

DISPERSING REPRESENTATIONS OF SEMI-SIMPLE SUBALGEBRAS OF COMPLEX MATRICES

LAURENT W. MARCOUX¹, HEYDAR RADJAVI, AND YUANHANG ZHANG²

ABSTRACT. In this paper we consider the problem of determining the maximum dimension of $P^\perp(\mathcal{A} \oplus \mathcal{B})P$, where $\mathcal{A}$ and $\mathcal{B}$ are unital, semi-simple subalgebras of the set $\mathbb{M}_n$ of $n \times n$ complex matrices, and $P \in \mathbb{M}_{2n}$ is a projection of rank n . We exhibit a number of equivalent formulations of this problem, including the one which occupies the majority of the paper, namely: determine the minimum dimension of the space $\mathcal{A} \cap S^{-1}\mathcal{B}S$, where S is allowed to range over the invertible group $\text{GL}(n, \mathbb{C})$ of $\mathbb{M}_n$. This problem in turn is seen to be equivalent to the problem of finding two automorphisms α and β of $\mathbb{M}_n$ for which the dimension of $\alpha(\mathcal{A}) + \beta(\mathcal{B})$ is maximised. It is this phenomenon which gives rise to the title of the paper.

1. INTRODUCTION

Given a positive integer n , we write $\mathbb{M}_n$ to denote the algebra of $n \times n$ complex matrices. Throughout the paper we shall conflate linear operators acting on the finite-dimensional Hilbert space $\mathbb{C}^n$ with their matrices relative to the standard orthonormal basis $\{e_k\}_{k=1}^n$ of $\mathbb{C}^n$, and so $\mathbb{M}_n$ also represents the space $\mathcal{B}(\mathbb{C}^n)$ of (necessarily bounded) linear operators on $\mathbb{C}^n$. This should not cause a problem. The norm on $\mathbb{M}_n$ is just the usual (Hilbert space) operator norm.

This article is concerned with a number of questions that arose while studying the following rather general problem.

PROBLEM 1. Let $n \geq 1$ be an integer and $\mathcal{L}$ be a unital subalgebra of $\mathbb{M}_{2n}$. To what extent is the structure of $\mathcal{L}$ determined by the set

$$\{P^\perp \mathcal{L} P : P = P^* = P^2 \in \mathbb{M}_{2n}, \text{ rank } P = n\}$$

of its “off-diagonal corners”?

Since we shall often be dealing with the set of such projections, it will be useful to establish a notation for them. Given $0 \leq k \leq m$ integers, let us write $\mathcal{P}_{k,m} = \{P = P^2 = P^* \in \mathbb{M}_m : \text{rank } P = k\}$. We also define $\mathcal{E}_{k,m} = \{E = E^2 \in \mathbb{M}_m : \text{rank } E = k\}$. In analogy to the situation with projections, given $E \in \mathcal{E}_{k,m}$, we shall use the notation $E^\perp$ to denote the idempotent $I_m - E \in \mathcal{E}_{m-k,m}$. Of course this is an algebraic, and not a geometric, notion of orthogonality.

At this level of generality, Problem 1 is too vague to fully answer, but it has opened several interesting lines of investigation. Consider, for example, the following question.

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PROBLEM 2. Let $\mathcal{A}$ and $\mathcal{B}$ be unital subalgebras of $\mathbb{M}_n$, and set $\mathcal{L} := \mathcal{A} \oplus \mathcal{B}$. Determine – as a function of the structure of $\mathcal{A}$ and $\mathcal{B}$ – $\max\{\dim P^\perp \mathcal{L} P : P \in \mathcal{P}_{n,2n}\}$.

Once again – at this level of generality, this problem seems quite difficult at the moment, though it is definitely more tractable than the first. As we shall see below, in many cases, definitive answers may be obtained. By restricting the nature of the algebras which we take into consideration, we arrive at a problem which is both interesting and within reach.

PROBLEM 2S. Let $\mathcal{A}$ and $\mathcal{B}$ be unital, *semi-simple* subalgebras of $\mathbb{M}_n$, and set $\mathcal{L} := \mathcal{A} \oplus \mathcal{B}$. Determine the maximal possible dimension of $P^\perp \mathcal{L} P$, where $P \in \mathcal{P}_{n,2n}$, as a function of the structure of $\mathcal{A}$ and $\mathcal{B}$.

It is a consequence of the Wedderburn-Artin Theorem that every semi-simple subalgebra of $\mathbb{M}_n$ is similar to a direct sum of (inflations of) full matrix algebras; in other words, to a C^* -subalgebra of $\mathbb{M}_n$. It is interesting and certainly not obvious that Problem 2S should be equivalent to the restricted problem.

PROBLEM 2SC. Let $\mathcal{A}$ and $\mathcal{B}$ be unital, C^* -subalgebras of $\mathbb{M}_n$, and set $\mathcal{L} := \mathcal{A} \oplus \mathcal{B}$. Determine the maximal possible dimension of $P^\perp \mathcal{L} P$, where $P \in \mathcal{P}_{n,2n}$, as a function of the structure of $\mathcal{A}$ and $\mathcal{B}$.

The fact that Problems 2S and 2SC are equivalent is a result of a surprising (at least to us) consequence of our investigations, namely that solving Problem 2S is actually equivalent to solving the following problem:

PROBLEM 3S. Let $\mathcal{A}$ and $\mathcal{B}$ be unital, semi-simple subalgebras of $\mathbb{M}_n$. Determine the maximum possible dimension of $\mathcal{A} + S^{-1}\mathcal{B}S$, where $S \in \text{GL}(n, \mathbb{C})$, the set of invertible operators in $\mathbb{M}_n$.

This can be thought of as a “*dispersion*” of the algebras $\mathcal{A}$ and $\mathcal{B}$ using automorphisms of their ambient space $\mathbb{M}_n$, and it is what gives rise to the title of this paper. In light of the equivalence of these two problems, we are interested in finding an invertible operator S which “separates” or “disentangles” $\mathcal{A}$ and $\mathcal{B}$ as much as possible so as to maximise the dimension of $\mathcal{A} + S^{-1}\mathcal{B}S$.

In Section 3 below, we establish the relatively simple result that solving Problem 3S is in turn equivalent to solving the following problem, and it will be this version of Problem 2S we shall most often directly address.

PROBLEM 4S. Let $\mathcal{A}$ and $\mathcal{B}$ be unital, semi-simple subalgebras of $\mathbb{M}_n$. Determine the minimum possible dimension of $\mathcal{A} \cap S^{-1}\mathcal{B}S$, where $S \in \text{GL}(n, \mathbb{C})$.

Problem 4S naturally suggests yet another three variants of the question.

PROBLEM 4SC. Let $\mathcal{A}$ and $\mathcal{B}$ be unital, C^* -subalgebras of $\mathbb{M}_n$. Determine the minimum possible dimension of $\mathcal{A} \cap S^{-1}\mathcal{B}S$, where $S \in \text{GL}(n, \mathbb{C})$.

PROBLEM 4U. Let $\mathcal{A}$ and $\mathcal{B}$ be unital, semi-simple subalgebras of $\mathbb{M}_n$. Determine the minimum possible dimension of $\mathcal{A} \cap U^*\mathcal{B}U$, where $U \in \mathbb{M}_n$ is a unitary operator.

PROBLEM 4UC. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^* -subalgebras of $\mathbb{M}_n$. Determine the minimum possible dimension of $\mathcal{A} \cap U^*\mathcal{B}U$, where $U \in \mathbb{M}_n$ is a unitary operator.

It is not hard to see that Problems 4S and 4SC are “equivalent”, in the sense that given unital, semi-simple subalgebras $\mathcal{A}$ and $\mathcal{B}$ of $\mathbb{M}_n$, (and keeping in mind that they are similar to C^* -subalgebras $\mathcal{A}_0$ and $\mathcal{B}_0$ of $\mathbb{M}_n$ respectively), the minimum value occurring in Problem 4S for $\mathcal{A}$ and $\mathcal{B}$ equals the minimum value occurring in Problem 4SC for $\mathcal{A}_0$ and $\mathcal{B}_0$. We do not know at this time whether the corresponding Problems 4U and 4UC are “equivalent”

in the same sense. In every instance in which we have been able to determine the minima occurring in Problems 4SC and 4UC, we have arrived at the same answer. We do not know if equality occurs for all pairs $\mathcal{A}$ and $\mathcal{B}$ of unital C^* -algebras, or if a counterexample exists in one of the cases where we have been unable to compute the exact value of the minimum dimension. Although we suspect that there exists a direct argument to show that these quantities are the same, despite considerable effort, a proof has thus far eluded us.

While we have been able to resolve Problem 4S (and therefore Problem 2S) in a great many cases, there are a few outstanding cases which have remain. Nevertheless, we have succeeded in completely resolving the following restricted version of the problem, and this will be the focus of Sections 4, 5, and 6 of the paper.

PROBLEM 5. Determine those pairs of $\mathcal{A}$ and $\mathcal{B}$ of unital, semi-simple subalgebras of $\mathbb{M}_n$ for which

$$\min\{\dim \mathcal{A} \cap S^{-1}\mathcal{B}S : S \in \mathrm{GL}(n, \mathbb{C})\} = 1.$$

Since $\mathcal{A}$ and $\mathcal{B}$ are both unital, this is clearly the minimum possible value that the quantity on the left-hand side of the equation can attain. As will be seen, the above equality holds if and only if

$$\max\{\dim P^\perp(\mathcal{A} \oplus \mathcal{B})P : P \in \mathcal{P}_{n,2n}\} = \dim \mathcal{A} + \dim \mathcal{B} - 1.$$

We feel that it is indeed the varied techniques of proof, as well as the equivalence of two apparently unrelated problems (namely Problems 2S and 3S) which are of greatest interest in the results we obtain below.

The problems we are considering in this paper are not isolated, but are rather a continuation of earlier work of the authors and some of their colleagues in analysing sets of operators through their compressions to subspaces of the spaces upon which they act. For example, in the paper [4], the authors considered the problem of estimating the dimension of a unital subalgebra $\mathcal{A} \subseteq \mathbb{M}_n$ in terms of the dimension of its essential off-diagonal corners $\mathfrak{L} := P^\perp \mathcal{A}|_{\mathrm{ran} P}$, where $\mathfrak{L}$ is said to be *essential* if $\cap\{\ker L : L \in \mathfrak{L}\} = \{0\}$ and $\cap\{\ker L^* : L \in \mathfrak{L}\} = \{0\}$, and where L^* denotes the adjoint of L . It was shown, for example, that if $P \in \mathcal{P}_{n,2n}$ and $\mathcal{A} \subseteq \mathbb{M}_{2n}$ is a unital algebra for which $\mathfrak{L} := P^\perp \mathcal{A}P$ is an essential corner of dimension one, then one could deduce the structure of $\mathcal{A}$ fully from its dimension, which can be shown to lie between 2 and 4. Those algebras $\mathcal{A}$ which admit an essential corner of dimension 2 were also described. In [2], the authors considered the problem of analysing single operators $T \in \mathbb{M}_n$ (as well as bounded operators on a complex, separable Hilbert space $\mathcal{H}$) by studying the ranks and norms of their off-diagonal corners $P^\perp TP$. For example, it was shown that for $n \geq 4$, an operator $T \in \mathbb{M}_n$ satisfies $\|P^\perp TP\| = \|PTP^\perp\|$ for all projections $P \in \mathbb{M}_n$ if and only if $T = \alpha I + \beta L$ for some $\alpha, \beta \in \mathbb{C}$, where $L \in \mathbb{M}_n$ is either a hermitian operator or a unitary operator, and that this set coincides with the set of operators T for which $\mathrm{rank} P^\perp TP = \mathrm{rank} PTP^\perp$ for all projections P . In [5], those integers j and k for which there exist normal matrices $D \in \mathbb{M}_n$ and a projection P such that $\mathrm{rank} P^\perp DP = j$ while $\mathrm{rank} PDP^\perp = k$ were characterised.

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ADDENDUM

1.1. M. Mastnak has now answered Question 2, which appears in the last section of this paper, showing the more general result that if $\mathcal{X}$ and $\mathcal{Y}$ are arbitrary *subspaces* of $\mathbb{M}_n$, then

$$\max\{\dim(\mathcal{X} + S^{-1}\mathcal{Y}S) : S \in \mathrm{GL}(n, \mathbb{C})\} = \max\{\dim(\mathcal{X} + U^*\mathcal{Y}U) : U \in \mathbb{M}_n \text{ unitary}\}.$$

He and the editor have kindly agreed to allow us to add this result to this paper in the form of an Appendix.

A consequence of this result is that Problems 4U and 4UC are indeed equivalent in the sense described above.

2. DEFINITIONS AND PRELIMINARY RESULTS

2.1. This section consists of three distinct “units”. In the first unit, we establish some of the notation we shall require for the rest of the paper, and we prove the equivalence of solving Problem 3S and Problem 4S. We also prove three “*local stability*” results. The first of these relates to the minimum dimension obtained in Problem 4S, and shows that if $S_0 \in \mathrm{GL}(n, \mathbb{C})$ implements the minimum value of $\{\dim(\mathcal{A} \cap S^{-1}\mathcal{B}S) : S \in \mathrm{GL}(n, \mathbb{C})\}$, then so does any invertible operator T sufficiently close to S_0 . Our second stability result shows that if $\mathcal{X} \subseteq \mathbb{M}_n$ is a subspace, and if $\mathcal{P}$ denotes the set of all orthogonal projections in $\mathbb{M}_n$, then the map $P \mapsto \dim(P^\perp \mathcal{X} P)$ is lower semicontinuous. Our third and final stability result shows that if $\{X_j\}_{j=1}^d$ is a finite subset of $\mathbb{M}_n$ and $\mathcal{L}_1 := (\{X_j\}_{j=1}^d)'$ is its commutant (i.e. the algebra of all operators in $\mathbb{M}_n$ commuting with each X_j , $1 \leq j \leq d$), then by choosing Y_j sufficiently close to X_j , $1 \leq j \leq d$, the resulting commutant $\mathcal{L}_2 := (\{Y_j\}_{j=1}^d)'$ is of no greater dimension than that of $\mathcal{L}_1$.

These stability results will prove instrumental in allowing us to do computations to explicitly determine the numerical values appearing in Problems 2S, 3S and 4S. The reason for this is that they allow us to replace a given invertible operator by an invertible operator which is very close to the original but which has a certain prescribed form which makes it amenable for computations. Similarly, the second stability result allows us to replace a given projection by one which has a specific form which allows us to explicitly compute the maximum occurring in Problem 2S.

In the second unit, we exhibit the specific forms for invertible matrices and projections which will serve as tools to facilitate these computations. In the case of invertible operators, the corresponding tool relates to what might be viewed as a “*local higher-order numerical range*” for a fixed operator $T \in \mathbb{M}_n$. More explicitly, we shall show that if $T \in \mathbb{M}_n$ is an operator of rank r and if $\varepsilon > 0$, then there exists an invertible operator $S \in \mathbb{M}_n$ satisfying $\|S - I_n\| < \varepsilon$ for which every compression of $S^{-1}TS$ to a standard subspace of dimension at most r will be invertible. (Recall that we have fixed an orthonormal basis $\{e_k\}_{k=1}^n$ for $\mathbb{C}^n$; a subspace $\mathcal{M}$ of $\mathbb{C}^n$ is said to be a *standard* subspace if it is spanned by a subset of this orthonormal basis.) This device will prove useful in Section 5 of the paper. In the case of a projection, we shall demonstrate a “canonical form” which will allow us to analyse its commutant, which will also play a key role in our investigations.

Finally, in the third unit, we demonstrate the equivalence of solving Problems 2S, 3S and 4S.

2.2. We begin with some helpful notation. Although the subspaces we shall deal with in the remainder of the paper will be unital, semi-simple subalgebras (and often just C^* -subalgebras) of $\mathbb{M}_n$, the definitions make sense in a more general setting and suggest some questions for later investigation.

2.3. Definition. Let $n \in \mathbb{N}$, $\mathcal{X}, \mathcal{Y}$ be subspaces of $\mathbb{M}_n$ and $\mathcal{L}$ be a subspace of $\mathbb{M}_{2n}$. We define

- $\text{MINT}_S(\mathcal{X}, \mathcal{Y}) = \min\{\dim(\mathcal{X} \cap T^{-1}\mathcal{Y}T) : T \in \text{GL}(n, \mathbb{C})\},$
- $\text{MINT}_U(\mathcal{X}, \mathcal{Y}) = \min\{\dim(\mathcal{X} \cap U^*\mathcal{Y}U) : U \in \mathbb{M}_n \text{ unitary}\},$
- $\text{MAXSP}_S(\mathcal{X}, \mathcal{Y}) = \max\{\dim(\mathcal{X} + S^{-1}\mathcal{Y}S) : S \in \text{GL}(n, \mathbb{C})\}, \text{ and}$
- $\text{MAXSP}_U(\mathcal{X}, \mathcal{Y}) = \max\{\dim(\mathcal{X} + U^*\mathcal{Y}U) : U \in \mathbb{M}_n \text{ unitary}\}.$
- $\text{MODD}_n(\mathcal{L}) = \max\{\dim P^\perp \mathcal{L} P : P \in \mathcal{P}_{n, 2n}\}.$
- $\text{MIDD}_n(\mathcal{L}) = \max\{\dim E^\perp \mathcal{L} E : E \in \mathcal{E}_{n, 2n}\}.$

It is clear from the definitions that $\text{MINT}_S(\mathcal{X}, \mathcal{Y}) \leq \text{MINT}_U(\mathcal{X}, \mathcal{Y})$, that $\text{MAXSP}_S(\mathcal{X}, \mathcal{Y}) \geq \text{MAXSP}_U(\mathcal{X}, \mathcal{Y})$, and that $\text{MODD}_n(\mathcal{L}) \leq \text{MIDD}_n(\mathcal{L})$. We shall be particularly interested in the relationship between $\text{MINT}_S(\mathcal{X}, \mathcal{Y})$ and $\text{MODD}_n(\mathcal{X} \oplus \mathcal{Y})$.

2.4. The notations $\text{MINT}_S(\mathcal{X}, \mathcal{Y})$ and $\text{MAXSP}_S(\mathcal{X}, \mathcal{Y})$ are meant to express the *minimal intersection* and the *maximal span* of the spaces under consideration. The notation $\text{MODD}_n(\mathcal{L})$ alludes to the *maximum off-diagonal dimension*, and the subscripts S and U in the definition of $\text{MINT}_S, \text{MINT}_U, \text{MAXSP}_S$ and MAXSP_U are meant to indicate whether we are dealing with similarities or unitary operators. The notation $\text{MIDD}_n(\mathcal{L})$ refers to the *maximal "Idempotent" off-diagonal dimension*. A moment's thought should also convince the reader that

$$\text{MINT}_S(\mathcal{X}, \mathcal{Y}) = \min\{\dim(R^{-1}\mathcal{X}R \cap T^{-1}\mathcal{Y}T) : R, T \in \text{GL}(n, \mathbb{C})\}.$$

Thus $\text{MINT}_S(\mathcal{X}, \mathcal{Y})$ is a measure of how much any two copies of $\mathcal{X}$ and $\mathcal{Y}$ must intersect, where *copy* refers to the image of $\mathcal{X}$ (resp. of $\mathcal{Y}$) under an algebraic automorphism α of $\mathbb{M}_n$. (It is a standard result that any such automorphism of $\mathbb{M}_n$ is *inner* - that is, $\alpha(X) = S^{-1}XS$ for all $X \in \mathbb{M}_n$, so that it is implemented by conjugation by an element $S \in \text{GL}(n, \mathbb{C})$.)

When $\mathcal{X}$ and $\mathcal{Y}$ are semi-simple subalgebras of $\mathbb{M}_n$, we may replace $\mathcal{X}$ and $\mathcal{Y}$ by any two algebras $\mathcal{A}$ (similar to $\mathcal{X}$) and $\mathcal{B}$ (similar to $\mathcal{Y}$) to find that $\text{MINT}_S(\mathcal{X}, \mathcal{Y}) = \text{MINT}_S(\mathcal{A}, \mathcal{B})$. In particular, when dealing with unital, semi-simple algebras below, using the classical Wedderburn-Artin Theorem [1, 8], we may (and do) assume without loss of generality that $\mathcal{X}$ and $\mathcal{Y}$ are C^* -algebras.

In a similar fashion, we find that $\text{MAXSP}_S(\mathcal{X}, \mathcal{Y})$ is a measure of how much $\mathcal{X}$ and $\mathcal{Y}$ can be *separated* by automorphisms of $\mathbb{M}_n$. That is, $\text{MAXSP}_S(\mathcal{X}, \mathcal{Y}) = \max\{\dim(\alpha(X) + \beta(Y)) : \alpha, \beta \text{ automorphisms of } \mathbb{M}_n\}.$

Analogously,

$$\text{MINT}_U(\mathcal{X}, \mathcal{Y}) = \min\{\dim(U^*\mathcal{X}U \cap V^*\mathcal{Y}V) : U, V \in \mathbb{M}_n \text{ unitary}\},$$

measures how much any two copies of $\mathcal{X}$ and $\mathcal{Y}$ must intersect, where this time, *copy* refers to the image of $\mathcal{X}$ (resp. $\mathcal{Y}$) under an isometric automorphism of $\mathbb{M}_n$; equivalently under a $*$ -automorphism of $\mathbb{M}_n$.

We begin with the following simple observation, which exhibits the equivalence of solving either of Problems 3S and 4S.

2.5. Remark. Let $n \in \mathbb{N}$ and $\mathcal{X}$ and $\mathcal{Y}$ be subspaces of $\mathbb{M}_n$. For any invertible operator $T \in \mathbb{M}_n$,

$$\dim(\mathcal{X} + T^{-1}\mathcal{Y}T) = \dim \mathcal{X} + \dim \mathcal{Y} - \dim(\mathcal{X} \cap T^{-1}\mathcal{Y}T),$$

and thus

$$\text{MINT}_S(\mathcal{X}, \mathcal{Y}) = \dim \mathcal{X} + \dim \mathcal{Y} - \text{MAXSP}_S(\mathcal{X}, \mathcal{Y}).$$

This is an immediate consequence of the well-known fact that given two subspaces $\mathcal{V}$ and $\mathcal{W}$ of a vector space $\mathcal{Z}$, we have that

$$\dim(\mathcal{V} + \mathcal{W}) = \dim \mathcal{V} + \dim \mathcal{W} - \dim(\mathcal{V} \cap \mathcal{W}).$$

That same well-known fact allows us to obtain a parallel result in the “unitary setting”:

$$\text{MINT}_U(\mathcal{X}, \mathcal{Y}) = \dim \mathcal{X} + \dim \mathcal{Y} - \text{MAXSP}_U(\mathcal{X}, \mathcal{Y}).$$

Before establishing the first of our three local stability results, we mention the following lemma (for which we do not claim any originality). Its proof is left to the reader.

2.6. Lemma. *Let $n \in \mathbb{N}$ and $\mathcal{V}$ be an n -dimensional complex normed linear space. Suppose that $\mathfrak{L}_1 = \{u_1, u_2, \dots, u_d\} \subseteq \mathcal{V}$ is linearly independent. Then there exists $\varepsilon > 0$ such that if $x_i \in \mathcal{V}$ satisfy $\|x_i - u_i\| < \varepsilon$, $1 \leq i \leq d$, then $\mathfrak{L}_2 := \{x_1, x_2, \dots, x_d\}$ is linearly independent.*

2.7. The next lemma provides us with our first “local stability” result. Presented with a similarity which implements the value of $\text{MINT}_S(\mathcal{A}, \mathcal{B})$, it will allow us to perturb that similarity to obtain a second similarity which still implements the value of $\text{MINT}_S(\mathcal{A}, \mathcal{B})$, but which also admits other desirable properties which make it amenable to analysis. In particular, in many cases it will allow us to explicitly compute the numerical value of $\text{MINT}_S(\mathcal{A}, \mathcal{B})$. A similar result holds for unitary operators and $\text{MINT}_U(\mathcal{A}, \mathcal{B})$.

