

Generic Absoluteness in Set Theory

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

In this thesis, we describe the state of the field of generic absoluteness, that is, the study of which statements retain their truth value under any forced generic extension or series of generic extensions of V , along with what large cardinal assumptions are necessary or sufficient to cause formulas to become generically absolute. We detail some of the tools used in generic absoluteness, such as homogeneously Suslin trees, direct limits of models, extenders, and the incredibly powerful Stationary Tower used for stationary tower forcing, along with the effect specific large cardinal properties have when combined with the stationary tower.

We describe various landmark generic absoluteness results including the following: Shoenfield's Σ_2^1 -Absoluteness theorem, which establishes that all Σ_2^1 statements retain their truth across all transitive models of ZF containing ω_1 . We also examine more advanced generic absoluteness results in the projective hierarchy, such as the equiconsistency projective absoluteness with the existence of ω -many strong cardinals, and that projective absoluteness follows from ω -many Woodin cardinals. Further up the hierarchy, we describe Σ_1^2 -absoluteness conditioned on CH , which states that assuming a proper class of measurable Woodin cardinals and CH holding in V , if ϕ is a Σ_1^2 statement and G is a generic filter for $\mathbb{P}$ some partial order such that $V[G] \models CH$, then $V \models \phi$ if and only if $V[G] \models \phi$. We also cover Σ_2^2 -absoluteness with generic $\diamond$, a generic form of Jensen's $\diamond$ principle, and its relation to Neeman Games in Ω -logic.

Furthermore, we cover the generic absoluteness properties of universally Baire sets, uB , such as a proper class of Woodin cardinals implying for $A \in uB$ and any generic extension $V[G]$ an elementary embedding $j : L(A, \mathbb{R}) \rightarrow L(A_G, \mathbb{R}_G)$ such that $j(A) = A$ and $j \restriction_{\mathbb{R}} = id$, along with the more recent axiom of Sealing, which seeks to 'seal' the truth of the universally Baire sets, along with examinations of the uB -powerset of uB , uB^p , such as Weak Sealing for uB^p , a strengthening of Sealing. We examine particular inner models of interest, such as Chang-like models and their unique generic absoluteness properties. Finally, we give proofs of the Shoenfield Σ_2^1 absoluteness theorem, and the fact that Σ_3^1 absoluteness follows from a class of measurable cardinals.

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Chapter 1

Introduction

1.1 History

Ever since Cantor invented the field of set theory in December 1873 by proving the set of real numbers was uncountable, the question of the size of $\mathbb{R}$ has been central to the study of the field. The pursuit of an answer to this question led to the discovery of cardinal and ordinal arithmetic, the push to unite all mathematics under a single theory and language, and the development of a system of axioms from which all previous mathematics could stem. This system, now referred to as ‘naïve set theory’, laid the groundwork for the modern field of set theory upon which nearly all mathematics is built.

After Bertrand Russel shook the world of set theory through his discovery of Russel’s Paradox in 1901, which used the language of naïve set theory to construct a contradictory statement “the set of all sets which do not contain themselves contains itself” set theorists began a push to develop a formal system of axioms powerful enough to prove and describe all of mathematics already formulated while not being so powerful contradictions that arose. Through the dedicated efforts of many minds over many years, the set theoretic community eventually settled on **Zermelo-Fraenel set theory with the Axiom of Choice (ZFC)**, finalized in 1922 with the addition of John von Neumann’s **Axiom of Regularity**. While the inclusion of Zermelo’s Axiom of Choice was controversial at the time and still is an occasional topic of discussion among set theorists to this very day,

partially due to increased interest in **Axiom of Determinacy (AD)**, *ZFC* remains the foundation upon which modern mathematics stands.

However, even though the formalization of mathematics in the language of set theory and the development of *ZFC* marked a historic change in both logic and mathematics in general, it brought logicians no closer to resolving the veracity of Cantor’s original question about the size of the reals, now named the **Continuum Hypothesis (CH)** and formulated as the statement “ $2^{\aleph_0} = \aleph_1$.” Set theorists’ hopes were further shaken in 1931 with the revelation of **Gödel’s incompleteness theorems**, which state firstly that any consistent formal theory of mathematics able to represent arithmetic is incomplete. That is, there exist statements φ *independent* of said theory. Worse yet, Gödel’s work also showed no such theory can prove its own consistency, or else it would be inconsistent.

Definition 1.1.1.

*A formula φ is **independent** of a theory T , if both the theories $T + \varphi$ and $T + \neg\varphi$ are consistent.*

If Russel’s paradox shook the world of set theory, Gödel’s incompleteness theorems rocked it to its core. Suddenly, any major unsolved problem could be independent of *ZFC*. Could the Continuum Hypothesis be one of them? In 1940, Gödel proved that *CH* was consistent with *ZFC* using his constructible universe L . Set theorists’ fears were confirmed in 1963 when Paul Cohen constructed a model of *ZFC* where $2^{\aleph_0} = \aleph_2$, finally proving *CH* is independent of *ZFC*. Ninety years since Cantor first announced his dream of a complete unified theory of the reals, Cohen’s result marked the final nail in the coffin, showing the question at the heart of the whole project was not even incorrect, but entirely unsolvable. Cantor’s dream was dead.

However, as the saying goes, when one door closes, another one opens. While the *original* goal of set theory was now unattainable, the theory had grown far beyond the scope of the continuum hypothesis. Ever since Gödel announced his incompleteness theorems, the focus of set theorists shifted away from trying to determine the truth or falsity of various conjectures, and towards addressing the problem of independence. The new ini-

tiative, named *Gödel's program*, seeks to remove independence from *ZFC* through natural extensions of the theory, and to make as much of the theory as possible *absolute*. Gödel believed the solution lay within the *Large Cardinal Hierarchy* but a result by Lévy and Solovay theorem later proved this to be erroneous [17].

As a result, while Cohen's proof of the independence of *CH* was a monumental result, its significance was not due to the result itself, but instead the method Cohen used to prove it. Cohen had developed a new method of constructing standard models of *ZF* and *ZFC* which he named *forcing* which entails extending the universe of sets V to a greater universe of containing a new 'generic object' G . The extended universe, $V[G]$, often referred to as a *generic extension*, can hence be 'forced' to satisfy a specific property. In Cohen's case, the property was that $2^{\aleph_0} = \aleph_2$. Instead of ending set theory, Cohen's work revolutionized it, quickly taking center stage as a primary method for studying independence alongside the large cardinal hierarchy and the *determinacy axioms*, and helped bring the field into the state in which it finds itself today.

The purpose of this thesis is to describe the field of *generic absoluteness* of statements about the subsets of $\mathbb{R}$. A formula φ is **generically absolute** if it does not change its truth value under any generic extension. First, however, we need to describe the three main components of set theory: the large cardinal hierarchy, the determinacy axioms, and forcing.

Chapter 2

Background

2.1 Notation

Using the standard notation for enumerating the ordinals and cardinals, we use the term ‘the reals’ to refer to elements of ω^ω , that is, the set of all functions $f : \omega \rightarrow \omega$, as ω^ω with the product topology is isomorphic to the space of irrational numbers $\mathbb{R} \setminus \mathbb{Q}$. Unless otherwise specified, we will use $\mathbb{R}$ to denote this space by set theory convention.

Furthermore, we use $\mathcal{P}(X)$ to denote the power set of X . For any cardinal κ , $[X]^\kappa$ denotes the collection of unordered subsets of X of cardinality κ , and $\mathcal{P}_\kappa(X) = \bigcup_{\lambda < \kappa} [X]^\lambda$ the set of all subsets of X of cardinality $< \kappa$.

Finally, if $j : X \rightarrow Y$ is a function, and $B \subseteq X$, we use the following notation: $j''B = \{j(x) : x \in B\}$.

2.2 Models of ZF

2.2.1 Models of Set Theory

The primary technique of proving independence is through the construction of *models*. That is, a theory T is consistent if and only if there exists a model for said theory. A **model** of set theory is a pair $\mathcal{M} = \langle M, \mathcal{L} \rangle$, where M is the *universe* of sets of the model, that is, the collection of all objects which qualify as sets in $\mathcal{M}$. $\mathcal{L}$ is a language which

contains the symbol “ $\in$ ”. When interpreted in $\mathcal{M}$, “ $\in$ ” becomes a binary relation over M representing set inclusion in $\mathcal{M}$. Usually, “ $\in$ ” is the only element of $\mathcal{L}$, but in defining certain models of ZF , new objects may be included. Usually, if $A \in \mathcal{L}$ such that $A \neq “\in”$, then A usually represents a unary relation defined by a set or class from an exterior model of set theory, that is, a model $\mathcal{N}$ which contains both every element of M and A as either a set or proper class. We use $A^{\mathcal{M}}$ to refer A ’s representation in $\mathcal{M}$, whether A is an element of $\mathcal{L}$ or whether A is a set defined by some formula. When interpreted in $\mathcal{M}$, the unary relation A is used to check whether elements of M would be in A when viewed as elements in $\mathcal{N}$. That is, for every $a \in M$, $\mathcal{M} \models a \in A^{\mathcal{M}}$ if, when viewed as an element of the exterior model $\mathcal{N}$, we have that $a \in A^{\mathcal{N}}$. We denote A being a part of the language by saying $\mathcal{M} = \langle M, A, \in \rangle$.

Not all models are sets. For example, the *universe of all sets*, denoted V by convention, is a model of set theory and is not a set at all, but a proper class. Furthermore, any model $\mathcal{M}$ in which M contains all the ordinals is also a proper class. Hence, we allow for the existence of *class models* of set theory.

If φ is a formula written using $\mathcal{L}$ with free variables $x_1, \dots, x_n$, then for any parameters $a_1, \dots, a_n$ in M , we say **$\mathcal{M}$ satisfies $\varphi(a_1, \dots, a_n)$** , denoted $\mathcal{M} \models \varphi(a_1, \dots, a_n)$, if the resulting sentence is true in $\mathcal{M}$. We say $\mathcal{M}$ is a **model of ZFC (or ZF)** if for every sentence φ in the theory of ZFC (or ZF), that $\mathcal{M} \models \varphi$.

For $\mathcal{M}$ a set-sized model of ZFC (that is, $\mathcal{M}$ is a set in V), the satisfaction relation is definable in V as a relation “ $\models_{\mathcal{M}}$ ” which exists as a set in V , however this is not the case for class-sized models. By a theorem of Tarski using Gödel’s second incompleteness theorem, there exists no formula which defines truth in V or any proper class model of set theory [38]. It is possible to define a satisfaction relation for class models for all sentences below a specific *complexity*, but this is not immediately relevant to this thesis. For more information, see Jech [9].

As mentioned earlier, a model of a theory T exists if and only if T is consistent as a theory. It is not known if ZFC (or even ZF) is consistent. Along with the above fact,

there is no way to construct a set in V which would verify that V satisfies the axioms of set theory and is consistent. As the basis of most of modern set theory, however, nearly all mathematicians assume ZFC 's consistency and that V is a class model of ZFC . In this thesis, we will treat V as a model of ZFC unless otherwise specified.

2.2.2 Countable Models of ZFC

The **Löwenheim-Skolem theorem** is a result in model theory stating “if there exists a model for a theory in a countable language (such as $\mathcal{L}$), then there exists a countable model for the same theory, and a model of size κ for every cardinal $\kappa \geq \aleph_0$ ” [32]. Hence, if one assumes that $V \models ZFC + \text{Con}(ZFC)$, that is that V both satisfies ZFC and the statement “ ZFC is consistent”, then there must exist a model of ZFC within V and by Löwenheim-Skolem, a countable model of ZFC . Since the consistency of ZFC is a common assumption among set theorists, the existence of a countable model of ZFC may not seem a far leap to make. However, the assertion that V satisfies the statement “ ZFC is consistent” is a far stronger assumption, as it purports the existence of set-sized models of ZFC within V . Furthermore, if ZFC is consistent, there may exist many proper class-sized models of ZFC but no set sized models, meaning $V \models ZFC$ but $V \not\models \text{Con}(ZFC)$. Notably, the theory $ZFC + \text{Con}(ZFC)$ does not violate Gödel’s second incompleteness theorem, as $\text{Con}(ZFC)$ is another axiom of the theory which is independent of ZFC is consistent.

While the previous discussion may seem to be irrelevant meta-mathematical minutiae, the philosophy of mathematical truth, satisfaction, and existence lie at the heart of set theory and it is worth noting that there are delicate assumptions that must be made when assuming the existence of such models which may result in troublesome implications. In this thesis, though we assume V is a model of ZFC , we will not assume there exist set-sized models of ZFC within V unless otherwise specified. The philosophy of mathematics behind these theories is a fascinating topic deserving of far more discussion than a single thesis, and is not the topic of this thesis in particular.

A problem with this assumption will arise when we describe Cohen’s method of forcing, as countable models of ZFC are the most common way to introduce forcing and describe how it works. Indeed, one of the primary ways mathematicians first learn about forcing is through Kunen’s book *Set Theory* [15], which describes it through countable models of ZF . However, countable models of ZF are not necessary for forcing to be logically sound, as one can describe it using Boolean models instead [9]. The reason the countable model formulation is used, however, is due to it being far more intuitive to most set theorists. We shall use the same method later in this paper even with the philosophical problems that come with it. As with much of modern set theory, one must be willing to dance with troublesome philosophical implications when working with set models of ZF .

Partially due to the issues described above, but mostly due to a preference for the utility of their results, set-sized models of ZFC are not the primary focus of study in set theory. Instead, set theorists are interested in what are called **standard models of ZFC** . That is, transitive models of ZFC which contain all the ordinals (Ord). A set or class A is **transitive** if for every $x \in A$, every element of x is also in A . For any set x , the smallest transitive set which contains all the elements of x is the **transitive closure of x** , denoted $TC(x)$. Standard models are desirable as they more closely resemble what mathematicians expect the universe V of all sets to look like and behave.

2.2.3 Inner models of ZF and Extensions of V

Standard models of ZF often fall into one of two types. Inner models and extensions of V .

Definition 2.2.1.

An **inner model** is a transitive class model of ZF contained within V which contains all the ordinals. An **extension of V** is a model of ZF which contains all of V , has the same ordinals as V , and which also contains sets which are not in V . Hence, V is an inner model of an extension of V .

Inner models are usually constructed using machinery provided by large cardinal ax-

ioms or sets constructible from subclasses of V , while extensions of V are usually constructed through forcing. Such extensions of ZF are referred to as *generic extensions* for reasons which will become clear later. There exist other, more sophisticated ways to construct standard models of ZF through processes such as extenders, mice, and direct limits of models. We will introduce some of these concepts as they become relevant to the thesis. There exists a powerful result about transitive models, which has to do with the *Lévy Hierarchy*.

Definition 2.2.2 (Lévy Hierarchy [11]).

*The **Lévy hierarchy** is defined as follows:*

1. A formula $\varphi(x)$ is Δ_0 if it is logically equivalent to a first-order formula $\psi(x)$ with bounded quantifiers. That is, if there exists a quantifier $\forall$ or $\exists$ in $\psi(x)$ then it must quantify over a set, not a proper class or all of V . All Δ_0 formulas are also Σ_0 and Π_0 .
2. A formula $\varphi(x)$ is Π_n if it is of the form $\neg\psi(x)$, where $\psi(x)$ is a Σ_n formula.
3. A formula $\varphi(x)$ is Σ_{n+1} if it is logically equivalent to a formula of the form $(\exists y_1 \exists y_2 \dots \exists y_m)\psi(x, y_1, y_2, \dots, y_m)$ where $\psi(x, y_1, \dots, y_m)$ is a Π_n formula.
4. A formula $\varphi(x)$ is Δ_n if it is both Σ_n and Π_n .

One important result of the Lévy hierarchy has to do with absoluteness among transitive models.

Theorem 2.2.3 (Lévy Absoluteness).

*All Σ_1 statements are **upwards absolute** among transitive models. That is, if φ is a Σ_1 sentence true in $\mathcal{M}$, then for any transitive model $\mathcal{N}$ of ZF containing $\mathcal{M}$ as a submodel, φ is true in $\mathcal{N}$. Similarly, all Π_1 sentences are **downwards absolute** among transitive models of ZF , and remain true in every transitive submodel of ZF . Therefore, all Δ_1 sentences are **absolute** across all transitive models of ZF . [9]*

In studying V and various other models of ZF , it is useful to organize the universe of sets based on set complexity using the ordinals. Here we make use of Von Neumann's **Axiom of Regularity**, which states that every nonempty set must contain an element which is disjoint from it. Significantly, this means that there is no infinite descending chain of inclusion $x_0 \ni x_1 \ni x_2 \ni \dots \ni x_n \ni \dots$ in V .

Definition 2.2.4.

For all ordinals λ , we define V_λ inductively as follows [9]:

1. If $\lambda = 0$, $V_\lambda = \emptyset$.
2. If $\lambda = \alpha + 1$ for some ordinal α , $V_\lambda = \mathcal{P}(V_\alpha)$.
3. If λ is a limit ordinal, $V_\lambda = \bigcup_{\gamma < \lambda} V_\gamma$.

This structuring of V is called the **Von Neumann Hierarchy**. By the Axiom of Regularity, for all $x \in V$, there exists an ordinal λ such that $x \in V_\lambda$. Furthermore, for all $\lambda \in \text{Ord}$, $\lambda \in V_{\lambda+1}$.

Besides V , the most prominent inner model is Gödel's constructible universe L , created as follows [5].

Definition 2.2.5.

For a set X , the set of **definable subsets of X** is the set:

$$\text{Def}(X) = \{\{x \in X : (X, \in) \models \varphi(x)\} : \varphi(x, y_1, \dots, y_n) \text{ is a first order formula and } y_1, \dots, y_n \in X\}.$$

For all ordinals λ we define L_λ inductively as follows:

1. If $\lambda = 0$, $L_\lambda = \emptyset$.
2. If $\lambda = \alpha + 1$ for some ordinal α , $L_\lambda = \text{Def}(L_\alpha)$.
3. If λ is a limit ordinal, $L_\lambda = \bigcup_{\gamma < \lambda} L_\gamma$.

$L = \bigcup_{\lambda \in \text{Ord}} L_\lambda$ is **Gödel's constructible universe** [9].

Similar to the Von Neumann hierarchy, every element of L is contained in some L_λ and every $\lambda \in \text{Ord}$ is in $L_{\lambda+1}$. Furthermore, L has the unique property that every set x in L is constructible. That is, x can be defined by a first order formula using a finite number of parameters lower in the hierarchy than x . L satisfies Choice, CH, and even GCH, the **Generalized Continuum Hypothesis**, which states “ $2^\kappa = \kappa^+$ for every cardinal κ ”. L is important since the statement “ x is constructible” is Δ_1 , and hence an absolute statement across all inner models of ZF [9]. Hence, since all inner models are closed under constructability, for all $\lambda \in \text{Ord}$, L_λ remains the same for all inner models of ZF. As a result, L not only the smallest inner model of ZF, meaning it is a subclass of every inner model of ZF, it is also absolute across all inner models of ZF, since it always contains the exact same sets and therefore satisfies the exact same statements. In other words, for every inner model $\mathcal{M} \subseteq V$, $L \subseteq \mathcal{M}$, and $L^\mathcal{M} = L^V$.

Gödel's L can also be used to determine the universe of *relative constructability* to some set A .

Definition 2.2.6.

Let $A \in V$ be a set. For a set X , the **definable subsets of X relative to A** is the set:

$$\text{Def}_A(X) = \{ \{x \in X : (X, \in) \models \varphi(x)\} : \varphi(x, y_1, \dots, y_n) \text{ is a first order formula} \\ \text{and } y_1, \dots, y_n \in X \cup \{A \cap X\} \}.$$

That is, $A \cap X$ can be used as a parameter and is a set in $\text{Def}_A(X)$. For all ordinals λ we define $L_\lambda[A]$ inductively as follows:

1. If $\lambda = 0$, $L_\lambda[A] = \emptyset$.
2. If $\lambda = \alpha + 1$ for some ordinal α , $L_\lambda[A] = \text{Def}_A(L_\alpha[A])$.
3. If λ is a limit ordinal, $L_\lambda[A] = \bigcup_{\gamma < \lambda} L_\gamma[A]$.

$L[A] = \bigcup_{\lambda \in \text{Ord}} L_\lambda[A]$ is the class of sets **constructible relative to A** , and has the same ordinal properties as V and L [9].

Note we do not necessarily have that $A \in L[A]$. However it is true that $A \cap L \in L[A]$ and $L[A] = L[A \cap L]$. $L[A]$ is the smallest inner model in V which contains $L \cap A$, that is, the set of all constructible subsets of A . In order to study a similarly small inner model which contains some set A , we define the *universe constructible from A* .

Definition 2.2.7.

Let $A \in V$ be a set. For each $\lambda \in \text{Ord}$ We define $\mathbf{L}_\lambda(\mathbf{A})$ inductively as follows:

1. If $\lambda = 0$, $L_\lambda(A) = TC(\{A\})$.
2. If $\lambda = \alpha + 1$ for some ordinal α , $L_\lambda(A) = Def(L_\alpha(A))$.
3. If λ is a limit ordinal, $L_\lambda(A) = \bigcup_{\gamma < \lambda} L_\gamma(A)$.

$\mathbf{L}(\mathbf{A}) = \bigcup_{\lambda \in \text{Ord}} L_\lambda(A)$ is the universe of sets **constructible from A** , and has the same ordinal properties as V , L , and $L[A]$ [9].

$L(A)$ is class of all sets which can be constructed using A or elements of A , and is the smallest inner model containing A . $L(A)$ is often referred to as the ‘theory of A ’ as it is the smallest inner model in which one can write each statement about members of $\{A, TC(A)\} \cup \text{Ord}$. Unlike L , however, $L(A)$ and $L[A]$ may change based on the theory of V and the inner model in which they are defined. Hence, a common goal of set theorists is to find theories T extending ZFC which make $L(A)$ absolute, hence ensuring the theory of A cannot change no matter which model of T is being used. In this case, we say that $L(A)$ becomes the *canonical model of A* .

These definitions are immediately useful when describing an incredibly important theorem from model theory, the *Reflection Principle*.

Theorem 2.2.8 (Reflection Principle, Theorem 12.14 in [9]).

Let $\varphi(x)$ be a formula. Let M_0 be a set. Then there exists an ordinal α such that $V_\alpha \supseteq M_0$

and such that

$$\varphi(y) \iff V_\alpha \models \varphi(y)$$

for all $y \in V_\alpha$. We say V_α *reflects* φ .

Finally, there exist models of ZF which may not be inner models, such as $H(\kappa)$ for some cardinal κ . In this case, $H(\kappa)$ is the class of **hereditarily $< \kappa$ -sized sets**. That is, the class of all sets whose transitive closure only contains sets of cardinality $< \kappa$. However, $H(\kappa)$ is not always a model of ZF , as it is only constructed as a subclass of V and may not satisfy all the relevant axioms. In particular, the **Power Set Axiom** states “Given any set X , there exists a set $\mathcal{P}(X)$ which contains all the subsets of X ”. Furthermore, by **Cantor’s Theorem** [2], for every set X , $|X| < 2^{|X|} = |\mathcal{P}(X)|$. The other axiom which may cause issues is the **Axiom of Union**, which states “For any set X , there exists a set $\bigcup X$ which contains exactly the elements of the elements of X ”. Hence, for $H(\kappa)$ to be a model of ZF , we must have that κ must be greater than any $< \kappa$ -sized unions of subsets of size $< \kappa$ and must be greater than $|2^\lambda|$ for any $\lambda < \kappa$. Such cardinals, however, may not even exist in V , a fact which leads us directly to the study of the *Large Cardinal Hierarchy*.

2.3 Large Cardinals

2.3.1 Inaccessible Cardinals

The Large Cardinal Hierarchy refers to a collection of axioms, known as “large cardinal axioms”, each stating the existence a cardinal with some specific “large cardinal property”. A large cardinal property is a property of a cardinal whose existence cannot be proven to be consistent with ZFC , but whose existence can be used to prove powerful results about V . The simplest example of a large cardinal property is an *inaccessible cardinal*. Recall the following concepts in cardinal arithmetic.

Definition 2.3.1.

Let κ be a cardinal. The **cofinality of κ** ($\text{cf}(\kappa)$) is the least cardinal λ such that there

exists some set $X \subseteq \kappa$ such that $|X| = \lambda$ and $\sup(X) = \kappa$. We say a cardinal is **regular** if $\text{cf}(\kappa) = \kappa$. Note that for all cardinals κ , $\text{cf}(\kappa)$ is always regular [9].

Definition 2.3.2.

An infinite cardinal κ is a **successor cardinal** if it is of the form $\aleph_{\alpha+1}$ for some $\alpha \in \text{Ord}$. Otherwise, κ is a **limit cardinal**. κ is a **strong limit cardinal** if for every cardinal $\lambda < \kappa$, $2^\lambda < \kappa$ [9].

Definition 2.3.3.

Let κ be an regular uncountable cardinal. $C \subseteq \kappa$ is **closed unbounded (club) in κ** if C is:

1. **Closed:** For all $X \subseteq C$, if $\sup(X) < \kappa$, then $\sup(X) \in C$.
2. **Unbounded in κ :** For all $\alpha \in \kappa$, there exists $\alpha \leq \gamma \in C$.

A set $S \subseteq \kappa$ is **stationary in κ** if for all club sets $C \subseteq \kappa$, $C \cap S \neq \emptyset$ [9].

With these definitions, we can define our first large cardinal property: inaccessible cardinals.

Definition 2.3.4.

An uncountable cardinal κ is **(strongly) inaccessible** if it is regular and strongly inaccessible.

An important property of set theory is that if a cardinal κ is inaccessible $\langle V_\kappa, \in \rangle$ is a model of ZFC . Hence, if one is able to prove that $ZFC + \text{“There exists an inaccessible cardinal”}$ is consistent, then there exists a model of ZFC with an inaccessible cardinal in V . However, *within* V one can find the set V_κ . One can show that if κ is inaccessible, then $V_\kappa \models ZFC$ [9]. Hence we would have that ZFC is able to prove its own consistency, contradicting Gödel’s second incompleteness theorem. Since all large cardinals are inaccessible, assuming the consistency of ZFC , one therefore cannot prove that ZFC is consistent with the existence of any large cardinal, at least using the machinery of ZFC

itself. Despite this, set theorists generally believe most large cardinal axioms are consistent with ZFC .

