Quadratically Dense Matroids

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This thesis is concerned with finding the maximum density of rank-n matroids in a minor-closed class. The extremal function of a non-empty minor-closed class $\mathcal{M}$ of matroids which excludes a rank-2 uniform matroid is defined by

$$h_{\mathcal{M}}(n) = \max(|M| \colon M \in \mathcal{M} \text{ is simple, and } r(M) \leq n).$$

The Growth Rate Theorem of Geelen, Kabell, Kung, and Whittle shows that this function is either linear, quadratic, or exponential in n.

In this thesis we prove a general result about classes with quadratic extremal function, and then use it to determine the extremal function for several interesting classes of representable matroids, for sufficiently large integers n. In particular, for each integer $t \geq 4$ we find the extremal function for all but finitely many n for the class of $\mathbb{C}$-representable matroids with no $U_{2,t}$-minor, and we find the extremal function for the class of matroids representable over finite fields $\mathrm{GF}(q)$ and $\mathrm{GF}(q')$ where $q-1$ divides $q'-1$ and q and q' are relatively prime.

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Chapter 1

Introduction

This thesis is concerned with minor-closed classes of matroids which exclude some rank-2 uniform matroid, and do not contain all GF(q)-representable matroids for any prime power q. The Growth Rate Theorem [12] proves that the number of points of a simple rank-n matroid in such a class is bounded above by a quadratic function of n.

The most interesting classes of this type are classes of representable matroids, and in this thesis we determine the extremal function for several natural classes of representable matroids for all but finitely many integers n. As the following two theorems illustrate, we prove results for both infinite fields and finite fields.

Theorem 1.0.1. *For each integer $t \geq 1$, if M is a simple $\mathbb{C}$-representable matroid of sufficiently large rank with no $U_{2,t+3}$-minor, then $|M| \leq t\binom{r(M)}{2} + r(M)$.*

Theorem 1.0.2. *If $\mathbb{F}_1$ and $\mathbb{F}_2$ are finite fields with different characteristic such that $|\mathbb{F}_1| - 1$ divides $|\mathbb{F}_2| - 1$, then each simple matroid M of sufficiently large rank representable over $\mathbb{F}_1$ and $\mathbb{F}_2$ satisfies $|M| \leq (|\mathbb{F}_1| - 1)\binom{r(M)}{2} + r(M)$.*

In both cases the bounds are best-possible and are attained by Dowling geometries, which are highly structured matroids arising naturally from complete graphs with group labels on the edges. In fact, for both of the above theorems we show that Dowling geometries over cyclic groups are the unique examples for which equality holds. This is no coincidence; these theorems are both corollaries of a much more general result, Theorem 1.7.6, whose technical statement we defer. In order to prove Theorem 1.7.6 we prove a result (Theorem 1.7.2) which has the following notable corollary.

Theorem 1.0.3. *For each integer $t \geq 2$, there is an integer c so that if M is a simple $\mathbb{R}$-representable matroid of sufficiently large rank with no $U_{2,t+3}$-minor, then $|M| \leq r(M)^2 + c \cdot r(M)$.*

The leading coefficient is best-possible since this class contains Dowling geometries over the group of size two, which have $r(M)^2$ elements.

In this chapter we introduce some basic definitions and terminology from matroid theory, provide an overview of important results in extremal matroid theory, give a general proof sketch for finding an extremal function, and state our main results.

1.1 Basics

A matroid is an object which describes the combinatorics of mathematical dependence, and was initially conceived as an abstraction of both linear dependence of sets of vectors in a vector space and edge-sets of circuits in a graph [35, 36, 37, 53]. In this section we introduce fundamental concepts in matroid theory which we will use throughout this thesis. All material in this section is standard, and much of it can be found in greater detail in [42].

A *matroid* M is a pair (E, r) where E is a finite set called the *ground set* and $r\colon 2^E \to \mathbb{Z}$ is the *rank function* of M, which satisfies three axioms:

- $0 \leq r(X) \leq |X|$ for all $X \subseteq E$,
- $r(X) \leq r(Y)$ for all $X \subseteq Y \subseteq E$, and
- $r(X) + r(Y) \geq r(X \cup Y) + r(X \cap Y)$ for all $X \subseteq Y \subseteq E$.

The *rank* of M is $r(E)$ and the *size* of M is $|E|$; for convenience we write $r(M)$ and $|M|$ for the rank and size of M, respectively.

Bases and Circuits

Since graphs and sets of vectors are the two canonical examples of matroids, nearly all matroid-theoretic terminology is borrowed from either graph theory or linear algebra. A set $X \subseteq E$ is *independent* in M if $r(X) = |X|$; otherwise it is *dependent*. A maximal independent set of M is a *basis* of M, and we say that $X \subseteq E$ is *spanning* in M if X contains a basis of M. A minimal dependent set of M is a *circuit*; the *girth* of M is the number of elements of a smallest circuit of M. A matroid is completely determined by its independent sets, bases, or circuits, and it can be useful to describe a matroid using one of these characterizations instead of providing the rank function.

Flats

The *closure* in M of $X \subseteq E$ is $\{e \in E\colon r(X \cup \{e\}) = r(X)\}$, and is denoted $\mathrm{cl}(X)$. A set $F \subseteq E$ is a *flat* of M if $F = \mathrm{cl}(F)$; a *point*, *line*, *plane*, or *hyperplane* is a flat of rank 1, 2, 3, or $r(M) - 1$, respectively. We write $\varepsilon(M)$ for the number of points of M. The *length* of a line of M is the number of points it contains, which is at least two. A line is *long* if it has length at least three.

Minors

We will write r_M for r and $E(M)$ for E when necessary to distinguish M from other matroids. For a set $Y \subseteq E(M)$, the *deletion* of Y in M is the matroid $M \backslash Y$ with ground set $E(M) - Y$ and rank function $r_{M\backslash Y}(X) = r_M(X)$ for all $X \subseteq E(M) - Y$. The *contraction* of Y in M is the matroid M/Y with ground set $E(M) - Y$ and rank function $r_{M/Y}(X) = r_M(X \cup Y) - r_M(Y)$. A matroid N is a *minor* of M if there are disjoint sets $C, D \subseteq E(M)$ so that $N = (M/C)\backslash D$. Using the definition of deletion and contraction one can show that $(M/C)\backslash D = (M\backslash D)/C$ for all disjoint sets $C, D \subseteq E(M)$. If $C = \varnothing$ then N is a *restriction* of M and we write $M|(E(M) - D)$ for $M\backslash D$, and if $D = \varnothing$ then N is a *contract-minor* of M.

Isomorphism

A matroid N is *isomorphic* to M if there is a bijection $\phi\colon E(N) \to E(M)$ so that $r_N(X) = r_M(\phi(X))$ for all $X \subseteq E(N)$; we write $M \cong N$. For a matroid N, we say that M has an *N-minor* if M has a minor isomorphic to N, and an *N-restriction* if M has a restriction isomorphic to N. A class $\mathcal{M}$ of matroids is *minor-closed* if it is closed under taking minors and under isomorphism.

Simplification

A *loop* of M is an element e so that $r(\{e\}) = 0$; each other element is a *nonloop*. We say that nonloops e and f are *parallel* in M if they are in a common point, so $r(\{e, f\}) = 1$. This defines an equivalence relation on the set of nonloops of M; the equivalence classes are called *parallel classes*. A parallel class is *nontrivial* if it has size at least two, and we say that a matroid is *simple* if it has no loops and no nontrivial parallel classes. The *simplification* of M, denoted $\operatorname{si}(M)$, is the matroid whose ground set is the set of points of M so that the rank of a set of points is the rank of their union in M. Note that $|\operatorname{si}(M)| = \varepsilon(M)$, the number of points of M. If $T \subseteq E$ is a transversal of the parallel classes of M, then $M|T$ is isomorphic to $\operatorname{si}(M)$; we say that $M|T$ is *a simplification* of M.

Duality

The *dual* of M, denoted M^*, is the matroid with ground set E and rank function r^* defined by $r^*(X) = |X| - r(M) + r_M(E - X)$ for all $X \subseteq E$. We say that $r^*(E)$ is the *corank* of M, and write $r^*(M)$ for convenience. One can check that the bases of M^* are precisely the complements of bases of M, which shows that $(M^*)^* = M$. A *coloop* of M is a loop of M^*, and a set $X \subseteq E$ is *cospanning* in M if X is spanning in M^*. Using r^* one can show that $e \in E$ is a coloop of M if and only if $r(M \backslash \{e\}) < r(M)$. One can also show that contraction is 'dual' to deletion, meaning that $(M \backslash e)^* = M^*/e$ and $(M/e)^* = M^* \backslash e$ for each $e \in E$.

The Extremal Function

The *extremal function* of a non-empty class of matroids $\mathcal{M}$ is denoted $h_{\mathcal{M}}$, and is defined by

$$h_{\mathcal{M}}(n) = \max(\varepsilon(M)\colon M \in \mathcal{M} \text{ and } r(M) \le n),$$

for all integers $n \ge 0$. Note that this definition is equivalent to the definition given in the abstract, since any matroid M has a simple restriction with $\varepsilon(M)$ elements. If this maximum fails to exist for some integer $n \ge 0$, then we set $h_{\mathcal{M}}(n) = \infty$. Roughly speaking, a simple matroid is 'dense' if the ratio of its size to its rank is large, and the extremal function gives the maximum density of simple matroids in $\mathcal{M}$. If $h_{\mathcal{M}}(n)$ is finite and $M \in \mathcal{M}$ is a rank-n matroid such that $\varepsilon(M) = h_{\mathcal{M}}(n)$, then M is *extremal* in $\mathcal{M}$. If $h_{\mathcal{M}}(n) = p(n)$ for sufficiently large n and some function p, we say that p is an *eventual extremal function* for $\mathcal{M}$.

Uniform Matroids

We now define an important class of matroids. Let a and b be integers with $0 \leq a \leq b$, let E be a set of size b, and define $r\colon 2^E \to \mathbb{Z}$ by $r(X) = \min(|X|, a)$. The matroid $M = (E, r)$ is a rank-a *uniform matroid* on b elements, and we write $M \cong U_{a,b}$. Rank-2 uniform matroids are informally called *lines*; their ground sets are lines in the previous sense. For each integer $\ell \geq 2$ we write $\mathcal{U}(\ell)$ for the class of matroids with no $U_{2,\ell+2}$-minor.

1.1.1 Representable Matroids

Let $\mathbb{F}$ be a field, and let A be a matrix with entries in $\mathbb{F}$ and columns indexed by a set E. For each set $X \subseteq E$ let $A[X]$ denote the submatrix of A consisting of all columns of A indexed by X. The matroid *represented* by A, denoted by $M(A)$, is the matroid with ground set E and rank function $r_{M(A)}(X) = \mathrm{rank}(A[X])$ for all $X \subseteq E$. One can check that matroid terms such as closure and basis generalize the corresponding notions for sets of vectors.

A matroid M is $\mathbb{F}$-*representable* if there is a matrix A with entries in $\mathbb{F}$ so that $M = M(A)$. If A' is obtained from A by performing elementary row operations, column scalings, and deleting all-zero rows, then $\mathrm{rank}(A[X]) = \mathrm{rank}(A'[X])$ for all $X \subseteq E$, so $M(A) = M(A')$. This shows that if M is a simple, rank-n $\mathbb{F}$-representable matroid, then there is a matrix A over $\mathbb{F}$ with n rows, no zero column and no parallel columns so that $M = M(A)$. One can also show that the class of $\mathbb{F}$-representable matroids is minor-closed by constructing matrices to represent contract-minors and restrictions; see Chapter 3 of [42].

For each prime power q, the maximum number of nonzero, pairwise nonparallel columns of a matrix over $\mathrm{GF}(q)$ with n rows is $\frac{q^n-1}{q-1}$, since each parallel class of the matroid $\mathrm{GF}(q)^n$ has size $q-1$. This implies that the extremal function for the class of $\mathrm{GF}(q)$-representable matroids is $\frac{q^n-1}{q-1}$ for all $n \geq 0$. In particular, the line $U_{2,q+2}$ is not representable over $\mathrm{GF}(q)$, so the class of $\mathrm{GF}(q)$-representable matroids is contained in $\mathcal{U}(q)$. Each simple rank-n extremal matroid for this class is represented by a matrix with n rows and precisely one column from each parallel class of vectors in $\mathrm{GF}(q)^n$, and is thus uniquely determined up to isomorphism. A simple rank-n $\mathrm{GF}(q)$-representable matroid M for which $|M| = \frac{q^n-1}{q-1}$ is a *projective geometry* over $\mathrm{GF}(q)$, and we write $M \cong \mathrm{PG}(n-1, q)$.

Matroids representable over $\mathrm{GF}(2)$ are particularly well-studied, and are called *binary* matroids. The rank-3 binary projective geometry $\mathrm{PG}(2,2)$ is a well-known matroid called the *Fano plane*, and is denoted by F_7. Binary matroids have the following beautiful characterization, proved by Tutte [50].

Theorem 1.1.1. *A matroid is binary if and only if it has no* $U_{2,4}$*-minor.*

We make use of this result several times in this thesis.

1.1.2 Graphic Matroids

Let $G = (V, E)$ be a graph. The *cycle matroid* of G is the matroid $M(G)$ with ground set E and rank function $r_{M(G)}(X) = |V| - n(X)$ for all $X \subseteq E$, where $n(X)$ is the number of components of the graph (V, X). One can check that the circuits of $M(G)$ are precisely the cycles of G, and that the bases of $M(G)$ are precisely the spanning forests of G. A matroid M is *graphic* if there is some graph G so that $M = M(G)$, and is a *clique* if there is some integer $n \geq 3$ so that $M = M(K_n)$, where K_n denotes the complete graph on n vertices. Note that $r(M(K_n)) = n - 1$, since each spanning tree of K_n has size $n - 1$.

Matroid terms such as loop, simplification, and circuit generalize the corresponding notions for graphs. In particular, matroid minors generalize graph minors, so for any graph G and any disjoint $C, D \subseteq E(G)$ we have $M(G/C\backslash D) = M(G)/C\backslash D$. One can show that any graphic matroid $M(G)$ is representable over any field, by considering the vertex-edge incidence matrix of G.

It is not hard to show that if M is graphic, then there is a connected graph G so that $M = M(G)$. This implies that if M is graphic and $r(M) = n$, then there is a graph G with $n + 1$ vertices so that $M = M(G)$. Therefore, if $\mathcal{G}$ denotes the class of graphic matroids, then $h_{\mathcal{G}}(n) = \binom{n+1}{2}$, and equality holds only for cliques.

1.1.3 Lines and Contraction

For any nonloop e of a matroid M, the lines of M which contain e provide information about M/e which is useful for studying the density of matroids in minor-closed classes.

Lemma 1.1.2. *If e is a nonloop of a matroid M, then each set $P \subseteq E(M/e)$ is a point of M/e if and only if $P \cup \{e\}$ is a line of M.*

This lemma implies that $(M/e)|T$ is a simplification of M/e if and only if T is a transversal of the lines of M through e so that no element of T is in a point of M with e. We can use Lemma 1.1.2 to derive the following formula for the number of points 'lost' when a nonloop e is contracted.

Lemma 1.1.3. *Let e be a nonloop of a matroid M, and let $\mathcal{L}$ denote the set of lines of M which contain e. Then*

$$\varepsilon(M) - \varepsilon(M/e) = 1 + \sum_{L \in \mathcal{L}} \big(\varepsilon(M|L) - 2\big).$$

Note that this lemma also holds if $\mathcal{L}$ is the set of long lines of M which contain e, since lines of length two do not change the value of the right-hand side. Lemma 1.1.3 has the following corollary for matroids with no $U_{2,\ell+2}$-restriction, which is useful for finding lots of long lines through a common point.

Lemma 1.1.4. *For all integers $\ell \geq 2$ and $m \geq 0$, if e is a nonloop of a matroid $M \in \mathcal{U}(\ell)$ and $\varepsilon(M) - \varepsilon(M/e) > 1 + (\ell - 1)m$, then there are at least $m + 1$ long lines of M through e.*

As we will see in this thesis, many long lines through a common point can give rise to interesting matroids. Most notably, a *spike* is any simple matroid M with an element e, called the *tip*, so that $\mathrm{si}(M/e)$ is a circuit and each parallel class of M/e has size two.

1.1.4 Connectivity

Matroid connectivity is a central topic of this thesis, and in particular we will be interested in maintaining connectivity properties while taking a minor. Let $M = (E, r)$ be a matroid. The *connectivity function* of M is the function $\lambda_M \colon 2^E \to \mathbb{Z}$ defined by $\lambda_M(X) = r(X) + r(E - X) - r(M)$ for all $X \subseteq E$. This function has the key property that it is submodular, meaning that $\lambda_M(A) + \lambda_M(B) \geq \lambda_M(A \cup B) + \lambda_M(A \cap B)$ for all sets $A, B \subseteq E$.

The connectivity function is used to describe the connectivity between pairs of sets in M. For disjoint sets $A, B \subseteq E$ we write $\kappa_M(A, B) = \min(\lambda(Z) \colon A \subseteq Z \text{ and } B \subseteq E - Z)$. If $B \subseteq C \subseteq E - A$, then it is clearly the case that $\kappa_M(A, B) \leq \kappa_M(A, C)$. It is also not hard to check that if N is a minor of M for which $A \cup B \subseteq E(N)$, then $\kappa_N(A, B) \leq \kappa_M(A, B)$. Tutte [52] proved the very useful result that $\kappa_M(A, B)$ is equal to the maximum value of $\lambda_N(A)$ over all minors N of M with ground set $A \cup B$.

Theorem 1.1.5 (Tutte's Linking Theorem)**.** *Let M be a matroid and let A and B be disjoint subsets of $E(M)$. Then M has a minor N such that $E(N) = A \cup B$ and $\lambda_N(A) = \kappa_M(A, B)$.*

The connectivity function is also used to define connectivity more globally. A *vertical k-separation* of M is a partition (X, Y) of E so that $r(X) + r(Y) - r(M) < k$ and $\min(r(X), r(Y)) \geq k$. We say that M is *vertically k-connected* if M has no vertical j-separation with $j < k$. This aligns with graph connectivity, meaning that a connected graph G is k-connected if and only if $M(G)$ is vertically k-connected. Equivalently, M is vertically k-connected if there is no partition (X, Y) of E so that $r(X) + r(Y) - r(M) < k - 1$ and $\max(r(X), r(Y)) < r(M)$.

While vertical connectivity is the main connectivity property in this thesis, we occasionally make use of a stronger property. We say that M is *round* if E is not the union of two hyperplanes. This implies that if $r(X) < r(M)$, then $E - X$ is spanning in M. In particular, M has no vertical k-separation with $k \geq 1$, so roundness can be thought of as 'infinite' vertical connectivity. Roundness is relevant in this thesis essentially because $M(K_n)$ is round for each integer $n \geq 3$, since the complement of each non-spanning set of edges contains a spanning tree. It is not hard to see that any matroid with a spanning round restriction is itself round; in particular, any matroid with a spanning clique restriction is round.

We also need some notation to capture how much subsets of E 'interact' in M. For sets

$A, B \subseteq E$, the *local connectivity* between A and B is $\sqcap_M(A, B) = r_M(A)+r_M(B)-r_M(A\cup B)$. Note that if A and B are disjoint, then $\sqcap_M(A, B) = r_M(A) - r_{M/B}(A) = r_M(B) - r_{M/A}(B)$. If $\sqcap_M(A, B) = 0$ we say that A and B are *skew* in M; roughly speaking, this means that A and B do not interact at all in M. More generally, sets $A_1, \dots, A_k \subseteq E$ are *mutually skew* in M if $r_M(A_1 \cup \dots \cup A_k) = \sum_i r_M(A_i)$.

1.1.5 Distance

In this thesis we need a way to measure the similarity between two matroids with the same ground set. An *extension* of a matroid M is a matroid M^+ with ground set $E(M) \cup \{e\}$ with $e \notin E(M)$ such that $M = M^+\backslash e$. If e is in a nontrivial parallel class of M^+, then M^+ is a *parallel extension* of M. We say that M^+ is a *trivial extension* of M if e is a loop or coloop of M^+, or e is parallel to an element of M; otherwise M^+ is a *nontrivial extension*. A *projection* of M is a matroid of the form M^+/e, where M^+ is an extension of M by e. A *lift* of M is a matroid N so that M is a projection of N.

For any integer $k \geq 0$, a *k-element projection* of M is a matroid obtained from M by a sequence of k projections. It is not hard to show that this is equivalent to a matroid of the form M^+/K, where $M^+\backslash K = M$ and $|K| = k$. Similarly, a *k-element lift* of M is a matroid N so that M is a k-element projection of N.

Intuitively, if a matroid N is a projection or a lift of M, then N and M are 'close'. For matroids M and N with the same ground set, the *distance* between M and N, denoted $\text{dist}(M, N)$, is the smallest integer k so that N can be obtained from M by a sequence of k operations each of which is a projection or lift. Note that $\text{dist}(M, N) = \text{dist}(N, M)$, and that $\text{dist}(M, N) \leq r(M) + r(N)$ since the matroid on ground set E consisting of only loops can be obtained from M by $r(M)$ projections.

1.2 Frame Matroids

Frame matroids are a broad generalization of graphic matroids, and they play a fundamental role in this thesis. A matroid M is *framed* by B, and B is a *frame* for M, if B is a basis of M and each element of M is spanned by a subset of B with at most two elements. A *frame matroid* is a matroid of the form $M\backslash B$ where M is a matroid framed by B. Intuitively, a matroid framed by B is graph-like with vertex set B because each element is spanned by at most two elements of B. We write $\mathcal{F}$ for the class of frame matroids.

An important special case of a frame matroid is a *B-clique*, which is a matroid framed by B so that each pair of elements of B is contained in a long line. For example, $M(K_n)$ is a B-clique for any spanning star B of K_n. Since each graphic matroid is a restriction of $M(K_n)$ for some $n \geq 3$, each graphic matroid is a frame matroid. However, matroids which are close to being graphic are not necessarily frame matroids, as shown by the following result of Zaslavsky [57].

Proposition 1.2.1. *The Fano plane F_7 is not a frame matroid.*

Both frame matroids and B-cliques behave well under taking minors. The class of frame matroids is minor-closed; if M is framed by B and $e \in E(M)$, then M/e is framed by any spanning subset of $B - \{e\}$. This implies that if M is a B-clique and N is a contract-minor of M, then there is some $B' \subseteq B$ so that N is a B'-clique.

Zaslavsky [56, 57] further developed the notion that frame matroids are graph-like by showing that each frame matroid is associated with a combinatorial object called a biased graph. We next define biased graphs, and then specialize to the case of group-labelled graphs in order to define Dowling geometries, arguably the most important class of matroids in this thesis.

1.2.1 Biased Graphs

A *theta graph* consists of two distinct vertices x and y, and three internally disjoint paths from x to y. A set of cycles $\mathcal{B}$ of a graph G satisfies the *theta property* if no theta subgraph of G contains exactly two cycles in $\mathcal{B}$. A *biased graph* is a pair $(G, \mathcal{B})$ where $\mathcal{B}$ is a collection of cycles of G which satisfies the theta property. The cycles in $\mathcal{B}$ are *balanced*, and the cycles not in $\mathcal{B}$ are *unbalanced*.

For each biased graph $(G, \mathcal{B})$, we define a matroid $M(G, \mathcal{B})$ with ground set $E(G)$ so that $C \subseteq E(G)$ is a circuit of $M(G, \mathcal{B})$ if and only if the edges of C form

- a balanced cycle,
- two vertex-disjoint unbalanced cycles with a path between them,
- two unbalanced cycles which share a single vertex, or
- a theta graph with all cycles unbalanced.

Circuits of the second and third types are called *loose handcuffs* and *tight handcuffs*, respectively. This set of subsets of $E(G)$ is the set of circuits of a matroid only because $\mathcal{B}$ satisfies the theta property. Since any matroid is completely determined by its collection of circuits, the matroid $M(G, \mathcal{B})$ is well-defined. In the special case that $\mathcal{B}$ is the set of all cycles of G we see that $M(G, \mathcal{B}) \cong M(G)$, the cycle matroid of G.

