Generalized Coherent States and Classical Limits in Quantum Mechanics

by

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A thesis

presented to the University of Waterloo

in fulfilment of the

thesis requirement for the degree of

Doctor of Philosophy

in

Applied Mathematics

Waterloo, Ontario, Canada, 2000

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Abstract

Coherent states in the harmonic oscillator have long been thought of as a bridge from quantum mechanics to classical mechanics. Many of their important properties (most notably those involving time evolution) are only valid in the harmonic oscillator and are at best approximations elsewhere. This thesis is an investigation into several means of generalizing coherent states for other systems and examines the resultant states for some systems. One system of interest is the harmonic oscillator with centripetal barrier for which coherent states of several disparate definitions coincide. Also studied is the spherical rotator, a system which is particularly amenable to defining annihilation operator coherent states. A third system under investigation is the hydrogen atom. This system serves as an arena for the development of an extension to a generalization due to Klauder [*J. Phys. A*, 29(12):L293–L298, 1996]. Klauder's construction is only applicable to systems without energy degeneracies and must be extended for application where degeneracies are present. The author provides a means for this extension and applies the complete construction to the hydrogen atom problem. As a demonstration of how this construction may be adapted, the author constructs Rydberg wave packets which are initially localized and exhibit full and fractional revivals in the long time evolution.

Acknowledgements

I am greatly indebted to my supervisors E.R. Vrscay and J. Paldus whose interest and enthusiasm in my work made completing this thesis an interesting and rewarding process. I owe a great deal of thanks to my wife, Jessica Meijer, who continues to be supportive and patient.

I am also grateful to the National Sciences and Engineering Research Council of Canada for part of the financial support of this thesis in the form of a postgraduate scholarship.

"Brains should have a shiny moist surface, pinkish color, and full plump consistency."

— M. Cunningham, *The Fanny Farmer Cookbook*, Alfred A. Knopf, New York, 13th edition, 1996; pg. 209.

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Chapter 1

Introduction

1.1 Context

The content of this thesis rests within the larger context of quantum classical correspondence. For the present purposes, quantum classical correspondence is concerned with how classical mechanics arises from quantum mechanics when moving continuously from the quantum to the classical regime. The various aspects of this thesis are concerned with this relationship, using generalized coherent states as a means of elucidating the connection.

Quantum mechanics first appeared as a physical theory in early part of the last century. For some time, physicists had been aware that the physical theories of the time failed to predict certain experimental results. Two prominent examples were blackbody radiation and the spectrum of light radiated from atoms. The most successful early theories to describe these phenomena involved assuming, by Planck in the case of blackbody radiation, that certain oscillators of frequency ν only take

on energies of values $nh\nu$ with h some universal constant in units of action (energy times time) now known as Planck's constant, and by Bohr in the case of radiated light, that electrons take on values of angular momentum equal to $nh/2\pi$. These two theories are united by assuming that the *action* of periodic orbits (*i.e.* the area enclosed by orbits in phase space) may only take on integer multiples of h, rather than values in the continuum as supposed by classical theory. In terms of modern quantum theory, this remains the basis of Bohr-Sommerfeld quantization, a semi-classical means of approximating energy levels.

By 1930, much of the skeleton of modern quantum theory had been established, though the theory continues to be fleshed out through the development of techniques and extensions. The spine of this skeleton is that given a particular system, experimentally amenable "observables" may be represented by linear, Hermitian operators on a Hilbert space, and a measurement of such an observable may result only with an eigenvalue of the operator. From this comes a Hamiltonian, an operator corresponding to a measurement of energy, whose eigenvalue spectrum in many cases is discrete, leading to the quantized energies and angular momenta of the previous paragraph.

As it turns out, Planck's constant is very small in terms of human experience ($h \sim 6.626 \times 10^{-34}$ Js, whereas the action of a grandfather clock pendulum is about 0.1 Js.). Accordingly, at the energy levels of our usual experience, the effects of quantization are seldom noticeable. It is only when the parameters of the system are on the order of h that quantization becomes important.

This notion of scale introduces an important question. Given a system, should

the architecture of classical mechanics or quantum mechanics be used? A simple-minded answer comes quickly and is alluded to above. If the parameters of the system are on the order of h, then use quantum mechanics; otherwise use classical mechanics. This is far from a perfect solution, and particularly in the case of separable systems in which the parameters in one separated set of variables are large and in another, small. Further, there is experimentally no corresponding "cutoff point." In the laboratory, the effects of quantization, which are large for experiments on a small scale, gradually disappear as the scale is increased. We demand the same of the theory.

In fact, this demand is an underlying principle of quantum mechanics introduced by Bohr in 1918 [64] and subsequently dogmatized: In some limit, depending on the system in question, the familiar classical laws emerge in some way from quantum, even though the treatment remains fully quantum. This is the so-called correspondence principle.

As an example, consider the double slit experiment. In this experiment, a beam of electrons at a certain energy is fired at a barrier in which there are two fine slits very close together. On a screen on the far side of the barrier, an interference pattern emerges with the passage of many electrons where one would classically expect a smooth distribution. As it turns out, the spacing between the slits determines how strongly the quantum behaviour is expressed. If this spacing is on the order of the wave length of the electrons (which is determined by Planck's constant), then quantum interference patterns will dominate the pattern on the screen. If this spacing is much larger, then the interference becomes too fine to resolve, and a

classical distribution emerges. Hence, quantum mechanical effects are not restricted to energy, action, or angular momentum quantization. In fact, this experiment involves a fixed energy within the continuum spectrum of the Hamiltonian. Also, the parameter of interest is not always an action variable.

Few general comments may be made about quantum classical correspondence. Depending on the system involved, the nature of the measurements involved, or the fundamental conception of what quantum classical correspondence means, different conclusions may be drawn. Thus, it is of foremost importance to be initially clear of what quantum classical correspondence means, and then be clear of how that relates to the system and how the system is manipulated. Not surprisingly, this is strongly entangled in the deeper question of quantum mechanical interpretation.

1.2 The Interpretation of Quantum Mechanics

A physical theory consists of elements and rules. These rules may be roughly divided into two categories: rules for combining elements of the theory and rules for relating elements of the theory to experiment. With respect to quantum mechanics, an example of the former is that the wave function $|\psi\rangle$ evolves according to the time dependent Schrödinger equation. An example of the latter is that a measurement of an observable x corresponding to the operator $\hat{x}$ results in an eigenvalue of $\hat{x}$. Different authors categorize some rules differently. However, the mathematical formalism belongs to the former, and the interpretation of the theory, which in the case of quantum mechanics is mostly concerned with the physical meaning of the wave function, belongs unambiguously in the latter.

Many books have been written about the interpretation of quantum mechanics. This is a controversial issue which in some circles takes on a flavour more of religion than of philosophy or physics. The present work is not intended to be a detailed presentation and comparison of many different interpretations, though some observations and ramifications will be discussed. Primarily, this section is intended to outline the interpretation within which the content of this thesis is set.

The underlying motivation for the interpretation used in this thesis is to be sure that conclusions are not overstated, to err on the side of caution. Thus, in some cases, a reader may wish to extend the present claims, to form conclusions which are beyond those herein contained, but the author will not do so since these issues remain open to debate.

The extreme view in this approach is that physical theories do not explain why systems behave the way they do; they only provide a means of predicting that behaviour. As an illustration, consider the classical principle of least action: A baseball traveling from point A at time t_1 to point B at time t_2 does not explore all paths or even just the neighbourhood of its actual path to ensure it is a path of least action. The action is only a tool introduced by theorists to permit a prediction of the path. To say otherwise is to suggest that while the ball is in flight, it explores all phase space but is only observed where the action is minimized. It is preposterous to suggest that baseballs explore all space while in flight, so we remove this aspect of the theory with Occam's razor and are left with the action as a tool for making predictions.

The early history of quantum mechanics does not depart from this approach

significantly. In Planck's description of blackbody radiation and Bohr's description of the atom, certain quantization assumptions lead to correct predictions of black body and hydrogen spectra respectively. It was not until Schrödinger published his wave equation that some physicists began to attach a greater meaning to their physical theories. This occurred with Born's exposition on the interpretation of quantum mechanics which developed into the Copenhagen Interpretation through the work of Bohr and Heisenberg.

There are two chief aspects to the Copenhagen Interpretation. First, the wave function is a complete, exhaustive description of an individual particle or system. Second, the wave function undergoes a discontinuous, unpredictable change at the instant of measurement from some initial state to an eigenstate of the operator corresponding to the property being measured. Different authors flavour these aspects differently. For example, some hold that the wave function physically represents the particle, so that the requisite spread due to uncertainty relations results in a particle physically spread out accordingly. Others say that the wave function is a representation of the knowledge of the experimentalist regarding the position (*etc.*) of the particle. However, even with these differences, all interpretations involving these two ideas will be, for the present purposes, considered as possible realizations of the Copenhagen Interpretation.

Now consider a few consequences of these ideas, some of the "paradoxes" of quantum mechanics. As part of the mathematical formalism, the sum of any two possible states for the system is also a possible state for the system (since Schrödinger's equation is linear). This is the so-called superposition principle, and

leads to the Schrödinger's cat paradox. This thought-experiment was introduced by Schrödinger to demonstrate how quantum superpositions may be amplified to yield counterintuitive superpositions of macroscopic states. A cat is put in a box with a vial of cyanide. Incorporated in the vial is a small radioactive substance, a Geiger counter, and a hammer. If the radioactive substance decays and triggers the Geiger counter, the hammer breaks the vial releasing the cyanide, killing the cat. After a certain interval at which there is an even chance that the substance has decayed, the wave function of the substance is in a superposition of decayed and not decayed, so that the cat's wave function is in a superposition of dead and not dead.

According to the Copenhagen Interpretation, prior to opening the door, the superposed state is an accurate and complete description of the state of the cat, so the cat is simultaneously dead and alive. The act of opening the door and "measuring the cat" collapses the wave function of the cat onto one of the familiar classical states of alive or dead. Though this example is often used by subscribers to the Copenhagen Interpretation as a means of demonstrating quantum weirdness, Schrödinger's original purpose was as an argument against the Copenhagen Interpretation.

Another important thought experiment was introduced by Einstein, Podolsky and Rosen [26] (EPR), and improved upon by Bohm in his textbook [18] who used spins rather than position and momenta. The experiment, briefly, is as follows. An initial, unstable state is prepared with a total spin and linear momentum of zero. The state decays and two particles emerge, traveling in opposite directions.

The particle which moves to the right (say) passes through an apparatus (a Stern-Gerlach machine) oriented in the z direction, performing a measurement of the spin in this direction without disturbing the particle moving to the left due to spatial separation. From the conservation of angular momentum, if spin up is detected on the right, the particle which went to the left is in a spin down state, and *vice versa*, with a perfect correlation. The z spin of the particle to the left is thus what Einstein *et al.* term an element of reality since it may be predicted with certainty.

Without changing how the initial state is prepared, the Stern-Gerlach machine may be reoriented to measure spin in the y direction, allowing one to predict with certainty the spin of the particle moving to the left if measured in the y direction: The y spin of the particle to the left is also an element of reality. However, both the z spin and y spin cannot be elements of reality since their corresponding operators do not commute, *unless the measurement on the right affects the particle on the left.* Thus, demanding the completeness of the wave-function introduces dependencies between spatially separated objects, what Einstein referred to as "spooky action at a distance." Admitting the incompleteness of quantum mechanics removes action at a distance.

To say that quantum mechanics is incomplete suggests that one has something in mind intended to "complete" the theory. This is not precisely the case. The most obvious means of completing quantum mechanics is through hidden variable theories. A hidden variable theory would say, for example, that an individual particle in an EPR experiment has an actual well defined spin prior to measurement. There have been several impossibility theorems intended to rule out different vari-

eties of hidden variable theories. The first was von Neumann's which turned out to be flawed in that one of the fundamental hypotheses was overstated [81]. A later theorem by Bell (involving "Bell's inequalities") ruled out local hidden variables. A theory which survives both of these theorems, due to Bohm [19, 20], is a non-local hidden variable theory, an extension of standard quantum theory. Certain features and implications of this theory are problematic (discussed later) leading the author to reject Bohm's theory, leaving him in the embarrassing situation of asserting the incompleteness of quantum mechanics without a suggestion on how to complete it.

An interpretation which permits this approach to quantum mechanics is the Statistical Interpretation, held by Einstein and formalized by Ballentine [6]. The distinguishing feature of this interpretation is that the wave function represents a conceptually infinite ensemble of similarly prepared systems. As an example of this identification, if the system in question is an electron bound to a proton, then the ensemble is the set of all such electrons that have been subjected to the same experimental procedures: heated, trapped, illuminated by laser light, and so forth. That is, the ensemble is a reflection of the preparation procedure. Just as with any collection, the properties of the collection as a whole may not be identified with the properties of an individual member of the collection or *vice versa*: The wave function need not provide a complete description of the individual system.

This relaxation does away with many of the paradoxes thought to be inherent in quantum mechanics. For an example, consider again the double slit experiment. A single electron coming through the apparatus and appearing on the screen at the far side represents an element of the ensemble, where the ensemble represents

all possible scenarios of electrons passing through the apparatus. In the Statistical Interpretation, the wave function has very little to say of the single event. Instead, after many electrons are sent through and the interference pattern becomes apparent, the statistical frequencies begin to approximate the probabilities as calculated from the wave function. So, according to this interpretation, one says, after many electrons have passed through the apparatus, "About half the electrons went through slit A and half went though slit B." After all, any wave like property of massive matter can only be seen by observing a large number of events.

The Statistical Interpretation is indeed a relaxation from the Copenhagen Interpretation, in the sense that the wave function may well represent an individual system. The author is prepared to entertain that possibility, though not prepared to admit its necessity. Also, this interpretation, though certainly not a hidden variable theory itself, admits the possibility of such theories such as that of Bohm not ruled out by Bell's inequalities.

1.3 Quantum Classical Correspondence

The Statistical Interpretation also illuminates the quantum classical correspondence principle. This principle states that, according to Bohm [18] "the laws of quantum physics must be so chosen that in the classical limit, where many quanta are involved, the quantum laws lead to the classical equations as an average." According to the Statistical Interpretation, the classical limit of the quantum wave packet is an ensemble of classical trajectories [7]. Take the harmonic oscillator for example, and the specific case of energy eigenstates in position space. In the limit of

quantum numbers, the probability distribution associated with these states become the time average of a single classical trajectory [50], *i.e.* the distribution which describes the probability of finding the particle at a given position at a random time. This classical distribution is equivalent to an ensemble of classical trajectories each at the given energy, distributed evenly in phase space, so that this limit may be equally well described by an appropriately chosen ensemble of classical trajectories.

Going back to the strict positivist approach, the purpose of the theory is to make predictions, even if those predictions are only of a statistical nature. The wave function is a reflection of the preparation of the system, and by evolving that wave function over a one second interval, one only obtains the probabilities associated with possible outcomes of a measurement one second later, and nothing else. That is, the calculations do not indicate what happened between preparation and measurement, only the result of the measurement. One must ask, therefore, what is the meaning of evolving a wave function continuously over an interval? Keeping within the strict positivist sense, given a preparation, the evolved wave function yields the probabilities associated with possible outcomes *if a measurement were to be made.* The measurement in this sense is conceptual, since the effect of the measurement on the system is not addressed. That is, a continuum of measurements may be considered without addressing the problem of successive measurements.

An actual measurement would disturb the system somehow, whether or not one accepts the idea that a measurement collapses the wave function into an eigenstate of the operator associated with the measurement. To relate the system with actual measurements to the system with conceptual measurements, therefore, the exper-

imental procedure becomes a repetitive process. The system is prepared in some prescribed manner and allowed to evolve for a certain duration until a measurement is made. Once the measurement is made the system continues to evolve, but without considering successive measurements, subsequent evolution is not relevant. Starting again, the preparation process begins and the newly prepared system is allowed to evolve for a slightly longer duration until a measurement is made. The system is repeatedly prepared and allowed to evolve for successively longer durations. Collectively, the results of these experiments give a sense of how the system evolves without the perturbing effects of measurement. In fact, this is a rough description of how actual experiments in the laboratory are carried out, such as the observation of wave function collapses and revivals [88].

Now, with regard to the Copenhagen Interpretation, consider the distinguishing feature, that the wave function is an exhaustive description of a single system. in connection with the correspondence principle, that the familiar classical laws emerge from the quantum equations in the so-called classical limit. What this suggests is that given an initial quantum state involving large quantum numbers (what Ballentine terms Ehrenfest's regime [7]), successive conceptual measurements in the above sense should yield the classical evolution of a single particle. It is clear that this is not the case: In the specific case of an eigenstate with large quantum number, this is a stationary state with stationary probabilities in time, whereas classical trajectories of large energy (or action) tend to move about somewhat. With more general initial states, the identification of the evolution of the state with the evolution of a single classical system is also seldom valid. The only states

which satisfy this requirement are the harmonic oscillator coherent states and states like them which satisfy minimum uncertainty in position and momentum jointly. Therefore, imposing in this context the quantum classical correspondence principle seems to select for coherent states in the large quantum number regime, even though there is no physical mechanism presupposed to make such a selection. Nor is such a selection necessarily unique.

This is the point at which it becomes necessary to relax the interpretation, to invoke Occam's razor and no longer demand that the wave function be a complete description of a single system, leading to the Statistical Interpretation of quantum mechanics [6]. In this interpretation the wave function represents a statistical description of the conceptually infinite set of similarly prepared systems. Then, when moving into the large quantum number regime, one deduces from this interpretation of the wave function that the classical limit of the wave function is an ensemble of classical trajectories. Thus, the quantum classical correspondence is not envisioned by identifying the Schrödinger equation with the Hamilton-Jacobi equation in the classical limit, but rather with the Liouville equation which describes the time evolution of distributions on classical phase space.

This identification has met with a great deal of success. First proposed by Wigner [82], the connection between phase space distributions evolved through the Liouville equation and the time evolution of the Wigner function lies at the heart of quantum-classical correspondence as seen through the Weyl-Wigner-Moyal formulation of quantum mechanics [52]. More recent proponents include Ballentine [7] and Fox and Elston [30] both of whom study the chaotic kicked pendulum. In both

studies, excellent agreement between classical and quantum evolution is observed when a classical distribution is compared to the quantum wave function. This agreement includes an initial phase of exponential growth of Δx at a rate corresponding to the relevant Liapunov exponent. In a more pedestrian example, in the harmonic oscillator (that is, for polynomial potentials of order less than three), quantum evolution may be described exactly by classical evolution in the Weyl-Wigner-Moyal formulation of quantum mechanics: Few researchers bother with this system in this formulation since the behaviour is essentially classical. (See also the discussion involving Ehrenfest's equations in Section 2.1.)

Each interpretation of quantum mechanics has its own version of the quantum classical correspondence principle. A third version, quite distinct from the above versions, is associated with Bohmian mechanics. This hidden variable theory, published by Bohm in the early 1950's, is not subject to such "impossibility theorems" due to von Neumann or Bell since it is a non-local theory: Action at a distance remains. In Bohm's theory, one writes the wave function as an amplitude function multiplied by a phase function. Substituting this into the time dependent Schrödinger equation and separating real and imaginary parts yields two equations: one which is a conservation equation for wave function normalization, and the other which takes on the form of the Hamilton-Jacobi equation. The phase function becomes Hamilton's principal function in a potential consisting of the original potential plus the so-called quantum potential which is proportional to $\hbar^2$. Bohmian particles then describe Bohmian trajectories which are classically evolved in this combined potential, classical plus quantum.

The Bohmian version of quantum classical correspondence is envisioned as some limit in which $\hbar$ becomes insignificant as compared to some characteristic action of the system. This limit is not particularly controversial since it is often invoked in many different systems. The difficulty comes with the assertion that the quantum potential therefore disappears in the classical limit and the evolution becomes simply classical evolution in the classical potential. As an example of this, consider the classical limit of the harmonic oscillator coherent states. This is certainly the oldest and arguably the best understood example of a classical limit. After all, the harmonic oscillator is not particularly quantum to begin with. In this case, the quadratic parts of the quantum and classical potentials cancel one another out, resulting in a linear total potential whose slope which oscillates in time in such a way that the total potential is a tangent line to the classical potential at the position of the peak of the coherent state. This is as true for states with low quantum numbers as in the classical limit. The quantum potential does not disappear in the classical limit even in the best understood case.

To compare quantum and classical time evolution according to the Statistical Interpretation, one identifies with the quantum wave function a distribution of classical trajectories in the classical limit. This is the principal thrust of Ballentine *et al.* [7] who observe that it is possible to construct a classical ensemble in comparison with a quantum wave function such that the difference between quantum evolution through the time dependent Schrödinger equation and classical time evolution through the Liouville equation is only seen in third and higher order corrections to Ehrenfest's equations. Most textbooks assume the difference to lie in

the second order term. This order of discrepancy may be attributed to the fact that the centroid of a classical distribution does not necessarily follow a classical trajectory.

1.4 The Use of Coherent States in Quantum Classical Correspondence

We now bring our attention to bear upon the question of how coherent states enter into the situation if we reject the assertion that a single classical trajectory necessarily emerges in the classical limit. Indeed, the useful classical limits of coherent states render the state a single classical trajectory in the classical limit (that is, δ-function distributions in position and momentum) so that expressing particular interest in the classical limit of coherent states may even be considered contradictory to the notion that ensembles emerge in general. However, we shall see that with additional properties attributed to the coherent states, classical distributions arise from the classical limits of arbitrary coherent states.

The specific mathematical tools necessary are that the states be complete and that in the appropriate classical limit, the overlap between distinct coherent states goes to zero, or more strongly, that the individual coherent states describe individual classical trajectories. With completeness, arbitrary states may be described as a superposition of coherent states. This is what leads to the Husimi distribution, a quantum mechanical phase space distribution function. In the classical limit, where the individual states take on the behaviour of individual classical trajectories, the

collection of states contributing to the arbitrary state form the members of the ensemble of classical trajectories, the classical limit of the arbitrary wave function.

This use of coherent states in connecting classical mechanics to quantum mechanics is by no means new. This was envisioned by Husimi [43] when the concept of the completeness of coherent states was still new. This was also used by Yaffe [84] in his discussion of quantum classical correspondence. (Yaffe's demonstration of how the Poisson brackets arise from the commutator in a $\hbar \to 0$ limit remains the clearest, and relies upon the completeness of coherent states.) Fox and Elston [29] promote this use of coherent states in a study of the kicked top. More recently, Fox [28] makes similar statements in terms of his Gaussian coherent states discussed in Chapter 5.

Chapter 2

Generalizations of Coherent States

2.1 The Harmonic Oscillator


he harmonic oscillator has many special and unique properties, which are responsible for why coherent states were first discussed for this system [70], and why it remains perhaps the most important simple model in physics. Whenever any smooth potential well is encountered, the simplest approximation is the harmonic oscillator with corrections handled through perturbation theory. Accordingly, one finds the harmonic oscillator in descriptions of both clock pendula and in the study of vibrational modes of diatomic molecules.

This model is particularly alluring due to the simple physics it exhibits. Classical periods are independent of amplitude, and distributions in classical phase space evolve in time by clockwise rotation. The quantum energy eigenlevels are equally

spaced (a property which, though rare, is not unique to the harmonic oscillator), which guarantees time-periodic behaviour of wave functions. In quantum optics, the harmonic oscillator arises in descriptions of the electromagnetic field. This role has little to do with the shape of the potential, but rests in the algebraic properties of the quantum system.

The harmonic oscillator is also one of the most dangerous models known to physics. The danger is in assuming that the many simple features it possesses are found elsewhere, and that when searching for simple behaviour (*i.e.* coherent states) the temptation is to search for the harmonic oscillator.

The harmonic oscillator Hamiltonian is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2, \tag{2.1}$$

where $\hat{p}$ and $\hat{x}$ are the momentum and position vectors respectively. In position space, the time independent Schrödinger equation is[1]

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2\hat{x}^2\right)\langle x|\psi\rangle = E\langle x|\psi\rangle, \tag{2.2}$$

and the equation in momentum space is given by

$$\left(-\frac{1}{2}m\omega^2\hbar^2\frac{d^2}{dp^2} + \frac{p^2}{2m}\right)\langle p|\psi\rangle = E\langle p|\psi\rangle, \tag{2.3}$$

[1]Regarding notation: Throughout this thesis, Dirac's bracket notation [76] is used with $|\psi\rangle$ denoting a vector in the Hilbert space, $\langle\psi|$ denoting its dual, and their inner product denoted by $\langle\psi|\phi\rangle$. Operators are "hatted" ($\hat{x}$) and c-numbers are plain (x).

making it completely symmetric in position and momentum. Any behaviour observed in position space will also be manifested in momentum space. For example, the eigenstates $|n\rangle$ in position space are given by

$$\langle x|n\rangle = \left(\frac{m\omega}{\pi\hbar(2^n n!)^2}\right)^{1/4} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)\exp\left(-\frac{m\omega}{2\hbar}x^2\right). \tag{2.4}$$

and in momentum space by

$$\langle p|n\rangle = \frac{i^n}{\sqrt{m\omega}}\left(\frac{\hbar}{\pi\hbar m\omega(2^n n!)^2}\right)^{1/4} H_n\left(\frac{p}{\sqrt{\hbar m\omega}}x\right)\exp\left(-\frac{p^2}{2\hbar m\omega}\right), \tag{2.5}$$

in which the H_n are the Hermite polynomials. The overall phase factor is retained to keep $\langle p|n\rangle$ the Fourier transform of $\langle x|n\rangle$.

The energies corresponding to these eigenstates are $E_n = \hbar\omega(n+\frac{1}{2})$, so that the energy levels have equal spacings, $\Delta E = \hbar\omega$, which has far reaching ramifications with respect to time evolution. In general, the time evolution of a state is obtained through the time evolution operator, which for a time-independent Hamiltonian is written

$$\hat{U}(t) = \exp(-i\hat{H}t/\hbar). \tag{2.6}$$

The time evolution of a state represented by energy eigenstates $|n\rangle$ is given by

$$\begin{aligned} |\psi(t)\rangle &= \hat{U}(t)|\psi(0)\rangle \\ &= \sum_n c_n \hat{U}(t)|n\rangle \\ &= \sum_n c_n e^{-iE_n t/\hbar}|n\rangle, \end{aligned} \tag{2.7}$$

so that if $E_n = E_0 + n\Delta E$,

$$|\psi(t)\rangle = e^{-iE_0 t/\hbar} \sum_n c_n e^{-in\Delta E t/\hbar} |n\rangle, \tag{2.8}$$

and at a time $kT = 2k\pi\hbar/\Delta E$ later with k integer,

$$\begin{aligned} |\psi(t+kT)\rangle &= e^{-iE_0 t/\hbar} e^{-iE_0 kT/\hbar} \sum_n c_n e^{-in\Delta E t/\hbar} e^{-i2kn\pi} |n\rangle \\ &= e^{-iE_0 kT/\hbar} |\psi(t)\rangle. \end{aligned} \tag{2.9}$$

The overall phase difference (equal to $e^{ik\pi} = (-1)^k$ in the harmonic oscillator) has no measurable effect. Otherwise, all wave functions in a system with such a spectrum are periodic with period T. In the harmonic oscillator, the period is given by $T = 2\pi/\omega$, equal to the classical period so that ω is equal to the classical angular frequency.

Another apparently classical property of arbitrary states in the harmonic oscillator is the time evolution of quantum expectation values. Given an initial state $|\psi(0)\rangle$ expressed as a superposition over Hamiltonian eigenstates.

$$\begin{aligned} \langle x \rangle &= \langle \psi(t) | \hat{x} | \psi(t) \rangle \\ &= \sum_{n,m=0}^{\infty} c_m^* c_n e^{-i(E_n - E_m)t/\hbar} \langle m | \hat{x} | n \rangle. \end{aligned} \tag{2.10}$$

(2.11)

Recalling the expression of $\hat{x}$ in terms of the creation and annihilation operators,

$$\langle m|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\delta_{m,n-1} + \sqrt{n+1}\delta_{m,n+1}\right), \tag{2.12}$$

so that

$$\begin{aligned}\langle x\rangle &= \sqrt{\frac{\hbar}{2m\omega}}\left[\sum_{n=1}^{\infty} c_{n-1}^{*}c_n e^{-i(E_n-E_{n-1})t/\hbar}\sqrt{n} + \sum_{n=0}^{\infty} c_{n+1}^{*}c_n e^{-i(E_n-E_{n+1})t/\hbar}\sqrt{n+1}\right] \\ &= \sqrt{\frac{2\hbar}{m\omega}}\mathrm{Re}\left(e^{-i\omega t}\sum_{n=0}^{\infty} c_n^{*}c_{n+1}\sqrt{n+1}\right) \\ &= \sqrt{\frac{2\hbar}{m\omega}}|A|\cos(\omega t+\phi),\end{aligned} \tag{2.13}$$

with the summation written as some complex number A, with some phase angle ϕ. With the appropriate initial conditions, a classical particle will evolve according to this expression. A similar calculation gives

$$\begin{aligned}\langle p\rangle &= \sqrt{2m\omega\hbar}\mathrm{Im}\left(e^{-i\omega t}\sum_{n=0}^{\infty} c_{n+1}^{*}c_n\sqrt{n+1}\right) \\ &= -\sqrt{2m\omega\hbar}|A|\sin(\omega t+\phi).\end{aligned} \tag{2.14}$$

This also represents classical evolution and should not be a surprise due to the Ehrenfest's relation [76]

$$\langle p\rangle = m\frac{d}{dt}\langle x\rangle. \tag{2.15}$$

The other of Ehrenfest's relations is more significant in this case,

$$\frac{d}{dt}\langle p\rangle = \left\langle -\frac{dV}{dx}\right\rangle. \tag{2.16}$$

Classical evolution would arise if the expectation value on the right was *inside* the evaluation of the derivative of the potential, so it is worthwhile to see how much the expression differs as a result. Since this is suggestive of a classical force, let $F(x) = -dV/dx$ and expand with a Taylor series about $x = \langle x \rangle$,

$$F(x) = F(\langle x \rangle) + \frac{dF}{dx}(x - \langle x \rangle) + \frac{1}{2}\frac{d^2F}{dx^2}(x - \langle x \rangle)^2 + \cdots, \tag{2.17}$$

where the derivatives of the force function are evaluated at $x = \langle x \rangle$. Now taking expectation values, the linear term in x vanishes and the quadratic term leads to $(\Delta x)^2$ which is *defined* by $(\Delta x)^2 = \langle (\hat{x} - \langle x \rangle)^2 \rangle$ so that

$$\frac{d}{dt}\langle p \rangle = F(\langle x \rangle) + \frac{1}{2}(\Delta x)^2 \left(\frac{d^2F}{dx^2}\right)_{x=\langle x \rangle} + \cdots. \tag{2.18}$$

In the harmonic oscillator, the force is linear with position so the second derivative vanishes. This leaves purely classical equations of evolution for the expectation values of $\hat{x}$ and $\hat{p}$, irrespective of the structure of the wave function in configuration space or any other consideration.

As a brief aside, note that in other systems where $\frac{d^2F}{dx^2} \neq 0$, Eq. (2.18) leads to initial classical behaviour if $(\Delta x)^2$ is small. This has led to, among other things, generalizations of coherent states which minimize uncertainty [58]. Also, it is not wholly accurate to say that the corrections in Eq. (2.18) are "quantum corrections" since similar corrections arise from purely classical calculations of classical distributions evolved in phase space through the Liouville equation as shown by Ballentine *et al.* [7].

The preceding comments are only the most prominent aspects of the harmonic oscillator rendering it singular in quantum and classical comparisons. Before embarking on any investigation of quantum classical correspondence it is important to initially establish precisely by how much they differ. In the harmonic oscillator, that difference is much smaller than it first appears. For this reason, particular caution must be exercised when generalizing from the harmonic oscillator to other systems when concerned with quantum classical correspondence.

2.1.1 The Harmonic Oscillator Coherent States

From Eqs. (2.13) and (2.14), it is already known that in the harmonic oscillator the time evolution of expectation values of arbitrary states follows classical trajectories. There is a significant discrepancy between the quantum and classical to be noted however: No consideration has been given to the energy of the classical trajectory as compared to the expectation value of the Hamiltonian in the quantum state. For a coherent state, the expectation values follow the classical trajectory of a particle whose energy is equal to $\langle H \rangle - E_0$.

In position space, the harmonic oscillator coherent states are given by [39]

$$\langle x|\alpha\rangle = \left(\frac{1}{2\pi(\Delta x)^2}\right)^{\frac{1}{4}} \exp\left(-\left[\frac{x-\langle x\rangle}{2(\Delta x)}\right]^2 + \frac{i\langle p\rangle x}{\hbar}\right). \tag{2.19}$$

Parameterized by the complex valued α,

$$\alpha = \frac{1}{2}\left(\frac{\langle x\rangle}{\Delta x} + \frac{i\langle p\rangle}{\Delta p}\right), \tag{2.20}$$

an expansion over eigenstates is given by

$$|\alpha\rangle = \exp\left(-\frac{|\alpha|^2}{2}\right) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{2.21}$$

Here, it appears that the uncertainties and expectation values enter as parameters of the state. Although they may be regarded as such, explicit calculations of these quantities through

$$\langle x \rangle = \langle \alpha | \hat{x} | \alpha \rangle, \quad (\Delta x)^2 = \langle \alpha | (\hat{x} - \langle x \rangle)^2 | \alpha \rangle, \tag{2.22}$$

and likewise for $\hat{p}$, yield the expected results. Eq. (2.21) follows not only by taking the direct product $\langle n|\alpha\rangle$ through Eqs. (2.4) and (2.19), but also by taking the Gaussian of Eq. (2.19) as the generating function of the Hermite polynomials found in Eq. (2.4).

Consider the time evolution of an individual state. Recall that the time evolution of an eigenstate $|n\rangle$ of the Hamiltonian $\hat{H}$ with energy E_n is given by

$$\hat{U}(t)|n\rangle = \exp\left(-i\hat{H}t/\hbar\right)|n\rangle \tag{2.23}$$

$$= \exp\left(-iE_n t/\hbar\right)|n\rangle, \tag{2.24}$$

where $\hat{U}(t)$ is the time evolution operator. With the harmonic oscillator eigenstate energies

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \tag{2.25}$$

and Eq. (2.21), the time evolution is given by

$$\begin{aligned}
\hat{U}(t)|\alpha\rangle &= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\hat{U}(t)|n\rangle && (2.26)\\
&= \exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\exp\left(-i\omega\left(n+\frac{1}{2}\right)t\right)|n\rangle && (2.27)\\
&= e^{-i\omega t/2}\exp\left(-\frac{|\alpha|^2}{2}\right)\sum_{n=0}^{\infty}\frac{(\alpha e^{-i\omega t})^n}{\sqrt{n!}}|n\rangle && (2.28)\\
&= e^{-i\omega t/2}|\alpha e^{-i\omega t}\rangle, && (2.29)
\end{aligned}$$

so that the time evolution of a coherent state may be expressed (up to an overall phase factor) by evolving the parameter α. This property follows from the α^n portion of the expansion Eq. (2.21) jointly with the evenly spaced energy eigenvalues.

Now, given that under time evolution $\alpha \to \alpha e^{i\omega t}$ from Eq. (2.20), the time evolution of the expectation values is given by

$$\begin{aligned}
\langle x(t)\rangle &= x_0\cos\omega t + \frac{p_0}{m\omega}\sin\omega t, && (2.30)\\
\langle p(t)\rangle &= p_0\cos\omega t - m\omega x_0\sin\omega t, && (2.31)
\end{aligned}$$

which are identical with the trajectory of a classical particle with initial position and momentum x_0 and p_0. Accordingly, the classical energy, given initial conditions, is given by

$$E = \frac{p_0^2}{2m} + \frac{1}{2}m\omega^2 x_0^2, \tag{2.32}$$

and the quantum expectation value is given by

$$\langle H\rangle = \frac{\langle p\rangle^2}{2m} + \frac{1}{2}m\omega^2\langle x\rangle^2 + \frac{1}{2}\hbar\omega, \tag{2.33}$$

differing from the classical expression by the ground state energy.