The axioms are referred to as the large cardinal ‘hierarchy’ because there appears to be a linear order in the ‘consistency strength’ of the cardinal axioms. For example, a supercompact cardinal implies the consistency of a Woodin cardinal, a Woodin cardinal implies the consistency of a strong cardinal, a strong cardinal implies the consistency of a measurable cardinal, and a measurable cardinal implies the consistency of an inaccessible cardinal. Note that if φ_1 is a stronger large cardinal property than φ_2 , it is not necessarily true the least κ which satisfies φ_1 is greater than the least κ which satisfies φ_2 . For example, *huge cardinals* are greater in consistency strength than supercompact cardinals, but the least huge cardinal is less than the least supercompact cardinal if both exist. Indeed, the existence of a huge cardinal does not even imply the existence of any supercompact cardinals, only that supercompactness is consistent with ZFC [9]. Set theorists use this ordering, the Large Cardinal Hierarchy, to rate the consistency strength of various statements in ZFC .

2.3.2 Measurable Cardinals

The quintessential large cardinal property is that of *measurability*, due to Gödel’s proof of the existence of a measurable cardinal implying $V \neq L$. However, before we can discuss measurable cardinals, we need to introduce some background concepts first, beginning with ultrafilters and their use in constructing ultrapowers.

Definition 2.3.5 (Filters and Ultrafilters [9]).

Let X be a set. A **filter on X** is a set $F \subseteq \mathcal{P}(X)$ such that:

- (i) $\emptyset \notin F$, $X \in F$.
- (ii) For all $A, B \subseteq X$, if $A \subseteq B$ and $A \in F$, then $B \in F$ (upwards-closed).
- (iii) If $A, B \in F$, then $A \cap B \in F$ (closed under finite intersection).

$U \subseteq \mathcal{P}(X)$ is an **ultrafilter on X** if it is a filter on X for all $A \subseteq X$, $A \notin U \implies X \setminus A \in U$. That is, for every $A \subseteq X$, exactly one of A or $X \setminus A$ is in U .

A **(two-valued) measure on X** is a function $\mu : \mathcal{P}(X) \rightarrow \{0, 1\}$ such that $\{A \subseteq X : \mu(A) = 1\}$ is an ultrafilter on X .

A filter (or ultrafilter) U is **nonprincipal (or free)** if U does not contain any singletons. A measure corresponding to a nonprincipal ultrafilter is called a **nontrivial measure**.

Let κ be an uncountable cardinal. A filter U is **κ -complete** if it is closed under intersections of size $< \kappa$. If U is closed under countable intersections, it is **σ -complete**. A measure corresponding to a κ -complete (σ -complete) ultrafilter is called a **κ -additive (σ -additive) measure**.

An ultrafilter U is **normal** if it is closed under diagonal intersections of size κ . That is, if for all sequences $\langle A_\alpha : \alpha < \kappa \rangle$ where each $A_\alpha \in U$, then

$$\Delta_{\alpha < \kappa} A_\alpha := \{\gamma < \kappa : \gamma \in \bigcap_{\alpha < \gamma} A_\alpha\} \in U.$$

A **normal measure** is a measure whose corresponding ultrafilter is normal.

Definition 2.3.6.

Let U be an ultrafilter on a set X , and let $\mathcal{M}$ be a model of ZFC. The **ultrapower of $\mathcal{M}$ by U** is a model with universe defined by $\text{Ult}_U(\mathcal{M}, \in) = M^X / \sim_U$, where $\sim_U$ is an equivalence relation defined on M^X , the set of functions $f : X \rightarrow M$ as follows:

$$f \sim_U g \iff \{x \in X : f(x) = g(x)\} \in U$$

Hence for $f, g \in M^X$, the model $Ult_U(\mathcal{M}, \in)$, equality and inclusion are defined as follows:

$$Ult_U(\mathcal{M}, \in) \models f = g \iff \{x \in X : f(x) = g(x)\} \in U$$

$$Ult_U(\mathcal{M}, \in) \models f \in g \iff \{x \in X : f(x) \in g(x)\} \in U$$

It is simple to check that this constructs a well-defined model in the language of set theory.

Theorem 2.3.7 (Łoś' Theorem [18]).

If $\varphi(x_1, \dots, x_n)$ is a formula in the language of set theory, X is a set, U is an ultrafilter on X , and $\mathcal{M}$ is a model of set theory, then

$$Ult_U(\mathcal{M}, \in) \models \varphi(f_1, \dots, f_n) \iff \{x \in X : \mathcal{M} \models \varphi(f_1(x), \dots, f_n(x))\} \in U$$

Hence, we have that if $\mathcal{M}$ is a transitive model of ZFC , then $Ult_U(\mathcal{M}, \in)$ is a model of ZFC as well. More interestingly, if U is a nonprincipal ultrafilter, $Ult_U(\mathcal{M}, \in)$ can have interesting properties compared to $\mathcal{M}$ which are then known to be consistent with ZFC . Furthermore, $\mathcal{M}$ need not only be a set model. One can also create ultrapowers of V . This is the main reason why measurable cardinals are useful to study.

Definition 2.3.8.

An uncountable cardinal κ is **measurable** if there exists a nonprincipal, κ -complete ultrafilter on κ .

First described by Ulam, Gödel used the existence of a measurable cardinal to prove that there exist nonconstructible sets. In other words, that $V \neq L$. We give a sketch of Gödel's argument here.

Proof. (Sketch)

Let κ be the least measurable cardinal and let U be a κ -complete nonprincipal ultrafilter on κ . $Ult_U(V, \in) = V^\kappa / \sim_U$ is a model of ZFC by Łoś' Theorem. Since U is σ -additive, we can apply Mostowski's Collapsing Theorem [21], and hence there exists a proper class

isomorphism $\pi : Ult_U(V, \in) \rightarrow M$, where M is a transitive model of ZFC containing Ord . Hence, M is an inner model of ZFC .

From the properties of the ultrapower $Ult_U(V, \in)$, describe the map $[\cdot]_X : V \rightarrow Ult_U(V, \in)$, where for all $y \in V$, $[y]_X$ is the function $f : X \rightarrow V$ such that $f(x) = y$. Since $X \in U$, $[\cdot]_X$ preserves the truth of statements in V , and hence is an elementary embedding $j : V \rightarrow M$ of the form $j(y) = \pi([y]_X)$. Furthermore, since j preserves the ordering of the ordinals, we must have that $\mathcal{M}$ contains all of Ord , and hence is an inner model. Since $\mathcal{M}$ is an inner model, we must have $L \subseteq M$.

If one assumes for contradiction that $V = L$, then since $L \subseteq M \subseteq V = L$ we must have $M = L$ and j is elementary embedding from L into itself. Therefore j must preserve the statement “ κ is the least measurable cardinal.” However, if we let $d \in Ult_U(V, \in)$ be the function that $d(\lambda) = \lambda$ for all $\lambda < \kappa$, then $Ult_U(V, \in) \models d < [\kappa]_X$ and for all $\lambda < \kappa$, $Ult_U(V, \in) \models [\lambda]_X < d$, as $\{\gamma < \kappa : \gamma > \lambda\} \in U$ for all $\lambda < \kappa$. Hence $\kappa \leq \pi(d) < \pi([\kappa]_X) = j(\kappa)$.

Hence we have that $\kappa, j(\kappa) \in M = L$, $\kappa < j(\kappa)$, and $L \models$ “ $j(\kappa)$ is the least measurable cardinal.” However, this contradicts κ being the least measurable cardinal. Hence, by contradiction, if there exists a measurable cardinal, then $V \neq L$. $\square$

Gödel’s proof uses an elementary embedding of V into itself which is not the identity, a **nontrivial elementary embedding**. If $j : V \rightarrow M$ is a nontrivial elementary embedding, then by Kunen [14], we have that M is an inner model such that $M \neq V$. Furthermore, there must exist some cardinal κ such that $j(\kappa) > \kappa$. The least such κ is the **critical point of j** , denoted $cp(j)$. Such nontrivial elementary embeddings are a large focus of set theory and especially inner model theory, and many large cardinal properties are defined in terms of such elementary embeddings.

Note that in the previous proof, the critical point of the elementary embedding is κ . Hence, if we discuss an ultrapower over a nonprincipal ultrafilter over ω , we would have that ω is a cardinal which is moved by j . Therefore, in the resulting model $\mathcal{M}$, the theory of $\omega^{\mathcal{M}}$ is different from ω in V , and would be a nonstandard model of arithmetic. Since

most set theorists studying large cardinals are trying to maintain the theory of ω in their models, they usually only study ultrapowers of measurable cardinals.

Furthermore, through the Axiom of Choice, $ZFC \models$ “There exists a nonprincipal ultrafilter on ω ”. However, no nonprincipal ultrafilter U on ω can be σ -complete, as for all $n \in \omega$, we must have $\omega \setminus \{n\} \in U$ by nonprincipality. By σ -completeness, $\emptyset = \bigcap_{n < \omega} (\omega \setminus \{n\}) \in U$, which is impossible. Hence, we cannot apply Mostowski’s Collapse Lemma to Ult_U , and we do not have a corresponding inner model. Therefore, the issue with nonprincipal ultrapowers over ω is not that they are impossible, but that they do not correspond to inner models, and they change the theory of ω .

Interestingly, there exists a nontrivial elementary embedding $j : V \rightarrow M$, where M is some inner model, if and only if there exists a measurable cardinal, where $\kappa = cp(j)$ is the measurable cardinal and the set $U \subseteq \mathcal{P}(\kappa)$ defined by $U = \{X \subseteq \kappa : \kappa \in j(X)\}$ is a normal nonprincipal κ -complete ultrafilter on κ [9]. This is a very powerful property when one attempts to examine the inner model M defined by the elementary embedding. Furthermore, Lemma 10.2 in Jech’s *Set Theory* [9] states that if there exists a nonprincipal σ -complete ultrafilter on some set X , then there exists a measurable cardinal. Finally, if κ is an ordinal and U is a κ -complete nonprincipal ultrafilter over κ , then $L[U]$ is an inner model extending L such that κ is a measurable cardinal and $U' := L[U] \cap U$ a κ -complete nonprincipal ultrafilter for κ in $L[U]$. If U is normal, then $L[U]$ is called a **κ -model** [11].

Large cardinals and inner models come together in the *Inner Model Program*, which seeks to build canonical L -like inner models for each large cardinal property [26]. Large cardinals can also be used with forcing by collapsing large cardinal powers to create models of ZFC with some desired property in order to find the consistency strength of the given property. One such example is Sealing, to be defined in Definition 5.1.10.

2.4 Forcing and Generic Absoluteness

One of the main tools used to further Gödel’s program is the technique known as **forcing**. Ever since Cohen used it to prove independence of the CH from ZFC in 1963 [11], forcing

has been a cornerstone of set theory due to its ability to extend V to well-founded standard models of ZF and ZFC .

There are two main ways to define forcing. One, first developed by Cohen in 1963 [3] and described by Kunen in his book *Set Theory* [15], uses partial orders. The other method, developed Solovay and Vopěnka and first used by Scott in 1967 [28], utilizes Boolean models constructed using complete Boolean algebras. This method can be found in Jech's *Set Theory* [9]. The partial order formulation is far more widely used as it is intuitive in ways the Boolean-valued model representation is not. However the Boolean-valued description of forcing has a more rigorous philosophical basis, not requiring countable and transitive models of ZFC [40]. The formulations are roughly interchangeable as every partially ordered set can be embedded into a complete Boolean algebra and every complete Boolean algebra B can be assigned a partial order $\leq$ through $a \leq b \iff a - b = 0$ for every $a, b \in B$. Furthermore, Boolean algebra approach begins to run into issues when dealing with proper class forcing, as the theories that fit partial orders into complete Boolean algebras don't necessarily apply to class-sized partial orders, so the partial order formulation is the preferred in general. We will give a brief overview of forcing using the partial order description. For a more comprehensive explanation of forcing, we recommend Kunen [15] and Jech [9].

Definition 2.4.1 (Forcing [15]).

*In the context of forcing, a **partial order** (or a **forcing** or **notion of forcing**) is a partially ordered set or class $\langle \mathbb{P}, \leq \rangle$ with a maximum element, $\mathbf{1}$. For $p, q \in \mathbb{P}$, **p extends (or refines) q** if $p \leq q$. The elements of $\mathbb{P}$ are often referred to as **forcing conditions**. For $p, q \in \mathbb{P}$:*

*p and q **compatible**, denoted $p \parallel q$ if $(\exists r \in \mathbb{P})[r \leq p \wedge r \leq q]$.*

*p and q are **incompatible** if $\neg(p \parallel q)$, denoted $p \perp q$.*

The elements of $\mathbb{P}$ are called forcing conditions since they can be said to ‘force’ set theoretic statements which are written in a language $\mathcal{L}^{\mathbb{P}}$ constructed using *names* from $\mathbb{P}$.

Definition 2.4.2.

Let $\mathbb{P}$ be a forcing. A binary relation τ is a $\mathbb{P}$ -**name** if and only if it satisfies $\forall(\sigma, p) \in \tau[(\sigma, p) \text{ is a } \mathbb{P}\text{-name} \wedge p \in \mathbb{P}]$. If $\mathcal{M}$ is a model of ZF and $\mathbb{P} \in M$, then $\mathbf{M}^{\mathbb{P}}$ is the class of all $\mathbb{P}$ -names in M . $V^{\mathbb{P}}$ is defined analogously. $\mathcal{L}^{\mathbb{P}}$ is the class language $V^{\mathbb{P}} \cup \{\in\}$, where $\in$ is a binary relation and each element of $V^{\mathbb{P}}$ is a constant. [15]

In order to interpret this language in the extension of V , one requires the use of a *generic filter* for $\mathbb{P}$

Definition 2.4.3.

Let $\langle \mathbb{P}, \leq \rangle$ be a notion of forcing. Let $D \subseteq \mathbb{P}$ be a set:

D is **dense in** $\mathbb{P}$ iff $(\forall p \in \mathbb{P})(\exists q \in D)[q \leq p]$.

D is an **antichain in** $\mathbb{P}$ iff $(\forall p, q \in D)[p \parallel q \rightarrow p = q]$.

For any $p \in \mathbb{P}$, we say D is an **antichain below** p iff D is an antichain in $\{q \in \mathbb{P} : q \leq p\}$ [15].

Definition 2.4.4.

Let $\langle \mathbb{P}, \leq \rangle$ be a notion of forcing. We say $G \subseteq \mathbb{P}$ is a **filter on** $\mathbb{P}$ iff the following hold:

1. $1 \in G$.
2. $(\forall p, q \in G)(\exists r \in G)[r \leq p \wedge r \leq q]$.
3. $(\forall p, q \in \mathbb{P})[(p \in G \wedge p \leq q) \rightarrow q \in G]$ (upwards-closed).

That is, a filter G is an upwards-closed nonempty subset of $\mathbb{P}$ such that for any two elements $p, q \in G$, then p and q are compatible, witnessed by some $r \in G$.

A filter G on $\mathbb{P}$ is **generic** if for all dense $D \subseteq \mathbb{P}$, $G \cap D \neq \emptyset$. For M a model of ZF and $\langle \mathbb{P}, \leq \rangle \in M$, a filter G is **$\mathcal{M}$ -generic for** $\mathbb{P}$ if for every $D \subseteq \mathbb{P}$, $D \in M$, $(\mathcal{M} \models D \text{ is dense in } \mathbb{P}) \implies (D \cap G \neq \emptyset)$. $M[G]$ is called the **generic extension of $\mathcal{M}$ by G** [15].

Notably, for a given $\mathbb{P} \in V$, there may not be a $G \in V$ such that G is V -generic for $\mathbb{P}$. However, V is a structure from the language $\{\in, \mathbb{P}_{\mathcal{L}}\}$, with $\mathbb{P}_{\mathcal{L}}$ as a constant representing $\mathbb{P}$.

However, in Kunen’s approach, it is assumed that $V \models ZFC + Con(ZFC)$, that is, there exists a countable model $\mathcal{M}$ of ZFC in V which contains $\mathbb{P}_{\mathcal{L}}^{\mathcal{M}}$, an analog of $\mathbb{P}$ in $\mathcal{M}$ ¹. By the *Generic Filter Existence Lemma* [15], for every $p \in \mathbb{P}_{\mathcal{L}}^{\mathcal{M}}$, there exists a $G_p \in V$ containing p such that G is $\mathcal{M}$ -generic for $\mathbb{P}_{\mathcal{L}}^{\mathcal{M}}$. Hence, for $p \in \mathbb{P}$ one may assume there is an G in some “external” universe V^+ extending V in which V may be countable such that G contains p and is V -generic for $\mathbb{P}$.

The metamathematics of this argument can cause some understandable concern among readers, as it may seem like one is simply assuming the existence of a greater model of ZFC . For a more detailed explanation, it is recommended one read Kunen [15], however even Kunen’s argument plays rather loose when it comes to switching between countable and transitive models of ZF . A more satisfyingly rigorous analogue of these arguments is made using Boolean-valued models in Jech [9] for those readers who wish to have a more solid philosophical foundation to believe the validity of the above claims.

Definition 2.4.5.

Let $\mathcal{M}$ be a transitive model of ZF , $\mathbb{P} \in \mathcal{M}$ be a notion of forcing, τ a $\mathbb{P}$ -name in $\mathcal{M}$, and $G \subseteq \mathbb{P}$ an $\mathcal{M}$ -generic filter for $\mathbb{P}$. Define τ_G recursively as follows:

$$\tau_G = \{\sigma_G : (\exists p \in G)[(\sigma, p) \in \tau]\}.$$

$M[G] = \{\tau_G : \tau \in M^{\mathbb{P}}\}$ is the collection of all elements in the universe ‘forced’ by G . That is, τ_G is the realization of the constant τ in $\mathcal{L}_{\mathbb{P}}^{\mathcal{M}}$ in $M[G]$. $V[G]$ is defined analogously.

For each $x \in M$, define $\check{x} = \{(y, 1) : y \in x\}$. Furthermore $\Gamma = \{(\check{p}, p) : p \in \mathbb{P}\}$. Note $\check{x}$ and Γ are both in $M^{\mathbb{P}}$. One finds that $\check{x}_G = x$ and $\Gamma_G = G$. Hence $V \cup \{G\} \subseteq V[G]$. Hence, $V[G]$ is an $\mathcal{L}_{\mathbb{P}}$ -structure extending V and containing G . However, the theory of $V[G]$ is not yet determined [15].

Definition 2.4.6.

¹Note that this is quite a powerful assumption to make, and the metamathematics around this approach are philosophically controversial. However, we will sidestep this issue by simply noting that forcing has a more rigorous representation using complete Boolean algebras, and Kunen’s method is being used here simply due to it being more intuitive.

Let $\mathbb{P}$ be a forcing in V , φ be any sentence written using $\mathcal{L}_{\mathbb{P}}$, and $p \in \mathbb{P}$. We say **p forces φ** (denoted $p \Vdash \varphi$) if and only if for every V -generic G for $\mathbb{P}$ containing p , $V[G] \models \varphi_G$, where φ_G is the interpretation of φ in $V[G]$ [15].

Clearly, if $p < q$ and $p \Vdash \varphi$, then $q \Vdash \varphi$, leading to the notion of $q < p$ meaning ‘ q is stronger than p ’. For $\tau \in V^{\mathbb{P}}$, $\text{dom}(\tau)$ denotes $\{\sigma \in V^{\mathbb{P}} : \exists r \in \mathbb{P}[(\sigma, r) \in \tau]\}$. From this notion as well, one gets that if G is V -generic for $\mathbb{P}$, $V[G] \models \varphi$ and $p \in G$, then there exists $q \in G, q \leq p$ such that $q \Vdash \varphi$.

For a discussion of the rules of the forcing relation, see Chapter IV of Kunen’s *Set Theory* [15], specifically Lemmas IV.2.30, IV.2.32, and IV.2.33. However, the most important result about forcing is the following.

Theorem 2.4.7 (The Forcing Theorem [9]).

Let $\mathbb{P}$ be a forcing in $\mathcal{M}$. If φ is a sentence written using $\mathcal{L}_{\mathbb{P}}^{\mathcal{M}}$, for every $\mathcal{M}$ -generic G for $\mathbb{P}$,

$$M[G] \models \varphi \iff (\exists p \in G)[p \Vdash \varphi]$$

The Forcing Theorem provides a method to determine the truth or falsity of statements written using $\mathcal{L}_{\mathbb{P}}$, and can be used to show if $\mathcal{M}$ is a transitive model of ZF , so is $M[G]$, and if $\mathcal{M}$ is a transitive model of ZFC , then so is $M[G]$. Furthermore, all generic extensions preserve all the hereditarily finite sets, that is, the sets with finite transitive closure. Therefore, for any generic extension $V[G]$, $\omega^{V[G]}$ is the same as ω^V .

As mentioned earlier, forcing is mainly used to show the consistency or independence of certain statements with ZF and ZFC by creating a generic extension $V[G]$ which satisfies the statement or statements in question.

Example 2.4.8.

The if κ is a regular cardinal and $\lambda > \kappa$ is a cardinal, the **collapse of λ to κ** , denoted $\text{Col}(\kappa, \lambda)$ is a forcing such that λ has cardinality κ in $V[G]$ for any $\text{Col}(\kappa, \lambda)$ -generic G .

$\text{Col}(\kappa, \lambda)$ consists of all partial functions $f : \kappa \rightarrow \lambda$ with finite domain ordered by reverse inclusion. That is, for all $a, b \in \text{Col}(\kappa, \lambda)$, we have $b < a \iff b \supseteq a$.

A generic filter G for $\mathbb{P}$ is a subset of $\text{Col}(\kappa, \lambda)$ such that for every $a, b \in G$, a and b agree on $\text{dom}(a) \cap \text{dom}(b)$. Hence, if we set $F = \bigcup G$, $F : \kappa \rightarrow \lambda$ is a partial function. Furthermore, we have for every $\beta \in \lambda$ and every $\gamma \in \kappa$, the sets $\{a \in \text{Col}(\kappa, \lambda) : \beta \in \text{range}(a)\}$ and $\{a \in \text{Col}(\kappa, \lambda) : \gamma \in \text{dom}(a)\}$ are both dense in $\text{Col}(\kappa, \lambda)$. Therefore, $F : \kappa \rightarrow \lambda$ is a total onto function in $V[G]$. Hence, $V[G] \models |\lambda| \leq \kappa$. Furthermore, $\text{Col}(\kappa, \lambda)$ satisfies the property that it is $< \kappa$ -closed. That is, for every sequence of decreasing chain of elements $p_1 > p_2 > p_3 > \dots > p_\eta > \dots$ of length δ in $\text{Col}(\kappa, \lambda)$, where $\delta < \kappa$, we have that there exists a lower bound for the chain in $\text{Col}(\kappa, \lambda)$. This property ensures that all cardinals $< \kappa$ are preserved in $V[G]$. Hence, $V[G] \models |\lambda| = \kappa$. This is most commonly used to create models of ZF where the collapsed cardinal is made countable through $\text{Col}(\omega, \lambda)$.

Cohen used a variation of the above collapse to prove the independence of the continuum hypothesis by constructing forcing poset which collapsed the set of all finite partial functions $f : \mathbb{N} \rightarrow \{0, 1\}$ to $\aleph_2$ [9].

Along with forcing, set theorists have divided a new system of proposed axioms referred to collectively as the **Forcing Axioms** which revolve around preserving fragments of generic filter for partial orders $\mathbb{P}$ such that $\mathbb{P}$ satisfies a specific condition, such as the *countable chain condition* or *properness*. The three main forcing axioms, listed in increasing consistency strength, are: **Martin's Axiom** (MA), the **Proper Forcing Axiom** (PFA), and **Martin's Maximum** (MM). We will discuss the forcing axioms and their enhancements in greater detail as they become relevant in this thesis.

Forcing is a fantastic tool, however it causes a problem for those set theorists seeking absoluteness due to its ability to prove the independence of statements from ZFC , going against the goals of Gödel's program. Hence, set theorists now seek to find theories extending ZFC which are able to 'lock in' the truth of statements or theories within ZFC when it comes to generic extensions. This is known as the study of **generic absoluteness**.

As a note on notation, if Γ refers to a class of type of object in V , for any V -generic G , we use Γ_G to refer to the class of the same type of object in $V[G]$. For example, $\mathbb{R}_G$ and $\mathcal{P}_G(\omega)$ refer to the set of all reals in $V[G]$ and the set of all subsets of ω in $V[G]$.

Furthermore, if $V[G]$ is a generic extension of V , $\mathbb{P} \in V[G]$ is a forcing, and H is a $V[G]$ -generic filter for $\mathbb{P}$, we use $V[G][H]$ or $V[G * H]$ to refer to the relevant generic extension of $V[G]$. We say G, H are **consecutive V -generic extensions**.

Definition 2.4.9.

Let $\varphi(x_1, \dots, x_n)$ be a sentence in the language of set theory with parameters taken from V and with $x_1, \dots, x_n$ free variables. $\varphi(x_1, \dots, x_n)$ is **generically absolute** if the truth value of $\varphi(a_1, \dots, a_n)$ for any $a_1, \dots, a_n \in V$ does not change under any generic extension of V . That is, for every V -generic G ,

$$(\forall a_1, \dots, a_n \in V)[V \models \varphi(a_1, \dots, a_n) \iff V[G] \models \varphi(a_1, \dots, a_n)].$$

Beyond fulfilling Gödel's program, generic absoluteness is desirable since if φ is a formula in some transitive model $\mathcal{M}$ of a theory T extending ZFC , such as $ZFC +$ There exists a strong cardinal. then if one can find some generic extension $M[G]$ for which φ 's truth value is known, φ 's truth value is then known for any generic extension of $\mathcal{M}$, providing powerful machinery to any mathematician studying the property φ . Other fields of mathematics, including descriptive set theory, operator algebras, functional analysis, and game theory can benefit from such results. Descriptive set theory in particular is tied to generic absoluteness, as will be seen shortly.

Beyond studying the generic absoluteness of specific statements, set theorists also seek to classify the consistency strength of generic absoluteness for theories within V . This is most often done by showing the existence of elementary embeddings between a relevant inner model in V with the same type of inner model in $V[G]$.

Definition 2.4.10.

Let Γ be a class of object within V (for example, $\Gamma = \mathbb{R}$). The **theory of Γ is generically absolute** if for all consecutive V -generics G, H there exists an elementary embedding

$$j_{G,H} : L(\Gamma_G) \rightarrow L(\Gamma_{G*H})$$

In this way, for any formula of set theory $\varphi(x_1, \dots, x_n)$, and any $a_1, \dots, a_n \in V[G]$,

$$L(\Gamma_G) \models \varphi(a_1, \dots, a_n) \iff L(\Gamma_{G*H}) \models \varphi(j_{G,H}(a_1), \dots, j_{G,H}(a_n))$$

In the above definition, any sentence in the language of set theory true of $L(\Gamma)$ in $V[G]$ is also true of $L(\Gamma)$ in $V[G][H]$. Note that since $\mathbb{P}$ can be the trivial forcing, V may be included among the $V[G]$ above, so V counts as one such generic extension. Also note that $L(\Gamma_{G,H})$ may contain new objects not in than $L(\Gamma_G)$. It is only the theories that remain the same. If Γ is a type of object whose theory is generically absolute, it means the model $L(\Gamma)$ is a **canonical model of Γ** within the theory under which V is described. Canonical models are an important facet of set theory as they approach a ‘true’ theory for the objects across all models of the extension of *ZFC*.

