

Quantum Error Correction and Quantum Metrology with Non-Markovian Noise

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

A portion of the contents of Chapter 2 and Chapter 3 were originally presented in the arXiv pre-print *Quantum Maps Between CPTP and HPTP* [1], of which Dr Ningping Cao is first author. This work was co-authored by Dr Maxwell Fitzsimmons, Zachary Mann, Dr Rajesh Pereira, and Dr Raymond Laflamme. Zachary Mann’s contributions to the research consist of the dephasing example presented in Section 2.5 of this thesis. All results from the original work presented in this thesis are indicated and accompanied by citation.

A portion of the content of Chapters 3 and 4 is modified from (and overlaps with) the arXiv pre-print *Quantum Error-Corrected Non-Markovian Metrology* [2]. Zachary Mann is the first author of the work. The work was co-authored by Dr Ningping Cao, Dr Raymond Laflamme, and Dr Sisi Zhou. The portions of this thesis present in the original work were contributions of Zachary Mann, derived with the assistance and following the suggestions of Dr. Cao, Dr. Laflamme, and Dr. Zhou.

Abstract

Quantum technologies have the potential to solve many important problems across science and industry. An important example is quantum computation. Quantum simulators promise to better model chemistry. Further, Shor’s factoring algorithm solves a problem in exponentially less time than what it would take now on our classical computers. This has led many to believe that quantum computers could bring exponential speedups to other difficult, real-world problems, such as optimization. Another example is quantum sensing, where quantum mechanical effects can be leveraged to increase measurement precision beyond the classical state of the art. This has many applications in both fundamental science, such as the LIGO experiment, and in industry, such as Nitrogen vacancy magnetometers. For these quantum technologies to reach their full potential, however, the barrier of noise must be overcome. Quantum effects usually live at very small system sizes or very cold temperatures, making them extra sensitive to thermal noise or small perturbations of the environment. A proposed solution to this problem, for both computation and sensing, is to use quantum error correction. Quantum error correction encodes a few quantum degrees of freedom into many physical degrees of freedom, building in redundancy. This redundancy allows for the detection and correction of unwanted errors in our protocol. Most of the literature on quantum error correction focuses on *Markovian* noise models, i.e., models where the noise is not temporally correlated. The temporally correlated, or *non-Markovian*, regime remains relatively unexplored. In this thesis, we explore quantum error correction for non-Markovian noise models. We first present a few of the many definitions and models for quantum non-Markovian phenomena present in the literature. We then generalize the Knill-Laflamme quantum error conditions to the *hidden Markov model*, an experimentally motivated model of non-Markovian noise. These conditions allow one to guarantee that a quantum error-correcting code will still do its job for more realistic noise models. Finally, we apply our notion of non-Markovian error correction to quantum sensing. We generalize previous Markovian results and derive conditions for guaranteeing Heisenberg limited precision scaling in the presence of temporally correlated noise using quantum error correction. The Heisenberg limit is the fundamental precision limit allowed by quantum mechanics for parameter estimation in a physical system. We also study the next-best achievable precision scaling when the Heisenberg limit is unattainable.

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Dedication

In loving memory of Raymond Laflamme.

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List of abbreviations

QFI	Quantum Fisher information
HL	Heisenberg limit
SQL	Standard quantum limit
QEC	Quantum error correction
HMM	Hidden Markov model
KL	Knill-Laflamme
CPTP	Completely positive trace-preserving
HNLS	Hamiltonian not in Lindblad span
HNES	Hamiltonian not in extended span
HNELS	Hamiltonian not in extended Lindblad span
FI	Fisher information, or classical Fisher information

Chapter 1

Introduction

First proposed by Yuri Manin in the East in 1980, and by Richard Feynman in the West in 1981 [3], a quantum computer leverages the quantum properties that appear in microscopic physical systems in order to perform complex computational tasks. When they were first proposed, the hope was to use the inherent complexity necessary for representing quantum systems as a computational resource. The first practical realization of these hopes came in 1994, when Peter Shor discovered an algorithm for factoring prime numbers in polynomial time [4], a problem which we only know how to solve in exponential time on a classical computer. This result has drawn much attention to the field, as an increasing number of researchers now work to find the next useful exponential speedup.

Although we patiently await the next great theoretical breakthrough, the experimental side of things mustn't be neglected. Indeed, quantum effects are usually observed at very small scales or very cold temperatures, making them very sensitive to noise that can arise from thermal or macroscopic perturbations in the environment. This often leads to the decoherence of a quantum system, i.e., when the quantum information is destroyed. When a system decoheres, it is no longer useful for quantum computation. To deal with the problem of decoherence, quantum error correction was proposed, with the first code correcting arbitrary single-qubit noise appearing in 1995 [5]. Quantum error correction works by encoding a few *logical* degrees of freedom into a large number of physical degrees of freedom [6]. Quantum error correction has been further demonstrated in the laboratory for small devices, including experiments where the error rates approach those necessary for implementing useful computations [7].

Although quantum error correction has historically been focused on quantum computation, it was proposed in 2014 that quantum error correction could also be used to increase

the precision in quantum sensing and quantum metrology [8].

In quantum metrology, many of the same quantum effects as quantum computation are leveraged to design sensing protocols that estimate physical parameters with lower variance than the best classical methods. There are many important applications of quantum sensing and metrology, such as gravitational wave detection [9], interferometry [10], and measurement of time with atomic clocks [11]. Much like quantum computation, metrological protocols greatly suffer from the presence of noise, and quantum error correction has the potential to remedy that [12, 13, 14].

In both quantum error correction and quantum metrology, quantum error correction has been studied under specific noise assumptions. In particular, a common assumption is that the noise is *Markovian*, which means that it displays no correlations in time. In this work, that assumption is dropped, and quantum error correction is studied for temporally correlated, or *non-Markovian*, noise. This thesis is organized as follows :

- In **Chapter 2**, different definitions and models of quantum non-Markovianity are studied. Parallels are drawn between the notions of quantum and classical non-Markovianity. The different definitions of non-Markovianity present in the literature are explained and compared, and different models for non-Markovian processes are explored. In particular, we generalize quantum error correction in a hidden Markov model, before applying it to quantum metrology. This section sets the table for the main results of this thesis, which can also be found in the paper *Quantum Error Corrected non-Markovian Metrology* [2], written by the author of this thesis in collaboration with Ningping Cao, Raymond Laflamme, and Sisi Zhou.
- In **Chapter 3**, quantum error correction is studied for noise modeled by a hidden Markov model. The main result of this chapter, Theorem 12, generalizes the quantum error correction conditions of [6] to the non-Markovian regime. Quantum error correction is also explored for non-CP maps, an alternative mathematical representation for quantum non-Markovian processes.
- In **Chapter 4**, quantum error correction for quantum metrology is explored. In particular, the HNLS conditions of [13, 14] are generalized to the hidden Markov model in Theorems 14, 20. These conditions, when satisfied, guarantee that quantum error correction can be used in a noisy system to achieve the optimal sensing precision permitted by quantum mechanics. Contrary to the Markovian conditions, which are necessary and sufficient, the derived non-Markovian conditions are only sufficient, leading to many interesting cases that are studied separately. Further, protocols are studied for obtaining the next-best measurement precision when error correction is

unable to rid us of decoherence. In the case where optimal scaling is impossible, we provide a conjecture on the next-best scaling, and support the conjecture with specific analytically solvable models and a large number of numerically computed examples.

These three chapters aim to paint a picture of non-Markovian quantum error correction and its applications in sensing. As previously mentioned, non-Markovian models remain relatively unexplored compared to their Markovian counterparts, despite many modern devices exhibiting non-Markovian noise effects. It is therefore important that protocols for these noise models be studied.

1.1 Notation

We use $\mathbb{C}$ to denote the set of complex numbers. We use $\mathbb{C}^k$ to denote a k -dimensional vector space over $\mathbb{C}$, with k finite. Let $i = \sqrt{-1}$, while the italic i will often be used as a label or index. We will use $\mathcal{H}_i$ to denote Hilbert space, where the index i will label the system. We will often use $\mathbb{C}^k$ and $\mathcal{H}_i$ interchangeably for finite-dimensional systems, where it is to be understood that we equip $\mathbb{C}^k$ with the standard Euclidean inner product and distance. We will use $\mathcal{B}(\mathbb{C}^d)$ to denote the set of square linear operators on the vector space $\mathbb{C}^d$.

We will use $\mathbb{1}$ to denote the identity operator, which for finite-sized systems is simply the identity matrix. The dimension of the $\mathbb{1}$ operator will always be clear from context, although we will sometimes use subscripts to specify the space it acts on. We will use X, Y, Z to denote the Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (1.1)$$

We will label the Bell states as follows

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle), \quad (1.2)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle), \quad (1.3)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle), \quad (1.4)$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle). \quad (1.5)$$

We will use $\mathcal{I}$ to denote either the identity channel, $\mathcal{I}(\rho) = \rho \forall \rho$, or the classical Fisher information depending on the context.

1.2 Quantum Metrology Preliminaries

In this section, we provide a short primer on quantum metrology.

1.2.1 Cramér-Rao Bounds for Parameter Estimation

Metrology is the study of precision measurement. The aim of the game is to construct measurement protocols that will estimate quantities of interest with the lowest possible variance. A fundamental result in classical metrology is the Cramér-Rao bound. Suppose ω is a parameter of the physical system, which we wish to estimate. The Cramér-Rao bound states that the variance of any unbiased estimator $\hat{\omega}$ of ω will be bounded from below by one over M times the Fisher information.

$$(\Delta\hat{\omega})^2 \geq \frac{1}{M\mathcal{I}(\omega)}. \quad (1.6)$$

Here, M denotes the number of independent, identically distributed (I.I.D.) samples we take and $\mathcal{I}(\omega)$ is the *Fisher information* (FI). The Fisher information is a property of the probability distribution $X(\omega)$ from which we are sampling to infer ω . It can be computed via the following equation.

$$\mathcal{I}(\omega) = \sum_{b_1, \dots, b_N} \left(\frac{\partial_\omega P(b_1, \dots, b_N)}{P(b_1, \dots, b_N)} \right)^2 P(b_1, \dots, b_N) = \mathbb{E} \left[\left(\frac{\partial_\omega P(b_1, \dots, b_N)}{P(b_1, \dots, b_N)} \right)^2 \right]. \quad (1.7)$$

With the rapid development of quantum theory, researchers realized that quantum effects such as superposition and entanglement could be leveraged to further increase the precision scaling, leading to the field of quantum metrology. In quantum metrology, the goal is to use quantum mechanical sensors to surpass classical scaling limits. There are many applications of quantum metrology and quantum sensing. For example, in gravitational wave detection [9], squeezed light is used to suppress certain noise sources while amplifying others. In quantum optics and interferometry [10], entangled states are used to yield a quadratic increase in precision versus standard classical techniques. Finally, quantum

metrology has been very important in the measurement of time, where researchers have been using atoms and their quantum behavior to build ever more precise clocks [11].

An important result in quantum metrology is the quantum Cramér-Rao bound [15], a generalization of the classical bound. The variance of any unbiased estimator $\hat{\omega}$ is once again bounded by below, however now the Fisher information of Equation 1.6 is replaced by the *quantum Fisher information* (QFI) $\mathcal{F}(\omega)$, which is a function of the quantum state which we will measure to infer ω .

$$(\Delta\hat{\omega})^2 \geq \frac{1}{M\mathcal{F}(\omega)}. \quad (1.8)$$

The Cramér-Rao bounds stated above are for the single parameter case, but can be generalized to multi-parameter estimation using the quantum Fisher information matrix [16]. For the rest of this section and this thesis, we will concern ourselves only with the case of single parameter estimation, in which ω is simply a scalar. To be more precise, ω will be the amplitude of a specific signal term of the Hamiltonian governing our system.

Recall that the classical Fisher information was computed for a probability distribution depending on ω and that the quantum Fisher information was computed for a quantum state, or density matrix, which depends on ω . If we fix a set of POVM elements and measure the state in this basis, we will obtain a probability distribution depending on ω and therefore a corresponding classical Fisher information. We define the quantum Fisher information as the maximization over all POVM measurements of the classical Fisher information of the associated probability distribution. In particular, we have that the Fisher information of any set of measurements is bounded from above by the quantum Fisher information

$$\mathcal{I}(\omega) \leq \mathcal{F}(\omega). \quad (1.9)$$

In practice, this optimization can be rather difficult. There exist many clever formulas for calculating the quantum Fisher information of a state ρ_ω directly. First, we define the symmetric logarithmic derivative as the self-adjoint operator L_ω satisfying the differential equation

$$\frac{L_\omega \rho_\omega + \rho_\omega L_\omega}{2} = \frac{\partial \rho_\omega}{\partial \omega}. \quad (1.10)$$

The quantum Fisher information can then be calculated using the following formula [17]

$$\mathcal{F}(\omega) = \text{Tr}(\rho_\omega L_\omega^2). \quad (1.11)$$

Another formula for computing the quantum Fisher information, which we will often employ in this thesis, is the following [17, 16]

$$\mathcal{F}(\rho) = 2 \sum_{i,j} \frac{|\langle \gamma_i | \partial_\omega \rho | \gamma_j \rangle|^2}{\gamma_i + \gamma_j}, \quad (1.12)$$

where γ_i and $|\gamma_i\rangle$ are the eigenvalue and eigenvector pairs of ρ , and $\partial_\omega \rho$ is the derivative of ρ with respect to the parameter ω . The sum is taken over i, j pairs such that $\gamma_i + \gamma_j > 0$.

The Cramér-Rao bound is saturated in the limit of many samples ($M \gg 1$) by maximum-likelihood estimation, and therefore an optimal estimator can in principle be constructed. This means that if we are interested in the absolute attainable measurement precision for estimating ω from a state ρ , we need only occupy ourselves with maximizing $\mathcal{F}(\omega)$. This will be our goal in Chapter 4.

1.2.2 Parallel and Sequential Strategies for Quantum Sensing

In this work, we will consider the task of sensing a Hamiltonian amplitude. We'll suppose our probes evolve under the Hamiltonian ωG , where ω is the parameter we wish to estimate. The probe will also be subject to some noise. The overall evolution of the probe under signal and noise for a duration t is given by the map $\Lambda_{\omega,t}$. There are many different paradigms for this problem, but they can generally be separated into two main categories, sequential and parallel [18, 13, 14, 19].

In a parallel strategy, N probes, as well as some auxiliary degrees of freedom, are prepared in an initial state. Usually, this state is taken to be entangled to obtain better scaling of the quantum Fisher information. Afterward, each probe then evolves for a time t under the dynamics $\Lambda_{\omega,t}$, yielding an output state ρ_ω which can then be measured to infer ω . This is the most standard setting for quantum metrology. Examples of parallel strategies include the GHZ strategy [20], in which N probes, for example spins, are entangled in a GHZ state to enhance phase sensitivity. Another example of this strategy is the 'N00N' state for interferometry [10]. The N00N state is a superposition of the two configurations where N photons are sent into one input of the interferometer, and 0 photons are sent into the other. In general, parallel strategies involve entangling many copies of the probe to increase the precision of the final measurement. Both the GHZ and N00N state strategies allow for a quadratic increase in precision over strategies where no entanglement is used. In this thesis, we will not discuss parallel strategies.

The second class of strategies is called sequential strategies. In a sequential strategy, we have access to a single probe and as many auxiliary degrees of freedom as we like. Once

again, we let this probe evolve for a time Δt . We then apply some control operations on the probe and auxiliary system, and let it time evolve again. We repeat the process until we obtain our desired output state. Contrary to parallel strategies, where we gain precision by scaling the number of entangled probes, in a sequential strategy, we instead increase precision by increasing the total exposure time to the signal before our final measurement, or by modifying the measurement scheme adaptively to collect information more efficiently. In this work, we will suppose that we can act on the probe and auxiliary system with a full set of quantum controls. Further, we will suppose that these control operations can be performed arbitrarily fast. In other words, the system may time evolve under $\Lambda_{\omega, \Delta t}$ for an infinitesimal time Δt before we act with the next round of control operations, which we suppose act instantaneously. It has been shown that for these controls, sequential strategies are more powerful than parallel strategies for quantum sensing [18, 19], as all parallel strategies can be simulated by a sequential one. This, however, comes at the cost of trading spatial resources for an increase in run-time. The sequential and parallel strategies are presented in Figure 1.1. Experimentally, sequential strategies have already been demonstrated in nitrogen-vacancy centers in diamond using quantum error correction [12], and in optical systems using adaptive measurement schemes [21, 22]. For quantum error correction, the protocol increases the coherence time of the probes, allowing for a longer exposure time to the signal. This allows for more phase to accumulate in the quantum state, leading to a final measurement containing more information. In the adaptive measurement strategy, measurements are repeated, where the next measurements are conditioned on the previous outcomes. In this work, we will be interested in sequential strategies. In particular, we will study the use of quantum error correction to increase the coherence time of the system, and an increase in sensing precision will arise from an increase in the total probing time.

1.2.3 Scaling Limits

Finally, we will be interested in two important metrological limits, the *standard quantum limit* (SQL) and the *Heisenberg limit* (HL). For a large class of quantum sensing scenarios, including interferometry and quantum phase estimation, the Heisenberg limit is the fundamental measurement precision allowed by the rules of quantum mechanics [19]. The bound is usually stated for parallel strategies. If N probes are used, the optimal functional scaling of the QFI is N^2 . This is the most common definition of the Heisenberg limit present in the literature. It is saturated using entangled probes, such as in the GHZ or N00N strategies. This is a quadratic increase in precision over strategies where N independent probes are used, in which case the QFI will scale linearly in N .

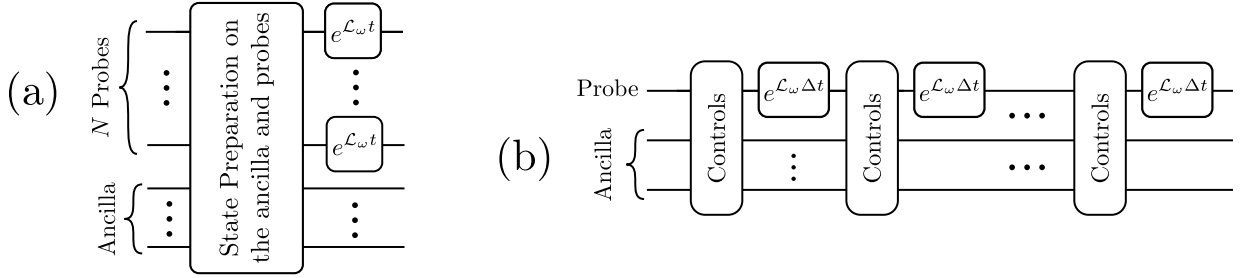


Figure 1.1: (a) Parallel strategy for quantum metrology. (b) Sequential strategy for quantum metrology. We allow for complete, fast, and accurate quantum controls and for Δt to be arbitrarily small. Here, we take the evolution to be generated by $\mathcal{L}_\omega$, i.e. $\Lambda_{\omega,t} = e^{\mathcal{L}_\omega t}$. Any strategy in the form (a) can be simulated by a strategy of the form (b).

As previously mentioned, sequential strategies with fast and accurate control can simulate parallel strategies [19, 18]. Therefore, the spatial resources can be traded for temporal resources. This has led to a Heisenberg bound in terms of *probing time* [23, 21, 13, 14, 12, 22, 24, 25]. Let T be the total exposure time of the probe to the signal. For sequential strategies, the Heisenberg bound then states that the best achievable QFI scaling is proportional T^2 . To be more precise, if we consider the circuit diagram of Figure 1.1 and if we define our sensing resource to be a block of time evolution $\Lambda_{\omega,t} = e^{\mathcal{L}_\omega t}$, then the Heisenberg bound states that the best possible QFI scaling is quadratic in the number of uses of this resource, i.e. the number of $e^{\mathcal{L}_\omega t}$ blocks appearing in the circuit diagram, regardless of if they are placed in a sequence or in parallel. It is important to note that this definition is not standard. Although increasing coherence time does give a quadratic gain in precision, it is not always referred to as the Heisenberg limit, as that name is sometimes reserved for the scaling of a parallel strategy. The difference in definitions becomes very important in systems where universal and fast quantum control is not available, in which case parallel strategies can no longer be traded for sequential strategies, as the sequential strategies can no longer simulate parallel strategies. Examples of such systems are spin or atom ensembles where not all degrees of freedom are individually addressable.

To saturate either of the above Heisenberg bounds, one needs to use quantum effects [19]. In the case of the parallel strategy, we use quantum entanglement. For the sequential strategy, we use superposition and coherence.

In the lab, noise and decoherence of the quantum system often make the Heisenberg limit impossible [13, 14]. We are instead reduced to the standard quantum limit, also called the shot noise limit, where for parallel strategies the quantum Fisher information has a scaling at most proportional to N and for sequential strategies to T . This scaling

limit can be seen as the classical limit, as for parallel strategies it corresponds to the optimal scaling one can achieve when quantum effects are not used [19]. For non-adaptive sequential strategies, decoherence of the quantum state forces one to measure before all the quantum information is destroyed, and then to restart the experiment. If the experiments are independent, then the QFI values of each experiment are summed, leading to the linear scaling in terms of duration T .

As previously mentioned, in this work, we will focus on systems where universal, fast, and accurate control is possible. We will therefore use the more general definition of Heisenberg limit scaling, and focus on the scaling of the QFI in terms of T . Candidates for quantum systems satisfying these assumptions are nitrogen-vacancy centers in diamond, where error-correction has already been performed to enhance coherence times for sensing [12], as well as other qubit technologies such as superconducting circuits. In order to saturate this temporal Heisenberg bound, we will derive quantum error correction protocols that extend the coherence time of a quantum system subjected to non-Markovian noise sources. We will focus on gains in sensing precision due to this increase in coherence time. We study sequential strategies as we are interested in metrology with non-Markovian, or temporally correlated noise. These non-Markovian effects are more important in sequential strategies, as the evolution time is longer and allows for more correlation to build up in the noise. For simplicity, we will consider a single probe. However, our results could potentially extend to hybrid strategies where we increase both the sensing time and the number of error-corrected probes. We leave such questions for future work.

Chapter 2

Quantum Non-Markovianity Background

In the laboratory, it is impossible to perfectly isolate a quantum system. Systems will usually couple to their environment, introducing unwanted interactions which can lead to noise and decoherence. In order to model these phenomena, we resort to the theory of open quantum systems. Contrary to the evolution of an isolated quantum system, the evolution of an open quantum system is not unitary and is often irreversible. In other words, information will flow from the system to the environment, and the system's entropy will increase.

There are many ways to model the evolution of an open quantum system. The models can be separated into two families, *Markovian* and *non-Markovian*. In classical probability theory, a Markov process is one where the next state of the system depends only on the current one. The process, therefore, has no memory of past states. There exist many definitions for a *quantum Markov process*, and many different methods for measuring quantum non-Markovianity. They all, however, try to capture the essence of the classical definition. Markovian quantum evolution will therefore depend only on the current state of the system, and not on past information that has been dissipated to the environment. A non-Markovian evolution is one for which this is not the case.

In this chapter, a few of the principal measures of quantum non-Markovianity will be reviewed. We first begin with a review of necessary aspects of the theory of open quantum systems. Next, non-Markovianity based on CP-divisibility, information backflow, and the process tensor will be studied. Further, we will present the hidden Markov model, which will allow us to study non-Markovian quantum error correction (Section 3.2) and error-

corrected non-Markovian quantum metrology (Section 4.3). For a more in-depth review of some of the concepts addressed in this chapter, we refer the reader to [26, 27, 28].

2.1 Review of Open Quantum Systems

We first begin with a review of certain concepts from the theory of open quantum systems. We briefly review the formalisms of completely-positive trace-preserving maps, GKSL master equations, and quantum combs.

2.1.1 Completely Positive Trace Preserving Maps

First, we review completely-positive trace-preserving maps, also called *quantum channels*. Consider an arbitrary quantum state ρ that is not entangled with any environment. If the system is closed, then the quantum state will evolve unitarily, i.e.

$$\rho_{\text{final}} = U \rho_{\text{initial}} U^\dagger \quad (2.1)$$

for some unitary operator U . For an open system, however, we will represent the evolution from one time step to the next by a *map* Φ .

$$\rho_{\text{final}} = \Phi(\rho_{\text{initial}}). \quad (2.2)$$

For our map to make sense physically, we require the following three properties :

1. **Linearity** : The map Φ should be a linear map. To gain some intuition for why this assumption makes sense, we consider the example of a mixed state. We prepare $\rho = \frac{1}{2}\rho_1 + \frac{1}{2}\rho_2$. We can interpret this as having prepared either ρ_1 or ρ_2 with 50% probability. After the evolution, we should therefore expect to obtain either the evolved ρ_1 or ρ_2 with 50% probability :

$$\Phi(\rho) = \Phi\left(\frac{1}{2}\rho_1 + \frac{1}{2}\rho_2\right) = \frac{1}{2}\Phi(\rho_1) + \frac{1}{2}\Phi(\rho_2). \quad (2.3)$$

2. **Trace Preserving (TP)** : The map Φ should be trace preserving. For a density operator ρ , the $\text{Tr}\rho = 1$ property reflects the fact that the probabilities of measurement outcomes sum to 1. We still want this to be the case after the evolution of our system, so we impose the trace preservation property.

3. **Complete Positivity (CP)** : The map Φ should be completely positive. Density operators are defined as positive semi-definite operators with a trace of 1. The latter property is guaranteed in the evolved density operator by the TP condition. A map that sends positive semi-definite operators to positive semi-definite operators is called *positive*.

However, positivity of the map is not enough. Consider a bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$ and a state ρ_{SE} of the system. If Φ acts solely on $\mathcal{H}_A$, then we want $[\Phi \otimes \mathcal{I}](\rho_{AB})$ to be a valid state as well. Here, $\mathcal{I}$ is the identity map, $\mathcal{I}(X) = X \forall X$. It turns out positivity is not strong enough to guarantee this for an arbitrary state ρ_{SE} on an arbitrary Hilbert space $\mathcal{H}_B$. The property that $[\Phi \otimes \mathcal{I}](\rho_{AB})$ is positive for an arbitrary positive semi-definite operator ρ_{AB} and Hilbert space $\mathcal{H}_B$ is known as *complete positivity*.

This property is not an explicit requirement for the map to be physical. As we will see in Section 2.3, it can be dropped when we are considering situations where there is unknown information on the system and its entanglement. However, when studying the full system with complete information, the dynamics should always be CP.

The quantum channel is a very general way of representing quantum processes. In fact, it is the most general model when we have complete information about the system and its dynamics. The name channel is due to the fact that it is a generalization of a classical channel. It takes quantum information as input, i.e., a density matrix, and yields a density matrix as an output, as guaranteed by the above properties.

Quantum channels can represent time evolution due to both continuous processes, such as Hamiltonian evolution, for example, and stochastic processes, where we take the transformation to be instantaneous. The latter case is often used when discussing quantum information processing in the circuit model. They are also often used to model situations where one does not have access to the intermediary dynamics, such as a noisy communication channel, for instance, where a state is transmitted.

There are many ways to mathematically represent these maps. We now present four of the principal representations: the Choi representation, the operator-sum representation, the natural representation, and the Stinespring dilation.

2.1.2 Choi Representation

The Choi representation [29, 30, 31] of quantum states arises from a correspondence between quantum states and quantum channels. To obtain the unique state corresponding to

a channel, we simply act with a channel on half of the (un-normalized) maximally entangled state supported on two copies of the Hilbert space of the input state

$$|\Phi\rangle = \sum_{i=1}^d |i\rangle |i\rangle. \quad (2.4)$$

As an equation, the Choi representation J_Φ of a channel Φ is given by

$$J_\Phi = (\Phi \otimes \mathcal{I})(|\Phi\rangle\langle\Phi|), \quad (2.5)$$

where $\mathcal{I}$ here denotes the identity channel acting on the other half of the maximally entangled state. To obtain the output of the channel Φ acting on a state ρ , we use [30, Equation 2.66] :

$$\Phi(\rho) = \text{Tr}_{\mathcal{H}_2}(J_\Phi(\mathbb{1}_{\mathcal{H}_1} \otimes \rho^T)), \quad (2.6)$$

where $\mathcal{H}_1, \mathcal{H}_2$ are the two copies of the Hilbert space where ρ resides. Unfortunately, the Choi representation does not offer any direct physical interpretation. However, it is often a very convenient representation to work with mathematically. This is because it is easy to verify certain properties of quantum channels using the Choi representation. Firstly, a linear map Φ is completely positive if and only if its Choi representation given by Equation 2.5 is positive-semidefinite [30, Theorem 2.22]. Second, Φ is a trace preserving map if and only if $\text{Tr}_{\mathcal{H}_2}(J_\Phi) = \mathbb{1}_{\mathcal{H}_1}$ [30, Theorem 2.26]. Finally, unlike the operator-sum and natural representations below, the Choi representation is unique up to the choice of basis for quantum states.

