Hybridizable discontinuous Galerkin methods for coupled flow and transport systems

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

In this thesis, we propose and analyze hybridizable discontinuous Galerkin methods for coupled flow and transport systems. Such systems may be used to model real-world scenarios in which a fluid contaminant travels through another medium. Common applications include environmental engineering problems and biochemical transport.

This thesis focuses on the Stokes/Darcy-transport and Navier–Stokes/Darcy-transport systems. We consider a two-way coupling between each flow and transport problem: the solution to the flow problem is directly involved in the transport problem, and the solution to the transport problem appears in the flow problem through a parameter function. In each of our considered systems, the flow problem is stationary while the transport problem is time-dependent. The resulting coupled flow and transport systems are quasi-stationary in the sense that the evolution of solutions to the flow problems over time is driven by the transport problem.

Our numerical schemes use a time-lagging method in which the flow and transport problems are decoupled and solved sequentially using hybridizable discontinuous Galerkin methods. This decoupling allows us to establish separate conditions on the discrete flow problem and on the discrete transport problem such that solutions to the combined scheme converge at optimal rates. Moreover, we show how existing results on related discrete flow problems and on the discrete transport problem may be exploited for efficient analysis of the coupled systems. We present this approach in a general setting, and illustrate its use through the specific examples of the Stokes/Darcy-transport and Navier–Stokes/Darcy-transport systems.

For all schemes, we establish the existence of unique numerical solutions over a considered time interval. We prove optimal rates of convergence in space and time, and provide numerical examples to support the theory.

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α The Beavers–Joseph constant 6

Γ_D^d The Dirichlet portion of Γ^d 4

Γ_N^d The Neumann portion of Γ^d 4

Γ^I The interface between the free flow and porous medium regions 4

Γ^j The external boundary of Ω^j 4

Γ_0^j The union of faces on Γ^j 8

c_I The injected concentration function 6

D The diffusion dispersion matrix in Ω^d 6

d The diffusion coefficient in Ω^s 6

$\tilde{D}$ A matrix with the value $D(u^d)$ in Ω^d, and $d\mathbb{I}$ in Ω^s 6

$\mathcal{F}^I$ The set of faces on Γ^I 8

$\mathcal{F}^j$ The set of faces on Γ^j 8

$\mathbb{I}$ The identity matrix 6

J The time interval 4

K A mesh element 8

κ The permeability matrix 5

k_c The polynomial degree used to define the discrete concentration spaces 9

k_f The polynomial degree used to define the discrete velocity and pressure spaces 9

μ The scalar, concentration-dependent fluid viscosity 4

n^j The outward unit normal to $\partial\Omega^j$ 4

$\partial\Omega$ The boundary of Ω 4

$\partial\Omega^j$ The boundary of Ω^j 4

$\mathcal{T}$ A triangulation of Ω 8

$\mathcal{T}^j$ A triangulation of Ω^j 8

τ^ℓ An orthonormal tangential vector 6

ϕ The scalar porosity 6

χ^d The characteristic function of Ω^d 6

Ω The spatial domain 4

Ω^d The porous medium region of Ω 4

Ω^s The free flow region of Ω 4

Chapter 1

Introduction

Coupled flow and transport systems may be used to model the propagation of a fluid substance through a medium. Important real-world scenarios fitting this description include environmental engineering problems such as seawater intrusion and groundwater contamination [2], and biochemical transport [16]. At its base, a coupled flow and transport system consists of two parts: a flow problem, which models the flow in the medium through which the propagation occurs; and a transport problem, which models the movement of the substance being propagated. The flow problem and the transport problem inherently affect each other, whence the term "coupled."

This thesis is centred on flow problems which themselves are coupled systems. Specifically, we focus on spatial domains composed of a free flow region and a porous medium region. We use the stationary Stokes and Navier–Stokes equations to model the flow in the free flow region, and the Darcy equations to model the flow in the porous medium region. We couple the models for the different spatial regions into one flow problem by imposing standard conditions at the interface between the regions: the conservation of mass, the balance of normal forces, and the Beavers–Joseph–Saffman condition [3, 34]. The transport equation is a time-dependent advection-diffusion equation which directly depends on the velocity of the fluid modelled by the flow problem. To make the coupling between the flow and transport problems two-way, we include parameters in the flow problems that depend on the concentration of the fluid modelled by the transport equation. Although the flow models considered here are stationary, the two-way coupling with the time-dependent transport equation causes the flows to evolve through time. We refer to such coupled flow and transport systems as quasi-stationary.

The existence of weak solutions to the quasi-stationary Stokes/Darcy-transport and quasi-stationary Navier–Stokes/Darcy-transport systems was established in the theoretical work [13]. In contrast, this thesis is devoted to the numerical solution of these systems by way of hybridizable discontinuous Galerkin (HDG) methods. The primary attractive feature of HDG methods is their ability to combine the benefits of continuous Galerkin (CG) and discontinuous Galerkin (DG) methods without the accompanying drawbacks of either method. Specifically, HDG methods provide the flexibility of DG methods at the low computational costs of CG methods [15]. HDG methods have been proposed and analyzed for a variety of problems closely related to those studied here: the stationary Stokes/Darcy system [12], the stationary Navier–Stokes/Darcy system [11], and the time-dependent Stokes/Darcy-transport system [9]. These methods and their analyses lay the groundwork for the HDG methods developed in this thesis. In particular, we use the framework of [9] to handle the nonlinearity in the coupling between the flow and transport problems. That is, we employ a decoupling algorithm in which we time-lag the concentration and solve the discrete flow problem and the discrete transport problem sequentially. This allows us to leverage results that have been previously proved separately for the flow problems and for the transport problem.

In this thesis, we first present a general approach to the development and error analysis of HDG methods for coupled flow and transport systems. As an initial demonstration of this approach, we apply it to the quasi-stationary Stokes/Darcy-transport system. Then, we move to the quasi-stationary Navier–Stokes/Darcy-transport system, which is more complex than the quasi-stationary Stokes/Darcy-transport system due to the addition of a nonlinear inertial term. For the quasi-stationary Navier–Stokes/Darcy-transport system, we present two different HDG methods: one in which we solve a nonlinear discretization of the flow problem in each iteration, and one in which we linearize the flow problem in all iterations but the first by time-lagging the convective velocity. For all schemes, we prove well-posedness and optimal rates of convergence in space and time. We also include numerical examples to support the theoretical claims.

We now mention existing works related to this topic. Numerical methods have been developed for various combinations of the models involved in the coupled flow and transport systems examined here. For instance, numerical methods for the Darcy/transport system include [44, 21, 36]. While numerical methods for the Navier–Stokes/transport system are not as numerous, a subgrid multiscale stabilized finite element method (FEM) was proposed in [24]. The numerical solution of quasi-stationary and time-dependent Stokes/Darcy-transport systems is well studied, for both the one-way coupling [39, 10, 25, 40, 19] and the two-way coupling [33, 42, 9, 37, 28, 43, 38]. Conversely, there exist few numerical studies on the more complex Navier–Stokes/Darcy-transport systems.

A DG method for a one-way coupled quasi-stationary Navier–Stokes/Darcy-transport system may be found in [32]. The author uses a primal DG method for both the Navier–Stokes and the Darcy equations. The transport equation is approximated by a DG method which upwinds the diffusive fluxes rather than averaging them as is done in standard DG methods. This eliminates the need for slope limiters, and results in improved robustness to physical conditions such as fractures and discontinuous permeability fields. To the extent of our knowledge, there is no existing numerical work on a two-way coupling of the Navier–Stokes/Darcy and transport problems.

The remainder of this thesis is structured as follows. In chapter 2, we define the specific coupled flow and transport systems studied here, and introduce common notation and results which will be used throughout the thesis. In chapter 3, we present a general approach to the error analysis of HDG schemes for coupled flow and transport systems. In chapter 4, we recall known results relating to the transport equation. In chapters 5 to 7, we propose schemes for the quasi-stationary Stokes/Darcy-transport and Navier–Stokes/Darcy-transport systems, prove the well-posedness thereof, and establish optimal error estimates. Chapter 5 is dedicated to the quasi-stationary Stokes/Darcy-transport system, and chapters 6 and 7 focus on schemes for the quasi-stationary Navier–Stokes/Darcy-transport system. In chapter 8, we provide numerical results. Chapter 9 is the conclusion.

Chapter 2

HDG schemes for coupled flow and transport systems

2.1 Coupled flow and transport systems

Each of our considered systems is defined on a bounded polygonal domain $\Omega \subset \mathbb{R}^{\dim}$, for $\dim = 2, 3$, over a time interval $J = [0, T]$. We assume that Ω is partitioned into a free flow region Ω^s and porous medium region Ω^d such that $\Omega = \Omega^s \cup \Omega^d$ and $\Omega^s \cap \Omega^d = \varnothing$. The interface between Ω^s and Ω^d is then given by $\Gamma^I := \partial\Omega^s \cap \partial\Omega^d$, where $\partial\Omega^j$ is the boundary of Ω^j. Letting $\partial\Omega$ denote the boundary of Ω, we define by $\Gamma^j := \partial\Omega \cap \partial\Omega^j$ the external boundary of Ω^j, for $j = s, d$. An example illustration with $\dim = 2$ is given in fig. 2.1. In the porous medium region, we partition Γ^d into a Neumann portion Γ^d_N and a Dirichlet portion Γ^d_D such that $\Gamma^d = \Gamma^d_N \cup \Gamma^d_D$ and $\Gamma^d_N \cap \Gamma^d_D = \varnothing$. While we assume that $|\Gamma^d_N| > 0$, we do not strictly require that $|\Gamma^d_D| > 0$ for the analyses in this thesis; we simply include this possibility for greater generality. As well, n^j denotes the outward unit normal to $\partial\Omega^j$. The superscript j may be dropped when there is no ambiguity.

The difference between the coupled flow and transport systems considered in this thesis lies in the momentum equation governing the flow in the free flow region. The quasi-stationary Stokes/Darcy-transport system uses the steady Stokes momentum equation

$$-\nabla \cdot (2\mu(c^s)\varepsilon(u^s)) + \nabla p^s = f^s(c^s) \qquad \text{in } \Omega^s \times J, \tag{2.1a}$$

where u^s is the velocity, p^s is the pressure, and $\varepsilon(u^s) := (\nabla u^s + (\nabla u^s)^T)/2$ is the strain rate tensor. The fluid viscosity μ and the source term f^s may depend on a concentration

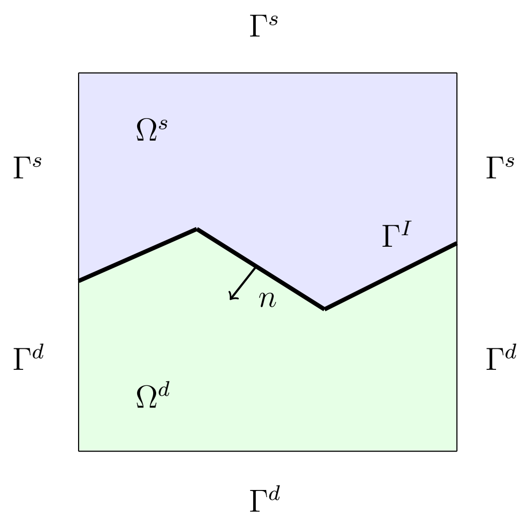


Figure 2.1: An example of a domain Ω partitioned into subdomains Ω^s and Ω^d.

c^s in the free flow region. Adding in an inertial term gives rise to the steady Navier–Stokes momentum equation used in the quasi-stationary Navier–Stokes/Darcy-transport system,

$$\nabla \cdot (u^s \otimes u^s) - \nabla \cdot (2\mu(c^s)\varepsilon(u^s)) + \nabla p^s = f^s(c^s) \qquad \text{in } \Omega^s \times J. \tag{2.1b}$$

The remaining equations, boundary conditions, and initial conditions are the same for both systems. The mass equation and boundary conditions in the free flow region are defined as

$$-\nabla \cdot u^s = 0 \qquad \text{in } \Omega^s \times J, \tag{2.1c}$$
$$u^s = 0 \qquad \text{on } \Gamma^s \times J. \tag{2.1d}$$

In the porous medium region, the flow is governed by the Darcy equations,

$$\mu(c^d)\kappa^{-1}u^d + \nabla p^d = \mu(c^d)\kappa^{-1}f^d(c^d) \qquad \text{in } \Omega^d \times J, \tag{2.1e}$$
$$-\nabla \cdot u^d = g_p - g_i \qquad \text{in } \Omega^d \times J, \tag{2.1f}$$

where u^d is the velocity, p^d is the piezometric head, κ is the permeability matrix, and the source and sink terms g_i and g_p are related to injection and production wells. For simplicity, we will refer to the piezometric head as the pressure for the remainder of this thesis. The fluid viscosity μ and the source term f^d may depend on a concentration c^d in the porous medium region. We take the boundary conditions

$$u^d \cdot n = 0 \qquad \text{on } \Gamma_N^d \times J, \tag{2.1g}$$

$$p^d = 0 \qquad \text{on } \Gamma_D^d \times J. \tag{2.1h}$$

In the case that $|\Gamma_D^d| = 0$, we assume for well-posedness that $\int_\Omega p \,\mathrm{d}x = 0$ and $\int_{\Omega^d} (g_p - g_i) \,\mathrm{d}x = 0$ at each $t \in J$. On the interface, we set $n = n^s$ and use the following conditions to couple together the flow problem in the free flow region and the flow problem in the porous medium region:

$$u^s \cdot n = u^d \cdot n \qquad \text{on } \Gamma^I \times J, \tag{2.2a}$$

$$p^d = p^s - 2\mu(c^s)\varepsilon(u^s)n \cdot n \qquad \text{on } \Gamma^I \times J, \tag{2.2b}$$

$$\alpha\mu(c^s)\kappa_\ell^{-1/2} u^s \cdot \tau^\ell = -2\mu(c^s)\varepsilon(u^s)n \cdot \tau^\ell, \quad 1 \le \ell \le \dim -1 \qquad \text{on } \Gamma^I \times J. \tag{2.2c}$$

In eq. (2.2c), $\kappa_\ell := \tau^\ell \cdot (\kappa \tau^\ell)$ where τ^ℓ are the orthonormal tangential vectors for $1 \le \ell \le \dim -1$, and $\alpha > 0$ is the experimentally-determined Beavers–Joseph constant. The interface condition eq. (2.2a) is the conservation of mass, eq. (2.2b) is the balance of normal forces, and eq. (2.2c) is the Beavers–Joseph–Saffman law [3, 34]. Finally, the transport equation for the concentration c is given by

$$\phi \partial_t c + \nabla \cdot (cu - \tilde{D}(u)\nabla c) = \chi^d (c_I g_i - c g_p) \qquad \text{in } \Omega \times J, \tag{2.3a}$$

$$c^s = c^d \qquad \text{on } \Gamma^I \times J, \tag{2.3b}$$

$$d\nabla c^s \cdot n = D(u^d)\nabla c^d \cdot n \qquad \text{on } \Gamma^I \times J, \tag{2.3c}$$

$$\tilde{D}(u)\nabla c \cdot n = 0 \qquad \text{on } \partial\Omega \setminus \Gamma_D^d \times J, \tag{2.3d}$$

$$c^d = 0 \qquad \text{on } \Gamma_D^d \times J. \tag{2.3e}$$

Here, χ^d is the characteristic function which takes the value 1 in Ω^d and 0 elsewhere. The scalar function ϕ is the space-dependent porosity of the porous medium in Ω^d and is set to 1 in Ω^s by convention. The matrix-valued function $\tilde{D}(u)$ is the diffusion dispersion matrix $D(u^d)$ in Ω^d and is set to $d\mathbb{I}$ in Ω^s, where d is the constant diffusion coefficient and $\mathbb{I}$ is the identity matrix. The function c_I is the injected concentration. To complete the model, we require an initial condition for the concentration,

$$c(x, 0) = c_0(x) \qquad \text{in } \Omega. \tag{2.3f}$$

We additionally make some assumptions on the functions involved in the systems. First, we assume that the initial concentration satisfies

$$c_0 \ge 0 \quad \text{a.e. in } \Omega, \qquad c_0 \in L^\infty(\Omega).$$

As well, we assume that μ is Lipschitz continuous in c with Lipschitz constant μ_L and that there exist positive constants ϕ_*, ϕ^*, μ_*, and μ^* such that

$$\phi_* \le \phi(x) \le \phi^* \qquad \forall x \in \Omega^d, \tag{2.4a}$$
$$\mu_* \le \mu(c) \le \mu^* \qquad \forall c \in \mathbb{R}. \tag{2.4b}$$

Next, we assume that there exist positive constants $D_{\min}$, $D_{\max}$, and D_L such that for all $u, v \in \mathbb{R}^{\dim}$,

$$D_{\min}(1+|u|)|\xi|^2 \le \xi^T D(u)\xi \le D_{\max}(1+|u|)|\xi|^2 \qquad \forall \xi \in \mathbb{R}^{\dim}, \tag{2.5a}$$
$$|D(u) - D(v)| \le D_L|u - v|, \tag{2.5b}$$

where $|\cdot|$ is the Euclidean norm. Similarly, we assume that κ is symmetric positive definite and that there exist positive constants κ_* and κ^* such that

$$\kappa_*|\xi|^2 \le \xi^T \kappa(x)\xi \le \kappa^*|\xi|^2 \qquad \forall \xi \in \mathbb{R}^{\dim},\ \forall x \in \bar{\Omega}^d. \tag{2.6}$$

We also assume that $g_p \ge 0$ and that $0 \le c_I \le 1$, for all $t \in J$. Finally, we assume that f^s and f^d are Lipschitz continuous in c with Lipschitz constants L_f^s and L_f^d.

2.2 Function spaces and norms

In this section, we introduce function spaces and norms that will be used throughout this thesis. We begin with some standard spaces and norms. Firstly, we have the Sobolev spaces $W^{m,p}(U)$ with corresponding norms $\|\cdot\|_{W^{m,p}(U)}$, where $1 \le p \le \infty$ and $0 \le m \le \infty$ are integers and U is a domain. We make some notational simplifications for frequently-used norms. When $p = 2$ and $m = 0$, we write $\|\cdot\|_U$ to indicate the norm on $L^2(U) := W^{0,2}(U)$. As well, $L_0^2(U)$ is used to denote the space of functions in $L^2(U)$ with zero mean. When $p = 2$ and $m \ne 0$, we write $\|\cdot\|_{m,U}$ to denote the norm on $H^m(U) := W^{m,2}(U)$. Similarly, we use $|\cdot|_{m,U}$ to denote the semi-norm on $H^m(U)$. In all other cases, we write $\|\cdot\|_{m,p,U}$ in place of $\|\cdot\|_{W^{m,p}(U)}$. Next, we define for scalars a and b and a Banach space X the Bochner spaces $L^2(a, b; X)$ and $L^\infty(a, b; X)$, with corresponding norms $\|f\|_{L^2(a,b;X)} := \left(\int_a^b \|f(t)\|_X^2 \,\mathrm{d}t\right)^{1/2}$ and $\|f\|_{L^\infty(a,b;X)} := \operatorname{ess\,sup}_{[a,b]} \|f(t)\|_X$. Further details on the above-mentioned spaces may be found in texts such as [20]. Lastly, we will use the discrete-in-time norm defined by $\|f\|_{\ell^2(0,T;X)} := \left(\Delta t \sum_{m=1}^n \|f^m\|_X^2\right)^{1/2}$ for an integer $n \ge 1$.

Before defining the discrete spaces used in this thesis, we first give a high-level overview of HDG spaces. To define these spaces, we discretize the spatial domain using shape-regular

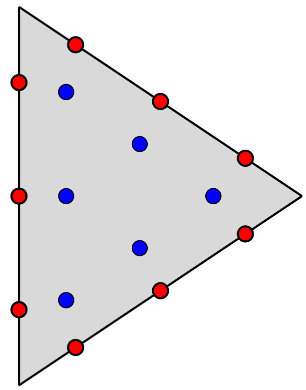


Figure 2.2: An example of a cell and its faces. Blue dots lie on the cell and represent degrees of freedom for cell functions. Red dots lie on the faces and represent degrees of freedom for face functions.

triangulations $\mathcal{T}^j := \{K\}$ of Ω^j, for $j = s, d$. We assume that $\mathcal{T}^s$ and $\mathcal{T}^d$ match at the interface, so that $\mathcal{T} := \mathcal{T}^s \cup \mathcal{T}^d$ is a triangulation of Ω. We refer to $K \in \mathcal{T}$ as a cell. For any cell $K \in \mathcal{T}$, any edge belonging to ∂K is called a face and is denoted by F. We use $\mathcal{F}^j$ and Γ_0^j to indicate the set and union of faces in Ω^j for $j = s, d$. When it is necessary to distinguish, we will use $\mathcal{F}_D^d$ and $\mathcal{F}_N^d$ to denote the sets of faces on Γ_D^d and Γ_N^d. As well, the notation $\mathcal{F}^I$ will be used to indicate the set of faces on Γ^I.

HDG methods include two types of functions: those that live on cells, and those that live on faces. This is illustrated in fig. 2.2. Intuitively, we will refer to the spaces used to define cell functions as cell spaces, and we will refer to the spaces used to define face functions as face spaces. The cell and face spaces used in our HDG methods consist of functions which are polynomial on their domains of definition. For this reason, it is useful to introduce the notation $P_r(U)$ to indicate the space of polynomials of degree r on a domain U.

We now define the specific spaces involved in the systems and HDG methods considered in this thesis. For the velocity, we require the spaces

$$
\begin{aligned}
V^s &:= \{v \in [H^2(\Omega^s)]^{\dim} \,:\, v = 0 \text{ on } \Gamma^s\}\,,\\
V^d &:= \{v \in [H^1(\Omega^d)]^{\dim} \,:\, v \cdot n = 0 \text{ on } \Gamma_N^d\}\,,\\
V &:= \{v \in H(\mathrm{div};\Omega) \,:\, v^s \in V^s, v^d \in V^d\}\,,\\
V_h &:= \{v_h \in [L^2(\Omega)]^{\dim} \,:\, v_h \in [P_{k_f}(K)]^{\dim}, \forall K \in \mathcal{T}\}\,,\\
V_h^j &:= \{v_h \in [L^2(\Omega^j)]^{\dim} \,:\, v_h \in [P_{k_f}(K)]^{\dim}, \forall K \in \mathcal{T}^j\}\,, \quad j = s,d,\\
\bar{V}_h &:= \{\bar{v}_h \in [L^2(\Gamma_0^s)]^{\dim} \,:\, \bar{v}_h \in [P_{k_f}(F)]^{\dim}, \forall F \in \mathcal{F}^s, \bar{v}_h = 0 \text{ on } \Gamma^s\}\,,
\end{aligned}
$$

where $k_f \geq 1$ is an integer. For the pressure, we require the spaces

$$
\begin{aligned}
Q &:= \{q \in L_0^2(\Omega) \,:\, q^s \in H^1(\Omega^s), q^d \in H^2(\Omega^d)\}\,,\\
Q_h &:= \{q_h \in L_0^2(\Omega) \,:\, q_h \in P_{k_f-1}(K), \forall K \in \mathcal{T}\}\,,\\
\bar{Q}_h^d &:= \{\bar{q}_h^d \in L^2(\Gamma_0^d) \,:\, \bar{q}_h^d \in P_{k_f}(F), \forall F \in \mathcal{F}^d\}
\end{aligned}
$$

when $|\Gamma_D^d| = 0$, and we require the spaces

$$
\begin{aligned}
Q &:= \{q \in L^2(\Omega) \,:\, q^s \in H^1(\Omega^s), q^d \in H^2(\Omega^d), q^d = 0 \text{ on } \Gamma_D^d\}\,,\\
Q_h &:= \{q_h \in L^2(\Omega) \,:\, q_h \in P_{k_f-1}(K), \forall K \in \mathcal{T}\}\,,\\
\bar{Q}_h^d &:= \{\bar{q}_h^d \in L^2(\Gamma_0^d) \,:\, \bar{q}_h^d \in P_{k_f}(F), \forall F \in \mathcal{F}^d, \bar{q}_h^d = 0 \text{ on } \Gamma_D^d\}
\end{aligned}
$$

otherwise. In all cases, we also require the spaces

$$
\begin{aligned}
Q_h^j &:= \{q_h \in L^2(\Omega^j) \,:\, q_h \in P_{k_f-1}(K), \forall K \in \mathcal{T}^j\}\,, \quad j = s,d,\\
\bar{Q}_h^s &:= \{\bar{q}_h^s \in L^2(\Gamma_0^s) \,:\, \bar{q}_h^s \in P_{k_f}(F), \forall F \in \mathcal{F}^s\}\,.
\end{aligned}
$$

For the concentration, we require the spaces

$$
\begin{aligned}
C &:= \{c \in H^2(\Omega)\}\,,\\
\bar{C}_h &:= \{\bar{c}_h \in L^2(\Gamma_0) \,:\, \bar{c}_h \in P_{k_c}(F), \forall F \in \mathcal{F}\}
\end{aligned}
$$

when $|\Gamma_D^d| = 0$, where $k_c \geq 0$ is an integer, and we require the spaces

$$
\begin{aligned}
C &:= \{c \in H^2(\Omega) \,:\, c = 0 \text{ on } \Gamma_D^d\}\,,\\
\bar{C}_h &:= \{\bar{c}_h \in L^2(\Gamma_0) \,:\, \bar{c}_h \in P_{k_c}(F), \forall F \in \mathcal{F}, \bar{c}_h = 0 \text{ on } \Gamma_D^d\}
\end{aligned}
$$

otherwise. In all cases, we also require the space

$$
C_h := \{c_h \in L^2(\Omega) \,:\, c_h \in P_{k_c}(K), \forall K \in \mathcal{T}\}\,.
$$

Letting $\bar{V}$, $\bar{Q}^j$, and $\bar{C}$ denote, respectively, the trace spaces of V restricted to Γ_0^s, Q^j restricted to Γ_0^j, and C restricted to Γ_0, we define $\boldsymbol{V} := V \times \bar{V}$, $\boldsymbol{Q} := Q \times \bar{Q}^s \times \bar{Q}^d$, and $\boldsymbol{C} := C \times \bar{C}$. We also denote $\boldsymbol{Q}^j := Q^j \times \bar{Q}^j$ for $j = s, d$ and $\boldsymbol{V}^s := V^s \times \bar{V}$. As well, we introduce trace operators $\gamma_V : V \to \bar{V}$, $\gamma_{Q^j} : Q^j \to \bar{Q}^j$, and $\gamma_C : C \to \bar{C}$. We may omit the subscript on γ when the function space on which the operator acts is clear from context. Then, we define the extended function spaces

$$
\begin{aligned}
&\boldsymbol{V}(h) := \boldsymbol{V}_h + \boldsymbol{V}, \quad \boldsymbol{V}^s(h) := \boldsymbol{V}_h^s + \boldsymbol{V}^s, \quad V^d(h) := V_h^d + V^d, \\
&\boldsymbol{Q}(h) := \boldsymbol{Q}_h + \boldsymbol{Q}, \quad \boldsymbol{C}(h) := \boldsymbol{C}_h + \boldsymbol{C},
\end{aligned}
$$

where $\boldsymbol{V}_h := V_h \times \bar{V}_h$, $\boldsymbol{V}_h^s := V_h^s \times \bar{V}_h$, $\boldsymbol{Q}_h := Q_h \times \bar{Q}_h^s \times \bar{Q}_h^d$, and $\boldsymbol{C}_h := C_h \times \bar{C}_h$. We furthermore define $\boldsymbol{Q}_h^j := Q_h^j \times \bar{Q}_h^j$ for $j = s, d$. Finally, we set $X := V \times Q$, $\boldsymbol{X}_h := \boldsymbol{V}_h \times \boldsymbol{Q}_h$, and $\boldsymbol{X}(h) := \boldsymbol{V}(h) \times \boldsymbol{Q}(h)$.

We will often write $\boldsymbol{v} := (v, \gamma_V(v)) \in \boldsymbol{V}(h)$ and $\boldsymbol{v}_h := (v_h, \bar{v}_h) \in \boldsymbol{V}_h$. Similar notation will be used for functions in $\boldsymbol{Q}(h)$, $\boldsymbol{C}(h)$, $\boldsymbol{V}_h^s$, $\boldsymbol{Q}_h$, $\boldsymbol{Q}_h^j$, and $\boldsymbol{C}_h$. We remark that for $\boldsymbol{q}_h := (q_h, \bar{q}_h^s, \bar{q}_h^d) \in \boldsymbol{Q}_h$, it is not guaranteed that $\bar{q}_h^s$ and $\bar{q}_h^d$ match at Γ^I. Furthermore, functions $\bar{v}_h \in \bar{V}_h$ are not defined on $\Gamma_0^d \setminus \Gamma^I$.

With our function spaces established, we now introduce norms. On $\boldsymbol{V}(h)$, we use the norms

$$
\begin{aligned}
|||\boldsymbol{v}|||_v^2 &:= |||\boldsymbol{v}|||_{v,s}^2 + |||\boldsymbol{v}|||_{v,d}^2 + \|\bar{v}^t\|_{\Gamma^I}^2, \\
|||\boldsymbol{v}|||_{v'}^2 &:= |||\boldsymbol{v}|||_{v',s}^2 + |||\boldsymbol{v}|||_{v,d}^2 + \|\bar{v}^t\|_{\Gamma^I}^2,
\end{aligned}
$$

where $w^t := w - (w \cdot n)n$ is the tangential component of a vector w and

$$
\begin{aligned}
|||\boldsymbol{v}|||_{v,s}^2 &:= \sum_{K \in \mathcal{T}^s} \left(\|\nabla v\|_K^2 + h_K^{-1} \|v - \bar{v}\|_{\partial K}^2 \right), \\
|||\boldsymbol{v}|||_{v',s}^2 &:= |||\boldsymbol{v}|||_{v,s}^2 + \sum_{K \in \mathcal{T}^s} h_K^2 |v|_{2,K}^2, \\
|||\boldsymbol{v}|||_{v,d}^2 &:= \|v\|_{\Omega^d}^2 + \|\nabla \cdot v\|_{\Omega^d}^2 \\
&\quad + \sum_{F \in \mathcal{F}^d \setminus (\mathcal{F}^I \cup \mathcal{F}_D^d)} h_F^{-1} \|[\![v \cdot n]\!]\|_F^2 + \sum_{K \in \mathcal{T}^d} h_K^{-1} \|(v - \bar{v}) \cdot n\|_{\partial K \cap \Gamma^I}^2 .
\end{aligned}
$$

Here, $[\![\cdot]\!]$ is the jump operator [18, Definition 1.17]. These norms have been previously used in [8], and they reduce to the norms used in [11] when $|\Gamma_D^d| = 0$. It was shown in [41, Eqn. 5.5] that $|||\cdot|||_{v,s}$ and $|||\cdot|||_{v',s}$ are equivalent norms on $\boldsymbol{V}_h$, which implies that $|||\cdot|||_v$ and

$|||\cdot|||_{v'}$ are equivalent on $\boldsymbol{V}_h$ with the same equivalence constants. Specifically, there exists a constant $c_e \geq 1$ such that

$$|||\boldsymbol{v}|||_{v,s} \leq |||\boldsymbol{v}|||_{v',s} \leq c_e |||\boldsymbol{v}|||_{v,s} \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_h, \tag{2.7}$$

$$|||\boldsymbol{v}|||_{v} \leq |||\boldsymbol{v}|||_{v'} \leq c_e |||\boldsymbol{v}|||_{v} \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_h. \tag{2.8}$$

We will also make use of

$$\|v_h\|_{1,h,\Omega^s} := |||(v_h, \{\!\{v_h\}\!\})|||_{v,s},$$

where $\{\!\{\cdot\}\!\}$ is the average operator [18, Definition 1.17]. Using the definitions of the average and jump operators, and applying the triangle inequality, Young's inequality, and [18, Lemma 1.43], we may write

$$\begin{aligned}
\sum_{K\in\mathcal{T}^s} h_K^{-1} \|v_h - \{\!\{v_h\}\!\}\|_{\partial K}^2 &= \frac{1}{4} \sum_{K\in\mathcal{T}^s} h_K^{-1} \sum_{F\in\partial K\setminus\partial\Omega^s} \|[\![v_h]\!]\|_F^2 \\
&\leq \frac{1}{2} \sum_{K\in\mathcal{T}^s} h_K^{-1} \sum_{F\in\partial K\setminus\partial\Omega^s} \left(\|v_h - \bar{v}_h\|_F^2 + \|v_h^{adj} - \bar{v}_h\|_F^2 \right) \\
&\leq \frac{1}{2} \sum_{K\in\mathcal{T}^s} h_K^{-1} \left(\|v_h - \bar{v}_h\|_{\partial K}^2 + \sum_{F\in\partial K\setminus\partial\Omega^s} \|v_h^{adj} - \bar{v}_h\|_F^2 \right) \\
&\leq c_\rho \sum_{K\in\mathcal{T}^s} h_K^{-1} \|v_h - \bar{v}_h\|_{\partial K}^2 ,
\end{aligned}$$

for a positive constant c_ρ related to the regularity of the mesh. In the above work, we have used v_h^{adj} to denote the cell function on the element $K^{adj} \neq K$ such that $\partial K \cap \partial K^{adj} = F$. It then follows that

$$\|v_h\|_{1,h,\Omega^s} \leq c_\rho |||\boldsymbol{v}_h|||_{v,s} \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^s. \tag{2.9}$$

In the analysis, we will make use of the following subspaces of $\boldsymbol{V}_h$:

$$\begin{aligned}
\boldsymbol{Z}_h^s &:= \{\boldsymbol{v}_h \in \boldsymbol{V}_h^s : b_h^s(\boldsymbol{q}_h^s, v_h) + b_h^{I,s}(\bar{q}_h^s, \bar{v}_h) = 0 \ \forall \boldsymbol{q}_h^s \in \boldsymbol{Q}_h^s\}, \\
\boldsymbol{Z}_h &:= \{\boldsymbol{v}_h \in \boldsymbol{V}_h : b_h(\boldsymbol{q}_h, \boldsymbol{v}_h) = 0 \ \forall \boldsymbol{q}_h \in \boldsymbol{Q}_h\}.
\end{aligned}$$

Definitions of the involved bilinear forms will be given in section 2.3. A noteworthy property of $\boldsymbol{Z}_h$ is that $|||\boldsymbol{v}_h|||_{v,d} = \|v_h\|_{\Omega^d}$ for functions $\boldsymbol{v}_h \in \boldsymbol{Z}_h$ [11, Lemma 6].