Furthermore, though the simplest calculations of uncertainties are through the annihilation operator definition of the coherent states given below, from Eq. (2.19) one can calculate uncertainties of $\hat{x}$ and $\hat{p}$ and find

$$(\Delta x)^2(\Delta p)^2 = \frac{\hbar^2}{4}. \tag{2.34}$$

That is, the state is a minimum uncertainty state and follows a classical trajectory for all time. This combination of properties prompted Schrödinger [70] to submit these as "quasiclassical" states, as some sort of bridge between from quantum to classical mechanics.

Another important property of the harmonic oscillator coherent states is that they satisfy a resolution of the identity. The resolution of the identity is given by

$$I = \frac{1}{\pi}\int d^2\alpha |\alpha\rangle\langle\alpha|, \qquad d^2\alpha = d\mathrm{Re}(\alpha)d\mathrm{Im}(\alpha), \tag{2.35}$$

where $\mathrm{Re}(\alpha)$ and $\mathrm{Im}(\alpha)$ stand for the real and imaginary parts of α respectively, and the integration is over the entire complex plane. Expressing the state as sums over eigenstates, Eq. (2.35) becomes

$$\begin{aligned} &\frac{1}{\pi}\int d^2\alpha |\alpha\rangle\langle\alpha| \qquad (2.36)\\ &= \frac{1}{\pi}\sum_{m,n}\int d^2\alpha \exp\left[-|\alpha|^2\right]\frac{(\alpha^*)^m\,\alpha^n}{\sqrt{m!}\sqrt{n!}}|m\rangle\langle n| \\ &= \frac{1}{\pi}\sum_{m,n}\frac{1}{\sqrt{m!}\sqrt{n!}}\int_0^\infty d|\alpha|\int_0^{2\pi} d\theta \exp\left[-|\alpha|^2\right]|\alpha|^{n+m+1}e^{i(n-m)\theta}|m\rangle\langle n| \end{aligned}$$

$$= \frac{1}{\pi}\sum_n \frac{2\pi}{n!}\left(\frac{n!}{2}\right)|n\rangle\langle n| = I. \tag{2.37}$$

Eq. (2.35) renders the set of harmonic oscillator coherent states a convenient basis for a representation of arbitrary states in the harmonic oscillator. Any state $|\psi\rangle$ may then be expressed using Eq. (2.35) as a superposition over the coherent states:

$$|\psi\rangle = \frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha|\psi\rangle. \tag{2.38}$$

Then, as with representations in terms of Hamiltonian eigenstates, the time evolution of $|\psi\rangle$ may be expressed via the time evolution of the coherent states.

$$|\psi(t)\rangle = \hat{U}(t)|\psi\rangle \tag{2.39}$$

$$= \frac{e^{-i\omega t/2}}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha e^{i\omega t}|\psi\rangle. \tag{2.40}$$

The particular value of this comes with the one to one correspondence between points in classical phase space and complex scalars α via Eq. (2.20). Indeed, $\frac{1}{\pi}|\langle\alpha|\psi\rangle|^2$ is the Husimi distribution, a means of representing quantum states on classical phase space. Note that a distribution given by $\frac{1}{\pi}|\langle\alpha e^{i\omega t}|\psi\rangle|^2$ rotates counterclockwise on the α plane with time, identical to how a classical distribution on phase space evolves in the harmonic oscillator through the Liouville equation.

2.1.2 Definitions of the States

When Schrödinger presented his quasiclassical states in 1926 [70], he merely gave their position space and eigenstate representations and worked out some of their

properties. It was not until 1963 that coherent states were defined by Glauber [39]. Motivated by their usefulness in quantum optics, Glauber gave three definitions for the coherent states, which all of which result in the same set of states.

They may be described as annihilation operator coherent states, whereby they are eigenstates of the harmonic oscillator annihilation operator (see Appendix C),

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{2.41}$$

Assuming an expansion in terms of Hamiltonian eigenstates and taking the inner product with $\langle n|$ yields the recurrence relation

$$c_{n+1}\sqrt{n+1} = \alpha c_n. \tag{2.42}$$

which is solved to yield

$$|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{2.43}$$

Normalization yields the expansion already given above, Eq. (2.21).

The harmonic oscillator coherent states may also be defined as the result of the application of a displacement operator on the ground state. To develop this construction, consider translation operators in position and momentum,

$$\hat{T}_x(x_0) = \exp(-i\hat{p}x_0/\hbar), \quad \hat{T}_p(p_0) = \exp(i\hat{x}p_0/\hbar). \tag{2.44}$$

In the case of the operator $\hat{T}_x$,

$$\begin{aligned}
\langle x|\hat{T}_x|\psi\rangle &= \int_{-\infty}^{\infty} dp\langle x|\hat{T}_x|p\rangle\langle p|\psi\rangle && (2.45)\\
&= \int_{-\infty}^{\infty} dp\langle x|p\rangle e^{-ipx_0/\hbar}\langle p|\psi\rangle && (2.46)\\
&= \int_{-\infty}^{\infty} dp\frac{e^{ip(x-x_0)/\hbar}}{\sqrt{2\pi\hbar}}\langle p|\psi\rangle && (2.47)\\
&= \int_{-\infty}^{\infty} dp\langle x-x_0|p\rangle\langle p|\psi\rangle && (2.48)\\
&= \langle x-x_0|\psi\rangle. && (2.49)
\end{aligned}$$

Similarly,

$$\langle p|\hat{T}_p(p_0)|\psi\rangle = \langle p-p_0|\psi\rangle. \tag{2.50}$$

Applying $\hat{T}_x(x_0)$ followed by $\hat{T}_p(p_0)$ thus results in translations in position and momentum by x_0 and p_0 respectively. This product is given by

$$\hat{T}_p(p_0)\hat{T}_x(x_0) = \exp\left(\frac{i}{\hbar}(p_0\hat{x} - x_0\hat{p}) - \frac{i}{2}\frac{x_0p_0}{\hbar}\right). \tag{2.51}$$

The third term in the exponential follows from the Baker-Campbell-Hausdorff formula: In the case that $[[\hat{A},\hat{B}],\hat{A}] = [[\hat{A},\hat{B}],\hat{B}] = 0$,

$$e^{\hat{A}}e^{\hat{B}} = e^{\hat{A}+\hat{B}+\frac{1}{2}[\hat{A},\hat{B}]}. \tag{2.52}$$

Since the current system of interest is the harmonic oscillator, it is convenient to

express the operator in terms of the annihilation operator

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right), \tag{2.53}$$

and its Hermitian conjugate $\hat{a}^\dagger$ (see Appendix C).

$$\hat{T}_p(p_0)\hat{T}_x(x_0) = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a} - \frac{i}{2}\frac{x_0 p_0}{\hbar}\right). \tag{2.54}$$

The complex scalar α is given by

$$\alpha = \sqrt{\frac{m\omega}{2\hbar}}\left(x_0 + \frac{i}{m\omega}p_0\right). \tag{2.55}$$

The third term in the exponential is an overall phase factor which does not affect whether the operator is unitary and so may be dropped in the definition of the displacement operator $\hat{D}(\alpha)$,

$$\hat{D}(\alpha) = \exp\left(\alpha\hat{a}^\dagger - \alpha^*\hat{a}\right). \tag{2.56}$$

The harmonic oscillator coherent states defined as displacement operator coherent states are simply the ground state acted upon by the displacement operator,

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle. \tag{2.57}$$

To see that the displacement operator definition corresponds to the annihilation

operator definition, note that

$$[\hat{a}, \hat{D}(\alpha)] = \alpha \hat{D}(\alpha). \tag{2.58}$$

Applying $\hat{a}$ to both sides of Eq. (2.57) yields

$$\begin{aligned} \hat{a}|\alpha\rangle &= \hat{a}\hat{D}(\alpha)|0\rangle & (2.59)\\ &= \left(\hat{D}(\alpha)\hat{a} + \alpha\hat{D}(\alpha)\right)|0\rangle & (2.60)\\ &= \alpha\hat{D}(\alpha)|0\rangle & (2.61)\\ &= \alpha|\alpha\rangle, & (2.62) \end{aligned}$$

where the second step follows from $\hat{a}|0\rangle = 0$. This reproduces the defining equation for annihilation operator coherent states.

The harmonic oscillator coherent states are also defined as minimum uncertainty coherent states. If two operators $\hat{A}$ and $\hat{B}$ commute according to $[\hat{A}, \hat{B}] = i\hat{C}$, then all states must satisfy the expectation value inequality (uncertainty relation) $(\Delta A)^2(\Delta B)^2 \geq \langle C\rangle^2/4$ [58]. Moreover, states $|\psi\rangle$ which satisfy equality in this relation must also satisfy the eigenvalue equation [58]

$$\frac{1}{2}\left(\frac{\hat{A}}{\Delta A} + i\frac{\hat{B}}{\Delta B}\right)|\psi\rangle = \frac{1}{2}\left(\frac{\langle A\rangle}{\Delta A} + i\frac{\langle B\rangle}{\Delta B}\right)|\psi\rangle. \tag{2.63}$$

In this equation, $\langle A\rangle$, $\langle B\rangle$, ΔA and ΔB stand for four parameters defining the state $|\psi\rangle$ which will have those expectation values. Note that ΔA and ΔB are related by the equality in the uncertainty relation leaving three independent parameters.

Since $[\hat{x},\hat{p}] = i\hbar$, letting $\hat{A} = \hat{x}$ and $\hat{B} = \hat{p}$ in the above yields $(\Delta x)^2(\Delta p)^2 \geq \hbar^2/4$, the infamous Heisenberg uncertainty relation. Since the momentum operator is realized as a first order differential operator in position space, Eq. (2.63) becomes a first order differential equation when projected into position space with the normalized Gaussian solution

$$\langle x|\psi\rangle = (2\pi(\Delta x)^2)^{-1/4} \exp\left(-\left(\frac{x-\langle x\rangle}{2\Delta x}\right)^2 + \frac{ix\langle p\rangle}{\hbar}\right), \tag{2.64}$$

where equality in the uncertainty relation was used to eliminate $(\Delta p)^2$. Eq. (2.64) serves to define what some authors call "squeezed states" since these states satisfy minimum uncertainty but are squeezed more in one uncertainty and less in the other than the traditional coherent states. The more general set of squeezed states (which depend on three parameters) are restricted to the coherent states (which depend on two parameters) by demanding that the state corresponding to $\langle x\rangle = 0$, $\langle p\rangle = 0$ is the ground state. This condition is motivated physically by demanding that coherent states of any energy may be obtained, down to and including the ground state energy. Any state with the ground state energy must be the ground state. Recall that the ground state of the harmonic oscillator is given by

$$\langle x|0\rangle = \left(\frac{a_0}{\sqrt{\pi}}\right)^{1/2} \exp\left(-\frac{1}{2}a_0^2x^2\right), \quad a_0 = \sqrt{\frac{m\omega}{\hbar}}, \tag{2.65}$$

which imposes the restriction

$$(\Delta x)^2 = \frac{1}{2a_0^2} = \frac{\hbar}{2m\omega}. \tag{2.66}$$

Accordingly, the equality in the uncertainty relation yields

$$(\Delta p)^2 = \frac{\hbar m\omega}{2}. \tag{2.67}$$

Using these known values for the uncertainties to rewrite the defining equation Eq. (2.63) yields

$$\sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i}{m\omega}\hat{p}\right)|\psi\rangle = \sqrt{\frac{m\omega}{2\hbar}}\left(\langle x\rangle + \frac{i}{m\omega}\langle p\rangle\right)|\psi\rangle. \tag{2.68}$$

The operator on the left hand side corresponds to the annihilation operator, and the scalar on the right hand side corresponds to α. This then repeats the earlier definition of the coherent states as eigenstates of the annihilation operator.

In addition to the three definitions of Glauber, a possible fourth has recently emerged in the form of Klauder's temporally stable construction [45]. This construction prescribes the eigenstate expansion in such a way that completeness and "temporal stability" follow immediately. To define the harmonic oscillator coherent states in this manner would be to simply state the eigenstate expansion Eq. (2.21), and hence discussion of this generalization is left to the next section.

2.2 Generalizations of Coherent States

There are many generalizations of the harmonic oscillator coherent states in the literature. Many approaches are as annihilation operator coherent states. A few general comments may be made regarding this variety of generalization.

2.2.1 Annihilation Operator Coherent States

In a discrete system, it is frequently possible to define operators to "jump" between Hamiltonian eigenstates. The creation and annihilation operators of the harmonic oscillator are the foremost examples. The second most common occurrences are the angular momentum ladder operators. Both of these examples have the useful property of being independent of the states upon which they act. In other systems, an operator that jumps from the state $|n\rangle$ to $|n-1\rangle$ may be dependent on n. Then, annihilation operator coherent states are defined as eigenstates of the operator that annihilates the ground state. Generalized annihilation operator coherent states of this description have been discussed by Nieto [58] and Ghosh [38].

With respect to the angular momentum ladder operators, these operators are the ladder operators associated with the Lie algebra so(3). In fact, Lie algebras with discrete representations generally have ladder operators. However, since SO(3) is a compact group, it has finite-dimensional unitary irreducible representations, meaning a finite number of angular momentum eigenstates pertaining to a particular representation. Hence, the ladder operators are nilpotent in the sense that $\hat{J}_-^k = 0$ for a finite k, and a nilpotent operator has no non-trivial eigenstates. The eigenstates of an operator closely related to the angular momentum ladder operators which is not nilpotent is discussed in Appendix F.

The situation is not so bleak with respect to non-compact algebras. The standard coherent states in fact arise as eigenstates of the annihilation operator of the Heisenberg-Weyl algebra, the dynamical algebra of the harmonic oscillator. The concept of coherent states arising as eigenstates of Lie algebraic ladder operators of

a non-compact algebra dates back to Barut and Girardello [9] who discuss in their paper the details of SO(2,1) annihilation operator eigenstates.

Note that creation operator eigenstates cannot exist. In the harmonic oscillator, consider the eigenstate equation

$$\hat{a}^\dagger|\beta\rangle = \beta|\beta\rangle. \tag{2.69}$$

If on the right, the nth state is lowest occupied state, on the left, the lowest occupied state is the $(n+1)$st, which can only happen if $\beta = 0$. That is, there are no non-trivial eigenstates of $\hat{a}^\dagger$, even though $\hat{a}^\dagger$ is not nilpotent.

2.2.2 Displacement Operator Coherent States

The most common generalization of harmonic oscillator coherent states seems to be as displacement operator coherent states, generalized by Perelomov for arbitrary Lie groups [67, 68]. (See, for example, the review article by Zhang, Feng and Gilmore [90], and papers collected by Klauder and Skagerstam [46].) Given the dynamical group of a Hamiltonian, this generalization provides a set of coherent states with much of the structure of the group coming through into the set of coherent states.

The concept of dynamical group is treated differently by different authors. Presently, the dynamical group [65] of a system is a group (usually a Lie group) such that all the states (of interest) are contained in a single irreducible representation of the group. The qualifier "of interest" is used because of cases such as the hydrogen atom for which SO(4,2) is the dynamical group: One irreducible repre-

sentation is valid for the bound portion, and a *different* SO(4,2) is appropriate for the scattering portion of the spectrum. A weaker sense of dynamical group requires that the Hamiltonian be expressible in terms of the generators of the dynamical group (elements of the associated Lie algebra).

Suppose the dynamical group G of the system in question is known. Let $|0\rangle$ be a fixed vector and $\hat{T}(g)$ the irreducible unitary representation of $g \in G$. Then, according to the Perelomov construction [68], the coherent states are given by

$$|g\rangle = \hat{T}(g)|0\rangle, \tag{2.70}$$

for $g \in G$. For some elements h of G, $|0\rangle$ may remain unaltered up to an overall phase factor. The collection of all such elements h of G forms the isotropy subgroup $H \subset G$ of $|0\rangle$. The set of coherent states is then restricted to the set $\{|x\rangle = \hat{T}(x)|0\rangle\}$, for x in the quotient space $X = G/H$.

These coherent states admit a resolution of the identity built upon the Haar measure. Let dg denote the normalized Haar measure [75] on G. The operator

$$\hat{B} = \int_G dg|g\rangle\langle g| \tag{2.71}$$

can be shown to commute with $\hat{T}(s)$ for all $s \in G$, so that by Schur's Lemma, $\hat{B}$ is proportional to the identity,

$$\hat{B} = b^{-1}\hat{1}, \tag{2.72}$$

for some constant b. If G is a compact group (for example, as for the degeneracy group of a bound system) then $b = \dim(T)$, which may be shown by taking the

trace of Eq. (2.71) over the finitely numerous eigenstates. Since the coherent states of interest are only those $|x\rangle$ for $x \in X$, the action of the subgroup H may be separated from G in Eq. (2.71) via [10]

$$\begin{aligned} \int_G dg|g\rangle\langle g| &= \int_X d\eta(gH) \int_H dh|gh\rangle\langle gh| \\ &= \text{vol}(H) \int_X d\eta(gH)|g\rangle\langle g|, \end{aligned} \tag{2.73}$$

in which vol(H) stands for the volume of H. The resolution of the identity is then given by

$$\hat{1}_n = b\ \text{vol}(H) \int_X d\eta(x)|x\rangle\langle x|, \tag{2.74}$$

In the harmonic oscillator, the dynamical group is the Heisenberg-Weyl group. The elements of the group may be labeled by the complex number α given in Eq. (2.55), and represented through $\hat{D}(\alpha)$ given by Eq. (2.56). Accordingly, this generalization provides the standard coherent states when applied to the harmonic oscillator.

2.2.3 Minimum Uncertainty Coherent States

Generalizations based upon the concept of minimum uncertainty usually stem from the $(\Delta x)^2$ term in the second of Ehrenfest's relation, Eq. (2.18). Without it (as in the harmonic oscillator) $\langle x\rangle$ and $\langle p\rangle$ evolve classically. However, if it is present, then $d\langle p\rangle/dt$ at least approximates classical evolution so long as $(\Delta x)^2$ is small. Also, recall the general expression pertaining to the time evolution of the expectation

value of any operator $\hat{O}$ [76],

$$\frac{d}{dt}\langle O\rangle = \frac{i}{\hbar}\langle[\hat{H},\hat{O}]\rangle + \left\langle\frac{\partial\hat{O}}{\partial t}\right\rangle, \tag{2.75}$$

where expectation values are taken in an arbitrary time evolved state $|\psi(t)\rangle$. With $\hat{O} = (\hat{x} - \langle x\rangle)^2$, one obtains $\langle O\rangle = (\Delta x)^2$. Accordingly, after some algebra

$$\frac{d}{dt}(\Delta x)^2 = \frac{1}{m}\langle(\hat{x} - \langle x\rangle)(\hat{p} - \langle p\rangle) + (\hat{p} - \langle p\rangle)(\hat{x} - \langle x\rangle)\rangle. \tag{2.76}$$

From this expression comes the order of magnitude result

$$\frac{d}{dt}\Delta x = \frac{1}{m}O(\Delta p). \tag{2.77}$$

so that the smaller Δp, the longer Δx remains small. Therefore, to provide for the longest interval of classical behaviour for $\langle x\rangle$ and $\langle p\rangle$, Δx and Δp must jointly be small. Of course, the best case of this is equality in Heisenberg's uncertainty relation, satisfied by the harmonic oscillator coherent states.

Nieto's generalization [57, 58, 59, 60, 55, 41, 61] follows this argument. This generalization, applied to potentials with one local minimum, defined operators through which the system appears (in senses described below) like the harmonic oscillator. In cases where the potential has more than one local minimum, the procedure may be adapted to produce several sets of coherent states, one corresponding to each minimum.

Classical trajectories about the minimum will be simple closed orbits in x-p phase space. There exists a one-to-one map from these orbits to elliptical orbits in

new coordinates X_c and $P_c = m\dot{X}_c$. With A and B denoting amplitudes in X_c and P_c,

$$\frac{X_c^2}{A^2} + \frac{P_c^2}{B^2} = 1, \tag{2.78}$$

so that

$$P_c^2 = (m\dot{X}_c)^2 = B^2\left(1 - \frac{X_c^2}{A^2}\right). \tag{2.79}$$

Eq. (2.79) is a first order differential equation for $X_c(t)$ yielding

$$X_c = A\sin(\omega t + \phi) \tag{2.80}$$

in which the angular frequency is given by $\omega = B/Am$. Elliptical orbits yield sinusoidal evolution. The mapping $X_c(x)$ itself may be found by constructing a differential equation as follows. Substitute $P_c = m\dot{X}_c = m\dot{x}dX_c/dx$ into Eq. (2.78) with $B = Am\omega$ to obtain

$$X_c' = \omega\left(\frac{m(A^2 - X_c^2)}{2(E - V(x))}\right)^{1/2}. \tag{2.81}$$

in which E is the energy of the classical trajectory, brought about by using $m\dot{x} = 2(E - V(x))$.

Having developed the classical map $X_c(x)$, quantum operators are now given by

$$\hat{X} = X_c(\hat{x}), \quad \text{and} \quad \hat{P} = \frac{1}{2}(X_c'(\hat{x})\hat{p} + \hat{p}X_c'(\hat{x})). \tag{2.82}$$

Define G as the commutator of these operators,

$$\hat{G} = -i[\hat{X}, \hat{P}], \tag{2.83}$$

and note that any state must satisfy the uncertainty relation

$$(\Delta X)^2(\Delta P)^2 \geq \frac{1}{4}\langle G\rangle^2. \tag{2.84}$$

Indeed, a state $|\psi\rangle$ which satisfies equality in the above also satisfies

$$\frac{1}{2}\left(\frac{\hat{X}}{\Delta X} + i\frac{\hat{P}}{\Delta P}\right)|\psi\rangle = \frac{1}{2}\left(\frac{\langle X\rangle}{\Delta X} + i\frac{\langle P\rangle}{\Delta P}\right)|\psi\rangle. \tag{2.85}$$

These above states $|\psi\rangle$ are the generalized minimum uncertainty coherent states by Nieto's construction. There are four parameters in the above expression: the expectation values and uncertainties for $\hat{X}$ and $\hat{P}$. One is eliminated by equality in the uncertainty relation Eq. (2.84). A further constraint is imposed whereby the ground state must be a possible coherent state, leaving the coherent states depending on two continuous parameters.

No resolution of the identity exists in general for coherent states by this construction. In at least one specific example besides the standard harmonic oscillator [56] (see Section 3.2), a resolution of the identity has been found though this is an isolated case. Several plausibility arguments exist that these states are complete, the strongest of which is that in all cases where the harmonic oscillator is the result of a certain limit, the same limit applied to these states results in the harmonic oscillator coherent states which are themselves complete.

This generalization applied to the harmonic oscillator trivially leads to the standard set of coherent states: Since orbits are already elliptical the appropriate map is the identity. Though this construction is generally applicable, it may be quite difficult to carry through to completion in a given system if the map X_c is dependent upon the energy E. In finding this map by solving Eq. (2.81), an arbitrary constant arises which in some cases may be chosen in such a way as to eliminate the energy from the map, but if this is not possible, the Hamiltonian H must be used in its place in the quantum operators $\hat{X}$ and $\hat{P}$.

2.2.4 Klauder's Construction

Recently, a construction of coherent states has been proposed by Klauder [45], which attacks the two properties of time evolution and completeness on a more basic level. This is the so-called temporally stable construction, though it should be emphasized that this stability only means that the coherent states evolve in time among themselves and has no bearing on their behaviour in configuration space.

Suppose the Hamiltonian is free of energy degeneracies, and the eigenstates $|n\rangle$ have energies E_n, $n = 0, 1, 2, \ldots$. Following Klauder, let $\rho(u) > 0$ be a positive density function defined on the positive real axis such that all the moments ρ_n of $\rho(u)$ exist. Then, the coherent states are given by

$$|s, \gamma\rangle = M(s^2) \sum_{n=0}^{\infty} \frac{s^n \exp(-i\gamma E_n/\hbar)}{\sqrt{\rho_n}} |n\rangle, \tag{2.86}$$

where $s \geq 0$ and γ is real, and $M(s^2)$ preserves normalization,

$$1 = \langle s, \gamma | s, \gamma \rangle = M^2(s^2) \sum_{n=0}^{\infty} \frac{s^{2n}}{\rho_n}. \tag{2.87}$$

Implicitly, a second condition on $\rho(u)$ appears such that all these sums must exist. Such functions exist: $\rho(u) = e^{-u}$ leads to the standard harmonic oscillator coherent states.

This construction builds coherent states from discretely labeled eigenstates. If the Hamiltonian has both discrete and continuous portions of the spectrum, then this construction applies only to the discrete portion. If there are only a finite number of discrete states (such as the bound states of the finite square well), then the upper limit on n may be changed to $n_{\max}$ as necessary. In this case, the conditions on $\rho(u)$ are relaxed considerably.

The time evolution of these states may be expressed by

$$\hat{U}(t)|s, \gamma\rangle = |s, \gamma + t\rangle, \tag{2.88}$$

which follows directly from Eq. (2.7). Note that this property of these states rests entirely in the $\exp(-i\gamma E_n/\hbar)$ portion of the eigenstate expansion.

Further, defining $k(u)$ by

$$k(u)M^2(u) = \rho(u), \tag{2.89}$$

these states satisfy the resolution of the identity

$$\hat{1} = \int d\mu(s,\gamma)|s,\gamma\rangle\langle s,\gamma|, \tag{2.90}$$

in which the integration is given by

$$\int d\mu(s,\gamma) = \lim_{\Gamma\to\infty} \frac{1}{2\Gamma} \int_0^\infty ds^2 k(s^2) \int_{-\Gamma}^{\Gamma} d\gamma, \tag{2.91}$$

where the limit is introduced to handle possibly incommensurate energy levels. Of course, if a continuous portion of the spectrum has been excluded, the so-called resolution of the identity should be regarded as a projection operator onto the discrete portion of the spectrum.

2.3 The Harmonic Oscillator Coherent State Classical Limit

2.3.1 The Limit

One of the more important aspects of the harmonic oscillator coherent states is seen through the so-called classical limit. An early use of this limit was by Bhaumik and Dutta-Roy [17] who used the limit in a comparison of quantum and classical time-dependent perturbation series in an anharmonically perturbed harmonic oscillator. Their calculations were only to first order, but they postulated that in principle higher orders are straightforward to obtain. Benoit, McRae and Vrscay [14], and

McRae and Vrscay [50] expanded upon this work, carrying out calculations to higher orders and in so doing, pointed out that the details are not as clear as earlier supposed.

In any case, the classical limit of a coherent state $|\alpha\rangle$ is taken by fixing the quantity $J = |\alpha|^2\hbar$, and taking $\hbar \to 0$. Of course, here we are not entertaining the possibility of manipulating a physical constant ($\hbar$) over which we have no control. we are merely considering how a system of a certain characteristic action behaves in relation to the size of $\hbar$. In other words, this is a limit of large characteristic action. ((characteristic length)^2/(characteristic time)), or a limit of increasing scale.

The effect of this limit on a coherent state is the following. Note firstly that $\Delta x\, \Delta p = O(\hbar)$ and that $\Delta x/\Delta p = 1/m\omega$, so that $\Delta x = O(\sqrt{\hbar})$ and $\Delta p = O(\sqrt{\hbar})$. Therefore, both uncertainties disappear in the limit. Not surprisingly, with the limit symbol standing for this classical limit,

$$\begin{aligned} \lim_{\text{classical}} |\langle x|\alpha\rangle|^2 &= \lim_{\text{classical}} \left(\frac{1}{2\pi(\Delta x)^2}\right)^{1/2} \exp\left(-\frac{1}{2}\left(\frac{x - \langle x(t)\rangle}{\Delta x}\right)^2\right) \\ &= \delta(x - \langle x(t)\rangle), \end{aligned} \tag{2.92}$$

where $\langle x(t)\rangle$ follows the classical motion of a particle with the same initial position and momentum. Similarly,

$$\begin{aligned} \lim_{\text{classical}} |\langle p|\alpha\rangle|^2 &= \lim_{\text{classical}} \left(\frac{1}{\pi m\omega\hbar}\right)^{1/2} \exp\left(-\left(\frac{(p - \langle p(t)\rangle)^2}{m\omega\hbar}\right)\right) \\ &= \delta(p - \langle p(t)\rangle), \end{aligned} \tag{2.93}$$

where $\langle p\rangle$ also follows the appropriate classical trajectory. These distributions in

position and momentum are the distributions which describe the classical motion of a single particle. That is, the classical limit of a coherent state is a single classical trajectory.

Note that distinct coherent states are non-orthogonal. The inner product is given by

$$\langle\alpha|\alpha'\rangle = \exp\left(-\frac{1}{2}(|\alpha|^2 + |\alpha'|^2 - 2\alpha^*\alpha')\right). \tag{2.94}$$

Taking the simultaneous classical limit on these states, that is, with $\alpha^2\hbar = J_1$ and $\alpha'^2\hbar = J_2$ (note that for this purpose, $J_{1,2}$ are complex to preserve phase information), one finds with $\alpha \neq \alpha'$

$$\begin{aligned}\lim_{\text{classical}} |\langle\alpha|\alpha'\rangle|^2 &= \lim_{\text{classical}} \exp\left(-\frac{1}{2}\left(|\alpha|^2 + \left|\frac{J_2}{J_1}\right||\alpha|^2 - 2\sqrt{\frac{J_2}{J_1}}|\alpha|^2\right)\right) \\ &= \lim_{\text{classical}} \exp\left(-\frac{1}{2}|\alpha|^2\left(1 + \left|\frac{J_2}{J_1}\right| - 2\sqrt{\frac{J_2}{J_1}}\right)\right) \\ &= 0, \end{aligned} \tag{2.95}$$

since the limit involves taking $|\alpha| \to \infty$ and the real part of the remainder of the exponent is positive. This result is also not surprising since, after all, this is the overlap of two non-coincident δ-functions.

2.3.2 The Limit in Phase Space

To be more precise, the δ-functions just mentioned must be δ-functions of some description on phase space, not in just position or momentum space since the classical limit of the overlap of two coherent states at the same position but with different momentum should still be zero. The initial urge is to take the product of

the two distributions, $\delta(q - \langle q(t)\rangle)\delta(p - \langle p(t)\rangle)$, and use this, but this lacks deeper underlying motivation. (In this section, the variable q will be used rather than x, due to the comparisons implied with classical mechanics.) Of course, this raises the topic of phase space distributions in quantum mechanics.

The topic of quantum mechanical distributions in phase space is large enough in its own right. Without going too deeply into things, there are three obvious choices at one's disposal. Given a wave function $|\psi\rangle$, one possibility is using $\rho(q,p) = |\langle q|\psi\rangle|^2|\langle p|\psi\rangle|^2$ termed the "joint probability distribution" for the present purposes. The advantage of this is that the marginal distributions equal the distributions in one of the variables. That is, integrating out the momentum distribution leaves the position distribution:

$$\int_{-\infty}^{\infty} \rho(q,p)dp = |\langle q|\psi\rangle|^2 \int_{-\infty}^{\infty} |\langle p|\psi\rangle|^2 dp = |\langle q|\psi\rangle|^2. \tag{2.96}$$

and similarly with q and p exchanged. The disadvantage is that this distribution does not allow one to build up areas in phase space independently: Classical distributions are seldom separable. (For example, if one tries to construct a state with only two localized populated areas in phase space in the first and third quadrants, one ends up with populations also in the second and fourth.) Despite this problem, this is a popular choice in the literature due to its simplicity.

Another popular choice is the Wigner distribution. This is the distribution which takes the place of the wave function in the Weyl-Wigner-Moyal formulation of quantum mechanics. Given an arbitrary state $|\psi\rangle$ Wigner [82] introduced the

function

$$\rho_W(q,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \langle q-\eta/2|\psi\rangle\langle\psi|q+\eta/2\rangle e^{ip\eta/\hbar}d\eta \tag{2.97}$$

as a means to calculate quantum corrections to virial coefficients. Weyl [80] introduced a correspondence scheme between functions on classical phase space and quantum mechanical operators, and Moyal [52] used this correspondence as applied to the density operator (of a pure state) $\hat{\rho} = |\psi\rangle\langle\psi|$, thereby obtaining the Wigner function, as a basis for a new formulation of quantum mechanics. The difficulty with Wigner's function as a distribution function is that, although it is always real, it may take on negative values, making it difficult to interpret as a probability distribution. However, the areas of these negative regions are always on the order of $\hbar$. Some authors regard this as non-negative "almost everywhere" and hence ignorable, whereas others regard negative values of the Wigner function as a "quantum signature." In any case, the Wigner function does recover the position and momentum distributions as marginal distributions.

The negative regions of the Wigner function are always coincident with regions of interference. Not surprisingly, then, if one smoothes the Wigner function by convolution with a Gaussian on phase space of minimum uncertainty, then positive values are regained. In fact, by "coarse graining" in this manner, one obtains the Husimi distribution [43], the third possibility. This connection to the Wigner function is surprising since the Husimi distribution is alternatively given by the overlap with coherent states: Given the completeness of harmonic oscillator coherent states,

$$|\psi\rangle = \frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha|\psi\rangle = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dq\int_{-\infty}^{\infty}dp\,|\alpha\rangle\langle\alpha|\psi\rangle, \tag{2.98}$$

the Husimi distribution of an arbitrary state $|\psi\rangle$ is defined by

$$\rho_H(q,p) = \frac{1}{2\pi\hbar}|\langle\alpha|\psi\rangle|^2. \tag{2.99}$$

where α depends on q and p through Eq. (2.20). The advantage of this distribution function is that it leads quickly to simple pictures in phase space. The principal disadvantage of this distribution is that the marginal distributions become only approximately true.

Now consider the phase space distributions of harmonic oscillator coherent states. Firstly, given that

$$|\langle q|\alpha\rangle|^2 = \left(\frac{1}{2\pi(\Delta q)^2}\right)^{1/2} \exp\left(-\frac{1}{2(\Delta q)^2}(q-\langle q\rangle)^2\right) \tag{2.100}$$

and that

$$|\langle p|\alpha\rangle|^2 = \left(\frac{1}{2\pi(\Delta p)^2}\right)^{1/2} \exp\left(-\frac{1}{2(\Delta p)^2}(p-\langle p\rangle)^2\right) \tag{2.101}$$

the first joint probability distribution given above is given by

$$\rho(q,p) = \frac{1}{\pi\hbar}\exp\left(-\frac{1}{2(\Delta q)^2}(q-\langle q\rangle)^2 - \frac{1}{2(\Delta p)^2}(p-\langle p\rangle)^2\right). \tag{2.102}$$

which is a Gaussian on phase space of minimum uncertainty. It is not difficult to see that in the classical limit, this becomes

$$\lim_{\text{classical}} \rho(q,p) = \delta(q-\langle q\rangle)\delta(p-\langle p\rangle). \tag{2.103}$$

The Wigner function of a coherent state is given by

$$\begin{aligned}\rho_W(q,p) &= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\langle q-\eta/2|\alpha\rangle\langle\alpha|q+\eta/2\rangle e^{ip\eta/\hbar}d\eta \\ &= \frac{1}{\pi\hbar}\exp\left(-\frac{1}{2(\Delta q)^2}(q-\langle q\rangle)^2-\frac{1}{2(\Delta p)^2}(p-\langle p\rangle)^2\right), \quad (2.104)\end{aligned}$$

identical to the one given earlier. Note that this Wigner function takes on no negative values. Indeed, this is the only strictly positive Wigner function. Since many authors consider this to be a classical state, the connection between "classical" and "positive" is made leaving negative values of the Wigner function characterizing "quantum." In the classical limit, the Wigner function is also the product of δ-functions.

The Husimi distribution of a coherent state $|\alpha\rangle$ is given by

$$\rho_H(q,p)=\frac{1}{2\pi\hbar}\exp\left(-\left(|\alpha|^2+|\alpha'|^2-2\mathrm{Re}(\alpha\alpha'^*)\right)\right). \quad (2.105)$$

With

$$\alpha'=\frac{1}{2}\left(\frac{q}{\Delta q}+i\frac{p}{\Delta p}\right), \qquad \alpha=\frac{1}{2}\left(\frac{\langle q\rangle}{\Delta q}+i\frac{\langle p\rangle}{\Delta p}\right) \quad (2.106)$$

the distribution becomes

$$\rho_H(q,p)=\frac{1}{2\pi\hbar}\exp\left(-\frac{1}{4(\Delta q)^2}(q-\langle q\rangle)^2-\frac{1}{4(\Delta p)^2}(p-\langle p\rangle)^2\right). \quad (2.107)$$

This is not a minimum uncertainty Gaussian on phase space: The width is scaled upwards by a factor of $\sqrt{2}$. After all, this represents the convolution of a Gaussian with itself, which has this effect. Even so, the classical limit of this distribution is

still the product of δ-functions on phase space. Note that this classical limit should be expected since it has already been pointed out that the overlap of distinct states disappears in the classical limit.