2.1.3 The Operator-Sum Representation

Before introducing operator sum-representation, we will introduce the vector-operator correspondence, which we denote by the double-ket [30]. It is the invertible linear transformation $|\cdot\rangle\rangle : \mathcal{B}(\mathbb{C}^n) \rightarrow \mathbb{C}^{n^2}$ sending operators to vectors as follows

$$|E_{ab}\rangle\rangle = e_a \otimes e_b, \quad (2.7)$$

where E_{ab} is the matrix of all zeros except for a one at indices a, b , and e_i is an n -dimensional vector of all zeros except for a one at index i . Put more simply, the “vectorized” density matrix is the vector obtained by taking each row of the density matrix as a column vector, and then stacking them. In particular, we have that for $|\psi\rangle, |\phi\rangle \in \mathbb{C}^n$,

$$||\psi\rangle\rangle\langle\phi|| = |\psi\rangle \otimes \overline{|\phi\rangle}, \quad (2.8)$$

where $\bar{\cdot}$ denotes complex conjugation. It will be convenient for later to consider the Pauli operators, which give the Bell states when passed through the vector-operator correspondence

$$|\mathbb{1}\rangle\rangle \langle\langle \mathbb{1}| = 2 |\Phi^+\rangle \langle\Phi^+|, \quad (2.9)$$

$$|X\rangle\rangle \langle\langle X| = 2 |\Psi^+\rangle \langle\Psi^+|, \quad (2.10)$$

$$|Z\rangle\rangle \langle\langle Z| = 2 |\Phi^-\rangle \langle\Phi^-|, \quad (2.11)$$

$$|ZX\rangle\rangle \langle\langle ZX| = 2 |\Psi^-\rangle \langle\Psi^-|. \quad (2.12)$$

An important identity we will use relating to the operator-vector correspondence is [30, Equation 1.132]

$$(A \otimes B) |X\rangle\rangle = |AXB^T\rangle\rangle. \quad (2.13)$$

The operator-sum representation consists of a set of operators $\{A_k\}_{k=1}^r$ such that we can write the channel

$$\Phi(\rho) = \sum_{k=1}^r A_k \rho A_k^\dagger. \quad (2.14)$$

Using the vector-operator correspondence, we can obtain the operator-sum representation from the Choi representation. Indeed, one can diagonalize the Choi representation into eigenvalue, eigenvector pairs $(\lambda_k, |\lambda_k\rangle)$ which can then be used to obtain the operators A_k for the operator-sum representation.

$$J_\Phi = \sum_{k=1}^r \lambda_k |\lambda_k\rangle \langle\lambda_k| = \sum_{k=1}^r |A_k\rangle\rangle \langle\langle A_k|, \quad (2.15)$$

with $|A_k\rangle\rangle = \sqrt{\lambda_k} |\lambda_k\rangle$, where the square root is well defined since J_Φ is positive semi-definite.

2.1.4 The Natural Representation

The natural representation of a quantum channel acts on vectorized states. If

$$\sigma = \Phi(\rho), \quad (2.16)$$

then the natural representation $\hat{\Phi}$ is the matrix such that

$$|\sigma\rangle\rangle = \hat{\Phi} |\rho\rangle\rangle. \quad (2.17)$$

It is a practical representation as it reduces open system dynamics to matrix multiplication. However, it is difficult to see whether a given matrix M corresponds to a valid quantum channel.

The natural representation can be constructed from the operator sum representation $\{A_k\}_{k=1}^r$, using [30, Equation 2.74]

$$\hat{\Phi} = \sum_{k=1}^r A_k \otimes \bar{A}_k. \quad (2.18)$$

As we will see in Section 2.1.6, the natural representation is practical when solving for the dynamics of master equations, which are differential equations governing the time evolution of the state. As such, we will often deal with natural representations which are the exponential of a *generator* $\mathcal{L}$:

$$\hat{\Phi} = \exp(\mathcal{L}). \quad (2.19)$$

2.1.5 The Stinespring Dilation

The Stinespring dilation, informally known as going to the church of the larger Hilbert space, is perhaps the most physically intuitive of the various channel representations. To construct the Stinespring representation, we begin by writing the quantum channel as a unitary that acts both on the system and environment degrees of freedom, which may be physically present or mathematical artifice. The application of the quantum channel is equivalent to unitary evolution on the joint system, before finally tracing out the environment degrees of freedom to obtain the output state.

$$\Phi(\rho) = \text{Tr}_E (U (\rho \otimes \rho_E) U^\dagger). \quad (2.20)$$

Without loss of generality, we can take the environment state to be pure,

$$\Phi(\rho) = \text{Tr}_E (U (\rho \otimes |\psi\rangle_E \langle\psi|_E) U^\dagger). \quad (2.21)$$

Let $\{|i\rangle\}_{i=1}^d$ be an orthonormal basis of the environment. Writing out the trace in this basis allows us to obtain the operator-sum representation from the Stinespring representation :

$$A_i = \langle i| U |\psi\rangle_E. \quad (2.22)$$

For the Stinespring dilation to yield a valid quantum channel, the system and environment must be in a product state, as in Equation 2.20. As we will see in Section 2.4, an entangled initial state leads to maps which are not completely positive.

2.1.6 Master Equations and Quantum Dynamical Maps

In Section 2.1.1, the representation of open quantum processes as CPTP maps was presented. These were in contrast to the unitaries, which generate evolution in the closed setting. For isolated quantum systems, the problem is often instead phrased in terms of the Hamiltonians, which generate families of unitaries by varying the evolution time parameter t . For open systems, it can be similarly useful to consider families of maps parametrized by a time parameter t .

The first generalization we will consider is *quantum dynamical maps*.

Definition 1 (Quantum Dynamical Maps). *A family of maps $\{\Phi_{t_2, t_1}\}_{t \geq t_0}$ parametrized by parameters $t_2 > t_1$ such that Φ_{t_2, t_1} is the map evolving the system from time steps t_1 to t_2 and such that*

$$\Phi_{t, t} = \mathcal{I} \quad (2.23)$$

is called a quantum dynamical map [27, Definition 7].

Quantum dynamical maps are rather general; however, it is often desirable to consider families of CPTP maps with an additional property.

Definition 2 (Semigroup of CPTP Maps). *We'll call the family $\{\Lambda_t\}_{t \geq 0}$ a semigroup of completely positive trace preserving maps [27, Definition 5] if*

1. Λ_t is completely positive and trace preserving for all t ,
2. Λ_t is a continuous function of t ,
3. $\Lambda_0 = \mathcal{I}$,
4. and $\Lambda_t \circ \Lambda_s = \Lambda_{t+s}$ for all t, s . This final property is the semigroup property, and it aims to generalize the classical Markov semigroup property for Markov operators.

The *generator*, $\mathcal{L}$, of the semi-group is the linear operator that satisfies

$$\frac{d}{dt} \hat{\Lambda}_t = \mathcal{L} \hat{\Lambda}_t \quad (2.24)$$

where we have written the equation in the natural representation. This allows us to rewrite the family of maps as

$$\hat{\Lambda}_t = e^{t\mathcal{L}}. \quad (2.25)$$

Let $\rho_t = \Lambda_t(\rho_0)$ where ρ_0 is the initial state of the dynamics. This allows us to write

$$\frac{d}{dt}\rho_t = \mathcal{L}(\rho_t), \quad (2.26)$$

or in the natural representation,

$$\frac{d}{dt}|\rho_t\rangle\rangle = \mathcal{L}|\rho_t\rangle\rangle. \quad (2.27)$$

An important result [32, 33] in the theory of open quantum systems is that a generator $\mathcal{L}$ generates a semigroup of CPTP maps if and only if

$$\mathcal{L}(\rho) = \sum_{i,j=0}^{d^2-1} C_{ij} F_i \rho F_j, \quad (2.28)$$

where d is the dimension of ρ , C_{ij} is a positive definite matrix, and the $\{F_i\}_{i=0}^{d^2-1}$ operators form an orthonormal basis of operators such that $F_0 = \frac{1}{\sqrt{d}}\mathbb{1}$ and $\text{Tr}(F_i) = 0$ for $i \geq 1$. Choosing the first operator to be $\mathbb{1}$ and the rest to be traceless allows us to introduce a Hamiltonian evolution term into the semigroup generator. Indeed, by setting

$$H = \frac{i}{2\sqrt{d}} \sum_{i=1}^{d^2-1} (C_{i0} F_i - C_{0i} F_i^\dagger), \quad (2.29)$$

and by diagonalizing the $1, \dots, d^2 - 1$ block of C_{ij} , we can obtain the diagonal form [27]

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_{k=1}^{d^2-1} \gamma_k \left(L_k^\dagger \rho L_k - \frac{1}{2} \{L_k^\dagger L_k, \rho\} \right), \quad (2.30)$$

where the γ_k are the eigenvalues of C_{ij} . The first term (commutator between ρ and H) corresponds to the standard Von Neumann equation for unitary evolution. The second term corresponds to the dissipative component of the dynamics, responsible for decoherence. Combining Equation 2.30 with Equation 2.26 yields the *Gorini–Kossakowski–Sudarshan–Lindblad master equation*, also called Lindbladian. We can pass this equation through the operator-vector correspondence to obtain the generator in the natural representation, allowing us to calculate the map with a simple matrix exponential (Equation 2.25). Using Equation 2.13 on the terms of the master equation, we obtain

$$-i[H_{SE}, |\psi\rangle\langle\psi|] \mapsto \left((-iH_{SE}) \oplus \overline{(-iH_{SE})} \right) |\psi\rangle\overline{|\psi\rangle}, \quad (2.31)$$

$$L_k |\psi\rangle\langle\psi| L_k^\dagger \mapsto (L_k \otimes \overline{L_k}) |\psi\rangle\overline{|\psi\rangle}, \quad (2.32)$$

$$\{L_k^\dagger L_k, |\psi\rangle\langle\psi|\} \mapsto \left(L_k^\dagger L_k \oplus \left(L_k^\dagger L_k \right)^T \right) |\psi\rangle\overline{|\psi\rangle}, \quad (2.33)$$

where here $\oplus$ denotes the Kronecker sum

$$A \oplus B = A \otimes \mathbb{1} + \mathbb{1} \otimes B. \quad (2.34)$$

The generator in the natural representation for a GKSL master equation in diagonal form is given by

$$\mathcal{L} = (-iH_{SE}) \oplus \overline{(-iH_{SE})} + \sum_{k=1}^r \left(L_k \otimes \overline{L_k} - \frac{1}{2} L_k^\dagger L_k \oplus (L_k^\dagger L_k)^T \right). \quad (2.35)$$

2.2 CP-divisibility

The first non-Markovianity metric we'll explore is that of CP-divisibility. These definitions of Markovianity for quantum processes wish to generalize the notion of Markov operators and Markov semigroups from classical probability theory. These definitions will focus more on the structure of the families of maps forming a quantum process, rather than on the flow of information or correlations.

Definition 3 (CP Divisibility). *Consider a quantum dynamical map Φ_{t_2, t_1} parametrized by $t_2 > t_1$. The quantum dynamical map is said to be CP-divisible [27, Definition 8] if for any $t_2 > t_1 > t_0$, there exists a CPTP map V such that*

$$\Phi_{t_2, t_0} = V_{t_2, t_1} \circ \Phi_{t_1, t_0}. \quad (2.36)$$

One can use the CP-divisibility property to define quantum non-Markovianity.

Definition 4 (Non-Markovianity based off CP-divisibility [27, 34]). *A Markovian process is a process which is CP-divisible, and a non-Markovian process is one which is not.*

All semigroups of CPTP maps are Markovian processes using this definition, with

$$V_{t_2, t_1} = e^{\mathcal{L}(t_2 - t_1)}. \quad (2.37)$$

One can also consider the slightly stronger definition of non-Markovianity presented in [35, 36].

Definition 5 (Non-Markovianity based off Semigroups [35, 36]). *A CPTP map is called Markovian if it is an element of a semigroup of CPTP maps as defined in Definition 2.*

2.3 Non-Markovianity based on Information backflow and non-CP dynamics

We next study definitions of Markovianity that focus more on the flow of information and the correlations between future and past states. In their seminal work [37], Breuer et al. proposed information backflow as an indicator of non-Markovianity. *Information backflow* is a phenomenon where information that has previously flowed into the environment flows back into the system. This can be seen as the environment mediating correlations between states at previous times and the future state of the system, motivating the definition of quantum non-Markovianity.

In the original work [37], the authors define information backflow based on the notion of *distinguishability* of quantum states. Suppose you are given two states ρ_1 and ρ_2 and must devise a measurement scheme to distinguish them. It turns out that the maximal probability of successfully distinguishing the states is given by

$$\frac{1}{2} [1 + T(\rho_1, \rho_2)], \quad (2.38)$$

where

$$T(\rho_1, \rho_2) = \frac{1}{2} \text{Tr} |\rho_1 - \rho_2| \quad (2.39)$$

is the *trace distance* between ρ_1 and ρ_2 [38, Chapter 9.2.1]. Information backflow is then defined as an increase in the distinguishability of two states, which is equivalent to an increase in the trace distance between two states.

Definition 6 (Non-Markovianity based off Information Backflow [37]). *Consider a quantum process parametrized by t . Consider initial states ρ_1 and ρ_2 , and let $\rho_1(t), \rho_2(t)$ be the evolved states under the process. We will say the process is non-Markovian if there exists ρ_1, ρ_2 , and t such that*

$$\frac{d}{dt} T(\rho_1(t), \rho_2(t)) > 0. \quad (2.40)$$

In other words, the process is non-Markovian if it can make two quantum states easier to distinguish.

It is important to note that for this definition of non-Markovianity, all completely-positive trace-preserving maps are Markovian. This is due to the fact that the trace distance is contractive under the application of a quantum channel Φ [38]

$$T(\Phi(\rho_1), \Phi(\rho_2)) \leq T(\rho_1, \rho_2). \quad (2.41)$$

In other words, the task of distinguishing two quantum states will always become more difficult after the application of a quantum channel. This is perhaps most intuitive when considering the Stinespring dilation of the channel. The environment contains no information that would help to distinguish the states, as it is independent of the input state; all useful information is contained in the system. After the unitary, the information is now scrambled between the system and the environment. As the unitary builds entanglement, the reduced state of the system will become more mixed, and therefore more random. This makes the task of distinguishing the states more difficult. Finally, all useful information contained in the environment is lost forever when we take the trace.

Since a quantum process must contain some non-CP dynamics to be non-Markovian in the sense of information backflow, this definition of non-Markovianity necessarily encompasses the definition based on CP-divisibility. A map can't be CP divisible if it is not CP. However, it is possible for a map to be Markovian in the sense of information backflow, but non-Markovian in the sense of CP-divisibility. Examples of this are P-divisible maps, in which the maps in the family are positive but not completely positive [39], or sequences of CPTP maps which do not share a common generator.

Finally, other metrics can be used when defining information backflow. Another natural choice could be the *fidelity* [38]

$$F(\rho_1, \rho_2) = \text{Tr} \sqrt{\rho_1^{1/2} \rho_2 \rho_1^{1/2}}. \quad (2.42)$$

The fidelity is also contractive under CPTP maps and can be related to the trace distance via the bound

$$T(\rho_1, \rho_2) \leq \sqrt{1 - F(\rho_1, \rho_2)}. \quad (2.43)$$

A third choice is the *quantum Fisher information*, which will be introduced in Section 1.2.

Finally, it is possible to define a quantitative measure of non-Markovianity based on information backflow [37]. Let $\{(a_i, b_i)\}$ be the set of intervals where there exists a ρ_1, ρ_2 such that $\frac{d}{dt}T(\rho_1(t), \rho_2(t)) > 0$. Non-Markovianity for a process Φ can be measured with the following formula :

$$\mathcal{N}_{IB}(\Phi) = \max_{\rho_1(0), \rho_2(0)} \sum_i [T(\rho_1(b_i), \rho_2(b_i)) - T(\rho_1(a_i), \rho_2(a_i))]. \quad (2.44)$$

This definition of non-Markovianity is broader than those based on CP-divisibility and the semigroup property. Indeed, Definition 6, based off information backflow, encompasses Definition 3, based off CP-divisibility, which encompasses Definition 5, based off semigroups of CPTP maps.

2.4 Quantum Linear Maps beyond CPTP

In light of the previous definition, non-Markovian quantum maps are maps which are not completely-positive. We offer a characterization of non-CP maps. The results of the characterization were originally presented in [1].

2.4.1 Positive Maps

The first and simplest relaxation from CP maps is to reduce the completely positive property to only being positive. Positive maps, however, are still Markovian according to Definition 6. Indeed, the proof of Equation 2.41 [38, Theorem 9.2] requires only that the map be positive, not completely positive, and so the trace distance is still contractive under the application of positive maps.

An example of a positive map that isn't CP is the transpose map,

$$\mathcal{T}(X) = X^T. \quad (2.45)$$

The map is positive, as the transpose of any positive semi-definite matrix is still positive semi-definite. However, it is not completely positive. This can be seen by considering the Peres-Horodecki criterion [40].

2.4.2 SP Maps

The next relaxation is the class of *semi-positive maps*. Semi-positive maps are a subset of *Hermitian preserving maps* (HP), which are maps that send Hermitian operators to Hermitian operators. SP maps are defined as follows :

Definition 7 (Definition 1 of [1]). *A Hermitian preserving map Φ is semi-positive (SP) if there exists an invertible density matrix ρ such that $\Phi(\rho)$ is also an invertible density matrix.*

By dropping the positivity requirement, we must introduce the notion of the *compatibility domain*. Indeed, for an SP map Φ there exists a set of density matrices such that if ρ is in this set, then $\Phi(\rho)$ is not positive, and therefore not a valid physical state. These density matrices are said to be incompatible with the dynamics. The compatibility domain is the set of density matrices that are mapped to valid density matrices under Φ .

We know that a Stinespring dilation of the form

$$\Lambda(\rho) = \text{Tr}_E (U \rho_E \otimes \rho U^\dagger) \quad (2.46)$$

leads to Λ being a CPTP map, no matter the choice of ρ_E, U . To obtain a non-CP map Φ with a similar dilation, the initial state must be entangled with the environment; in other words

$$\Phi(\rho) = \text{Tr}_E (U \rho_{SE} U^\dagger) \quad (2.47)$$

where $\text{Tr}_E (\rho_{SE}) = \rho$. The entanglement between ρ and the environment mediates the information backflow that occurs when Φ is applied. This motivates a more enlightening definition of SP maps, which comes from [1, Theorem 1]. The map Φ is semi-positive if and only if there exists CPTP maps Ψ, Ξ such that

$$\Phi = \Xi \circ \Psi^{-1}. \quad (2.48)$$

The inverse of a CPTP map is always SP [1, Corollary 1], and here the composition of CP and SP maps leads to an overall SP map. Let ρ_0 be a density matrix and let $\rho = \Psi(\rho_0)$. We see that the compatibility domain of the map Φ is the image of the map Ψ . Further,

$$\Xi(\rho_0) = \Phi(\rho). \quad (2.49)$$

In other words, the non-Markovian process $\Phi(\rho)$ can be seen as a time slice of the completely-positive, but not CP-divisible, process Ξ . We can clarify the notion of “compatible entanglement” by taking the Stinespring dilation of Equation 2.48. Let U_C be the Stinespring dilation of Ψ , acting on an initial state $\rho_E^0 \otimes \rho_0$. Let $U \circ U_C$ be the initial state of Ξ acting on the same initial state. For ρ to be in the compatibility domain of Φ , there must exist a ρ_0 such that

$$\rho_{SE} = U_C \rho_E \otimes \rho_0 U_C^\dagger. \quad (2.50)$$

The state ρ must be consistent, or compatible, with the entangled state ρ_{SE} of Equations 2.47 and 2.50.

2.4.3 Semi-Nonnegative Maps

We now consider the most general relaxation for non-Markovian quantum linear maps; *semi-nonnegative maps*. Semi-nonnegative (SN) maps are defined as follows :

Definition 8 (Definition 2 of [1]). *A map Φ is semi-nonnegative if there exists a density matrix ρ such that $\Phi(\rho)$ is also a valid density matrix.*

This is the most general relaxation because a map that is not SN will send any density matrix to a matrix that is not positive. Much like SP maps, SN maps can be characterized in terms of a context unitary/channel. Indeed, [1, Theorem 2] states that if Φ is a SNTP map, then there exists CPTP maps Ψ and Ξ such that

$$\Psi \circ \Phi = \Xi. \quad (2.51)$$

The difference between Equations 2.48 and 2.51 is that for SN maps, the CPTP map Ψ need not be invertible. However, the Stinespring dilation of Equation 2.51 can be taken in the same manner as in the SP case. Figure 2.1 gives a diagram representing the situation.

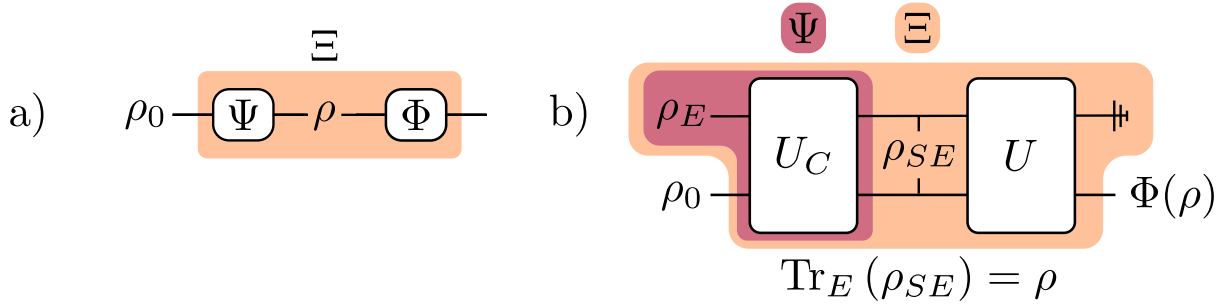


Figure 2.1: a) Composition of maps defining the SNTP map (Equation 2.51). If the CPTP map Ψ is invertible, then Φ is also a SPTP map. b) Stinespring dilations of Ψ and Ξ (Equation 2.50). We see that the input state ρ to the map Φ can be entangled with the environment. This entanglement can lead to information backflow.

Finally, it is important to note that the set of SNTP maps contains the set of all SPTP maps. The set of all SPTP maps contains the set of all positive trace-preserving maps. Finally, the set of all positive maps contains the set of all CPTP maps [1, Proposition 1].

$$\text{CPTP Maps} \subset \text{Positive TP Maps} \subset \text{SPTP Maps} \subset \text{SNTP Maps} \subset \text{HPTP Maps}. \quad (2.52)$$

2.4.4 The Operator-Sum Representation

Both SP and SN maps are still Hermitian-preserving maps. All Hermitian preserving maps admit an operator sum representation of the following form :

$$\Phi(\rho) = \sum_{k=1}^r \sigma(k) A_k \rho A_k^\dagger. \quad (2.53)$$

We call A_k the Kraus operators, and $\sigma(k) : \mathbb{N} \rightarrow \{-1, 1\}$ is a function that assigns an overall sign of ± 1 to the terms of the sum. As previously mentioned, the map Φ is CP if and only if $\sigma(k) = 1 \ \forall \ k$. Further [30, Theorem 2.26] states that the map Φ is trace preserving if and only if

$$\sum_{k=1}^r \sigma(k) A_k^\dagger A_k = \mathbb{1}. \quad (2.54)$$

2.4.5 Choi Representation

The Choi representation can be defined for non-CP maps in exactly the same way as for CP maps. Similarly, the operator sum representation for non-CP maps can be obtained from the spectral decomposition of the non-CP Choi matrix. The principal difference is that for non-CP maps, there is no longer a duality between maps and quantum states. In particular, the Choi representations for non-CP maps are not positive operators. They are, however, hermitian.

2.4.6 Non-CP Stinespring Dilations and the Tomography Problem

As we have seen in Equation 2.50 and Figure 2.1, it is possible to dilate non-CP maps to unitary evolution on a larger space, similarly to Stinespring dilation for CPTP maps. There is, however, a problem with the consistency of such a dilation. This problem has previously been discussed in [41, Section II, A]. We demonstrate the problem with the following example.

Suppose an experimentalist wishes to prepare a fiducial state of $|1\rangle_S \langle 1|_S$ on a qubit in the laboratory. We label the system of the qubit S . In reality however, they end up preparing the state $|\Psi_{SE,0}\rangle = p|0\rangle_E \otimes |0\rangle_S + (1-p)|1\rangle_E \otimes |1\rangle_S$, where system E is a two-level system representing the environment. Let $\rho_{SE}^0 = |\Psi_{SE,0}\rangle \langle \Psi_{SE,0}|$. Further, let

$$\rho_0 = \text{Tr}_E(\rho_{SE}^0) = \begin{pmatrix} p & 0 \\ 0 & 1-p \end{pmatrix} \quad (2.55)$$

The experimentalist can then act with a unitary U_C on system S to transform the initial state, as the environment is inaccessible. This transforms the state of the full system as follows :

$$\rho_{SE} = (\mathbb{1}_E \otimes U_C) \rho_{SE}^0 (\mathbb{1}_E \otimes U_C)^\dagger \quad (2.56)$$

The experimentalist only sees $\rho = \text{Tr}_E(\rho_{SE})$, as they can only probe the system. Afterwards, the system evolves according to a unitary U acting as the noise, before the environment is finally traced out. A diagram depicting the scenario is presented in Figure 2.2.

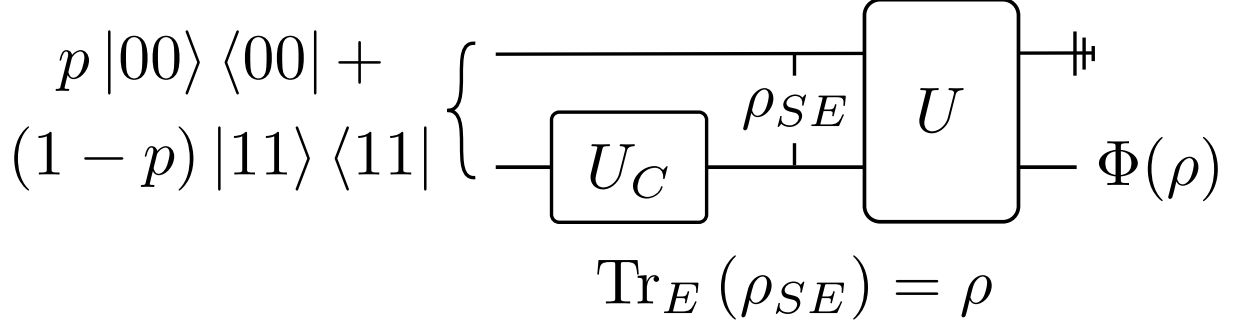


Figure 2.2: Circuit for demonstrating the problem with consistent tomography for non-CP maps. The problem arises when we assume a constant dilation for the application of the map to different states.

We now wish to compute the non-CP map for the evolution from ρ until after the unitary U and the tracing-out of the environment. For CPTP maps, where the initial state ρ_{SE}^0 is separable, this evolution will consistently give the same CPTP map for ρ , regardless of U_C . For an entangled initial state, however, such as the one chosen, the resulting map will not be CP. Its operator sum representation will depend on U_C . In other words, for different choices of initial state ρ , we end up with different maps Φ . This makes process tomography impossible.

To show we end up with different maps, we will take U to be the control- Z gate $U = |0\rangle_E \langle 0|_E \otimes \mathbb{1} + |1\rangle_E \langle 1|_E \otimes Z$. Acting on the initial state, we obtain :

$$(\mathbb{1} \otimes U_C) |\Psi_{SE,0}\rangle = |\Psi_{SE}\rangle = \sqrt{p} |0\rangle \otimes U_C |0\rangle + \sqrt{1-p} |1\rangle \otimes U_C |1\rangle. \quad (2.57)$$

Which, as a density matrix, gives :

$$\rho = U_C \rho_0 U_C^\dagger = p U_C |0\rangle \langle 0| U_C^\dagger + (1-p) U_C |1\rangle \langle 1| U_C^\dagger. \quad (2.58)$$

We then compute the evolution via U and trace out the environment.

$$\Phi(\rho) = \text{Tr}_E(U |\Psi_{SE}\rangle \langle \Psi_{SE}| U^\dagger), \quad \rho = \text{Tr}_E(|\Psi_{SE}\rangle \langle \Psi_{SE}|). \quad (2.59)$$

We now calculate the operator sum representation for this map, which is given by Equation 2.59.

$$U |\Psi_{SE}\rangle = \sqrt{p} |0\rangle \otimes U_C |0\rangle + \sqrt{1-p} |1\rangle \otimes ZU_C |1\rangle. \quad (2.60)$$

$$\Phi(\rho) = \text{Tr}_E (U |\Psi_{SE}\rangle \langle \Psi_{SE}| U^\dagger) = pU_C |0\rangle \langle 0| U_C^\dagger + (1-p)ZU_C |1\rangle \langle 1| U_C^\dagger Z \quad (2.61)$$

We wish to write the right-hand side of this equation in terms of the input ρ . Therefore, we consider the two following maps :

$$\Xi_0(\sigma) = \frac{1}{2p-1} (p\sigma - (1-p)X\sigma X), \quad \Xi_0(\rho_0) = |0\rangle \langle 0|, \quad (2.62)$$

$$\Xi_1(\sigma) = \frac{1}{1-2p} ((1-p)\sigma - pX\sigma X), \quad \Xi_1(\rho_0) = |1\rangle \langle 1|. \quad (2.63)$$

These maps are non-CP since they have negative coefficients in their diagonal operator sum representations. These maps allow us to write :

$$\Phi(\rho) = pU_C \Xi_0(\rho_0) U_C^\dagger + (1-p)ZU_C \Xi_1(\rho_0) U_C^\dagger Z \quad (2.64)$$

Where $\rho = U_C \rho_0 U_C^\dagger$ implies $\rho_0 = U_C^\dagger \rho U_C$. We can therefore write :

$$\Phi(\rho) = pU_C \Xi_0(U_C^\dagger \rho U_C) U_C^\dagger + (1-p)ZU_C \Xi_1(U_C^\dagger \rho U_C) U_C^\dagger Z \quad (2.65)$$

Expanding the Ξ_i in their operator sum representation, we finally obtain the operator sum representation for Φ ,

$$\begin{aligned} \Phi(\rho) = & \frac{p}{2p-1} \left[p\rho - (1-p)U_C X U_C^\dagger \rho U_C X U_C^\dagger \right] \\ & + \frac{1-p}{1-2p} \left[(1-p)Z\rho Z - pZU_C X U_C^\dagger \rho U_C X U_C^\dagger Z \right]. \end{aligned} \quad (2.66)$$

Here is where the tomography problem appears. Despite fixed initial correlations in the $|\Psi_{SE,0}\rangle$ state, because of the entanglement, the map affecting ρ will still depend on how ρ was prepared. This dependence is captured by the dependence on U_C in the operator sum representation for $\Phi(\rho)$. This is where the non-Markovian character of the noise comes in; the actions U_C of previous time-steps will have an effect. Unfortunately, the Choi matrix for Φ will also depend on U_C , making a consistent tomographic reconstruction of the Choi matrix impossible. To show the dependence on U_C , consider the Choi matrices for $U_C = \mathbb{1}$ and $U_C = H$, where here H refers to the Hadamard gate and not the Hamiltonian. The

Choi matrix is given by :

$$\begin{aligned}\Phi(\rho) = & \frac{p^2}{2p-1} |\mathbb{1}\rangle\rangle \langle\langle \mathbb{1}| - \frac{p(1-p)}{2p-1} |U_C X U_C^\dagger\rangle\rangle \langle\langle U_C X U_C^\dagger| \\ & + \frac{(1-p)^2}{1-2p} |Z\rangle\rangle \langle\langle Z| - \frac{p(1-p)}{1-2p} |Z U_C X U_C\rangle\rangle \langle\langle Z U_C X U_C|. \end{aligned} \quad (2.67)$$

For $U_C = \mathbb{1}$, the Choi matrix is given by :

$$J_\Phi = \begin{pmatrix} 1 & 0 & 0 & \frac{p^2+(1-p)^2}{2p-1} \\ 0 & 0 & -2\frac{p(1-p)}{2p-1} & 0 \\ 0 & -2\frac{p(1-p)}{2p-1} & 0 & 0 \\ \frac{p^2+(1-p)^2}{2p-1} & 0 & 0 & 1 \end{pmatrix}. \quad (2.68)$$

For $U_C = H$, we use the relation $HXH = Z$ and $HZH = X$ to obtain the Choi matrix :

$$J_\Phi = \begin{pmatrix} 1 & 0 & 0 & \frac{p^2+2p(1-p)+(1-p)^2}{2p-1} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{p^2+2p(1-p)-(1-p)^2}{2p-1} & 0 & 0 & 1 \end{pmatrix}. \quad (2.69)$$

From the same initial correlations, we obtain two different Choi maps for the non-CP processes depending on how we unitarily prepare the state. Standard process tomography will therefore not work in this setup, as the Choi representation will depend on the state preparation.