Finally, set theorists seek to study generic absoluteness by ranking set-theoretic statements about the reals by their complexity. It is easy to see that Δ_1 statements in the Lévy hierarchy are generically absolute, as they are upwards-absolute among transitive models. However, many basic operations of set theory are not Δ_0 in the Lévy hierarchy. For example, the power set operation is Σ_1 . Hence, it can be difficult to determine where statements rank in the Lévy hierarchy. Hence, in order to approach more complicated statements, we introduce a new way to rank statements by their complexity.

2.5 Descriptive Set Theory and the Analytic and Projective Hierarchies

From Δ_0 in the Lévy hierarchy, one can define the **arithmetical hierarchy**, the first step in our hierarchies of formulas.

Definition 2.5.1.

The **arithmetical hierarchy** [9] is defined as follows:

1. A formula $\varphi(x)$ is Σ_1^0 if it is logically equivalent to a first-order formula over the integers of the form $(\exists y \in \omega)\psi(x, y)$ where $\psi(x, y)$ is Δ_0 .

2. A formula $\varphi(x)$ is Π_n^0 if it is of the form $\neg\psi(x)$, where $\psi(x)$ is a Σ_n^0 formula.
3. A formula $\varphi(x)$ is Σ_{n+1}^0 if it is logically equivalent to a formula of the form $(\exists y \in \omega)\psi(x, y)$, where $\psi(x, y)$ is a Π_n^0 formula.
4. A formula $\varphi(x)$ is Δ_n^0 if it is both Σ_n^0 and Π_n^0 .

The variables for formulas in the arithmetic hierarchy are generally considered to range over ω .

Due to the absoluteness of ω under forcing, every statement in the arithmetical hierarchy is generically absolute. Hence one needs to check more complicated formulas.

Definition 2.5.2.

Let $\varphi(x)$ be a formula in the language of set theory. Let $a \in \mathbb{R}$.

1. $\varphi(x)$ is Σ_1^1 if it is logically equivalent to $(\exists y \in \mathbb{R})\psi(x, y)$, where $\psi(x, y)$ is Σ_n^0 for some $n < \omega$. $\varphi(x)$ is $\Sigma_1^1(a)$ if it is Σ_1^1 in terms of a , that is, for some $n < \omega$ there exists some Σ_n^0 set theoretic formula $\psi(x, y, z)$ such that $\varphi(x)$ is logically equivalent to $(\exists y \in \mathbb{R})\varphi(x, y, a)$.
2. $\varphi(x)$ is $\Pi_n^1 [\Pi_n^1(a)]$ if it is of the form $\neg\psi(x)$, where $\psi(x)$ is a $\Sigma_n^1 [\Sigma_n^1(a)]$ formula.
3. $\varphi(x)$ is $\Sigma_{n+1}^1 [\Sigma_{n+1}^1(a)]$ if it is logically equivalent to a formula of the form $(\exists y \in \mathbb{R})\psi(x, y)$, where $\psi(x, y)$ is a $\Pi_n^1 [\Pi_n^1(a)]$ formula.
4. $\varphi(x)$ is $\Delta_n^1 [\Delta_n^1(a)]$ if it is both Σ_n^1 and $\Pi_n^1 [\Sigma_n^1(a)]$ and $\Pi_n^1(a)$.

The Σ_n^1 is called the **analytic hierarchy of formulas**, or alternatively the **lightface hierarchy** [9]. The lightface hierarchy is logically equivalent to the formulas which can be expressed in second-order logic of the integers.

The **boldface hierarchy** is defined very similar to the lightface hierarchy. A formula $\varphi(x)$ is Σ_n^1 if there exists an $a \in \mathbb{R}$ such that $\varphi(x) \in \Sigma_n^1(a)$. That is, $\Sigma_n^1 = \bigcup_{a \in \mathbb{R}} \Sigma_n^1(a)$. In the same vein, $\Pi_n^1 = \bigcup_{a \in \mathbb{R}} \Pi_n^1(a)$, $\Delta_n^1 = \bigcup_{a \in \mathbb{R}} \Delta_n^1(a)$.

The variables for formulas in the lightface and boldface hierarchy are generally considered to range over $\mathbb{R}$.

The notation of the lightface and boldface hierarchies are not ideal, as authors occasionally do not clarify which hierarchy is being used, which usually leads to confusion. Recently, it has become customary to use Σ_n^1 to denote the boldface hierarchy when unspecified. To avoid such confusion in this thesis, however, we will use the light and bold notation as has been customary in the past.

Definition 2.5.3.

Let $\varphi(x)$ be a formula in the language of set theory, and let A be a set in V .

1. $\varphi(x)$ is $\Sigma_1^2 [\Sigma_1^2(A)/\Sigma_1^2]$ if it is logically equivalent to $(\exists Y \subseteq \mathbb{R})\psi(x, Y)$, where $\psi(x, Y)$ is $\Sigma_n^1 [\Sigma_n^1(A)/\Sigma_n^1]$ for some $n < \omega$.
2. $\varphi(x)$ is $\Pi_n^2 [\Pi_1^2(A)/\Pi_1^2]$ if it is of the form $\neg\psi(x)$, where $\psi(x)$ is a $\Sigma_n^2 [\Sigma_n^2(A)/\Sigma_n^2]$ formula.
3. $\varphi(x)$ is $\Sigma_{n+1}^2 [\Sigma_{n+1}^2(A)/\Sigma_{n+1}^2]$ if it is logically equivalent to a formula of the form $(\exists Y \subseteq \mathbb{R})\psi(x, Y)$, where $\psi(x, Y)$ is a $\Pi_n^2 [\Pi_n^2(A)/\Pi_n^2]$ formula.
4. $\varphi(x)$ is $\Delta_n^2 [\Delta_n^2(A)/\Delta_n^2]$ if it is both $\Sigma_n^2 [\Sigma_n^2(A)/\Sigma_n^2]$ and $\Pi_n^2 [\Pi_n^2(A)/\Pi_n^2]$ [9].

The variables for formulas in this hierarchy are generally considered to range over $\mathcal{P}(\mathbb{R})$.

It using the above hierarchies that set theorists attempt to classify generic absoluteness by the complexity of formulas, and while the consistency strength of Σ_n^0 and Σ_n^1 absoluteness are known for all $n < \omega$, the study of Σ_1^2 and Σ_2^2 absoluteness sentences are at the forefront of the field.

The boldface hierarchy is especially relevant to a subset of set theory named *descriptive set theory*, the study of definable subsets of Polish spaces through the **projective hierarchy**.

The Baire space $\mathbb{R}$, that is ω^ω under the product topology is a **Polish space**: it is a separable, completely metrizable topological space. Another powerful property of $\mathbb{R}$ is

every other Polish space can be embedded into $\mathbb{R}$ using a Borel function, and so one can make statements about all Polish spaces by studying the definable subsets of Baire space. Every Polish space X has the **perfect set property**: either X is countable, or $|X| = 2^{\aleph_0}$. Hence, Polish spaces satisfy a local version of *CH*. The central object of descriptive set theory, however, is the **projective hierarchy of subsets of $\mathbb{R}$** .

Definition 2.5.4.

*If X is a Polish space, $A \subseteq X$ is **analytic** if and only if it is the continuous image of the Baire space. That is, there exists a continuous $f : \mathbb{R} \rightarrow X$ such that $f[\mathbb{R}] = A$.*

Analytic sets are the base upon which descriptive set theory is built. It is provable in *ZFC* that analytic sets are measurable with respect to every complete probability measure on $\mathbb{R}$, have the perfect set property, and have the property of Baire [12]. Hence, they are very ‘nice’ sets to work with. However, they are still complicated enough for there to exist very strange analytic sets. For example, not all analytic sets are Borel, so $\mathcal{B}(\mathbb{R}) \subsetneq \Sigma_1^1(\mathbb{R})$. However, analytic sets are only the beginning of the projective hierarchy.

Definition 2.5.5.

Let X be a Polish space. Let $A \subseteq X$

1. *A is Σ_1^1 if it is an analytic set.*
2. *A is Π_n^1 if it is the complement of a Σ_n^1 set. That is, $X \setminus A \in \Sigma_n^1(X)$*
3. *A is Σ_{n+1}^1 if there exists a Π_n^1 set $B \subseteq X \times \mathbb{R}$ such that A is the projection of B onto X . That is,*

$$A = \{x \in X : (\exists y \in \mathbb{R})[(x, y) \in B]\}$$

We will show that the projective hierarchy and the boldface hierarchy relate in the following way: For all $A \subseteq \mathbb{R}$:

$$A \text{ is } \Sigma_n^1 \text{ in the projective hierarchy} \iff$$

$$A = \{x \in \mathbb{R} : \varphi(x)\} \text{ where } \varphi \text{ is a } \Sigma_n^1 \text{ statement in the boldface hierarchy.}$$

If the above holds, then clearly the same statement then holds for $\mathbf{\Pi}_n^1$ subsets of $\mathbb{R}$ and $\mathbf{\Pi}_n^1$ formulas in the boldface hierarchy.

The projective sets are the closure of Borel subsets of $\mathbb{R}$ under countable union, countable intersection, set complements, and continuous images. While the theory of analytic sets is well-known, the theory of projective sets is far more obscure. For example, the statements “every projective set is Lebesgue measurable”, “every projective set has the perfect set property”, “every projective set has the property of Baire”, and “every projective set has the Ramsey property” are each independent of ZFC and each requires a large cardinal axiom to prove. Therefore, if one could construct a theory in which the theory of projective sets is generically absolute, it would be a massive boon to those studying descriptive set theory.

While there are many equivalent definitions for an analytic set, for our purposes we are only interested in subsets of $\mathbb{R}$. Hence, we will use an equivalent definition for analytic subsets of $\mathbb{R}$ defined in terms of *trees* on $\omega \times \omega$.

Definition 2.5.6 (Tree on $\omega \times \lambda$ [9]).

Let λ be an infinite cardinal. A **tree on $\omega \times \lambda$** is a subset $T \subseteq \omega^{<\omega} \times \lambda^{<\omega}$ such that T is closed under initial segments, that is, for all $(s, t) \in T$, $|s| = |t|$ and $(s \restriction i, t \restriction i) \in T$ for each $i \in |s| < \omega$.

Suppose T is a tree on $\omega \times \lambda$. For $s \in \omega^{<\omega}$ the **restriction of T to s** , denoted T_s , is the set

$$T_s = \{t \in \lambda^{<\omega} : (s, t) \in T\}.$$

For any $x \in \mathbb{R}$, the **restriction of T to x** , denoted T_x is the set

$$T_x = \bigcup \{T_{x \restriction n} : n \in \omega\}.$$

Note T_x is a tree on λ for every $x \in \mathbb{R}$. That is, it is a subset of $\lambda^{<\omega}$ closed under initial sequences.

For each $x \in \mathbb{R}$, $T_x \subseteq \lambda^{<\omega}$ and is a tree on λ . The **set of branches of T** , denoted

$[T]$, is the set

$$[T] = \{(x, t) : x \in \mathbb{R} \wedge t \in \lambda^\omega \wedge \forall n \in \omega \ (x \restriction n, t \restriction n) \in T\}.$$

The **projection of T on $\mathbb{R}$** , denoted $p[T]$ is the set

$$p[T] = \{x \in \mathbb{R} : \exists f \in \lambda^\omega \ (x, f) \in [T]\}.$$

A set $A \subseteq \mathbb{R}$ is **λ -Suslin** if there exists some tree T on $\omega \times \lambda$ such that $A = p[T]$.

Theorem 2.5.7.

Let $A \subseteq \mathbb{R}$. A is analytic only if it is the projection of a tree T on $\omega \times \omega$. That is, it is ω -Suslin [9].

Note from this equivalent definition of analytic sets, a set $A \subseteq \mathbb{R}$ is analytic (Σ_1^1) if and only if it is defined by a Σ_1^1 formula from the boldface hierarchy. Furthermore, from the earlier definition of the projective hierarchy, a set $A \subseteq \mathbb{R}$ is Σ_n^1 (Π_n^1) if and only if there exists a Σ_n^1 (Π_n^1) formula $\varphi(x)$ from the boldface hierarchy such that $A = \{x \in \mathbb{R} : \varphi(x) \text{ is satisfied}\}$.

Hence, projective absoluteness is the same as generic absoluteness of the theory of all projective sets. The consistency strength of projective absoluteness, therefore, has wide-reaching effects throughout topology, functional analysis, and operator algebras. However, there remains one aspect central to not only descriptive set theory but to set theory as a whole: the problem of determinacy.

2.6 Determinacy

The last major component upon which modern set theory is built is a collection of axioms which revolve around the existence of winning strategies to infinite two-player games played around subset a subset of the reals, called the determinacy game. These axioms are referred to as the **Determinacy Hierarchy**.

Definition 2.6.1.

Let $A \in \mathcal{P}(\mathbb{R})$. The **determinacy game for A** is the following infinite two-player game:

Starting with player I, player I and player II each take turns choosing an element from ω to create an infinite sequence of elements from ω : Player I wins if the resulting sequence

I	a_0	a_2	a_4	$\dots$
II	a_1	a_3	$\dots$	

Figure 2.1: The steps of the determinacy game.

$\langle a_n : n < \omega \rangle$ is in A . Otherwise, player II wins. The set A is determined if either player has a winning strategy for the game on A .

In ZFC , it can be shown every Borel set is determined. However, the statement “all analytic sets are determined” requires the large cardinal property “for every $x \in \mathbb{R}$, $x^\#$ exists” ($x^\#$ is pronounced ‘ x sharp’). We will give more detail on sharps with Definition 4.1.2. However, to prove the determinacy of Σ_n^1 subsets of $\mathbb{R}$ for any $n > 1$, one requires a new type of large cardinal property.

Definition 2.6.2.

A cardinal κ is a **Woodin cardinal** if for every $f : \kappa \rightarrow \kappa$, there exists an ordinal $\delta < \kappa$ such that for all $\alpha \in \delta$, $f(\alpha) \in \delta$ and an elementary embedding $j : V \rightarrow M$ with critical point δ and such that $V_{j(f)(\delta)} \subseteq M$.

Woodin cardinals are a stronger large cardinal property than measurable cardinals. However, a Woodin cardinal need not be a measurable cardinal. If κ a Woodin cardinal, then $\{\lambda < \kappa : \lambda \text{ is a measurable cardinal}\}$ is stationary in κ [9]. Woodin cardinals already have a powerful generic absoluteness property.

Theorem 2.6.3 (Woodin [16]).

Suppose that κ is a Woodin cardinal, and let $\mathcal{M}$ be a transitive model of ZFC containing V_κ as an element. Then κ is a Woodin cardinal in $\mathcal{M}$.

Furthermore, it follows from the existence of n -many Woodin cardinals with a measurable cardinal above them all implies determinacy for the Σ_{n+1}^1 sets [16]. Therefore,

the determinacy of all projective sets is implied by the existence of a measurable cardinal above ω -many Woodin cardinals. Woodin cardinals are central to the study of generic absoluteness and will appear continuously throughout this thesis. In fact, Woodin cardinals are central to determining the consistency strength of the first axiom in the determinacy hierarchy.

Axiom 2.6.4.

*The **Axiom of Determinacy** (AD) is the statement:*

For all $A \subseteq \mathbb{R}$, A is determined.

AD is clearly an important axiom in descriptive set theory. However, AD contradicts Choice, and therefore can never be studied in models of ZFC . One of the primary uses of AD is examining whether certain inner models of V satisfy AD to identify which subsets of the reals are determined. For example, if V is such that $L(\mathbb{R}) \models AD$, then every $A \subseteq \mathbb{R}$ such that A is constructible from the reals in V is determined. Furthermore, the winning strategy for any determinacy game of $L(\mathbb{R})$ is also constructible from the reals. Hence, by proving determinacy in similar inner models, set theorists are able to show the determinacy of an entire theory of sets is consistent with or implied by certain assumptions such as the existence of large cardinals. AD is also important to generic absoluteness due to the following equiconsistency result:

Theorem 2.6.5 (Woodin [41]).

The theory $ZF + AD$ is equiconsistent with $ZFC +$ “There exist infinitely many Woodin Cardinals.”

Hence, if AD is consistent with ZF , we know there exist (class) models of ZFC with infinitely many Woodin cardinals. Conversely, if we assume that infinitely many Woodin cardinals exist, we can construct a model of ZF in which AD holds. Furthermore, a theorem of Martin-Steel-Woodin [19] states “if there exists infinitely many Woodin cardinals with a measurable cardinal on top, then $L(\mathbb{R}) \models AD$.” Hence the statement

$L(\mathbb{R}) \models AD$ has a consistency strength of at most “infinitely many Woodins with a measurable on top.”

The next level in the determinacy hierarchy is AD^+ , first described by Woodin. To define AD^+ , we must first define Θ , ∞ -Borel sets, and $DC_{\mathbb{R}}$.

Axiom 2.6.6.

The **Axiom of Dependent Choice** DC is the statement “If R is a relation on some set X such that for every $a \in X$, there exists $b \in X$ such that aRb (that is, R is **total**), then there exists a sequence $\langle a_n : n < \omega \rangle$ such that $a_n R a_{n+1}$ for all $n < \omega$.”

The **Axiom of Dependent Choice for the Reals**, denoted $DC_{\mathbb{R}}$ is the same statement as above, except X is replaced by $\mathbb{R}$.

Both DC and $DC_{\mathbb{R}}$ are weaker statements than Choice, however, as far as we know, they do not contradict AD . Furthermore, they provide some semblance of the power of infinite choice in models which satisfy DC or $DC_{\mathbb{R}}$. Furthermore, for much of mathematics, DC and $DC_{\mathbb{R}}$ are more than sufficient to fill the role normally filled by Choice.

Definition 2.6.7 (Woodin [41]).

A set $A \subseteq \mathbb{R}$ is ∞ -**Borel** if there exists $S \subseteq \text{Ord}$, $\alpha \in \text{Ord}$, and a formula $\varphi(x_0, x_1)$ such that:

$$A = \{y \in \mathbb{R} : L_{\alpha}[S, y] \models \varphi[S, y]\}$$

As described by Woodin [41], there is an equivalent definition for ∞ -Borel sets in ZF , however this definition is descriptive more than rigorous and is difficult to explain formally. Essentially, $A \subseteq \mathbb{R}$ is ∞ -Borel if and only if A has an effective transfinite Borel code. However, under the assumption of AD , there exists a far more accessible definition of ∞ -Borel sets.

Theorem 2.6.8 (Woodin [41]).

Assuming $ZF + AD$, let $A \subseteq \mathbb{R}$. Then the following are equivalent:

- a) A is ∞ -Borel

b) *There exists $S \subseteq Ord$ such that $A \in L(S, \mathbb{R})$*

That is, under AD , the set of ∞ -Borel is the set of all sets constructible from the reals and some subset of the ordinals.

In models of ZF , Θ is used to refer to the first $\lambda \in Ord$ which is not a surjective image of $\mathbb{R}$. In models of ZFC , this is just $(2^{\aleph_0})^+$. However, in models of $ZF + AD$, Θ may vary wildly. The value of $cf(\Theta)$ is important to study models of ZF , as it allows set theorists to determine whether or not the model satisfies various axioms such DC , AD , and $V = L(\mathbb{R})$.

Definition 2.6.9.

Under $ZF + DC_{\mathbb{R}}$, AD^+ is the conjunction of the following statements:

1. *For every $A \subseteq \mathbb{R}$ A is ∞ -Borel*
2. *For any $\lambda < \Theta$ and any continuous $\pi : \lambda^\omega \rightarrow \omega^\omega$, for each $A \subseteq \mathbb{R}$, $\pi^{-1}[A]$ is determined. That is, in the variant of the determinacy game where players choose from elements of λ , rather than ω , at least one player in the game for $\pi^{-1}[A]$ has a winning solution.*

Finally $AD_{\mathbb{R}}$ the same statement of AD , except we consider subsets of $\mathbb{R}^\omega$, and in the determinacy game, the players take turns choosing elements of $\mathbb{R}$, rather than ω . $AD_{\mathbb{R}}$ is a stronger axiom than AD^+ , which is in turn a stronger axiom than AD . Hence the name ‘determinacy hierarchy’.

Now that we have gone over the important background of set theory and with knowledge of the value and significance that generic absoluteness has to set theory in hand, we are ready to begin our overview of the major generic absoluteness results, and to give a description of the current state of the study of the field.

Chapter 3

Projective Sets

As mentioned earlier, the study of the consistency strength of generic absoluteness for various levels of the projective hierarchy and the projective hierarchy as whole has been a major focus of study among set theorists due the wide-ranging implications in the study of the real numbers and especially descriptive set theory. However, we first must distinguish between two types of absoluteness in the projective hierarchy.

Definition 3.0.1.

Let V be a model of ZF .

1. **One-Step generic absoluteness for Σ_n^1** means for every $\varphi(x) \in \Sigma_n^1$, any $a \in \mathbb{R}$ and any V -generic G

$$V \models \varphi(a) \iff V[G] \models \varphi(a).$$

2. **Two-Step generic absoluteness for Σ_n^1** is the statement “For every V -generic G , one-step generic absoluteness for Σ_n^1 holds in $V[G]$ ”

One-step generic absoluteness for Σ_n^1 is equivalent to generic absoluteness of $\bigcup_{a \in \mathbb{R}^V} \Sigma_n^1(a)$ formulas, formulas of the form $\varphi(x, a)$, where $\varphi \in \Sigma_n^1$ and $a \in \mathbb{R}^V$, while two-step generic absoluteness for Σ_n^1 formulas is generic absoluteness for all Σ_n^1 formulas in all generic extensions.

The nuance here is for a V -generic G , $V[G]$ may contain reals and subsets of reals

which are not in V . That is $\mathbb{R} \subsetneq \mathbb{R}_G$ or $\mathcal{P}(\mathbb{R}) \subsetneq \mathcal{P}_G(\mathbb{R}_G)$. Hence it may be possible to find theories with Σ_n^1 generic absoluteness for V but not for some generic extension $V[G]$, as more reals can be used when evaluating Σ_n^1 formulas.

Two-step generic absoluteness for Σ_n^1 for every $n < \omega$ is the same as projective absoluteness, that is the theory of $L(\bigcup_{n < \omega} \Sigma_n^1)$. Clearly, two-step absoluteness is stronger and more desirable to set theorists, but does not always follow from one-step. However, by the following theorem,

Theorem 3.0.2 (Hauser [8]).

The following theories are equiconsistent:

1. $ZFC + 1\text{-step projective absoluteness}$.
2. $ZFC + \text{projective absoluteness}$.

The consistency strength of one-step and two-step projective absoluteness is the same. We will go into more detail about the mathematics behind this result later, but for now it gives us a goal to work towards.

3.1 Shoenfield absoluteness

The first and most famous generic absoluteness result for the projective hierarchy comes an theorem far stronger than generic absoluteness, known as **Shoenfield's Absoluteness Theorem**.

Theorem 3.1.1 (Shoenfield's Absoluteness Theorem (1961) [9]).

Let $a \in \mathbb{R}$. Every $\Sigma_2^1(a)$ and $\Pi_2^1(a)$ relation is absolute for all transitive models M of ZF such that $\omega_1 \in M$ and $a \in M$.

Shoenfield's Absoluteness Theorem, instead of referring to all generic extensions of M , proves the absoluteness across all models of ZF containing ω_1 , not just ZFC . As a result, for any standard model M of ZF , two-step generic absoluteness for Σ_2^1 and Π_2^1 formulas holds. In fact, it shows all Σ_2^1 and Π_2^1 formulas are absolute for L . As a result,

every Σ_2^1 or Π_2^1 real is constructible. First demonstrated in his 1961 paper *The Problem of Predicativity* [30], a new way of proving the result has been developed by showing all Σ_2^1 sets are ω_1 -Suslin. We will give a proof of Shoenfield's Absoluteness theorem near the end of this thesis.

Shoenfield absoluteness was a landmark result, as a great deal of important unsolved problems, such as $P = NP$ and the Riemann Hypothesis can be represented by Σ_2^1 sentences. In fact, the aforementioned problems are far less complex than Σ_2^1 . Furthermore, Shoenfield absoluteness is simply true in ZF , it does not require any large cardinals to show, and hence nearly all mathematicians are able to benefit from it without any of the potential philosophical qualms with assuming a large cardinal.

The following are examples of Σ_2^1 or Π_2^1 formulas:

1. “ x is a well-founded tree.” (Π_1^1).
2. “ x is well-ordering on $\mathbb{N}$ ” (Π_1^1).
3. “ $f \in C([0, 1])$ and f satisfies the mean value theorem” (Π_2^1) [12].
4. “ $((f_n)_{n < \omega}, f) \in C([0, 1])^\omega \times C([0, 1])$ and there exists a subsequence $(f_{n_k})_{k < \omega} \subseteq (f_n)_{n < \omega}$ such that f_{n_k} converges pointwise to f .” (Σ_2^1) [12].

However, while the absoluteness of these formulas can be useful, a more practical application of Shoenfield Absoluteness is to show that a given Π_1^1 or Σ_2^1 set is nonempty in V or $V[G]$.

Example 3.1.2.

A theorem of Mazurkiewicz [12] states “The set of differentiable functions on $C([0, 1])$ is a Π_1^1 set”. Hence, if $\varphi(x)$ is a Π_1^1 formula, then the statement “ $\psi(x) : (\exists x \in C([0, 1]))[\varphi(x) \wedge x \text{ is a differentiable function on } C([0, 1])]$ ” is Σ_2^1 . Hence, if in some generic extension $V[G]$ the set of $x \in C([0, 1])$ such that $V[G] \models \varphi(x)$ is nonempty, we know the set is also nonempty in V . The following are more examples of Π_1^1 sets whose nonemptiness become generically absolute through Shoenfield.

1. *In the study of continuous functions on $[0, 1]$, $\{(f_n)_{n < \omega} \in C([0, 1]) : (f_n) \text{ converges pointwise}\}$ and for $x \in [0, 1]$, $\{(f_n)_{n < \omega} \in C([0, 1]) : (f_n) \text{ converges pointwise to } x\}$ are all Π_1^1 .*
2. *In topology, using $\mathbb{R}$ representing the real number line, the set of compact $K \subseteq \mathbb{R}^2$ with no holes, the set path connected compact $K \subseteq \mathbb{R}^2$, and the set of Jordan Curves $J \subseteq \mathbb{R}^2$ are all Π_1^1 .*
3. *In the study of compact spaces, the set of quasiperiodic homeomorphisms of the Cantor set $\mathcal{C}$ is Π_1^1 .*
4. *In the study of Banach spaces, the class of non-universal separable Banach spaces and the class of separable Banach spaces with separable dual space are both Π_1^1 [12].*

More examples of such sets can be listed and appear all over mathematics and we could go on, but the point has been made. Hence, set theorists began looking for results which proved similar absoluteness for formula further up in the projective hierarchy.