2.8. Proposition. *Let $n \in \mathbb{N}$, and let $\mathcal{X}, \mathcal{Y}$ be subspaces of $\mathbb{M}_n$. Choose $S_0 \in \text{GL}(n, \mathbb{C})$ such that*

$$\dim(\mathcal{X} + S_0^{-1}\mathcal{Y}S_0) = \text{MAXSP}_S(\mathcal{X}, \mathcal{Y}),$$

then there exists $\delta > 0$ such that if $S \in \mathbb{M}_n$ and $\|S - S_0\| < \delta$ implies that S is invertible and

$$\dim(\mathcal{X} + S^{-1}\mathcal{Y}S) = \text{MAXSP}_S(\mathcal{X}, \mathcal{Y}).$$

It follows that if $T_0 \in \mathbb{M}_n$ is invertible and

$$\dim(\mathcal{X} \cap T_0^{-1}\mathcal{Y}T_0) = \text{MINT}_S(\mathcal{X}, \mathcal{Y}).$$

Then there exists $\delta > 0$ such that if $T \in \mathbb{M}_n$ and $\|T - T_0\| < \delta$, then T is invertible and

$$\dim(\mathcal{X} \cap T^{-1}\mathcal{Y}T) = \text{MINT}_S(\mathcal{X}, \mathcal{Y}).$$

Proof. For invertible S , consider the linear map

$$T_S : \mathcal{X} \oplus \mathcal{Y} \rightarrow \mathbb{M}_n,$$

defined by $T_S(X, Y) = X + S^{-1}YS$. Note that

$$\dim(\mathcal{X} + S^{-1}\mathcal{Y}S) = \text{rank}(T_S).$$

Since rank is lower semicontinuous, and since T_S depends continuously on S , we conclude that $\text{rank}(T_S) = \text{rank}(T_{S_0})$ for S sufficiently close to S_0 .

The second assertion follows immediately from the above argument and the facts (outlined in Remark 2.5) that

$$\dim(\mathcal{X} + T^{-1}\mathcal{Y}T) = \dim \mathcal{X} + \dim \mathcal{Y} - \dim(\mathcal{X} \cap T^{-1}\mathcal{Y}T),$$

and thus

$$\text{MAXSP}_S(\mathcal{X}, \mathcal{Y}) = \dim \mathcal{X} + \dim \mathcal{Y} - \text{MINT}_S(\mathcal{X}, \mathcal{Y}).$$

□

Proposition 2.9 is mentioned only for completeness, and it will not be used in the sequel. Its proof is a mild adaptation of the proof of Proposition 2.8 and is left to the reader.

2.9. Proposition. *Let $n \in \mathbb{N}$, and let $\mathcal{X}, \mathcal{Y}$ be subspaces of $\mathbb{M}_n$. Choose $U_0 \in \mathbb{M}_n$ unitary such that*

$$\dim(\mathcal{X} \cap U_0^{-1} \mathcal{Y} U_0) = \text{MINT}_U(\mathcal{X}, \mathcal{Y}).$$

Then there exists $\delta > 0$ such that if $U \in \mathbb{M}_n$ is unitary and $\|U - U_0\| < \delta$, then

$$\dim(\mathcal{X} \cap U^{-1} \mathcal{Y} U) = \text{MINT}_U(\mathcal{X}, \mathcal{Y}).$$

We are now ready to prove our second local stability result.

2.10. Lemma. *Let $n \in \mathbb{N}$ and let $\mathcal{X}$ be a subspace of $\mathbb{M}_n$. Denote by $\mathcal{P} := \{P \in \mathbb{M}_n : P = P^* = P^2\}$ the set of orthogonal projections in $\mathbb{M}_n$ and by $\mathcal{E} := \{E \in \mathbb{M}_n : E^2 = E\}$ the set of all idempotents in $\mathbb{M}_n$. The maps*

$$\begin{array}{ccc} \varrho : \mathcal{P} & \rightarrow & \mathbb{N} \cup \{0\} \\ P & \mapsto & \dim(P^\perp \mathcal{X} P) \end{array} \quad \text{and} \quad \begin{array}{ccc} \tau : \mathcal{E} & \rightarrow & \mathbb{N} \cup \{0\} \\ E & \mapsto & \dim(E^\perp \mathcal{X} E) \end{array}$$

are lower-semicontinuous.

Proof. Let $\chi : \mathbb{M}_n \rightarrow \mathbb{N} \cup \{0\}$ be defined by

$$\chi(A) = \dim((I - A)\mathcal{X}A).$$

Note that χ is lower semicontinuous. This is because

$$\dim((I - A)\mathcal{X}A) = \text{rank}(S_A),$$

where S_A is the linear map

$$S_A : \mathcal{X} \rightarrow \mathbb{M}_n, \quad X \mapsto (I - A)XA.$$

The assertions of the lemma about ϱ and τ now follow by restricting χ to $\mathcal{P}$ and $\mathcal{E}$ respectively.

□

2.11. We finish this unit with a stability result which we shall require in Section 5. As we have mentioned in the introduction, solving Problem 2S is equivalent to solving Problem 4SC, which we may now reformulate as the problem of determining $\text{MINT}_S(\mathcal{A}, \mathcal{B})$ when $\mathcal{A}$ and $\mathcal{B}$ are unital C^* -subalgebras of $\mathbb{M}_n$.

In the finite-dimensional setting, if $\mathcal{A} \subseteq \mathbb{M}_n$ is a unital C^* -algebra, then $\mathcal{A}'$, the commutant of $\mathcal{A}$, is again a C^* -algebra and $\mathcal{A} = \mathcal{A}'' := (\mathcal{A}')'$. (This is the finite-dimensional version of the well-known Double Commutant Theorem of von Neumann (see, e.g. [6].) Since every finite-dimensional C^* -algebra is obviously finitely-generated as an algebra, we may choose generators $\{X_j\}_{j=1}^d$ for $\mathcal{A}'$, and thereby present $\mathcal{A}$ as $\mathcal{A}'' = (\{X_j\}_{j=1}^d)'$. The next lemma shows that if we “slightly” perturb each X_j to obtain an operator Y_j , $1 \leq j \leq d$, then the resulting algebra $\mathcal{B} := (\{Y_j\}_{j=1}^d)'$ has dimension no greater than that of $\mathcal{A}$. In fact, the lemma is even more general than this. We refer the reader to the paper [3] which investigates non-selfadjoint subalgebras of $\mathbb{M}_n$ which coincide with their double commutants, where a similar analysis may be applied.

2.12. Lemma. *Let $\{X_1, X_2, \dots, X_d\}$ be a finite subset of $\mathbb{M}_n$, and consider its commutant*

$$\mathcal{L}_1 := \{X_1, X_2, \dots, X_d\}'.$$

There exists $\varepsilon > 0$ such that if $\{Y_1, Y_2, \dots, Y_d\} \subseteq \mathbb{M}_n$ satisfies

$$\|Y_j - X_j\| < \varepsilon, \quad \text{for all } 1 \leq j \leq d,$$

then

$$\dim \{Y_1, Y_2, \dots, Y_d\}' \leq \dim \mathcal{L}_1.$$

Proof. First observe that $\{X_1, X_2, \dots, X_d\}' = \{X_1\}' \cap \{X_2\}' \cap \dots \cap \{X_d\}'$.

The defining equations for the algebra $\mathcal{L}_1 := \{X_1, X_2, \dots, X_d\}'$, namely that $L \in \mathcal{L}_1$ if and only if $X_j L - L X_j = 0$, $1 \leq j \leq d$, give rise to a homogeneous system of dn^2 equations in the n^2 matrix entries of $L \in \mathbb{M}_n$. Now fix an order for these matrix entries and consider the $dn^2 \times n^2$ matrix of coefficients M for this system whose entries are from among those of the matrices X_j . It is clear that a sufficiently small perturbation of the entries of M can only increase its rank, that is, it can only decrease the dimension of the solution space.

From this we conclude that there is an $\varepsilon > 0$ with the desired property. $\square$

MATRIX FORMS FOR PROJECTIONS AND INVERTIBLE OPERATORS

2.13. We now turn our attention to describing matrix forms for projections and invertible operators. These are the forms which will facilitate computations in the subsequent sections of the paper. For self-adjoint operators $H, K \in \mathbb{M}_n$, we use the standard notation $H \leq K$ to indicate that $K - H$ is a positive semi-definite operator (i.e. it is self-adjoint with non-negative eigenvalues, while $H < K$ means that $K - H$ is a positive definite operator (i.e. it is self-adjoint with strictly positive eigenvalues).

2.14. Lemma. *Let $m, n \in \mathbb{N}$ and $P = \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \in \mathbb{M}_{m+n}$ be a projection, where $X \in \mathbb{M}_m$ satisfies $0 < X < I_m$ and $Z \in \mathbb{M}_n$ satisfies $0 < Z < I_n$. Then $m = n$ and there exist commuting, positive operators $0 < M, N < I_n$ satisfying $M^2 + N^2 = I_n$, and a unitary operator $V \in \mathbb{M}_n$ such that*

$$P = \begin{bmatrix} M^2 & MNV \\ V^*MN & V^*N^2V \end{bmatrix}.$$

Proof. First note that $P = P^2$ implies that $X - X^2 = YY^*$ and $Z - Z^2 = Y^*Y$. Since $\text{rank } YY^* = \text{rank } Y^*Y$, we see that $\text{rank } (X - X^2) = \text{rank } (Z - Z^2)$. But $0 < X < I_m$ and $0 < Z < I_n$, so that $X - X^2 \in \mathbb{M}_m$ and $Z - Z^2 \in \mathbb{M}_n$ are invertible. Hence

$$m = \text{rank } (X - X^2) = \text{rank } (Z - Z^2) = n.$$

Next, since $0 < X < I_n$, we may write $X = M^2$ for some $0 < M < I_n$. Using the polar decomposition, we write $Y = |Y^*|V$ for some unitary operator $V \in \mathbb{M}_n$, and define $N = (I - M^2)^{\frac{1}{2}}$. Clearly $NM = MN$ and $M^2 + N^2 = I_n$. Also, from above we have that

$$|Y^*|^2 = X - X^2 = M^2 - M^4 = M^2(I - M^2) = M^2N^2,$$

and thus $|Y^*| = MN$ (the latter being positive since $MN = NM$). In particular, $Y = |Y^*|V = MNV$ is invertible.

The equation $P^2 = P$ also implies that $XY + YZ = Y$, or equivalently that $YZ = (I - X)Y$. Since Y is invertible, this then implies that

$$\begin{aligned} Z &= Y^{-1}(I - X)Y \\ &= V^*|Y^*|^{-1}(I - M^2)|Y^*|V \\ &= V^*(MN)^{-1}(N^2)(MN)V \\ &= V^*N^2V. \end{aligned}$$

□

2.15. Lemma. *Let $r \in \mathbb{N}$ and $P = \begin{bmatrix} I_r & 0 \\ 0 & 0_r \end{bmatrix} \in \mathbb{M}_{2r}$. Given $\varepsilon > 0$, there exists a projection $Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^* & Q_4 \end{bmatrix} \in \mathbb{M}_{2r}$ such that $\|P - Q\| < \varepsilon$, $0 < Q_1, Q_4 < I_r$, and $\sigma(Q_1 \oplus Q_4)$ consists of $2r$ distinct eigenvalues.*

Proof. Let $0 < \varepsilon < 1$. Choose $d_1 < d_2 < \dots < d_r < 1$ such that $\sqrt{1 - d_i^2} < \frac{\varepsilon}{4}$, $1 \leq i \leq r$ and set $D := \text{diag}(d_i)_{i=1}^r$. Set $Q_1 = D^2$, $Q_4 = (I_r - D^2)$, and $Q_2 = D(I_r - D^2)^{\frac{1}{2}}$.

That $Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^* & Q_4 \end{bmatrix}$ is a projection is a routine calculation which is left to the reader. Now,

$$\|Q_1 - I_r\| = \|D^2 - I_r\| = \|Q_4 - 0_r\| = \max\{|d_i^2 - 1| : 1 \leq i \leq r\} < \left(\frac{\varepsilon}{4}\right)^2 < \frac{\varepsilon}{4}.$$

Also,

$$\|Q_2\| \leq \|D\| \|(I_r - D^2)^{\frac{1}{2}}\| \leq 1 \cdot (\max\{\sqrt{1 - d_i^2} : 1 \leq i \leq r\}) \leq \frac{\varepsilon}{4}.$$

Thus

$$\|P - Q\| \leq \|Q_1 - I_r\| + 2\|Q_2\| + \|Q_4 - 0\| < \varepsilon.$$

Finally, it is clear that $Q_1 = D^2$ has r distinct eigenvalues all of which lie in the interval $(\frac{3}{4}, 1)$, while $\sigma(Q_4) = \sigma(I_r - D^2) = 1 - \sigma(D^2) \subseteq (0, \frac{1}{4})$, so that $\sigma(Q_1 \oplus Q_4)$ has cardinality $2r$.

□

We remark in passing that it is straightforward to see that we can always choose the values $d_1, d_2, \dots, d_r < 1$ in the above lemma to avoid (i.e. be disjoint from) any pre-chosen finite set $\Lambda \subseteq [0, \infty)$.

2.16. Lemma. *Let $n \in \mathbb{N}$ and $P \in \mathbb{M}_2(\mathbb{M}_n)$ be a projection of rank n . Given $\varepsilon > 0$ there exist commuting operators $0 < M, N < I_n$ satisfying $M^2 + N^2 = I_n$ and $V \in \mathbb{M}_n$ unitary such that*

$$R := \begin{bmatrix} M^2 & MNV \\ V^*MN & V^*N^2V \end{bmatrix}$$

defines a projection for which $\|R - P\| < \varepsilon$ and $\sigma(M^2 \oplus N^2)$ has cardinality $2n$.

Proof. Let $0 < \varepsilon < 1$. Suppose that $P = \begin{bmatrix} P_1 & P_2 \\ P_2^* & P_4 \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ is a general projection of rank n . Then $0 \leq P_1, P_4 \leq I_n$, and we can write $P_1 = P_1^\circ \oplus X_0$, $P_4 = Z_0 \oplus P_4^\circ$, where

$P_1^\circ \in \mathbb{M}_m$, $P_4^\circ \in \mathbb{M}_{m_1}$ are projections for some $0 \leq m, m_1 \leq n$, and $0 < X_0 < I_{n-m}$, $0 < Z_0 < I_{n-m_1}$. Thus

$$P = \begin{bmatrix} P_1^\circ & 0 & 0 & 0 \\ 0 & X_0 & Y_0 & 0 \\ 0 & Y_0^* & Z_0 & 0 \\ 0 & 0 & 0 & P_4^\circ \end{bmatrix}.$$

Applying Lemma 2.14 above to the subprojection $Q_0 := \begin{bmatrix} X_0 & Y_0 \\ Y_0^* & Z_0 \end{bmatrix}$, we see that $n - m = n - m_1$ and thus $m_1 = m$. By that same lemma, there exist commuting operators $0 < M_0, N_0 < I_{n-m}$ and $V_0 \in \mathbb{M}_{n-m}$ unitary such that $M_0^2 + N_0^2 = I_{n-m}$ and

$$Q_0 = \begin{bmatrix} M_0^2 & M_0 N_0 V_0 \\ V_0^* M_0 N_0 & V_0^* N_0^2 V_0 \end{bmatrix}.$$

Furthermore, it is not hard to see from this that $\text{rank } Q_0 = n - m$. Let $0 < M_1 < I_{n-m}$ be an operator with $n - m$ distinct eigenvalues such that $\sigma(M_1^2) \cap \sigma(I_{n-m} - M_1^2) = \emptyset$,

$$\max(\|M_1 - M_0\|, \|M_1^2 - M_0^2\|, \|M_1(I_{n-m} - M_1^2)^{\frac{1}{2}} - M_0(I_{n-m} - M_0^2)^{\frac{1}{2}}\|) < \frac{\varepsilon}{4},$$

and set $N_1 = (I_{n-m} - M_1^2)^{\frac{1}{2}}$. Then an easy calculation shows that

$$R_0 := \begin{bmatrix} M_1^2 & M_1 N_1 V_0 \\ V_0^* M_1 N_1 & V_0^* N_1^2 V_0 \end{bmatrix}$$

is a projection of rank $n - m$ such that $\|R_0 - Q_0\| < \varepsilon$. Moreover, $\sigma(M_1^2 \oplus V_0^* N_1^2 V_0)$ has cardinality $2(n - m)$.

The fact that $\text{rank } Q_0 = n - m$ implies that the rank of the projection

$$\begin{bmatrix} P_1^\circ & 0 \\ 0 & P_4^\circ \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_m)$$

is m . But then $r := \text{rank } P_1^\circ = m - \text{rank } P_4^\circ$, and so without loss of generality, we may write

$$\begin{bmatrix} P_1^\circ & 0 \\ 0 & P_4^\circ \end{bmatrix} = \begin{bmatrix} I_r & 0 & 0 & 0 \\ 0 & 0_{m-r} & 0 & 0 \\ 0 & 0 & I_{m-r} & 0 \\ 0 & 0 & 0 & 0_r \end{bmatrix}.$$

Next, by applying Lemma 2.15 to each of $I_r \oplus 0_r$ and $0_{m-r} \oplus I_{m-r}$, it is not hard to see that we can find diagonal matrices D_1 and D_4 such that

$$\sigma(D_1^2 \oplus (I_{m-r} - D_2^2) \oplus D_2^2 \oplus (I_r - D_1^2))$$

has cardinality $2m$; it is disjoint from $\sigma(M_1^2 \oplus N_1^2)$, and

$$R^\circ := \begin{bmatrix} D_1^2 & 0 & 0 & D_1(I_r - D_1^2)^{\frac{1}{2}} \\ 0 & I_{m-r} - D_2^2 & D_2(I_{m-r} - D_2^2)^{\frac{1}{2}} & 0 \\ 0 & D_2(I_{m-r} - D_2^2)^{\frac{1}{2}} & D_2^2 & 0 \\ D_1(I_r - D_1^2)^{\frac{1}{2}} & 0 & 0 & I_r - D_1^2 \end{bmatrix}$$

defines a projection satisfying $\|R^\circ - \begin{bmatrix} P_1^\circ & 0 \\ 0 & P_4^\circ \end{bmatrix}\| < \varepsilon$.

Setting $M := D_1 \oplus (I_{m-r} - D_2^2)^{\frac{1}{2}} \oplus M_1$, $N := (I_r - D_1^2)^{\frac{1}{2}} \oplus D_2 \oplus N_1$, and

$$V = \begin{bmatrix} 0 & 0 & I_r \\ 0 & I_{m-r} & 0 \\ V_0 & 0 & 0 \end{bmatrix}$$

completes the proof. The verification of this is left to the reader. $\square$

2.17. Having established a useful form for projections, we now turn to approximating invertible operators by invertible operators – each of whose principal submatrices are again invertible. In fact, we shall establish a more general result, namely that if $T \in \mathbb{M}_n$ is an operator of rank r , then for all $\varepsilon > 0$ there exists a unitary operator $U \in \mathbb{M}_n$ satisfying $\|U - I_n\| < \varepsilon$ for which U^*TU has the property that each of its $k \times k$ principal submatrices has rank equal to the minimum of k and r , $1 \leq k \leq n$.

2.18. Definition. Let $1 \leq k \leq n \in \mathbb{N}$ and $T = [t_{i,j}] \in \mathbb{M}_n$. If $\Omega \subset \{1, 2, \dots, n\}$ is a set with k elements, the $k \times k$ matrix

$$T_\Omega := [t_{i,j}]_{i,j \in \Omega}$$

is called the **principal submatrix** of T of order k , **determined by** Ω .

We may also extend this definition to the infinite-dimensional setting as follows: if $T \in \mathcal{B}(\mathcal{H})$ is a bounded operator acting on the infinite-dimensional, separable, complex Hilbert space $\mathcal{H}$, and if $\{e_n\}_{n=1}^\infty$ is an orthonormal basis for $\mathcal{H}$, then we may write $T = [t_{i,j}]$, where $t_{i,j} = \langle Te_j, e_i \rangle$. Given a finite subset $\Omega \subseteq \mathbb{N}$ with k elements, we define

$$T_\Omega := [t_{i,j}]_{i,j \in \Omega},$$

and refer to this once again as the **principal submatrix** of T of order k , **determined by** Ω .

2.19. Lemma. Let $n \in \mathbb{N}$ and $T \in \mathbb{M}_n$. If $1 \leq k \leq \text{rank } T$, $\varepsilon > 0$ and $\Omega = \{1, 2, \dots, k\}$, then there exists a unitary $V \in \mathbb{M}_n$ such that

- (a) $\|V - I_n\| < \varepsilon$, and
- (b) $(V^*TV)_\Omega \in \mathbb{M}_k$ is invertible.

Proof.

Since the claim is clearly true for $k = \text{rank } T$, we shall assume that $k < \text{rank } T$.

STEP 1.

If $T \in \mathbb{M}_n(\mathbb{C})$ has rank $k < n$, then there exists an orthogonal projection P of rank $n - 1$ such that PTP has rank k .

To prove this, pick an orthonormal basis $\{e_1, \dots, e_n\}$ for $\mathbb{C}^n$ relative to which T has upper triangular form and its (n, n) entry is zero. If the algebraic multiplicity of zero in the spectrum of T is one, then the projection P onto $\{e_n\}^\perp$ clearly has the desired property. If the algebraic multiplicity of zero is two or more, we can assume with no loss of generality that the $(1, 1)$ entry of T is also zero. In this case, let P be the projection onto $\{e_1 + e_n\}^\perp$. To show that $\text{rank } PTP = k$, it suffices to verify that if $x \in \mathbb{C}^n$ and $Px \in \ker PTP$, then $Px \in \ker T$. To this end, note that

$$PTPx = 0 \Rightarrow TPx = \alpha(e_1 + e_n)$$

for some $\alpha \in \mathbb{C}$. But since the range of T is contained in $\{e_n\}^\perp$, we conclude that $\alpha = 0$, and thus $Px \in \ker T$, as claimed.

STEP 2.

The statement (b) now follows easily from Step 1 by induction, except for the proximity of V to the identity operator.

STEP 3.

Write V as e^{iA} where $A = A^*$. Consider the matrix

$$(V^{-t}TV^t)_\Omega = (e^{-itA}Te^{itA})_\Omega$$

and note that we have just shown that the matrix is invertible for $t = 1$. Since its determinant is an analytic function of t (and thus its zeros are isolated), we deduce that the determinant is nonzero in a sufficiently small neighborhood of zero. $\square$

2.20. Theorem. *Let $A \in \mathbb{M}_n$ be any matrix of rank r and $\varepsilon > 0$. There exists a unitary matrix $U \in \mathbb{M}_n$ with $\|U - I_n\| < \varepsilon$ such that all principal submatrices of U^*AU of orders less than or equal to r are invertible.*

Proof. Let $\Omega \subseteq \{1, 2, \dots, n\}$ be a set of k elements, where $k \leq r$. Define $\Lambda = \{1, 2, \dots, k\}$. Choose a permutation unitary $W \in \mathbb{M}_n$, such that for any $T \in \mathbb{M}_n$, $(W^*TW)_\Lambda = T_\Omega$ as matrices in $\mathbb{M}_k$.

Let $\varepsilon > 0$. By Lemma 2.19 above, we can find a unitary $V \in \mathbb{M}_n$ such that $\|V - I_n\| < \varepsilon$ and $(V^*W^*AWV)_\Lambda$ is invertible. Therefore, $(WV^*W^*AWVW^*)_\Omega$ is invertible, and $\|WVW^* - I_n\| < \varepsilon$.