We now turn to phase space distributions of superpositions of coherent states. The simplest collection (besides the singleton) consists of two distinct states. By examining in detail the nature of distributions of pairs of coherent states, a great deal may be extrapolated to general ensembles of coherent states.

If $|\psi\rangle = N(|\alpha_1\rangle + |\alpha_2\rangle)$, with N securing normalization, (a complex factor applied to the second coherent state does not change matters appreciably) then

$$\begin{aligned}
|\langle q|\psi\rangle|^2 = & \\
& N^2\left(\frac{1}{2\pi(\Delta q)^2}\right)^{1/2}\left[\exp\left(-\frac{1}{2(\Delta q)^2}(q-\langle q_1\rangle)^2\right)+\exp\left(-\frac{1}{2(\Delta q)^2}(q-\langle q_2\rangle)^2\right)\right. \\
+ \quad & \left.2\mathrm{Re}\left(\exp\left(-\frac{1}{4(\Delta q)^2}\left((q-\langle q_1\rangle)^2+(q-\langle q_2\rangle)^2\right)+iq(\langle p_1\rangle-\langle p_2\rangle)/\hbar\right)\right)\right] \\
= \quad & N^2\left(\frac{1}{2\pi(\Delta q)^2}\right)^{1/2}\left[\exp\left(-\frac{1}{2(\Delta q)^2}(q-\langle q_1\rangle)^2\right)+\exp\left(-\frac{1}{2(\Delta q)^2}(q-\langle q_2\rangle)^2\right)\right. \\
+ \quad & \left.2\exp\left(-\frac{1}{2(\Delta q)^2}\left((q-q_+)^2+q_-^2\right)\right)\cos\left(q(\langle p_1\rangle-\langle p_2\rangle)/\hbar\right)\right], \qquad (2.108)
\end{aligned}$$

where $q_\pm = (\langle q_1\rangle \pm \langle q_2\rangle)/2$. Unless $\langle q_1\rangle = \langle q_2\rangle$, the interference term disappears in the classical limit, and what remains in the limit are two δ-functions positioned on the locations of the contributing coherent states. If $\langle q_1\rangle = \langle q_2\rangle$, then by hypothesis, $\langle p_1\rangle \neq \langle p_2\rangle$. In this case, the exponential part of the interference term equals one at $q = q_+$ but the cosine part rapidly oscillates. In the usual sense, the limit does not exist, but asymptotically the oscillations render this term null.

The entire preceding paragraph is equally valid for momentum, if one makes the appropriate substitutions of momenta for positions. Thus, for the simple approach to a joint probability distribution, one finds that the classical limit of two distinct coherent states becomes

$$\begin{aligned}\lim_{\text{classical}} \rho(q,p) &= \frac{1}{4}(\delta(q-\langle q_1\rangle)\delta(p-\langle p_1\rangle)+\delta(q-\langle q_2\rangle)\delta(p-\langle p_1\rangle) \\ &\quad +\delta(q-\langle q_1\rangle)\delta(p-\langle p_2\rangle)+\delta(q-\langle q_2\rangle)\delta(p-\langle p_2\rangle)). \quad (2.109)\end{aligned}$$

This distribution will have four peaks in phase space rather than the intended two, unless $\langle q_1\rangle = \langle q_2\rangle$ or $\langle p_1\rangle = \langle p_2\rangle$. Clearly this is not a desirable feature.

The Wigner function for two coherent states is a little more complicated. Again with $|\psi\rangle = N(|\alpha_1\rangle + |\alpha_2\rangle)$, the Wigner function is given by

$$\begin{aligned}\rho_W(q,p) &= N^2(\rho_{\alpha_1}(q,p)+\rho_{\alpha_2}(q,p)) \\ &\quad +\frac{N^2}{2\pi\hbar}\int_{-\infty}^{\infty}\langle q-\eta/2|\alpha_1\rangle\langle\alpha_2|q+\eta/2\rangle e^{ip\eta/\hbar}d\eta + \text{c.c.}, \quad (2.110)\end{aligned}$$

in which $\rho_{\alpha_1}(q,p)$ is here intended to stand for the Wigner function of the state $|\alpha_1\rangle$ and *mutatis mutandis* elsewhere. The interference term written out above is

$$\begin{aligned}&\frac{N^2}{\pi\hbar}\exp\left[-\frac{1}{2(\Delta q)^2}(q-q_+)^2-\frac{1}{2(\Delta p)^2}(p-p_+)^2\right. \quad (2.111)\\ &+\left.\frac{i}{2\hbar}(2(\langle p_1\rangle-\langle p_2\rangle)q-2(\langle q_1\rangle-\langle q_2\rangle)p+(\langle q_1\rangle-\langle q_2\rangle)(\langle p_1\rangle+\langle p_2\rangle))\right].\end{aligned}$$

Note that at the midpoint between the coherent states in phase space (*i.e.* at (q_+,p_+)), the interference term takes on its maximum amplitude, as seen through

the real part of the exponential. The imaginary part gives rise to oscillations only observed in the complete Wigner function at this midpoint. In the classical limit of this state, the interference term does not disappear at the the midpoint, but the (phase-spatial) frequency of the oscillations diverges: Strictly speaking, the classical limit of this Wigner function does not exist. With a small amount of averaging, this interference pattern is smoothed out, leaving δ-functions at the locations of the coherent states.

The Husimi distribution of the sum of coherent states is much simpler. Again with $|\psi\rangle = N(|\alpha_1\rangle + |\alpha_2\rangle)$, one finds

$$\begin{aligned} \rho_H(q,p) &= \frac{N^2}{2\pi\hbar}|\langle\alpha|(|\alpha_1\rangle + |\alpha_2\rangle)|^2 \\ &= \frac{N^2}{2\pi\hbar}\left(|\langle\alpha|\alpha_1\rangle|^2 + |\langle\alpha|\alpha_2\rangle|^2 + 2\mathrm{Re}\left(\langle\alpha|\alpha_1\rangle\langle\alpha_2|\alpha\rangle\right)\right). \end{aligned} \quad (2.112)$$

The interference term disappears in the classical limit and δ-functions remain at the locations of the component coherent states in phase space.

Summarizing these calculations, the classical limit of a coherent state is a δ-function on phase space irrespective of how the identification to a phase space distribution function is made from the wave function. If a state is constructed consisting of two distinct coherent states, the view in phase space depends on the phase space probability distribution used. If the distribution used is the product of the position and momentum distributions, then additional peaks arise. If the Wigner function is used, then only two peaks are found, each of which is one of minimum uncertainty (roughly speaking), but an interference term arises which takes on negative values, and in the classical limit does not converge. If the Husimi

distribution is used, the peaks are no longer of minimum uncertainty, but the interference term vanishes in the classical limit.

If more than two distinct coherent states contribute to the wave function, then the behaviour may be described in terms of the two-coherent state situation. The product distribution gives rise to a great deal of additional structure, the Wigner function gives interference terms between coherent states which take on negative values and (possibly) has no classical limit, and the Husimi distribution widens peaks and has a classical limit. Among these three possibilities, and with the criteria of an acceptable classical limit, the Husimi distribution appears to be of the greatest use, even though it only provides approximations to marginal distributions.

Thus we approach the concept of the classical limit of an arbitrary wave function. Recall that in the Statistical Interpretation of quantum mechanics the wave function describes an ensemble of similarly prepared systems. Given that arbitrary wave functions may be represented in terms of basis functions, or elements of some complete set of states, it is wrong to identify elements of the ensemble with basis vectors. In particular, using a representation in terms of coherent states, elements of the ensemble may not be identified with individual coherent states.

According to the Statistical Interpretation, the classical limit of a wave function is an ensemble of classical trajectories [7]. This concept has met with a great deal of success. The coherent states are somewhat anomalous, then, in that they describe individual classical trajectories in the classical limit. Thus when the classical limit is applied to an arbitrary wave function, the result is a state represented in terms of non-overlapping coherent states, each of which describes a single classical trajectory.

That is, the arbitrary state becomes an ensemble of classical trajectories.

Chapter 3

Some Comparisons

3.1 Annihilation Operator Coherent States and Minimum Uncertainty

Many coherent state generalizations hearken back to the annihilation operator definition of coherent states. Why is this so? Glauber's original motivation [39] was involved in the representation of electric field operators providing considerable calculational convenience, but this does not necessarily carry over into other systems. The construction can be convenient in terms of calculating properties, but, as with any other state construction, the method must be reconciled to the purpose in mind. That is, calculational convenience should not override physical motivation. Bearing this in mind, a few general comments may be made.

Suppose $\hat{O}$ is a non-Hermitian operator. Suppose also that the Hermitian and

anti-Hermitian parts of $\hat{O}$, found through

$$\hat{A} = \frac{1}{2}(\hat{O} + \hat{O}^\dagger), \quad \hat{B} = -\frac{i}{2}(\hat{O} - \hat{O}^\dagger), \tag{3.1}$$

do not commute,

$$[\hat{A}, \hat{B}] = -\frac{i}{2}[\hat{O}^\dagger, \hat{O}] = i\hat{C}, \tag{3.2}$$

which stands as a definition for the Hermitian operator $\hat{C}$. The operators $\hat{O}$ and $\hat{O}^\dagger$ may be recovered from $\hat{A}$ and $\hat{B}$ through

$$\hat{O} = \hat{A} + i\hat{B}, \quad \hat{O}^\dagger = \hat{A} - i\hat{B}. \tag{3.3}$$

According to the usual custom, since $\hat{A}$ and $\hat{B}$ do not commute, all states must satisfy the uncertainty relation

$$(\Delta A)^2(\Delta B)^2 \geq \frac{1}{4}|\langle C\rangle|^2. \tag{3.4}$$

Now suppose that $\hat{O}$ has eigenstates $|\omega\rangle$ with (in general) complex eigenvalues ω,

$$\hat{O}|\omega\rangle = \omega|\omega\rangle. \tag{3.5}$$

With expectation values calculated in such states,

$$\langle A\rangle = \mathrm{Re}(\omega), \quad \langle B\rangle = \mathrm{Im}(\omega), \tag{3.6}$$

and

$$\langle A^2\rangle = \frac{1}{4}\langle \hat{O}^2 + 2\hat{O}^\dagger\hat{O} + (\hat{O}^\dagger)^2 + 2\hat{C}\rangle = \mathrm{Re}(\omega)^2 + \frac{1}{2}\langle C\rangle, \tag{3.7}$$

$$\langle B^2\rangle = -\frac{1}{4}\langle \hat{O}^2 - 2\hat{O}^\dagger\hat{O} + (\hat{O}^\dagger)^2 - 2\hat{C}\rangle = \mathrm{Im}(\omega)^2 + \frac{1}{2}\langle C\rangle. \tag{3.8}$$

The uncertainties of ΔA and ΔB are given by

$$(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2 = \frac{1}{2}\langle C\rangle, \tag{3.9}$$

$$(\Delta B)^2 = \langle B^2\rangle - \langle B\rangle^2 = \frac{1}{2}\langle C\rangle, \tag{3.10}$$

rendering the uncertainty product minimized,

$$(\Delta A)^2(\Delta B)^2 = \frac{1}{4}|\langle C\rangle|^2. \tag{3.11}$$

With $\hat{O}$ possibly standing for an annihilation operator, the relationship between annihilation operator coherent states and minimum uncertainty coherent states becomes apparent.

Working in the opposite direction, start with two Hermitian operators $\hat{A}$ and $\hat{B}$, and construct the operator $\hat{O}$ by considering

$$\hat{O} = \hat{A} + i\eta\hat{B}, \tag{3.12}$$

with η some real parameter. The eigenvalues of $\hat{O}$ for an eigenstate $|\omega\rangle$ must be given by

$$\hat{O}|\omega\rangle = (\langle A\rangle + i\eta\langle B\rangle)\,|\omega\rangle \tag{3.13}$$

where $\langle A \rangle$ and $\langle B \rangle$ enter as two more parameters to describe the state $|\omega\rangle$. Again with $\hat{C} = -i[\hat{A}, \hat{B}]$, one finds

$$(\Delta A)^2 = \frac{\eta}{2}\langle C \rangle \quad \text{and} \quad (\Delta B)^2 = \frac{1}{2\eta}\langle C \rangle \tag{3.14}$$

where expectation values are calculated in the eigenstate $|\omega\rangle$ of $\hat{O}$, so that the uncertainty relation between $\hat{A}$ and $\hat{B}$ is again saturated,

$$(\Delta A)^2(\Delta B)^2 = \frac{1}{4}\langle C \rangle^2. \tag{3.15}$$

Thus, to find states which minimize the uncertainty product of a certain pair of operators, it is sufficient to form an eigenvalue-eigenstate problem with Eq. (3.13). The eigenstates in Eq. (3.13) minimize the uncertainty product.

To compare the minimum uncertainty and annihilation operator constructions of the harmonic oscillator coherent states, in Eq. (3.13), take $\hat{A} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x}$ and $\hat{B} = \hat{p}/\sqrt{2\hbar m\omega}$ where ω is here the parameter to the harmonic oscillator potential. Then, any value of η in Eq. (3.13) yields a minimum uncertainty state in $\hat{x}$ and $\hat{p}$. This defines what some authors call squeezed states [62]. The coherent states are regained when $\eta = 1$. This casts the operator $\hat{O}$ precisely into the form of the annihilation operator.

3.1.1 SO(2,1) as an Example

The above derivation is valid for any operator $\hat{O}$ provided it meets the given criteria, but gains import in the context of annihilation operator constructions of coherent

states. The two most obvious sets of operators for which this applies are those of the harmonic oscillator and those of the Lie algebra so(2,1). In fact, as noted by Barut and Girardello [9], annihilation operator coherent states may be constructed in association with any non-compact Lie algebra. The standard ladder operators for angular momentum are very useful but do not have eigenstates due to the finite spectrum, so the above derivation does not apply.

From Appendix A, the algebra so(2,1) is spanned by the operators $\hat{T}_{1,2,3}$, and, as in the case of so(3), the ladder operators are given by $\hat{T}_{\pm} = \hat{T}_1 \pm i\hat{T}_2$. With b_0 standing for the lowest eigenvalue of $\hat{T}_3$ the action of $\hat{T}_-$ on an eigenstate $|b\rangle$ of $\hat{T}_3$ is given by

$$\hat{T}_-|b_0, b\rangle = \sqrt{(b - b_0)(b + b_0 - 1)}|b_0, b - 1\rangle, \tag{3.16}$$

for $b \neq b_0$ and $\hat{T}_-|b_0, b_0\rangle = 0$.

Suppose the coherent state $|b_0, t\rangle$ is an eigenstate of $\hat{T}_-$,

$$\hat{T}_-|b_0, t\rangle = t|b_0, t\rangle, \tag{3.17}$$

and that $|b_0, t\rangle$ may be represented by a summation over the eigenstates of $\hat{T}_3$,

$$|b_0, t\rangle = \sum_{b=b_0}^{\infty} c_b|b_0, b\rangle. \tag{3.18}$$

Eqs. (3.16) and (3.18) jointly lead to the recursion relation

$$c_{b+1}\sqrt{(b - b_0 + 1)(b + b_0)} = tc_b. \tag{3.19}$$

Leaving c_{b_0} as a normalization constant, this recursion relation is solved to yield

$$c_{b_0+k} = c_{b_0} t^k \left(\frac{\Gamma(2b_0)}{k!\Gamma(2b_0+k)} \right)^{1/2}. \tag{3.20}$$

Demanding normalization yields

$$|b_0, t\rangle = \left(\frac{|t|^{2b_0-1}}{I_{2b_0-1}(2|t|)} \right)^{1/2} \sum_{k=0}^{\infty} \frac{t^k}{\sqrt{k!\Gamma(2b_0+k)}} |b_0, b_0+k\rangle, \tag{3.21}$$

where $I_\nu(z)$ is the modified Bessel function [40].

By virtue of the construction at the beginning of this section, the annihilation operator coherent states Eq. (3.21) satisfy the minimum uncertainty criterion

$$(\Delta T_1)^2 (\Delta T_2)^2 = \frac{1}{4} |\langle T_3 \rangle|^2. \tag{3.22}$$

Of course, whether Eq. (3.22) may be given physical significance is another story, depending on the realization of so(2,1) under consideration.

In any case, regarding the distribution of the eigenvalue of $\hat{T}_3$, one finds

$$\langle T_3 \rangle = \frac{1}{2} + \frac{|t|}{2} \frac{I_{2b_0-2} + I_{2b_0}}{I_{2b_0-1}}, \tag{3.23}$$

and

$$\langle T_3^2 \rangle = \frac{1}{4} + \frac{3}{4}|t| \frac{I_{2b_0-2} + I_{2b_0}}{I_{2b_0-1}} + \frac{1}{4}|t|^2 \left(2 + \frac{I_{2b_0-3} + I_{2b_0+1}}{I_{2b_0-1}} \right), \tag{3.24}$$

where the arguments of all the modified Bessel functions are $2|t|$. These follow from

the relations [40]

$$\sum_{k=0}^{\infty} \frac{|t|^{2(b_0+k)}}{k!\Gamma(2b_0+k)} = |t| I_{2b_0-1}(2|t|), \tag{3.25}$$

and

$$\frac{d}{dx} I_\nu(2x) = I_{\nu-1}(2x) + I_{\nu+1}(2x). \tag{3.26}$$

The asymptotic behaviour of the modified Bessel function is given by [40], for large values of the argument,

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \left[1 - \frac{(\nu+\frac{1}{2})(\nu-\frac{1}{2})}{2z} + \cdots \right], \tag{3.27}$$

so that for large $|t|$, one finds

$$\langle T_3 \rangle \sim \frac{1}{2} + |t| \quad \text{and} \quad (\Delta T_3)^2 \sim \frac{|t|}{2}. \tag{3.28}$$

Since this construction is not tied to any realization of so(2,1) it is interesting to compare this with the Perelomov construction of coherent states for the same algebra. Simply stating Perelomov's result [68] (since this is not the point of this section)

$$|b_0, \zeta\rangle = (1-|\zeta|^2)^{b_0} \sum_{k=0}^{\infty} \sqrt{\frac{\Gamma(2b_0+k)}{k!\Gamma(2b_0)}} \zeta^k |b_0, b_0+k\rangle. \tag{3.29}$$

Clearly these are not equivalent to the annihilation operator states Eq. (3.21). Also, the region of convergence for the sum is $|\zeta| < 1$. Performing the same calculations as for the previous states,

$$\langle T_3 \rangle = b_0 + \frac{2b_0|\zeta|^2}{1-|\zeta|^2}, \tag{3.30}$$

and

$$\langle T_3^2 \rangle = b_0^2 + \frac{4b_0^2|\zeta|^2}{1-|\zeta|^2} + \frac{2b_0|\zeta|^2}{(1-|\zeta|^2)^2} + \frac{4b_0^2|\zeta|^4}{(1-|\zeta|^2)^2}. \tag{3.31}$$

This renders the uncertainty in $\hat{T}_3$

$$(\Delta T_3)^2 = \frac{2b_0|\zeta|^2}{(1-|\zeta|^2)^2}. \tag{3.32}$$

A significant difference between Eq. (3.28) and Eq. (3.32) is their behaviour as $\langle T_3 \rangle$ becomes very large. In Eq. (3.28), this follows from taking $|t|$ large so that $(\Delta T_3)^2 \sim \frac{1}{2}\langle T_3 \rangle$. For the states constructed à la Perelomov, large $\langle T_3 \rangle$ is achieved through taking $|\zeta| \to 1$. In this limit, a comparison between Eq. (3.30) and Eq. (3.32) indicates that $\Delta T_3 \sim \langle T_3 \rangle$, indicating much broader distributions. With so(2,1) acting as the spectrum generating algebra for the hydrogen atom problem, some authors [49] cite the broad distribution as a reason for rejecting the Perelomov states in the hydrogen atom problem in favour of an annihilation operator construction as by Barut and Girardello [9], these comments being in the context of a search for "classical" states of the hydrogen atom. However, such assertions are meaningless in the absence of a framework for the comparison of quantum and classical behaviour. If that framework involves comparing quantum wave functions with classical *distributions*, the width in n is not relevant.

3.2 Minimum Uncertainty and Klauder's Construction

Another connection of interest [23] is found in a particular system between Nieto's construction of minimum uncertainty coherent states and Klauder's construction of temporally stable coherent states. The potential in question is sometimes called the harmonic oscillator with centripetal barrier [56]. Though the configuration space of this potential has only one degree of freedom it carries this name due to the relationship between this potential and the radial part of the three dimensional harmonic oscillator with non-zero angular momentum.

Nieto's generalization of the harmonic oscillator coherent states (see Section 2.2.3) focuses principally on the fact that harmonic oscillator coherent states are minimum uncertainty states, satisfying equality in Heisenberg's uncertainly relation for the standard $\hat{x}$ and $\hat{p}$ operators. In this generalization, the coherent states satisfy minimum uncertainty for a new pair of operators $\hat{X}$ and $\hat{P}$ through which the potential "appears" like the harmonic oscillator. The thrust in defining states according to this scheme is to provide states which remain coalesced for the longest possible time, and hence follow corresponding classical trajectories for the longest possible time via Ehrenfest's relations.

Klauder's generalization [45] focuses on the form of the eigenstate expansion which guarantees that the harmonic oscillator coherent states remain temporally stable: that the time evolution may be described by evolving the parameters to the coherent state itself. This generalization prescribes the form of the eigenstate

expansion in any (non-degenerate) potential to permit the definition of temporally stable states in other systems.

Klauder's generalization also provides a resolution of the identity for the set of generalized coherent states. It is important to point out that Nieto's generalization does not provide a resolution of the identity in general, although there are several plausibility arguments to suggest that one should exist: If a resolution of the identity exists, it must be found case by case.

The potential of concern is given by

$$V(x) = U_0 \left(\frac{1}{z} - z\right)^2, \tag{3.33}$$

in which the length scale is via $z = ax$ for some positive a, and the energy scale is expressed through

$$U_0 = \frac{\hbar^2 a^2}{2m}\lambda(\lambda+1), \tag{3.34}$$

where λ is introduced here as a convenience for later calculations. In position space, the Hamiltonian eigenstates are given by

$$\langle x|n\rangle = \left(\frac{2a\nu\Gamma(n+1)}{\Gamma(n+\lambda+\frac{3}{2})}\right)^{1/2} e^{-y/2} y^{\lambda+1} L_n^{(\lambda+1/2)}(y), \quad n = 0, 1, 2\ldots, \tag{3.35}$$

where $\nu = \sqrt{\lambda(\lambda+1)}$, $y = \nu z^2$ and L_n are the Laguerre polynomials. These eigenstates each have energies

$$E_n = \frac{\hbar^2 a^2}{2m}\left(\nu(4n+2\lambda+3) - 2\nu^2\right). \tag{3.36}$$

To see the relationship between this system and the radial part of the three dimensional harmonic oscillator, observe that the time-independent Schödinger equation for the latter system is given by

$$\left(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{r}^2\right)|\psi\rangle = E|\psi\rangle, \tag{3.37}$$

or

$$\left(\frac{\hat{p}_r^2}{2m} + \frac{L^2}{2m\hat{r}^2} + \frac{1}{2}m\omega^2\hat{r}^2\right)|\psi\rangle = E|\psi\rangle. \tag{3.38}$$

Recognizing that the eigenstate $|\psi\rangle$ is also an eigenstate of $\hat{L}^2$ with eigenvalue $\hbar^2\ell(\ell+1)$, the equation becomes

$$\left(\frac{\hat{p}_r^2}{2m} + \frac{\hbar^2\ell(\ell+1)}{2m\hat{r}^2} + \frac{1}{2}m\omega^2\hat{r}^2\right)|\psi\rangle = E|\psi\rangle. \tag{3.39}$$

From this expression one finds the effective radial potential

$$V_r(r) = \frac{\hbar^2\ell(\ell+1)}{2mr^2} + \frac{1}{2}m\omega^2r^2, \tag{3.40}$$

which pertains to a particular total angular momentum, $\hbar^2\ell(\ell+1)$. Thus, the connection is made through

$$V_r(x) = V(x) + 2U_0 \tag{3.41}$$

with $V(x)$ given by Eq. (3.33), and the identifications

$$\lambda \to \ell, \quad \text{and} \quad \hbar a^2\nu \to m\omega. \tag{3.42}$$

With these identifications these two systems are one and the same.

Now constructing the coherent states according to Nieto's recipe [56], classical particles in the potential Eq. (3.33) oscillate through

$$a^2x(t)^2-\left(\frac{E}{2U_0}+1\right)=\left(\left(\frac{E}{2U_0}+1\right)^2-1\right)^{1/2}\sin(\omega(t+\phi)),\quad \omega=2a\sqrt{\frac{2U_0}{m}}, \tag{3.43}$$

in which the two arbitrary constants are embodied in the energy and the phase angle ϕ. Accordingly, the map through which classical trajectories are sinusoidal is the left hand side of Eq. (3.43). The new position and momentum operators are given by

$$\hat{X}=a^2\hat{x}^2-\left(\frac{\hat{H}}{2U_0}+1\right),\quad \hat{P}=a^2(\hat{x}\hat{p}+\hat{p}\hat{x}). \tag{3.44}$$

The generalized minimum uncertainty states are thus solutions of the problem

$$\left(\frac{\hat{X}}{\Delta X}+i\frac{\hat{P}}{\Delta P}\right)|\beta\rangle=C|\beta\rangle \tag{3.45}$$

for some complex valued C. Even though the operator $\hat{X}$ involves the Hamiltonian, the exact solution may be obtained and is given by [56]

$$|\beta\rangle=\left(\frac{2a\nu^{1/2}e^{-\nu u}}{I_{\lambda+1/2}(\nu|C|)}\right)^{1/2}e^{-y/2}y^{1/4}I_{\lambda+1/2}(2(\nu Cy/2)^{1/2}), \tag{3.46}$$

with ν and y given as before. This may be expressed as a superposition over eigenstates given by

$$|\beta\rangle=\left(\frac{e^{i\mathrm{Re}(\beta)}\beta^{\lambda+1/2}}{I_{\lambda+1/2}(2|\beta|)}\right)^{1/2}\sum_{n=0}^{\infty}\frac{(-\beta)^n}{\sqrt{\Gamma(n+1)\Gamma(n+\lambda+\frac{3}{2})}}|n\rangle, \tag{3.47}$$

in which I_ν is the modified Bessel function and β is reworked from C,

$$\beta = \frac{\nu C}{2} = \frac{\nu}{2}\langle X\rangle + \frac{i}{4}\langle P\rangle. \tag{3.48}$$

A resolution of the identity has been found to be [56]

$$\hat{1} = \int d^2\beta f(|\beta|)|\beta\rangle\langle\beta|, \tag{3.49}$$

in which $d^2\beta = d\mathrm{Re}(\beta)d\mathrm{Im}(\beta)$ and the integration is over the entire complex plane. The function $f(\rho)$ is given by

$$f(\rho) = \frac{K_{\lambda+1/2}(2\rho)I_{\lambda+1/2}(2\rho)}{8\pi}. \tag{3.50}$$

where K_μ is the other modified Bessel function.

As an initial argument in expressing these coherent states in terms of Klauder's construction, note that in their respective eigenstate expansions Eqs. (3.47) and (2.86), the change in phase of the coefficients between successive terms is constant, in the first case due to successive powers of a complex number, and in the latter since the eigenenergies, Eq. (3.36), are equally spaced.

If Nieto's states given above may be described in terms of Klauder's construction, then the normalizing function $M(s^2)$ in Eq. (2.86) must be at least proportional to the leading part of Eq. (3.47). Also, Klauder's $k(s^2)$ of Eq. (2.89) must be at least proportional to the function $f(|\beta|)$ given by Nieto, and if so, $\rho(u)$ is provided by Eq. (2.89). A consistency check will then be provided by calculating the moments ρ_n which should correspond to the denominator within the sum of

Eq. (3.47), *i.e.*,

$$\rho_n \propto \Gamma(n+1)\Gamma(n+\lambda+3/2). \tag{3.51}$$

To begin, the integration over the complex β plane may be expressed as

$$\int d^2\beta = \int_0^\infty |\beta| d|\beta| \int_0^{2\pi} d\theta \tag{3.52}$$

where θ is being used as the phase angle of β. Accordingly, the expression Eq. (3.49) may be rewritten

$$\hat{1} = \int_0^\infty d|\beta| \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{|\beta| K_{\lambda+1/2}(2|\beta|) I_{\lambda+1/2}(2|\beta|)}{4} |\beta\rangle\langle\beta|. \tag{3.53}$$

Since the dependence of β on θ is 2π-periodic, integration as written with respect to θ corresponds to integration with respect to γ, so that the two may be identified.

Relating normalization functions,

$$M^2(s^2) = \frac{|\beta|^{\lambda+1/2}}{I_{\lambda+1/2}(2|\beta|)}, \tag{3.54}$$

from Eq. (3.47). Any further progress requires the assumption $s^2 = |\beta|^p$ for some power p so that

$$M^2(u) = \frac{u^{(\lambda+1/2)/p}}{I_{\lambda+1/2}(2u^{1/p})}. \tag{3.55}$$

Now identifying $k(u)$ with the rest of the measure in Eq. (3.53) with integration over θ disregarded yields

$$k(u) = \frac{u^{2/p-1} K_{\lambda+1/2}(2u^{1/p}) I_{\lambda+1/2}(2u^{1/p})}{4p}. \tag{3.56}$$

From Eq. (2.89), Eq. (3.56) leads to

$$\rho(u) = \frac{1}{4p} u^{\lambda/p+5/2p-1} K_{\lambda+1/2}(2u^{1/p}). \tag{3.57}$$

The moments of this function are given by

$$\rho_n = \frac{1}{4p} \int_0^\infty u^{n+\lambda/p+5/2p-1} K_{\lambda+1/2}(2u^{1/p}) du. \tag{3.58}$$

This is a known integral [40] which yields

$$\rho_n = \frac{1}{16} \Gamma(\frac{p}{2}n + \lambda + \frac{3}{2}) \Gamma(\frac{p}{2}n + 1). \tag{3.59}$$

The reader will note that this does correspond to the appropriate part of Nieto's construction, Eq. (3.51), with $p = 2$.

3.3 A Further Connection Between Sections 3.1.1 and 3.2

The minimum uncertainty coherent states in the harmonic oscillator with centripetal barrier, shown to be Klauder coherent states, are also SO(2,1) annihilation operator coherent states. The forms of the eigenstate expansions, Eqs. (3.21) and (3.47), are certainly suggestive of one another, and to see why this is the case, note the following.

As detailed in Section A.3, so(2,1) may be used as a spectrum generating algebra for the isotropic harmonic oscillator and hence the harmonic oscillator with centripetal barrier. To develop this connection, the time independent Schrödinger equation for the harmonic oscillator with centripetal barrier is given by

$$\left(\frac{\hat{p}^2}{2m} + U_0\left(\frac{1}{a\hat{x}} - a\hat{x}\right)^2\right)|\psi\rangle = E|\psi\rangle, \tag{3.60}$$

or

$$\frac{1}{2}\left(\frac{1}{2}\hat{p}^2 + \frac{1}{2}\hbar^2\nu^2\hat{x}^{-2} + \frac{1}{2}\hbar^2a^4\nu^2\hat{x}^2\right)|\psi\rangle = \frac{m}{2}\left(E + 2U_0\right)|\psi\rangle, \tag{3.61}$$

where the ν arises from

$$U_0 = \frac{\hbar^2a^2}{2m}\nu^2. \tag{3.62}$$

This must be connected to Eq. (A.42) in terms of two operators which commute by $[\hat{R}, \hat{P}] = 1$. This may be accomplished by setting

$$\hat{R} = \hbar^{-1/2}\lambda\hat{x}, \quad \hat{P} = \hbar^{-1/2}\lambda^{-1}\hat{p}, \tag{3.63}$$

which casts Eq. (3.61) into the form

$$\frac{1}{2}\left(\frac{1}{2}\hat{P}^2+\frac{1}{2}\nu^2\hat{R}^{-2}+\frac{1}{2}\hbar^2a^4\nu^2\lambda^{-4}\hat{R}^2\right)|\psi\rangle=\frac{m}{2\hbar\lambda^2}\left(E+2U_0\right)|\psi\rangle. \tag{3.64}$$

The coefficient of $\hat{R}^2$ is changed to unity by setting

$$\lambda^4=\frac{1}{2}\hbar^2a^4\nu^2, \tag{3.65}$$

putting the Schrödinger equation into the form

$$\frac{1}{2}\left(\frac{1}{2}\hat{P}^2+\frac{1}{2}\nu^2\hat{R}^{-2}+\hat{R}^2\right)|\psi\rangle=\frac{m}{\sqrt{2}\hbar^2a^2\nu}\left(E+2U_0\right)|\psi\rangle. \tag{3.66}$$

This may be identified with Eq. (A.42) provided

$$n=2,\quad \xi=\frac{1}{2}\nu^2,\quad \frac{1}{2}DB^{-1/2}=\frac{m}{\sqrt{2}\hbar^2a^2\nu}\left(E+2U_0\right). \tag{3.67}$$

From the work in Appendix A, this may be considered an eigenvalue equation for the $\hat{T}_3$ operator in some realization of so(2,1). With this identification, one finds

$$E=\frac{\sqrt{2}\hbar^2a^2\nu}{m}\left(b_0+k\right)-2U_0, \tag{3.68}$$

in which k is some integer and

$$b_0=\frac{1}{2}\pm\frac{1}{4}\sqrt{1+8\nu^2}. \tag{3.69}$$

The negative root is rejected since this results in a spectrum bounded above, leaving

the positive root as representative of the ground state, and $k = 0, 1, 2 \ldots$. With this in mind, the expression Eq. (3.68) for the eigenenergy agrees with Eq. (3.36).

From Appendix A, the ladder operator $\hat{T}_-$ of the Lie algebra so(2,1) alluded to above is given, in terms of the original operators $\hat{x}$ and $\hat{p}$, by

$$\begin{aligned} \hat{T}_- &= \hat{T}_1 - i\hat{T}_2 \\ &= \frac{\nu}{2\sqrt{2}}\left(\frac{1}{\hbar^2 a^2 \nu^2}\hat{p}^2 + \frac{1}{a^2\hat{x}^2} - a^2\hat{x}^2\right) - \frac{1}{4\hbar}(\hat{x}\hat{p} + \hat{p}\hat{x}). \end{aligned} \tag{3.70}$$

Compare this with the expression for $\hat{X}$ and $\hat{P}$ in Nieto's construction, Eq. (3.44), expressed in terms of $\hat{x}$ and $\hat{p}$.

$$\begin{aligned} \hat{X} &= -\frac{1}{2}\left(\frac{\hat{p}^2}{\hbar^2 a^2 \nu^2} + \frac{1}{a^2\hat{x}^2} - a^2\hat{x}^2\right) \\ \hat{P} &= a^2(\hat{x}\hat{p} + \hat{p}\hat{x}). \end{aligned} \tag{3.71}$$

Matching the real and imaginary parts, the annihilation operator coherent state definition for this system

$$\hat{T}_-|t\rangle = t|t\rangle \tag{3.72}$$

corresponds with the minimum uncertainty equation, Eq. (3.45): Coherent states of these constructions are identical.

Chapter 4

Coherent States for the Spherical Rotator

4.1 The Spherical Rotator


common problem in the literature is the pendulum, alias the rotator. Typically, the physical system in question is planar, a point mass constrained to a certain radius from a point on a plane. Like other simple systems, ideas are frequently set in the context of the rotator for pedagogical reasons [69], and the planar rotator is a popular subject of perturbative studies [25]. When subjected to periodic kicking, this classical system becomes chaotic, and figures prominently in the study of quantum classical correspondence in the context of quantum chaos [29, 7, 22].

An obvious generalization of this model is to permit the mass to move in any direction on the surface of a sphere of a certain radius. The spherical rotator is frequently invoked to describe the low energy rotational modes of a diatomic molecule

when the molecular bonding is such that the inter-atomic distance may be regarded as fixed. Not surprisingly, when subjected to a periodic impulse, the spherical rotator becomes chaotic and has also been the subject of studies of quantum classical correspondence [44].