2.5 Hidden Markov Model

We now introduce the model we will use to study quantum error correction and quantum metrology. The model consists of environment degrees of freedom, $\mathcal{H}_E$, which are inaccessible to the experimenter. This environment is coupled to experimentally accessible degrees of freedom, $\mathcal{H}_S$. Together, the environment and accessible degrees of freedom evolve under the following master equation :

$$\frac{d\rho}{dt} = -i[H_{PE}] + \sum_{k=1}^r \left(L_k \rho L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_k, \rho\} \right). \quad (2.70)$$

We suppose as well that the experimenter can perform controls arbitrarily fast and with perfect precision. We call this model the *hidden Markov model*. Excluding possible controls, which would add possibly time-dependent terms to Equation 2.70, we see that the evolution on the full system is governed by a Lindblad master equation. Therefore, when considering the full system, the dynamics are CP, CP-divisible, and are obtained from a semigroup generator. They are therefore Markovian under both Definition 3, Definition 5, and Definition 6. However, when we consider the reduced dynamics on the experimentally accessible component S , the dynamics appear non-Markovian for all definitions considered. We demonstrate this fact with the following example, which was previously derived by the author of this thesis in [1].

Consider a qubit as our accessible system and a two-level system as our inaccessible environment. In other words, $\mathcal{H}_S \simeq \mathcal{H}_E \simeq \mathbb{C}^2$. We'll suppose that the jump operators L_k are all identically zero, and that the Hamiltonian generating the unitary evolution is given by $\chi Y \otimes Z$, where χ here is a scalar amplitude. Finally, we suppose an initial environment state of $|0\rangle\langle 0|$.

This choice of master equation yields dynamics generated by the unitary

$$U_{SE} = \exp(-i\chi t Y \otimes Z), \quad (2.71)$$

which, when acting on $|0\rangle\langle 0| \otimes \rho$, gives a dephasing channel.

$$\Phi_{0,t}(\rho) = \cos^2(\chi t) \rho + \sin^2(\chi t) Z \rho Z, \quad (2.72)$$

where Φ_{t_1,t_2} is the map that evolves the state from times t_1 to t_2 . Note that the map $\Phi_{0,t}$ is CPTP, since the initial state is separable. It can be shown that Equation 2.72's inverse is given by

$$\Phi_{0,t}^{-1}(\rho) = \frac{1}{\cos^4(\chi t) - \sin^4(\chi t)} \cos^2(\chi t) \rho - \frac{1}{\cos^4(\chi t) - \sin^4(\chi t)} \sin^2(\chi t) Z \rho Z. \quad (2.73)$$

We now consider the map that takes the system from times t to $t + \delta$. This map is not always CPTP; it will sometimes be SPTP. The map can be constructed from the CPTP maps $\Phi_{0,t}$ and $\Phi_{0,t+\delta}$ following the recipe of Equation 2.48 for SPTP maps

$$\Phi_{t,t+\delta} = \Phi_{0,t+\delta} \circ \Phi_{0,t}^{-1}. \quad (2.74)$$

By composing the operator sum representations, we obtain

$$\begin{aligned} \Phi_{t,t+\delta}(\rho) = \frac{1}{\cos^4(\chi t) - \sin^4(\chi t)} & \left[(\cos^2(\chi(t+\delta)) \cos^2(\chi t) - \sin^2(\chi(t+\delta)) \sin^2(\chi t)) \rho \right. \\ & \left. + (\sin^2(\chi(t+\delta)) \cos^2(\chi t) - \cos^2(\chi(t+\delta)) \sin^2(\chi t)) Z \rho Z \right], \end{aligned} \quad (2.75)$$

which can be simplified to

$$\Phi_{t,t+\delta}(\rho) = \frac{1}{\cos(2\chi t)} [\cos(2\chi t + 2\chi\delta) \cos(\chi\delta)\rho + \sin(2\chi t + 2\chi\delta) \sin(\chi\delta) Z\rho Z]. \quad (2.76)$$

The Choi representation can be computed for $\Phi_{t,t+\delta}$,

$$J_{\Phi_\delta} = \begin{pmatrix} 1 & 0 & 0 & \frac{\cos(2\chi(t+\delta))}{\cos(2\chi t)} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{\cos(2\chi(t+\delta))}{\cos(2\chi t)} & 0 & 0 & 1 \end{pmatrix}. \quad (2.77)$$

The eigenvalues of this matrix are

$$\left\{ 0, 0, 1 + \left| \frac{\cos(2\chi(t+\delta))}{\cos(2\chi t)} \right|, 1 - \left| \frac{\cos(2\chi(t+\delta))}{\cos(2\chi t)} \right| \right\}. \quad (2.78)$$

The map is completely positive if and only if the Choi matrix is positive semi-definite, which is equivalent to having nonnegative eigenvalues. Therefore, the condition for the map $\Phi_{t,t+\delta}$ to be positive is

$$|\cos(2\chi(t+\delta))| \leq |\cos(2\chi t)|. \quad (2.79)$$

In the limit of $\delta \rightarrow 0$ for $\delta > 0$, this inequality is equivalent to

$$\lim_{\delta \rightarrow 0} \frac{|\cos(2\chi(t+\delta))| - |\cos(2\chi t)|}{\delta} \leq 0. \quad (2.80)$$

We recognize the derivative of $\cos(2\chi t)$, allowing us to rewrite the CP condition as

$$\Phi_{t,t+\delta} \text{ is completely-positive} \Leftrightarrow \frac{d}{dt} |\cos(2\chi t)| \leq 0. \quad (2.81)$$

This condition for the map being CP is the same condition for the map being non-Markovian according to Definition 6. Calculating the derivative, we obtain

$$\frac{d}{dt} |\cos(2\chi t)| = \begin{cases} -2\chi \sin(2\chi t), & t \in \left(-\frac{\pi}{4\chi}, \frac{\pi}{4\chi}\right) + \frac{\pi}{\chi}\mathbb{Z} \\ 2\chi \sin(2\chi t), & t \in \left(\frac{\pi}{4\chi}, \frac{3\pi}{4\chi}\right) + \frac{\pi}{\chi}\mathbb{Z} \end{cases}. \quad (2.82)$$

Using this, we can calculate the regions where the dynamics are CP. We have that the dynamics are CP if and only if t is contained in

$$\bigcup_{k \in \mathbb{Z}} \left[\frac{\pi}{\chi}k, \frac{\pi}{4\chi} + \frac{\pi}{\chi}k \right] \cup \left[\frac{\pi}{2\chi} + \frac{\pi}{\chi}k, \frac{3\pi}{4\chi} + \frac{\pi}{\chi}k \right]. \quad (2.83)$$

These regions are represented by the white areas of Figure 2.3. The grey areas correspond to intervals where the dynamics are non-CP.

Further, consider now the initial states $\rho_1 = |+\rangle\langle+|$ and $\rho_2 = |-\rangle\langle-|$. Without loss of generality, we will set $\chi = 1$. We will calculate the trace distance between these states as a function of time, where both states are evolving according to the map $\Phi_{0,t}$.

$$\begin{aligned} T(\Phi_{0,t}(\rho_1), \Phi_{0,t}(\rho_2)) &= \text{Tr} \frac{1}{2} |\Phi_{0,t}(\rho_1) - \Phi_{0,t}(\rho_2)| \\ &= \frac{1}{2} \text{Tr} |(\cos^2(t) - \sin^2(t)) |+\rangle\langle+| + (\sin^2(t) - \cos^2(t)) |-\rangle\langle-|| \\ &= |\cos^2(t) - \sin^2(t)| = |\cos(2t)|. \end{aligned} \quad (2.84)$$

The evolution of the trace distance as a function of time is shown in Figure 2.3. We now consider the difference in trace distance over a maximal interval of non-Markovianity. We consider the open interval $(t_1, t_2) = (\frac{\pi}{4}, \frac{\pi}{2})$, for which Equation 2.83 tells us the dynamics are non-CP. Using Equation 2.84, we see that at t_1 the trace distance is minimal and zero, and at t_2 the trace distance is maximal and equal to 1. From this, we conclude that ρ_1 and ρ_2 are the states that maximize the non-Markovianity given by Equation 2.44, and that therefore the non-Markovianity over this interval is $\mathcal{N}_{IB}(\Phi) = 1$.

2.6 Quantum Combs and Process Tensors

The hidden Markov model is in fact a special case of a more general framework for non-Markovian quantum dynamics, known as *process tensors* [28], or quantum combs. Contrary to the other models considered in this Section, which can be viewed as maps and transformations on quantum states, the process tensor is a map that acts directly on quantum channels. Rather than take as input a quantum state, process tensors take as input CPTP maps, or more generally, quantum instruments, which represent the control operations that an experimentalist would perform in the lab.

The quantum comb has a very similar structure to the hidden Markov model of Section 2.5. Rather than time-evolution from a master equation, however, we time evolve with a sequence of CPTP maps acting on the system of interest and an environment of arbitrary dimension. A diagram representing the quantum comb is presented in Figure 3.1. The mathematical object we use to describe quantum combs is a tensor network encompassing the initial state $\rho_E \otimes |\psi\rangle\langle\psi|$ (which more generally we allow to be entangled) as well as the CPTP maps Λ_i . A tensor network is a tensor with specific structure that often

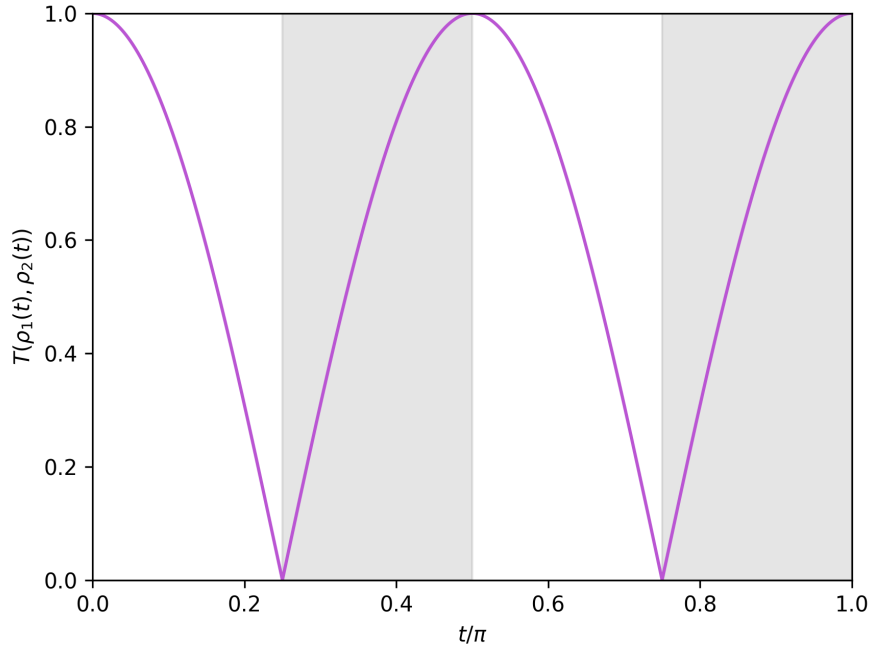


Figure 2.3: Trace distance between $\rho_1 = |+\rangle\langle+|$ and $\rho_2 = |-\rangle\langle-|$ evolving under the map $\Phi_{0,t}$ (Equation 2.72). Grey areas correspond to time intervals where the dynamics are non-CP (Equation 2.83) and therefore non-Markovian according to Definition 6. During the full evolution, the non-Markovianity can be quantified by Equation 2.44, yielding $\mathcal{N}_{IB}(\Phi_{0,t}) = 2$.

allows it to be represented using fewer parameters. This tensor network can then be contracted with a tensor network describing the control operations $\mathcal{C}_i$ (which more generally we allow to be correlated) to yield the output state of the overall process. The quantum comb model is fully general and encapsulates the hidden Markov model. In this thesis, we will restrict ourselves to hidden Markov models and therefore will not go into the details of these tensor networks. For more information on quantum combs, we invite the reader to consult [41, 28, 42].

As for the hidden Markov model, a quantum comb can exhibit non-Markovianity under all of the definitions studied in this chapter. Further, there exists a consistent experimental tomography protocol allowing to reconstruct quantum combs (and therefore hidden Markov models) [43, 44, 45], avoiding the tomography problems of the non-CP maps studied in Section 2.4.6. The trade-off for resolving the tomography problem is to model the noise with complete information, leading to a consistent model. This, unfortunately, requires many more parameters to describe the mathematical model. In fact, exponentially more parameters in the number of time-steps of the comb are required.

In [28], a definition of non-Markovianity based on the process tensor is introduced. As previously mentioned, the quantum comb can be represented by a tensor network. Similarly to the Choi representation of channels, there exists a correspondence between process tensors and states. Formally, a process tensor is said to be Markovian if its Choi representation is a separable state [28, Section VI A 1]. An equivalent definition, which is less math-heavy, is given by Definition 9.

Definition 9 (Non-Markovianity for Process Tensors (Informal)). *A process tensor is said to be Markovian if it can be represented without any environmental degrees of freedom. In other words, it can be represented by an initial state and a sequence of CPTP maps.*

A diagram representing this definition is presented in Figure 2.4.

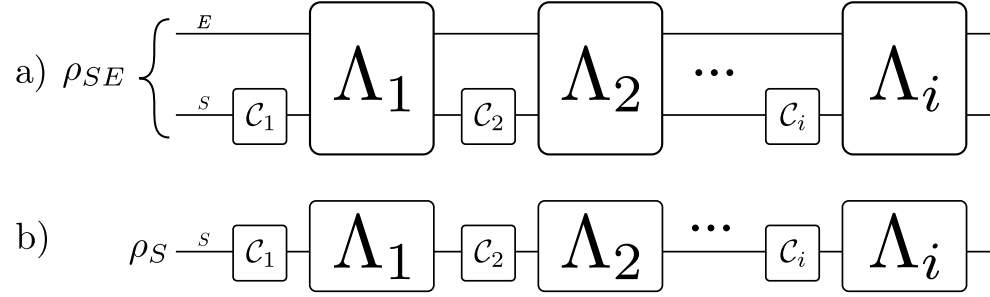


Figure 2.4: a) Non-Markovian Quantum Comb. A wire for the environmental degrees of freedom is required to model the process. b) Markovian quantum comb. No environmental degrees of freedom are required to represent the process. The process can be represented by an initial state ρ_S and a sequence of CPTP maps Λ_i .

Chapter 3

Quantum Error Correction in the non-Markovian Regime

3.1 Prior Work on non-Markovian Quantum Error Correction

Quantum error correction has recently been considered for non-Markovian noise models represented by quantum combs. In [46], the authors present the strategic code formalism. This formalism is very general, and captures dynamical codes as well as static codes. Further, the authors provide conditions that one can check to verify that the quantum error correction protocol will be able to correct errors from arbitrary quantum combs. Their formalism is strictly more general and encapsulates that which we will present in Section 3.2. By simplifying from general quantum combs to the hidden Markov model, we derive in Section 3.2 alternative operational conditions for non-Markovian quantum error correction which are much easier to check. Further, by imposing more structure on the combs, the results of Section 3.2 can be used to verify error correction in non-Markovian models for arbitrary long times, a necessary ingredient for the study of quantum metrology in Chapter 4.

Quantum error correction conditions for the HPTP map representation of non-Markovian noise have been previously explored in [1]. We will study these conditions in Section 3.3. Quantum error correction for HPTP maps has also been studied by Biswas et al. in [47]. In their work, they study QEC for HPTP maps and also study approximate QEC for HPTP maps. They give examples where the non-Markovian noise model is generated by a master

equation with time-dependent jump operators, and compare the performance of various recovery protocols.

3.2 Quantum Error Correction for the hidden Markov model

We now define quantum error-correcting codes for the hidden Markov model. As before, we assume that we have an experimentally addressable system S interacting with an inaccessible environment E , and that they are jointly affected by CPTP maps. We will generalize slightly from the definition of the hidden Markov model presented in Section 2.5, and work instead with quantum combs (Section 2.6). Instead of a Lindblad master equation governing the dynamics, we will suppose a collection of CPTP maps $\{\Lambda_i\}_{i=1}^N$, where N is the number of time steps for which we wish to protect our information. N can be infinite as well. After our first control operation $\mathcal{C}_1$ on S , Λ_1 acts on SE . In general, the i -th control operation $\mathcal{C}_i$ applied to S is followed up by Λ_i . For convenience, we will suppose the initial state is a product state. A diagram representing this setup is presented in Figure 3.1.

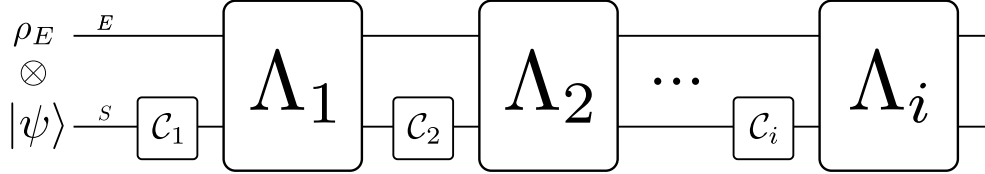


Figure 3.1: Quantum Comb for non-Markovian Quantum Error Correction

What we will demand of our quantum error correction code is that it preserves the local information on S . In other words, if our QECC encodes a logical state $|\psi\rangle$ on S , then after each round of noise we can apply a local recovery operation $\mathcal{R}$ to re-obtain $|\psi\rangle$. After the i -th round of error correction, we have a product state of the form $\rho_E^{(i)} \otimes |\psi\rangle \langle \psi|$. A diagram representing this condition is presented in Figure 3.2. We present the formal definition of a QECC with these properties in Definition 10. This definition generalizes [48, Definition 2.1] to accommodate our hidden Markov model. The rest of this Section will follow the statements and proofs of [48, 49] quite closely, with modifications made where necessary to accommodate the non-Markovianity.

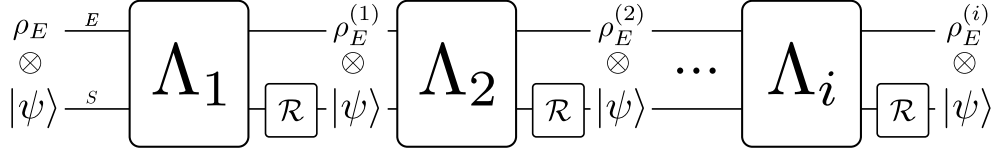


Figure 3.2: Quantum Error Correction for the Hidden Markov Model

Definition 10 (QECC for the Hidden Markov Model). *Let U be an isometry from $\mathbb{C}^k$ to $\mathcal{H}_S$, of dimension d_S , and let $\mathcal{E}$ be a set of linear operators mapping $\mathcal{H}_E \otimes \mathcal{H}_S$ to $\mathcal{H}_{E'} \otimes \mathcal{H}_{S'}$. Let $P = UU^\dagger$ be the code space projector and let $\{|i\rangle_E\}_{i=1}^{d_E}$ (resp. $\{|i\rangle_{E'}\}_{i=1}^{d_{E'}}$) be an orthonormal basis for $\mathcal{H}_E$ (resp. $\mathcal{H}_{E'}$).*

We will say the $(U, \mathcal{E})$ is a quantum error correcting code (QECC) for $\mathcal{E}$ if there exists a CPTP map $\mathcal{D} : \mathcal{B}(\mathbb{C}^{d'_S}) \rightarrow \mathcal{B}(\mathbb{C}^k)$ such that $\forall |\psi\rangle \in \mathbb{C}^k, \forall |\phi\rangle \in \mathbb{C}^{d_E}$ and $\forall E \in \mathcal{E}$,

$$(\mathcal{D} \otimes \mathbb{1}_{d'_E}) (E(U \otimes \mathbb{1}_{d_E}) |\psi\rangle \langle \psi| \otimes |\phi\rangle \langle \phi| (U^\dagger \otimes \mathbb{1}_{d_E}) E^\dagger) = c_{E,\psi,\phi} |\psi\rangle \langle \psi| \otimes \alpha_{E,\psi,\phi}, \quad (3.1)$$

where $\alpha_{E,\psi,\phi}$ is a density matrix in $\mathbb{C}^{d'_E} \times \mathbb{C}^{d'_E}$. The errors E will be called correctable.

In order to correspond this definition to the setup depicted in Figure 3.2, we take $\mathcal{E}$ to be the set of all Kraus operators for the different channels Λ_i . In other words, if Λ_i admits the Kraus representation

$$\Lambda_i(X) = \sum_{k=1}^r A_k^{(i)} X A_k^{(i)\dagger}, \quad (3.2)$$

then $\mathcal{E} = \bigcup_{i=1}^N \left\{ A_k^{(i)} \right\}_{k=1}^r$.

We can obtain an equivalent definition by taking the Stinespring dilation of the decoding operation $\mathcal{D}$, which we shall label by V . V is an isometry. Equation 3.1 becomes :

$$(V \otimes \mathbb{1}_{d'}) E(U \otimes \mathbb{1}_d) |\psi\rangle \otimes |\phi\rangle = |\psi\rangle \otimes \sqrt{c_{E,\psi,\phi}} \left| \tilde{S}_{E,\psi,\phi} \right\rangle, \quad (3.3)$$

The state $\left| \tilde{S}_{E,\psi,\phi} \right\rangle$ contains the environment degrees of freedom, as well as degrees of freedom associated with the outcome of the syndrome measurements. Let $d_{\tilde{S}}$ be the dimension of the Stinespring dilation, i.e. $\left| \tilde{S}_{E,\psi,\phi} \right\rangle \in \mathbb{C}^{d_{\tilde{S}}}$.

As for standard error correcting codes, linear combinations of correctable errors remain correctable. This is equivalent to the Markovian case [48, Theorem 2.4].

Proposition 11. *If $(U, \mathcal{E})$ is a QECC, then $(U, \text{span}(\mathcal{E}))$ is a QECC.*

Proof. We follow the proof given by [48, 49], modifying when necessary to accommodate the non-Markovian model. Let $F, E \in \mathcal{E}$. Using the definition of the QEC code (Equation 3.3), $\forall |\psi\rangle \in \mathbb{C}^k$ and $\forall |\phi\rangle \in \mathbb{C}^{d_E}$ we have that

$$(V \otimes \mathbb{1}_{d'_E})E(U \otimes \mathbb{1}_{d_E})|\psi\rangle|\phi\rangle = \sqrt{c_{E,\psi,\phi}}|\psi\rangle|\tilde{S}_{E,\psi,\phi}\rangle, \quad (3.4)$$

$$(V \otimes \mathbb{1}_{d'_E})F(U \otimes \mathbb{1}_{d_E})|\psi\rangle|\phi\rangle = \sqrt{c_{F,\psi,\phi}}|\psi\rangle|\tilde{S}_{F,\psi,\phi}\rangle. \quad (3.5)$$

Using the linearity of the operators, one obtains :

$$(V \otimes \mathbb{1}_{d'_E})(aE + bF)(U \otimes \mathbb{1}_{d_E})|\psi\rangle|\phi\rangle = |\psi\rangle \left(a\sqrt{c_{E,\psi,\phi}}|\tilde{S}_{E,\psi,\phi}\rangle + b\sqrt{c_{F,\psi,\phi}}|\tilde{S}_{F,\psi,\phi}\rangle \right) \quad (3.6)$$

and so $aE + bF$, with $a, b \in \mathbb{R}$, is also a correctable error. □

In principle, it can be very difficult to check when Equation 3.1 holds. In their seminal paper [6], Knill and Laflamme derived equivalent conditions, known as the Knill-Laflamme (KL) conditions, that are often easier to verify. We now generalize those conditions to the hidden Markov model.

Theorem 12 (KL Conditions for the HMM). *Let U be an isometry from $\mathbb{C}^k$ to $\mathcal{H}_S$, of dimension d_S , and let $\mathcal{E}$ be a set of linear operators mapping $\mathcal{H}_E \otimes \mathcal{H}_S$ to $\mathcal{H}_{E'} \otimes \mathcal{H}_{S'}$. Let $P = UU^\dagger$ be the code space projector and let $\{|i\rangle_E\}_{i=1}^{d_E}$ (resp. $\{|i\rangle_{E'}\}_{i=1}^{d'_{E'}}$) be an orthonormal basis for $\mathcal{H}_E$ (resp. $\mathcal{H}_{E'}$). We have that $(U, \mathcal{E})$ is a quantum error correcting code (Definition 10) if and only if we have*

$$P \langle l|_E E_i^\dagger |m\rangle_{E'} \langle n|_{E'} E_j |k\rangle_E P = c_{ij}^{lmnk} P, \quad (3.7)$$

for all $E_i, E_j \in \mathcal{E}$, where c_{ij}^{lmnk} are scalar constants.

Proof. We first provide a proof that the conditions are necessary. Let $|\psi\rangle, |\phi\rangle$ be linearly independent states of $\mathbb{C}^k$ and let E_i, E_j be errors in $\mathcal{E}$. Let $|\chi_1\rangle, |\chi_2\rangle \in \mathbb{C}^{d_E}$ be orthogonal states. Assuming a QECC, Equation 3.3 tells us

$$(V \otimes \mathbb{1}_{d'_E})E_i(U \otimes \mathbb{1}_{d_E})|\psi\rangle|\chi_1\rangle = \sqrt{c_{E_i,\psi,\chi_1}}|\psi\rangle|\tilde{S}_{E_i,\psi,\chi_1}\rangle, \quad (3.8)$$

$$(V \otimes \mathbb{1}_{d'}) E_j (U \otimes \mathbb{1}_d) |\phi\rangle |\chi_2\rangle = \sqrt{c_{E_j, \phi, \chi_2}} |\phi\rangle \left| \tilde{S}_{E_j, \psi, \chi_2} \right\rangle. \quad (3.9)$$

Let

$$|\eta\rangle = \alpha |\psi\rangle + \beta |\psi^\perp\rangle \quad (3.10)$$

with $|\psi^\perp\rangle$ orthogonal to $|\psi\rangle$. We have

$$(V \otimes \mathbb{1}_{d'_E}) E_i (U \otimes \mathbb{1}_{d_E}) |\eta\rangle |\chi_1\rangle = \sqrt{c_{E_i, \eta, \chi_1}} |\eta\rangle \left| \tilde{S}_{E_i, \eta, \chi_1} \right\rangle. \quad (3.11)$$

First projecting with $|\psi\rangle \langle\psi|$ on Equations 3.8 and 3.11, then dividing the second equation by α , the LHS of the equations become equal. Equating them, we obtain

$$\sqrt{c_{E_i, \psi, \chi_1}} \left| \tilde{S}_{E_i, \psi, \chi_1} \right\rangle = \sqrt{c_{E_i, \eta, \chi_1}} \left| \tilde{S}_{E_i, \eta, \chi_1} \right\rangle \quad \forall \eta. \quad (3.12)$$

From this equation, we conclude that the syndrome and residual environment state don't depend on the choice of $|\psi\rangle$;

$$c_{E_i, \psi, \chi_1} \equiv c_{E_i, \chi_1}. \quad (3.13)$$

We take the inner product between Equation 3.8 and Equation 3.11 with $|\xi_l\rangle_S \langle\xi_k|_S$ sandwiched between them, where $|\xi_l\rangle_S, |\xi_k\rangle_S$ are orthogonal basis elements of $\mathbb{C}^{d_S}$. Relabeling χ_1 to χ_m and χ_2 to χ_n , we obtain

$$\begin{aligned} & \langle\phi| \langle\chi_m| (U^\dagger \otimes \mathbb{1}_{d_E}) E_j^\dagger (V^\dagger \otimes \mathbb{1}_{d'_E}) \cdot (\mathbb{1}_k \otimes |\xi_l\rangle_S \langle\xi_k|_S) (V \otimes \mathbb{1}_{d'_E}) E_i (U \otimes \mathbb{1}_{d_E}) |\psi\rangle |\chi_n\rangle \\ &= \sqrt{c_{E_i, \chi_n} c_{E_j, \chi_m}} \left\langle \tilde{S}_{E_j, \chi_m} \right| \xi_l \rangle \left\langle \xi_k \right| \tilde{S}_{E_i, \chi_n} \rangle \langle\phi| \psi\rangle. \end{aligned} \quad (3.14)$$

By defining

$$c_{ij}^{mlk} = \sqrt{c_{E_i, \chi_m} c_{E_j, \chi_n}} \left\langle \tilde{S}_{E_j, \chi_n} \right| \xi_l \rangle \times \left\langle \xi_k \right| \tilde{S}_{E_i, \chi_m} \rangle \langle\phi| \psi\rangle, \quad (3.15)$$

the equation is reduced to

$$P \langle\chi_m| E_i^\dagger |\chi_l\rangle \langle\chi_k| E_j |\chi_n\rangle P = c_{ij}^{mlk} P, \quad (3.16)$$

which is Equation 3.7. This completes the proof that the conditions are necessary.