Set

$$\mathcal{G} := \{U \in \mathbb{M}_n : U \text{ is unitary and } \|U - I_n\| < \varepsilon\}.$$

We have shown that for each non-empty subset Ω of $\{1, 2, \dots, n\}$, the subset

$$\mathcal{G}_\Omega := \{U \in \mathcal{G} : (U^*AU)_\Omega \text{ is invertible}\}$$

is non-empty. It is easy to check that $\mathcal{G} \setminus \mathcal{G}_\Omega$ is a relatively closed, nowhere-dense subset of $\mathcal{G}$. By the Baire Category Theorem, the (finite) union

$$\cup \{\mathcal{G} \setminus \mathcal{G}_\Omega : \Omega \subseteq \{1, 2, \dots, n\}\}$$

is nowhere dense; i.e. there is a $U \in \mathcal{G}$ such that $(U^*AU)_\Omega$ is invertible for all such Ω . $\square$

The above result admits a number of generalisations. Although we shall not require the next four results in the remainder of this paper, we feel that they are of sufficient interest in their own right to merit inclusion here.

2.21. Theorem. *Let $\mathcal{H}$ be an infinite-dimensional, separable Hilbert space with orthonormal basis $(e_n)_{n=1}^\infty$. Suppose that $A \in \mathcal{B}(\mathcal{H})$ is an operator of rank $r \in \mathbb{N} \cup \{\infty\}$. Given $\varepsilon > 0$ and a finite subset $\Omega \subseteq \mathbb{N}$ of cardinality at most r , there exists a unitary operator $W \in \mathcal{B}(\mathcal{H})$ with $\|W - I\| < \varepsilon$ and such that the principal submatrix $(W^*AW)_\Omega$ is invertible.*

Proof. Let $\varepsilon > 0$, and write the matrix for A relative to $(e_n)_{n=1}^\infty$ as $[a_{i,j}]_{i,j \geq 1}$. For each $m \geq 1$, we set

$$\mathcal{H}_m = \text{span} \{e_i : 1 \leq i \leq m\}.$$

Choose $N \geq 1$ sufficiently large as to ensure that $B := [a_{i,j}]_{1 \leq i,j \leq N}$ has rank r and $\Omega \subset \{1, 2, \dots, N\}$. By Theorem 2.20 above, we can find a unitary matrix $U \in \mathbb{M}_N$ with $\|U - I_N\| < \varepsilon$ such that all principal submatrices of U^*BU of orders less than or equal to r are invertible.

Set $W = U \oplus I_{\mathcal{H}_N}^\perp$. Then W is unitary, $\|W - I\| < \varepsilon$ and $(W^*AW)_\Omega = (U^*BU)_\Omega$ is invertible.

□

2.22. Theorem. *Let $\mathcal{H}$ be an infinite-dimensional, separable Hilbert space with orthonormal basis $(e_n)_{n=1}^\infty$. Let $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ be a countable set of operators, each of which has infinite rank. Given $\varepsilon > 0$, then there exists a unitary U with $\|U - I\| < \varepsilon$ such that $\Omega \subseteq \mathbb{N}$ finite and $A \in \mathcal{S}$ implies that the principal submatrix of $(U^*AU)_\Omega$ is invertible.*

Proof. Fix $\varepsilon > 0$ and let

$$\mathcal{G} := \{W \in \mathcal{B}(\mathcal{H}) : W \text{ is unitary and } \|W - I\| < \varepsilon\}.$$

By Theorem 2.21, for each $A \in \mathcal{S}$, and for each non-empty finite subset Ω of $\mathbb{N}$, the subset

$$\mathcal{G}_{A,\Omega} := \{W \in \mathcal{G} : (W^*AW)_\Omega \text{ is invertible}\}$$

is non-empty. It is easy to check that $\mathcal{G} \setminus \mathcal{G}_{A,\Omega}$ is a relatively closed, nowhere-dense subset of $\mathcal{G}$. Note that $\mathcal{S}$ is countable and the set of finite subsets of $\mathbb{N}$ is also countable. By the Baire Category Theorem, the union

$$\cup\{\mathcal{G} \setminus \mathcal{G}_{A,\Omega} : A \in \mathcal{S}, \Omega \subseteq \mathbb{N}, |\Omega| < \infty\}$$

is nowhere dense; i.e. there is a $U \in \mathcal{G}$ such that $(U^*AU)_\Omega$ is invertible for all such Ω , and for all $A \in \mathcal{S}$.

□

An easy adaptation of the above proof yields the following Corollary.

2.23. Corollary. *If $\mathcal{S}$ is a countable subset of the invertible group $GL(n, \mathbb{C})$ of $\mathbb{M}_n$, and if $\varepsilon > 0$, then there exists a unitary operator $U \in \mathbb{M}_n$ with $\|U - I\| < \varepsilon$ such that every finite principal submatrix of U^*AU is invertible for all $A \in \mathcal{S}$.*

2.24. Proposition. *Let $\mathcal{H}$ be a complex separable Hilbert space with the orthonormal basis $(e_n)_{n=1}^\infty$, and let $A = [a_{i,j}]_{i,j \geq 1} \in \mathcal{B}(\mathcal{H})$ be a bounded operator of finite rank r . Given $\varepsilon > 0$, there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ with $\|U - I\| < \varepsilon$ such that for all k , every principal compression of U^*AU of order k has rank equal to $\min(k, r)$.*

Proof. Fix $\varepsilon > 0$ and let

$$\mathcal{G} := \{W \in \mathcal{B}(\mathcal{H}) : W \text{ is unitary and } \|W - I\| < \varepsilon\}.$$

By Theorem 2.21, for each non-empty finite subset Ω of $\mathbb{N}$ with $|\Omega| \leq r$, the subset

$$\mathcal{G}_\Omega := \{W \in \mathcal{G} : (W^*AW)_\Omega \text{ is invertible}\}$$

is non-empty. Once again, it is easy to check that $\mathcal{G} \setminus \mathcal{G}_\Omega$ is a relatively closed, nowhere-dense subset of $\mathcal{G}$. Note that $\{\Omega : \Omega \subset \mathbb{N}, 1 \leq |\Omega| \leq r\}$ is countable. By the Baire Category Theorem,

$$\cup\{\mathcal{G} \setminus \mathcal{G}_\Omega : \Omega \subseteq \mathbb{N}, |\Omega| \leq r\}$$

is nowhere dense; i.e. there is a $U \in \mathcal{G}$ such that $(U^*AU)_\Omega$ is invertible for all such Ω with $|\Omega| \leq r$.

For $\Omega \subset \mathbb{N}$ with $|\Omega| = k > r$, fix a subset $\Omega_0 \subseteq \Omega$ with $|\Omega_0| = r$. Then

$$r = |\Omega_0| = \text{rank}(U^*AU)_{\Omega_0} \leq \text{rank}(U^*AU)_\Omega \leq \text{rank } U^*AU = \text{rank } A = r,$$

completing the proof.

□

We shall also two more canonical forms for (most) idempotents. It is worth comparing the next proposition with Lemma 2.16.

2.25. Proposition. *Let $E_0 \in \mathcal{E}_{n,2n}$ be an idempotent of rank n . Then there exists $\varepsilon > 0$ and an idempotent $E \in \mathcal{E}_{n,2n}$ of the form*

$$E = \begin{bmatrix} K^2 & KLT \\ T^{-1}KL & T^{-1}L^2T \end{bmatrix} = \begin{bmatrix} K \\ T^{-1}L \end{bmatrix} \begin{bmatrix} K & LT \end{bmatrix},$$

where T, K and $L \in \mathbb{M}_n$ are invertible and $K^2 + L^2 = I_n$, such that $\|E - E_0\| < \varepsilon$.

Consequently,

$$I_{2n} - E = \begin{bmatrix} L^2 & -KLT \\ -T^{-1}KL & T^{-1}K^2T \end{bmatrix} = \begin{bmatrix} L \\ -T^{-1}K \end{bmatrix} \begin{bmatrix} L & -KT \end{bmatrix}.$$

Proof. Let $\varepsilon > 0$. By Theorem 2.20, we see that we can find $E \in \mathcal{E}_{n,2n}$ such that E is unitarily equivalent to E_0 , $\|E - E_0\| < \varepsilon$, and E can be decomposed as

$$E = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n),$$

where A and D are invertible. But then so are B and C , since $\text{rank } E = n$ implies that $D = CA^{-1}B$. It follows that we can find $K, L \in \mathbb{M}_n$ such that $K^2 = A$, $L^2 = I_n - K^2$, and $KL = LK$. Thus $K^2 + L^2 = I_n$, and we may rewrite

$$E = \begin{bmatrix} K^2 & KX \\ YK & YX \end{bmatrix}$$

where $X = K^{-1}B$ and $Y = CK^{-1}$. The fact that $E = E^2$ is an idempotent of rank n implies that

$$K^2 + XY = I_n,$$

so that $XY = L^2$. Setting $T = L^{-1}X$, we get

$$E = \begin{bmatrix} K^2 & KLT \\ T^{-1}KL & T^{-1}L^2T \end{bmatrix}.$$

The remaining claims follow from routine computations. □

In fact, using Proposition 2.25, we are able to retrieve the result from Lemma 2.16 as follows:

2.26. Lemma. *Let $n \in \mathbb{N}$ and $P = \begin{bmatrix} P_1 & P_2 \\ P_2^* & P_4 \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n)$ be a projection of rank n . Given $\varepsilon > 0$ there exist commuting operators $0 < M, N < I_n$ satisfying $M^2 + N^2 = I_n$ and $V \in \mathbb{M}_n$ unitary such that*

$$R := \begin{bmatrix} M^2 & MNV \\ V^*MN & V^*N^2V \end{bmatrix}$$

defines a projection for which $\|R - P\| < \varepsilon$ and $\sigma(M^2 \oplus N^2)$ has cardinality $2n$.

Proof. Let $\varepsilon > 0$. By Theorem 2.20, we see that we can find $Q \in \mathcal{P}_{n,2n}$ such that Q is unitarily equivalent to P , $\|Q - P\| < \frac{\varepsilon}{2}$, and Q can be decomposed as

$$Q = \begin{bmatrix} X & Y \\ Y^* & Z \end{bmatrix} \in \mathbb{M}_2(\mathbb{M}_n),$$

where X and Z are invertible. Since $0 < X < I_n$, we may write $X = M_1^2$ for some $0 < M_1 < I_n$. Using the polar decomposition, we write $Y = |Y^*|V$ for some unitary operator $V \in \mathbb{M}_n$, and define $N_1 = (I - M_1^2)^{\frac{1}{2}}$. Clearly $N_1 M_1 = M_1 N_1$ and $M_1^2 + N_1^2 = I_n$. Also, from above we have that

$$|Y^*|^2 = X - X^2 = M_1^2 - M_1^4 = M_1^2(I - M_1^2) = M_1^2 N_1^2,$$

and thus $|Y^*| = M_1 N_1$ (the latter being positive since $M_1 N_1 = N_1 M_1$). In particular, $Y = |Y^*|V = M_1 N_1 V$ is invertible.

The equation $Q^2 = Q$ also implies that $XY + YZ = Y$, or equivalently that $YZ = (I - X)Y$. Since Y is invertible, this then implies that

$$\begin{aligned} Z &= Y^{-1}(I - X)Y \\ &= V^*|Y^*|^{-1}(I - M_1^2)|Y^*|V \\ &= V^*(M_1 N_1)^{-1}(N_1^2)(MN)V \\ &= V^* N_1^2 V. \end{aligned}$$

Finally, we could find two commuting invertible positive operators $M, N \in \mathbb{M}_n$, such that

- (1) $\|M_1^2 - M^2\| + \|N_1^2 - N^2\| + 2\|M_1 N_1 - MN\| < \frac{\varepsilon}{2}$;
- (2) $\sigma(M^2 \oplus N^2)$ has cardinality $2n$.

We complete the proof by defining

$$R := \begin{bmatrix} M^2 & MNV \\ V^*MN & V^*N^2V \end{bmatrix}.$$

□

The second canonical form we shall require (in Section 5) is as follows.

2.27. Proposition. *Let $m \in \mathbb{N}$ and*

$$E = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}$$

be an idempotent of rank m in $\mathbb{M}_2(\mathbb{M}_m)$. If E_1 , $I - E_1$ and E_4 are invertible, then there is an invertible $M \in \text{GL}(m, \mathbb{C})$ such that

$$E = \begin{bmatrix} E_1 & (I - E_1)M \\ M^{-1}E_1 & M^{-1}(I - E_1)M \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & M^{-1} \end{bmatrix} \begin{bmatrix} E_1 & I - E_1 \\ E_1 & I - E_1 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}.$$

Proof. Since E has rank m , and since both E_1 and E_4 are invertible, it follows that

$$E_4 = E_3 E_1^{-1} E_2,$$

and thus E_2 and E_3 are invertible as well. The equation $E^2 = E$ now yields $E_2 E_4 = (I - E_1) E_2$, hence $(I - E_1)$ is invertible. Letting $M := (I - E_1)^{-1} E_2$, we obtain the desired form for E .

□

THE EQUIVALENCE OF SEVERAL FORMS OF THE PROBLEM

2.28. We now turn our attention to proving the equivalence of solving Problem 2SC and solving Problem 3S. In fact, the following proposition will yield a result somewhat more general than just this.

2.29. Proposition. *Let $n \in \mathbb{N}$ and $\mathcal{X}, \mathcal{Y}$ be subspaces of $\mathbb{M}_n$. Then*

$$\text{MODD}_n(\mathcal{X} \oplus \mathcal{Y}) = \text{MAXSP}_S(\mathcal{X}, \mathcal{Y}) = \text{MIDD}_n(\mathcal{X} \oplus \mathcal{Y}).$$

Proof. Clearly it suffices to consider the case where at least one of $\mathcal{X}$ and $\mathcal{Y}$ is non-zero. We begin by proving the first equality.

Choose a projection $P_0 \in \mathcal{P}_{n,2n}$ such that

$$\text{MODD}_n(\mathcal{X} \oplus \mathcal{Y}) = \dim P_0^\perp(\mathcal{X} \oplus \mathcal{Y})P_0.$$

By Lemma 2.10, there exists $\varepsilon > 0$ such that if $P \in \mathcal{P}_{n,2n}$ is a projection and $\|P - P_0\| < \varepsilon$, then

$$\text{MODD}_n(\mathcal{X} \oplus \mathcal{Y}) = \dim P^\perp(\mathcal{X} \oplus \mathcal{Y})P.$$

By Lemma 2.16, there exists a projection $R \in \mathbb{M}_2(\mathbb{M}_n)$ of the form

$$R = \begin{bmatrix} M^2 & MNV \\ V^*MN & V^*N^2V \end{bmatrix} = \begin{bmatrix} M & NV \end{bmatrix}^* \begin{bmatrix} M & NV \end{bmatrix},$$

with $0 < M, N < I_n$, $M^2 + N^2 = I_n$ and $V \in \mathbb{M}_n$ unitary such that $\|R - P_0\| < \varepsilon$. In particular,

$$\text{MODD}_n(\mathcal{X} \oplus \mathcal{Y}) = \dim R^\perp(\mathcal{X} \oplus \mathcal{Y})R.$$

Observe that

$$R^\perp = I - R = \begin{bmatrix} N^2 & -MNV \\ -V^*MN & V^*M^2V \end{bmatrix} = \begin{bmatrix} N & -MV \end{bmatrix}^* \begin{bmatrix} N & -MV \end{bmatrix}.$$

Thus – keeping in mind that $Y \in \mathcal{Y}$ if and only if $-Y \in \mathcal{Y}$, that $\begin{bmatrix} M & NV \end{bmatrix}^*$ is an isometry and that $\begin{bmatrix} N & -MV \end{bmatrix}$ is a co-isometry – we get

$$\begin{aligned} \text{MODD}_n(\mathcal{X} \oplus \mathcal{Y}) &= \dim R^\perp(\mathcal{X} \oplus \mathcal{Y})R \\ &= \dim \begin{bmatrix} N & -MV \end{bmatrix} \begin{bmatrix} \mathcal{X} & 0 \\ 0 & \mathcal{Y} \end{bmatrix} \begin{bmatrix} M \\ V^*N \end{bmatrix} \\ &= \dim \{NXM - MVYV^*N : X \in \mathcal{X}, Y \in \mathcal{Y}\} \\ &= \dim \{N(X - N^{-1}MVYV^*NM^{-1})M : X \in \mathcal{X}, Y \in \mathcal{Y}\} \\ &= \dim \{X - N^{-1}MVYV^*NM^{-1} : X \in \mathcal{X}, Y \in \mathcal{Y}\} \\ &= \dim \{X + T^{-1}(-Y)T : X \in \mathcal{X}, Y \in \mathcal{Y}\} \\ &= \dim (\mathcal{X} + T^{-1}\mathcal{Y}T), \end{aligned}$$

where $T = V^*NM^{-1}$. Hence

$$\text{MODD}_n(\mathcal{X} \oplus \mathcal{Y}) \leq \text{MAXSP}_S(\mathcal{X}, \mathcal{Y}).$$

Conversely, suppose that $T_0 \in \mathbb{M}_n$ is invertible is chosen such that

$$\text{MAXSP}_S(\mathcal{X}, \mathcal{Y}) = \dim (\mathcal{X} + T_0^{-1}\mathcal{Y}T_0).$$

By Proposition 2.8, there exists $\varepsilon > 0$ such that if $T \in \mathbb{M}_n$ and $\|T - T_0\| < \varepsilon$, then T is invertible and

$$\dim (\mathcal{X} + T^{-1}\mathcal{Y}T) = \text{MAXSP}_S(\mathcal{X}, \mathcal{Y}).$$

Writing the polar decomposition of T_0 as $T_0 = V|T_0|$, we can easily find a positive, invertible operator K such that $\|K - |T_0|\| < \varepsilon$ and K possesses n distinct eigenvalues, say $0 < k_1 < k_2 < \dots < k_n$. Set $T := VK$, and note that $K = |T|$.

We claim that there exist $M, N \in \mathbb{M}_n$ such that $0 < M, N < I_n$ and $|T| = NM^{-1}$.

Without loss of generality, we may suppose that $|T| = K = \text{diag}(k_1, k_2, \dots, k_n)$. Set

$$N = \text{diag} \left(\frac{k_1}{\sqrt{1+k_1^2}}, \frac{k_2}{\sqrt{1+k_2^2}}, \dots, \frac{k_n}{\sqrt{1+k_n^2}} \right),$$

and

$$M = \text{diag} \left(\frac{1}{\sqrt{1+k_1^2}}, \frac{1}{\sqrt{1+k_2^2}}, \dots, \frac{1}{\sqrt{1+k_n^2}} \right).$$

Then $0 < M, N < I_n$ and $M^2 + N^2 = I_n$, and $MN = NM = |T|$. Thus $T = VNM^{-1}$.

If we set $R = \begin{bmatrix} M^2 & MNV \\ V^*MN & V^*N^2V \end{bmatrix}$, then R is a projection of rank n in $\mathbb{M}_{2n}$ and the computation above shows that

$$\dim R^\perp(\mathcal{X} \oplus \mathcal{Y})R = \dim(\mathcal{X} + T^{-1}\mathcal{Y}T) = \text{MAXSP}_S(\mathcal{X}, \mathcal{Y}).$$

Thus

$$\text{MAXSP}_S(\mathcal{X}, \mathcal{Y}) \leq \text{MODD}_n(\mathcal{X} \oplus \mathcal{Y}),$$

from which equality of these two quantities now follows.

Next, let us prove the second equality.

From the definition, it is clear that $\text{MODD}_n(\mathcal{X} \oplus \mathcal{Y}) \leq \text{MIDD}_n(\mathcal{X} \oplus \mathcal{Y})$. Combining this with the first equality, we get

$$\text{MAXSP}_S(\mathcal{X}, \mathcal{Y}) \leq \text{MIDD}_n(\mathcal{X} \oplus \mathcal{Y}).$$

It therefore suffices to prove the reverse inequality. The argument is not unlike the previous argument. Let $E_0 \in \mathcal{E}_{n,2n}$ be an idempotent which implements the value of $\text{MIDD}_n(\mathcal{X} \oplus \mathcal{Y})$. By Lemma 2.10 and Proposition 2.25, we may replace E_0 by a unitarily equivalent idempotent E of the form

$$E = \begin{bmatrix} K^2 & KLT \\ T^{-1}KL & T^{-1}L^2T \end{bmatrix}$$

where K, L and $T \in \text{GL}(n, \mathbb{C})$ satisfy $KL = LK$ and $K^2 + L^2 = I_n$, and such that

$$\text{MIDD}_n(\mathcal{X} \oplus \mathcal{Y}) = \dim E^\perp(\mathcal{X} \oplus \mathcal{Y})E = \begin{bmatrix} K \\ T^{-1}L \end{bmatrix} \begin{bmatrix} K & LT \end{bmatrix}.$$

Observe that $\begin{bmatrix} K \\ T^{-1}L \end{bmatrix}$ and $\begin{bmatrix} L & -KT \end{bmatrix}$ are left- and right-invertible respectively, and thus

$$\begin{aligned} \text{MIDD}_n(\mathcal{X} \oplus \mathcal{Y}) &= \dim E^\perp(\mathcal{X} \oplus \mathcal{Y})E \\ &= \dim \begin{bmatrix} L & -KT \end{bmatrix} (\mathcal{X} \oplus \mathcal{Y}) \begin{bmatrix} K \\ T^{-1}L \end{bmatrix} \\ &= \dim \{LXK - KTYT^{-1}L : X \in \mathcal{X}, Y \in \mathcal{Y}\} \\ &= \dim \{X - (L^{-1}KT)Y(L^{-1}KT)^{-1} : X \in \mathcal{X}, Y \in \mathcal{Y}\} \\ &= \dim(\mathcal{X} + (L^{-1}KT)\mathcal{Y}(L^{-1}KT)^{-1}) \\ &\leq \text{MAXSP}_S(\mathcal{X}, \mathcal{Y}). \end{aligned}$$

This completes the proof. $\square$

2.30. Proposition. *Let $n \in \mathbb{N}$ and $\mathcal{X}, \mathcal{Y}$ be subspaces of $\mathbb{M}_n$. Let $S_0 \in \text{GL}(2n, \mathbb{C})$.*

- (a) $\text{MIDD}_n(S_0^{-1}(\mathcal{X} \oplus \mathcal{Y})S_0) = \text{MIDD}_n(\mathcal{X} \oplus \mathcal{Y})$.
- (b) *As a consequence, if $\mathcal{X} = R_1^{-1}\mathcal{X}_0R_1$ and $\mathcal{Y} = R_2^{-1}\mathcal{Y}_0R_2$ for some $R_1, R_2 \in \text{GL}(n, \mathbb{C})$, then*

$$\text{MODD}_n(\mathcal{X} \oplus \mathcal{Y}) = \text{MODD}_n(\mathcal{X}_0 \oplus \mathcal{Y}_0).$$

Proof.

- (a) Since $\mathcal{E}_{n,2n} = S_0^{-1}\mathcal{E}_{n,2n}S_0$, we find that

$$\begin{aligned} \text{MIDD}_n(\mathcal{X} \oplus \mathcal{Y}) &= \max\{\dim E^\perp(\mathcal{X} \oplus \mathcal{Y})E : E \in \mathcal{E}_{n,2n}\} \\ &= \max\{\dim(S_0^{-1}E^\perp S_0) S_0^{-1}(\mathcal{X} \oplus \mathcal{Y})S_0 (S_0^{-1}ES_0) : E \in \mathcal{E}_{n,2n}\} \\ &= \text{MIDD}_n(S_0^{-1}(\mathcal{X} \oplus \mathcal{Y})S_0). \end{aligned}$$

- (b) Set $R := R_1 \oplus R_2$, so that $R \in \text{GL}(2n, \mathbb{C})$ and $\mathcal{X}_0 \oplus \mathcal{Y}_0 = R^{-1}(\mathcal{X} \oplus \mathcal{Y})R$. Then by Proposition 2.29 together with item (a) above, we obtain

$$\begin{aligned} \text{MODD}_n(\mathcal{X} \oplus \mathcal{Y}) &= \text{MIDD}_n(\mathcal{X} \oplus \mathcal{Y}) \\ &= \text{MIDD}_n(R^{-1}(\mathcal{X} \oplus \mathcal{Y})R) \\ &= \text{MIDD}_n(\mathcal{X}_0 \oplus \mathcal{Y}_0) \\ &= \text{MODD}_n(\mathcal{X}_0 \oplus \mathcal{Y}_0) \end{aligned}$$

□

2.31. Remarks.

- (a) In light of the comments in Remark 2.5, we now see that if $\mathcal{A}$ and $\mathcal{B}$ are unital, semi-simple subalgebras of $\mathbb{M}_n$, then

$$\text{MODD}_n(\mathcal{A} \oplus \mathcal{B}) = \text{MAXSP}_S(\mathcal{A}, \mathcal{B}) = \dim \mathcal{A} + \dim \mathcal{B} - \text{MINT}_S(\mathcal{A}, \mathcal{B}).$$

As such, Problems 2S, 3S, and 4S are all equivalent.