The Hamiltonian for the spherical rotator with moment of inertia I is given by

$$\hat{H} = \frac{\hat{J}^2}{2I}, \tag{4.1}$$

i.e. proportional to the total angular momentum squared,

$$\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2. \tag{4.2}$$

Note that the angular momentum components all commute with the total angular momentum, $[\hat{J}_k, \hat{J}^2] = 0$, which provides the $\hat{J}_k$ as generators of the degeneracy group. This group is of course SO(3) or SU(2) (see Appendix A). Forming the irreducible representations of the group through simultaneous eigenstates of $\hat{J}^2$ and $\hat{J}_3$, one finds that

$$\hat{J}^2|j,m\rangle = \hbar^2 j(j+1)|j,m\rangle, \quad \hat{J}_3|j,m\rangle = \hbar m|j,m\rangle, \tag{4.3}$$

where $j = 0, \frac{1}{2}, 1, \ldots$ and $m = -j, -j+1, \ldots, j$, so that the eigenstate energies are given by

$$E_j = \frac{\hbar^2}{2I} j(j+1). \tag{4.4}$$

The allowed values for m implies that the dimensions of the irreducible represen-

tations are $2j+1$, which is thus the degeneracy of the energy level labeled by the half-integer and integer j.

A position space representation is required so the half-integer j values must be eliminated to avoid double-valued eigenfunctions. Then, the eigenstates are given by the spherical harmonics Y_j^m,

$$\langle\theta,\phi|j,m\rangle = Y_j^m(\theta,\phi) = (-1)^m\sqrt{\frac{2j+1}{4\pi}\frac{(j-m)!}{(j+m)!}}P_j^m(\cos\theta)e^{im\phi}, \qquad (4.5)$$

in which the P_j^m are the associated Legendre functions. This restriction on j carries through to m, of course, and prohibits the situation $\langle\theta,\phi|j,m\rangle = -\langle\theta,\phi+2\pi|j,m\rangle$,

4.2 Angular Momentum Coherent States

4.2.1 On SO(3)

The Lie group SO(3) is the degeneracy group of the spherical rotator. This group may be generated by the components of the orbital angular momentum $\hat{J}_k$, with $k = 1,2,3$. Beginning with these, the irreducible representations are spanned by sets of angular momentum eigenstates $|j,m\rangle$ pertaining to a particular integer j. The most important construction of coherent states for these sets of states is due to Perelomov [67, 68]. These have been variously termed angular momentum coherent states, Bloch coherent states or atomic coherent states.

Reviewing the generalizations given in Chapter 2, be aware firstly that despite

readily available ladder operators,

$$\hat{J}_{\pm} = \hat{J}_1 \pm i\hat{J}_2, \tag{4.6}$$

which satisfy

$$[\hat{J}_+, \hat{J}_-] = 2\hbar\hat{J}_3, \quad \text{and} \quad [\hat{J}_3, \hat{J}_{\pm}] = \pm\hbar\hat{J}_{\pm}, \tag{4.7}$$

no eigenstates of these operators exist besides the null vector. This is simply because the spectrum is finite. (A contrived method around this problem is considered in Appendix F.)

Following Perelomov [68], and moving temporarily into atomic units, note that the angular momentum operators satisfy

$$[\hat{J}_j, \hat{J}_k] = i\epsilon_{jk\ell}\hat{J}_\ell \tag{4.8}$$

where $\epsilon_{jk\ell}$ is the completely antisymmetric Levi-Civita tensor. This is the hallmark of the generators of the Lie group SO(3), locally isomorphic to SU(2), to which Perelomov's generalization of coherent states will now be applied. Recall that this generalization requires a fiducial vector, and a homogeneous space X formed by the quotient of the group SU(2) with the isotropy subgroup H for the fiducial vector. Choosing $|j, -j\rangle$ as the fiducial vector, the elements of $H \sim$ U(1) are of the form $\exp(ia\hat{J}_3)$ with a real. Elements of X are hence of the form [68]

$$\begin{aligned} \hat{D}(\zeta) &= \exp(i(\alpha\hat{J}_1 + \beta\hat{J}_2)) \\ &= \exp(\xi\hat{J}_+ - \xi^*\hat{J}_-) \end{aligned}$$

$$= \exp(\zeta \hat{J}_+) \exp(\eta \hat{J}_3) \exp(-\zeta^* \hat{J}_-) \tag{4.9}$$

in which α and β are real, $\xi = \frac{1}{2}(\beta + i\alpha)$, $\xi = -|\xi| e^{-i\phi}$, $\zeta = -\tan|\xi| e^{-i\phi}$ and $\eta = \ln(1 + |\zeta|^2)$. Accordingly, the coherent states are given by

$$\begin{aligned} |j, \zeta\rangle &= \hat{D}(\zeta)|j, -j\rangle \\ &= \sum_{m=-j}^{j} \left[\frac{(2j)!}{(j+m)!(j-m)!}\right]^{1/2} \frac{\zeta^{j+m}}{(1+|\zeta|^2)^j} |j, m\rangle. \end{aligned} \tag{4.10}$$

With respect to the angular parameterization, the normalized Haar measure [75] on SU(2) for a function f is given by

$$\int_{SU(2)} f(g) dg = \frac{1}{2\pi^2} \int_0^{\pi} \sin^2\psi d\psi \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\phi f(\psi, \theta, \phi). \tag{4.11}$$

Then, integrating the projection operators $|j, \zeta\rangle\langle j, \zeta|$ over X with respect to the measure induced by Eq. (4.11) gives

$$\begin{aligned} \hat{1}_j &= \frac{2j+1}{\pi} \int \frac{d^2\zeta}{(1+|\zeta|^2)^2} |j, \zeta\rangle\langle j, \zeta| \\ &= \int d\nu_j(\zeta) |j, \zeta\rangle\langle j, \zeta|, \end{aligned} \tag{4.12}$$

where $d^2\zeta = d\mathrm{Re}(\zeta) d\mathrm{Im}(\zeta)$, and integration is over the entire complex plane projected from the unit sphere with the transformation

$$\zeta = -\tan\frac{\theta}{2} e^{-i\phi}. \tag{4.13}$$

This construction of angular momentum coherent states does not yield minimum

uncertainty states for the customary angular momentum operators. Reinserting $\hbar$ into the calculations, in the state $|j,\zeta\rangle$,

$$\langle J_+\rangle = \frac{2j\hbar\zeta^*}{1+|\zeta|^2}, \quad \langle J_-\rangle = \frac{2j\hbar\zeta}{1+|\zeta|^2}, \tag{4.14}$$

so that with $\hat{J}_1 = \frac{1}{2}(\hat{J}_+ + \hat{J}_-)$ and $\hat{J}_1 = \frac{1}{2i}(\hat{J}_+ - \hat{J}_-)$,

$$\langle J_1\rangle = \frac{2j\hbar\mathrm{Re}(\zeta)}{1+|\zeta|^2}, \quad \langle J_2\rangle = -\frac{2j\hbar\mathrm{Im}(\zeta)}{1+|\zeta|^2}, \quad \langle J_3\rangle = j\hbar\frac{-1+|\zeta|^2}{1+|\zeta|^2}, \tag{4.15}$$

where the $\hat{J}_3$ calculation follows from Eq. (4.10). Also, for the squared components,

$$\langle J_+^2\rangle = \frac{2j(2j-1)\hbar^2\zeta^{*2}}{(1+|\zeta|^2)^2}, \quad \langle J_-^2\rangle = \frac{2j(2j-1)\hbar^2\zeta^2}{(1+|\zeta|^2)^2}, \quad \langle J_+\hat{J}_-\rangle = \hbar^2\frac{4j^2|\zeta|^2+2j|\zeta|^4}{(1+|\zeta|^2)^2}, \tag{4.16}$$

which provides some support for

$$\begin{aligned}\langle J_1^2\rangle &= \frac{1}{4}\langle \hat{J}_+^2 + 2\hbar\hat{J}_+\hat{J}_- + \hat{J}_-^2 - 2\hat{J}_3\rangle \\ &= \frac{\hbar^2}{4(1+|\zeta|^2)^2}\left(2j(2j-1)\left(\zeta^{*2}+\zeta^2\right) + 8j^2|\zeta|^2 + 2j(|\zeta|^4-1)\right).\end{aligned} \tag{4.17}$$

Following through to the calculation of the uncertainty,

$$(\Delta J_1)^2 = \frac{\hbar^2}{4(1+|\zeta|^2)^2}\left(4j^2\mathrm{Re}(\zeta)^2 - 2j\left(\zeta^{*2}+\zeta^2\right) + 2j(|\zeta|^4-1)\right). \tag{4.18}$$

Calculations proceed along similar lines for $\hat{J}_2$, but will not be drawn to their conclusion here. The uncertainty product becomes increasingly complicated and diminshingly significant.

Though the preceding calculation did not meet with success, SO(3) coherent states do satisfy a certain minimum uncertainty condition. Note that in the angular momentum eigenstate $|j,m\rangle$

$$(\Delta J_1)^2 = \langle J_1^2\rangle = \frac{\hbar^2}{2}(j^2 - m^2 + j) \tag{4.19}$$

$$= (\Delta J_2)^2 \tag{4.20}$$

so that the uncertainty relation

$$(\Delta J_1)^2(\Delta J_2)^2 \geq \frac{\hbar^2}{4}\langle J_3\rangle \tag{4.21}$$

is saturated in the extremal states $m = \pm j$. Also, SO(3) is the rotation group, so the states $|j,\zeta\rangle$ are transformed by a rotation from $|j,-j\rangle$. Accordingly, states $|j,\zeta\rangle$ are angular momentum minimum uncertainty states for the rotated angular momentum operators

$$\hat{\underline{J}}_\ell = \hat{D}(\zeta)\hat{J}_\ell\hat{D}^{-1}(\zeta), \quad \ell = 1,2,3, \tag{4.22}$$

where $\hat{D}(\zeta)$ is the displacement operator given by Eq. (4.9). Expectation values may be taken from the fiducial vector,

$$\langle j,\zeta|\hat{\underline{J}}_\ell|j,\zeta\rangle = \langle j,-j|\hat{J}_\ell|j,-j\rangle, \tag{4.23}$$

and

$$\langle j,\zeta|\hat{\underline{J}}_\ell^2|j,\zeta\rangle = \langle j,\zeta|\hat{D}(\zeta)\hat{J}_\ell\hat{D}^{-1}(\zeta)\hat{D}(\zeta)\hat{J}_\ell\hat{D}^{-1}(\zeta)|j,\zeta\rangle = \langle j,-j|\hat{J}_\ell^2|j,-j\rangle, \quad (4.24)$$

and so on. The uncertainty product is thus

$$(\Delta\underline{J}_1)^2(\Delta\underline{J}_2)^2 = \frac{\hbar^2}{4}\langle\underline{J}_3\rangle^2 = \frac{\hbar^2}{4}j^2. \quad (4.25)$$

This is precisely why one of the extremal states was chosen as the fiducial vector. Also, expectation values for the rotated angular momentum operators are given by

$$\langle\underline{J}_1\rangle = \langle\underline{J}_2\rangle = 0, \quad \langle\underline{J}_3\rangle = -j\hbar. \quad (4.26)$$

Finally, calculating the expectation values, Eq. (4.15), in terms of the transformation onto the unit sphere, Eq. (4.13), used to describe the resolution of the identity, one finds

$$\langle J_1\rangle = -j\hbar\sin\theta\cos\phi, \quad \langle J_2\rangle = -j\hbar\sin\theta\sin\phi, \quad \langle J_3\rangle = -j\hbar\cos\theta, \quad (4.27)$$

which, since the fiducial state gives the above triplet as $-j\hbar(0,0,1)$, represents a reorientation of the initial state in the direction of (θ,ϕ).

4.2.2 The Spherical Rotator

Generalized coherent states for the spherical rotator have been proposed by Atkins and Dobson [3] and Bhaumik, Nag and Dutta-Roy [16], among others. Both of

these propositions are generalized annihilation operator coherent states arising from Schwinger's bosonic treatment [72] of angular momentum (see Appendix B).

Atkins and Dobson describe their states as simultaneous eigenstates of the singletons $\hat{a}_+$ and $\hat{a}_-$ (with complex eigenvalues), and hence include the half-integer j states. This definition renders them equivalent to the direct product of two harmonic oscillator coherent states, and so their various properties follow quickly. For example, they satisfy a resolution of the identity given by the standard form Eq. (2.35) written twice, once for each eigenvalue. As well, they have a classical limit which may be exploited to study the quantum classical correspondence of angular momentum. Note that convenient time evolution does not follow (under the action of the Hamiltonian $\hat{H} = \hat{J}^2/2I$) since the eigenvalues of $\hat{J}^2$ are quadratic, unlike the equally spaced eigenvalues of the harmonic oscillator Hamiltonian. However, the main difficulty with the states of Atkins and Dobson is the lack of a position space representation due to the inclusion of the half integer j states.

Bhaumik *et al.* omit the half integer j states. They define their states as simultaneous eigenstates of $\hat{K}_-$ and $\hat{I}_-$, the ladder operators of bosonic construction discussed in Appendix B. It is unclear why they define their states in terms of $\hat{K}_-$ and not $\hat{L}_-$, thereby retaining the symmetry between positive and negative m. Also, their expression for $\hat{I}_-$ is one which only when modified leads to a generator for so(2,1) or su(1,1).

Briefly, their states are defined by satisfying jointly the two eigenvector-eigenvalue equations

$$\hat{I}_-|\beta,\gamma\rangle = \beta|\beta,\gamma\rangle, \quad \hat{K}_-|\beta,\gamma\rangle = \gamma|\beta,\gamma\rangle. \tag{4.28}$$

with β and γ complex numbers. From Eqs. (B.20) and (B.18), each of the above gives a two-term recursion relation which is simply solved to give

$$|\beta,\gamma\rangle = \frac{1}{\sqrt{\cosh\xi}} \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \frac{\beta^m \gamma^{j-m}}{\sqrt{(j+m)!(j-m)!}} |j,m\rangle. \tag{4.29}$$

in which $\xi = |\beta|(1+|\alpha|^2)$ and $\gamma = \alpha\beta$. In terms of α and β, this becomes

$$|\alpha;\beta\rangle = \frac{1}{\sqrt{\cosh\xi}} \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \frac{\alpha^{j-m} \beta^j}{\sqrt{(j+m)!(j-m)!}} |j,m\rangle. \tag{4.30}$$

Bhaumik *et al.* [16] also report a resolution of the identity satisfied by their states. Unfortunately, theirs does not appear to be valid, an observation not present in the literature. A resolution of the identity which these states do satisfy is given by

$$\hat{1} = \int \frac{d^2\alpha}{\pi} \int \frac{d^2\beta}{2\pi} e^{-\xi} \cosh\xi |\alpha;\beta\rangle\langle\alpha;\beta|, \tag{4.31}$$

where d^2z stands for $d\mathrm{Re}(z)d\mathrm{Im}(z)$. The expression given by Bhaumik *et al.* lacks the $\cosh\xi$ which eliminates the normalization constant of Eq. (4.30), and a factor of $\frac{1}{2}$ associated with the integration over β.

In any case, it is somewhat more convenient to define them slightly differently. Consider states which are simultaneous eigenstates of $\hat{I}_-$ and $\hat{L}_-$. With

$$\hat{I}_- = \frac{1}{2}\hat{a}_+\hat{a}_+, \quad \hat{L}_- = \frac{1}{2}\hat{a}_-\hat{a}_-, \tag{4.32}$$

it is further convenient to identify the eigenvalues according to

$$\hat{I}_-|\alpha,\beta\rangle = \frac{1}{2}\alpha^2|\alpha,\beta\rangle, \quad \hat{L}_-|\alpha,\beta\rangle = \frac{1}{2}\beta^2|\alpha,\beta\rangle, \tag{4.33}$$

with $\mathrm{Re}(\alpha) \geq 0$ and $\mathrm{Re}(\beta) \geq 0$ to avoid the duplication of states. (See Appendix B for more details regarding the bosonic treatment of angular momentum.) Then, expressing the state as a superposition over angular momentum eigenstates $|j,m\rangle$ with integer j, the recursion relations

$$\sqrt{(j+m+2)(j+m+1)}c_{j+1,m+1} = \alpha^2 c_{jm}, \tag{4.34}$$

$$\sqrt{(j-m+2)(j-m+1)}c_{j+1,m-1} = \beta^2 c_{jm} \tag{4.35}$$

follow through Eqs. (B.20) and (B.22). Starting at c_{00} and applying successively Eq. (4.34), one obtains

$$c_{kk} = \frac{\alpha^{2k}c_{00}}{\sqrt{(2k)!}}. \tag{4.36}$$

Then applying Eq. (4.35) successively to c_{kk} yields

$$c_{k+\ell,k-\ell} = \frac{\alpha^{2k}\beta^{2\ell}c_{00}}{\sqrt{(2k)!(2\ell)!}}. \tag{4.37}$$

Changing the labels to the standard j and m and imposing the usual normalization yields

$$|\alpha,\beta\rangle = \frac{1}{\sqrt{\cosh\xi}}\sum_{j,m}\frac{\alpha^{j+m}\beta^{j-m}}{\sqrt{(j+m)!(j-m)!}}|j,m\rangle, \tag{4.38}$$

where $\xi = |\alpha|^2 + |\beta|^2$. Note the use of the comma to distinguish these states from

Eq. (4.31). The states Eq. (4.38) satisfy the resolution of the identity

$$\hat{1} = 4 \int_{\mathrm{Re}(\alpha)\geq 0} \frac{d^2\alpha}{\pi} \int_{\mathrm{Re}(\beta)\geq 0} \frac{d^2\beta}{\pi} e^{-\xi} \cosh \xi |\alpha, \beta\rangle\langle\alpha, \beta|, \tag{4.39}$$

in which, as indicated, the integrals are only over the non-negative real portion of the complex planes of α and β.

Before describing some of the properties of these states, it is worthwhile to connect them to the standard SO(3) coherent states, given by Eq. (4.10). Suppose $|\alpha; \zeta\rangle$ is a superposition of SO(3) coherent states which is also an eigenstate of $\hat{I}_-$. Then

$$\hat{I}_- |\alpha; \zeta\rangle = \hat{I}_- \sum_{j=0}^{\infty} c_j |j, \zeta\rangle = \frac{1}{2}\alpha^2 |\alpha; \zeta\rangle. \tag{4.40}$$

Carrying the operator through the summation in the usual fashion in conjunction with Eq. (B.20) yields the two term recursion relation

$$c_{j+1} = \frac{1 + |\zeta|^2}{\zeta^2} \frac{\alpha^2 c_j}{\sqrt{(2j+2)(2j+1)}}, \tag{4.41}$$

so that

$$c_j = \frac{(1 + |\zeta|^2)^j}{\zeta^{2j}} \frac{\alpha^{2j} c_0}{\sqrt{(2j)!}}. \tag{4.42}$$

Assembling the state $|\alpha; \zeta\rangle$,

$$|\alpha; \zeta\rangle = \frac{1}{\sqrt{\cosh \xi}} \sum_{j,m} \frac{\alpha^{2j} \zeta^{-j+m}}{\sqrt{(j+m)!(j-m)!}} |j, m\rangle, \quad \xi = |\alpha|^2 (1 + |\zeta|^{-2}), \tag{4.43}$$

and comparing with Eq. (4.38), indicates that $|\alpha; \zeta\rangle = |\alpha, \beta\rangle$ through the identi-

fication $\alpha = \zeta\beta$. A similar recipe exists for constructing superpositions of SO(3) coherent states which are also eigenstates of $\hat{L}_-$.

Not surprisingly, since $[\hat{I}_-, \hat{K}_-] = 0$, the states $|\alpha, \beta\rangle$ may also be constructed as an eigenstate of $\hat{K}_-$ formed of SO(3) coherent states. Suppose

$$\hat{K}_-|\kappa;\zeta\rangle = \kappa|\kappa;\zeta\rangle, \quad \text{with} \quad |\kappa;\zeta\rangle = \sum_{j=0}^{\infty} c_j|j,\zeta\rangle. \tag{4.44}$$

This gives the recursion relation

$$c_{j+1}\sqrt{(2j+1)(2j+2)}\frac{\zeta}{1+|\zeta|^2} = \kappa c_j, \tag{4.45}$$

which may be solved to yield

$$c_j = \frac{(1+|\zeta|^2)^j}{\zeta^j}\frac{\kappa^j c_0}{\sqrt{(2j)!}}. \tag{4.46}$$

Assembling the state $|\kappa;\zeta\rangle$ in conjunction with Eq. (4.10) and demanding normalization gives

$$|\kappa;\zeta\rangle = \frac{1}{\sqrt{\cosh\xi}}\sum_{j,m}\frac{\zeta^m\kappa^j}{\sqrt{(j+m)!(j-m)!}}|j,m\rangle, \quad \xi = \frac{|\kappa|}{|\zeta|}(1+|\zeta|^2). \tag{4.47}$$

With the identifications $\kappa = \alpha\beta$, and $\alpha = \beta\zeta$, Eq. (4.47) corresponds to Eq. (4.38) including the value of ξ.

Properties of the States

Due to the work in Section 3.1, the states $|\alpha,\beta\rangle$ defined by Eq. (4.38) are minimum uncertainty states for some pairs of operators. In fact, since they are eigenstates of $\hat{I}_-$, $\hat{L}_-$ and $\hat{K}_-$ they the satisfy minimum uncertainty products

$$(\Delta I_1)^2(\Delta I_2)^2 = \frac{\hbar^2}{4}\langle I_3\rangle^2, \quad (\Delta L_1)^2(\Delta L_2)^2 = \frac{\hbar^2}{4}\langle L_3\rangle^2, \quad (\Delta K_1)^2(\Delta K_2)^2 = \frac{\hbar^2}{4}\langle K_3\rangle^2. \tag{4.48}$$

Through the constructions of Eq. (4.43) and Eq. (4.47), it is not difficult to work out these uncertainties,

$$(\Delta I_1)^2 = (\Delta I_2)^2 = \frac{\hbar^2}{4}\left(|\alpha|^2\tanh\xi + \frac{1}{2}\right). \tag{4.49}$$

$$(\Delta L_1)^2 = (\Delta L_2)^2 = \frac{\hbar^2}{4}\left(|\beta|^2\tanh\xi + \frac{1}{8}\right). \tag{4.50}$$

$$(\Delta K_1)^2 = (\Delta K_2)^2 = \frac{\hbar^2}{4}(\xi\tanh\xi + 1). \tag{4.51}$$

However, these operators are derived from abstract ladder operators in some abstract space. Minimum uncertainty in these operators does not correspond to anything physically significant, or if it does, it is far from obvious. Only in the fortuitous case of $\hat{J}_\pm$ do the operators correspond to physical angular momentum, and it is to these operators we now turn our attention.

The simplest calculations of these expectation values in a coherent state $|\alpha,\beta\rangle$ are through an expression in terms of superpositions of SO(3) coherent states, for example, Eq. (4.43). Accordingly, expectation values of the angular momentum

components are

$$\langle J_1 \rangle = \hbar\xi \tanh \xi \frac{\mathrm{Re}(\zeta)}{1+|\zeta|^2}, \tag{4.52}$$

$$\langle J_2 \rangle = -\hbar\xi \tanh \xi \frac{\mathrm{Im}(\zeta)}{1+|\zeta|^2}, \tag{4.53}$$

$$\langle J_3 \rangle = -\hbar\frac{\xi}{2} \tanh \xi \frac{1-|\zeta|^2}{1+|\zeta|^2}. \tag{4.54}$$

In terms of the angles θ and ϕ given by Eq. (4.13) these are

$$(\langle J_1 \rangle, \langle J_2 \rangle, \langle J_3 \rangle) = -\hbar\frac{\xi}{2} \tanh \xi (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{4.55}$$

As with the SO(3) coherent states, these states are not minimum uncertainty states for the standard angular momentum operators. However, looking at the rotated angular momentum operators as mentioned before, some progress can be made.

To this point, the treatment of angular momentum operators has been somewhat cavalier. In representing the operators as pairs of bosonic creation and annihilation operators, direct products should be used. For example, $\hat{J}_+ = \hat{a}_+^\dagger \otimes \hat{a}_-$, and $\hat{I}_+ = \frac{1}{2}(\hat{a}_+^\dagger \hat{a}_+^\dagger) \otimes \hat{1}_-$, and so on. Similarly, when expressing the standard angular momentum operators $\hat{J}_\pm$ without regard to their bosonic realization, but applying them to various states of differing j, the operator actually in use is the direct sum of the operator over all the representations:

$$\hat{\mathcal{J}}_\ell = \bigoplus_{j=0}^{\infty} \hat{J}_{j,\ell}, \tag{4.56}$$

in which the subscript j labels the representation, and $\ell = 1, 2, 3$. The more care-

ful treatment is introduced now to accommodate the rotation of these operators through Eq. (4.22), since the displacement operator exists only within representations of SO(3). The rotation of $\hat{\mathcal{J}}_\ell$ is given by

$$\underline{\hat{J}}_\ell = \left(\bigoplus_{j=0}^{\infty} \hat{D}_j(\zeta)\right)\left(\bigoplus_{j=0}^{\infty} \hat{J}_{j,\ell}\right)\left(\bigoplus_{j=0}^{\infty} \hat{D}_j^{-1}(\zeta)\right) = \bigoplus_{j=0}^{\infty} \hat{D}_j \hat{J}_{j,\ell} \hat{D}_j^{-1} = \bigoplus_{j=0}^{\infty} \underline{\hat{J}}_{j,\ell}. \quad (4.57)$$

Of course, all this means is that a cavalier treatment of the rotation of these operators continues to be legitimate.

Accordingly, due to Eq. (4.26), the expectation values of $\underline{\hat{J}}_\ell$ in the state $|\alpha, \beta\rangle$ are given by

$$\langle \underline{J}_1 \rangle = \langle \underline{J}_2 \rangle = 0. \quad \langle \underline{J}_3 \rangle = -\hbar \frac{\xi}{2} \tanh \xi, \quad (4.58)$$

where the calculations were carried out by expressing the state as a superposition of SO(3) coherent states, via Eq. (4.42). As for the squared quantities,

$$\langle \underline{J}_1^2 \rangle = \langle \underline{J}_2^2 \rangle = \hbar^2 \frac{\xi}{4} \tanh \xi, \quad (4.59)$$

so that the coherent state $|\alpha, \beta\rangle$ is a minimum uncertainty state for the rotated angular momentum operators,

$$(\Delta \underline{J}_1)^2 (\Delta \underline{J}_2)^2 = \langle \underline{J}_1^2 \rangle \langle \underline{J}_2^2 \rangle = \hbar^4 \frac{\xi^2}{16} \tanh^2 \xi = \frac{\hbar^2}{4} \langle \underline{J}_3 \rangle. \quad (4.60)$$

Returning to the expectation values of $\hat{J}_1$, $\hat{J}_2$ and $\hat{J}_3$, consider the complex

parameters α and β in a polar notation.

$$\alpha = |\alpha|e^{i\theta_1}, \quad \beta = |\beta|e^{i\theta_2}, \tag{4.61}$$

which translates the angular momentum expectation values into

$$\langle J_1 \rangle = \hbar|\alpha||\beta|\cos(\theta_2 - \theta_1)\tanh\xi, \tag{4.62}$$

$$\langle J_2 \rangle = \hbar|\alpha||\beta|\sin(\theta_2 - \theta_1)\tanh\xi, \tag{4.63}$$

$$\langle J_3 \rangle = \frac{\hbar}{2}(|\beta|^2 - |\alpha|^2)\tanh\xi, \tag{4.64}$$

$$\langle J^2 \rangle = \frac{\hbar^2}{4}(3\xi\tanh\xi + \xi^2) \tag{4.65}$$

With these expressions and the identifications

$$\mathsf{J} = -\frac{\hbar}{2}\xi\tanh\xi, \tag{4.66}$$

$$\cos\theta = \frac{|\alpha|^2 - |\beta|^2}{\xi}, \tag{4.67}$$

$$\sin\theta = -2\frac{|\beta||\alpha|}{\xi}, \tag{4.68}$$

$$\phi = \theta_2 - \theta_1, \tag{4.69}$$

the expectation values become

$$(\langle J_1 \rangle, \langle J_2 \rangle, \langle J_3 \rangle) = \mathsf{J}(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta), \tag{4.70}$$

in agreement with the earlier expression Eq. (4.55).

To calculate the expectation values of the Cartesian coordinate operators, a cer-

tain approximation must be made which requires ξ to be sufficiently large that the state $|\alpha,\beta\rangle$ is in effect absent from states of low j. The details of this approximation are discussed in Appendix D.

From the appendix, these expectation values are given by

$$\langle x\rangle = \frac{1}{2}(\alpha^2+\alpha^{*2}-\beta^2-\beta^{*2})\frac{\tanh\xi}{\xi}. \tag{4.71}$$

$$\langle y\rangle = \frac{i}{2}(\alpha^2-\alpha^{*2}+\beta^2-\beta^{*2})\frac{\tanh\xi}{\xi}. \tag{4.72}$$

$$\langle z\rangle = (\alpha\beta+\alpha^*\beta^*)\frac{\tanh\xi}{\xi}. \tag{4.73}$$

Making the further identification

$$\psi = \theta_2+\theta_1, \tag{4.74}$$

these expectation values may be expressed by

$$\langle x\rangle = (\cos\phi\cos\theta\cos\psi+\sin\phi\sin\psi)\tanh\xi, \tag{4.75}$$

$$\langle y\rangle = (\sin\phi\cos\theta\cos\psi-\cos\phi\sin\psi)\tanh\xi, \tag{4.76}$$

$$\langle z\rangle = -(\sin\theta\cos\psi)\tanh\xi, \tag{4.77}$$

The significance of these angles are shown in Figure 4.1. The angular momentum vector $\vec{J}$ makes an angle θ with the z-axis and a line on the x-y plane below it makes an angle ϕ with the x-axis. The vector $\vec{r}_0$ signifies the location of the state with $\psi=0$, which is below $\vec{J}$ by a right angle. The location $\vec{r}$ of the coherent state is found by rotating $\vec{r}_0$ an angle ψ clockwise while looking down $\vec{J}$ and remaining

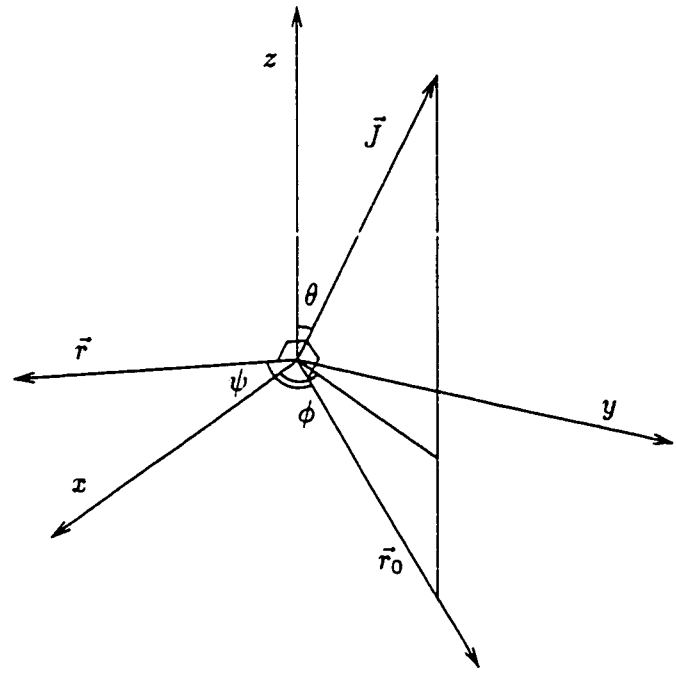


Figure 4.1: The angles involved in the location of the coherent state.

on a plane normal to $\vec{J}$. Note that the set of points corresponding to $\psi \in [0, 2\pi]$ is a geodesic on the sphere, the path one expects a classical particle to take with an angular momentum oriented parallel to Eq. (4.70).

Lastly, consider time evolution. These coherent states do not evolve in time among themselves. However, it is always possible to express the time evolution in terms of the evolution of coherent state parameters plus corrections. Such an effort is conferred meaning when the corrections are small.

Given a value j_0, one may write

$$j(j+1) = -j_0^2 + (2j_0 + 1)j + (j - j_0)^2. \tag{4.78}$$

Therefore, with $|\psi(0)\rangle = |\alpha,\beta\rangle$, the evolved state $|\psi(t)\rangle$ is given by

$$\begin{aligned}
|\psi(t)\rangle &= \frac{1}{\sqrt{\cosh\xi}}\sum_{j,m}\frac{\alpha^{j+m}\beta^{j-m}}{\sqrt{(j+m)!(j-m)!}}e^{-i\frac{\hbar}{2I}j(j+1)t}|j,m\rangle \\
&= \frac{e^{i\frac{\hbar}{2I}j_0^2 t}}{\sqrt{\cosh\xi}}\sum_{j,m}\frac{\alpha^{j+m}\beta^{j-m}}{\sqrt{(j+m)!(j-m)!}} \\
&\quad \times \exp\left(-i\frac{\hbar}{2I}\left((2j_0+1)j+(j-j_0)^2\right)t\right)|j,m\rangle \\
&= e^{i\frac{\hbar}{2I}j_0^2 t}|\alpha e^{-i\omega t/2}, \beta e^{-i\omega t/2}\rangle \\
&\quad + \frac{e^{i\frac{\hbar}{2I}j_0^2 t}}{\sqrt{\cosh\xi}}\sum_{j,m}\frac{(\alpha e^{-i\omega t/2})^{j+m}\beta e^{-i\omega/2t})^{j-m}}{\sqrt{(j+m)!(j-m)!}}\left(-i\frac{\hbar}{2I}(j-j_0)^2 t\right)|j,m\rangle \\
&\quad + \ldots,
\end{aligned} \tag{4.79}$$

in which successive terms arise from continued expansion of the exponential, and $\omega = \frac{\hbar}{2I}(2j_0+1)$.

The value of j_0 to be used is that which minimizes the corrections. Taking j_0 at the centre of the distribution in j gives $j_0 = \frac{1}{2}\xi\tanh\xi$. Since the expansion of an exponential was required to obtain Eq. (4.79), one would not expect this correction to be small. However, at $t = 0$ the first correction term (denoted $|\Psi^{(1)}\rangle$) vanishes. To see for how long it remains small, note that

$$\begin{aligned}
&\langle\Psi^{(1)}(t)|\Psi^{(1)}(t)\rangle \\
&= \frac{\hbar^2 t^2}{2I}\left(j_0^4 - 2\xi\tanh\xi j_0^3 + \frac{3}{2}\left[\xi\tanh\xi+\xi^2\right]j_0^2 - \frac{1}{2}\left[3\xi^2+(\xi+\xi^3)\tanh\xi\right]j_0\right. \\
&\quad \left. + \frac{1}{16}\left[7\xi^2+\xi^4+(\xi+6\xi^3)\tanh\xi\right]\right).
\end{aligned} \tag{4.80}$$

The proper way to choose j_0 is to to minimize Eq. (4.80) over j_0 as a parameter.

Since the expression in question is a quartic in j_0, this approach is not very fruitful. However, the derivative of the quartic has one real root and two complex roots (by Maple). Since the coefficient of j_0^4 is positive, the real root must correspond to a single, global, minimum of the quartic. Not surprisingly, the expression for this root is extravagantly complicated, but does consist of terms tending to zero plus $\frac{1}{2}\xi\tanh\xi$ with increasing ξ. Inserting this value for j_0 into Eq. (4.80) yields

$$\begin{aligned}\langle\Psi^{(1)}|\Psi^{(1)}\rangle &= \frac{\hbar^2 t^2}{2I}\left(\frac{\xi^4}{16}\frac{3\tanh^2\xi+1}{\cosh^2\xi}+\frac{\xi^3}{8}\left[\tanh^2\xi-\tanh\xi\right]\right.\\ &\quad\left.+\frac{\xi^2}{16}\left[7-4\tanh^2\xi\right]+\frac{\xi}{16}\tanh\xi\right). \qquad (4.81)\end{aligned}$$

Note that with increasing ξ, the quartic and cubic terms vanish exponentially. The quadratic term is the largest remaining term, tending to $3\xi^2/16$ for large values of ξ.

A Classical Limit

Large values of ξ have been alluded to in several contexts already. To formalize this concept, consider the following limiting process hereafter designated the classical limit for the angular momentum coherent states $|\alpha,\beta\rangle$,

$$\hbar\to 0, \quad \mathrm{J}=\text{constant}\sim -\frac{1}{2}\hbar\xi. \qquad (4.82)$$

The first direct implication is that $\xi\to\infty$ simultaneous to $\hbar\to 0$.