3.2 Σ_3^1 absoluteness

Unfortunately, Shoenfield absoluteness does not extend any further than Σ_2^1 and Π_2^1 formula from the existence of $0^\#$, a Δ_3^1 subset of the natural numbers which cannot exist in L , but whose existence is implied by the existence of a measurable cardinal. Hence, absoluteness over all models of ZF cannot hold for Σ_3^1 formulas. $0^\#$ is an important concept in set theory and will be discussed in greater detail in a later section.

Furthermore, even 1-step Σ_3^1 generic absoluteness for V cannot be proven in ZFC due to a result of Jensen.

Theorem 3.2.1 (Jensen-Solovay [10]).

There exists an L -generic extension $L[a]$ such that a is a nonconstructable Δ_3^1 real in $L[a]$.

A real $a \in \mathbb{R}$ is a Δ_3^1 real if its existence is asserted by a formula Δ_3^1 formula. Hence there exists a Σ_3^1 statement which cannot be satisfied in L but which is satisfied in $L[a]$,

and so 1-step Σ_3^1 generic absoluteness implies $V \neq L$. Hence to ensure Σ_3^1 absoluteness, a large cardinal property is required. In fact, even the consistency of 1-step Σ_3^1 generic absoluteness requires a large cardinal property, *reflectiveness*.

Definition 3.2.2.

A cardinal κ is Σ_2 -**reflecting** [9] if the following hold:

1. κ is inaccessible.
2. For all Σ_2^0 formulas, for all $a \in V_\kappa$, $V_\kappa \models \varphi(a) \iff V \models \varphi(a)$

Σ_2 -reflecting cardinals are a far weaker large cardinal axiom than measurable cardinals, and are weaker than the statement $V \neq L$, which is weaker than $0^\#$.

Theorem 3.2.3 (Feng-Magidor-Woodin [7]).

The following statements are equiconsistent:

1. One-step Σ_3^1 generic absoluteness holds.
2. There exists a Σ_2 -reflecting cardinal.

However, this is not strong enough to prove one-step Σ_3^1 generic absoluteness in V , merely its consistency. In the same paper Feng, Magidor, and Woodin proved the following:

Theorem 3.2.4 (Feng-Magidor-Woodin [7]).

If there exists a proper class of measurable cardinals, then two-step Σ_3^1 -absoluteness holds.

The term ‘proper class of measurable cardinals’ refers to the measurable cardinals being unbounded in *Ord*. Hence, two-step Σ_3^1 absoluteness is not proven by a specific large cardinal property, but by the existence of so many cardinals of a certain type, they cannot be described by a set. We will give a proof of this theorem later in this thesis. Besides the absoluteness property, two-step Σ_3^1 generic absoluteness has effects on the theory of V itself.

Theorem 3.2.5 (Feng-Magidor-Woodin [7]).

The following are equivalent:

1. *Two-step Σ_3^1 generic absoluteness holds.*
2. *Every set has a sharp.*
3. *Every Δ_2^1 set is universally Baire.*

Similar to Shoenfield absoluteness, there are a wide variety of important unsolved Σ_3^1 statements which could benefit from this result. However, of more interest to the set-theoretic community, and especially descriptive set theorists is the question of *projective absoluteness*.

3.3 Projective Absoluteness

By the previous results, one would expect for each layer of the projective hierarchy to require a new type of large cardinal axiom. However, a paper by Kai Hauser [8] gives a powerful characterization for the consistency of two-step Σ_n^1 generic absoluteness for all $3 < n < \omega$. These results revolve around a new large cardinal property called *strong cardinals*.

Definition 3.3.1.

A cardinal κ is a **strong cardinal** if for every set $x \in V$, there exists an elementary embedding $j : V \rightarrow M$ with critical point κ such that $x \in M$ [16]

Strongness is a more powerful property than measurability, and a weaker property than being a Woodin cardinal. We begin with a result by Woodin.

Theorem 3.3.2.

Let $\kappa_1 < \kappa_2 < \dots < \kappa_n$ be strong cardinals. If G is a V -generic filter for $\text{Col}(\omega, 2^{2^{\kappa_n}})$, then $V[G]$ satisfies two-step Σ_{n+3}^1 generic absoluteness. [13]

Hence, the consistency strength of two-step Σ_{n+3}^1 absoluteness is $n-3$ strong cardinals. Hauser builds on this with the following result.

Theorem 3.3.3 (Hauser [8]).

Assume there exists no proper class inner model with a Woodin cardinal. If two-step Σ_{n+3}^1 generic absoluteness holds for $n \geq 1$, then there exist at least n strong cardinals. Furthermore, if projective absoluteness holds, there exist ω -many Woodin cardinals.

Hence, combining Kai and Woodin's results, we are able to determine the exact consistency strength of projective absoluteness.

Theorem 3.3.4 (The Projective Generic Absoluteness Theorem [8]).

The following theories are equiconsistent for all $1 \leq n < \omega$:

1. $ZFC + \text{"There exist } n \text{ strong cardinals"}$.
2. $ZFC + \text{"Two-step } \Sigma_{3+n}^1 \text{ generic absoluteness holds."}$

Furthermore, the following two theories are equiconsistent:

1. $ZFC + \text{"There exist } \omega\text{-many strong cardinals"}$.
2. $ZFC + \text{"Two-step projective generic absoluteness holds."}$

Hence, the consistency strength of projective generic absoluteness is known exactly, along with the consistency strength of two-step generic Σ_n^1 -absoluteness. Now set theorists can construct a model of ZFC for which projective generic absoluteness holds. However, while we now know the consistency strength of projective absoluteness, we have not yet established a sufficient large cardinal property to prove projective absoluteness. In order to discuss such a result, we first note that the existence of Woodin cardinals has a particularly powerful effect on the theory of projective sets, as they can be used to construct a sufficient condition for projective absoluteness.

Theorem 3.3.5 (Martin-Steel [20]).

Assume there exist ω -many Woodin cardinals. Then projective absoluteness holds.

Hence, Woodin cardinals are of interest to descriptive set theorists. However, a variation of the assumption in Theorem 3.3.5 can be used to show a far stronger result.

However, the new assumption revolves around iterable models, which we will describe in the next chapter.

Theorem 3.3.6 (Projective Absoluteness [6][36]).

Assume there is an iterable countable transitive model of ZFC which contains ω -many Woodin cardinals. Then two-step projective absoluteness holds.

This gives us a final large cardinal property which forces projective absoluteness for subsets of the reals. Since the assumption does not require a proper class of any kind of cardinal, it can be described stating the existence of specific set in V . The advantage of this is that one can now potentially build a canonical model for projective absoluteness. This canonical model can then be considered the “true” theory of the projective sets, fulfilling a major component of Gödel’s program.

Following this result, there are myriad directions in which to go, and many questions to answer. Is there a larger subset of $\mathcal{P}(\mathbb{R})$ for which one can prove a form of generic absoluteness? What kind of generic absoluteness results can one find about Σ_1^2 formulas? What about Σ_2^2 ? Are there any inner models for which generic absoluteness can be shown more easily? We will cover results pertaining to all these questions in this thesis. However, we first need to cover more advanced concepts and objects within set theory.

Chapter 4

Advanced Concepts

4.1 Statements of Interest

In set theory, there exist myriad statements about specific cardinals, sets, or trees which hold prominent place in studying models of set theory. We shall detail a few of them here.

4.1.1 Sharps

As mentioned earlier, $0^\#$ is a Δ_3^1 set of natural numbers whose existence in a model $\mathcal{M}$ of ZF has profound effects on what can be said about the constructability of the model.

Definition 4.1.1.

Let λ be a limit ordinal. Let $\mathcal{M} = (M, E)$ be a model elementarily equivalent to $(L_\lambda, \in)$.

A set $I \subseteq \text{Ord}^\mathcal{M}$ is a **set of indiscernibles for $\mathcal{M}$** if for every formula φ ,

$$\mathcal{M} \models \varphi(x_1, \dots, x_n) \iff \mathcal{M} \models \varphi(y_1, \dots, y_n).$$

Whenever $x_1 < \dots < x_n$ and $y_1 < \dots < y_n$ are elements of I . [9]

These indiscernibles are used to describe $0^\#$, which exists if and only if the following hold in V , which are properties described by Silver's Theorem.

Definition 4.1.2.

If V is such that:

1. If κ and λ are uncountable cardinals and $\kappa < \lambda$, then $(L_\kappa, \in)$ is an elementary submodel of $(L_\lambda, \in)$.
2. There exists a unique club class of ordinals I containing all uncountable cardinals such that for every uncountable cardinal κ :
 - (a) $|I \cap \kappa| = \kappa$.
 - (b) $|I \cap \kappa|$ is a set of indiscernibles for $(L_\kappa, \in)$.
 - (c) Every $a \in L_\kappa$ is definable in $(L_\kappa, \in)$ using only elements from $I \cap \kappa$. [9]

That is, $I \cap L_\lambda$ is a class of indiscernibles for every L_λ , with λ an uncountable cardinal, and every $a \in L$ is definable from $I \cap \kappa$, where κ is least uncountable cardinal greater than or equal to the rank of a in L .

If the above conditions hold for V , since $L_{\aleph_\omega}$ is an elementary submodel of L by the reflection theorem, and there exist infinite elements from I in $L_{\aleph_\omega}$, namely $\aleph_{n+1}$ for all $n < \omega$, we define $0^\#$ as follows [9].

$$0^\# := \{\varphi : L_{\aleph_\omega} \models \varphi(\aleph_1, \dots, \aleph_n)\}.$$

$0^\#$ is hence a set of formulas, but since the set of all formulas expressible in the language of set theory is countable, $0^\#$ can instead be represented as a subset of the natural numbers. Since $\mathcal{P}(\omega) \simeq 2^{\aleph_0}$ can also be equated with $\mathbb{R}$, $0^\#$ also can be considered a real. In fact, $0^\#$ can be described by a Δ_3^1 formula, and is hence a Δ_3^1 real. The most important fact about $0^\#$, however, comes from the following theorem of Kunen [9].

Theorem 4.1.3.

The following are equivalent:

1. $0^\#$ exists.

2. *There exists a nontrivial elementary embedding $j : L \rightarrow L$.* [9]

Note the embedding mentioned above need not be definable in L , but instead definable in the larger universe V . As a result, the existence of $0^\#$ is inconsistent with the statement “ $V = L$ ”. Furthermore, the existence of $0^\#$ is equivalent to the failure of a powerful statement named *Jensen’s Covering Lemma*.

Axiom 4.1.4 (Jensen’s Covering Lemma [4]).

For every uncountable set of ordinals X , there exists a constructible set $X \subseteq Y$ such that $|X| = |Y|$.

Hence, the nonexistence of $0^\#$ is equivalent to the idea that the cardinal structure of V can be approximated by the cardinal structure of L . $0^\#$ therefore takes a place in the large cardinal hierarchy as the ‘dividing line’ between large cardinals which are consistent with $V = L$ and large cardinals which imply $V \neq L$.

Similar to $0^\#$, for every set $x \in V$, one defines $x^\#$ by replacing L with $L(x)$ in the above definition of $0^\#$. That is, if $L_\lambda(x)$ satisfies the conditions listed for L_λ in the definition of $0^\#$ for every uncountable λ , then

$$x^\# = \{\varphi : L_{\aleph_\omega}(x) \models \varphi(\aleph_1, \dots, \aleph_n)\}.$$

Notably, every $x^\#$ is absolute for all transitive models of ZF containing $x^\#$ [9]. Hence, since the existence of $x^\#$ also implies that every element of $L[x]$ can be defined from a particular class of ordinals, the sharps constitute an important absoluteness property when they exist. In fact, $\mathbb{R}^\#$, when it exists, can be used to show a generic absoluteness property for $L(\mathbb{R})$, given sufficiently many Woodin cardinals [16]. We will discuss this more in our section on Sealing.

4.1.2 Jensen’s $\diamond$ Principle and the Nonstationary Ideal.

Another important principle in set theory is Jensen’s $\diamond$ principle, pronounced the **Diamond Principle**.

Axiom 4.1.5 (Jensen's $\diamond$).

There exists a sequence of sets $S = \langle S_\alpha : \alpha < \omega_1 \rangle$ with each $S_\alpha \subseteq \alpha$ such that for every $X \subseteq \omega_1$, the set $\{\alpha < \omega_1 : X \cap \alpha = S_\alpha\}$ is a stationary subset of ω_1 . We call S a $\diamond$ -sequence [24].

For each uncountable cardinal κ and stationary $E \subseteq \kappa$ there exists a variation of $\diamond$, $\diamond(E)$ which pertains to a sequence of sets indexed by E , however only $\diamond$ is considered in this thesis. Furthermore, there exists a strengthening of $\diamond$.

Axiom 4.1.6 (Jensen's $\diamond^+$).

A sequence $\langle S_\gamma : \gamma < \omega_1 \rangle$ such that for all $\gamma < \omega_1$, S_γ is countable and $S_\gamma \subseteq \mathcal{P}(\gamma)$ is a $\diamond^+$ -sequence if for every $A \subseteq \omega_1$, there exists a club set $C_A \subseteq \omega_1$ such that $\forall \alpha \in C_A$, $A \cap \alpha \in S_\alpha$ and $C_A \cap \alpha \in S_\alpha$ [24].

Jensen's $\diamond^+$ states "There exists a $\diamond^+$ -sequence.

$\diamond$ and $\diamond^+$ are important notions for Σ_2^2 generic absoluteness as they are Σ_2^2 statements whose existence has important consequences for the determinacy of *Neeman games*, and through that, Σ_2^2 absoluteness. Moreover, $V = L$ implies $\diamond$ and $\diamond$ implies CH . Hence, $\diamond$ and $\diamond^+$ are statements independent of ZFC .

Definition 4.1.7.

Let κ be a cardinal such that $cf(\kappa)$ is uncountable. The **nonstationary ideal on κ** is the ideal $I = \{X \subseteq \kappa : X \text{ is nonstationary in } \kappa\}$. The nonstationary ideal on ω_1 , denoted I_{NS} , is an item of special significance, and is usually simply referred to as **the nonstationary ideal** [9].

I_{NS} is an object of special significance in set theory as its properties can be used to tell set theorists much about V .

Definition 4.1.8.

Let κ be a cardinal. We say an ideal I is **κ -saturated** if every pairwise disjoint subset of elements in I has cardinality $< \kappa$. An ideal $I \subseteq \kappa$ is **saturated** if it is κ^+ -saturated [9].

The saturation of I_{NS} has profound effects on the theory of ZFC .

Theorem 4.1.9 (Shelah [29], Steel [37]).

The following are equiconsistent:

1. I_{NS} is saturated.
2. There exists a Woodin cardinal.

Another result about I_{NS} refers to its density. An ideal I of κ is **$\aleph_1$ -dense** if there exists a dense subset of the filter $F = \mathcal{P}(\kappa) \setminus I$ of cardinality $\aleph_1$.

Theorem 4.1.10 (Woodin [41]).

The following are equiconsistent:

1. I_{NS} is ω_1 -dense.
2. $L(\mathbb{R}) \models AD$.

I_{NS} is another object which is relevant to determining the feasibility of Σ_2^2 absoluteness, as $\Sigma_2^2(I_{NS})$ absoluteness causes problems when viewed through the lens of Ω -logic, which will be described in the next chapter.

4.2 Advanced Model Construction Methods

Further exploration of generic absoluteness requires study of more advanced concepts in set theory, specifically, models which cannot be constructed via the standard forcing or ultrafilter method.

4.2.1 Inner Models of Large Cardinals

A major focus of set theory revolves around the Inner Model Program, which is the project attempting to construct a minimal *inner model for φ* for each large cardinal property φ . A **minimal inner model for φ** can be considered the smallest inner model which satisfies φ . In essence, one wants the inner model to be as close to L as possible. A **canonical**

inner model is an inner model whose theory remains the same within whatever universe of ZF in which it can be represented. Therefore, finding the canonical inner model of a large cardinal property is even more desirable.

The quintessential canonical inner model is Gödel’s constructible universe L , however through new methods such as fine-structure theory, inner models have been constructed for large cardinal properties as powerful as Woodin cardinals [41].

Example 4.2.1.

Let κ be a measurable cardinal, and U be a normal κ -complete nonprincipal ultrafilter over κ . $L[U]$, the universe of sets constructible relative to U , is an example of a canonical inner model for a large cardinal property. Let $U' = U \cap L[U]$. We find that $L[U] \models “U' \text{ is a } \kappa\text{-complete normal principal ultrafilter on } \kappa”$. Hence, κ is measurable in $L[U]$. Furthermore, one can show that κ is the only measurable cardinal in $L[U]$ and $L[U] \models GCH$. There is no smaller model which contains a measurable cardinal, hence $L[U]$ is the minimal inner model for the existence of a measurable cardinal [9].

4.2.2 Generic Ultrapowers

Up to this point, we’ve only described ultrapowers of V using ultrafilters that exist within V . However, there is a way to create ultrapowers of V from *outside* V . If $\mathcal{M}, \mathcal{N}$ are models of ZFC such that $M \subseteq N$ and there exists $U \in N$ such that if U were in M , it would satisfy the properties of an ultrafilter, then U is a **$\mathcal{M}$ -ultrafilter**. All properties one would use to describe ultrafilters (e.g. normal, nonprincipal, κ -complete, etc) when applied to $\mathcal{M}$ -ultrafilters refer to it having that property according to the members of $\mathcal{M}$.

Definition 4.2.2.

*Let κ be a regular uncountable cardinal and let F be a filter (a set filter, not one for some forcing) on κ . We can view the subsets of κ which are in F as a partially ordered set when ordered by inclusion. That is, $\kappa = \mathbb{1}$, and for $X, Y \in F$, $X \leq Y \iff X \subseteq Y$. Let G be a V -generic filter for $\mathbb{P} = \langle F, \subseteq \rangle$. We call G a **V -generic ultrafilter extending F** in $V[G]$ [9].*

Lemma 4.2.3.

Let F be a filter on a regular uncountable cardinal κ and G be a V -generic ultrafilter from F . Then the following hold [9]:

1. G is a V -ultrafilter on κ extending F .
2. If F is κ -complete, then G is a κ -complete V -ultrafilter.
3. If F is normal, G is V -normal.
4. If F contains every cofinite subset of κ , then G is a nonprincipal V -ultrafilter.

Hence, since $G \in V[G]$, one can construct a **V-generic ultrapower** of V within $V[G]$ using G . This ultrapower is denoted Ult_G . By Loś' theorem, Ult_G is elementarily equivalent to V . However, Ult_G might not be well-founded. For an ideal I on κ and Ult_G is well-founded, where G is the generic ultrafilter extending the dual filter $F = \{X \in \mathcal{P}(\kappa) : \kappa \setminus X \in I\}$, then I is a **precipitous** ideal. Generic ultrapowers are important in the study of the stationary tower, which we will discuss later in this chapter. Most importantly, generic ultrapowers from precipitous ideals provide a way to construct otherwise impossible to create inner models using ultrafilters from outside V .

4.2.3 Direct Limits of Models

These methods almost all use the concept of the *direct limit* of a directed system of models to construct new inner models of ZF and ZFC .

Definition 4.2.4.

Let I be a directed set. A set of models $\langle M_i : i \in I \rangle$ indexed by a directed set I is a **directed system of models** if and only if for all $i \leq j \in I$, there exist an elementary embedding $j_{i,j} : M_i \rightarrow M_j$ such that

1. $j_{i,i}$ is the identity map.
2. For all $i < j < k$ in I , $j_{i,k} = j_{j,k} \circ j_{i,j}$.

Theorem 4.2.5 (Direct Limit [9]).

If $\langle M_i : i \in I \rangle$ is a directed system of models, there exists a model unique up to isomorphism, and elementary embeddings $j_i : M_i \rightarrow \varinjlim M_i$ such that $M = \bigcup_{i \in I} j_i[M_i]$ and $j_n = j_m \circ j_{n,m}$ for all $n \leq m$ in I . This model is known as the **direct limit** of $\langle M_i : i \in I \rangle$.

Note that the direct limit may not be well-founded, so additional work needs to be done for each direct limit to prove correspondence to an inner model in V .

4.2.4 Iterated Ultrapowers

If V contains a measurable cardinal κ , then there exists a nontrivial elementary embedding $j : V \rightarrow M$ such that $j(\kappa) > \kappa$ and $j(\kappa)$ is measurable in M . However, this also means in M , there exists another nontrivial embedding into a further inner model. This process can continue indefinitely in a way creating a sequence of ‘iterated’ ultrapowers.

Definition 4.2.6.

Let U be a κ -complete nonprincipal ultrafilter on κ in V . The **iterated ultrapower of U** is defined as follows for every $\alpha < \omega$:

- a) $(Ult^{(0)}, E^{(0)}) = (V, \in)$.
- b) $(Ult^{(\alpha+1)}, E^{(\alpha+1)})$ is the ultrapower of $Ult^{(\alpha)}$ over $U^{Ult^{(\alpha)}}$.

This sequence of inner models has the additional property that for all $\alpha < \beta < \omega$, if we set $i_{\alpha,\beta} = j_{\beta-1} \circ \dots \circ j_{\alpha+1} \circ j_\alpha$, where j_γ is the canonical elementary embedding $j : Ult^{(\gamma)} \rightarrow Ult^{(\gamma+1)}$, then $(Ult^{(\alpha)}, i_{\alpha,\beta})$ is a directed system of inner models. Hence, one can find the direct limit of $(Ult^{(\alpha)}, i_{\alpha,\beta})_{\alpha < \beta < \omega}$. Furthermore, one can show the direct limit is well-founded. This allows us to define transfinite inductive set to the iterated ultrapower, namely:

- c) For any limit ordinal α , $(Ult^{(\alpha)}, E^{(\alpha)})$ is the direct limit $\varinjlim_{\gamma < \alpha} (Ult^{(\gamma)}, E^{(\gamma)})$.

Hence we can iterate the ultrapower of V through every ordinal in V . Gaifman showed that if U is a κ -complete normal nonprincipal ultrafilter on κ , then $Ult^{(\alpha)}$ is well-founded for every $\alpha \in Ord$ [9].

However, this result only works for ultrafilters and ultrapowers of V . One can attempt to do the same for ultrapowers or generic $\mathcal{M}$ -ultrapowers for any model $\mathcal{M}$ of ZFC . In this case, however, some iterates may not be well-founded.

Definition 4.2.7.

Let $\mathcal{M}$ be a model of ZFC and let U be any $\mathcal{M}$ -ultrafilter. If for all α we let $\mathcal{M}^{(\alpha+1)}$ be the inner model induced by $U^{(\alpha)}$, where $U^{(\alpha)} = j_\alpha(U)$, j_α being is the canonical elementary embedding $j_\alpha : \mathcal{M} \rightarrow U^{(\alpha)}$, and if for all limit α we let $\mathcal{M}^{(\alpha)}$ be the direct limit of all $\mathcal{M}^{(\beta)}$ for $\beta < \alpha$, then the sequence $\langle \mathcal{M}^{(\alpha)}, U^{(\alpha)}, \in \rangle$ is **iterable** and U is an **iterable $\mathcal{M}$ -ultrafilter** if $\mathcal{M}^{(\alpha)}$ is well-founded for every $\alpha \in \text{Ord}$.

If we let κ be a measurable cardinal and U be a κ -complete nonprincipal normal ultrafilter over κ , then $L[U]$ from Example 4.2.1 is an example of an iterable model. Hence, the core model of a measurable cardinal already is iterable.

Iterated and iterable ultrapowers play an important part in **mice**, sophisticated iterated objects used by set theorists to construct inner models for various large cardinal properties.

4.2.5 Towers of Measures

Towers of Measures are constructions created utilizing infinite chains of coherent measures to create a new model of ZF from a direct limit.

Definition 4.2.8 (Towers of Measures [16]).

Let X be a nonempty set. Denote $\mathbf{m}(X)$ as the set of σ -additive measures (σ -complete ultrafilters) on $P(X)$. Let $X = Y^{<\omega}$ and $\mu \in \mathbf{m}(Y^{<\omega})$. Since μ is countably complete, there is a unique $n < \omega$ where $\mu(Y^n) = 1$.

Let $\mu_1, \mu_2 \in \mathbf{m}(Y^{<\omega})$, and $\mu_1(Y^{n_1}) = \mu_2(Y^{n_2}) = 1$. We say μ_1 **projects to** μ_2 if and only if $n_1 < n_2$ and for all $A \subseteq Y^{n_1}$,

$$\mu_1(A) = 1 \iff \mu_2(\{s \in Y^{n_2} : s \restriction n_1 \in A\}) = 1.$$

Recall for each $\mu \in m(X)$, there exists a canonical embedding $j_\mu : V \rightarrow M_\mu$ of the universe V into an inner model M_μ through an ultrapower of V by Loś' Theorem.

If $\mu_1, \mu_2 \in m(Y^{<\omega})$ are such that μ_1 projects to μ_2 , one can construct a canonical elementary embedding

$$j_{\mu_1, \mu_2} : M_{\mu_1} \rightarrow M_{\mu_2}$$

such that

$$j_{\mu_2} = j_{\mu_1, \mu_2} \circ j_{\mu_1}.$$

Definition 4.2.9.

Let $S = \langle \mu_k : k < \omega \rangle$ be a sequence of measures from $m(Y^{<\omega})$ such that for each $n < \omega$, $\mu_n(Y^n) = 1$. S is a **tower of measures** if for all $n < k$, μ_n projects to μ_k . The tower S is **countably complete** if for any sequence $\langle A_k : k < \omega \rangle$ such that for all $k < \omega$, $A_k \subseteq Y^k$ and $\mu_k(A_k) = 1$, there exists $f \in Y^\omega$ such that $f \restriction k \in A_k$ for all $k < \omega$ [16].

Towers of measures provide directed and elementary embeddings to an ω -sequence of inner models. Since each tower of measures S corresponds to a directed system $\langle M_{\mu_n} : n < \omega \rangle$ of models of ZFC , the direct limit of S is a proper class model of ZFC .

Fact 4.2.10.

Let S be a tower of measures. The direct limit of S is well-founded if and only if S is countable complete [16].

Hence, if S is countably complete, the transitive collapse of the direct limit of S is hence an inner model extending M_{μ_n} for all $\mu_n \in S$, and we are able to find a new inner model for each countably complete tower of measures which may not be able to be constructed via either forcing or a single ultrapower embedding.

We use towers of measures to define a new notion of trees on $\omega \times Z$, *weakly homogeneous trees*.