- (b) Again, if $\mathcal{A}$ and $\mathcal{B}$ are unital, semi-simple subalgebras of $\mathbb{M}_n$, then by the Wedderburn-Artin Theorem [1, 8], there exist $R_1, R_2 \in \text{GL}(n, \mathbb{C})$ such that $\mathcal{A}_0 := R_1^{-1}\mathcal{A}R_1$ and $\mathcal{B}_0 := R_2^{-1}\mathcal{B}R_2$ are unital C^* -algebras. By Proposition 2.30 (b) and item (a) above, we conclude that

$$\begin{aligned} \text{MODD}_n(\mathcal{A} \oplus \mathcal{B}) &= \text{MODD}_n(\mathcal{A}_0 \oplus \mathcal{B}_0) \\ &= \dim \mathcal{A}_0 + \dim \mathcal{B}_0 - \text{MINT}_S(\mathcal{A}_0, \mathcal{B}_0) \\ &= \dim \mathcal{A} + \dim \mathcal{B} - \text{MINT}_S(\mathcal{A}_0, \mathcal{B}_0). \end{aligned}$$

In other words, when trying to calculate $\text{MODD}_n(\mathcal{A} \oplus \mathcal{B})$ for semi-simple, unital algebras $\mathcal{A}, \mathcal{B}$, it suffices to consider the case where they are in fact C^* -algebras. Moreover, the problem reduces to calculating $\text{MINT}_S(\mathcal{A}, \mathcal{B})$ for unital C^* -subalgebras of $\mathbb{M}_n$.

3. A CASE-BY-CASE ANALYSIS

3.1. In light of the notation established in the previous section, we may now reformulate (two of) the problems from Section 1 as follows:

PROBLEM 2S: Let $\mathcal{A}$ and $\mathcal{B}$ be unital, semi-simple subalgebras of $\mathbb{M}_n$. Determine

$$\text{MODD}_n(\mathcal{A} \oplus \mathcal{B}).$$

PROBLEM 4SC: Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^* -subalgebras of $\mathbb{M}_n$. Determine

$$\text{MINT}_S(\mathcal{A}, \mathcal{B}).$$

From Remarks 2.31 (b), we now know that these problems are equivalent. Our focus in the sequel is to determine for exactly which pairs $\mathcal{A}, \mathcal{B}$ of unital, semi-simple subalgebras of $\mathbb{M}_n$ we can maximise $\text{MODD}_n(\mathcal{A} \oplus \mathcal{B})$; that is, for which pairs $\mathcal{A}, \mathcal{B}$ as above do we have $\text{MODD}_n(\mathcal{A} \oplus \mathcal{B}) = \dim \mathcal{A} + \dim \mathcal{B} - 1$?

This therefore reduces to the problem of determining for which pairs $(\mathcal{A}, \mathcal{B})$ of unital C^* -algebras of $\mathbb{M}_n$ we have $\text{MINT}_S(\mathcal{A}, \mathcal{B}) = 1$. As one might imagine, the structure theory of C^* -algebras will be exceedingly useful in approaching this problem. Having said that, when n is large, $\mathbb{M}_n$ admits a great many C^* -subalgebras, and unsurprisingly, we will have to have a strategy to consider all cases. With that in mind, we now establish the notation we shall use in the remainder of the paper. Here, $n \in \mathbb{N}$ refers to the dimension of the underlying Hilbert space $\mathbb{C}^n$.

- We suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital C^* -subalgebras of $\mathbb{M}_n$, and we write

$$\mathcal{A} = \mathbb{M}_{p_1}^{(\mu_1)} \oplus \mathbb{M}_{p_2}^{(\mu_2)} \oplus \cdots \oplus \mathbb{M}_{p_k}^{(\mu_k)},$$

where

$$p_1 \geq p_2 \geq \cdots p_k \geq 1,$$

and if $p_i = p_{i+1}$, then $\mu_i \geq \mu_{i+1}$.

- Similarly, we write

$$\mathcal{B} = \mathbb{M}_{q_1}^{(\nu_1)} \oplus \mathbb{M}_{q_2}^{(\nu_2)} \oplus \cdots \oplus \mathbb{M}_{q_l}^{(\nu_l)},$$

where

$$q_1 \geq q_2 \geq \cdots q_l \geq 1,$$

and if $q_j = q_{j+1}$, then $\nu_j \geq \nu_{j+1}$.

- We then define

$$\kappa_{\mathcal{A}} = \mu_1 + \mu_2 + \cdots + \mu_k$$

$$\kappa_{\mathcal{B}} = \nu_1 + \nu_2 + \cdots + \nu_l.$$

- **Important hypothesis.** Without loss of generality, we may and we shall assume that $p_1 \leq q_1$.

3.2. Strategy. We shall proceed by a case-by-case analysis, depending upon the sizes and multiplicities of the blocks appearing in the decompositions of $\mathcal{A}$ and $\mathcal{B}$ above. For the benefit of the reader, we provide a quick overview of the cases below.

We shall demonstrate that – except for the Case 2(a)(ii) and Case 4(a) below – in each of the following cases,

$$\text{MINT}_U(\mathcal{A}, \mathcal{B}) = \text{MINT}_S(\mathcal{A}, \mathcal{B}) = 1.$$

CASE 1. $p_1 + q_1 \leq n$, $\kappa_{\mathcal{A}} = 2$, $\mu_1 = 2$.

Thus $\mathcal{A} = \mathbb{M}_{p_1}^{(2)}$. Now

$$n = p_1 + p_1 \leq p_1 + q_1 \leq n$$

implies that n is even and $p_1 = q_1 = n/2$. Thus $\mathcal{A} = \mathbb{M}_{n/2}^{(2)}$.

(a) If $\kappa_{\mathcal{B}} = 2$, then either

(i) $\nu_1 = 1 = \nu_2$ and $\mathcal{B} = \mathbb{M}_{n/2} \oplus \mathbb{M}_{n/2}$. The result will follow from Theorem 4.2 below; or

(ii) $\nu_1 = 2$ and $\mathcal{B} = \mathbb{M}_{n/2}^{(2)} = \mathcal{A}$. The result will follow from Corollary 4.3 below.

(b) If $\kappa_{\mathcal{B}} \geq 3$, then the result will follow by appealing to Theorem 4.9 with the roles of $\mathcal{A}$ and $\mathcal{B}$ interchanged.

CASE 2. $p_1 + q_1 \leq n$, $\kappa_{\mathcal{A}} = 2$, $\mu_1 = 1 = \mu_2$.

Thus $\mathcal{A} = \mathbb{M}_{p_1} \oplus \mathbb{M}_{p_2}$. Now

$$n = p_1 + p_2 \leq q_1 + p_2 \leq q_1 + p_1 \leq n$$

implies that $p_1 = p_2 = q_1 = n/2$. Thus $\mathcal{A} = \mathbb{M}_{n/2} \oplus \mathbb{M}_{n/2}$.

(a) If $\kappa_{\mathcal{B}} = 2$, then either

(i) $\nu_1 = 2$ and $\mathcal{B} = \mathbb{M}_{n/2}^{(2)}$. This is just Case 1(a)(i) above, with $\mathcal{A}$ and $\mathcal{B}$ interchanged. Thus the result follows from Theorem 4.2; or

(ii) $\nu_1 = 1 = \nu_2$ and $\mathcal{B} = \mathbb{M}_{n/2} \oplus \mathbb{M}_{n/2} = \mathcal{A}$. It will follow from Corollary 5.6 below that

$$\text{MINT}_S(\mathcal{A}, \mathcal{A}) = \text{MINT}_U(\mathcal{A}, \mathcal{A}) = \frac{n}{2}.$$

(b) If $\kappa_{\mathcal{B}} \geq 3$, then we shall appeal to Theorem 4.9 (interchanging the roles of $\mathcal{A}$ and $\mathcal{B}$) to find that

$$\text{MINT}_U(\mathcal{A}, \mathcal{B}) = \text{MINT}_S(\mathcal{A}, \mathcal{B}) = 1.$$

CASE 3. $p_1 + q_1 \leq n$, $\kappa_{\mathcal{A}} \geq 3$.

Let $\mathcal{C} := \mathbb{M}_{q_1} \oplus \mathbb{M}_{n-q_1}$, and observe that $\mathcal{B} \subseteq \mathcal{C}$.

(a) If $q_1 \leq n/2$, then we are in a position to appeal directly to Theorem 4.9 with $q = q_1$ to conclude that

$$1 \leq \text{MINT}_S(\mathcal{A}, \mathcal{B}) \leq \text{MINT}_U(\mathcal{A}, \mathcal{B}) \leq \text{MINT}_U(\mathcal{A}, \mathcal{C}) = \text{MINT}_S(\mathcal{A}, \mathcal{C}) = 1.$$

(b) If $q_1 > n/2$, then observe that $p_1 + q_1 \leq n$ implies that $p_1 \leq n - q_1 < n/2$. Letting $q = n - q_1$ in Theorem 4.9, we conclude as in item (a) above that

$$\text{MINT}_U(\mathcal{A}, \mathcal{B}) = \text{MINT}_S(\mathcal{A}, \mathcal{B}) = 1.$$

CASE 4. $p_1 + q_1 > n$. Since $p_1 \leq q_1$, it follows that $q_1 > \frac{n}{2}$, and thus $\nu_1 = 1$ in this case.

(a) If there exists $1 \leq i_0 \leq k$ such that $q_1 > n - \frac{p_{i_0}}{\mu_{i_0}}$, then

$$\text{MINT}_S(\mathcal{A}, \mathcal{B}) > 1.$$

This is the content of Corollary 6.6.

(b) If the condition that $p_i + q_1 > n$ implies that $q_1 \leq n - \frac{p_i}{\mu_i}$, then

$$\text{MINT}_S(\mathcal{A}, \mathcal{B}) = 1.$$

This will be proven in Theorem 6.15.

4. RESULTS – CASES 1, 2 AND 3.

4.1. In this Section we shall establish most of the results that we require to handle the first three cases outlined in our strategy in Paragraph 3.2. The one exception is Case 2(a)(ii), which will be handled separately in Section 5.

4.2. Theorem. *Let $n \in 2\mathbb{N}$, and suppose that $\mathcal{A} = \mathbb{M}_{n/2}^{(2)}$ and $\mathcal{B} = \mathbb{M}_{n/2} \oplus \mathbb{M}_{n/2}$. Then*

$$\text{MINT}_U(\mathcal{A}, \mathcal{B}) = \text{MINT}_S(\mathcal{A}, \mathcal{B}) = 1.$$

Proof. Observe first that $\mathcal{A} = \{E_{11}, E_{12}, E_{21}, E_{22}\}'$, where $E_{ij} \in \mathbb{M}_2(\mathbb{C}I_{n/2})$ are

$$E_{11} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}, \quad E_{21} = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}, \quad \text{and} \quad E_{22} = \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix}.$$

Let $M = \text{diag}(m_1, m_2, \dots, m_{n/2})$, where $0 < m_i < 1$ for all $1 \leq i \leq n/2$, and $m_i = m_j$ if and only if $i = j$. Set $N = \text{diag}(\sqrt{1 - m_1^2}, \sqrt{1 - m_2^2}, \dots, \sqrt{1 - m_{n/2}^2})$, so that

- $0 < M, N < I_{n/2}$,
- $MN = NM$, and
- $M^2 + N^2 = I_{n/2}$.

It follows that

$$Q_0 := \begin{bmatrix} M^2 & MN \\ MN & N^2 \end{bmatrix},$$

is a projection of rank $n/2$.

Since N^2 is not a scalar operator, it follows from Theorem 2 of [7] that there exists a unitary operator $V \in \mathbb{M}_{n/2}$ such that all entries of V^*N^2V are non-zero. Set $W = I_{n/2} \oplus V$, so that $W \in \mathbb{M}_n$ is unitary, and let

$$Q := W^*VW = \begin{bmatrix} M^2 & MNV \\ V^*MN & V^*N^2V \end{bmatrix}.$$

Clearly Q is a projection of rank $n/2$ as well, and as such, $\mathcal{B}$ is unitarily equivalent to $\{Q\}'$, say

$$\{Q\}' = U^*BU,$$

where $U \in \mathbb{M}_n$ is unitary.

Consider

$$\mathcal{A} \cap U^*BU = \{E_{11}, E_{12}, E_{21}, E_{22}, Q\}'.$$

It is not hard to see that

$$\{E_{11}, E_{12}, E_{21}, E_{22}, Q\}' = (\mathbb{M}_2(\mathcal{Q}))',$$

where

$$\mathcal{Q} := \text{Alg}(M^2, MNV, V^*MN, V^*N^2V)$$

is the subalgebra of $\mathbb{M}_{n/2}$ generated by the operator-entries of Q . Since all of the entries of M^2 are distinct, $\mathcal{Q}$ includes all diagonal operators in $\mathbb{M}_{n/2}$. Since none of the entries of V^*N^2V are zero, it now follows that

$$\mathcal{Q} = \mathbb{M}_{n/2}.$$

Thus

$$\mathcal{A} \cap U^*\mathcal{B}U = (\mathbb{M}_2(\mathcal{Q}))' = (\mathbb{M}_2(\mathbb{M}_{n/2}))' = \mathbb{M}'_n = \mathbb{C}I_n,$$

whence

$$1 \leq \text{MINT}_S(\mathcal{A}, \mathcal{B}) \leq \text{MINT}_U(\mathcal{A}, \mathcal{B}) \leq \dim(\mathcal{A} \cap U^*\mathcal{B}U) = 1.$$

Hence

$$\text{MINT}_U(\mathcal{A}, \mathcal{B}) = \text{MINT}_S(\mathcal{A}, \mathcal{B}) = 1.$$

□

4.3. Corollary. *Let $n \in 2\mathbb{N}$ and suppose that $\mathcal{A} = \mathbb{M}_{n/2}^{(2)}$. Then*

$$\text{MINT}_U(\mathcal{A}, \mathcal{A}) = \text{MINT}_S(\mathcal{A}, \mathcal{A}) = 1.$$

Proof. Note that $\mathcal{A} \subseteq \mathcal{B} = \mathbb{M}_{n/2} \oplus \mathbb{M}_{n/2}$, and thus if $U \in \mathbb{M}_n$ is the unitary from Theorem 4.2 such that

$$\dim(\mathcal{A} \cap U^*\mathcal{B}U) = 1,$$

then

$$1 \leq \dim(\mathcal{A} \cap U^*\mathcal{A}U) \leq \dim(\mathcal{A} \cap U^*\mathcal{B}U) = 1,$$

whence

$$1 \leq \text{MINT}_S(\mathcal{A}, \mathcal{A}) \leq \text{MINT}_U(\mathcal{A}, \mathcal{A}) \leq 1.$$

From this the corollary now follows.

□

We now turn our attention to resolving Case 1 (b) from Paragraph 3.2. This will require some auxiliary results.

4.4. Lemma. *Let $t_1, t_2, n \in \mathbb{N}$, and suppose that $1 \leq t_1, t_2 \leq n \leq t_1 + t_2$. If $D \in \mathbb{M}_n$ is positive, invertible, and diagonal, then there exist positive operators $M_1, M_2 \in \mathbb{M}_n$ such that*

- (i) $\text{rank } M_k = t_k, k = 1, 2$;
- (ii) $M_1 + M_2 = D$; and
- (iii) all entries of M_1 are non-zero.

Proof.

CASE 1. $D = I_n$.

- If $\min(t_1, t_2) < n$, set $A_1 = I_{n-t_2} \oplus \frac{1}{2}I_{(t_1+t_2)-n} \oplus 0_{n-t_1}$. It is clear that $0 \leq A_1$ has rank t_1 , and that A_1 is not scalar. Moreover, $A_2 := I_n - A_1 \geq 0$ has rank t_2 , and A_2 is not scalar.
- If $t_2 = t_1 = n$, we instead define $A_1 = \frac{1}{4}I_1 \oplus \frac{1}{2}I_{n-1}$ and $A_2 = \frac{3}{4}I_1 \oplus \frac{1}{2}I_{n-1}$. Again, $0 \leq A_1, A_2 \leq I_n$ are non-scalar, $\text{rank } A_i = n = t_i, i = 1, 2$, and $A_1 + A_2 = I_n$.

Since in either case A_1 is non-scalar, by Theorem 2 of [7], there exists a unitary U such that all entries of $M_1 := U^* A_1 U$ are non-zero. Set $M_2 := U^* A_2 U$. It is routine to verify that these operators satisfy conditions (i) - (iii) for $D = I_n$, as required.

CASE 2. $0 < D \in \mathbb{M}_n$ is diagonal.

From CASE 1 above, there exist $0 \leq N_1, N_2 \in \mathbb{M}_n$ such that

- $\text{rank } N_k = t_k, k = 1, 2$;
- $N_1 + N_2 = I_n$; and
- all entries of N_1 are non-zero.

Set $M_k := D^{\frac{1}{2}} N_k D^{\frac{1}{2}}, k = 1, 2$. It is routine to verify that M_1, M_2 satisfy conditions (i), (ii) and (iii) above. □

4.5. Lemma. *Let $p, k \in \mathbb{N}$ with $k \geq 2$. Suppose that $r_i \in \mathbb{N}, 1 \leq i \leq k$ satisfy*

$$r_k \leq r_{k-1} \leq \cdots \leq r_2 \leq r_1 \leq p.$$

Suppose furthermore that $p \leq \sum_{i=1}^k r_i$ and that $0 < D \in \mathbb{M}_p$ is diagonal.

Then there exist $0 \leq N_i \in \mathbb{M}_p$ such that $\text{rank } N_i = r_i, 1 \leq i \leq k$, all of the entries of N_1 are non-zero, and

$$N_1 + N_2 + \cdots + N_k = D.$$

Proof. Let $t_1 := r_1 \leq p$ and $t_2 := \min\{\sum_{i=2}^k r_i, p\}$.

By Lemma 4.4, we can find $0 \leq N_1, M \in \mathbb{M}_p$ such that $\text{rank } N_1 = t_1, \text{rank } M = t_2$, all entries of N_1 are non-zero and

$$N_1 + M = D.$$

Fix a unitary operator $U \in \mathbb{M}_p$ such that

$$U^* M U = \text{diag}(m_1, m_2, \dots, m_{t_2}, 0, 0, \dots, 0),$$

with $m_1 \geq m_2 \geq \cdots \geq m_{t_2}$. (Of course, if $t_2 = p$, then no 0 terms are present.)

It is not hard to see that we can define, for $2 \leq i \leq k$, diagonal, positive semi-definite operators $0 \leq X_i := \text{diag}(x_1^{(i)}, x_2^{(i)}, \dots, x_p^{(i)})$, such that $\text{rank } X_i = r_i, 2 \leq i \leq k$, and $\sum_{i=2}^k X_i = U^* M U$.

Indeed, for each $2 \leq i \leq k$, choose subsets $\Lambda_i \subseteq \{1, 2, \dots, t_2\}$ such that $|\Lambda_i| = r_i$, and $\cup_{i=2}^k \Lambda_i = \{1, 2, \dots, t_2\}$. Define

$$y_{ij} = \begin{cases} 1 & \text{if } j \in \Lambda_i, \\ 0 & \text{otherwise.} \end{cases}$$

It is routine to verify that the following choice of $x_j^{(i)}$'s works:

$$x_j^{(i)} = \begin{cases} \frac{m_j}{\sum_{l=2}^k y_{lj}} & \text{if } j \in \Lambda_i, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, let $N_i = U X_i U^*, 2 \leq i \leq k$. Then

$$\sum_{i=2}^k N_i = \sum_{i=2}^k U X_i U^* = M,$$

whence

$$D = \sum_{i=1}^k N_i.$$

□

4.6. Lemma. *Let $n, p, q \in \mathbb{N}$ with $p \leq q \leq \frac{n}{2}$, and let $0 < X < I_p$. If B be any $p \times (n-p)$ matrix with*

$$X^2 + BB^* = X,$$

then there exists a projection $Q \in \mathbb{M}_n$ of the form

$$\begin{bmatrix} X & B \\ B^* & Z \end{bmatrix},$$

such that $\text{rank } Q = q$.

Proof. Set $0 < M := X^{\frac{1}{2}}$, and $0 < N = (I_p - M^2)^{\frac{1}{2}}$. Observe that M and N commute and that

$$R := \begin{bmatrix} X & MN \\ MN & N^2 \end{bmatrix} = \begin{bmatrix} M^2 & MN \\ MN & N^2 \end{bmatrix}$$

is a projection in $\mathbb{M}_{2p}$ whose rank is p . Moreover, $BB^* = (MN)^2$. As such, there exists a unitary operator $U \in \mathbb{M}_{n-p}$ such that

$$BU = \begin{bmatrix} MN & 0 \end{bmatrix}.$$

Let $V = I_p \oplus U \in \mathbb{M}_n$, so that V is unitary and let $Q_0 \in \mathbb{M}_{n-2p}$ be a projection of rank $q-p$. Then $R \oplus Q_0 \in \mathbb{M}_n$ is a projection of rank q and thus so is

$$Q := V(R \oplus Q_0)V^* = \begin{bmatrix} X & B \\ B^* & U(N^2 \oplus Q_0)U^* \end{bmatrix}.$$

Setting $Z := U(N^2 \oplus Q_0)U^*$ completes the proof.

□

4.7. Lemma. *Let $n, r \in \mathbb{N}$ with $r \geq 2$, and suppose that we are given integers t_i , $1 \leq i \leq r$, p and q satisfying*

$$1 \leq t_r \leq t_{r-1} \leq \cdots \leq t_1 \leq p \leq q \leq \frac{n}{2}, \quad \text{and} \quad p \leq \sum_{i=1}^r t_i.$$

Suppose that $X = \text{diag}(x_1, x_2, \dots, x_p) \in \mathbb{M}_p$ satisfies $0 < X < I_p$.

Then there exists a projection of $Q \in \mathbb{M}_n$ of rank q whose first $p \times n$ block row is

$$\begin{bmatrix} X & B_1 & B_2 & \cdots & B_r \end{bmatrix},$$

where each $B_i \in \mathbb{M}_{p \times t_i}$ is of rank t_i , $1 \leq i \leq r$. Furthermore, all entries of $B_1 B_1^*$ are non-zero.

Proof. Let $D = X - X^2 > 0$. Let $N_1, \dots, N_r \in \mathbb{M}_p$ be the operators obtained as in Lemma 4.5 so that $\text{rank } N_i = t_i$, $1 \leq i \leq r$, each N_i is nonnegative, all of the entries of N_1 are non-zero, and

$$N_1 + N_2 + \cdots + N_r = D.$$

It is not hard to see that for each such i , there is an operator $B_i \in \mathbb{M}_{p \times t_i}$ such that $N_i = B_i B_i^*$. Clearly $\text{rank } B_i = t_i$ for each $1 \leq i \leq r$. Set $B := \begin{bmatrix} B_1 & B_2 & \cdots & B_r \end{bmatrix}$. Applying Lemma 4.6 now produces the desired projection $Q \in \mathbb{M}_n$.

□

4.8. Theorem. Let $p, q, r, n \in \mathbb{N}$ and suppose that $3 \leq r \leq n$. Let $\{P_1, P_2, \dots, P_r\}$ be a set of mutually orthogonal non-zero projections in $\mathbb{M}_n$ and set $p_i := \text{rank } P_i$, $1 \leq i \leq r$. Suppose furthermore that

$$1 \leq p_r \leq p_{r-1} \leq \dots \leq p_1 \leq q \leq \frac{n}{2},$$

and that $\sum_{i=1}^r P_i = I_n$. Then there exists a projection $Q \in \mathbb{M}_n$ of rank q such that

$$\dim \{P_1, P_2, \dots, P_r, Q\}' = 1.$$

Proof. Set $\mathcal{A} := \{P_1, P_2, \dots, P_r\}'$. Relative to the decomposition $\mathbb{C}^n = \oplus_{i=1}^r \text{ran } P_i$, we may write a typical member of $\mathcal{A}$ as

$$A = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_r \end{bmatrix},$$

where $A_i \in \mathbb{M}_{p_i}$ is arbitrary.