It is not hard to see that $M(G, \mathcal{B})$ is a frame matroid. If $(G', \mathcal{B}')$ is obtained from $(G, \mathcal{B})$ by adding an unbalanced loop at each vertex, then $M(G, \mathcal{B})$ is a restriction of $M(G', \mathcal{B}')$ and this set of unbalanced loops is a frame for $M(G', B')$. This construction is due to Zaslavsky [56], who also proved that the converse is true.

Theorem 1.2.2. *If M is a frame matroid, then there is a biased graph $(G, \mathcal{B})$ so that $M \cong M(G, \mathcal{B})$.*

Working with biased graphs can be easier than working with frame matroids. For example, it is easy to see that each long line of $M(G, \mathcal{B})$ is either a set of edges incident to at most two vertices of G, or a balanced triangle of G. This shows that each long line of a matroid M framed by B either spans two elements of B, or has length exactly three. The following lemma states some properties of frame matroids which are easy to prove by considering an associated biased graph.

Lemma 1.2.3. *Let M be a matroid framed by B. Let $b \in B$, and let f be an element of M which is not parallel to any element of B. Then*

(i) each line of M of length at least four spans two elements of B,

(ii) f is on at most one line of length at least four, and

(iii) each long line of M through b contains some $b' \in B - \{b\}$.

Note that (iii) implies that b is not the tip of a spike restriction of M, since $B - \{b\}$ is independent in M/b.

1.2.2 Dowling Geometries

Perhaps the most important families of biased graphs arise from group-labelled graphs. We define group-labeled graphs and the associated biased graphs, following [2]. A *group-labeling* of a graph G consists of an orientation of the edges of G and a function $\phi\colon E(G) \to \Gamma$ for some multiplicative group Γ. Roughly speaking, a cycle C of G is balanced if there is a simple closed walk around C so that the product of the group labels is the identity, but we make this more precise. We use ϕ to label each walk on G, and use this to define the balanced cycles of G.

For each walk W on G with edge sequence $e_1, e_2, \ldots, e_k$, define $\epsilon_i(W)$ by

$$\epsilon_i(W) = \begin{cases} 1 & \text{if } e_i \text{ is traversed forward in } W, \\ -1 & \text{if } e_i \text{ is traversed backward in } W, \end{cases}$$

and define $\phi(W) = \prod_{i=1}^{k} \phi(e_i)^{\epsilon_i(W)}$.

It is not hard to see that for each cycle C of G, either every simple closed walk W around C satisfies $\phi(W) = 1$, or there is no simple closed walk W around C so that $\phi(W) = 1$. Thus, we define $\mathcal{B}_\phi$ to be the set of all cycles C for which there is some simple closed walk W around C so that $\phi(W) = 1$. This set $\mathcal{B}_\phi$ of cycles satisfies the theta property, so $(G, \mathcal{B}_\phi)$ is a biased graph. The frame matroid of a biased graph constructed in this way from a graph labeled by Γ is called a Γ-*frame matroid*. In the work of Zaslavsky these matroids are called Γ-*gain-graphic*.

Dowling geometries are simple Γ-frame matroids of a given rank with as many elements as possible. Let Γ be a finite group of size at least two with identity 1, and let G be a graph with $k \geq 3$ vertices, a single loop at each vertex with label in $\Gamma - \{1\}$, and exactly $|\Gamma|$ parallel edges between each pair of vertices, so that between each pair of vertices each edge has the same orientation and each element of Γ appears as a label. The frame matroid obtained from this group-labelled graph is the rank-k *Dowling geometry* over the group Γ, and is denoted $\mathrm{DG}(k, \Gamma)$. If $|\Gamma| = 1$ then we define $\mathrm{DG}(k, \Gamma)$ to be the frame matroid constructed from the graph K_k with a loop at each vertex so that all nonloop cycles are unbalanced. It is not hard to show that $\mathrm{DG}(k, \{1\}) \cong M(K_{k+1})$, which implies that each graphic matroid is the frame matroid of a $\{1\}$-labeled graph. Dowling geometries were first introduced by Dowling in [4], although the definition presented here is due to Zaslavsky [56].

Note that the set B of loops of G is a basis of $\mathrm{DG}(k, \Gamma)$ since it does not contain any circuits. In fact, $\mathrm{DG}(k, \Gamma)$ is framed by B, since each edge of G is on a path between two loops. Also, $|\mathrm{DG}(k, \Gamma)| = |\Gamma|\binom{k}{2} + k$ since G has k loops and $|\Gamma|$ edges between each pair of vertices. The class of Dowling geometries with group Γ has the useful property that $\mathrm{si}(\mathrm{DG}(k, \Gamma))/e \cong \mathrm{DG}(k-1, \Gamma)$ for each $e \in E(\mathrm{DG}(k, \Gamma))$. It is also not hard to see that each simple rank-k Γ-frame matroid is a restriction of $\mathrm{DG}(k, \Gamma)$, which is analogous to the fact that each simple rank-k $\mathrm{GF}(q)$-representable matroid is a restriction of $\mathrm{PG}(k-1, q)$.

In [4] Dowling proved that $\mathrm{DG}(k, \Gamma)$ and $\mathrm{DG}(k, \Gamma')$ are isomorphic matroids if and only if Γ and Γ' are isomorphic groups. He also proved the following beautiful theorem, which we use to prove applications of our main result.

Theorem 1.2.4. *The Dowling geometry* $\mathrm{DG}(k, \Gamma)$ *is representable over a field* $\mathbb{F}$ *if and only if* Γ *is a subgroup of the multiplicative group of* $\mathbb{F}$.

If Γ is a subgroup of the multiplicative group of $\mathbb{F}$, then $\mathrm{DG}(k, \Gamma)$ has a natural representation over $\mathbb{F}$. Let $B = \{b_1, \dots, b_k\}$ be a frame for $\mathrm{DG}(k, \Gamma)$, and for each $\alpha \in \Gamma$ and $1 \leq i < j \leq k$, let $(i, j)_\alpha$ denote the element of $\mathrm{DG}(k, \Gamma)$ spanned by $\{b_i, b_j\}$ and labeled with α in the corresponding Γ-labeled graph. Then $\mathrm{DG}(k, \Gamma)$ is represented by the matrix A over $\mathbb{F}$ so that B indexes an identity submatrix of A, and for each $\alpha \in \Gamma$ and $1 \leq i < j \leq k$, the column of A indexed by $(i, j)_\alpha$ has a 1 in the row indexed by b_i and a α^{-1} in the row indexed by b_j. Since every finite subgroup of the multiplicative group of a field is cyclic, Theorem 1.2.4 implies that $\mathrm{DG}(k, \Gamma)$ is representable only if Γ is cyclic. The converse is true as well, since every cyclic group is a subgroup the multiplicative group of some field.

Dowling geometries also have the attractive property that if Γ' is a subgroup of Γ, then $\mathrm{DG}(k, \Gamma')$ is a restriction of $\mathrm{DG}(k, \Gamma)$. This is because if G is a Γ-labelled graph whose frame matroid is $\mathrm{DG}(k, \Gamma)$, then the restriction of the graph to edges labeled by the subgroup Γ' gives a Γ'-labelled graph whose frame matroid is $\mathrm{DG}(k, \Gamma')$. In particular, since $\mathrm{DG}(k, \{1\}) \cong M(K_{k+1})$ and every group has a subgroup of size one, every Dowling geometry has a spanning clique restriction. Since cliques are round and every matroid with a spanning round restriction is round, Dowling geometries are round. As we shall see in Chapter 2, Dowling geometries are occasionally easier to work with if we delete the frame B. We define $\mathrm{DG}^-(k, \Gamma) = \mathrm{DG}(k, \Gamma)\backslash B$, where B is a frame of $\mathrm{DG}(k, \Gamma)$.

1.3 The Growth Rate Theorem

This thesis is concerned with finding the extremal functions of minor-closed classes of matroids. One reason to study extremal functions is that they hint at the structure of matroids in minor-closed classes. This phenomenon is stunningly illustrated by the Growth Rate Theorem, which shows that the extremal function of a minor-closed class of matroids is either infinite, exponential, quadratic or linear. The Growth Rate Theorem is a combination of three results, and in this section we state these results and discuss their implications for the structure of matroids in minor-closed classes. Before this, we give an example of how extremal functions indicate structure in minor-closed classes of graphs.

We have seen that the class $\mathcal{G}$ of graphic matroids has extremal function $h_{\mathcal{G}}(n) = \binom{n+1}{2}$, which is a quadratic function of n. The following classical result of Mader [34] shows that the situation is quite different when we exclude any clique as a minor.

Theorem 1.3.1. *Let $t \geq 3$ be an integer, and let $\mathcal{G}_t$ denote the class of graphic matroids with no $M(K_t)$-minor. Then there is a constant c_t so that $h_{\mathcal{G}_t}(n) \leq c_t n$ for all $n \geq 0$.*

This dichotomy of extremal functions suggests that graphs with no K_t-minor are somehow more structured than graphs in general, since all of their minors have at most linearly many edges. This structure was famously described by Robertson and Seymour over the course of 23 papers collectively known as the Graph Minors Project [45].

The extremal functions of minor-closed classes of matroids exhibit similar behavior, although the dichotomies are more pronounced since matroids are more general objects than graphs. In perhaps the first extremal result for general minor-closed classes of matroids, Kung [31] showed that minor-closed classes of matroids have drastically different behavior depending on whether or not they exclude a rank-2 uniform matroid.

Theorem 1.3.2. *For each integer $\ell \geq 2$, the class $\mathcal{U}(\ell)$ of matroids with no $U_{2,\ell+2}$-minor satisfies $h_{\mathcal{U}(\ell)}(n) \leq \frac{\ell^n - 1}{\ell - 1}$ for all $n \geq 0$.*

Since rank-2 uniform matroids can have arbitrarily many elements, this theorem implies for a minor-closed class $\mathcal{M}$ that $h_{\mathcal{M}}(n)$ is finite for $n \geq 2$ if and only if there is some integer $\ell \geq 2$ so that $U_{2,\ell+2} \notin \mathcal{M}$. This major dichotomy suggests that matroids with no $U_{2,\ell+2}$-minor have some underlying structure. Indeed, Geelen, Gerards, and Whittle conjecture in [10] that highly vertically connected matroids which exclude a uniform minor are a bounded distance from either a frame matroid, the dual of a frame matroid, or a matroid representable over a finite field of bounded size. This conjecture is likely very difficult to prove, as it generalizes the ongoing Matroid Minors Project for matroids representable over finite fields [10, 21].

Another result of Kung [31] shows that there is a dichotomy among minor-closed classes of matroids which exclude $U_{2,\ell+2}$ for some integer $\ell \geq 2$, namely that the extremal function either has a polynomial upper bound or an exponential lower bound. This was strengthened by Geelen and Kabell [11], who showed that classes of matroids representable over finite fields are the fundamental reason for this dichotomy.

Theorem 1.3.3. *For each integer $\ell \geq 2$ and each minor-closed class $\mathcal{M}$ of matroids for which $U_{2,\ell+2} \notin \mathcal{M}$, there is a constant $c_{\mathcal{M}}$ so that either*

- $h_{\mathcal{M}}(n) \leq n^{c_{\mathcal{M}}}$ *for all $n \geq 0$, or*
- *there is a prime power q so that $\frac{q^n-1}{q-1} \leq h_{\mathcal{M}}(n) \leq c_{\mathcal{M}} q^n$ for all $n \geq 0$, and $\mathcal{M}$ contains all* $\mathrm{GF}(q)$*-representable matroids.*

Classes of the second type are called *base-q exponentially dense.* The fact that an algebraic object such as a finite field shows up in a purely combinatorial theorem is surprising, and indicates that exponentially dense classes may have more underlying structure.

There there are two more major dichotomies among minor-closed classes of matroids; the first was proved by Geelen and Whittle [19].

Theorem 1.3.4. *For each integer $\ell \geq 2$ and each minor-closed class $\mathcal{M}$ of matroids for which $U_{2,\ell+2} \notin \mathcal{M}$, there is a constant $c_{\mathcal{M}}$ so that either*

- $h_{\mathcal{M}}(n) \leq c_{\mathcal{M}} n$ *for all $n \geq 0$, or*
- $h_{\mathcal{M}}(n) \geq \binom{n+1}{2}$ *for all $n \geq 0$ and $\mathcal{M}$ contains all graphic matroids.*

Classes of the first type are called *linearly dense.* When stated differently it is clear that Theorem 1.3.4 generalizes Theorem 1.3.1.

Theorem 1.3.5. *There is a function $\alpha_{1.3.5} \colon \mathbb{Z}^2 \to \mathbb{Z}$ so that for all integers $\ell, t \geq 2$, if $M \in \mathcal{U}(\ell)$ satisfies $\varepsilon(M) > \alpha_{1.3.5}(\ell, t) r(M)$, then M has an $M(K_{t+1})$-minor.*

This theorem says that even among general matroids with no $U_{2,\ell+2}$-minor, it is hard to avoid having a clique minor, so matroids with no $U_{2,\ell+2}$-minor and no $M(K_t)$-minor should have some structural description. However, since this class of matroids contains all graphic matroids with no $M(K_t)$-minor, any structural description would likely be quite complex.

The last ingredient of the Growth Rate Theorem is a result of Geelen, Kung, and Whittle

[12] which shows that if $\mathcal{M}$ is minor-closed and $h_{\mathcal{M}}(n) \le n^c$ for some constant c, then in fact $h_{\mathcal{M}}(n) \le c'n^2$ for some constant c', so the dichotomy exhibited in Theorem 1.3.3 is in fact much sharper. This result, combined with Theorems 1.3.2 and 1.3.4, gives the Growth Rate Theorem [12].

Theorem 1.3.6 (Growth Rate Theorem)**.** *If $\mathcal{M}$ is a minor-closed class of matroids, then there exists a constant $c_{\mathcal{M}}$ so that either*

1. *$h_{\mathcal{M}}(n) = \infty$ for all $n \ge 2$ and $\mathcal{M}$ contains all rank-2 uniform matroids, or*
2. *there is a prime power q such that $\frac{q^n-1}{q-1} \le h_{\mathcal{M}}(n) \le c_{\mathcal{M}} q^n$ for all $n \ge 0$, and $\mathcal{M}$ contains all $\mathrm{GF}(q)$-representable matroids, or*
3. *$\binom{n+1}{2} \le h_{\mathcal{M}}(n) \le c_{\mathcal{M}} n^2$ for all $n \ge 0$ and $\mathcal{M}$ contains all graphic matroids, or*
4. *$h_{\mathcal{M}}(n) \le c_{\mathcal{M}} n$ for all $n \ge 0$.*

This theorem implies that matroids in minor-closed classes have a good deal of structure. Classes of the third type are called *quadratically dense*, and are the focus of this thesis. We highlight two corollaries of Theorem 1.3.6 for quadratically dense classes.

Corollary 1.3.7. *A minor-closed class $\mathcal{M}$ of matroids is quadratically dense if and only if $\mathcal{M}$ contains all graphic matroids and there are integers $\ell \ge 2$ and $n \ge 3$ so that $U_{2,\ell+2} \notin \mathcal{M}$ and $\mathcal{M}$ contains no rank-n projective geometry.*

Many interesting classes of representable matroids are quadratically dense. The following result uses the fact that a rank-3 projective geometry of characteristic p is only representable over fields of characteristic p, which we prove in Chapter 7.

Corollary 1.3.8. *Let $\mathcal{F}$ be a family of fields having no common finite subfield, and let $\ell \ge 2$ be an integer. Then the class of matroids representable over all fields in $\mathcal{F}$ and with no $U_{2,\ell+2}$-minor is quadratically dense.*

This result is of particular interest when $\mathcal{F}$ consists of either the complex numbers, the real numbers, or a pair of finite fields of different characteristic. In this thesis we find the extremal function for any class of matroids in Corollary 1.3.8 up to a linear error term, and determine the extremal function almost exactly in several special cases.

1.4 Classification of Extremal Functions

The Growth Rate Theorem shows that the extremal function of any minor-closed class of matroids which excludes a line is either exponential, quadratic, or linear. Perhaps within each of these classes there are more dichotomies that can be found through the study of extremal functions. This motivates the following problem.

Problem 1.4.1. *Classify the functions which can occur as the eventual extremal function of a minor-closed class of matroids.*

This problem is difficult, and it seems that linearly dense classes may be a bit too wild to admit a complete answer to Problem 1.4.1, as indicated by results of Eppstein [5]. This is true in part because extremal members of linearly dense classes need not be highly connected in any way, and indeed can arise from random constructions, even for graphs. As we will see in Section 1.5, even the very natural class of graphic matroids with no $M(K_t)$-minor is not well-behaved, since the extremal matroids arise from random graphs.

While the situation for linearly dense classes is bleak, the story for exponentially dense classes is quite the opposite. Geelen and Nelson [17] were amazingly able to give a complete answer to Problem 1.4.1 for exponentially dense classes.

Theorem 1.4.2. *Let q be a prime power. If $\mathcal{M}$ is a base-q exponentially dense minor-closed class of matroids, then there are nonnegative integers k and $d \leq \frac{q^{2k}-1}{q-1}$ so that $h_{\mathcal{M}}(n) = \frac{q^{n+k}-1}{q-1} - qd$ for all sufficiently large n. Moreover, each extremal matroid of large enough rank is, up to simplification, a k-element projection of a projective geometry.*

They were able to prove that extremal matroids in exponentially dense classes are *weakly round*, which is a strong connectivity property similar to roundness, and this was key in their proof.

The results for exponentially dense classes indicate that there is some hope for solving Problem 1.4.1 for quadratically dense classes. However, there is currently very little known about extremal functions of quadratically dense classes; the following seemingly basic conjecture from [10] is wide open in general.

Conjecture 1.4.3. *If $\mathcal{M}$ is a quadratically dense minor-closed class of matroids, then there are real numbers a, b and c so that $h_{\mathcal{M}}(n) = an^2 + bn + c$, for all sufficiently large n.*

Grace and Van Zwam [21] proved this for classes representable over a fixed finite field, using the powerful Matroid Minors Structure Theorem, whose proof is in the process of being written [10].

Proving Conjecture 1.4.3 is crucial for solving Problem 1.4.1 for quadratically dense classes, and a reasonable first step towards a proof is to determine which matroids can be among the densest matroids in quadratically dense classes. These are conjectured in [10] to be matroids which admit a B-clique after contracting a bounded set T.

More precisely, a matroid M is an (α, t)-*frame matroid* if it has a basis $V \cup T$ with $|T| = t$ such that

- for each $e \in E(M) - (V \cup T)$ the unique circuit of $M|(V \cup T \cup \{e\})$ contains at most two elements of V, and

- for each $u, v \in V$ there are exactly α elements that are in the span of $T \cup \{u, v\}$ but not in the span of either $T \cup \{u\}$ or $T \cup \{v\}$.

Note that M/T is a V-clique, and if $T = \varnothing$, then M is a matroid framed by V so that each pair of elements of V spans a $U_{2,\alpha+2}$-restriction. Geelen, Gerards, and Whittle make the following conjecture in [10].

Conjecture 1.4.4. *If $\mathcal{M}$ is a quadratically dense minor-closed class of matroids, then there are integers $\alpha \geq 1$ and $t \geq 0$ so that*

- $h_{\mathcal{M}}(n) = \alpha\binom{n}{2} + O(n)$*, and*
- *for each integer $n > t$, $\mathcal{M}$ contains an (α, t)-frame matroid of rank n.*

Since $\varepsilon(M) = \alpha\binom{r(M)}{2} + O(r(M))$ for any (α, t)-frame matroid M, this conjecture says that (α, t)-frame matroids are close to being the densest matroids in every quadratically dense class. A proof of this conjecture would show that the leading coefficient from Conjecture 1.4.3 is always an integer, and in this thesis we prove Conjecture 1.4.4 in an important special case.

1.5 Interesting Minor-Closed Classes

All of the results in Sections 1.3 and 1.4 are concerned with finding approximate bounds on the extremal functions for very general minor-closed classes of matroids. Whenever possible we would like to precisely determine the extremal function of a minor-closed class, particularly if the class is 'interesting' or 'natural' in some way. Indeed, this was stated by Kung [31] to be one of the fundamental problems of extremal matroid theory.

Problem 1.5.1. *Let $\mathcal{M}$ be an 'interesting' minor-closed class of matroids. Determine $h_{\mathcal{M}}(n)$, and characterize the extremal matroids of $\mathcal{M}$.*

Instances of this problem tend to be difficult; there are only a handful of classes for which Problem 1.5.1 has been completely solved. In this section we highlight notable instances that have been solved, focusing mostly on quadratically dense classes. Many of these interesting classes arise from fields, in particular finite fields.

As in Section 1.3, our first example comes from the class of graphic matroids with no $M(K_t)$-minor. It turns out that the constant c_t given by the original proof of Theorem 1.3.1 is 2^{t-3}, and is far from exact. Kostochka [27, 28] and Thomason [48] independently proved that the correct order of magnitude for c_t is $\sqrt{t \log t}$, and Thomason later gave an asymptotically best-possible bound in [49].

Theorem 1.5.2. *Let $t \geq 3$ be an integer, and let $\mathcal{G}_t$ denote the class of graphic matroids with no $M(K_t)$-minor. Then $h_{\mathcal{G}_t}(n) = ((\alpha + o_t(1))\sqrt{t \log t})n$ for all $n \geq 0$, where $\alpha = .319\ldots$ is an explicit constant.*

Thomason also proved that this upper bound is asymptotically tight for random graphs.

For exponentially dense minor-closed classes, Theorem 1.3.2 gives an easy upper bound on $h_{\mathcal{U}(\ell)}$, but it is not always tight. Geelen and Nelson [13] were able to find an eventual extremal function for $\mathcal{U}(\ell)$.

Theorem 1.5.3. *Let $\ell \geq 2$ be an integer, and let q be the largest prime power less than or equal to ℓ. Then $h_{\mathcal{U}(\ell)}(n) = \frac{q^n-1}{q-1}$ for all sufficiently large n.*

This shows that Theorem 1.3.2 is tight if and only if ℓ is a prime power. Just as in Theorem 1.3.3, it is surprising that finite fields play such a large role in these purely combinatorial classes. Nelson [39] was also able to find an eventual extremal function for the class of $\mathrm{GF}(q^2)$-representable matroids with no $\mathrm{PG}(k, q^2)$-minor, and for the class of matroids representable over $\mathrm{GF}(q^2)$ and $\mathrm{GF}(q^j)$ for each odd integer j.

Many interesting minor-closed classes of matroids turn out to be quadratically dense, for two main reasons. The first is that any class of matroids representable over a family $\mathcal{F}$ of fields which contains fields of different characteristic and a finite field is quadratically dense. In particular, any class of matroids representable over a pair of finite fields of different characteristic is quadratically dense. The second reason is that any minor-closed class of $\mathrm{GF}(p)$-representable matroids which does not contain all $\mathrm{GF}(p)$-representable matroids is quadratically dense, for any prime p. Classes of this type are particularly well-studied for binary matroids.

We first state some results for classes representable over a family of fields $\mathcal{F}$. A matroid is *regular* if it is representable over every field, and the extremal function for the class of regular matroids follows from a classical result of Heller [23].

Theorem 1.5.4. *The class of regular matroids has extremal function $h(n) = \binom{n+1}{2}$ for $n \geq 0$.*

Tutte [50, 51] proved that a matroid is regular if and only if it is representable over GF(2) and a field of characteristic other than two, so given Theorem 1.5.4 we may assume that $\mathrm{GF}(2) \notin \mathcal{F}$.

A matroid is *near-regular* if it is representable over all fields, except possibly GF(2), and a matroid is *sixth-root-of-unity* if it is representable over GF(3) and GF(4). Whittle [54] showed that a matroid is near-regular if and only if it is representable over GF(3) and GF(8). Even though the class of sixth-root-of-unity matroids properly contains the class of near-regular matroids, Oxley, Vertigan, and Whittle [43] proved that these classes have the same extremal function.

Theorem 1.5.5. *The class of sixth-root-of-unity matroids has extremal function* $h(n) = \binom{n+2}{2} - 2$ *for* $n \geq 4$*, and the extremal matroids are near-regular and are projections of cliques.*

Another well-studied class is the class of matroids representable over GF(3) and GF(5), called *dyadic* matroids. Kung and Oxley [29, 33] solved Problem 1.5.1 for the class of dyadic matroids.

Theorem 1.5.6. *The class of dyadic matroids has extremal function* $h(n) = n^2$ *for* $n \geq 0$*, and the extremal matroids are Dowling geometries over* $\mathrm{GF}(3)^\times$*.*

The authors comment that their proof also gives the following result for $\mathbb{R}$-representable matroids.