Definition 4.2.11 (Weakly Homogeneous Tree [16]).

Let λ be an ordinal and δ be a cardinal, $\lambda > 0$. A tree T on $\omega \times \lambda$ is **δ -weakly homogeneous** if there is a countable set $\tau \subseteq m(\lambda^{<\omega})$ of δ -complete measures such that for all

$x \in \mathbb{R}$,

$$x \in p[T] \iff (\exists \mu \in \tau^\omega)[\mu \text{ is countably complete} \wedge (\forall k < \omega)[\mu_k(T_{x \restriction k}) = 1]].$$

T is **$< \delta$ -weakly homogeneous** if it is γ -weakly homogeneous for all $\gamma < \delta$, and **weakly homogeneous** if it is ω_1 -weakly homogeneous.

Definition 4.2.12.

Let λ, δ be ordinals. A tree T on $\omega \times \lambda$ is **δ -homogeneous** if there exists a partial function $\pi : \omega^{<\omega} \rightarrow m(\delta^{<\omega})$ such that:

1. For each $s \in \text{dom}(\pi)$, $\pi(s)$ is a δ -complete measure with $\pi(s)(T_s) = 1$.
2. For each $x \in \mathbb{R}$, $x \in p[T]$ if and only if $x \restriction k \in \text{dom}(\pi)$ for each $k < \omega$ and $\langle \pi(x \restriction k) : k < \omega \rangle$ is a countably complete tower.

A tree is **$< \delta$ -homogeneous** if it is γ -homogeneous for all $\gamma < \delta$, and **homogeneous** if it is ω_1 -homogeneous [16].

Definition 4.2.13.

Let δ be a cardinal. A set $A \subseteq \mathbb{R}$ is **δ -[weakly] homogeneously Suslin** if $A = p[T]$ for some δ -[weakly] homogeneous tree T on $\omega \times \lambda$ for some ordinal λ . $A \subseteq \mathbb{R}$ is **$< \delta$ -[weakly] homogeneously Suslin** if it is γ -[weakly] homogeneously Suslin for all $\gamma < \delta$. $A \subseteq \mathbb{R}$ is **∞ -[weakly] homogeneously Suslin** if it is δ -[weakly] homogeneously Suslin for all cardinals δ , and $A \subseteq \mathbb{R}$ is **[weakly] homogeneously Suslin** if it is ω_1 -[weakly] homogeneously Suslin [16].

By result from Steel [19], every homogeneously Suslin set is determined. In addition, weakly homogeneous trees have a powerful property in the realm of generic absoluteness, which needs to be explained in terms of absolutely complementing trees.

Definition 4.2.14.

Let (S, T) be trees on $\omega \times \lambda$ for some ordinal λ . Let κ be a cardinal. (S, T) are **κ -absolutely complementing** if for all forcings $\mathbb{P}$ such that $|\mathbb{P}| < \kappa$ and for all V -generic

G for $\mathbb{P}$,

$$V[G] \models p[S] = \mathbb{R} \setminus p[T]$$

Theorem 4.2.15.

Let δ be a cardinal and let T be a δ -weakly homogeneous tree. There exists a tree S for T such that (T, S) are δ -absolutely complementing [35].

Hence, ∞ -weakly homogeneously Suslin sets have a powerful regularity property, as for all δ -weakly homogeneously Suslin $A \subseteq \mathbb{R}$, if we let (T, S) be δ -absolutely complementing trees such that $p[T] = A$ for any V -generic G for a forcing $\mathbb{P}$ of cardinality $< \delta$ one can define an analogue of A in $V[G]$, $A_G = (p[T_G])_G$. It can be shown that $S_G = S$ [16], and so we have that $(p[T_G])_G = \mathbb{R} \setminus (p[S_G])_G = \mathbb{R} \setminus (p[S])_G = (p[T])_G$. When examining the effects of Woodin cardinals, homogeneously and weakly homogeneously Suslin sets become central to studying *universally Baire* sets of reals, which will be discussed the next chapter.

4.2.6 Extenders

The final way of constructing a direct limit of models we will discuss in this thesis are known as *extenders*. We begin with the direct definition of an extender, then explain how one can be constructed using strong cardinals. First, however, one needs to describe coherent ultrafilters on $[\kappa]^{<\omega}$.

Definition 4.2.16 (Coherence [9]).

Let λ be a regular cardinal and $\{E_a : a \in [\lambda]^{<\omega}\}$ be a collection of ultrafilters such that for each $a \in [\lambda]^{<\omega}$, E_a is an ultrafilter on $[\lambda]^{|a|}$. For every $a, b \in [\lambda]^{<\omega}$ such that $a \subset b$, $b = \{\alpha_1, \dots, \alpha_n\}$, $\alpha_1 < \dots < \alpha_n$, and let $1 \leq i_1 < \dots < i_m \leq n$ such that $a = \{\alpha_{i_1}, \dots, \alpha_{i_m}\}$. Define $\pi_{b,a} : [\lambda]^{|b|} \rightarrow [\lambda]^{|a|}$ such that

$$\pi_{b,a}(\{\beta_1, \dots, \beta_n\}) = \{\beta_{i_1}, \dots, \beta_{i_m}\}, \quad (\beta_1 < \beta_2 < \dots < \beta_n).$$

The E_a 's are **coherent** if for all $b \supset a$, for all $X \in \mathcal{P}([\lambda]^{|a|})$

$$X \in E_a \iff \{s : \pi_{b,a}(s) \in X\} \in E_b.$$

Note that if one defines $i_{a,b} : \text{Ult}_{E_a} \rightarrow \text{Ult}_{E_b}$ such that $i_{a,b}([f]_{E_a}) = [f \circ \pi_{b,a}]_{E_b}$, then if the E_a 's are coherent, $(E_a, i_{a,b})$ is a directed system.

Definition 4.2.17 (Extender [9]).

Let κ be a cardinal and λ be an ordinal such that $\kappa \leq \lambda$. A set $E = \{E_a : a \in [\lambda]^{<\omega}\} \subseteq V_{\kappa+2}$ is a **(κ, λ) -extender** if the following hold:

1. For all $a \in [\lambda]^{<\omega}$, E_a is a κ -complete nonprincipal ultrafilter on $[\kappa]^{|a|}$ such that
 - (a) At least one E_a is not κ^+ -complete
 - (b) For each $\alpha \in \kappa$, at least one E_a contains the set $\{s \in [\kappa]^{|a|} : \alpha \in s\}$
2. The E_a 's are coherent.
3. For all $f : [\kappa]^{|a|} \rightarrow V$, if $\{s \in [\kappa]^{|a|} : f(s) \in \max(s)\} \in E_a$, then for some $b \supset a$,

$$\{t \in [\kappa]^{|b|} : (f \circ \pi_{b,a})(t) \in t\} \in E_b.$$
4. The direct limit $\text{Ult}_E := \varinjlim E_a$ is well-founded.

Clearly, if extenders exist, they can be used to construct inner models which cannot be constructed using the previous methods. Finding a reliable way to build extenders, however, is not as simple. In this instance, strong cardinals come to the rescue.

Theorem 4.2.18.

Let κ be a strong cardinal and let λ be an ordinal such that $\kappa \leq \lambda$. Let $j : V \rightarrow M$ be an elementary embedding with critical point κ such that $j(\kappa) > \lambda$. For each $a \in [\lambda]^{<\omega}$, let

$$E_a = \{X \in \mathcal{P}([\kappa]^{<\omega}) : a \in j(X)\}.$$

Let $E = \{E_a : a \in [\lambda]^{<\omega}\}$. Then E is a (κ, λ) -extender, called the **(κ, λ) -extender derived from j** . [9]

Extenders are an important part of modern set theory, and, along with iterable models, are used in constructing the aforementioned mice.

Furthermore, extenders are used to construct an object called the **Extender Algebra**, described by Farah in [6]. However, The Extender Algebra revolves around infinitary logic (that is, logic which allows infinite conjunction, and infinite strings of quantifiers) and there isn't time or space in this thesis to go into detail about it. Using the extender algebra and the **Genericity Iteration Theorem** [36], one can show a sufficient condition for projective absoluteness, as described in Theorem 3.3.6. In fact, the same conditions also imply both absoluteness and determinacy for $L(\mathbb{R})$, and hence is a far stronger criterion than simply the existence of ω -many Woodin cardinals.

4.3 The Stationary Tower

In this section we discuss a powerful construction used in forcing called *the Stationary Tower*, a forcing built around a generalization of closed unbounded (club) and stationary sets. Normally defined only for subsets of ordinals, the stationary tower is defined by sets which are *stationary in themselves*. The following definitions are from Larson [16].

Definition 4.3.1.

Let X be a set such that $\bigcup X \neq \emptyset$. A set $C \subseteq \mathcal{P}(X)$ is **closed unbounded (club)** in $\mathcal{P}(X)$ if there exists a function $F : [X]^{<\omega} \rightarrow X$ such that

$$C = \{z : F[[z]^{<\omega}] \subseteq z\}.$$

Definition 4.3.2.

Let X be a set such that $\bigcup X \neq \emptyset$. A set $C \subseteq \mathcal{P}(X)$ is **stationary in $\mathcal{P}(X)$** if for every function $F : [X]^{<\omega} \rightarrow X$ there exists a $z \in C$ such that $F[[z]^{<\omega}] \subseteq z$. X is **stationary** if it is stationary in $\mathcal{P}(\bigcup X)$.

Definition 4.3.3 (The Stationary Tower [16]).

Let κ be an inaccessible cardinal. The **Stationary Tower (up to κ)** is the set $\mathbb{P} = \{X \in$

$V_\kappa : X \text{ is stationary}$ with the following partial order: For $x, y \in \mathbb{P}$,

$$X \geq Y \iff \bigcup X \subseteq \bigcup Y \wedge (\forall z \in Y)[Z \cap (\bigcup X) \in X]$$

That is, Y is stronger than X if Y 'extends' X in terms of being a stationary set. We denote the stationary tower up to κ with $\mathbb{P}_{<\kappa}$. Note that for any two inaccessible cardinals $\kappa \leq \lambda$, $\mathbb{P}_{<\kappa} \subseteq \mathbb{P}_{<\lambda}$. Therefore, it is possible to define a proper class forcing which encompasses every stationary tower, $\mathbb{P}_\infty$.

$$\mathbb{P}_\infty := \bigcup_{\kappa \in \text{Ord}} \mathbb{P}_{<\kappa}$$

There also exists a variant of the stationary tower, named the *countable tower*.

Definition 4.3.4 (Countable Tower [16]).

Let κ be an inaccessible cardinal. If $\mathbb{P}_{<\kappa}$ is the stationary tower up to κ , then we define the **countable tower (up to κ)**. For every ordinal $\lambda < \kappa$, let $P_\lambda = \{X \cap V_\lambda : X \in \mathcal{P}_{\omega_1}(V_\kappa)\}$

$$\mathbb{Q}_{<\kappa} = \{a \in \mathbb{P}_{<\kappa} : a \subset P_{\sup((\cup a) \cap \kappa)}\}$$

One can find that $\mathbb{Q}_{<\kappa}$ has similar structural properties as $\mathbb{P}_{<\kappa}$, except for every $X \in \mathbb{Q}_{<\kappa}$, every element of X is countable.

Fact 4.3.5 ([16]).

Let κ be an inaccessible cardinal and G be V -generic for $\mathbb{P}_{<\kappa}$. Also let $X \in V_\kappa$ be nonempty. If we denote

$$U_X = \{\{Y \cap X : Y \in a\} : a \in G \wedge X \subseteq \bigcup a\}$$

Then U_X is a V -ultrafilter on $\mathcal{P}(X)$ extending the closed unbounded filter on $\mathcal{P}(X)$.

Moreover, for every $X, Y \in V_\kappa$ such that $X \subseteq Y$, there exists an elementary embedding $j_{X,Y} : M_X \rightarrow M_Y$, where M_X, M_Y are the corresponding inner models generated by collapsing $\text{Ult}_{U_X}, \text{Ult}_{U_Y}$. Hence, $(M_X, j_{X,Y})$ a directed system over $V_\kappa \setminus \{\emptyset\}$. We use (M_G, E_G) to denote the direct limit of $(M_X, j_{X,Y})_{X \in V_\kappa \setminus \{\emptyset\}}$, and $j_G : V \rightarrow M_G$ to represent

the corresponding elementary embedding.

Furthermore, for every $a \in \mathbb{P}_{<\kappa}$,

$$a \in G \iff j_G[\cup a] \in j(a).$$

Hence, each V -generic G over $\mathbb{P}_{<\kappa}$ can be used to create a model of ZF which is the direct limit of V -generic ultrapowers, all constructed from G in $V[G]$. The corresponding model (M_G, E_G) has powerful regular structural properties. However, it is not necessarily well-founded. This is where Woodin cardinals become relevant to the study of the stationary tower.

Theorem 4.3.6 ([16]).

Let δ be a Woodin cardinal. If G is V -generic for $\mathbb{P}_{<\delta}$, and $j_G : V \rightarrow (M_G, E_G)$ be the corresponding elementary embedding to the direct limit model of ZF generated by G in the previous lemma, then (M_G, E_G) is well-founded and closed under sequences of length $< \delta$.

Hence, with δ a Woodin cardinal, one can construct an inner model taken from the direct limit of generic V -ultrapowers in $V[G]$ for any V -generic G for $\mathbb{P}_{<\delta}$. Moreover, V -generic extensions on $\mathbb{P}_{<\delta}$ preserve that δ is Woodin [16], so it may be possible to construct iterated stationary tower forcing models. It is also notable that if there exist multiple Woodin cardinals, then the relevant direct limit models of the Woodin cardinals are coherent, and hence can create yet more direct limits of inner models if there exist infinite Woodin cardinals.

Woodin cardinals also have similar powerful regularity aspects in $\mathbb{Q}_{<\delta}$, most of which have to do with managing the existence of *semi-proper* subsets of $\mathbb{Q}_{<\delta}$. These concepts, however, are not as relevant to this thesis, so we will not go into depth about their inner workings.

In summary, the stationary tower represents an incredibly powerful system of forcings which, when combined with Woodin cardinals, is able to construct powerful models which we will find are able to lock in theories far larger than just the projective sets. We

will show just how far the stationary tower can go in constructing generic absoluteness results when we get to the section on universally Baire sets and Sealing.

4.4 Forcing Axioms

The final topics we will discuss in this section are the *forcing axioms*. These are axioms which are extensions to the Baire Category Theorem which have to do with preserving fragments of V -generic filters for forcings satisfying specific conditions. The forcing axioms each have their own consistency strength and can be used to prove specific forms of generic absoluteness, depending on what other large cardinal properties are assumed. We list the three main axioms in order of increasing consistency strength.

Axiom 4.4.1 (Martin's Axiom (MA)).

Let $\mathbb{P}$ be a forcing such that $\mathbb{P}$ satisfies the countable chain condition ($c.c.c.$), that is, there does not exist an uncountable antichain for $\mathbb{P}$. Let D be a collection of dense subsets of $\mathbb{P}$ such that $|D| < 2^{\aleph_0}$. Then there exists a filter G for $\mathbb{P}$ which is D -generic. That is, for all $D_\alpha \in D$, $D \cap G \neq \emptyset$.

For all $\kappa < 2^{\aleph_0}$, we use $MA(\kappa)$ to refer to the same statement, except $|D| \leq \kappa$. $< 2^{\aleph_0}$ is the largest possible value for such versions of MA , and hence is the one usually referred to by set theorists.

Definition 4.4.2.

*A forcing $\mathbb{P}$ is **proper** if for every uncountable cardinal λ , any stationary subset of $[\lambda]^\omega$ remains stationary in $V[G]$ for any V -generic G for $\mathbb{P}$.*

Axiom 4.4.3 (Proper Forcing Axiom (PFA)).

Let $\mathbb{P}$ be a proper forcing. Let D be a collection of dense subsets of $\mathbb{P}$ such that $|D| \leq \aleph_1$. Then there exists a filter G on $\mathbb{P}$ which is D -generic.

Axiom 4.4.4 (Martin's Maximum (MM)).

If $\mathbb{P}$ is a forcing which preserves stationary subsets of ω_1 , and $D = \langle D_\alpha : \alpha < \omega_1 \rangle$ is a collection of dense subsets of $\mathbb{P}$, then there exists a D -generic filter G for $\mathbb{P}$ in V .

The effect of the forcing axioms is that for any generic extension $V[G]$ of a forcing $\mathbb{P}$ which satisfies the relevant conditions, a filter F which represents a fragment of G can be found in V for every collection of dense subsets of $\mathbb{P}$ of cardinality $\leq \aleph_1$ (or $2^{\aleph_0}$, in the case of MA). These D -generic objects are useful for proving results within V from filters in $V[G]$. MM is the strongest of the forcing axioms. That is, there is no larger proper class of forcings where such a fragment of the generic filter can be preserved. MM is consistent relative to a supercompact cardinal, and MM implies $2^{\aleph_0} = \aleph_2$ and hence contradicts CH . However, while MM defines the largest possible class of forcings for which a forcing axiom can be defined, there exist stronger versions of MM .

Axiom 4.4.5 (Martin's Maximum⁺⁺ (MM^{++})).

If $\mathbb{P}$ is a forcing which preserves stationary subsets of ω_1 , $D = \langle D_\alpha : \alpha < \omega_1 \rangle$ is a collection of dense subsets of $\mathbb{P}$, and $\{\tau_i : i < \omega_1\}$ is a collection of names for stationary subsets of ω_1 , then there exists a D -generic filter G on $\mathbb{P}$ such that $\tau_i^G = \{\gamma : (\exists p \in G)[p \Vdash \gamma \in \tau_i]\}$ is stationary for every $i < \omega_1$.

A recent groundbreaking result by Asperó and Schindler [1] shows MM^{++} implies that Woodin's (*) axiom holds. Woodin's (*) axiom pertains to the behavior of forcings using $\mathbb{P}_{\max}$, a partial order defined over iterated ultrapower models of $ZFC + MA(\omega_1)$. $\mathbb{P}_{\max}$ was long considered a axiomatic rival to MM^{++} , due to it having a seemingly similar consistency strength along with different but desirable implications. The details of $\mathbb{P}_{\max}$ are beyond the scope of this thesis. However, Schindler's and Asperó's result unifying both the theories into one is such a significant result that it bears mentioning.

Chapter 5

Further Generic Absoluteness Results

5.1 Universally Baire Sets and Sealing

In referencing the earlier questions about moving beyond projective absoluteness, one may ask ‘What is the largest subset of $\mathcal{P}(\mathbb{R})$ whose theory can be made generically absolute?’

In pursuit of this question, we introduce the *universally Baire sets*.

5.1.1 Universally Baire Sets

Definition 5.1.1.

Let A be a subset of the reals, λ an infinite cardinal. A is **λ -universally Baire** if for every topological space X with a regular open basis of cardinality $\leq \lambda$, for every continuous function $f : X \rightarrow \omega^\omega$, $f^{-1}[A]$ has the property of Baire. That is, there is some open set D such that $D \Delta f^{-1}[A]$ is meager. A is **universally Baire** if it is λ -universally Baire for every infinite λ . We use uB to denote the set of all universally Baire sets $\{A \subseteq \mathbb{R} : A \text{ is universally Baire}\}$.

From this definition, one can show that universally Baire sets have the property of Baire and are Lebesgue measurable [7]. Furthermore, uB is a σ -algebra, and hence every

Σ_1^1 set is universally Baire. Feng, Magidor, and Woodin proved an equivalent definition of universally Baire sets which comes with powerful absoluteness properties.

Theorem 5.1.2 (Feng-Magidor-Woodin [7]).

Let κ be an uncountable cardinal, $A \subseteq \mathbb{R}$ is κ -universally Baire if and only if there exists a pair of κ -absolutely complementing trees (S, T) such that $p[S] = A$.

Corollary 5.1.3 (Feng-Magidor-Woodin [7]).

$A \subseteq \mathbb{R}$ is universally Baire if and only if for every uncountable cardinal κ , there exists a pair of κ -absolutely complementing trees (S, T) such that $p[S] = A$.

Hence, for all $A \in uB$, A is κ -Suslin for some infinite cardinal κ . Furthermore for every uncountable cardinal λ , all $< \lambda$ -weakly homogeneously Suslin sets are λ -universally Baire. Using these trees, for every $A \in uB$ and V -generic G for a set forcing we are able to define an analogue for A in $V[G]$.

Definition 5.1.4 (Feng-Magidor-Woodin [7]).

Let A be κ -universally Baire and (S, T) be a pair of κ -absolutely complementing trees for A . Let G be V -generic for some partially ordered set $\mathbb{P}$ of size $< \kappa$. We denote $A_G^{S, T} = (p[S])^{V[G]}$.

Recall by the definition of $p[T]$, we have that for all $x \in \mathbb{R}$, $x \in p[T] \iff T_x$ is ill-founded. Hence, if κ is an uncountable cardinal, $A \in uB$, and if (S, T) and (U, W) are both κ -absolutely complementing trees such that $p[S] = p[U] = A$ and G is a generic object for a forcing of cardinality $< \kappa$, we have for every x in $\mathbb{R}_G$

$$x \in A_G^{S, T} \iff x \in (p[T])_G \tag{5.1}$$

$$\iff V[G] \models T_x \text{ is ill founded.} \tag{5.2}$$

$$\iff V[G] \models S_x \text{ is well-founded.} \tag{5.3}$$

$$\iff V[G] \models W_x \text{ is well-founded.} \tag{5.4}$$

$$\iff V[G] \models U_x \text{ is ill founded.} \tag{5.5}$$

The second-to-last equivalence comes the fact that well-foundedness generically absolute. In fact, it is Δ_1 in the Lévy hierarchy [9]. Hence, the statement “ S_x is well-founded $\iff W_x$ is well-founded” is also Δ_1 . We will refer to this fact when we prove Shoenfield absoluteness. Hence, we can shorten the notation to of the analogue of A in $V[G]$ to simply A_G . Furthermore, for all $\lambda > \kappa$, we have any pair of λ -complementing trees is also κ -complementing. So for any V -generic G for a forcing $\mathbb{P}$ of size $< \lambda$, A_G is can be defined by the λ -complementing trees as well. This preserves a well-defined notion of A throughout all generic extensions of V .

However, just because for every $A \in uB$, there exists an analogue A_G for every V -generic G , the theory of A_G and A may not be the same. To address this, Woodin expanded upon a powerful generic absoluteness result about $L(\mathbb{R})$.

Theorem 5.1.5 (Steel-Woodin [16]).

Suppose there exists a proper class of Woodin cardinals. Then the following hold.

1. $L(\mathbb{R}) \models AD$. That is, every subset of $\mathbb{R}$ constructible from $TC(\{Ord, \mathbb{R}\})$ is determined by a strategy that is also constructible from $TC(\{Ord, \mathbb{R}\})$.
2. For any consecutive V -set generics G and H , there exists an elementary embedding

$$j : L(\mathbb{R}_G) \rightarrow L(\mathbb{R}_{G*H})$$

This is a powerful result which follows from the following three theorems, all taken from Larson’s book *The Stationary Tower* [16].

Theorem 5.1.6 (3.3.9 from *The Stationary Tower* [16]).

Suppose δ is a limit of Woodin cardinals and there exists a measurable cardinal above δ . Then $\mathbb{R}^\#$ is $< \delta$ -weakly homogeneously Suslin.

The proof of this uses stationary tower forcing and Theorem 4.3.6. Furthermore, note that if $\mathbb{R}^\#$ exists, every set of reals in $L(\mathbb{R})$ can be defined using $\mathbb{R}^\#$ and a subclass of the ordinals. Hence, the theorem actually shows every set of reals in $L(\mathbb{R})$ is $< \delta$ -weakly

homogeneously Suslin. However, one property of Woodin cardinals is for every Woodin κ , the set of measurable cardinals below κ is stationary. Hence, if there exists a class of Woodin cardinals, every limit of Woodin cardinals has a measurable above it.

Theorem 5.1.7 (3.3.12 from *The Stationary Tower* [16]).

Suppose δ is a limit of Woodin cardinals. The following are equivalent for every $A \subseteq \mathbb{R}$:

1. *A is $< \delta$ -weakly homogeneously Suslin.*
2. *A is $< \delta$ homogeneously Suslin.*
3. *A is $< \delta$ universally Baire.*

Therefore, if there exists a class of Woodin cardinals, every subset of the reals in $A \in L(\mathbb{R})$ is both universally Baire and ∞ -homogeneously Suslin, and hence determined. Furthermore, the strategy witnessing A is determined is also in $L(\mathbb{R})$.

The elementary embedding of $L(\mathbb{R})$ comes from a corollary of the same application of stationary tower forcing.

Theorem 5.1.8 (Corollary 3.1.16 from *The Stationary Tower* [16]).

If δ is a limit of Woodin cardinals and there exists a measurable cardinal above δ , then no generic extension for a filter $\mathbb{P} \in V_\delta$ can change the theory of $L(\mathbb{R})$ with real parameters.

This, combined with the fact that if δ is a Woodin cardinal, and G is V -generic for some $\mathbb{P} \in V_\delta$, then δ remains Woodin in $V[G]$, if there exists a proper class of Woodin cardinals in V , then there exists a proper class of Woodin cardinals in $V[G]$ for any V -generic G . Therefore, in every generic extension of V , the theory $L(\mathbb{R})$ remains unchanged assuming a class of Woodin cardinals. Woodin modified the proof of the original theorem to show that it extends to the theory of sets definable from $\mathbb{R}$ and an arbitrary $A \in uB$.

Theorem 5.1.9 (Steel-Woodin [34][16]).

Suppose there exists a proper class of Woodin cardinals. Let $A \in uB$. The following hold:

1. $L(A, \mathbb{R}) \models AD$.

2. For any G set generic in V and any H set-generic in $V[G]$, there exists an elementary embedding

$$j : L(A_G, \mathbb{R}_G) \rightarrow L(A_{G*H}, \mathbb{R}_{G*H})$$

such that $j(A_G) = A_{G*H}$

Every projective set can be written using a formula with terms in $L(\mathbb{R})$, so both projective absoluteness and projective determinacy is implied by a proper class of Woodin cardinals. Hence, this is a far stronger statement than projective absoluteness. Indeed, by a result of Steel [35], every set of reals that can be defined using a generically absolute formula is universally Baire [22]. Hence, the universally Baire sets are the largest subset of the reals can have a generic absoluteness property.

However, Woodin's result only applies to one universally Baire set at a time. In response, the desire to extend generic absoluteness to all universally Baire sets led to a new absoluteness statement: *Sealing*, which is currently near the forefront of the study of generic absoluteness.

5.1.2 Sealing

Definition 5.1.10 (Woodin).

