F}$, there is exactly one HNT of F and exactly one VNT of $\hat{F}$. Further, the HNT τ of F in the transformation φ is the VNT τ of $\hat{F}$.

Proof. Let τ be the nested tubing of F with the maximal number of tubes such that for each tube t_i, $\partial t_i = v_i$. Because the boundary of each tube is a single vertex, then each boundary is a horizontal forest, and Definition 5.8 is satisfied, and τ is a HNT of F. Thus τ is a HNT of F. Now we claim that there are no other tubings of F that are HNTs of F.

Let $\tau' \neq \tau$ be a nested tubing of F. Then there is at least one tube t_i such that ∂t_i is not a single vertex. Suppose for the sake of contradiction that ∂t_i is a horizontal forest. Then there are two vertices u, v in ∂t_i that are not adjacent in the ladder, implying there exists a vertex w such that, without loss of generality, $u \prec w \prec v$. Then the tube maximal tube nested in t_i is not an upset because v is not contained in it. Therefore, τ' is not a valid nested tubing.

Now, let τ be the nested tubing of $\hat{F}$ with the maximal number of tubes such that for each tube t_i, $\partial t_i = v_i$. Because the boundary of each tube is a single vertex, then each boundary is a horizontal forest with exactly one vertex, conditions (i) and (ii) of Definition 5.7 are satisfied, and τ is a VNT of $\hat{F}$. Now we claim that there are no other tubings of $\hat{F}$ that are VNTs of $\hat{F}$. Let $\tau' \neq \tau$ be a nested tubing of $\hat{F}$. Then there is at least one tube t_i of τ' with a boundary of more than one vertex. Because $\hat{F}$ is a horizontal forest, then ∂t_i is a horizontal forest, and (i) of Definition 5.7 fails. Thus τ' is not a VNT of $\hat{F}$.

Finally, clearly the HNT τ of F maps to the VNT τ of $\hat{F}$ because they each have the property that $\partial t_i = v_i$.

□

Lemma 5.23. Let F be a ladder and let $\hat{F}$ be the horizontal forest $\varphi(F)$. Then:

1. Every VNT of F is an HNT of $\hat{F}$
2. Every HNT of $\hat{F}$ is a VNT of F

Proof. Let τ be a VNT of F. Then by Lemma 5.21, τ can be any nested tubing of F, and thus is also a nested tubing of $\hat{F}$ by Lemma 5.18 and therefore is an HNT of $\hat{F}$. The equivalent is true for τ an HNT of $\hat{F}$. By Lemma 5.21, τ can be any nested tubing of $\hat{F}$, and by Lemma 5.19, is also a nested tubing of F.

□

Lemma 5.22 and Lemma 5.23 tell us that for a forest F which is either a horizontal forest or a ladder, $\varphi(F)$ is a bijection between VNTs of F and HNTs of $\hat{F}$.

Corollary 5.24. Let F be either a ladder or a horizontal forest. Then there is a bijection between VNTs of F and HNTs of $\hat{F}$.

Ladders and horizontal forests are fundamental types of forests that have this duality between vertical nested tubings and horizontal nested tubings, hence the results from this subsection will be useful in the following subsection.

5.6.2 Leftward Forests

As we've seen in examples (such as in Figure 5.14), not all rooted forests map to rooted forests in φ. But there is a distinct class of forests that do, which we call leftward forests. In fact, ladders and horizontal forests are the most basic leftward forests. Next we define this class of forests that we transform using φ. First, we introduce a few definitions.

Definition 5.25 (Heir, Spare)**.** A vertex v is an *heir* if it is a minimal element in the $<_r$ order. Every vertex that is not an heir is called a *spare*.

Definition 5.26 (Leftward Forest)**.** A *leftward forest* is a forest such that only an heir may have children. We call the class of all leftward forests $\mathcal{LF}$.

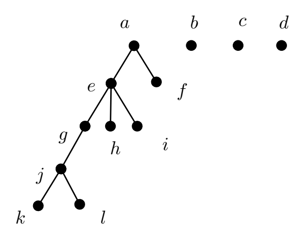


Figure 5.16: Example of a leftward forest. The heirs are a, e, g, j, k and the spares are b, c, d, f, h, i, l.