Theorem 1.5.7. *The class of* $\mathbb{R}$*-representable matroids with no* $U_{2,5}$*-minor has extremal function* $h(n) = n^2$ *for* $n \geq 0$*, and the extremal matroids are Dowling geometries over* $\mathrm{GF}(3)^\times$*.*

Whittle [54] proved that if $\mathcal{F}$ is a family of fields with $\mathrm{GF}(3) \in \mathcal{F}$, and $\mathcal{M}$ is the class of matroids representable over all fields in $\mathcal{F}$, then there is some $q \in \{2, 3, 4, 5, 7, 8\}$ so that $\mathcal{M}$ is the class of matroids representable over GF(3) and GF(q). He also proved that each 3-connected matroid representable over GF(3) and GF(7) is either a dyadic matroid or a sixth-root-of-unity matroid. Combined with Theorems 1.5.5 and 1.5.6, this implies that the extremal function for the class of matroids representable over GF(3) and GF(7) is $h(n) = n^2$, for all sufficiently large n. Thus, if $\mathrm{GF}(3) \in \mathcal{F}$, then the extremal function for the class of matroids representable over all fields in $\mathcal{F}$ is known.

We now highlight some results for proper minor-closed subclasses of GF(p)-representable matroids. The following theorem is a consequence of results proved independently by Sauer [46] and Shelah [47].

Theorem 1.5.8. *The class of binary matroids with no* $\mathrm{PG}(2, 2)$*-minor has extremal function* $h(n) = \binom{n+1}{2}$ *for* $n \geq 0$*.*

Since all regular matroids are binary and $\mathrm{PG}(2, 2)$ is not regular, the class of binary matroids with no $\mathrm{PG}(2, 2)$-minor contains all regular matroids, so Theorem 1.5.8 implies Theorem 1.5.4. Theorem 1.5.8 can be further strengthened by considering another natural class of binary matroids. The *affine geometry* $\mathrm{AG}(n, q)$ is obtained from $\mathrm{PG}(n, q)$ by deleting a hyperplane. Since $\mathrm{PG}(2, 2)$ is a minor of $\mathrm{AG}(3, 2)$, the following result of Kung, Mayhew, Pivotto, and Royle [32] implies Theorem 1.5.8.

Theorem 1.5.9. *The class of binary matroids with no* $\mathrm{AG}(3, 2)$*-minor has extremal function* $h(n) = \binom{n+1}{2}$ *for* $n \geq 6$*, and the extremal matroids are cliques.*

Theorems 1.5.4-1.5.9 are nearly all instances of Problem 1.5.1 which have been solved for quadratically dense classes, although it seems like the Matroid Minors Structure Theorem will lead to many new results [20, 22, 41]. In this thesis we prove a result which simultaneously generalizes Theorems 1.5.6, 1.5.7, and 1.5.9, for sufficiently large integers n.

1.6 Extremal Function Proofs

We now describe a general strategy for finding an eventual extremal function for a minor-closed class of matroids, and then discuss how this strategy has been used in the past and how we apply it in this thesis. This proof sketch in its full complexity was invented in [13], and a slightly simpler precursor can be found in [43]. There are generally three steps to show that $h_{\mathcal{M}}(n) \le f(n)$ for some function f.

Proof Strategy:

(I) If $h_{\mathcal{M}}(n) > f(n)$ for sufficiently large integers n, find some $M \in \mathcal{M}$ with $\varepsilon(M) > f(r(M))$ so that M has a connectivity property, a structured minor G, and a restriction X which is incompatible with G.

(II) Use the connectivity property of M to find a minor N with a structured minor G' of G as a spanning restriction such that $N|X = M|X$.

(III) Use the structure of G' and X to find a minor of N which is not in $\mathcal{M}$, giving a contradiction.

Step (I) deals with exploiting the density of matroids in $\mathcal{M}$. If f is quadratic or exponential in n, then $\mathcal{M}$ has extremal matroids with some connectivity property essentially because densities combine only linearly from piecing together two matroids of smaller rank. The restriction X usually arises by assuming M is minor-minimal with $\varepsilon(M) > f(r(M))$. This implies that for each element e of M there are many long lines of M through e, which provides useful structure. Step (II) deals with exploiting the connectivity property of M to find a minor N in which G' and X are forced to interact. We say that X is a 'certificate', since G' and X together certify that N has a minor which is not in $\mathcal{M}$.

This proof strategy was used to prove most of the results in Section 1.5. If $\mathcal{M}$ is a base-q quadratically dense class, then G is usually a projective geometry over GF(q), and X is a bounded-size collection of lines of length $q+2$, as in [13] and [39]. Then X is incompatible with G because none of these lines is contained in G, since $U_{2,q+2}$ is not GF(q)-representable. For exponentially dense classes, the connectivity property is roundness. Roundness is easy to work with because it is preserved under contraction, and is stronger than vertical k-connectivity for any k. Step (II) works well for exponentially dense classes since typically M is round and X has bounded size.

For quadratically dense classes, the minor G is generally a Dowling geometry, and the restriction X is a collection of spike-like matroids. Then X is incompatible with G because Dowling geometries do not contain spikes of rank at least five, as we will see in Chapter 3. This restriction X may not have bounded size, which makes step (II) more difficult. The connectivity property of M is generally vertical k-connectivity for some k (see [43] and [32]), and this property is unfortunately not preserved under contraction. One of the main technical contributions of this thesis is a pair of results, Theorems 4.0.1 and 4.5.7, which should make step (II) easy for a large family of quadratically dense classes.

While step (II) can be difficult for quadratically dense classes, there are two results of Geelen and Nelson which are very helpful for steps (I) and (III). The first shows that in any quadratically dense class, there exist extremal matroids which are highly vertically connected and have some structure [16].

Theorem 1.6.1. *Let $\mathcal{M}$ be a quadratically dense minor-closed class of matroids and let $p(x)$ be a real quadratic polynomial with positive leading coefficient. If $h_{\mathcal{M}}(n) > p(n)$ for infinitely many integers $n \geq 0$, then for all integers $r, s \geq 1$ there exists $M \in \mathcal{M}$ satisfying $\varepsilon(M) > p(r(M))$ and $r(M) \geq r$ such that either*

- *M has a spanning clique restriction, or*
- *M is vertically s-connected and has an s-element independent set S so that $\varepsilon(M) - \varepsilon(M/e) > p(r(M)) - p(r(M) - 1)$ for each $e \in S$.*

The key ingredients in the proof are Theorem 1.3.5 and the fact that quadratic functions are concave-up, and the structure in the second outcome essentially arises by taking a minor-minimal matroid with large density. Theorem 1.6.1 is essentially all we need for step (I) of the general proof strategy, although in Chapter 3 we will refine the structure given by the second outcome. In Chapter 3 we prove a slight strengthening of Theorem 1.6.1 by replacing 'infinitely many integers n' with 'a sufficiently large integer n', so that we can obtain explicit bounds for our main result. In [16] the authors apply Theorem 1.6.1 with $s = 4$; this thesis will see the first application of this theorem at full strength.

The second result of Geelen and Nelson shows that any matroid with a spanning clique restriction in a quadratically dense class is a bounded distance from a frame matroid [18].

Theorem 1.6.2. *There is a function $h_{1.6.2}\colon \mathbb{Z}^2 \to \mathbb{Z}$ so that for all integers $\ell, n \geq 2$ and any $M \in \mathcal{U}(\ell)$ with a spanning B-clique restriction and no rank-n projective-geometry minor, there is a set $\hat{B} \subseteq B$ and a $\hat{B}$-clique N such that $\mathrm{dist}(M, N) \leq h_{1.6.2}(\ell, n)$. Moreover, there are disjoint sets $C_1, C_2 \subseteq E(M)$ with $r_M(C_1 \cup C_2) \leq h_{1.6.2}(\ell, n)$ such that*

- $\mathrm{si}(N)$ *is isomorphic to a restriction of M/C_1,*
- *for all $X \subseteq E(M) - (C_1 \cup C_2)$, if $(M/(C_1 \cup C_2))|X$ is simple, then $N|X = (M/C_1)|X$.*

The second condition implies that $\varepsilon(N) \geq \varepsilon(M/(C_1 \cup C_2))$, by taking $(M/(C_1 \cup C_2))|X$ to be a simplification of $M/(C_1 \cup C_2)$.

This is a deep result, and is vital for our main proof. It is clearly useful when the first outcome of Theorem 1.6.1 holds, but even when the second outcome holds we will find a minor with a spanning clique restriction so that we can apply this theorem. Since every quadratically dense class excludes a line and all projective geometries of some fixed rank, Theorem 1.6.2 should be useful in step (III) of the general proof sketch for many quadratically dense classes.

1.7 This Thesis

This thesis is primarily motivated by the following problem.

Problem 1.7.1. *Classify the functions which can occur as the eventual extremal function of a quadratically dense minor-closed class of matroids.*

As we saw in Section 1.4, we can approach this problem by trying to prove Conjecture 1.4.4. In this thesis we prove Conjecture 1.4.4 in the case that $t = 0$, which means that there is some integer $n \geq 3$ so that $\mathcal{M}$ contains no $(\alpha, 1)$-frame matroid of rank-n. To ensure that $\mathcal{M}$ contains no $(\alpha, 1)$-frame matroid of rank-n, we exclude matroids called 'doubled cliques'.

For all integers $n \geq 3$, a rank-n *doubled clique* is a simple matroid M with an element e so that $\mathrm{si}(M/e) \cong M(K_n)$ and each parallel class of M/e has size two. The main result of this thesis shows that Dowling geometries are the densest matroids in quadratically dense classes which exclude all rank-n doubled cliques for some integer $n \geq 3$, up to a linear error term. This theorem and its applications are due to joint work with Peter Nelson and Jim Geelen that has not yet been published.

Theorem 1.7.2. *There is a function $f_{1.7.2} : \mathbb{Z}^4 \to \mathbb{Z}$ so that for all integers $\ell \geq 2$, $t \geq 1$ and $n, k \geq 3$, if $\mathcal{M}$ is a minor-closed class of matroids such that $U_{2,\ell+2} \notin \mathcal{M}$, then either*

- *$\mathcal{M}$ contains a rank-n doubled clique, or*
- *$\mathcal{M}$ contains $\mathrm{DG}(k, \Gamma)$ for some group Γ with $|\Gamma| \geq t$, or*
- *each $M \in \mathcal{M}$ satisfies $\varepsilon(M) \leq (t-1)\binom{r(M)}{2} + f_{1.7.2}(\ell, n, k, t) \cdot r(M)$.*

This theorem has the following corollary, which shows that there are dichotomies among minor-closed classes which exclude a line and all rank-n doubled cliques.

Theorem 1.7.3. *For all integers $\ell \geq 2$ and $n \geq 3$, if $\mathcal{M}$ is a minor-closed class of matroids such that $U_{2,\ell+2} \notin \mathcal{M}$ and $\mathcal{M}$ contains no rank-n doubled clique, then there is a constant $c_{\mathcal{M}}$ so that either*

- *$h_{\mathcal{M}}(r) \le c_{\mathcal{M}} \cdot r$ for all $r \ge 0$, or*
- *there is a group Γ so that $|\Gamma|\binom{r}{2} + r \le h_{\mathcal{M}}(r) \le |\Gamma|\binom{r}{2} + c_{\mathcal{M}} \cdot r$ for all $r \ge 0$, and $\mathcal{M}$ contains all Γ-frame matroids.*

This result is analogous to Theorem 1.3.3, with Γ-frame matroids playing the role of $\mathrm{GF}(q)$-representable matroids.

The proof of Theorem 1.7.2 takes up the bulk of this thesis. In Chapter 2 we prove some properties of Dowling geometries which we need for both Theorem 1.7.2 and its applications. Then Chapters 3-5 follow the proof strategy given in Section 1.6, where Chapter 3 corresponds to step (I), Chapter 4 corresponds to step (II), and Chapter 5 corresponds to step (III). After proving Theorem 1.7.2, we prove some applications in Chapters 6 and 7.

1.7.1 The Main Proof

Chapters 3-5 of this thesis are devoted to proving Theorem 1.7.2. In Chapter 3 we refine the structure given by the second outcome of Theorem 1.6.1 by showing that it leads to any of three distinct structures, each of which is incompatible with cliques. We define these three structures in more detail in Chapter 3, but we roughly describe them here so that we can informally state the main result of Chapter 3.

The first structure is a *stack*, which is essentially a collection of mutually skew restrictions of bounded rank so that each is either not a frame matroid, or has a $U_{2,t+2}$-minor. The second is a collection of nearly skew small spikes with a common tip. The third, and most important, is a collection of matroids called *porcupines*, which generalize spikes. The main result of Chapter 3, Theorem 3.6.1, shows that the second outcome of Theorem 1.6.1 leads to either a large stack, a large collection of nearly skew spikes with common tip, or a large collection of porcupines. This theorem, together with Theorem 1.6.1, gives step (I) of the general proof sketch from Section 1.6. In Chapter 5, we show that each of these restrictions in the span of a clique leads to either a rank-n doubled-clique minor, or a $\mathrm{DG}(k, \Gamma)$-minor with $|\Gamma| \ge t$.

The most difficult part of the proof of Theorem 1.7.2 is in step (II) of the general proof strategy, where we use vertical connectivity to find a minor with a spanning clique restriction, and a restriction consisting of a stack, nearly skew spikes, or porcupines. The following theorem is the main result of Chapter 4, and performs the bulk of the work in the proof of Theorem 1.7.2. We state this more formally in Chapter 4.

Theorem 1.7.4. *Let $\ell, m, n \ge 2$ and $k \ge 1$ be integers, let $M \in \mathcal{U}(\ell)$ be a matroid with no rank-n doubled-clique minor, and let Γ be a finite group. If M has sufficiently large vertically connectivity, a $\mathrm{DG}(r, \Gamma)$-minor G with r sufficiently large, and a size-k independent set such* *that each element is the tip of a porcupine restriction, then M has a minor N of rank at least m such that*

- *N has $\mathrm{DG}(r(N), \Gamma)$-restriction, and*
- *N has a size-k independent set such that each element is the tip of a porcupine restriction.*

The proof of Theorem 1.7.4 mostly relies on properties of porcupines, but a key step in the proof utilizes the clique-like structure of Dowling geometries. The main tool we use for maintaining connectivity is the notion of 'tangles', which we introduce in detail in Chapter 4. We also need the following generalization of Tutte's Linking Theorem for a nested collection of sets, which we prove in Chapter 4.

Theorem 1.7.5. *Let M be a matroid, $m \geq 1$ be an integer, and $Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y_m \subseteq E(M) - X$. Then M has a minor N with ground set $X \cup Y_m$ such that $\kappa_N(X, Y_i) = \kappa_M(X, Y_i)$ for each $i \in [m]$, while $N|X = M|X$ and $N|Y_1 = M|Y_1$.*

Finally, in Chapter 5 we prove Theorem 1.7.2, essentially by applying Theorems 1.6.1, 3.6.1 and 1.7.4, in that order.

1.7.2 Applications

In Chapters 6 and 7, we use Theorem 1.7.2 to solve Problem 1.5.1 for several interesting classes of matroids. The following theorem is the main result of Chapter 6.

Theorem 1.7.6. *For all integers $t \geq 1$, $\ell \geq 2$, and $k, n \geq 3$, if $\mathcal{M}$ is a minor-closed class of matroids so that $U_{2,\ell+2} \notin \mathcal{M}$, then either*

- *$\mathcal{M}$ contains a rank-n doubled clique, or*
- *$\mathcal{M}$ contains a nontrivial extension of $\mathrm{DG}(k, \Gamma)$ with $|\Gamma| \geq t$, or*
- *each $M \in \mathcal{M}$ with sufficiently large rank satisfies $\varepsilon(M) \leq t\binom{r(M)}{2} + r(M)$. Moreover, if $r(M)$ is sufficiently large and $\varepsilon(M) = t\binom{r(M)}{2} + r(M)$, then $\mathrm{si}(M)$ is isomorphic to a Dowling geometry.*

Then only new ingredient required for the proof of the upper bound is an analogue of Theorem 1.6.1 for classes which contain no rank-n doubled clique, which finds a matroid with a spanning Dowling-geometry restriction with large group size instead of a spanning clique restriction. Otherwise the proof uses the machinery developed in Chapters 3 and 4, in particular Theorems 3.6.1 and 1.7.4.

In Chapter 7 we show that Theorems 1.7.2 and 1.7.6 apply to classes of representable matroids. The key idea is that every large-rank doubled clique in $\mathcal{U}(\ell)$ has a rank-3 minor called a Reid geometry, which is only representable over fields of characteristic p for some prime p. This implies that large-rank doubled cliques are not representable over infinite fields, or fields of two different characteristics. In fact, this is also true for nontrivial extensions of Dowling geometries, which gives the following corollary of Theorem 1.7.6.

Theorem 1.7.7. *Let $\mathcal{F}$ be a family of fields having no common subfield, and let $t \geq 1$ be an integer. Then the class $\mathcal{M}$ of matroids representable over all fields in $\mathcal{F}$ and with no $U_{2,t+3}$-minor satisfies $h_{\mathcal{M}}(n) \leq t\binom{n}{2}+n$ for sufficiently large n. Moreover, if $\varepsilon(M) = t\binom{r(M)}{2}+r(M)$ and $r(M)$ is sufficiently large, then $\mathrm{si}(M)$ is isomorphic to a Dowling geometry.*

Whenever the size of the largest common subgroup (up to isomorphism) of the multiplicative groups of fields in $\mathcal{F}$ has size t, Dowling geometries over that group give a matching lower bound for the extremal function. There are two notable cases for which this occurs. The first was conjectured independently by Nelson [38] and Kapadia [25].

Theorem 1.7.8. *For each integer $t \geq 1$, the class of $\mathbb{C}$-representable matroids with no $U_{2,t+3}$-minor has extremal function $h(n) = t\binom{n}{2} + n$ for sufficiently large n. Moreover, if n is sufficiently large and equality holds for M, then $\mathrm{si}(M)$ is isomorphic to a Dowling geometry.*

The second was conjectured by Geelen, Gerards and Whittle in [10].

Theorem 1.7.9. *If $\mathbb{F}_1$ and $\mathbb{F}_2$ are finite fields with different characteristic such that $\mathbb{F}_1^{\times}$ is a subgroup of $\mathbb{F}_2^{\times}$, then the class of matroids representable over $\mathbb{F}_1$ and $\mathbb{F}_2$ has extremal function $h(n) = |\mathbb{F}_1^{\times}|\binom{n}{2} + n$ for sufficiently large n. Moreover, if n is sufficiently large and equality holds for M, then $\mathrm{si}(M)$ is isomorphic to a Dowling geometry.*

As we saw in Section 1.5, this result was already known when $\mathbb{F}_1 = \mathrm{GF}(2)$, and in many cases with $\mathbb{F}_1 = \mathrm{GF}(3)$, and was open in all other cases.

In Chapter 7 we also state a notable direct corollary of Theorem 1.7.2. Although it does not give a precise extremal function, it determines the correct leading coefficient of the extremal function of any quadratically dense class of matroids representable over a family of fields.

Theorem 1.7.10. *Let $\ell \geq 2$ and $\alpha \geq 1$ be integers so that $\ell > \alpha$. Let $\mathcal{F}$ be a family of fields having no common subfield so that α is the size of the largest common subgroup (up to isomorphism) of size less than ℓ, of the multiplicative groups of the fields in $\mathcal{F}$. Then the class $\mathcal{M}$ of matroids representable over all fields in $\mathcal{F}$ and with no $U_{2,\ell+2}$-minor has extremal function $h_{\mathcal{M}}(n) = \alpha\binom{n}{2} + O(n)$.*

Since the largest finite subgroup of the multiplicative group of the real numbers has size two, Theorem 1.7.10 has the following corollary.

Theorem 1.7.11. *For each integer $\ell \geq 3$, the class of $\mathbb{R}$-representable matroids with no*

$U_{2,\ell+2}$-minor has extremal function $h(n) = 2\binom{n}{2} + O(n)$ for all sufficiently large n.

The previous best bound on the leading coefficient for $\ell \geq 4$ was $\ell^{2^\ell - 1} - \ell^{2^\ell - 2}$, proved by Kung [30]. Theorem 1.7.11 determines the correct leading coefficient, but we would still like to determine this extremal function precisely for this natural class of matroids. To do so, we need to determine which matroids are the densest $\mathbb{R}$-representable matroids with no $U_{2,\ell+2}$-minor for $\ell \geq 4$. More generally, we would like to determine the extremal matroids for any minor-closed class which excludes a line and all rank-n doubled cliques. There is some evidence for the following conjecture, which is an analogue of Theorem 1.4.2.

Conjecture 1.7.12. *Let $\ell \geq 2$ and $n \geq 3$ be integers, and let $\mathcal{M}$ be a minor-closed class of matroids so that $U_{2,\ell+2} \notin \mathcal{M}$, and $\mathcal{M}$ contains no rank-n doubled clique. Then there is an integer $k \geq 0$ so that each simple extremal matroid of $\mathcal{M}$ of sufficiently large rank is a k-element projection of a Dowling geometry.*

In Chapter 8 we discuss the evidence for this conjecture, and other natural questions for minor-closed classes with no rank-n doubled clique. Finally, we discuss the possibility of using the techniques of this thesis to prove Conjecture 1.4.4.

Chapter 2

Dowling Geometries

In this chapter we prove some properties of Dowling geometries which are fundamental in this thesis. In Section 2.1 we provide sufficient conditions for a frame matroid to have a Dowling-geometry minor, using a well-known result of Kahn and Kung. In Section 2.2 we prove a result (Lemma 2.2.4) concerned with finding a $\mathrm{DG}^-(m,\Gamma)$-restriction, which is vital for the proof of Theorem 1.7.2.

2.1 Finding a Dowling-Geometry Minor

In this section we only consider clique-like frame matroids, since they provide a good deal of control when taking minors. Recall that a *B-clique* is a matroid M framed by B so that each pair of elements of B is contained in a long line of M.

The following beautiful theorem of Kahn and Kung [26] characterizes when a simple B-clique is a Dowling geometry, and is the foundation for all results in this section. The proof presented here is a slight modification of the proof in [26]. We remark that the condition that $r(M) \geq 4$ is necessary; it is possible to construct rank-3 matroids which satisfy the theorem hypotheses from Latin squares, and not all Latin squares arise as the Cayley table of a group.

Theorem 2.1.1. *For each integer $t \geq 1$, if M is a simple B-clique of rank at least four so that $|\operatorname{cl}_M(\{b_1, b_2\})| = t + 2$ for all distinct $b_1, b_2 \in B$, and*

(*) for all distinct $b_1, b_2, b_3 \in B$ and elements $a \in \operatorname{cl}_M(\{b_1, b_2\}) - B$ and $b \in \operatorname{cl}_M(\{b_1, b_3\}) - B$, there is some $c \in \operatorname{cl}_M(\{b_2, b_3\}) - B$ for which $\{a, b, c\}$ is a circuit of M,

then there exists a group Γ with $|\Gamma| = t$ so that $M \cong \mathrm{DG}(r(M), \Gamma)$.

Proof. Let $B = \{b_1, b_2, \dots, b_m\}$, and for all $1 \leq i < j \leq m$ let $L_{ij} = \operatorname{cl}_M(\{b_i, b_j\}) - B$. For all $1 \leq i < j < k \leq m$ let $t_{ijk} = \operatorname{cl}_M(\{b_i, b_j, b_k\})$. We will refer to t_{ijk} as a *facet* of M. Note that the element c from (*) is necessarily unique; otherwise there is a rank-2 set $\{a, b, c, c'\}$ which spans $\{b_1, b_2, b_3\}$. We will refer to 3-element circuits of M as *triangles*. A triangle of M is *basic* if it is spanned by some pair of elements of B, and *nonbasic* otherwise. Our first claim is a general property of rank-4 frame matroids.