On $\boldsymbol{Q}(h)$, we use the norm

$$|||\boldsymbol{q}|||_p^2 := \sum_{j=s,d} |||\boldsymbol{q}^j|||_{p,j}^2,$$

where

$$\|\!|\boldsymbol{q}^j|\!\|_{p,j}^2 := \|q\|_{\Omega^j}^2 + \sum_{K\in\mathcal{T}^j} h_K \|\bar{q}^j\|_{\partial K}^2, \quad j = s, d.$$

This norm has previously been used for HDG discretizations of related problems such as [12, 9, 8].

On $\boldsymbol{C}(h)$, we use the semi-norm

$$\|\!|\boldsymbol{c}|\!\|_c^2 := \sum_{K\in\mathcal{T}} \left(\|\nabla c\|_K^2 + h_K^{-1} \|c - \bar{c}\|_{\partial K}^2 \right),$$

which has previously been used in the works [9, 10].

2.3 HDG forms

In this section, we set out various forms that will be used in the HDG discretizations of the flow and transport problems. The forms presented here are well-established and have been used in works such as [12, 9, 8]. We begin with some forms that are bilinear in $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(h)$, given $\boldsymbol{c} \in \boldsymbol{C}(h)$:

$$\begin{aligned}
a_h^s(c; \boldsymbol{u}, \boldsymbol{v}) :=& \sum_{K\in\mathcal{T}^s} \int_K 2\mu(c)\varepsilon(u) : \varepsilon(v)\,\mathrm{d}x + \sum_{K\in\mathcal{T}^s} \int_{\partial K} \frac{2\beta_f}{h_K}\mu(c)(u-\bar{u})\cdot(v-\bar{v})\,\mathrm{d}s \\
& - \sum_{K\in\mathcal{T}^s} \int_{\partial K} 2\mu(c)\varepsilon(u)n^s\cdot(v-\bar{v})\,\mathrm{d}s - \sum_{K\in\mathcal{T}^s} \int_{\partial K} 2\mu(c)\varepsilon(v)n^s\cdot(u-\bar{u})\,\mathrm{d}s, \\
a_h^d(c; u, v) :=& \sum_{K\in\mathcal{T}^d} \int_K \mu(c)\kappa^{-1} u\cdot v\,\mathrm{d}x, \\
a_h^I(\bar{c}; \bar{u}, \bar{v}) :=& \sum_{\ell=1}^{\dim -1} \int_{\Gamma^I} \alpha\mu(\bar{c})\kappa_\ell^{-1/2}(\bar{u}\cdot\tau^\ell)(\bar{v}\cdot\tau^\ell)\,\mathrm{d}s,
\end{aligned}$$

where $\beta_f > 0$ is a penalty parameter. We use these forms to define

$$a_h^L(\boldsymbol{c}; \boldsymbol{u}, \boldsymbol{v}) := a_h^s(c; \boldsymbol{u}, \boldsymbol{v}) + a_h^d(c; u, v) + a_h^I(\bar{c}; \bar{u}, \bar{v}).$$

Next, we define a form which is bilinear in $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}^s(h)$, given $w \in V^s(h)$:

$$\begin{aligned}
t_h(w; \boldsymbol{u}, \boldsymbol{v}) :=& -\sum_{K\in\mathcal{T}^s} \int_K u\otimes w : \nabla v\,\mathrm{d}x + \sum_{K\in\mathcal{T}^s} \int_{\partial K} \frac{1}{2} w\cdot n(u+\bar{u})\cdot(v-\bar{v})\,\mathrm{d}s \\
& + \sum_{K\in\mathcal{T}^s} \int_{\partial K} \frac{1}{2}|w\cdot n|(u-\bar{u})\cdot(v-\bar{v})\,\mathrm{d}s + \int_{\Gamma^I} (w\cdot n)\bar{u}\cdot\bar{v}\,\mathrm{d}s.
\end{aligned}$$

This form corresponds to the nonlinear inertial term of eq. (2.1b). With this in hand, we define the following bilinear form for $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(h)$, given $\boldsymbol{c} \in \boldsymbol{C}(h)$ and $w \in V^s(h)$:

$$\begin{aligned} a_h(\boldsymbol{c}; w; \boldsymbol{u}, \boldsymbol{v}) &:= a_h^s(c; \boldsymbol{u}, \boldsymbol{v}) + a_h^d(c; u, v) + a_h^I(\bar{c}; \bar{u}, \bar{v}) + t_h(w; \boldsymbol{u}, \boldsymbol{v}) \\ &= a_h^L(\boldsymbol{c}; \boldsymbol{u}, \boldsymbol{v}) + t_h(w; \boldsymbol{u}, \boldsymbol{v}). \end{aligned}$$

Although this form is technically new, it differs from the analogous form of [11] only by its dependence on the concentration $\boldsymbol{c}$ through the viscosity parameter. Next, we define forms which couple together a velocity $\boldsymbol{v} \in \boldsymbol{V}(h)$ and pressure $\boldsymbol{q}^j \in \boldsymbol{Q}^j(h)$, for $j = s, d$:

$$\begin{aligned} b_h^j(\boldsymbol{q}^j, v) &:= -\sum_{K \in \mathcal{T}^j} \int_K q^j \nabla \cdot v \,\mathrm{d}x + \sum_{K \in \mathcal{T}^j} \int_{\partial K} \bar{q}^j v \cdot n^j \,\mathrm{d}s, \\ b_h^{I,j}(\bar{q}^j, \bar{v}) &:= -\int_{\Gamma^I} \bar{q}^j \bar{v} \cdot n^j \,\mathrm{d}s. \end{aligned}$$

Combining these, we obtain for $\boldsymbol{q} \in \boldsymbol{Q}(h)$ and $\boldsymbol{v} \in \boldsymbol{V}(h)$ the form

$$b_h(\boldsymbol{q}, \boldsymbol{v}) := \sum_{j=s,d} b_h^j(\boldsymbol{q}^j, v) + \sum_{j=s,d} b_h^{I,j}(\bar{q}^j, \bar{v}).$$

Next, we define bilinear forms for $\boldsymbol{c}, \boldsymbol{\psi} \in \boldsymbol{C}(h)$, given $u \in V(h)$:

$$\begin{aligned} \boldsymbol{B}_h(u; \boldsymbol{c}, \boldsymbol{\psi}) &:= \boldsymbol{B}_h^a(u; \boldsymbol{c}, \boldsymbol{\psi}) + \boldsymbol{B}_h^d(u; \boldsymbol{c}, \boldsymbol{\psi}), \\ \boldsymbol{B}_h^a(u; \boldsymbol{c}, \boldsymbol{\psi}) &:= -\sum_{K \in \mathcal{T}} \int_K c u \cdot \nabla \psi \,\mathrm{d}x + \sum_{K \in \mathcal{T}} \int_{\partial K} (c u \cdot n)(\psi - \bar{\psi}) \,\mathrm{d}s \\ &\quad - \sum_{K \in \mathcal{T}} \int_{\partial K^{\mathrm{in}}} u \cdot n (c - \bar{c})(\psi - \bar{\psi}) \,\mathrm{d}s, \\ \boldsymbol{B}_h^d(u; \boldsymbol{c}, \boldsymbol{\psi}) &:= \sum_{K \in \mathcal{T}} \int_K \tilde{D}(u) \nabla c \cdot \nabla \psi \,\mathrm{d}x + \sum_{K \in \mathcal{T}} \int_{\partial K} \frac{\beta_c}{h_K} [\tilde{D}(u) n](c - \bar{c}) \cdot (\psi - \bar{\psi}) n \,\mathrm{d}s \\ &\quad - \sum_{K \in \mathcal{T}} \int_{\partial K} \big(\tilde{D}(u) \nabla c \cdot n\big)(\psi - \bar{\psi}) \,\mathrm{d}s - \sum_{K \in \mathcal{T}} \int_{\partial K} \big(\tilde{D}(u) \nabla \psi \cdot n\big)(c - \bar{c}) \,\mathrm{d}s, \end{aligned}$$

where $\beta_c > 0$ is a penalty parameter and ∂K^{in} is the portion of ∂K where $u \cdot n < 0$. The forms $\boldsymbol{B}_h^a(\cdot; \cdot, \cdot)$ and $\boldsymbol{B}_h^d(\cdot; \cdot, \cdot)$ correspond to the advective and diffusive parts of the discrete transport equation.

Lastly, we note that eqs. (2.1a) and (2.1b) have the same right-hand side. This right-hand side can be summed with the right-hand side of eq. (2.1e) to express the right-hand sides of the momentum equations over the full domain Ω. This has the following HDG discretization:

$$\int_{\Omega^s} f^s(c_h) \cdot v_h \,\mathrm{d}x + \int_{\Omega^d} \mu(c_h)\kappa^{-1} f^d(c_h) \cdot v_h \,\mathrm{d}x. \tag{2.10a}$$

Similarly, we can sum the right-hand sides of eqs. (2.1c) and (2.1f) to express the right-hand sides of the mass equations over Ω, with the following HDG discretization:

$$\sum_{K\in\mathcal{T}^d} \int_K q_h^d (g_p - g_i) \,\mathrm{d}x. \tag{2.10b}$$

2.3.1 Properties

Here we recall established results on the various HDG forms and prove straightforward extensions thereof. Firstly, the continuity of the terms in $a_h^L(\cdot;\cdot,\cdot)$ follows from [11, Eqn 16] and the assumptions eqs. (2.4b) and (2.6): there exists a constant $c_{ac}^s > 0$ independent of h such that for all $\boldsymbol{c} \in \boldsymbol{C}(h)$ and $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(h)$,

$$|a_h^s(c; \boldsymbol{u}, \boldsymbol{v})| \le c_{ac}^s \mu^* |||\boldsymbol{u}|||_{v',s} |||\boldsymbol{v}|||_{v',s}, \tag{2.11a}$$

$$|a_h^d(c; u, v)| \le \mu^* \kappa_*^{-1} \|u\|_{\Omega^d} \|v\|_{\Omega^d}, \tag{2.11b}$$

$$|a_h^I(\bar{c}; \bar{u}, \bar{v})| \le \alpha\mu^* \kappa_*^{-1/2} \|\bar{u}^t\|_{\Gamma^I} \|\bar{v}^t\|_{\Gamma^I}. \tag{2.11c}$$

As a result,

$$|a_h^L(\boldsymbol{c}; \boldsymbol{u}, \boldsymbol{v})| \le c_{ac}^L \mu^* |||\boldsymbol{u}|||_{v'} |||\boldsymbol{v}|||_{v'} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(h), \tag{2.12}$$

with $c_{ac}^L = 2c_e^2 \max\left(c_{ac}^s, \kappa_*^{-1}, \alpha\kappa_*^{-1/2}\right)$.

The continuity of $t_h(\cdot;\cdot,\cdot)$ was shown in [11, Lemma 4]: for any $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(h)$, there exists a constant $c_w > 0$ such that

$$|t_h(w_1; \boldsymbol{u}, \boldsymbol{v}) - t_h(w_2; \boldsymbol{u}, \boldsymbol{v})| \le c_w \|w_1 - w_2\|_{1,h,\Omega^s} |||\boldsymbol{u}|||_{v,s} |||\boldsymbol{v}|||_{v,s}, \tag{2.13}$$

provided that $w_1, w_2 \in V^s(h) \cap H(\mathrm{div}; \Omega^s)$ such that $\nabla \cdot w_j = 0$ on each $K \in \mathcal{T}^s$ for $j = 1, 2$. By combining the result on $a_h^L(\cdot;\cdot,\cdot)$ with the result on $t_h(\cdot;\cdot,\cdot)$, we obtain the continuity of $a_h(\cdot;\cdot;\cdot,\cdot)$ as given in the following lemma.

Lemma 2.3.1. *Let $w \in V^s(h) \cap H(div; \Omega^s)$ such that $\nabla \cdot w = 0$ on each $K \in \mathcal{T}^s$. Furthermore, let $\boldsymbol{c} \in \boldsymbol{C}(h)$. For all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}(h)$, we have that*

$$|a_h(\boldsymbol{c}; w; \boldsymbol{u}, \boldsymbol{v})| \le c_{ac} \mu^* |||\boldsymbol{u}|||_{v'} |||\boldsymbol{v}|||_{v'}, \tag{2.14a}$$

with $c_{ac} = 2c_e^2 \max(c_w(\mu^*)^{-1} \|w\|_{1,h,\Omega^s} + c_{ac}^s, \kappa_*^{-1}, \alpha\kappa_*^{-1/2})$. *If we additionally have that* $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}_h$, *then*

$$|a_h(\boldsymbol{c}; w; \boldsymbol{u}, \boldsymbol{v})| \le c_{ac}\mu^* |||\boldsymbol{u}|||_v |||\boldsymbol{v}|||_v. \tag{2.14b}$$

Proof. Taking the same steps as the proof of [11, Lemma 5] and using eqs. (2.11a) to (2.11c) and (2.13), we obtain

$$\begin{aligned}|a_h(\boldsymbol{c}; w; \boldsymbol{u}, \boldsymbol{v})| \le{}& c_w \|w\|_{1,h,\Omega^s} |||\boldsymbol{u}|||_{v,s} |||\boldsymbol{v}|||_{v,s} + c_{ac}^s \mu^* |||\boldsymbol{u}|||_{v',s} |||\boldsymbol{v}|||_{v',s} \\ &+ \mu^* \kappa_*^{-1} \|u\|_{\Omega^d} \|v\|_{\Omega^d} + \alpha\mu^*\kappa_*^{-1/2} \|\bar{u}^t\|_{\Gamma^I} \|\bar{v}^t\|_{\Gamma^I} \\ \le{}& 2\max(c_w \|w\|_{1,h,\Omega^s} + \mu^* c_{ac}^s, \mu^*\kappa_*^{-1}, \alpha\mu^*\kappa_*^{-1/2}) |||\boldsymbol{u}|||_{v'} |||\boldsymbol{v}|||_{v'}.\end{aligned} \tag{2.15}$$

From here, eq. (2.14a) follows from the fact that $c_e \ge 1$. On the other hand, eq. (2.14b) may be concluded by applying eq. (2.8) to eq. (2.15). □

The coercivity of the terms in $a_h^L(\cdot; \cdot, \cdot)$ is proved by a modification to the proof of [29, Lemma 4.2]. It gives the existence of a constant $c_{ae}^s > 0$ independent of h and a constant $\beta_0^f > 0$ such that for $\beta_f > \beta_0^f$ and for any $\boldsymbol{c} \in \boldsymbol{C}(h)$, there holds for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ that

$$a_h^s(c; \boldsymbol{v}_h, \boldsymbol{v}_h) \ge \mu_* c_{ae}^s |||\boldsymbol{v}_h|||_{v,s}^2, \tag{2.16a}$$

$$a_h^d(c; v_h, v_h) \ge \mu_*(\kappa^*)^{-1} \|v_h\|_{\Omega^d}^2, \tag{2.16b}$$

$$a_h^I(\bar{c}; \bar{v}_h, \bar{v}_h) \ge \alpha\mu_*(\kappa^*)^{-1/2} \|\bar{v}_h^t\|_{\Gamma^I}^2. \tag{2.16c}$$

Consequently,

$$a_h^L(\boldsymbol{c}; \boldsymbol{v}_h, \boldsymbol{v}_h) \ge c_{ae}^L \mu_* |||\boldsymbol{v}_h|||_v^2 \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \text{ s.t. } |||\boldsymbol{v}_h|||_{v,d} = \|\boldsymbol{v}_h\|_{\Omega^d}, \tag{2.17}$$

with $c_{ae}^L = \min\left(c_{ae}^s, (\kappa^*)^{-1}, \alpha(\kappa^*)^{-1/2}\right)$. These results allow us to prove the coercivity of $a_h(\cdot; \cdot; \cdot, \cdot)$ by the following modification to the proof of [11, Lemma 6].

Lemma 2.3.2. *Let* $w \in V^s(h) \cap H(div; \Omega^s)$ *such that* $\nabla \cdot w = 0$ *on each* $K \in \mathcal{T}^s$ *and* $\|w \cdot n\|_{\Gamma^I} \le \mu_* c_{ae}^s \delta (c_{pq}^2 + c_{si,4}^2)^{-1}$ *with* $0 < \delta < 1$. *Then, for* $\beta_f > \beta_0^f$ *and for any* $\boldsymbol{c} \in \boldsymbol{C}(h)$,

$$a_h(\boldsymbol{c}; w; \boldsymbol{v}_h, \boldsymbol{v}_h) \ge c_{ae}\mu_* |||\boldsymbol{v}_h|||_v^2 \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \ \textit{s.t.}\ |||\boldsymbol{v}_h|||_{v,d} = \|\boldsymbol{v}_h\|_{\Omega^d}, \tag{2.18}$$

where $c_{ae} = \min\left(c_{ae}^s(1-\delta), (\kappa^*)^{-1}, \alpha(\kappa^*)^{-1/2}\right) > 0$.

Proof. In [11, Eqns 26-28], it was shown that there exist positive constants c_{pq} and $c_{si,4}$ such that

$$t_h(w;\boldsymbol{v}_h,\boldsymbol{v}_h) \geq -(c_{pq}^2 + c_{si,4}^2)\left\|w\cdot n\right\|_{\Gamma^I} |||\boldsymbol{v}_h|||_{v,s}^2.$$

For $\beta_f > \beta_0^f$, it then follows by eq. (2.16), the assumption on $\left\|w\cdot n\right\|_{\Gamma^I}$, and that $|||\boldsymbol{v}_h|||_{v,d} = \left\|v_h\right\|_{\Omega^d}$ for $\boldsymbol{v}_h \in \boldsymbol{Z}_h$ that

$$\begin{aligned}
&a_h(\boldsymbol{c};w;\boldsymbol{v}_h,\boldsymbol{v}_h)\\
&\geq \left(\mu_* c_{ae}^s - (c_{pq}^2 + c_{si,4}^2)\left\|w\cdot n\right\|_{\Gamma^I}\right)|||\boldsymbol{v}_h|||_{v,s}^2 + \mu_*(\kappa^*)^{-1}\left\|v_h\right\|_{\Omega^d}^2 + \alpha\mu_*(\kappa^*)^{-1/2}\left\|\bar{v}_h^t\right\|_{\Gamma^I}^2\\
&\geq \mu_* c_{ae}^s(1-\delta)|||\boldsymbol{v}_h|||_{v,s}^2 + \mu_*(\kappa^*)^{-1}\left\|v_h\right\|_{\Omega^d}^2 + \alpha\mu_*(\kappa^*)^{-1/2}\left\|\bar{v}_h^t\right\|_{\Gamma^I}^2\\
&\geq c_{ae}\mu_*|||\boldsymbol{v}_h|||_v^2,
\end{aligned}$$

as desired. □

The boundedness of $b_h(\cdot,\cdot)$ was shown in [11, Lemma 3] for the case of $|\Gamma_D^d| = 0$. By following the same steps and using that $\bar{q}_h^d = 0$ on Γ_D^d, we obtain the same result when $|\Gamma_D^d| > 0$: there exists a positive constant c_{bc} such that for all $(\boldsymbol{v},\boldsymbol{q}) \in \boldsymbol{X}(h)$,

$$|b_h(\boldsymbol{q},\boldsymbol{v})| \leq c_{bc}|||\boldsymbol{v}|||_v|||\boldsymbol{q}|||_p. \tag{2.19}$$

The inf-sup condition was proved in [11, Lemma 2] for the case of $|\Gamma_D^d| = 0$ and in [8, Appendix A] for the case of $|\Gamma_D^d| > 0$. It states the existence of a positive constant c_{bb}, independent of h, such that for all $\boldsymbol{q}_h \in \boldsymbol{Q}_h$,

$$c_{bb}|||\boldsymbol{q}_h|||_p \leq \sup_{\substack{\boldsymbol{v}_h\in\boldsymbol{V}_h\\ \boldsymbol{v}_h\neq 0}} \frac{b_h(\boldsymbol{q}_h,\boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_v}. \tag{2.20}$$

Finally, eq. (2.10a) may be bounded using the Lipschitz continuity of f^s and f^d, eqs. (2.4b) and (2.6), and the Cauchy-Schwarz inequality as

$$\begin{aligned}
&\int_{\Omega^s} f^s(c_h)\cdot v_h \,\mathrm{d}x + \int_{\Omega^d}\mu(c_h)\kappa^{-1}f^d(c_h)\cdot v_h\,\mathrm{d}x\\
&\leq \left(L_f^s\left\|c_h\right\|_{\Omega^s} + L_f^d\mu^*\kappa_*^{-1}\left\|c_h\right\|_{\Omega^d} + \left\|f_0^s\right\|_{\Omega^s} + \mu^*\kappa_*^{-1}\left\|f_0^d\right\|_{\Omega^d}\right)\left\|v_h\right\|_{\Omega},
\end{aligned} \tag{2.21}$$

where $f_0^j := f^j(0)$ is constant with respect to the concentration but may vary in time. Likewise, eq. (2.10b) may be bounded using the Cauchy-Schwarz inequality as

$$\sum_{K\in\mathcal{T}^d}\int_K q_h^d(g_p - g_i)\,\mathrm{d}x \leq \left\|q_h^d\right\|_{\Omega^d}\left\|g_p - g_i\right\|_{\Omega^d}. \tag{2.22}$$

2.4 HDG schemes

2.4.1 Semi-discrete schemes

Using the forms defined in section 2.3, we may discretize the flow and transport problems at a fixed $0 < t \leq T$ to obtain semi-discrete schemes. In each flow problem, we discretize the momentum equations and the mass equations separately. The semi-discretization of the steady Stokes and Darcy momentum equations is defined as in [12]: given $\boldsymbol{c}_h(t) \in \boldsymbol{C}_h$, find $(\boldsymbol{u}_h(t), \boldsymbol{p}_h(t)) \in \boldsymbol{X}_h$ such that

$$\begin{aligned} &a_h^L(\boldsymbol{c}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{p}_h, \boldsymbol{v}_h) \\ &= \int_{\Omega^s} f^s(c_h) \cdot v_h \,\mathrm{d}x + \int_{\Omega^d} \mu(c_h)\kappa^{-1} f^d(c_h) \cdot v_h \,\mathrm{d}x \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \end{aligned} \tag{2.23}$$

The semi-discretization of the steady Navier–Stokes and Darcy momentum equations is defined as in [11]: given $\boldsymbol{c}_h(t) \in \boldsymbol{C}_h$, find $(\boldsymbol{u}_h(t), \boldsymbol{p}_h(t)) \in \boldsymbol{X}_h$ such that

$$\begin{aligned} &a_h(\boldsymbol{c}_h; u_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{p}_h, \boldsymbol{v}_h) \\ &= \int_{\Omega^s} f^s(c_h) \cdot v_h \,\mathrm{d}x + \int_{\Omega^d} \mu(c_h)\kappa^{-1} f^d(c_h) \cdot v_h \,\mathrm{d}x \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \end{aligned} \tag{2.24}$$

As the considered systems differ only in their momentum equations, the same semi-discrete mass equations and transport equation may be used for the HDG discretizations of both systems. The semi-discretization of the mass equations is defined as in [12]: find $(\boldsymbol{u}_h(t), \boldsymbol{p}_h(t)) \in \boldsymbol{X}_h$ such that

$$b_h(\boldsymbol{q}_h, \boldsymbol{u}_h) = \sum_{K \in \mathcal{T}^d} \int_K q_h^d (g_p - g_i) \,\mathrm{d}x \qquad \forall \boldsymbol{q}_h \in \boldsymbol{Q}_h. \tag{2.25}$$

By following the steps of [12, Section 3.3], we can show that any velocity solution $\boldsymbol{u}_h \in \boldsymbol{V}_h$ satisfying eq. (2.25) has the following properties:

$$-\nabla \cdot u_h = \chi^d \Pi_Q (g_p - g_i) \qquad \forall x \in K, \ \forall K \in \mathcal{T}, \tag{2.26a}$$

$$[\![u_h \cdot n]\!] = 0 \qquad \forall x \in F, \ \forall F \in \mathcal{F} \setminus (\mathcal{F}^I \cup \mathcal{F}_D^d), \tag{2.26b}$$

$$\bar{u}_h \cdot n^j = u_h \cdot n^j \qquad \forall x \in F, \ \forall F \in \mathcal{F}^I, j = s, d. \tag{2.26c}$$

The semi-discrete equation for the transport equation is defined as in [9]: given $u_h(t) \in V_h$, find $\boldsymbol{c}_h(t) \in \boldsymbol{C}_h$ such that

$$\sum_{K\in\mathcal{T}} \int_K \phi \partial_t c_h \psi_h \,\mathrm{d}x + \boldsymbol{B}_h(u_h; \boldsymbol{c}_h, \boldsymbol{\psi}_h) + \sum_{K\in\mathcal{T}^d} \int_K g_p c_h \psi_h \,\mathrm{d}x = \sum_{K\in\mathcal{T}^d} \int_K c_I g_i \psi_h \,\mathrm{d}x \qquad \forall \boldsymbol{\psi}_h \in \boldsymbol{C}_h. \tag{2.27}$$

It can be shown by standard arguments that the semi-discrete schemes are consistent. That is, if (u, p, c) satisfies the quasi-stationary Stokes/Darcy-transport system eqs. (2.1a), (2.1c) to (2.1h), (2.2) and (2.3), then $(\boldsymbol{u}(t), \boldsymbol{p}(t), \boldsymbol{c}(t))$ satisfies the semi-discrete scheme eqs. (2.23), (2.25) and (2.27), where $\boldsymbol{u} := (u, \gamma(u))$, $\boldsymbol{p} := (p, \gamma(p^s), \gamma(p^d))$, and $\boldsymbol{c} := (c, \gamma(c))$. Likewise, if (u, p, c) satisfies the quasi-stationary Navier–Stokes/Darcy-transport system eqs. (2.1b) to (2.1h), (2.2) and (2.3), then $(\boldsymbol{u}(t), \boldsymbol{p}(t), \boldsymbol{c}(t))$ satisfies the semi-discrete scheme eqs. (2.24), (2.25) and (2.27). A semi-discrete scheme with $(\boldsymbol{u}, \boldsymbol{p}, \boldsymbol{c})$ in place of $(\boldsymbol{u}_h, \boldsymbol{p}_h, \boldsymbol{c}_h)$ will be referred to as a consistency equation throughout this thesis. Further details on proving consistency for similar or identical discretizations can be found in [12, 10, 11].

2.4.2 Fully-discrete schemes

The semi-discrete schemes introduced in section 2.4.1 discretize the flow and transport equations in space at a fixed time t. To fully discretize each coupled flow and transport system in time and space, we divide the time interval J into N partitions of uniform size $\Delta t = T/N$ and construct a discrete sequence of time steps $\{t^n\}_{n=0}^N$, where $t^n := n\Delta t$. We use the notation $f^n := f(t^n)$ to indicate the evaluation of a function f at the n-th time step. We define discrete first derivatives of time by the backward difference operator $d_t f^n := (f^n - f^{n-1})/\Delta t$, for $n \geq 1$.

All fully-discrete schemes considered in this thesis follow a common structure, which is a generalization of the fully-discrete scheme from [9]. In this structure, we decouple the flow and transport problems by lagging the concentration behind the velocity and pressure in time. For the Navier–Stokes equations, we may also choose to lag behind the convective velocity; this possibility will be explored in later chapters. To present the fully-discrete schemes in general terms, we let $\mathcal{L}_f$ and $\mathcal{G}_f$ denote bilinear and linear forms discretizing a flow problem, and let $\mathcal{L}_t$ and $\mathcal{G}_t$ denote bilinear and linear forms discretizing a transport problem. The general scheme structure is as follows. Set $\boldsymbol{c}_h^0 = (\Pi_C c_0, \bar{\Pi}_C c_0)$, where Π_C and $\bar{\Pi}_C$ are the L^2-projection operators onto C_h and $\bar{C}_h$, respectively. For $n = 1, \ldots, N$,

(i) Find $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ such that for all $(\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h$,

$$\mathcal{L}_f(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}_h^{n-1}; \boldsymbol{u}_h^n, \boldsymbol{p}_h^n, \boldsymbol{v}_h, \boldsymbol{q}_h) = \mathcal{G}_f(\boldsymbol{v}_h, \boldsymbol{q}_h). \tag{2.28}$$

(ii) Find $\boldsymbol{c}_h^n \in \boldsymbol{C}_h$ such that for all $\boldsymbol{\psi}_h \in \boldsymbol{C}_h$,

$$\mathcal{L}_t(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}_h^n; \boldsymbol{c}_h^n, \boldsymbol{\psi}_h) = \mathcal{G}_t(\boldsymbol{\psi}_h). \tag{2.29}$$

The dependence of $\mathcal{L}_f$ on the time-lagged velocity $\boldsymbol{u}_h^{n-1}$ in eq. (2.28) is facultative and occurs in only one of the specific schemes proposed in this thesis. As we do not have an initial $\boldsymbol{u}_h^0$, the details of executing step (i) at $n = 1$ for this case will be provided in the relevant chapter.

2.5 Useful results

We close this chapter by gathering standard results which will be used in the analysis. An additional list of results referred to by name only is given in appendix A. In the remainder of this chapter, we use c and C, with various subscripts, to denote positive constants independent of h and Δt. To begin, from [18, Remark 1.47] and [5, Theorem 1.6.6] we have the following discrete and continuous trace inequalities on any $K \in \mathcal{T}$, where k is a non-negative integer:

$$\|v\|_{\partial K} \le C_t h_K^{-1/2} \|v\|_K \qquad \forall v \in P_k(K), \tag{2.30a}$$

$$\|v\|_{\partial K}^2 \le C_{ct}^2 \big(h_K^{-1} \|v\|_K^2 + h_K \|v\|_{1,K}^2\big) \qquad \forall v \in H^1(K). \tag{2.30b}$$

We may also bound the interface trace by the velocity in the Navier–Stokes region due to the following results from [22, Theorem 4.4] and [5, Theorem 1.6.6]:

$$\|v_h^s\|_{0,r,\Gamma^I} \le c_{si,r} |||\boldsymbol{v}_h|||_{v,s} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^s, \quad r \ge 2, \tag{2.30c}$$

$$\|v\|_{\Gamma^I} \le C_{si,2} \|v\|_{1,\Omega^s} \qquad \forall v \in H^1(\Omega^s). \tag{2.30d}$$

The following Poincaré-type bounds can be found in [11, Eqn 13] and [9, Eqn 29]:

$$\|v_h\|_{\Omega^s} \le c_p |||\boldsymbol{v}_h|||_{v,s} \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^s, \tag{2.30e}$$

$$\|v\|_{\Omega^s}^2 \le C_p |||\boldsymbol{v}|||_{v,s}^2 \qquad \forall \boldsymbol{v} = (v, \mu) \in H^1(\mathcal{T}^s) \times \bar{V}_h. \tag{2.30f}$$

Finally, the following Korn's inequality in our context is given in [9, Eqn 30]:

$$\sum_{K \in \mathcal{T}^s} \big(\|\varepsilon(v_h)\|_K^2 + h_K^{-1} \|v_h - \bar{v}_h\|_{\partial K}^2 \big) \ge C_{ko} |||\boldsymbol{v}_h|||_{v,s}^2 \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h^s. \tag{2.30g}$$

Chapter 3

Error analysis for HDG discretizations of coupled flow and transport systems

The HDG schemes considered in this thesis will all share the general structure described in section 2.4.2. As a result, the error analyses for all of our HDG schemes will proceed in a similar manner. To avoid repetition in the coming chapters, we provide here an abstraction of the approach used to prove optimal convergence rates in all schemes. To do this, we frame the error analysis in terms of sufficient conditions that must be met separately by the velocity and pressure solutions to the discrete flow equation eq. (2.28) and by the concentration solution to the discrete transport equation eq. (2.29) at each time step. The goal of the error analysis in later chapters will then be to prove that the numerical solutions to each of our schemes meet these sufficient conditions. Throughout this section and the error analysis sections of chapters 5 to 7, we use C to denote a constant which may have different values at each appearance but is always independent of h and Δt.