In a universe V of ZF , **Sealing** holds if and only if the following are satisfied for every V -generic G for some set forcing $\mathbb{P}$:

1. $L(uB_G, \mathbb{R}_G) \models AD^+$.
2. $\mathcal{P}(\mathbb{R}) \cap L(uB_G, \mathbb{R}_G) = uB_G$
3. For every $V[G]$ -generic H for some set forcing $\mathbb{P} \in V[G]$, there exists an elementary embedding

$$j : L(uB_G, \mathbb{R}_G) \rightarrow L(uB_{G*H}, \mathbb{R}_{G*H})$$

such that for all $A \in uB_G$, $j(A) = A_H$

Sealing is a significantly more powerful absoluteness property than any previously mentioned in this thesis. Not only does it state that the theory of $L(uB, \mathbb{R})$ is generically absolute, but the second property assures every $A \in \mathcal{P}(\mathbb{R})^V$ which is ordinal-constructible uB_G and $\mathbb{R}_G$ is also universally Baire in $V[G]$. In essence, this ‘Seals’ the theory of universally Baire sets. In order to describe the consistency strength of Sealing, we need to introduce a new kind of cardinal.

Definition 5.1.11.

*A cardinal κ is **supercompact** if for every ordinal γ , there exists an elementary embedding $j : V \rightarrow M$ with critical point κ such that $j(\kappa) > \gamma$, and $M^\gamma \subseteq M$. That is, if $A = \langle x_\alpha : \alpha < \gamma \rangle \in V$ is such that $x_\alpha \in M$ for all $\alpha < \gamma$, then $A \in M$. In other words, M contains all its γ -sequences. Supercompactness is a stronger property than being a Woodin cardinal and every supercompact cardinal is Woodin.*

Theorem 5.1.12 (Woodin [16]).

Suppose there exists a proper class of Woodin cardinals and that κ is a supercompact cardinal. Let G be V -generic for $\text{Col}(\omega, 2^\kappa)$. Then $V[G] \models \text{Sealing}$.

Once again, Woodin came to this result using stationary tower forcing, and a detailed proof is given in Larson’s book [16].

As Sealing is an absoluteness property for the largest possible general classification of subsets of the reals for which a result is possible, the study of $L(uB, \mathbb{R})$ with Sealing is an active area of research among set theorists as many questions still remain unanswered. The most obvious one still being “Is there a large cardinal whose existence implies Sealing?”

An answer of ‘Yes’ has powerful implications for inner model theory, as if κ is a cardinal whose existence implies Sealing, the theory of the inner model of κ would be vastly different from any inner model previously known. In particular, unlike all other known inner models for large cardinals, $L(uB, \mathbb{R})$ would not contain a well-ordering of the reals. This is referred to as the *Sealing Dichotomy* [22].

So far, all large cardinals statements which imply Sealing purport the existence of a proper class of Woodin cardinals, which cannot be represented by a large cardinal property.

However, such a class is not necessary for Sealing to be consistent.

Theorem 5.1.13 (Sargsyan-Trang [27]).

Assume there exists a Woodin cardinal which is itself a limit of Woodin cardinals. Then Sealing is consistent.

Sargsyan and Trang proved this using a technique called *Core Model Induction beyond* $L(\mathbb{R})$ from a starting universe that assumes *PFA* holds for V and Jensen’s Covering Lemma holds for L , which together combine to imply the existence of $0^\#$. In this case, Sargsyan and Trang used Core Model Induction to find *maximal models of determinacy* within the universe [22]. Impressively, it does not require a class of Woodin cardinals, which was originally thought to be a necessary condition to ensure the theory of $L(\mathbb{R})$ is generically absolute. Sargsyan’s and Trang’s result hints towards an enticing possibility that through usage of more advanced techniques such as HOD mice and core model induction the consistency strength of many statements can be shown to be far lower than previously believed.

Examination of what sort of statements imply Sealing is still an active field of study. For example, the behavior of $L(uB)$ may provide a way to determine if Sealing holds, as in the following conjecture.

Conjecture 5.1.14.

The following theories are equivalent.

1. *There is a class of Woodin cardinals and all generic extensions satisfy the statement “The Axiom of Choice fails in $L(uB, \mathbb{R})$ ”.*
2. *Sealing holds.*

If the conjecture holds, it means that the implicative strength of a class of Woodin cardinals is far greater than previously believed, and that either AD^+ or Choice holds for $L(uB, \mathbb{R})$. Furthermore, it would mean that with a class of Woodin cardinals, the theory of $L(uB, \mathbb{R})$ only allows for generic non-absoluteness when it satisfies the Axiom

of Choice. Now is an exciting time to study Sealing and the effects. As possible new characterizations of Sealing and the whatever outcome of the Sealing Dichotomy turns out to be true would reveal a great deal about the workings of the large cardinal axioms and set theory in general.

5.1.3 The uB -Powerset and Extensions to Sealing

We now discuss the uB -powersets, a type of object only described in models which have Sealing, as well its possible effects on the universe and its relation to stronger versions of Sealing which seek to make nice the sets definable the uB -powerset of uB .

Definition 5.1.15 (Müller-Sargsyan [22]).

*If we let X be a set, the **uB -powerset of X** , denoted $\mathcal{A}_\infty(X)$, is the set of all $Y \subseteq X$ such that whenever G is V -generic for $\text{Col}(\omega, \max\{|X|, |uB|\})$, Y is ordinal definable in $L(uB_G, \mathbb{R}_G)$ from parameters belonging to the set $\{X, j_G''uB\} \cup j_G''uB$, where $j : L(uB, \mathbb{R}) \rightarrow L(uB_G, \mathbb{R}_G)$ is the canonical embedding described in Sealing. We denote $\mathcal{A}_\infty(uB)$ as uBp .*

The behavior of $\mathcal{A}_\infty(X)$ is of particular interest to those studying Sealing, as it may also pave the way to characterize Sealing itself. Not in the following conjecture, $cf(X)$ is least κ such that there exists a sequence $\langle X_\alpha : \alpha < \kappa \rangle$ of subsets of κ , such that for each $\alpha < \kappa$, $|X_\alpha| < |X|$, such that $\bigcup_{\alpha < \kappa} X_\alpha = X$.

Conjecture 5.1.16.

Assume the existence of a class of Woodin cardinals. If, for every set $X \subseteq uB$, $cf(\mathcal{A}_\infty(X)) = \omega$, then Sealing holds.

In fact, when studying Sealing in models satisfying MM , we may be able to show that $\mathcal{A}_\infty(X)$ is a canonical object for any set $X \supseteq uB$.

Conjecture 5.1.17.

If MM holds, then $cf(\mathcal{A}_\infty(X)) \in \{\omega, \omega_1\}$ for any $X \supseteq uB$.

The uB -powerset of uB is also an object of specific interest to those exploring Sealing, as it forms the basis of a stronger version of Sealing, called Weak Sealing for uBp .

Definition 5.1.18 (Weak Sealing for the uB -Powerset [22]).

*The statement “**Weak Sealing holds for the uB powerset**” is the conjunction of the following:*

1. *Sealing holds.*
2. $L(uBp) \models AD^+$.
3. *If G, G' are consecutive V -generics such that $V[G][G'] \models |(2^{2^\omega})^{V[G]}| = \aleph_0$, then there exists an elementary embedding $k : L(uBp_G) \rightarrow L(uBp_{G*G'})$ such that $k \restriction L(uB_G, \mathbb{R}_G) = j_{G, G'}$.*

If Weak Sealing for the uB powerset holds, the theory of $L(uBp)$ becomes generically absolute, as for any two consecutive V -generics G, G' , one can find a generic extension $V[H]$ of both $V[G]$ and $V[G][G']$ which collapses both $\mathcal{P}(\mathcal{P}(\mathbb{R}))_G$ and $\mathcal{P}(\mathcal{P}(\mathbb{R}))_{G*H}$ into being countable [22].

Similar to Woodin’s result about a supercompact cardinal and a class of Woodins implying the consistency of Sealing, Müller and Sargsyan found the consistency strength of Weak Sealing for the uB -powerset.

Theorem 5.1.19 (Müller-Sargsyan [23]).

Let κ be a supercompact cardinal and there exist a proper class of inaccessible limits of Woodin cardinals. Let G be V -generic for $Col(\omega, 2^{2^\kappa})$. Then Weak Sealing for the uB powerset holds in $V[G]$.

A fascinating aspect of such extensions created by collapsing powers of supercompact cardinals is the behavior of Θ in the relevant inner models. For example, in the generic extension described by Müller-Sargsyan, $\Theta^{L(uBp_G)}$ is a regular cardinal in $L(uBp_G)$ [22]. Similarly, in the generic extension Woodin created for Sealing by collapsing 2^κ with κ supercompact to being countable, $\Theta^{L(uB_G, \mathbb{R}_G)}$ is regular in $L(uB_G, \mathbb{R}_G)$ [22]. Furthermore, in

AD^+ , since $< \Theta$ determinacy holds, this reveals that the respective inner models $L(uBp_G)$ and $L(uB_G, \mathbb{R}_G)$ are incredibly far from satisfying choice, as for all cardinals λ below some regular cardinal, any continuous preimage from $\mathbb{R}$ into λ^ω is determined.

A stronger version of the theorem is **Sealing for the uB -powerset**, which is the same statement, except the third condition applies to all generic extensions, not just the ones that collapse 2^{2^ω} . Whether or not Sealing for the uB -powerset holds any model, or even the model described in Theorem 5.1.19 is still an open question [22].

The model $L(uBp)$ is of particular interest due the possibility that one might be able to force MM^{++} over the relevant models which satisfy AD^+ , something that has been a major source of interest for set theorists for years. However, in order to approach the possibility of finding such a model, one needs to discuss specific extensions to $L(uBp)$ which we call *Chang-like* models [23], which we will describe in a later section.

5.2 Beyond Projective Absoluteness

A natural question to ask after finding the consistency strength of projective absoluteness whether generic absoluteness goes any farther up the hierarchy. That is, are there next steps in the hierarchies which can be studied for generic absoluteness, that is, for Σ_1^2 statements and beyond? The answer, as it turns out, is a conditional 'yes'.

5.2.1 Σ_1^2 Absoluteness

The most famous Σ_1^2 statement, the Continuum Hypothesis, cannot be determined by large cardinals. Hence, one cannot find a large cardinal property which implies the existence of one-step Σ_1^2 absoluteness as one can with the projective hierarchy. Instead, a weaker form of generic absoluteness, *conditional generic absoluteness* is the primary object of study, one which is implied by a proper class of Woodin cardinals.

Theorem 5.2.1 (Σ_1^2 Absoluteness Conditioned on CH [42]).

Suppose there exists a proper class of measurable Woodin cardinals and CH . Let φ be a

Σ_1^2 sentence. Suppose $\mathbb{P}$ is a forcing such for any V -generic G over $\mathbb{P}$, $V[G] \models CH$. Then

$$V \models \varphi \iff V[G] \models \varphi.$$

This result follows from the following theorems in Larson's book:

Theorem 5.2.2 (3.2.1 from [16]).

Let $\varphi(x)$ be a Σ_1^2 formula and let $r \in \mathbb{R}$. Suppose that κ is a measurable Woodin cardinal. Then if $\mathbb{P}$ and $\mathbb{P}'$ are forcings in V_κ such that for any V -generic G for $\mathbb{P}$, $V[G] \models \varphi(r)$ and for any V -generic G' for $\mathbb{P}'$, $V[G'] \models CH$, then for any V -generic H for $\mathbb{P}'$, $V[H] \models \varphi(r)$.

Theorem 5.2.3 (3.3.10 from [16]).

Assume CH . If δ is a measurable Woodin cardinal, then every Σ_1^2 set of reals is $< \delta$ -weakly homogeneously Suslin.

Hence, a class of measurable Woodin cardinals not only implies Σ_1^2 absoluteness conditioned on CH , but also that every Σ_1^2 set is uB and determined.

Generic absoluteness condition on CH is not a general generic absoluteness result. Instead, CH becomes a condition upon which the absoluteness rests for all problems expressible as Σ_1^2 sentences. This is the most powerful form of generic absoluteness known for Σ_1^2 formulas.

5.2.2 Σ_2^2 Absoluteness and Strong Logics

In examining Σ_2^2 absoluteness, we seek to define it in terms of a conditional statement, like with Σ_1^2 absoluteness. In this case, the statement is Jensen's $\diamond$. No generic absoluteness result for Σ_2^2 conditioned on $\diamond$ has yet been discovered. In fact, there is good reason to believe that Σ_2^2 absoluteness does not hold from any generic absoluteness property [42]. Therefore, instead of using standard model-theoretic satisfaction to describe Σ_2^2 absoluteness all major generic absoluteness results for Σ_2^2 are expressed in terms of *strong logics*.

Definition 5.2.4 (Ω^* -Logic [42]).

Suppose there exists a proper class of Woodin cardinals and that φ is a sentence. Then for a theory T ,

$$T \vdash_{\Omega^*} \varphi$$

if for all ordinals α and for all V generic G for some forcing $\mathbb{P}$, if $V[G]_\alpha \models T$, then $V_\alpha[G] \models \varphi$.

In this way, Σ_1^2 absoluteness conditioned on CH can be reformulated as “Assuming a proper class of measurable Woodin cardinals, for every Σ_1^2 sentence φ , either $ZFC + CH \vdash_{\Omega^*} \varphi$ or $ZFC + CH \vdash_{\Omega^*} \neg\varphi$.”

Ω^* -logic is another way to describe conditional generic absoluteness and regular generic absoluteness. Using the universally Baire sets, there exists a different strong logic: Ω -logic.

Definition 5.2.5.

Suppose $A \in uB$ and M is a transitive set such that if $\mathcal{M} = (M, \in)$, then $\mathcal{M} \models ZFC$. $\mathcal{M}$ is **A-closed** if for every forcing $\mathbb{P} \in M$, if G is V -generic for $\mathbb{P}$, then $A_G \cap M[G] \in M[G]$.

Definition 5.2.6 (Ω -Logic [42]).

Assume there exists a proper class of Woodin cardinals, φ is a sentence, and T is a theory. Then $T \vdash_{\Omega} \varphi$ if there exists a set $A \in uB$ such that if M is an A -closed countable transitive set model of ZFC , then $\mathcal{M} \models \varphi$.

Both Ω -logic and Ω^* -logic are definable in ZFC and generically absolute. Woodin showed there exists a Ω -logic variant to Σ_1^2 absoluteness conditioned on CH assuming the existence of specific iterable models.

Theorem 5.2.7 (Woodin [42]).

Assume there exists a proper class of measurable Woodin cardinals. If there exists an iterable model of ZFC with measurable Woodin cardinals, then for each Σ_1^2 sentence φ , either $ZFC + CH \vdash_{\Omega} \varphi$ or $ZFC + CH \vdash_{\Omega} \neg\varphi$.

This theorem by Woodin led to his Ω Conjecture:

Conjecture 5.2.8 (Ω Conjecture [42]).

Assume there exists a proper class of Woodin cardinals and that φ is a Π_2^0 sentence. Then $ZFC \vdash_{\Omega^*} \varphi$ if and only if $ZFC \vdash_{\Omega} \varphi$.

This conjecture plays an important role in Σ_2^2 absoluteness due to an equivalent definition for Σ_2^2 formulas.

Fact 5.2.9.

A formula $\varphi(x)$ is Σ_2^2 if and only if there exists some Σ_2^0 formula $\psi(x)$ such that $\varphi(x)$ is provably equivalent in ZFC to the statement

$$x \in H(\mathfrak{c}^+) \wedge \langle H(\mathfrak{c}^+), \in \rangle \models \psi(x).$$

If $X \subseteq c$, then $\varphi(x)$ is $\Sigma_2^2(X)$ if and only if there exists some Σ_2^0 formula $\psi(x)$ such that $\varphi(x)$ is provably equivalent in ZFC to the statement

$$x \in H(\mathfrak{c}^+) \wedge \langle H(\mathfrak{c}^+), X, \in \rangle \models \psi(x).$$

Recall $H(\mathfrak{c}^+)$ refers to the class of hereditarily continuum-sized sets.

Woodin used the above fact to show a limit to Ω -logic. Recall I_{NS} refers to the nonstationary ideal.

Theorem 5.2.10.

Assume there exists a proper class of Woodin cardinals and ψ is a sentence. If for each $\Sigma_2^2(I_{NS})$ sentence φ , either $ZFC + \psi \vdash_{\Omega} \varphi$ or $ZFC + \psi \vdash_{\Omega} \neg\varphi$ holds, then $ZFC + \psi$ is Ω -inconsistent. [42]

Hence one cannot have $\Sigma_2^2(I_{NS})$ absoluteness with Ω -logic in ZFC . Therefore, if the Ω Conjecture holds, generic absoluteness for $\Sigma_2^2(I_{NS})$ sentences is impossible for any theory containing ZFC . Hence, if the Ω Conjecture is true, Σ_2^2 absoluteness is the best possible bound for generic absoluteness. Even if the Ω Conjecture is false, it seems likely that Σ_2^2

may be the best-possible conditional form of generic absoluteness, due to the relation of the Σ_2^2 statement $\diamond$ to specific *Neeman games*.

5.2.3 Σ_2^2 absoluteness and Neeman games.

Definition 5.2.11 (Neeman Games [42]).

For a formula $\varphi(x_1, \dots, x_n)$, define X_φ as the set of all $a \subseteq \{0, 1\}^{\omega_1}$ such that there exists a club set $C \subseteq \omega_1$ such that

$$\langle H(\omega_1), a, \in \rangle \models \varphi(\gamma_1, \dots, \gamma_n) \text{ for all } \gamma_1 < \dots < \gamma_n \in C.$$

The **Neeman game** for φ is the determinacy game of X_φ . That is, Player I and Player II take turns choosing elements from $\{0, 1\}$ for ω_1 steps. If the resulting sequence is in X_φ , then Player I wins. Otherwise, Player II wins.

Let $A \subseteq \mathbb{R}$. $X_{(\varphi, A)}$ is the set of $a \subseteq \{0, 1\}^{\omega_1}$ such that there exists a club set $C \subseteq \omega_1$ such that

$$\langle H(\omega_1), a, A, \in \rangle \models \varphi(\gamma_1, \dots, \gamma_n) \text{ for all } \gamma_1 < \dots < \gamma_n \in C.$$

The **A-Neeman game** for φ is the determinacy game of $X_{(\varphi, A)}$. Note that in Neeman and A-Neeman games, there are ω_1 -many steps, not ω -many.

The strength of the statement “All Neeman games are determined” is at most the existence of a Woodin limit of Woodin cardinals [42]. However, the determinacy of all A-Neeman games does not have a known consistency strength, and relates strongly to a form of generic absoluteness for Σ_2^2 .

Axiom 5.2.12 ($\diamond_G$ [42]).

For each Σ_2^2 sentence, $V \models \varphi$ if and only if for every V -generic G for $\text{Col}(\omega_1, \mathbb{R})$, $V[G] \models \varphi$.

Woodin describes $\diamond_G$ as a generic form of Jensen’s $\diamond$ [42]. Jensen’s $\diamond$ is the most significant of the Σ_2^2 sentences, which takes the place of CH when describing Σ_2^2 absoluteness conditioned on some formula in Ω -logic.

Theorem 5.2.13.

Assuming a proper class of supercompact cardinals. the following are equivalent [42]:

1. For each $A \in uB$, $ZFC + \Diamond_G \vdash_\Omega$ “All A -Neeman games are determined”.
2. For all $A \in uB$, for each Σ_2^2 formula $\varphi(x)$, either $ZFC + \Diamond_G \vdash_\Omega \varphi(A)$ or $ZFC + \Diamond_G \vdash_\Omega \neg\varphi(A)$.

The hope is to replace the instances of $\Diamond_G$ in the above result with simply $\Diamond$. In fact, one direction of the resulting conjecture has already been proven,

Theorem 5.2.14 ([42]).

Assume there exists a proper class of supercompact cardinals and for each $A \in uB$, $ZFC + \Diamond \vdash_\Omega$ “All A -Neeman games are determined”. Then for all $A \in uB$, for each Σ_2^2 formula $\varphi(x)$,

$$ZFC + \Diamond \vdash_\Omega \varphi(A) \vee ZFC + \Diamond \vdash_\Omega \neg\varphi(A).$$

However, if one extends the study to *stationary* A -Neeman games, which relate to $\Sigma_2^2(I_{NS})$ absoluteness, one runs into problems in inner model theory. First, we state some more definitions.

Definition 5.2.15 (Stationary Neeman Games [42]).

For a formula $\varphi(x_1, \dots, x_n)$, define Y_φ as the set of all $a \subseteq \{0, 1\}^{\omega_1}$ such that there exists a stationary set $S \subseteq \omega_1$ such that

$$\langle H(\omega_1), a, \in \rangle \models \varphi(\gamma_1, \dots, \gamma_n) \text{ for all } \gamma_1 < \dots < \gamma_n \in S.$$

The **Stationary Neeman game** for φ is the determinacy game of Y_φ . The **Stationary A -Neeman game** is defined analogously to the A -Neeman game.

Stationary Neeman games provide results about the conditional $\Sigma_2^2(I_{NS})$ - Ω -absoluteness analogous to the earlier results for the Ω -absoluteness of Σ_2^2 . Hence, if the Ω Conjecture holds, it is incredibly unlikely that stationary Neeman determinacy can be proven by some

large cardinal property. One Neeman and one Stationary Neeman game of particular interest, the **Canonical Function Game** and the **Stationary Canonical Function Game**, described by Woodin in [42], have powerful implications in how $\diamond$ and $\diamond^+$ relate to inner model theory.

Lemma 5.2.16 ([42]).

Suppose $\diamond$ holds. Then Player II has a winning strategy in the Canonical Function Game. Suppose $\diamond^+$ holds. Then Player II has a winning strategy in the Stationary Canonical Function Game.

However, a theorem about huge cardinals seems to work in opposition to the above lemma.

Definition 5.2.17.

*A cardinal κ is a **huge cardinal** if there exists an elementary embedding $j : V \rightarrow M$ with critical point κ such that*

$$M^{j(\kappa)} \subseteq M.$$

Hugeness is a far stronger property than supercompactness, only below superhuge, n -huge, and I3, I2, I1 and I0 cardinals, which are currently the strongest large cardinal properties believed consistent with Choice.

Theorem 5.2.18.

Suppose there exists a huge cardinal. Then there exists a forcing $\mathbb{P}$ such that for any V -generic G for $\mathbb{P}$, Player I has a winning strategy for the Stationary Canonical Function Game.

Theorem 5.2.18 and Lemma 5.2.16 strongly hint that there is no large cardinal property φ which is a conditional form of $\Sigma_2^2(I_{NS})$ generic absoluteness conditioned on $\diamond^+$, as this would also force the determinacy of the Stationary Neeman Games, and specifically the Stationary Canonical Function Game. If there was such a large cardinal property, then that property would have to be stronger than hugeness, and would force $\diamond^+$ to always

be false, causing our desired generic absoluteness result to fall into irrelevancy. In fact, the existence of a measurable cardinal implies there exists a generic extension $V[G]$ in which the stationary canonical function game is not determined [42]. Hence, if the Ω Conjecture does hold, set theorists need to be on careful in exploring the study of $\Sigma_2^2(I_{NS})$ absoluteness.

5.3 The Chang Model

In this section, we focus on a particular inner model of interest. The *Chang Model*.

Definition 5.3.1 (The Chang Model [16]).

The **Chang model**, denoted $L(Ord^\omega)$ is constructed as follows:

$$L(Ord^\omega) = \bigcup_{\gamma \in Ord} L(\mathcal{P}_{\omega_1}(\gamma)).$$

$L(Ord^\omega)$ is the smallest inner model containing every countable sequence of ordinals in V .

Originally, however, the Chang model was described as the analogue of Gödel's constructible L under $\mathcal{L}_{\omega_1\omega_1}$ logic. That is, logic which allowed for countably infinite blocks of conjunctions, disjunctions and quantifiers, where $L(Ord^\omega)$ was constructed in the same manner as Gödel's L , except at the successor stage, all sets which can be defined in $\mathcal{L}_{\omega_1\omega_1}$ in the previous step are included in the next layer of the Chang Hierarchy.

From these equivalent definitions, every element of the Chang model is Σ_2^0 definable within $L(Ord^\omega)$ from a countable sequence of the ordinals.

Theorem 5.3.2.

If there exists uncountably many measurable cardinals, then the Axiom of Choice fails in $L(Ord^\omega)$.

The proof of this does not show that $L(Ord^\omega) \models AD$, but instead shows that if $\langle \kappa_\alpha : \alpha < \omega_1 \rangle$ is a strictly increasing sequence of measurable cardinals, then if $\lambda = \sup\{\kappa_\alpha : \alpha < \omega_1\}$, there does not exist a well-ordering of λ^ω . In fact, using the same

technique as the proof, for any cardinal κ , if there exist κ^+ measurable cardinals, then

$\bigcup_{\kappa < \gamma \in Ord} L(\mathcal{P}_{\kappa^+}(\alpha))$ does not satisfy Choice either [16].

5.3.1 Generic Absoluteness in the Chang Model

The Chang model is of particular interest as it has powerful regularity properties implied by large cardinals. Furthermore, since $\mathbb{R} = \omega^\omega$, every real in V is contained in the Chang model. That is $L(\mathbb{R}) \subseteq L(Ord^\omega)$. The study of the generic properties of the Chang model begins with one of the earliest applications of forcing. Specifically, the following theorem of Solovay [33].

Theorem 5.3.3.

Let κ be an inaccessible cardinal and G be V -generic for $Col(\omega, < \kappa)$. Then every set of reals in $V[G]$ which is definable from a countable sequence of ordinals is Lebesgue measurable, satisfies the perfect set property, and has the property of Baire.

Notably, after taking the Lévy collapse of an inaccessible cardinal, the resulting Chang model within $V[G]$ has every set of reals satisfy the relevant properties. This pairs nicely with the following theorem.

Theorem 5.3.4 ([16]).

Let κ be a Woodin cardinal which is the limit of Woodin cardinals. Let G be V -generic for $Col(\omega, < \kappa)$. Then there exists an elementary embedding

$$j : L(Ord^\omega) \rightarrow L((Ord^\omega)_G).$$

That is, the theory of $L(Ord^\omega)$ is unchanged by any V -generic G for forcings $\mathbb{P}$ such that $|\mathbb{P}| < \kappa$. Furthermore, in $V[G]$, κ is still a Woodin limit of Woodin cardinals. Hence, in $V[G]$, the theory of the Chang model cannot be changed by forcings of cardinality $< \kappa$ either. Hence, the existence of a Woodin limit of Woodin cardinals implies the theory of $L(Ord^\omega)$ is generically absolute over forcings of cardinality less than κ . Once again, this theorem was proved by Woodin using stationary tower forcing, however in this case he

used the countable tower, and the properties a Woodin cardinal κ has when forcing with $\mathbb{Q}_{<\kappa}$.