The conception of this limit is in direct analogy with that of the harmonic oscillator coherent states in which $\hbar\to 0$ with $\hbar|\alpha|^2$ fixed. With $\xi=|\alpha|^2+|\beta|^2$,

the limit Eq. (4.82) is simply the same limit, applied jointly to the α and β parts of the angular momentum coherent state.

Under the action of this limit, the uncertainties become

$$(\Delta I_1) \to 0, \quad (\Delta I_2) \to 0, \tag{4.83}$$

as well as for $\hat{L}_{1,2}$ and $\hat{K}_{1,2}$. Also, the uncertainties of the rotated angular momentum operators disappear,

$$(\Delta \underline{J}_1) \to 0, \quad (\Delta \underline{J}_2) \to 0. \tag{4.84}$$

Since $\tanh \xi \to 1$ (which it does quite quickly),

$$\langle x \rangle \to (\cos\phi \cos\theta \cos\psi + \sin\phi \sin\psi), \tag{4.85}$$

$$\langle y \rangle \to (\sin\phi \cos\theta \cos\psi - \cos\phi \sin\psi), \tag{4.86}$$

$$\langle z \rangle \to -(\sin\theta \cos\psi), \tag{4.87}$$

so that $\langle r \rangle^2 = \langle x \rangle^2 + \langle y \rangle^2 + \langle z \rangle^2 \to 1$. The significance of this is that since the surface of the sphere is convex, any distribution whatsoever of the wave function on the surface results in $\langle r \rangle < 1$. Therefore, in the limit, with $\langle r \rangle = 1$, the wave function must be localized on a point.

As for time evolution, the leading behaviour is given by $|\alpha e^{i\omega t/2}, \beta e^{i\omega t/2}\rangle$ so that under time evolution, recalling the identifications of Eqs. (4.66) to (4.69) and Eq. (4.74), the angles θ and ϕ are constants of the motion, and ψ behaves like

$\psi_0 + \omega t$. In the classical limit, $\omega = J/I$ for a moment of inertia I, *i.e.* precisely the behaviour one would expect from a classical particle on the surface of a sphere.

In the classical limit, the correction term $|\Phi^{(1)}\rangle$ does not vanish. Given a specific time and enacting the classical limit,

$$\langle \Psi^{(1)} | \Psi^{(1)} \rangle = \frac{3t^2}{16I}. \tag{4.88}$$

From some numerical work, this does not improve significantly when using the true minimum of the quartic calculated through the symbolic computations of Maple rather than the approximate minimum $\frac{1}{2}\xi \tanh \xi$. However, the classical evolution of a localized particle on a sphere takes place along a localized particle moving along the geodesic of a sphere, which is exactly what is given by $|\alpha e^{i\omega t/2}, \beta e^{i\omega t/2}\rangle$. The discrepancy must arise from the destructive effects of expanding exponentials.

Chapter 5

Klauder's Construction with Energy Degeneracies

Klauder's construction for generalized coherent states is not directly applicable to energy degenerate systems. In the presence of energy degeneracies, the resolution of the identity fails. This chapter addresses the issue of how to overcome this difficulty, and applies the resultant construction to the hydrogen atom problem.

5.1 Background

Coherent states for the hydrogen atom have been contemplated by researchers since the first mention by Schrödinger of the harmonic oscillator coherent states [70]. Schrödinger proposed that non-spreading wave functions which follow classical motion in the Coulomb potential should exist, but the technical difficulties over-

whelmed him at that time. Indeed, the technical difficulties have kept the topic alive for seventy five years. All the published constructions of hydrogen atom coherent states have only been "coherent" in a weaker sense than that conceived by Schrödinger.

One of the earlier studies was by Brown [21] who constructed states which traveled strictly on circular orbits, representing only a small, specific set of classical orbits. Brown's construction was *ad hoc* in nature, though it deserves mention being an early contribution.

Many of the hydrogenic coherent state constructions are based on constructing the Perelomov coherent states for SO(4,2), the dynamical group of the hydrogen atom [8]. The first of these was Mostowski [51]. Mostowski (and subsequently de Prunelé [24]) claims that the initial motion is along a classical trajectory, but the wave function does spread with time and cannot be described by the evolution of the coherent state parameters.

McAnally and Bracken [49] construct SO(4,2) coherent states, using Barut and Girardello's conception of coherent states [9] as eigenstates of annihilation operators. McAnally and Bracken argue that the Perelomov construction for the hydrogen atom is inappropriate due to the enormous number of contributing eigenlevels in the large quantum number limit (see Section 3.1.1). They also take issue with some of Mostowski's calculations and provide a more detailed description of their calculations and conclusions.

Zlatev *et al.* [91] (making no reference to McAnally and Bracken's construction) provide a further presentation of Perelomov's construction of SO(4,2) coher-

ent states, exploiting a bosonic realization of the group [83]. Their conclusion is that SO(4,2) coherent states cannot have a classical limit, at least in their terms.

Gerry [35] also exploits a bosonic representation of SO(4,2) in a construction of hydrogen atom coherent states, writing the states directly in terms of harmonic oscillator coherent states expressed by the bosonic operators. Gerry uses this construction to derive Kepler's third law without resorting to large n limits. Gerry and Kiefer [37] also present a group theoretical construction of hydrogenic coherent states. Using the spectrum generating subgroup SO(2,1), Gerry and Kiefer construct coherent states which do not disperse when evolved in a so-called "fictitious time" corresponding to evolution in a four dimensional harmonic oscillator mapped to the Coulomb problem through the Kustaanheimo-Stiefel transformation.

This Kustaanheimo-Stiefel transformation has been the basis of several other constructions of hydrogenic coherent states. Another work by Gerry [36] is such a paper. Gerry reports that these coherent states remain localized while evolving in the same sense as Gerry and Kiefer's states cited above. Bhaumik, Dutta-Roy and Ghosh [15] almost simultaneously published a similar construction, though their states are evolved in real time. Similarly, Nuori [63] constructs coherent states for the d-dimensional Coulomb problem by mapping onto a harmonic oscillator of appropriate dimensionality. Ghosh [38] constructs generalized annihilation operator coherent states for the three dimensional Coulomb potential by mapping the system through squared parabolic coordinates onto two dimensional harmonic oscillators.

Nieto's generalization for minimum uncertainty coherent states [58] (see Section 2.2.3) may also be adapted to three-dimensional radial problems, and was applied

to the hydrogen atom [55]. This approach relies upon expressing the wave function as the product of radial and angular parts.

Numerous publications involving hydrogenic coherent states have come from a group centred at the University of Rochester engaged in the experimental generation of Rydberg states in atoms [85, 87, 88, 86, 89]. Theoretical calculations show [32, 66] that a picosecond laser pulse acting on certain atoms excites a valence electron into a localized superposition of Rydberg states. These states are regarded as coherent in the sense that they are initially localized and initially exhibit classical behaviour. The long time evolution of these states is dominated by a series of fractional revivals [4, 54], intensely non-classical behaviour. The experimental results bear these predictions out.

From this group, Gaeta and Stroud [31] consider coherent states similar to those of Brown [21], examining in detail the issue of wave function revivals. Mallalieu and Stroud [48] also examine these states as evolved through a semi-classical propagator.

Two somewhat similar studies make use of SO(4) as the degeneracy group of the hydrogen atom. Gay, Delande and Bommier [33] discuss superpositions of SO(4) coherent states described as the direct product of two SO(3) coherent states, though in different terms from what appears in this chapter. Nauenberg [53] constructs SO(4) states by first considering states associated with the often overlooked SO(3) subgroup generated by the operators $\hat{A}_1$, $\hat{A}_2$ and $\hat{J}_3$. Whereas Gay *et al.* do not even consider the superposition of different SO(4) coherent states, Nauenberg, stating that general results are reasonably independent of how they are superposed, performs his calculations with a Gaussian superposition.

Recently, several papers have appeared constructing "temporally stable" hydrogenic coherent states based on the general procedure of Klauder. Klauder's original paper [45], contains such a construction, making use of a somewhat *ad hoc* construction of SO(4) coherent states which, among other things, disrupts normalization. Klauder's paper was followed by one by Majumdar and Sharatchandra [47], who construct the SO(4) coherent states more carefully, though in doing so they report a resolution of the identity which cannot be valid. Unfortunately, this problem undermines the entire basis of their construction. A central claim, developed in an unpublished preprint [73], is that there is a one to one correspondence between coherent states by Klauder's construction and their measures in the resolution of the identity. This claim has been shown to be false by Sixdeniers *et al.* [74] who, as an example in their construction of coherent states, give multiple sets of coherent states all leading to the same measure.

Another take on temporally stable coherent states comes from Fox [27] who also uses careful construction of SO(4) coherent states. Hearkening back to Nauenberg's construction [53], Fox employs a Gaussian superposition of these states parameterized in such a way as to provide a resolution of the identity. As such, this approach may be applied to systems besides the hydrogen atom, though it does not result in the standard coherent states when applied to the harmonic oscillator itself. Fox uses the Gaussian superposition as one way of overcoming the criticisms of Bellomo and Stroud [12, 13], who claim that the temporally stable approach to constructing coherent states does not support wave function revivals. Another way is discussed in Section 5.3.4.

Of course, in the low quantum number regime, Fox's states are only Gaussian in the principal quantum number distribution, not in configuration space. Fox shows [27] that in a limit involving large quantum numbers, the configuration space distribution does become Gaussian, and that the inevitable dephasing of the azimuthal angle (spreading of the wave packet about the classical orbit) occurs relatively slowly.

The literature on the topic of constructing hydrogenic coherent states is crowded to say the least. It appears that each author has his own idea as to what "coherent" means, stated or unstated. For the hydrogen atom problem, Schrödinger's idea of "coherent" meaning minimum uncertainty wave packets which follow classical trajectories for all time, the prospects are quite dim. After all, the potential at large distances from the origin is very flat, so that finding non-dispersing hydrogenic wave packets is as likely as finding such packets for free electrons. In any case, with the present task of defining temporally stable hydrogen coherent states, certain problems exist with published constructions thus far. The following sections remedy this situation.

5.2 Handling the Degeneracy

As observed in Section 2.2.4, if the Hamiltonian $\hat{H}$ contains any energy degeneracy, the resolution of the identity fails. In order to circumvent this problem, the solution is to select one eigenstate from each energy level so that $\delta_{E_n,E_m} = \delta_{n,m}$, but which eigenstate? Clearly, a standard "number state" will not suffice since the resolution of the identity will then only project onto the contributing states. The selected

states must be in some sense representative of all the states in the energy eigenlevel.

In fact, the minimum requirement for the selected eigenstate is that it be from among a set of states which satisfy a "resolution of the identity" on the entire degenerate subspace of the Hilbert space, and that the measure involved in the resolution of the identity is independent of the subspace. Recall that any linear combination of degenerate eigenstates is still an eigenstate.

Energy degeneracies arise in the presence of symmetries in the system. Symmetries are embodied in quantum mechanics by sets of commuting operators. These commuting operators are the Casimir operators and generators of the degeneracy group, where for the present purposes, the degeneracy group of a system is a group such that one unitary irreducible representation contains all the eigenstates of a single degenerate energy level. The obvious, though not unique, construction of these sets of states is Perelomov's construction of coherent states (see Section 2.2.2), applied to the degeneracy group.

To accomplish this, replace Eq. (2.86) with

$$\begin{aligned} &|s,\gamma,x\rangle \\ &\quad = N(s^2)\sum_{n=0}^{\infty}\frac{s^n\exp(-i\gamma e_n)}{\sqrt{\rho_n}}\sqrt{d_n}|n,x\rangle, \end{aligned} \tag{5.1}$$

where d_n is the degeneracy of the nth energy level, $|n,x\rangle$ are the Perelomov coherent states for the degeneracy group G, and the normalizing factor $N(s^2)$ is given by

$$1 = \langle s,\gamma,x|s,\gamma,x\rangle = N^2(s^2)\sum_{n=0}^{\infty}\frac{s^{2n}d_n}{\rho_n}. \tag{5.2}$$

From Eq. (2.74), in each energy degenerate subspace of the Hilbert space, the Perelomov coherent states satisfy the resolution of the identity

$$\hat{1}_n = d_n \mathrm{vol}(H) \int_X d\eta(x) |n, x\rangle\langle n, x|, \tag{5.3}$$

in which H is the isotropy subgroup relative to the fiducial vector in the construction of the Perelomov coherent states, $X = G/H$ is the quotient space formed by the degeneracy group with the isotropy subgroup, and the measure $d\eta$ is induced from the Haar measure on the degeneracy group. The states Eq. (5.1) therefore satisfy the resolution of the identity

$$\hat{1} = \int d\mu(s, \gamma, x) |s, \gamma, x\rangle\langle s, \gamma, x|, \tag{5.4}$$

with

$$\int d\mu(s, \gamma, x) = \lim_{\Gamma \to \infty} \frac{1}{2\Gamma} \int_0^\infty ds^2 k(s^2) \int_{-\Gamma}^{\Gamma} d\gamma \ \mathrm{vol}(H) \int_X d\eta(x). \tag{5.5}$$

Since the states $|n, x\rangle$ are formed by superpositions over states which share a common energy eigenvalue e_n, they are also eigenstates of the Hamiltonian and so evolve simply in time. Accordingly, the states $|s, \gamma, x\rangle$ preserve the temporal stability property of the non-degenerate construction.

The salient feature of Eq. (5.1) is the somewhat unobtrusive but nevertheless important factor $\sqrt{d_n}$ which accommodates the d_n of Eq. (5.3). This factor, though

associated with the measure over the degenerate subspaces in the resolution of the identity, cannot remain with the measure unlike Majumdar and Sharatchandra's expression [47], since it is dependent on the energy level. If, like Klauder [45] or Fox [27], the factor is bundled in with the state $|n, x\rangle$ normalization is disrupted. If this factor is incorporated in the present manner with normalization imposed afterwards, then neither of these problems arise. Furthermore, this admits the democratic interpretation that the state Eq. (5.1) consists of a superposition of energy eigenstates, each eigenstate being a coherent state of the degeneracy group, *weighted by the degeneracy of the group.* In this conception, all energy eigenstates are weighted equally up to the factor of s^{2n}/ρ_n, whereas the other constructions noted favour eigenstates in levels of low degeneracy.

This completes the general construction. Note that the function $\rho(u)$ remains unspecified so that specific problems may be tackled by specific choices of this function. To illustrate this construction and this degree of freedom, we now turn to the hydrogen atom problem.

5.3 As Applied to the Hydrogen Atom

The group theoretical treatment of the hydrogen atom is standard in the literature [8, 79, 1]. For the hydrogen atom problem, there are two pertinent realizations of the degeneracy group SO(4). One uses the elements of the angular momentum vector, $\hat{L}_j$, and a scaled quantum Runge-Lenz vector, $\hat{A}_j$, as generators of the group, whereas the other decouples these six generators into two sets, $\hat{M}_j = \frac{1}{2}(\hat{L}_j + \hat{A}_j)$, and $\hat{N}_j = \frac{1}{2}(\hat{L}_j - \hat{A}_j)$. In the second representation, one finds that SO(4) =

SO(3) $\otimes$ SO(3), so that, loosely speaking, a Perelomov coherent state for SO(4) may be given by the direct product of two SO(3) coherent states.

5.3.1 Coherent States of the Degeneracy Group

The SO(4) coherent states are quite interesting in their own right. Such coherent states have been discussed in the past. The present construction is most similar to that of Gay, Delande and Bommier [33] although they are not interested in the completeness of the states and obtain their expressions by considering special cases. Though he uses a different construction, much the same may be said of Nauenberg's construction [53].

The SO(3) coherent states with the fiducial vector $|j,-j\rangle$, $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$, are given by Eq. (4.10),

$$|j,\zeta\rangle = \sum_{m=-j}^{j} \left[\frac{(2j)!}{(j+m)!(j-m)!}\right]^{1/2} \frac{\zeta^{j+m}}{(1+|\zeta|^2)^j}|j,m\rangle, \tag{5.6}$$

and the resolution of the identity for these states is written

$$\hat{1}_j = \frac{2j+1}{\pi}\int \frac{d^2\zeta}{(1+|\zeta|^2)^2}|j,\zeta\rangle\langle j,\zeta|, \tag{5.7}$$

where $d^2\zeta = d\mathrm{Re}(\zeta)d\mathrm{Im}(\zeta)$, and integration is over the entire complex plane projected from the unit sphere with the transformation $\zeta = -\tan\frac{\theta}{2}e^{-i\phi}$. Expectation values of the operator components are given by (in atomic units)

$$\langle\hat{\mathbf{J}}\rangle = \frac{2j}{1+|\zeta|^2}\left(\mathrm{Re}(\zeta), -\mathrm{Im}(\zeta), \frac{1}{2}(|\zeta|^2-1)\right). \tag{5.8}$$

In the hydrogenic realization, the representations of each copy of SO(3) are of equal dimension ($n = 2j + 1$) so the dimensions of the relevant representations of SO(4) are $d_n = n^2$, $n = 1, 2, 3, \ldots$. The SO(4) coherent states are thus

$$|n, \zeta_1, \zeta_2\rangle = |j, \zeta_1\rangle|j, \zeta_2\rangle, \tag{5.9}$$

where the individual SO(3) coherent states are given by the expansion in Eq. (5.6). This leads to the direct product states $|j, m_1\rangle|j, m_2\rangle$ which may be expressed in terms of the standard angular momentum states via Clebsch-Gordon coefficients.

To examine the properties of these states, with $\hat{\mathbf{L}} = \hat{\mathbf{M}} + \hat{\mathbf{N}}$ and $\hat{\mathbf{A}} = \hat{\mathbf{M}} - \hat{\mathbf{N}}$ one has from Eq. (4.15)

$$\begin{aligned}
\langle \hat{L}_1 \rangle &= \frac{2j\mathrm{Re}(\zeta_1)}{1+|\zeta_1|^2} + \frac{2j\mathrm{Re}(\zeta_2)}{1+|\zeta_2|^2}, \\
\langle \hat{L}_2 \rangle &= -\frac{2j\mathrm{Im}(\zeta_1)}{1+|\zeta_1|^2} - \frac{2j\mathrm{Im}(\zeta_2)}{1+|\zeta_2|^2}, \\
\langle \hat{L}_3 \rangle &= \frac{j(|\zeta_1|^2-1)}{1+|\zeta_1|^2} + \frac{j(|\zeta_2|^2-1)}{1+|\zeta_2|^2},
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
\langle \hat{A}_1 \rangle &= \frac{2j\mathrm{Re}(\zeta_1)}{1+|\zeta_1|^2} - \frac{2j\mathrm{Re}(\zeta_2)}{1+|\zeta_2|^2}, \\
\langle \hat{A}_2 \rangle &= -\frac{2j\mathrm{Im}(\zeta_1)}{1+|\zeta_1|^2} + \frac{2j\mathrm{Im}(\zeta_2)}{1+|\zeta_2|^2}, \\
\langle \hat{A}_3 \rangle &= \frac{j(|\zeta_1|^2-1)}{1+|\zeta_1|^2} - \frac{j(|\zeta_2|^2-1)}{1+|\zeta_2|^2},
\end{aligned} \tag{5.11}$$

so that by choosing ζ_1 and ζ_2 appropriately, one may define states of a specified angular momentum and Laplace-Runge-Lenz vector. Of course, these six elements above are not all independent. For example, $\langle \hat{\mathbf{L}} \rangle$ and $\langle \hat{\mathbf{A}} \rangle$ must be perpendicular,

since $\hat{\mathbf{L}} \cdot \hat{\mathbf{A}} = \hat{M}^2 - \hat{N}^2$, and in this realization, $\hat{M}^2 - \hat{N}^2 = 0$. In fact, this is simply one of the Casimir operators of SO(4). The other Casimir operator provides the other constraint,

$$\langle \hat{M}^2 + \hat{N}^2 \rangle = \frac{1}{2}\langle \hat{L}^2 + \hat{A}^2 \rangle = 2j(j+1) = n^2 - 1, \tag{5.12}$$

leaving four degrees of freedom among the six components, specified by the real and imaginary parts of ζ_1 and ζ_2. Since the fiducial vectors are states of minimal fluctuations, these states are states of minimal fluctuation in terms of the rotated four dimensional angular momentum vectors.

In order to visualize a few examples of SO(4) coherent states, it is useful to consider those with $\langle \hat{\mathbf{L}} \rangle$ parallel to the z-axis. From Eq. (5.10), this is accomplished by setting $\zeta_2 = -\zeta_1$. With this identification, the Laplace-Runge-Lenz vector is on the x-y plane. Since the system is rotationally invariant, the general properties of these states can be completely characterized by considering only those states with the Laplace-Runge-Lenz vector parallel to the x-axis. From Eq. (5.11), this is accomplished by setting $\mathrm{Im}(\zeta_1) = \mathrm{Im}(\zeta_2) = 0$. These identifications reduce the problem to one real degree of freedom, $\mathrm{Re}(\zeta_1) = \eta$. In terms of this parameter, the expectation values are given by

$$\langle \hat{\mathbf{L}} \rangle = \frac{2j}{1+\eta^2}\left(0, 0, \eta^2 - 1\right), \tag{5.13}$$

$$\langle \hat{\mathbf{A}} \rangle = \frac{2j}{1+\eta^2}\left(2\eta, 0, 0\right). \tag{5.14}$$

Since these are states with minimal fluctuations about these values, the quantum

wave functions should be close to the classical trajectories described by these vectors. Thus, the eccentricity, proportional to the magnitude of $\langle \hat{\mathbf{A}} \rangle$, is given by $\epsilon = 2\eta/(1+\eta^2)$, depicted in Figure 5.1. Note that $\epsilon \leq 1$ for all η, consistent with this being a bound state.

Describing the quantum wave function in terms of the corresponding classical orbit, $\eta = 0$ leads to a circular orbit ($\epsilon = 0$). This also follows since only the extremal state is preserved in the coherent state, Eq. (5.9), with this choice of parameter. With η increasing, the orbit elongates in the x direction. Since the energy is fixed, the semi-major axis is constant, but the semi-minor axis shrinks according to $b = a\sqrt{1-\epsilon^2}$. When $\eta = 1$, the eccentricity becomes one, and the angular momentum goes to zero. The quantum state given by these parameters will be dominated by interference, with the incoming and outgoing portions of the wave function interfering with one another. As η exceeds one, the eccentricity reduces, and the angular momentum changes sign.

It is well known that the orbit of a particle in the Coulomb potential in position space is an ellipse with one focus at the origin. Using the position space representation of eigenstates, a sample SO(4) coherent state is depicted in Figure 5.2. It is less well known that the orbit in momentum space is a circle, displaced from the origin a distance proportional to the eccentricity. Using the momentum space representation of eigenstates [42], the same state as in Figure 5.2 is depicted in momentum space in Figure 5.3.

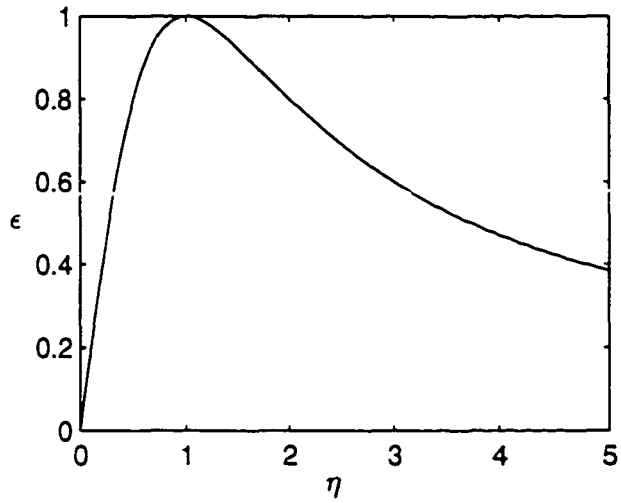


Figure 5.1: The eccentricity as a function of η.

5.3.2 Hydrogenic Coherent States

It is now straightforward to construct the coherent states for the full system. The coherent states for the hydrogen atom problem by this construction are given by

$$|s,\gamma,\zeta_1,\zeta_2\rangle = N(s^2)\sum_{n=0}^{\infty}\frac{s^n\exp(-i\gamma e_{n+1})(n+1)}{\sqrt{\rho_n}}|n+1,\zeta_1,\zeta_2\rangle. \tag{5.15}$$

The states Eq. (5.15) satisfy the resolution of the identity

$$\hat{1}_B = \frac{1}{\pi^2}\int d\mu(s,\gamma)\int\frac{d^2\zeta_1 d^2\zeta_2}{(1+|\zeta_1|^2)^2(1+|\zeta_2|^2)^2}|s,\gamma,\zeta_1,\zeta_2\rangle\langle s,\gamma,\zeta_1,\zeta_2|, \tag{5.16}$$

where the subscripted B is included to emphasize that this is more appropriately regarded as a projection operator into the bound portion of the Hilbert space. In the specific example of $\rho(u) = e^{-u}$, with moments $\rho_n = n!$, explicit form may be

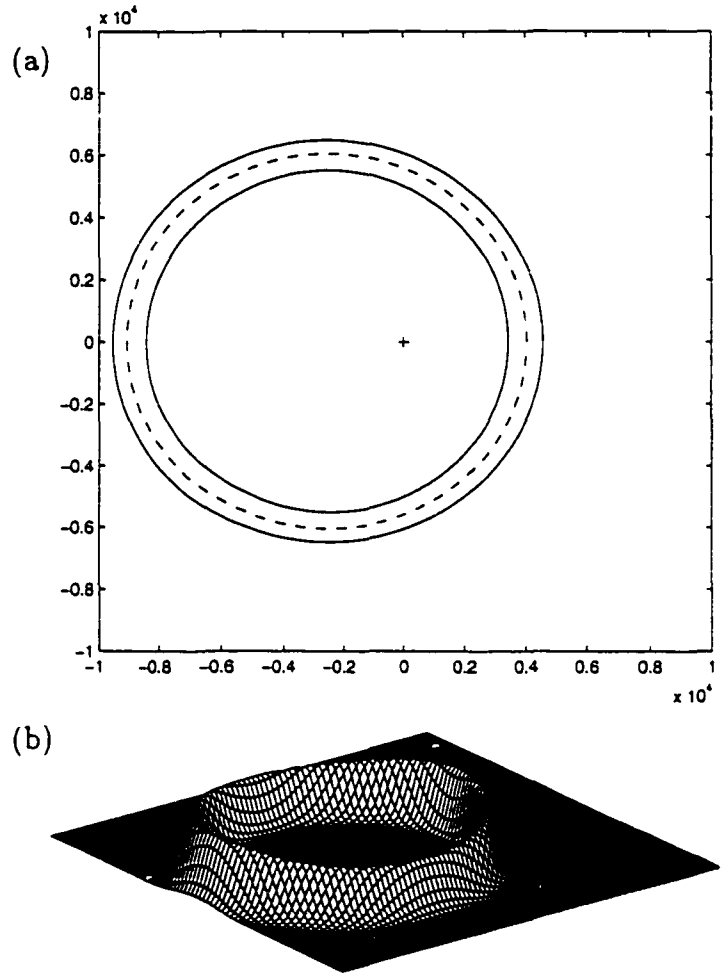


Figure 5.2: An SO(4) coherent state, on the 81st energy level, with the angular momentum parallel to the z-axis and with an eccentricity of $\epsilon = 0.385$, shown on a cut through the state on the x-y plane. (a) The solid lines are a half-height contour and the dashed line is an ellipse with the given eccentricity, *i.e.* the classical orbit with the corresponding parameters. (b) This is the same state on the x-y plane with $|\langle \mathbf{r}|n, \zeta_1, \zeta_2\rangle|^2$ plotted on the vertical axis.

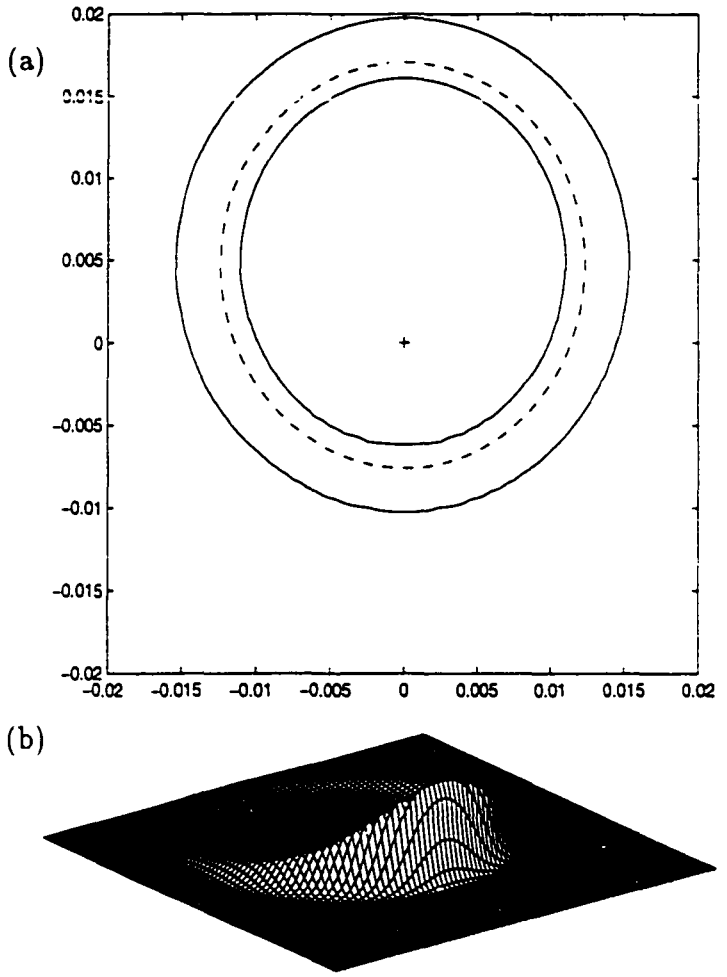


Figure 5.3: An SO(4) coherent state in momentum space, on the 81st energy level, with the angular momentum parallel to the z-axis and with an eccentricity of $\epsilon = 0.385$, shown on a cut through the state on the p_x-p_y plane. (a) The solid lines are a half-height contour and the dashed line is a circle displaced from the origin a distance proportional to the eccentricity, *i.e.* the classical orbit with the corresponding parameters. (b) This is the same state on the p_x-p_y plane with $|\langle \mathbf{p}|n, \zeta_1, \zeta_2\rangle|^2$ plotted on the vertical axis.

given to $N(s^2)$ and $k(u)$ by

$$N(s^2) = e^{-s^2/2}(1 + 3s^2 + s^4)^{-1/2} \tag{5.17}$$

and

$$k(u) = 1 + 3u + u^2. \tag{5.18}$$

5.3.3 Some Clarification

At this point, a few observations are in order. Primarily, the term "temporal stability" in no way refers to the time evolution of the structure in configuration space. Only through a rather generous interpretation does this construction "positively" solve the long-standing problem of forming non-dispersing wave packets for the hydrogen atom. Temporal stability refers strictly to the mathematical property that the states evolve in time among themselves. With this property in mind, some authors [47] have grossly overstated the nature of the configuration space time evolution, while other authors [12, 13] have studied in detail the long-time evolution of individual states, even though there is no underlying physical basis either to provide for spatial coherence, or to presume states of this description are found in the laboratory at all. The question of how to prepare these states in the laboratory remains very much open.

Much of the study of generalized coherent states rests more in the mathematical than the physical nature of mathematical physics. Glauber's motivation in this study of coherent states [39] was not so much that coherent states are found in the laboratory, but that they provide a representation in which otherwise difficult cal-

culations become feasible. Glauber noted that certain electric field operators have representations as sums over the modal annihilation operators. In diagonalizing these operators, one arrives at states which are eigenstates of these annihilation operators. Restricted to a single mode, this corresponds to the annihilation operator definition of harmonic oscillator coherent states. Hence as annihilation operator coherent states, they arise from a representation, a point of mathematical convenience, not as a conclusion from the physics of the problem. In any case, generalizations of annihilation operator coherent states have appeared widely, though the physical motivation to study such definitions in any context besides as representations is unclear.

Glauber also showed how these states may be constructed through the action of a displacement operator on the ground state. This definition was generalized by Perelomov [67], a generalization which has been widely successful. This success is founded upon the properties of the dynamical group coming through into the set of coherent states, not from an assertion (which few researchers make) that an individual state by such a construction matches a state by some preparation in the laboratory. This success is of a mathematical, not physical, nature, again resting upon the use of these states as a representation.

Of Glauber's original three definitions, the approach which appears to invest the most physics is the minimum uncertainty construction. Indeed, squeezed states, a generalization of this construction, are used as descriptions of physical aspects of certain quantum optical experiments in the laboratory. Nieto *et al.* [61] have also developed a generalization which minimizes the uncertainty product of a pair

of "natural" operators. Ehrenfest's relations then lead to the initial evolution of the quantum expectation values approximating classical evolution. Though this approach is strongest in terms of an underlying physical motivation, these states still lack (in general) any physical hypothesis which selects for states of this description in the laboratory. As an aside, the Nieto construction, though seen from time to time, is not as widely used as the Perelomov construction for perhaps two reasons. Firstly, though it is generally applicable in principle, many systems are intractable to carry through to completion (when the Hamiltonian enters into the "natural" operators). Secondly, it appears a certain tradeoff is at work: This approach is considerably less mathematically endowed than Perelomov's approach.

Now consider Klauder's construction. All of the attractions are mathematical in nature. As initially presented, no reservations are made for coherence in configuration space (*i.e.* semi-classical behaviour) and there is no general physical mechanism which would result in finding these states in the laboratory. However, a certain degree of freedom remains in the construction, and two suggestions have separately appeared that a fourth requirement will simultaneously eliminate the degree of freedom and ensure for the behaviour in configuration space [47, 34]. It is likely that a fourth requirement, if it exists, will be physical in nature. The requirement postulated by Majumdar and Sharatchandra [47] is that the measure found in the resolution of the identity corresponds to the "canonical" measure on classical phase space. They further assert that the measure uniquely identifies the set of coherent states. This assertion is false, as shown by Sixdeniers *et al.* [74] who demonstrate multiple measures corresponding to the same set of coherent states.

Also, though it is convenient from a mathematical point of view, it is unclear why the measures should correspond to one another at all from a physical point of view, or even if a meaningful identification (one to one) can always be made between individual coherent states and points in classical phase space.

A fourth requirement is also postulated by Gazeau and Klauder [34] which is motivated by an attempt to formalize the connection between the quantum parameters to the coherent state and the classical action-angle variables. Unfortunately, their requirement results in an angle variable whose rate of change with time is independent of the action, a rather special circumstance restricted to the harmonic oscillator and a small number of other systems. This is a severe limitation in terms of semi-classical behaviour, since this is clearly at odds with how the angle variable evolves in, say, the hydrogen atom problem.

A degree of freedom also remains in Fox's construction [27] of Gaussian generalized coherent states, namely the width of the Gaussian in question. Note that in this context, the distribution in energy level, not configuration space, is Gaussian. Fox does not give any criteria which are intended to specify a suitable width. As with the Klauder's construction, this degree of freedom may be optimized according to the aim in mind.

Hence, in the absence of an acceptable fourth criterion (none is herein proposed), we carry on. This limits the construction to a mathematical tool, though an interesting mathematical tool it is. Note that the time dependent Schrödinger equation for a time independent Hamiltonian in this coherent state representation

becomes (in atomic units)

$$\frac{\partial}{\partial\gamma}\langle s,\gamma|\psi\rangle = -\frac{\partial}{\partial t}\langle s,\gamma|\psi\rangle \tag{5.19}$$

whose solution is trivial. With this expression representing the unperturbed solution, this would make for an interesting starting point in a study of time dependent perturbation theory. That is, coherent states are useful when considered as an ensemble of states, not as individual states.

Speaking now in the specific case, some authors [12, 13] have suggested that the temporally stable construction of coherent states does not support the possibility of exhibiting full or fractional revivals as described by Averbukh and Perelman [4] or Nauenberg [53]. Firstly, before one decides whether a state is to be found in the laboratory, one should first postulate a physical mechanism for the preparation of these states: The presence or absence of phenomenon which is, after all, universal is not relevant. Secondly, these authors did not exploit the degree of freedom which remains in the construction. Without supplying a physical motivation which would lead to finding these states in the laboratory, we shall see that by exploiting this degree of freedom wave functions may be formed by the present construction which exhibit the full panoply of revivals.