We wish to find a projection Q of rank q such that $\mathcal{A} \cap \{Q\}' = \mathbb{C}I_n$. Set $p = p_1$ and choose $X = \text{diag}(x_1, x_2, \dots, x_p) \in \mathbb{M}_p$ such that $0 < X < I_p$ and $x_j \neq x_k$ if $j \neq k$.

Since $p_1 + \sum_{i=2}^r p_i = n$ and $p_1 \leq \frac{n}{2}$, it follows that $p_1 \leq \sum_{i=2}^r p_i$. Applying Lemma 4.7, we obtain a projection $Q \in \mathbb{M}_n$ of rank q whose first $p \times n$ block row is

$$[X \quad B_2 \quad B_3 \quad \dots \quad B_r],$$

where each $B_i \in \mathbb{M}_{p \times p_i}$ has rank p_i , $2 \leq i \leq r$ and all of the entries of $B_2 B_2^*$ are non-zero. Thus

$$Q = \begin{bmatrix} X & B_2 & \dots & B_r \\ B_2^* & Q_{22} & \dots & Q_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ B_r^* & Q_{r2}^* & \dots & Q_{rr} \end{bmatrix}.$$

We will complete the proof of the theorem if we can show that $AQ = QA$ implies that $A \in \mathbb{C}I_n$.

The equation $AQ = QA$ implies that $A_1 X = X A_1$, and that $A_1 B_i = B_i A_i$ and $A_i B_i^* = B_i^* A_1$ for $2 \leq i \leq r$. In particular,

$$A_1 B_2 B_2^* = B_2 A_2 B_2^* = B_2 B_2^* A_1.$$

Thus A_1 commutes with both X and $B_2 B_2^*$. But X is a diagonal operator with distinct diagonal values, and thus $A_1 X = X A_1$ implies that A_1 is also diagonal. On the other hand, all of the entries of $B_2 B_2^*$ are non-zero, which then forces A_1 to be scalar, say $A_1 = \alpha I_p$.

For each $2 \leq i \leq r$, the equation $\alpha B_i = A_1 B_i = B_i A_i$, combined with the fact that $\text{rank } B_i = p_i$ implies that $A_i = \alpha I_{p_i}$.

In conclusion, $A = \oplus_{i=1}^r \alpha I_{p_i} = \alpha I_n \in \mathbb{C}I_n$, as required. $\square$

The following result now follows immediately from Theorem 4.8.

4.9. Theorem. Let $n, q, k, p_1, p_2, \dots, p_k, \mu_1, \mu_2, \dots, \mu_k \in \mathbb{N}$. Suppose that

$$p_k \leq p_{k-1} \leq \dots \leq p_1 \leq q \leq \frac{n}{2},$$

and that

$$\mathcal{A} = \mathbb{M}_{p_1}^{(\mu_1)} \oplus \mathbb{M}_{p_2}^{(\mu_2)} \oplus \dots \oplus \mathbb{M}_{p_k}^{(\mu_k)}$$

is a unital C^* -subalgebra of $\mathbb{M}_n$, where $\sum_{i=1}^k \mu_i \geq 3$.

Let $\mathcal{B} \subseteq \mathbb{M}_q \oplus \mathbb{M}_{n-q}$ be a unital C^* -algebra. Then

$$\text{MINT}_U(\mathcal{A}, \mathcal{B}) = \text{MINT}_S(\mathcal{A}, \mathcal{B}) = 1.$$

Proof. It is easy to see from the structure of $\mathcal{A}$ that there exist $r := \sum_{i=1}^k \mu_i \geq 3$ mutually orthogonal, diagonal projections $P_1, P_2, \dots, P_r$ such that $\text{rank } P_1 = p_1$, $\sum_{i=1}^r P_i = I_n$, and $\mathcal{A} \subseteq \{P_1, P_2, \dots, P_r\}'$. (In fact, equality occurs precisely when $\mu_i = 1$ for all i .)

Meanwhile, clearly there exists a projection $R \in \mathbb{M}_n$ with $\text{rank } R = q$ such that $\mathcal{B} \subseteq R'$.

By Theorem 4.8, we can find a projection $Q \in \mathbb{M}_n$ of rank q such that

$$\{P_1, P_2, \dots, P_r, Q\}' = \mathbb{C}I_n.$$

Let $U \in \mathbb{M}_n$ be any unitary for which $Q = U^*RU$. Then $U^*\mathcal{B}U \subseteq \{Q\}'$, whence

$$\begin{aligned} \text{MINT}_U(\mathcal{A}, \mathcal{B}) &\leq \dim(A \cap U^*\mathcal{B}U) \\ &\leq \dim(\{P_1, P_2, \dots, P_r\}' \cap \{Q\}') \\ &= \dim(\{P_1, P_2, \dots, P_r, Q\}') \\ &= 1. \end{aligned}$$

But since $\mathcal{A}$ and $\mathcal{B}$ are unital, it is easy to see that $\text{MINT}_S(\mathcal{A}, \mathcal{B}) \geq 1$, and thus

$$1 \leq \text{MINT}_S(\mathcal{A}, \mathcal{B}) \leq \text{MINT}_U(\mathcal{A}, \mathcal{B}) \leq 1$$

implies that $\text{MINT}_U(\mathcal{A}, \mathcal{B}) = \text{MINT}_S(\mathcal{A}, \mathcal{B}) = 1$, as claimed. □

5. RESULTS – THE CASE WHERE $\mathcal{A} = \mathcal{B}$, AND CASE 2(A)(II).

5.1. In this section, we shall consider a special case of Problem 4S where the algebras $\mathcal{A}$ and $\mathcal{B}$ coincide. Corollary 5.6 is required to complete Case 2(a)(ii) from Paragraph 3.2. As it turns out, the technical difficulties that arise from Case 4 of Paragraph 3.2 disappear when $\mathcal{A} = \mathcal{B}$.

5.2. Strategy. Mirroring our strategy to attack the general problem as outlined in Paragraph 3.2, we proceed to consider this special version of the problem by a case-by-case analysis, depending upon the structure of $\mathcal{A}$.

CASE 1. $\kappa_{\mathcal{A}} = 2, \quad \mu_1 = 2.$

Thus $n = 2p_1$ and $\mathcal{A} = \mathbb{M}_{p_1}^{(2)}$. By Corollary 4.3,

$$\text{MINT}_U(\mathcal{A}, \mathcal{A}) = \text{MINT}_S(\mathcal{A}, \mathcal{A}) = 1.$$

CASE 2. $\kappa_{\mathcal{A}} = 2, \quad \mu_1 = 1 = \mu_2.$

Then $\mathcal{A} = \mathbb{M}_{p_1} \oplus \mathbb{M}_{p_2}$. By Corollary 5.5 below,

$$\text{MINT}_U(\mathcal{A}, \mathcal{A}) = \text{MINT}_S(\mathcal{A}, \mathcal{A}) = (p_1 - p_2)^2 + p_2.$$

CASE 3. $\kappa_{\mathcal{A}} \geq 3$, $p_1 \leq \frac{n}{2}$.

Let $q_1 = p_1$ and let $\mathcal{C} := \mathbb{M}_{q_1} \oplus \mathbb{M}_{n-q_1}$. Observe that $\mathcal{A} \subseteq \mathcal{C}$. Note also that

$$p_k \leq p_{k-1} \leq \cdots \leq p_1 = q_1 \leq \frac{n}{2}.$$

By Theorem 4.9,

$$\text{MINT}_U(\mathcal{A}, \mathcal{A}) = \text{MINT}_S(\mathcal{A}, \mathcal{A}) = 1.$$

CASE 4. $\kappa_{\mathcal{A}} \geq 3$, $p_1 > \frac{n}{2}$.

In this case, by Theorem 5.9,

$$\text{MINT}_S(\mathcal{A}, \mathcal{A}) = \text{MINT}_U(\mathcal{A}, \mathcal{A}) = (n - 2m)^2 + 1,$$

where $m := \sum_{i=2}^{\kappa_{\mathcal{A}}} p_i \mu_i = n - p_1$.

5.3. In order to handle CASE 2 above, we shall prove a more general result, namely, we shall calculate the explicit value of $\text{MINT}_S(\mathcal{A}, \mathcal{B})$ when $\mathcal{A} = \mathbb{M}_p \oplus \mathbb{M}_{n-p}$ and $\mathcal{B} = \mathbb{M}_q \oplus \mathbb{M}_{n-q}$. Even in this simple setting, the formula is non-trivial, and this will hopefully give the reader an idea of the level of complexity that would be involved in calculating $\text{MINT}_S(\mathcal{A}, \mathcal{B})$ for arbitrary C^* -subalgebras of $\mathbb{M}_n$.

5.4. Theorem. *Let $1 \leq p \leq q < 2q \leq n$ be integers. If $\mathcal{A} = \mathbb{M}_p \oplus \mathbb{M}_{n-p}$ and $\mathcal{B} = \mathbb{M}_q \oplus \mathbb{M}_{n-q}$, then*

$$\text{MINT}_S(\mathcal{A}, \mathcal{B}) = \text{MINT}_U(\mathcal{A}, \mathcal{B}) = (q - p)^2 + (n - p - q)^2 + p.$$

Proof.

STEP 1. $\text{MINT}_U(\mathcal{A}, \mathcal{B}) \leq (q - p)^2 + (n - p - q)^2 + p$.

Let $Q \in \mathcal{P}_{q,n}$ be the projection $Q = I_{q-p} \oplus 0_{n-q-p} \oplus I_p \oplus 0_p$ and define $P \in \mathcal{P}_{p,n}$ via

$$P = \begin{bmatrix} 0_{n-2p} & 0 & 0 \\ 0 & M^2 & MN \\ 0 & MN & N^2 \end{bmatrix},$$

where $0 < \alpha < \frac{1}{2}$, $M := \text{diag}(\alpha, \alpha^2, \dots, \alpha^p)$ and $N = (I_p - M^2)^{\frac{1}{2}}$.

Then $\mathcal{A}$ is unitarily equivalent to $\{P\}'$ and $\mathcal{B}$ is unitarily equivalent to $\{Q\}'$, so that

$$\text{MINT}_U(\mathcal{A}, \mathcal{B}) \leq \{P\}' \cap \{Q\}' = (\text{Alg}(P, Q))'.$$

Let $X \in (\text{Alg}(P, Q))'$. Since $XQ = QX$, relative to the decomposition $\mathbb{C}^n = \mathbb{C}^{q-p} \oplus \mathbb{C}^{n-p-q} \oplus \mathbb{C}^p \oplus \mathbb{C}^p$ we find that

$$X = \begin{bmatrix} X_{11} & 0 & X_{13} & 0 \\ 0 & X_{22} & 0 & X_{24} \\ X_{31} & 0 & X_{33} & 0 \\ 0 & X_{42} & 0 & X_{44} \end{bmatrix}.$$

A routine calculation shows that since QP, QPQ and $PQ \in \text{Alg}(P, Q)$, we actually have that for all $\alpha_i \in \mathbb{C}$, $1 \leq i \leq 4$,

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_1 I_p & \alpha_2 I_p \\ 0 & 0 & \alpha_3 I_p & \alpha_4 I_p \end{bmatrix} \in \text{Alg}(P, Q).$$

From this it easily follows that X_{13}, X_{31}, X_{24} and X_{42} are all zero (of their respective sizes), and that $X_{33} = X_{44}$. But $QPQ = 0_{q-p} \oplus 0_{n-q-p} \oplus M^2 \oplus 0_p$, and thus $X(QPQ) = (QPQ)X$ implies that X_{33} is diagonal.

Thus

$$X = \begin{bmatrix} X_{11} & 0 & 0 & 0 \\ 0 & X_{22} & 0 & 0 \\ 0 & 0 & X_{33} & 0 \\ 0 & 0 & 0 & X_{33} \end{bmatrix},$$

where $X_{33} \in \mathbb{M}_p$ is diagonal and $X_{11} \in \mathbb{M}_{q-p}$, $X_{22} \in \mathbb{M}_{n-p-q}$ are arbitrary. It is routine to check that every operator X of that form lies in $\{P, Q\}'$, and so

$$\text{MINT}_U(\mathcal{A}, \mathcal{B}) \leq \dim \{P, Q\}' = (q-p)^2 + (n-p-q)^2 + p.$$

STEP 2. $\text{MINT}_S(\mathcal{A}, \mathcal{B}) \geq (q-p)^2 + (n-p-q)^2 + p.$

Since $\mathcal{A} = \{P\}'$ for some projection $P \in \mathcal{P}_{p,n}$, for any invertible operator $S \in \mathbb{M}_n$, $S^{-1}\mathcal{A}S = \{S^{-1}PS\}'$. That is, any algebra similar to $\mathcal{A}$ is of the form $\{G\}'$ for some idempotent operator $G \in \mathbb{M}_n$ of rank p . Choose a projection $Q \in \mathcal{P}_{q,n}$ such that $\mathcal{B} = \{Q\}'$. Thus, we could find an idempotent $E \in \mathbb{M}_n$ of rank p such that

$$\dim\{E, Q\}' = \text{MINT}_S(\mathcal{A}, \mathcal{B}).$$

Our goal is to show that

$$\dim\{E, Q\}' \geq (q-p)^2 + (n-p-q)^2 + p.$$

By Lemma 2.12, there exists $\varepsilon > 0$ such that if $\|T - E\| < \varepsilon$, then $\dim\{T, Q\}' \leq \dim\{E, Q\}'$. Hence, for any idempotent $F \in \mathbb{M}_n$ of rank p with $\|F - E\| < \varepsilon$,

$$\dim\{F, Q\}' = \dim\{E, Q\}'.$$

In fact, clearly, $\dim\{F, Q\}' \leq \dim\{E, Q\}'$; on the other side, since $F = S^{-1}PS$ for some invertible operator $S \in \mathbb{M}_n$, $\dim\{F, Q\}' = \dim(S^{-1}\mathcal{A}S \cap \mathcal{B}) \geq \dim\{E, Q\}'$.

Relative to the decomposition

$$\mathbb{C}^n = \text{ran} Q \oplus \ker Q,$$

we may write

$$E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}.$$

By Theorem 2.20, we may assume without loss of generality that E_{11} and E_{22} are both of rank p . Consequently, $\dim(\text{ran} Q \cap \ker E) = q - p$, and $\dim(\ker Q \cap \ker E) = n - q - p$.

Relative to the decomposition

$$\mathbb{C}^n = (\text{ran} Q \cap \ker E) \oplus (\ker Q \cap \ker E) \oplus (\text{ran} Q \ominus (\text{ran} Q \cap \ker E)) \oplus (\ker Q \ominus (\ker Q \cap \ker E)),$$

we have that $Q = I_{q-p} \oplus 0_{n-q-p} \oplus I_p \oplus 0_p$. Thus the commutant of Q is easy to describe, namely

$$\{Q\}' = \begin{bmatrix} W_1 & 0 & X_1 & 0 \\ 0 & W_2 & 0 & X_2 \\ Y_1 & 0 & Z_1 & 0 \\ 0 & Y_2 & 0 & Z_2 \end{bmatrix},$$

where W_i, X_i, Y_i , and Z_i are arbitrary (in their respective spaces), $i = 1, 2$.

Next, observe that relative to the decomposition

$$\mathbb{C}^n = [(\text{ran} Q \cap \ker E) \oplus (\ker Q \cap \ker E)] \oplus [(\text{ran} Q \ominus (\text{ran} Q \cap \ker E)) \oplus (\ker Q \ominus (\ker Q \cap \ker E))],$$

we may write

$$E = \begin{bmatrix} 0 & RE_0 \\ 0 & E_0 \end{bmatrix},$$

where E_0 is an idempotent of rank p in $\mathbb{M}_{2p}$.

We are interested in the dimension of $\{E, Q\}'$, and thus we are interested in the solution space of the equation:

$$\begin{bmatrix} RE_0Y & RE_0Z \\ E_0Y & E_0Z \end{bmatrix} = \begin{bmatrix} 0 & WRE_0 + XE_0 \\ 0 & YRE_0 + ZE_0 \end{bmatrix}, \quad (*)$$

where

$$W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix}, \quad X = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix}, \quad Y = \begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix}, \quad Z = \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix}.$$

By Theorem 2.20, we can choose $F_0 \in \mathbb{M}_{2p}$ unitarily equivalent to E_0 with $\|F_0 - E_0\| < \frac{\varepsilon}{\|R\|+1}$ such that every principal submatrix of F_0 of order at most p is invertible. Thus, replacing E_0 by F_0 and E by $F := \begin{bmatrix} 0 & RF_0 \\ 0 & F_0 \end{bmatrix}$ if necessary, and by applying Proposition 2.27, we may assume without loss of generality that E_0 is of the form

$$E_0 = \begin{bmatrix} I & 0 \\ 0 & M^{-1} \end{bmatrix} \begin{bmatrix} A & I - A \\ A & I - A \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}$$

for some A and $M \in \text{GL}(p, \mathbb{C})$, with $I - A$ also an element of $\text{GL}(p, \mathbb{C})$.

Again, by Lemma 2.12, we may perturb A (and thus E_0 and hence E) slightly so as to ensure that A and $I - A$ have $2p$ distinct, non-zero eigenvalues between them.

Next, the equation $E_0Y = 0$ from $(*)$ together with the fact that Y is of the form

$$Y = \begin{bmatrix} Y_1 & 0 \\ 0 & Y_2 \end{bmatrix}$$

imply that $Y = 0$. This substantially simplifies the remaining equations in $(*)$: indeed, we now have

$$\begin{bmatrix} 0 & RE_0Z \\ 0 & E_0Z \end{bmatrix} = \begin{bmatrix} 0 & WRE_0 + XE_0 \\ 0 & ZE_0 \end{bmatrix},$$

which means that we are looking for simultaneous solutions to the equations

$$E_0Z = ZE_0$$

and

$$(X + WR - RZ)E_0 = 0.$$

Now a tedious but easy calculation shows that the equation $E_0Z = ZE_0$ implies that A commutes with Z_1 and that $Z_2 = M^{-1}Z_1M$. Since A has p distinct eigenvalues, it follows that

$$\dim \{Z = Z_1 \oplus Z_2 \in \mathbb{M}_{2p} : E_0Z = ZE_0\} = \dim \{A\}' = p.$$

Next, fix $W = \begin{bmatrix} W_1 & 0 \\ 0 & W_2 \end{bmatrix} \in \mathbb{M}_{q-p} \oplus \mathbb{M}_{n-q-p}$, and also fix $Z \in \{A\}'$. Set $\begin{bmatrix} K_1 & K_2 \\ K_3 & K_4 \end{bmatrix} := WR - RZ$. A second tedious but easy calculation shows that we can always solve the equation $(X + WR - RZ)E_0 = 0$ by setting

$$X_1 := -(K_1 + K_2M^{-1}) \quad \text{and} \quad X_2 := -(K_3M + K_4).$$

It follows that the dimension of the space of solutions to the equation

$$\begin{bmatrix} RE_0Y & RE_0Z \\ E_0Y & E_0Z \end{bmatrix} = \begin{bmatrix} 0 & WRE_0 + XE_0 \\ 0 & YRE_0 + ZE_0 \end{bmatrix}$$

is exactly $p + (q - p)^2 + (n - q - p)^2$.

Consequently,

$$\text{MINT}_S(\mathcal{A}, \mathcal{B}) = \dim\{E, Q\}' = p + (q - p)^2 + (n - q - p)^2.$$

STEP 3.

Finally, note that

$(q - p)^2 + (n - q - p)^2 + p = \text{MINT}_S(\mathcal{A}, \mathcal{B}) \leq \text{MINT}_U(\mathcal{A}, \mathcal{B}) \leq (q - p)^2 + (n - q - p)^2 + p$,
completing the proof. □

5.5. Corollary. *Let $p_2 \leq p_1$ and $n := p_1 + p_2 \in \mathbb{N}$. Let $\mathcal{A} = \mathbb{M}_{p_1} \oplus \mathbb{M}_{p_2} \subseteq \mathbb{M}_n$. Then*

$$\text{MINT}_U(\mathcal{A}, \mathcal{A}) = \text{MINT}_S(\mathcal{A}, \mathcal{A}) = (p_1 - p_2)^2 + p_2.$$

5.6. Corollary. *Let $n \in \mathbb{N}$ be even and $p_1 := \frac{n}{2}$. Let $\mathcal{A} = \mathbb{M}_{p_1} \oplus \mathbb{M}_{p_1}$. Then*

$$\text{MINT}_S(\mathcal{A}, \mathcal{A}) = \text{MINT}_U(\mathcal{A}, \mathcal{A}) = p_1 = \frac{n}{2}.$$

5.7. Proposition. *Let $p_2 \leq p_1 - 1 \in \mathbb{N}$ and $n := p_1 + p_2$. Let $\mathcal{A} = \mathbb{M}_{p_1} \oplus \mathbb{C}I_{p_2}$. Then*

$$\text{MINT}_S(\mathcal{A}, \mathcal{A}) \geq (p_1 - p_2)^2 + 1.$$

Proof. First let us choose an invertible operator $T \in \text{GL}(n, \mathbb{C})$ such that

$$\text{MINT}_S(\mathcal{A}, \mathcal{A}) = \dim(\mathcal{A} \cap T^{-1}\mathcal{A}T).$$

We begin by showing that we can reduce the problem to the case where $\mathbb{C}^{p_1} = \mathcal{N} \oplus \mathcal{M}$, $\dim \mathcal{M} = p_2$ and relative to the decomposition $\mathbb{C}^n = \mathcal{N} \oplus \mathcal{M} \oplus \mathbb{C}^{p_2}$,

$$T = \begin{bmatrix} I_{p_1-p_2} & 0 & 0 \\ 0 & I_{p_2} & I_{p_2} \\ 0 & T_{32} & I_{p_2} \end{bmatrix}.$$

This will be achieved through a number of steps involving small perturbations and multiplication by “admissible” similarities, i.e. invertible operators lying in $\mathbb{M}_{p_1} \oplus \mathbb{M}_{p_2}$. The reason for allowing multiplication by such similarities will be made clear in STEP 2 below.

STEP 1. Relative to the decomposition $\mathbb{C}^n = \mathbb{C}^{p_1} \oplus \mathbb{C}^{p_2}$, let us write

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix}.$$

By Theorem 2.20 and Proposition 2.8, we may (and we do) assume without loss of generality that both T_1 and T_4 are invertible.

STEP 2. Observe that if $X = \begin{bmatrix} X_1 & 0 \\ 0 & X_4 \end{bmatrix}$ and $Y = \begin{bmatrix} Y_1 & 0 \\ 0 & Y_4 \end{bmatrix}$ are invertible elements of $\mathbb{M}_{p_1} \oplus \mathbb{M}_{p_2}$, then

- $\mathcal{A} = X\mathcal{A}Y$; and thus

•

$$\begin{aligned}
\text{MINT}_S(\mathcal{A}, \mathcal{A}) &= \dim(\mathcal{A} \cap T^{-1}\mathcal{A}T) \\
&= \dim(Y^{-1}(\mathcal{A} \cap T^{-1}(X^{-1}\mathcal{A}X)T)Y) \\
&= \dim(Y^{-1}\mathcal{A}Y \cap (XTY)^{-1}\mathcal{A}(XTY)) \\
&= \dim(\mathcal{A} \cap (XTY)^{-1}\mathcal{A}(XTY)).
\end{aligned}$$

In other words, we are free to replace T by any operator of the form XTY , where X and Y are of the above form.