3.1 Projection operators and error splitting

The error analysis will require projections onto various spaces. For $S \in \{Q, Q^s, Q^d\}$, the notation Π_S is used to denote the L^2-projection operator onto S_h; the notation $\bar{\Pi}_S$ is used to denote the L^2-projection operator onto $\bar{S}_h$. The notation $\bar{\Pi}_V$ is also used to denote the L^2-projection operator onto $\bar{V}_h$. To project onto $V_h \cap H(\text{div}; \Omega)$, we use the BDM interpolator $\Pi_V : [H^1(\Omega)]^{\dim} \to V_h \cap H(\mathrm{div};\Omega)$, which satisfies the following properties for $u \in [H^{k_f+1}(K)]^{\dim}$ and $K \in \mathcal{T}$ [23, Lemma 7]:

$$\|u - \Pi_V u\|_{m,K} \le C h_K^{\ell-m} \|u\|_{\ell,K} \qquad \text{for } m = 0,1,2,\ m \le \ell \le k_f + 1, \tag{3.1a}$$

$$\|\nabla \cdot (u - \Pi_V u)\|_{m,K} \le C h_K^{\ell-m} \|\nabla \cdot u\|_{\ell,K} \qquad \text{for } m = 0,1,\ m \le \ell \le k_f, \tag{3.1b}$$

$$\int_K q(\nabla \cdot u - \nabla \cdot \Pi_V u)\,\mathrm{d}x = 0 \qquad \forall q \in P_{k_f-1}(K), \tag{3.1c}$$

$$\int_F \bar{q}(n \cdot u - n \cdot \Pi_V u)\,\mathrm{d}s = 0 \qquad \forall \bar{q} \in P_{k_f}(F),\ F \subset \partial K. \tag{3.1d}$$

The error analysis for the concentration solution requires the continuous interpolation operator $\mathcal{I}c \in C_h \cap \mathcal{C}^0(\overline{\Omega})$ [5]. On faces, we take $\bar{\mathcal{I}}c := \mathcal{I}c|_{\Gamma^0} \in \bar{C}_h$.

It will be convenient to split $\varrho^n - \varrho_h^n$ into an interpolation error $e_\varrho^{I,n}$ and an approximation error $e_\varrho^{h,n}$ for $n \ge 0$ and $\varrho = u, p, c$. Similarly for $\varrho = u^s, p^j, c$, with $j = s, d$, we will split each error $\gamma(\varrho^n) - \bar{\varrho}_h^n$ into an interpolation error $\bar{e}_\varrho^{I,n}$ and an approximation error $\bar{e}_\varrho^{h,n}$. That is,

$$u^n - u_h^n = e_u^{I,n} - e_u^{h,n}, \qquad \gamma(u^n) - \bar{u}_h^n = \bar{e}_u^{I,n} - \bar{e}_u^{h,n}, \tag{3.2a}$$

$$p^n - p_h^n = e_p^{I,n} - e_p^{h,n}, \qquad \gamma(p^{jn}) - \bar{p}_h^{jn} = \bar{e}_{p,j}^{I,n} - \bar{e}_{p,j}^{h,n}, \qquad j = s, d, \tag{3.2b}$$

$$c^n - c_h^n = e_c^{I,n} - e_c^{h,n}, \qquad \gamma(c^n) - \bar{c}_h^n = \bar{e}_c^{I,n} - \bar{e}_c^{h,n}, \tag{3.2c}$$

where

$$\begin{aligned} e_u^{I,n} &:= u^n - \Pi_V u^n, & e_u^{h,n} &:= u_h^n - \Pi_V u^n, \\ e_p^{I,n} &:= p^n - \Pi_Q p^n, & e_p^{h,n} &:= p_h^n - \Pi_Q p^n, \\ e_c^{I,n} &:= c^n - \mathcal{I}c^n, & e_c^{h,n} &:= c_h^n - \mathcal{I}c^n, \end{aligned}$$

and

$$\begin{aligned} \bar{e}_u^{I,n} &:= \gamma(u^{sn}) - \bar{\Pi}_V u^n, & \bar{e}_u^{h,n} &:= \bar{u}_h^n - \bar{\Pi}_V u^n, \\ \bar{e}_{p,j}^{I,n} &:= \gamma(p^{jn}) - \bar{\Pi}_Q^j p^{jn}, & \bar{e}_{p,j}^{h,n} &:= \bar{p}_h^{jn} - \bar{\Pi}_Q^j p^{jn}, \\ \bar{e}_c^{I,n} &:= \gamma(c^n) - \mathcal{I}c^n, & \bar{e}_c^{h,n} &:= \bar{c}_h^n - \mathcal{I}c^n. \end{aligned}$$

Our chosen interpolation operators have some desirable approximation properties. By a minor modification to the proof of [8, Lemma 7], we can show that for any $n \ge 1$,

$$|||\boldsymbol{e}_u^{I,n}|||_{v'}^2 \le C h^{2(\ell-1)} \|u^n\|_{\ell,\Omega}^2 \qquad \forall\ 2 \le \ell \le k_f + 1.$$

Resultingly, we have under the condition $u \in L^\infty(0,T;H^{k_f+1}(\Omega))$ that

$$|||\boldsymbol{e}_u^{I,n}|||_{v'}^2 \le Ch^{2k_f}. \tag{3.3}$$

Similarly, if $p \in L^\infty(0,T;H^{k_f}(\Omega))$, then

$$|||\boldsymbol{e}_p^{I,n}|||_p^2 \le Ch^{2k_f}. \tag{3.4}$$

By properties of the operator $\mathcal{I}c$ [5, Section 4.4], we have for any $n \ge 1$ that

$$\sum_{K\in\mathcal{T}} \|e_c^{I,n}\|_{r,K}^2 \le Ch^{2(\ell-r)} \|c^n\|_{\ell,\Omega}^2 \qquad \forall\ 2 \le \ell \le k_c+1,\ 0 \le r \le \ell. \tag{3.5}$$

Since $e_c^{I,n} = \bar{e}_c^{I,n}$ on faces for any $n \ge 1$, taking $\ell = k_c+1$ and $r = 1$ in eq. (3.5) shows that

$$|||\boldsymbol{e}_c^{I,n}|||_c^2 = \sum_{K\in\mathcal{T}} \|\nabla e_c^{I,n}\|_K^2 \le \sum_{K\in\mathcal{T}} \|e_c^{I,n}\|_{1,K}^2 \le Ch^{2k_c} \|c^n\|_{k_c+1,\Omega}^2 .$$

If $c \in L^\infty(0,T;H^{k_c+1}(\Omega))$, we therefore have that

$$|||\boldsymbol{e}_c^{I,n}|||_c^2 \le Ch^{2k_c}. \tag{3.6}$$

3.2 Main error results for coupled systems

In this section, we consider schemes of the form eqs. (2.28) and (2.29) and assume interpolation operators which satisfy eqs. (3.1), (3.3) and (3.6). We also assume that $\boldsymbol{c}_h^0$ is chosen so that the approximation error in the initial conditions satisfies

$$\|e_c^{h,0}\|_\Omega^2 \le Ch^{2k_c}, \tag{3.7a}$$

$$|||\boldsymbol{e}_c^{h,0}|||_c^2 \le Ch^{2k_c}. \tag{3.7b}$$

Next, we assume that the error in the time-lagged concentration can be bounded in terms of the concentration error as

$$\Delta t \sum_{m=1}^{n} \sum_{K\in\mathcal{T}^s} h_K^2 \|c^m - c_h^{m-1}\|_{1,K}^2 \le C\left((\Delta t)^2 + \Delta t \sum_{m=1}^{n} \sum_{K\in\mathcal{T}^s} h_K^2 \|c^{m-1} - c_h^{m-1}\|_{1,K}^2\right), \tag{3.7c}$$

$$\Delta t \sum_{m=1}^{n} \|c^m - \bar{c}_h^{m-1}\|_{\Gamma^I}^2 \le C\left((\Delta t)^2 + h^{2k_c} + \Delta t \sum_{m=1}^{n} |||\boldsymbol{e}_c^{h,m-1}|||_c^2\right), \tag{3.7d}$$

$$\Delta t \sum_{m=1}^{n} \sum_{K\in\mathcal{T}} \|c^m - c_h^{m-1}\|_K^2 \le C\left((\Delta t)^2 + \Delta t \sum_{m=1}^{n} \sum_{K\in\mathcal{T}} \|c^{m-1} - c_h^{m-1}\|_K^2\right). \tag{3.7e}$$

We also assume that the approximation errors for the concentration satisfy bounds of the form

$$\begin{aligned} &\|e_c^{h,n}\|_\Omega^2 + \Delta t \sum_{m=1}^n |||\boldsymbol{e}_c^{h,m}|||_c^2 \\ &\le C\bigg((\Delta t)^2 + h^{2k_c} + \Delta t \sum_{m=1}^n \bigg(\|u^m - u_h^m\|_\Omega^2 + \sum_{K\in\mathcal{T}} h_K \|u^m - u_h^m\|_{\partial K}^2 \bigg)\bigg), \end{aligned} \tag{3.7f}$$

so that combining the assumptions on the interpolation errors and the approximation errors by way of eq. (3.2c) and the triangle inequality leads to the bound

$$\begin{aligned} &\|c^n - c_h^n\|_\Omega^2 + \Delta t \sum_{m=1}^n |||\boldsymbol{c}^m - \boldsymbol{c}_h^m|||_c^2 \\ &\le C\bigg((\Delta t)^2 + h^{2k_c} + \Delta t \sum_{m=1}^n \bigg(\|u^m - u_h^m\|_\Omega^2 + \sum_{K\in\mathcal{T}} h_K \|u^m - u_h^m\|_{\partial K}^2 \bigg)\bigg). \end{aligned} \tag{3.7g}$$

For the velocity, we assume that the approximation errors satisfy bounds of the form

$$\begin{aligned} &\Delta t \sum_{m=1}^n |||\boldsymbol{e}_u^{h,m}|||_v^2 \\ &\le C\bigg((\Delta t)^2 + h^{2k_f} + h^{2k_c} + \Delta t \sum_{m=1}^n \|c^m - c_h^{m-1}\|_{\Omega^d}^2 + \Delta t \sum_{m=1}^n \|c^m - \bar{c}_h^{m-1}\|_{\Gamma^I}^2 \\ &\quad + \Delta t \sum_{m=1}^n \sum_{K\in\mathcal{T}^s} \big(\|c^m - c_h^{m-1}\|_K^2 + h_K^2 \|c^m - c_h^{m-1}\|_{1,K}^2 \big)\bigg). \end{aligned} \tag{3.7h}$$

Using eq. (3.2a) and the triangle inequality to combine the assumptions on the interpolation errors and the approximation errors, we obtain the bound

$$\begin{aligned} &\Delta t \sum_{m=1}^n |||\boldsymbol{u}^m - \boldsymbol{u}_h^m|||_v^2 \\ &\le C\bigg((\Delta t)^2 + h^{2k_f} + h^{2k_c} + \Delta t \sum_{m=1}^n \|c^m - c_h^{m-1}\|_{\Omega^d}^2 + \Delta t \sum_{m=1}^n \|c^m - \bar{c}_h^{m-1}\|_{\Gamma^I}^2 \\ &\quad + \Delta t \sum_{m=1}^n \sum_{K\in\mathcal{T}^s} \big(\|c^m - c_h^{m-1}\|_K^2 + h_K^2 \|c^m - c_h^{m-1}\|_{1,K}^2 \big)\bigg). \end{aligned} \tag{3.7i}$$

We now present the main result.

Theorem 3.2.1. *Consider a fully-discrete scheme in the form eqs.* (2.28) *and* (2.29). *Suppose that eq.* (3.7) *is satisfied. Then, for each $n \geq 1$, there holds*

$$\Delta t \sum_{m=1}^{n} |||\boldsymbol{u}^m - \boldsymbol{u}_h^m|||_v^2 \leq C\big((\Delta t)^2 + h^{2k_f} + h^{2k_c}\big), \tag{3.8a}$$

$$\Delta t \sum_{m=1}^{n} \sum_{K\in\mathcal{T}} h_K \, \|u^m - u_h^m\|_{\partial K}^2 \leq C\big((\Delta t)^2 + h^{2k_f} + h^{2k_c}\big), \tag{3.8b}$$

$$\|c^n - c_h^n\|_\Omega^2 + \Delta t \sum_{m=1}^{n} |||\boldsymbol{c}^m - \boldsymbol{c}_h^m|||_c^2 \leq C\big((\Delta t)^2 + h^{2k_f} + h^{2k_c}\big). \tag{3.8c}$$

Proof. We proceed by induction on n. First, using eq. (3.2c), Minkowski's and Young's inequalities, eqs. (3.6), (3.7a) and (3.7b), we see that

$$\sum_{K\in\mathcal{T}} \|c^0 - c_h^0\|_K^2 \leq 2 \sum_{K\in\mathcal{T}} \big(\|e_c^{I,0}\|_K^2 + \|e_c^{h,0}\|_K^2 \big) \leq Ch^{2k_c}, \tag{3.9a}$$

$$\sum_{K\in\mathcal{T}^s} h_K^2 \, \|c^0 - c_h^0\|_{1,K}^2 \leq 2 \sum_{K\in\mathcal{T}^s} h_K^2 \big(\|e_c^{I,0}\|_{1,K}^2 + \|e_c^{h,0}\|_{1,K}^2 \big) \leq Ch^{2k_c}. \tag{3.9b}$$

Inserting eq. (3.9b) into eq. (3.7c), eq. (3.7b) into eq. (3.7d), and eq. (3.9a) into eq. (3.7e) then shows that

$$\Delta t \sum_{K\in\mathcal{T}^s} h_K^2 \, \|c^1 - c_h^0\|_{1,K}^2 \leq C\big((\Delta t)^2 + h^{2k_c}\big), \tag{3.10a}$$

$$\Delta t \, \|c^1 - \bar{c}_h^0\|_{\Gamma^I}^2 \leq C\big((\Delta t)^2 + h^{2k_c}\big), \tag{3.10b}$$

$$\Delta t \sum_{K\in\mathcal{T}} \|c^1 - c_h^0\|_K^2 \leq C\big((\Delta t)^2 + h^{2k_c}\big). \tag{3.10c}$$

Inserting eq. (3.10) into eq. (3.7i) with $n = 1$ proves the base case for eq. (3.8a). As well, inserting eq. (3.10) into eq. (3.7h) with $n = 1$ gives the bound

$$\Delta t |||\boldsymbol{e}_u^{h,1}|||_v^2 \leq C\big((\Delta t)^2 + h^{2k_f} + h^{2k_c}\big).$$

Using eq. (3.2a), Minkowski's and Young's inequalities, the trace inequalities eqs. (2.30a) and (2.30b), and eqs. (2.30e) and (3.1), we thus obtain

$$\begin{aligned} \Delta t \sum_{K\in\mathcal{T}} h_K \, \|u^1 - u_h^1\|_{\partial K}^2 &\leq C\Delta t \sum_{K\in\mathcal{T}} \big(\|e_u^{I,1}\|_K^2 + h_K^2 \, \|e_u^{I,1}\|_{1,K}^2 \big) + C\Delta t |||\boldsymbol{e}_u^{h,1}|||_v^2 \\ &\leq C\big((\Delta t)^2 + h^{2k_f} + h^{2k_c}\big), \end{aligned}$$

which is the base case of eq. (3.8b). Applying eq. (2.30e) and inserting the base cases of eqs. (3.8a) and (3.8b) into eq. (3.7g) with $n = 1$ proves the base case for eq. (3.8c).

Moving to the inductive step, we assume that eq. (3.8) holds for some $k \geq 1$. We wish to prove that eq. (3.8) holds for $k+1$. To begin, we have from eqs. (3.7c) and (3.9b) and the inductive hypothesis on eq. (3.8c) that

$$\begin{aligned}
&\Delta t \sum_{m=1}^{k+1} \sum_{K \in \mathcal{T}^s} h_K^2 \left\| c^m - c_h^{m-1} \right\|_{1,K}^2 \\
&\leq C\bigg((\Delta t)^2 + \Delta t \sum_{K \in \mathcal{T}^s} h_K^2 \left\| c^0 - c_h^0 \right\|_{1,K}^2 + \Delta t \sum_{m=1}^{k} \sum_{K \in \mathcal{T}^s} h_K^2 \left\| c^m - c_h^m \right\|_{1,K}^2 \bigg) \\
&\leq C\big((\Delta t)^2 + h^{2k_f} + h^{2k_c} \big).
\end{aligned} \tag{3.11}$$

Likewise, we have from eqs. (3.7e) and (3.9a) and the inductive hypothesis on eq. (3.8c) that

$$\begin{aligned}
&\Delta t \sum_{m=1}^{k+1} \sum_{K \in \mathcal{T}} \left\| c^m - c_h^{m-1} \right\|_K^2 \\
&\leq C\bigg((\Delta t)^2 + \Delta t \sum_{K \in \mathcal{T}} \left\| c^0 - c_h^0 \right\|_K^2 + \Delta t \sum_{m=1}^{k} \sum_{K \in \mathcal{T}} \left\| c^m - c_h^m \right\|_K^2 \bigg) \\
&\leq C\big((\Delta t)^2 + h^{2k_f} + h^{2k_c} \big).
\end{aligned} \tag{3.12}$$

Furthermore, we have from eqs. (2.30e), (3.7b), (3.7d) and (3.7f), and the inductive hypotheses on eqs. (3.8a) and (3.8b) that

$$\begin{aligned}
&\Delta t \sum_{m=1}^{k+1} \left\| c^m - \bar{c}_h^{m-1} \right\|_{\Gamma^I}^2 \\
&\leq C\bigg((\Delta t)^2 + h^{2k_c} + \Delta t \||\boldsymbol{e}_c^{h,0}\||_c^2 + \Delta t \sum_{m=1}^{k} \||\boldsymbol{e}_c^{h,m}\||_c^2 \bigg) \\
&\leq C\bigg((\Delta t)^2 + h^{2k_c} + \Delta t \sum_{m=1}^{k} \bigg(\left\| u^m - u_h^m \right\|_\Omega^2 + \sum_{K \in \mathcal{T}} h_K \left\| u^m - u_h^m \right\|_{\partial K}^2 \bigg) \bigg) \\
&\leq C\big((\Delta t)^2 + h^{2k_f} + h^{2k_c} \big).
\end{aligned} \tag{3.13}$$

Inserting eqs. (3.11) to (3.13) into eq. (3.7i) proves the inductive result for eq. (3.8a). As well, inserting eqs. (3.11) to (3.13) into eq. (3.7h) reveals that

$$\Delta t \sum_{m=1}^{k+1} |||\boldsymbol{e}_u^{h,m}|||_v^2 \le C\big((\Delta t)^2 + h^{2k_f} + h^{2k_c}\big).$$

Using eq. (3.2a), Minkowski's and Young's inequalities, the trace inequalities eqs. (2.30a) and (2.30b), eqs. (2.30e) and (3.1), we therefore have that

$$\begin{aligned}
&\Delta t \sum_{m=1}^{k+1} \sum_{K\in\mathcal{T}} h_K \, \|u^m - u_h^m\|_{\partial K}^2 \\
&\le C\Delta t \sum_{m=1}^{k+1} \sum_{K\in\mathcal{T}} \big(\|e_u^{I,m}\|_K^2 + h_K^2 \, \|e_u^{I,m}\|_{1,K}^2 \big) + C\Delta t \sum_{m=1}^{k+1} |||\boldsymbol{e}_u^{h,m}|||_v^2 \\
&\le C\big((\Delta t)^2 + h^{2k_f} + h^{2k_c}\big),
\end{aligned}$$

which proves the inductive result on eq. (3.8b). Finally, applying eq. (2.30e) and inserting the inductive results on eqs. (3.8a) and (3.8b) into eq. (3.7g) proves the inductive result on eq. (3.8c). □

In chapter 4, we will show how results from [9] imply that the concentration error for the specific transport equation considered here satisfies eqs. (3.7a) to (3.7g), under a regularity condition. It remains to show that the solutions to the flow problems satisfy eq. (3.7h), from which eq. (3.7i) will immediately follow thanks to our choice of projection operators. This will be the main focus of the error analysis in chapters 5 to 7.

The observant reader may notice that main result theorem 3.2.1 does not discuss the pressure error. In fact, the assumption eq. (3.7h) can only be met by a discrete flow problem which is pressure robust, meaning that the velocity error is unaffected by the pressure error. The discrete flow problems considered in this thesis are indeed all pressure robust, as previously established in [12, 11, 8]. Since the transport equation does not explicitly involve the pressure, a pressure-robust discrete flow problem implies that the concentration error depends only on the velocity error.

3.3 Auxiliary results for error analysis

With the aim of avoiding repetition in future chapters, we collect here some results that are used recurrently in the error analyses of our schemes. The results of this section are specific to the HDG schemes that will be shown in chapters 5 to 7. In particular, we assume that the projection operators are those defined in section 3.1. As well, we assume that $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ is the numerical solution to eq. (2.28) such that $\boldsymbol{u}_h^n$ satisfies eq. (2.26) for $n \geq 1$. We also assume that $(u, p) \in X$ is the exact flow solution to the coupled flow and transport system discretized by eqs. (2.28) and (2.29).

We begin with results which will be used to derive error equations. The first result, proved in [9, Lemma 6.2], holds for any $\boldsymbol{u}_h^n$ which satisfies eq. (2.26):

$$b_h(\boldsymbol{e}_p^{I,n}, \boldsymbol{v}_h) = 0 \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{3.14a}$$

$$b_h(\boldsymbol{q}_h, \boldsymbol{e}_u^{I,n}) = 0 \qquad \forall \boldsymbol{q}_h \in \boldsymbol{Q}_h. \tag{3.14b}$$

The proof depends on eq. (2.26) and properties of Π_V, Π_Q, and $\bar{\Pi}_Q^j$. A consequence of eqs. (3.2a), (3.2b) and (3.14) is that for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$ and any $\boldsymbol{q}_h \in \boldsymbol{Q}_h$,

$$b_h(\boldsymbol{p}_h^n - \boldsymbol{p}^n, \boldsymbol{v}_h) + b_h(\boldsymbol{q}_h, \boldsymbol{u}_h^n - \boldsymbol{u}^n) = b_h(\boldsymbol{e}_p^{h,n}, \boldsymbol{v}_h) + b_h(\boldsymbol{q}_h, \boldsymbol{e}_u^{h,n}). \tag{3.15}$$

Next, by adding and subtracting $\boldsymbol{u}^n$ and using (3.2a), we may write

$$a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}_h^n, \boldsymbol{v}_h) = a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{e}_u^{h,n}, \boldsymbol{v}_h) - a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{e}_u^{I,n}, \boldsymbol{v}_h) + a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}^n, \boldsymbol{v}_h), \tag{3.16}$$

and by adding and subtracting $\boldsymbol{u}_h^n$ and using (3.2a), we may write

$$t_h(u^n; \boldsymbol{u}^n, \boldsymbol{v}_h) = t_h(u^n; \boldsymbol{e}_u^{I,n}, \boldsymbol{v}_h) - t_h(u^n; \boldsymbol{e}_u^{h,n}, \boldsymbol{v}_h) + t_h(u^n; \boldsymbol{u}_h^n, \boldsymbol{v}_h). \tag{3.17}$$

By adding and subtracting $\boldsymbol{\Pi}_V u^n$ and $t_h(u^{n-1}; \boldsymbol{\Pi}_V u^n, \boldsymbol{v}_h)$, and using (3.2a), we also have as in [8, Theorem 5.1] that

$$\begin{aligned} t_h(u_h^{n-1}; \boldsymbol{u}_h^n, \boldsymbol{v}_h) - t_h(u^n; \boldsymbol{u}^n, \boldsymbol{v}_h) = {} & t_h(u_h^{n-1}; \boldsymbol{e}_u^{h,n}, \boldsymbol{v}_h) - t_h(u^n; \boldsymbol{e}_u^{I,n}, \boldsymbol{v}_h) \\ & + \left[t_h(u_h^{n-1}; \boldsymbol{\Pi}_V u^n, \boldsymbol{v}_h) - t_h(u^{n-1}; \boldsymbol{\Pi}_V u^n, \boldsymbol{v}_h)\right] \\ & + \left[t_h(u^{n-1}; \boldsymbol{\Pi}_V u^n, \boldsymbol{v}_h) - t_h(u^n; \boldsymbol{\Pi}_V u^n, \boldsymbol{v}_h)\right]. \end{aligned} \tag{3.18}$$

Now, we give some results that will be used to bound the approximation errors. To obtain bounds on $\boldsymbol{e}_u^{h,n}$, we will want to apply the coercivity results eq. (2.17) and lemma 2.3.2 with $\boldsymbol{v}_h = \boldsymbol{e}_u^{h,n}$. The following lemma shows that this choice of $\boldsymbol{v}_h$ satisfies the conditions of eq. (2.17) and lemma 2.3.2.

Lemma 3.3.1. *Suppose that $\boldsymbol{u}_h^n$ satisfies eq. (2.26) and that u satisfies eqs. (2.1e) and (2.1f) in Ω^d. Then, for $n \geq 1$,*

$$|||\boldsymbol{e}_u^{h,n}|||_{v,d} = \|\boldsymbol{e}_u^{h,n}\|_{\Omega^d}.$$

Proof. By the definition of $e_u^{h,n}$, eq. (2.26a), properties of Π_V and Π_Q, and eq. (2.1f),

$$\|\nabla \cdot e_u^{h,n}\|_{\Omega^d}^2 = \|\Pi_Q(g_p^n - g_i^n) + \Pi_Q \nabla \cdot u^n\|_{\Omega^d}^2 = 0. \tag{3.19}$$

Also, eq. (3.14) implies that $\boldsymbol{e}_u^{I,n} \in \boldsymbol{Z}_h$, so that $|||\boldsymbol{e}_u^{I,n}|||_{v,d} = \|\boldsymbol{e}_u^{I,n}\|_{\Omega^d}$. In turn, this implies that

$$\sum_{F \in \mathcal{F}^d \setminus (\mathcal{F}^I \cup \mathcal{F}_D^d)} h_F^{-1} \|[\![e_u^{I,n} \cdot n]\!]\|_F^2 = \sum_{K \in \mathcal{T}^d} h_K^{-1} \|(e_u^{I,n} - \bar{e}_u^{I,n}) \cdot n\|_{\partial K \cap \Gamma^I}^2 = 0.$$

For each $F \in \mathcal{F}^d \setminus (\mathcal{F}^I \cup \mathcal{F}_D^d)$, we therefore have by eq. (3.2a), Minkowski's inequality, eq. (2.26b), and the single-valuedness of $u^n \cdot n$ on cell faces that

$$\|[\![e_u^{h,n} \cdot n]\!]\|_F \le \|[\![e_u^{I,n} \cdot n]\!]\|_F + \|[\![u_h^n \cdot n]\!]\|_F + \|[\![u^n \cdot n]\!]\|_F = 0. \tag{3.20}$$

For each $K \in \mathcal{T}^d$, we have by eq. (3.2a), Minkowski's inequality, eq. (2.26c) and that $u^n = \gamma(u^n)$ on cell faces that

$$\begin{aligned} &\|(e_u^{h,n} - \bar{e}_u^{h,n}) \cdot n\|_{\partial K \cap \Gamma^I} \\ &\le \|(e_u^{I,n} - \bar{e}_u^{I,n}) \cdot n\|_{\partial K \cap \Gamma^I} + \|(u_h^n - \bar{u}_h^n) \cdot n\|_{\partial K \cap \Gamma^I} + \|(u^n - \gamma(u^n)) \cdot n\|_{\partial K \cap \Gamma^I} \\ &= 0. \end{aligned} \tag{3.21}$$

Combining eqs. (3.19) to (3.21) in the definition of $|||\boldsymbol{e}_u^{h,n}|||_{v,d}$ gives the result. □

A couple of other useful results for bounding the approximation errors were proved in [9, Eqns 66 – 73]. They are proved there with $\boldsymbol{v}_h = \boldsymbol{e}_u^{h,n}$, but hold for general $\boldsymbol{v}_h \in \boldsymbol{V}_h$ as presented here. Firstly,

$$\begin{aligned} &\left[a_h^L(\boldsymbol{c}^n; \boldsymbol{u}^n, \boldsymbol{v}_h) - a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}^n, \boldsymbol{v}_h)\right] \\ &\le 2C\mu_L \|\nabla u^n\|_{0,\infty,\Omega^s} \left(\sum_{K \in \mathcal{T}^s} \left(\|c^n - c_h^{n-1}\|_K^2 + h_K^2 \|c^n - c_h^{n-1}\|_{1,K}^2 \right) \right)^{1/2} |||\boldsymbol{v}_h|||_v \\ &\quad + \mu_L \kappa_*^{-1} \|c^n - c_h^{n-1}\|_{\Omega^d} \|u^n\|_{0,\infty,\Omega^d} |||\boldsymbol{v}_h|||_v \\ &\quad + \mu_L \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I} \left(\sum_{\ell=1}^{\dim -1} \gamma^\ell \right)^{1/2} \|u^n\|_{0,\infty,\Omega^s} |||\boldsymbol{v}_h|||_v. \end{aligned} \tag{3.22}$$

Secondly,

$$\int_{\Omega^s} \left[f^s(c_h^{n-1}) - f^s(c^n)\right] \cdot v_h \,\mathrm{d}x \le CL_f^s \|c^n - c_h^{n-1}\|_{\Omega^s} |||\boldsymbol{v}_h|||_v. \tag{3.23}$$

Thirdly,

$$\begin{aligned}&\int_{\Omega^d} \left[\mu(c_h^{n-1})\kappa^{-1} f^d(c_h^{n-1}) - \mu(c^n)\kappa^{-1} f^d(c^n)\right] \cdot v_h \,\mathrm{d}x \\ &\le \left(\kappa_*^{-1}\mu^* L_f^d + \mu_L \kappa_*^{-1} \left\| f^d(c^n)\right\|_{0,\infty,\Omega^d}\right) \left\| c^n - c_h^{n-1}\right\|_{\Omega^d} \left\|\left| \boldsymbol{v}_h \right|\right\|_v.\end{aligned} \tag{3.24}$$

The next two results were proved in [8, Theorem 5.1, Appendix C]:

$$\begin{aligned}&[t_h(u^n; \boldsymbol{\Pi}_V u^n, \boldsymbol{e}_u^{h,n}) - t_h(u^{n-1}; \boldsymbol{\Pi}_V u^n, \boldsymbol{e}_u^{h,n})] \\ &\le C(\Delta t)^{1/2} \left\|\partial_t u\right\|_{L^2(t^{n-1},t^n;H^1(\Omega^s))} \left\| u^n\right\|_{1,\Omega^s} \left\|\left| \boldsymbol{e}_u^{h,n} \right|\right\|_{v,s},\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}&[t_h(u^{n-1}; \boldsymbol{\Pi}_V u^n, \boldsymbol{e}_u^{h,n}) - t_h(u_h^{n-1}; \boldsymbol{\Pi}_V u^n, \boldsymbol{e}_u^{h,n})] \\ &\le C\Big(h^{k_f} \left\| u^{n-1}\right\|_{k_f+1,\Omega^s} \left\|\nabla u^n\right\|_{\Omega^s} + h^{2k_f} \left\| u^n\right\|_{k_f+1,\Omega^s} \left\| u^{n-1}\right\|_{k_f+1,\Omega^s} \\ &\quad + \left\| u^n\right\|_{1,3,\Omega^s} \left\| e_u^{h,n-1}\right\|_{\Omega^s}\Big) \left\|\left| \boldsymbol{e}_u^{h,n} \right|\right\|_v.\end{aligned} \tag{3.26}$$

Finally, from eq. (2.12) and the equivalence of $\left\|\left|\cdot\right|\right\|_v$ and $\left\|\left|\cdot\right|\right\|_{v'}$ on $\boldsymbol{V}_h$, we have that for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{e}_u^{I,n}, \boldsymbol{v}_h) \le c_{ac}^L \mu^* \left\|\left| \boldsymbol{e}_u^{I,n} \right|\right\|_{v'} \left\|\left| \boldsymbol{v}_h \right|\right\|_v, \tag{3.27}$$

$$a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{e}_u^{h,n}, \boldsymbol{v}_h) \le c_{ac}^L \mu^* \left\|\left| \boldsymbol{e}_u^{h,n} \right|\right\|_{v} \left\|\left| \boldsymbol{v}_h \right|\right\|_v. \tag{3.28}$$

Likewise, we have from eq. (2.14a) and the equivalence of $\left\|\left|\cdot\right|\right\|_v$ and $\left\|\left|\cdot\right|\right\|_{v'}$ on $\boldsymbol{V}_h$ that for any $\boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$a_h(\boldsymbol{c}_h^{n-1}; u^n; \boldsymbol{e}_u^{I,n}, \boldsymbol{v}_h) \le c_{ac} \mu^* \left\|\left| \boldsymbol{e}_u^{I,n} \right|\right\|_{v'} \left\|\left| \boldsymbol{v}_h \right|\right\|_v, \tag{3.29}$$

with $c_{ac} = 2c_e^2 \max(c_w(\mu^*)^{-1} \left\| u^n\right\|_{1,h,\Omega^s} + c_{ac}^s, \kappa_*^{-1}, \alpha\kappa_*^{-1/2})$.

Chapter 4

The discrete transport equation

In all of the fully-discrete schemes considered here, eq. (2.29) will have the definition

$$\sum_{K\in\mathcal{T}}\int_K \phi d_t c_h^n \psi_h \,\mathrm{d}x + \boldsymbol{B}_h(u_h^n; \boldsymbol{c}_h^n, \boldsymbol{\psi}_h) + \sum_{K\in\mathcal{T}^d}\int_K g_p^n c_h^n \psi_h \,\mathrm{d}x = \sum_{K\in\mathcal{T}^d}\int_K c_I g_i^n \psi_h \,\mathrm{d}x, \tag{4.1}$$

which corresponds to discretizing eq. (2.27) in time according to eq. (2.29). The only difference between eq. (4.1) and the fully-discrete transport equation [9, Eqn 40] is the slight modification to the space $\boldsymbol{C}_h$ here to allow for more general boundary conditions. As a consequence, many results from [9] are applicable to the analyses in this thesis. We assemble the relevant results in this section.