5.3.4 Dynamics

Having thus constructed the states emphasizing, among other things, time evolution, it is now interesting to consider the behaviour of these states as evolved in time. Other authors have defined hydrogen atom coherent states with a variety

of constructions and with various reports of evolution in "fictitious" time [36], or evolution along circular [21, 31] or Keplerian elliptical orbits with possible, eventual state revivals [33, 53]. Coherent states also may be constructed by the present recipe which travel along elliptical orbits and exhibit fractional revivals.

According to Averbukh and Perelman [4], fractional revivals are a universal phenomenon exhibited by wave functions provided third order corrections and higher do not contribute significantly to a polynomial approximation to the energy eigenvalues over contributing energy eigenstates. Expanding about $n = \bar{n}$ (using the more compact $\bar{n} \equiv \langle n \rangle$), the hydrogen atom energy levels are

$$e_n = -\frac{1}{2n^2} = \tag{5.20}$$
$$-\frac{1}{2\bar{n}^2} + \frac{1}{\bar{n}^3}(n-\bar{n}) - \frac{3}{2\bar{n}^4}(n-\bar{n})^2 + \frac{2}{\bar{n}^5}(n-\bar{n})^3 + \cdots .$$

In this expansion the first term leads to an overall (time dependent) phase, and so can be ignored for the purposes of this discussion. Starting at time $t = 0$, the first order term leads to initial classical behaviour. Note that this term contributes integer multiples of 2π in phase after a time of $t = 2\pi\bar{n}^3$. (Observe that with circular states, $\langle r \rangle = (2n^2 + n)/2$ which in conjunction with this time leads to the quantum version of Kepler's third law.) Classically, a particle in the Coulomb potential with an energy of $E = -(2\bar{n}^2)^{-1}$ has a semi-major axis of $\bar{n}^2$ and hence a period of $T_{Cl} = 2\pi\bar{n}^3$, corresponding to the above. (This may further be established through Bohr-Sommerfeld quantization.) After one half-period, phase differences equal integer multiples of π. Hence, after one classical period, one expects the state to be more or less reformed, depending on the dephasing for which the quadratic

and larger terms are responsible at the edges of the number distribution. The quadratic contributions at the edges of the distribution ($n = \bar{n} \pm \Delta n$) are small after one period provided $3\pi(\Delta n)^2 \ll \bar{n}$, which is the first indication that a narrow wave packet is less inclined to disperse in configuration space. This argument is usually phrased in terms of the fact that the energy levels are more evenly spaced in the hydrogen atom spectrum at large quantum numbers.

The same argument may be applied to the quadratic term in the expansion Eq. (5.20). Again starting at time $t = 0$, the linear term gives the initial near-periodicity provided the dephasing by the quadratic term is small. Eventually, this dephasing will not be small. However, after a sufficiently long time, the dephasing contributions will be on the order of π. In fact, at some time T_r with

$$\frac{3}{2\bar{n}^4}(n-\bar{n})^2 T_r = k^2\pi, \quad T_r = \frac{2\pi}{3}\bar{n}^4, \tag{5.21}$$

for some integer k, the phase contributions will be integer multiples of π. (Since phase angles are equivalent up to multiples of 2π, the k^2 term acts equivalently to k, which is equivalent to the sequence $\{0, 1, 0, \ldots\}$.) This represents the initial state evolved one half period from the initial time so that if the initial behaviour is of a more or less coherently moving, localized wave packet, the behaviour in the neighbourhood of T_r will be likewise, provided the cubic term is sufficiently small at the edges of the distribution, *i.e.*,

$$4\pi(\Delta n)^3 \ll 3\bar{n}, \tag{5.22}$$

which asserts that the number distribution must be narrow if wave function revivals are to be observed. Note that $T_r = T_{rev}/2$ in Averbukh and Perelman's notation, who consider integer multiples of 2π, rather than π as above.

These considerations may be carried on *ad infinitum* with successive terms in the energy level expansion, yielding super-revivals and so on, but the same general behaviour will result for successively longer times: An initially localized wave packet which has long since dispersed will reappear out of nowhere. This phenomenon has generated a great deal of interest since there is no classical counterpart. Classically, a smooth distribution of trajectories will disperse and remain so.

At the heart of the coherent states described above lies the function $\rho(u)$. Given the distribution function $\rho(u) = e^{-u}$ with moments $\rho_n = n!$, the hydrogenic coherent states, Eq. (5.15) with Eq. (5.17), leads to a eigenlevel distribution characterized by

$$\begin{aligned} \langle n \rangle &= \frac{e^{-s^2}}{s^4 + 3s^2 + 1} \sum_{n=0}^{\infty} \frac{n s^{2n} (n+1)^2}{n!} \\ &= s^2 \frac{s^4 + 5s^2 + 4}{s^4 + 3s^2 + 1}. \end{aligned} \tag{5.23}$$

This summation follows from the usual collection of techniques involving differentiating by s, shifting indices, and comparing with the expansion of $\exp(-s^2)$. By similar techniques, one obtains

$$(\Delta n)^2 = s^2 \frac{s^8 + 6s^6 + 14s^4 + 10s^2 + 4}{s^8 + 6s^6 + 11s^4 + 6s^2 + 1}, \tag{5.24}$$

so that, taking leading order behaviour, $\Delta n \sim \sqrt{\langle n \rangle}$. Substituting into the above

necessary condition for a revival at $t = T_r$ gives $4\pi\sqrt{\langle n\rangle}/3 \ll 1$ which is never satisfied since $\langle n\rangle \geq 1$.

However, the function $\rho(u)$ is a "degree of freedom" in the construction, and may be chosen according to application in mind. Accordingly, consider instead $\rho(u) = \exp(-u^\alpha)$ for some constant $\alpha > 0$ with a view to constructing wave packets which exhibit strong revivals. The moments of this function are

$$\rho_n = \int_0^\infty u^n \exp(-u^\alpha) du = \frac{1}{\alpha}\Gamma(\frac{n+1}{\alpha}). \tag{5.25}$$

Many of the expressions involved in subsequent calculations may be handled using properties of the functions of Mittag-Leffler [11], though they will be treated instead by comparisons to expressions following from $\rho(u) = e^{-u}$. In fact, using $\rho(u) = \exp(-u^\alpha)$ results in a set of coherent states closely related to those described by Sixdeniers *et al.* [74].

Expressions for $\langle n\rangle$ and $(\Delta n)^2$ may be approximated by recognizing the scalings necessary to map expressions with $\alpha = 1$ onto those with general α:

$$n + 1 \to (n+1)/\alpha, \quad s \to s^\alpha. \tag{5.26}$$

Hence, one obtains to leading order (large values of s will eventually be involved)

$$\langle n\rangle \sim \alpha s^{2\alpha}, \quad \Delta n \sim \alpha s^\alpha, \tag{5.27}$$

so that $\Delta n \sim \sqrt{\alpha\langle n\rangle}$. Substituting this into the minimal condition for the first full

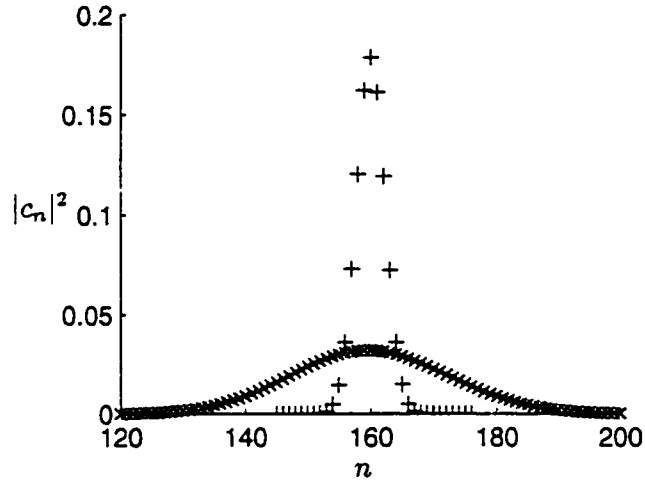


Figure 5.4: Distributions in energy level for $\alpha = 1$ ($\times$) and $\alpha = 1/32$ ($+$).

revival gives

$$4\pi\alpha^{3/2}\sqrt{\langle n\rangle} \ll 3, \tag{5.28}$$

which may be satisfied if α is chosen sufficiently small. Without discussing the effect of changing α very much further, there will be a tradeoff between large and small α: Large α will introduce many significantly contributing energy levels for a given $\langle n\rangle$ yielding good spatial localization, but weak or non-existent revivals, whereas small α yields strong revivals of poorly localized states. Note that in this construction a small width in n follows from an appropriate choice for $\rho(u)$, whereas the same may be accomplished by Fox's construction [27] by simply specifying the width to be narrow.

As a typical example, consider the state depicted in Figures 5.5 to 5.8. The energy level distribution is shown in Figure 5.4. For this state, $\alpha = 1/32$ and $s =$

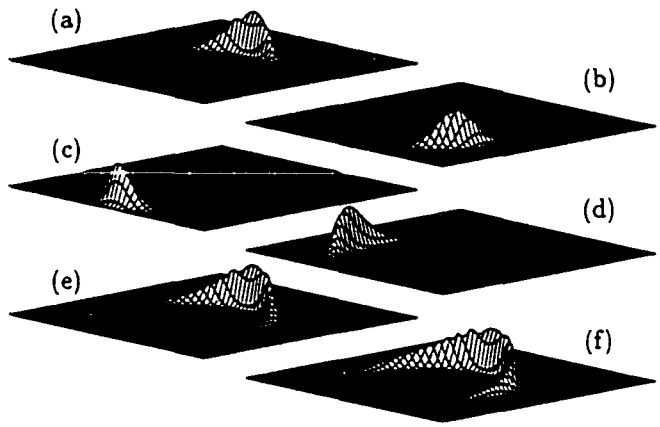


Figure 5.5: A hydrogenic coherent state depicted on the x-y plane, evolved shown at times (a) $t = 0$, (b) $t = T_{Cl}/4$, (c) $t = T_{Cl}/2$, (d) $t = 3T_{Cl}/4$, (e) $t = T_{Cl}$, and (f) $t = 2T_{Cl}$, where T_{Cl} is the classical period.

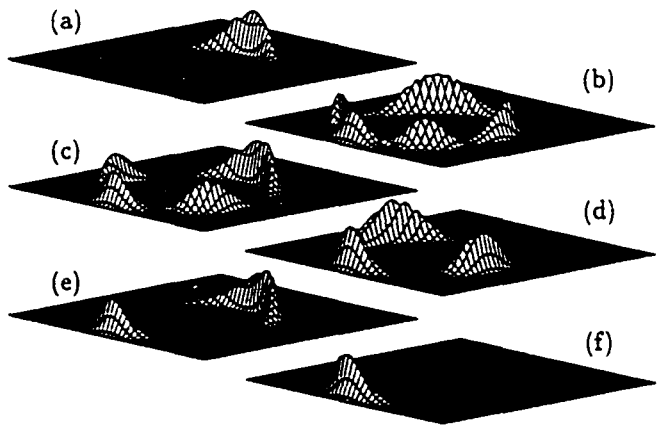


Figure 5.6: A hydrogenic coherent state depicted on the x-y plane, evolved shown at times (a) $t = 0$, (b) $t = T_r/5$, (c) $t = T_r/4$, (d) $t = T_r/3$, (e) $t = T_r/2$, and (f) $t = T_r$.

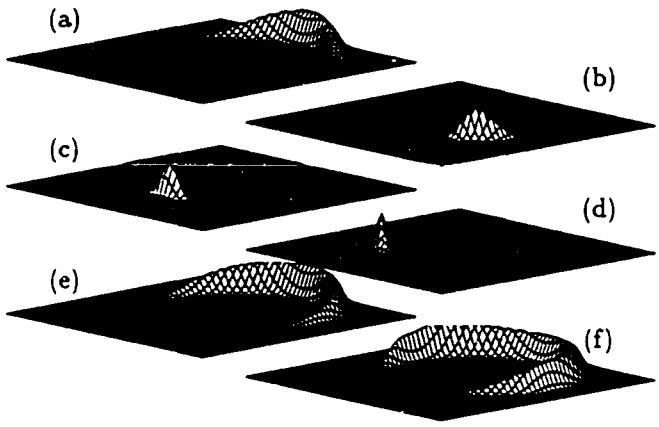


Figure 5.7: A hydrogenic coherent state depicted on the p_x-p_y plane in momentum space, evolved shown at times (a) $t = 0$, (b) $t = T_{Cl}/4$, (c) $t = T_{Cl}/2$, (d) $t = 3T_{Cl}/4$, (e) $t = T_{Cl}$, and (f) $t = 2T_{Cl}$.

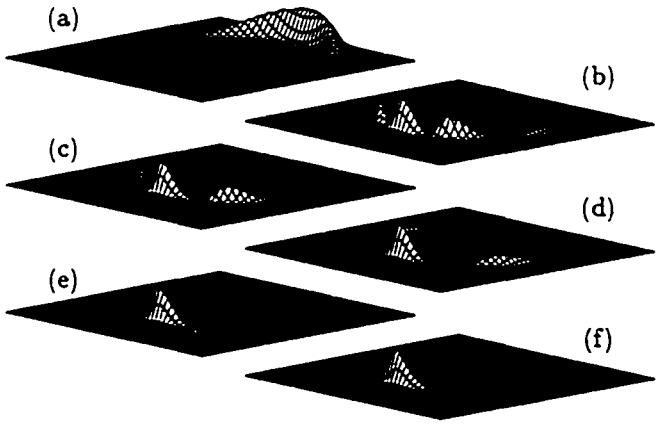


Figure 5.8: A hydrogenic coherent state depicted on the p_x-p_y plane in momentum space, evolved shown at times (a) $t = 0$, (b) $t = T_r/5$, (c) $t = T_r/4$, (d) $t = T_r/3$, (e) $t = T_r/2$, and (f) $t = T_r$.

2.209×10^{59}. This results in a state centred at $\langle n \rangle = 160$ with a width of $\Delta n = \sqrt{5}$ for which one expects a full revival at $t = T_r = 1.373 \times 10^9$. The chosen parameters $\zeta_{1,2}$ to the SO(4) coherent state provide an elliptical orbit with eccentricity $\epsilon = 0.385$, the major axis parallel to the x-axis, and the angular momentum parallel to the z-axis, so that the state is constrained to a narrow region about the x-y plane in position space and the p_x-p_y plane in momentum space. The vertical axes are amplitudes of the wave functions in position and momentum space, calculated at the times indicated on a square grid 80000 units in width centred at the origin in position, and 0.02 units in width in momentum space.

The evolution of this state is as expected. Initially localized, the state evolves semi-classically. The wave function then spreads out but remains close to the ellipse. As the expected times for the various fractional revivals arrive, the state exhibits the expected revival including the full revival at $t = T_r$, even though an examination of the minimal condition for the first revival gives the dubious result $0.29 \ll 1$. Revivals are observed in position space as well as momentum space.

Figure 5.10 depicts the autocorrelation function for the same state as above, exhibiting the typical pattern characterizing revivals (compare with Figure 2 of Parker and Stroud [66]). Compare Figure 5.10 with Figure 5.9 in which the autocorrelation function is depicted for a similar state calculated with $\alpha = 1$. This latter calculation is analogous to the calculations carried out by Bellomo and Stroud [12, 13], clearly showing that no revival is apparent at the expected revival time of $T_r = 1.373 \times 10^9$.

The minimal condition, Eq. (5.28), for observing revivals is worthy of a further point. Note that with a fixed α, states with larger $\langle n \rangle$ are less likely to exhibit

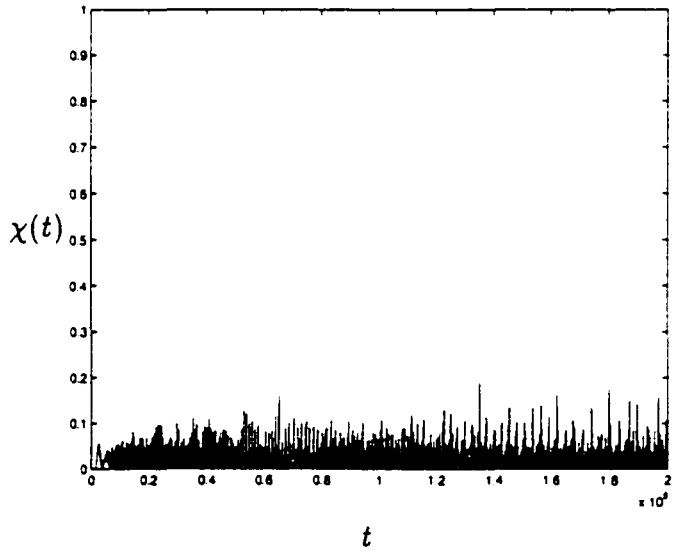


Figure 5.9: Autocorrelation function $\chi(t) = |\langle\psi(0)|\psi(t)\rangle|^2$ at time t of a coherent state with $\alpha = 1$. The parameters were so chosen to result with a state of the same $\langle n\rangle$, same eccentricity and so forth as depicted in Figures 5.5 to 5.8. The revival time is $T_r = 1.373 \times 10^9$.

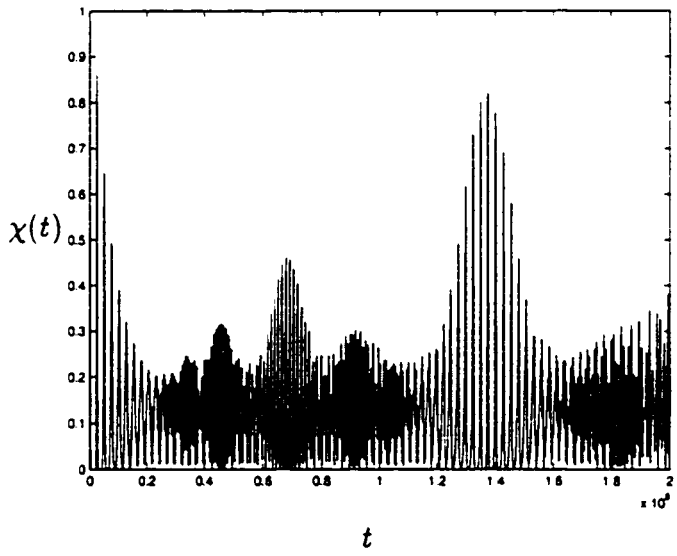


Figure 5.10: Autocorrelation function $\chi(t) = |\langle\psi(0)|\psi(t)\rangle|^2$ at time t of the state depicted in Figures 5.5 to 5.8. This state employs $\alpha = 1/32$ and hence has a narrow distribution in n. The revival time is $T_r = 1.373 \times 10^9$.

revivals. Strictly speaking, this is not due to the idea that one expects classical behaviour in the large quantum number regime, and revivals are a quantum phenomenon. This suppression of revivals has more to do with the relationship between α and $\langle n \rangle$ in this definition of coherent state. One could easily imagine a sequence of states with increasing $\langle n \rangle$ and decreasing α such that revivals continue to be observed in the large quantum number regime. In any case, note that for increasing $\langle n \rangle$, the revival time is of order $O(\langle n \rangle^4)$ — after very long times. Though the cubic term in the energy level expansion is smaller at larger $\langle n \rangle$, the dephasing that results from it is larger due to the long time involved.

Commenting again on the assertion by some authors that "temporally stable" states for the hydrogen atom can not exhibit this phenomenon [12, 13], their calculations involved, in present terms, $\alpha = 1$, hence wide distributions in n which exhibited no appreciable revivals over the time frames calculated. Wave function revivals are a universal phenomenon depending on the nature of the energy eigenlevel spacings, and in the case of the hydrogen atom, the width Δn. By exploiting the fact that one may choose $\rho(u)$ to one's liking, Δn may be reduced such that the resultant states do revive. Further, one study [12] used values of s leading to $\langle n \rangle$ equal to 25 and 400. With respect to the second calculation, from Eq. (5.21) $T_r \sim 5.36 \times 10^{10}$, though their calculation only extends to $t = 5 \times 10^9$. For the other study [13], values of $\langle n \rangle$ exhibited were 10 and 200. A full revival should be found at about $t = 3.35 \times 10^9$ though their calculation only extends to $t = 5 \times 10^8$. As shown in Figure 5.9, even if calculations are carried out over the appropriate time frame, no revivals are apparent.

5.4 Conclusion

This approach to defining temporally stable coherent states in degenerate systems, in particular the hydrogen atom problem, is similar in spirit to those of Majumdar and Sharatchandra [47], Klauder [45], and Fox [27]. Majumdar and Sharatchandra make explicit use of the SO(4) degeneracy group, but treat the factor d_n differently. Although Klauder's construction is specific to the hydrogen atom problem rather than the present general approach, a similar term appears in that construction which disrupts normalization. Fox uses a somewhat different construction which avoids the use of the moments of some function $\rho(u)$. Even so, one may construct coherent states in degenerate systems related to Fox's construction as the present construction is related to Klauder's original work.

The general construction of Eq. (5.1) provides states with many useful properties. They form a complete set of states (in the bound portion of the spectrum) and evolve in time among themselves. This makes them a clear candidate for use in representations of time evolved, bound states. Further, there is a freedom in their definition which stems from the choice of $\rho(u)$. If $\rho(u)$ is appropriately chosen, hydrogen atom coherent states may be defined which exhibit the full range of phenomena exhibited by other approaches: initial semi-classical behaviour, interference between the head and tail of the state as it disperses about the Keplerian ellipse, localization on the Keplerian ellipse and wave function revivals at predictable times.

The salient difference between these coherent states and other constructions is the natural and explicit manner in which the energy degeneracies are treated herein, via Perelomov's group theoretical construction of generalized coherent states.

Appendix A

SO(3) and SO(2,1)

Throughout this thesis, frequent use is made of the compact Lie group SO(3) and the closely associated non-compact SO(2,1). In the present context, the frequent occurrence of SO(3) is due to the fact that the quantum angular momentum operators $\hat{J}_k$ can act as generators for the group, *i.e.* as a basis for the associated Lie algebra so(3). In that sense, much of what is said also goes for SU(2), to which SO(3) is locally isomorphic. The group SO(2,1) arises in two separate contexts: being generated by the "hyperbolic" angular momentum that comes out of the bosonic realization of angular momentum, and also as a spectrum generating algebra for the hydrogen atom and the isotropic harmonic oscillator. Accordingly, the algebra receives more attention than the group, and again, much of what is said of SO(2,1) also applies to SU(1,1), to which it is locally isomorphic.

A.1 SO(3)

In Euclidean three dimensional space, the classical angular momentum vector is given by $\mathbf{J} = \mathbf{x} \times \mathbf{p}$. Similarly, the quantum vector is given by $\hat{\mathbf{J}}' = \hat{\mathbf{x}} \times \hat{\mathbf{p}}$. Using the standard commutation relations $[\hat{x}_j, \hat{p}_k] = i\hbar\delta_{jk}$, where the δ_{jk} is the Kronecker delta and $j, k = 0, 1, 2, 3$, one may verify the angular momentum commutation relation

$$[\hat{J}'_j, \hat{J}'_k] = i\hbar\epsilon_{jk\ell}\hat{J}'_\ell. \tag{A.1}$$

where $\epsilon_{jk\ell}$ is the completely antisymmetric Levi-Civita tensor. Introducing the scaling

$$\hat{J}_j = \hat{J}'_j/\hbar \tag{A.2}$$

gives the commutation relation

$$[\hat{J}_j, \hat{J}_k] = i\epsilon_{jk\ell}\hat{J}_\ell. \tag{A.3}$$

This is the standard commutation relation for the Lie algebra so(3) whose Casimir operator is $\hat{J}^2 = \hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2$, the total angular momentum squared. Since $\hat{J}^2$ commutes with all the components, we choose by convention to consider joint eigenstates of $\hat{J}^2$ and $\hat{J}_3$.

Most textbooks on quantum mechanics note these commutation relations, but few point out that these are the commutation relations for the Lie algebra so(3) [76]. Much of what follows is thus fairly standard in the literature, but frequently, the connection to Lie algebras and Lie groups is frankly ignored.

To assist in working out the nature of these irreducible representations of so(3), consider the raising and lowering operators

$$\hat{J}_{\pm} = \hat{J}_1 \pm i\hat{J}_2 \tag{A.4}$$

which satisfy the commutation relations

$$[\hat{J}_3, \hat{J}_{\pm}] = \pm\hat{J}_{\pm}, \qquad [\hat{J}_+, \hat{J}_-] = 2\hat{J}_3. \tag{A.5}$$

Since $\hat{J}^2$ and $\hat{J}_3$ commute, they share eigenstates. Suppose $|m\rangle$ is an eigenstate of $\hat{J}_3$ with eigenvalue m and that $\hat{J}_{\pm}|m\rangle \neq 0$. Then, using the first of the above commutation relations

$$\hat{J}_3\hat{J}_{\pm}|m\rangle = (\hat{J}_{\pm}\hat{J}_3 \pm \hat{J}_{\pm})|m\rangle = (m \pm 1)\hat{J}_{\pm}|m\rangle. \tag{A.6}$$

so that $\hat{J}_{\pm}|m\rangle$ is an eigenstate of $\hat{J}_3$ whose eigenvalue is ± 1, that is, $|m \pm 1\rangle$. Since so(3) is a compact simple Lie algebra, unitary irreducible representations are finite dimensional. Therefore, there must be a largest eigenvalue, say j, of $\hat{J}_3$. This state will be annihilated by the creation operator:

$$\hat{J}_+|j\rangle = 0. \tag{A.7}$$

Operating on the left with $\hat{J}_-$ and expanding,

$$\hat{J}_-\hat{J}_+|l\rangle \;=\; (\hat{J}_1^2 - i(\hat{J}_2\hat{J}_1 - \hat{J}_1\hat{J}_2) + \hat{J}_2^2)|j\rangle$$

$$= (\hat{J}^2 - \hat{J}_3^2 - \hat{J}_3)|j\rangle = 0. \tag{A.8}$$

which rearranges to

$$\hat{J}^2|j\rangle = j(j+1)|j\rangle. \tag{A.9}$$

With m' now standing for the lowest eigenvalue (so that $\hat{J}_-|m'\rangle = 0$), a similar argument gives

$$\hat{J}^2|m'\rangle = m'(m'-1)|m'\rangle, \tag{A.10}$$

which signifies that $j(j+1) = m'(m'-1)$. The two solutions for m' are $m' = -j$ and $m' = j+1$, but the second solution is discarded due to the assumption that m' is the lowest eigenvalue, *i.e.* not greater than the largest eigenvalue.

The implication of these calculations is that the eigenvalues of $\hat{J}_3$ range from some largest value j to a lowest value $-j$, spaced along unit intervals. Note that this is a finite range, a reflection of the fact that SO(3) is a compact Lie group. Since whole numbers come in only two varieties (even and odd), j can only assume integer or half integer values. Since j identifies the representation, it will be used as a label along with m: These eigenstates will henceforth be denoted $|j,m\rangle$.

As observed above, the state $\hat{J}_\pm|j,m\rangle$ is proportional to $|j,m\pm 1\rangle$. The constants of proportionality (matrix elements) $c_\pm$ can be determined by considering the expression

$$\langle j,m|\hat{J}_\pm\hat{J}_\mp|j,m\rangle = \langle j,m|(\hat{J}^2 - \hat{J}_3^2 \pm \hat{J}_3)|j,m\rangle. \tag{A.11}$$

Since $\hat{J}_\pm$ is the Hermitian conjugate of $\hat{J}_\mp$, acting to the left with $\hat{J}_\pm$ gives $|c_\mp|^2$ on

the left hand side of the above. On the right, expanding the operators and acting to the right yields

$$|c_{\mp}|^2 = j(j+1) - m^2 \pm m. \tag{A.12}$$

Taking $c_{\pm}$ real and positive, and factoring the right,

$$\hat{J}_+|j,m\rangle = \sqrt{(j-m)(j+m+1)}|j,m+1\rangle, \tag{A.13}$$

$$\hat{J}_-|j,m\rangle = \sqrt{(j-m+1)(j+m)}|j,m-1\rangle. \tag{A.14}$$

Different sources express the matrix elements in different forms; the present form is intentionally chosen in relation to the bosonic realizations of angular momentum in Chapter 4 and Appendix B.

A.2 SO(2,1)

Much of this section is carried out in analogy to the previous section, though in the absence of any specific realization. Also, note that whereas discussions of angular momentum are standard in the literature, discussions of so(2,1) are existent [83, 1, 65] though much less common. To begin, consider the generators $\hat{T}_j$ for $j = 1, 2, 3$, which satisfy

$$[\hat{T}_1, \hat{T}_2] = -i\hat{T}_3, \quad [\hat{T}_2, \hat{T}_3] = i\hat{T}_1, \quad [\hat{T}_3, \hat{T}_1] = i\hat{T}_2, \tag{A.15}$$

differing from so(3) by the one sign. Continuing the analogy, representations will be explored in term of simultaneous eigenstates of $\hat{T}_3$ and the Casimir operator.

The Casimir operator is given by

$$\hat{T}^2 = \hat{T}_3^2 - \hat{T}_1^2 - \hat{T}_2^2. \tag{A.16}$$

(This is not simply copied from so(3) with a judicious change of sign, but is the result of the standard construction involving the Cartan-Killing form,

$$\kappa = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \tag{A.17}$$

which, since it is not negative definite, indicates that the group is not compact.) Let

$$\hat{T}_\pm = \hat{T}_1 \pm i\hat{T}_2, \tag{A.18}$$

so that

$$[\hat{T}_3, \hat{T}_\pm] = \pm\hat{T}_\pm, \quad \text{and} \quad [\hat{T}_+, \hat{T}_-] = -2\hat{T}_3. \tag{A.19}$$

Suppose $|b\rangle$ is an eigenstate of $\hat{T}_3$ with eigenvalue b. Then

$$\hat{T}_3\hat{T}_\pm|b\rangle = \hat{T}_\pm(\hat{T}_3 \pm 1) = (b \pm 1)\hat{T}_\pm|b\rangle, \tag{A.20}$$

which demonstrates how $\hat{T}_\pm$ acts as a ladder operator between eigenstates of $\hat{T}_3$.

Without any assumptions on the nature of the realizations or representations (in particular, whether or not the representations are unitary), the foregoing calculations give the general structure of the spectrum of $\hat{T}_3$. Like that of SO(3), the

spectrum will be given by $b_0 + k$ for k some integer. Without making additional assumptions, k remains unrestricted. To impose some restrictions, realizations of interest will be Hermitian, leading to unitary representations of the group. In this case, the eigenvalues b of $\hat{T}_3$ are real, and so is the eigenvalue of the Casimir operator. Further, $\hat{T}_\pm^\dagger = \hat{T}_\mp$, so that

$$\langle b|\hat{T}_+\hat{T}_-|b\rangle \geq 0, \quad \langle b|\hat{T}_-\hat{T}_+|b\rangle \geq 0. \tag{A.21}$$

Another restriction introduced is that there will be an eigenstate $|b_0\rangle$ annihilated by the operator $\hat{T}_-$: $\hat{T}_-|b\rangle = 0$. Operating on the left of this expression by $\hat{T}_+$ yields

$$\hat{T}_+\hat{T}_-|b_0\rangle = (\hat{T}_3^2 - \hat{T}^2 - \hat{T}_3)|b_0\rangle = 0. \tag{A.22}$$

Rearranging,

$$\hat{T}^2|b_0\rangle = b_0(b_0 - 1)|b_0\rangle. \tag{A.23}$$

which identifies the eigenvalue of the Casimir operator.

The matrix elements of the ladder operators may now be determined in analogy to those of so(3). Provided $\hat{T}_-|b\rangle \neq 0$, for some $|b\rangle$ and noting that $\hat{T}_+^\dagger = \hat{T}_-$, the same recipe may be used as for SO(3):

$$\begin{aligned} |c_-|^2 &= \langle b|\hat{T}_+\hat{T}_-|b\rangle = \langle b|(\hat{T}_3^2 - \hat{T}^2 - \hat{T}_3)|b\rangle \\ &= (b - b_0)(b + b_0 - 1), \end{aligned} \tag{A.24}$$

and

$$\begin{aligned} |c_+|^2 &= \langle b|\hat{T}_-\hat{T}_+|b\rangle = \langle b|(\hat{T}_3^2 - \hat{T}^2 + \hat{T}_3)|b\rangle \\ &= (b - b_0 + 1)(b + b_0), \end{aligned} \tag{A.25}$$

so that

$$\hat{T}_-|b\rangle = \sqrt{(b - b_0)(b + b_0 - 1)}|b - 1\rangle, \tag{A.26}$$

$$\hat{T}_+|b\rangle = \sqrt{(b - b_0 + 1)(b + b_0)}|b + 1\rangle. \tag{A.27}$$

Supposing there to be a largest eigenvalue b_1 for which $\hat{T}_+|b_1\rangle = 0$ yields

$$\hat{T}_-\hat{T}_+|b_1\rangle = (\hat{T}_3^2 - \hat{T}^2 + \hat{T}_3)|b_1\rangle = 0. \tag{A.28}$$

which leads to $b_1(b_1 + 1) = b_0(b_0 - 1)$. Solving, $b_1 = -b_0$ or $b_1 = b_0 - 1$. Clearly, the latter solution must be discarded since $b_1 \geq b_0$ by hypothesis. With respect to the former solution, note that since the realizations are Hermitian, the operators $\hat{T}_\pm\hat{T}_\mp$ are positive definite. So,

$$\langle b|\hat{T}_+\hat{T}_-|b\rangle = \langle b|(\hat{T}_3^2 - \hat{T}^2 - \hat{T}_3)|b\rangle \geq 0, \tag{A.29}$$

so that $(b - b_0)(b + b_0 - 1) \geq 0$. The first factor is always greater or equal to zero so $b + b_0 \geq 1$. This must be true for all values of b, and in particular, $b = b_0$ so that $b_0 \geq \frac{1}{2}$. Therefore, with $b_0 > 0$, the solution $b_1 = -b_0$ for the largest eigenvalue must also be discarded so that there can be no largest eigenvalue: A spectrum

bounded below is unbounded above.

If the assumption was made instead that the spectrum is bounded above similar calculations would indicate that the spectrum in unbound below. This is a fundamental difference between SO(3) and SO(2,1). Whereas the former is a compact group with finitely dimensional representations, the latter is non-compact rendering the unitary irreducible representations infinitely dimensional.

Hence, the spectrum of the operator $\hat{T}_3$, generator of the group SO(2,1), is given by $b = b_0 + k$, where $k = 0, 1, 2, \ldots$.

A.3 A Useful Realization of so(2,1)

To ease the discussion of both the isotropic harmonic oscillator and the hydrogen atom problem, the following realization of so(2,1) may be applied, with a prudent selection of parameters, to either of these problems. This development follows Čížek and Paldus [78], employing a scaling transformation, rather than the "tilting" transformation found elsewhere [8].

Stemming from the basic commutation relation (in atomic units)

$$[\hat{r}, \hat{p}_r] = i, \tag{A.30}$$

between radial distance, $\hat{r}^2 = \hat{x}^2 + \hat{y}^2 + \hat{z}^2$ and its conjugate momentum $\hat{p}_r$, scaled radial and radial momentum operators may be defined by

$$\hat{R} = \lambda \hat{r}, \quad \hat{P} = \lambda^{-1} \hat{p}_r \tag{A.31}$$

which satisfy the same commutation relation as above, with some constant λ to be determined later. Using the new operators, three generators may be defined by

$$\hat{W}_1 = \hat{R}^n, \tag{A.32}$$

$$\hat{W}_2 = n^{-1}\left(\hat{R}\hat{P} - \frac{i}{2}(n-1)\right), \tag{A.33}$$

$$\hat{W}_3 = n^{-2}\hat{R}^{2-n}\hat{P}^2 + \xi\hat{R}^{-n}, \tag{A.34}$$

in which ξ is some object which commutes with both $\hat{r}$ and $\hat{p}_r$. These operators commute according to

$$[\hat{W}_1, \hat{W}_2] = i\hat{W}_1, \quad [\hat{W}_2, \hat{W}_3] = i\hat{W}_3, \quad [\hat{W}_1, \hat{W}_3] = 2i\hat{W}_2. \tag{A.35}$$

Since they are closed under commutation, they span some algebra. To see precisely what algebra this is, let

$$\hat{T}_{1,3} = \frac{1}{2}(\hat{W}_3 \mp \hat{W}_1) \tag{A.36}$$

$$\hat{T}_2 = \hat{W}_2. \tag{A.37}$$

Through these new operators, the former commutation relations translate into

$$[\hat{T}_1, \hat{T}_2] = -i\hat{T}_3, \quad [\hat{T}_2, \hat{T}_3] = i\hat{T}_1, \quad [\hat{T}_3, \hat{T}_1] = i\hat{T}_2, \tag{A.38}$$

exposing the algebra to be so(2,1). According to the usual construction, concern will be focused on joint eigenstates of the Casimir operator and $\hat{T}_3$. The Casimir

operator of the algebra in this realization is given by

$$\begin{aligned}\hat{C} &= \hat{T}_3^2 - \hat{T}_1^2 - \hat{T}_2^2 \\ &= \hat{W}_1\hat{W}_3 - \hat{W}_2(\hat{W}_2 + i) \\ &= \xi + \frac{1}{4n^2}(1 - n^2),\end{aligned} \tag{A.39}$$

independent of the scaling factor λ.