2.1.1.1. *For all $1 \leq i < j < k < h \leq m$, if $\{a, b, c\}$, $\{c, d, e\}$, and $\{a, f, e\}$ are nonbasic triangles on three distinct facets out of t_{ijk}, t_{ijh}, t_{ikh}, and t_{jkh}, then $\{b, d, f\}$ is a nonbasic triangle on the fourth facet.*

Proof. Without loss of generality, we may assume that $(i, j, k, h) = (1, 2, 3, 4)$ and $\{a, b, c\}$, $\{c, d, e\}$ and $\{a, f, e\}$ are nonbasic triangles of t_{123}, t_{134}, and t_{124}, respectively. This implies that $a \in L_{12}$, $b \in L_{23}$, $c \in L_{13}$, $d \in L_{34}$, $e \in L_{14}$, and $f \in L_{24}$. We will show that

$\{b,d,f\}$ is a triangle of t_{234}. In M/b both $\{c,d,e\}$ and $\{a,f,e\}$ are circuits and a and c are parallel, so $r_{M/b}(\{a,c,d,e,f\}) = 2$. If d and f are not parallel in M/b, then $\{b_2,b_3,b_4\} \subseteq \mathrm{cl}_{M/b}(\{d,f\})$ since $b \in L_{23}$. But then $b_1 \subseteq \mathrm{cl}_{M/b}(\{d,f,e\})$ since $e \in L_{14}$, so $\{a,c,d,e,f\}$ spans $\{b_1,b_2,b_3,b_4\}$ in M/b, which contradicts that $r_{M/b}(\{a,c,d,e,f\}) = 2$. Thus, d and f are parallel in M/b, so $\{b,d,f\}$ is a nonbasic triangle of M. □

Let Γ be a set of size t, and let $\epsilon \in \Gamma$. We will label $E(M) - B$ by elements of Γ according to three rules, which force $\epsilon \in \Gamma$ to be the group identity element. Rules (A) and (B) look very similar, but we need both in order to show that ϵ commutes with all elements of Γ once we define a group operation on Γ.

2.1.1.2. *There exists a function $f\colon (E(M) - B) \to \Gamma$ so that for all $1 \le i < j < k \le m$,*

(A) if $\{a,b,c\}$ is a triangle in t_{ijk} not contained in L_{ij}, L_{ik}, or L_{jk} such that $b \in L_{jk}$ and $f(b) = \epsilon$, then $f(a) = f(c)$, and

(B) if $\{a,b,c\}$ is a triangle in t_{ijk} not contained in L_{ij}, L_{ik}, or L_{jk} such that $b \in L_{ij}$ and $f(b) = \epsilon$, then $f(a) = f(c)$, and

(C) f restricted to $L_{i'j'}$ is a bijection for all $1 \le i' < j' \le m$.

Proof. For each $2 \le j \le m$, arbitrarily choose an element $a \in L_{1j}$ and set $f(a) = \epsilon$. Arbitrarily assign labels to elements of L_{12} so that f restricted to L_{12} is a bijection. We use the following three steps to define f, relying on the fact that M satisfies (*).

(1) For each $2 \le j < k \le m$, let $a \in L_{jk}$ be the element in a triangle with the elements of L_{1j} and L_{1k} labelled ϵ, and set $f(a) = \epsilon$. Note that this element $a \in L_{jk}$ exists since M satisfies (*).

(2) For each $3 \le k \le m$, let $a \in L_{1k}$ be the element in a triangle with the element of L_{12} labelled α and the element of L_{2k} labelled ϵ, and set $f(a) = \alpha$. This shows that t_{12k} satisfies (A) for all $3 \le k \le m$.

(3) For each $2 \le j < k \le m$, let $a \in L_{jk}$ be the element in a triangle with the element of L_{1j} labelled ϵ and the element of L_{1k} labelled α, and set $f(a) = \alpha$. This show that t_{1jk} satisfies (B) for all $2 \le j < k \le m$.

We now have a function $f\colon (E(M) - B) \to \Gamma$, and we will show that it satisfies (A), (B), and (C). If f restricted to L_{1k} is not a bijection for some $3 \le k \le m$, then there is some $a \in L_{1k}$ and $e \in L_{2k}$ which are in triangles with two distinct elements of L_{12}, which contradicts the uniqueness of c in (*). Similarly, if f restricted to L_{jk} is not a bijection for some $2 \le j < k \le m$, then there is some $a \in L_{jk}$ and $e \in L_{1j}$ which are in triangles with two distinct elements of L_{1k}, which contradicts the uniqueness of c in (*). Thus, f satisfies (C).

We now prove a sequence of claims to show that f satisfies (A) and (B). For each $\alpha \in \Gamma$ and $1 \leq i < j \leq m$, let α_{ij} denote the element in L_{ij} such that $f(\alpha_{ij}) = \alpha$.

(i) By (1) and 2.1.1.1 with (a, b, c, d, e, f) all labelled ϵ, the set $\{\epsilon_{ij}, \epsilon_{ik}, \epsilon_{jk}\}$ is a triangle of M for all $1 \leq i < j < k \leq m$.

(ii) By (2), f satisfies (A) in t_{12k} for all $3 \leq k \leq m$.

(iii) By (i), (ii) and 2.1.1.1 with $(a, b, c, d, e, f) = (\alpha_{12}, \alpha_{1j}, \epsilon_{2j}, \epsilon_{jk}, \epsilon_{2k}, \alpha_{1k})$ we find that $\{\alpha_{1j}, \epsilon_{jk}, \alpha_{1k}\}$ is a triangle, so f satisfies (A) in t_{1jk} for all $2 \leq j < k \leq m$.

(iv) By (3), f satisfies (B) in t_{1jk} for all $2 \leq j < k \leq m$.

(v) By (i), (iv) and 2.1.1.1 with $(a, b, c, d, e, f) = (\alpha_{1k}, \alpha_{ik}, \epsilon_{1i}, \epsilon_{ij}, \epsilon_{1j}, \alpha_{jk})$ we find that $\{\epsilon_{ij}, \alpha_{jk}, \alpha_{ik}\}$ is a triangle, so (B) holds for all $2 \leq i < j < k \leq m$.

(vi) By (iv), (iii) and 2.1.1.1 with $(a, b, c, d, e, f) = (\epsilon_{1i}, \alpha_{ij}, \alpha_{1j}, \epsilon_{jk}, \alpha_{1k}, \alpha_{ik})$, we find that $\{\alpha_{ij}, \epsilon_{jk}, \alpha_{ik}\}$ is a triangle, so (A) holds for all $2 \leq i < j < k \leq m$.

Thus, f satisfies (A) by (iii) and (vi) and f satisfies (B) by (iv) and (v). □

Now for each facet t_{ijk} we define a binary operation $\circ_{ijk} : \Gamma \times \Gamma \to \Gamma$ by $\circ_{ijk}(\alpha, \beta) = \gamma$ if $\{\alpha_{ij}, \beta_{jk}, \gamma_{ik}\}$ is a triangle in M. These operations are well-defined since M satisfies (*).

2.1.1.3. $\circ_{ijk} = \circ_{i'j'k'}$ *for all* $1 \leq i < j < k \leq m$ *and* $1 \leq i' < j' < k' \leq m$.

Proof. We will show that any two facets with two common indices have the same binary operation. Without loss of generality, it suffices to show that $\circ_{123} = \circ_{124}$ and $\circ_{134} = \circ_{234}$ and $\circ_{124} = \circ_{134}$. We first show that $\circ_{123} = \circ_{124}$. Let $\alpha, \beta \in \Gamma$. By 2.1.1.1 with $(a, b, c, d, e, f) = (\beta_{23}, (\alpha \circ_{123} \beta)_{13}, \alpha_{12}, (\alpha \circ_{124} \beta)_{14}, \beta_{24}, \epsilon_{34})$, we find that $\{\epsilon_{34}, (\alpha \circ_{123} \beta)_{13}, (\alpha \circ_{124} \beta)_{14}\}$ is a triangle. This shows that $\alpha \circ_{123} \beta = \alpha \circ_{124} \beta$ since rule (A) holds for t_{134}.

Similarly, 2.1.1.1 with $(a, b, c, d, e, f) = (\alpha_{13}, \epsilon_{12}, \alpha_{23}, (\alpha \circ_{234} \beta)_{24}, \beta_{34}, (\alpha \circ_{134} \beta)_{14})$ shows that $\circ_{134} = \circ_{234}$. Lastly, 2.1.1.1 with $(a, b, c, d, e, f) = (\alpha_{12}, \alpha_{13}, \epsilon_{23}, \beta_{34}, \beta_{24}, (\alpha \circ_{124} \beta)_{14})$ shows that $\circ_{124} = \circ_{134}$. Thus, any two facets with two common indices have the same binary operation, and this implies that all facets have the same binary operation on Γ. □

Thus, there is a single binary operation $\circ$ on Γ defined by $\circ = \circ_{123}$.

2.1.1.4. $(\Gamma, \circ)$ *is a group.*

Proof. Clearly $\alpha \circ \epsilon = \epsilon \circ \alpha = \alpha$ for all $\alpha \in \Gamma$, since (A) and (B) hold for t_{123}, so ϵ is the identity. Also, the inverse of α is the unique element β such that $\{\alpha_{12}, \beta_{23}, \epsilon_{13}\}$ is a triangle of M, and this element exists by (*). For all $\alpha, \beta, \gamma \in \Gamma$, by 2.1.1.1 with

$$(a, b, c, d, e, f) = (\beta_{23}, \alpha_{12}, (\alpha \circ \beta)_{13}, ((\alpha \circ \beta) \circ \gamma)_{14}, \gamma_{34}, (\beta \circ \gamma)_{24})$$

we find that $\{\alpha_{12}, (\beta \circ \gamma)_{24}, ((\alpha \circ \beta) \circ \gamma)_{14}\}$ is a triangle of M. Thus, $\alpha \circ (\beta \circ \gamma) = (\alpha \circ \beta) \circ \gamma$, so $\circ$ is a group operation on Γ. Note that we require $r(M) \geq 4$ in order to prove associativity. □

We will now show that M is isomorphic to a Dowling geometry, using biased graphs. Let $(G, \mathcal{B})$ be a biased graph on $r(M)$ vertices whose frame matroid is M such that there is an unbalanced loop at each vertex. Let $(G, \mathcal{B}')$ be the group-labelled graph obtained from G by labelling each $e \in E(M) - B$ by $f(e)$. We may assume that if $e \in L_{ij}$, then e is oriented from b_i to b_j. Note that the frame matroid of $(G, \mathcal{B}')$ is $\mathrm{DG}(r(M), (\Gamma, \circ))$, which satisfies (*).

2.1.1.5. *$\mathcal{B} = \mathcal{B}'$.*

Proof. We first show that $\{a, b, c\} \in \mathcal{B}$ if and only if $\{a, b, c\} \in \mathcal{B}'$. Let $a \in L_{ij}$, $b \in L_{jk}$, and $c \in L_{ik}$. If $\{a, b, c\} \in \mathcal{B}$, then $f(a) \circ f(b) = f(c)$, and due to the orientation of edges of G, the value of the cycle is $f(a) \circ f(b) \circ (f(c))^{-1} = \epsilon$, so $\{a, b, c\} \in \mathcal{B}'$. If $\{a, b, c\} \in \mathcal{B}'$, then $f(a) \circ f(b) \circ (f(c))^{-1} = \epsilon$, and so $f(a) \circ f(b) = f(c)$ and thus $\{a, b, c\} \in \mathcal{B}$. Therefore, $\mathcal{B}$ and $\mathcal{B}'$ contain the same triangles of G.

We now show that $\mathcal{B} \subseteq \mathcal{B}'$. Let $C \in \mathcal{B}$ have minimum size so that $C \notin \mathcal{B}'$. Then $|C| \geq 4$ since $\mathcal{B}$ and $\mathcal{B}'$ have no cycles of size less than three since the corresponding frame matroids are simple, and $\mathcal{B}$ and $\mathcal{B}'$ have the same triangles. Without loss of generality assume that $C = \{a_1, a_2, \ldots, a_{k-1}, a_k\}$ such that $a_j \in L_{j,j+1}$ for each $j \in [1, k-1]$, and $a_k \in L_{1k}$.

By (*) for M there is some $d \in L_{13}$ so that $\{a_1, a_2, d\} \in \mathcal{B}$. Since $\mathcal{B}$ satisfies the theta property, $\{d, a_3, a_4, \ldots, a_k\} \in \mathcal{B}$. This cycle has size less than k and is thus in $\mathcal{B}'$. Then since $\{a_1, a_2, d\} \in \mathcal{B}'$ and $\mathcal{B}'$ satisfies the theta property, $C \in \mathcal{B}'$. The same argument can be applied to show that $\mathcal{B}' \subseteq \mathcal{B}$. □

Since $(G, \mathcal{B}) = (G, \mathcal{B}')$ and the frame matroid of $(G, \mathcal{B}')$ is $\mathrm{DG}(r(M), (\Gamma, \circ))$, we have shown that $M \cong \mathrm{DG}(r(M), (\Gamma, \circ))$. □

In this thesis we only need Theorem 2.1.1 for the following corollary, which says that if a frame matroid is not a Dowling geometry, then there is some element which we can contract to increase density.

Corollary 2.1.2. *Let M be a simple B-clique of rank at least four such that $|\mathrm{cl}_M(\{b, b'\})| = t + 2$ for all $b, b' \in B$. Then M is isomorphic to a Dowling geometry if and only if there is no $e \in E(M)$ so that $\varepsilon((M/e)|\mathrm{cl}_{M/e}(\{b, b'\})) \geq t + 3$ for some $b, b' \in B$.*

Proof. If M is isomorphic to a Dowling geometry, then $\mathrm{si}(M/e)$ is isomorphic to a Dowling geometry, so $\varepsilon((M/e)|\,\mathrm{cl}_{M/e}(\{b,b'\})) \le t+2$ for all $b, b' \in B$ and $e \in E(M)$.

If M is not isomorphic to a Dowling geometry, then by Lemma 2.1.1 there are distinct $b_1, b_2, b_3 \in B$ and elements $a \in \mathrm{cl}_M(\{b_2, b_3\}) - B$ and $b \in \mathrm{cl}_M(\{b_1, b_2\}) - B$ such that $\mathrm{cl}_M(\{a,b\}) = \{a,b\}$. Now $(M/a)|\,\mathrm{cl}_M(\{b_1,b_3\}) \cong M|\,\mathrm{cl}_M(\{b_1,b_3\}) \cong U_{2,t+2}$, and since $\{a,b\}$ is a flat of M we have $(M/a)|(\mathrm{cl}_M(\{b_1,b_3\}) \cup \{b\}) \cong U_{2,t+3}$. □

We also need a straightforward lemma about B-cliques which is essentially equivalent to the statement that a complete graph with a matching J of size $\binom{k}{2}$ has a K_k-minor with edge-set J.

Lemma 2.1.3. *Let $k, t \ge 3$ be integers, let M be a B-clique and let $\mathcal{X}$ be a collection of $\binom{k}{2}$ pairwise disjoint 2-element subsets of B. Then there is some $B' \subseteq B$ with $|B'| = k$ and a B'-clique minor N of M such that $r_N(X) = r_M(X) = 2$ for all $X \in \mathcal{X}$, and for all distinct $b_1, b_2 \in B'$ there is some $X \in \mathcal{X}$ so that $\mathrm{cl}_M(X) \subseteq \mathrm{cl}_N(\{b_1,b_2\})$.*

Proof. We write $\cup\mathcal{X}$ for $\cup_{X\in\mathcal{X}} X$. We may assume that $B = \cup\mathcal{X}$. Let $(B_1, B_2, \dots, B_k)$ be a partition of $\cup\mathcal{X}$ so that $|B_i| = k-1$ for each $i \in [k]$, and for each $1 \le i < j \le k$ there is some $x \in \mathcal{X}$ such that $|X \cap B_i| = 1$ and $|X \cap B_j| = 1$. This amounts to arbitrarily putting X into bijection with $E(K_k)$. For each $i \in [k]$, fix some $b_i \in B_i$ and let $C_i \subseteq \mathrm{cl}_M(B_i) - B$ be a set of size $k-2$ so that for each $b \in B_i - \{b_i\}$ there is some $e \in (C_i \cap \mathrm{cl}_M(\{b_i, b\})) - B$. Such a set C_i exists since M is a B-clique. Then $B_i \cap \mathrm{cl}_M(C_i) = \varnothing$ or else C_i spans B_i. Since $|B_i| = |C_i| + 1$ we have $r_{M/C_i}(B_i) = 1$.

Let $N = M/(\cup_i C_i)$, and let B' be a transversal of $(B_1, B_2, \dots, B_k)$ which is independent in N. Then B' spans N since $B = \cup\mathcal{X}$, so N is a B'-clique. Let $b_1, b_2 \in B'$ with $b_1 \ne b_2$, and assume without loss of generality that $b_1 \in B_1$ and $b_2 \in B_2$. There is some $X \in \mathcal{X}$ such that $|X \cap B_1| = 1$ and $|X \cap B_2| = 1$. Let $b_1' \in X \cap B_1$ and $b_2' \in X \cap B_2$, and note that b_j and b_j' are parallel in N for each $j \in \{1,2\}$ since $B_i \cap \mathrm{cl}_M(C_i) = \varnothing$ and $r_{M/C_i}(B_i) = 1$ for each $i \in [k]$. Thus, $r_N(X) = r_M(X) = 2$, and $\mathrm{cl}_M(X) \subseteq \mathrm{cl}_N(X) = \mathrm{cl}_N(\{b_1, b_2\})$. □

We now combine Corollary 2.1.2 and Lemma 2.1.3 to show that any B-clique in $\mathcal{U}(\ell)$ with very large rank contains a big Dowling-geometry minor with group size as large as possible. The idea is that we contract elements and increase density until we have a B-clique minor such that each pair of elements of B spans a $U_{2,\ell+1}$-restriction, and this must be a Dowling geometry by Corollary 2.1.2.

Lemma 2.1.4. *Let $\ell - 1 \ge t \ge 1$ and $k \ge 3$ be integers. If $M \in \mathcal{U}(\ell)$ is a B-clique and $\mathcal{X}$ is a collection of pairwise disjoint 2-element subsets of B such that $|\mathcal{X}| \ge k^{2^\ell}$ and each $X \in \mathcal{X}$ satisfies $\varepsilon(M|\,\mathrm{cl}_M(X)) \ge t+2$, then there is some $B' \subseteq B$ and a B'-clique minor of M which is isomorphic to $\mathrm{DG}(k, \Gamma)$ with $|\Gamma| \ge t$.*

Proof. Define a function $f_1 \colon \mathbb{Z}^3 \to \mathbb{Z}$ by $f_1(\ell, \ell, k) = 1$, and $f_1(\ell, t, k) = \binom{(k+1)\cdot f_1(\ell, t+1, k)}{2}$ for $1 \le t < \ell$. Define $f_{2.1.4}(\ell, k) = f_1(\ell, 1, k)$, and note that $f_{2.1.4}(\ell, k) \ge f_1(\ell, t, k)$ for all $1 \le t < \ell$. One can show using induction that $f_{2.1.4}(\ell, k) \le k^{2^\ell - 2}$. Fix $\ell \ge 2$, and let t be maximal so that there exists a B-clique $M \in \mathcal{U}(\ell)$ with a collection $\mathcal{X}$ of pairwise disjoint 2-element subsets of B such that $|\mathcal{X}| \ge f_{2.1.4}(\ell, k)$ and each $X \in \mathcal{X}$ satisfies $\varepsilon(M|\operatorname{cl}_M(X)) \ge t + 2$, but there is no $B' \subseteq B$ and a B'-clique minor of M which is isomorphic to $\mathrm{DG}(k, \Gamma)$ with $|\Gamma| \ge t$.

By Lemma 2.1.3, there is some $B' \subseteq B$ with $|B'| = (k+1) \cdot f_1(\ell, t+1, k)$ and a B'-clique minor N of M such that $\varepsilon(N|\operatorname{cl}_N(\{b_1, b_2\})) \ge t + 2$ for all distinct $b_1, b_2 \in B'$. If $t = \ell - 1$ then N is a rank-$(k+1)$ Dowling geometry by Corollary 2.1.2, so M is not a counterexample. Thus, $t < \ell - 1$. Let $(B_1, B_2, \dots, B_h)$ be a partition of B' such that $h = f_1(\ell, t+1, k)$ and $|B_i| = k + 1$ for each $i \in [h]$. Since M is a counterexample and $r_M(B_i)k + 1 \ge 4$, for each $i \in [h]$ there is some $e_i \in \operatorname{cl}_N(B_i)$ such that N/e_i has a $U_{2,t+3}$-restriction spanned by two elements of B_i, by Corollary 2.1.2. By the maximality of t, there is some $B'' \subseteq B'$ and a B''-clique minor of $N/(\cup_i e_i)$ which is isomorphic to $\mathrm{DG}(k, \Gamma)$ with $|\Gamma| \ge t+1$, which contradicts that M is a counterexample. □

As an easy corollary we can bound the size of any simple B-clique in $\mathcal{U}(\ell)$ with no Dowling-geometry minor with large group size.

Corollary 2.1.5. *For all integers $\ell \ge t \ge 2$ and $k \ge 3$, if $M \in \mathcal{U}(\ell)$ is a B-clique with no $\mathrm{DG}(k, \Gamma)$-minor with $|\Gamma| \ge t$, then $\varepsilon(M) \le (t-1)\binom{r(M)}{2} + 2\ell k^{2^\ell} \cdot r(M)$.*

Proof. Let $m = k^{2^\ell}$. Let $M \in \mathcal{U}(\ell)$ be a B-clique, and let $\mathcal{X}$ be a maximum-size collection of pairwise disjoint 2-element subsets of B so that each $X \in \mathcal{X}$ satisfies $\varepsilon(M|\operatorname{cl}_M(X)) \ge t + 2$. By Lemma 2.1.4 we have $|\mathcal{X}| < m$, so by the maximality of $|\mathcal{X}|$ there are at most $2(m-2) \cdot r(M)$ pairs $b, b' \in B$ such that $\varepsilon(M|\operatorname{cl}_M(\{b, b'\})) \ge t + 2$. Since M is a B-clique and has no $U_{2,\ell+2}$-restriction, each pair $b, b' \in B$ satisfies $\varepsilon(M|(\operatorname{cl}_M(\{b, b'\}) - \{b, b'\})) \le \ell - 1$. Thus, $\varepsilon(M) \le (t-1)\binom{r(M)}{2} + (2(m-2)\ell + 1)r(M)$. □

2.2 Finding a Dowling-Geometry Restriction

Recall that $\mathrm{DG}^-(k, \Gamma) = \mathrm{DG}(k, \Gamma) \backslash B$, where B is a frame of $\mathrm{DG}(k, \Gamma)$. The main result of this section shows that each matroid in $\mathcal{U}(\ell)$ with no rank-n doubled-clique minor and a $\mathrm{DG}(r, \Gamma)$-minor has a $\mathrm{DG}^-(k, \Gamma)$-restriction if r is sufficiently large. The key is the following lemma, which provides sufficient conditions for a $\mathrm{DG}^-(k, \Gamma)$-minor of a matroid M to be a restriction of M.

Before stating the lemma we define a special type of circuit of $\mathrm{DG}^-(k, \Gamma)$. Let G be a

Dowling geometry over a group Γ, and let B be a frame for G. We say that a circuit C of $G\backslash B$ is *balanced* if $\mathrm{cl}_G(C) \cap B = \varnothing$. The balanced circuits are precisely the circuits of G which correspond to balanced cycles of a Γ-labeled graph associated with G.

Lemma 2.2.1. *Let $k \geq 4$ be an integer, let Γ be a finite group, and let M be a matroid with a $\mathrm{DG}^-(k, \Gamma)$-minor G. If each balanced circuit of G of size at most four is also a circuit of M, then G is a restriction of M.*

Proof. Let G_1 be a Dowling geometry with frame $B = \{b_1, b_2, \ldots, b_k\}$ so that $G_1\backslash B = G$. Assume for a contradiction that $r(G) < r_M(E(G))$. Let H be a Γ-labeled graph so that $V(H) = B$ and the associated frame matroid is G, and each edge between vertices b_i and b_j with $i < j$ is oriented from b_i to b_j. For each $\alpha \in \Gamma$ and $1 \leq i < j \leq k$, let $(i,j)_\alpha$ denote the element of G spanned by $\{b_i, b_j\}$ and labeled by α in the Γ-labeling of H, and let E_α denote the set of elements of G labeled by α. Let ϵ denote the identity element of Γ.

2.2.1.1. $M|E_\epsilon = G|E_\epsilon$.