Choose $X = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & I_{p_2} \end{bmatrix}$ and $Y = \begin{bmatrix} I_{p_1} & 0 \\ 0 & T_4^{-1} \end{bmatrix}$. Then

$$\text{MINT}_S(\mathcal{A}, \mathcal{A}) = \dim(\mathcal{A} \cap R^{-1}\mathcal{A}R),$$

where

$$R = \begin{bmatrix} I_{p_1} & R_2 \\ R_3 & I_{p_2} \end{bmatrix},$$

and where $R_2 = T_1^{-1}T_2T_4^{-1}$ and $R_3 = T_3T_4^{-1}$. By replacing T by R if necessary, we may (and do) assume without loss of generality that T is of the form

$$T = \begin{bmatrix} I_{p_1} & T_2 \\ T_3 & I_{p_2} \end{bmatrix}.$$

Furthermore, the set of operators of rank p_2 in $\mathbb{M}_{p_1 \times p_2}$ and in $\mathbb{M}_{p_2 \times p_1}$ is easily seen to be dense. It follows once again from our stability result Proposition 2.8 that we may further assume that T_2 and T_3 have rank p_2 .

STEP 3. Since $\text{rank } T_2 = p_2$, we can find invertible operators $V_1 \in \mathbb{M}_{p_1}$ and $W_4 \in \mathbb{M}_{p_2}$ such that $V_1T_2W_4 = \begin{bmatrix} 0 \\ I_{p_2} \end{bmatrix} \in \mathbb{M}_{p_1 \times p_2}$. Set $V = \begin{bmatrix} V_1 & 0 \\ 0 & W_4^{-1} \end{bmatrix}$ and $W = \begin{bmatrix} V_1^{-1} & 0 \\ 0 & W_4 \end{bmatrix}$. By the argument in STEP 2, we see that $\text{MINT}_S(\mathcal{A}, \mathcal{A})$ is implemented by

$$S := VTW = \begin{bmatrix} I_{p_1} & V_1T_2W_4 \\ W_4^{-1}T_3V_1^{-1} & I_{p_2} \end{bmatrix} = \begin{bmatrix} I_{p_1-p_2} & 0 & 0 \\ 0 & I_{p_2} & I_{p_2} \\ S_{31} & S_{32} & I_{p_2} \end{bmatrix}.$$

Thus, by replacing T by S if necessary, we may (and do) assume without loss of generality that T is of the form

$$T = \begin{bmatrix} I_{p_1-p_2} & 0 & 0 \\ 0 & I_{p_2} & I_{p_2} \\ T_{31} & T_{32} & I_{p_2} \end{bmatrix}.$$

Yet another application of our stability result Proposition 2.8 allows us to assume that T_{32} is invertible.

STEP 4. We are almost there! Let

$$Z = \begin{bmatrix} I_{p_1-p_2} & 0 & 0 \\ Z_{21} & I_{p_2} & 0 \\ 0 & 0 & I_{p_2} \end{bmatrix},$$

where $Z_{21} := -T_{32}^{-1}T_{31}$. By the argument in STEP 2 above, we may (and do) replace T with $Z^{-1}TZ$ so as to assume *a priori* that T is of the form

$$\begin{bmatrix} I_{p_1-p_2} & 0 & 0 \\ 0 & I_{p_2} & I_{p_2} \\ 0 & T_{32} & I_{p_2} \end{bmatrix}.$$

STEP 5. We now know that $\text{MINT}_S(\mathcal{A}, \mathcal{A}) = \dim(\mathcal{A} \cap T^{-1}\mathcal{A}T)$, where T is of the form stipulated in STEP 4.

Note that for any $A_{11} \in \mathbb{M}_{p_1-p_2}$ and any $\alpha \in \mathbb{C}$,

$$A := \begin{bmatrix} A_{11} & 0 & 0 \\ 0 & \alpha I_{p_2} & 0 \\ 0 & 0 & \alpha I_{p_2} \end{bmatrix} \in \mathcal{A} \cap \{T\}',$$

and thus $A \in \mathcal{A} \cap T^{-1}\mathcal{A}T$. It follows that

$$\text{MINT}_S(\mathcal{A}, \mathcal{A}) = \dim(\mathcal{A} \cap T^{-1}\mathcal{A}T) \geq \dim \mathbb{M}_{p_1-p_2} + 1 = (p_1 - p_2)^2 + 1,$$

completing the proof. $\square$

5.8. Proposition. *Let $1 \leq p_3 \leq p_2 \leq p_1 \in \mathbb{N}$ and suppose that $p_1 + p_2 + p_3 = n < 2p_1$. Let $\mathcal{A} = \mathbb{M}_{p_1} \oplus \mathbb{M}_{p_2} \oplus \mathbb{M}_{p_3}$. Set $m := p_2 + p_3$. Then*

$$\text{MINT}_U(\mathcal{A}, \mathcal{A}) \leq (p_1 - m)^2 + 1 = (n - 2m)^2 + 1.$$

Proof. Since $2m < n$, we can decompose $\mathbb{C}^n = \mathbb{C}^{n-2m} \oplus \mathbb{C}^m \oplus \mathbb{C}^m$. Let $R \in \mathbb{M}_n$ be a projection of rank p_2 all of whose entries are non-zero. This is possible by [7].

Define $P_2 := 0_{n-2m} \oplus R \oplus 0_m$, and $P_3 := 0_{n-2m} \oplus (I_m - R) \oplus 0_m$, so that P_2 and $P_3 \in \mathbb{M}_n$ are projections of ranks p_2 and p_3 respectively. Furthermore, P_2 and P_3 are mutually orthogonal, and thus

$$\mathcal{A} \simeq \{P_2, P_3\}' = C^*(P_2, P_3)',$$

say $\mathcal{A} = U^*C^*(P_2, P_3)'U$ for some unitary operator $U \in \mathbb{M}_n$.

Let $T_0 := P_2 + P_3 = 0_{n-2m} \oplus I_m \oplus 0_m$.

Next, choose $0 < c_1 < c_2 < \dots < c_m < 1$, and set $C = \text{diag}(c_1, c_2, \dots, c_m) \in \mathbb{M}_m$, noting that $0 < C < I_m$. Define $s_i = \sqrt{1 - c_i^2}$, $1 \leq i \leq m$, and $S = \text{diag}(s_1, s_2, \dots, s_m) \in \mathbb{M}_m$. Observe that $SC = CS$ and

$$Q_0 := \begin{bmatrix} 0_{n-2m} & 0 & 0 \\ 0 & C^2 & CS \\ 0 & CS & S^2 \end{bmatrix}$$

is a projection of rank $m = p_2 + p_3$ in $\mathbb{M}_n$. As such, we may write $Q_0 = Q_2 + Q_3$ for mutually orthogonal projections Q_2 and Q_3 of ranks p_2 and p_3 respectively. Then

$$\mathcal{A} \simeq \{Q_2, Q_3\}' = C^*(Q_2, Q_3)',$$

say $\mathcal{A} = V^*C^*(Q_2, Q_3)'V$ for some unitary operator $V \in \mathbb{M}_n$.

It is not hard to see that (since all diagonal entries of C are distinct),

$$C^*(T_0, Q_0) = \left\{ \begin{bmatrix} 0_{n-2m} & 0 & 0 \\ 0 & D_1 & D_2 \\ 0 & D_3 & D_4 \end{bmatrix} : D_i \in \mathcal{D}_m, 1 \leq i \leq 4 \right\} \subseteq C^*(P_2, P_3, Q_0),$$

where $\mathcal{D}_m = \{D \in \mathbb{M}_m : D \text{ is diagonal}\}$.

Recalling that $P_2 = \begin{bmatrix} 0_{n-2m} & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & 0_m \end{bmatrix}$ and that all entries of R are non-zero, it is not

hard to see that

$$C^*(P_2, P_3, Q_0) = 0_{n-2m} \oplus \mathbb{M}_{2m}.$$

Thus $Q_2, Q_3 \in C^*(P_2, P_3, Q_0)$ and so

$$C^*(P_2, P_3, Q_2, Q_3) = 0_{n-2m} \oplus \mathbb{M}_{2m}.$$

But then

$$(U\mathcal{A}U^*) \cap (V\mathcal{A}V^*) = C^*(P_2, P_3)' \cap C^*(Q_2, Q_3)' = C^*(P_2, P_3, Q_2, Q_3)' = \mathbb{M}_{n-2m} \oplus \mathbb{C}I_{2m},$$

whence

$$\text{MINT}_U(\mathcal{A}, \mathcal{A}) \leq \dim(\mathcal{A} \cap (U^*V\mathcal{A}V^*U)) = \dim((U\mathcal{A}U^*) \cap (V\mathcal{A}V^*)) = (n - 2m)^2 + 1.$$

□

5.9. Theorem. *Let $1 \leq p_k \leq p_{k-1} \leq \dots \leq p_2 \leq p_1 \in \mathbb{N}$, $\mu_1, \mu_2, \dots, \mu_k \in \mathbb{N}$, and $\mathcal{A} = \mathbb{M}_{p_1}^{(\mu_1)} \oplus \mathbb{M}_{p_2}^{(\mu_2)} \oplus \dots \oplus \mathbb{M}_{p_k}^{(\mu_k)}$. Set $n := \sum_{i=1}^k p_i \mu_i$, $\kappa := \mu_1 + \mu_2 + \dots + \mu_k$ and suppose that $\kappa \geq 3$ and $p_1 > n/2$. Then*

$$\text{MINT}_S(\mathcal{A}, \mathcal{A}) = \text{MINT}_U(\mathcal{A}, \mathcal{A}) = (n - 2m)^2 + 1,$$

where $m := \sum_{i=2}^k p_i \mu_i = n - p_1$.

Proof. First note that $p_1 > n/2$ implies that $\mu_1 = 1$. Thus $m := \sum_{i=2}^k p_i \mu_i = n - p_1$, as claimed. Set $q_1 := p_1$. Next,

- if $\mu_k = 1$, set $q_2 := p_2 \mu_2 + p_3 \mu_3 + \dots + p_{k-1} \mu_{k-1}$, and $q_3 = p_k$.
- if $\mu_k > 1$, set $q_2 := p_2 \mu_2 + p_3 \mu_3 + \dots + p_{k-1} \mu_{k-1} + p_k(\mu_k - 1)$, and $q_3 = p_k$.

It is clear that $\mathcal{A} \subseteq \mathcal{B} := \mathbb{M}_{q_1} \oplus \mathbb{M}_{q_2} \oplus \mathbb{M}_{q_3}$. Furthermore, $m := q_2 + q_3 = n - q_1$, and $n < 2q_1$. By Proposition 5.8,

$$\text{MINT}_U(\mathcal{A}, \mathcal{A}) \leq \text{MINT}_U(\mathcal{B}, \mathcal{B}) \leq (n - 2m)^2 + 1.$$

Let $\mathcal{C} := \mathbb{M}_{q_1} \oplus \mathbb{C}I_m$, and observe that $\mathcal{C} \subseteq \mathcal{A}$. By Proposition 5.7,

$$(q_1 - m)^2 + 1 = (n - 2m)^2 + 1 \leq \text{MINT}_S(\mathcal{C}, \mathcal{C}) \leq \text{MINT}_S(\mathcal{A}, \mathcal{A}) \leq \text{MINT}_U(\mathcal{A}, \mathcal{A}) \leq (n - 2m)^2 + 1,$$

from which the desired equality ensues.

□

6. RESULTS - CASE 4

6.1. We now turn our attention to obtaining the results required for Case 4 of the strategy outlined in Section 3. Our first lemma is relatively straightforward, and its proof is left to the reader.

6.2. Lemma. *Suppose that $\mu \in \mathbb{N}$ and $1 \leq r < p \leq \mu p \leq n$ are integers. Let $C \in \mathbb{M}_n$ and write $C = [C_{ij}]_{i,j=1}^{\mu+1}$ relative to the decomposition $\mathbb{C}^n = (\mathbb{C}^p)^{(\mu)} \oplus \mathbb{C}^{n-p\mu}$.*

If $\text{rank } C = r = \text{rank } C_{11}$, then there exists $\{M_j, N_j : 1 \leq j \leq \mu\} \subseteq \mathbb{M}_p$, $M_{\mu+1} \in \mathbb{M}_{n-\mu p, p}$, and $N_{\mu+1} \in \mathbb{M}_{p, n-\mu p}$ such that

$$C_{ij} = M_i C_{11} N_j, \quad 1 \leq i, j \leq \mu + 1.$$

Furthermore, without loss of generality we may assume that $M_1 = I_p = N_1$.

6.3. Lemma. *Suppose that $\mu \in \mathbb{N}$ and $1 \leq r \leq \mu r < p \leq \mu p \leq n$ are integers. Let $A \in \mathbb{M}_p$ be an operator of rank r . Suppose furthermore that $\{M_j, N_j : 1 \leq j \leq \mu\} \subseteq \mathbb{M}_p$. Then there exists a rank-one operator $X_0 \in \mathbb{M}_p$ such that*

$$X_0 M_j A = 0 = A N_j X_0, \quad 1 \leq j \leq \mu.$$

Consequently,

$$X_0 M_i A N_j = 0 = M_i A N_j X_0 \quad \text{for all } 1 \leq i, j \leq \mu.$$

Proof. Let $\mathcal{M} := \text{span}\{\text{ran } M_j A : 1 \leq j \leq \mu\}$, and let $\mathcal{N} := \cap\{\ker A N_j : 1 \leq j \leq \mu\}$. Observe that $\dim \mathcal{M} \leq \mu r < p$, while $\dim \mathcal{N} \geq p - \mu r > 0$.

Choose norm-one vectors $x \in \mathcal{M}^\perp$ and $y \in \mathcal{N}$, and set $X_0 := y \otimes x^*$. Then for all $z \in \mathbb{C}^p$, and any $1 \leq j \leq \mu$,

- $X_0 M_j A z = (y \otimes x^*)(M_j A z) = \langle M_j A z, x \rangle y = 0 y = 0$, so $X_0 M_j A = 0$; while
- $A N_j X_0 = A N_j (y \otimes x^*) = (A N_j y) \otimes x^* = 0 \otimes x^* = 0$.

The last statement follows immediately from this. □

6.4. Remark. Suppose that we are given $C \in \mathbb{M}_n$ of rank r as in Lemma 6.2, and that we have chosen the appropriate M_j, N_j as in that lemma so as to obtain $C_{ij} = M_i C_{11} N_j$, $1 \leq i, j \leq \mu + 1$. If we choose $X_0 \in \mathbb{M}_p$ such that $X_0 M_j C_{11} = 0 = C_{11} N_j X_0$, $1 \leq i, j \leq \mu$, and if we set $X := X_0^{(\mu)} \oplus 0_{n-\mu p}$, then the last statement of Lemma 6.3 implies that

$$XC = 0 = CX.$$

The next theorem is the first of the two main results of this section.

6.5. Theorem. *Suppose that $\mu \in \mathbb{N}$ and $1 \leq r \leq \mu r < p \leq \mu p \leq n < p + q$ are integers, where $q := n - r$. If $\mathcal{A}_0 := \mathbb{M}_p^{(\mu)} \oplus \mathbb{C}I_{n-\mu p}$ and $\mathcal{B}_0 := \mathbb{M}_q \oplus \mathbb{C}I_{n-q}$, then*

$$\text{MINT}_S(\mathcal{A}_0, \mathcal{B}_0) > 1.$$

Proof. With respect to the decomposition $\mathbb{C}^n = \mathbb{C}^q \oplus \mathbb{C}^{n-q}$, consider

$$D := \begin{bmatrix} 0 & 0 \\ 0 & D_4 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 0 & 0 \\ 0 & D_4 L_4 D_4 \end{bmatrix},$$

where $D_4 \in \mathbb{M}_r$ is a diagonal operator with r distinct non-zero eigenvalues, and $L_4 \in \mathbb{M}_r$ is a matrix, all of whose entries are non-zero.

Fix $S_0 \in \text{GL}(n, \mathbb{C})$ such that

$$\dim(\mathcal{A}_0 \cap S_0^{-1} \mathcal{B}_0 S_0) = \text{MINT}_S(\mathcal{A}_0, \mathcal{B}_0).$$

Recall from Proposition 2.8 that there exists $\varepsilon > 0$ such that if $S \in \mathbb{M}_n$ satisfies $\|S - S_0\| < \varepsilon$, then S is invertible and

$$\dim(\mathcal{A}_0 \cap S^{-1} \mathcal{B}_0 S) = \text{MINT}_S(\mathcal{A}_0, \mathcal{B}_0).$$

By Proposition 2.24, there exists $U \in \mathbb{M}_n$ unitary with $\|U - I_n\| < \frac{\varepsilon}{\|S_0\|}$ such that if $C := U^* S_0^{-1} D S_0 U$, then every principal submatrix of C of order $1 \leq k \leq n$ has rank equal to $\min(k, r)$. Since $\|S_0 - S_0 U\| \leq \|S_0\| \|I_n - U\| < \varepsilon$, we conclude that with $T := S_0 U$,

$$\dim(\mathcal{A}_0 \cap T^{-1} \mathcal{B}_0 T) = \text{MINT}_S(\mathcal{A}_0, \mathcal{B}_0).$$

Moreover, a routine calculation shows that $\mathcal{B}_0 = \{D, L\}'$, and thus

$$T^{-1}\mathcal{B}_0T = \{T^{-1}DT, T^{-1}LT\}'.$$

Clearly $\mathcal{A}_0$ and $T^{-1}\mathcal{B}_0T$ are unital, and thus $\mathcal{A}_0 \cap T^{-1}\mathcal{B}_0T$ contains $\{\mathbb{C}I_n\}$. To prove the theorem, therefore, it suffices to find a non-scalar operator in $\mathcal{A}_0 \cap T^{-1}\mathcal{B}_0T$.

Let us write $C = [C_{ij}]_{i,j=1}^{\mu+1}$ relative to the decomposition of $\mathbb{C}^n$ as $\mathbb{C}^n = (\mathbb{C}^p)^{(\mu)} \oplus \mathbb{C}^{n-\mu p}$. It is not hard to see that

- $C = U^*S_0^{-1}DS_0U$ has rank r (since D does), and
- C_{11} also has rank r since it is a principal submatrix of order $p > r$.

By Lemma 6.2, one concludes that there exist $\{M_j, N_j : 1 \leq j \leq \mu\} \subseteq \mathbb{M}_p$, $M_{\mu+1} \in \mathbb{M}_{n-\mu p, p}$, and $N_{\mu+1} \in \mathbb{M}_{p, n-\mu p}$ such that

$$C_{ij} = M_i C_{11} N_j, \quad 1 \leq i, j \leq \mu + 1.$$

By Lemma 6.3, there exists a rank-one operator $X_0 \in \mathbb{M}_p$ such that $X_0 M_i C_{11} = 0 = C_{11} N_j X_0$, $1 \leq i, j \leq \mu$, and from Remark 6.4, one sees that by setting $X = X_0^{(\mu)} \oplus 0_{n-\mu p}$, one has that

$$XC = 0 = CX.$$

Now $X \in \mathbb{M}_p^{(\mu)} \oplus 0_{n-\mu p} \subseteq \mathcal{A}_0$ is not scalar (since $p > r \geq 1$). Not only does X lie in $\{C\}'$, but in fact

$$\begin{aligned} X(T^{-1}LT) &= X(T^{-1}DT)(T^{-1}D^{-1}LT) \\ &= XC(T^{-1}D^{-1}LT) \\ &= 0 \\ &= (T^{-1}LD^{-1}T)CX \\ &= (T^{-1}LD^{-1}T)(T^{-1}DT)X \\ &= (T^{-1}LT)X. \end{aligned}$$

In other words, $X \in \{C, T^{-1}LT\}' = \{T^{-1}DT, T^{-1}LT\}' = T^{-1}\mathcal{B}_0T$.

Thus

$$\text{MINT}_S(\mathcal{A}_0, \mathcal{B}_0) = \dim(\mathcal{A}_0 \cap T^{-1}\mathcal{B}_0T) \geq 2.$$

□

6.6. Corollary. *Suppose that $\mathcal{A}$ and $\mathcal{B}$ are unital C^* -subalgebras of $\mathbb{M}_n$, and write*

$$\mathcal{A} = \mathbb{M}_{p_1}^{(\mu_1)} \oplus \mathbb{M}_{p_2}^{(\mu_2)} \oplus \cdots \oplus \mathbb{M}_{p_k}^{(\mu_k)},$$

where $p_1 \geq p_2 \geq \cdots p_k \geq 1$, and if $p_i = p_{i+1}$, then $\mu_i \geq \mu_{i+1}$. Similarly, write

$$\mathcal{B} = \mathbb{M}_{q_1}^{(\nu_1)} \oplus \mathbb{M}_{q_2}^{(\nu_2)} \oplus \cdots \oplus \mathbb{M}_{q_l}^{(\nu_l)},$$

where $q_1 \geq q_2 \geq \cdots q_s \geq 1$, and if $q_j = q_{j+1}$, then $\nu_j \geq \nu_{j+1}$.

If there exists $1 \leq i_0 \leq k$ such that $q_1 > n - \frac{p_{i_0}}{\mu_{i_0}}$, then

$$\text{MINT}_S(\mathcal{A}, \mathcal{B}) > 1.$$

Proof. Having chosen $1 \leq i_0 \leq k$ as in the statement of the Corollary, observe that $\mathcal{A}$ contains a C^* -subalgebra $\mathcal{A}_0$ unitarily equivalent to $\mathbb{M}_{p_{i_0}}^{(\mu_{i_0})} \oplus \mathbb{C}I_{n-\mu_{i_0}p_{i_0}}$. Similarly, $\mathcal{B}$ contains a C^* -subalgebra $\mathcal{B}_0$ unitarily equivalent to $\mathbb{M}_{q_1} \oplus \mathbb{C}I_{n-q_1}$. By Theorem 6.5,

$$1 < \text{MINT}_S(\mathcal{A}_0, \mathcal{B}_0) \leq \text{MINT}_S(\mathcal{A}, \mathcal{B}).$$

□

There remains only one case to consider, namely the case where no such $1 \leq i_0 \leq k$ as in the statement of Corollary 6.6 exists.

6.7. Lemma. *Suppose that $1 \leq r < p$ are integers and that $p \leq 2r$. Let $D_1, D_2 \in \mathbb{M}_r$ be positive, diagonal, invertible matrices, and suppose that D_1 has r distinct eigenvalues. Then there exists a unitary operator $U \in \mathbb{M}_p$ such that if $C_1 = D_1 \oplus 0_{p-r}$ and $C_2 = D_2 \oplus 0_{p-r}$, then*

$$\{C_1, C_2 U\}' = \mathbb{C}I_p.$$

Proof. Write $\mathbb{C}^p = \mathbb{C}^r \oplus \mathbb{C}^{p-r}$. Choose a unitary $U = \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} \in \mathbb{M}_p$ such that

- all entries of U are non-zero;
- $U_1 \in \mathbb{M}_r$ is invertible; and
- U_2 has rank $p - r$ (i.e. U_2 has trivial kernel).

(To see that such a unitary operator U exists, it suffices to consider the discrete Fourier Transform $U := \frac{1}{\sqrt{p}}[\omega^{(i-1)(j-1)}]_{1 \leq i, j \leq p}$, where ω is a primitive p^{th} root of unity in $\mathbb{C}$.) Suppose that $X \in \{C_1, C_2 U\}'$.

Relative to the decomposition $\mathbb{C}^p = \mathbb{C}^r \oplus \mathbb{C}^{p-r}$ we may write $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix}$. It is routine to verify that $X \in C_1'$ if and only if X_1 is diagonal and X_2, X_3 are both 0 (in their respective spaces). Thus

$$X = \begin{bmatrix} D_X & 0 \\ 0 & X_4 \end{bmatrix},$$

where $D_X \in \mathbb{M}_r$ is diagonal.