4.1 Compatibility

Suppose that we are given a solution to eq. (2.28) which is divergence-conforming and mass-conserving at $n \geq 1$. That is, the velocity solution $\boldsymbol{u}_h^n$ satisfies eq. (2.26). We will later see that this is true of all of the numerical velocity solutions produced by schemes in this thesis. We can show similarly to [10, Section 3.4] that with $k_c = k_f - 1$, eq. (4.1) preserves the constant solution. Explicitly, we have for any $n \geq 1$ that

$$\sum_{K\in\mathcal{T}}\int_K \phi d_t c_I \psi_h \,\mathrm{d}x + \boldsymbol{B}_h(u_h^n; (c_I, c_I), \boldsymbol{\psi}_h) + \sum_{K\in\mathcal{T}^d}\int_K g_p^n c_I \psi_h \,\mathrm{d}x = \sum_{K\in\mathcal{T}^d}\int_K c_I g_i^n \psi_h \,\mathrm{d}x,$$

supposing that c_I is constant in space and in time. This shows that with $k_c = k_f - 1$, any fully-discrete scheme in which eq. (2.29) is defined by eq. (4.1) meets the compatibility criteria of [17], provided that the solution to eq. (2.28) satisfies eq. (2.26). This ensures that the accuracy and conservation properties of the discrete transport problem are maintained when the discrete flow problem and the discrete transport problem are combined in the fully-discrete scheme. The importance of maintaining these properties was illustrated in [10, Section 7.2] by way of numerical examples.

4.2 Existence and uniqueness of the concentration solution

Due to eq. (2.5a) and that for any $\boldsymbol{\psi}_h \in \boldsymbol{C}_h$, $\bar{\psi}_h = 0$ on Γ_D^d whenever $|\Gamma_D^d| > 0$, the following results from [9, Theorem 6.10, Eqn 95] can be proved with a minor adjustment to take into account the different boundary conditions when $|\Gamma_D^d| > 0$.

Lemma 4.2.1. *Given $\boldsymbol{u}_h^n \in \boldsymbol{V}_h$ satisfying eq. (2.26), define for $\boldsymbol{\psi}_h \in \boldsymbol{C}_h$*

$$|||\boldsymbol{\psi}_h|||_{c,u_h^n}^2 := \sum_{K\in\mathcal{T}} \big(\|\tilde{D}(u_h^n)^{1/2}\nabla\psi_h\|_K^2 + h_K^{-1} \|(1+\chi^d|u_h^n|)^{1/2}(\psi_h - \bar{\psi}_h)\|_{\partial K}^2 \big).$$

There exists a constant β_0^c such that if $\beta_c > \beta_0^c$, then for all $\boldsymbol{\psi}_h \in \boldsymbol{C}_h$,

$$\begin{aligned}\boldsymbol{B}_h(u_h^n;\boldsymbol{\psi}_h,\boldsymbol{\psi}_h) &\geq C_{tr}|||\boldsymbol{\psi}_h|||_{c,u_h^n}^2 + \frac{1}{2}\sum_{K\in\mathcal{T}^d}\int_K \nabla\cdot u_h^n\psi_h^2 \,\mathrm{d}x \\ &\geq C|||\boldsymbol{\psi}_h|||_c^2 + \frac{1}{2}\sum_{K\in\mathcal{T}^d}\int_K \nabla\cdot u_h^n\psi_h^2 \,\mathrm{d}x,\end{aligned}$$

where C_{tr} and C are positive constants that depend on d, $D_{\min}$, and $D_{\max}$.

This allows us to prove the existence of a unique concentration solution as in [9, Theorem 6.11], given $\boldsymbol{u}_h^n \in \boldsymbol{V}_h$ which satisfies eq. (2.26). As in [9, Theorem 6.11], the proof of the following theorem depends on lemma 4.2.1; the assumptions eq. (2.4a), $g_p^n \geq 0$, and $0 \leq c_I \leq 1$; the property eq. (2.26a); and the discrete Grönwall inequality [26, Lemma 27].

Theorem 4.2.2. *Let $c_0 \in L^2(\Omega)$ and $g_i, g_p \in L^2(0,T;L^\infty(\Omega^d))$. Let $n \geq 1$ and suppose that $\beta_c > \beta_0^c$. If $d_m\Delta t < 1$ for all m, where $d_m := \phi_*^{-1}\big(1 + C\,\|g_p^m - g_i^m\|_{0,\infty,\Omega^d}\big)$ and C is a positive constant, then there exists a unique solution $\boldsymbol{c}_h^n \in \boldsymbol{C}_h$ to eq. (2.27), given that $c_h^0 = \Pi_C c_0$ and that $\boldsymbol{u}_h^n \in \boldsymbol{V}_h$ satisfies eq. (2.26). Furthermore, with $K_n := \sum_{m=1}^n d_m/(1-\Delta t d_m)$, there holds*

$$\phi_*\,\|c_h^n\|_\Omega^2 + C\Delta t\sum_{m=1}^n |||\boldsymbol{c}_h^m|||_c^2 \leq e^{K_n\Delta t}\big(\phi_*\,\|c_0\|_\Omega^2 + \|g_i\|_{\ell^2(0,T;L^2(\Omega^d))}^2\big). \tag{4.2}$$

As we will later show more concretely, the discrete flow equation eq. (2.28) for each of our considered schemes is such that the velocity solution $\boldsymbol{u}_h^n$ satisfies eq. (2.25). Resultingly, any $\boldsymbol{u}_h^n$ which solves one of our discrete flow equations will also satisfy eq. (2.26), so that theorem 4.2.2 holds when eq. (2.29) is defined by eq. (4.1) in the scheme from section 2.4.2.

4.3 The concentration error

In this section, we verify that solutions to eq. (4.1) meet the conditions of eq. (3.7) relating to the numerical concentration solution. To start, it can be shown that if $c_0 \in H^{k_c+1}(\Omega)$, then eqs. (3.7a) and (3.7b) are fulfilled by the choice $\boldsymbol{c}_h^0 = (\Pi_C c_0, \bar{\Pi}_C c_0)$, due to properties of $\mathcal{I}c$ and the L^2-projection operator [18, Lemma 1.59].

Next, the following result, which was proved in [9, Theorem 6.4], provides bounds of the form eqs. (3.7c) to (3.7e).

Lemma 4.3.1. *Let $c_0^s \in H^{k_c+1}(\Omega^s)$, $c^s \in L^2(0,T;H^{k_c+1}(\Omega^s))$, and $c^d \in H^1(0,T;L^2(\Omega^d))$ such that $\partial_t c^s \in L^2(0,T,H^1(\Omega^s))$, and $\bar{c} = \gamma(c)$ on Γ_0. For each $n \geq 1$, the following estimates hold:*

$$\Delta t \sum_{m=1}^{n} \sum_{K \in \mathcal{T}^s} h_K^2 \left\| c^m - c_h^{m-1} \right\|_{1,K}^2 \leq C\Bigg(h^2 (\Delta t)^2 \left\| \partial_t c \right\|_{L^2(0,T;H^1(\Omega^s))}^2 + \Delta t \sum_{m=1}^{n} \sum_{K \in \mathcal{T}^s} h_K^2 \left\| c^{m-1} - c_h^{m-1} \right\|_{1,K}^2 \Bigg),$$

$$\Delta t \sum_{m=1}^{n} \left\| \bar{c}^m - \bar{c}_h^{m-1} \right\|_{\Gamma^I}^2 \leq C\Bigg(\Delta t \sum_{m=1}^{n} \left\|\!\left| \boldsymbol{e}_c^{h,m-1} \right|\!\right\|_c^2 + (\Delta t)^2 \left\| \partial_t c \right\|_{L^2(0,T;H^1(\Omega^s))}^2 + h^{2k_c+1}\left(\Delta t \left\| c_0 \right\|_{k_c+1,\Omega^s}^2 + \left\| c \right\|_{\ell^2(0,T;H^{k_c+1}(\Omega^s))}^2 \right) \Bigg),$$

$$\Delta t \sum_{m=1}^{n} \sum_{K \in \mathcal{T}} \left\| c^m - c_h^{m-1} \right\|_K^2 \leq C\Bigg((\Delta t)^2 \left\| \partial_t c \right\|_{L^2(0,T;L^2(\Omega))}^2 + \Delta t \sum_{m=1}^{n} \sum_{K \in \mathcal{T}} \left\| c^{m-1} - c_h^{m-1} \right\|_K^2 \Bigg).$$

Moreover, taking the steps used in the proof of [9, Theorem 6.13] without the assumptions of [9, Corollary 6.7] on the velocity error, we deduce that if Δt is sufficiently small relative to $1/\left\|g_p - g_i\right\|_{L^\infty(0,T;L^\infty(\Omega^d))}$, then

$$
\begin{aligned}
&\|e_c^{h,n}\|_\Omega^2 + \Delta t \sum_{m=1}^{n} |||\boldsymbol{e}_c^{h,m}|||_c^2 \\
&\leq C\bigg((\Delta t)^2 \left\|\partial_{tt} c\right\|_{L^2(0,T;L^2(\Omega))}^2 + h^{2k_c} \Delta t \left\|c\right\|_{L^\infty(0,T;H^{k_c+1}(\Omega))}^2 \\
&\quad + h^{2k_c} \left\|c_0\right\|_{k_c,\Omega}^2 + h^{2k_c} \left\|\partial_t c\right\|_{L^2(0,T;H^{k_c}(\Omega))}^2 \\
&\quad + \bigg[\Delta t \big(1 + \left\|c\right\|_{L^\infty(0,T;L^\infty(\Omega))}^2 + \left\|c\right\|_{L^\infty(0,T;W^{1,\infty}(\Omega^d))}^2 \big) \\
&\qquad \times \sum_{m=1}^{n} \bigg(\left\|u^m - u_h^m\right\|_\Omega^2 + \sum_{K\in\mathcal{T}} h_K \left\|u^m - u_h^m\right\|_{\partial K}^2 \bigg)\bigg]\bigg),
\end{aligned}
$$

subject to regularity assumptions on u which will be given in the next theorem. This shows the satisfaction of the condition eq. (3.7f). Proceeding as in the proof of [9, Corollary 6.14] gives the following bound on the concentration error, which is in the form of eq. (3.7g).

Theorem 4.3.2. *Suppose that $u^s \in L^\infty(0,T;W^{1,\infty}(\Omega^s))$, $u^d \in L^\infty(0,T;W^{1,\infty}(\Omega^d))$, $c_0 \in H^{k_c}(\Omega)$, $c \in L^2(0,T;H^{k_c+1}(\Omega)) \cap L^\infty(0,T;W^{1,\infty}(\Omega))$, $\partial_t c \in L^2(0,T;H^{k_c}(\Omega))$, $\partial_{tt} c \in L^2(0,T;L^2(\Omega))$, and $g_i, g_p \in L^\infty(0,T;L^\infty(\Omega^d))$. Then, for each $n \geq 1$,*

$$
\begin{aligned}
&\|c^n - c_h^n\|_\Omega^2 + \Delta t \sum_{m=1}^{n} |||\boldsymbol{c}_h^m - \boldsymbol{c}^m|||_c^2 \\
&\leq C\bigg((\Delta t)^2 \left\|\partial_{tt} c\right\|_{L^2(0,T;L^2(\Omega))}^2 + h^{2k_c} \Delta t \left\|c\right\|_{L^\infty(0,T;H^{k_c+1}(\Omega))}^2 \\
&\quad + h^{2k_c} \left\|c_0\right\|_{k_c,\Omega}^2 + h^{2k_c} \left\|c\right\|_{L^\infty(0,T;H^{k_c}(\Omega))}^2 + h^{2k_c} \left\|\partial_t c\right\|_{L^2(0,T;H^{k_c}(\Omega))}^2 \\
&\quad + \bigg[\Delta t \big(1 + \left\|c\right\|_{L^\infty(0,T;L^\infty(\Omega))}^2 + \left\|c\right\|_{L^\infty(0,T;W^{1,\infty}(\Omega^d))}^2 \big) \\
&\qquad \times \sum_{m=1}^{n} \bigg(\left\|u^m - u_h^m\right\|_\Omega^2 + \sum_{K\in\mathcal{T}} h_K \left\|u^m - u_h^m\right\|_{\partial K}^2 \bigg)\bigg]\bigg).
\end{aligned}
$$

Chapter 5

Quasi-stationary Stokes/Darcy-transport

This chapter is devoted to the quasi-stationary Stokes/Darcy-transport system. The flow problem for this system is defined by the Stokes equations eqs. (2.1a), (2.1c) and (2.1d), the Darcy equations eqs. (2.1e) to (2.1h), and the interface conditions eq. (2.2). The transport problem is defined by eq. (2.3).

To obtain a fully-discrete scheme for this system, we temporally discretize the semi-discrete schemes eqs. (2.23) and (2.25) according to eq. (2.28). We combine the resulting flow problem with eq. (4.1) in keeping with the general structure laid out in section 2.4.2. The scheme is stated as follows. Set $\boldsymbol{c}_h^0 = (\Pi_C c_0, \bar{\Pi}_C c_0)$, where Π_C and $\bar{\Pi}_C$ are the L^2-projection operators onto C_h and $\bar{C}_h$. Then, for $n = 1, \dots, N$,

(i) Find $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ such that for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$\begin{aligned} &a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}_h^n, \boldsymbol{v}_h) + b_h(\boldsymbol{p}_h^n, \boldsymbol{v}_h) \\ &= \int_{\Omega^s} f^s(c_h^{n-1}) \cdot v_h \, \mathrm{d}x + \int_{\Omega^d} \mu(c_h^{n-1}) \kappa^{-1} f^d(c_h^{n-1}) \cdot v_h \, \mathrm{d}x, \end{aligned} \tag{5.1a}$$

and for all $\boldsymbol{q}_h \in \boldsymbol{Q}_h$,

$$b_h(\boldsymbol{q}_h, \boldsymbol{u}_h^n) = \sum_{K \in \mathcal{T}^d} \int_K q_h^d (g_p^n - g_i^n) \, \mathrm{d}x. \tag{5.1b}$$

(ii) Find $\boldsymbol{c}_h^n \in \boldsymbol{C}_h$ such that eq. (4.1) holds for all $\boldsymbol{\psi}_h \in \boldsymbol{C}_h$.

5.1 Well-posedness

Due to eqs. (2.12), (2.17) and (2.19) to (2.22), the well-posedness of eq. (5.1) is a direct result of [4, Theorem 4.2.3]. We find that eq. (5.1) is guaranteed to have a unique solution $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ at time $n \geq 1$ such that

$$\begin{aligned} \left\lvert\!\left\lvert\!\left\lvert \boldsymbol{u}_h^n \right\rvert\!\right\rvert\!\right\rvert_v \leq{}& (c_{ae}^L \mu_*)^{-1} \big(L_f^s \left\| c_h^{n-1} \right\|_{\Omega^s} + L_f^d \mu^* \kappa_*^{-1} \left\| c_h^{n-1} \right\|_{\Omega^d} \big) \\ &+ (c_{ae}^L \mu_*)^{-1} \big(\left\| f_0^{sn} \right\|_{\Omega^s} + \mu^* \kappa_*^{-1} \left\| f_0^{dn} \right\|_{\Omega^d} \big) \\ &+ 2 c_{ac}^L \mu^* (c_{ae}^L \mu_* c_{bb})^{-1} \left\| g_p^n - g_i^n \right\|_{\Omega^d} \end{aligned}$$

and

$$\begin{aligned} \left\lvert\!\left\lvert\!\left\lvert \boldsymbol{p}_h^n \right\rvert\!\right\rvert\!\right\rvert_p \leq{}& 2 c_{ac}^L \mu^* (c_{ae}^L \mu_* c_{bb})^{-1} \big(L_f^s \left\| c_h^{n-1} \right\|_{\Omega^s} + L_f^d \mu^* \kappa_*^{-1} \left\| c_h^{n-1} \right\|_{\Omega^d} \big) \\ &+ 2 c_{ac}^L \mu^* (c_{ae}^L \mu_* c_{bb})^{-1} \big(\left\| f_0^{sn} \right\|_{\Omega^s} + \mu^* \kappa_*^{-1} \left\| f_0^{dn} \right\|_{\Omega^d} \big) \\ &+ 2 (c_{ac}^L \mu^*)^2 (c_{ae}^L \mu_* c_{bb}^2)^{-1} \left\| g_p^n - g_i^n \right\|_{\Omega^d} . \end{aligned}$$

Thanks to eq. (4.2), the terms involving c_h^{n-1} may be bounded independently of c_h^{n-1} via

$$\begin{aligned} & L_f^s \left\| c_h^{n-1} \right\|_{\Omega^s} + L_f^d \mu^* \kappa_*^{-1} \left\| c_h^{n-1} \right\|_{\Omega^d} \\ \leq{}& \big(L_f^s + L_f^d \mu^* \kappa_*^{-1} \big) \phi_*^{-1/2} e^{K_{n-1} \Delta t/2} \left(\phi_* \left\| c_0 \right\|_\Omega^2 + \Delta t \sum_{m=1}^{n-1} \left\| g_i^m \right\|_{\Omega^d}^2 \right)^{1/2} . \end{aligned} \tag{5.2}$$

5.2 Error analysis

Our first step in the error analysis is to derive the error equation.

Lemma 5.2.1. *Let $n \geq 1$ and suppose that $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ satisfies eq. (5.1) for a given $\boldsymbol{c}_h^{n-1} \in \boldsymbol{C}_h$. Then, for all $(\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h$, there holds*

$$\begin{aligned} & a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{e}_u^{h,n}, \boldsymbol{v}_h) - a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{e}_u^{I,n}, \boldsymbol{v}_h) + a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}^n, \boldsymbol{v}_h) - a_h^L(\boldsymbol{c}^n; \boldsymbol{u}^n, \boldsymbol{v}_h) \\ & + b_h(\boldsymbol{e}_p^{h,n}, \boldsymbol{v}_h) + b_h(\boldsymbol{q}_h, \boldsymbol{e}_u^{h,n}) \\ ={}& \int_{\Omega^s} \left[f^s(c_h^{n-1}) - f^s(c^n) \right] \cdot v_h \,\mathrm{d}x \\ & + \int_{\Omega^d} \left[\mu(c_h^{n-1}) \kappa^{-1} f^d(c_h^{n-1}) - \mu(c^n) \kappa^{-1} f^d(c^n) \right] \cdot v_h \,\mathrm{d}x, \end{aligned} \tag{5.3}$$

where (u, p, c) is the solution to the quasi-stationary Stokes/Darcy-transport system and $\boldsymbol{u}^n := (u(t^n), \gamma(u(t^n)))$, $\boldsymbol{p}^n := (p(t^n), \gamma(p^s(t^n)), \gamma(p^d(t^n)))$, and $\boldsymbol{c}^n := (c(t^n), \gamma(c(t^n)))$.

Proof. Let $n \geq 1$. Let $(\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h$. Subtracting the consistency equation for eq. (5.1a) from eq. (5.1a) gives

$$\begin{aligned}
&a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}_h^n, \boldsymbol{v}_h) - a_h^L(\boldsymbol{c}^n; \boldsymbol{u}^n, \boldsymbol{v}_h) + b_h(\boldsymbol{p}_h^n - \boldsymbol{p}^n, \boldsymbol{v}_h) \\
&= \int_{\Omega^s} \left[f^s(c_h^{n-1}) - f^s(c^n) \right] \cdot v_h \, \mathrm{d}x \\
&\quad + \int_{\Omega^d} \left[\mu(c_h^{n-1}) \kappa^{-1} f^d(c_h^{n-1}) - \mu(c^n) \kappa^{-1} f^d(c^n) \right] \cdot v_h \, \mathrm{d}x.
\end{aligned} \tag{5.4}$$

Subtracting the consistency equation for eq. (5.1b) from eq. (5.1b) gives

$$b_h(\boldsymbol{q}_h, \boldsymbol{u}_h^n - \boldsymbol{u}^n) = 0. \tag{5.5}$$

Summing eqs. (5.4) and (5.5) and inserting eqs. (3.15) and (3.16) into the result yields eq. (5.3). □

With the error equation established, we now prove bounds on the velocity errors.

Lemma 5.2.2. *Suppose that (u, p, c) is the solution to the quasi-stationary Stokes/Darcy-transport system. Suppose that $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ satisfies eq. (5.1) for $n \geq 1$, with $\boldsymbol{c}_h^{n-1} \in \boldsymbol{C}_h$ given. Then,*

$$\begin{aligned}
|||\boldsymbol{e}_u^{h,n}|||_v \leq C \Bigg[& |||\boldsymbol{e}_u^{I,n}|||_{v'} + \|u^n\|_{0,\infty,\Omega^s} \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I} \\
& + \left(1 + \|\nabla u^n\|_{0,\infty,\Omega^s}\right) \Bigg(\sum_{K \in \mathcal{T}^s} \left(\|c^n - c_h^{n-1}\|_K^2 + h_K^2 \|c^n - c_h^{n-1}\|_{1,K}^2 \right) \Bigg)^{1/2} \\
& + \left(1 + \|u^n\|_{0,\infty,\Omega^d} + \|f^d(c^n)\|_{0,\infty,\Omega^d}\right) \|c^n - c_h^{n-1}\|_{\Omega^d} \Bigg].
\end{aligned}$$

Proof. Let $n \geq 1$. Setting $(\boldsymbol{v}_h, \boldsymbol{q}_h) = (\boldsymbol{e}_u^{h,n}, -\boldsymbol{e}_p^{h,n})$ in eq. (5.3) and rearranging, we find that

$$\begin{aligned}
a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{e}_u^{h,n}, \boldsymbol{e}_u^{h,n}) = {} & a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{e}_u^{I,n}, \boldsymbol{e}_u^{h,n}) + \left[a_h^L(\boldsymbol{c}^n; \boldsymbol{u}^n, \boldsymbol{e}_u^{h,n}) - a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}^n, \boldsymbol{e}_u^{h,n}) \right] \\
& + \int_{\Omega^s} \left[f^s(c_h^{n-1}) - f^s(c^n) \right] \cdot e_u^{h,n} \, \mathrm{d}x \\
& + \int_{\Omega^d} \left[\mu(c_h^{n-1}) \kappa^{-1} f^d(c_h^{n-1}) - \mu(c^n) \kappa^{-1} f^d(c^n) \right] \cdot e_u^{h,n} \, \mathrm{d}x.
\end{aligned}$$

We bound below the left-hand side using eq. (2.17), which is valid due to lemma 3.3.1, and bound above the right-hand side using eqs. (3.22) to (3.24) and (3.27). Dividing through by $|||\boldsymbol{e}_u^{h,n}|||_v$ on both sides and combining all constants gives the result. □

Corollary 5.2.3. *Suppose that*

$$\begin{aligned} u &\in L^\infty(0,T;H^{k_f+1}(\Omega)) \cap L^\infty(0,T;L^\infty(\Omega^s)) \cap L^\infty(0,T;L^\infty(\Omega^d)), \\ \nabla u &\in L^\infty(0,T;L^\infty(\Omega^s)), \\ f^d(c) &\in L^\infty(0,T;L^\infty(\Omega^d)). \end{aligned}$$

Then for $n \geq 1$,

$$\begin{aligned} \|\!|\!| \boldsymbol{u}^n - \boldsymbol{u}_h^n \|\!|\!|_v^2 \leq C\Bigg(& h^{2k_f} + \|c^n - c_h^{n-1}\|_{\Omega^d}^2 + \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I}^2 \\ & + \sum_{K\in\mathcal{T}^s} \left(\|c^n - c_h^{n-1}\|_K^2 + h_K^2 \|c^n - c_h^{n-1}\|_{1,K}^2 \right) \Bigg). \end{aligned}$$

Proof. From the assumptions on the functions and lemma 5.2.2, we can write

$$\begin{aligned} \|\!|\!| \boldsymbol{e}_u^{h,n} \|\!|\!|_v \leq C\Bigg[& \|\!|\!| \boldsymbol{e}_u^{I,n} \|\!|\!|_{v'} + \|c^n - c_h^{n-1}\|_{\Omega^d} + \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I} \\ & + \left(\sum_{K\in\mathcal{T}^s} \left(\|c^n - c_h^{n-1}\|_K^2 + h_K^2 \|c^n - c_h^{n-1}\|_{1,K}^2 \right) \right)^{1/2} \Bigg]. \end{aligned} \tag{5.6}$$

Using eq. (3.2a), the triangle inequality, and eqs. (3.3) and (5.6), it follows that

$$\begin{aligned} \|\!|\!| \boldsymbol{u}^n - \boldsymbol{u}_h^n \|\!|\!|_v &\leq \|\!|\!| \boldsymbol{e}_u^{I,n} \|\!|\!|_{v'} + \|\!|\!| \boldsymbol{e}_u^{h,n} \|\!|\!|_v \\ &\leq C\Bigg[h^{k_f} + \|c^n - c_h^{n-1}\|_{\Omega^d} + \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I} \\ &\quad + \left(\sum_{K\in\mathcal{T}^s} \left(\|c^n - c_h^{n-1}\|_K^2 + h_K^2 \|c^n - c_h^{n-1}\|_{1,K}^2 \right) \right)^{1/2} \Bigg]. \end{aligned}$$

The result is obtained by squaring both sides and applying the discrete Hölder's inequality. □

Replacing n by m in the result of corollary 5.2.3, summing from $m = 1$ to $m = n$, and multiplying by Δt, we immediately see that eq. (3.7i) is satisfied. Squaring the result of lemma 5.2.2 and proceeding similarly, we also see that eq. (3.7h) is satisfied under the assumptions of corollary 5.2.3. We conclude the error analysis of our scheme with some bounds on the pressure errors.

Lemma 5.2.4. *Suppose that (u, p, c) is the solution to the quasi-stationary Stokes/Darcy-transport system. Suppose that $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ satisfies eq. (5.1) for $n \geq 1$, with $\boldsymbol{c}_h^{n-1} \in \boldsymbol{C}_h$ given. Then,*

$$\begin{aligned}
|||\boldsymbol{e}_p^{h,n}|||_p \leq C\bigg(& h^{k_f} + \|u^n\|_{0,\infty,\Omega^s} \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I} \\
& + \big(1 + \|\nabla u^n\|_{0,\infty,\Omega^s}\big)\bigg(\sum_{K\in\mathcal{T}^s} \big(\|c^n - c_h^{n-1}\|_K^2 + h_K^2 \|c^n - c_h^{n-1}\|_{1,K}^2 \big)\bigg)^{1/2} \\
& + \big(1 + \|u^n\|_{0,\infty,\Omega^d} + \|f^d(c^n)\|_{0,\infty,\Omega^d}\big) \|c^n - c_h^{n-1}\|_{\Omega^d} \bigg).
\end{aligned}$$

Proof. Setting $\boldsymbol{q}_h = 0$ in eq. (5.3) and rearranging reveals that

$$\begin{aligned}
b_h(\boldsymbol{e}_p^{h,n}, \boldsymbol{v}_h) = {} & a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{e}_u^{I,n}, \boldsymbol{v}_h) - a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{e}_u^{h,n}, \boldsymbol{v}_h) \\
& + \big[a_h^L(\boldsymbol{c}^n; \boldsymbol{u}^n, \boldsymbol{v}_h) - a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}^n, \boldsymbol{v}_h)\big] + \int_{\Omega^s} \big[f^s(c_h^{n-1}) - f^s(c^n)\big] \cdot v_h \,\mathrm{d}x \\
& + \int_{\Omega^d} \big[\mu(c_h^{n-1})\kappa^{-1} f^d(c_h^{n-1}) - \mu(c^n)\kappa^{-1} f^d(c^n)\big] \cdot v_h \,\mathrm{d}x.
\end{aligned}$$

Applying eqs. (3.22) to (3.24), (3.27) and (3.28), dividing through by $\boldsymbol{v}_h \neq 0$, and combining all constants, we see that for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ with $\boldsymbol{v}_h \neq 0$,

$$\begin{aligned}
\frac{b_h(\boldsymbol{e}_p^{h,n}, \boldsymbol{v}_h)}{|||\boldsymbol{v}_h|||_v} \leq C\bigg(& |||\boldsymbol{e}_u^{I,n}|||_{v'} + |||\boldsymbol{e}_u^{h,n}|||_v + \|u^n\|_{0,\infty,\Omega^s} \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I} \\
& + \big(1 + \|\nabla u^n\|_{0,\infty,\Omega^s}\big)\bigg(\sum_{K\in\mathcal{T}^s} \big(\|c^n - c_h^{n-1}\|_K^2 + h_K^2 \|c^n - c_h^{n-1}\|_{1,K}^2 \big)\bigg)^{1/2} \\
& + \big(1 + \|u^n\|_{0,\infty,\Omega^d} + \|f^d(c^n)\|_{0,\infty,\Omega^d}\big) \|c^n - c_h^{n-1}\|_{\Omega^d} \bigg).
\end{aligned}$$

Inserting in lemma 5.2.2 and eq. (3.3), taking the supremum over $\boldsymbol{v}_h \in \boldsymbol{V}_h$ with $\boldsymbol{v}_h \neq 0$, and applying eq. (2.20) completes the proof. □

Using eq. (3.2b) and the triangle inequality, we can combine lemma 5.2.4 and eq. (3.4) analogously to corollary 5.2.3 to show that

$$
\begin{aligned}
|||\boldsymbol{p}^n - \boldsymbol{p}_h^n|||_p \le C\Bigg(& h^{k_f} + \|u^n\|_{0,\infty,\Omega^s} \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I} \\
& + \big(1 + \|\nabla u^n\|_{0,\infty,\Omega^s}\big)\Bigg(\sum_{K\in\mathcal{T}^s} \big(\|c^n - c_h^{n-1}\|_K^2 + h_K^2 \|c^n - c_h^{n-1}\|_{1,K}^2\big)\Bigg)^{1/2} \\
& + \big(1 + \|u^n\|_{0,\infty,\Omega^d} + \|f^d(c^n)\|_{0,\infty,\Omega^d}\big) \|c^n - c_h^{n-1}\|_{\Omega^d}\Bigg).
\end{aligned} \tag{5.7}
$$

Squaring both sides leads to an upper bound identical to that of corollary 5.2.3, subject to the same regularity conditions. Taking similar steps to those used in the proof of theorem 3.2.1 therefore shows that the pressure has the same convergence rate as the velocity.

Chapter 6

Quasi-stationary Navier–Stokes/Darcy-transport: nonlinear scheme

In this chapter and the next, we develop HDG methods for the quasi-stationary Navier–Stokes/Darcy-transport system. The flow problem for this system is defined by the Navier–Stokes equations eqs. (2.1b) to (2.1d), the Darcy equations eqs. (2.1e) to (2.1h), and the interface conditions eq. (2.2). The transport problem is defined by eq. (2.3).

We will consider two fully-discrete schemes for this system. Both schemes have the structure described in section 2.4.2, with eq. (2.29) defined by eq. (4.1). In both schemes, eq. (2.28) is a temporal discretization of eqs. (2.24) and (2.25). The differentiating trait is whether or not the resulting eq. (2.28) is linear in $\boldsymbol{u}_h^n$. This chapter focuses on the nonlinear case, which has a fully-discrete scheme defined as follows. Set $\boldsymbol{c}_h^0 = (\Pi_C c_0, \bar{\Pi}_C c_0)$, where Π_C and $\bar{\Pi}_C$ are the L^2-projection operators onto C_h and $\bar{C}_h$. For $n = 1, \ldots, N$,

(i) Find $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ such that for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$\begin{aligned} &a_h(\boldsymbol{c}_h^{n-1}; u_h^n; \boldsymbol{u}_h^n, \boldsymbol{v}_h) + b_h(\boldsymbol{p}_h^n, \boldsymbol{v}_h) \\ &= \int_{\Omega^s} f^s(c_h^{n-1}) \cdot v_h \,\mathrm{d}x + \int_{\Omega^d} \mu(c_h^{n-1}) \kappa^{-1} f^d(c_h^{n-1}) \cdot v_h \,\mathrm{d}x, \end{aligned} \tag{6.1a}$$

and for all $\boldsymbol{q}_h \in \boldsymbol{Q}_h$,

$$b_h(\boldsymbol{q}_h, \boldsymbol{u}_h^n) = \sum_{K \in \mathcal{T}^d} \int_K q_h^d (g_p^n - g_i^n) \,\mathrm{d}x. \tag{6.1b}$$

(ii) Find $\boldsymbol{c}_h^n \in \boldsymbol{C}_h$ such that eq. (4.1) holds for all $\boldsymbol{\psi}_h \in \boldsymbol{C}_h$.