We now prove that the conditions are sufficient. Let

$$\tilde{E}_{i,n,k} = \langle n|_{E'} E_i |k\rangle_E. \quad (3.17)$$

The KL conditions for the hidden Markov model are equivalent to the standard KL conditions for the extended error set $\{\tilde{E}_{i,n,k}\}$ indexed by i, n, k . These extended errors act on the accessible system S only.

Let $|\chi\rangle \in \mathbb{C}^{d_E}$ be a state of the environment and $|\psi\rangle \in \mathbb{C}^{d_s}$ be a state in the code space. Expanding $|\chi\rangle$ over an orthonormal basis $\{|l\rangle\}_{l=1}^{d_E}$ of $\mathcal{H}_E$, we obtain

$$|\chi\rangle = \sum_{l=0}^{d_E} c_l |l\rangle_E. \quad (3.18)$$

Let $E_i \in \mathcal{E}$ be an error. We have that

$$\begin{aligned} E_i |\psi\rangle |\chi\rangle &= \left(\sum_{n=0}^{d'_E} \sum_{k=0}^{d_E} \langle n|_{E'} E_i |k\rangle_E \otimes |n\rangle_{E'} \langle k|_E \right) \left(|\psi\rangle \otimes \sum_{l=0}^{d_E} c_l |l\rangle_E \right) \\ &= \sum_{n=0}^{d'_E} \left[\left(\sum_{k=0}^{d_E} c_k \tilde{E}_{i,n,k} |\psi\rangle \right) \otimes |n\rangle_{E'} \right]. \end{aligned} \quad (3.19)$$

If the non-Markovian KL conditions hold, then the standard KL conditions hold for $\tilde{E}_{i,n,k}$, and so there exists a CPTP map $\mathcal{D}$ that acts solely on the probe such that

$$\begin{aligned} (D \otimes \mathbb{1}_{d'_E}) &\left(\sum_{n,n'=0}^{d'_E} \left[\left(\sum_{k,k'=0}^{d_E} c_k c_{k'}^* \tilde{E}_{i,n,k} |\psi\rangle \langle \psi| \tilde{E}_{i,n',k'}^\dagger \right) \right. \right. \\ &\left. \left. \otimes |n\rangle_{E'} \langle n'|_{E'} \right] \right) = |\psi\rangle \langle \psi| \otimes \sum_{n,n'=0}^{d'_E} \alpha_{nn'} |n\rangle_{E'} \langle n'|_{E'} \end{aligned} \quad (3.20)$$

where $\alpha_{nn'}$ are the coefficients of the residual non-normalized state on the environment degrees of freedom. This Equation is true due to the fact that $\mathcal{D}$ map can correct for linear combinations of $\tilde{E}_{i,n,k}$. Composing with a partial trace map of the environment, we obtain Equation 3.1 defining a QECC.

□

Using these generalized KL conditions, one can check that a code is able to correct all the errors that can be induced on the system by operators E with an environment state ρ_E . It is important to note that this generalization of a quantum error-correcting code is perhaps too strong. Consider Figure 3.2. Our definition of a QEC code guarantees that $|\psi\rangle$ will be preserved for any possible ρ_E . In practice, however, we are not likely to encounter any ρ_E . We need only correct for the extended errors $\tilde{E}_{i,n,k}$ that occur during the evolution for a given initial state. In principle, it is possible for the set of these errors as well as their span to be smaller than the set of errors for *any* initial states.

Verifying which errors occur and which do not during the time evolution is a hard problem. It requires computing information about the trajectories of the evolution for arbitrary control operations on the accessible system. For this reason, we will restrict ourselves to the simpler case of considering arbitrary environment states. This is also the main difference between our formalism and the strategic code formalism of [46]. This simplification will lead to sufficient but not necessary conditions when deriving metrological scaling limits in Chapter 4.

3.2.1 A Small Example

We'll now consider an example of quantum error correction in a spin system to highlight the differences between the Markovian and non-Markovian cases. We'll consider a system where $\mathcal{H}_P$ consists of an ensemble of individually addressable spin-half degrees of freedom (or qubits), and the environment $\mathcal{H}_E$ consists of inaccessible spin-1 degrees of freedom. For simplicity, we'll assume each spin-half particle interacts with a few spin-1 particles, where the interaction Hamiltonian is given by

$$H_{PE} = S_z \otimes Z + S_x \otimes X, \quad (3.21)$$

and where Z, X are the Pauli matrices acting on the spin-half particle and S_X, S_z are the spin-1 operators

$$S_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \quad (3.22)$$

Let $|-1\rangle_E, |0\rangle_E$, and $|1\rangle_E$ be the $-1, 0$, and 1 eigenstates of S_z respectively.

We first consider the Markovian regime. Following the prescription of [26], we make two approximations. We first suppose that the spin-half degrees of freedom couple weakly to the bath of environmental spins, such that the bath is weakly affected by the interaction. We also assume that the environmental spins decohere to the $|0\rangle_E$ state much faster than the spin-half degrees of freedom. A Markovian master equation for the dynamics can then be derived as follows.

We'll consider N environmental spin-1 degrees of freedom. We'll consider a single spin-half, as we'll assume the noise on each spin-half is uncorrelated. The total Hamiltonian of the system can be written as

$$H = \sum_{i=1}^N H_{E,i} + \sum_{i=1}^N \tau_i H_{PE,i} \quad (3.23)$$

where

$$H_{PE,i} = S_x^{(i)} \otimes X + S_z^{(i)} \otimes Z. \quad (3.24)$$

Here, the superscript is used to index the spins the operator acts on. We'll neglect any terms of the Hamiltonian acting solely on the spin-half degree of freedom, as they can always be offset by our fast and accurate control. This spin-1 Hamiltonian is inspired by the Hamiltonian of nitrogen vacancy centers in diamond [50, 51]. The full interaction Hamiltonian can be written

$$H_I = \sum_{i=1}^N \sum_{A \in \{X,Z\}} \tau_i S_A^{(i)} \otimes A. \quad (3.25)$$

We set

$$H_{E,i} = \Delta (S_z^{(i)})^2 + \alpha S_z^{(i)}, \quad (3.26)$$

where Δ is the zero-field splitting and α is the Zeeman splitting. τ_i is the strength of the coupling between environmental spin i and the spin-half, which we assume to be small. If $\Delta \gg \alpha$, then $|0\rangle_E$ is the ground state of this Hamiltonian. Following the prescription of [37], we can obtain a Markovian master equation acting on the spin-half by making the Born-Markov approximation. In this approximation, we assume that the state can be written as

$$\rho(t) \approx \rho_P(t) \otimes (|0\rangle_E \langle 0|_E)^{\otimes N}. \quad (3.27)$$

In other words, the environment spins are approximated to always be in the state $(|0\rangle_E \langle 0|_E)^{\otimes N}$ due to their weak coupling and fast decoherence. The master equation is given by Eq. (3.143) of [37].

$$\frac{d\rho_P}{dt} = \sum_{A,Q \in \{X,Z\}} \gamma_{AQ} \left(A \rho_P Q - \frac{1}{2} \{QA, \rho_P\} \right). \quad (3.28)$$

We have neglected the Lamb shift Hamiltonian as it can be offset by our controls, leaving only the dissipative terms. The decay rates γ_{AQ} are given by Eq. (3.139) of [26]

$$\gamma_{AQ} = \sum_{i,j=1}^N \int_{\mathbb{R}} dt e^{i\omega t} \text{Tr} \left[e^{itH_{E,i}} S_A^{(i)} e^{-itH_{E,i}} S_Q^{(j)} (|0\rangle_E \langle 0|_E)^{\otimes N} \right]. \quad (3.29)$$

In particular, we see that $\gamma_{AQ} = 0$ if A or Q is equal to Z . And so, Equation 3.27 reduces to

$$\frac{d\rho_P}{dt} = \gamma_{XX} (X \rho_P X - \rho). \quad (3.30)$$

We recognize the equation for dephasing along the X axis. The corresponding channel has Kraus operators proportional to $\mathbb{1}$ and X .

This master equation generates a noise channel with Kraus operators proportional to $\mathbb{1}$ and X . Therefore, A repetition code [52, 53] that only corrects X errors would be sufficient to protect the logical information.

We next consider the non-Markovian regime. In this regime, we suppose the spin-half degrees of freedom interact with few coherent spin-1 degrees of freedom. For simplicity, we suppose that there is no dissipation. In the non-Markovian regime, since the environment state evolves in time, we must consider the set of extended errors

$$K_{i,j} = \langle i|_E e^{-itH_{PE}} |j\rangle. \quad (3.31)$$

Since we consider arbitrarily fast control, we can consider the errors to first order in t . We obtain the operators

$$K_{0,0} = \mathbb{1}, \quad K_{j,j} = \mathbb{1} - ijtZ, \quad (3.32)$$

$$K_{-1,0} = K_{0,-1} = K_{1,0} = K_{0,1} = \frac{-it}{\sqrt{2}}X. \quad (3.33)$$

We see that to protect the logical information, a repetition code no longer suffices due to the presence of Z errors. Because of the additional error operators introduced by the non-Markovianity, a more capable code is required.

3.2.2 Computation versus Metrology

Currently, the vast majority of quantum error correction research focuses on stabilizer codes. For qubits, stabilizer codes have three important parameters: the number of physical qubits n , the number of logical qubits k encoded in the physical qubits, and finally the distance d . A distance d stabilizer code can correct error operators which can be written as products of $\lfloor (d-1)/2 \rfloor$ arbitrary single qubit errors [48].

As previously shown, the non-Markovian quantum error correction conditions for the hidden Markov model can be rephrased as the standard Knill-Laflamme conditions acting on an extended error set. The non-Markovianity will therefore be problematic for quantum error correction if the memory effects in the system cause the errors to be of high weight. When this is not the case and when the non-Markovian noise is also local, it has been shown that a fault-tolerance threshold is still possible [54, 55]. Consider the previous example of each qubit interacting with a spin-1 particle. This model is local, and so the errors are

independent between qubits. The only difference due to the non-Markovianity is the form of the errors, which can be either X or Z operators rather than just X .

In certain systems mentioned above, this is unlikely to be a problem. For example, in superconducting qubits, one expects the interaction with the two-level chip defects to be relatively local. Therefore, even though the memory and coherence of the defect may change the form of the error, it will not affect the weight of the error. Major sources of error in superconducting experiments have instead been crosstalk or high-weight errors due to cosmic rays [56, 57].

For quantum sensing and metrology, however, non-Markovianity has the potential to be much more destructive. Consider the previous example, where we wish to use the spin-1/2 particles to detect a magnetic field. Suppose the magnetic field acts independently on each spin as a Z operator. In this case, we do not want our code to detect and correct single qubit Z errors, as then our signal would be discarded. Our code will not have a distance, and low-weight errors with the same form as the signal will be detrimental. This will be explored in detail in the next section.

3.3 Quantum Error Correction for SPTP Maps

In the previous subsection, quantum error correction conditions for the hidden Markov model were derived (Theorem 12). As seen in Section 2.4.6, a fixed hidden Markov model can correspond to many different SPTP maps depending on the choice of controls. Likewise, a fixed SPTP map can correspond to the evolution of many different hidden Markov models for a particular choice of control operations. It is therefore interesting to derive quantum error correction conditions for SPTP maps, to complement the conditions derived for hidden Markov models. It turns out the standard Knill-Laflamme conditions of [6] can be applied to correct the errors of SPTP maps, as demonstrated in Theorem 6 of Cao et al. [1]. We repeat here the theorem as well as its proof.

Theorem 13 (Knill-Laflamme conditions for non-CP maps). *Let U be an isometry from $\mathbb{C}^k$ to $\mathcal{H}_S$, of dimension d_S , and let $\mathcal{E}$ be an SPTP map with operator sum representation*

$$\mathcal{E}(\rho) = \sum_{k=1}^r \sigma(k) E_k \rho E_k^\dagger. \quad (3.34)$$

If the following conditions hold

$$P E_i E_j^\dagger P = \alpha_{ij} P, \quad (3.35)$$

where $i, j \in \{1, 2, \dots, r\}$ and α_{ij} is a Hermitian matrix, then there exists a CPTP map $\mathcal{R}$ such that

$$U^\dagger (\mathcal{R} \circ \mathcal{E} (U |\psi\rangle \langle\psi| U^\dagger)) U = |\psi\rangle \langle\psi| \quad (3.36)$$

for all $|\psi\rangle \in \mathbb{C}^k$. In other words, there exists a recovery channel $\mathcal{R}$ that corrects the errors of $\mathcal{E}$.

Proof. The key to the proof is to consider the CPTP map $\mathcal{E}'$ defined by

$$\mathcal{E}'(\rho) = \sum_{k=1}^r E_k \rho E_k^\dagger. \quad (3.37)$$

Equation 3.35 can then be interpreted as the standard KL conditions for the map $\mathcal{E}'$, and a recovery operation can be constructed [6, 48, 49]. For this proof, we closely follow the construction of [38, Theorem 10.1]. In particular, the matrix α_{ij} can be diagonalized, introducing

$$F_k = \sum_i u_{ik} E_i, \quad (3.38)$$

such that

$$P F_k^\dagger F_l P = \delta_{kl} d_{kk} P. \quad (3.39)$$

We can compute the polar decomposition of F_k ,

$$F_k P = \sqrt{d_{kk}} U_k P, \quad (3.40)$$

such that [38, Equation 10.20-10.23]

$$U_k^\dagger P_k F_l \sqrt{\rho} = \delta_{kl} \sqrt{d_{kk}} \sqrt{\rho}. \quad (3.41)$$

The recovery operations are given by

$$\mathcal{R}(\cdot) = \sum_k U_k^\dagger P_k \cdot P_k U_k. \quad (3.42)$$

Let $\rho = |\bar{\psi}\rangle \langle\bar{\psi}|$, where $|\bar{\psi}\rangle = U |\psi\rangle$. We now compute the action of the recovery operation

on $\mathcal{E}(\rho)$.

$$(\mathcal{R} \circ \mathcal{E})(\rho) = \sum_{k=1}^r \sum_{j=1}^r \sigma(j) U_k^\dagger P_k E_j \rho E_j^\dagger P_k U_k \quad (3.43)$$

$$= \sum_{k=1}^r \sum_{j=1}^r \sum_{i=1}^r \sigma(j) u_{ij}^* U_k^\dagger P_k F_i \rho F_i^\dagger P_k U_k \quad (3.44)$$

$$= \sum_{k=1}^r \sum_{j=1}^r \sum_{i=1}^r \sigma(j) u_{ij}^* \delta_{ki} d_k k \rho \quad (3.45)$$

$$= \left(\sum_{k=1}^r \sum_{j=1}^r \sum_{i=1}^r \sigma(j) u_{ij}^* \delta_{ki} d_k k \right) \rho \propto \rho. \quad (3.46)$$

Since both the recovery operation and the SPTP map are trace preserving, and $\text{Tr}(\rho) = 1$, we deduce that the proportionality constant of Equation 3.46 is 1, completing the proof. $\square$

Chapter 4

Error Corrected non-Markovian Quantum Metrology

In this chapter, we present an application of quantum error-correcting codes for the hidden Markov model in the fields of quantum sensing and quantum metrology. First, we'll review the state of the art on quantum error correction for quantum metrology, before generalizing to the non-Markovian case using our hidden Markov model. Further, conditions for achieving the various scaling limits in the non-Markovian regime will be presented, helping to chart the landscape of non-Markovian quantum sensing.

This section will mainly cover the contents of the paper *Quantum Error Corrected non-Markovian Metrology* [2], written by the author of this thesis in collaboration with Ningping Cao, Raymond Laflamme, and Sisi Zhou.

4.1 Background

4.1.1 Prior Work on Quantum Error Correction for Quantum Metrology

One proposal for achieving the Heisenberg limit despite the presence of noise is to use quantum error correction. Quantum error correction for quantum metrology was first proposed by Kessler et al. in 2014 [8]. There have since been many works studying quantum error correction for metrology, including but not limited to [58, 8, 59, 18, 23, 60, 61, 62]. In particular, a quantum error correction protocol for quantum sensing of magnetic fields

using NV centers in diamond has even been performed in experiment [12]. In [13, 14], the authors derive sufficient and necessary conditions for when quantum error correction can be used to obtain Heisenberg limit precision scaling despite the presence of noise. However, their HNLS conditions only apply to the case of Markovian noise. In this chapter, our goal will be to generalize these conditions to the non-Markovian regime and decide for which non-Markovian noise models quantum error correction can be used to achieve the Heisenberg limit.

4.1.2 Prior Work on non-Markovian Quantum Metrology

Non-Markovian metrology has been studied in many different contexts. It has been studied with non-Markovian bosonic environments [63, 64], where the non-Markovianity and the ability to perform fast measurement allow QFI scaling as $N^{3/2}$, called the Zeno limit. Further, in [65, 66], the authors studied Ramsey spectroscopy protocols for quantum sensing under non-Markovian noise. In [67], the authors study non-Markovian magnetic field sensing and show the SQL can be surpassed. In [68], quantum sensing is studied for non-equilibrium environments with non-Markovian behavior.

There have also been a few works with similar models to our hidden Markov model. In particular, in [69, 70, 71, 72], the authors consider noise and signal modeled by quantum combs and optimize the quantum Fisher information, or derive bounds on the quantum Fisher information. In our work, we simplify the model from a general quantum comb to the hidden Markov model presented in Section 2.5, allowing us to derive asymptotic scaling bounds with operational conditions which are much simpler to check.

Non-Markovian noise is an important source of decoherence in many experimental quantum sensing platforms. In superconducting circuits, a major source of decoherence is coupling between qubits and two-level quantum states in the environment. These two level systems are coherent with non-negligible coherence times [73, 74, 75], allowing them to serve as a memory mediating temporal correlations in the noise. Further, in many practical cases, few two-level systems are required to accurately model the noise [76, 77, 78]. Some theorize that these two-level systems are responsible for the $1/f$ spectral profile of the noise present across many different solid-state platforms [79, 80].

In nitrogen vacancy centers in diamond, for example, a major source of decoherence is coupling between the electron of the center and the surrounding carbon or nitrogen nuclear spins. The surrounding spins are coherent [81], allowing them to serve as a memory. The noise in nitrogen vacancy center experiments can also be accurately modeled using a relatively small number of these environmental spins [82, 83].

4.2 Metrology in the Hidden Markov Model

We will once again consider the hidden Markov model of Section 2.5, with some slight modifications to better fit the context of quantum sensing. In particular, we add a Hamiltonian to generate the signal we wish to sense.

We consider a quantum system consisting of a probe and inaccessible degrees of freedom, which we call the environment. Let $\mathcal{H}_P$ denote the d_P -dimensional Hilbert space of the probe, and $\mathcal{H}_E$ represent the d_E -dimensional Hilbert space of the environment. Further, we suppose that we have access to noiseless auxiliary degrees of freedom. Let $\mathcal{H}_A$ denote the Hilbert space of the auxiliary system. The total state of the quantum system is denoted by ρ , where $\rho \in \mathcal{H}_E \otimes \mathcal{H}_P \otimes \mathcal{H}_A$. In this work, we will consider the case of finite-dimensional systems $d_E, d_P < \infty$. The state of the probe and auxiliary system is given by the reduced density matrix $\rho_{PA} = \text{Tr}_E(\rho)$, obtained by tracing out the environment. We suppose the state ρ evolves in time according to a Lindblad master equation of the form :

$$\frac{d\rho}{dt} = -i[H_{PE}, \rho] - i[\omega\tilde{G}, \rho] - i[\omega\tilde{H}_E, \rho] + \sum_{k=1}^r \left(L_k \rho L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_k, \rho\} \right). \quad (4.1)$$

We suppose $\tilde{G}$ is of the form $\mathbb{1}_E \otimes G \otimes \mathbb{1}_A$, in other words, G only acts on the probe degrees of freedom. Similarly, we suppose $\tilde{H}_E$ is of the form $H_E \otimes \mathbb{1}_P \otimes \mathbb{1}_A$, and only acts on the environment. Further, we assume G is non-trivial, i.e., G is not proportional to $\mathbb{1}_P$. We have that G , H_E , and H_{PE} are Hermitian operators. The r different L_k operators are called the jump operators, also known as Lindblad operators. H_{PE} and the L_k act solely on the probe and environment degrees of freedom, acting trivially on the auxiliary system.

The real parameter ω is what we wish to estimate. Both $\omega\tilde{G}$ and ωG will be referred to as the signal Hamiltonian acting on the probe, or simply the signal on the probe. The Hamiltonian ωH_E corresponds to the evolution of the environment dependent on the parameter ω , or the signal on the environment. In this model, we suppose that the signal acts independently on probe and environment degrees of freedom.

The H_{PE} and L_k terms correspond to the undesired noise affecting the system. The effect of H_{PE} is referred to as the unitary coupling. The jump operators correspond to the dissipative portion of the noise. These operators only act non-trivially on the environment and probe degrees of freedom, and act trivially on the auxiliary degrees of freedom. In this model, we suppose that the noise does not depend on the signal parameter ω .

Finally, we assume access to an unlimited supply of noiseless auxiliary degrees of freedom, enabling the generation of perfect copies of any desired probe state. In experiment,

this assumption has been well approximated by considering an electron state in a nitrogen vacancy sensor as the probe, and by using addressable ^{13}C carbon nuclear spins, which are more robust to noise and have longer coherence times, as the auxiliary system [12].

Additionally, we assume that all control operations on both the probes and auxiliary system can be performed with arbitrary speed and precision. For error correction to be feasible, the coherence times of our probe must greatly exceed the duration of control operations. As for the non-Markovianity, it has been shown that for certain superconducting systems interacting with environmental two-level states, the coherence times of the two-level states are of the same order as those of the qubit [84, 74, 85, 73, 75]. This would allow for non-Markovian effects to develop across many rounds of control operations.

As seen previously in Section 2.5, the model is Markovian when considering all degrees of freedom, but when we restrict ourselves to $\mathcal{H}_P \otimes \mathcal{H}_A$ it is non-Markovian. In particular, the model exhibits information backflow of the quantum Fisher information (See Section 2.3).

The parameter we wish to estimate is the amplitude ω of the signal Hamiltonian. We will consider the three following timescales :

1. The first time scale is the number of *shots*, which we will denote by M . This corresponds to the M of Equation 1.8. A shot consists of preparing probe, auxiliary system, and environment with a fixed initial state and then acting with a sequence of controls. The controls are the same for each shot, and in between shots, the environment is completely reset.
2. The second time scale is the number of *calls to the signal Hamiltonian*, which we denote by N . This timescale is somewhat arbitrary; however, we define it for convenience. It corresponds to the number of times we allow our probe to interact with the signal Hamiltonian before, for example, swapping it out for a fresh quantum state or destructively measuring it.
3. The final time scale is the *exposure time to the Hamiltonian*, which we label t . For each call to the signal Hamiltonian, we allow probe and environment to jointly evolve for a duration t before moving on to the next call.

We define the total shot time as $T := Nt$. The total experiment time is given by MT . The setup and the various time scales are presented in Figure 4.1. We will investigate Heisenberg Limit scaling of the shot time T .

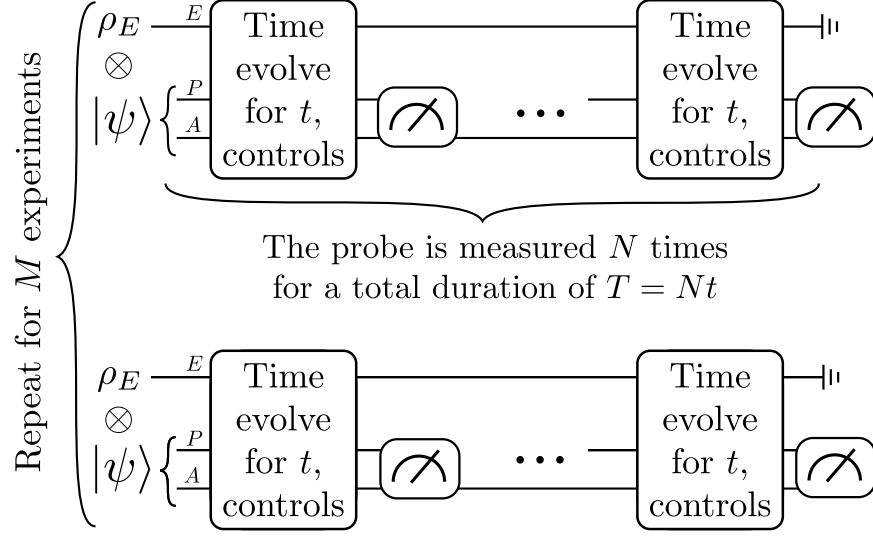


Figure 4.1: Subsystems and time scales for quantum metrology in the hidden Markov model.

4.3 Achieving Heisenberg limited precision in the Hidden Markov Model

4.3.1 The “Hamiltonian not in Extended Span” Conditions for Achieving the Heisenberg Limit

In [13], the authors present sufficient and necessary conditions, called “Hamiltonian not in Lindblad span” (HNLS), for achieving the HL asymptotically. When these conditions are met, the probe and a noiseless auxiliary system can be encoded into a specialized error-correcting code. For this code, all of the noise becomes correctable errors, and a portion of the signal remains as a logical operation. This logical component of the signal leads to HL scaling within the codespace. However, the HNLS conditions were originally derived for Markovian noise, where no environmental degrees of freedom are present, and the master equation applies solely to the probe. Further, these conditions are necessary to achieve the HL, as non-correctable errors inevitably lead to complete decoherence of the system and the signal over long timescales.

⁴ We generalize these conditions to the HMM. The HNLS conditions were derived under

the assumption that error correction can be performed perfectly and arbitrarily fast. We will keep the same assumptions.

Theorem 14 (Hamiltonian not in extended span (HNES) Conditions). *Let $\{|\phi_i\rangle\}_{i=1}^{d_E}$ be an orthonormal basis for $\mathcal{H}_E$. Let $\mathcal{S}$ be the linear span of the operators $\mathbb{1}_P$, $\langle\phi_i|H_{PE}|\phi_m\rangle$, $\langle\phi_i|L_k|\phi_m\rangle$, $\langle\phi_i|L_k^\dagger|\phi_m\rangle$ and $\langle\phi_i|L_k^\dagger|\phi_m\rangle\langle\phi_n|L_j|\phi_l\rangle$ ($\forall k, j \in \{1, \dots, r\}$, where r is the number of jump operators; $i, m, n, l \in \{1, \dots, d_E\}$).*

If $G \notin \mathcal{S}$, then HL scaling can be achieved for the parameter ω . The condition $G \notin \mathcal{S}$ will be referred to as the “Hamiltonian not in extended span” (HNES) conditions. The extended span refers to the possible errors that can affect the reduced state of the probe for arbitrary choices of environment state.