Proof. If not, then let C' be a circuit of $G|E_\epsilon$ of minimum size so that C' is independent in M. Since each circuit of $G|E_\epsilon$ is a balanced circuit of G we have $|C'| > 4$. Let C_1 and C_2 be distinct circuits of $G|E_\epsilon$ of size less than $|C'|$ so that C_1, C_2, C' are the cycles of a theta subgraph of H; these circuits may be obtained by adding a chord to C'. Then $C' \subseteq C_1 \cup C_2$ and $r_G(C_1 \cup C_2) = |C_1 \cup C_2| - 2$. By the minimality of $|C'|$, both C_1 and C_2 are circuits of M. Then $M|(C_1 \cup C_2)$ has at least two circuits, so $r_M(C_1 \cup C_2) \leq |C_1 \cup C_2| - 2 \leq r_G(C_1 \cup C_2)$. This implies that $M|C' = G|C'$, a contradiction. $\square$

If $|\Gamma| = 1$, then 2.2.1.1 shows that $M|E(G) = G$, so we may assume that $|\Gamma| \geq 2$. This implies that $r_M(E_\epsilon) < r(G) < r_M(E(G))$.

2.2.1.2. *Each $\gamma \in \Gamma - \{\epsilon\}$ satisfies $M|(E_\epsilon \cup E_\gamma) = G|(E_\epsilon \cup E_\gamma)$, and $E_\gamma \subseteq \mathrm{cl}_M(E_\epsilon \cup \{(1,2)_\gamma\})$.*

Proof. We show that $(i,j)_\gamma \in \mathrm{cl}_M(E_\epsilon \cup \{(1,2)_\gamma\})$ for all $1 \leq i < j \leq k$, which implies that

$$r_M(E_\epsilon \cup E_\gamma) \leq r_M(E_\epsilon) + 1 = r_G(E_\epsilon) + 1 = r_G(E_\epsilon \cup E_\gamma),$$

where the first equality holds by 2.2.1.1. We first prove this in the case that $2 < i < j \leq k$. The set $\{(1,2)_\gamma, (2,j)_\epsilon, (i,j)_\gamma, (1,i)_\epsilon\}$ is a balanced circuit of G of size at most four and is thus a circuit of M, which implies that $(i,j)_\gamma \in \mathrm{cl}_M(E_\epsilon \cup \{(1,2)_\gamma\})$. Similarly, the balanced circuit $\{(1,2)_\gamma, (2,j)_\epsilon, (1,j)_\gamma\}$ shows that $(1,j)_\gamma \in \mathrm{cl}_M(E_\epsilon \cup \{(1,2)_\gamma\})$ for all $3 \leq j \leq k$. Using this, the balanced circuit $\{(1,2)_\epsilon, (2,j)_\gamma, (1,j)_\gamma\}$ shows that $(2,j)_\gamma \in \mathrm{cl}_M(E_\epsilon \cup \{(1,2)_\gamma\})$ for all $3 \leq j \leq k$. $\square$

Fix some $\gamma \in \Gamma - \{\epsilon\}$. We will show that $E(G) \subseteq \mathrm{cl}_M(E_\epsilon \cup E_\gamma)$; then

$$r_M(E(G)) \le r_M(E_\epsilon \cup E_\gamma) = r_G(E_\epsilon \cup E_\gamma) = r(G),$$

where the first equality holds by 2.2.1.2. By 2.2.1.2 it suffices to show that each $\beta \in \Gamma - \{\epsilon, \gamma\}$ satisfies $(1,2)_\beta \in \mathrm{cl}_M(E_\epsilon \cup E_\gamma)$. Let $\beta \in \Gamma - \{\epsilon, \gamma\}$, and note that $C_1 = \{(1,2)_\beta, (2,4)_\epsilon, (3,4)_\beta, (1,3)_\epsilon\}$ and $C_2 = \{(1,2)_\beta, (2,4)_\beta, (3,4)_\beta, (1,3)_\beta\}$ are both balanced circuits of G. Also, the set $\{(1,3)_\beta, (1,3)_\epsilon, (2,4)_\epsilon, (2,4)_\beta\}$ is independent in G since it contains no balanced cycle of H, and no handcuff or theta subgraph of H. Thus,

$$r_G(C_1 \cup C_2) \ge 4 = |C_1 \cup C_2| - 2 \ge r_M(C_1 \cup C_2),$$

where the last inequality holds because $M|(C_1 \cup C_2)$ contains two distinct circuits, and so $M|(C_1 \cup C_2) = G|(C_1 \cup C_2)$. The set $C = \{(1,3)_\beta, (1,3)_\epsilon, (1,2)_\beta, (2,4)_\beta, (2,4)_\epsilon\}$ is a loose handcuff of H which contains no balanced cycle of H, and is thus a circuit of G. Since $C \subseteq C_1 \cup C_2$ and $M|(C_1 \cup C_2) = G|(C_1 \cup C_2)$ it follows that C is a circuit of M. Then $(1,2)_\beta \in \mathrm{cl}_M(C - \{(1,2)_\beta\})$, since $M|C$ is a circuit. Since $(C - \{(1,2)_\beta\}) \subseteq E_\epsilon \cup E_\gamma$, this implies that $(1,2)_\beta \in \mathrm{cl}_M(E_\epsilon \cup E_\gamma)$, as desired. $\square$

To prove the main result of this section we need to be able to recognize when a matroid has a rank-n doubled-clique minor.

Lemma 2.2.2. *Let $\ell \ge 2$ and $n \ge 3$ be integers, and let $M \in \mathcal{U}(\ell)$ be a simple matroid with an element e so that M/e has a spanning B-clique restriction. If there is a collection $\mathcal{S}$ of $n^{2^{\ell+1}}$ pairwise disjoint 2-subsets of B so that each $S \in \mathcal{S}$ spans a nontrivial parallel class of M/e which is disjoint from S, then M has a rank-n doubled-clique minor with tip e.*

Proof. Note that $n^{2^{\ell+1}} \ge \binom{n^{2^\ell}}{2}$, and assume that $|\mathcal{S}| = \binom{n^{2^\ell}}{2}$. Let X and Y be disjoint transversals of the parallel classes of M/e spanned by sets in $\mathcal{S}$, so $|X| = |Y| = \binom{n^{2^\ell}}{2}$.

2.2.2.1. *There is some $C \subseteq E(M/e)$ and $X' \subseteq X$ so that $(M/e/C)|X' \cong M(K_n)$ and $e \notin \mathrm{cl}_M(C)$.*

Proof. By Lemma 2.1.3 with $\mathcal{X} = \mathcal{S}$ and $k = n^{2^\ell}$, there is some $B_1 \subseteq B$ and a B_1-clique minor M_1 of M/e so that $|B_1| = n^{2^\ell}$ and for all $b, b' \in B_1$ with $b \neq b'$ there is some $x \in X$ for which $\{b, b', x\}$ is a circuit of M_1. Then $M_1|(B_1 \cup X)$ is a B_1-clique, so by Lemma 2.1.4 with $k = n$ there is some $B_2 \subseteq B_1$ and a B_2-clique minor M_2 of $M_1|(B_1 \cup X)$ so that $M_2 \cong M(K_{n+1})$.

Then B_2 corresponds to a spanning star of K_{n+1}, so $M_2 \backslash B_2 \cong M(K_n)$. Since $M_1|(B_1 \cup X)$ is a B_1-clique, each element of $B_1 - B_2$ is parallel in M_2 to an element of B_2. Since M_2 is simple and $E(M_2) \subseteq B_1 \cup X$, this implies that $E(M_2 \backslash B_2) \subseteq X$. Thus, the claim holds by taking $X' = E(M_2 \backslash B_2)$, and $C \subseteq E(M)$ to be a set so that M_2 is a restriction of $M/e/C$. □

Let $Y' \subseteq Y$ be the set of elements on a line of M through e and an element of X'. In M/C there is no line through e which contains two elements of X', since $(M/e/C)|X'$ is simple. Therefore, $(M/C)|(X' \cup Y' \cup \{e\})$ is simple, and is thus a doubled clique by 2.2.2.1. □

We also need to be able to recover the frame of a $\mathrm{DG}^-(k, \Gamma)$-restriction so that we can apply Lemma 2.2.2. The following lemma shows that we can recover $\mathrm{DG}(k, \Gamma)$ from $\mathrm{DG}^-(k+2, \Gamma)$ by contracting two elements.

Lemma 2.2.3. *Let G_1 be a matroid so that $G_1 \cong \mathrm{DG}(r(G_1), \Gamma)$, and let B be a frame of G_1. If $G = G_1 \backslash B$, then each pair of elements of B spans a subset C of $E(G)$ so that $|C| \leq 2$ and G/C has a $\mathrm{DG}(r(G/C), \Gamma)$-restriction.*

Proof. If $|\Gamma| = 1$, then $G \cong \mathrm{DG}(r(G), \Gamma)$ and the lemma holds with $C = \varnothing$, so we may assume that $|\Gamma| \geq 2$. Let b_1 and b_2 be distinct elements of B, and let e_1 and e_2 be distinct elements in $\mathrm{cl}_{G_1}(\{b_1, b_2\}) \cap E(G)$. We will show that each $b \in B - \{b_1, b_2\}$ is parallel to an element of G in $G_1/\{e_1, e_2\}$. Let $b \in B - \{b_1, b_2\}$, and let $x \in \mathrm{cl}_{G_1}(\{b, b_1\}) \cap E(G)$. Then $r_{G_1}(\{e_1, e_2, b, x\}) \leq 3$ since this set is spanned by $\{b_1, b_2, b\}$ in G_1. Also, $\{b, x\}$ is disjoint from $\mathrm{cl}_{G_1}(\{e_1, e_2\})$, since $\{e_1, e_2\}$ and $\{b_1, b_2\}$ span the same flat of G_1 and $\{b, x\}$ is disjoint from $\mathrm{cl}_{G_1}(\{b_1, b_2\})$. Thus, $\{b, x\}$ is a parallel pair of $G_1/\{e_1, e_2\}$, and so $\mathrm{si}(G/\{e_1, e_2\})$ is isomorphic to $\mathrm{si}(G_1/\{e_1, e_2\})$. Since $\mathrm{si}(G_1/\{e_1, e_2\}) \cong \mathrm{DG}(r(G) - 2, \Gamma)$, this implies that $G/\{e_1, e_2\}$ has a $\mathrm{DG}(r(G) - 2, \Gamma)$-restriction. □

We now prove the main result of this section, which lets us move from a $\mathrm{DG}^-(r, \Gamma)$-minor of a matroid M to a $\mathrm{DG}(m, \Gamma)^-$-restriction whenever $M \in \mathcal{U}(\ell)$ has no rank-n doubled-clique minor, and r is sufficiently large.

Lemma 2.2.4. *Let $\ell \geq 2$ and $m, n \geq 3$ and $d \geq 0$ be integers, and let Γ be a finite group. There is a function $f_{2.2.4} \colon \mathbb{Z}^3 \to \mathbb{Z}$ so that if $M \in \mathcal{U}(\ell)$ is a matroid with no rank-n doubled-clique minor, and with a $\mathrm{DG}^-(m + f_{2.2.4}(\ell, n, d), \Gamma)$-minor G for which $r_M(E(G)) - r(G) \leq d$, then $M|E(G)$ has a $\mathrm{DG}^-(m, \Gamma)$-restriction.*

Proof. Let $m_1 = n^{2^{\ell+1}}$, and define $f_{2.2.4}(\ell, n, d) = 8(d+2)m_1 + 2$. Assume for a contradiction that the lemma is false. Let M be a counterexample so that M/C_0 has a $\mathrm{DG}^-(m + 8(d + 2)m_1 + 2, \Gamma)$-restriction G_0, and $|C_0|$ is minimal over all counterexamples. If there is some $e \in C_0 - \mathrm{cl}_M(E(G_0))$, then the lemma holds for M/e if and only if it holds for M, so $C_0 \subseteq \mathrm{cl}_M(E(G_0))$. This implies that $r_M(C_0) = r_M(E(G_0)) - r(G_0) \leq d$. Minimality of $|C_0|$ also implies that C_0 is independent in M.

By Lemma 2.2.3 there is a set $C_1 \subseteq E(G_0)$ so that $|C_1| \leq 2$ and G_0/C_1 has a $\mathrm{DG}(m + 8(d+2)m_1, \Gamma)$-restriction G. Let $C = C_0 \cup C_1$, and note that $r_M(C) \leq d+2$ and that C is independent in M. Let B be a frame for G, and let H be a Γ-labeled graph so that $V(H) = B$ and the associated frame matroid is $G \backslash B$. We say that two circuits of G are *vertex-disjoint* if the corresponding cycles of H are vertex-disjoint. Let $\mathcal{C}$ be a maximum-size collection of pairwise vertex-disjoint balanced circuits of G, each of size at most four, which are independent in M. If $|\mathcal{C}|$ is large, then we can find a rank-n doubled-clique minor.

2.2.4.1. $|\mathcal{C}| \leq (d+2)(m_1 - 1)$.

Proof. Assume for a contradiction that $|\mathcal{C}| > (d+2)(m_1-1)$. Since each set $Y \in \mathcal{C}$ is independent in M, each Y satisfies $\sqcap_M(Y, C) = 1$. We first contract all but one element of C. For each $Y \in \mathcal{C}$ there is some $c_Y \in C$ so that $\sqcap_M(Y, C - \{c_Y\}) = 0$; otherwise there is a pair c, c' in C so that $\{c, c'\} \subseteq \mathrm{cl}_{M/(C-\{c,c'\})}(Y)$, and so $\sqcap_M(Y, C) > 1$ since C is independent in M. Since $|\mathcal{C}| > (d+2)(m_1-1)$ and $|C| \leq d+2$, there is some $c \in C$ and some $\mathcal{C}_1 \subseteq \mathcal{C}$ so that $|\mathcal{C}_1| = m_1$ and each $Y \in \mathcal{C}_1$ satisfies $\sqcap_M(Y, C - \{c\}) = 0$. Let $M_1 = (M/(C - \{c\}))|(E(G) \cup \{c\})$. Then $M_1/c \cong G$, and each $Y \in \mathcal{C}_1$ is independent in M_1.

For each $Y \in \mathcal{C}_1$, let $Y' \subseteq Y$ so that $|Y'| = |Y| - 2$. Let $Z = \cup_{Y \in \mathcal{C}_1} Y'$, and let M_2 be a simplification of M_1/Z. Let $B_1 \subseteq B$ be a frame for G/Z. Then $\mathrm{cl}_G(B_1)$ is a spanning Dowling-geometry restriction of M_2/c, and for each $Y \in \mathcal{C}_1$ the set $(Y - Y') \cup \{c\}$ is a circuit of M_2, since $Y - Y'$ is a parallel pair of G/Z but not of M_2. Also, for each $Y \in \mathcal{C}_1$ the parallel class of $Y - Y'$ in G/Z is disjoint from B_1, since Y is a balanced circuit of G. So for each $Y \in \mathcal{C}_1$ there is some $B_Y \subseteq B_1$ so that $|B_Y| = 2$ and $Y - Y' \subseteq \mathrm{cl}_{M_2/c}(B_Y)$. Note that the sets B_Y are pairwise disjoint subsets of B_1, since the circuits in $\mathcal{C}_1$ are pairwise vertex-disjoint. Thus, M_2 is a simple matroid with an element c so that M_2/c has a spanning B_1-clique restriction, and there is a collection $\mathcal{X}$ of m_1 pairwise disjoint 2-subsets of B_1 for which each $X \in \mathcal{X}$ spans a nontrivial parallel class of M_2/c which contains neither element of X. But then M_2 has a rank-n doubled-clique minor by Lemma 2.2.2, a contradiction. □

Let $B_1 \subseteq B$ be a set of minimum size so that $(\cup_{Y \in \mathcal{C}} Y) \subseteq \mathrm{cl}_G(B_1)$, and let $B_2 = B - B_1$. Then $|B_1| \leq 8|\mathcal{C}| \leq 8(d+2)(m_1-1)$, so $|B_2| \geq m+1 \geq 4$. By the maximality of $|\mathcal{C}|$, each balanced circuit of $\mathrm{cl}_G(B_2)$ of size at most four is a circuit of M. Then Lemma 2.2.1 shows that $G|(\mathrm{cl}_G(B_2) - B_2)$ is a restriction of M. □

Chapter 3

Exploiting Density

This chapter is concerned with step (I) of the general growth-rates-proof sketch outlined in Section 1.6, in which we try to exploit the density of a matroid M to find a restriction which is incompatible with Dowling geometries. We will be working with matroids M which have an s-element independent set S so that each $e \in S$ satisfies $\varepsilon(M) - \varepsilon(M/e) > p(r(M)) - p(r(M) - 1)$, for some quadratic polynomial p. We show that such a matroid admits one of three distinct structures, each of which will lead to either a doubled-clique minor or a Dowling-geometry minor. To find these structures we need p to have a large linear coefficient, and we need $|S|$ to be large; we exploit both 'local' and 'global' density of M.

In Section 3.2 we introduce these three structures, and then prove some properties in Sections 3.3 and 3.5. In Section 3.6 we prove Theorem 3.6.1, which is the main result of this chapter. Before any of this, in Section 3.1 we prove a slight strengthening of Theorem 1.6.1. We apply the Growth Rate Theorem in Section 3.1, but this chapter is otherwise self-contained. Given a collection $\mathcal{Y}$ of sets, we will write $\cup\mathcal{Y}$ for $\cup_{Y \in \mathcal{Y}} Y$, for convenience.

3.1 An Upgraded Connectivity Reduction

We now prove a strengthening of Theorem 1.6.1. The proof closely follows [16], but we separate out two lemmas which we will use in Chapter 6. The arguments are very delicate, since they involve careful examination of vertical separations.

The first lemma is a property of matroids in quadratically dense classes, and it will lead us towards the second outcome of Theorem 1.6.1. It essentially follows from the Growth Rate Theorem (Theorem 1.3.6), and the fact that quadratic functions with positive leading coefficient are concave up.

Lemma 3.1.1. *There is a function $\nu_{3.1.1} \colon \mathbb{R}^7 \to \mathbb{Z}^+$ so that for all integers $\ell, k \geq 2$ and $r, s \geq 1$ and any real quadratic polynomial $p(x) = ax^2 + bx + c$ with $a > 0$, if $M \in \mathcal{U}(\ell)$ has no rank-k projective-geometry minor, $r(M) > 0$, and $\varepsilon(M) > p(r(M)) + \nu_{3.1.1}(a, b, c, \ell, k, r, s) \cdot r(M)$, then M has a vertically s-connected minor N such that*

- *$r(N) \geq r$ and $\varepsilon(N) > p(r(N)) + \nu_{3.1.1}(a, b, c, \ell, k, r, s) \cdot r(N)$, and*
- *$\varepsilon(N) - \varepsilon(N/e) > p(r(N)) - p(r(N) - 1) + \nu_{3.1.1}(a, b, c, \ell, k, r, s)$ for each $e \in E(N)$.*

Proof. Fix ℓ, k, r, s, and p. Let n_0 be a positive integer such that $p(x) > p(x - 1) \geq 0$ for all real $x \geq n_0$. By Theorem 1.3.6 there is a real number $\alpha > 0$ such that $\varepsilon(M) \leq \alpha p(r(M))$ for all matroids $M \in \mathcal{U}(\ell)$ with no rank-k projective-geometry minor and $r(M) \geq n_0$. Let $n_1 \geq \max(r, s, n_0)$ be an integer so that

$$a(\alpha + 2s)(x + y) + ((\alpha + 1)b + \alpha|c|)s + c - as^2 \leq 2axy$$

for all real $x, y \geq n_1$. Finally, define $\nu_{3.1.1}(a,b,c,\ell,k,r,s) = \nu = \max(-b, \ell^{n_1}, \ell^{n_1} - \min_{x\in\mathbb{R}} p(x))$. Note that the polynomial $p(x) + \nu x$ satisfies $p(x) + \nu x \geq \ell^{n_1}$ for all $x \in \mathbb{R}$, and is nondecreasing for all $x > 0$.

Let $M \in \mathcal{U}(\ell)$ with no rank-k projective-geometry minor, $r(M) > 0$ and $\varepsilon(M) > p(r(M)) + \nu r(M)$. Let N be a minimal minor of M such that $r(N) > 0$ and $\varepsilon(N) > p(r(N)) + \nu r(N)$. Note that N is simple. Since $r(N) > 0$ we have $\varepsilon(N) \geq \ell^{n_1}$. This implies that $r(N) \geq n_1$, since $N \in \mathcal{U}(\ell)$. By the minor-minimality of N, each $e \in E(N)$ satisfies $\varepsilon(N) - \varepsilon(N/e) > p(r(N)) - p(r(N)-1) + \nu$. Since $r(N) \geq n_1 \geq \max(r,s)$, N is not vertically s-connected or else the lemma holds.

Let (A, B) be a partition of $E(N)$ so that $r_N(A) \leq r_N(B) < r(N)$ and $r_N(A) + r_N(B) < r(N) + s - 1$. Let $r_A = r_N(A)$ and $r_B = r_N(B)$ and $r_N = r(N)$. If $r_A < n_1$, then $|A| < \ell^{n_1}$, so

$$\begin{aligned} |B| &= |N| - |A| && (1)\\ &> p(r_N) + \nu r_N - \ell^{n_1} && (2)\\ &\geq p(r_N - 1) + \nu(r_N - 1) && (3)\\ &\geq p(r_B) + \nu r_B, && (4) \end{aligned}$$

which contradicts the minor-minimality of N. Line (3) holds because $\nu \geq \ell^{n_1}$ and $p(x) \geq p(x-1)$ for all $x \geq n_1$, and line (4) holds because $a > 0$ and $\nu + b \geq 0$. Thus, $r_B \geq r_A \geq n_1$.

We now show that $x = r_A$ and $y = r_B$ contradicts the definition of n_1. Since $r_N \geq n_1 \geq n_0$ we have $p(r_N) \geq 0$, so $\nu r_N < |N| \leq \alpha p(r_N)$. Since $r_N \geq 1$, this implies that

$$\nu \leq \alpha\Big(a r_N + b + \frac{c}{r_N}\Big) \leq \alpha\Big(a(r_A + r_B) + b + |c|\Big).$$

Using the partition (A, B), we have

$$p(r_A + r_B - s) + \nu(r_A + r_B - s) \leq p(r_N) + \nu r_N < |A| + |B| \leq p(r_A) + \nu r_A + p(r_B) + \nu r_B,$$

where the first inequality holds because $r_A + r_B - s \leq r_N$ and $a > 0$ and $\nu + b \geq 0$, and the third inequality holds by the minor-minimality of N. Then expanding p and simplifying gives

$$s(\nu + b) + c - as^2 + 2as(r_A + r_B) > 2r_A r_B.$$

Combining this with our upper bound for ν gives

$$a(\alpha + 2s)(r_A + r_B) + ((\alpha + 1)b + \alpha|c|)s + c - as^2 > 2r_A r_B,$$

which contradicts that $r_B \geq r_A \geq n_1$. □

The second lemma is mostly a property of Dowling geometries, and relies on the fact that the closure of any subset of the frame of a Dowling geometry is itself a Dowling geometry. The proof is taken almost verbatim from Claim 6.1.1 in [16].

Lemma 3.1.2. *There is a function $n_{3.1.2}\colon \mathbb{R}^6 \to \mathbb{Z}$ so that for all integers $\ell \geq 2$ and $r, s \geq 1$ and any real quadratic polynomial $q(x) = ax^2 + bx + c$ with $a > 0$, if $M \in \mathcal{U}(\ell)$ satisfies $\varepsilon(M) > q(r(M))$ and has a $\mathrm{DG}(n_{3.1.2}(a, b, c, \ell, r, s), \Gamma)$-minor, then M has a minor N such that $r(N) \geq r$ and $\varepsilon(N) > q(r(N))$ and either*

(a) N has a $\mathrm{DG}(r(N), \Gamma)$-restriction, or

(b) N has an s-element independent set S so that each $e \in S$ satisfies $\varepsilon(N) - \varepsilon(N/e) > q(r(N)) - q(r(N) - 1)$.

Proof. Define $n_{3.1.2}(a, b, c, \ell, r, s) = (s(s-1)+1)n_2$, where $n_2 \geq r+1$ is an integer such that $q(x) - q(y) \geq \ell^s$ for all real x, y with $x \geq n_2$ and $x - 1 \geq y \geq 0$.