6.1 Well-posedness

Proving the well-posedness of the flow problem eq. (6.1) is more complicated than proving the well-posedness of eq. (5.1) due to the nonlinearity introduced by the term $t_h(u_h^n; \boldsymbol{u}_h^n, \boldsymbol{v}_h)$ in eq. (6.1a). Nevertheless, we may obtain the existence and uniqueness of a solution at each $n \geq 1$ by a similar approach to that used in [11], with small adjustments to take into account the concentration-dependent viscosity. We begin by proving the existence and uniqueness of solutions to a related linear problem: given $\boldsymbol{c}_h^{n-1} \in \boldsymbol{C}_h$ and $w_h \in V_h^s$ which satisfies the conditions of lemma 2.3.2, find $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ such that for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$\begin{aligned} &a_h(\boldsymbol{c}_h^{n-1}; w_h; \boldsymbol{u}_h^n, \boldsymbol{v}_h) + b_h(\boldsymbol{p}_h^n, \boldsymbol{v}_h) \\ &= \int_{\Omega^s} f^s(c_h^{n-1}) \cdot v_h \,\mathrm{d}x + \int_{\Omega^d} \mu(c_h^{n-1})\kappa^{-1} f^d(c_h^{n-1}) \cdot v_h \,\mathrm{d}x, \end{aligned} \tag{6.2a}$$

and for all $\boldsymbol{q}_h \in \boldsymbol{Q}_h$,

$$b_h(\boldsymbol{q}_h, \boldsymbol{u}_h^n) = \sum_{K \in \mathcal{T}^d} \int_K q_h^d (g_p^n - g_i^n) \,\mathrm{d}x. \tag{6.2b}$$

For this problem, we turn to [4, Theorem 4.2.3]. As a consequence of eqs. (2.14b) and (2.19) to (2.22) and lemma 2.3.2, the problem eq. (6.2) has a unique solution such that

$$\begin{aligned} |||\boldsymbol{u}_h^n|||_v \leq\ & (c_{ae}\mu_*)^{-1}\big(L_f^s \,\|c_h^{n-1}\|_{\Omega^s} + L_f^d \mu^* \kappa_*^{-1} \,\|c_h^{n-1}\|_{\Omega^d}\big) \\ &+ (c_{ae}\mu_*)^{-1}\big(\,\|f_0^{sn}\|_{\Omega^s} + \mu^*\kappa_*^{-1}\,\|f_0^{dn}\|_{\Omega^d}\big) \\ &+ 2c_{ac}\mu^*(c_{ae}\mu_* c_{bb})^{-1}\,\|g_p^n - g_i^n\|_{\Omega^d} \end{aligned} \tag{6.3a}$$

and

$$\begin{aligned} |||\boldsymbol{p}_h^n|||_p \leq\ & 2c_{ac}\mu^*(c_{ae}\mu_* c_{bb})^{-1}\big(L_f^s \,\|c_h^{n-1}\|_{\Omega^s} + L_f^d \mu^* \kappa_*^{-1} \,\|c_h^{n-1}\|_{\Omega^d}\big) \\ &+ 2c_{ac}\mu^*(c_{ae}\mu_* c_{bb})^{-1}\big(\,\|f_0^{sn}\|_{\Omega^s} + \mu^*\kappa_*^{-1}\,\|f_0^{dn}\|_{\Omega^d}\big) \\ &+ 2(c_{ac}\mu^*)^2(c_{ae}\mu_* c_{bb}^2)^{-1}\,\|g_p^n - g_i^n\|_{\Omega^d}. \end{aligned} \tag{6.3b}$$

Although we recall here that c_{ac} depends on w_h, eqs. (6.3a) and (6.3b) may be bounded independently of c_h^{n-1} using eq. (5.2). In what follows, we define for each $n \geq 1$ the quantity

$$\begin{aligned} M^n :=\ & (c_{ae}\mu_*)^{-1}\big(L_f^s + L_f^d\mu^*\kappa_*^{-1}\big)\phi_*^{-1/2} e^{K_{n-1}\Delta t/2}\left(\phi_*\,\|c_0\|_\Omega^2 + \Delta t \sum_{m=1}^{n-1} \|g_i^m\|_{\Omega^d}^2\right)^{1/2} \\ &+ (c_{ae}\mu_*)^{-1}\big(\,\|f_0^{sn}\|_{\Omega^s} + \mu^*\kappa_*^{-1}\,\|f_0^{dn}\|_{\Omega^d}\big) + 2c_f\mu^*(c_{ae}\mu_* c_{bb})^{-1}\,\|g_p^n - g_i^n\|_{\Omega^d}, \end{aligned}$$

with $c_f = 2c_e^2 \max\left(c_w(\mu^*)^{-1}c_\rho c_{si,2}^{-1}\mu_* c_{ae}^s \delta(c_{pq}^2 + c_{si,4}^2)^{-1} + c_{ac}^s, \kappa_*^{-1}, \alpha\kappa_*^{-1/2}\right)$. This is the right-hand side of eq. (6.3a) after inserting in the upper bound eq. (5.2) and replacing c_{ac} by c_f. Notably, M^n depends only on the data and problem parameters. Then, for each $n \geq 1$, we consider the space

$$\boldsymbol{R}_h^{sn} := \{\boldsymbol{v}_h \in \boldsymbol{Z}_h^s : |||\boldsymbol{v}_h|||_{v,s} \leq M^n\}.$$

Elements of these spaces have a couple of useful properties, which we show next.

Lemma 6.1.1. *Suppose that $\boldsymbol{w}_h^{sn} \in \boldsymbol{R}_h^{sn}$, for $n \geq 1$. If*

$$M^n \leq c_{si,2}^{-1}\mu_* c_{ae}^s \delta(c_{pq}^2 + c_{si,4}^2)^{-1}, \tag{6.4}$$

then

1. *$\boldsymbol{w}_h^{sn}$ fulfills the conditions of lemma 2.3.2, and*
2. *the continuity constant from eqs. (2.14a) and (2.14b) with $w = w_h^{sn}$ is such that $c_{ac} \leq c_f$.*

Proof. First, considering $\boldsymbol{q}_h^s = (q_h^s, 0)$ in the definition of $\boldsymbol{Z}_h^s$ shows that $\nabla \cdot w_h^{sn} = 0$ on each $K \in \mathcal{T}^s$. Next, we have from eq. (2.30c) with $r = 2$ and eq. (6.4) that

$$\|w_h^{sn}\|_{\Gamma^I} \leq c_{si,2}|||\boldsymbol{w}_h^n|||_{v,s} \leq c_{si,2}M^n \leq \mu_* c_{ae}^s \delta(c_{pq}^2 + c_{si,4}^2)^{-1},$$

which shows that $\boldsymbol{w}_h^{sn}$ satisfies the conditions of lemma 2.3.2. The property $c_{ac} \leq c_f$ comes from eq. (2.9) and eq. (6.4). □

Our next step in proving the well-posedness of eq. (6.1a) is to consider at each $n \geq 1$ a fixed point operator $\boldsymbol{\Psi}_h^n : \boldsymbol{R}_h^{sn} \to \boldsymbol{R}_h^{sn}$ defined as $\boldsymbol{\Psi}_h^n(\boldsymbol{w}_h^{sn}) := \boldsymbol{u}_h^{sn}$ for all $\boldsymbol{w}_h^{sn} \in \boldsymbol{R}_h^{sn}$. Here, $\boldsymbol{u}_h^n$ is the velocity solution to the linear problem eq. (6.2) and $\boldsymbol{u}_h^{sn}$ is the restriction of $\boldsymbol{u}_h^n$ to Ω^s. With any given $\boldsymbol{c}_h^{n-1} \in \boldsymbol{C}_h$, this mapping is well-defined under the assumption eq. (6.4). To see this, we consider $\boldsymbol{w}_h^{sn} \in \boldsymbol{R}_h^{sn}$. As a result of lemma 6.1.1 and eq. (5.2), there exists a unique solution $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n)$ to eq. (6.2) such that $|||\boldsymbol{u}_h^n|||_v \leq M^n$. Furthermore, $\boldsymbol{u}_h^{sn} \in \boldsymbol{Z}_h^s$ due to eq. (2.26). With this operator defined, we now use fixed point theorems to prove the existence of a unique solution to eq. (6.1a) at $n \geq 1$.

Theorem 6.1.2. *Let $n \geq 1$. In addition to eq. (6.4), suppose that*

$$M^n < c_{ae}\mu_*(c_w c_\rho)^{-1}, \tag{6.5}$$

and let $\boldsymbol{c}_h^{n-1} \in \boldsymbol{C}_h$ be given. Then, $\boldsymbol{\Psi}_h$ has a unique fixed point in $\boldsymbol{R}_h^{sn}$.

Proof. Suppose that $\boldsymbol{w}_{h,m}^{sn} \in \boldsymbol{R}_h^{sn}$ is given, for $m = 1, 2$. Let $(\boldsymbol{u}_{h,m}^n, \boldsymbol{p}_{h,m}^n) \in \boldsymbol{X}_h$ be the corresponding unique solutions to eq. (6.2). Subtracting eq. (6.2) with $m = 2$ from eq. (6.2) with $m = 1$, and setting $\boldsymbol{v}_h = \boldsymbol{u}_{h,1}^n - \boldsymbol{u}_{h,2}^n$ and $\boldsymbol{q}_h = \boldsymbol{p}_{h,2}^n - \boldsymbol{p}_{h,1}^n$, we obtain

$$a_h(\boldsymbol{c}_h^{n-1}; w_{h,1}^{sn}; \boldsymbol{u}_{h,1}^n, \boldsymbol{u}_{h,1}^n - \boldsymbol{u}_{h,2}^n) = a_h(\boldsymbol{c}_h^{n-1}; w_{h,2}^{sn}; \boldsymbol{u}_{h,2}^n, \boldsymbol{u}_{h,1}^n - \boldsymbol{u}_{h,2}^n).$$

Using the linearity of $a_h(\cdot;\cdot;\cdot,\cdot)$ in the third argument, we can therefore write

$$\begin{aligned}
&a_h(\boldsymbol{c}_h^{n-1}; w_{h,1}^{sn}; \boldsymbol{u}_{h,1}^n - \boldsymbol{u}_{h,2}^n, \boldsymbol{u}_{h,1}^n - \boldsymbol{u}_{h,2}^n) \\
&= a_h(\boldsymbol{c}_h^{n-1}; w_{h,2}^{sn}; \boldsymbol{u}_{h,2}^n, \boldsymbol{u}_{h,1}^n - \boldsymbol{u}_{h,2}^n) - a_h(\boldsymbol{c}_h^{n-1}; w_{h,1}^{sn}; \boldsymbol{u}_{h,2}^n, \boldsymbol{u}_{h,1}^n - \boldsymbol{u}_{h,2}^n) \\
&= t_h(w_{h,2}^{sn}; \boldsymbol{u}_{h,2}^n, \boldsymbol{u}_{h,1}^n - \boldsymbol{u}_{h,2}^n) - t_h(w_{h,1}^{sn}; \boldsymbol{u}_{h,2}^n, \boldsymbol{u}_{h,1}^n - \boldsymbol{u}_{h,2}^n).
\end{aligned}$$

Combining this with lemmas 2.3.2 and 6.1.1 and eqs. (2.9), (2.13) and (2.26), and using that $|||\cdot|||_{v,s} \leq |||\cdot|||_v$, we find that

$$\begin{aligned}
c_{ae}\mu_* |||\boldsymbol{u}_{h,1}^n - \boldsymbol{u}_{h,2}^n|||_{v,s}^2 &\leq a_h(\boldsymbol{c}_h^{n-1}; w_{h,1}^{sn}; \boldsymbol{u}_{h,1}^n - \boldsymbol{u}_{h,2}^n, \boldsymbol{u}_{h,1}^n - \boldsymbol{u}_{h,2}^n) \\
&\leq |t_h(w_{h,1}^{sn}; \boldsymbol{u}_{h,2}^n, \boldsymbol{u}_{h,1}^n - \boldsymbol{u}_{h,2}^n) - t_h(w_{h,2}^{sn}; \boldsymbol{u}_{h,2}^n, \boldsymbol{u}_{h,1}^n - \boldsymbol{u}_{h,2}^n)| \\
&\leq c_w c_\rho |||\boldsymbol{w}_{h,1}^{sn} - \boldsymbol{w}_{h,2}^{sn}|||_{v,s} |||\boldsymbol{u}_{h,2}^n|||_{v,s} |||\boldsymbol{u}_{h,1}^n - \boldsymbol{u}_{h,2}^n|||_{v,s}.
\end{aligned}$$

Dividing through by $|||\boldsymbol{u}_{h,1}^n - \boldsymbol{u}_{h,2}^n|||_{v,s}$ and using that $\boldsymbol{u}_{h,m}^{sn} =: \boldsymbol{\Psi}_h^n(\boldsymbol{w}_{h,m}^{sn})$, we may write

$$|||\boldsymbol{\Psi}_h^n(\boldsymbol{w}_{h,1}^{sn}) - \boldsymbol{\Psi}_h^n(\boldsymbol{w}_{h,2}^{sn})|||_{v,s} \leq c_w c_\rho (c_{ae}\mu_*)^{-1} M^n |||\boldsymbol{w}_{h,1}^{sn} - \boldsymbol{w}_{h,2}^{sn}|||_{v,s},$$

which shows that $\boldsymbol{\Psi}_h^n$ is continuous and therefore has a fixed point by Brouwer's fixed point theorem. Applying the assumption eq. (6.5) further shows that $\boldsymbol{\Psi}_h^n$ is a contraction, so that Banach's fixed point theorem guarantees the uniqueness of the fixed point. □

Consider now the condition

$$M^n < \min(c_{si,2}^{-1}\mu_* c_{ae}^s \delta(c_{pq}^2 + c_{si,4}^2)^{-1}, c_{ae}\mu_*(c_w c_\rho)^{-1}), \tag{6.6}$$

which implies the fulfillment of eqs. (6.4) and (6.5). If eq. (6.6) is met at $n \geq 1$, then theorem 6.1.2 gives the existence of unique velocity and pressure solutions $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ to eq. (6.1) such that

$$\begin{aligned}
|||\boldsymbol{u}_h^n|||_v \leq{}& (c_{ae}\mu_*)^{-1}\big(L_f^s \,\|c_h^{n-1}\|_{\Omega^s} + L_f^d \mu^* \kappa_*^{-1} \,\|c_h^{n-1}\|_{\Omega^d}\big) \\
&+ (c_{ae}\mu_*)^{-1}\big(\,\|f_0^{sn}\|_{\Omega^s} + \mu^*\kappa_*^{-1}\,\|f_0^{dn}\|_{\Omega^d}\big) \\
&+ 2c_f\mu^*(c_{ae}\mu_* c_{bb})^{-1}\,\|g_p^n - g_i^n\|_{\Omega^d}
\end{aligned} \tag{6.7a}$$

and

$$\begin{aligned}\|\!|\boldsymbol{p}_h^n|\!\|_p \leq 2c_f\mu^*(c_{ae}\mu_*c_{bb})^{-1}\big(L_f^s\,\|c_h^{n-1}\|_{\Omega^s} + L_f^d\mu^*\kappa_*^{-1}\,\|c_h^{n-1}\|_{\Omega^d}\big)\\ + 2c_f\mu^*(c_{ae}\mu_*c_{bb})^{-1}\big(\,\|f_0^{sn}\|_{\Omega^s} + \mu^*\kappa_*^{-1}\,\|f_0^{dn}\|_{\Omega^d}\big)\\ + 2(c_f\mu^*)^2(c_{ae}\mu_*c_{bb}^2)^{-1}\,\|g_p^n - g_i^n\|_{\Omega^d}\,.\end{aligned} \tag{6.7b}$$

We note that the right-hand sides of eqs. (6.7a) and (6.7b) match the respective right-hand sides of eqs. (6.3a) and (6.3b), with c_f in place of the w_h-dependent c_{ac}. Supposing that $\boldsymbol{c}_h^{n-1}$ is the solution to eq. (4.1), these right-hand sides may further be bounded independently of the concentration with the help of eq. (5.2).

6.2 Error analysis

We begin our error analysis by establishing the error equation.

Lemma 6.2.1. *Let $n \geq 1$ and suppose that $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ satisfies eq. (6.1) for a given $\boldsymbol{c}_h^{n-1} \in \boldsymbol{C}_h$. Then, for all $(\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h$, there holds*

$$\begin{aligned}&a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{e}_u^{h,n}, \boldsymbol{v}_h) - a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{e}_u^{I,n}, \boldsymbol{v}_h) + a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}^n, \boldsymbol{v}_h) - a_h^L(\boldsymbol{c}^n; \boldsymbol{u}^n, \boldsymbol{v}_h)\\ &+ t_h(u_h^n; \boldsymbol{u}_h^n, \boldsymbol{v}_h) - t_h(u^n; \boldsymbol{e}_u^{I,n}, \boldsymbol{v}_h) + t_h(u^n; \boldsymbol{e}_u^{h,n}, \boldsymbol{v}_h) - t_h(u^n; \boldsymbol{u}_h^n, \boldsymbol{v}_h)\\ &+ b_h(\boldsymbol{e}_p^{h,n}, \boldsymbol{v}_h) + b_h(\boldsymbol{q}_h, \boldsymbol{e}_u^{h,n})\\ =& \int_{\Omega^s} \big[f^s(c_h^{n-1}) - f^s(c^n)\big]\cdot v_h \,\mathrm{d}x\\ &+ \int_{\Omega^d} \big[\mu(c_h^{n-1})\kappa^{-1}f^d(c_h^{n-1}) - \mu(c^n)\kappa^{-1}f^d(c^n)\big]\cdot v_h\,\mathrm{d}x,\end{aligned} \tag{6.8}$$

where (u, p, c) is the solution to the quasi-stationary Navier–Stokes/Darcy-transport system and $\boldsymbol{u}^n := (u(t^n), \gamma(u(t^n)))$, $\boldsymbol{p}^n := (p(t^n), \gamma(p^s(t^n)), \gamma(p^d(t^n)))$, and $\boldsymbol{c}^n := (c(t^n), \gamma(c(t^n)))$.

Proof. Let $n \geq 1$. Let $(\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h$. Subtracting the consistency equation for eq. (6.1a) from eq. (6.1a) gives

$$\begin{aligned}&a_h(\boldsymbol{c}_h^{n-1}; u_h^n; \boldsymbol{u}_h^n, \boldsymbol{v}_h) - a_h(\boldsymbol{c}^n; u^n; \boldsymbol{u}^n, \boldsymbol{v}_h) + b_h(\boldsymbol{p}_h^n - \boldsymbol{p}^n, \boldsymbol{v}_h)\\ =& \int_{\Omega^s}\big[f^s(c_h^{n-1}) - f^s(c^n)\big]\cdot v_h\,\mathrm{d}x\\ &+ \int_{\Omega^d}\big[\mu(c_h^{n-1})\kappa^{-1}f^d(c_h^{n-1}) - \mu(c^n)\kappa^{-1}f^d(c^n)\big]\cdot v_h\,\mathrm{d}x.\end{aligned} \tag{6.9}$$

Subtracting the consistency equation for eq. (6.1b) from eq. (6.1b) gives

$$b_h(\boldsymbol{q}_h, \boldsymbol{u}_h^n - \boldsymbol{u}^n) = 0. \tag{6.10}$$

Summing eqs. (6.9) and (6.10), splitting $a_h(\cdot;\cdot;\cdot,\cdot)$ into linear and nonlinear parts, and inserting eqs. (3.15) to (3.17) yields the result. $\square$

Next, we prove bounds on the velocity and pressure errors.

Lemma 6.2.2. *Suppose that (u, p, c) is the solution to the quasi-stationary Navier–Stokes/Darcy-transport system. Suppose that eq. (6.6) holds and that $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ satisfies eq. (6.1) for $n \geq 1$, with $\boldsymbol{c}_h^{n-1} \in \boldsymbol{C}_h$ given. If $M^n < \frac{1}{2} c_{ae}\mu_*(c_w c_\rho)^{-1}$ and u^n satisfies the conditions on w of lemma 2.3.2, then*

$$\begin{aligned}
|||\boldsymbol{e}_u^{h,n}|||_v \leq C\Bigg[&\left(1 + \|u^n\|_{1,h,\Omega^s}\right)|||\boldsymbol{e}_u^{I,n}|||_{v'} + \|u^n\|_{0,\infty,\Omega^s}\,\|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I} \\
&+ \left(1 + \|\nabla u^n\|_{0,\infty,\Omega^s}\right)\Bigg(\sum_{K\in\mathcal{T}^s}\left(\|c^n - c_h^{n-1}\|_K^2 + h_K^2\,\|c^n - c_h^{n-1}\|_{1,K}^2\right)\Bigg)^{1/2} \\
&+ \left(1 + \|u^n\|_{0,\infty,\Omega^d} + \|f^d(c^n)\|_{0,\infty,\Omega^d}\right)\|c^n - c_h^{n-1}\|_{\Omega^d}\Bigg].
\end{aligned}$$

Proof. Let $n \geq 1$. Setting $(\boldsymbol{v}_h, \boldsymbol{q}_h) = (\boldsymbol{e}_u^{h,n}, -\boldsymbol{e}_p^{h,n})$ in eq. (6.8) and rearranging shows that

$$\begin{aligned}
&a_h(\boldsymbol{c}_h^{n-1}; u^n; \boldsymbol{e}_u^{h,n}, \boldsymbol{e}_u^{h,n}) \\
&= a_h(\boldsymbol{c}_h^{n-1}; u^n; \boldsymbol{e}_u^{I,n}, \boldsymbol{e}_u^{h,n}) + \left[a_h^L(\boldsymbol{c}^n; \boldsymbol{u}^n, \boldsymbol{e}_u^{h,n}) - a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}^n, \boldsymbol{e}_u^{h,n})\right] \\
&\quad + \left[t_h(u^n; \boldsymbol{u}_h^n, \boldsymbol{e}_u^{h,n}) - t_h(u_h^n; \boldsymbol{u}_h^n, \boldsymbol{e}_u^{h,n})\right] + \int_{\Omega^s}\left[f^s(c_h^{n-1}) - f^s(c^n)\right]\cdot e_u^{h,n}\,\mathrm{d}x \\
&\quad + \int_{\Omega^d}\left[\mu(c_h^{n-1})\kappa^{-1}f^d(c_h^{n-1}) - \mu(c^n)\kappa^{-1}f^d(c^n)\right]\cdot e_u^{h,n}\,\mathrm{d}x.
\end{aligned}$$

By construction, $\boldsymbol{u}_h^n \in \boldsymbol{R}_h^{sn}$. Under the assumption on M^n, we thus have that

$$|||\boldsymbol{u}_h^n|||_{v,s} < \frac{1}{2}c_{ae}\mu_*(c_w c_\rho)^{-1}.$$

From eqs. (2.9), (2.13) and (3.2a), and the triangle inequality, it follows that

$$\begin{aligned}
|t_h(u^n; \boldsymbol{u}_h^n, \boldsymbol{e}_u^{h,n}) - t_h(u_h^n; \boldsymbol{u}_h^n, \boldsymbol{e}_u^{h,n})| &\leq c_w\,\|e_u^{I,n} - e_u^{h,n}\|_{1,h,\Omega^s}\,|||\boldsymbol{u}_h^n|||_{v,s}|||\boldsymbol{e}_u^{h,n}|||_{v,s} \\
&< \frac{1}{2}c_{ae}\mu_*|||\boldsymbol{e}_u^{I,n} - \boldsymbol{e}_u^{h,n}|||_{v,s}|||\boldsymbol{e}_u^{h,n}|||_{v,s} \\
&\leq \frac{1}{2}c_{ae}\mu_*|||\boldsymbol{e}_u^{I,n}|||_{v'}|||\boldsymbol{e}_u^{h,n}|||_v + \frac{1}{2}c_{ae}\mu_*|||\boldsymbol{e}_u^{h,n}|||_v^2. \qquad (6.11)
\end{aligned}$$

By the assumption on u^n and lemma 3.3.1, we can use lemma 2.3.2 to bound below the left-hand side. To bound above the right-hand side, we use eqs. (3.22) to (3.24), (3.29) and (6.11). Moving the term involving $\|\!|\boldsymbol{e}_u^{h,n}|\!\|_v^2$ to the left-hand side, dividing both sides through by $\|\!|\boldsymbol{e}_u^{h,n}|\!\|_v$, and combining all constants yields the result. $\square$

Remark 6.2.1. *Conditions for the existence of a weak solution such that u satisfies the conditions on w of lemma 2.3.2 were proved in [13], in the case that the right-hand sides of eqs. (2.1a), (2.1b), (2.1e) and (2.3a) are independent of the concentration.*

Corollary 6.2.3. *In addition to the hypotheses of corollary 5.2.3, suppose that*

$$\nabla u \in L^\infty(0,T;L^2(\Omega^s)).$$

Then for $n \geq 1$,

$$\|\!|\boldsymbol{u}^n - \boldsymbol{u}_h^n|\!\|_v^2 \leq C\bigg(h^{2k_f} + \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I}^2 + \|c^n - c_h^{n-1}\|_{\Omega^d}^2 + \sum_{K\in\mathcal{T}^s} \left(\|c^n - c_h^{n-1}\|_K^2 + h_K^2 \|c^n - c_h^{n-1}\|_{1,K}^2 \right)\bigg).$$

Proof. We note that since $\{\!\{u^n\}\!\} = u^n$ on $F \in \mathcal{F}^s$ [18, Lemma 1.23], then $\|u^n\|_{1,h,\Omega^s} = \|\nabla u^n\|_{\Omega^s}$. Equipped with this knowledge, we can proceed as in the proof of corollary 5.2.3, using lemma 6.2.2 in place of lemma 5.2.2. $\square$

Lemma 6.2.4. *Suppose that (u,p,c) is the solution to the quasi-stationary Navier–Stokes/Darcy-transport system. Suppose that eq. (6.6) holds and that $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ satisfies eq. (6.1) for $n \geq 1$, with $\boldsymbol{c}_h^{n-1} \in \boldsymbol{C}_h$ given. Then,*

$$\begin{aligned}\|\!|\boldsymbol{e}_p^{h,n}|\!\|_p \leq C\big(1 + \|u^n\|_{1,h,\Omega^s}\big)\bigg(&\big(1 + \|u^n\|_{1,h,\Omega^s}\big) h^{k_f} + \|u^n\|_{0,\infty,\Omega^s} \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I} \\ &+ \big(1 + \|\nabla u^n\|_{0,\infty,\Omega^s}\big)\bigg(\sum_{K\in\mathcal{T}^s} \left(\|c^n - c_h^{n-1}\|_K^2 + h_K^2 \|c^n - c_h^{n-1}\|_{1,K}^2 \right)\bigg)^{1/2} \\ &+ \big(1 + \|u^n\|_{0,\infty,\Omega^d} + \|f^d(c^n)\|_{0,\infty,\Omega^d}\big) \|c^n - c_h^{n-1}\|_{\Omega^d} \bigg).\end{aligned}$$

Proof. Setting $\boldsymbol{q}_h = 0$ in eq. (6.8), applying eqs. (2.13), (2.14b), (3.22) to (3.24) and (3.29), and combining all constants, we are left with

$$
\begin{aligned}
& b_h(\boldsymbol{e}_p^{h,n}, \boldsymbol{v}_h) \\
& \le C\Bigg(\big(1 + \|u^n\|_{1,h,\Omega^s}\big) |||\boldsymbol{e}_u^{I,n}|||_{v'} + \big(1 + \|u^n\|_{1,h,\Omega^s}\big) |||\boldsymbol{e}_u^{h,n}|||_v \\
& \quad + \|u^n - u_h^n\|_{1,h,\Omega^s} |||\boldsymbol{u}_h^n|||_{v,s} + \|u^n\|_{0,\infty,\Omega^s} \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I} \\
& \quad + \big(1 + \|\nabla u^n\|_{0,\infty,\Omega^s}\big) \Big(\sum_{K \in \mathcal{T}^s} \big(\|c^n - c_h^{n-1}\|_K^2 + h_K^2 \|c^n - c_h^{n-1}\|_{1,K}^2 \big) \Big)^{1/2} \\
& \quad + \big(1 + \|u^n\|_{0,\infty,\Omega^d} + \|f^d(c^n)\|_{0,\infty,\Omega^d}\big) \|c^n - c_h^{n-1}\|_{\Omega^d} \Bigg) |||\boldsymbol{v}_h|||_v. \qquad (6.12)
\end{aligned}
$$

Recalling that $\boldsymbol{u}_h^n \in \boldsymbol{R}_h^{sn}$, and using eqs. (2.9) and (3.2a) and the triangle inequality, we have under the assumption eq. (6.6) that

$$
\|u^n - u_h^n\|_{1,h,\Omega^s} |||\boldsymbol{u}_h^n|||_{v,s} \le M^n |||\boldsymbol{u}^n - \boldsymbol{u}_h^n|||_{v,s} < c_{ae}\mu_*(c_w c_\rho)^{-1}\big(|||\boldsymbol{e}_u^{I,n}|||_{v'} + |||\boldsymbol{e}_u^{h,n}|||_v\big). \quad (6.13)
$$

To conclude the proof, we first insert eqs. (3.3) and (6.13) and lemma 6.2.2 into eq. (6.12), combine the constants, and divide through by $|||\boldsymbol{v}_h|||_v$. Then, taking the supremum over $\boldsymbol{v}_h \in \boldsymbol{V}_h$ with $\boldsymbol{v}_h \neq 0$ and applying eq. (2.20), we arrive at the result. □

With the same steps used to obtain eq. (5.7), we can also show that

$$
\begin{aligned}
|||\boldsymbol{p}^n - \boldsymbol{p}_h^n|||_p \le C\big(1 + \|u^n\|_{1,h,\Omega^s}\big) \Bigg(& \big(1 + \|u^n\|_{1,h,\Omega^s}\big) h^{k_f} + \|u^n\|_{0,\infty,\Omega^s} \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I} \\
& + \big(1 + \|\nabla u^n\|_{0,\infty,\Omega^s}\big) \Big(\sum_{K \in \mathcal{T}^s} \big(\|c^n - c_h^{n-1}\|_K^2 + h_K^2 \|c^n - c_h^{n-1}\|_{1,K}^2 \big) \Big)^{1/2} \\
& + \big(1 + \|u^n\|_{0,\infty,\Omega^d} + \|f^d(c^n)\|_{0,\infty,\Omega^d}\big) \|c^n - c_h^{n-1}\|_{\Omega^d} \Bigg). \qquad (6.14)
\end{aligned}
$$

Similarly to section 5.2, we can use lemma 6.2.2 and corollary 6.2.3 to show that eqs. (3.7h) and (3.7i) hold, subject to the regularity conditions of corollary 6.2.3. We can then use eq. (6.14) to show that the pressure converges at the same rate as the velocity.

Chapter 7

Quasi-stationary Navier–Stokes/Darcy-transport: linear scheme

In the fully-discrete scheme for the quasi-stationary Navier–Stokes/Darcy-transport system proposed in chapter 6, the discrete flow problem eq. (6.1) is nonlinear in $\boldsymbol{u}_h^n$. By time-lagging the convective velocity, we can remove this nonlinearity to obtain an alternative scheme, which is the focus of this chapter. We note that the quasi-stationary Navier–Stokes/Darcy-transport system does not have any initial condition for u that can be used to set the time-lagged $\boldsymbol{u}_h^{n-1}$ when $n = 1$. To handle this, we solve the nonlinear flow problem eq. (6.1) at $n = 1$, then proceed with the linear flow problem for $n \geq 2$.

The scheme is formally defined as follows. Set $\boldsymbol{c}_h^0 = (\Pi_C c_0, \bar{\Pi}_C c_0)$, where Π_C and $\bar{\Pi}_C$ are the L^2-projection operators onto C_h and $\bar{C}_h$. Find $(\boldsymbol{u}_h^1, \boldsymbol{p}_h^1) \in \boldsymbol{X}_h$ which satisfies eq. (6.1). Find $\boldsymbol{c}_h^1 \in \boldsymbol{C}_h$ which satisfies eq. (4.1) for all $\boldsymbol{\psi}_h \in \boldsymbol{C}_h$. For $n = 2, \ldots, N$,

(i) Find $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ such that for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$\begin{aligned} &a_h(\boldsymbol{c}_h^{n-1}; u_h^{n-1}; \boldsymbol{u}_h^n, \boldsymbol{v}_h) + b_h(\boldsymbol{p}_h^n, \boldsymbol{v}_h) \\ &= \int_{\Omega^s} f^s(c_h^{n-1}) \cdot v_h \,\mathrm{d}x + \int_{\Omega^d} \mu(c_h^{n-1})\kappa^{-1} f^d(c_h^{n-1}) \cdot v_h \,\mathrm{d}x, \end{aligned} \tag{7.1a}$$

and for all $\boldsymbol{q}_h \in \boldsymbol{Q}_h$,

$$b_h(\boldsymbol{q}_h, \boldsymbol{u}_h^n) = \sum_{K \in \mathcal{T}^d} \int_K q_h^d (g_p^n - g_i^n) \,\mathrm{d}x. \tag{7.1b}$$

(ii) Find $\boldsymbol{c}_h^n \in \boldsymbol{C}_h$ such that eq. (4.1) holds for all $\boldsymbol{\psi}_h \in \boldsymbol{C}_h$.

7.1 Well-posedness

To prove the well-posedness of the flow problem eq. (7.1), we first consider $n \geq 2$ and suppose that we are given $\boldsymbol{u}_h^{n-1}$ which satisfies the hypotheses on w of lemma 2.3.2. Provided that $\beta_f > \beta_0^f$, we have by [4, Theorem 4.2.3] and eqs. (2.14b) and (2.19) to (2.22) and lemma 2.3.2 that there exists a unique solution $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ to eq. (7.1) such that

$$\begin{aligned} |||\boldsymbol{u}_h^n|||_v \leq (c_{ae}\mu_*)^{-1}\big(L_f^s \, \|c_h^{n-1}\|_{\Omega^s} + L_f^d \mu^* \kappa_*^{-1} \, \|c_h^{n-1}\|_{\Omega^d}\big) \\ + (c_{ae}\mu_*)^{-1}\big(\, \|f_0^{sn}\|_{\Omega^s} + \mu^* \kappa_*^{-1} \, \|f_0^{dn}\|_{\Omega^d}\big) \\ + 2(c_{ac}^{n-1}\mu^*)(c_{ae}\mu_* c_{bb})^{-1} \, \|g_p^n - g_i^n\|_{\Omega^d} \end{aligned} \tag{7.2a}$$

and

$$\begin{aligned} |||\boldsymbol{p}_h^n|||_p \leq 2c_{ac}^{n-1}\mu^*(c_{ae}\mu_* c_{bb})^{-1}\big(L_f^s \, \|c_h^{n-1}\|_{\Omega^s} + L_f^d \mu^* \kappa_*^{-1} \, \|c_h^{n-1}\|_{\Omega^d}\big) \\ + 2c_{ac}^{n-1}\mu^*(c_{ae}\mu_* c_{bb})^{-1}\big(\, \|f_0^{sn}\|_{\Omega^s} + \mu^* \kappa_*^{-1} \, \|f_0^{dn}\|_{\Omega^d}\big) \\ + 2(c_{ac}^{n-1}\mu^*)^2(c_{ae}\mu_* c_{bb}^2)^{-1} \, \|g_p^n - g_i^n\|_{\Omega^d}, \end{aligned} \tag{7.2b}$$

with $c_{ac}^{n-1} = 2c_e^2 \max(c_w(\mu^*)^{-1} \, \|u_h^{n-1}\|_{1,h,\Omega^s} + c_{ac}^s, \kappa_*^{-1}, \alpha\kappa_*^{-1/2})$. We aim to show that $\boldsymbol{u}_h^n$ also satisfies the hypotheses of lemma 2.3.2 on w, so that there exists a unique solution to eq. (7.1) at the next time step. The existence and uniqueness of solutions at all $n \geq 1$ will then follow by induction. To this end, we remove the dependence of c_{ac}^{n-1} on u_h^{n-1} in the following lemma.