Relative to the same decomposition of $\mathbb{C}^p$, we may also write

$$C_2 U = \begin{bmatrix} D_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} U_1 & U_2 \\ U_3 & U_4 \end{bmatrix} = \begin{bmatrix} D_2 U_1 & D_2 U_2 \\ 0 & 0 \end{bmatrix}.$$

Then $X(C_2 U) = (C_2 U)X$ implies that

$$\begin{bmatrix} D_X(D_2 U_1) & D_X(D_2 U_2) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} (D_2 U_1)D_X & (D_2 U_2)X_4 \\ 0 & 0 \end{bmatrix}.$$

Since D_2 is positive, invertible, and diagonal, all entries of $D_2 U_1$ are non-zero. Since D_X is diagonal, the equation $D_X(D_2 U_1) = (D_2 U_1)D_X$ implies that D_X is scalar, say $D_X = \beta I_r$ for some $\beta \in \mathbb{C}$.

The second equation $D_X(D_2 U_2) = (D_2 U_2)X_4$ now becomes $(D_2 U_2)(\beta I_{p-r} - X_4) = 0$. Since U_2 has trivial kernel and D_2 is invertible, this in turn forces $X_4 = \beta I_{p-r}$, whence $X = \beta I_p$. This completes the proof.

□

6.8. Lemma. *Let $1 \leq r < 2r \leq p$ be integers and suppose that $D_1, D_2 \in \mathbb{M}_r$ are positive, diagonal, invertible matrices. Suppose furthermore that D_1 has r distinct eigenvalues. Let $V \in \mathbb{M}_r$ be a unitary operator, all of whose entries are non-zero. Decompose $\mathbb{C}^p = \mathbb{C}^r \oplus \mathbb{C}^r \oplus \mathbb{C}^{p-2r}$, and relative to this decomposition, define the operators*

$$T_1 := \begin{bmatrix} D_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad T_2 = \begin{bmatrix} D_2 V & D_2 V & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

If $X \in \mathbb{M}_p$ commutes with both T_1 and T_2 , then we can find $\alpha \in \mathbb{C}$ such that relative to the above decomposition of $\mathbb{C}^n$, we have

$$X = \begin{bmatrix} \alpha I_r & 0 & 0 \\ 0 & \alpha I_r & 0 \\ 0 & X_{32} & X_{33} \end{bmatrix}.$$

Proof. The fact that X commutes with both T_1 and T_2 implies that X commutes with everything in $\text{Alg}\{T_1, T_2\}$, the algebra generated by T_1 and T_2 . Since that algebra contains T_1 , it contains every operator of the form $Z := Z_{11} \oplus 0_{p-r}$, where $Z_1 \in \mathbb{M}_r$ is diagonal. Furthermore, it contains ZT_2 for every such Z . From this we deduce that X is of the form

$$X = \begin{bmatrix} X_{11} & 0 & 0 \\ 0 & X_{22} & X_{23} \\ 0 & X_{32} & X_{33} \end{bmatrix},$$

where $X_{11} \in \mathbb{M}_r$ is again diagonal. The equation $T_2 X = X T_2$ implies that $X_{11}(D_2 V) = (D_2 V)X_{11}$. But every entry of V (and hence of $D_2 V$) is non-zero, and thus X_{11} must be scalar, say $X_{11} = \alpha I_r$ for some $\alpha \in \mathbb{C}$.

The equation $X T_2 = T_2 X$ next implies that

$$\begin{bmatrix} \alpha D_2 V & \alpha D_2 V & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \alpha D_2 V & D_2 V X_{22} & D_2 V X_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

from which we deduce that $X_{23} = 0$ (since $D_2 V$ is invertible) and $D_2 V(\alpha I_r - X_{22}) = 0$, whence $X_{22} = \alpha I_r$ (again, since $D_2 V$ is invertible). This completes the proof. $\square$

The usefulness of the above lemma might not be obvious at first glance. If we express it in operator theoretic terms, however, it becomes stronger. What we have shown is the following. Let $\mathcal{M}$ and $\mathcal{N}$ be two mutually orthogonal subspaces of $\mathbb{C}^p$, each of dimension r , and let $\mathcal{K} = \mathbb{C}^p \ominus (\mathcal{M} \oplus \mathcal{N})$. Given any two operators T_1, T_2 which admit the above decomposition relative to $\mathbb{C}^p = \mathcal{M} \oplus \mathcal{N} \oplus \mathcal{K}$, it follows that for any operator $X \in \{T_1, T_2\}'$, there exists a scalar $\alpha \in \mathbb{C}$ such that $(X - \alpha I_p)^*|_{\mathcal{M} \oplus \mathcal{N}} = 0$. We now put this formulation of Lemma 6.8 to use.

6.9. Lemma. *Let $1 \leq r < p$ and $2 \leq \mu$ be integers, and suppose that $p \leq \mu r$. Set $\lambda := \lceil \frac{p}{r} \rceil$, the least integer being larger than or equal to $\frac{p}{r}$. For each $1 \leq j \leq \lambda$, let $0 < D_j \in \mathbb{M}_r$ be a diagonal matrix, and suppose that D_1 has r distinct eigenvalues. If we define*

$$C_j := D_j \oplus 0_{p-r} \in \mathbb{M}_p,$$

then there exist unitary operators $U_1, U_2, \dots, U_\lambda \in \mathbb{M}_p$ such that

$$\{C_1 U_1, C_2 U_2, \dots, C_\lambda U_\lambda\}' = \mathbb{C} I_p.$$

Furthermore, we may assume that $U_1 = I_p$.

Proof. The case where $\lambda = 2$ was dealt with in Lemma 6.7. Let us therefore argue the case where $\lambda \geq 3$.

Let $\mathfrak{B} := \{e_j\}_{j=1}^p$ denote the standard orthonormal basis for $\mathbb{C}^p$. For $1 \leq j \leq \mu - 1$, set $\mathcal{M}_j := \text{span}\{e_{(j-1)r+i}\}_{i=1}^r$, and set $\mathcal{M}_\lambda := \text{span}\{e_{p-r+i}\}_{i=1}^r$. Then each $\mathcal{M}_j$ is an r -dimensional subspace of $\mathbb{C}^p$.

Let $V \in \mathbb{M}_r$ be a unitary operator, all of whose entries relative to $\mathfrak{B}$ are non-zero. Set $U_1 = I_p$ and for $2 \leq j \leq \lambda$, define the unitary operator whose block-matrix relative to the decomposition $\mathbb{C}^p = \mathcal{M}_1 \oplus \mathcal{M}_j \oplus (\mathcal{M}_1 \oplus \mathcal{M}_j)^\perp$ is

$$U_j := \frac{1}{\sqrt{2}} \begin{bmatrix} V & V & 0 \\ -V & V & 0 \\ 0 & 0 & \sqrt{2} I_{p-2r} \end{bmatrix}.$$

Relative to the same decomposition of $\mathbb{C}^p$, we see that

$$C_j U_j = \frac{1}{\sqrt{2}} \begin{bmatrix} D_j V & D_j V & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By Lemma 6.8 above and the comments that follow it, for each $2 \leq j \leq \lambda$, there exists $\alpha_j \in \mathbb{C}$ such that

$$(X - \alpha_j I_p)^*|_{\mathcal{M}_1 \oplus \mathcal{M}_j} = 0.$$

But then $(X - \alpha_{j_1} I_p)^*|_{\mathcal{M}_1} = 0 = (X - \alpha_{j_2} I_p)^*|_{\mathcal{M}_1}$ implies that $\alpha_{j_1} = \alpha_{j_2}$ for all $2 \leq j_1, j_2 \leq \lambda$. Denote this common value of α_j , $2 \leq j \leq \lambda$ by α . We then obtain

$$(X - \alpha I_p)^*|_{\mathcal{M}} = 0,$$

where $\mathcal{M} = \text{span}\{\mathcal{M}_j\}_{j=1}^\lambda$. But $\mathcal{M} = \mathbb{C}^p$, and therefore $X - \alpha I_p = 0$, i.e. $X = \alpha I_p$. □

6.10. Lemma. Let $1 \leq r < w \leq p$ be integers and suppose that $w \leq 2r$. Let $0 < D_1, D_2 \in \mathbb{M}_r$ be diagonal matrices each with r distinct eigenvalues, and let $F_j \in \mathbb{M}_{p \times w} = \mathcal{B}(\mathbb{C}^w, \mathbb{C}^p)$, $j = 1, 2$ be the operator whose block-matrix relative to the decompositions $\mathbb{C}^w = \mathbb{C}^r \oplus \mathbb{C}^{w-r}$ and $\mathbb{C}^p = \mathbb{C}^r \oplus \mathbb{C}^{p-r}$ is

$$F_j := \begin{bmatrix} D_j & 0 \\ 0 & 0 \end{bmatrix}.$$

Then with $U_1 := I_w$, there exists a unitary operator $U_2 \in \mathbb{M}_w$ such that if $\beta \in \mathbb{C}$, $X \in \mathbb{M}_w$, and

$$\beta(F_j U_j) = (F_j U_j)X, \quad j = 1, 2,$$

then

$$X = \beta I_w.$$

Proof. Let us choose a unitary operator $U \in \mathbb{M}_w$ such that relative to the decomposition $\mathbb{C}^w = \mathbb{C}^r \oplus \mathbb{C}^{w-r}$,

$$U_2 = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix},$$

where V_1 is invertible and $\text{rank } V_2 = w - r$ (equivalently, $\ker V_2 = \{0\}$).

Suppose that $\beta \in \mathbb{C}$ and $X = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \in \mathbb{M}_w$ satisfy $\beta(F_j U_j) = (F_j U_j)X$, $j = 1, 2$.

Setting $j = 1$ yields

$$\begin{bmatrix} \beta D_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} D_1 X_1 & D_1 X_2 \\ 0 & 0 \end{bmatrix}.$$

Since D_1 is invertible, $X_2 = 0$. Furthermore, $\beta D_1 = D_1 X_1$ implies that $D_1(\beta I_r - X_1) = 0$, whence $X_1 = \beta I_r$ (again, because D_1 is invertible).

Now setting $j = 2$ results in the equation

$$\begin{bmatrix} \beta D_2 V_1 & \beta D_2 V_2 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \beta D_2 V_1 + D_2 V_2 X_3 & D_2 V_2 X_4 \\ 0 & 0 \end{bmatrix}.$$

Since $D_2 V_2$ has trivial kernel, the equations $(D_2 V_2)(\beta I_{w-r} - X_4) = 0$ and $(D_2 V_2)X_3 = 0$ imply that $X_3 = 0$ and $X_4 = \beta I_{w-r}$, from which we conclude that $X = \beta I_w$. $\square$

6.11. Lemma. *Let $1 \leq r < w \leq p$ and $2 \leq \mu$ be integers and suppose that $w \leq \mu r$. Set $\eta := \lceil \frac{w}{r} \rceil$. For each $1 \leq j \leq \eta$, let $0 < D_j \in \mathbb{M}_r$ be a diagonal matrix with r distinct eigenvalues, and define $F_j \in \mathbb{M}_{p \times w} = \mathcal{B}(\mathbb{C}^w, \mathbb{C}^p)$ to be the operator whose block-matrix relative to the decompositions $\mathbb{C}^w = \mathbb{C}^r \oplus \mathbb{C}^{w-r}$ and $\mathbb{C}^p = \mathbb{C}^r \oplus \mathbb{C}^{p-r}$ is*

$$F_j := \begin{bmatrix} D_j & 0 \\ 0 & 0 \end{bmatrix}.$$

Then there exist unitary operators $U_1, U_2, \dots, U_\eta \in \mathbb{M}_w$ such that if $\beta \in \mathbb{C}$, $X \in \mathbb{M}_w$, and

$$\beta(F_j U_j) = (F_j U_j)X \text{ for all } 1 \leq j \leq \eta,$$

then

$$X = \beta I_w.$$

Moreover, we may assume that $U_1 = I_w$.

Proof. Having dealt with the case where $\eta = 2$ in Lemma 6.10, we concentrate on the case where $\eta \geq 3$.

The argument is an adaptation of the proofs of Lemma 6.8 and Lemma 6.9.

Let $\mathfrak{B} := \{e_i\}_{i=1}^p$ denote the standard orthonormal basis for $\mathbb{C}^p$, and identify $\mathbb{C}^w$ with the subspace $\text{span}\{e_i\}_{i=1}^w$ of $\mathbb{C}^p$. For $1 \leq j \leq \eta - 1$, set $\mathcal{M}_j := \text{span}\{e_{(j-1)r+i}\}_{i=1}^r$ and $\mathcal{M}_\eta := \text{span}\{e_{w-r+i}\}_{i=1}^r$, so that each $\mathcal{M}_j$ is an r -dimensional subspace of $\mathbb{C}^w \subseteq \mathbb{C}^p$, and $\text{span}\{\mathcal{M}_j\}_{j=1}^\eta = \mathbb{C}^w$. Define also $\mathcal{N}_j := \mathbb{C}^w \ominus (\mathcal{M}_1 \oplus \mathcal{M}_j)$ and $\mathcal{N}_j^\circ = \mathbb{C}^p \ominus (\mathcal{M}_1 \oplus \mathcal{M}_j)$, $2 \leq j \leq \eta$.

Let $V \in \mathbb{M}_r = \mathcal{B}(\mathcal{M}_1)$ be a unitary operator, all of whose entries relative to the orthonormal basis $\{e_i\}_{i=1}^r$ for $\mathcal{M}_1$ are non-zero. Define $U_1 = I_w$, and for $2 \leq j \leq \eta$, define (relative to the decomposition $\mathbb{C}^w = \mathcal{M}_1 \oplus \mathcal{M}_j \oplus \mathcal{N}_j$),

$$U_j := \frac{1}{\sqrt{2}} \begin{bmatrix} V & V & 0 \\ -V & V & 0 \\ 0 & 0 & \sqrt{2} I_{p-2r} \end{bmatrix}.$$

Then $F_1 U_1 = F_1$, and for $2 \leq j \leq \eta$, relative to $\mathbb{C}^w = \mathcal{M}_1 \oplus \mathcal{M}_j \oplus \mathcal{N}_j$ and $\mathbb{C}^p = \mathcal{M}_1 \oplus \mathcal{M}_j \oplus \mathcal{N}_j^\circ$ we may write

$$F_j U_j = \frac{1}{\sqrt{2}} \begin{bmatrix} D_j V & D_j V & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{M}_{p \times w} = \mathcal{B}(\mathbb{C}^w, \mathbb{C}^p).$$

Let $X \in \mathbb{M}_w$, and relative to $\mathbb{C}^w = \mathcal{M}_1 \oplus \mathcal{M}_j \oplus \mathcal{N}_j$, write

$$X = \begin{bmatrix} X_{11} & X_{1j} & Z_{1j} \\ X_{j1} & X_{jj} & Z_{2j} \\ Z_{3j} & Z_{4j} & Z_{5j} \end{bmatrix}.$$

Suppose that $\beta \in \mathbb{C}$ and $\beta(F_j U_j) = (F_j U_j)X$ for all $1 \leq j \leq \eta$.

For each such $2 \leq j \leq \eta$, the equation $\beta F_1 = \beta(F_1 U_1) = (F_1 U_1)X = F_1 X$ reduces to (writing both operators with respect to $\mathbb{C}^w = \mathcal{M}_1 \oplus \mathcal{M}_j \oplus \mathcal{N}_j$ and $\mathbb{C}^p = \mathcal{M}_1 \oplus \mathcal{M}_j \oplus \mathcal{N}_j^\circ$)

$$\begin{bmatrix} \beta D_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} D_1 X_{11} & D_1 X_{1j} & D_1 Z_{1j} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since D_1 is invertible, we conclude from $D_1(\beta I_r - X_{11}) = 0$ that $X_{11} = \beta I_r$ and that $X_{1j} = Z_{1j} = 0$ for all $2 \leq j \leq \eta$. Stated otherwise, $X(\mathbb{C}^w \ominus \mathcal{M}_1) \subseteq (\mathbb{C}^w \ominus \mathcal{M}_1)$.

The equations $\beta(F_j U_j) = (F_j U_j)X$ now reduce to

$$\begin{bmatrix} \beta D_j V & \beta D_j V & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} D_j V(\beta I_r) + D_j V X_{j1} & D_j V X_{jj} & D_j V Z_{2j} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

From these (and the invertibility of each $D_j V$) we deduce that $X_{j1} = 0$ and $Z_{2j} = 0$ for all $2 \leq j \leq \eta$, and $X_{jj} = \beta I_r$.

Finally, combining these observations with the fact that $\text{span}\{\mathcal{M}_j\}_{j=1}^\eta = \mathbb{C}^w$ yields $X = \beta I_w$.

□

The proof of the next result is routine.

6.12. Lemma. *Let $w \leq r < p$ be positive integers, and suppose that $Y \in \mathbb{M}_{r \times w}$ is a left-invertible operator. Set*

$$Z := \begin{bmatrix} Y \\ 0 \end{bmatrix} \in \mathbb{M}_{p \times w}.$$

If $\gamma \in \mathbb{C}$ and $\gamma Z = ZX$ for some $X \in \mathbb{M}_w$, then $X = \gamma I_w$.

6.13. Lemma. *Let $1 \leq p_1 < 2p_1 \leq n$ be integers and $A \in \mathbb{M}_{p_1}$ be such that $0 \leq A \leq I_{p_1}$. Let $r := \text{rank } A$. If $B \in \mathbb{M}_{p_1 \times (n-p_1)}$ satisfies*

$$A^2 + BB^* = Q,$$

where Q is the orthogonal projection of $\mathbb{C}^{p_1}$ onto the range of A , then

$$P_0 := \begin{bmatrix} A^2 & AB \\ B^* A & B^* B \end{bmatrix}$$

is a projection of rank r .

Proof. Clearly P_0 is self-adjoint. Observe also that $QA = A$ and since $BB^* \leq Q$, we also have that $\text{ran } B \subseteq \text{ran } Q$, whence $QB = B$. But then

$$\begin{aligned} P_0^2 &= \begin{bmatrix} A \\ B^* \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} A \\ B^* \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} \\ &= \begin{bmatrix} A \\ B^* \end{bmatrix} Q \begin{bmatrix} A & B \end{bmatrix} \\ &= \begin{bmatrix} A \\ B^* \end{bmatrix} \begin{bmatrix} QA & QB \end{bmatrix} \\ &= \begin{bmatrix} A \\ B^* \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} \\ &= P_0, \end{aligned}$$

proving that P_0 is a projection.

Thus using the fact that $\text{rank } TT^* = \text{rank } T^*T$, we have

$$\text{rank } P_0 = \text{rank} \begin{bmatrix} A \\ B^* \end{bmatrix} \begin{bmatrix} A & B \end{bmatrix} = \text{rank} \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} A \\ B^* \end{bmatrix} = \text{rank } A = r.$$

□

Our second main result of this section is the following.

6.14. Theorem. *Let*

$$\mathcal{A} = \mathbb{M}_{p_1}^{(\mu_1)} \oplus \mathbb{M}_{p_2}^{(\mu_2)} \oplus \cdots \oplus \mathbb{M}_{p_k}^{(\mu_k)},$$

where

$$p_1 \geq p_2 \geq \cdots \geq p_k \geq 1,$$

and if $p_i = p_{i+1}$, then $\mu_i \geq \mu_{i+1}$. Set $\kappa_{\mathcal{A}} = \sum_{i=1}^k \mu_i$. Let

$$\mathcal{B} = \mathbb{M}_q \oplus \mathbb{M}_{n-q},$$

where $p_1 \leq q < n < p_1 + q$. Suppose that whenever $p_i + q > n$, we must have $q \leq n - \frac{p_i}{\mu_i}$. Then

$$\text{MINT}_U(\mathcal{A}, \mathcal{B}) = 1,$$

Proof. Set $r = n - q$. Clearly we only need to show the existence of a projection $P \in \mathcal{P}_{r,n}$ such that

$$\mathcal{A} \cap \{P\}' = \mathbb{C}I_n.$$

STEP 1. Choose $m \in \mathbb{N}$, such that

$$p_m + q > n, \quad p_{m+1} + q \leq n.$$

Since $p_{m+1} \leq n - q = r$, we know $p_i \leq r$, $m + 1 \leq i \leq k$.

Set $p = p_1$. For $1 \leq i \leq m$, set

$$\lambda_i = \lceil \frac{p_i}{r} \rceil.$$

For $1 \leq i \leq m$ we have that $p_i + q > n$, and so by hypothesis,

$$q \leq n - \frac{p_i}{\mu_i}.$$

Hence

$$\mu_i \geq \lceil \frac{p_i}{r} \rceil = \lambda_i \geq 2, \quad 1 \leq i \leq m.$$

STEP 2. By Lemma 6.13, we can determine a projection $P_0 \in \mathcal{P}_{r,n}$ by specifying an $0 \leq A \leq I_{p_1}$ of rank r and an operator $B \in \mathbb{M}_{p_1 \otimes (n-p_1)}$ satisfying

$$A^2 + BB^* = Q,$$

where Q is the orthogonal projection of $\mathbb{C}^{p_1}$ onto the range of A . We shall first define A , and then define B by determining its decomposition relative to the decomposition $\mathbb{C}^n = (\mathbb{C}^{p_1})^{(\mu_1-1)} \oplus (\mathbb{C}^{p_2})^{(\mu_2)} \oplus \dots \oplus (\mathbb{C}^{p_k})^{(\mu_k)}$.

To that end, let $0 < D_{11} < I_r$ be a diagonal operator with r distinct eigenvalues. It is not hard to see (and we leave it for the reader to verify) that we can then choose positive diagonal operators

- $0 < D_{ij} < I_r$, $1 \leq i \leq m$, $1 \leq j \leq \lambda_i$, $(i, j) \neq (1, 1)$, such that $D_{11}D_{ij}$ has r distinct eigenvalues for all such i and j , and
- $0 < B_s^\circ < I_{p_s}$, $m+1 \leq s \leq k$, each with p_s distinct eigenvalues

such that

$$\sum_{i=1}^m \sum_{j=1}^{\lambda_i} D_{ij}^2 + \sum_{s=m+1}^k (B_s^\circ \oplus 0_{r-p_s})^2 = I_r.$$

Define $A := D_{11} \oplus 0_{p_1-r} \in \mathbb{M}_{p_1}$. Next, we define

- $B_1 := \begin{bmatrix} B_{12} & B_{13} & \cdots & B_{1\lambda_1} & 0 \end{bmatrix} \in \mathbb{M}_{p_1 \times (\mu_1-1)p_1}$, where for each $2 \leq j \leq \lambda_1$, we define $B_{1j} = \begin{bmatrix} D_{1j} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}_{p_1}$;
- $B_i := \begin{bmatrix} B_{i1} & B_{i2} & \cdots & B_{i\lambda_i} & 0 \end{bmatrix} \in \mathbb{M}_{p_1 \times \mu_i p_i}$, $2 \leq i \leq m$, where for each $1 \leq j \leq \lambda_i$, we define $B_{ij} = \begin{bmatrix} D_{ij} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{M}_{p_1 \times p_i}$; and
- $B_s := \begin{bmatrix} B_{s1} & 0 \end{bmatrix} \in \mathbb{M}_{p_1 \times \mu_s p_s}$, $m+1 \leq s \leq k$, where $B_{s1} := \begin{bmatrix} B_s^\circ \\ 0 \end{bmatrix} \in \mathbb{M}_{p_1 \times p_s}$. (In essence, we are first adding $p_1 - p_s$ zero rows to B_s° to obtain B_{s1} , and then adding $(\mu_s - 1)p_s$ zero columns to B_{s1} to obtain a $p_1 \times \mu_s p_s$ operator whose “top left hand corner” is B_s° , and whose remaining entries are all zero.)

Finally, we define

$$B = \begin{bmatrix} B_1 & B_2 & \cdots & B_m & B_{m+1} & \cdots & B_k \end{bmatrix}.$$

A routine calculation shows that

$$A^2 + BB^* = \sum_{i=1}^m \sum_{j=1}^{\lambda_i} (D_{ij} \oplus 0_{p_1-r})^2 + \sum_{s=m+1}^k (B_s^\circ \oplus 0_{p_1-p_s})^2 = I_r \oplus 0_{p_1-r},$$

the projection onto the range of A . As we have already noted, by Lemma 6.13,

$$P_0 := \begin{bmatrix} A^2 & AB \\ B^*A & B^*B \end{bmatrix} \in \mathbb{M}_n$$

is a projection of rank r .