Let $M \in \mathcal{U}(\ell)$ satisfy $\varepsilon(M) > q(r(M))$ and have a $\mathrm{DG}(n_{3.1.2}(a, b, c, \ell, r, s), \Gamma)$-minor N_1. Let M_1 be a minimal minor of M so that $\varepsilon(M_1) > q(r(M_1))$ and N_1 is a minor of M_1, and let C be an independent set in M_1 so that N_1 is a spanning restriction of M_1/C. We may assume that $|C| < s$ or else M_1 and C satisfy (b), by the minimality of M_1.

Let $i \geq 0$ be minimal so that there is a minor M_2 of M_1 for which $\varepsilon(M_2) > q(r(M_2))$, and there exists $X \subseteq E(M_2)$ such that $r_{M_2}(X) \leq i$ and M_2/X has a $\mathrm{DG}((is+1)n_2, \Gamma)$-restriction N_2. Note that $(i, M_2, X) = (s-1, M_1, C)$ is a candidate since $|C| \leq s-1$, so this choice is well-defined. We consider two cases depending on whether $i = 0$.

Suppose that $i > 0$ and let $Y_1, Y_2, \ldots, Y_s, Z$ be mutually skew sets in N_2 so that $N_2|Y_j \cong \mathrm{DG}(n_2, \Gamma)$ for each $j \in [s]$ and $N_2|Z \cong \mathrm{DG}(((i-1)s+1)n_2, \Gamma)$; these sets can be chosen to be the closures in N_2 of disjoint subsets of a frame for N_2. If $M_2|Y_j = N_2|Y_j$ for some $j \in [s]$, then M_2 has a $\mathrm{DG}(n_2, \Gamma)$-restriction, which contradicts that $i > 0$ and i is minimal. Thus, $M_2|Y_j \neq N_2|Y_j$ for each j, implying that $r_{M_2/Y_j}(X) \leq r_{M_2}(X) - 1 \leq i - 1$ for each j.

Let $Y = Y_1 \cup \cdots \cup Y_s$ and let J be a maximal subset of Y such that $\varepsilon(M_2/J) > q(r(M_2/J))$. Let $M_3 = M_2/J$. If $Y_j \subseteq J$ for some j, then $r_{M_3}(X) \leq i - 1$ and $(M_3/X)|Z = N_2|Z \cong \mathrm{DG}((i-1)s+1)n_2, \Gamma)$, contradicting the minimality of i. Therefore, $Y - J$ contains a transversal T of $(Y_1, \ldots, Y_s)$. Note that T is an s-element independent set of N_2/J and therefore of $M_2/J = M_3$. Moreover, by the maximality of J, each $e \in T$ satisfies $\varepsilon(M_3) - \varepsilon(M_3/e) > q(r(M_3)) - q(r(M_3) - 1)$. Since $r(M_3) \geq r(N_2|Z) \geq n_2 - 1 \geq r$, (b) holds for M_3 and T.

If $i = 0$, then N_2 is a $\mathrm{DG}(n_2, \Gamma)$-restriction of M_2. Let M_4 be a minimal minor of M_2 such that $\varepsilon(M_4) > q(r(M_4))$ and N_2 is a restriction of M_4. If N_2 is spanning in M_4 then (a) holds.

Otherwise, by minimality we have $\varepsilon(M_4|\operatorname{cl}_{M_4}(E(N_2)) \le q(r(N_2))$, so since $r(M_4) \ge n_2$ we have

$$\varepsilon(M_4 \backslash \operatorname{cl}_{M_4}(E(N_2))) > q(r(M_4)) - q(r(N_2)) \ge \ell^s.$$

Therefore, there is an s-element independent set S of M_4 which is disjoint from $\operatorname{cl}_{M_4}(E(N_2))$. Since N_2 is a restriction of M_4/e for each $e \in S$, it follows from the minor-minimality of M_4 that M_4 and S satisfy (b). □

We now combine Lemmas 3.1.1 and 3.1.2 to prove the main result of this section, which is a slight strengthening of Theorem 1.6.1.

Theorem 3.1.3. *There is a function $r_{3.1.3}\colon \mathbb{R}^6 \to \mathbb{Z}$ so that for all integers $\ell, k \ge 2$ and $r, s \ge 1$ and any real polynomial $p(x) = ax^2 + bx + c$ with $a > 0$, if $M \in \mathcal{U}(\ell)$ satisfies $r(M) \ge r_{3.1.3}(a, b, c, \ell, r, s)$ and $\varepsilon(M) > p(r(M))$, then M has a minor N with $\varepsilon(N) > p(r(N))$ and $r(N) \ge r$ such that either*

(1) N has a spanning clique restriction, or

(2) N is vertically s-connected and has an s-element independent set S so that $\varepsilon(N) - \varepsilon(N/e) > p(r(N)) - p(r(N) - 1)$ for each $e \in S$.

Proof. We first define the function $r_{3.1.3}$. Let $\nu = \nu_{3.1.1}(a, b, c, \ell, \max(r, s), r, s)$, and define $\hat{r}_1$ to be an integer so that

$$(2s + 1)a(x + y) + s(\nu + b) + c - as^2 \le 2axy$$

and $p(x-s) \le p(x-s+1)$ for all real $x, y \ge \hat{r}_1$. Let f be a function which takes in an integer m and outputs an integer $f(m) \ge \max(r, 2m, 2\hat{r}_1)$ such that $p(x) - p(x-1) \ge ax + \ell^{\max(m, \hat{r}_1)}$ for all real $x \ge f(m)$. Define $r_{\lceil \nu/a \rceil} = 1$, and for each $i \in \{0, 1, 2, \dots, \lceil \nu/a \rceil - 1\}$ recursively define r_i to be an integer so that $p(x) > \alpha_{1.3.5}(\ell, n_{3.1.2}(a, b, c, \ell, f(r_{i+1}), s)) \cdot x$ for all real $x \ge r_i$. Finally, define $r_{3.1.3}(a, b, c, \ell, r, s) = r_0$.

Let $M \in \mathcal{U}(\ell)$ such that $r(M) \ge r_0$ and $\varepsilon(M) > p(r(M))$. We may assume that M has no rank-$\max(r, s)$ projective-geometry minor; otherwise outcome (2) holds. Let $\mathcal{M}$ denote the class of minors of M. We may assume that $h_{\mathcal{M}}(n) \le p(n) + \nu n$ for all $n \ge 1$, or else (2) holds by Lemma 3.1.1. The following claim essentially finds some ν' so that the coefficient of the linear term of $h_{\mathcal{M}}(n)$ is in the interval $[\nu' + b - a, \nu' + b + a]$.

3.1.3.1. *There is some $0 \le \nu' < \nu$ and $i \ge 0$ such that $h_{\mathcal{M}}(n) > p(n) + \nu' n$ for some $n \ge r_i$, and $h_{\mathcal{M}}(n) \le p(n) + (\nu' + a)n$ for all $n \ge r_{i+1}$.*

Proof. We will break up the real interval $[0, \nu]$ into subintervals of size a. Define $\nu_i = ai$ for all $i \in \{0, 1, 2, \ldots, \lceil \frac{\nu}{a} \rceil\}$. Let $i \geq 0$ be minimal so that $h_{\mathcal{M}}(n) \leq p(n) + \nu_{i+1} n$ for all $n \geq r_{i+1}$. This choice of i is well-defined, because $i = \lceil \nu/a \rceil - 1$ is a valid choice since $\nu_{\lceil \nu/a \rceil} \geq \nu$ and $h_{\mathcal{M}}(n) \leq p(n) + \nu n$ for all $n \geq 1 = r_{\lceil \nu/a \rceil}$.

If $i > 0$, then $h_{\mathcal{M}}(n) > p(n) + \nu_i$ for some $n \geq r_i$ by the minimality of i. If $i = 0$, then M certifies that $h_{\mathcal{M}}(n) > p(n)$ for some $n \geq r_0$. Thus, there is some $i \geq 0$ such that $h_{\mathcal{M}}(n) > p(n) + \nu_i n$ for some $n \geq r_i$, and $h_{\mathcal{M}}(n) \leq p(n) + \nu_{i+1} n$ for all $n \geq r_{i+1}$. Since $\nu_i + a = \nu_{i+1}$, we may choose $\nu' = \nu_i$. Note that $\nu_i = ai < \nu$ since $i \leq \lceil \frac{\nu}{a} \rceil - 1$. □

By 3.1.3.1, M has a minor M_1 with $r(M_1) \geq r_i$ and $\varepsilon(M_1) > p(r(M_1)) + \nu' r(M_1)$. Since $r(M_1) \geq r_i$, we have $p(r(M_1)) > \alpha_{1.3.5}(\ell, n_{3.1.2}(a, b, c, \ell, f(r_{i+1}), s)) \cdot r(M_1)$, so M_1 has a $\mathrm{DG}(n_{3.1.2}(a, b, c, \ell, f(r_{i+1}), s), \{1\})$-minor by Theorem 1.3.5. Then by Lemma 3.1.2 with $r = f(r_{i+1})$ and $q = p + \nu'$, M_1 has a minor N such that $r(N) \geq f(r_{i+1})$ and $\varepsilon(N) > p(r(N)) + \nu' r(N)$, and N either has a spanning clique restriction or an s-element independent set so that $\varepsilon(N) - \varepsilon(N/e) > p(r(N)) - p(r(N) - 1) + \nu' r(N)$ for each $e \in S$. We may assume that N is simple. Since $f(r_{i+1}) \geq r$ and $\nu' \geq 0$ we may assume that N is not vertically s-connected, or else the theorem holds.

Let (A, B) be a partition of $E(N)$ so that $r_N(A) \leq r_N(B) < r(N)$ and $r_N(A) + r_N(B) - r(N) < s - 1$. Let $r_N = r(N)$ and $r_A = r_N(A)$ and $r_B = r_N(B)$. We first show that $r_A \geq \max(\hat{r}_1, r_{i+1})$. If not, then $r_B \geq r_N - r_A \geq \max(r_{i+1}, \hat{r}_1)$, using that $r_N \geq f(r_{i+1}) \geq \max(2r_{i+1}, 2\hat{r}_1)$. Also,

$$\begin{aligned} |B| = |N| - |A| &> p(r_N) + \nu' r_N - \ell^{\max(\hat{r}_1, r_{i+1})} && (1) \\ &\geq p(r_N - 1) + (\nu' + a) r_N && (2) \\ &\geq p(r_B) + (\nu' + a) r_B. && (3) \end{aligned}$$

Line (1) holds because $r_A < \max(\hat{r}_1, r_{i+1})$ and $M \in \mathcal{U}(\ell)$, and line (2) holds because $r_N \geq f(r_{i+1})$. Line (3) holds because $r_B \geq \hat{r}_1$, so $p(r_B) \leq p(r_N - 1)$ since $r_B \leq r_N - 1$. But then $r_B \geq r_{i+1}$ and $|B| > p(r_B) + (\nu' + a) r_B$, which contradicts 3.1.3.1 and the choice of ν'. Thus, $r_B \geq r_A \geq \max(\hat{r}_1, r_{i+1})$. Then

$$p(r_A + r_B - s) + \nu'(r_A + r_B - s) \leq p(r_N) + \nu' r_N < |A| + |B| \leq p(r_A) + p(r_B) + (\nu' + a)(r_A + r_B),$$

where the first inequality holds because $r_A + r_B - s \leq r_N$ and $p(x - s) \leq p(x - s + 1)$ for all $x \geq \hat{r}_1$, and the last inequality holds by 3.1.3.1 because $r_B \geq r_A \geq r_{i+1}$. Expanding $p(x) = ax^2 + bx + c$ and simplifying, we have

$$(2s + 1) a (r_A + r_B) + s(\nu' + b) + c - as^2 > 2a r_A r_B,$$

which contradicts that $r_A \geq \hat{r}_1$, since $\nu' < \nu$. □

3.2 Porcupines and Stacks

In this section we define three structures that arise from the second outcome of Theorem 3.1.3. The first structure is a collection of bounded-size restrictions which are not a restriction of any $\mathrm{DG}(k, \Gamma)$ with $|\Gamma| < t$. More generally, for any collection $\mathcal{O}$ of matroids and integers $b \geq 2$ and $h \geq 1$, a matroid M is an $(\mathcal{O}, b, h)$-*stack* if there are disjoint sets $P_1, P_2, \dots, P_h \subseteq E(M)$ such that

- $\cup_i P_i$ spans M, and
- for each $i \in [h]$ the matroid $(M/(P_1 \cup \dots \cup P_{i-1}))|P_i$ has rank at most b and is not in $\mathcal{O}$.

Stacks were used to find the extremal functions for exponentially dense minor-closed classes in [15] and [17] with $\mathcal{O}$ equal to the class of $\mathrm{GF}(q)$-representable matroids; our definition generalizes the original definition from [14]. We say that a matroid M is an $\mathcal{O}$-*stack* if there are integers $b \geq 2$ and $h \geq 1$ so that M is an $(\mathcal{O}, b, h)$-stack.

In this thesis we always take $\mathcal{O}$ to be $\mathcal{F} \cap \mathcal{U}(t)$ where $\mathcal{F}$ is the class of frame matroids; thus $\mathcal{F} \cap \mathcal{U}(t)$ is the class of frame matroids with no $U_{t+2,2}$-minor. Stacks are helpful because every matroid with a spanning clique restriction and a large enough $\mathcal{F} \cap \mathcal{U}(t)$-stack restriction either has a $\mathrm{DG}(k, \Gamma)$-minor with $|\Gamma| \geq t$, or is not a bounded distance from a frame matroid, as we prove in Chapter 5.

The second structure is a collection $\mathcal{S}$ of mutually skew sets in M/e so that for each $R \in \mathcal{S}$, the matroid $M|(R \cup \{e\})$ is a spike with tip e. We prove in Chapter 5 that such a restriction in the span of a clique admits a doubled-clique minor. This structure also has the useful property that if $|\mathcal{S}| \geq 2$, then $M|(\cup \mathcal{S} \cup \{e\})$ is not a frame matroid, which we prove in Section 3.4.

The third structure is a large independent set such that each element is the tip of many large spike restrictions. More precisely, for any integer $g \geq 3$ a g-*preporcupine* is a matroid P with an element f such that

- each line of P through f has at least three points, and
- $\mathrm{si}(P/f)$ has girth at least g.

We say that f is the *tip* of P, and we write $d(P)$ for $r^*(\mathrm{si}(P/f))$, the corank of $\mathrm{si}(P/f)$.

It will often be convenient to work with the following more specific matroid: a g-*porcupine* P is a simple g-preporcupine with tip f such that each line of P through f has exactly three points, and $\mathrm{si}(P/f)$ has no coloops. Clearly every g-preporcupine has a g-porcupine restriction, but porcupines are more restricted; every g-porcupine P with $d(P) = 0$ consists of a single element, and every g-porcupine P with $d(P) = 1$ is a spike.

More generally, a matroid P is a *(pre)porcupine* if there is an integer $g \geq 3$ so that P is a g-(pre)porcupine. Porcupines are helpful because the union of enough porcupines with independent tips is not a bounded distance from a frame matroid, as we prove in Chapter 5.

3.3 Nearly Skew Spikes with Common Tip

The main result of this section is Proposition 3.3.3, which provides sufficient conditions for a matroid to have a collection of nearly skew small spike restrictions with common tip. We start by introducing some notation which we will use for the remainder of this chapter.

For a matroid M and $f \in E(M)$, let $\delta(M,f) = \varepsilon(M) - \varepsilon(M/f)$, and let $\mathcal{L}_M(f)$ denote the set of long lines of M through f. Recall from Lemma 1.1.3 that

$$\delta(M,f) = 1 + \sum_{L \in \mathcal{L}_M(f)} \Big(\varepsilon(M|L) - 2\Big),$$

so $\delta(M,f) = \delta\big(M|(\cup \mathcal{L}_M(f)), f\big)$. We use this to prove the following lemma, which gives a formula for $\delta(M,f) - \delta(M/C,f)$ in terms of the long lines of M/C through f.

Lemma 3.3.1. *If M is a matroid with $C \subseteq E(M)$ and $f \in E(M) - \mathrm{cl}_M(C)$ so that $\mathrm{cl}_M(C) \cup \{f\}$ is a flat of M, then*

$$\delta(M,f) - \delta(M/C,f) = \sum_{L \in \mathcal{L}_{M/C}(f)} \Big(\delta(M|L,f) - \varepsilon\big((M/C)|L\big) + 1\Big).$$

Proof. We may assume that M is simple, by the definition of $\delta(M,f)$. Since $\mathrm{cl}_M(C) \cup \{f\}$ is a flat of M, each long line of M through f is disjoint from C and is therefore contained in a long line of M/C through f. Since M is simple and $\mathrm{cl}_M(C) \cup \{f\}$ is a flat of M, for each $L, L' \in \mathcal{L}_{M/C}(f)$ with $L \neq L'$ we have $L \cap L' = \{f\}$. Thus, each long line of M through f is contained in precisely one set in $\mathcal{L}_{M/C}(f)$, so

$$\delta(M,f) = 1 + \sum_{L \in \mathcal{L}_{M/C}(f)} \Big(\varepsilon(M|L) - \varepsilon\big((M|L)/f\big) - 1\Big) = 1 + \sum_{L \in \mathcal{L}_{M/C}(f)} \Big(\delta(M|L,f) - 1\Big).$$

Combining this with the fact that $\delta(M/C,f) = 1 + \sum_{L \in \mathcal{L}_{M/C}(f)} (\varepsilon\big((M/C)|L\big) - 2)$ gives the desired result. □

We now use Lemma 3.3.1 to find a large collection of small spikes with common tip e such that each spike spans some other element f.

Lemma 3.3.2. *For all integers $b \geq 1$ and $\ell \geq 2$, if $\{e, f\}$ is a 2-element independent set of a matroid $M \in \mathcal{U}(\ell)$ and $\delta(M, f) - \delta(M/e, f) \geq \ell^{2b+4}$, then there is a collection $\mathcal{S}$ of b mutually skew sets in $M/\{e, f\}$ so that for each $S \in \mathcal{S}$, the matroid $M|(S \cup \{e\})$ is a spike of rank at most four with tip e.*

Proof. We may assume that M is simple. Let $M_1 = M \backslash (\mathrm{cl}_M(\{e, f\}) - \{e, f\})$. Then $\{e, f\}$ is a flat of M_1, and $\delta(M_1, f) - \delta(M_1/e, f) \geq \ell^{2b+3}$. Let $\mathcal{L} = \mathcal{L}_{M_1/e}(f)$. We first prove a claim to help find long lines through e.

3.3.2.1. *For each $L \in \mathcal{L}$, if $\delta(M_1|L, f) - \varepsilon\big((M_1/e)|L\big) + 1 > 0$, then $M_1|\big((L \cup \{e\}) - \{f\}\big)$ has at least two long lines through e.*

Proof. Assume for a contradiction that $M_1|(L \cup \{e\})$ has at most one long line through e, and let $m \geq 2$ be the length of the longest line of $M_1|(L \cup \{e\})$ through e. Since $\{e, f\}$ is a flat of M_1, each line of $M_1|(L \cup \{e\})$ through e contains at most one element of each line of $M_1|L$ through f. This implies that there are at least $m - 1$ lines of $M_1|L$ through f, so $\varepsilon\big((M_1|L)/f\big) \geq m - 1$, and thus $\delta(M_1|L, f) \leq \varepsilon(M_1|L) - (m - 1)$. Also, $\varepsilon\big((M_1/e)|L\big) = \varepsilon(M_1|L) - (m - 2)$ since there is only one long line of $M_1|(L \cup \{e\})$ through e. Combining these facts gives

$$\delta(M_1|L, f) - \varepsilon\big((M_1/e)|L\big) + 1 \leq \big(\varepsilon(M_1|L) - (m - 1)\big) - \big(\varepsilon(M_1|L) - (m - 2)\big) + 1 = 0,$$

a contradiction. □

Each $L \in \mathcal{L}$ satisfies

$$\delta(M_1|L, f) - \varepsilon\big((M_1/e)|L\big) + 1 \leq \varepsilon(M_1|L) \leq \ell^3$$

by Theorem 1.3.2, since $r_{M_1}(L) \leq 3$ and $M_1 \in \mathcal{U}(\ell)$. Since $\delta(M_1, f) - \delta(M_1/e, f) \geq \ell^3 \ell^{2b}$, by Lemma 3.3.1 with $C = \{e\}$ there are at least ℓ^{2b} sets $L \in \mathcal{L}$ for which $\delta(M_1|L, f) - \varepsilon\big((M_1/e)|L\big) + 1 > 0$. Since each $L \in \mathcal{L}$ is a point of $M_1/\{e, f\}$ and $M_1 \in \mathcal{U}(\ell)$, there is some $\mathcal{L}' \subseteq \mathcal{L}$ such that $|\mathcal{L}'| = 2b$, $r_{M_1/\{e,f\}}(\cup \mathcal{L}') = 2b$, and each $L \in \mathcal{L}'$ satisfies $\delta(M_1|L, f) - \varepsilon\big((M_1/e)|L\big) + 1 > 0$.

For each $L \in \mathcal{L}'$, the matroid $M_1|((L \cup \{e\}) - \{f\})$ contains at least two long lines through e, by 3.3.2.1. Let $\mathcal{L}' = \{L_1, \dots, L_{2b}\}$. Then each $i \in [b]$ satisfies $r_{M_1}(L_{2i-1} \cup L_{2i}) = 4$, and there are at least four long lines through e in $M_1|((L_{2i-1} \cup L_{2i} \cup \{e\}) - \{f\})$. For each $i \in [b]$, let P_i denote the union of the long lines through e in $M_1|((L_{2i-1} \cup L_{2i} \cup \{e\}) - \{f\})$. Then $r((M_1|P_i)/e) \leq 3$ and $\varepsilon((M_1|P_i)/e) \geq 4$, so $\mathrm{si}((M_1|P_i)/e)$ contains a circuit. Since $M_1|P_i$ is simple and each parallel class of $(M_1|P_i)/e$ has size at least two, for each $i \in [b]$ there is some $S_i \subseteq P_i - \{e\}$ so that $M_1|(S_i \cup \{e\})$ is a spike with tip e. Since $r_{M_1/\{e,f\}}((\cup \mathcal{L}') - \{f\}) = 2b$, the sets $\{S_i \colon i \in [b]\}$ are mutually skew in $M_1/\{e, f\}$. □

If Lemma 3.3.2 applies for enough elements f, then we can find many nearly skew small spikes with common tip e. This is one of the three structures which arise from the second outcome of Theorem 3.1.3.

Proposition 3.3.3. *Let $m \geq 1$ and $\ell \geq 2$ be integers, and let $M \in \mathcal{U}(\ell)$ be a matroid with a $3m$-element independent set X and $e \in E(M) - X$ such that $M|(X \cup \{e\})$ is simple and each $f \in X$ satisfies $\delta(M, f) - \delta(M/e, f) \geq \ell^{6m}$. Then there is a collection $\mathcal{S}$ of m mutually skew sets in M/e so that for each $S \in \mathcal{S}$, the matroid $M|(S \cup \{e\})$ is a spike of rank at most four with tip e.*

Proof. Let $\mathcal{S}$ be maximum-size collection of mutually skew sets in M/e so that for each $S \in \mathcal{S}$, the matroid $M|(S \cup \{e\})$ is a spike of rank at most four with tip e, and assume for a contradiction that $k < m$. Then $r_M(\cup\mathcal{S}) \leq 3m - 2$, so there is some $f \in X - \mathrm{cl}_M(\cup\mathcal{S})$. Let $B \cup \{e\}$ be a basis for $M|(\cup\mathcal{S})$, so $|B| \leq 3m - 3$ and $B \cup \{e, f\}$ is independent.