Note that horizontal forests are leftward forests because no vertices have children. Ladders are leftward forests because every vertex is an heir.

Now we take the forest from Figure 5.16 and show its transformation under φ in Figure 5.17.

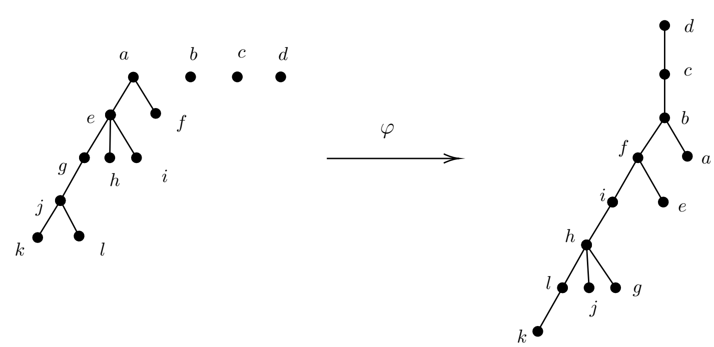


Figure 5.17: Forest from Figure 5.16 and its dual in the transformation φ.

This next lemma is about a property of leftward forests. It will be useful in the proof for the lemma that follows, which says that we need a forest to be a leftward forest in this transformation φ.

Lemma 5.27. For a forest $F \in \mathcal{LF}$, let y be the youngest heir. Then y is the youngest heir in $\varphi(F)$.

Proof. If y is the youngest heir in F, then for all heirs v_i in F, $y \prec v_i$ and for all spares u_i in F, $y <_r u_i$. Therefore, in $\varphi(F)$, $y <_r v_i$ and $y \prec u_i$. Because $F \in \mathcal{LF}$, there are no other vertices. So $y \in \varphi(F)$ is a minimal element in both orderings $<_r$ and $\prec$. □

Lemma 5.28. Given a plane rooted forest F, $\varphi(F)$ is a plane rooted forest if and only if F is a leftward forest. Furthermore, $\varphi(F)$ is also a leftward forest.

Proof. We begin this proof by using two claims. First we show that a leftward forest maps to a leftward forest in φ and second we show that leftward forests are the only forests that map to leftward forests in φ.

Claim 1: If F is any leftward forest, then $\varphi(F)$ is also a leftward forest.

Suppose F is a leftward forest. Let $S = V(F)$ and let S_* be the set of exactly all the heirs, so S_* and y the youngest heir. Therefore S_* is a ladder in F and $S \backslash S_*$ is a horizontal forest in F. Therefore, by Lemma 5.17, $S \backslash S_*$ is a ladder and S_* is a horizontal forest disjoint from $S \backslash S_*$ in $\varphi(F)$. We aim to show that $\varphi(F) \in \mathcal{LF}$, so we check that $(S \backslash S_*) \cup y$ are $<_r$ minimal in $\varphi(F)$ and that no element of $S_* \backslash y$ has a child in $\varphi(F)$. This will give that $(S \backslash S_*) \cup y$ are the heirs of $\varphi(F)$, $S_* \backslash y$ are the spares of $\varphi(F)$, and that $\varphi(F)$ is a leftward forest. First, we know that in F, $S \backslash S_*$ are spares, so $(S \backslash S_*) \cup y$ are leaves, therefore they are $\prec$-minimal in F. Thus, in $\varphi(F)$, they are $<_r$-minimal, and therefore are heirs in $\varphi(F)$. Next, we show that no element in $S_* \backslash y$ has a child in $\varphi(F)$. In F, $S_* \backslash y$ are heirs, so they are $<_r$-minimal in F. Therefore, in $\varphi(F)$, they are $\prec$-minimal, thus they are leaves and have no children in $\varphi(F)$. Hence we have that for $F \in \mathcal{LF}$, $\varphi(F) \in \mathcal{LF}$.

Claim 2: For every forest F that is not a leftward forest, $\varphi(F)$ is not a rooted forest.