Lemma 7.1.1. *Let $n \geq 2$. Suppose that*

$$|||\boldsymbol{u}_h^{n-1}|||_v \leq (c_w c_\rho)^{-1}\mu^* \min\big(\kappa_*^{-1}, \alpha\kappa_*^{-1/2}\big). \tag{7.3}$$

Then,

$$c_{ac}^{n-1} \leq 2c_e^2 \max(\kappa_*^{-1} + c_{ac}^s, \alpha\kappa_*^{-1/2} + c_{ac}^s).$$

Proof. Subject to eq. (7.3), we have by eq. (2.9) that

$$c_w(\mu^*)^{-1} \, \|u_h^{n-1}\|_{1,h,\Omega^s} + c_{ac}^s \leq \kappa_*^{-1} + c_{ac}^s$$

and

$$c_w(\mu^*)^{-1} \, \|u_h^{n-1}\|_{1,h,\Omega^s} + c_{ac}^s \leq \alpha\kappa_*^{-1/2} + c_{ac}^s.$$

The result follows by the definition of c_{ac}^{n-1} and that $c_{ac}^s > 0$. $\square$

We now prove the main existence and uniqueness result.

Theorem 7.1.2. *Suppose that $\beta_f > \beta_0^f$, and assume that eq. (6.6) is satisfied at $n = 1$. Let $(\boldsymbol{u}_h^1, \boldsymbol{p}_h^1) \in \boldsymbol{X}_h$ be the solution to eq. (6.1) at $n = 1$. Suppose that*

$$\begin{aligned}
&(c_{ae}\mu_*)^{-1}\big(L_f^s \left\|\Pi_C c_0\right\|_{\Omega^s} + L_f^d \mu^* \kappa_*^{-1} \left\|\Pi_C c_0\right\|_{\Omega^d}\big) \\
&\quad + (c_{ae}\mu_*)^{-1}\big(\|f_0^{s1}\|_{\Omega^s} + \mu^*\kappa_*^{-1} \|f_0^{d1}\|_{\Omega^d} \big) + 2c_f\mu^*(c_{ae}\mu_* c_{bb})^{-1} \|g_p^1 - g_i^1\|_{\Omega^d} \\
&\le \min\big(\mu_*\delta c_{ae}^s (c_{si,2})^{-1}(c_{pq}^2 + c_{si,4}^2)^{-1}, (c_w c_\rho)^{-1}\mu^*\kappa_*^{-1}, (c_w c_\rho)^{-1}\mu^*\alpha\kappa_*^{-1/2}\big),
\end{aligned} \tag{7.4}$$

and that for all $n \ge 2$,

$$\begin{aligned}
&(c_{ae}\mu_*)^{-1}\big(L_f^s + L_f^d\mu^*\kappa_*^{-1}\big)\phi_*^{-1/2} e^{K_{n-1}\Delta t/2}\left(\phi_* \left\|c_0\right\|_\Omega^2 + \Delta t \sum_{m=1}^{n-1} \|g_i^m\|_{\Omega^d}^2\right)^{1/2} \\
&\quad + (c_{ae}\mu_*)^{-1}\big(\|f_0^{sn}\|_{\Omega^s} + \mu^*\kappa_*^{-1} \|f_0^{dn}\|_{\Omega^d} \big) \\
&\quad + 4c_e^2\mu^* \max(\kappa_*^{-1} + c_{ac}^s, \alpha\kappa_*^{-1/2} + c_{ac}^s)(c_{ae}\mu_* c_{bb})^{-1} \|g_p^n - g_i^n\|_{\Omega^d} \\
&\le \min\big(\mu_*\delta c_{ae}^s (c_{si,2})^{-1}(c_{pq}^2 + c_{si,4}^2)^{-1}, (c_w c_\rho)^{-1}\mu^*\kappa_*^{-1}, (c_w c_\rho)^{-1}\mu^*\alpha\kappa_*^{-1/2}\big).
\end{aligned} \tag{7.5}$$

Then, for all $n \ge 2$, there exists a unique solution $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ to eq. (7.1) such that

$$\|\!|\boldsymbol{u}_h^n|\!\|_v \le \min\big(\mu_*\delta c_{ae}^s (c_{si,2})^{-1}(c_{pq}^2 + c_{si,4}^2)^{-1}, (c_w c_\rho)^{-1}\mu^*\kappa_*^{-1}, (c_w c_\rho)^{-1}\mu^*\alpha\kappa_*^{-1/2}\big).$$

Proof. We proceed by induction on n. Since $(\boldsymbol{u}_h^1, \boldsymbol{p}_h^1) \in \boldsymbol{X}_h$ is the solution to eq. (6.1) at $n = 1$, and since $\boldsymbol{c}_h^0 = (\Pi_C c_0, \bar{\Pi}_C c_0)$, it follows from eqs. (6.7a) and (7.4) that

$$\|\!|\boldsymbol{u}_h^1|\!\|_v \le \min\big(\mu_*\delta c_{ae}^s (c_{si,2})^{-1}(c_{pq}^2 + c_{si,4}^2)^{-1}, (c_w c_\rho)^{-1}\mu^*\kappa_*^{-1}, (c_w c_\rho)^{-1}\mu^*\alpha\kappa_*^{-1/2}\big). \tag{7.6}$$

By eqs. (2.30c) and (7.6) with $r = 2$, u_h^{s1} satisfies the conditions on w of lemma 2.3.2:

$$\|u_h^{s1} \cdot n\|_{\Gamma^I} \le \mu_* c_{ae}^s \delta (c_{pq}^2 + c_{si,4}^2)^{-1}.$$

Furthermore, we have from eq. (7.6) and lemma 7.1.1 that

$$c_{ac}^1 \le 2c_e^2 \max(\kappa_*^{-1} + c_{ac}^s, \alpha\kappa_*^{-1/2} + c_{ac}^s).$$

Thus, by eqs. (5.2), (7.2a) and (7.5), there exists a unique solution $(\boldsymbol{u}_h^2, \boldsymbol{p}_h^2) \in \boldsymbol{X}_h$ to eq. (7.1) such that

$$\begin{aligned}
\|\!|\boldsymbol{u}_h^2|\!\|_v &\le (c_{ae}\mu_*)^{-1}\big(L_f^s + L_f^d\mu^*\kappa_*^{-1}\big)\phi_*^{-1/2} e^{K_1\Delta t/2}\big(\phi_* \left\|c_0\right\|_\Omega^2 + \Delta t \|g_i^1\|_{\Omega^d}^2\big)^{1/2} \\
&\quad + (c_{ae}\mu_*)^{-1}\big(\|f_0^{s2}\|_{\Omega^s} + \mu^*\kappa_*^{-1} \|f_0^{d2}\|_{\Omega^d} \big) \\
&\quad + 4c_e^2\mu^* \max(\kappa_*^{-1} + c_{ac}^s, \alpha\kappa_*^{-1/2} + c_{ac}^s)(c_{ae}\mu_* c_{bb})^{-1} \|g_p^2 - g_i^2\|_{\Omega^d} \\
&\le \min\big(\mu_*\delta c_{ae}^s (c_{si,2})^{-1}(c_{pq}^2 + c_{si,4}^2)^{-1}, (c_w c_\rho)^{-1}\mu^*\kappa_*^{-1}, (c_w c_\rho)^{-1}\mu^*\alpha\kappa_*^{-1/2}\big).
\end{aligned}$$

This proves the base case. Now, assume for some $k \geq 2$ that there exists a unique solution $(\boldsymbol{u}_h^k, \boldsymbol{p}_h^k) \in \boldsymbol{X}_h$ to eq. (7.1) such that

$$|||\boldsymbol{u}_h^k|||_v \leq \min\big(\mu_* \delta c_{ae}^s (c_{si,2})^{-1} (c_{pq}^2 + c_{si,4}^2)^{-1}, (c_w c_\rho)^{-1} \mu^* \kappa_*^{-1}, (c_w c_\rho)^{-1} \mu^* \alpha \kappa_*^{-1/2}\big).$$

By eq. (2.30c) with $r = 2$, this implies that u_h^{sk} satisfies the conditions on w of lemma 2.3.2. It also implies that $\boldsymbol{u}_h^k$ satisfies eq. (7.3), so that lemma 7.1.1 gives the bound

$$c_{ac}^k \leq 2 c_e^2 \max(\kappa_*^{-1} + c_{ac}^s, \alpha \kappa_*^{-1/2} + c_{ac}^s).$$

By eq. (7.2a), there then exists a unique solution $(\boldsymbol{u}_h^{k+1}, \boldsymbol{p}_h^{k+1}) \in \boldsymbol{X}_h$ to eq. (7.1) such that

$$\begin{aligned}
|||\boldsymbol{u}_h^{k+1}|||_v \leq{} & (c_{ae}\mu_*)^{-1} \big(L_f^s \, \|c_h^k\|_{\Omega^s} + L_f^d \mu^* \kappa_*^{-1} \, \|c_h^k\|_{\Omega^d}\big) \\
& + (c_{ae}\mu_*)^{-1} \big(\, \|f_0^{s,k+1}\|_{\Omega^s} + \mu^* \kappa_*^{-1} \, \|f_0^{d,k+1}\|_{\Omega^d}\big) \\
& + 4 c_e^2 \mu^* \max(\kappa_*^{-1} + c_{ac}^s, \alpha \kappa_*^{-1/2} + c_{ac}^s)(c_{ae}\mu_* c_{bb})^{-1} \, \|g_p^{k+1} - g_i^{k+1}\|_{\Omega^d}.
\end{aligned}$$

It follows from eqs. (5.2) and (7.5) that

$$|||\boldsymbol{u}_h^{k+1}|||_v \leq \min\big(\mu_* \delta c_{ae}^s (c_{si,2})^{-1} (c_{pq}^2 + c_{si,4}^2)^{-1}, (c_w c_\rho)^{-1} \mu^* \kappa_*^{-1}, (c_w c_\rho)^{-1} \mu^* \alpha \kappa_*^{-1/2}\big),$$

which completes the proof. □

7.2 Error analysis

As usual, our error analysis starts with the derivation of the error equation.

Lemma 7.2.1. *Let $n \geq 1$ and suppose that $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ satisfies eq. (7.1) for a given $\boldsymbol{c}_h^{n-1} \in \boldsymbol{C}_h$. Then, for all $(\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h$, there holds*

$$\begin{aligned}
& a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{e}_u^{h,n}, \boldsymbol{v}_h) - a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{e}_u^{I,n}, \boldsymbol{v}_h) + a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}^n, \boldsymbol{v}_h) - a_h^L(\boldsymbol{c}^n; \boldsymbol{u}^n, \boldsymbol{v}_h) \\
& \quad + t_h(u_h^{n-1}; \boldsymbol{e}_u^{h,n}, \boldsymbol{v}_h) - t_h(u^n; \boldsymbol{e}_u^{I,n}, \boldsymbol{v}_h) + \big[t_h(u_h^{n-1}; \boldsymbol{\Pi}_V u^n, \boldsymbol{v}_h) - t_h(u^{n-1}; \boldsymbol{\Pi}_V u^n, \boldsymbol{v}_h)\big] \\
& \quad + \big[t_h(u^{n-1}; \boldsymbol{\Pi}_V u^n, \boldsymbol{v}_h) - t_h(u^n; \boldsymbol{\Pi}_V u^n, \boldsymbol{v}_h)\big] + b_h(\boldsymbol{e}_p^{h,n}, \boldsymbol{v}_h) + b_h(\boldsymbol{q}_h, \boldsymbol{e}_u^{h,n}) \\
={} & \int_{\Omega^s} \big[f^s(c_h^{n-1}) - f^s(c^n)\big] \cdot v_h \,\mathrm{d}x \\
& \quad + \int_{\Omega^d} \big[\mu(c_h^{n-1})\kappa^{-1} f^d(c_h^{n-1}) - \mu(c^n)\kappa^{-1} f^d(c^n)\big] \cdot v_h \,\mathrm{d}x, \qquad (7.7)
\end{aligned}$$

where (u, p, c) is the solution to the quasi-stationary Navier–Stokes/Darcy-transport system and $\boldsymbol{u}^n := (u(t^n), \gamma(u(t^n)))$, $\boldsymbol{p}^n := (p(t^n), \gamma(p^s(t^n)), \gamma(p^d(t^n)))$, and $\boldsymbol{c}^n := (c(t^n), \gamma(c(t^n)))$.

Proof. Let $n \geq 1$. Let $(\boldsymbol{v}_h, \boldsymbol{q}_h) \in \boldsymbol{X}_h$. Subtracting the consistency equation for eq. (7.1a) from eq. (7.1a) gives

$$
\begin{aligned}
&a_h(\boldsymbol{c}_h^{n-1}; u_h^{n-1}; \boldsymbol{u}_h^n, \boldsymbol{v}_h) - a_h(\boldsymbol{c}^n; u^n; \boldsymbol{u}^n, \boldsymbol{v}_h) + b_h(\boldsymbol{p}_h^n - \boldsymbol{p}^n, \boldsymbol{v}_h) \\
&= \int_{\Omega^s} \left[f^s(c_h^{n-1}) - f^s(c^n) \right] \cdot v_h \, \mathrm{d}x \\
&\quad + \int_{\Omega^d} \left[\mu(c_h^{n-1}) \kappa^{-1} f^d(c_h^{n-1}) - \mu(c^n) \kappa^{-1} f^d(c^n) \right] \cdot v_h \, \mathrm{d}x.
\end{aligned}
\tag{7.8}
$$

Subtracting the consistency equation for eq. (7.1b) from eq. (7.1b) gives

$$
b_h(\boldsymbol{q}_h, \boldsymbol{u}_h^n - \boldsymbol{u}^n) = 0. \tag{7.9}
$$

Summing eqs. (7.8) and (7.9), splitting $a_h(\cdot; \cdot; \cdot, \cdot)$ into linear and nonlinear parts, and inserting eqs. (3.15), (3.16) and (3.18) yields the result. $\square$

Now, we prove bounds on the velocity and pressure errors.

Lemma 7.2.2. *Suppose that (u, p, c) is the solution to the quasi-stationary Navier–Stokes/Darcy-transport system. Suppose that $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ satisfies eq. (7.1) for $n \geq 1$, with $\boldsymbol{c}_h^{n-1} \in \boldsymbol{C}_h$ given. Then,*

$$
\begin{aligned}
\| \! | \boldsymbol{e}_u^{h,n} \| \! |_v \leq C \Bigg[&\left(1 + \|u^n\|_{1,h,\Omega^s}\right) \| \! | \boldsymbol{e}_u^{I,n} \| \! |_{v'} + \|u^n\|_{1,3,\Omega^s} \|e_u^{h,n-1}\|_{\Omega^s} + \|u^n\|_{0,\infty,\Omega^s} \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I} \\
&+ h^{k_f} \|u^{n-1}\|_{k_f+1,\Omega^s} \|\nabla u^n\|_{\Omega^s} + h^{2k_f} \|u^n\|_{k_f+1,\Omega^s} \|u^{n-1}\|_{k_f+1,\Omega^s} \\
&+ (\Delta t)^{1/2} \|\partial_t u\|_{L^2(t^{n-1},t^n; H^1(\Omega^s))} \|u^n\|_{1,\Omega^s} \\
&+ \left(1 + \|\nabla u^n\|_{0,\infty,\Omega^s}\right) \left(\sum_{K \in \mathcal{T}^s} \left(\|c^n - c_h^{n-1}\|_K^2 + h_K^2 \|c^n - c_h^{n-1}\|_{1,K}^2 \right) \right)^{1/2} \\
&+ \left(1 + \|u^n\|_{0,\infty,\Omega^d} + \|f^d(c^n)\|_{0,\infty,\Omega^d}\right) \|c^n - c_h^{n-1}\|_{\Omega^d} \Bigg].
\end{aligned}
$$

Proof. Let $n \geq 1$. Setting $(\boldsymbol{v}_h, \boldsymbol{q}_h) = (\boldsymbol{e}_u^{h,n}, -\boldsymbol{e}_p^{h,n})$ in eq. (7.7) and rearranging, we find that

$$
\begin{aligned}
&a_h(\boldsymbol{c}_h^{n-1}; u_h^{n-1}; \boldsymbol{e}_u^{h,n}, \boldsymbol{e}_u^{h,n}) \\
&= a_h(\boldsymbol{c}_h^{n-1}; u^n; \boldsymbol{e}_u^{I,n}, \boldsymbol{e}_u^{h,n}) + \left[a_h^L(\boldsymbol{c}^n; \boldsymbol{u}^n, \boldsymbol{e}_u^{h,n}) - a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}^n, \boldsymbol{e}_u^{h,n})\right] \\
&\quad + \left[t_h(u^{n-1}; \boldsymbol{\Pi}_V u^n, \boldsymbol{e}_u^{h,n}) - t_h(u_h^{n-1}; \boldsymbol{\Pi}_V u^n, \boldsymbol{e}_u^{h,n})\right] \\
&\quad + \left[t_h(u^n; \boldsymbol{\Pi}_V u^n, \boldsymbol{e}_u^{h,n}) - t_h(u^{n-1}; \boldsymbol{\Pi}_V u^n, \boldsymbol{e}_u^{h,n})\right] \\
&\quad + \int_{\Omega^s} \left[f^s(c_h^{n-1}) - f^s(c^n)\right] \cdot e_u^{h,n} \,\mathrm{d}x \\
&\quad + \int_{\Omega^d} \left[\mu(c_h^{n-1})\kappa^{-1} f^d(c_h^{n-1}) - \mu(c^n)\kappa^{-1} f^d(c^n)\right] \cdot e_u^{h,n} \,\mathrm{d}x.
\end{aligned}
$$

By theorem 7.1.2, and eq. (2.30c) with $r = 2$, u_h^{n-1} satisfies the conditions on w of lemma 2.3.2. Thanks to lemma 3.3.1, we can use lemma 2.3.2 to bound below the left-hand side. We use eqs. (3.22) to (3.26) and (3.29) to bound above the right-hand side. Dividing through by $|||\boldsymbol{e}_u^{h,n}|||_v$ and combining all constants gives the result. □

Following the same steps as corollaries 5.2.3 and 6.2.3, and making use of lemma 7.2.2, we also have the following result.

Corollary 7.2.3. *In addition to the hypotheses of corollaries 5.2.3 and 6.2.3, suppose that*

$$
\begin{aligned}
u &\in L^\infty(0, T; W^{1,3}(\Omega^s)), \\
\partial_t u &\in L^2(t^{n-1}, t^n; H^1(\Omega^s)),
\end{aligned}
$$

for $n \geq 1$. Then,

$$
\begin{aligned}
|||\boldsymbol{u}^n - \boldsymbol{u}_h^n|||_v^2 \leq C\Big(&\Delta t + h^{2k_f} + \|e_u^{h,n-1}\|_{\Omega^s}^2 + \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I}^2 + \|c^n - c_h^{n-1}\|_{\Omega^d}^2 \\
&+ \sum_{K \in \mathcal{T}^s} \left(\|c^n - c_h^{n-1}\|_K^2 + h_K^2 \|c^n - c_h^{n-1}\|_{1,K}^2 \right)\Big).
\end{aligned}
$$

Due to their dependence on $\boldsymbol{e}_u^{h,n-1}$, the error bounds of lemma 7.2.2 and corollary 7.2.3 do not immediately lead to bounds matching the forms eqs. (3.7h) and (3.7i). In the next result, however, we show that with $(\boldsymbol{u}_h^1, \boldsymbol{p}_h^1)$ chosen as the solution to eq. (6.1) at $n = 1$, the solutions to eq. (7.1) indeed satisfy error estimates of the form eq. (3.7i) for $n \geq 2$. Error estimates of the form eq. (3.7h) then follow from eq. (3.2a), the triangle inequality, and eq. (3.3).

Corollary 7.2.4. *Let $n \geq 2$, and suppose that $(\boldsymbol{u}_h^1, \boldsymbol{p}_h^1)$ is the solution to eq. (6.1) at $n = 1$. Given sufficient smoothness of u, c, and f^d, the velocity solution $\boldsymbol{u}_h^n$ to eq. (7.1) satisfies eq. (3.7i) within the context of the fully-discrete scheme.*

Proof. We proceed by strong induction on n. Since $(\boldsymbol{u}_h^1, \boldsymbol{p}_h^1)$ is the solution to eq. (6.1), then with sufficiently smooth u, c, and f^d, we have by lemma 6.2.2 and eq. (3.3) that

$$\begin{aligned}\Delta t \| \! | \boldsymbol{e}_u^{h,1} | \! \|_v^2 \leq C\Big(& h^{2k_f} + \Delta t \left\| c^1 - c_h^0 \right\|_{\Omega^d}^2 + \Delta t \left\| c^1 - \bar{c}_h^0 \right\|_{\Gamma^I}^2 \\ & + \Delta t \sum_{K \in \mathcal{T}^s} \left(\left\| c^1 - c_h^0 \right\|_K^2 + h_K^2 \left\| c^1 - c_h^0 \right\|_{1,K}^2 \right) \Big).\end{aligned}$$

As this is in the form eq. (3.7h), we are guaranteed from theorem 3.2.1 that

$$\Delta t \| \! | \boldsymbol{u}^1 - \boldsymbol{u}_h^1 | \! \|_v^2 \leq C\big((\Delta t)^2 + h^{2k_f} + h^{2k_c}\big).$$

We also recall from section 4.3 that, subject to a regularity condition, the concentration error for eq. (4.1) satisfies eqs. (3.7a) to (3.7g). Thus, by corollary 7.2.3 and eqs. (2.30e), (3.2a) and (3.3), we have that

$$\begin{aligned}\Delta t \| \! | \boldsymbol{u}^2 - \boldsymbol{u}_h^2 | \! \|_v^2 &\leq C\Big((\Delta t)^2 + h^{2k_f} + \Delta t \left\| e_u^{h,1} \right\|_{\Omega^s}^2 + \Delta t \left\| c^2 - c_h^1 \right\|_{\Omega^d}^2 + \Delta t \left\| c^2 - \bar{c}_h^1 \right\|_{\Gamma^I}^2 \\ &\quad + \Delta t \sum_{K \in \mathcal{T}^s} \left(\left\| c^2 - c_h^1 \right\|_K^2 + h_K^2 \left\| c^2 - c_h^1 \right\|_{1,K}^2 \right) \Big) \\ &\leq C\Big((\Delta t)^2 + h^{2k_f} + \Delta t \left\| u^1 - u_h^1 \right\|_{\Omega^s}^2 + \Delta t \left\| e_u^{I,1} \right\|_{\Omega^s}^2 + \Delta t \left\| c^2 - \bar{c}_h^1 \right\|_{\Gamma^I}^2 \\ &\quad + \Delta t \left\| c^2 - c_h^1 \right\|_{\Omega^d}^2 + \Delta t \sum_{K \in \mathcal{T}^s} \left(\left\| c^2 - c_h^1 \right\|_K^2 + h_K^2 \left\| c^2 - c_h^1 \right\|_{1,K}^2 \right) \Big) \\ &\leq C\Big((\Delta t)^2 + h^{2k_f} + h^{2k_c} + \Delta t \left\| c^2 - c_h^1 \right\|_{\Omega^d}^2 + \Delta t \left\| c^2 - \bar{c}_h^1 \right\|_{\Gamma^I}^2 \\ &\quad + \Delta t \sum_{K \in \mathcal{T}^s} \left(\left\| c^2 - c_h^1 \right\|_K^2 + h_K^2 \left\| c^2 - c_h^1 \right\|_{1,K}^2 \right) \Big),\end{aligned}$$

which proves the base case. Next, we assume that eq. (3.7i) holds for $n = 2, \ldots, k$, where $k \geq 2$ is arbitrary. Theorem 3.2.1 implies that

$$\Delta t \sum_{m=2}^{k} \| \! | \boldsymbol{u}^m - \boldsymbol{u}_h^m | \! \|_v^2 \leq C\big((\Delta t)^2 + h^{2k_f} + h^{2k_c}\big),$$

so that we have from corollary 7.2.3, Minkowski's and Young's inequalities, and eqs. (2.30e) and (3.3) that

$$\begin{aligned}
&\Delta t |||\boldsymbol{u}^{k+1} - \boldsymbol{u}_h^{k+1}|||_v^2 \\
&\le C\bigg((\Delta t)^2 + h^{2k_f} + \Delta t |||\boldsymbol{e}_u^{I,k}|||_{v'}^2 + \Delta t |||\boldsymbol{u}^k - \boldsymbol{u}_h^k|||_v^2 + \Delta t \left\| c^{k+1} - c_h^k \right\|_{\Omega^d}^2 \\
&\quad + \Delta t \left\| c^{k+1} - \bar{c}_h^k \right\|_{\Gamma^I}^2 + \Delta t \sum_{K \in \mathcal{T}^s} \left(\left\| c^{k+1} - c_h^k \right\|_K^2 + h_K^2 \left\| c^{k+1} - c_h^k \right\|_{1,K}^2 \right) \bigg) \\
&\le C\bigg((\Delta t)^2 + h^{2k_f} + h^{2k_c} + \Delta t \left\| c^{k+1} - c_h^k \right\|_{\Omega^d}^2 \\
&\quad + \Delta t \left\| c^{k+1} - \bar{c}_h^k \right\|_{\Gamma^I}^2 + \Delta t \sum_{K \in \mathcal{T}^s} \left(\left\| c^{k+1} - c_h^k \right\|_K^2 + h_K^2 \left\| c^{k+1} - c_h^k \right\|_{1,K}^2 \right) \bigg). \qquad (7.10)
\end{aligned}$$

Summing eq. (7.10) with eq. (3.7i) at $n = k$ shows that eq. (3.7i) holds at $n = k + 1$. This completes the proof. □

Lemma 7.2.5. *Suppose that (u, p, c) is the solution to the quasi-stationary Navier–Stokes/Darcy-transport system. Suppose that $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ satisfies eq. (7.1) for $n \ge 1$, with $\boldsymbol{c}_h^{n-1} \in \boldsymbol{C}_h$ given. Then,*

$$\begin{aligned}
|||\boldsymbol{e}_p^{h,n}|||_p \le C\big(1 + \left\| u_h^{n-1} \right\|_{1,h,\Omega^s}\big) \bigg(&\left\| u^n \right\|_{1,3,\Omega^s} \left\| e_u^{h,n-1} \right\|_{\Omega^s} + \left\| u^n \right\|_{1,\Omega^s} |||\boldsymbol{e}_u^{h,n-1}|||_{v,s} \\
&+ h^{k_f} \big(1 + \left\| u^n \right\|_{1,\Omega^s} + \left\| u^{n-1} \right\|_{k_f+1,\Omega^s} \left\| \nabla u^n \right\|_{\Omega^s}\big) + h^{2k_f} \left\| u^n \right\|_{k_f+1,\Omega^s} \left\| u^{n-1} \right\|_{k_f+1,\Omega^s} \\
&+ (\Delta t)^{1/2} \left\| \partial_t u \right\|_{L^2(t^{n-1},t^n;H^1(\Omega^s))} \left\| u^n \right\|_{1,\Omega^s} + \left\| u^n \right\|_{0,\infty,\Omega^s} \left\| c^n - \bar{c}_h^{n-1} \right\|_{\Gamma^I} \\
&+ \big(1 + \left\| \nabla u^n \right\|_{0,\infty,\Omega^s}\big) \bigg(\sum_{K \in \mathcal{T}^s} \left(\left\| c^n - c_h^{n-1} \right\|_K^2 + h_K^2 \left\| c^n - c_h^{n-1} \right\|_{1,K}^2 \right) \bigg)^{1/2} \\
&+ \big(1 + \left\| u^n \right\|_{0,\infty,\Omega^d} + \left\| f^d(c^n) \right\|_{0,\infty,\Omega^d}\big) \left\| c^n - c_h^{n-1} \right\|_{\Omega^d} \bigg).
\end{aligned}$$

Proof. Setting $\boldsymbol{q}_h = 0$ in eq. (7.7), applying eqs. (2.13), (2.14b), (3.22) to (3.24) and (3.29), and combining all constants, we find that

$$
\begin{aligned}
b_h(\boldsymbol{e}_p^{h,n}, \boldsymbol{v}_h) \le C\Bigg(& \big(1 + \|u^n\|_{1,h,\Omega^s}\big) \|\!|\boldsymbol{e}_u^{I,n}|\!\|_{v'} + \big(1 + \|u_h^{n-1}\|_{1,h,\Omega^s}\big) \|\!|\boldsymbol{e}_u^{h,n}|\!\|_v \\
& + \big(\|u^{n-1} - u_h^{n-1}\|_{1,h,\Omega^s} + \|u^n - u^{n-1}\|_{1,h,\Omega^s}\big) \|\!|\boldsymbol{\Pi}_V u^n|\!\|_{v,s} \\
& + \big(1 + \|\nabla u^n\|_{0,\infty,\Omega^s}\big) \Big(\sum_{K \in \mathcal{T}^s} \big(\|c^n - c_h^{n-1}\|_K^2 + h_K^2 \|c^n - c_h^{n-1}\|_{1,K}^2 \big) \Big)^{1/2} \\
& + \|u^n\|_{0,\infty,\Omega^s} \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I} \\
& + \big(1 + \|u^n\|_{0,\infty,\Omega^d} + \|f^d(c^n)\|_{0,\infty,\Omega^d}\big) \|c^n - c_h^{n-1}\|_{\Omega^d} \Bigg) \|\!|\boldsymbol{v}_h|\!\|_v. \qquad (7.11)
\end{aligned}
$$

Next, we note that that $\|\!|\boldsymbol{\Pi}_V u^n|\!\|_{v,s} \le C \|u^n\|_{1,\Omega^s}$ by properties of Π_V and $\bar{\Pi}_V$ [30, Eqn 28] and that $\|u^n - u^{n-1}\|_{1,h,\Omega^s} = \|\nabla(u^n - u^{n-1})\|_{\Omega^s}$ due to the smoothness of u^n and u^{n-1}. Furthermore, it was shown in [8, Eqn B.3] that for sufficiently smooth g,

$$
\|g^n - g^{n-1}\|_{\Omega^s} \le (\Delta t)^{1/2} \|\partial_t g\|_{L^2(t^{n-1},t^n;L^2(\Omega^s))}. \qquad (7.12)
$$

Using eqs. (2.9), (3.2a) and (7.12) and the triangle inequality, it follows that

$$
\begin{aligned}
& \big(\|u^{n-1} - u_h^{n-1}\|_{1,h,\Omega^s} + \|u^n - u^{n-1}\|_{1,h,\Omega^s}\big) \|\!|\boldsymbol{\Pi}_V u^n|\!\|_{v,s} \\
& \le C\big(\|\!|\boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1}|\!\|_{v,s} + \|\nabla(u^n - u^{n-1})\|_{\Omega^s}\big) \|u^n\|_{1,\Omega^s} \\
& \le C\big(\|\!|\boldsymbol{e}_u^{I,n-1}|\!\|_{v'} + \|\!|\boldsymbol{e}_u^{h,n-1}|\!\|_{v,s} + (\Delta t)^{1/2} \|\partial_t u\|_{L^2(t^{n-1},t^n;H^1(\Omega^s))}\big) \|u^n\|_{1,\Omega^s}. \qquad (7.13)
\end{aligned}
$$

Next, we insert eqs. (3.3) and (7.13) and lemma 7.2.2 into eq. (7.11), combine constants, and divide through by $\|\!|\boldsymbol{v}_h|\!\|_v$. Taking the supremum over $\boldsymbol{v}_h \in \boldsymbol{V}_h$ with $\boldsymbol{v}_h \ne 0$ and applying eq. (2.20) gives the result. □

With the same steps used to obtain eq. (5.7), we can also show that

$$
\begin{aligned}
\|\!|\boldsymbol{p}^n - \boldsymbol{p}_h^n|\!\|_p \le C\big(1 + \|u_h^{n-1}\|_{1,h,\Omega^s}\big)\Bigg(& \|u^n\|_{1,3,\Omega^s} \|e_u^{h,n-1}\|_{\Omega^s} + \|u^n\|_{1,\Omega^s} \|\!|\boldsymbol{e}_u^{h,n-1}|\!\|_{v,s} \\
& + h^{k_f}\big(1 + \|u^n\|_{1,\Omega^s} + \|u^{n-1}\|_{k_f+1,\Omega^s} \|\nabla u^n\|_{\Omega^s}\big) + \|u^n\|_{0,\infty,\Omega^s} \|c^n - \bar{c}_h^{n-1}\|_{\Gamma^I} \\
& + (\Delta t)^{1/2} \|\partial_t u\|_{L^2(t^{n-1},t^n;H^1(\Omega^s))} \|u^n\|_{1,\Omega^s} + h^{2k_f} \|u^n\|_{k_f+1,\Omega^s} \|u^{n-1}\|_{k_f+1,\Omega^s} \\
& + \big(1 + \|\nabla u^n\|_{0,\infty,\Omega^s}\big) \Big(\sum_{K \in \mathcal{T}^s} \big(\|c^n - c_h^{n-1}\|_K^2 + h_K^2 \|c^n - c_h^{n-1}\|_{1,K}^2 \big) \Big)^{1/2} \\
& + \big(1 + \|u^n\|_{0,\infty,\Omega^d} + \|f^d(c^n)\|_{0,\infty,\Omega^d}\big) \|c^n - c_h^{n-1}\|_{\Omega^d} \Bigg).
\end{aligned}
$$

To obtain convergence rates on the pressure, we can proceed as in sections 5.2 and 6.2.