Proof. The proof generalizes the HNLS conditions for Heisenberg-limit scaling from [13], with modifications to accommodate the HMM. It will therefore be very similar to the original proof, with some slight changes. The evolution of the state ρ for a duration Δt is given by the first-order expansion of the master equation

$$\begin{aligned} \Phi_{\Delta t}(\rho) = & \rho + \left(-i[H_{PE}, \rho] - i[\omega\tilde{G}, \rho] - i[\omega\tilde{H}_E, \rho] \right) \Delta t \\ & + \Delta t \sum_{k=1}^r \left(L_k \rho L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_k, \rho\} \right) + O(\Delta t^2). \end{aligned} \quad (4.2)$$

To reduce clutter, we use the following notation

$$A^{(i,j)} := \langle\phi_i|A|\phi_j\rangle, \quad (4.3)$$

where A is a linear operator on $\mathcal{H}_E \otimes \mathcal{H}_P$. This leaves us with an operator acting on P . Let $\mathcal{S}_H$ denote the subspace of Hermitian matrices in $\mathcal{S}$, which is spanned by $L_k^{(i,j)} + L_k^{(i,j)\dagger}$, $i(L_k^{(i,j)} - L_k^{(i,j)\dagger})$, $(L_k^\dagger L_m)^{(i,j)} + h.c.$, $i((L_k^\dagger L_m)^{(i,j)} - h.c.)$, $i(H_{PE}^{(i,j)} - H_{PE}^{(i,j)\dagger})$, and $H_{PE}^{(i,j)} + h.c.$. Remark that $\mathcal{S}_H \subseteq \mathcal{S}$. G is Hermitian and can therefore be decomposed into $G = G_{\parallel} + G_{\perp}$ where $G_{\parallel} \in \mathcal{S}_H$ and $G_{\perp} \perp \mathcal{S}_H$. The assumption that $G \notin \mathcal{S}$ implies that $G_{\perp}$ is nonzero. $G_{\perp}$ must have trace zero in order to be orthogonal to $\mathbb{1}_P$ under the trace inner product. The spectral decomposition for $G_{\perp}$ is therefore

$$G_{\perp} = \frac{1}{2} (\text{tr}|G_{\perp}|) (\rho_0 - \rho_1) \quad (4.4)$$

where $|G_{\perp}| := \sqrt{G_{\perp}^2}$, and ρ_0 and ρ_1 are density matrices. We consider $|C_0\rangle$ and $|C_1\rangle$ to be normalized purifications of ρ_0 and ρ_1 on $\mathcal{H}_P \otimes \mathcal{H}_A$ where $\mathcal{H}_A$ consists of auxiliary

degrees of freedom that we append to our system. We further construct $|C_0\rangle$ and $|C_1\rangle$ to have orthonormal supports on $\mathcal{H}_A$. To be explicit, assuming $\rho_0 = \sum_{i=1}^{d_1} (\rho_0)_{ii} |i\rangle_P \langle i|_P$ and $\rho_1 = \sum_{i=d_1+1}^{d_P} (\rho_1)_{ii} |i\rangle_P \langle i|_P$ are in the diagonalized forms,

$$\begin{aligned} |C_0\rangle &= \sum_{i=1}^{d_1} \sqrt{(\rho_0)_{ii}} |i\rangle_P |i\rangle_A, \\ |C_1\rangle &= \sum_{i=d_1+1}^{d_P} \sqrt{(\rho_1)_{ii}} |i\rangle_P |i\rangle_A, \end{aligned}$$

where $\{|i\rangle_P\}$ and $\{|i\rangle_A\}$ are orthonormal bases of the probe and the auxiliary system, respectively, where the auxiliary system is assumed to be as large as the probe. The support of ρ_0 and ρ_1 are $\text{span}\{|i\rangle_P\}_{i=1}^{d_1}$ and $\text{span}\{|i\rangle_P\}_{i=d_1+1}^{d_P}$ respectively. This will be the code space, with the code space projector given by

$$P = |C_0\rangle \langle C_0| + |C_1\rangle \langle C_1|. \quad (4.5)$$

The orthogonality of the states on the auxiliary system space implies that

$$\langle C_0| B \otimes \mathbb{1}_A |C_1\rangle = \langle C_1| B \otimes \mathbb{1}_A |C_0\rangle = 0 \quad (4.6)$$

where B is any linear operator on $\mathcal{H}_P$. We also use the relation :

$$\text{tr}((|C_0\rangle \langle C_0| - |C_1\rangle \langle C_1|)(B \otimes \mathbb{1}_A)) = \frac{2 \text{tr}(G_\perp B)}{\text{tr}|G_\perp|}. \quad (4.7)$$

Although a smaller auxiliary system, such as a qubit, is enough to guarantee Equation 4.6, it would not be enough to guarantee Equation 4.7, which requires d_A dimensions.

Equation 4.7 is zero for any $B \in \mathcal{S}$, allowing us to write

$$\langle C_0| B \otimes \mathbb{1}_A |C_0\rangle = \langle C_1| B \otimes \mathbb{1}_A |C_1\rangle. \quad (4.8)$$

These equations imply that

$$P L_k^{(i,j)} P = \lambda_k^{ij} P, \quad (4.9)$$

$$P \left(L_k^\dagger L_m \right)^{(i,j)} P = \mu_{km}^{ij} P, \quad (4.10)$$

where this equation is due to the fact that $\left(L_k^\dagger L_m \right)^{(i,j)}$ is in the linear span of $L_k^{(i,j)\dagger} L_{k'}^{(i',j')}$. By assumption on the span $\mathcal{S}$, we also have that

$$P L_k^{\dagger(i,j)} L_m^{(i',j')} P = \zeta_{km}^{ij i' j'} P. \quad (4.11)$$

Further,

$$PH_{PE}^{(i,j)}P = \chi^{ij}P. \quad (4.12)$$

We see from Equation 4.7 that

$$PGP \neq \text{const.} \times P. \quad (4.13)$$

Let $\rho = \rho_E \otimes \rho_{PA}$ be a state in the code space, i.e. $P\rho_{PA}P = \rho_{PA}$. Let $\rho_E = \sum_{i=0}^{d_E} \alpha_i |\phi_i\rangle \langle \phi_i|$. We can take ρ_E to be diagonal without loss of generality since the choice of basis $\{|\phi_i\rangle\}_{i=1}^{d_E}$ is arbitrary. We will use P and $\mathbb{1}_E \otimes P$ interchangeably; it will be clear from the context which operator we are referring to.

$$\begin{aligned} P\Phi_{\Delta t}(\rho)P &= \rho - i\omega \left[P\tilde{G}P, \rho \right] \Delta t - i\omega [H_E, \rho_E] \otimes \rho_{PA} \Delta t \\ &\quad + \left(\sum_i \alpha_i \sum_{k=1}^r A_{ik} \otimes \rho_{PA} \right) \Delta t + O(\Delta t^2), \end{aligned} \quad (4.14)$$

where

$$A_{ik} := \sum_{i',j'} \lambda_k^{i'i} \left(\lambda_k^{j'i} \right)^* |\phi_{i'}\rangle \langle \phi_{j'}| + \sum_{i'} \left((\mu_{kk}^{i'i} - i\chi^{i'i}) |\phi_{i'}\rangle \langle \phi_i| + h.c. \right). \quad (4.15)$$

We consider as well the projector $P_{\perp} = \mathbb{1} - P$. We have that

$$P_{\perp}\Phi_{\Delta t}(\rho)P_{\perp} = \Delta t \sum_i \alpha_i \sum_{k=1}^r B_{ik} (|\phi_i\rangle \langle \phi_i| \otimes \rho_{PA}) B_{ik}^{\dagger} + O(\Delta t^2), \quad (4.16)$$

where $B_{ik} = \sum_j |\phi_j\rangle \langle \phi_i| \otimes \left(L_k^{(j,i)} - \lambda_k^{ji} \mathbb{1}_{PA} \right)$. After measurement and neglecting Δt^2 terms, the state is given by

$$\begin{aligned} P\Phi_{\Delta t}(\rho)P + P_{\perp}\Phi_{\Delta t}(\rho)P_{\perp} &= \rho - i\omega \left[P\tilde{G}P, \rho \right] \Delta t - i\omega [H_E, \rho_E] \otimes \rho_{PA} \Delta t \\ &\quad + \left(\sum_i \alpha_i \sum_{k=1}^r A_{ik} \otimes \rho_{PA} \right) \Delta t + \Delta t \sum_i \alpha_i \sum_{k=1}^r B_{ik} (|\phi_i\rangle \langle \phi_i| \otimes \rho_{PA}) B_{ik}^{\dagger}. \end{aligned} \quad (4.17)$$

We see that for all the terms except for the last, fast and frequent measurement is enough to project away the noise and keep the state in the code space. This can be seen as a manifestation of the quantum Zeno effect. However, for the last term, the operators B_{ik} act non-trivially on the probe and can therefore potentially lead to dephasing as the system

evolves. Hence, measurement is not enough, and the recovery operation is required here as well.

Equation 4.11 is exactly that of the generalized KL conditions for the HMM in Lemma 12, where we take the errors to be the linear operators acting on the state in Equation 4.16. Therefore, there exists a recovery operation $\mathcal{R}$ such that

$$\mathcal{R}(P_{\perp}\Phi_{\Delta t}(\rho)P_{\perp}) = \Delta t \sigma_E \otimes \rho_{PA} + O(\Delta t^2), \quad (4.18)$$

where $\Delta t \sigma_E$ is a residual operator supported on the environment degrees of freedom after applying the recovery operation to Equation 4.18.

Combining Equation 4.19 and Equation 4.18, we obtain the final state after QEC :

$$\begin{aligned} P\Phi_{\Delta t}(\rho)P + \mathcal{R}(P_{\perp}\Phi_{\Delta t}(\rho)P_{\perp}) &= \rho - i\omega [P\tilde{G}P, \rho] \Delta t \\ &\quad - i\omega [H_E, \rho_E] \otimes \rho_{PA} \Delta t + \Delta t \sigma'_E \otimes \rho_{PA} + O(\Delta t^2), \end{aligned} \quad (4.19)$$

where $\sigma'_E = \sigma_E + \sum_i \alpha_i \sum_{k=1}^r A_{ik}$. $\sigma'_E \otimes \rho_{PA}$ can also be written as

$$\sigma'_E \otimes \rho_{PA} = \mathcal{D} \left(-i [H_{PE}, \rho] + \sum_{k=1}^r \left(L_k \rho L_k^{\dagger} - \frac{1}{2} \{ L_k^{\dagger} L_k, \rho \} \right) \right) \quad (4.20)$$

where $\mathcal{D}(\cdot) = P \cdot P + \mathcal{R}(P_{\perp} \cdot P_{\perp})$. Since the argument of the RHS of Equation 4.20 has trace zero, then σ'_E must have trace zero due to the fact that $\mathcal{D}$ is a CPTP map and that $\text{Tr}(\rho_{PA}) = 1$. We trace out the environment to obtain the reduced dynamics on the probe, which are unitary.

$$\text{tr}_E[\mathcal{R}(P_{\perp}\Phi_{\Delta t}(\rho)P_{\perp}) + P\Phi_{\Delta t}(\rho)P] = \rho_{PA} - i\omega [P(G \otimes \mathbb{1}_A)P, \rho_{PA}] \Delta t. \quad (4.21)$$

We also neglect the second-order terms in Δt . From here on out, the proof is equivalent to the one presented in [13]. We summarize the final steps. Let λ_0, λ_1 be the eigenvalues of $P(G \otimes \mathbb{1}_A)P$ associated with the eigenstates $|C_0\rangle$ and $|C_1\rangle$ respectively. We can see that the codewords are eigenstates of the effective Hamiltonian from Equation 4.6. By choosing an initial state of

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|C_0\rangle + |C_1\rangle) \quad (4.22)$$

on the probe and auxiliary system, one can show that the QFI scales as

$$\mathcal{F}(\rho_{PA}(t)) = t^2 (\lambda_0 - \lambda_1)^2. \quad (4.23)$$

Equation 4.13 guarantees that this QFI is nonzero for $t > 0$. We therefore have that if the HNES conditions are satisfied, the QFI scales as a constant multiplied by t^2 , and so we obtain the HL in T . $\square$

Note that the above QEC protocol to achieve the HL assumes QEC, as quantum controls, is performed fast and frequently throughout the state evolution. It corresponds to the situation in Figure 4.1 where the probe is only measured at the end of the evolution, where $N = 1$ and $T = t$.

Further, although the above protocol yields HL scaling, it is not optimal in general. In particular, the constant factors in the expression of the QFI may be suboptimal. To achieve better scaling, the code may be optimized in the same manner as [14].

Finally, from the point of view of information backflow, this error correction protocol eliminates all non-Markovianity in the system. Since the QEC protocol ensures that the system and environment are always in a product state, information backflow from the environment is not possible. Although this prevents potential correlations in the system from introducing harmful effects, it also prevents any potential sensing advantages due to the non-Markovianity, such as, for example, using the environment as a memory protected from decoherence [75].

4.3.2 Ruling out the Heisenberg Limit

In Section 4.3.1 we derived sufficient conditions for achieving the Heisenberg limit. Using the “Hamiltonian not in Lindblad span” (HNLS) conditions of [13] on the full system, we can obtain conditions for when the HL is unattainable.

Theorem 15 (Unattainability of the HL by violating the HNLS conditions [13]). *Let $\mathcal{S}_{PE}$ be the linear span of $\mathbb{1}_{PE}$, L_k , $L_k^\dagger$, $L_k^\dagger L_j \forall k, j$. If $(\mathbb{1}_E \otimes G + H_E \otimes \mathbb{1}_P) \in \mathcal{S}_{PE}$, then the SQL is the best achievable scaling. In other words, $\mathcal{F} = O(T)$.*

The proof is as follows. The HNLS conditions of [13] tell us that if $\mathbb{1}_E \otimes G + H_E \otimes \mathbb{1}_P \in \mathcal{S}_{PE}$, then the SQL is the best achievable asymptotic scaling with controls that act on the full system, i.e., probe, auxiliary system, and environment. In our analysis, we restricted ourselves to control operations acting only on the probe and auxiliary system, which is a strict subset of the set of full system controls. Therefore, the SQL is necessarily the best achievable asymptotic scaling for the local controls.

Further, we can relate these conditions to the previously derived HNLS conditions (Theorem 14). Consider

$$\mathbb{1}_E \otimes G + H_E \otimes \mathbb{1}_P \in \mathcal{S}_{PE} \Rightarrow \mathbb{1}_E \otimes G + H_E \otimes \mathbb{1}_P = \sum_{A \in \mathcal{S}_{PE}} c_A A, \quad (4.24)$$

where $c_A \in \mathbb{C}$. We simply trace out the environment to obtain the following:

$$G = \frac{1}{d_E} \sum_i \sum_{A \in \mathcal{S}_{PE}} c_A \langle \phi_i | A | \phi_i \rangle - \text{tr}(H_E) \mathbb{1}_P. \quad (4.25)$$

From this equation, we see that $(\mathbb{1}_E \otimes G + H_E \otimes \mathbb{1}_P) \in \mathcal{S}_{PE}$ implies $G \in \mathcal{S}$. We can obtain a simple yet useful result by taking the contra-positive to this statement. If $G \notin \mathcal{S}$ then $(\mathbb{1}_E \otimes G + H_E \otimes \mathbb{1}_P) \notin \mathcal{S}_{PE}$.

These containment conditions and their relations to the scaling of the QFI are presented in Figure 4.2.

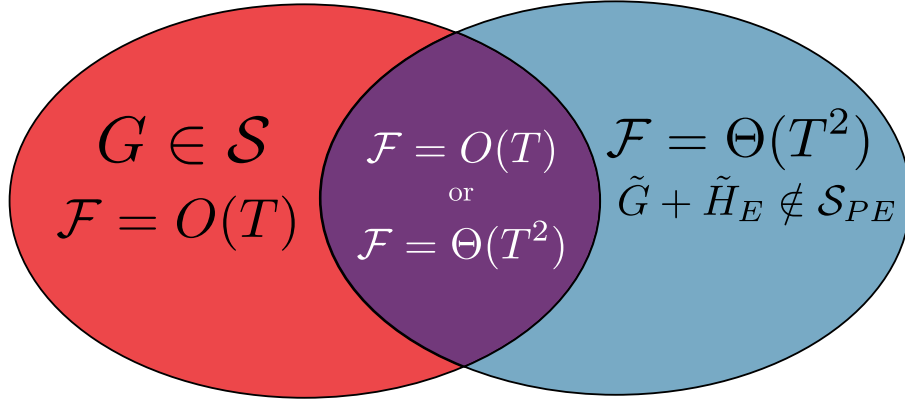


Figure 4.2: Relationship between the HNES and HNLS conditions, and the respective QFI scalings they guarantee.

4.3.3 Unitary Evolution

We now consider the special case of when the evolution of the system is unitary. We will also consider the additional assumption of the signal acting only on the probe, $\tilde{H}_E = 0$. In this scenario, the master equation governing the evolution of the HMM (Equation 4.1) reduces to

$$\frac{d\rho}{dt} = -i[H_{PE}, \rho] - i[\omega\tilde{G}, \rho]. \quad (4.26)$$

When the HNES conditions are satisfied, the QFI scales as t^2 . For unitary evolution with the signal acting only on the probe, we can guarantee that the QFI scales as a constant times t^2 multiplied by some periodic envelope function, even when the HNES conditions are

not satisfied. Once again, let $N = 1$. In particular, we can show that $\limsup_{t \rightarrow \infty} \mathcal{F}/t^2 > 0$ when $d_E < \infty$. We can therefore estimate the parameter with t^{-2} precision as long as we make sure to measure the system at the times when the value of the envelope is above some nonzero constant.

Theorem 16. *Suppose the evolution of the system is unitary and that the signal doesn't act on the environment; in other words, the evolution of the HMM is generated by Equation 4.26. Suppose $\mathcal{H}_E$ has finite dimension, i.e., $d_E < \infty$. Under these assumptions, there is a QEC protocol that achieves $\limsup_{t \rightarrow \infty} \mathcal{F}/t^2 > 0$. More precisely, it is possible to achieve a QFI that scales as t^2 multiplied by an almost periodic envelope function. By measuring at correct times, we recover the HL.*

Proof. We begin by defining a QEC code in the same manner as in the proof of Theorem 14 with the extended span $\mathcal{S}$ replaced by $\text{span}\{\mathbb{1}\}$. $|C_0\rangle$ and $|C_1\rangle$ in this case correspond to the positive and negative parts of $G - \frac{\text{tr}(G)}{d_P} \mathbb{1}_P$ (the eigenstates with eigenvalues zero can be assigned in either the positive or negative parts without affecting the calculations), whose supports are denoted by projectors P_0 and P_1 . $P_0 + P_1 = \mathbb{1}_P = \mathbb{1}_A$. To guarantee HL scaling with this QEC code, it is not necessary to act with the recovery operation. Fast and accurate syndrome measurements will be sufficient. The QEC serves to transform the noise and enforce commutation with the signal, facilitating the QFI computation.

Note that since our only controls consist of rapid measurement, the protocol can be seen as a special case of the quantum Zeno effect. Indeed, by measuring fast and frequently, we project the noise and signal into a desired subspace so that it is all of the same form. We can't measure away all of the noise, however. If the noise and signal are of the same form, then measuring away the noise also destroys the signal. Instead, we will rely on the fact that the dynamics are not dissipative. The non-Markovianity is necessary here, as it allows information lost to the environment to flow back into the system at later times.

For an initial state in the code space, the resulting dynamics are governed by the master equation

$$\frac{d\rho}{dt} = \lim_{\Delta t \rightarrow 0} \frac{P\Phi_{\Delta t}(\rho)P + P_{\perp}\Phi_{\Delta t}(\rho)P_{\perp}}{\Delta t} = -i[PH_{PE}P, \rho] - i[\omega P\tilde{G}P, \rho], \quad (4.27)$$

where P is the projection operator onto the code space. Expanding the evolution to first order, we obtain

$$P\Phi_{\Delta t}(\rho)P + P_{\perp}\Phi_{\Delta t}(\rho)P_{\perp} = \rho - i\left[P(\omega\tilde{G} + H_{PE})P, \rho\right]\Delta t + O(\Delta t^2). \quad (4.28)$$

We neglect the second-order terms in Δt . This master equation corresponds to unitary evolution generated by the following effective Hamiltonian

$$H_{\text{eff}} = \omega P \tilde{G} P + P H_{PE} P. \quad (4.29)$$

First, we show that the two terms of the Hamiltonian commute. Since, from Equation 4.6, $P B_{PE} P = P_0 B_{PE} P_0 + P_1 B_{PE} P_1$ for any operator B_{PE} acting on the probe and the environment, we have

$$\left[P H_{PE} P, P \tilde{G} P \right] = \left[P_0 H_{PE} P_0, P_0 \tilde{G} P_0 \right] + \left[P_1 H_{PE} P_1, P_1 \tilde{G} P_1 \right] \quad (4.30)$$

where $P_0 = |C_0\rangle \langle C_0|$ and $P_1 = |C_1\rangle \langle C_1|$. Since $\tilde{G}$ only has support on the probe degrees of freedom, we have that $\left[\langle C_i | \tilde{G} | C_i \rangle, \langle C_j | H_{PE} | C_j \rangle \right] = 0 \ \forall i, j$. Therefore, the terms cancel and we obtain

$$\left[P H_{PE} P, P \tilde{G} P \right] = 0. \quad (4.31)$$

We will now show that the HL is achieved. We prepare the following state on the probe and auxiliary system

$$|\psi_{PA}\rangle = \frac{1}{\sqrt{2}} (|C_0\rangle + |C_1\rangle), \quad (4.32)$$

where $|C_0\rangle, |C_1\rangle$ are the code words of the error correcting code. Let $|\psi_E\rangle$ be an arbitrary state on the environment degrees of freedom. We can take $|\psi_E\rangle$ to be pure without loss of generality by considering a purification of the environment. The total state of the system is

$$|\Psi_{PAE}\rangle = |\psi_E\rangle \frac{1}{\sqrt{2}} (|C_0\rangle + |C_1\rangle). \quad (4.33)$$

We can rewrite this state as a sum over the joint eigenstates of $P H_{PE} P \otimes \mathbb{1}_A$ and $\mathbb{1}_E \otimes P \tilde{G} P \otimes \mathbb{1}_A$

$$|\Psi_{PEA}\rangle = \sum_{k=1}^{d_E} c_k^0 |\psi_k^0\rangle |C_0\rangle + \sum_{l=1}^{d_E} c_l^1 |\psi_l^1\rangle |C_1\rangle, \quad (4.34)$$

where the joint eigenstates of these operators are necessarily product states between the codewords and states $|\psi_i^j\rangle$ (defined below) on the environment. Below, we will drop the tensor products with identity on the environment and auxiliary system from the notation to reduce clutter; it will be clear from the context which operators act on which subsystems. To prove Equation 4.34 follows, we know from Equation 4.6 that

$$\langle C_0 | P H_{PE} P | C_1 \rangle = \langle C_0 | P \tilde{G} P | C_1 \rangle = 0. \quad (4.35)$$

This allows us to write the terms of the effective Hamiltonian in the following form

$$PH_{PE}P = H_{PE}^0 \otimes |C_0\rangle\langle C_0| + H_{PE}^1 \otimes |C_1\rangle\langle C_1|, \quad (4.36)$$

$$P\tilde{G}P = \lambda_0 \mathbb{1} \otimes |C_0\rangle\langle C_0| + \lambda_1 \mathbb{1} \otimes |C_1\rangle\langle C_1|, \quad (4.37)$$

where λ_0, λ_1 are the (non-trivial) eigenvalues of $P\tilde{G}P$ as defined in the previous section and H_{PE}^0, H_{PE}^1 are operators acting on the environment. From this form, it is clear that the joint eigenvectors of $PH_{PE}P$ and $P\tilde{G}P$ are the tensor products between the eigenvectors of the blocks H_{PE}^i and the projectors $|C_i\rangle\langle C_i|$.

We now consider the time evolution for our chosen probe state. Since the terms of the Hamiltonian commute, we can independently consider their generated time evolution. We begin with the signal

$$e^{-it\omega P\tilde{G}Pt} |\Psi_{PEA}\rangle = \sum_{k=1}^{d_E} c_k^0 e^{-it\omega\lambda_0} |\psi_k^0\rangle |C_0\rangle + \sum_{l=1}^{d_E} c_l^1 e^{-it\omega\lambda_1} |\psi_l^1\rangle |C_1\rangle. \quad (4.38)$$

For the second term, we obtain

$$|\Psi_{PEA}(t)\rangle = e^{-itPH_{PE}P} e^{-itP\omega\tilde{G}P} |\Psi_{PEA}\rangle \quad (4.39)$$

$$= \sum_{k=1}^{d_E} c_k^0 e^{-it\omega\lambda_0} e^{-it\psi_k^0} |\psi_k^0\rangle |C_0\rangle + \sum_{l=1}^{d_E} c_l^1 e^{-it\omega\lambda_1} e^{-it\psi_l^1} |\psi_l^1\rangle |C_1\rangle, \quad (4.40)$$

where ψ_i^j is the eigenvalue of $PH_{PE}P$ associated with the eigenstate $|\psi_i\rangle |C_i\rangle$. We trace out the environment to obtain the reduced state of the probe and auxiliary system

$$\rho_{PA} = \begin{pmatrix} e^{-i\omega t(\lambda_0 - \lambda_1)} \alpha(t) & \frac{1}{2} \\ e^{i\omega t(\lambda_0 - \lambda_1)} \alpha(t)^* & \frac{1}{2} \end{pmatrix}. \quad (4.41)$$

We obtain a 2 by 2 matrix as the state is contained in the two-dimensional code space, which is spanned by the states $|C_0\rangle, |C_1\rangle$. We know that the diagonals are $\frac{1}{2}$ since the initial state had amplitudes of $\frac{1}{\sqrt{2}}$ in $|C_0\rangle$ and $|C_1\rangle$ which are both eigenstates of $P\tilde{G}P$ and $PH_{PE}P$. Therefore, the probabilities of measurement in that basis do not change. The value of α is given by

$$\alpha = \sum_{k=1}^{d_E} \sum_{l=1}^{d_E} c_k^0 c_l^{1*} e^{-it(\psi_k^0 - \psi_l^1)} \langle \psi_l^1 | \psi_k^0 \rangle. \quad (4.42)$$

Let $\Delta\lambda = \lambda_0 - \lambda_1$. The QFI of a mixed state can be calculated Equation 1.12.

One can show, by direct calculation, that the eigenvalues of ρ_{PA} are given by

$$\gamma_{\pm} = \frac{1}{2} \pm |\alpha(t)|. \quad (4.43)$$

Since the reduced density matrix has positive eigenvalues between zero and one, we can deduce that $|\alpha(t)| \in [-\frac{1}{2}, \frac{1}{2}]$. We rewrite $\alpha(t)$ in the polar form as $\alpha(t) = |\alpha(t)|e^{-i\phi_{\alpha}(t)}$. We will drop the explicit t dependence from the notation to reduce clutter. Further, the eigenvectors of ρ_{PA} are given by

$$|\gamma_{\pm}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm e^{i\omega t \Delta\lambda - i\phi_{\alpha}} \end{pmatrix}. \quad (4.44)$$

Inserting these values into Equation 1.12 gives

$$\mathcal{F}(\rho_{PA}) = 4t^2 (\Delta\lambda)^2 |\alpha(t)|^2. \quad (4.45)$$

To show the HL, we need to show that

$$\limsup_{t \rightarrow \infty} |\alpha| = \text{constant}. \quad (4.46)$$

We already know that $\limsup_{t \rightarrow \infty} |\alpha| \leq \frac{1}{2}$ from the earlier derived condition. We can rewrite α by letting $\bar{c}_m = c_k^0 c_l^{1*} \langle \psi_l | \psi_k \rangle$ and $\phi_m = \psi_k^0 - \psi_l^1$, where $m = (k, l)$ runs over the indices of l and k . The equation for $\alpha(t)$ becomes

$$\alpha(t) = \sum_m \bar{c}_m e^{-it\phi_m}, \quad (4.47)$$

where m runs over d_E^2 different values. It is important to note that the $\bar{c}_m$ do not depend on t . For $t = 0$ we have $\rho_{PA} = |\psi_{PA}\rangle \langle \psi_{PA}|$, and so $\alpha(0) = \frac{1}{2}$. Our goal is to prove

$$\limsup_{t \rightarrow \infty} |\alpha(t)| = \frac{1}{2}. \quad (4.48)$$

This condition is very similar to quantum state revival in a closed quantum system, which is guaranteed by the Poincaré recurrence theorem [86, 87, 88]. One can prove it by factoring the time evolution to separate signal and noise, and considering the recurrence times of the noise Hamiltonian. In fact, the revival of the QFI is directly implied from the theory of almost periodicity [88, 89], which we argue below.

Remark $\alpha(t)$ is a uniformly convergent Fourier series as it contains finite d_E^2 terms, indicating its almost periodicity [88, 87, 89]. Then $\alpha(t)$, as an almost periodic function, satisfies $\forall \epsilon > 0$ and $\forall t_0$, there exists an infinite many values of τ such that :

$$|\alpha(t_0 + \tau) - \alpha(t_0)| < \epsilon. \quad (4.49)$$

The HL is achievable so long as we measure the system at proper times where $\alpha(t)$ is above a certain positive constant. Further, these values of τ are well spread out between $-\infty$ and $+\infty$ such that one cannot construct arbitrarily long intervals between two values. The almost periodicity implies that if we pick a random t from a sufficiently large interval, there is a non-zero probability that $|\alpha(t)|$ is above a positive constant. Therefore, even if we have no knowledge of the environmental state, we can still find a proper t that achieves the HL by picking randomly. $\square$

Similar to Section 4.3.1, the above protocol achieves $\limsup_{T \rightarrow 0} \mathcal{F}/T^2 > 0$ taking $T = t$ and $N = 1$ in Figure 4.1, where QEC is performed fast and frequently, and no intermediate measurements on the probe system are performed.

As mentioned previously, measuring at random times will yield HL scaling; however, the constant factors in the scaling will likely be suboptimal. In practice, it is very difficult to obtain the exact revival times for a system, even with knowledge of the dynamics. For a guaranteed precision scaling, it is therefore better to consider the HNES conditions and to use error correction to remove the envelope function when possible.

It is worth noting, however, that for systems with small environment dimensions and simple couplings, such as the previously discussed superconducting qubits interacting with two-level systems and nitrogen vacancy centers interacting with nearby spins, the envelope functions found in experiment are relatively simple with short revival times [73, 74, 81, 90].

4.3.4 Diagonal Interaction and Relaxations to the “Hamiltonian not in Extended Span” Conditions

We begin by defining the diagonal interaction. This is a special case of the model where the components of operators supported on the environment degrees of freedom all commute pairwise.

Definition 17 (Diagonal Interaction). *We call a HMM in Equation 4.1 a HMM with diagonal interactions if and only if there exists an orthonormal basis $\{|\phi_i\rangle\}_{i=1}^{d_E}$ in $\mathcal{H}_E$ such*

that

$$H_E = \sum_i h_E^i |\phi_i\rangle \langle \phi_i| \otimes \mathbb{1}_{PA}, \quad (4.50)$$

$$H_{PE} = \sum_i |\phi_i\rangle \langle \phi_i| \otimes H_{PE}^{(i,i)}, \quad (4.51)$$

$$L_k = \sum_i |\phi_i\rangle \langle \phi_i| \otimes L_k^{(i,i)}, \quad \forall k. \quad (4.52)$$

For HMMs with diagonal interactions, the quantum dynamics can be simplified through the following lemma.