The obscure motivation for these derivations will now be laid bare. The generators $\hat{T}_k$ have an abundance of parameters which may be chosen to suit the problem at hand. A suitable choice permits this realization of so(2,1) to act as a spectrum generating algebra for a variety of problems, including the N-dimensional isotropic harmonic oscillator and the N-dimensional hydrogen atom.

Suppose that the time-independent Schrödinger equation may be arranged into the form

$$\lambda^n\hat{W}_3 + \lambda^{-n}B\hat{W}_1|\psi\rangle = D|\psi\rangle. \tag{A.40}$$

Expanding in terms of $\hat{R}$ and $\hat{P}$,

$$\frac{1}{2}(n^{-2}\hat{R}^{2-n}\hat{P}^2 + \xi\hat{R}^{-n} + B\lambda^{-2n}\hat{R}^n)|\psi\rangle = \frac{1}{2}D\lambda^{-n}|\psi\rangle. \tag{A.41}$$

Now choosing λ so that $B\lambda^{-2n} = 1$, the above becomes

$$\frac{1}{2}(n^{-2}\hat{R}^{2-n}\hat{P}^2 + \xi\hat{R}^{-n} + \hat{R}^n)|\psi\rangle = \frac{1}{2}DB^{-\frac{1}{2}}|\psi\rangle, \tag{A.42}$$

which may be expressed in the convenient form

$$\hat{T}_3|\psi\rangle = \frac{1}{2}DB^{-\frac{1}{2}}|\psi\rangle. \tag{A.43}$$

The eigenspectrum of $\hat{T}_3$ and hence of the Schrödinger equation is now clear. From the work of Section A.2, the lowest eigenvalue b_0 of $\hat{T}_3$ is given by

$$b_0(b_0 - 1) = \xi + \frac{1}{4n^2}(1 - n^2), \tag{A.44}$$

or

$$b_0 = \frac{1}{2} \pm \frac{1}{2n}\sqrt{1 + 4n^2\xi}, \tag{A.45}$$

where the sign is to be determined later, with subsequent eigenvalues separated by unit intervals:

$$\frac{1}{2}DB^{-\frac{1}{2}} = b_0 + k, \tag{A.46}$$

with $k = 0, 1, 2 \ldots$.

Appendix B

A Boson Calculus for Angular Momentum

This appendix is a brief discussion of boson calculus of angular momentum (including and beyond so(3)) as introduced by Schwinger [72]. Note that Schwinger discusses the $\hat{K}_{\pm}$ operators but does not invoke the Lie algebraic connection. His intention was to describe the standard theory of quantum angular momentum in terms of particular bilinear combinations of bosonic creation and annihilation operators. Bhaumik *et al.* [16] also discuss the $\hat{K}_{\pm}$ operators as well as the $\hat{I}_{\pm}$ and $\hat{L}_{\pm}$ though in a slightly different form, and they also do not make mention of the so(2,1) structure of the $\hat{K}_{\pm}$, $\hat{I}_{\pm}$ and $\hat{L}_{\pm}$ operators.

The bosonic realization of the generators of U(n) as $\hat{a}_i^{\dagger}\hat{a}_j$ for $i, j = 1, \ldots, n$ is well known [65] of which the present work is only the $n = 2$ example. These generators are a manifestation of the fact that U(n) may act as the degeneracy group of the n-dimensional isotropic harmonic oscillator [5]. The bosonic realizations of so(2,1)

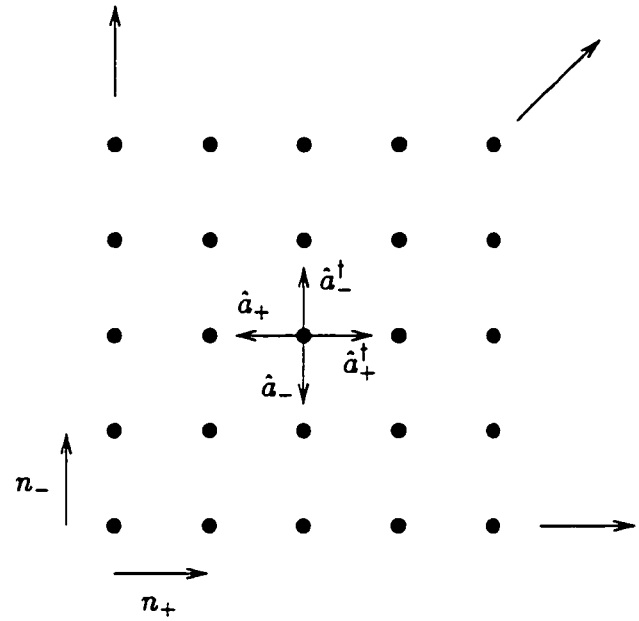


Figure B.1: Bosonic creation and annihilation operators.

are quite common, though Barut [10] alludes to them and Perelomov [68] makes direct reference to them in connection to his construction of coherent states. In any case, the following development suffices for the present purposes.

Following Schwinger [72], consider the direct sum of two Heisenberg-Weyl algebras, one to be spanned by the bosonic creation and annihilation operators $\hat{a}_+^\dagger$, and $\hat{a}_+$ and the other by $\hat{a}_-$, and $\hat{a}_-^\dagger$, as diagrammed in Figure B.1. These operators, spanning different copies of the usual algebra, satisfy the commutation relations

$$[\hat{a}_+, \hat{a}_+^\dagger] = [\hat{a}_-, \hat{a}_-^\dagger] = 1, \tag{B.1}$$

with all other commutators equal to zero. The operators $\hat{n}_\pm = \hat{a}_\pm^\dagger \hat{a}_\pm$ give the

occupation numbers, so that

$$\hat{a}_-|n_-;n_+\rangle = \sqrt{n_-}|n_- - 1;n_+\rangle, \tag{B.2}$$

$$\hat{a}_-^\dagger|n_-;n_+\rangle = \sqrt{n_- + 1}|n_- + 1;n_+\rangle, \tag{B.3}$$

$$\hat{n}_-|n_-;n_+\rangle = n_-|n_-;n_+\rangle, \tag{B.4}$$

as well as

$$\hat{a}_+|n_-;n_+\rangle = \sqrt{n_+}|n_-;n_+ - 1\rangle, \tag{B.5}$$

$$\hat{a}_+^\dagger|n_-;n_+\rangle = \sqrt{n_+ + 1}|n_-;n_+ + 1\rangle, \tag{B.6}$$

$$\hat{n}_+|n_-;n_+\rangle = n_+|n_-;n_+\rangle. \tag{B.7}$$

The ground state is given by $|0;0\rangle$ with $\hat{a}_-|0;0\rangle = \hat{a}_+|0;0\rangle = 0$, and any state may be obtained by applying the appropriate number of creation operators:

$$|n_-;n_+\rangle = \frac{\hat{a}_-^{\dagger\, n_-}\hat{a}_+^{\dagger\, n_+}}{\sqrt{n_-!n_+!}}|0;0\rangle. \tag{B.8}$$

The collection of states can thus be imagined as the lattice points on a unit grid in the first quadrant with n_- along the vertical axis and n_+ along the horizontal as in Figure B.1. Now instead of the states $|n_-;n_+\rangle$, consider $|j,m\rangle$ with $j = (n_+ + n_-)/2$ and $m = (n_+ - n_-)/2$ so that $n_+ = j+m$ and $n_- = j-m$. This rotates the structure onto its point with j vertical and m horizontal.

The states $|j,m\rangle$ have been so called not simply to create confusion with angular momentum eigenstates. There is a specific connection which may be drawn between

the bosonic creation and annihilation operators and those of angular momentum. The identification is made thusly:

$$\hat{J}_+ = \hat{a}_+^\dagger \hat{a}_-, \text{ and } \hat{J}_- = \hat{a}_-^\dagger \hat{a}_+. \tag{B.9}$$

Using Eq. (4.6) to obtain $\hat{J}_1$, $\hat{J}_2$, and Eq. (4.7) to obtain $\hat{J}_3$,

$$\hat{J}_1 = \frac{1}{2}(\hat{a}_+^\dagger \hat{a}_- + \hat{a}_-^\dagger \hat{a}_+) \tag{B.10}$$

$$\hat{J}_2 = -\frac{i}{2}(\hat{a}_+^\dagger \hat{a}_- - \hat{a}_-^\dagger \hat{a}_+) \tag{B.11}$$

$$\hat{J}_3 = \frac{1}{2}(\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-). \tag{B.12}$$

After some algebra, one may verify that these satisfy the standard commutation relations for angular momentum. It follows, therefore, that the action of the usual ladder operators $\hat{J}_\pm = \hat{J}_1 \pm i\hat{J}_2$ on a state $|j,m\rangle$ must agree with the usual result

$$\hat{J}_+|j,m\rangle = \sqrt{(j-m)(j+m+1)}|j,m+1\rangle, \tag{B.13}$$

$$\hat{J}_-|j,m\rangle = \sqrt{(j-m+1)(j+m)}|j,m-1\rangle, \tag{B.14}$$

irrespective of whether the calculation is carried out in terms of angular momentum ladder operators or as bosonic operators in conjunction with Eq. (B.4) and (B.7). This is indeed the case.

The standard angular momentum ladder operators only allow one to move between eigenstates pertaining to a particular j. The bosonic operators provide some additional ammunition, allowing one to move between states of different j.

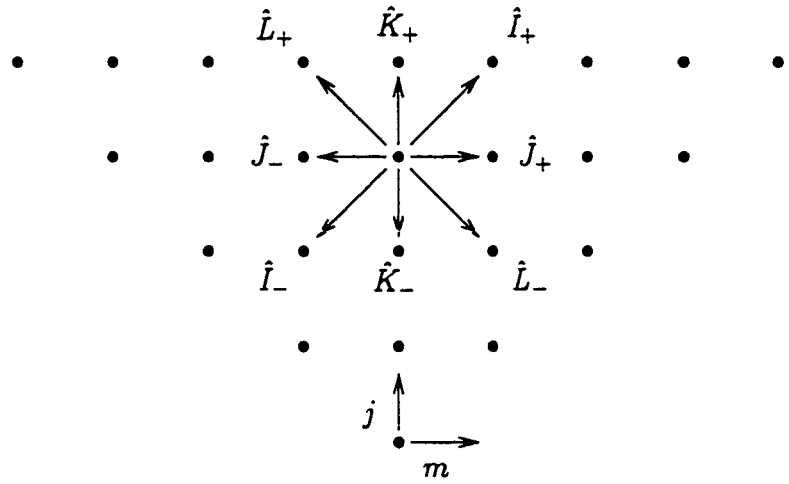


Figure B.2: Angular momentum ladder operators, acting on states $|j,m\rangle$ with integer j.

In particular, consider the creation and annihilation pair $\hat{K}_\pm$ in Figure B.2 with $\hat{K}_+ = \hat{a}_+^\dagger \hat{a}_-^\dagger$ and $\hat{K}_- = \hat{a}_+ \hat{a}_-$. The state $\hat{K}_\pm |j,m\rangle$ is proportional to $|j \pm 1, m\rangle$. Then, define $\hat{K}_3$ according to $[\hat{K}_+, \hat{K}_-] = -2\hat{K}_3$, and note that $[\hat{K}_3, \hat{K}_\pm] = \pm \hat{K}_\pm$. Extracting $\hat{K}_{1,2}$ from $\hat{K}_\pm = \hat{K}_1 \pm \hat{K}_2$ makes the collection $\hat{K}_{1,2,3}$ a realization of the Lie algebra so(2,1), worked out in greater detail below.

Similarly, the operators $\hat{I}_+ = \hat{a}_+^\dagger \hat{a}_+^\dagger/2$ and $\hat{I}_- = \hat{a}_+ \hat{a}_+/2$ act diagonally as diagrammed in Figure B.2, and $\hat{L}_+ = \hat{a}_-^\dagger \hat{a}_-^\dagger/2$ and $\hat{L}_- = \hat{a}_- \hat{a}_-/2$, move along the opposite diagonal. Both of these pairs generate a realization of so(2,1). These operators are suggested by Bhaumik *et al.* [16] although their forms differ from these by the factor of $\frac{1}{2}$, and they make no allusion to any Lie algebraic structure.

In anticipation that a position space representation will be required, states of non-integer j are now ruled out. Employing only the eight ladder operators $\hat{J}_\pm$,

$\hat{K}_\pm$, $\hat{I}_\pm$, and $\hat{L}_\pm$, one remains on the depleted lattice (shown in Figure B.2). The question now arises "What does this depleted lattice represent?" This lattice is the direct sum of the two copies of so(2,1) spanned by $\hat{I}_{\pm,3}$ and $\hat{L}_{\pm,3}$, merged with $\hat{J}_\pm$, together with whatever operators are necessary to close the commutation relations. This last requirement introduces $\hat{K}_\pm$ for a total of ten generators. The depleted lattice is not, of course, a sub-algebra of the direct sum of two Heisenberg-Weyl algebras since the new algebra is spanned by bilinear combinations of the bosonic operators, not linear combinations.

There is a certain impulse to connect the operators $\hat{I}_\pm$, $\hat{K}_\pm$ or $\hat{L}_\pm$ with the quantum Laplace-Runge-Lenz vector which may act as a ladder operator of sorts between angular momentum eigenstates of differing j. There are two problems with such a connection. Firstly, the action of the Laplace-Runge-Lenz vector renders a rather complicated superposition of angular momentum eigenstates [1], not a scaled pure eigenstate as in the cases of $\hat{I}_\pm$, $\hat{K}_\pm$ and $\hat{L}_\pm$. Secondly, the Laplace-Runge-Lenz vector comes out of a realization of SO(4), a compact Lie group, not a realization of a noncompact group as with $\hat{I}_\pm$, $\hat{K}_\pm$ and $\hat{L}_\pm$.

Returning to the set of states as diagrammed in Figure B.1, the horizontal and vertical axes represent separately two irreducible representations of the Heisenberg-Weyl group, as do any horizontally or vertically aligned collection of points. Due to the algebraic structure of pairs of bosonic creation and annihilation operators, collections of states on a diagonal line connecting the horizontal to the vertical axis form representations of SO(3) (the $\hat{J}_\pm$ operators form the pertinent realization). These are finite collections of states, since SO(3) is a compact group. Collections

of states on diagonal lines starting on horizontal or vertical axes and moving diagonally upwards and to the right (through the operators $\hat{K}_\pm$) form representations of SO(2,1). These are collections are infinitely numerous, and SO(2,1) is non-compact.

The ladder operators are given by (with their actions on angular momentum eigenstates also noted)

$$\hat{J}_+ = \hat{a}_+^\dagger \hat{a}_-, \quad \hat{J}_+|j,m\rangle = \sqrt{(j-m)(j+m+1)}|j,m+1\rangle, \tag{B.15}$$

$$\hat{J}_- = \hat{a}_-^\dagger \hat{a}_+, \quad \hat{J}_-|j,m\rangle = \sqrt{(j-m+1)(j+m)}|j,m-1\rangle, \tag{B.16}$$

$$\hat{K}_+ = \hat{a}_+^\dagger \hat{a}_-^\dagger, \quad \hat{K}_+|j,m\rangle = \sqrt{(j+m+1)(j-m+1)}|j+1,m\rangle, \tag{B.17}$$

$$\hat{K}_- = \hat{a}_+ \hat{a}_-, \quad \hat{K}_-|j,m\rangle = \sqrt{(j+m)(j-m)}|j-1,m\rangle, \tag{B.18}$$

$$\hat{I}_+ = \frac{1}{2}\hat{a}_+^\dagger \hat{a}_+^\dagger, \quad \hat{I}_+|j,m\rangle = \frac{1}{2}\sqrt{(j+m+1)(j+m+2)}|j+1,m+1\rangle, \tag{B.19}$$

$$\hat{I}_- = \frac{1}{2}\hat{a}_+ \hat{a}_+, \quad \hat{I}_-|j,m\rangle = \frac{1}{2}\sqrt{(j+m)(j+m-1)}|j-1,m-1\rangle, \tag{B.20}$$

$$\hat{L}_+ = \frac{1}{2}\hat{a}_-^\dagger \hat{a}_-^\dagger, \quad \hat{L}_+|j,m\rangle = \frac{1}{2}\sqrt{(j-m+1)(j-m+2)}|j+1,m-1\rangle, \tag{B.21}$$

$$\hat{L}_- = \frac{1}{2}\hat{a}_- \hat{a}_-, \quad \hat{L}_-|j,m\rangle = \frac{1}{2}\sqrt{(j-m)(j-m-1)}|j-1,m+1\rangle. \tag{B.22}$$

With $\hat{T}_\pm$ standing for all of the above pairs of ladder operators, the first two components are obtained by inverting $\hat{T}_\pm = \hat{T}_1 \pm i\hat{T}_2$. This gives

$$\hat{J}_1 = \frac{1}{2}(\hat{a}_+^\dagger \hat{a}_- + \hat{a}_-^\dagger \hat{a}_+), \quad \hat{J}_2 = -\frac{i}{2}(\hat{a}_+^\dagger \hat{a}_- - \hat{a}_-^\dagger \hat{a}_+), \tag{B.23}$$

$$\hat{K}_1 = \frac{1}{2}(\hat{a}_+^\dagger \hat{a}_-^\dagger + \hat{a}_+ \hat{a}_-), \quad \hat{K}_2 = -\frac{i}{2}(\hat{a}_+^\dagger \hat{a}_-^\dagger - \hat{a}_+ \hat{a}_-), \tag{B.24}$$

$$\hat{I}_1 = \frac{1}{4}(\hat{a}_+^\dagger \hat{a}_+^\dagger + \hat{a}_+ \hat{a}_+), \quad \hat{I}_2 = -\frac{i}{4}(\hat{a}_+^\dagger \hat{a}_+^\dagger - \hat{a}_+ \hat{a}_+), \tag{B.25}$$

$$\hat{L}_1 = \frac{1}{4}(\hat{a}_-^\dagger \hat{a}_-^\dagger + \hat{a}_- \hat{a}_-), \quad \hat{L}_2 = -\frac{i}{4}(\hat{a}_-^\dagger \hat{a}_-^\dagger - \hat{a}_- \hat{a}_-). \tag{B.26}$$

Then, since $[\hat{T}_1, \hat{T}_2] = \frac{1}{2}[\hat{T}_+, \hat{T}_-]$, the third components are derived according to

$$\hat{J}_3 = \frac{1}{2}[\hat{J}_+, \hat{J}_-] = \frac{1}{2}(\hat{a}_+^\dagger \hat{a}_+ + \hat{a}_-^\dagger \hat{a}_-), \tag{B.27}$$

$$\hat{K}_3 = -\frac{1}{2}[\hat{K}_+, \hat{K}_-] = \frac{1}{2}(\hat{a}_+^\dagger \hat{a}_+ + \hat{a}_-^\dagger \hat{a}_- + 1), \tag{B.28}$$

$$\hat{I}_3 = -\frac{1}{2}[\hat{I}_+, \hat{I}_-] = \frac{1}{2}\hat{a}_+^\dagger \hat{a}_+ + \frac{1}{4}, \tag{B.29}$$

$$\hat{L}_3 = -\frac{1}{2}[\hat{L}_+, \hat{L}_-] = \frac{1}{2}\hat{a}_-^\dagger \hat{a}_- + \frac{1}{4}. \tag{B.30}$$

With these definitions and identifications, the following commutation relations may be verified, where $\hat{T}_\ell$ stands for any of the four sets of operators listed above.

$$[\hat{T}_3, \hat{T}_1] = i\hat{T}_2, \quad [\hat{T}_2, \hat{T}_3] = i\hat{T}_1. \tag{B.31}$$

In the case of the $\hat{J}_\ell$ this is an affirmation that they may act as basis elements of su(2) or so(3). For all the other sets of operators, this completes the description of the basis elements of so(2,1), the incidental sign arising from the definitions given in Eqs. (B.28) to (B.30).

Now turning to calculating the Casimir operators, they may be constructed according to the standard forms of those for so(3) and so(2,1). In terms of the number operators, they are given by

$$\begin{aligned} \hat{J}^2 &= \hat{J}_+\hat{J}_- + \hat{J}_3^2 - \hat{J}_3 \\ &= \frac{1}{4}(\hat{n}_+ + \hat{n}_-)^2 + \frac{1}{2}(\hat{n}_+ + \hat{n}_-) \\ &= \frac{1}{2}(\hat{n}_+ + \hat{n}_-)\left(\frac{1}{2}(\hat{n}_+ + \hat{n}_-) + 1\right), \end{aligned} \tag{B.32}$$

$$\begin{aligned}
\hat{K}^2 &= -\hat{K}_+\hat{K}_- + \hat{K}_3^2 - \hat{K}_3 \\
&= \frac{1}{4}(\hat{n}_+ - \hat{n}_-)^2 - \frac{1}{4}, \qquad \text{(B.33)} \\
\hat{I}^2 &= -\hat{I}_+\hat{I}_- + \hat{I}_3^2 - \hat{I}_3 = -\frac{3}{16}, \qquad \text{(B.34)} \\
\hat{L}^2 &= -\hat{L}_+\hat{L}_- + \hat{L}_3^2 - \hat{L}_3 = -\frac{3}{16}. \qquad \text{(B.35)}
\end{aligned}$$

Clearly, the eigenvalue of $\hat{J}^2$ is $j(j+1)$ as usual, independent of m as it must be. From this, one may conclude that the largest and smallest eigenvalues of $\hat{J}_3$ are $\pm j$ as usual.

The eigenvalue of $\hat{K}^2$ is $m^2 - 1/4$, independent of j, reflecting how the roles of j and m are exchanged in $\hat{K}_\pm$ from $\hat{J}_\pm$. From Section A.2, the lowest eigenvalue of $\hat{K}_3$ is given by $b_0(b_0-1) = m^2 - 1/4$. Solving, $b_0 = \pm m + 1/2$. However, looking at Eq. (B.28), the eigenvalues of $\hat{K}_3$ may be expressed as $j + 1/2$. For any given state $|j,m\rangle$, $j \geq |m|$, so that the lowest value of j, given an m, is $|m|$. This induces the choice of sign for b_0 which may be expressed $b_0 = |m| + 1/2$. The spectrum of $\hat{K}_3$, as expected, is $b = |m| + 1/2 + k = j + 1/2$ for non-negative integer k, so that $j = |m| + k$.

The eigenvalues of $\hat{I}^2$ and $\hat{L}^2$ are both independent of j and m. Looking at $\hat{I}^2$, this is a reflection that the operators do not involve $\hat{a}_-$ or $\hat{a}_-^\dagger$ whatsoever, and hence relies only on one copy of the Heisenberg-Weyl algebra, existing irrespective of the direct sum. Again from Section A.2 and Eq. (B.34), the lowest eigenvalue of $\hat{I}_3$ is given by $b_0(b_0-1) = -3/16$ so that $b_0 = 1/4$ or $3/4$. Comparing with Eq. (B.29), this choice merely stems from whether one takes the ground state of the representation with $n_+ = 0$ (as with the ground state $|0,0\rangle$) or $n_+ = 1$ (as

with $|j,m\rangle = |1,0\rangle$), so that both representations exist on the depleted lattice. Analogous statements hold for the $\hat{L}$ operators with respect to n_-.

Appendix C

The Harmonic Oscillator

The harmonic oscillator is a quantum system which corresponds to the classical problem of a mass on a spring with one degree of freedom. The Hamiltonian for a particle of mass m and spring constant $k = m\omega^2$ is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2. \tag{C.1}$$

In general, the time evolution of states is via the time dependent Schrödinger equation,

$$i\hbar\frac{d}{dt}|\psi\rangle = \hat{H}|\psi\rangle, \tag{C.2}$$

but since the Hamiltonian is time independent, time evolution may be more simply expressed via the time evolution operator,

$$|\psi(t)\rangle = \hat{U}(t)|\psi\rangle = e^{-i\hat{H}t/\hbar}|\psi\rangle. \tag{C.3}$$

Accordingly, if the state is an eigenstate of the Hamiltonian with energy E, the time evolution is particularly simple,

$$|\psi(t)\rangle = e^{-i\hat{E}t/\hbar}|\psi\rangle. \tag{C.4}$$

The collection of eigenstates of the Hamiltonian is a complete set of states and hence arbitrary states may be expressed as superpositions over states. This allows the time evolution of arbitrary states to be expressed in terms of the time evolution of eigenstates. Thus attention is now turned to solving the eigenvalue eigenstate problem

$$\hat{H}|n\rangle = E_n|n\rangle. \tag{C.5}$$

To this end, recall that $[\hat{x}, \hat{p}] = i\hbar$. Indeed, the objects

$$e_0 = i, \quad e_1 = i\hat{p}/\sqrt{m\omega\hbar}, \quad e_2 = i\hat{x}\sqrt{\frac{m\omega}{\hbar}}, \tag{C.6}$$

may be regarded as the basis of a nilpotent three dimensional Lie algebra characterized by the commutation relation

$$[e_1, e_2] = e_0, \tag{C.7}$$

with the two other commutators equal to zero. This real Lie algebra is the Heisenberg-Weyl algebra [68] often denoted $\mathcal{W}_1$. A general element g of this algebra is given

by (summing over repeated indices)

$$g = s_k e_k, \tag{C.8}$$

with implied summation and in which the s_k are real numbers. Now consider the new pair of operators

$$\hat{a} = \frac{1}{\sqrt{2}}(e_1 - ie_2), \tag{C.9}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}(-e_1 - ie_2). \tag{C.10}$$

or,

$$e_1 = \frac{1}{\sqrt{2}}\left(\hat{a} - \hat{a}^\dagger\right). \tag{C.11}$$

$$e_2 = \frac{i}{\sqrt{2}}\left(\hat{a} + \hat{a}^\dagger\right). \tag{C.12}$$

In terms of these, the element g is given by

$$g = is_0 + \alpha\hat{a} - \alpha^*\hat{a}^\dagger, \tag{C.13}$$

in which $\alpha = (s_2 + is_1)/\sqrt{2}$. Their commutator is given by

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{C.14}$$

Expressing the Hamiltonian in terms of the operators $\hat{a}$ and its Hermitian adjoint

gives

$$\hat{H} = -\hbar\omega\left(e_1^2 + e_2^2\right)/2 \tag{C.15}$$

$$= \hbar\omega\left(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right)/2 \tag{C.16}$$

$$= \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right), \tag{C.17}$$

so that finding eigenstates of $\hat{a}^\dagger\hat{a}$ yields eigenstates of $\hat{H}$. Suppose that $|n\rangle$ is an eigenstate of $\hat{a}^\dagger\hat{a}$ with eigenvalue n. Now, due to the commutation relation, Eq. (C.14),

$$(\hat{a}^\dagger\hat{a})\hat{a}^\dagger|n\rangle = \hat{a}^\dagger(1 + \hat{a}^\dagger\hat{a})|n\rangle = (n+1)\hat{a}^\dagger|n\rangle, \tag{C.18}$$

So that $\hat{a}^\dagger|n\rangle$ is also an eigenstate of $\hat{a}^\dagger\hat{a}$ but with eigenvalue $n+1$. Similarly, if $n \neq 0$, then for $\hat{a}$ one finds

$$(\hat{a}^\dagger\hat{a})\hat{a}|n\rangle = (n-1)\hat{a}|n\rangle. \tag{C.19}$$

Hence, the operators $\hat{a}$ and $\hat{a}^\dagger$ annihilate and create units of the eigenvalue of $\hat{a}^\dagger\hat{a}$, and are called the annihilation and creation operators respectively. Examining the action of the annihilation operator more carefully, $\hat{a}$ acting on $|n\rangle$ is proportional the $|n-1\rangle$:

$$\hat{a}|n\rangle = c_n|n-1\rangle. \tag{C.20}$$

Taking the inner product of the above with its own Hermitian adjoint yields

$$\langle n|\hat{a}^\dagger\hat{a}|n\rangle = |c_n|^2. \tag{C.21}$$

Since $|n\rangle$ is an eigenstate of $\hat{a}^\dagger\hat{a}$, we obtain

$$c_n = \sqrt{n}, \tag{C.22}$$

where the positive square root is taken as convention. Similarly, one finds

$$\hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{C.23}$$

Knowing that the Hamiltonian spectrum is bounded below indicates that the spectrum of $\hat{a}^\dagger\hat{a}$ is also bounded below. Hence, there is a state $|\psi\rangle$ which is annihilated entirely by an application of $\hat{a}$:

$$\hat{a}|\psi\rangle = 0. \tag{C.24}$$

Operating on this equation by $\hat{a}^\dagger$ gives the eigenvalue of this state as $n = 0$, and will hereafter be called the ground state, $|0\rangle$. Operating on $|0\rangle$ n times yields the nth eigenstate of $\hat{a}^\dagger\hat{a}$,

$$\frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle = |n\rangle. \tag{C.25}$$

To determine the position space representations of these states, Eq. (C.24), with $\hat{a}$ in terms of $\hat{x}$ and $\hat{p}$, projected into position space yields

$$\left(\frac{m\omega}{\hbar}x + \frac{d}{dx}\right)\langle x|0\rangle = 0. \tag{C.26}$$

This first order differential equation yields the solution

$$\langle x|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right), \tag{C.27}$$

which has been normalized to $\langle 0|0\rangle = 1$. The nth state is then obtained through Eq. (C.25).

$$\left(\hat{a}^\dagger\right)^n |0\rangle = \sqrt{n!}|n\rangle. \tag{C.28}$$

Defining the quantity $a_0 = \sqrt{m\omega/\hbar}$ (an inverse length scale), the creation operator in position space becomes

$$\hat{a}^\dagger = \frac{1}{a_0\sqrt{2}}\left(a_0^2 x - \frac{d}{dx}\right). \tag{C.29}$$

To apply this operator to some function $f(x)$, observe that

$$\frac{1}{a_0\sqrt{2}}\left(a_0^2 x - \frac{d}{dx}\right) f = -\frac{1}{a_0\sqrt{2}} e^{\frac{1}{2}a_0^2x^2} \frac{d}{dx}\left(e^{-\frac{1}{2}a_0^2x^2} f\right). \tag{C.30}$$

With the ground state in terms of a_0 given by

$$\langle x|0\rangle = \left(\frac{a_0}{\sqrt{\pi}}\right)^{1/2} e^{-\frac{1}{2}a_0^2x^2}, \tag{C.31}$$

the successor is obtained,

$$\langle x|\hat{a}^\dagger|0\rangle = -\left(\frac{a_0}{2\sqrt{\pi}}\right)^{1/2} \frac{1}{a_0} e^{\frac{1}{2}a_0^2x^2} \frac{d}{dx}\left(e^{-a_0^2x^2}\right), \tag{C.32}$$

and indeed, in general one finds

$$\langle x| \left(\hat{a}^\dagger\right)^n |0\rangle = (-1)^n \left(\frac{a_0}{2^n\sqrt{\pi}}\right)^{1/2} \frac{1}{a_0^n} e^{\frac{1}{2}a_0^2x^2} \frac{d^n}{dx^n}\left(e^{-a_0^2x^2}\right) \quad \text{(C.33)}$$

$$= \left(\frac{a_0}{2^n\sqrt{\pi}}\right)^{1/2} \frac{1}{a_0^n} e^{-\frac{1}{2}a_0^2x^2} \left((-1)^n e^{a_0^2x^2} \frac{d^n}{dx^n}\left(e^{-a_0^2x^2}\right)\right) \quad \text{(C.34)}$$

$$= \left(\frac{a_0}{2^n\sqrt{\pi}}\right)^{1/2} e^{-\frac{1}{2}a_0^2x^2} H_n(a_0x), \quad \text{(C.35)}$$

in which the parenthetically enclosed expression of Eq. (C.34) is recognized as a standard definition [2] of the Hermite polynomials, H_n. From Eq. (C.25) the eigenstates may be written

$$\langle x|n\rangle = \left(\frac{a_0}{n!2^n\sqrt{\pi}}\right)^{1/2} e^{-\frac{1}{2}a_0^2x^2} H_n(a_0x), \quad \text{(C.36)}$$

Appendix D

The Spherical Rotator

D.1 Angular Momentum Matrix Elements

In order to calculate $\langle x \rangle$, $\langle y \rangle$, or $\langle z \rangle$ in a state composed of angular momentum eigenstates, the matrix elements $\langle j', m' | \hat{x} | j, m \rangle$ are required as well as those for $\hat{y}$ and $\hat{z}$. To find these matrix elements, one can consult the literature [77], or undertake the calculation one's self. The rest of this section is devoted to the latter approach. In this section, references to equation numbers in Gradshteyn and Ryzhik's *Table of Integrals, Series, and Products* [40] are given by, for example, GR 8.735.5.