Let $F \notin \mathcal{LF}$ be a forest. Suppose v is some vertex in F. Then because F is not a leftward forest, there exists some spare, u with a child u'. Then we have the following orders: $u' \prec u$, $v <_r u$, and $v <_r u'$. Therefore, in $\varphi(F)$, the orders become $v \prec u$, $v \prec u'$, and $u' <_r u$. Then $\varphi(F)$ is not a rooted forest because v has two parents.

Therefore, if F is not a leftward forest, then $\hat{F}$ is not a leftward forest. Thus, both conditions are proved, and we can conclude that $\varphi(F)$ if a leftward forest if and only if F is a leftward forest. □

From now on, for a leftward forest F, we will let $\hat{F}$ be the leftward forest resulting from applying φ to F.

Corollary 5.29. For a forest $F \in \mathcal{LF}$, every heir except y in F becomes a spare in $\hat{F}$ and every spare in F becomes an heir in $\hat{F}$.

This follows directly from the proof above, Lemma 5.28.

Thus, because we have this duality, we can extend this duality to tubings on leftward forests.

Lemma 5.30. Let $F \in \mathcal{LF}$. A subset $t_i \in V(F)$ is a tube if and only if for every vertex $v_i \in t_i$, we have that

1. all vertices $v_j \succ v_i$ are in t_i
2. all vertices $u_j >_r v_i$ are in t_i.

In particular, τ is a nested tubing of F if all the tubes in τ satisfy these properties.

Proof. ($\Longrightarrow$) Suppose t_i is a tube of F. Let $F_* = F \cup v_*$ be the forest with virtual vertex v_* and $t_* = t_i \cup v_*$. Then by definition, t_i is an upset of $(V(F_*), \prec)$. Thus for some vertex $v_i \in t_*$, every vertex v_j such that $v_i \prec v_j$ is also in t_i, therefore condition (1) is satisfied. For the second condition, suppose not, that is, suppose there exists some $u_j >_r v_i$ such that $u_j \notin t_*$. Thus u_j is a spare with parent $v_k \in F_*$ such that $v_k \prec v_j \preceq v_i$. Without loss of generality, suppose $v_k \in t_*$, because, if not, we can choose v_k in place of u_j. We know that $v_j \preceq v_i$. Thus $v_j, u_j \in b_{v_k}$ but $u_j \notin t_* \cap b_{v_k}$, so $t_* \cap b_{v_k}$ is not an upset of b_{v_k}, so t_i is not a tube, a contradiction, so condition (2) must hold.

($\Longleftarrow$) Suppose t_i is some subset of the vertices of $F_* = (F \cup v_*)$ such that (1) and (2) hold all some vertices $v_i \in t_i$. We wish to show that t_* is a tube and thus t_i is a tube. First, we know that for all vertices $v_j \prec v_{j-1} \prec \cdots \prec v_1 \prec v_*$ are in t_*. Therefore t_* is an upset of $(V(F_*), \prec)$. For the second condition, let v_k be the parent of v_i, and we have that all vertices rightward of v_i are in t_*, implying that all of the children of v_k rightward of v_i are in t_*. Therefore $b_{v_k} \cap t_i$ is an upset of b_{v_k}. This same argument applies for all v_i in the tube t_*, including for the $<_r$-maximal elements in t_i, because for them, v_k would be v_*, and $b_{v_*} \cap t_*$ is an upset of b_{v_*}.

□

We use this lemma to prove the following result which gives us a bijection between nested tubings of F and nested tubings of $\hat{F}$.

Lemma 5.31. Let F be a forest such that $F \in \mathcal{LF}$ and let τ be a nested tubing of F. Then τ is also a nested tubing of $\hat{F}$.