By Lemma 3.3.2 with $b = 3m - 2$, there is a collection $\mathcal{P}$ of $3m - 2$ mutually skew sets in $M/\{e, f\}$ so that for each $P \in \mathcal{P}$, the matroid $M|(P \cup \{e\})$ is a spike of rank at most four with tip e. Recall that for sets A_1 and A_2 of a matroid N, we write $\sqcap_N(A_1, A_2)$ for $r_N(A_1) + r_N(A_2) - r_N(A_1 \cup A_2)$. Since the sets in $\mathcal{P}$ are mutually skew in $M/\{e, f\}$, we can show using submodularity of the rank function of M and induction on $|\mathcal{P}|$ that

$$\sqcap_{M/\{e,f\}}(B, \cup\mathcal{P}) \geq \sum_{P \in \mathcal{P}} \sqcap_{M/\{e,f\}}(B, P).$$

Thus, there is some $P \in \mathcal{P}$ such that $\sqcap_{M/\{e,f\}}(B, P) = 0$, or else $\sqcap_{M/\{e,f\}}(B, \cup\mathcal{P}) \geq |\mathcal{P}| > |B|$, a contradiction. Then

$$\sqcap_{M/e}(\cup\mathcal{S}, P) = \sqcap_{M/e}(B, P) \leq \sqcap_{M/\{e,f\}}(B, P) = 0,$$

because B is a basis for $(M/e)|(\cup\mathcal{S})$ and $f \notin \mathrm{cl}_{M/e}(B)$. But then $\mathcal{S} \cup \{P\}$ contradicts the maximality of $|\mathcal{S}|$. □

3.4 Star-Partitions

Recall that for any integer $g \geq 3$, a *g-preporcupine* is a matroid P with an element f so that each line of P through f has at least three points, and $\mathrm{si}(P/f)$ has girth at least g. A *g-porcupine* P is a simple g-preporcupine such that each line of P through f has exactly three points, and $\mathrm{si}(P/f)$ has no coloops.

In this section we will explore the notion that preporcupines are incompatible with frame matroids. We show that frame matroids do not contain spikes of rank at least five; this implies that frame matroids can only have preporcupine restrictions with very specific structure.

This is described by the main result of this section, Proposition 3.4.5, which roughly says that every preporcupine either has a bounded-size restriction which is not a frame matroid, or has structure similar to that of a frame matroid. Before proving this result, we prove several easy lemmas which describe the structure of frame matroid preporcupines. We will use Lemmas 3.4.1-3.4.4 to identify bounded-size restrictions which are not frame matroids.

The basic idea is that if a matroid M is both a frame matroid and a preporcupine with tip f, then $\mathrm{si}(M/f)$ is a restriction of a *star*, which is a matroid S with a basis $B \cup \{t\}$ such that $E(S) = \cup_{b \in B} \mathrm{cl}_S(\{t, b\})$. We say that t is the *tip* of S.

Lemma 3.4.1. *If M is both a frame matroid and a preporcupine with tip f such that $\mathrm{si}(M/f)$ contains a circuit, then all but at most one line through f has length three, and $\mathrm{si}(M/f)$ is a restriction of a star. Moreover, if M has a line L of length at least four through f then for each $e \in L - \mathrm{cl}_M(\{f\})$, the matroid $\mathrm{si}(M/\{e, f\})$ contains no circuit.*

Proof. We may assume that M is simple. Let N be a matroid framed by $B = \{b_1, b_2, \ldots, b_r\}$ so that $N \backslash B = M$. We may assume without loss of generality that $f \in \mathrm{cl}_N(\{b_1, b_2\})$, by relabeling B. Since $\mathrm{si}(M/f)$ contains a circuit, f is not parallel to b_1 or b_2 by Lemma 1.2.3 (iii). Then each element of M is in $\mathrm{cl}_N(\{b_1, b_i\})$ for some $i \geq 2$ or $\mathrm{cl}_N(\{b_2, b_i\})$ for some $i \geq 3$, or else that element is not on a long line through f.

Let $(M/f)|T$ be a simplification of M/f. Each element of T is in $\mathrm{cl}_{N/f}(\{b_2, b_i\})$ for some $i \geq 3$. Since $\{b_2, b_3, \ldots, b_r\}$ is independent in N/f, the matroid $(M/f)|T$ is a restriction of a star with tip b_2. If M has a line L of length at least four through f, then $L \subseteq \mathrm{cl}_N(\{b_1, b_2\})$ by Lemma 1.2.3 (i). Then for each $e \in L - \{f\}$, f is parallel to b_1 or b_2 in N/e, so $\mathrm{si}(M/\{e, f\})$ contains no circuit by Lemma 1.2.3 (iii). □

To make use of Lemma 3.4.1 we will need some properties of restrictions of stars. Stars have the property that each circuit is contained in the union of at most two lines through the tip. We would also like to describe this property for restrictions of stars, but it is trickier if the restriction does not contain the tip. To deal with this, we define a *star-partition* of a matroid M to be a pair $(X, \mathcal{L})$ such that

- $r_M(X) \leq 1$,
- $\mathcal{L}$ partitions $E(M) - X$,
- $L \cup X$ is a flat of M of rank at most two for each $L \in \mathcal{L}$, and
- $r_M(L \cup L') \leq 3$ for all distinct $L, L' \in \mathcal{L}$.

The idea is that any matroid with a star-partition $(X, \mathcal{L})$ so that each small circuit is contained in $L \cup L' \cup X$ for some $L, L' \in \mathcal{L}$ is 'star-like'. The set X is either empty or contains the tip of the star, and each set L is a line through the tip (excluding the tip). We cannot define a star-partition to actually be a partition, since it may be the case that X is empty.

Lemmas 3.4.3 and 3.4.4 combine to show that any restriction of a star has a star-partition $(X, \mathcal{L})$ so that each circuit is contained in $L \cup L' \cup X$ for some $L, L' \in \mathcal{L}$. We first show that stars have this property.

Lemma 3.4.2. *Let S be a simple star with tip t and basis $B \cup \{t\}$, and let*

$$\mathcal{L} = \{L \subseteq E(S) - \{t\} \colon L \cup \{t\} \text{ is a line of } S\}.$$

Then $(\{t\}, \mathcal{L})$ is a star-partition of S so that for each circuit C of S there are sets $L, L' \in \mathcal{L}$ for which $C \subseteq \{t\} \cup L \cup L'$. In particular, S has no circuit of size at least five.

Proof. Clearly $(\{t\}, \mathcal{L})$ is a star-partition of S, since each line in $\mathcal{L}$ spans t. Let C be a circuit of S, and note that $C - \{t\}$ is a union of circuits of S/t. Since $\mathrm{si}(S/t)$ is independent, the only circuits of S/t are parallel pairs, so $C - \{t\}$ is a union of parallel pairs of S/t. Then since $r_{S/t}(C) \geq |C| - 2$, the set $C - \{t\}$ intersects at most two parallel classes of S/t. □

Lemma 3.4.2 and Lemma 3.4.1 together imply that spikes of rank at least five are not frame matroids, since every spike is a preporcupine.

If a restriction of a star contains the tip of the star then we can easily find a star-partition.

Lemma 3.4.3. *Let M be a simple restriction of a star so that $x \in E(M)$ is on two long lines of M, and let $\mathcal{L} = \{L \subseteq E(M) - \{x\} \colon L \cup \{x\} \text{ is a line of } M\}$. Then $(\{x\}, \mathcal{L})$ is a star-partition of M such that for each circuit C of M there are sets $L, L' \in \mathcal{L}$ so that $C \subseteq \{x\} \cup L \cup L'$.*

Proof. Let S be a star with tip t such that $S|A = M$. If $x \in A$ is on two long lines of $S|A$, then $x = t$ since each element of S other than t is on at most one long line of S. Thus, the result holds by applying Lemma 3.4.2 to S. □

If a restriction of a star does not contain the tip of the star then we can still find a star-partition. Moreover, for each circuit there are two elements of that circuit which define a star-partition.

Lemma 3.4.4. *Let M be a simple restriction of a star such that no element of M is on two long lines of M, and let C be a circuit of M. Then for each $e \in C$ there is some $e' \in C - \{e\}$ such that*

$$\Big(\varnothing, \{\mathrm{cl}_M(\{e, e'\})\} \cup \{P \colon P \text{ is a parallel class of } M/\{e, e'\}\}\Big)$$

is a star-partition of M, and for each circuit C' of M there are sets $L, L' \in \{\mathrm{cl}_M(\{e, e'\})\} \cup \{P \colon P \text{ is a parallel class of } M/\{e, e'\}\}$ so that $C' \subseteq L \cup L'$.

Proof. Let S be a star with tip $\{t\}$ such that $S|A = M$, and let $e \in C$. Since $C - \{t\}$ is a union of circuits of S/t, there is some $e' \in C$ so that $\{e, e'\}$ is a parallel pair of S/t. Let $\mathcal{P}$ denote the collection of parallel classes of $M/\{e, e'\}$. If $(\varnothing, \{\mathrm{cl}_M(\{e, e'\})\} \cup \mathcal{P})$ is not a star-partition of M, then there is some $P \in \mathcal{P}$ which is not a flat of M, so $\mathrm{cl}_M(P) \cap \mathrm{cl}_M(\{e, e'\}) \neq \varnothing$. But then there is an element of M which is on two long lines of M, a contradiction.

Since $t \in \mathrm{cl}_S(\{e, e'\})$ and $\mathrm{si}(S/t)$ is independent, each set in $\{\mathrm{cl}_M(\{e, e'\})\} \cup \mathcal{P}$ is contained in a set in

$$\{L \subseteq E(S) - \{t\} \colon L \cup \{t\} \text{ is a line of } S\}.$$

Thus, for each circuit C' of M there exist $L, L' \in \{\mathrm{cl}_M(\{e, e'\})\} \cup \mathcal{P}$ so that $C' \subseteq L \cup L'$, by applying Lemma 3.4.2 to S. $\square$

We now use these properties of stars to prove an extension of Lemma 3.4.1 to matroids for which each bounded-rank restriction is a frame matroid. The first outcome will help build a stack for which each piece is not a frame matroid, and the second outcome will help find a g-porcupine.

Proposition 3.4.5. *Let M be a preporcupine with tip f, and let $(M/f)|T$ be a simplification of M/f. If $(M/f)|T$ has a circuit of size at most four, then for each integer $g \geq 4$, either*

(1) M has a restriction of rank at most $3g$ which is not a frame matroid, or

(2) $(M/f)|T$ has a star-partition $(X, \mathcal{L})$ so that each line of M through f and an element of $T - X$ has length three, and for each circuit C' of $(M/f)|T$ of size less than g, there are sets $L, L' \in \mathcal{L}$ so that $C' \subseteq (X \cup L \cup L')$.

Proof. Assume that (1) does not hold for M. Then M satisfies the following property:

(1') For each set $X \subseteq E(M)$ of rank at most $3g$, the matroid $M|X$ is a frame matroid.

We will show that M satisfies (2). We may assume that M is simple. Let $M' = (M/f)|T$ and let C be a circuit of M' of size at most four.

3.4.5.1. *Each line of M through f has length three.*

Proof. If a line L_1 of M through f has length at least four, let $x \in L_1 \cap T$. Let $X = \{x\}$ and $\mathcal{L} = \{L \subseteq E(M') - \{x\} \colon L \cup \{x\} \text{ is a line of } M'\}$. Then $(X, \mathcal{L})$ is a star-partition of M', since each line in $\mathcal{L}$ spans x. We show that $(X, \mathcal{L})$ satisfies (2). If there is some $e \in T - X$ such that the line L_2 of M through e and f has length at least four, then $\mathrm{cl}_M(C \cup \{f, x, e\})$ is not a frame matroid by Lemma 3.4.1. Since $r_M(C \cup \{f, x, e\}) \leq 7 \leq 3g$, this contradicts (1'). Thus, each line of M through f and an element of $T - X$ has length three.

If M' has a circuit C' of size less than g which is not contained in $X \cup L \cup L'$ for any $L, L' \in \mathcal{L}$, then $C' - \{x\}$ intersects at least three parallel classes of M'/x. Since $r_{M'/x}(C' - \{x\}) \geq |C'| - 2$, some parallel class of $(M'/x)|(C' - \{x\})$ has size one. Then since $C' - \{x\}$ is a union of circuits of M'/x, the matroid $(M'/x)|(C' - \{x\})$ has a circuit which intersects at least three parallel classes of M'/x, so $\mathrm{si}(M'/x)$ has a circuit of size less than g. Since $|L_1| \geq 4$ and $x \in L_1$, the matroid $M|\,\mathrm{cl}_M(C \cup C' \cup \{f, x\})$ is not frame by Lemma 3.4.1. Since $r_M(C \cup C' \cup \{f, x\}) \leq g + 6 \leq 3g$, this contradicts (1'). Therefore, $(X, \mathcal{L})$ is a star-partition of M' which satisfies (2). □

The next claim allows us to apply Lemma 3.4.4.

3.4.5.2. *No element of M' is on two long lines of M'.*

Proof. Suppose for a contradiction that $x \in E(M')$ is on two long lines of M'. Let $X = \{x\}$ and $\mathcal{L} = \{L \subseteq E(M') - \{x\} \colon L \cup \{x\} \text{ is a line of } M'\}$. Then $(X, \mathcal{L})$ is a star-partition of M', and there are distinct $L_1, L_2 \in \mathcal{L}$ such that $|L_1| \geq 2$ and $|L_2| \geq 2$. By 3.4.5.1, each line of M through f and an element of $\cup\mathcal{L}$ has length three. Assume for a contradiction that $(X, \mathcal{L})$ does not satisfy (2). Then there is some circuit C' of M' of size less than g which is not contained in $X \cup L \cup L'$ for any $L, L' \in \mathcal{L}$.

Let $\mathcal{L}' = \{L \in \mathcal{L} \colon L \cap C' \neq \varnothing\} \cup \{L_1, L_2\}$, and note that $C' \subseteq \cup(\mathcal{L}' \cup \{X\})$. If $M'|(\cup(\mathcal{L}' \cup \{X\}))$ is a restriction of a star, then each circuit is contained in $X \cup L \cup L'$ for some $L, L' \in \mathcal{L}'$ by Lemma 3.4.3. But this contradicts the existence of C', so $M'|(\cup(\mathcal{L}' \cup \{X\}))$ is not a restriction of a star. But then the set of elements of M on a long line through f and an element of $\{x\} \cup (\cup\mathcal{L}')$ is not a frame matroid by Lemma 3.4.1. Since $r_M(\{f, x\} \cup (\cup\mathcal{L}')) \leq g + 2 \leq 3g$, this contradicts (1'). Thus, $(X, \mathcal{L})$ satisfies (2). □

Fix some element $e \in C$. For each $e' \in C - \{e\}$, let $\mathcal{P}_{e,e'}$ denote the collection of parallel classes of $M'/\{e, e'\}$.

3.4.5.3. *There is some $e' \in C - \{e\}$ so that the pair $(\varnothing, \{\mathrm{cl}_{M'}(\{e, e'\})\} \cup \mathcal{P}_{e,e'})$ is a star-partition of M' such that for each circuit C' of M' of size less than g, there are $L, L' \in \{\mathrm{cl}_{M'}(\{e, e'\})\} \cup \mathcal{P}_{e,e'}$ for which $C' \subseteq L \cup L'$.*

Proof. Assume for a contradiction that the claim is false. For each $e' \in C - \{e\}$, the pair $(\varnothing, \{\mathrm{cl}_{M'}(\{e, e'\})\} \cup \mathcal{P}_{e,e'})$ is a star-partition of M' by 3.4.5.2, since these sets are nonempty pairwise coplanar flats of rank at most two. So for each $e' \in C - \{e\}$ there is a circuit $C_{e'}$ of M' of size less than g not contained in $L \cup L'$ for any $L, L' \in \{\mathrm{cl}_{M'}(\{e, e'\})\} \cup \mathcal{P}_{e,e'}$. Let $J = \mathrm{cl}_{M'}(C) \cup (\cup_{e' \in C - \{e\}} C_{e'})$, and for each $e' \in C - \{e\}$ let $\mathcal{P}'_{e,e'}$ denote the collection of parallel classes of $(M'|J)/\{e, e'\}$.

If $M'|J$ is a restriction of a star, then by 3.4.5.2 and Lemma 3.4.4 there is some $e' \in C-\{e\}$ for which the pair $(\{\mathrm{cl}_{M'}(\{e,e'\})\} \cup \mathcal{P}'_{e,e'}, \varnothing)$ is a star-partition of $M'|J$ such that each circuit of $M'|J$ is contained in $L \cup L'$ for some $L, L' \in \{\mathrm{cl}_{M'}(\{e,e'\})\} \cup \mathcal{P}'_{e,e'}$. This contradicts the existence of $C_{e'}$, so $M'|J$ is not a restriction of a star. Then $\mathrm{cl}_M(J \cup \{f\})$ is not a frame matroid, by Lemma 3.4.1. Since $|C-\{e\}| \leq 3$ we have $r_{M'}(J) \leq 3(g-2)+3$. But then $r_M(J \cup \{f\}) \leq 3g$, which contradicts (1'). □

Let e' be given by 3.4.5.3. Then by 3.4.5.1 and 3.4.5.3, outcome (2) holds with $X = \varnothing$ and $\mathcal{L} = \{\mathrm{cl}_{M'}(\{e,e'\})\} \cup \mathcal{P}_{e,e'}$, a contradiction. □

3.5 Porcupines and Frame Matroids

Recall that $\mathcal{F} \cap \mathcal{U}(t)$ is the class of frame matroids with no $U_{2,t+2}$-minor. In this section we apply Proposition 3.4.5 to a collection of preporcupines with independent tips, and show that we can either find a small matroid which is not in $\mathcal{F} \cap \mathcal{U}(t)$, or a large independent set so that each element is the tip of a g-porcupine.

Before proving this we need two straightforward lemmas to help find a $U_{2,t+2}$-minor. The first deals with a rank-4 frame matroid.

Lemma 3.5.1. *For each integer $t \geq 2$, if M is both a frame matroid and a porcupine with tip f such that* $\mathrm{si}(M/f)$ *has rank three and is the disjoint union of a line of length t and a line of length two, then M has a $U_{2,t+2}$-minor.*

Proof. We may assume that M is simple. Let N be a matroid framed by $B = \{b_1, b_2, b_3, b_4\}$ such that $N \backslash B = M$. Let $(M/f)|T$ be a simplification of M/f. We may assume that $f \in \mathrm{cl}_N(\{b_1, b_2\})$, by relabeling B. Note that f is not parallel to b_1 or b_2, by Lemma 1.2.3 (iii), since f is the tip of a spike restriction of M.

Each element of N/f is spanned by $\{b_2, b_3\}$ or $\{b_2, b_4\}$, or else that element is not on a long line of M through f. Since M is a porcupine, the matroid $(M/f)|T$ has no coloops, and therefore $\mathrm{cl}_{(N/f)|T}(\{b_2, b_3\})$ and $\mathrm{cl}_{(N/f)|T}(\{b_2, b_4\})$ each contain at least two elements in $E(M)$ which are not parallel to b_2 in N/f. Let L_1 and L_2 be the lines of $(M/f)|T$ of length t and two, respectively, whose union is T.

If $t \geq 3$, assume without loss of generality that $L_1 \subseteq \mathrm{cl}_{(N/f)|T}(\{b_2, b_3\})$, since $\{b_2, b_3\}$ and $\{b_2, b_4\}$ span the only long lines of $(N/f)|T$. Then $L_2 \subseteq \mathrm{cl}_{(N/f)|T}(\{b_2, b_4\})$, and no element of $(N/f)|T$ is parallel to b_2 since L_1 and L_2 are flats of $(N/f)|T$. Then since $|L_1| = t$ and each element of L_1 is on a long line of $M \backslash b_2$ through f, there are t long lines of M through f so that each contains one element in $\mathrm{cl}_N(\{b_1, b_3\})$ and one element in $\mathrm{cl}_N(\{b_2, b_3\})$. Similarly, $\{b_1, b_4\}$ and $\{b_2, b_4\}$ each span two elements of $E(M)$ since $|L_2| = 2$.

Let $e, e' \in \mathrm{cl}_N(\{b_2, b_4\}) \cap E(M)$ with $e \neq e'$, and let $x \in \mathrm{cl}_N(\{b_2, b_3\}) \cap E(M)$. Let $Y = \mathrm{cl}_N(\{b_1, b_3\}) \cap E(M)$, so $|Y| = t$. Note that $\{f, b_1\}$ and $\{x, b_3\}$ are parallel pairs in $N/\{e, e'\}$ since $b_2 \in \mathrm{cl}_N(\{e, e'\})$, and f and x are not parallel in $N/\{e, e'\}$. Thus, $\{f, x\} \subseteq \mathrm{cl}_{M/\{e,e'\}}(Y) - Y$, so $(M/\{e, e'\})|(Y \cup \{f, x\}) \cong U_{2,t+2}$, as desired. If $t = 2$ then $\{b_2, b_3\}$ and $\{b_2, b_4\}$ each span two points of M/f which are not parallel to b_2, and the same argument applies. □

The second lemma deals with a rank-3 frame matroid.

Lemma 3.5.2. *For each integer $t \geq 2$, if M is a both a frame matroid and a preporcupine with tip f such that $\mathrm{si}(M/f) \cong U_{2,t+1}$ and M has a line of length $t + 1$ through f, then M has a $U_{2,t+2}$-minor.*

Proof. We may assume that M is simple. If $t = 2$ then M is a rank-3 spike, and if M is binary then M is isomorphic to the Fano plane, which is a contradiction since the Fano plane is not a frame matroid by Proposition 1.2.1. Thus, M is not binary, so M has a $U_{2,4}$-minor and the lemma holds for $t = 2$. Therefore, we may assume that $t \geq 3$. Let N be a matroid framed by $B = \{b_1, b_2, b_3\}$ so that $N \backslash B = M$. Assume without loss of generality that $f \in \mathrm{cl}_N(\{b_1, b_2\})$, and note that b_3 is not parallel to an element of $E(M)$ by Lemma 1.2.3 (iii) since b_3 is not on any long line with f. Since $t \geq 3$ and M has a line of length $t + 1$ through f we have $|E(M) \cap \mathrm{cl}_N(\{b_1, b_2\})| = t + 1$ by Lemma 1.2.3 (i).

If b_i is not parallel to any element in $E(M)$, then for each $e \in \mathrm{cl}_M(\{b_i, b_3\})$ the matroid M/e has a $U_{2,t+2}$-restriction. So b_1 and b_2 are each parallel in N to an element in $E(M)$, say b'_1 and b'_2, respectively. Then since f is on t long lines of N through f other than $\mathrm{cl}_N(\{b_1, b_2\})$, we have $|E(M) \cap \mathrm{cl}_N(\{b_i, b_3\})| = t + 1$ for each $i \in \{1, 2\}$. Since b_3 is not parallel to an element in $E(M)$, we see that M/b'_1 has a $U_{2,t+2}$-restriction. □

We now prove a result which does the bulk of the work in the proof of the main result of this chapter, Theorem 3.6.1. The proof of this proposition relies on Proposition 3.4.5 to find star-partitions. Recall that $\mathcal{F} \cap \mathcal{U}(t)$ is the class of frame matroids with no $U_{2,t+2}$-minor. Also, if P is a porcupine with tip f, then we write $d(P)$ for the corank of $\mathrm{si}(P/f)$.

Proposition 3.5.3. *For all integers $t \geq 2, h \geq 0$ and $g \geq 4$, if M is a matroid with a size-$(2h+1)$ independent set S so that $\varepsilon(M) - \varepsilon(M/f) > (t-1)(r(M) - 1 + h) + 1$ for each $f \in S$, then M has either*

(1) a restriction of rank at most $3g$ which is not in $\mathcal{F} \cap \mathcal{U}(t)$, or

(2) a size-$(h+1)$ independent set such that each element is the tip of a g-porcupine restriction P of M with $d(P) = h + 1$.

Proof. Assume that (1) and (2) do not hold for M, and that M is simple. Since (1) does not hold, M has no $U_{2,t+2}$-restriction. Also, M has no spike restriction of rank at least five and at most $3g$, since Lemma 3.4.2 and Lemma 3.4.1 together imply that spikes of rank at least five are not frame matroids. For convenience, we say that a g-(pre)porcupine P is an (s, g)-*(pre)porcupine* if $d(P) = s$.

If $f \in S$ is not the tip of a spike of rank less than g, then f is on at least $r(M) + h$ long lines, or else

$$\delta(M, f) = \varepsilon(M) - \varepsilon(M/f) \leq 1 + (t-1)(r(M) + h - 1),$$

by Lemma 1.1.3. But if $f \in S$ is not the tip of a spike of rank less than g and is on at least $r(M) + h$ long lines of M, then f is the tip of an $(h+1, g)$-porcupine. Since (2) does not hold, there is some $S_1 \subseteq S$ such that $|S_1| \geq |S| - h \geq h + 1$ and each element of S_1 is the tip of a spike of rank less than g in M. Each of these spikes has rank at most four, or else (1) holds.