Corollary 5.3.5.

Let κ be a Woodin limit of Woodin cardinals. Then no forcing in V_κ can change the theory of the Chang model with real parameters.

Furthermore, combining with the first theorem in this section, we have a sufficient large cardinal axiom for all the reals in $L(\text{Ord}^\omega)$ having desirable properties.

Corollary 5.3.6.

Assume there exists a Woodin limit of Woodin cardinals. Then every $A \in \mathcal{P}(\mathbb{R}) \cap L(\text{Ord}^\omega)$ is Lebesgue measurable, satisfies the perfect set property, and has the property of Baire.

The above theorem also dovetails into a large cardinal hypothesis which implies the generic absoluteness of the Chang model, that is, the theory of the sets definable from countable sequences of ordinals.

Theorem 5.3.7.

Suppose there exists a proper class of Woodin limits of Woodin cardinals. Then for all consecutive V -generic G, H for set forcings, there exists an elementary embedding

$$j : L((\text{Ord}^\omega)_G) \rightarrow L((\text{Ord}^\omega)_{G*H})$$

That is, the theory $L(\text{Ord}^\omega)$ with real parameters cannot be changed by generic extensions from set forcings.

The fact that the theory does not change from consecutive generic extensions is that δ being a Woodin cardinal is absolute among generic extensions for forcings in V_δ . Since this relies on a class of Woodin limits of Woodin cardinals, the fact that there exists such a proper class does not change under any set generic extension.

Fascinatingly, the previous two corollaries still hold when one replace “A Woodin limit of Woodin cardinals” with “A strongly compact cardinal”.

Definition 5.3.8.

A cardinal κ is **strongly compact** if for each λ there exists an elementary embedding $j : V \rightarrow M$ with critical point κ such that $M^\kappa \subseteq M$ and such that there exists $X \in M$ such that $|X|^M < j(\kappa)$ and $j[\lambda] \subseteq X$.

Strong compactness is implied by supercompactness, and is a stronger large cardinal property than being a Woodin cardinal. All strongly compact cardinals are measurable.

Hence, we have a more succinct large cardinal hypothesis implying Chang model generic absoluteness.

5.3.2 Chang-like Models and Absoluteness

Due to the absoluteness properties of the Chang model, a new field of study has emerged examining models with a similar structure to the Chang model, but containing more complicated objects.

One such model is $L(Ord^{\omega_1})$, which is the smallest transitive inner model containing all ω_1 -length sequences of ordinals.

A recent result by Viale [39] categorizes conditions under which stationary set preserving forcing cannot change the theory of $L(Ord^{\omega_1})$. Viale's result revolves around MM^{+++} , a strengthening of MM^{++} . The particulars of MM^{+++} are beyond the scope of this thesis, but revolve around category forcing on the category of stationary set preserving forcings. A summary by Asperó and Schindler describes it sufficiently for our purposes:

“It $[MM^{+++}]$ says a class $\mathbb{T}$ of towers of ideals with certain nice structural properties is dense in the category of stationary set preserving forcings; in other words, for every stationary set preserving forcing $\mathbb{P}$, there is a tower $\mathcal{T} \in \mathbb{T}$ such that $\mathbb{P}$ completely embeds into $\mathcal{T}$ in such a way that any V -generic extension for the quotient forcing preserves stationary sets. The axiom MM^{+++} implies MM^{++} .” [1]

The consistency strength of MM^{+++} is that of an *almost super-huge cardinal* [1].

Definition 5.3.9.

A cardinal κ is **almost super-huge** if for every $\gamma \in Ord$ there exists an elementary

embedding $j : V \rightarrow M$ with critical point κ , $\gamma < j(\kappa)$, and $M^{<j(\kappa)} \subseteq M$

Almost super-hugeness is a slightly weaker property than hugeness and is stronger than supercompactness.

Furthermore, if there exists a class of almost super-huge cardinals, then whether or not MM^{+++} holds in the ω_1 -Chang model is provable in the theory of the model, which is incredibly important when considering Viale's theorem.

Theorem 5.3.10 (Viale [39]).

Assume a class of Σ_2^0 -reflecting cardinals exists. The theory of $L(\text{Ord}^{\omega_1})$ conditioned on MM^{+++} cannot be changed by stationary set preserving forcing. That is, assuming MM^{+++} and a class of Σ_2^0 -reflecting cardinals, whenever G is V -generic for a forcing $\mathbb{P}$ such that $\mathbb{P}$ preserves stationary sets and such that $V[G] \models MM^{+++}$, then $L(\text{Ord}^{\omega_1})$ is elementary equivalent to $L((\text{Ord}^{\omega_1})_G)$

This result, along with the preexisting generic absoluteness properties for the Chang model have driven set theorists to search for models in which one can reduce the study of $L(\text{Ord}^{\omega_1})$ under MM^{+++} to a definable universe, such as with Müller and Sargsyan's paper on absoluteness results for Chang models [23], or for *Chang-like* models, such as the *Chang-type uB -powerset model*.

The absoluteness properties of Chang-type models are also of interest when trying to find ways to force MM^{++} over the relevant models which satisfy AD^+ , as we mentioned in our discussion on $L(uBp)$. The issue is, $L(uBp)$ is probably not a large enough inner model to suffice for such a search, and by a conjecture of Müller-Sargsyan, a model of interest would be the Chang-like model $L(\mathcal{P}_{\omega_1}(uBp))$. By Müller-Sargsyan, however, the only way $L(\mathcal{P}_{\omega_1}(uBp))$ would be able to be studied is if $cf(\sup(\text{Ord} \cap uBp)) = \omega$ [22].

Finally Chang-like models are relevant to a variant of Sealing, known as the *Sealing Theorem for φ* , where φ is some formula. For the following definition, for any formula ϕ , define $M_\phi = \{x : V \models \phi(x)\}$, and use φ to denote some formula.

Definition 5.3.11 (The Sealing theorem for φ [22]).

*The **Sealing Theorem holds for φ** (or for M_φ) if the following are true:*

1. M_φ is a transitive model of AD^+ which contains ever $x \in \mathbb{R}$.
2. For all V -generic G , $V[G] \models \mathcal{P}(\mathbb{R}) \cap M_\varphi \in uB$.
3. For all consecutive V -generic G, H , there exists an elementary embedding $j : (M_\varphi)_G \rightarrow (M_\varphi)_{G*H}$ such that for every $A \in \mathcal{P}(\mathbb{R})^{(M_\varphi)_G}$, $j(A) = A_H$. That is, the elementary embedding follows the generic extension representations of the universally Baire sets.

Definition 5.3.12.

Let X be a set. η_X denote the least ordinal such that there does not exist a surjective $f : \mathcal{P}(\mathbb{R}) \rightarrow \eta_X$ which is ordinal definable from X . Let $\eta = (\eta_{uB})^{L(uB, \mathbb{R})}$. $C - uBp = L(\eta^\omega, uBp)$ is the **Chang-type uB powerset model** [22].

Conjecture 5.3.13.

Let κ be a supercompact cardinal and let there exist a proper class of inaccessible limits of Woodin cardinals. Suppose G is V -generic for $Col(\omega, 2^{2^\kappa})$, and $\eta_G := (\eta_{uB_G})^{L(uB_G, \mathbb{R}_G)}$. Then $cf(\eta_G) = \omega$ and the Sealing Theorem holds for $C - uBp$ in $V[G]$. [22]

Chapter 6

Proofs

6.1 Shoenfield Absoluteness

In this section we prove Shoenfield's absoluteness theorem. However, first we need to state the following basic facts about Borel sets and closed sets in $\mathbb{R}^n$ (where we are considering $\mathbb{R}$ as the Baire space).

Fact 6.1.1 (Proposition 12.6 in [11]).

A set $A \subseteq \mathbb{R}^n$ is Σ_n^1 if and only if for some $a \in \mathbb{R}$, it is defined by some $\Sigma_n^0(a)$ formula R such that

$$(x_1, \dots, x_n) \in A \iff R(x_1, \dots, x_n, a).$$

A similar result is true for Π_n^0 sets and $\Pi_n^0(a)$ formulas.

Hence, by our definition of the projective hierarchy, $A \subseteq \mathbb{R}$ is analytic if and only if there exists some $n < \omega$ and some Σ_n^0 relation R such that

$$x \in A \iff \exists(y \in \mathbb{R})[R(x, y)].$$

That is, A is defined by some Σ_1^1 formula.

Fact 6.1.2 (Proposition 12.10 in [11]).

A set $C \subseteq \mathbb{R}^n$ is closed if and only if there exists some tree T on ω^n such that $C = [T]$.

Hence we have that since $A \subseteq \mathbb{R}$ is Σ_1^1 if and only if there exists some closed $C \in \mathbb{R}^2$ such that $A = \{x \in \mathbb{R} : (\exists y \in \mathbb{R})[(x, y) \in C]\}$ (this is a basic fact of descriptive set theory [12]), we have that since $C = [T]$ for some tree T on $\omega \times \omega$, $A = p[T]$, and hence is ω -Suslin. This gives us the equivalent characterization of analytic sets in Theorem 2.5.7.

We therefore have that every Π_1^1 set $A \subseteq \mathbb{R}$ is equal to $\mathbb{R} \setminus B$ for some analytic B . Therefore, since there exists a tree T on $\omega \times \omega$ such that $x \in B \iff "T_x \text{ is ill-founded}."$ We have

$$x \in A \iff "T_x \text{ is well-founded}."$$

This is the **Normal Form Theorem for Π_1^1** in Jech [9].

We need one more fact to prove our version of Shoenfield, taken from Kanamori, based on the *Kleene-Brouwer ordering*.

Definition 6.1.3 (Kleene-Brouwer Ordering [11]).

Let λ be an ordinal. The **Kleene-Brouwer on $\lambda^{<\omega}$** is the ordering defined as follows. For all $s, t \in \lambda^{<\omega}$,

$$s <_{KB} t \iff (s \supseteq t) \vee (s_i < t_i \text{ for the least } i < \omega \text{ such that } s_i \neq t_i).$$

This is a strict linear ordering which extends to $\text{Ord}^{<\omega}$.

Fact 6.1.4 (Luzin-Sierpinski [31]).

A tree T on an ordinal λ is well-founded if and only if $<_{KB} \restriction T$ is a well-ordering of T . Therefore, T is well-founded if and only if there exists an order-preserving function $f : T \rightarrow \lambda + 1$ constructible from T .

This is because under $<_{KB}$, T is well-founded if and only if there exists no infinitely decreasing chain in T .

Theorem 6.1.5 (Theorem 25.18 from [9]).

Every Σ_2^1 set is ω_1 -Suslin. If A is $\Sigma_2^1(a)$, then there exists a tree $T \in L[a]$ such that $A = p[T]$.

We prove this fact for Σ_2^1 statements using Jech's approach [9].

Proof.

Let $a \in \mathbb{R}$ and let $A \subseteq \mathbb{R}$ be $\Sigma_2^1(a)$. By our previous work, there exists a tree T on $\omega \times \omega \times \omega$ such that

$$x \in A \iff (\exists y \in \mathbb{R})[T_{x,y} \text{ is well-founded.}]$$

However, a tree S is well-founded if and only if it admits an order-preserving map $f : S \rightarrow \omega_1$. Hence, we must have that

$$x \in A \iff \exists(y \in \mathbb{R})[(\exists f : \omega^{<\omega} \rightarrow \omega_1)[f \upharpoonright T_{x,y} \text{ is order-preserving.}]]$$

Let $\langle s_n : n < \omega \rangle$ be an Δ_1^0 enumeration of $\omega^{<\omega}$ such that for all $n < \omega$, $|s_n| \geq n$. For every partial function $f : \omega \rightarrow V$, use $f^* : \omega^{<\omega} \rightarrow V$ to denote the corresponding partial function such that $f^*(s_n) = f(n)$. Then that the previous equivalence can be rewritten as

$$x \in A \iff (\exists y \in \mathbb{R})[(\exists f : \omega \rightarrow \omega_1)[f^* \upharpoonright T_{x,y} \text{ is order-preserving.}]]$$

If we for every $s, t \in \omega^\omega$ we define the set $T(s, t) = \{u \in \omega^{<\omega} : (s \upharpoonright_{|u|}, t \upharpoonright_{|u|}, u) \in T\}$, we can use the above notions to construct a new tree, T' , on $\omega \times \omega \times \omega_1$ such that for all $(s, t, f) \in \omega^{<\omega} \times \omega^{<\omega} \times \omega_1^{<\omega}$,

$$(s, t, h) \in T' \iff (|s| = |t| \wedge |s| = |h|) \wedge (h^* \upharpoonright T(s, t) \text{ is order-preserving}).$$

It is easy to see that T' is closed under subsequences, making it a tree on $\omega^2 \times \omega_1$. Note that if $x, y \in \mathbb{R}$, $h \in \omega_1^n$ are such that $(x \upharpoonright n, y \upharpoonright n, h) \in T'$ then $h^* \upharpoonright T_{x,y}$ is an order preserving partial function on $T_{x,y}$, as whenever $s, t \in \text{dom}(h^*) \cap T_{x,y}$, we have that under the previous enumeration $s = u_i, t = u_j$ for some $i, j < n$, and so $|s|, |t| < n$. Therefore $s, t \in T(x \upharpoonright n, y \upharpoonright n)$. Hence for every $f \in \omega_1^\omega$

$$f \text{ is a branch of } T'_{x,y} \iff \forall(n < \omega)[(f \upharpoonright n)^* \upharpoonright T_{x,y} \text{ is order-preserving}].$$

Which is equivalent to

$$f \text{ is a branch of } T'_{x,y} \iff f^* \restriction T_{x,y} \text{ is order-preserving.}$$

Hence, we have the following equivalence

$$\begin{aligned} x \in A &\iff (\exists y \in \mathbb{R})(\exists f \in \omega_1^\omega)[f \text{ is a branch of } T'(x, y)] \\ &\iff (\exists y \in \mathbb{R})(\exists f \in \omega_1^\omega)(\forall n < \omega)[(x \restriction n, y \restriction n, f \restriction n) \in T'] \end{aligned}$$

Hence $A = p[T']$. However, T' is a tree on $\omega^2 \times \omega_1$. Therefore, if we let $g : \omega \times \omega_1 \rightarrow \omega_1$ be a constructible injective map, we can define a new tree T'' on $\omega \times \omega_1$ such that $(x, y) \in T'' \iff (x, z, w) \in T' \wedge y = g(z, w)$, we have that T'' is a tree on $\omega \times \omega_1$ which is constructible from T such that $p[T''] = A$.

Since $T \in L[a]$, we must have that $T'' \in L[a]$, and so A is ω_1 -Suslin from a tree which exists in $L[a]$. [9] □

Note that in the above proof, we do not state that A is ω_1 -Suslin in $L[a]$, merely that there exists a tree T'' in $L[a]$ on $\omega \times \omega_1^V$ such that $V \models A = p[T'']$.

Theorem 6.1.6 (The Shoenfield Absoluteness Theorem [11]).

For any $a \in \mathbb{R}$, any $\Sigma_2^1(a)$ relation is absolute across all inner models $\mathcal{M}$ of ZF such that $a \in \mathcal{M}$.

To prove this version of Shoenfield Absoluteness, we use a modification of Kanamori's proof in [11].

Proof.

Let $A \subseteq \mathbb{R}$ be $\Sigma_2^1(a)$. Let $T \subseteq \omega^2 \times \omega$ such that for any $x \in \mathbb{R}$,

$$x \in A \iff (\exists y \in \mathbb{R})[T_{x,y} \text{ is well-founded}].$$

Since we know from Fact 6.1.1 in this section this tree is constructible from a , we may

assume $T^{\mathcal{M}} = T$. Let $x \in M$ and let $\mathcal{M} \models x \in A$, so there exists some $y \in \mathbb{R}^{\mathcal{M}}$ such that $\mathcal{M} \models T_{x,y}$ is well-founded. Hence, there must exist in M an order-preserving map $f : T_{x,y} \rightarrow \omega_1$. This map must therefore also exist in V . As a result, $T_{x,y}$ is well-founded in V , so $V \models x \in A$.

All that is left is to prove that if $V \models x \in A$, then $\mathcal{M} \models x \in A$. By Theorem 6.1.5, there exists a tree $S \in L[a]$ on $\omega \times \omega_1$ such that

$$x \in A \iff S_x \text{ is ill-founded.}$$

Therefore, if $x \in \mathcal{M}$ and $V \models x \in A$, since $T \in L[a] \subseteq M$, we have that $T_x^{\mathcal{M}} = T_x$. Since T_x is ill-founded in V , there exists no order-preserving map $f : T_x \rightarrow \omega_1$ in M , hence $\mathcal{M} \models T_x$ is ill-founded. Therefore, $\mathcal{M} \models x \in A$ [11]. $\square$

Since every $\Sigma_2^1(a)$ relation is absolute across all inner models containing A , we have the same result for $\Pi_2^1(a)$ sets. Thus, we have shown Shoenfield Absoluteness.

Note that the above proof did not use the Axiom of Choice or Dependent Choice, only the constructability of the trees in question are required. Hence we have for all $a \in \mathbb{R}$, every $\Sigma_2^1(a)$ statement is absolute across all inner models of ZF containing a , not just models of ZFC or $ZF + DC$.

There exist many corollaries to Shoenfield Absoluteness theorem, one of which is mentioned by Jech:

Corollary 6.1.7 (Corollary 25.21 in [9]).

Every set $A \subseteq \omega$ which is definable by a $\Sigma_2^1(a)$ formula is in $L[a]$. Every Σ_2^1 real is constructible.

Hence, we must have that $0^\#$ is at least Δ_3^1 in complexity.

6.2 Σ_3^1 Absoluteness

In this section we detail a proof for the following theorem about Σ_3^1 formulas.

Theorem 6.2.1.

Assume there exists a class of measurable cardinals in V . Then two-step Σ_3^1 absoluteness holds in V . [11]

We follow the approach given by Kanamori in *The Higher Infinite* [11].

Definition 6.2.2.

Let κ, λ be cardinals, δ be an ordinal, and $f : [\kappa]^{<\lambda} \rightarrow \delta$ be a total function. We say $X \subseteq \kappa$ is **homogeneous for f** if for all $x, y \in [X]^{<\lambda}$, $f(x) = f(y)$. That is, f maps every element of $[X]^{<\lambda}$ to the same value.

We also will use the following theorem of Rowbottom

Fact 6.2.3 (Rowbottom [25]).

Suppose κ is a measurable cardinal and U is a normal κ -complete ultrafilter over κ . Then if we have $f : [\kappa]^{<\omega} \rightarrow \gamma$ for some $\gamma < \kappa$, then there exists some $X \in U$ such that X is homogeneous for f .

Essentially, this is the statement “measurable cardinals are Ramsey cardinals” which is a well-known fact of the large cardinal hierarchy.

Next we prove an important property about measurable cardinals and forcing.

Lemma 6.2.4 (Proposition 10.15 from [11]).

Suppose that κ is a measurable cardinal and $\mathbb{P}$ is a forcing of size $< \kappa$. Then

$$\mathbb{1} \Vdash \kappa \text{ is measurable.}$$

Proof. (Kanamori [11]).

Let U be a κ -complete ultrafilter over κ , and

$$\dot{U} := \{Y \subseteq \kappa : (\exists X \in \check{U})[X \subseteq Y]\}$$

This is the formula for the filter generated by U in $V[G]$ given any V -generic G for $\mathbb{P}$.

First we state the following fact:

(*) If $p \Vdash \dot{\tau} : \kappa \rightarrow \check{V}$, then there exists $q \leq p, Z \in U$, and $f : Z \rightarrow V$ such that $q \Vdash \dot{\tau} \restriction \check{Z} = \check{f}$.

To prove this claim, for all $r \leq p$, let $X_r = \{\gamma < \kappa : \exists x(r \Vdash \dot{\tau}(\gamma) = \check{x})\}$. Since $p \Vdash \dot{\tau} : \kappa \rightarrow \check{V}$, we must have that $\bigcup_{r \leq p} X_r = \kappa$, and therefore, since $|\mathbb{P}| < \kappa$, there must exist some $q \leq p$ such that $X_q \in U$ by the κ -completeness of U . Let $f : X_q \rightarrow V$ be defined such that for all $\gamma \in X_q, f(\gamma) = x$ such that $q \Vdash \dot{\tau}(\gamma) = \check{x}$. Since $q \Vdash \dot{\tau}$ is a function, the corresponding x to γ must be unique. Hence we must have $q \Vdash \dot{\tau} \restriction \check{X}_q = \check{f}$ and (*) is satisfied.

It is clear $\dot{U}$ is a filter. Furthermore, since every element of U has cardinality κ due to κ -completeness and nonprincipality, we must have that $\dot{U}$ is nonprincipal as well.

In order to see $\dot{U}$ is an *ultrafilter* and is κ -complete, note we need only show if $\gamma < \kappa$ and $\mathbb{1} \Vdash \dot{\tau} : \kappa \rightarrow \gamma$, then there must exist some $\alpha < \gamma$ such that $\mathbb{1} \Vdash \dot{\tau}^{-1}(\{\alpha\}) \in \dot{U}$. Ultra-ness comes from the case where $\gamma = 2$, and κ -completeness comes from this being equivalent to the statement “for every partition P of κ into γ pieces, where $\gamma < \kappa$, there must exist some $X \in P \cap \dot{U}$ ”.

Let γ and $\dot{\tau}$ be as above. By (*), there exists some dense $D \subseteq \mathbb{P}$ such that for all $q \in D$, there must exist some $Z_q \in U$ and a function $f_q : Z_q \rightarrow \gamma$ such that $q \Vdash \dot{\tau} \restriction \check{Z}_q = \check{f}_q$. If we take $Z = \bigcap_{q \in D} Z_q$. Then for all $q \in D, q \Vdash \dot{\tau} \restriction Z = \check{f}_q$. However, since we have that

$$q \Vdash \dot{\tau} \text{ is a function}$$

the f_q must all agree on Z . Let $f = f_q \restriction Z$ for every $q \in D$. By κ -completeness of U , we hence have that there exists some $\alpha < \gamma$ such that $f^{-1}(\{\alpha\}) \in U$. Let $A = f^{-1}(\{\alpha\})$. By our earlier work, for any $q \in D, q \Vdash \check{A} \cap \check{Z} \subseteq \dot{\tau}^{-1}(\{\alpha\})$, and so $q \Vdash \dot{\tau}^{-1}(\{\alpha\}) \in \dot{U}$.

Since D is dense in $\mathbb{P}$, we therefore have that for any V -generic G for $\mathbb{P}$, $V[G] \models \dot{\tau}^{-1}(\{\alpha\}) \in \dot{U}$, and hence $\dot{U}$ is a nonprincipal κ -complete ultrafilter for κ in $V[G]$. Therefore $V[G] \models \kappa$ is measurable. Furthermore, also note $\dot{U}$ is normal if U is normal from (*) [11]. □

Now that the groundwork has been laid, we can proceed to prove the desired result. We use a similar, yet far more sophisticated method to the proof of Shoenfield absoluteness.

Recall from our proof of Shoenfield absoluteness, if $B \subseteq \mathbb{R}$ is Σ_2^1 , then there exists a tree T on $\omega^2 \times \omega$ such that for all $x \in \mathbb{R}$, $x \in B \iff (\exists y \in \mathbb{R})[T_{x,y} \text{ is well-founded}]$. Moreover, T can be defined using a Δ_0^0 formula using some $a \in \mathbb{R}$ as a parameter. We may assume without loss of generality that $(\emptyset, \emptyset, \emptyset) \in T$.

First, however, we need to define another ordering using T based on the Kleene-Brouwer ordering.

Definition 6.2.5 ([11]).

Let T be a tree on $\omega^2 \times \omega$ such that $(\emptyset, \emptyset, \emptyset) \in T$. Let $\langle s_n : n < \omega \rangle$ be a Δ_0^0 enumeration of $\omega^{<\omega}$ such that for all $n < \omega$, $|s_n| \leq n$. For any $x, y \in \mathbb{R}$, define the ordering $<_{T,x,y}$ on ω as follows:

$$\begin{aligned} n <_{T,x,y} m &\iff s_n \notin T_{x,y} \wedge s_m \notin T_{x,y} \wedge n < m \quad \vee \\ &\quad s_n \notin T_{x,y} \wedge s_m \in T_{x,y} \quad \vee \\ &\quad s_n \in T_{x,y} \wedge s_m \in T_{x,y} \wedge s_n <_{KB} s_m \end{aligned}$$

From the above definition, we have the following lemma.

Lemma 6.2.6 (Lemma 15.1 from [11]).

Let $B \subseteq \mathbb{R}$ be $\Sigma_2^1(a)$ for some $a \in \mathbb{R}$. Let T be a tree on $\omega^2 \times \omega$ which is $\Delta_0^0(a)$ such that $(\emptyset, \emptyset, \emptyset) \in T$ and $x \in B \iff (\exists y \in \mathbb{R})[T_{x,y} \text{ is well-founded}]$. Then for all $x \in \mathbb{R}$,

$$x \in B \iff (\exists y \in \mathbb{R})[<_{T,x,y} \text{ is a well-ordering on } \omega].$$

Proof.

First, note that $(\emptyset, \emptyset, \emptyset)$ is the maximum of $<_{T,x,y}$. Assume $x \notin B$, then $T_{x,y}$ is not well-founded for any $y \in \mathbb{R}$ and therefore $<_{KB}$ is not a well-ordering on $T_{x,y}$ for any $y \in \mathbb{R}$. Hence, for there is does not exist a $y \in \mathbb{R}$ such that $<_{T,x,y}$ is a well-ordering on

$\{n < \omega : s_n \in T\}$.

Conversely, assume $x \in B$. Then $<_{KB}$ is not a well-ordering on $T_{x,y}$ for some $y \in \mathbb{R}$. Hence, $<_{T,x,y}$ is a well-ordering on $\{n < \omega : s_n \in T\}$. Since $\omega \setminus \{n < \omega : s_n \in T\}$ is ordered with the standard ordinal ordering and every $n \in \{n < \omega : s_n \in T\}$ is greater than any $m \notin \{n < \omega : s_n \in T\}$, $<_{T,x,y}$ is a well-ordering of ω . $\square$

Next, we produce another tree for $B \subseteq \mathbb{R}$ using a regular uncountable cardinal κ . Eventually, we will need κ to be measurable, but it is not necessary for this definition.

Definition 6.2.7 ([11]).

For every $s, t \in \omega^{<\omega}$ such that $|s| = |t|$, define

$$T_{s,t} = \{v \in \omega^{<\omega} : (\exists m < |s|)[(s \upharpoonright m, t \upharpoonright m, v) \in T]\}$$

And define $<_{T,s,t}$ as the ordering on $\{n < \omega : n < |s|\}$ defined identically to $<_{T,x,y}$ in Definition 6.2.5 except $T_{s,t}$ replaces $T_{x,y}$.