Lemma 18. *Consider a quantum state ρ as an input state to a HMM with diagonal interactions. The output state after evolution of time t is $\Phi_t(\rho)$ where $\Phi_t(\cdot)$ denotes the evolution channel after time t . We have*

$$\text{tr}_E(\Phi_t(\rho)) = \text{tr}_E(\Phi_t((\mathcal{D}_E \otimes \mathbb{1}_{PA})(\rho))), \quad (4.53)$$

where $\mathcal{D}_E$ is the completely dephasing channel on $\mathcal{H}_E$ such that $\mathcal{D}_E(\cdot) = \sum_k |\phi_k\rangle \langle \phi_k| (\cdot) |\phi_k\rangle \langle \phi_k|$. In addition, Equation 4.53 still holds when quantum controls are applied on the probe and auxiliary system during the evolution.

Proof. To see Equation 4.53 holds, we observe it is sufficient to prove the commuting relation between $\mathcal{D}_E$ and Φ_t , because

$$\text{tr}_E(\Phi_t(\rho)) = \text{tr}_E((\mathcal{D}_E \otimes \mathbb{1}_{PA})(\Phi_t(\rho))). \quad (4.54)$$

The fact that $\mathcal{D}_E \otimes \mathbb{1}_{PA}$ commutes with Φ_t can be proven through noticing $\mathcal{D}_E \otimes \mathbb{1}_{PA}$ commutes with the differential in Equation 4.1 for diagonal interactions. Finally, $\mathcal{D}_E \otimes \mathbb{1}_{PA}$ also commutes with any quantum operations acting on the probe and auxiliary system because they act on different systems. $\square$

Lemma 18 implies that for diagonal interactions, it is always sufficient to consider input states that are diagonal in the environmental basis $\{|\phi_i\rangle\}_{i=1}^d$, and the off-diagonal terms will not contribute to the final measurement outcomes at the probe and auxiliary system. Below, we will assume the quantum states under consideration $\rho \in \mathcal{H}_E \otimes \mathcal{H}_P \otimes \mathcal{H}_A$ are diagonal in $\mathcal{H}_E$, i.e.,

$$\rho = \sum_i \alpha_i |\phi_i\rangle \langle \phi_i| \otimes \rho^{(i,i)}, \quad (4.55)$$

where $\rho^{(i,i)}$ are density operators on $\mathcal{H}_{PA}$ and α_i are non-negative constants summing up to one. As we are considering HMMs with diagonal interactions, only $\rho^{(i,i)}$ will evolve over time under the master equation, and it is sufficient to track the evolution of $\rho^{(i,i)}$.

Further, when studying diagonal interactions, we can ignore the contribution from $\tilde{H}_E$ as it will not make a difference to the QFI.

Lemma 19. *Consider a quantum state ρ in the form of Equation 4.55 as the input state to a HMM with diagonal interactions. We have that*

$$\text{tr}_E(\Phi_t(\rho)) = \text{tr}_E(\hat{\Phi}_t(\rho)), \quad (4.56)$$

where $\hat{\Phi}_t(\cdot)$ denotes the evolution channel after time t where we ignore the contributions of $\tilde{H}_E$. In other words, it's the evolution under the equation

$$\frac{d\rho}{dt} = -i[H_{PE}, \rho] - i[\omega\tilde{G}, \rho] + \sum_{k=1}^r \left(L_k \rho L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_k, \rho\} \right). \quad (4.57)$$

Proof. First, notice that $\tilde{H}_E$ commutes with all other operators of the master equation. This allows us to factor the evolution as follows

$$\Phi_t(\cdot) = \hat{\Phi}_t \left(e^{-it\tilde{H}_E}(\cdot) e^{it\tilde{H}_E} \right) = e^{-it\tilde{H}_E} \hat{\Phi}_t(\cdot) e^{-it\tilde{H}_E}. \quad (4.58)$$

For a ρ in the form of Equation 4.55, time evolution from H_E has no effect.

$$e^{-it\tilde{H}_E} \rho e^{it\tilde{H}_E} = \sum_i \alpha_i e^{-itH_E} |\phi_i\rangle \langle \phi_i| e^{itH_E} \otimes \rho_{PA} \quad (4.59)$$

$$= \sum_i \alpha_i e^{-ith_E^i} |\phi_i\rangle \langle \phi_i| e^{ith_E^i} \otimes \rho_{PA} \quad (4.60)$$

$$= \sum_i \alpha_i |\phi_i\rangle \langle \phi_i| \otimes \rho_{PA} = \rho. \quad (4.61)$$

And so, we see that the evolution due to $\tilde{H}_E$ has no effect, and only the evolution due to $\hat{\Phi}_t$ must be considered. $\square$

When the interaction is diagonal, with less restrictive conditions than the HNES conditions, we can guarantee that the QFI scales as a constant times t^2 multiplied by some periodic envelope function, and that $\limsup_{t \rightarrow \infty} \mathcal{F}/t^2 > 0$ when $d_E < \infty$. This is the same scaling as the case of unitary evolution seen in 4.3.3 and guaranteed by Theorem 16.

Theorem 20 (Hamiltonian not in extended Lindblad span (HNELS) Conditions). *Consider the case of a diagonal interaction. Let $\{|\phi_i\rangle\}_{i=1}^{d_E}$ be the orthonormal basis for $\mathcal{H}_E$ from Definition 17. Let $\mathcal{H}_E$ have finite dimension, i.e $d_E < \infty$. Let $\mathcal{S}^\circ$ be the linear span of the operators $\mathbb{1}_P$, $\langle\phi_i|L_k|\phi_i\rangle$, $\langle\phi_i|L_k^\dagger|\phi_i\rangle$, and $\langle\phi_i|L_k^\dagger|\phi_i\rangle\langle\phi_m|L_j|\phi_m\rangle$ ($\forall k, j \in \{1, 2, \dots, r\}$, $\forall i, m \in \{1, 2, \dots, d_E\}$). $\mathcal{S}^\circ$ is similar to $\mathcal{S}$, except we exclude the contributions from H_{PE} and only consider diagonal elements.*

If $G \notin \mathcal{S}^\circ$, then $\limsup_{t \rightarrow \infty} \mathcal{F}/t^2 > 0$. Further, it is possible to achieve a QFI that scales as t^2 multiplied by a periodic envelope function. We will refer to the containment $G \notin \mathcal{S}^\circ$ as the “Hamiltonian not in extended Lindblad span” (HNELS) conditions.

Proof. We consider ρ in diagonal form (Equation 4.55) since we trace out the environment at the end and may thus apply Lemma 18 and Lemma 19. For a ρ of this form, the master equation governing the HMM takes the form

$$\frac{d\rho}{dt} = \sum_i \alpha_i |\phi_i\rangle \langle\phi_i| \otimes \left(-i\omega [G, \rho^{(i,i)}] - i \left[H_{PE}^{(i,i)}, \rho^{(i,i)} \right] + \right. \quad (4.62)$$

$$\left. \sum_{j=1}^r \left(L_j^{(i,i)} \rho^{(i,i)} \left(L_j^{(i,i)} \right)^\dagger - \frac{1}{2} \left\{ \left(L_k^{(i,i)} \right)^\dagger L_k^{(i,i)}, \rho^{(i,i)} \right\} \right) \right) \quad (4.63)$$

We employ the same construction for a QEC code as in Section 4.3.1, except we swap $\mathcal{S}$ for $\mathcal{S}^\circ$. For an initial state in the code space, the resulting dynamics are governed by the master equation

$$\begin{aligned} \frac{d\rho}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{P\Phi_{\Delta t}(\rho)P + \mathcal{R}(P_\perp \Phi_{\Delta t}(\rho)P_\perp)}{\Delta t} \\ &= \sum_i \alpha_i |\phi_i\rangle \langle\phi_i| \otimes \left(-i\omega [PGP, \rho^{(i,i)}] - i \left[PH_{PE}^{(i,i)} P, \rho^{(i,i)} \right] + \sum_{j=1}^r \left(PL_j^{(i,i)} \rho^{(i,i)} \left(L_j^{(i,i)} \right)^\dagger P \right. \right. \\ &\quad \left. \left. + \mathcal{R} \left(P_\perp L_j^{(i,i)} \rho^{(i,i)} \left(L_j^{(i,i)} \right)^\dagger P_\perp \right) - \frac{1}{2} \left\{ P \left(L_k^{(i,i)} \right)^\dagger L_k^{(i,i)} P, \rho^{(i,i)} \right\} \right) \right). \end{aligned} \quad (4.64)$$

where P is the projection operator onto the code space. Since $G \notin \mathcal{S}^\circ$, we have that PGP acts non-trivially in the code-space. Further, by design of the QEC code, the last three terms of Equation 4.64 give $\rho^{(i,i)}$ up to a scalar. This scalar must be zero since the jump operator portion of the master equation has trace zero, and the channel $P \cdot P + \mathcal{R}(P \cdot P)$ is trace preserving. Therefore, the evolution is given by

$$\frac{d\rho}{dt} = \sum_i \alpha_i |\phi_i\rangle \langle\phi_i| \otimes -i \left[P(\omega G + H_{PE}^{(i,i)})P, \rho^{(i,i)} \right], \quad (4.65)$$

which can be rewritten as

$$\frac{d\rho}{dt} = -i \left[P \left(\omega \tilde{G} + \sum_i |\phi_i\rangle \langle \phi_i| \otimes H_{PE}^{(i,i)} \right) P, \rho \right]. \quad (4.66)$$

We see from Equation 4.66 that after quantum error correction, for a diagonal interaction, the overall evolution of the system and environment is unitary. Applying Theorem 16 therefore yields the desired QFI scaling.

Note that in this proof, a first code was used to obtain unitary dynamics on the full system. In order to prove the scaling of Theorem 16, the probe and auxiliary system were encoded in a QECC that imposed commutation between the signal and coupling Hamiltonian. It is not necessary here to concatenate the two codes, as the first code used in the proof already has all the properties necessary to prove the HL scaling. $\square$

4.3.5 Dependence on the initial state and the trajectory

As shown in Figure 4.2, there exists a middle ground where $G \in \mathcal{S}$ and $\tilde{G} \notin \mathcal{S}_{PE}$. In this region, Theorem 14 does not guarantee that the Heisenberg limit is achievable, and Theorem 15 does not rule it out. It turns out that in this region it is impossible to derive a general scaling argument based solely on the operators appearing in Equation 4.1. To decide the scaling, one must also take into account the initial environment state, and by extension, the trajectories that can be generated with this state and arbitrary controls on the probe and auxiliary system. We show this with the following example, for which we can be limited by either HL or SQL depending on the initial environment state.

We wish to sense the parameter ω with signal Hamiltonian $\tilde{G} = \mathbb{1}_E \otimes Z$. We consider environment and probe, which are both qubits, so that $\mathcal{H}_E \otimes \mathcal{H}_P = \mathbb{C}^2 \otimes \mathbb{C}^2$. Further, we assume no unitary coupling between the qubits, i.e. $H_{PE} = 0$, and no signal on the environment $H_E = 0$. We consider a model with the following two jump operators

$$L_1 = |0\rangle \langle 0| \otimes \mathbb{1}, \quad L_2 = |1\rangle \langle 1| \otimes Z. \quad (4.67)$$

Here, Z refers to the Pauli σ_Z matrix. We first consider the situation where the environment is initially in the $|0\rangle$ state, such that the initial density matrix is given by $\rho_{PE}(0) = |0\rangle \langle 0| \otimes \rho_{P,0}$ where $\rho_{P,0}$ is the initial probe state. One can see that throughout the evolution, the state will always remain in a product state. The environment remains unchanged and stays in the $|0\rangle \langle 0|$ state. We can therefore trace out the environment and

look only at the dynamics on the probe. The reduced density matrix on the probe evolves under the equation

$$\frac{d\rho_{PA}}{dt} = -i\omega [G, \rho_{PA}]. \quad (4.68)$$

For this choice of initial state, the probe evolution is entirely generated by the signal Hamiltonian, and the evolution is Markovian and unitary. This guarantees the attainability of the HL.

Similarly, consider the case where the initial environment state is $|1\rangle$. Once again, the state always remains in a product state throughout the evolution, and in particular, the environment always remains in the $|1\rangle\langle 1|$ state and does not change. We can trace out the environment and look at the evolution of the probe. In this case, the probe evolves under the following master equation

$$\frac{d\rho_{PA}}{dt} = -i\omega [G, \rho_{PA}] + Z\rho_{PA}Z - \rho_{PA}. \quad (4.69)$$

The reduced dynamics are Markovian, and therefore the results of [13] can be applied. For these dynamics, the HL cannot be achieved using any quantum controls due to violation of the HNLS conditions. In this case, the SQL is the best achievable scaling since the signal Hamiltonian and the jump operator are both given by the same matrix.

4.4 Achieving the Standard Quantum Limit with a Prepare and Measure Strategy

In the previous Sections, we asked the question of whether quantum error correction could be used to achieve Heisenberg limit scaling of the quantum Fisher information. Further, in the cases where we can definitely rule out the Heisenberg limit, we know the scaling of the QFI is upper bounded by the SQL.

In the Markovian case, the standard quantum limit can always be achieved by considering a prepare-and-measure strategy. The prepare and measure strategy is one in which we prepare a fiducial probe state $|\psi\rangle$, allow it to evolve under the action of both Hamiltonian and noise for a duration t , and then finally either measure it or shelve the state. This sequence is repeated for N rounds, for a total experiment duration of $T = Nt$. The SQL is achieved, as the quantum Fisher information scales linearly in the number of rounds N , and the duration t is a constant. The question arises of whether the same strategy can achieve the SQL in the non-Markovian regime. Indeed, in the Markovian case, the system

is completely reset between rounds, and so the measurement outcomes between rounds are not correlated. In other words, the scaling in N is interchangeable with the scaling in M for I.I.D. experiments seen in Equation 1.8. In the non-Markovian regime, the environment now mediates correlations between the measurement outcomes of different rounds of prepare and measure. The rounds of prepare and measure, therefore, can not be treated as I.I.D. experiments, and so SQL is not a priori guaranteed. A diagram presenting the strategy in the hidden Markov model is shown in Figure 4.3.

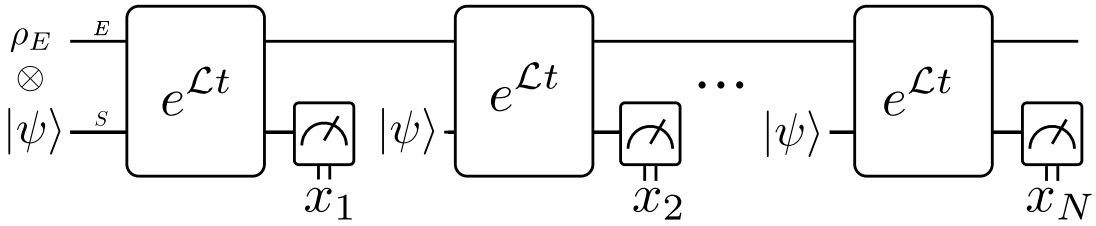


Figure 4.3: Prepare and measure strategy in the hidden Markov model. The evolution $e^{\mathcal{L}t}$ corresponds to evolution under Equation 4.1 in the natural representation.

In this thesis, we will not prove that the SQL is always achievable; however, we will leave it as conjecture and present evidence.

Conjecture 21. *The standard quantum limit is always achievable for the hidden Markov model using a prepare and measure strategy, for a correct choice of initial state and measurement basis.*

4.4.1 Achieving the Standard Quantum Limit when the Interaction is Diagonal

We'll consider a diagonal interaction as defined by Definition 17, as we did when considering the HNELS conditions of Section 4.3.4. We simplify the dynamics further by applying QEC on the probe and auxiliary system to impose commutation of interactions on the probe degrees of freedom as well, allowing us to directly compute a QFI lower bound in a basis where all operators are simultaneously diagonal. To obtain the QFI lower bound, we will measure the probe at each time step and calculate the classical Fisher information (FI) of the probability distribution for the measurement outcomes. As mentioned above, we'll employ the prepare-and-measure strategy. The FI of the joint probability distribution is a lower bound for the QFI of the quantum state of all output probes [17]. Consequently,

observing SQL scaling in the FI ensures a QFI scaling no less than SQL. In our case, the joint probability distribution for the measurement outcomes is a mixture of binomial distributions. The FI of such a distribution scales linearly in N asymptotically. We shall use these arguments to prove the following theorem.

Theorem 22. *Consider the case of a diagonal interaction (Definition 17). With an interaction of this form, SQL scaling of the QFI is always achievable in N using a combination of a prepare-and-measure strategy and QEC, for suitable choices of t .*

Proof. The proof is as follows. Here, we define the QEC code in the same manner as in the proof of Theorem 16, with $\mathcal{S}$, $|C_0\rangle$, and $|C_1\rangle$ defined in the same way. Contrary to the unitary case, in this section, we will need to perform the recovery operation during the QEC protocol. Although the code here cannot correct any non-trivial errors, it is sufficient to transform the dynamics on the probe system to commuting operations, with the recovery operation being

$$\mathcal{R}(\cdot) = |C_0\rangle \langle C_0| \text{tr}(\mathbb{1}_P \otimes (P_0)_A(\cdot)) + |C_1\rangle \langle C_1| \text{tr}(\mathbb{1}_P \otimes (P_1)_A(\cdot)). \quad (4.70)$$

For an initial state in the code space, the dynamics are governed by the following equation

$$\begin{aligned} \frac{d\rho}{dt} &= \lim_{\Delta t \rightarrow 0} \frac{P\Phi_{\Delta t}(\rho)P + \mathcal{R}(P_{\perp}\Phi_{\Delta t}(\rho)P_{\perp})}{\Delta t} \\ &= -i[PH_{PE}P, \rho] - i[\omega P\tilde{G}P, \rho] + \sum_{i=1}^r \left(PL_i\rho L_iP + \mathcal{R}(P_{\perp}L_i\rho L_i^{\dagger}P_{\perp}) - \frac{1}{2}\{PL_i^{\dagger}L_iP, \rho\} \right), \end{aligned} \quad (4.71)$$

where P is the projection operator onto the code space. Applying Lemma 19 allows us to ignore the signal on the environment. Assuming an initial state in the form of Equation 4.55 and using Equation 4.70, we have

$$\begin{aligned} \frac{d\rho^{(i,i)}}{dt} &= -i[\omega P(G \otimes \mathbb{1}_A)P, \rho^{(i,i)}] \\ &\quad - i\left[PH_{PE}^{(i,i)}P + \frac{\text{Im}(a_i)}{2}\bar{Z}, \rho^{(i,i)}\right] + \frac{\text{Re}(a_i)}{2}(\bar{Z}\rho^{(i,i)}\bar{Z} - \rho^{(i,i)}), \end{aligned} \quad (4.72)$$

where the logical operator $\bar{Z} = |C_0\rangle \langle C_0| - |C_1\rangle \langle C_1|$ and

$$\begin{aligned} a_i &= \sum_k \frac{1}{2} \left(\langle C_0| \left(L_k^{\dagger} L_k \right)^{(i,i)} |C_0\rangle + \langle C_1| \left(L_k^{\dagger} L_k \right)^{(i,i)} |C_1\rangle \right) - \langle C_0| L_k^{(i,i)} |C_0\rangle \langle C_0| (L_k^{\dagger})^{(i,i)} |C_0\rangle. \end{aligned} \quad (4.73)$$

It is clear that $\rho^{(i,i)}$ is evolving under Pauli-Z rotations and Markovian dephasing noise, and the strengths of the Pauli-Z rotation and the dephasing noise are different for different i .

We consider the prepare-and-measure strategy. At each time step, we prepare the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|C_0\rangle + |C_1\rangle)$. For the measurement basis, we choose

$$|\pm_i\rangle = \frac{1}{\sqrt{2}}(|C_0\rangle \pm i|C_1\rangle), \quad (4.74)$$

which we associate to the outcomes 1 and -1 respectively. Let X_n denote the random variable associated with the measurement at the n -th time step. We'll denote by b_n the outcome of the measurement for the n -th step. Consider a given measurement sequence with outcomes $b_1, \dots, b_N$. The probability of this event occurring is given by the joint distribution $P(X_N = b_N, \dots, X_1 = b_1)$.

We will first consider the time evolution between two arbitrarily chosen steps of the prepare-and-measure protocol. As in the proof of Theorem 20, we will factor the evolution into two steps. The first step will be evolution under the master equation

$$\frac{d\rho^{(i,i)}}{dt} = -i \left[PH_{PE}^{(i,i)} P + \frac{\text{Im}(a_i)}{2} \bar{Z}, \rho^{(i,i)} \right] + \frac{\text{Re}(a_i)}{2} (\bar{Z} \rho^{(i,i)} \bar{Z} - \rho^{(i,i)}), \quad (4.75)$$

which corresponds to the evolution due only to the noise. We can integrate the equation to obtain the CPTP map Λ_t corresponding to the time evolution for a duration t :

$$\Lambda_t(|\psi\rangle\langle\psi|) = A_0 |\psi\rangle\langle\psi| A_0^\dagger + A_1 |\psi\rangle\langle\psi| A_1^\dagger, \quad (4.76)$$

where

$$A_0 = \sqrt{\frac{1 + \exp(-\text{Re}(\Gamma_i)t)}{2}} \exp\left(-\frac{i}{2} \text{Im}(\Gamma_i)t\bar{Z}\right), \quad (4.77)$$

and where

$$A_1 = \sqrt{\frac{1 - \exp(-\text{Re}(\Gamma_i)t)}{2}} \bar{Z} \exp\left(-\frac{i}{2} \text{Im}(\Gamma_i)t\bar{Z}\right). \quad (4.78)$$

Here, $\Gamma_i = i \text{tr}(\bar{Z} P H_{PE}^{(i,i)} P) + a_i$. We have taken our initial probe state to be $|\psi\rangle$. Further, we know that for $t = 0$, Λ_t leaves the state unchanged and acts as the identity channel, as

$A_0(t=0) = \mathbb{1}_{PA}$ and $A_1(t=0) = 0$. After evolving under this channel, the next step is the unitary evolution due to the signal.

$$e^{-i\omega t P(G \otimes \mathbb{1}_A)P} \Lambda_t(|\psi\rangle \langle \psi|) e^{i\omega t P(G \otimes \mathbb{1}_A)P} = \sum_{k=0}^1 e^{-i\omega t P(G \otimes \mathbb{1}_A)P} A_k |\psi\rangle \langle \psi| A_k^\dagger e^{i\omega t P(G \otimes \mathbb{1}_A)P}. \quad (4.79)$$

The final step is to measure in the chosen basis. The probability of obtaining an outcome of ± 1 is given by

$$P(X_n = b_n | \rho_E = |\phi_i\rangle \langle \phi_i|) = \langle \pm_i | e^{-i\omega t P(G \otimes \mathbb{1}_A)P} \Lambda_t(|\psi\rangle \langle \psi|) e^{i\omega t P(G \otimes \mathbb{1}_A)P} | \pm_i \rangle. \quad (4.80)$$

First, we see that the probability of obtaining $b_n = 1$ is independent of n due to the fact that for an initial environment state of $|\phi_i\rangle \langle \phi_i|$, the environment remains invariant and does not mediate correlations between measurements. Let $p_i := P(X_n = 1 | \rho_E = |\phi_i\rangle \langle \phi_i|)$ and $q_i = 1 - p_i$ be the probability of obtaining $b_n = -1$. The probability of obtaining $+1$ is given by

$$p_i = \frac{1}{2} + \frac{1}{2} \exp(-\operatorname{Re}(\Gamma_i)t) \sin((\operatorname{Im}(\Gamma_i) + \omega(\lambda_0 - \lambda_1))t), \quad (4.81)$$

where λ_0, λ_1 are the eigenvalues of $P(G \otimes \mathbb{1}_A)P$ associated to $|C_0\rangle, |C_1\rangle$. Further, $\lambda_0 \neq \lambda_1$ as we assume G acts non-trivially on the probe. The derivative of the probability with respect to the sensing parameter ω is given by

$$\partial_\omega p_i = \frac{1}{2} \exp(-\operatorname{Re}(\Gamma_i)t) (\lambda_0 - \lambda_1) t \cos((\operatorname{Im}(\Gamma_i) + \omega(\lambda_0 - \lambda_1))t). \quad (4.82)$$

Since the measurement outcomes are independent, the joint probability can be expressed as the product of the marginals

$$P(X_N = b_N, \dots, X_1 = b_1 | \rho_E = |\phi_i\rangle \langle \phi_i|) = \prod_{n=1}^N p_i^{(1+b_n)/2} (1 - p_i)^{(1-b_n)/2}. \quad (4.83)$$

For an arbitrary diagonal environment state, such as in Equation 4.55, we sum the joint probabilities (Equation 4.83) for the different environment basis states

$$P(X_N = b_N, \dots, X_1 = b_1) = \sum_{i=1}^{d_E} \alpha_i \prod_{n=1}^N p_i^{(1+b_n)/2} (1 - p_i)^{(1-b_n)/2}. \quad (4.84)$$

As mentioned previously, the order of the outcomes doesn't matter, and we recognize a sum of binomial distributions, also called a binomial mixture. For a measurement sequence $b_1, \dots, b_N$, let N_1 count the +1 outcomes in the sequence. The associated probability distribution for N_1 is

$$P(N_1 = n_1) = \sum_{i=1}^{d_E} \alpha_i \binom{N}{n_1} p_i^{n_1} (1 - p_i)^{N-n_1}. \quad (4.85)$$

From this point on, we will assume without loss of generality that all the p_i are distinct when we evaluate them at the true value of ω , and where a specific value of t has already been fixed. We may do this since similar values of p can be combined into single terms of the sum. Suppose, for example, that $p_2 = p_3$, then we can set $\alpha_2 \mapsto \alpha_2 + \alpha_3$, and remove the index 3 from the sum.

To show that this strategy achieves the SQL, one must show that the FI of this binomial mixture scales linearly in N asymptotically. We now show this.

The FI, which we denote by $\mathcal{I}$, can be calculated using Equation 1.7. We first consider the numerator of Equation 1.7 and take the derivative of Equation 4.85,

$$\partial_\omega P(N_1 = n_1) = \sum_{i=1}^{d_E} \alpha_i \binom{N}{n_1} p_i^{n_1} (1 - p_i)^{N-n_1} \frac{n_1 - N p_i}{p_i(1 - p_i)} (\partial_\omega p_i). \quad (4.86)$$

We will choose $t > 0$ such that $p_i \neq 0$ or 1. We can see from Equation 4.81 that for $t = 0$, $p_i = 0.5$. Further, the probability depends continuously on the variable t . There therefore exists a choice $t > 0$ such p_i is not too far from $\frac{1}{2}$, for all i . Similarly, we will choose t such that $\partial_\omega p_i \neq 0$. We can see from Equation 4.82 that for $t > 0$ but small, the derivative is given by

$$\partial_\omega p_i = \frac{1}{2} \exp(-\text{Re}(\Gamma_i)t) (\lambda_0 - \lambda_1)t + O(t^2), \quad (4.87)$$

and so for a choice of small t the derivative doesn't vanish. Below, we assume t is small enough so that for all i , $0 < p_i < 1$ and $\partial_\omega p_i \neq 0$.

The full FI can be written as

$$\mathcal{I}(\omega) = \sum_{n_1=0}^N \frac{\left(\sum_{i=1}^{d_E} \alpha_i (\partial_\omega p_i) p_i^{n_1} (1 - p_i)^{N-n_1} \frac{n_1 - N p_i}{p_i(1 - p_i)} \right)^2}{\left(\sum_{i=1}^{d_E} \alpha_i p_i^{n_1} (1 - p_i)^{N-n_1} \right)^2} \cdot \left(\binom{N}{n_1} \sum_{i=1}^{d_E} \alpha_i p_i^{n_1} (1 - p_i)^{N-n_1} \right). \quad (4.88)$$

To show the asymptotic linear scaling of the FI, we will need to prove three important bounds. Let $i_* = \operatorname{argmax} \{p_i \mid \alpha_i > 0\}$ and let $p_* = p_{i_*}$. Such a p_i always exists since we've assumed $\sum_i \alpha_i = 1$, and so there must be at least one nonzero α_i . To prove the SQL is always achievable, we will need the following three bounds.

Lemma 23. $\forall \eta_1 > 0, \epsilon_1 > 0$, and $\forall p_i \neq p_*, \exists N_1^{(1)}$ such that $\forall N > N_1^{(1)}$, and $\forall n_1$ satisfying

$$\frac{n_1}{N} > \frac{\log \left(\frac{1-p_i}{1-p_*} \right)}{\log \left(\frac{p_*(1-p_i)}{p_i(1-p_*)} \right)} + \epsilon_1, \quad (4.89)$$

we have that

$$\left| \alpha_* p_*^{n_1} (1-p_*)^{N-n_1} \cdot \frac{n_1 - Np_*}{p_*(1-p_*)} (\partial_\omega p_*) \right| > \eta_1 \left| \alpha_i p_i^{n_1} (1-p_i)^{N-n_1} \cdot \frac{n_1 - Np_i}{p_i(1-p_i)} (\partial_\omega p_i) \right|. \quad (4.90)$$

Proof. We can rearrange the terms of Equation 4.90 to obtain the following equation

$$\left| \frac{p_*(1-p_i)}{p_i(1-p_*)} \right|^{n_1} \left| \frac{1-p_*}{1-p_i} \right|^N \left| \frac{n_1 - Np_*}{n_1 - Np_i} \right| > \eta_1 \frac{\alpha_i p_*(1-p_*)}{\alpha_* p_i(1-p_i)} \left| \frac{\partial_\omega p_i}{\partial_\omega p_*} \right|. \quad (4.91)$$

We'll denote the right side of this equation by $\hat{\eta}$. Remark $\hat{\eta}$ depends on neither N or n_1 . Taking the natural log of both sides, we obtain

$$n_1 \log \left| \frac{p_*(1-p_i)}{p_i(1-p_*)} \right| + N \log \left| \frac{1-p_*}{1-p_i} \right| + \log \left| \frac{n_1 - Np_*}{n_1 - Np_i} \right| > \log \hat{\eta}. \quad (4.92)$$

First, we let $n_1 = Nk_1$. We divide the previous equation by N and shuffle terms around to obtain

$$k_1 > \left(\log \left| \frac{p_*(1-p_i)}{p_i(1-p_*)} \right| \right)^{-1} \cdot \left(\log \left| \frac{1-p_i}{1-p_*} \right| + \frac{1}{N} \log \left| \frac{k_1 - p_i}{k_1 - p_*} \right| + \frac{\log \hat{\eta}}{N} \right). \quad (4.93)$$

There exists an $N_1^{(1)}$ such that $N > N_1^{(1)}$ implies that the last two terms when expanding the right side of the equation are smaller than ϵ_1 . This yields the inequality

$$k_1 > \frac{\log \left(\frac{1-p_i}{1-p_*} \right)}{\log \left(\frac{p_*(1-p_i)}{p_i(1-p_*)} \right)} + \epsilon_1. \quad (4.94)$$

And so, the condition for Equation 4.90 to hold is that

$$k_1 = \frac{n_1}{N} > \frac{n_1^{(1)}}{N}, \quad N > N_1^{(1)}, \quad (4.95)$$

where we let

$$\frac{n_1^{(1)}}{N} = \frac{\log\left(\frac{1-p_i}{1-p_*}\right)}{\log\left(\frac{p_*(1-p_i)}{p_i(1-p_*)}\right)} + \epsilon_1. \quad (4.96)$$

□

In the proof of this theorem, we will also use the following property of $n_1^{(1)}$.