Proof. We have from Lemma 5.30 that for a vertex v_i in a tube t_i, then all vertices $v_j, u_j \in F$ such that $v_j \succ v_i$ and all vertices $u_j >_r v_i$ are also in t_i. Then in $\hat{F}$, all vertices $v_j >_r v_i$ and all vertices $u_j \succ v_i$ are in t_*. Then by Lemma 5.30, τ is a nested tubing of $\hat{F}$. ☐

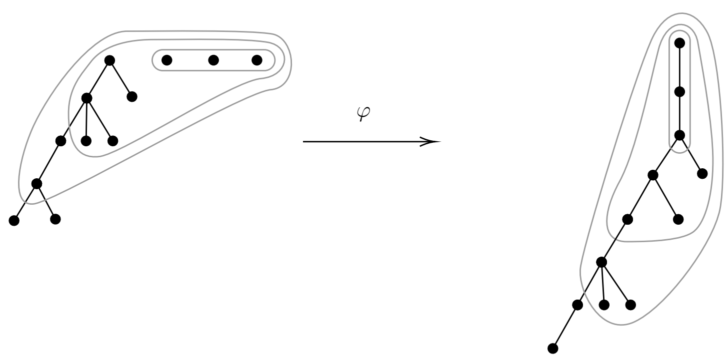


Figure 5.18: An example of a nested tubing on the forest from Figure 5.17. We can see that the all of the tubes in this tubing are also tubes in $\varphi(F)$.

Lemma 5.32. Let τ be a nested tubing of *any* plane rooted forest F. For every $t_i \in \tau$, if every ∂t_i is a ladder, then τ is a a VNT.

Proof. Proof by definition of VNTs (5.7.) For all $t_i \in \tau$, the following two conditions must be satisfied. The first condition is that ∂t_i is not a horizontal nested forest with $n > 1$ vertices. Because the boundary of each tube is a ladder, then the only way it can be a horizontal forest is if it has one vertex. So the first condition is satisfied. The second condition only relies on the boundary being a forest with more than one tree. Because the boundary of each tube is a ladder, there is only one root, so it is only the child of a single vertex, whether v_* or a vertex of a subtube, thus satisfying condition two. ☐

Theorem 5.33. Let F be a forest in $\mathcal{LF}$.

1. For any HNT, τ, of F, τ is a VNT of $\varphi(F)$.

2. It is not true that for any VNT, τ_V, of F, τ_V is an HNT of $\varphi(F)$.

Proof.

1. by the definition of HNT (5.8), the boundary of every tube $t_i \in \tau$ is a horizontal forest. Because all horizontal forests in F translate to ladders in $\hat{F}$, then for τ of $\hat{F}$, the boundary of every tube is a a ladder. By Lemma 5.32, τ of $\hat{F}$ is a VNT.

2. See the following counterexample.

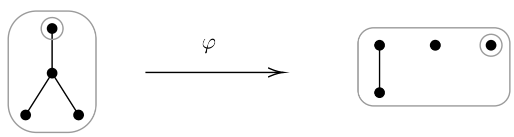


Figure 5.19: This example shows a valid VNT on F, and the same tubing on $\hat{F}$, which is not a valid HNT.

□

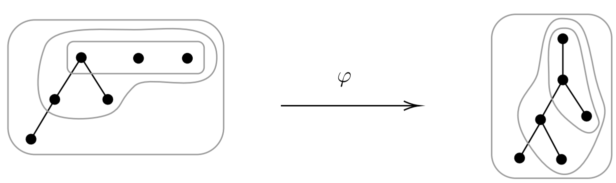


Figure 5.20: An example that shows a horizontal nested tubing of T becoming a vertical nested tubing in $\varphi(T)$.

Forests that are not leftward forests will result in posets containing cycles and/or non-rooted trees. This obviously will not be applicable to the VNTs and HNTs of plane rooted forests. But possibly, these nested tubings can be extended to the poset perspective, rather than just trees.

Chapter 6

Conclusion

6.1 In Conclusion

We have studied two different types of tubings of forests of rooted trees, binary tubings of rooted trees and Magnus tubings. This thesis discusses just a few interesting ways that we can use tubed rooted trees in combinatorics. As discussed in Chapter 1, rooted trees have applications in mathematics and computer science. Rooted trees are particularly interesting in their use in enumeration, and tubings of rooted trees add further interesting application in enumeration.