Let κ be a regular uncountable cardinal. Define $T^{*,\kappa}$ as follows.

$$T^{*,\kappa} = \{(s, t, u) \in \bigcup_{m < \omega} (\omega^m \times \omega^m \times \kappa^m) : (\forall i, j < |s|)[u_i < u_j \iff i \leq_{T,s,t} j]\}$$

Lemma 6.2.8 (Lemma 15.2 from [11]).

Let $B \subseteq \mathbb{R}$ be $\Sigma_2^1(a)$ for some $a \in \mathbb{R}$. Let T be a tree on $\omega^2 \times \omega$ which is $\Delta_0^0(a)$ such that $(\emptyset, \emptyset, \emptyset) \in T$ and $x \in B \iff (\exists y \in \mathbb{R})[T_{x,y} \text{ is well-founded}]$. Let κ be a regular uncountable cardinal. Then for all $x \in \mathbb{R}$,

$$x \in B \iff (\exists y \in \mathbb{R})(\exists g \in \kappa^\omega)[(x, y, g) \in [T^{*,\kappa}].$$

Proof.

First, we note a few things about $<_{T,s,t}$. Firstly, $(\emptyset, \emptyset, \emptyset)$ is the minimum of $<_{T,s,t}$. Next, let $s, t, s', t' \in \omega^{<\omega}$ such that $|s| = |t|$, $|s'| = |t'| = m < |s|$, and $s' = s \upharpoonright m$, $t' = t \upharpoonright m$. Then by the description of $T_{s,t}$, we have that for all $i < m$, $s_i \in T_{s',t'} \iff s_i \in T_{s,t}$, and

so $<_{T,s,t} \supseteq <_{T,s',t'}$. Since $<_{T,s,t}$ is defined analogously to $<_{T,x,y}$ for $x, y \in \mathbb{R}$, we also have that $i <_{T,x,y} j \iff (\forall m < \omega)[i <_{T,x \upharpoonright m, y \upharpoonright m} j]$. Hence, $<_{T,x,y} = \bigcup_{m < \omega} <_{T,x \upharpoonright m, y \upharpoonright m}$ [11].

By Lemma 6.2.6 and the fact $i <_{T,x,y} j \iff (\forall m < \omega)[i <_{T,x \upharpoonright m, y \upharpoonright m} j]$, we have

$$x \in B \iff (\exists y \in \mathbb{R})(\exists g \in \kappa^\omega)[(x, y, g) \in [T^{*,\kappa}]]$$

as such a g exists if and only if $g \upharpoonright m$ is map preserving the order of $<_{T,x \upharpoonright m, y \upharpoonright m}$ for all $m < \omega$, which is only possible if each $<_{T,x \upharpoonright m, y \upharpoonright m}$ is a well order on $\{i < \omega : i < m\}$, which is the case if and only if $<_{T,x,y}$ is a well-order. Hence by Lemma 6.2.6, we have that the lemma is true [11]. $\square$

Therefore, if $A \subseteq \mathbb{R}$ is Π_2^1 , we have that $A = \mathbb{R} \setminus B$, where B is Σ_2^1 so for every regular uncountable cardinal κ , there exists a tree $T^{*,\kappa}$ on $\omega \times \omega \times \kappa$ such that

$$\begin{aligned} x \in A &\iff x \notin B \\ &\iff (\forall y \in \mathbb{R})(\forall g \in \kappa^\omega)(\exists m < \omega)[(x \upharpoonright m, y \upharpoonright m, g \upharpoonright m) \notin T^{*,\kappa}] \\ &\iff T_x^{*,\kappa} \text{ is well-founded.} \end{aligned}$$

We then define an analogue of Fact ??, where the the order-preserving map has range κ , rather than κ^+ .

Lemma 6.2.9 ([11]).

For all $x \in \mathbb{R}$, $T_x^{*,\kappa}$ as described in Definition 6.2.7 above is well-founded if and only if there exists an order-preserving map $f : T_x^{*,\kappa} \setminus \{(\emptyset, \emptyset)\} \rightarrow \kappa$ using $\supseteq$ as the order for $T_x^{*,\kappa}$.

Proof [11].

We begin our proof by defining an antilexicographical version of $<_{KB}$ on $T_x^{*,\kappa}$. This will be denoted $<_{KB^2}$. Recall that $(s, t, u) \in T^{*,\kappa}$ if and only if $|s| = |t| = |u|$ and u as a function over $\{i < \omega : i < |s|\}$ is order-isomorphic to $<_{T,s,t}$. Hence, we may assume that $u(0)$ is

the maximum value of the range of u . We now define $<_{KB^2}$. For all $(t, u), (t', u') \in T_x^{*,\kappa}$,

$$\begin{aligned} (t, u) <_{KB^2} (t', u') &\iff (t = \emptyset = u \wedge |t'| = |u'| > 0) \vee \\ &\quad (t \supset t' \wedge u \supset u') \vee \\ &\quad (\exists m < |t|)[(\forall n < m)[t_n = t'_n \wedge u_n = u'_n] \wedge (u_m < u'_m \vee [u_m = u'_m \wedge t_m < t'_m])]. \end{aligned}$$

From this ordering, whenever $(t, u) <_{KB^2} (t', u')$, then either $t = u = \emptyset$ or $u(0) \leq u'(0)$. Since $u(0)$ is the maximum of the range of u , we must have that $\{(t', u') \in T_x^{*,\kappa} : (t', u') < (t, u)\}$ has cardinality at most $|\omega^{<\omega} \times (u(0) + 1)^{<\omega}| \leq \aleph_0 \times |u(0) + 1|^{<\omega}$. Since κ is regular, $|u(0) + 1| = \gamma < \kappa$, and $\aleph_0 \times \gamma^{<\omega} < \kappa$.

Let $Pre(t, u)$ denote the order-type of the set of predecessors of (t, u) in $<_{KB^2}$. Then we have the order-type of $<_{KB^2}$ is less than or equal to $\sum_{\gamma < \kappa} Pre(0, \gamma) = \kappa$. However, in $<_{KB^2}$, $(\emptyset, \emptyset)$ is the minimum element, which risks that κ is the value to which $(\emptyset, \emptyset)$ is mapped, and the order-type being $\kappa + 1$. Therefore, we can drop $(\emptyset, \emptyset)$, and find that the order of $T_x^{*,\kappa} \setminus \{(\emptyset, \emptyset)\}$ is preserved by $<_{KB^2}$.

Hence, there exists an order-preserving map $f : T_x^{*,\kappa} \setminus \{(\emptyset, \emptyset)\} \rightarrow \kappa$. Conversely, if $T_x^{*,\kappa}$ is ill-founded, there exists an infinitely decreasing chain, and hence no such order-preserving map. $\square$

Corollary 6.2.10 (Lemma 15.3 from [11]).

Let $A \subseteq \mathbb{R}$ be Π_2^1 and κ be a regular cardinal. There exists a tree $T^{*,\kappa}$ on $\omega \times \omega \times \kappa$ such that for all $x \in \mathbb{R}$,

$$x \in A \iff (\exists f : T_x^{*,\kappa} \rightarrow \kappa)[f \text{ is an order-preserving map.}]$$

We now construct yet another tree, however first we need another definition. For κ a regular cardinal, $s, t \in \omega^{<\omega}$ such that $|s| = |t|$, let

$$T_{s,t}^{*,\kappa} = \{u \in \kappa^{|s|} : (s, t, u) \in T^{*,\kappa}\}$$

Note that $T_{s,t}^{*,\kappa}$ is not a tree, as it only contains elements from $\kappa^{|s|}$. This may seem counter-intuitive, yet it gives the following useful property:

Fact 6.2.11.

Let $s, t \in \omega^{<\omega}$ such that $|s| = |t|$. Let $A \subseteq \mathbb{R}$ be $\mathbf{\Pi}_2^1$ and κ be a regular cardinal. Let $T^{*,\kappa}$ be as in Definition 6.2.7. Then there exists a bijective map $\tau : T_{s,t}^{*,\kappa} \rightarrow [\kappa]^{|s|}$ given by $\tau(u) = \text{range}(u)$.

This is due to the fact that $<_{T_{s,t}}$ is a strict linear ordering. Therefore for all $u, v \in T_{s,t}^{*,\kappa}$ we know $\text{range}(u) \in [\kappa]^{|s|}$ and for any $u \neq v$, $\text{range}(u) \neq \text{range}(v)$. Finally, since we are only referring to finite linear orders, for every $\{\lambda_1, \dots, \lambda_{|s|}\} \in [\kappa]^{|s|}$, there exists $u \in T_{s,t}^{*,\kappa}$ such that $\text{range}(u) = \{\lambda_1, \dots, \lambda_{|s|}\}$. In other words, for each $s, t \in \omega^{<\omega}$, $T_{s,t}^{*,\kappa}$ represents a copy of $[\kappa]^{|s|}$. This is where measurability becomes relevant.

Definition 6.2.12 ([11]).

Let $T^{*,\kappa}$ be as in Definition 6.2.7. For any $s \in \omega^{<\omega}$. We say a sequence $\langle h_i : i < |s| \rangle$ of functions are **orderly for s** if and only if for all $i, j < |s|$:

1. $h_i : T_{s \restriction |s_i|, s_i}^{*,\kappa} \rightarrow \kappa$
2. If $s_i \supset s_j$, $s_j \neq \emptyset$, and $u \in T_{s \restriction |s_i|, s_i}^{*,\kappa}$, then $h_i(u) < h_j(u \restriction |s_j|)$

Where s_i, s_j refers to the enumeration of $\omega^{<\omega}$ from Definition 6.2.5.

We define $T^{+,\kappa}$ as follows:

$$T^{+,\kappa} = \{(s, \langle h_i : i < |s| \rangle) : s \in \omega^{<\omega} \wedge \langle h_i : i < |s| \rangle \text{ is orderly for } s\}.$$

$T^{+,\kappa}$ is a tree where instead of ordinals, we are taking a tree over a set of functions. If there exists an order preserving map with respect to $f : T_x^{*,\kappa} \setminus \{(\emptyset, \emptyset)\} \rightarrow \kappa$, where the order is $<_{KB^2}$, it is simple to use that map to construct an orderly sequence $\langle h_i : i < m \rangle$ for $x \restriction m$ for each $m < \omega$ by setting for each $u \in T_{x \restriction |s_i|, s_i}^{*,\kappa}$, $h_i(u) = f(s_i, u)$, and the resulting sequence, by the properties of $<_{KB^2}$ ends up being orderly for $x \restriction m$ for every m .

Conversely, if there exists an $x \in \mathbb{R}$ such that for every $m < \omega$, we have an orderly sequence for $x \restriction m$, then it is clear that the chain of sequences can be converted into an order preserving map $f : T_x^{*,\kappa} \setminus \{(\emptyset, \emptyset)\} \rightarrow \kappa$. Hence, we have the following lemma holds true.

Lemma 6.2.13 (Lemma 15.4 from [11]).

Let A and $T^{+, \kappa}$ be as in Definition 6.2.12. For every $x \in \mathbb{R}$,

$$x \in A \iff (\exists h \in \Pi_{i < \omega} \kappa^{T_{x \restriction |s_i|, s_i}^{*, \kappa}})(\forall m < \omega)[(x \restriction m, h \restriction m) \in T^{+, \kappa}].$$

Let κ be measurable. We have that since for every $s, t \in \omega^{<\omega}$ such that $|s| = |t|$, $T_{s,t}^{*, \kappa}$ is bijective with $[\kappa]^{|s|}$, we may associate the h_i 's with the the ordinals they represent in well-founded ultrapowers over $T_{s,t}^{*, \kappa}$.

Let U be a normal nonprincipal κ -complete ultrafilter over κ . For each $s, t \in \omega^{<\omega}$ such that $|s| = |t|$, we define $U_{s,t}$

$$U_{s,t} = \{X \subseteq T_{s,t}^{*, \kappa} : (\exists H \in U)([H]^{|s|} \subseteq \text{range}'' X)\}.$$

By Rowbottom, we have that for all $n < \omega$ and $Y \subseteq [\kappa]^n$, there exists some $H \in U$ such that either $[H]^n \subseteq Y$ or $Y \subseteq [\kappa]^n \setminus [H]^n$. Hence $U_{s,t}$ is a κ -complete ultrafilter over $T_{s,t}^{*, \kappa}$. We also define the following elementary embeddings:

$$j_{s,t} : V \rightarrow M_{s,t}$$

Where $M_{s,t}$ is the transitive collapse of $Ult_{U_{s,t}}(V)$.

Note that in the above work, $h_0 : \{\emptyset\} \rightarrow \kappa$, $U_{\emptyset, \emptyset} = \{\{\emptyset\}\}$, $M_{\emptyset, \emptyset} = V$, and $j_{\emptyset, \emptyset} = id$ by convention, and do not play any significant role in the proof.

For each pair $(s, \langle h_i : i < |s| \rangle)$, we associate each h_i with the value assigned $[h_i]$ by the transitive collapse of $M_{s \restriction |s_i|, s_i}$. Note that this value is an ordinal since $h_i(u)$ is an ordinal

for all $u \in T_{s,t}^{*,\kappa}$. Therefore, we are able to turn $T^{+,\kappa}$ into a tree on $\omega \times \lambda_U$, where

$$\lambda_U = \sup\{j_{s,t}(\kappa) : s, t \in \omega^{<\omega} \wedge |s| = |t|\}.$$

Fact 6.2.14 ([11]).

Let $s, t \in \omega^{<\omega}$, $|s| = |t| = n$. Let $m \leq n$ and define $s' = s \restriction m$, $t' = t \restriction m$. Let $X \subseteq T_{s,t}^{*,\kappa}$.

Then

$$X \in U_{s,t} \iff \{u \in T_{s',t'}^{*,\kappa} : u \restriction m \in X\} \in U_{s',t'}.$$

Proof [11].

Let $H \in U$. We have $[H]^m \subseteq \text{range}'' X$ if and only if for every $v \in [H]^n$, we have that $[v]^n \subseteq \text{range}'' X$. $\square$

We hence can define new elementary embeddings between the $M_{s,t}$.

$$j_{(s,t),(s',t')} : M_{s,t} \rightarrow M_{s',t'}$$

$$j_{(s,t),(s',t')}([f]_{U_{s,t}}) = [\{u, f(u \restriction m) : u \in T_{s',t'}^{*,\kappa}\}]_{U_{s',t'}}.$$

Such that $j_{(s,t),(s',t')} \circ j_{s,t} = j_{s',t'}$. Hence, we actually have that $\langle M_{s,t}, j_{(s,t),(s',t')} \rangle$ is a directed system.

We can now define one last tree, $T^{\circ,\kappa}$, on $\omega \times \lambda_U$ for the sake of our proof. For all $(s, \delta) \in \omega^{<\omega} \times \gamma_U^{<\omega}$,

$$(s, \delta) \in T^{\circ,\kappa} \iff |s| = |\delta| \wedge (\forall i, j < |s|)[s_i \supset s_j \implies \delta_i < j_{(s \restriction |s_j|, s_j), (s \restriction |s_i|, s_i)}(\delta_j)].$$

We now get to the main theorem we use for our proof.

Theorem 6.2.15 (Proposition 15.5 from [11]).

Let A be a Π_2^1 statement, and let $T^{\circ,\kappa}$ be as described above, then

$$A = p[T^{\circ,\kappa}]$$

Proof [11].

From our previous work and Lemma 6.2.13, we have that if $x \in A$, then $x \in p[T^{\circ, \kappa}]$.

Hence, all we need is to show that if $x \in p[T^{\circ, \kappa}]$, then $x \in A$.

Let $x \in p[T^{\circ, \kappa}]$, with $\langle x, g \rangle \in [T^{\circ, \kappa}]$. Set $\langle h_i : i < \omega \rangle$ such that $h_i : T_{x \restriction |s_i|, s_i}^{*, \kappa} \rightarrow \kappa$ and g_i is the image of $[h_i]_{U_{x \restriction |s_i|, s_i}}$ in $M_{x \restriction |s_i|, s_i}$ for all $i < \omega$.

We then have from the properties of ultrapowers and the definition of $T^{\circ, \kappa}$ that for all $i, j < \omega$ such that $s_i \supset s_j$ and $s_j \neq \emptyset$,

$$\{u \in T_{x \restriction |s_i|, s_i}^{*, \kappa} : h_i(u) < h_j(u \restriction |s_j|)\} \in U_{x \restriction |s_i|, s_i}$$

Hence, by our construction of $U_{s,t}$ we have that there exists some $X_{i,j} \in U$ such that

$$\text{range}(u) \in [X_{i,j}]^{|s_i|} \implies h_i(u) < h_j(u \restriction |s_j|)$$

Let $Y = \bigcap_{i,j < \omega \wedge s_i \supset s_j} X_{i,j}$. Since U is κ -complete, we must have that $Y \in U$.

If $x \notin A$, then let $B = \mathbb{R} \setminus A$. B is a Σ_2^1 statement, and by Lemma 6.2.6, there exists $y \in \mathbb{R}$ such that $<_{T,x,y}$ is a well-order, where T is the tree for B used to describe $T^{\circ, \kappa}$.

Since $Y \in U$ and U is nonprincipal, we must have that Y is uncountable so there exists a function $d : \omega \rightarrow Y$ such that for all $i, j < \omega$, we must have

$$d(i) < d(j) \iff i <_{T,x,y} j.$$

Hence, $d \restriction m \in T_{x \restriction |m, y \restriction m|}^{*, \kappa}$ for all $m < \omega$, and therefore since $Y \subseteq X_{i,j}$, for all $i, j < \omega$, we have that since $\text{range}(d \restriction m) \in [X_{i,j}]^m$, we have that for all s_i, s_j such that $y \supset s_i \supset s_j$, and $s_j \neq \emptyset$, then $h_i(d \restriction |s_i|) < h_j(d \restriction |s_j|)$. This is an infinite descending sequence of ordinals, causing a contradiction. Therefore, $p[T^{\circ, \kappa}] \implies x \in A$, proving $A = p[T^{\circ, \kappa}]$. $\square$

We finally arrive to the final step in our proof of Theorem 6.2.1, where we prove how the measurability of κ provides a generic absoluteness property for forcings of size $< \kappa$.

Theorem 6.2.16 (Theorem 15.6 from [11]).

Let κ be a measurable cardinal. Suppose that $A \subseteq \mathbb{R}$ is $\mathbf{\Pi}_2^1$. Let $\mathbb{P}$ be a forcing such that $|\mathbb{P}| < \kappa$. Then for all V -generic G over $\mathbb{P}$,

$$V \models (\exists x \in \mathbb{R})[x \in A] \iff V[G] \models (\exists x \in \mathbb{R})[x \in A]$$

Proof.

Let $T^{\circ, \kappa}$ be the associated tree as defined above using U and κ , where U is a normal nonprincipal κ -complete ultrafilter over κ . From the proof of Lemma 6.2.4, we have that if we set

$$\dot{U} = \{Y \subseteq \kappa : (\exists x \in \check{U})[X \subseteq Y]\}$$

Then $\dot{U}$ is a κ -complete ultrafilter over κ in $V[G]$. Using Fact 6.2.3 (Rowbottom), we can show a variant of $(*)$ in the same proof:

Claim 1 ([11]).

Let $n < \omega$ and $p \Vdash \dot{\tau} : [\kappa]^n \rightarrow \check{V}$, then there exists some $q \leq p$, $Z \in U$, and some $f : [Z]^n \rightarrow V$ such that $q \Vdash \dot{\tau} \upharpoonright [\check{Z}]^n = \check{f}$.

Proof of Claim 1.

For all $r \leq p$, let $X_r = \{\gamma \in [\kappa]^n : \exists x(r \Vdash \dot{\tau}(\gamma) = \check{x})\}$. Since $p \Vdash \dot{\tau} : [\kappa]^n \rightarrow \check{V}$, we must have that $\bigcup_{r \leq p} X_r = [\kappa]^n$. Let $v : [\kappa]^n \rightarrow \mathbb{P}$ be such that for each $\gamma \in [\kappa]^n$, $v(\gamma) \leq p$ and $\gamma \in X_{v(\gamma)}$. By Fact 6.2.3 (Rowbottom), we have that there exists some $Z \in U$ such that there exists $q \leq p$ where $v(\gamma) = q$ for all $\gamma \in [Z]^n$, and so $[Z]^n \subseteq X_q$. Let $f : [Z]^n \rightarrow V$ be defined such that for all $\gamma \in [Z]^n$, $f(\gamma) = x$ such that $q \Vdash \dot{\tau}(\gamma) = \check{x}$. We can always find such an x by the construction of X_q , as $\exists x(q \Vdash \dot{\tau}(\gamma) = \check{x})$ for all $\gamma \in X_q$. Therefore, since $q \Vdash \text{“}\dot{\tau} \text{ is a function”}$, the corresponding x to γ must be unique. Hence we must have $q \Vdash \dot{\tau} \upharpoonright [\check{Z}]^n = \check{f}$ and the claim is satisfied. $\square$

Hence, by the claim and using the same logic as was used for Lemma 6.2.4, for any V -generic G for $\mathbb{P}$, and any $s, t \in \omega^{<\omega}$ such that $|s| = |t|$, we have that if we define $\dot{U}_{s,t}^{V[G]}$ from $\dot{U}^{V[G]}$ in $V[G]$ in the same way that $U_{s,t}$ is defined from U in V , then $\dot{U}_{s,t}^{V[G]}$

is a κ -complete ultrafilter over $(T_{s,t}^{*,\kappa})_G$ in $V[G]$ such that $U_{s,t} \subseteq \dot{(U_{s,t})}^{V[G]}$. So for any $f : T_{s,t}^{*,\kappa} \rightarrow Ord$ in V , $\mathbb{1} \Vdash \check{f} : \check{T}_{s,t}^{*,\kappa} \rightarrow Ord$, and therefore we have that there exists a dense $D \subseteq \mathbb{P}$ such that for all $q \in D$, $q \Vdash \check{f} : \check{T}_{s,t}^{*,\kappa} \rightarrow Ord$. Since $T_{s,t}^{*,\kappa}$ is bijective with $[\kappa]^{|s|}$ and $[\kappa]^n$ is invariant under forcing with G , we may assume $(\check{T}_{s,t}^{*,\kappa})_G = T_{s,t}^{*,\kappa}$, and we can replace it temporarily with $[\kappa]^{|s|}$. By our claim, we have that for every $q \in D$, there exist Z_q, f_q such that $f_q : [Z_q]^{|s|} \rightarrow \kappa$, and $q \Vdash \check{f} \restriction [\check{Z}_q]^{|s|} = \check{f}_q$. Since for each $q \in D$, $q \Vdash \check{f}$ is a function" we must have that if $Z = \bigcap_{q \in D} Z_q$, then in $V[G]$ we must have all the f_q agree over Z . Furthermore, $Z \in U$ by κ -completeness. Let $g = f_q \restriction Z$. From our description of f , $V[G] \models (f \restriction Z) = (g \restriction Z)$, as $D \cap G \neq \emptyset$. Therefore since $[Z]^{|s|}$ represents a set in $U_{s,t}$, it is also in $\dot{U}_{s,t}$, and since the functions are equal on an element in both ultrafilters, they must map to the same element under both ultrapowers' transitive collapse. In other words, we have that $j_{s,t}$ and $(j_{s,t})_G$ agree on all $f : T_{s,t}^{*,\kappa} \rightarrow Ord$. Therefore, for all $s, t, s', t' \in \omega^{<\omega}$, such that $|s| = |t| = n$, $|s'| = |t'| = m$ such that $m \leq n$, and $t \restriction m = t'$, $s \restriction m = s'$, then we have that $j_{(s',t'),(s,t)}$ and $(j_{(s',t'),(s,t)})_G$ agree on the ordinals.

Therefore, since $T^{\circ,\kappa}$ and $(T^{\circ,\kappa})_G$ are each defined by the action of $j_{(s',t'),(s,t)}$ and $(j_{(s',t'),(s,t)})_G$ on the ordinals, $T^{\circ,\kappa} = (T^{\circ,\kappa})_G$.

We use this absoluteness property to finish our proof in the same way we finished the proof of Theorem ???. Let $x \in \mathbb{R}$ be such that $V \models x \in A$, then $x \in p[T^{\circ,\kappa}] \subseteq (p[T^{\circ,\kappa}])_G$, so $V[G] \models x \in A$. Conversely, if $V \models A = \emptyset$, then we must have that $p[T^{\circ,\kappa}] = \emptyset$, so $[T^{\circ,\kappa}] = \emptyset$, ergo $T^{\circ,\kappa}$ must be well-founded. Therefore, there exists some order-preserving map $d : T^{\circ,\kappa} \rightarrow Ord$. Since $T^{\circ,\kappa} = (T^{\circ,\kappa})_G$, we hence have the same order-preserving map exists in $V[G]$ for $(T^{\circ,\kappa})_G$, and hence $V[G] \models T^{\circ,\kappa}$ is well-founded, and therefore there exists no $x \in \mathbb{R}_G$ such that $V[G] \models x \in A$.

Hence we have proven

$$(\exists x \in \mathbb{R})[x \in A] \iff V[G] \models (\exists x \in \mathbb{R})[x \in A]$$

for all V -generic G for $\mathbb{P}$ such that $|\mathbb{P}| < \kappa$. □

Note that similar logic to the above proof can be similarly applied to $A \subseteq \mathbb{R}^n$ for any $n < \omega$, however for the sake of simpler notation, we only proved it in the case where $n = 1$.

Proof of Theorem 6.2.1.

Assume there exists a class of measurable cardinals. Let $\varphi(x)$ be a Σ_3^1 relation. We have that this is equivalent to there existing some $A \subseteq \mathbb{R}^2$ such that A is Π_2^1 and $\varphi(x)$ is equivalent to $(\exists y \in \mathbb{R})[(x, y) \in A]$. Then for any forcing $\mathbb{P}$, we have that there exists some measurable $\kappa > |\mathbb{P}|$, and by Theorem 6.2.16, we must have that

$$V \models (\exists y \in \mathbb{R})[(x, y) \in A] \iff V[G] \models (\exists y \in \mathbb{R})[(x, y) \in A]$$

for any V -generic G for $\mathbb{P}$. Hence $\varphi(x)$ is absolute with regards to V -generic extensions over $\mathbb{P}$. Since $\mathbb{P}$ was arbitrary, we therefore have one-step Σ_3^1 absoluteness holds in V .

Furthermore, by Lemma 6.2.4, for all measurable $\kappa > |\mathbb{P}|$,

$$V[G] \models \kappa \text{ is measurable.}$$

So $V[G]$ also has a class of measurable cardinals, and hence for any generic extension of V , one-step Σ_3^1 generic absoluteness also holds. Hence we have two-step Σ_3^1 generic absoluteness in V . $\square$

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