STEP 3.

Let $B_1^\circ := \begin{bmatrix} A & B_1 \end{bmatrix} \in \mathbb{M}_{p_1 \times \mu_1 p_1}$. By Lemma 6.9, there exist unitary operators $U_{11} = I_{p_1}, U_{12}, \dots, U_{1\lambda_1} \in \mathbb{M}_{p_1}$ such that

$$\{A^2 = A^2 U_{11}, AB_{12} U_{12}, AB_{13} U_{13}, \dots, AB_{1\lambda_1} U_{1\lambda_1}\}' = \mathbb{C} I_{p_1}.$$

For each $2 \leq i \leq m$, by Lemma 6.11 there exist unitaries $U_{i1}, U_{i2}, \dots, U_{i\lambda_i} \in \mathbb{M}_{p_i}$ such that if $\beta \in \mathbb{C}$, $X \in \mathbb{M}_{p_i}$, and

$$\beta(AB_{ij}U_{ij}) = (AB_{ij}U_{ij})X \quad \text{for all } 1 \leq j \leq \lambda_i,$$

then

$$X = \beta I_{p_i}.$$

We define a unitary $U \in \mathbb{M}_n$ as follows:

- for $1 \leq i \leq m$, $\lambda_i + 1 \leq j \leq \mu_i$, define $U_{ij} = I_{p_i}$;
- for $m + 1 \leq i \leq k$, $1 \leq j \leq \mu_i$, define $U_{ij} = I_{p_i}$.

Define

$$U = (\oplus_{j=1}^{\mu_1} U_{1j}) \oplus \dots \oplus (\oplus_{j=1}^{\mu_m} U_{mj}) \oplus (\oplus_{j=1}^{\mu_{m+1}} U_{m+1,j}) \oplus \dots \oplus (\oplus_{j=1}^{\mu_k} U_{kj}).$$

Then U is a unitary operator in $\mathbb{M}_n$ with $U_{11} = I_{p_1}$. Set $P := U^* P_0 U$, so that $P \in \mathcal{P}_{r,n}$.

STEP 4. With respect to the decomposition

$$\mathbb{C}^n = (\mathbb{C}^{p_1})^{(\mu_1)} \oplus (\mathbb{C}^{p_2})^{(\mu_2)} \oplus \dots \oplus (\mathbb{C}^{p_k})^{(\mu_k)},$$

we can decompose any $T \in \mathbb{M}_n$ as a $\kappa_{\mathcal{A}} \times \kappa_{\mathcal{A}}$ block-matrix. (We remind the reader that $\kappa_{\mathcal{A}} = \sum_{i=1}^k \mu_i$.)

Suppose that $X \in \mathcal{A} \cap \{P\}'$. Being in $\mathcal{A}$, X is of the form

$$X = X_1^{(\mu_1)} \oplus X_2^{(\mu_2)} \oplus \dots \oplus X_k^{(\mu_k)},$$

where $X_i \in \mathbb{M}_{p_i}$, $1 \leq i \leq k$.

Since $X \in \{P\}'$, after comparing the entries indexed from $(1, 1)$ to $(1, \lambda_1)$ in the $\kappa_{\mathcal{A}} \times \kappa_{\mathcal{A}}$ block-matrix forms of XP and PX , we find that

$$\{A^2 = A^2 U_{11}, AB_{12} U_{12}, AB_{13} U_{13}, \dots, AB_{1\lambda_1} U_{1\lambda_1}\}' = \mathbb{C} I_{p_1}.$$

Choose $\beta \in \mathbb{C}$ such that $X_1 = \beta I_{p_1}$.

Fix $2 \leq i \leq m$. By comparing the entries indexed from $(1, (\sum_{l=1}^{i-1} \mu_l) + 1)$ to $(1, (\sum_{l=1}^{i-1} \mu_l) + \lambda_i)$ in the $\kappa_{\mathcal{A}} \times \kappa_{\mathcal{A}}$ block-matrix forms of XP and PX , we obtain:

$$X_1(AB_{ij}U_{ij}) = \beta(AB_{ij}U_{ij}) = (AB_{ij}U_{ij})X_i, \quad 1 \leq j \leq \lambda_i.$$

From the argument in STEP 3, we conclude that $X_i = \beta I_{p_i}$.

Finally, suppose that $m + 1 \leq s \leq k$. By comparing the $(1, (\sum_{l=1}^{s-1} \mu_l) + 1)$ entry of XP and PX , we see that

$$\beta(AB_{s1}) = X_1(AB_{s1}) = (AB_{s1})X_s.$$

By Lemma 6.12, $X_s = \beta I_{p_s}$.

It follows that $X = \beta I_n$, completing the proof. $\square$

6.15. Theorem. *Let*

$$\mathcal{A} = \mathbb{M}_{p_1}^{(\mu_1)} \oplus \mathbb{M}_{p_2}^{(\mu_2)} \oplus \dots \oplus \mathbb{M}_{p_k}^{(\mu_k)},$$

where

$$p_1 \geq p_2 \geq \dots \geq p_k \geq 1,$$

and if $p_i = p_{i+1}$, then $\mu_i \geq \mu_{i+1}$. *Let*

$$\mathcal{B} = \mathbb{M}_{q_1} \oplus \mathbb{M}_{q_2}^{(\nu_2)} \oplus \dots \oplus \mathbb{M}_{q_k}^{(\nu_k)},$$

where $p_1 \leq q_1 < n < p_1 + q_1$. Suppose that whenever $p_i + q_1 > n$, we must have $q_1 \leq n - \frac{p_i}{\mu_i}$. Then

$$\text{MINT}_U(\mathcal{A}, \mathcal{B}) = 1,$$

Proof. Define $\mathcal{B}_{\max} = \mathbb{M}_{q_1} \oplus \mathbb{M}_{n-q_1}$. Noting that $\mathcal{B}$ is a unital subalgebra of $\mathcal{B}_{\max}$, by Theorem 6.14 above,

$$1 \leq \text{MINT}_U(\mathcal{A}, \mathcal{B}) \leq \text{MINT}_U(\mathcal{A}, \mathcal{B}_{\max}) = 1.$$

□

7. AN EXAMPLE AND SOME REMAINING QUESTIONS

7.1. We have seen that it is possible to determine when $\text{MODD}_n(\mathcal{A} \oplus \mathcal{B}) = 1$, where $\mathcal{A}, \mathcal{B} \subseteq \mathbb{M}_n$ are semi-simple algebras. A precise determination of $\text{MODD}_n(\mathcal{A} \oplus \mathcal{B})$ appears to be a delicate problem, despite our having obtained answers in a great many cases. Determining $\text{MODD}_n(\mathcal{L})$ when $\mathcal{L} \subseteq \mathbb{M}_{2n}$ is a general (unital, even semi-simple) algebra seems quite difficult at the moment. We provide but two examples, in the hope that it may lead others to attack this problem.

7.2. Proposition. Suppose that $m, n \in \mathbb{N}$ with $n \leq m < 2n$. If $\mathcal{A} = \mathbb{M}_m \oplus \mathbb{C}I_{2n-m}$, then

$$\text{MODD}_n(\mathcal{A}) = n^2.$$

Proof. Arguing as we did in Proposition 2.29, if $P = \frac{1}{2} \begin{bmatrix} I_n & I_n \\ I_n & I_n \end{bmatrix} \in \mathbb{M}_{2n}$, then

$$\dim P^\perp \mathcal{A} P = \dim \begin{bmatrix} I_n & -I_n \end{bmatrix} \mathcal{A} \begin{bmatrix} I_n \\ I_n \end{bmatrix}.$$

In our case, relative to $\mathbb{C}^{2n} = \mathbb{C}^n \oplus \mathbb{C}^n$, we may write an arbitrary element of $\mathcal{A}$ as

$$A = \begin{bmatrix} W & X_0 \\ Y_0 & Z_0 \end{bmatrix},$$

where $W \in \mathbb{M}_n$, $X \in \mathbb{M}_{(m-n) \times n}$, $Y \in \mathbb{M}_{n \times (m-n)}$ and $Z \in \mathbb{M}_{m-n}$ are arbitrary and where where $X_0 := \begin{bmatrix} X & 0 \end{bmatrix}$, $Y_0 := \begin{bmatrix} Y \\ 0 \end{bmatrix}$ and $Z_0 := \begin{bmatrix} Z & 0 \\ 0 & \alpha I_{2n-m} \end{bmatrix}$.

Then

$$\begin{aligned} \text{MODD}_n(\mathcal{A}) &\geq \dim \left\{ \begin{bmatrix} I_n & -I_n \end{bmatrix} \begin{bmatrix} W & X_0 \\ Y_0 & Z_0 \end{bmatrix} \begin{bmatrix} I_n \\ I_n \end{bmatrix} : W, X_0, Y_0, Z_0 \text{ as above} \right\} \\ &\geq \dim \{W : W \in \mathbb{M}_n\} \\ &= n^2. \end{aligned}$$

But clearly $\text{MODD}_n(\mathcal{A}) \leq n^2$, because that is the dimension of $P^\perp \mathbb{M}_{2n} P$, so that

$$\text{MODD}_n(\mathcal{A}) = n^2.$$

□

7.3. Corollary. *Suppose that $m, n \in \mathbb{N}$ with $n \leq m < 2n$. If $\mathcal{A} \subseteq \mathbb{M}_{2n}$ is a C^* -algebra and $\mathbb{M}_m \oplus \mathbb{C}I_{2n-m} \subseteq \mathcal{A}$, then*

$$\text{MODD}_n(\mathcal{A}) = n^2.$$

7.4. Proposition. *Let $a \geq b \geq c > 0$ be integers, and suppose that $a + b + c = 2n$ for some positive integer n . If $\mathcal{A} := \mathbb{M}_a(\mathbb{C}) \oplus \mathbb{M}_b(\mathbb{C}) \oplus \mathbb{M}_c(\mathbb{C})$, then*

$$\text{MODD}_n(\mathcal{A}) = n^2.$$

Proof. First observe that if $a \geq n$, then by Corollary 7.3, $\text{MODD}_n(\mathcal{A}) = n^2$.

For the rest of the proof, we assume that $1 < a < n$.

Let us define $w := n - a$ and $z := n - b$. The fact that $a + w = n = b + z$ means that $(a - z) + w = b$, which we shall use below. Then we may rewrite

$$\mathcal{A} \simeq \begin{bmatrix} \mathbb{M}_a & 0 & 0 & 0 \\ 0 & \mathbb{M}_{w \times w} & \mathbb{M}_{w \times z} & 0 \\ 0 & \mathbb{M}_{z \times w} & \mathbb{M}_{z \times z} & 0 \\ 0 & 0 & 0 & \mathbb{M}_b \end{bmatrix}.$$

Consider $P = \frac{1}{2} \begin{bmatrix} I_n & I_n \\ I_n & I_n \end{bmatrix}$. As we have seen several times before,

$$\dim(I - P)\mathcal{A}P = \dim \begin{bmatrix} I_n & -I_n \end{bmatrix} \mathcal{A} \begin{bmatrix} I_n \\ I_n \end{bmatrix}.$$

Let us write $\mathbb{M}_a = \begin{bmatrix} \mathbb{M}_{z \times z} & \mathbb{M}_{z \times (a-z)} \\ \mathbb{M}_{(a-z) \times z} & \mathbb{M}_{(a-z) \times (a-z)} \end{bmatrix}$.

We write an element of $\mathbb{M}_a$ relative to this decomposition as $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$. Elements of $\begin{bmatrix} \mathbb{M}_{w \times w} & \mathbb{M}_{w \times z} \\ \mathbb{M}_{z \times w} & \mathbb{M}_{z \times z} \end{bmatrix}$ we write as $C := \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$. Recall that $(a - z) + w = b$, and so given an element of $\mathbb{M}_b$, we may write it as $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$ relative to $\mathbb{C}^b = \mathbb{C}^{(a-z)} \oplus \mathbb{C}^w$. So for elements of $\mathcal{A}$ of the form

$$T := A \oplus C \oplus B = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & W & X & 0 & 0 \\ 0 & 0 & Y & Z & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{11} & B_{12} \\ 0 & 0 & 0 & 0 & B_{21} & B_{22} \end{bmatrix},$$

we find that

$$\begin{aligned} \begin{bmatrix} I_n & -I_n \end{bmatrix} T \begin{bmatrix} I_n \\ I_n \end{bmatrix} &= \begin{bmatrix} I_n & -I_n \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ X & 0 & W \\ Z & 0 & Y \\ 0 & B_{11} & B_{12} \\ 0 & B_{21} & B_{22} \end{bmatrix} \\ &= \begin{bmatrix} A_{11} - Z & A_{12} & -Y \\ A_{21} & A_{22} - B_{11} & -B_{12} \\ X & -B_{21} & W - B_{22} \end{bmatrix}. \end{aligned}$$

Since every term $A_{ij}, B_{i,j}, W, X, Y$ and Z is arbitrary, we see that

$$(I - P)\mathcal{A}P = \mathbb{M}_n,$$

whence $\text{MODD}_n(\mathcal{A}) = n^2$.

□

7.5. We definitely feel that there is more information to be gleaned about subalgebras (or subspaces) $\mathcal{A}$ of $\mathbb{M}_n$ (alternatively subalgebras or subspaces $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ when $\mathcal{H}$ is infinite-dimensional) from their off-diagonal corners $P^\perp \mathcal{A} P$. Below we mention just a few of the questions which remain open.

Question 1.

Let $\mathcal{X}$ and $\mathcal{Y}$ be subspaces of $\mathbb{M}_n$. Find $\text{MODD}_n(\mathcal{X} \oplus \mathcal{Y})$, $\text{MINT}_S(\mathcal{X}, \mathcal{Y})$ and $\text{MINT}_U(\mathcal{X}, \mathcal{Y})$.

The next problem is especially dear to us, given the considerable effort we expended to try to solve it. In all cases where we have been able to compute both quantities, they are equal.

Question 2.

Find a direct proof that $\text{MINT}_S(\mathcal{A}, \mathcal{B}) = \text{MINT}_U(\mathcal{A}, \mathcal{B})$ for all unital C^* -subalgebras of $\mathbb{M}_n$, or find a counterexample to show that this does not hold. Note that the only unsolved case is Case 4A.

Question 3.

Let $m, n \in \mathbb{N}$ and $\mathcal{A} \subseteq \mathbb{M}_m$, $\mathcal{B} \subseteq \mathbb{M}_n$ be unital, semisimple subalgebras. Find

$$\text{MODD}_{m,n}(\mathcal{A}, \mathcal{B}) := \max\{\dim P^\perp(\mathcal{A} \oplus \mathcal{B})P : P = P^2 = P^* \in \mathbb{M}_{m+n}, \text{rank } P = m\}.$$

Addendum. M. Mastnak has answered Question 2 by finding a direct proof that works not only for unital C^* -algebras, but in fact for all subspaces of $\mathbb{M}_n$. His solution is presented in the Appendix to the current paper.

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DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO, CANADA N2L 3G1

Email address: `Laurent.Marcoux@uwaterloo.ca`

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO, CANADA N2L 3G1

Email address: `hradjavi@uwaterloo.ca`

SCHOOL OF MATHEMATICS, JILIN UNIVERSITY, CHANGCHUN 130012, P.R. CHINA

Email address: `zhangyuanhang@jlu.edu.cn`

APPENDIX TO “DISPERSING REPRESENTATIONS OF SEMI-SIMPLE SUBALGEBRAS OF COMPLEX MATRICES”

M. MASTNAK¹

Question 2 of Section 7 of the above paper asks whether there exists a direct proof of the conjecture that $\text{MINT}_S(\mathcal{A}, \mathcal{B}) = \text{MINT}_U(\mathcal{A}, \mathcal{B})$ for all unital C^* -subalgebras of $\mathbb{M}_n$. Below we establish a more general result, namely: $\text{MINT}_S(\mathcal{X}, \mathcal{Y}) = \text{MINT}_U(\mathcal{X}, \mathcal{Y})$ whenever $\mathcal{X}$ and $\mathcal{Y}$ are arbitrary subspaces of $\mathbb{M}_n$.

As a consequence, we show that Problems 4U and 4UC from Section 1 of the above paper are equivalent, as conjectured there.

The following observation about multivariable polynomials is surely known. We sketch an idea of the proof for the sake of completeness.

Lemma A.1. *Let $p \in \mathbb{C}[t_1, \dots, t_n]$ be a nonzero polynomial in n variables. Then there exist complex numbers $z_1, z_2, \dots, z_n$ of modulus one such that $p(z_1, \dots, z_n) \neq 0$.*

Proof. Let $d_1, \dots, d_n$ be such that $p(d_1, \dots, d_n) \neq 0$. Define $p_1(t) = p(t, d_2, \dots, d_n)$. Since p_1 is a nonzero polynomial (as $p_1(d_1) \neq 0$) it has at most finitely many roots, so there is an $z_1 \in \mathbb{C}$ of modulus one such that $p_1(z_1) \neq 0$. Now define $p_2(t) = p(z_1, t, d_3, \dots, d_n)$ and find $z_2 \in \mathbb{C}$ of modulus one such that $p(z_1, z_2, d_3, \dots, d_n) \neq 0$. The desired result now follows from an obvious (finite) induction process. □

Lemma A.2. *Let $D = \text{diag}(d_1, d_2, \dots, d_n) \in \mathbb{M}_n$ and let $A_1, \dots, A_k, B_1, \dots, B_k \in \mathbb{M}_n$ be such that $A_1DB_1, A_2DB_2, \dots, A_kDB_k$ are linearly independent. Then there exist complex numbers of modulus one $z_1, z_2, \dots, z_n$ such that for $F = \text{diag}(z_1, z_2, \dots, z_n)$, the matrices $A_1FB_1, A_2FB_2, \dots, A_kFB_k$ are also linearly independent.*

Proof. Unwind matrices into column vectors in $\mathbb{C}^{n^2}$ and consider

$$M := \begin{bmatrix} A_1DB_1 & A_2DB_2 & \cdots & A_kDB_k \end{bmatrix}$$

as an $n^2 \times k$ matrix. Linear independence means that some $k \times k$ sub-determinant of M is nonzero. Now define the polynomial $p(t_1, \dots, t_n)$ in n indeterminates to be the sub-determinant corresponding to the matrix obtained from M by replacing the matrix D by $T = \text{diag}(t_1, \dots, t_n)$. Since p is a nonzero polynomial ($p(d_1, \dots, d_n) \neq 0$), it follows from Lemma A.1 that there are values $z_1, z_2, \dots, z_n$ of modulus one such that $p(z_1, z_2, \dots, z_n) \neq 0$. Then for $F = \text{diag}(z_1, z_2, \dots, z_n)$ we have that $A_1FB_1, \dots, A_kFB_k$ are linearly independent. □

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Proposition A.3. *Let $\mathcal{X}, \mathcal{Y}$ be subspaces of $\mathbb{M}_n$. Then for every invertible $A \in \mathbb{M}_n$ there exists a unitary $U \in \mathbb{M}_n$ such that $\dim(\mathcal{X} + A^{-1}\mathcal{Y}A) \leq \dim(\mathcal{X} + U^*\mathcal{Y}U)$.*

In particular, $\text{MAXSP}_U(\mathcal{X}, \mathcal{Y}) \geq \text{MAXSP}_S(\mathcal{X}, \mathcal{Y})$ for all subspaces $\mathcal{X}$ and $\mathcal{Y}$ of $\mathbb{M}_n$, and therefore

$$\text{MINT}_S(\mathcal{X}, \mathcal{Y}) = \text{MINT}_U(\mathcal{X}, \mathcal{Y}).$$

Proof. Fix an invertible $A \in \mathbb{M}_n$ and choose $X_1, X_2, \dots, X_r \in \mathcal{X}$, $Y_1, Y_2, \dots, Y_s \in \mathcal{Y}$ with $r + s = \dim(\mathcal{X} + A^{-1}\mathcal{Y}A)$ such that $X_1, X_2, \dots, X_r, A^{-1}Y_1A, A^{-1}Y_2A, \dots, A^{-1}Y_sA$ are linearly independent. Note that this means also that the matrices $AX_1, \dots, AX_r, Y_1A, \dots, Y_sA$ are linearly independent. Let $A = PU$ be the polar decomposition of A where P is positive and U is unitary. With no loss of generality we assume that P is diagonal. For $i = 1, \dots, r$ let $A_i = I$ and $B_i = UX_i$. For $j = 1, \dots, s$ let $A_{r+j} = Y_j$ and let $B_{r+j} = U$. Then by Lemma A.2 above there exists a unitary diagonal matrix F such that $A_1FB_1, \dots, A_{r+s}FB_{r+s}$ are linearly independent. But this then implies that for the unitary $V = FU$, we have that $VX_1, \dots, VX_r, Y_1V, \dots, Y_sV$ are linearly independent, and hence

$$\dim(\mathcal{X} + V^*\mathcal{Y}V) = \dim(V\mathcal{X} + \mathcal{Y}V) \geq r + s = \dim(\mathcal{X} + A^{-1}\mathcal{Y}A).$$

From this the last statement easily follows. □

A.4. It immediately follows from Proposition A.3 that Problems 4S and 4U of Section 1 of the above paper are equivalent, as are Problems 4SC and 4UC. In fact, Problems 4U and 4UC are also equivalent.

Indeed, let $\mathcal{X}$ and $\mathcal{Y}$ be subspaces of $\mathbb{M}_n$ and $R_{\mathcal{X}}$ and $R_{\mathcal{Y}} \in \mathbb{M}_n$ be invertible. Set $\mathcal{X}_0 := R_{\mathcal{X}}^{-1}\mathcal{X}R_{\mathcal{X}}$ and $\mathcal{Y}_0 := R_{\mathcal{Y}}^{-1}\mathcal{Y}R_{\mathcal{Y}}$.

Now observe that

$$\begin{aligned} \text{MINT}_U(\mathcal{X}, \mathcal{Y}) &= \min\{\dim(\mathcal{X} \cap U^*\mathcal{Y}U) : U \in \mathbb{M}_n \text{ unitary}\} \\ &= \min\{\dim(\mathcal{X}_0 \cap (R_{\mathcal{X}}U^*R_{\mathcal{Y}})\mathcal{Y}_0(R_{\mathcal{Y}}^{-1}UR_{\mathcal{X}}^{-1})) : U \in \mathbb{M}_n \text{ unitary}\} \\ &\geq \min\{\dim(\mathcal{X}_0 \cap T^{-1}\mathcal{Y}_0T) : T \in \text{GL}(n, \mathbb{C})\} \\ &= \text{MINT}_S(\mathcal{X}_0, \mathcal{Y}_0) \\ &= \text{MINT}_U(\mathcal{X}_0, \mathcal{Y}_0). \end{aligned}$$

By symmetry, $\text{MINT}_U(\mathcal{X}_0, \mathcal{Y}_0) \geq \text{MINT}_U(\mathcal{X}, \mathcal{Y})$, and therefore $\text{MINT}_U(\mathcal{X}_0, \mathcal{Y}_0) = \text{MINT}_U(\mathcal{X}, \mathcal{Y})$.

If $\mathcal{A}$ and $\mathcal{B}$ are unital, semi-simple subalgebras of $\mathbb{M}_n$, there exist invertible elements $R_{\mathcal{A}}$ and $R_{\mathcal{B}} \in \mathbb{M}_n$ such that $\mathcal{A}_0 := R_{\mathcal{A}}^{-1}\mathcal{A}R_{\mathcal{A}}$ and $\mathcal{B}_0 := R_{\mathcal{B}}^{-1}\mathcal{B}R_{\mathcal{B}}$ are C^* -algebras. The above implies that

$$\text{MINT}_U(\mathcal{A}, \mathcal{B}) = \text{MINT}_U(\mathcal{A}_0, \mathcal{B}_0).$$

[MITJA MASTNAK] DEPARTMENT OF MATHEMATICS AND COMPUTING SCIENCE, SAINT MARY’S UNIVERSITY, 923 ROBIE ST, HALIFAX, NOVA SCOTIA, CANADA B3N 1Z9

Email address: mmastnak@cs.smu.ca