Chapter 8

Numerical results

In the numerical experiments of this chapter, we choose polynomial degrees to satisfy the relation $k_c = k_f - 1$, which ensures compatibility as discussed in section 4.1. We set the penalty parameters to $\beta_f = 8k_f^2$ and $\beta_c = 8k_c^2$ [1, 31]. The implementation is done in Python using the finite element library NGSolve [35].

8.1 Example 1: rates of convergence

We measure the velocity error with the norm $\|u - u_h\|_E^2 := \|\nabla(u - u_h)\|_{\Omega^s}^2 + \|u - u_h\|_{\Omega^d}^2$ as is done in [22, 8]. Supposing that the data and the exact solution are sufficiently smooth, the results of sections 5.2 and 6.2 and chapter 7 suggest that for any of our schemes and for any $n \geq 1$, we should observe that

$$\left(\sum_{m=1}^{n} \|\!|u^m - u_h^m|\!\|_E^2\right)^{1/2} \leq C(\Delta t)^{-1/2}\big(\Delta t + h^{k_f} + h^{k_c}\big), \tag{8.1a}$$

$$\left(\sum_{m=1}^{n} \|p^m - p_h^m\|_\Omega^2\right)^{1/2} \leq C(\Delta t)^{-1/2}\big(\Delta t + h^{k_f} + h^{k_c}\big), \tag{8.1b}$$

$$\left(\sum_{m=1}^{n} \|\nabla(c^m - c_h^m)\|_\Omega^2\right)^{1/2} \leq C(\Delta t)^{-1/2}\big(\Delta t + h^{k_f} + h^{k_c}\big). \tag{8.1c}$$

We provide the results of refining in space and in time separately. When focusing on spatial refinement, we take the time interval $J = [0, 0.1]$ and use a fixed time step of

$\Delta t = 0.2 h_{\max_r}^{2k_c}$, where $h_{\max_r}$ is the spatial mesh size of the most refined spatial mesh considered. This ensures that the convergence rates are not limited by the time step. More specifically, eq. (8.1) tells us that

$$\left(\sum_{m=1}^{N} |||u^m - u_h^m|||_E^2\right)^{1/2} \leq C h_{\max_r}^{-k_c}\left(h_{\max_r}^{2k_c} + h^{k_f} + h^{k_c}\right) \leq C h^{k_c} + C h_{\max_r}^{-k_c} h^{k_c}, \tag{8.2a}$$

$$\left(\sum_{m=1}^{N} \|p^m - p_h^m\|_\Omega^2\right)^{1/2} \leq C h_{\max_r}^{-k_c}\left(h_{\max_r}^{2k_c} + h^{k_f} + h^{k_c}\right) \leq C h^{k_c} + C h_{\max_r}^{-k_c} h^{k_c}, \tag{8.2b}$$

$$\left(\sum_{m=1}^{N} \|\nabla(c^m - c_h^m)\|_\Omega^2\right)^{1/2} \leq C h_{\max_r}^{-k_c}\left(h_{\max_r}^{2k_c} + h^{k_f} + h^{k_c}\right) \leq C h^{k_c} + C h_{\max_r}^{-k_c} h^{k_c}, \tag{8.2c}$$

where N is the fixed integer such that $0.1 = N\Delta t$ is the final time step. Because $h_{\max_r}^{-k_c}$ is constant, this indicates convergence at a rate of k_c. In our displayed results, the notation $\|e\|$ is to be interpreted as $\left(\sum_{m=1}^{N} \|e^m\|^2\right)^{1/2}$ when refining spatially, for an error e and norm $\|\cdot\|$.

When focusing on temporal refinement, we take the time interval $J = [0, 1]$ and use a fixed mesh size of $h = (\Delta t)_{\max_r}$, where $(\Delta t)_{\max_r}$ is the time step of the most refined temporal mesh considered. Assuming that $k_c \geq 1$, eq. (8.1) gives the estimates

$$|||u^n - u_h^n|||_E \leq C(\Delta t)^{-1/2}\left(\Delta t + (\Delta t)^{k_f} + (\Delta t)^{k_c}\right) \leq C(\Delta t)^{1/2}, \tag{8.3a}$$

$$\|p^n - p_h^n\|_\Omega \leq C(\Delta t)^{-1/2}\left(\Delta t + (\Delta t)^{k_f} + (\Delta t)^{k_c}\right) \leq C(\Delta t)^{1/2}, \tag{8.3b}$$

$$\|\nabla(c^n - c_h^n)\|_\Omega \leq C(\Delta t)^{-1/2}\left(\Delta t + (\Delta t)^{k_f} + (\Delta t)^{k_c}\right) \leq C(\Delta t)^{1/2}, \tag{8.3c}$$

at any $n \geq 1$. In our numerical examples, we compute convergence rates based on the errors at the final time, as this does not move when the temporal mesh is refined. In our displayed results, the notation $\|e\|$ is to be interpreted as $\|e^N\|$ when refining temporally, for an error e and norm $\|\cdot\|$.

For our first example, we consider the domain $\bar{\Omega} = [0, 1] \times [-0.5, 0.5]$ with $\bar{\Omega}^s = [0, 1] \times [0, 0.5]$ and $\bar{\Omega}^d = [0, 1] \times [-0.5, 0]$, so that $\Gamma^I = [0, 1] \times \{0\}$. For each of the considered systems, we set the source terms so that the exact solution is given by

$$u^s = \begin{bmatrix} \pi x \cos(\pi x y) + 1 \\ -\pi y \cos(\pi x y) - 2x \end{bmatrix}, \qquad u^d = \begin{bmatrix} \pi x \sin(\pi y) + \cos(\pi x) + 2 \\ -\pi y \cos(\pi x y) - 2x \end{bmatrix},$$

$$p^s = \sin(3x)\cos(4y) + \sin(2\pi x y), \qquad p^d = \cos(3xy),$$

$$c = 0.25 + 0.25(\sin(2\pi(x - t))\cos(2\pi(y - t)) + 1).$$

We define our concentration-dependent viscosity as

$$\mu(\nu_0, \nu_1; c) = \nu_0 \left(\left(\frac{\nu_0}{\nu_1} \right)^{1/4} c + (1 - c) \right)^{-4},$$

which is the quarter-power mixing rule [27], with $\nu_0 = 0.9$, $\nu_1 = 1.3$ as in [33, 9, 28]. We set the constant diffusion coefficient in Ω^s to be $d = 1e^{-2}$, and define the diffusion dispersion matrix in Ω^d as

$$D(d; u) = \big(d - \sin(\pi y)(1 + |u^d|^2)\big)\mathbb{I}.$$

As well, we set $\alpha = 1$ and $\phi = 0.4$. We consider two different permeabilities, $\kappa = \mathbb{I}$ and $\kappa = 1e^3\mathbb{I}$. We take polynomial degrees $k_f = 2$ and $k_c = 1$.

In tables 8.1 and 8.2, we display results for the quasi-stationary Stokes/Darcy-transport system obtained using the HDG scheme of chapter 5. In tables 8.3 and 8.4, we display results for the quasi-stationary Navier–Stokes/Darcy-transport system obtained using the nonlinear HDG scheme of chapter 6. In tables 8.5 and 8.6, we display results for the quasi-stationary Navier–Stokes/Darcy-transport system obtained using the linear HDG scheme of chapter 7.

Our first observation is that the errors produced by the nonlinear HDG scheme of chapter 6 are indistinguishable from those produced by the linear HDG scheme of chapter 7 when refining spatially, and marginally different when refining temporally. This is logical, since the schemes differ in the choice of whether or not to time-lag the convective velocity. Next, eq. (8.2) suggests that we should expect convergence rates of 1 when refining spatially. In practice, we see from tables 8.1, 8.3 and 8.5 that many of the numerical solutions converge at rates even better than predicted, with some solutions converging at rates close to 2. When refining temporally, we similarly see from tables 8.2, 8.4 and 8.6 that the numerical solutions converge more rapidly than the rate of 0.5 predicted by eq. (8.3). Our last observation is that the values of $\|\nabla \cdot u_h + \chi^d(g_p - g_i)\|_\Omega$ in all of the tables are quite small, which supports our claim that the methods satisfy eq. (2.26a).

8.2 Example 2: realistic application

Our second example is inspired by a problem which was examined first in [10, Section 7.3] and later in [9, Section 7.3]. Here, the domain is defined as $\bar{\Omega} = [0, 1]^2$ with $\bar{\Omega}^s = [0, 1] \times [0.5, 1]$ and $\bar{\Omega}^d = [0, 1] \times [0, 0.5]$, so that $\Gamma^I = [0, 1] \times \{0.5\}$. We partition Γ^s as $\Gamma^s = \Gamma^s_1 \cup \Gamma^s_2 \cup \Gamma^s_3$, where

$$\Gamma^s_1 := \{x \in \Gamma^s : x_1 = 0\}, \quad \Gamma^s_2 := \{x \in \Gamma^s : x_1 = 1\}, \quad \Gamma^s_3 := \{x \in \Gamma^s : x_2 = 1\}.$$

DOFs	$\|u-u_h\|_E$	Rate	$\|\nabla(c-c_h)\|_\Omega$	Rate	$\|\nabla\cdot u_h+\chi^d(g_p-g_i)\|_\Omega$
$\kappa=\mathbb{I}$					
745	8.33e+00	–	2.65e+01	–	7.49e-06
2833	2.95e+00	1.50	1.75e+01	0.60	1.15e-07
11041	7.71e-01	1.94	8.79e+00	1.00	1.79e-09
43585	2.19e-01	1.81	4.40e+00	1.00	6.41e-11
$\kappa=1e^3\mathbb{I}$					
745	8.33e+00	–	2.65e+01	–	7.49e-06
2833	2.95e+00	1.50	1.75e+01	0.60	1.15e-07
11041	7.64e-01	1.95	8.79e+00	1.00	1.43e-08
43585	1.93e-01	1.98	4.40e+00	1.00	5.81e-08

(a) Energy-norm errors.

DOFs	$\|u-u_h\|_\Omega$	Rate	$\|p-p_h\|_\Omega$	Rate	$\|c-c_h\|_\Omega$	Rate
$\kappa=\mathbb{I}$						
745	4.75e-01	–	1.87e+01	–	8.94e-01	–
2833	1.28e-01	1.90	5.14e+00	1.86	3.82e-01	1.23
11041	3.00e-02	2.09	1.09e+00	2.23	9.30e-02	2.04
43585	1.43e-02	1.07	2.89e-01	1.92	2.25e-02	2.05
$\kappa=1e^3\mathbb{I}$						
745	4.73e-01	–	1.87e+01	–	8.94e-01	–
2833	1.27e-01	1.90	5.13e+00	1.86	3.82e-01	1.23
11041	2.72e-02	2.22	1.08e+00	2.25	9.30e-02	2.04
43585	6.55e-03	2.05	2.47e-01	2.13	2.25e-02	2.05

(b) L^2-norm errors.

Table 8.1: Convergence rates for the quasi-stationary Stokes/Darcy-transport system for Example 1 with spatial refinement and $k_f=2$.

Δt	$\|u-u_h\|_E$	Rate	$\|\nabla(c-c_h)\|_\Omega$	Rate	$\|\nabla\cdot u_h+\chi^d(g_p-g_i)\|_\Omega$
$\kappa=\mathbb{I}$					
0.2500	9.98e-02	–	7.66e-01	–	5.36e-12
0.1250	6.36e-02	0.65	4.80e-01	0.67	5.78e-12
0.0625	3.92e-02	0.70	3.06e-01	0.65	5.50e-12
0.0312	2.28e-02	0.78	1.82e-01	0.75	5.69e-12
$\kappa=1e^3\mathbb{I}$					
0.2500	1.04e-01	–	7.66e-01	–	5.40e-09
0.1250	6.35e-02	0.71	4.79e-01	0.68	5.80e-09
0.0625	3.86e-02	0.72	3.05e-01	0.65	5.78e-09
0.0312	2.25e-02	0.78	1.82e-01	0.75	5.68e-09

(a) Energy-norm errors.

Δt	$\|u-u_h\|_\Omega$	Rate	$\|p-p_h\|_\Omega$	Rate	$\|c-c_h\|_\Omega$	Rate
$\kappa=\mathbb{I}$						
0.2500	4.38e-02	–	2.46e-01	–	6.90e-02	–
0.1250	2.99e-02	0.55	1.28e-01	0.94	4.07e-02	0.76
0.0625	1.65e-02	0.86	6.78e-02	0.92	2.57e-02	0.66
0.0312	8.69e-03	0.93	3.60e-02	0.92	1.51e-02	0.77
$\kappa=1e^3\mathbb{I}$						
0.2500	4.42e-02	–	2.44e-01	–	6.93e-02	–
0.1250	2.99e-02	0.56	1.22e-01	1.00	4.06e-02	0.77
0.0625	1.64e-02	0.86	6.30e-02	0.95	2.57e-02	0.66
0.0312	8.65e-03	0.93	3.34e-02	0.92	1.51e-02	0.77

(b) L^2-norm errors.

Table 8.2: Convergence rates for the quasi-stationary Stokes/Darcy-transport system for Example 1 with temporal refinement and $k_f=2$.

DOFs	$\|u-u_h\|_E$	Rate	$\|\nabla(c-c_h)\|_\Omega$	Rate	$\|\nabla\cdot u_h+\chi^d(g_p-g_i)\|_\Omega$
$\kappa=\mathbb{I}$					
745	8.27e+00	–	2.65e+01	–	7.49e-06
2833	2.95e+00	1.48	1.75e+01	0.60	1.15e-07
11041	7.71e-01	1.94	8.79e+00	1.00	1.79e-09
43585	2.17e-01	1.83	4.40e+00	1.00	6.42e-11
$\kappa=1e^3\mathbb{I}$					
745	8.27e+00	–	2.65e+01	–	7.49e-06
2833	2.95e+00	1.49	1.75e+01	0.60	1.15e-07
11041	7.65e-01	1.95	8.79e+00	1.00	1.43e-08
43585	1.93e-01	1.98	4.40e+00	1.00	5.81e-08

(a) Energy-norm errors.

DOFs	$\|u-u_h\|_\Omega$	Rate	$\|p-p_h\|_\Omega$	Rate	$\|c-c_h\|_\Omega$	Rate
$\kappa=\mathbb{I}$						
745	4.66e-01	–	1.88e+01	–	8.94e-01	–
2833	1.27e-01	1.87	5.13e+00	1.88	3.82e-01	1.23
11041	2.92e-02	2.12	1.09e+00	2.23	9.30e-02	2.04
43585	1.26e-02	1.21	2.82e-01	1.95	2.25e-02	2.05
$\kappa=1e^3\mathbb{I}$						
745	4.65e-01	–	1.88e+01	–	8.94e-01	–
2833	1.27e-01	1.88	5.13e+00	1.87	3.82e-01	1.23
11041	2.71e-02	2.22	1.08e+00	2.25	9.30e-02	2.04
43585	6.54e-03	2.05	2.47e-01	2.13	2.25e-02	2.05

(b) L^2-norm errors.

Table 8.3: Convergence rates for the nonlinear scheme for the quasi-stationary Navier–Stokes/Darcy-transport system for Example 1 with spatial refinement and $k_f=2$.

Δt	$\|u-u_h\|_E$	Rate	$\|\nabla(c-c_h)\|_\Omega$	Rate	$\|\nabla\cdot u_h+\chi^d(g_p-g_i)\|_\Omega$
$\kappa=\mathbb{I}$					
0.2500	9.68e-02	–	7.66e-01	–	5.73e-12
0.1250	6.19e-02	0.65	4.80e-01	0.67	6.10e-12
0.0625	3.80e-02	0.70	3.06e-01	0.65	5.63e-12
0.0312	2.21e-02	0.79	1.82e-01	0.75	5.82e-12
$\kappa=1e^3\mathbb{I}$					
0.2500	1.00e-01	–	7.66e-01	–	5.41e-09
0.1250	6.17e-02	0.70	4.79e-01	0.68	5.76e-09
0.0625	3.75e-02	0.72	3.05e-01	0.65	5.64e-09
0.0312	2.18e-02	0.78	1.82e-01	0.75	5.65e-09

(a) Energy-norm errors.

Δt	$\|u-u_h\|_\Omega$	Rate	$\|p-p_h\|_\Omega$	Rate	$\|c-c_h\|_\Omega$	Rate
$\kappa=\mathbb{I}$						
0.2500	4.37e-02	–	2.40e-01	–	6.91e-02	–
0.1250	2.99e-02	0.55	1.25e-01	0.95	4.07e-02	0.76
0.0625	1.65e-02	0.86	6.47e-02	0.95	2.57e-02	0.66
0.0312	8.68e-03	0.93	3.39e-02	0.93	1.51e-02	0.77
$\kappa=1e^3\mathbb{I}$						
0.2500	4.40e-02	–	2.36e-01	–	6.93e-02	–
0.1250	2.98e-02	0.56	1.18e-01	1.00	4.06e-02	0.77
0.0625	1.64e-02	0.86	6.03e-02	0.97	2.56e-02	0.66
0.0312	8.65e-03	0.93	3.16e-02	0.93	1.51e-02	0.77

(b) L^2-norm errors.

Table 8.4: Convergence rates for the nonlinear scheme for the quasi-stationary Navier–Stokes/Darcy-transport system for Example 1 with temporal refinement and $k_f=2$.

DOFs	$\|u-u_h\|_E$	Rate	$\|\nabla(c-c_h)\|_\Omega$	Rate	$\|\nabla\cdot u_h+\chi^d(g_p-g_i)\|_\Omega$
$\kappa=\mathbb{I}$					
745	8.27e+00	–	2.65e+01	–	7.49e-06
2833	2.95e+00	1.48	1.75e+01	0.60	1.15e-07
11041	7.71e-01	1.94	8.79e+00	1.00	1.79e-09
43585	2.17e-01	1.83	4.40e+00	1.00	6.42e-11
$\kappa=1e^3\mathbb{I}$					
745	8.27e+00	–	2.65e+01	–	7.49e-06
2833	2.95e+00	1.49	1.75e+01	0.60	1.15e-07
11041	7.65e-01	1.95	8.79e+00	1.00	1.43e-08
43585	1.93e-01	1.98	4.40e+00	1.00	5.80e-08

(a) Energy-norm errors.

DOFs	$\|u-u_h\|_\Omega$	Rate	$\|p-p_h\|_\Omega$	Rate	$\|c-c_h\|_\Omega$	Rate
$\kappa=\mathbb{I}$						
745	4.66e-01	–	1.88e+01	–	8.94e-01	–
2833	1.27e-01	1.87	5.13e+00	1.88	3.82e-01	1.23
11041	2.92e-02	2.12	1.09e+00	2.23	9.30e-02	2.04
43585	1.26e-02	1.21	2.82e-01	1.95	2.25e-02	2.05
$\kappa=1e^3\mathbb{I}$						
745	4.65e-01	–	1.88e+01	–	8.94e-01	–
2833	1.27e-01	1.88	5.13e+00	1.87	3.82e-01	1.23
11041	2.71e-02	2.22	1.08e+00	2.25	9.30e-02	2.04
43585	6.54e-03	2.05	2.47e-01	2.13	2.25e-02	2.05

(b) L^2-norm errors.

Table 8.5: Convergence rates for the linear scheme for the quasi-stationary Navier–Stokes/Darcy-transport system for Example 1 with spatial refinement and $k_f=2$.

Δt	$\|u - u_h\|_E$	Rate	$\|\nabla(c - c_h)\|_\Omega$	Rate	$\|\nabla \cdot u_h + \chi^d(g_p - g_i)\|_\Omega$
$\kappa = \mathbb{I}$					
0.2500	9.82e-02	–	7.66e-01	–	5.75e-12
0.1250	6.23e-02	0.66	4.80e-01	0.67	5.80e-12
0.0625	3.81e-02	0.71	3.06e-01	0.65	5.60e-12
0.0312	2.21e-02	0.79	1.82e-01	0.75	5.54e-12
$\kappa = 1e^3\mathbb{I}$					
0.2500	1.02e-01	–	7.66e-01	–	5.46e-09
0.1250	6.22e-02	0.71	4.79e-01	0.68	5.80e-09
0.0625	3.76e-02	0.72	3.05e-01	0.65	5.81e-09
0.0312	2.19e-02	0.78	1.82e-01	0.75	5.48e-09

(a) Energy-norm errors.

Δt	$\|u - u_h\|_\Omega$	Rate	$\|p - p_h\|_\Omega$	Rate	$\|c - c_h\|_\Omega$	Rate
$\kappa = \mathbb{I}$						
0.2500	4.38e-02	–	2.47e-01	–	6.91e-02	–
0.1250	2.99e-02	0.55	1.28e-01	0.95	4.07e-02	0.76
0.0625	1.65e-02	0.86	6.56e-02	0.96	2.57e-02	0.66
0.0312	8.68e-03	0.93	3.41e-02	0.94	1.51e-02	0.77
$\kappa = 1e^3\mathbb{I}$						
0.2500	4.41e-02	–	2.44e-01	–	6.93e-02	–
0.1250	2.98e-02	0.56	1.21e-01	1.01	4.06e-02	0.77
0.0625	1.64e-02	0.86	6.11e-02	0.99	2.56e-02	0.66
0.0312	8.65e-03	0.93	3.19e-02	0.94	1.51e-02	0.77

(b) L^2-norm errors.

Table 8.6: Convergence rates for the linear scheme for the quasi-stationary Navier–Stokes/Darcy-transport system for Example 1 with temporal refinement and $k_f = 2$.

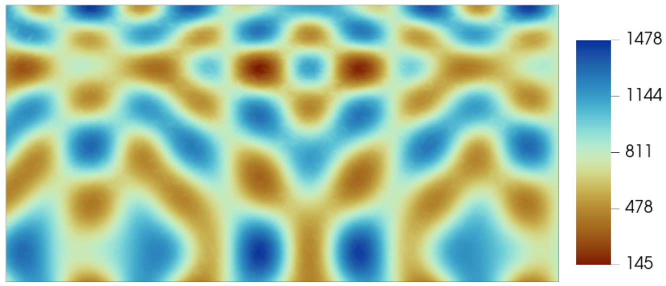


Figure 8.1: The permeability of the porous medium for Example 2.

We partition Γ^d as $\Gamma^d = \Gamma^d_1 \cup \Gamma^d_2$, where

$$\Gamma^d_1 := \{x \in \Gamma^d : x_1 = 0 \text{ or } x_1 = 1\}, \quad \Gamma^d_2 := \{x \in \Gamma^d : x_2 = 0\}.$$

We take the boundary conditions

$$\begin{aligned}
&u^s = \big(x_2(3/2 - x_2)/10, 0\big) && \text{on } \Gamma^s_1,\\
&\big(-2\mu(c^s)\varepsilon(u^s) + p^s\mathbb{I}\big)n = 0 && \text{on } \Gamma^s_2,\\
&u^s \cdot n = 0 \text{ and } \big(-2\mu(c^s)\varepsilon(u^s)n\big)\cdot\tau = 0 && \text{on } \Gamma^s_3,\\
&u^d \cdot n = 0 && \text{on } \Gamma^d_1,\\
&p^d = -0.5 && \text{on } \Gamma^d_2,
\end{aligned}$$

where τ is the unit tangential vector on Γ^s_3. We take the permeability

$$\kappa = 700\big(1 + 0.5\big(\sin(10\pi x_1)\cos(20\pi x_2^2) + \cos^2(6.4\pi x_1)\sin(9.2\pi x_2)\big)\big) + 100,$$

which is displayed in fig. 8.1. The viscosity is set according to the quarter-power mixing rule identically to in section 8.1. We set the constant diffusion coefficient in Ω^s to be $d = 1e^{-6}$, and define the diffusion dispersion matrix in Ω^d as

$$D(u) = \phi d_m \mathbb{I} + d_l|u|\mathbb{T} + d_t|u|(\mathbb{I} - \mathbb{T}),$$

where $\mathbb{T} = uu^T/|u|^2$. The longitudinal dispersivity d_l, the transverse dispersivity d_t, and the molecular diffusivity d_m are all set to $1e^{-5}$. As well, we set $\alpha = 0.5$ and $\phi = 0.4$. The source terms $f^s(c)$, $f^d(c)$, g_p, and g_i are all set to 0. The initial condition for the concentration is chosen as

$$c_0(x) = \begin{cases} 0.95 & \text{if } \sqrt{(x_1 - 0.2)^2 + (x_2 - 0.7)^2} < 0.1,\\ 0.05 & \text{otherwise.}\end{cases}$$

To take into account the different boundary conditions on Γ_2^s and Γ_3^s, we replace eqs. (5.1), (6.1) and (7.1) with modified versions, which are derived in appendix B. To define these modified schemes succinctly, we introduce a binary function Ξ and a positive integer η. We also define a parameter λ which has the value 1 on inflow faces and 0 elsewhere. The modified scheme can then be written as follows: find $(\boldsymbol{u}_h^n, \boldsymbol{p}_h^n) \in \boldsymbol{X}_h$ such that for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$
\begin{aligned}
&a_h^L(\boldsymbol{c}_h^{n-1}; \boldsymbol{u}_h^n, \boldsymbol{v}_h) + \Xi t_h(u_h^\eta; \boldsymbol{u}_h^n, \boldsymbol{v}_h) + \Xi(1-\lambda) \int_{\Gamma_2^s \cup \Gamma_3^s} (\bar{u}_h^n \cdot n^s)(\bar{u}_h^n \cdot \bar{v}_h) \,\mathrm{d}s \\
&+ b_h(\boldsymbol{p}_h^n, \boldsymbol{v}_h) - \int_{\Gamma^s} \bar{p}_h^{sn} n^s \cdot \bar{v}_h \,\mathrm{d}s + \int_{\Gamma_3^s} \bar{p}_h^{sn} n^s \cdot \bar{v}_h \,\mathrm{d}s \\
&- \int_{\Gamma_3^s} \big(2\mu(c_h^{n-1})\varepsilon(u_h^n) n^s \cdot n^s\big) n^s \cdot \bar{v}_h \,\mathrm{d}s + \int_{\Gamma_3^s} \frac{2\beta_f}{h_K} \mu(c_h^{n-1}) \big((u_h^n - \bar{u}_h^n) \cdot n^s\big) n^s \cdot \bar{v}_h \,\mathrm{d}s = 0,
\end{aligned}
$$

and for all $\boldsymbol{q}_h \in \boldsymbol{Q}_h$,

$$
b_h(\boldsymbol{q}_h, \boldsymbol{u}_h^n) - \int_{\Gamma^s} \bar{q}_h^s \bar{u}_h^n \cdot n^s \,\mathrm{d}s = 0.
$$

When $\Xi = 0$, this is the modification of eq. (5.1). When $\Xi = 1$, this is the modification of either eq. (6.1) or eq. (7.1), depending on the choice of η. Specifically, when $n \geq 2$, taking $\eta = n - 1$ gives the modification of eq. (7.1), while taking $\eta = n$ gives the modification of eq. (6.1). When $n = 1$, η must be chosen as n.

This example is included for exploratory purposes: we are primarily interested in observing qualitative features of the numerical solutions to the coupled flow and transport systems. To this end, we fix the spatial and temporal meshes and plot the numerical solutions at various points within the time interval $J = [0, 15]$. We choose $h = 1/40$ and $\Delta t = 1e^{-2}$, and take polynomial degrees $k_f = 2$ and $k_c = 1$, which keeps the runtimes manageable. Although the numerical solutions produced by our HDG method for the Stokes/Darcy-transport system are different from those produced by our HDG methods for the Navier–Stokes/Darcy-transport system, all numerical solutions produced by our schemes demonstrate the same main points. For this reason, we include plots only for the Navier–Stokes/Darcy-transport system. All plots are generated using VisIt [14].

In fig. 8.2, we compare the velocity fields for the Navier–Stokes/Darcy-transport system produced by the modifications to eq. (6.1) and eq. (7.1). We see that there is minimal, if any, difference between the numerical solutions. Furthermore, we see that the flow generally avoids areas of the porous medium with low permeability. In fig. 8.3, we plot the velocity fields produced by the linear scheme at the initial time and at the final time on the same plot. We see that there is a small but observable difference in the velocity fields. As the

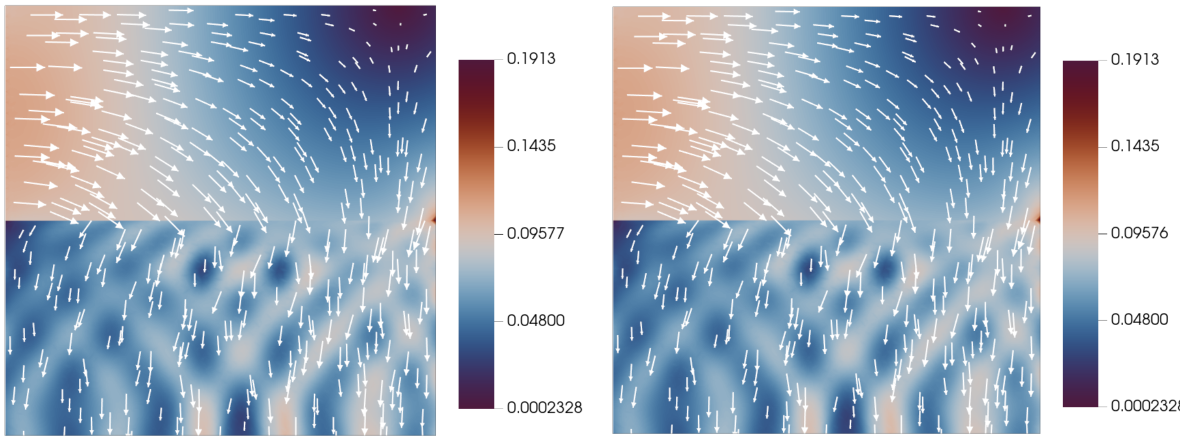


Figure 8.2: The numerical velocity fields produced by the nonlinear scheme (left) and the linear scheme (right) to the Navier–Stokes/Darcy-transport system for Example 2 at the final time. The background colour represents the magnitude of the numerical velocity solution.

source terms for this example are constant with respect to the concentration, this change in the velocity over time is completely driven by the concentration-dependent viscosity.

In fig. 8.4, we plot the numerical concentration of the contaminant produced by the linear scheme at various times in the interval. As in [9, Section 7.3], the contaminant plume stays compact while in the free flow region, and spreads out once it reaches the porous medium region. In fig. 8.5, we plot the numerical pressure solution produced by the linear scheme at various times in the interval. Comparing each plot of fig. 8.5 with the corresponding plot of fig. 8.4 at the same time clearly illustrates the influence of the concentration on the pressure in the free flow region.

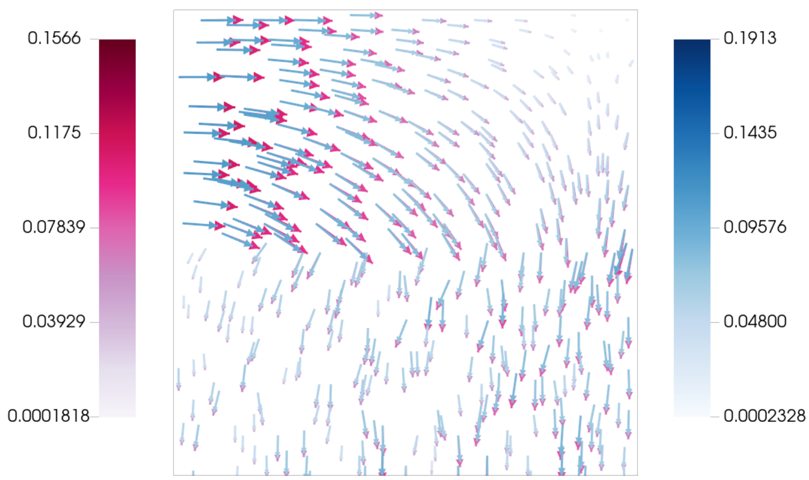
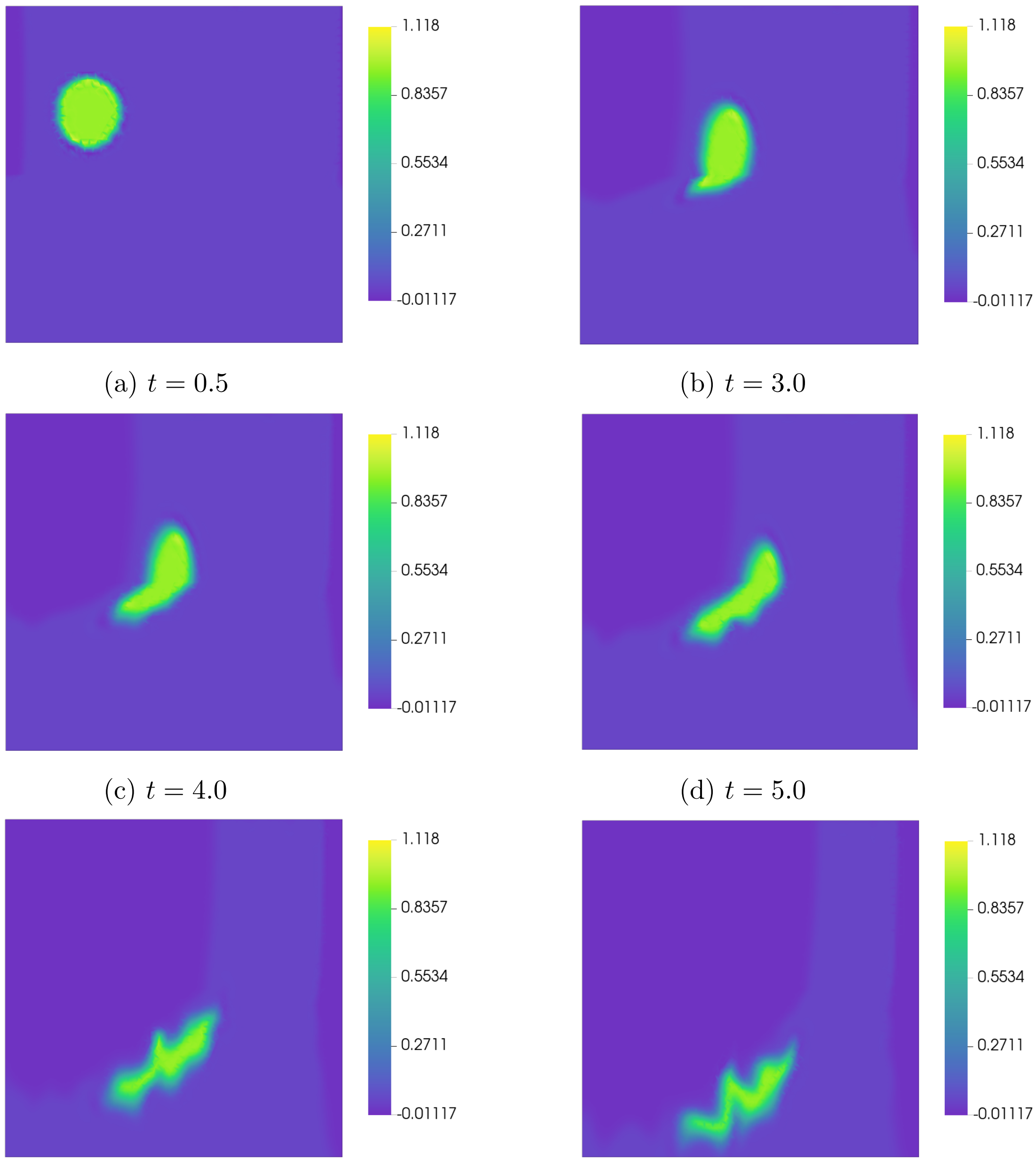


Figure 8.3: The numerical velocity fields produced by the linear scheme for the Navier–Stokes/Darcy-transport system for Example 2 at the initial time (pink) and final time (blue).