Lemma 24. *For $p_i < p_*$ and $0 < p_i, p_* < 1$, we have*

$$\frac{\log\left(\frac{1-p_i}{1-p_*}\right)}{\log\left(\frac{p_*(1-p_i)}{p_i(1-p_*)}\right)} < p_*. \quad (4.97)$$

Proof. We can rewrite this equation as

$$\log\left(\frac{1-p_i}{1-p_*}\right) - p_* \log\left(\frac{p_*(1-p_i)}{p_i(1-p_*)}\right) < 0. \quad (4.98)$$

Taking the exponential of both sides, we obtain the inequality

$$\exp\left(\log\left(\frac{1-p_i}{1-p_*}\right) - p_* \log\left(\frac{p_*(1-p_i)}{p_i(1-p_*)}\right)\right) < 1. \quad (4.99)$$

We will now prove this inequality. First, remark that for $p_i = p_*$, we have that

$$\exp\left(\log\left(\frac{1-p_*}{1-p_*}\right) - p_* \log\left(\frac{p_*(1-p_*)}{p_*(1-p_*)}\right)\right) = 1. \quad (4.100)$$

We now consider the derivative of the left-hand side with respect to p_* .

$$\frac{\partial}{\partial p_*} \exp\left(\log\left(\frac{1-p_i}{1-p_*}\right) - p_* \log\left(\frac{p_*(1-p_i)}{p_i(1-p_*)}\right)\right) = \quad (4.101)$$

$$-\left(\frac{1-p_i}{1-p_*}\right)^{1-p_*} \left(\frac{p_i}{p_*}\right)^{p_*} \left[\log\left(\frac{p_*}{p_i}\right) + \log\left(\frac{1-p_i}{1-p_*}\right)\right]. \quad (4.102)$$

We see that the derivative is strictly negative for all $p_i < p_*$ such that $0 < p_i, p_* < 1$. In other words, for $p_* > p_i$, as we increase p_* , the function defined by the left side of Equation 4.99 decreases from the value of 1 at $p_i = p_*$. Therefore, the inequality of Equation 4.99 must hold.

□

Equivalently, by the density of real numbers, this lemma states that there exists an $\epsilon > 0$ such that

$$\frac{\log\left(\frac{1-p_i}{1-p_*}\right)}{\log\left(\frac{p_*(1-p_i)}{p_i(1-p_*)}\right)} < p_* - \epsilon. \quad (4.103)$$

Note that the right-hand side of this equation satisfies

$$\frac{\log\left(\frac{1-p_i}{1-p_*}\right)}{\log\left(\frac{p_*(1-p_i)}{p_i(1-p_*)}\right)} + \epsilon_1 > p_i. \quad (4.104)$$

Lemma 25. $\forall \eta_2 > 0, \epsilon_2 > 0$, and $\forall p_i \neq p_*, \exists N_1^{(2)}$ such that $\forall N > N_1^{(2)}$, and $\forall n_1$ satisfying

$$\frac{n_1}{N} > \frac{\log\left(\frac{1-p_i}{1-p_*}\right)}{\log\left(\frac{p_*(1-p_i)}{p_i(1-p_*)}\right)} + \epsilon_2, \quad (4.105)$$

we have that

$$\alpha_* p_*^{n_1} (1 - p_*)^{N - n_1} > \eta_2 \alpha_i p_i^{n_1} (1 - p_i)^{N - n_1}. \quad (4.106)$$

Proof. We can rearrange the equation by following the same steps as the previous derivation to obtain

$$k_1 > \frac{\log\left(\frac{1-p_i}{1-p_*}\right)}{\log\left(\frac{p_*(1-p_i)}{p_i(1-p_*)}\right)} + \frac{\log \eta_2}{N}. \quad (4.107)$$

For all $\epsilon_2 > 0$, there exists $N_1^{(2)}$ such that for $N > N_1^{(2)}$, the last term is smaller than ϵ_2 . This gives us

$$k_1 > \frac{\log\left(\frac{1-p_i}{1-p_*}\right)}{\log\left(\frac{p_*(1-p_i)}{p_i(1-p_*)}\right)} + \epsilon_2. \quad (4.108)$$

Once again, the choice of $N_1^{(2)}$ is somewhat arbitrary, as we can choose to suppress ϵ_2 more or less in exchange for lowering the value of $n_1^{(2)}$, which we define as

$$\frac{n_1^{(2)}}{N} = \frac{\log\left(\frac{1-p_i}{1-p_*}\right)}{\log\left(\frac{p_*(1-p_i)}{p_i(1-p_*)}\right)} + \epsilon_2. \quad (4.109)$$

Once again, we must choose $N_1^{(2)}$ big enough such that $n_1^{(2)} < N$ and such that there exists integer values of n_1 such that $n_1^{(2)} < n_1 < N$. □

Lemma 26. $\forall 1 > \eta_3 > 0, \epsilon_3 > 0, \exists N_1^{(3)}$ such that $\forall N > N_1^{(3)}, \exists n_1^{(3)} > (p_* - \epsilon_3)N$ such that $n_1 > n_1^{(3)}$ implies that

$$\sum_{n_1 > n_1^{(3)}} \binom{N}{n_1} \frac{\left(p_*^{n_1}(1-p_*)^{N-n_1} \frac{n_1-Np_*}{p_*(1-p_*)}\right)^2}{p_*^{n_1}(1-p_*)^{N-n_1}} > \eta_3 \sum_{n_1=1}^N \binom{N}{n_1} \frac{\left(p_*^{n_1}(1-p_*)^{N-n_1} \frac{n_1-Np_*}{p_*(1-p_*)}\right)^2}{p_*^{n_1}(1-p_*)^{N-n_1}}. \quad (4.110)$$

Proof. We can decompose the RHS into partial sums,

$$\text{RHS} = \eta_3 \sum_{n_1 \leq n_1^{(3)}} \binom{N}{n_1} \frac{\left(p_*^{n_1}(1-p_*)^{N-n_1} \frac{n_1-Np_*}{p_*(1-p_*)}\right)^2}{p_*^{n_1}(1-p_*)^{N-n_1}} \quad (4.111)$$

$$+ \eta_3 \sum_{n_1 > n_1^{(3)}} \binom{N}{n_1} \frac{\left(p_*^{n_1}(1-p_*)^{N-n_1} \frac{n_1-Np_*}{p_*(1-p_*)}\right)^2}{p_*^{n_1}(1-p_*)^{N-n_1}}. \quad (4.112)$$

We can bound the first term,

$$\eta_3 \sum_{n_1 \leq n_1^{(3)}} \binom{N}{n_1} \frac{\left(p_*^{n_1}(1-p_*)^{N-n_1} \frac{n_1-Np_*}{p_*(1-p_*)}\right)^2}{p_*^{n_1}(1-p_*)^{N-n_1}} \quad (4.113)$$

$$= \eta_3 \sum_{n_1 \leq n_1^{(3)}} \binom{N}{n_1} p_*^{n_1}(1-p_*)^{N-n_1} \left(\frac{n_1-Np_*}{p_*(1-p_*)}\right)^2 \quad (4.114)$$

$$\leq \eta_3 \frac{N^2}{(p_*(1-p_*))^2} \sum_{n_1 \leq n_1^{(3)}} \binom{N}{n_1} p_*^{n_1}(1-p_*)^{N-n_1} \quad (4.115)$$

This final sum can now be bounded from above using the Chernoff bound [91],

$$\sum_{n_1 \leq n_1^{(3)}} \binom{N}{n_1} p_*^{n_1} (1 - p_*)^{N - n_1} \leq \exp \left(-ND \left(\frac{n_1^{(3)}}{N} \parallel p_* \right) \right), \quad (4.116)$$

where $D \left(\frac{n_1^{(3)}}{N} \parallel p_* \right)$ is the Kullback-Leibler (KL) divergence for a binomial distribution.

For any $\epsilon_3 > 0$, the KL divergence is positive when $\frac{n_1^{(3)}}{N} < p_* - \epsilon_3$. For a positive KL divergence, the partial sum is exponentially suppressed in the limit of large N . In other words, $\forall \epsilon_3 > 0, \exists N_1^{(3)}$ such that $N > N_1^{(3)}$ implies

$$\eta_3 \frac{N^2}{p_*(1 - p_*)} \sum_{n_1 \leq n_1^{(3)}} p_*^{n_1} (1 - p_*)^{N - n_1} < \epsilon_3, \quad (4.117)$$

and so

$$\text{RHS} < \eta_3 \sum_{n_1 > n_1^{(3)}} \frac{\left(p_*^{n_1} (1 - p_*)^{N - n_1} \frac{n_1 - N p_*}{p_*(1 - p_*)} \right)^2}{p_*^{n_1} (1 - p_*)^{N - n_1}} + \epsilon_3. \quad (4.118)$$

Taking $\epsilon_3 < 1 - \eta_3$ proves the inequality of Equation 4.110.

□

We can combine these three bounds to show linear scaling. Let $p_2 = \max \{p_i \mid \alpha_i \neq 0, p_i < p_*\}$ be the second largest p_i . If such a p_i doesn't exist, then there is only one binomial distribution in the mixture. For that particular case, we know that the FI of a binomial distribution scales linearly in N , completing the proof that the SQL is achievable. Indeed the FI of a binomial distribution with parameter p_* is given by

$$\mathcal{I}(\omega) = \frac{N}{p_*(1 - p_*)} (\partial_\omega p_*)^2. \quad (4.119)$$

We now consider the case where p_2 is well defined. We choose $\epsilon_1, \epsilon_2, \epsilon_3$ (as in Lemmas 23, 25, and 26 respectively) small enough such that

$$\frac{\log \left(\frac{1 - p_2}{1 - p_*} \right)}{\log \left(\frac{p_*(1 - p_2)}{p_2(1 - p_*)} \right)} + \max \{\epsilon_1, \epsilon_2\} < p_* - \epsilon_3. \quad (4.120)$$

We show this inequality holds in Lemma 24. Then, for any $\eta_1, \eta_2 > d_E$ and $1 > \eta_3 > 0$, we know that for any N large enough there exists a corresponding n_1^* such that

$$p_2 < \frac{\log\left(\frac{1-p_2}{1-p_*}\right)}{\log\left(\frac{p_*(1-p_2)}{p_2(1-p_*)}\right)} + \max\{\epsilon_1, \epsilon_2\} < \frac{n_1^*}{N} < p_* - \epsilon_3, \quad (4.121)$$

and such that Equation 4.90, Equation 4.106, and Equation 4.110 hold. Applying these bounds allows us to derive the following inequalities for the FI.

We first remove a few terms of the sum of Equation 4.88. Since all the terms are positive, this leads to a lower bound.

$$\mathcal{I}(\omega) \geq \sum_{n_1 > n_1^*}^N \binom{N}{n_1} \frac{\left(\sum_{i=1}^{d_E} \alpha_i (\partial_\omega p_i) p_i^{n_1} (1-p_i)^{N-n_1} \frac{n_1 - N p_i}{p_i(1-p_i)}\right)^2}{\left(\sum_{i=1}^{d_E} \alpha_i p_i^{n_1} (1-p_i)^{N-n_1}\right)}. \quad (4.122)$$

Next, we apply Lemma 23 to lower bound the numerator.

$$\mathcal{I}(\omega) \geq \sum_{n_1 > n_1^*} \binom{N}{n_1} \cdot \frac{\left((1 - d_E \eta_1^{-1}) \alpha_* (\partial_\omega p_*) p_*^{n_1} (1-p_*)^{N-n_1} \frac{n_1 - N p_*}{p_*(1-p_*)}\right)^2}{\sum_{i=1}^{d_E} \alpha_i p_i^{n_1} (1-p_i)^{N-n_1}}, \quad (4.123)$$

where we have upper bounded each of the terms associated with p_i with $p_i \neq p_*$ by η_1^{-1} times the p_* term. We give each of these terms a sign opposite to that of the p_* term, so that we obtain a lower bound. We now apply Lemma 25 to bound the denominator.

$$\mathcal{I}(\omega) \geq \sum_{n_1 > n_1^*} \binom{N}{n_1} \cdot \frac{\left((1 - d_E \eta_1^{-1}) \alpha_* (\partial_\omega p_*) p_*^{n_1} (1-p_*)^{N-n_1} \frac{n_1 - N p_*}{p_*(1-p_*)}\right)^2}{(1 + d_E \eta_2^{-1}) \alpha_* p_*^{n_1} (1-p_*)^{N-n_1}}. \quad (4.124)$$

Similarly, we bound the denominator by bounding all the non- p_* terms by η_2^{-1} times the p_* term, and assigning them the same sign to obtain a lower bound. We can rewrite this bound as

$$\mathcal{I}(\omega) \geq \frac{(1 - d_E \eta_1^{-1})^2}{1 + d_E \eta_2^{-1}} \cdot \sum_{n_1 > n_1^*} \binom{N}{n_1} \frac{\left(\alpha_* (\partial_\omega p_*) p_*^{n_1} (1-p_*)^{N-n_1} \frac{n_1 - N p_*}{p_*(1-p_*)}\right)^2}{\alpha_* p_*^{n_1} (1-p_*)^{N-n_1}}. \quad (4.125)$$

Finally, we apply Lemma 26 to the inequality to obtain

$$\mathcal{I}(\omega) \geq \frac{(1 - d_E \eta_1^{-1})^2}{1 + d_E \eta_2^{-1}} \eta_3 \cdot \sum_{n_1=0}^N \binom{N}{n_1} \frac{\left(\alpha_* (\partial_\omega p_*) p_*^{n_1} (1 - p_*)^{N-n_1} \frac{n_1 - N p_*}{p_* (1 - p_*)} \right)^2}{\alpha_* p_*^{n_1} (1 - p_*)^{N-n_1}}. \quad (4.126)$$

We recognize the formula for the FI of a binomial distribution, and can therefore substitute the known result, Equation 4.119, to obtain a final lower bound for the FI of the binomial mixture.

$$\mathcal{I}(\omega) \geq \eta_3 \frac{(1 - d_E \eta_1)^2}{1 + d_E \eta_2} \alpha_*^2 (\partial_\omega p_*)^2 \frac{N}{p_* (1 - p_*)}. \quad (4.127)$$

We see that the final lower bound scales linearly in N , thus completing the proof that the SQL is achievable. □

4.4.2 Monte Carlo Estimation of the Fisher Information

We now consider the more general case, where the interaction need not be diagonal. We will use numerical methods to compute the Fisher information for a finite number of prepare and measure rounds and argue that the scaling appears linear, and therefore should saturate the standard quantum limit asymptotically. We will consider a large variety of models. In particular, the first model we consider will be a Heisenberg model-type interaction between two qubits with added dephasing noise. We will then consider randomly generated hidden Markov models.

We first discuss the numerical methods that we will employ. Consider the joint quantum state of N probes. If these probes are qudits, then the Hilbert space scales as d_P^N and so the computation of the QFI using exact diagonalisation and Equation 1.12 quickly becomes intractable as N grows large. To simplify the problem, we instead measure the probes at each time step and compute the FI, which, as mentioned in Section 1.2, lower bounds the QFI.

For N rounds of prepare and measure, we can associate the measurements to random variables $\{X_1, \dots, X_N\}$ which form a stochastic process. These random variables can be correlated, with the correlations mediated by the environment. It has been shown that if the process is stationary and its Markov order is finite, then the classical Fisher

information will scale linearly in N [92], saturating the SQL. The hidden Markov model under consideration does not generally possess these two properties.

We compute the classical Fisher information using Equation 1.7. Since the same probe state is prepared at every time step, the i -th measurement outcome depends solely on the residual state of the environment after the previous measurement. Consequently, it suffices to track only the $d_E \times d_E$ density matrix of the environment and its derivative throughout the time evolution. This approach enables the computation of the probabilities and their derivatives for the measurement outcomes $\{x_1, \dots, x_N\}$ using constant memory, thereby avoiding the exponential memory requirements associated with storing the full density matrix for N probes. We trade the memory overhead for exponential time complexity. If each random variable X_i takes on n_{out} possible different values, then there are $(n_{\text{out}})^N$ different measurement outcomes which must all be calculated separately to compute the FI exactly.

To get around this exponential scaling, we use random sampling to approximate the FI. Our methods are inspired by previous works using Monte Carlo to calculate the FI [93, 94]. We consider the following estimator for the FI:

$$\hat{\mathcal{I}}(\omega) = \frac{1}{S} \sum_{s=1}^S \left(\frac{\partial_{\omega} P(x_1^{(s)}, \dots, x_N^{(s)})}{P(x_1^{(s)}, \dots, x_N^{(s)})} \right)^2, \quad (4.128)$$

where $\{x_1^{(s)}, \dots, x_N^{(s)}\}$ is a sample from $\{X_1, \dots, X_N\}$ and the s index differentiates the samples. This estimator is unbiased :

$$\mathbb{E} [\hat{\mathcal{I}}(\omega)] = \frac{1}{S} \sum_{s=1}^S \mathbb{E} \left[\left(\frac{\partial_{\omega} P(x_1^{(s)}, \dots, x_N^{(s)})}{P(x_1^{(s)}, \dots, x_N^{(s)})} \right)^2 \right] = \frac{1}{S} \sum_{i=1}^S \mathcal{I}(\omega) = \mathcal{I}(\omega). \quad (4.129)$$

The variable S is the total number of samples. Each sample is obtained by computing the time-evolution for N time steps. At each measurement step, we randomly decide the outcome. For measurements with two possible outcomes, we randomly sample from a Bernoulli distribution with parameter $p = P(X_n = x_n | X_{n-1} = x_{n-1}, \dots)$, where $P(X_n = x_n | X_{n-1} = x_{n-1}, \dots)$ is given by the trace of the non normalized environment density matrix after projecting the probe into the state associated with a +1 outcome.

This method provides an approximate value of the FI. For a fixed N , the variance of this estimator scales as $O(S^{-1})$ and so convergence is guaranteed for a large number of samples. The constant hidden by the big- O may depend on N and t . We a priori do not

know this functional dependence. The variance is given by :

$$\text{Var} [\hat{\mathcal{I}}(\omega)] = \frac{1}{S^2} \sum_{i=1}^S \text{Var} \left[\left(\frac{\partial_\omega P(x_1^{(s)}, \dots, x_N^{(s)})}{P(x_1^{(s)}, \dots, x_N^{(s)})} \right)^2 \right] \quad (4.130)$$

$$= \frac{1}{S^2} \sum_{s=1}^S \left(\mathbb{E} \left[\left(\frac{\partial_\omega P(x_1^{(s)}, \dots, x_N^{(s)})}{P(x_1^{(s)}, \dots, x_N^{(s)})} \right)^4 \right] - \mathcal{I}(\omega)^2 \right) \quad (4.131)$$

$$\leq \frac{1}{S^2} \sum_{s=1}^S \mathbb{E} \left[\left(\frac{\partial_\omega P(x_1^{(s)}, \dots, x_N^{(s)})}{P(x_1^{(s)}, \dots, x_N^{(s)})} \right)^4 \right]. \quad (4.132)$$

Equation 4.132 can be estimated by taking the argument of Equation 4.128, which we already computed for the FI, and squaring it. These can be used to obtain a heuristic estimate for the variance of $\hat{\mathcal{I}}(\omega)$. It is heuristic because we do not a priori know the convergence of this estimator as N grows large. Although not rigorous, this estimator serves as evidence for convergence when small. Indeed, we see in Figure 4.4 that the error bars are hardly visible on the plots, and Figure 4.4 b) shows that for small system sizes the exact and estimated values of the Fisher information are in excellent agreement.

We will now go over the technical details of the simulation as well as provide some pseudo-code. As mentioned previously, the simulation is conducted by sampling measurement outcomes and then computing the corresponding probabilities and derivatives of the probabilities. We will work in the vectorized representation. The time dynamics will therefore be governed by

$$\hat{\Lambda}_t = e^{t\mathcal{L}_{SE} + \omega t\mathcal{L}_\omega}, \quad (4.133)$$

where $\mathcal{L}_{SE}$ and $\mathcal{L}_\omega$ correspond to the semi-group generators corresponding to the noise and signal terms of Equation 2.70, respectively. The noise semi-group generator is given by Equation 2.35, and the signal semi-group generator is given by

$$\mathcal{L}_\omega = (-iG) \oplus (i\bar{G}) + (-iH_E) \oplus (i\bar{H}_E). \quad (4.134)$$

Consider the first round of prepare and measure. The initial state of the system is $\rho_E^0 \otimes \rho_\psi$, where $\rho_\psi = |\psi\rangle\langle\psi|$ will be the state we prepare in each round. We then time-evolve for a duration t before performing a projective measurement in the basis $\{|\phi_0\rangle, \dots, |\phi_{d_P}\rangle\}$. The resulting state of the environment after measuring $|\phi_{x_1}\rangle$ is given by

$$|\rho_E^{x_1}\rangle\rangle = \left(\mathbb{1}_E \otimes \langle\phi_{x_1}| \otimes \mathbb{1}_E \otimes \overline{\langle\phi_{x_1}|} \right) \Lambda_t |\rho_E^0 \otimes \rho_\psi\rangle\rangle. \quad (4.135)$$

Working in the vectorized representation makes it easier to take the derivatives, as we can simply apply the product rule for matrix multiplication.

$$\frac{\partial}{\partial \omega} |\rho_E^{x_1}\rangle\rangle = \left(\mathbb{1}_E \otimes \langle \phi_{x_1} | \otimes \mathbb{1}_E \otimes \overline{\langle \phi_{x_1} |} \right) \left(\frac{\partial}{\partial \omega} \Lambda_t \right) |\rho_E^0 \otimes \rho_\psi\rangle\rangle, \quad (4.136)$$

where

$$\frac{\partial}{\partial \omega} \Lambda_t = \frac{\partial}{\partial \omega} \exp(t\mathcal{L}_{SE} + t\omega\mathcal{L}_\omega) \quad (4.137)$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \frac{\partial}{\partial \omega} (\mathcal{L}_{SE} + \omega\mathcal{L}_\omega)^k \quad (4.138)$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \sum_{i=0}^{k-1} (\mathcal{L}_{SE} + \omega\mathcal{L}_\omega)^i \mathcal{L}_\omega (\mathcal{L}_{SE} + \omega\mathcal{L}_\omega)^{k-i-1}. \quad (4.139)$$

If $\mathcal{L}_{SE}$ and $\mathcal{L}_\omega$ commute, then Equation 4.139 simplifies to

$$\frac{\partial}{\partial \omega} \Lambda_t = t\mathcal{L}_\omega \Lambda_t. \quad (4.140)$$

In the models studied in this section, however, this is not the case, and so Equation 4.139 will be computed numerically and approximately by taking the first 100 terms of the sum. We can then compute the probability of obtaining the outcome x_i as well as the derivative of the probability.

$$P(x_i) = \text{Tr}(\rho_E^{x_1}), \quad (4.141)$$

$$\frac{\partial}{\partial \omega} P(x_i) = \text{Tr} \left(\frac{\partial}{\partial \omega} \rho_E^{x_1} \right) = \langle \mathbb{1} | \frac{\partial}{\partial \omega} |\rho_E^{x_1}\rangle\rangle. \quad (4.142)$$

For the general case, if $|\rho_E^{x_1 x_2 \dots x_{k-1}}\rangle$ is the state of the environment after $k-1$ rounds of prepare and measure with measurement outcomes $x_1, \dots, x_{k-1}$, then state of the environment after measuring x_k in the k -th round is given by

$$|\rho_E^{x_1 x_2 \dots x_k}\rangle\rangle = \left(\mathbb{1}_E \otimes \langle \phi_{x_k} | \otimes \mathbb{1}_E \otimes \overline{\langle \phi_{x_k} |} \right) \Lambda_t |\rho_E^{x_1 x_2 \dots x_{k-1}} \otimes \rho_\psi\rangle\rangle, \quad (4.143)$$

and its derivative by

$$\begin{aligned} \frac{\partial}{\partial \omega} |\rho_E^{x_1 x_2 \dots x_k}\rangle\rangle &= \left(\mathbb{1}_E \otimes \langle \phi_{x_k} | \otimes \mathbb{1}_E \otimes \overline{\langle \phi_{x_k} |} \right) \left(\frac{\partial}{\partial \omega} \Lambda_t \right) |\rho_E^{x_1 x_2 \dots x_{k-1}} \otimes \rho_\psi\rangle\rangle \\ &+ \left(\mathbb{1}_E \otimes \langle \phi_{x_k} | \otimes \mathbb{1}_E \otimes \overline{\langle \phi_{x_k} |} \right) \Lambda_t \left(\frac{\partial}{\partial \omega} |\rho_E^{x_1 x_2 \dots x_{k-1}} \otimes \rho_\psi\rangle\rangle \right). \end{aligned} \quad (4.144)$$

And the probabilities can be calculated as before :

$$P(x_1, \dots, x_k) = \text{Tr}(\rho_E^{x_1 x_2 \dots x_k}), \quad (4.145)$$

$$\frac{\partial}{\partial \omega} P(x_1, \dots, x_k) = \text{Tr}\left(\frac{\partial}{\partial \omega} \rho_E^{x_1 x_2 \dots x_k}\right). \quad (4.146)$$

These probabilities can then be used to calculate the Fisher information at time step k using Equation 4.128. Further, we can use these probabilities to directly sample from measurement trajectories. For the models we will study numerically in the rest of this section, the probe is a two-level system, and the measurements are projective Pauli. There are therefore only two outcomes. We choose the value of $x_k \in \{0, 1\}$ by sampling from a Bernoulli distribution (coin flip) with parameter $P(x_k = 1)$. We can sample from the joint distribution $x_1, \dots, x_N$ by first deciding x_1 and then recursively constructing the probabilities with Equation 4.145 and deciding $x_2, x_3, \dots$ and so on by repeating the process. Algorithm 1 contains the pseudo code for the simulation algorithm used to estimate the Fisher information for the different time steps. It outputs the estimates for the Fisher information at different time steps as well as the variances.

4.4.3 Numerics for noise with a Heisenberg-type Interaction

We first consider the prepare-and-measure strategy for a model where the operators of the master equation do not commute. We once again model the probes and environment as qubits $\mathcal{H}_P \simeq \mathcal{H}_E \simeq \mathbb{C}^2$. We consider a signal given by $G = Z$ and a single jump operator $L = \mathbb{1}_E \otimes Z$. We consider a Heisenberg-type interaction for the coupling Hamiltonian

$$H_{PE} = \gamma (X \otimes X + Z \otimes Z + Y \otimes Y) \quad (4.147)$$

where γ is the coupling strength. We will consider two cases of the model. In the first case, there is no signal acting on the environment and $H_E = 0$. In the second case, there is a signal acting on the environment and $H_E = Z$.

We fix here the measurement basis to be the eigenstates of the Pauli X operator, and we prepare each probe in the state $|+_i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$. We fix the parameters to be $\gamma = 1$, $\omega = 0$, and $t = 0.25$. The initial state of the environment is obtained by sampling pure states from the Haar measure. The same five random states are used for both the $H_E = 0$ and $H_E = Z$ models. The Fisher information is then calculated using 25000 samples.