With integral j values, the angular momentum eigenstates expressed on the surface of a unit sphere are given by the spherical harmonics

$$\langle \theta, \phi | j, m \rangle = Y_j^m(\theta, \phi) = (-1)^m \sqrt{\frac{2j+1}{4\pi} \frac{(j-m)!}{(j+m)!}} P_j^m(\cos\theta) e^{im\phi}, \qquad \text{(D.1)}$$

in which the functions P_j^m are the associated Legendre functions. Accordingly, expressing $\hat{x} = \sin\hat{\theta}\cos\hat{\phi}$,

$$\begin{aligned}
&\langle j', m'|\hat{x}|j, m\rangle \\
&= \int_0^{\pi} d\theta \int_0^{2\pi} \sin\theta d\phi \langle j', m'|\theta, \phi\rangle\langle\theta, \phi|\sin\hat{\theta}\cos\hat{\phi}|j, m\rangle \\
&= (-1)^{m'+m}\sqrt{\frac{(2j+1)(2j'+1)}{(4\pi)^2}\frac{(j-m)!(j'-m')!}{(j+m)!(j'+m')!}} \\
&\quad \times \int_0^{\pi} d\theta \int_0^{2\pi} \sin\theta d\phi \sin\theta\cos\phi P_{j'}^{m'}(\cos\theta) P_j^m(\cos\theta) e^{i(m-m')\phi}. \qquad \text{(D.2)}
\end{aligned}$$

Integrating over ϕ,

$$\int_0^{2\pi} \cos\phi e^{i(m-m')\phi} d\phi = \pi(\delta_{m',m+1} + \delta_{m',m-1}), \qquad \text{(D.3)}$$

in which the δ_{ab} is the Kronecker delta. Applying this to Eq. (D.2), one obtains

$$\begin{aligned}
\langle j', m'|\hat{x}|j, m\rangle &= (-1)\sqrt{\frac{(2j+1)(2j'+1)}{16}\frac{(j-m)!(j'-m-1)!}{(j+m)!(j'+m+1)!}} \\
&\quad \times \int_0^{\pi} d\theta \sin^2\theta P_{j'}^{m+1}(\cos\theta) P_j^m(\cos\theta)\delta_{m',m+1} \\
&\quad + \sqrt{\frac{(2j+1)(2j'+1)}{16}\frac{(j-m)!(j'-m+1)!}{(j+m)!(j'+m-1)!}} \\
&\quad \times \int_0^{\pi} d\theta \sin^2\theta P_{j'}^{m-1}(\cos\theta) P_j^m(\cos\theta)\delta_{m',m-1}. \qquad \text{(D.4)}
\end{aligned}$$

Examining the first integral, note that through GR 8.735.5,

$$\sqrt{1-x^2}P_{j'}^{m+1}(x)$$

$$= \frac{(j'-m)(j'-m+1)}{2j'+1} P^m_{j'+1}(x) - \frac{(j'+m)(j'+m+1)}{2j'+1} P^m_{j'-1}(x), \quad \text{(D.5)}$$

so that

$$\begin{aligned} I_1 &= \int_0^{\pi} d\theta \sin^2\theta P^{m+1}_{j'}(\cos\theta) P^m_j(\cos\theta) \\ &= \int_{-1}^{1} \sqrt{1-x^2} P^{m+1}_{j'}(x) P^m_j(x) dx \\ &= \frac{(j'-m)(j'-m+1)}{2j'+1} \int_{-1}^{1} P^m_{j'+1}(x) P^m_j(x) dx \\ &\quad - \frac{(j'+m)(j'+m+1)}{2j'+1} \int_{-1}^{1} P^m_{j'-1}(x) P^m_j(x) dx. \end{aligned} \quad \text{(D.6)}$$

Carrying out the integration through GR 7.112.1 and simplifying the factorials,

$$I_1 = \frac{2(j+m)!}{(2j+1)(2j-1)(j-m-2)!} \delta_{j',j-1} - \frac{2(j+m+2)!}{(2j+3)(2j+1)(j-m)!} \delta_{j',j+1} \quad \text{(D.7)}$$

Similarly, the second integral of Eq. (D.4) is given by

$$I_2 = \frac{2}{(2j+3)(2j+1)} \frac{(j+m)!}{(j-m)!} \delta_{j',j+1} - \frac{2}{(2j+1)(2j-1)} \frac{(j+m)!}{(j-m)!} \delta_{j',j-1}, \quad \text{(D.8)}$$

where use was made instead of GR 8.733.4 to raise the upper index of the associated Legendre function, rather than lowering it as through GR 8.735.5. Assembling the entire matrix element, and using the Kronecker-δ to write primed indices in terms of unprimed indices, Eq. (D.2) becomes

$$\langle j', m' | \sin\theta \cos\phi | j, m \rangle$$

$$= \frac{1}{2}\left(\sqrt{\frac{(j+m)(j+m-1)}{(2j+1)(2j-1)}}\delta_{j',j-1}\delta_{m',m-1} - \sqrt{\frac{(j-m+2)(j-m+1)}{(2j+3)(2j+1)}}\delta_{j',j+1}\delta_{m',m-1}\right.$$
$$\left. - \sqrt{\frac{(j-m)(j-m-1)}{(2j+1)(2j-1)}}\delta_{j',j-1}\delta_{m',m+1} + \sqrt{\frac{(j+m+2)(j+m+1)}{(2j+3)(2j+1)}}\delta_{j',j+1}\delta_{m',m+1}\right). \tag{D.9}$$

To calculate the matrix element for $\hat{y}$, a similar procedure is used, though the integration over ϕ becomes

$$\int_0^{2\pi} \sin\phi e^{i(m-m')\phi}d\phi = i\pi(\delta_{m',m-1} - \delta_{m',m+1}), \tag{D.10}$$

so that the calculation follows analogously with certain sign changes. This yields

$$\langle j', m'|\sin\theta\sin\phi|j, m\rangle$$
$$= \frac{i}{2}\left(\sqrt{\frac{(j+m)(j+m-1)}{(2j+1)(2j-1)}}\delta_{j',j-1}\delta_{m',m-1} - \sqrt{\frac{(j-m+2)(j-m+1)}{(2j+3)(2j+1)}}\delta_{j',j+1}\delta_{m',m-1}\right.$$
$$\left. + \sqrt{\frac{(j-m)(j-m-1)}{(2j+1)(2j-1)}}\delta_{j',j-1}\delta_{m',m+1} - \sqrt{\frac{(j+m+2)(j+m+1)}{(2j+3)(2j+1)}}\delta_{j',j+1}\delta_{m',m+1}\right). \tag{D.11}$$

For the z component, the integration over ϕ yields

$$\int_0^{2\pi} e^{i(m-m')\phi}d\phi = 2\pi\delta_{m',m}, \tag{D.12}$$

so that

$$\begin{aligned}
&\langle j', m' | \cos\hat{\theta} | j, m \rangle \\
&= \sqrt{\frac{(2j+1)(2j'+1)(j-m)!(j'-m)!}{4(j+m)!(j'+m)!}} \\
&\quad \times \int_0^\pi d\theta \sin\theta \cos\theta P_{j'}^m(\cos\theta) P_j^m(\cos\theta) \delta_{m',m} \\
&= \sqrt{\frac{(2j+1)(2j'+1)(j-m)!(j'-m)!}{4(j+m)!(j'+m)!}} \int_{-1}^1 x P_{j'}^m(x) P_j^m(x) dx \; \delta_{m',m}. \quad \text{(D.13)}
\end{aligned}$$

The expression given by GR 8.733.2 yields

$$x P_{j'}^m(x) = \frac{j'-m+1}{2j'+1} P_{j'+1}^m(x) + \frac{j'+m}{2j'+1} P_{j'-1}^m(x). \quad \text{(D.14)}$$

so that integrating again through GR 7.112.1, simplifying the factorials and expressing primed terms through unprimed terms,

$$\begin{aligned}
&\langle j', m' | \cos\hat{\theta} | j, m \rangle = \\
&\sqrt{\frac{(j+m)(j-m)}{(2j+1)(2j-1)}} \delta_{j',j-1} \delta_{m',m} + \sqrt{\frac{(j+m+1)(j-m+1)}{(2j+3)(2j+1)}} \delta_{j',j+1} \delta_{m',m}. \quad \text{(D.15)}
\end{aligned}$$

D.2 Some Expectation Values

The matrix elements in the preceding section may be used to calculate the expectation values of $\hat{x}$, $\hat{y}$ and $\hat{z}$ in the coherent state $|\alpha, \beta\rangle$ defined in Section 4.2.2. In all

the following cases, the expectation values are calculated through (see Eq. (4.38))

$$\langle\alpha',\beta'|\hat{O}|\alpha,\beta\rangle = \frac{1}{\cosh\xi}\sum_{j,m,j',m'}\frac{\alpha^{j+m}\beta^{j-m}\alpha^{*j'+m'}\beta^{*j'-m'}}{\sqrt{(j+m)!(j-m)!(j'+m')!(j'-m')!}}\langle j',m'|\hat{O}|j,m\rangle, \quad \text{(D.16)}$$

in which $\xi = |\alpha|^2 + |\beta|^2$.

Taking the first of the four pieces of Eq. (D.9) and carrying the calculation through,

$$\begin{aligned} a_1 &= \frac{1}{\cosh\xi}\sum_{j,m,j',m'}\frac{\alpha^{j+m}\beta^{j-m}\alpha^{*j'+m'}\beta^{*j'-m'}}{\sqrt{(j+m)!(j-m)!(j'+m')!(j'-m')!}} \\ &\quad\times\sqrt{\frac{(j+m)(j+m-1)}{(2j+1)(2j-1)}}\delta_{j',j-1}\delta_{m',m-1} \\ &= \frac{\alpha^2}{\cosh\xi}\sum_{j=1}^{\infty}\sum_{m=-j+2}^{j}\frac{|\alpha|^{2(j+m-2)}|\beta|^{2(j-m)}}{(j+m-2)!(j-m)!}\frac{1}{\sqrt{(2j+1)(2j-1)}} \\ &= \frac{\alpha^2}{\cosh\xi}\sum_{j=0}^{\infty}\frac{\xi^{2j}}{(2j)!}\frac{1}{\sqrt{(2j+3)(2j+1)}}. \end{aligned} \quad \text{(D.17)}$$

At this point, the calculation can be carried no further without an approximation. The appropriate approximation is $(2j+3)^{-1/2} \to (2j+1)^{-1/2}$, which is clearly questionable for the initial values of j in the sum. This sum is justified only with the joint assumption that ξ is sufficiently large and that the overwhelming bulk of the state $|\alpha,\beta\rangle$ is weighted away from the low-j angular momentum eigenstates. Thus adopting this approximation, the summation quickly follows so that

$$a_1 \sim \frac{\alpha^2}{\xi}\tanh\xi. \quad \text{(D.18)}$$

To obtain the second piece, the same approximation must be adopted yielding

$$a_2 \sim \frac{\beta^{*2}}{\xi} \tanh \xi. \tag{D.19}$$

The third and forth pieces yield

$$a_3 \sim \frac{\beta^2}{\xi} \tanh \xi, \quad a_4 \sim \frac{\alpha^{*2}}{\xi} \tanh \xi. \tag{D.20}$$

In all cases, it is necessary to approximate as above. These results together yield

$$\langle x \rangle \sim \frac{1}{2}(\alpha^2 + \alpha^{*2} - \beta^2 - \beta^{*2}) \frac{\tanh \xi}{\xi}, \tag{D.21}$$

and

$$\langle y \rangle \sim \frac{i}{2}(\alpha^2 - \alpha^{*2} + \beta^2 - \beta^{*2}) \frac{\tanh \xi}{\xi}. \tag{D.22}$$

Not surprisingly, this approximation must also be adopted in the calculation of $\langle z \rangle$. Using the techniques carried out for $\langle x \rangle$ and $\langle y \rangle$, Eq. (D.15) leads to

$$\langle z \rangle \sim (\alpha\beta + \alpha^*\beta^*) \frac{\tanh \xi}{\xi}. \tag{D.23}$$

The approximation involved is precisely

$$\frac{1}{\cosh \xi} \sum_{j=0}^{\infty} \frac{\xi^{2j+1}}{(2j)!\sqrt{(2j+3)(2j+1)}} \sim \tanh \xi. \tag{D.24}$$

The left hand and right hand sides of this expression are plotted in Figure D.1. The approximation is poor for small values of ξ but improves as ξ increases.

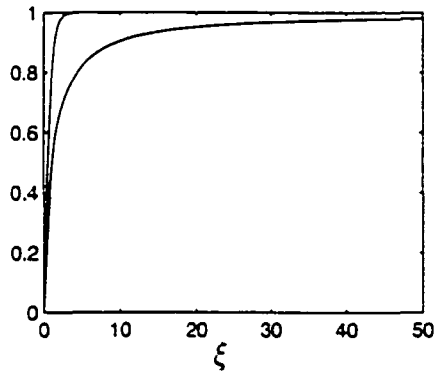


Figure D.1: The left hand side of Eq. (D.24) versus the right hand side. The upper curve is $\tanh \xi$ whereas the lower curve is the sum.

Appendix E

The Hydrogen Atom

E.1 The Group Structure

Group theoretical treatments of the hydrogen atom are standard in the literature [8, 78, 79, 1, 65]. This section is provided for reference, and to draw particular attention to aspects of the development relevant for the present purposes.

E.1.1 The Degeneracy Group SO(4)

Since the potential $V(r)$ is spherically symmetric, it commutes with the angular momentum operators, $[\hat{L}^2, V(\hat{r})] = [\hat{\mathbf{L}}, V(\hat{r})] = 0$, where the bold face indicates the operator vector, so that the degeneracy group at least contains as a subgroup the Lie group SO(3). The dimension of the irreducible representations being $2\ell + 1$, this accounts for a $(2\ell + 1)$-fold degeneracy for an energy level $E_{\ell,n'}$ pertaining to a total angular momentum quantum number ℓ and radial quantum number n'. However, as it turns out, many of these $E_{\ell,n'}$ correspond to one another increasing

the degeneracy of the nth energy level to n^2 where $n = n' + \ell + 1$, a degeneracy not accounted for by SO(3).

The reason for this so-called accidental degeneracy stems from the conservation of a certain operator quantity noted by Pauli in 1926, a prodigious year for quantum mechanics. This quantity is the quantum mechanical analogue of the classical Laplace-Runge-Lenz vector

$$\mathbf{A} = \mathbf{p} \times \mathbf{L} - Z\frac{\mathbf{r}}{r}, \tag{E.1}$$

given by

$$\hat{\mathbf{A}}' = \frac{1}{2}\left(\hat{\mathbf{p}} \times \hat{\mathbf{L}} - (\hat{\mathbf{L}} \times \hat{\mathbf{p}}\right) - Z\frac{\hat{\mathbf{r}}}{\hat{r}}. \tag{E.2}$$

the elements of which commute with the Hamiltonian. Also, the elements of angular momentum and the Laplace-Runge-Lenz vector commute according to

$$[\hat{L}_j, \hat{L}_k] = i\epsilon_{jkl}\hat{L}_l, \tag{E.3}$$

$$[\hat{L}_j, \hat{A}'_k] = i\epsilon_{jkl}\hat{A}'_l, \tag{E.4}$$

$$[\hat{A}'_j, \hat{A}'_k] = (-2\hat{H})i\epsilon_{jkl}\hat{L}_l, \tag{E.5}$$

where $\hat{H}$ is the hydrogen Hamiltonian. These commutation relations almost define the span of these six elements as some algebra, except they are not closed under commutation due to the factor of $\hat{H}$. Restricting the action of the group to a subspace of the Hilbert space corresponding to a single energy level E_n, the vector $\hat{A}'$ may be scaled to remove the $\hat{H}$. This has one of two consequences. If a continuum energy level is used ($E_n > 0$), the negative sign in Eq. (E.5) remains yielding so(3,1), the non-compact algebra associated with the Lorenz group. If a bound energy level

is used ($E_n < 0$), the algebra is compact, and becomes so(4). Since the present interest is with bound states, only the latter scaling will be employed. This scaling is given by

$$\hat{\mathbf{A}} = \hat{\mathbf{A}}'/\sqrt{-2E_n}, \tag{E.6}$$

making SO(4) the degeneracy group of the bound portion of the hydrogen atom problem.

E.1.2 The Spectrum Generating Algebra

Continuing from Section A.3, the realization of so(2,1) developed there may be connected to the hydrogen atom problem. The Hamiltonian

$$\hat{H} = \frac{1}{2}\hat{p}^2 - \frac{Z}{\hat{r}}, \tag{E.7}$$

may be rewritten rendering the Schrödinger equation (with eigenvalues for $\hat{H}$ written as E)

$$(\hat{r}\hat{p}_r^2 + \hat{r}^{-1}\hat{L}^2 - 2\hat{r}E)|\psi\rangle = 2Z|\psi\rangle. \tag{E.8}$$

so that constant quantities are on the right and operator quantities are on the left. The states $|\psi\rangle$ will be not only eigenstates of $\hat{T}_3$, but also eigenstates of $\hat{L}^2$ since $[\hat{T}_3, \hat{L}^2] = [\hat{T}_3, \hat{L}_3] = 0$. Therefore, Eq. (E.8) may be rewritten

$$(\hat{r}\hat{p}_r^2 + \hat{r}^{-1}\ell(\ell+1) - 2\hat{r}E)|\psi\rangle = 2Z|\psi\rangle. \tag{E.9}$$

Eq. (E.9) identifies with Eq. (A.42) provided

$$n = 1, \tag{E.10}$$

$$\xi = \ell(\ell+1), \tag{E.11}$$

$$B = -2E, \tag{E.12}$$

$$D = 2Z. \tag{E.13}$$

From Eq. (A.45), the lowest eigenvalue is thus

$$b_0^{(\pm)} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1+4\ell(\ell+1)} = \begin{cases} -\ell \\ \\ \ell+1 \end{cases}. \tag{E.14}$$

Now from Eq. (A.46),

$$\frac{1}{2}DB^{-\frac{1}{2}} = Z(-2E_n)^{-\frac{1}{2}} = b_0^{(\pm)} + n', \qquad n' = 0, 1, 2, \ldots. \tag{E.15}$$

Firstly, we expect $E_n < 0$ since the right hand side of Eq. (E.15) is real. Secondly, if that is the case, then the left hand side is positive, so the negative root $b_0^{(-)}$ must be discarded. Solving Eq. (E.15) for E_n therefore yields the famous Bohr formula,

$$E_n = -\frac{Z^2}{2n^2}, \qquad n = \ell + 1 + n'. \tag{E.16}$$

Therefore, so(2,1) is generates the spectrum of the hydrogen atom with the realization

$$\hat{T}_{1,3} = \frac{1}{2}(\hat{r}\hat{p}_r^2 + \hat{L}^2\hat{r}^{-1} \mp \hat{r}), \tag{E.17}$$

$$\hat{T}_2 = \hat{r}\hat{p}_r. \tag{E.18}$$

Appendix F

Angular Momentum Annihilation Operator Eigenstates

It seems rather clear that an annihilation operator acting on a finite spectrum can have no eigenstates: An application of the annihilation operator on the any state annihilates the weight associated with the ground state, and no corresponding weight enters into the uppermost state. For a similar reason, a creation operator can have no eigenstate, acting on a finite or infinite spectrum bounded below. However, in a somewhat contrived manner, an annihilation-like operator may be constructed which has eigenstates on a finite spectrum, such as angular momentum states of a fixed total angular momentum.

Let

$$\hat{A}_- = \hat{J}_- + \frac{d}{(2j)!}|j,j\rangle\langle j,-j|, \tag{F.1}$$

for some complex number d, the $|j,m\rangle$ are the angular momentum states, simul-

taneous eigenstates of $\hat{J}^2$ and $\hat{J}_3$, and the $(2j)!$ was added for future convenience. Though $\hat{A}_-$ is j-dependent, it is not necessary to clutter the notation by adding the subscript. The second part of the operator $\hat{A}_-$ transfers the coefficient of $|j,-j\rangle$ to $|j,j\rangle$, so that one end of the set of states $|j,m\rangle$ is connected to the other when seen through $\hat{A}_-$. Accordingly, $\hat{A}_-$ transfers weights one step counterclockwise among the states, now tied into a loop. Note that the Hermitian conjugate $\hat{A}_-^\dagger = \hat{A}_+$ shifts in the opposite direction.

Suppose the eigenstate $|a\rangle$ of $\hat{A}_-$ with eigenvalue a can be expressed as a superposition over the $|j,m\rangle$. Then, recalling that

$$\hat{J}_-|j,m\rangle = \sqrt{(j-m+1)(j+m)}|j,m-1\rangle, \tag{F.2}$$

an application of $\hat{A}_-$ to $|a\rangle$ yields

$$\begin{aligned}
\hat{A}_-|a\rangle &= \sum_{m=-j}^{j} c_m \hat{A}_-|j,m\rangle \\
&= \sum_{m=-j}^{j} c_m\sqrt{(j-m+1)(j+m)}|j,m-1\rangle + \frac{d}{(2j)!}\sum_{m=-j}^{j} c_m|j,j\rangle\langle j,-j|j,m\rangle \\
&= \sum_{m=-j}^{j-1} c_{m+1}\sqrt{(j-m)(j+m+1)}|j,m\rangle + \frac{d}{(2j)!}c_{-j}|j,j\rangle \\
&= a\sum_{m=-j}^{j} c_m|j,m\rangle.
\end{aligned} \tag{F.3}$$

From the last equality above follows the recursion relation

$$c_{m+1} = \frac{ac_m}{\sqrt{(j-m)(j+m+1)}}, \text{ for } m = j,\ldots,j-1; \quad \frac{d}{(2j)!}c_{-j} = ac_j. \tag{F.4}$$

Leaving c_{-j} as a normalization constant, the first part of the recursion relation gives

$$c_m = c_{-j}\frac{a^{j+m}}{\sqrt{(2j)!}}\sqrt{\frac{(j-m)!}{(j+m)!}}. \tag{F.5}$$

Consistency with the second part demands that

$$c_j = c_{-j}\frac{a^{2j}}{(2j)!} = c_{-j}\frac{d}{a(2j)!}. \tag{F.6}$$

The states are therefore given by

$$|a\rangle = c_{-j}\sum_{m=-j}^{j}\frac{a^{j+m}}{\sqrt{(2j)!}}\sqrt{\frac{(j-m)!}{(j+m)!}}|j,m\rangle, \tag{F.7}$$

with normalization

$$1 = |c_j|^2\sum_{k=0}^{2j}\frac{(2j-k)!|a|^{2k}}{(2j)!k!}. \tag{F.8}$$

The restriction Eq. (F.6) has several interesting consequences, the first of which is that, unlike other annihilation operator constructions, there are only a finite number of eigenvalues. These eigenvalues are the $2j+1$ roots of $a^{2j+1} = d$, that is, arranged evenly about a circle on the complex plane centred on the origin, equal in number to the number of angular momentum eigenstates.

Since the states $|a\rangle$ are eigenstates of some operator, contrived as it may be, they satisfy a certain minimum uncertainty product for a pair of operators following the conclusions of Section 3.1. The uncertainty product in question is given by

$$(\Delta A_1)^2(\Delta A_2)^2 = \frac{1}{4}\langle A_3\rangle^2 \tag{F.9}$$

in which

$$\hat{A}_1 = \hat{J}_1 + \frac{1}{2(2j)!}(d|j,j\rangle\langle j,-j| + d^*|j,-j\rangle\langle j,j|), \tag{F.10}$$

$$\hat{A}_2 = \hat{J}_2 - \frac{i}{2(2j)!}(d|j,j\rangle\langle j,-j| - d^*|j,-j\rangle\langle j,j|), \tag{F.11}$$

$$\hat{A}_3 = -i[\hat{A}_1,\hat{A}_2] = J_3 + \cdots, \tag{F.12}$$

where, unfortunately, the remainder of $\hat{A}_3$ has been omitted since it is a large expression devoid of physical interpretation. For that reason, derivations along these lines end now. The existence of these states is interesting, as is the spectrum of $\hat{A}_-$. However, they have been constructed simply because it is possible to do so, and have no clear application.

Bibliography

[1] B. G. Adams, J. Čížek, and J. Paldus. Lie algebraic methods and their applications to simple quantum systems. In *Advances in Quantum Chemistry, Vol 19*. Academic Press, New York, 1988.

[2] G. Arfken. *Mathematical Methods for Physicists*. Academic Press, Inc., San Diego, third edition, 1985.

[3] P. W. Atkins and J. C. Dobson. Angular momentum coherent states. *Proc. R. Soc. London*, A321:321, 1971.

[4] I. S. Averbukh and N. F. Perelman. Fractional revivals: Universality in the long-term evolution of quantum wave packets beyond the correspondence principle dynamics. *Phys. Lett. A*, 139(9):449–453, 1989.

[5] G. A. Baker. Degeneracy of the n-dimensional, isotropic harmonic oscillator. *Phys. Rev.*, 103(4):1119–1120, 1956.

[6] L. E. Ballentine. The statistical interpretation of quantum mechanics. *Rev. of Mod. Phys.*, 42(4):358–381, 1970.

[7] L. E. Ballentine, Y. Yand, and J. P. Zibin. Inadequacy of Ehrenfest's theorem to characterize the classical regime. *Phys. Rev. A*, 50(4):2854–2859, 1994.

[8] A. O. Barut and G. L. Bornzin. $SO(4,2)$-formulation of the symmetry bradking in relativistic Kepler problems with or without magnetic charges. *J. Math. Phys.*, 12(5):841–846, 1971.

[9] A. O. Barut and L. Girardello. New "coherent" states associated with non-compact groups. *Commun. Math. Phys.*, 21:41–55, 1971.

[10] A. O. Barut and R. Raczka. *Theory of Group Representations and Applications*. Polish Scientific Publishers, Warszawa, second edition, 1980.

[11] H. Bateman and A. Erdélyi. *Higher Transcendental Functions*. R. E. Krieger Pub. Co., Malabar, Fl., USA, 1981.

[12] P. Bellomo and C. R. Stroud Jr. Dispersion of klauder's temporally stable coherent states for the hydrogen atom. *J. Phys. A*, 31:L445–L450, 1998.

[13] P. Bellomo and C. R. Stroud Jr. Comment on 'Coherent states for the hydrogen atom'. *Phys. Rev. A*, 59(1):900–903, 1999.

[14] P. M. Benoit, S. M. McRae, and E. R. Vrscay. Classical-quantum correspondence in perturbation theory. In J.M Yuan, D. H. Feng, and G. M. Zaslavsky, editors, *Quantum Dynamics of Chaotic Systems: Proceedings of the Third Drexel Symposium on Quantum Nonintegrability*, Langhorne, PA, 1993. Gordon and Breach Science Publishers.

[15] D. Bhaumik, B. Dutta-Roy, and G. Ghosh. Classical limit of the hydrogen atom. *J. Phys. A*, 19:1355–1364, 1986.

[16] D. Bhaumik. T. Nag, and B. Dutta-Roy. Coherent states for angular momentum. *J. Phys. A*, 8(12):1868–1874, 1975.

[17] K. Bhaumik and B. Dutta-Roy. The classical nonlinear oscillator and the coherent state. *J. Math. Phys.*, 16(5):1131–1133, 1974.

[18] D. Bohm. *Quantum Theory.* Prentice-Hall, Inc., New York, 1951.

[19] D. Bohm. A suggested interpretation of the quantum theory in terms of "hidden" variables. i. *Phys. Rev.*, 85(2):166–179, 1952.

[20] D. Bohm. A suggested interpretation of the quantum theory in terms of "hidden" variables. II. *Phys. Rev.*, 85(2):180–193, 1952.

[21] L. M. Brown. Classical limit of the hydrogen atom. *Am. J. Phys.*, 41:525, 1973.

[22] G. Casati, B. V. Chirikov, F. M. Izraelev, and J. Ford. Stochastic behaviour of a quantum pendulum under periodic perturbation. In G. Casati and J. Ford, editors, *Lecture Notes in Physics* **93***: Stochastic Behaviour in Classical and Quantum Hamiltonian Systems.* Springer-Verlag, New York, 1977.

[23] M. G. A. Crawford. The correspondence of two definitions of coherent states in a particular system. *J. Phys. A*, 32:L215–L219, 1999.

[24] E. de Prunelé. O(4,2) coherent states and the hydrogenic atoms. *Phys. Rev. A*, 42(2):2542–2549, 1990.

[25] L. Oliveria e Silva and J. T. Mendonça. Asymmetric pendulum. *Phys. Rev. A*, 46(10):6700–6707, 1992.

[26] A. Einstein, B. Podolsky, and N. Rosen. Can quantum-mechanical description of physical reality be considered complete? *Phys. Rev.*, 47:777–780, 1935.

[27] R. F. Fox. Generalized coherent states. *Phys. Rev. A*, 59(5):3241–3255, 1999.

[28] R. F. Fox and M. H. Choi. Generalized coherent states and quantum-classical correspondence. *Phys. Rev. A*, 61(3):032107–1–11, 2000.

[29] R. F. Fox and T. C. Elston. Chaos and a quantum-classical correspondence in the kicked top. *Phys. Rev. E*, 50(4):2553–2563, 1994.

[30] R. F. Fox and T. C. Elston. Chaos and the quantum-classical correspondence in the kicked pendulum. *Phys. Rev. E*, 49(5):3683–3696, 1994.

[31] Z. D. Gaeta and C. R. Stroud Jr. Classical and quantum-mechanical dynamics of a quasiclassical state of the hydrogen atom. *Phys. Rev. A*, 42(11):6308–6313, 1990.

[32] Z. D. Gaeta, M. W. Noel, and Jr. C. R. Stroud. Excitation of the classical-limit state of an atom. *Phys. Rev. Lett.*, 73(5):636–639, 1994.

[33] J. C. Gay, D Delande, and Antoine Bommier. Atomic quantum states with maximum localization of classical elliptical orbits. *Phys. Rev. A*, 39(12):6587–6590, 1989.

[34] J. P. Gazeau and J. R. Klauder. Coherent states for systems with discrete and continuous spectrum. *J. Phys. A*, 32:123–132, 1999.

[35] C. C. Gerry. On coherent states for the hydrogen atom. *J. Phys. A.*, 17:L737–L740, 1984.

[36] C. C. Gerry. Coherent states and the Kepler-Coulomb problem. *Phys. Rev. A*, 33(1):6–11, 1986.

[37] C. C. Gerry and J. Kiefer. Radial coherent states for the coulomb problem. *Phys. Rev. A*, 37(3):665–671, 1998.

[38] G. Ghosh. Generalized annihilation operator coherent states. *J. Math. Phys.*, 39(3):1366–1372, 1998.

[39] R. J. Glauber. Coherent and incoherent states of the radiation field. *Phys. Rev.*, 131(6):2766–2788, 1963.

[40] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series, and Products.* Academic Press, New York, fourth edition, 1965.

[41] V. P. Gutschick and M. M. Nieto. Coherent states for general potentials: V. Time evolution. *Phys. Rev. D*, 22(2):403–418, 1980.

[42] R. N. Hill. Hydrogenic wave functions. In G. W. F. Drake, editor, *Atomic, Molecular and Optical Physics Handbook.* AIP Press, Woodbury, New York, 1996.

[43] K. Husimi. *Proc. Phys. Math. Soc. Japan*, 22:264, 1940.

[44] G. Junker, H. Karl, and H. Leschke. Classical and quantum dynamics of the n-dimensional kicked rotator. *Physics Letters A*, 226:155–166, 1997.

[45] J. R. Klauder. Coherent states for the hydrogen atom. *J. Phys. A*, 29(12):L293–L298, 1996.

[46] J. R. Klauder and B.-S. Skagerstam, editors. *Coherent States: Applications in Physics and Mathematical Physics.* World Scientific, Singapore, 1985.

[47] P. Majumdar and H. S. Sharatchandra. Coherent states for the hydrogen atom. *Phys. Rev. A*, 56(5):R3322–R3325, 1997.

[48] M. Mallalieu and Jr. C. R. Stroud. Semiclassical dynamics of circular-orbit Rydberg wave packets. *Phys. Rev. A*, 49(4):2329–2339, 1994.

[49] D. S. McAnally and A. J. Bracken. Quasiclassical states of the Coulomb system and so(4,2). *J. Phys. A*, 23:2027–2055, 1990.

[50] S. M. McRae and E. R. Vrscay. Perturbation theory and the classical limit of quantum mechanics. *J. Math. Phys.*, 38(6):2899, 1997.

[51] J. Mostowski. On the classical limit of the Kepler problem. *Lett. Math. Phys.*, 2:1–5, 1977.

[52] J. E. Moyal. Quantum mechanics as a statistical theory. *Proc. Cambridge Phil. Soc.*, 45:99–124, 1949.

[53] M. Nauenberg. Quantum wave packets on Kepler elliptic orbits. *Phys. Rev. A*, 40(2):1133–1136, 1989.

[54] M. Nauenberg. Autocorrelation function and quantum recurrence of wavepackets. *J. Phys. B*, 23:L385–L390, 1990.

[55] M. M. Nieto. Coherent states for general potentials: IV. Three-dimensional systems. *Phys. Rev. D*, 22(2):391–401, 1980.

[56] M. M. Nieto. Resolution of the identity for minimum-uncertainty coherent states: An example related to charged-boson coherent states. *Phys. Rev. D*, 30(4):770–773, 1984.

[57] M. M. Nieto and L. M. Simmons, Jr. Coherent states for general potentials. *Phys. Rev. Lett.*, 41(4):207–210, 1978.

[58] M. M. Nieto and L. M. Simmons, Jr. Coherent states for general potentials: I. Formalism. *Phys. Rev. D*, 20(6):1321–1331, 1979.

[59] M. M. Nieto and L. M. Simmons, Jr. Coherent states for general potentials: II. Confining one-dimensional examples. *Phys. Rev. D*, 20(6):1332–1341, 1979.

[60] M. M. Nieto and L. M. Simmons, Jr. Coherent states for general potentials: III. Nonconfining one-dimensional examples. *Phys. Rev. D*, 20(6):1342–1350, 1979.

[61] M. M. Nieto, L. M. Simmons, Jr., and V. P. Gutschick. Coherent states for general potentials: VI. Conclusions about the classical motion and the WKB approximation. *Phys. Rev. D*, 23(4):927–933, 1981.

[62] M. M. Nieto and D. R. Truax. Squeezed states for general systems. *Phys. Rev. Lett.*, 71(18):2843–2846, 1993.

[63] D. Nouri. Generalized coherent states for the d-dimensional Coulomb problem. *Phys. Rev. A*, 60(2):1702, 1999.

[64] A. Pais. *Niels Bohr's Times: In Physics, Philosophy and Polity.* Oxford University Press, Oxford, 1991.

[65] J. Paldus. Dynamical groups. In G. W. F. Drake, editor, *Atomic, Molecular and Optical Physics Handbook.* AIP Press, Woodbury, New York, 1996.

[66] J. Parker and C. R. Stroud Jr. Coherence and decay of rydberg wave packets. *Phys. Rev. Lett.*, 56(7):716–719, 1986.

[67] A. M. Perelomov. Coherent states for arbitrary Lie group. *Commun. Math. Phys.*, 26:222–236, 1972.

[68] A. M. Perelomov. *Generalized Coherent States and Their Applications.* Springer-Verlag, London, 1986.

[69] W. S. Porter. An interesting coherent state of the rotator. *Am. J. Phys.*, 61(11):1050–1051, 1993.

[70] E. Schrödinger. Der stetige übergang von der Mikro-zur Makromechanik. *Die Naturwissenschaften*, 14:664–666, 1926. An English translation can be found in Schrödinger's *Collected Papers on Wave Mechanics* [71].

[71] E. Schrödinger. *Collected Papers on Wave Mechanics*. Chelsea Publishing Company, New York, third edition, 1982.

[72] J. Schwinger. On angular momentum. In L. C. Biedenharn and H. Van Dam, editors, *Quantum Theory of Angular Momentum*. Academic Press, New York, 1965.

[73] H. S. Sharatchandra. Coherent states for the anharmonic oscillator and classical phase space trajectories. Preprint quant-ph/9707032, 1997.

[74] J.-M. Sixdeniers, K. A. Penson, and A. I. Solomon. Mittag-Leffler coherent states. *J. Phys. A*, 32:7543–7563, 1999.

[75] M. Sugiura. *Unitary Representations and Harmonic Analysis*. North Holland, New York, 1990.

[76] J. S. Townsend. *A Modern Approach to Quantum Mechanics*. McGraw-Hill, Inc., New York, 1992.

[77] D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii. *Quantum Theory of Angular Momentum*. World Scientific, New Jersey, 1988.

[78] J. Čížek and J. Paldus. An algebraic approach to bound states of simple one-electron systems. *Int. J. Quantum Chem.*, 12:875–896, 1977.

[79] J. Čížek and E. R. Vrscay. Large order perturbation theory in the context of atomic and molecular physics — interdisciplinary aspects. *Int. J. Quant. Chem.*, 21:27–68, 1982.

[80] H. Weyl. *The Theory of Groups and Quantum Mechanics.* Dover, London, 1931.

[81] D. Wick. *The Infamous Boundary: Seven Decades of Controversy in quantum physics.* Birkhauser, Boston, 1995.

[82] E. Wigner. On the quantum correction for thermodynamic equilibrium. *Phys. Rev.*, 40:749–759, 1932.

[83] B. G. Wybourne. *Classical Groups for Physicists.* John Wiley and Sons, Toronto, 1974.

[84] G. Yaffe. Large N limits as classical mechanics. *Rev. Mod. Phys.*, 54(2):407–435, 1982.

[85] J. A. Yeazell and Jr. C. R. Stroud. Observation of spatially localized atomic electron wave packets. *Phys. Rev. Lett.*, 60(15):1494–1497, 1988.

[86] J. A. Yeazell and Jr. C. R. Stroud. Observation of fractional revivals in the evolution of a Rydberg atomic wave packet. *Phys. Rev. A*, 43(9):5153–5156, 1991.

[87] J. A. Yeazell, M. Mallaieu, J. Parker, and Jr. C. R. Stroud. Classical periodic motion of atomic-electron wave packets. *Phys. Rev. A*, 40(9):5040–5043, 1989.

[88] J. A. Yeazell, M. Mallalieu, and Jr. C. R. Stroud. Observation of the collapse and revival of a Rydberg electronic wave packet. *Phys. Rev. Lett.*, 64(17):2007–2010, 1990.

[89] J. A. Yeazell, G. Raithel, L. Marmet, H. Held, and H. Walther. Observation of wave packet motion along quasi-Landau orbits. *Phys. Rev. Lett.*, 70(19):2884–2887, 1993.

[90] W.-M. Zhang, D. H. Feng, and R. Gilmore. Coherent states: Theory and some applications. *Rev. Mod. Phys.*, 62(4):867–927, 1990.

[91] I. Zlatev, W.-M. Zhang, and D. H. Feng. Possibility that Schrödinger's conjecture for the hydrogen-atom coherent states is not attainable. *Phys. Rev. A*, 50(3):R1973–R1975, 1994.

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