(a) $t = 0.5$

(b) $t = 3.0$

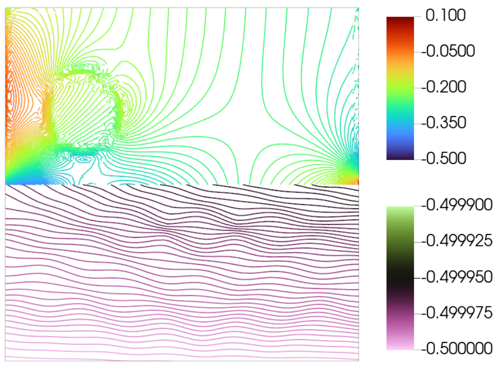
(c) $t = 4.0$

(d) $t = 5.0$

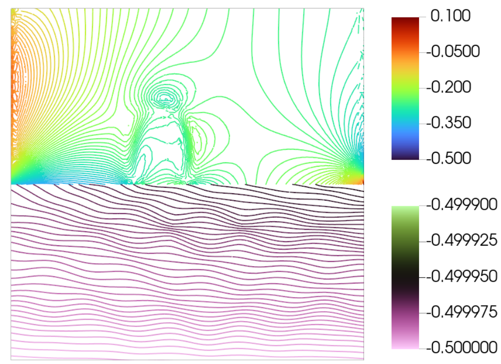
(e) $t = 8.0$

(f) $t = 10.0$

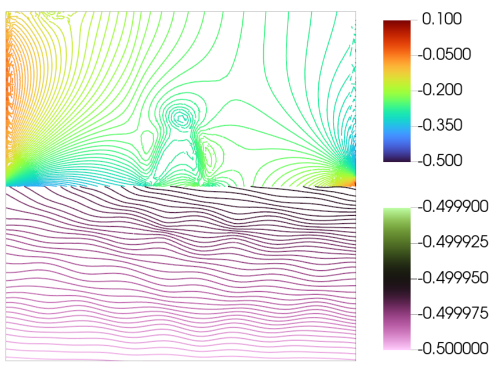
Figure 8.4: The numerical concentration solution produced by the linear scheme for the Navier–Stokes/Darcy-transport system for Example 2 at various times.



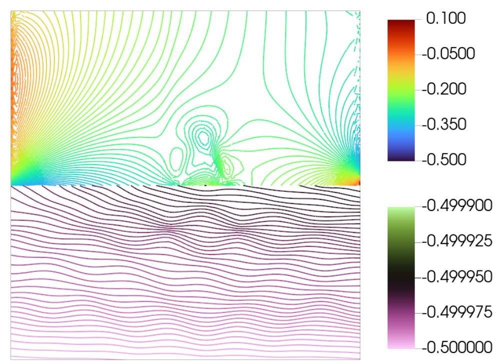
(a) $t = 0.5$



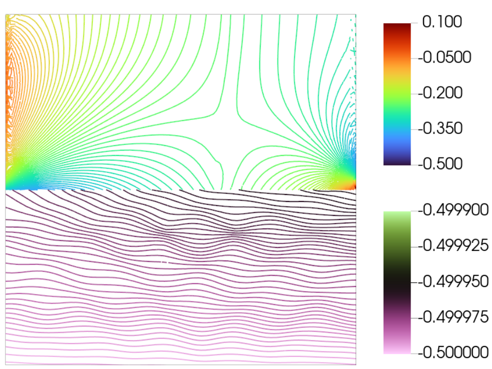
(b) $t = 3.0$



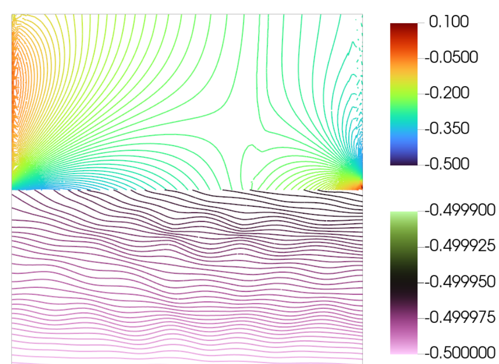
(c) $t = 4.0$



(d) $t = 5.0$



(e) $t = 8.0$



(f) $t = 10.0$

Figure 8.5: Contours for the numerical pressure solution produced by the linear scheme for the Navier–Stokes/Darcy-transport system for Example 2 at various times.

Chapter 9

Conclusions and future work

In this thesis, we studied HDG methods for coupled flow and transport systems. We began by abstracting the work of [9] to obtain a general algorithm which may be used to decouple the discrete flow and transport problems. We then provided a set of sufficient conditions for the discrete flow and transport problems to separately meet at each time step, in order to achieve optimal convergence rates when the discrete problems are combined according to the algorithm. We demonstrated practical applications of this general framework by proposing and analyzing HDG schemes for the quasi-stationary Stokes/Darcy-transport and Navier–Stokes/Darcy-transport systems. More specifically, we proved that each of our schemes admits a unique numerical solution at each time step, and that the numerical solutions converge to the true solutions as the spatial and temporal meshes are refined. We also provided numerical results to support the theoretical findings.

One recommendation for future work is to extend this analysis to the time-dependent Navier–Stokes/Darcy-transport system. The primary anticipated difficulty is in proving the well-posedness of the discrete flow problem. As we saw with the quasi-stationary Navier–Stokes/Darcy-transport system analyzed here, it is necessary for the numerical velocity solution to remain small at each time step in order to guarantee the existence of a unique numerical solution to the discrete flow problem at the next time step. The time derivative of the velocity involved in the time-dependent Navier–Stokes/Darcy-transport system presents one more avenue through which the velocity may evolve from one time step to the next. Consequently, it will be important to control this time derivative to ensure that a unique numerical solution to the discrete flow problem exists at each time step.

A potential approach to this challenge would be to modify the HDG method for the time-dependent Navier–Stokes/Darcy system of [8] to take into account the concentration-dependent viscosity. The resulting method could then be used as the discrete flow problem in a scheme with the general structure described in section 2.4.2. This would allow some of the results of [8] to be leveraged by way of chapter 3.

A more broad extension to this work would be to consider different types of coupled flow and transport systems. For example, HDG methods for the Stokes/Biot and Navier–Stokes/Biot problems were presented in [6, 7]. These coupled flow problems differ from the Stokes/Darcy and Navier–Stokes/Darcy flow problems in that the porous medium is deformable. In order to capture this, they contain additional variables which represent the solid displacement and the total pressure. Although there is little existing literature on Stokes/Biot-transport and Navier–Stokes/Biot-transport systems, they may be interesting to study from a numerical perspective. It may be possible to utilize the results of [6, 7] for this purpose by modifying the HDG methods presented therein to include the concentration. Then, the modified methods could be combined with a discrete transport equation in a scheme with a simular structure to that given in section 2.4.2.

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APPENDICES

Appendix A

Named inequalities

Some of the standard inequalities below are applications of more general results; for simplicity, we write them here only in the forms in which they are used in this thesis. Full statements can be found in the appendices of [20].

$$
\begin{aligned}
ab &\leq \frac{1}{2}a^2 + \frac{1}{2}b^2, \quad a, b > 0 && \text{(Young's inequality)},\\
ab &\leq \epsilon a^2 + \frac{1}{4\epsilon}b^2, \quad a, b > 0, \epsilon > 0 && \text{(Cauchy's inequality with } \epsilon > 0\text{)},\\
\|u + v\|_U &\leq \|u\|_U + \|v\|_U, \quad u, v \in L^2(U) && \text{(Minkowski's inequality)},\\
\left|\sum_{k=1}^{n} a_k b_k\right| &\leq \left(\sum_{k=1}^{n} a_k^2\right)^{1/2}\left(\sum_{k=1}^{n} b_k^2\right)^{1/2} && \text{(discrete Hölder's inequality)},\\
\int_U |uv| \,\mathrm{d}x &\leq \|u\|_U \|v\|_U, \quad u, v \in L^2(U) && \text{(Cauchy-Schwarz inequality)}.
\end{aligned}
$$

Appendix B

Generalized boundary conditions

In this section, we derive an HDG method for the (Navier–)Stokes/Darcy problem with the boundary conditions considered in section 8.2 and with all source terms set to 0. In the free flow region, the flow is governed by

$$
\begin{aligned}
\Xi\nabla\cdot(u^s\otimes u^s)-\nabla\cdot(2\mu(c^s)\varepsilon(u^s))+\nabla p^s&=0 && \text{in } \Omega^s\times J, && \text{(B.1a)}\\
-\nabla\cdot u^s&=0 && \text{in } \Omega^s\times J, && \text{(B.1b)}\\
u^s&=U^s && \text{on } \Gamma_1^s\times J, && \text{(B.1c)}\\
\big(-2\mu(c^s)\varepsilon(u^s)+p^s\mathbb{I}\big)n&=0 && \text{on } \Gamma_2^s\times J, && \text{(B.1d)}\\
u^s\cdot n=0 \text{ and } \big(-2\mu(c^s)\varepsilon(u^s)n\big)\cdot\tau^\ell&=0 && \text{on } \Gamma_3^s\times J,\quad 1\le\ell\le\dim-1, && \text{(B.1e)}
\end{aligned}
$$

where $\Xi=0$ for the Stokes problem and $\Xi=1$ for the Navier–Stokes problem. In the porous medium region, the flow is governed by

$$
\begin{aligned}
\mu(c^d)\kappa^{-1}u^d+\nabla p^d&=0 && \text{in } \Omega^d\times J, && \text{(B.1f)}\\
-\nabla\cdot u^d&=0 && \text{in } \Omega^d\times J, && \text{(B.1g)}\\
u^d\cdot n&=0 && \text{on } \Gamma_1^d\times J, && \text{(B.1h)}\\
p^d&=P^d && \text{on } \Gamma_2^d\times J. && \text{(B.1i)}
\end{aligned}
$$

The interface conditions are defined by eq. (2.2).

The face spaces for the velocity and the pressure in the porous medium region are modified to

$$
\begin{aligned}
\bar{V}_h &:= \{\bar{v}_h\in[L^2(\Gamma_0^s)]^{\dim} : \bar{v}_h\in[P_{k_f}(F)]^{\dim},\forall F\in\mathcal{F}^s,\bar{v}_h=0 \text{ on } \Gamma_1^s\},\\
\bar{Q}_h^d &:= \{\bar{q}_h^d\in L^2(\Gamma_0^d) : \bar{q}_h^d\in P_{k_f}(F),\forall F\in\mathcal{F}^d,\bar{q}_h^d=0 \text{ on } \Gamma_2^d\}.
\end{aligned}
$$

All other HDG spaces maintain their definitions from section 2.2.

We define the momentum flux as

$$\sigma^s := \Xi\sigma_a^s + \sigma_d^s,$$

where

$$\sigma_a^s := u^s \otimes u^s \quad \text{and} \quad \sigma_d^s := p^s\mathbb{I} - 2\mu(c^s)\varepsilon(u^s).$$

To define the numerical flux, we introduce a parameter λ which takes the value 1 on inflow faces and 0 elsewhere. Explicitly,

$$\lambda = \begin{cases} 1, & \text{if } u_h^s \cdot n < 0, \\ 0, & \text{if } u_h^s \cdot n \geq 0 \end{cases}.$$

On cell faces, the numerical flux is then defined as

$$\hat{\sigma}_h^s := \Xi\hat{\sigma}_{a,h}^s + \hat{\sigma}_{d,h}^s,$$

where

$$\hat{\sigma}_{a,h}^s := u_h^s \otimes u_h^s + (\bar{u}_h - u_h^s) \otimes \lambda u_h^s$$

and

$$\hat{\sigma}_{d,h}^s := \bar{p}_h^s\mathbb{I} - 2\mu(c_h^s)\varepsilon(u_h^s) + \frac{2\beta_f}{h_K}\mu(c_h^s)(u_h^s - \bar{u}_h) \otimes n^s,$$

where β_f is a penalty parameter.

B.1 Free flow region

B.1.1 Local momentum problem

First, using integration by parts and the definition of the strain rate tensor, it can be shown that

$$\begin{aligned} \int_K \nabla \cdot (2\mu(c^s)\varepsilon(u^s)) \cdot v_h^s \,\mathrm{d}x &= \int_{\partial K} 2\mu(c^s)\varepsilon(u^s)n \cdot v_h^s \,\mathrm{d}s - \int_K 2\mu(c^s)\varepsilon(u^s) : \nabla v_h^s \,\mathrm{d}x \\ &= \int_{\partial K} 2\mu(c^s)\varepsilon(u^s)n \cdot v_h^s \,\mathrm{d}s - \int_K 2\mu(c^s)\varepsilon(u^s) : \varepsilon(v_h^s) \,\mathrm{d}x. \end{aligned}$$

Furthermore, using the product rule to write $\partial_j(u_i^s u_j^s)v_{h,i}^s = \partial_j(u_i^s u_j^s v_{h,i}^s) - (u_i^s u_j^s)\partial_j(v_{h,i}^s)$ and applying the divergence theorem, we find that

$$\begin{aligned}\int_K \nabla\cdot(u^s\otimes u^s)\cdot v_h^s\,\mathrm{d}x &= \int_K \partial_j(u_i^s u_j^s)v_{h,i}^s\,\mathrm{d}x\\ &= \int_{\partial K}(u_i^s u_j^s v_{h,i}^s)n_j^s\,\mathrm{d}x - \int_K (u_i^s u_j^s)\partial_j(v_{h,i}^s)\,\mathrm{d}x\\ &= \int_{\partial K}(u^s\otimes u^s):(v_h^s\otimes n^s)\,\mathrm{d}x - \int_K (u^s\otimes u^s):\nabla v_h^s\,\mathrm{d}x.\end{aligned}$$

Multiplying eq. (B.1a) by $v_h^s \in V_h^s$, integrating over an element $K \in \mathcal{T}^s$, integrating by parts, and summing over all $K \in \mathcal{T}^s$ thus gives

$$\begin{aligned}&\Xi\sum_{K\in\mathcal{T}^s}\int_{\partial K}\sigma_a^s:(v_h^s\otimes n^s)\,\mathrm{d}x - \Xi\sum_{K\in\mathcal{T}^s}\int_K (u^s\otimes u^s):\nabla v_h^s\,\mathrm{d}x\\ &+\sum_{K\in\mathcal{T}^s}\int_K 2\mu(c^s)\varepsilon(u^s):\varepsilon(v_h^s)\,\mathrm{d}x + \sum_{K\in\mathcal{T}^s}\int_{\partial K}\sigma_d^s n^s\cdot v_h^s\,\mathrm{d}s\\ &-\sum_{K\in\mathcal{T}^s}\int_K p^s\nabla\cdot v_h^s\,\mathrm{d}x = 0.\end{aligned}$$

It can also be shown that

$$\hat{\sigma}_{a,h}^s n^s\cdot v_h^s = \hat{\sigma}_{a,h}^s:(v_h^s\otimes n^s) = \frac{1}{2}\big[(u_h^s\cdot n^s)(u_h^s+\bar{u}_h) + |u_h^s\cdot n^s|(u_h^s-\bar{u}_h)\big]\cdot v_h^s.$$

Replacing u^s, p^s, and c^s with the corresponding cell variables on cells, and replacing σ_a^s and σ_d^s with the corresponding numerical fluxes on faces therefore gives

$$\begin{aligned}&\Xi\sum_{K\in\mathcal{T}^s}\int_{\partial K}\frac{1}{2}\big[(u_h^s\cdot n^s)(u_h^s+\bar{u}_h)+|u_h^s\cdot n^s|(u_h^s-\bar{u}_h)\big]\cdot v_h^s\,\mathrm{d}x\\ &-\sum_{K\in\mathcal{T}^s}\int_{\partial K}2\mu(c_h^s)\varepsilon(u_h^s)n^s\cdot v_h^s\,\mathrm{d}s - \Xi\sum_{K\in\mathcal{T}^s}\int_K (u_h^s\otimes u_h^s):\nabla v_h^s\,\mathrm{d}x\\ &+\sum_{K\in\mathcal{T}^s}\int_K 2\mu(c_h^s)\varepsilon(u_h^s):\varepsilon(v_h^s)\,\mathrm{d}x + \sum_{K\in\mathcal{T}^s}\int_{\partial K}\frac{2\beta_f}{h_K}\mu(c_h^s)(u_h^s-\bar{u}_h)\cdot v_h^s\,\mathrm{d}s\\ &-\sum_{K\in\mathcal{T}^s}\int_K p_h^s\nabla\cdot v_h^s\,\mathrm{d}x + \sum_{K\in\mathcal{T}^s}\int_{\partial K}\bar{p}_h^s n^s\cdot v_h^s\,\mathrm{d}s = 0.\end{aligned}$$

B.1.2 Global momentum problem

We first note that by eqs. (2.2b) and (2.2c),

$$\begin{aligned}\int_{\Gamma^I} \sigma_d^s n^s \cdot \bar{v}_h \,\mathrm{d}s &= \int_{\Gamma^I} (n^s \cdot \sigma_d^s n^s)(\bar{v}_h \cdot n^s) \,\mathrm{d}s + \sum_{\ell=1}^{\dim -1} \int_{\Gamma^I} (\tau^\ell \cdot \sigma_d^s n^s)(\bar{v}_h \cdot \tau^\ell) \,\mathrm{d}s \\ &= \int_{\Gamma^I} p^d n^s \cdot \bar{v}_h \,\mathrm{d}s + \sum_{\ell=1}^{\dim -1} \int_{\Gamma^I} (\alpha\mu(c^s)\kappa_\ell^{-1/2} u^s \cdot \tau^\ell)(\bar{v}_h \cdot \tau^\ell) \,\mathrm{d}s,\end{aligned}$$

Asserting the continuity of the normal flux, and using eqs. (B.1d) and (B.1e) and that $\bar{v}_h = 0$ on Γ_1^s, we consequently have that

$$\begin{aligned}\sum_{K \in \mathcal{T}^s} \int_{\partial K} \sigma^s n^s \cdot \bar{v}_h \,\mathrm{d}s &= \int_{\Gamma_2^s} \sigma^s n^s \cdot \bar{v}_h \,\mathrm{d}s + \int_{\Gamma_3^s} \sigma^s n^s \cdot \bar{v}_h \,\mathrm{d}s + \int_{\Gamma^I} \sigma^s n^s \cdot \bar{v}_h \,\mathrm{d}s \\ &= \Xi \int_{\Gamma_2^s} (u^s \otimes u^s) n^s \cdot \bar{v}_h \,\mathrm{d}s + \Xi \int_{\Gamma_3^s} (u^s \otimes u^s) n^s \cdot \bar{v}_h \,\mathrm{d}s \\ &\quad + \int_{\Gamma_3^s} \left(\sigma_d^s n^s \cdot n^s\right) n^s \cdot \bar{v}_h \,\mathrm{d}s + \Xi \int_{\Gamma^I} (u^s \otimes u^s) n^s \cdot \bar{v}_h \,\mathrm{d}s \\ &\quad + \int_{\Gamma^I} p^d n^s \cdot \bar{v}_h \,\mathrm{d}s + \sum_{\ell=1}^{\dim -1} \int_{\Gamma^I} (\alpha\mu(c^s)\kappa_\ell^{-1/2} u^s \cdot \tau^\ell)(\bar{v}_h \cdot \tau^\ell) \,\mathrm{d}s.\end{aligned}$$

Replacing u, p, and c with the face variables and σ_a^s and σ_d^s with the numerical fluxes, and rearranging, we have that

$$\begin{aligned}0 &= \Xi \sum_{K \in \mathcal{T}^s} \int_{\partial K} \hat{\sigma}_{a,h}^s n^s \cdot \bar{v}_h \,\mathrm{d}s + \sum_{K \in \mathcal{T}^s} \int_{\partial K} \bar{p}_h^s n^s \cdot \bar{v}_h \,\mathrm{d}s \\ &\quad + \sum_{K \in \mathcal{T}^s} \int_{\partial K} \frac{2\beta_f}{h_K} \mu(c_h^s)(u_h^s - \bar{u}_h) \cdot \bar{v}_h \,\mathrm{d}s - \sum_{K \in \mathcal{T}^s} \int_{\partial K} 2\mu(c_h^s)\varepsilon(u_h^s) n^s \cdot \bar{v}_h \,\mathrm{d}s \\ &\quad - \Xi \int_{\Gamma_2^s} (\bar{u}_h \otimes \bar{u}_h) n^s \cdot \bar{v}_h \,\mathrm{d}s - \Xi \int_{\Gamma_3^s} (\bar{u}_h \otimes \bar{u}_h) n^s \cdot \bar{v}_h \,\mathrm{d}s \\ &\quad - \int_{\Gamma_3^s} \left(\hat{\sigma}_{d,h}^s n^s \cdot n^s\right) n^s \cdot \bar{v}_h \,\mathrm{d}s - \Xi \int_{\Gamma^I} (\bar{u}_h \otimes \bar{u}_h) n^s \cdot \bar{v}_h \,\mathrm{d}s \\ &\quad - \int_{\Gamma^I} \bar{p}_h^d n^s \cdot \bar{v}_h \,\mathrm{d}s - \sum_{\ell=1}^{\dim -1} \int_{\Gamma^I} (\alpha\mu(\bar{c}_h^s)\kappa_\ell^{-1/2} \bar{u}_h \cdot \tau^\ell)(\bar{v}_h \cdot \tau^\ell) \,\mathrm{d}s\end{aligned}$$

Rewriting $\hat{\sigma}^s_{a,h} n^s \cdot \bar{v}_h = \frac{1}{2}\big[(u^s_h \cdot n^s)(u^s_h + \bar{u}_h) + |u^s_h \cdot n^s|(u^s_h - \bar{u}_h)\big] \cdot \bar{v}_h$, and adding the integral of $(\bar{u}_h \otimes \bar{u}_h) n^s \cdot \bar{v}_h$ on the inflow portions of Γ^s_2 and Γ^s_3, then rewriting $(\bar{u}_h \otimes \bar{u}_h) n^s \cdot \bar{v}_h = (\bar{u}_h \cdot n^s)(\bar{u}_h \cdot \bar{v}_h)$, we get that

$$
\begin{aligned}
0 = {} & \Xi \sum_{K \in \mathcal{T}^s} \int_{\partial K} \frac{1}{2}\big[(u^s_h \cdot n^s)(u^s_h + \bar{u}_h) + |u^s_h \cdot n^s|(u^s_h - \bar{u}_h)\big] \cdot \bar{v}_h \,\mathrm{d}s \\
& + \sum_{K \in \mathcal{T}^s} \int_{\partial K} \bar{p}^s_h n^s \cdot \bar{v}_h \,\mathrm{d}s - \int_{\Gamma^s_3} \bar{p}^s_h n^s \cdot \bar{v}_h \,\mathrm{d}s \\
& + \sum_{K \in \mathcal{T}^s} \int_{\partial K} \frac{2\beta_f}{h_K} \mu(c^s_h)(u^s_h - \bar{u}_h) \cdot \bar{v}_h \,\mathrm{d}s - \sum_{K \in \mathcal{T}^s} \int_{\partial K} 2\mu(c^s_h)\varepsilon(u^s_h) n^s \cdot \bar{v}_h \,\mathrm{d}s \\
& + \int_{\Gamma^s_3} \big(2\mu(c^s_h)\varepsilon(u^s_h) n^s \cdot n^s\big) n^s \cdot \bar{v}_h \,\mathrm{d}s - \int_{\Gamma^s_3} \frac{2\beta_f}{h_K} \mu(c^s_h)\big((u^s_h - \bar{u}_h) \cdot n^s\big) n^s \cdot \bar{v}_h \,\mathrm{d}s \\
& - \Xi(1-\lambda) \int_{\Gamma^s_2 \cup \Gamma^s_3} (\bar{u}_h \cdot n^s)(\bar{u}_h \cdot \bar{v}_h) \,\mathrm{d}s - \Xi \int_{\Gamma^I} (\bar{u}_h \cdot n^s)(\bar{u}_h \cdot \bar{v}_h) \,\mathrm{d}s \\
& - \int_{\Gamma^I} \bar{p}^d_h n^s \cdot \bar{v}_h \,\mathrm{d}s - \sum_{\ell=1}^{\dim -1} \int_{\Gamma^I} (\alpha \mu(\bar{c}^s_h) \kappa_\ell^{-1/2} \bar{u}_h \cdot \tau^\ell)(\bar{v}_h \cdot \tau^\ell) \,\mathrm{d}s.
\end{aligned}
$$

B.1.3 Local mass problem

Multiplying eq. (B.1b) by $q^s_h \in Q^s_h$, integrating over an element $K \in \mathcal{T}^s$, summing over all $K \in \mathcal{T}^s$, and replacing u^s by u^s_h gives

$$
-\sum_{K \in \mathcal{T}^s} \int_K q^s_h \nabla \cdot u^s_h \,\mathrm{d}x = 0.
$$

B.1.4 Global mass problem

To enforce the normal continuity of u^s_h over cell faces, we impose

$$
\begin{aligned}
\sum_{K \in \mathcal{T}^s} \int_{\partial K} \bar{q}^s_h u^s_h \cdot n^s \,\mathrm{d}s &= \sum_{K \in \mathcal{T}^s} \int_{\partial K} \bar{q}^s_h \bar{u}^s_h \cdot n^s \,\mathrm{d}s \\
&= \int_{\Gamma^s} \bar{q}^s_h \bar{u}^s_h \cdot n^s \,\mathrm{d}s + \int_{\Gamma^I} \bar{q}^s_h \bar{u}^s_h \cdot n^s \,\mathrm{d}s.
\end{aligned}
$$

B.2 Porous medium region

B.2.1 Local momentum problem

Multiplying eq. (B.1f) by a test function $v_h^d \in V_h^d$, integrating over an element $K \in \mathcal{T}^d$, integrating by parts, summing over all $K \in \mathcal{T}^d$, and replacing u, p, and c with the corresponding cell variables on cells and the face variables on faces gives

$$\sum_{K \in \mathcal{T}^d} \int_K \mu(c_h^d)\kappa^{-1} u_h^d \cdot v_h^d \, \mathrm{d}x - \sum_{K \in \mathcal{T}^d} \int_K p_h^d \nabla \cdot v_h^d \, \mathrm{d}x + \sum_{K \in \mathcal{T}^d} \int_{\partial K} \bar{p}_h^d v_h^d \cdot n^d \, \mathrm{d}s = 0.$$

B.2.2 Local mass problem

Multiplying eq. (B.1g) by a test function $q_h^d \in Q_h^d$, integrating over an element $K \in \mathcal{T}^d$, summing over all $K \in \mathcal{T}^d$, and replacing u^d with u_h^d gives

$$-\sum_{K \in \mathcal{T}^d} \int_K q_h^d \nabla \cdot u_h^d \, \mathrm{d}x = 0.$$

B.2.3 Global mass problem

To enforce the normal continuity of u_h^s over cell faces, we impose

$$\sum_{K \in \mathcal{T}^d} \int_{\partial K} \bar{q}_h^d u_h^d \cdot n^d \, \mathrm{d}s = \sum_{K \in \mathcal{T}^d} \int_{\partial K} \bar{q}_h^d \bar{u}_h^d \cdot n^d \, \mathrm{d}s.$$

Using that $\bar{q}_h^d = 0$ on Γ_2^d, and imposing eqs. (2.2a) and (B.1h) on $\bar{u}_h$ and u_h^d, we obtain

$$\sum_{K \in \mathcal{T}^d} \int_{\partial K} \bar{q}_h^d u_h^d \cdot n^d \, \mathrm{d}s = \int_{\Gamma^I} \bar{q}_h^d \bar{u}_h \cdot n^d \, \mathrm{d}s.$$

B.3 Combined regions

Subtracting the global momentum problem in the free flow region from the local momentum problem in the free flow region and adding a consistent symmetrizing term gives

$$
\begin{aligned}
& a_h^s(c_h^s; \boldsymbol{u}_h^s, \boldsymbol{v}_h^s) + \Xi t_h(u_h^s; \boldsymbol{u}_h^s, \boldsymbol{v}_h^s) + a_h^I(\bar{c}_h^s; \bar{u}_h, \bar{v}_h) \\
& + b_h^s(\boldsymbol{p}_h^s, v_h^s) + b_h^{I,s}(\bar{p}_h^s, \bar{v}_h) - \int_{\Gamma^s} \bar{p}_h^s n^s \cdot \bar{v}_h \,\mathrm{d}s + b_h^{I,d}(\bar{p}_h^d, \bar{v}_h) \\
& + \int_{\Gamma_3^s} \bar{p}_h^s n^s \cdot \bar{v}_h \,\mathrm{d}s + \Xi(1-\lambda) \int_{\Gamma_2^s \cup \Gamma_3^s} (\bar{u}_h \cdot n^s)(\bar{u}_h \cdot \bar{v}_h) \,\mathrm{d}s \\
& - \int_{\Gamma_3^s} \big(2\mu(c_h^s)\varepsilon(u_h^s) n^s \cdot n^s\big) n^s \cdot \bar{v}_h \,\mathrm{d}s + \int_{\Gamma_3^s} \frac{2\beta_f}{h_K} \mu(c_h^s)\big((u_h^s - \bar{u}_h) \cdot n^s\big) n^s \cdot \bar{v}_h \,\mathrm{d}s = 0.
\end{aligned}
$$

The local momentum problem in the porous medium region can be written as

$$
a_h^d(c_h^d; u_h^d, v_h^d) + b_h^d(\boldsymbol{p}_h^d, v_h^d) = 0.
$$

Summing the local and global mass equations in both regions gives

$$
b_h(\boldsymbol{q}_h, \boldsymbol{u}_h) - \int_{\Gamma^s} \bar{q}_h^s \bar{u}_h \cdot n^s \,\mathrm{d}s = 0.
$$

Combining all problems in both regions leads to the final method: find $(\boldsymbol{u}_h, \boldsymbol{p}_h) \in \boldsymbol{X}_h$ such that for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$,

$$
\begin{aligned}
& a_h^L(\boldsymbol{c}_h; \boldsymbol{u}_h, \boldsymbol{v}_h) + \Xi t_h(u_h^s; \boldsymbol{u}_h^s, \boldsymbol{v}_h^s) + \Xi(1-\lambda) \int_{\Gamma_2^s \cup \Gamma_3^s} (\bar{u}_h \cdot n^s)(\bar{u}_h \cdot \bar{v}_h) \,\mathrm{d}s \\
& + b_h(\boldsymbol{p}_h, \boldsymbol{v}_h) - \int_{\Gamma^s} \bar{p}_h^s n^s \cdot \bar{v}_h \,\mathrm{d}s + \int_{\Gamma_3^s} \bar{p}_h^s n^s \cdot \bar{v}_h \,\mathrm{d}s \\
& - \int_{\Gamma_3^s} \big(2\mu(c_h^s)\varepsilon(u_h^s) n^s \cdot n^s\big) n^s \cdot \bar{v}_h \,\mathrm{d}s + \int_{\Gamma_3^s} \frac{2\beta_f}{h_K} \mu(c_h^s)\big((u_h^s - \bar{u}_h) \cdot n^s\big) n^s \cdot \bar{v}_h \,\mathrm{d}s = 0,
\end{aligned}
$$

and for all $\boldsymbol{q}_h \in \boldsymbol{Q}_h$,

$$
b_h(\boldsymbol{q}_h, \boldsymbol{u}_h) - \int_{\Gamma^s} \bar{q}_h^s \bar{u}_h \cdot n^s \,\mathrm{d}s = 0.
$$