Figure 4.4 shows the classical Fisher information as a function of the number of prepare-and-measure rounds for the model where $H_E = 0$. The FI curves are calculated for five

Algorithm 1 Monte Carlo estimation of the Fisher information

```

 $s \leftarrow 0$ 
 $\text{FI} \leftarrow [0, 0, \dots, 0]$ 
 $\text{SampleFI} \leftarrow [0, 0, \dots, 0]$ 
while  $s < S$  do
     $\text{SampleFI} \leftarrow []$ 
     $\text{VariancesSampleFI} \leftarrow []$ 
     $|\rho_E \otimes \rho_\psi\rangle \leftarrow |\rho_E^0 \otimes \rho_\psi\rangle$ 
     $n \leftarrow 0$ 
    while  $n < N$  do
         $|\rho\rangle \leftarrow \Lambda_t |\rho_E \otimes \rho_\psi\rangle$ 
        if  $n \geq 1$  then
             $|\rho'\rangle \leftarrow \left( \frac{\partial}{\partial \omega \Lambda_t} \right) |\rho_E \otimes \rho_\psi\rangle + \Lambda_t |\partial_\omega \rho_E \otimes \rho_\psi\rangle$ 
        else
             $|\rho'\rangle \leftarrow \left( \frac{\partial}{\partial \omega \Lambda_t} \right) |\rho_E \otimes \rho_\psi\rangle$ 
        end if
         $\rho_P \leftarrow \text{Tr}_E(\rho)$ 
         $\rho'_P \leftarrow \text{Tr}_E(\rho')$ 
         $\alpha \leftarrow \frac{1}{\text{Tr}(\rho_P)} \langle \phi_1 | \rho_P | \phi_1 \rangle$ 
        if  $\text{ContinuousUniform}(0, 1) \leq \alpha$  then
             $b = 1$ 
        else
             $b = 0$ 
        end if
         $|\rho_E \otimes \rho_\psi\rangle \leftarrow \left( \mathbb{1}_E \otimes |\psi\rangle \langle \phi_b| \otimes \mathbb{1}_E \otimes \overline{|\psi\rangle \langle \phi_b|} \right) |\rho\rangle$ 
         $|\partial_\omega \rho_E \otimes \rho_\psi\rangle \leftarrow \left( \mathbb{1}_E \otimes |\psi\rangle \langle \phi_b| \otimes \mathbb{1}_E \otimes \overline{|\psi\rangle \langle \phi_b|} \right) |\rho'\rangle$ 
         $\text{SampleFI.insert} \left[ \frac{1}{S} \left( \frac{\frac{\partial}{\partial \omega} \langle \phi_b | \rho'_P | \phi_b \rangle}{\langle \phi_b | \rho_P | \phi_b \rangle} \right)^2 \right]$ 
         $\text{VariancesSampleFI.insert} \left[ \frac{1}{S^2} \left( \frac{\frac{\partial}{\partial \omega} \langle \phi_b | \rho'_P | \phi_b \rangle}{\langle \phi_b | \rho_P | \phi_b \rangle} \right)^4 \right]$ 
    end while
     $\text{FI} \leftarrow \text{FI} + \text{SampleFI}$  (Elementwise)
     $\text{VariancesFI} \leftarrow \text{FI} + \text{VariancesSampleFI}$  (Elementwise)
     $s = s + 1$ 
end while

```

pure Haar random initial environment states $\{\rho_0, \dots, \rho_4\}$. For $N \leq 15$, the FI is computed both exactly and by random sampling. At larger N , the values are exclusively from random sampling.

Figure 4.4 a) shows the FI values up to $N = 60$ as well as linear regression curves for the data. For each initial environment state ρ_i , a linear regression is performed on the values of the FI to show the linear (SQL) scaling. The results of the linear regressions are presented in Table 4.1. The linear regression gives us the slope of the linear relation along with the Pearson correlation coefficient r_c , which measures the linear correlation between N and the FI. A higher r_c indicates a stronger correlation, with $r_c \in [-1, 1]$. The r_c coefficients of Table 4.1 are all very close to 1, which strongly indicates that the relationship between the Fisher information and N is linear. Further, the values for the slopes are all above zero. Therefore, we see that the SQL is achieved up to depths of 60. We conjecture that this trend holds in the limit of $N \rightarrow \infty$.

Figure 4.4 b) shows the FI values up to $N = 15$ from both exact and random sampling methods. The curves in this figure serve only to guide the eye. This figure shows the agreement between the sampling results and exact computations.

Figure 4.5 and Table 4.2 show the same information but for the $H_E = Z$ model, where the signal also acts on the environment. Similarly to Figure 4.4 b), Figure 4.5 b) shows excellent agreement between the exact and sampled Fisher information. In Figure 4.5 a), the presence of signal on the environment seems to lower the values of the FI. Further, the scaling of the FI does not seem to be linear. This is reflected by the values of r_c obtained for the model in Table 4.2, which are much lower than those of Table 4.1. In particular, the values of r_c are not high enough to confidently conclude that the SQL is achieved.

Table 4.1: Linear regressions for the Fisher information as a function of the number of probes for the Heisenberg-type model with $H_E = 0$. Associated curves are shown in Figure 4.4.

i	Slope	r_c Coefficient
0	0.05755739026318657	0.9987758048158253
1	0.057886505716965755	0.997690310855459
2	0.05267854285583423	0.9980036537841903
3	0.05734837817591377	0.9990203746173694
4	0.0643685351834392	0.9956504727114459

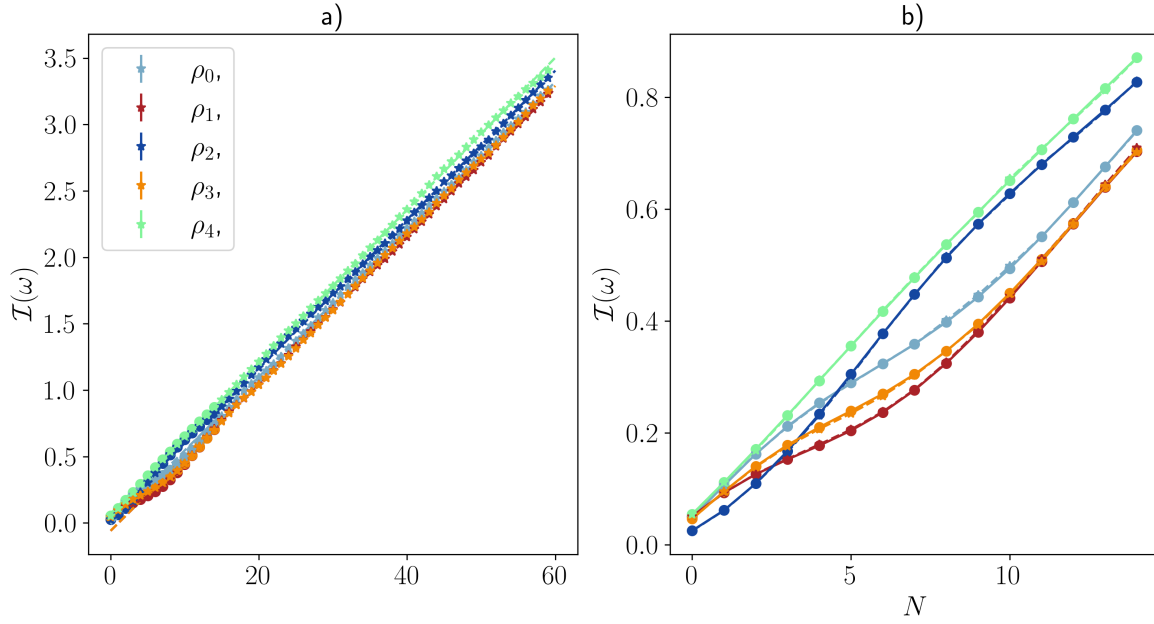


Figure 4.4: FI for the Heisenberg model with $H_E = 0$. Solid dots were calculated exactly, and stars were calculated by random sampling. Different colours correspond to different initial environment states ρ_i sampled from Haar random states. (a) FI for system sizes up to $N = 60$. Dashed lines show linear regression on the data. (b) Comparison of exact computation and Monte Carlo sampling FI values up to $N = 15$. Lines serve to guide the eye. Dashed and solid lines interpolate the random sampling and exact FI values respectively.

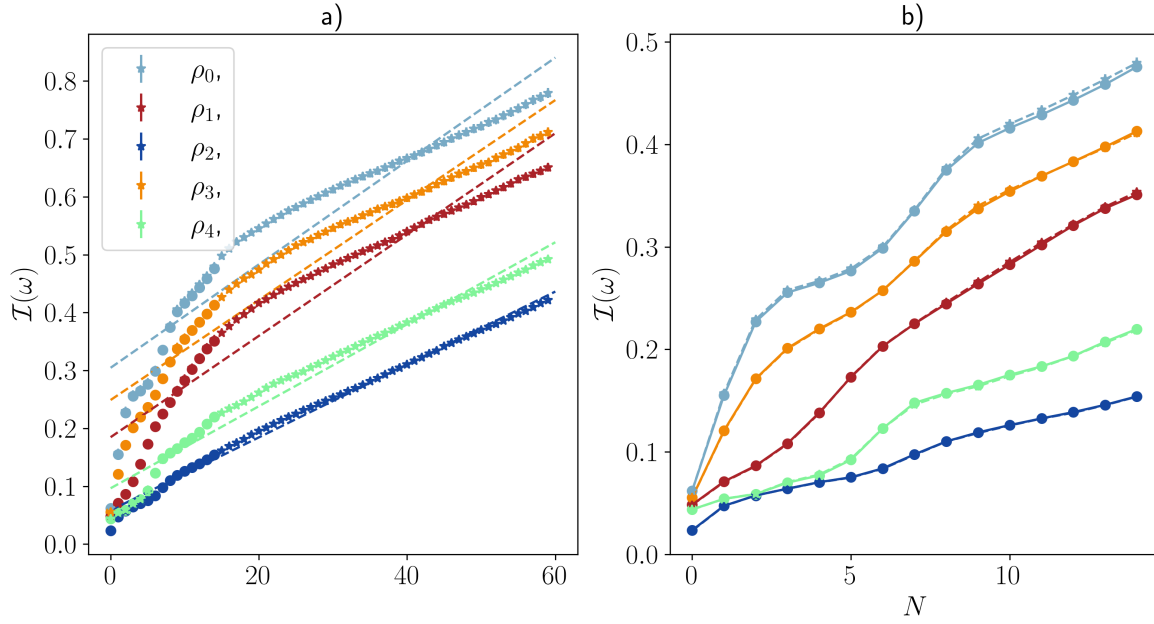


Figure 4.5: FI for the Heisenberg model with $H_E = Z$. Solid dots were calculated exactly, and stars were calculated by random sampling. Different colours correspond to different initial environment states ρ_i sampled from Haar random states. (a) FI for system sizes up to $N = 60$. Dashed lines show linear regression on the data. (b) Comparison of exact computation and Monte Carlo sampling FI values up to $N = 15$. Lines serve to guide the eye. Dashed and solid lines interpolate the random sampling and exact FI values respectively.

4.4.4 Numerics for Random Master Equations

To provide further evidence that the SQL is always achievable, we numerically study the prepare-and-measure strategy for a model where the noise operators of the master equation are randomly generated. We again model the probe and environment by qubits, $\mathcal{H}_P \simeq \mathcal{H}_E \simeq \mathbb{C}^2$. The protocol does not require any auxiliary degrees of freedom. We once again consider the signal to be $G = Z$ and suppose the parameter we are sensing is $\omega = 0$. For each round of the protocol, we prepare the probe in the state $|+_i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, allow the system to evolve for a time $t = 0.1$, before finally measuring in the eigenbasis of the X operator. As in the Heisenberg case, we consider two cases of the model, $H_E = 0$ and $H_E = Z$.

We consider a model with three jump operators. The jump operators are obtained by sampling random matrices from the Ginibre ensemble [95, 96]. For the coupling Hamiltonian H_{PE} , we sample a random matrix A from the Ginibre ensemble. We then set $H_{PE} = \frac{1}{2}(A + A^\dagger)$. Finally, the initial environment state is taken to be a Haar random pure state.

For a given master equation, we compute the FI as a function of the number of rounds N from $N = 1$ to $N = 30$. The values of the FI are calculated using the random sampling method of Section 4.4.2 with 25000 samples. A linear fit is then applied to the resulting curve, yielding a slope and an r_c coefficient. These quantities were computed for 1050 different random master equations. The same 1050 master equations are used for the $H_E = 0$ and $H_E = Z$ models.

The distributions of the r_c values and slopes for the $H_E = 0$ models are presented in Figure 4.6 a) and Figure 4.6 b), respectively. Firstly, we see from Figure 4.6 a) that all of the obtained r_c values were very close to 1, with the lowest being 0.9820 rounded to

Table 4.2: Linear regressions for the Fisher information as a function of the number of probes for the Heisenberg-type model with $H_E = Z$. Associated curves are shown in Figure 4.5.

i	Slope	r_c Coefficient
0	0.00893164136192775	0.937906503836237
1	0.008752382336452712	0.9550822417044361
2	0.006298895354174299	0.9968012417020177
3	0.00863620471038918	0.9498835201202465
4	0.007083260388947953	0.9857235070906991

the fourth digit. There is therefore a strong linear relationship between the FI and N for all of the random models studied. Therefore, it appears that linear scaling of the FI up to small fluctuations is a general property with this choice of strategy. Further, the minimum slope obtained was approximately $1.48466 \cdot 10^{-5}$. Although the minimal slope is quite small, it is still nonzero and of a few orders of magnitude above machine precision. All of the randomly generated models, therefore, have approximately linear FI scaling with a nonzero slope. This is strong evidence that the prepare-and-measure strategy can generically achieve the SQL. It is important to note that since we chose the same values of t , the same measurement basis, and the same initial state for all of the random noise models considered, the values of the FI obtained may be far from optimal, even among prepare-and-measure strategies. We can therefore expect an increase in performance by fine-tuning these parameters for each random model considered.

Figure 4.7 presents the same data for the $H_E = Z$ case of the random models. The minimum r_c value among the models is 0.8058 rounded to the fourth digit, and the minimum slope is $3.2925e-06$. We see that the majority of the r_c values of Figure 4.7 a) are clustered between 0.99 and 1, indicating that for the majority of the random models, the relationship is strongly linear and therefore the scaling is probably SQL. However, there are three noticeable outliers. It is therefore unclear whether the SQL is always achieved in the presence of an environment signal, $H_E = Z$.

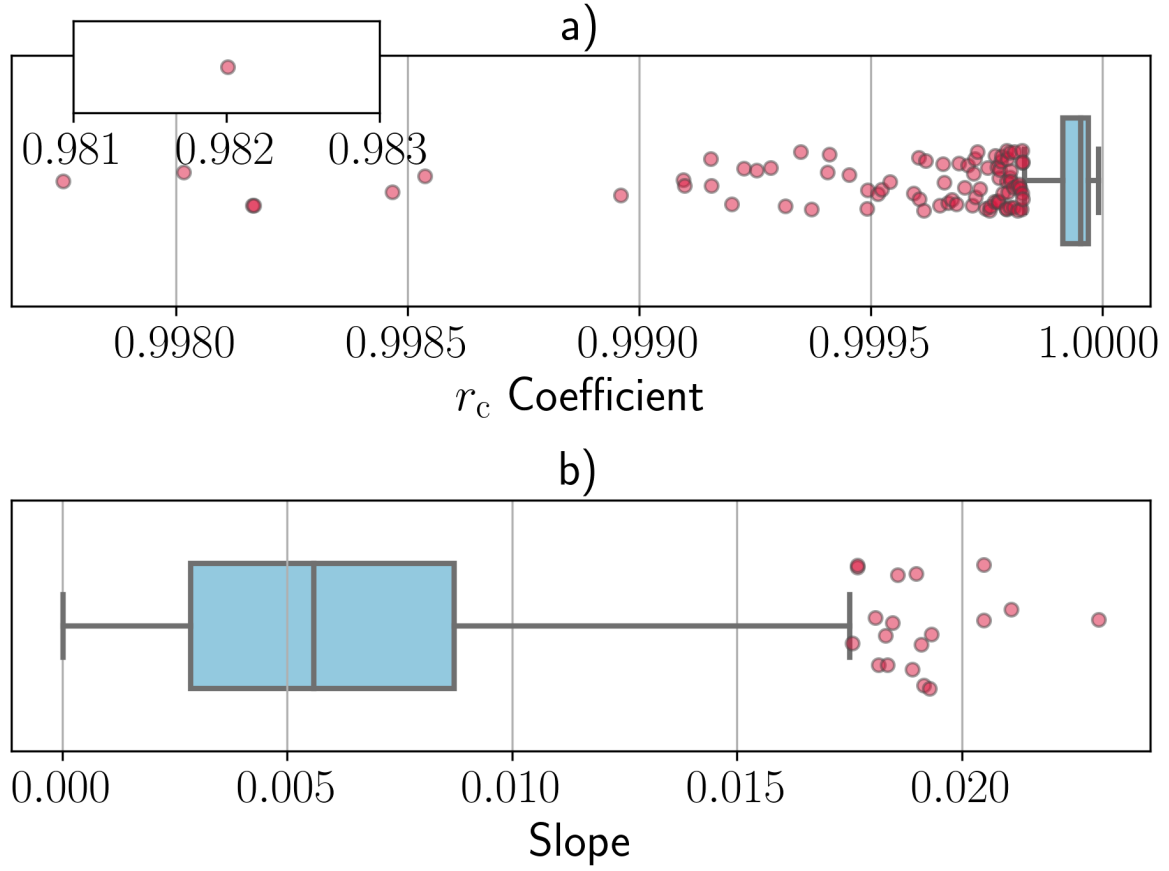


Figure 4.6: Statistics for the linear fits to the FI of 1050 random models with $H_E = 0$. The FI of each data point is obtained with a prepare-and-measure strategy with noise from a random master equation. The boxes extend from the first to third quartile, split by the median. The whiskers extend from the quartiles to 1.5 times the inter-quartile range. (a) Slopes of the linear fits. The sub-plot in the top-left corner shows an isolated outlier, and has been separated for aesthetic purposes. (b) Pearson coefficients r_c of the linear fits.

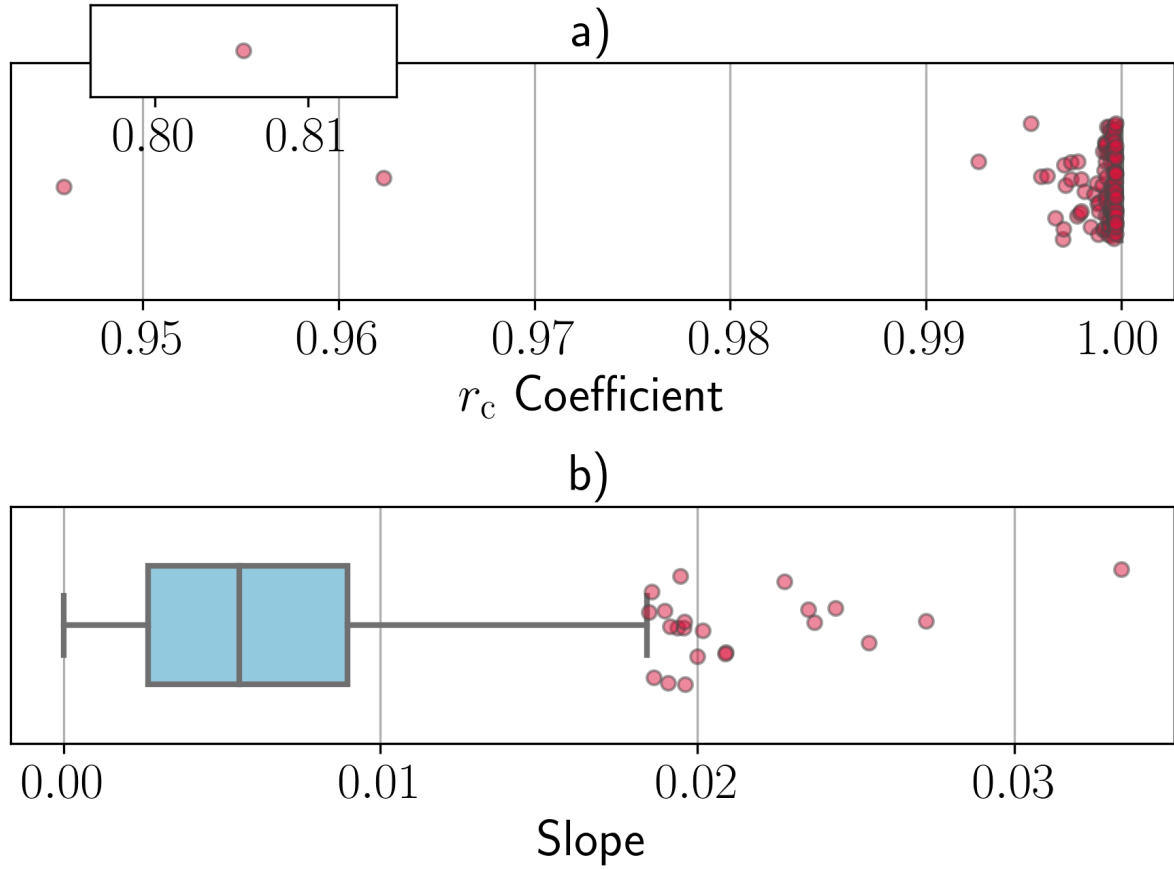


Figure 4.7: Statistics for the linear fits to the FI of 1050 random models with $H_E = Z$. The FI of each data point is obtained with a preparation and measure strategy with noise from a random master equation. (a) Slopes of the linear fits. The sub-plot in the top-left corner shows an isolated outlier and has been separated for aesthetic purposes. (b) Pearson coefficients r_c of the linear fits.

Chapter 5

Conclusion

In conclusion, we have studied

1. Quantum non-Markovianity, and the various models used to represent quantum non-Markovian processes.
2. Quantum error correction for non-Markovian noise, where the chosen model for non-Markovianity was the *hidden Markov model*.
3. Quantum metrology and quantum sensing with non-Markovian noise, and in particular, conditions for when optimal scaling can be achieved.

First, quantum non-Markovianity was explored in Chapter 2. Four important definitions of non-Markovianity were presented; quantum non-Markovianity based on the CP-divisibility of the process (Definition 3), quantum non-Markovianity based off of semigroup structure (Definition 5), quantum non-Markovianity based on information backflow (Definition 6), and non-Markovianity based on quantum combs (Definition 9). Different models for non-Markovian processes, including non-CP linear maps (Section 2.3) and hidden Markov models (Section 2.5), were discussed. Non-CP maps provide a mathematically compact way to represent dynamics that are non-Markovian under the first three definitions. However, they suffer from the drawback that they provide no information about the *context*. Not every initial state is compatible with every map, and the non-CP map may depend on how the input state was prepared. This means that there is no experimental tomography protocol that consistently constructs a non-CP map representing the dynamics. To remedy this issue, we instead considered hidden Markov models, and more generally,

quantum combs. Quantum combs contain the complete information of the dynamics. The trade-off is that they require exponentially more parameters to represent. The models can exhibit non-Markovianity under all four definitions presented in the chapter, and can be consistently reconstructed using the tomography protocols of [43, 44].

Next, non-Markovian quantum error correction was explored in Chapter 3. Quantum error correction conditions for perfect state recovery were derived in Section 3.2 for noise modeled by the hidden Markov model of Section 2.5. In particular, the Knill-Laflamme error correction conditions of [6] were generalized to the hidden Markov model and quantum combs in Theorem 12. These error correction conditions reduce to the standard Markovian conditions for an enlarged set of extended errors on the system, which correspond to matrix elements of the full error operators that act on both system and environment. Finally, quantum error correction conditions for non-CP maps were also considered. In particular, the results of [1] were presented, which include Knill-Laflamme conditions for non-CP maps (Theorem 13). Once again, these conditions reduce to the standard error correction conditions, where we construct the recovery operation for a quantum channel that is obtained by removing the negative signs from the operator sum representation of the non-CP map, which we wish to correct.

Finally, the application of non-Markovian error correction to quantum metrology was explored in Chapter 4. Quantum metrology was studied in noisy systems, where the noise was governed by a hidden Markov model. There are three timescales to consider when studying non-Markovian quantum metrology (Section 4.2). The first is the number of experiments where the full system, both probe and environment, is reset. We call each run an *experiment* or a *shot*. The optimal measurement precision, the Heisenberg limit, can't be achieved at this timescale as we don't allow for entanglement between runs. Since we also take experiments to be independent and identically distributed, the precision limit at this timescale is the standard quantum limit. We are instead interested in T , which is the total duration of an experiment where we may control and measure the probe, but leave the environment untouched. T can be subdivided into the number of times we destructively measure the probe, N , and the duration t between such measurements.

First, in Section 4.3, it was studied how quantum error correction can be used to obtain the Heisenberg limit in T despite the presence of noise. In Section 4.3.1, the HNLS conditions of [13, 14] were generalized to the non-Markovian regime. The HNLS conditions (Theorem 14) were presented. These conditions, when satisfied, guarantee that quantum error correction can be used to achieve the Heisenberg limit. Contrary to their Markovian counterparts, however, these conditions are sufficient but not necessary, as shown by example in Section 4.3.5. For systems where there are no dissipative terms, only unitary, in the master equation, it was shown that the Heisenberg limit can be achieved

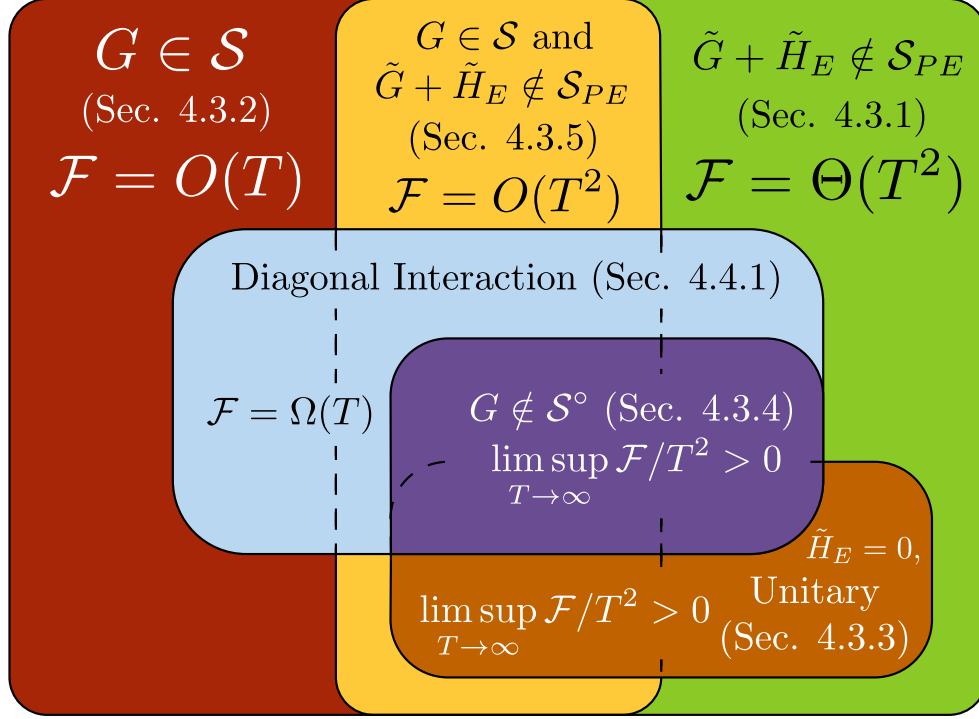


Figure 5.1: Summary of results of Chapter 4.

trivially by measuring at the correct times (Section 4.3.3, Theorem 16). For a special model where the operators of the master equation commute locally on the environment degrees of freedom, which we call a *diagonal interaction* (Definition 17), it was shown that the HNLS conditions could be weakened. Indeed, the less restrictive HNLS conditions (Section 4.3.4, Theorem 20) guarantee that the Heisenberg limit can still be achieved when measuring at the correct times, similarly to the unitary case. By direct application of [13, Theorem 1], the HNLS conditions were used to rule out Heisenberg limit scaling for certain classes of models (Section 4.3.2, Theorem 15). A diagram summarizing the results of the different sections is presented in Figure 5.1.

Following the study of Heisenberg limited scaling, the attainability of the standard quantum limit in T was considered in Section 4.4. To achieve this goal, we consider a specific sensing strategy known as prepare and measure. The strategy consists of preparing a specific probe state, letting it evolve without applying any controls, and then measuring. We also consider a slightly modified strategy where the second step is replaced by quantum error correction. We impose a measurement basis and compute the classical Fisher

information of the resulting probability distribution of outcomes. Since the Fisher information is a lower bound of the quantum Fisher information, observing linear scaling in T of the Fisher information guarantees that the standard quantum limit is achieved. First, it was shown in Section 4.4.1, Theorem 22, that the standard quantum limit can always be achieved in T for diagonal interactions. This provides strong evidence for the conjecture that the standard quantum limit is always achievable in T (Conjecture 21). To provide more evidence for the conjecture, examples where the classical Fisher information was computed numerically were presented in Section 4.4.2. An algorithm for estimating the Fisher information was presented (Algorithm 1). This algorithm allows for computations of a large number of prepare and measure rounds. The Fisher information was first considered for a Heisenberg-type interaction in Section 4.4.3. For that model, when the signal acts only on the probe and not the environment, there is strong linear correlation between T and the computed values of the Fisher information, showing strong evidence of standard quantum limit scaling. When the signal also acts on the environment, the numerics are inconclusive due to strong oscillations in the curves. The same approach was applied to randomly generated models in Section 4.4.4. For the random models, we also observe a strong linear correlation with a nonzero slope when the signal acts only on the probe. When the signal acts on the environment as well, most of the models have linear scaling aside from a few important outliers. Overall, these numerical examples provide strong evidence that the conjecture holds, and that prepare and measure achieves the standard quantum limit.

Many open problems and lines of inquiry remain. First, we considered quantum error correction for the hidden Markov models where the errors are perfectly corrected. The case of approximate error correction, where the state is only recovered up to some fidelity $1 - \epsilon$, is yet to be studied. As previously mentioned, it has been studied for non-CP maps in [47], but is yet to be considered for quantum combs. There are many important differences between approximate error correction in the Markovian and non-Markovian cases. In the non-Markovian case, information backflow allows the state to revive without application of a recovery operation, as seen for the case of quantum sensing in Section 4.3.3. This effect can perhaps be leveraged to increase the fidelity at later time steps, or perhaps it will be detrimental, similarly to the case of correlations decreasing the Fisher information of the state, as seen in [92]. Second, a proof of Conjecture 21 remains elusive. Finally, when considering quantum metrology, we have only considered the case where the signal Hamiltonian affects only the probe and the environment independently. More generally, there may be some effects of the signal on the unitary coupling and dissipative terms. The question arises of whether these effects are good or bad for quantum sensing, as well as how they affect the theorems of Chapter 4.

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