Binary tubings of rooted trees are one type of tubing we have studied, resulting in a nice bijection to connected chord diagrams. Chord diagrams provide a framework for solving some series expansions of Dyson-Schwinger equations, and this bijection introduces a new series expansion of Dyson-Schwinger equations. Both rooted trees and chord diagrams can be built by recursively adding a sub-tree, or sub-diagram into another at any of its insertion places. We have seen in Chapter 2, that rooted trees and connected chord diagrams have the same number of insertion places for a fixed number of vertices and chords. This leads nicely into our bijection between tubed rooted trees and chord diagrams, where we recursively build new trees and chord diagrams by inserting smaller ones into each other using our defined map. This result is central to the topic in Chapter 4, where we defined a map between connected chord diagrams corresponding to rooted trees.

Re-rooting is a process on a rooted tree where we pick any other root to be the root of the tree, and with tubed rooted trees, we just keep the tubes the same before and after re-rooting. This is a well-known operation on a tree in combinatorics and computer science, however we have examined through the perspective of chord diagrams using the bijection from Chapter 3. We have defined an algorithm which takes a chord diagram to another chord diagram which correspond to a tree and the tree after re-rooting. This helps us better understand underlying properties about the chord diagrams that are not necessarily intuitive from the outset.

Finally, we studied Magnus tubings of forests and how they compute the coefficients of the Magnus expansion, specifically vertical nested tubings and horizontal nested tubings. The Magnus expansion is a powerful tool that solves linear differential equations with time-dependent coefficients. Mencattini and Quesney [27] introduced a new type of tubing that uses two partial orderings on forests. We studied their work and expanded on it by treating these forests as plane posets. We studied how we can leverage this duality of forests in a transformation, which we called φ. We discovered that only a special class of forests, called leftward forests map to forests using this transformation φ. This lead us to some results about the duality between vertical nested tubings and horizontal nested tubings.

6.2 Future Work

When re-rooting tubed rooted trees and their chord diagrams, we need a lot of information about the tree before we start. We need to know what the ing of the whole tree is after analyzing the tree itself, rather than the chord diagram. We also need to know what the buds are of the tubes for this process. It would be very useful if it was not necessary to look at the tree at all; that is, given a chord diagram that corresponds to some tubed rooted tree, is there an algorithm to all the chords without looking at the tree? Additionally, a quick way to find all the buds of the tubes directly from the chord diagram would be very useful.

In this project, we only looked at one notion of re-rooting trees, which is when we take the next vertex in pre-order to be the next root. However, in the general sense, re-rooting means *any* root can be taken to be the new root. It's not immediately obvious that it's easy to re-root a chord diagram when any of the roots is arbitrarily chosen rather than the leftmost child. Obviously, we can do so with this method, but we have to do it iteratively through every root until we reach the root we want. This does not give a lot of mathematical insight to this re-rooting. Thus, something to explore would be to see if there is an easier way to do that, other than iteratively. In our notion of re-rooting, we chose the leftmost child. Finding an analogous algorithm where we choose the rightmost child instead appears to be relatively easy.

In a more general sense, it would be interesting to explore other ways that manipulating a tree would change the chord diagram, like what we did in this research with re-rooting. For example, swapping tubings or finding the chord diagram of the mirror image of a plane rooted tree.

Relating to the Magnus tubings, we would like to find a better connection between vertical nested tubings and horizontal nested tubings. The hope when we started the plane poset portion of Chapter 5 was to find that there is some map between the two tubings using this notion of plane posets. Clearly, that did not work. As we saw in that chapter, we need a forest to be a leftward forest for it to also be a forest in the plane poset transformation. However, one open question we have is how we can interpret VNTs and HNTs of a poset, rather than of a forest. When we have a plane forest F that is *not* a leftward forest, then in the transformation it becomes a poset that is not a rooted forest. So for some tubing of F, can we tweak the definitions of the Magnus tubings so that we can have Magnus tubings on the poset $\varphi(F)$?

Finally, we are interested in some way to connect binary tubings and Magnus tubings, but this may be very difficult. It would be nice to have something to connect them because both are ways of indexing series expansions used in physics and in some instances have some overlap.

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