For each $f \in S_1$, let $P_f = M|(\cup_{L \in \mathcal{L}_M(f)} L)$, and let $(P_f/f)|T_f$ be a simplification of P_f/f. Note that $\delta(P_f, f) = \delta(M, f)$.

3.5.3.1. *There is some $f \in S_1$ for which $(P_f/f)|T_f$ has a star-partition $(X, \mathcal{L})$ such that each line of P_f through f and an element of $T_f - X$ has length three, and $|\mathcal{L}| \leq r(M) + h - 2$.*

Proof. By Proposition 3.4.5, for each $f \in S_1$, the matroid $(P_f/f)|T_f$ has a star-partition $(X, \mathcal{L})$ so that each circuit of $(P_f/f)|T_f$ of size less than g is contained in $X \cup L \cup L'$ for some $L, L' \in \mathcal{L}$, and each line of P_f through f and an element of $T_f - X$ has length three.

Since $(P_f/f)|T_f$ has a circuit of size at most four, there is some $L \in \mathcal{L}$ such that $|L| \geq 2$. Let $T' \subseteq T_f$ such that $|T'| = |\mathcal{L}| + 1$, while T' contains a transversal of $\mathcal{L}$ and $|T \cap L| = 2$. If $|T'| \geq r(M) + h$, then the set of elements of P_f on a long line of P_f through f and an element of T' is an $(h+1, g)$-preporcupine with tip f, and thus contains an $(h+1, g)$-porcupine with tip f. Since (2) does not hold and $|S_1| - h \geq 1$ the claim holds. □

Note that $(t-1)(r(M) - 1 + h) + 1 = (t-1)(r(M) + h - 2) + t$, and let $f \in S_1$ and $(X, \mathcal{L})$ be given by 3.5.3.1. Let $m = \sum_{x \in X}(|\operatorname{cl}_M(\{x, f\})| - 2)$, and note that $m \leq t - 1$ since M

has no $U_{2,t+2}$-restriction. Then

$$\begin{aligned}(t-1)(r(M)+h-2)+t &\le \delta(P_f/f)-1 && (1)\\ &= \sum_{e\in T_f}\big(|\operatorname{cl}_M(\{e,f\})|-2\big) && (2)\\ &= m+\sum_{L\in\mathcal{L}}|L| && (3)\\ &\le m+(t-1)(r(M)+h-2)-(t-1)|\mathcal{L}|+\sum_{L\in\mathcal{L}}|L|. && (4)\end{aligned}$$

Line (2) holds by Lemma 1.1.3. Line (3) holds because each line of P_f through f and an element of $T_f - X$ has length three by 3.5.3.1, and line (4) holds since $|\mathcal{L}| \le r(M)+h-2$. Thus,

$$t-m+(t-1)|\mathcal{L}| \le \sum_{L\in\mathcal{L}}|L| \qquad (5)$$

so there is some $L_1 \in \mathcal{L}$ such that $|L_1| \ge t$ since $m \le t-1$. By Lemma 3.5.1 each $L \in \mathcal{L}-\{L_1\}$ satisfies $|L| = 1$, or else (1) holds. Then we have $\sum_{L\in\mathcal{L}}|L| = |L_1| + |\mathcal{L}| - 1$.

We have two cases to consider. If $m > 0$, then $X \neq \varnothing$ and thus $|L_1| = t$ or else $(P_f/f)|(L_1\cup X)$ has a $U_{2,t+2}$-restriction by the definition of a star-partition. Then $m = t-1$ or else (5) does not hold. But then by Lemma 3.5.2, M has a rank-3 restriction which is not in $\mathcal{U}(t)$ and (1) holds.

Now assume that $m = 0$. Since P_f/f has no $U_{2,t+2}$-restriction, $|L_1| \le t+1$. Then $t = 2$ and $|L_1| = 3$, or else (5) does not hold. But then the set of elements of P_f on a line through f and an element of L_1 is a rank-3 spike with no $U_{2,4}$-minor. Thus, P_f has an F_7-restriction, and (1) holds since F_7 is not a frame matroid, by Proposition 1.2.1. □

3.6 The Proof

We now prove the main result of this chapter, which refines the structure found in the second outcome of Theorem 3.1.3. The case $h = 0$ is particularly important and has a more exact flavor; it says that if M has an element with density loss greater than any element of a Dowling geometry with group size less than t, then M has a highly structured restriction which is not contained in any Dowling geometry with group size less than t. The proof only uses Proposition 3.3.3 and Proposition 3.5.3.

Theorem 3.6.1. *Let $\ell \ge t \ge 2$ and $h \ge 0$ be integers, and let $M \in \mathcal{U}(\ell)$ be a matroid with a size-2^{15h} independent set S so that each $f \in S$ satisfies $\varepsilon(M) - \varepsilon(M/f) > (t-1)(r(M)-1) + \ell^{28h}$. Then there is a set $C \subseteq E(M)$ of rank at most $h2^{h+7}$ so that M/C has either*

(1) an $(\mathcal{F} \cap \mathcal{U}(t), 15 \cdot 2^h, h+1)$-stack restriction,

(2) an element e and a collection $\mathcal{S}$ of $h+2$ mutually skew sets in $M/(C \cup \{e\})$ so that for each $R \in \mathcal{S}$, the matroid $(M/C)|(R \cup \{e\})$ is a spike of rank at most four with tip e, or

(3) a size-$(h+1)$ independent set so that each element is the tip of a $5 \cdot 2^h$-porcupine restriction P of M/C with $d(P) = h+1$.

Proof. The constants in the theorem statement are larger than we need, to make the statement more readable. Using crude estimates for $h \geq 1$ we have $2^{15h} \geq 2h + h(15 \cdot 2^h)(3(h+2)+1)$, and

$$\ell^{28h} \geq (t-1)h + 1 + h(15 \cdot 2^h)\ell^{6(h+2)},$$

and these inequalities also hold for $h = 0$. Assume for a contradiction that none of (1), (2), (3) hold. Let $k \geq 0$ be maximal so that M has an $(\mathcal{F} \cap \mathcal{U}(t), 15 \cdot 2^h, k)$-stack restriction C_0. Then $k \leq h$ since (1) does not hold, so $r_M(C_0) \leq h(15 \cdot 2^h)$. The following claim uses Proposition 3.3.3 to reduce to a situation in which we can apply Proposition 3.5.3.

3.6.1.1. *There is some $S_1 \subseteq S$ with $|S_1| \geq 2h+1$ so that S_1 is independent in M/C_0, and each $f \in S_1$ satisfies $\delta(M/C_0, f) > (t-1)(r(M) - 1 + h) + 1$.*

Proof. Let $c_1, c_2, \ldots, c_b$ be a basis of C_0, and let $m = h+2$. Let $S_1 \subseteq S$ be a maximum-size independent set in M/C_0 for which each $f \in S_1$ satisfies $\delta(M/C_0, f) > (t-1)(r(M)-1+h)+1$, and assume for a contradiction that $|S_1| \leq 2h$. Then there is some $S' \subseteq S$ with $|S'| \geq |S| - b - 2h \geq b(3m)$ so that S' is independent in M/C_0, and each $f \in S'$ satisfies

$$\delta(M, f) - \delta(M/C_0, f) \geq b\ell^{6m}.$$

Then for each $f \in S'$ there is some $i \in [b]$ such that

$$\delta(M/\{c_1, \ldots, c_{i-1}\}, f) - \delta(M/\{c_1, \ldots, c_i\}, f) \geq \ell^{6m},$$

since $r_M(C_0) = b$. Since $|S'| \geq b(3m)$, there is some $i \in [b]$ which is chosen for at least $3m$ elements of S'. Let $S'' \subseteq S'$ denote this set of $3m$ elements. By Proposition 3.3.3 with $(M, X, e) = (M/\{c_1, \ldots, c_{i-1}\}, S'', c_i)$, outcome (2) holds with $C = \{c_1, \ldots, c_{i-1}\}$ and $e = c_i$, a contradiction. □

By Proposition 3.5.3 with $(M, S, g) = (M/C_0, S_1, 5 \cdot 2^h)$, either k is not maximal or (3) holds with $C = C_0$. □

Chapter 4

Exploiting Connectivity

In this chapter we prove the following result, which allows us to complete step (II) of the general growth-rates-proof sketch outlined in Section 1.6. Recall that if P is a porcupine with tip f, then we write $d(P)$ for the corank of $\mathrm{si}(P/f)$.

Theorem 4.0.1. *Let $\ell \geq 2$ and $k, s \geq 1$ and $g, m, n \geq 3$ be integers, and let Γ be a finite group. There are functions $s_{4.0.1}\colon \mathbb{Z}^3 \to \mathbb{Z}$ and $r_{4.0.1}\colon \mathbb{Z}^6 \to \mathbb{Z}$ so that if $M \in \mathcal{U}(\ell)$ is a vertically $s_{4.0.1}(k, s, g)$-connected matroid with no rank-n doubled-clique minor, and with a $\mathrm{DG}(r_{4.0.1}(\ell, m, k, n, s, g), \Gamma)$-minor G and a size-k independent set such that each element is the tip of a g-porcupine restriction P with $d(P) = s$, then M has a minor N of rank at least m so that*

- *N has a $\mathrm{DG}(r(N), \Gamma)$-restriction, and*
- *N has a size-k independent set such that each element is the tip of a g-porcupine restriction P with $d(P) = s$.*

This is the most difficult step in the proof of Theorem 1.7.2, due to the fact that the matroid M has bounded vertical connectivity, but each porcupine restriction can have arbitrarily large rank. To deal with this we need some technical properties of porcupines, which we prove in Section 4.2. We also need Theorem 4.1.4, Tutte's Linking Theorem for nested sets, which we prove in Section 4.1. Much of the difficulty of Theorem 4.0.1 is present even in the special case that G is a restriction of M, and we prove this case in Section 4.3.

When G is not a restriction of M, we use structures called tangles to maintain connectivity between certain pairs of sets as we take a minor. We introduce tangles in Section 4.4, and prove a key property of tangles and Dowling geometries in Section 4.5. Finally, we prove Theorem 4.0.1 in Section 4.6. In Sections 4.5 and 4.6 we use Lemma 2.2.4 to move from a Dowling-geometry minor to a Dowling-geometry restriction, but this chapter is otherwise self-contained.

4.1 A Generalization of Tutte's Linking Theorem

In this section we prove a generalization of Tutte's Linking Theorem. The following lemma was proved in [8], and follows from the submodularity of the connectivity function.

Lemma 4.1.1. *Let e be an element of a matroid M, and let X, Y be subsets of $M - \{e\}$. Then*

$$\lambda_{M \backslash e}(X) + \lambda_{M/e}(Y) \geq \lambda_M(X \cap Y) + \lambda_M(X \cup Y \cup \{e\}) - 1.$$

We use this to prove a lemma about a collection of nested sets in a matroid.

Lemma 4.1.2. *Let M be a matroid, $m \geq 1$ be an integer, and $Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y_m \subseteq E(M) - X$. If $E(M) \neq \mathrm{cl}_M(X) \cup \mathrm{cl}_M(Y_m)$, then there is some $e \in E(M) - (\mathrm{cl}_M(X) \cup \mathrm{cl}_M(Y_m))$*

so that $\kappa_{M/e}(X, Y_i) = \kappa_M(X, Y_i)$ *for all* $i \in [m]$.

Proof. Let $|E(M) - (\mathrm{cl}_M(X) \cup \mathrm{cl}_M(Y_m))|$ be minimal so that the claim is false, and let $e \in E(M) - (\mathrm{cl}_M(X) \cup \mathrm{cl}_M(Y_m))$. Since the claim is false for M, there is some $k \in [m]$ so that $\kappa_{M/e}(X, Y_k) < \kappa_M(X, Y_k)$. Let $(E - e - Z^k, Z^k)$ be a partition of $E(M) - \{e\}$ so that $(\mathrm{cl}_M(X) - Y_m) \subseteq E - Z^k$ and $Y_k \subseteq Z^k$, while $\lambda_{M/e}(Z_k) = \kappa_{M/e}(X, Y_k)$; such a partition exists because each set $A \subseteq E(M/e)$ satisfies $\lambda_{M/e}(A) \geq \lambda_{M/e}(\mathrm{cl}_{M/e}(A))$. Note that $e \in \mathrm{cl}_M(Z^k)$, or else $\lambda_M(Z^k) = \lambda_{M/e}(Z^k) < \kappa_M(X, Y_k)$, which contradicts the defintion of $\kappa_M(X, Y_k)$.

Now, if $|E(M) - (\mathrm{cl}_M(X) \cup \mathrm{cl}_M(Y_m))| = 1$, then $Z^k \subseteq \mathrm{cl}_M(Y_m)$, since $(\mathrm{cl}_M(X) - Y_m) \subseteq E - Z^k$. But then since $e \in \mathrm{cl}_M(Z^k)$, we have $e \in \mathrm{cl}_M(Y_m)$, which contradicts the choice of e. Thus, $|E(M) - (\mathrm{cl}_M(X) \cup \mathrm{cl}_M(Y_m))| > 1$, so by minimality there is some $j \in [m]$ so that $\kappa_{M\backslash e}(X, Y_j) < \kappa_M(X, Y_j)$.

First assume that $j \leq k$, so $Y_j \subseteq Y_k$. Let $(E - e - Z_j, Z_j)$ be a partition of $E(M) - \{e\}$ such that $X \subseteq E - e - Z_j$ and $Y_j \subseteq Z_j$, while $\lambda_{M\backslash e}(Z_j) < \kappa_M(X, Y_j)$. Note that $X \subseteq (E - Z_j) \cap (E - Z^k) = E - (Z_j \cup Z^k \cup \{e\})$ and $Y_j \subseteq Z_j \cap Z^k$. Then

$$\begin{aligned}
\kappa_M(X, Y_j) - 1 + \kappa_M(X, Y_k) - 1 &\geq \lambda_{M\backslash e}(Z_j) + \lambda_{M/e}(Z^k) && (1)\\
&\geq \lambda_M(Z_j \cap Z^k) + \lambda_M(Z_j \cup Z^k \cup \{e\}) - 1 && (2)\\
&\geq \kappa_M(X, Y_j) + \kappa_M(X, Y_k) - 1, && (3)
\end{aligned}$$

a contradiction. Line (2) follows from Lemma 4.1.1 applied to Z_j and Z^k, and line (3) holds because $Y_j \subseteq Z_j \cap Z^k$ and $X \subseteq E(M) - (Z_j \cup Z^k \cup \{e\})$. If $k \leq j$, then we apply the same argument with M^*, since $\kappa_{M/e}(A, B) = \kappa_{M^*\backslash e}(A, B)$ for each $e \in E(M)$ and disjoint $A, B \subseteq E(M) - \{e\}$. □

We will use the following strengthening of Tutte's Linking Theorem, which was proved by Geelen, Gerards, and Whittle in [9], and will be invoked several times in this thesis.

Theorem 4.1.3. *Let* X *and* Y *be disjoint sets of elements of a matroid* M*. Then* M *has a minor* N *with ground set* $X \cup Y$ *such that* $\kappa_N(X, Y) = \kappa_M(X, Y)$*, while* $N|X = M|X$ *and* $N|Y = M|Y$*.*

We now prove a generalization of Tutte's Linking Theorem, which allows us find a minor which preserves connectivity between a set and a nested collection of sets.

Theorem 4.1.4. *Let* M *be a matroid,* $m \geq 1$ *be an integer, and* $Y_1 \subseteq Y_2 \subseteq \cdots \subseteq Y_m \subseteq E(M) - X$*. Then* M *has a minor* N *with ground set* $X \cup Y_m$ *such that* $\kappa_N(X, Y_i) = \kappa_M(X, Y_i)$ *for each* $i \in [m]$*, while* $N|X = M|X$ *and* $N|Y_1 = M|Y_1$*.*

Proof. Let N be a minimal minor of M such that $X \cup Y_m \subseteq E(N)$, while $\kappa_N(X, Y_i) = \kappa_M(X, Y_i)$ for each $i \in [m]$, $N|X = M|X$, and $N|Y_1 = M|Y_1$. Assume for a contradiction that $E(N) \neq X \cup Y_m$. Take $e \in E(N) - (X \cup Y_m)$. If $E(N) = \mathrm{cl}_N(X) \cup \mathrm{cl}_N(Y_1)$, then $\kappa_{N\backslash e}(X, Y_i) = \kappa_N(X, Y_i)$ for each $i \in [m]$ by Theorem 4.1.3 applied with $Y = Y_1$, which contradicts the minor-minimality of N.

Let $j \in [m]$ be maximal so that $E(N) \neq \mathrm{cl}_N(X) \cup \mathrm{cl}_N(Y_j)$. By Lemma 4.1.2 applied to X and $Y_1, \ldots, Y_j$, there is some $e \in E(N) - (\mathrm{cl}_N(X) \cup \mathrm{cl}_N(Y_j))$ so that $\kappa_{N/e}(X, Y_i) = \kappa_N(X, Y_i)$ for each $1 \leq i \leq j$. There is some integer $i \in [m]$ so that $\kappa_{N/e}(X, Y_i) < \kappa_N(X, Y_i)$, or else N/e contradicts the minimality of N. Then $i > j$, so $E(N) = \mathrm{cl}_N(X) \cup \mathrm{cl}_N(Y_i)$ by the maximality of j. Then each set Z with $X \subseteq Z \subseteq E(N) - Y_i$ and $\lambda_N(Z) = \kappa_N(X, Y_i)$ satisfies $Z \subseteq \mathrm{cl}_N(X)$. Since $e \notin \mathrm{cl}_N(X)$ this implies that $\lambda_{N/e}(Z - \{e\}) = \lambda_N(Z)$ for each such set Z, and thus $\kappa_{N/e}(X, Y_i) = \kappa_N(X, Y_i)$, a contradiction. Thus, $E(N) = X \cup Y$, and N is the desired minor of M. □

We conclude this section by proving a lemma concerning κ_M.

Lemma 4.1.5. *Let Y and J be disjoint sets of elements of a matroid M, and let*

$$D = \{e \in E(M) - (J \cup Y) \colon \kappa_{M/e}(A, J) < \kappa_M(A, J) \text{ for some } A \subseteq Y\}.$$

Then $\kappa_M(Y \cup D, J) = \kappa_M(Y, J)$.

Proof. Let $E = E(M)$, and let $\kappa = \kappa_M$. Clearly $\kappa(Y \cup D, J) \geq \kappa(Y, J)$; assume for a contradiction that $\kappa(Y \cup D, J) > \kappa(Y, J)$. Let D_1 be a maximal subset of D so that $\kappa(Y \cup D_1, J) = \kappa(Y, J)$, and let $e \in D - D_1$. Since $e \in D$, there is some $A \subseteq Y$ so that $\kappa_{M/e}(A, J) < \kappa_M(A, J)$, which implies that $\kappa(A \cup \{e\}, J) = \kappa(A, J)$. Then there is some $Z_1 \subseteq E$ so that $A \cup \{e\} \subseteq Z_1 \subseteq E - J$ and $\lambda(Z_1) = \kappa(A, J)$. Similarly, there is some $Z_2 \subseteq E$ so that $Y \cup D_1 \subseteq Z_2 \subseteq E - J$ and $\lambda(Z_2) = \kappa(Y \cup D_1, J) = \kappa(Y, J)$. Then $(Y \cup D_1 \cup \{e\}) \subseteq Z_1 \cup Z_2$, and

$$\begin{aligned}
\lambda(Z_1 \cup Z_2) &\leq \lambda(Z_1) + \lambda(Z_2) - \lambda(Z_1 \cap Z_2) \\
&= \kappa(A, J) + \kappa(Y, J) - \lambda(Z_1 \cap Z_2) \\
&\leq \kappa(A, J) + \kappa(Y, J) - \kappa(A, J) \\
&= \kappa(Y, J),
\end{aligned}$$

where the third lines holds because $A \subseteq Z_1 \cap Z_2$ and $J \subseteq E - (Z_1 \cap Z_2)$. Thus, $\kappa(Y \cup D_1 \cup \{e\}, J) = \kappa(Y, J)$, which contradicts the maximality of D_1. □

4.2 Prickles

In this section we define some terminology related to porcupines, and then prove several lemmas which we will use in the proof of Theorem 4.0.1. Recall that a g-porcupine is a simple matroid P with an element t so that each line of P through t has length three, and $\mathrm{si}(P/t)$ has no coloops and girth at least g. We write $d(P)$ for the corank of $\mathrm{si}(P/t)$. Note that a porcupine is a spike if and only if $d(P) = 1$, and that if $d(P) = 0$ then P consists of only the tip. If $d(P) = 0$ then P is a *trivial porcupine*.

For porcupines P and P', we say that P' is a *subporcupine* of P, and write $P' \preceq P$, if $E(P') \subseteq E(P)$ and P' and P have the same tip t. If $E(P')$ is a proper subset of $E(P)$ then we write $P' \prec P$; this implies that $d(P') < d(P)$, since $\mathrm{si}(P'/t)$ has no coloops. Note that P' is a restriction of P, and that each line of P' through t is a line of P through t. Also, note that every porcupine P has a unique trivial subporcupine.

If P_1 and P_2 are subporcupines of P, then $P|(E(P_1) \cup E(P_2))$ is a porcupine, which we denote by $P_1 \cup P_2$. Note that if $P_1 \not\preceq P_2$ and $P_2 \not\preceq P_1$ and neither P_1 nor P_2 is equal to P, then $d(P_1 \cup P_2) > \max(d(P_1), d(P_2))$, since $\mathrm{si}(P_1/t)$ and $\mathrm{si}(P_2/t)$ have no coloops. We say that P' is a *retract* of P if P' and P have the same tip, $E(P') \subseteq E(P)$, and $d(P') = d(P)$. Whenever we work with retracts it will be the case that P is a restriction of a matroid M and P' is a restriction of a minor N of M. Since $d(P') = d(P)$, one can think of P' as being a copy of P which we recover in the minor N.

This notation for porcupines extends to collections of porcupines. In the animal kingdom a collection of porcupines is called a prickle, and we use the same terminology in this thesis. A *prickle* is a pair $\mathbf{R} = (R, \mathcal{P})$ where R is a matroid and $\mathcal{P}$ is a collection of pairwise disjoint porcupine restrictions of R such that $\cup_{P \in \mathcal{P}} E(P) = E(R)$. If each porcupine in $\mathcal{P}$ is a g-porcupine, then we say that $\mathbf{R}$ is a *g-prickle*. For a matroid M, we say that $\mathbf{R}$ is a *prickle of M* if R is a restriction of M, and we write $E(\mathbf{R})$ for $E(R)$. Define $d(\mathbf{R}) = \sum_{P \in \mathcal{P}} d(P)$, and note that if $d(\mathbf{R}) = 0$ then each porcupine in $\mathcal{P}$ is simply a tip.

We say that a prickle $(R', \mathcal{P}')$ is a *subprickle* of $(R, \mathcal{P})$, and write $(R', \mathcal{P}') \preceq (R, \mathcal{P})$, if there is a bijection $\psi \colon \mathcal{P} \to \mathcal{P}'$ so that for each $P \in \mathcal{P}$, the porcupine $\psi(P)$ is a subporcupine of P. If there is some $P \in \mathcal{P}$ so that $\psi(P) \neq P$, then we write $(R', \mathcal{P}') \prec (R, \mathcal{P})$; this implies that $d(R', \mathcal{P}') < d(R, \mathcal{P})$ since $d(\psi(P)) < d(P)$. Since the porcupines in $\mathcal{P}$ are pairwise disjoint this also implies that $E(R')$ is a proper subset of $E(R)$.

It is important to note that each subprickle of $(R, \mathcal{P})$ contains the tip of each porcupine in $\mathcal{P}$, which implies that the unique subprickle $(R', \mathcal{P}')$ of $(R, \mathcal{P})$ with $d(R', \mathcal{P}') = 0$ is simply the collection of tips of porcupines in $\mathcal{P}$; we say that $(R', \mathcal{P}')$ is the *trivial subprickle* of $(R, \mathcal{P})$. If $(R_1, \mathcal{P}_1)$ and $(R_2, \mathcal{P}_2)$ are subprickles of a prickle $(R, \mathcal{P})$, then $(R_1 \cup R_2, \mathcal{P}_1 \cup \mathcal{P}_2)$ denotes the subprickle of $(R, \mathcal{P})$ such that each porcupine in $\mathcal{P}_1 \cup \mathcal{P}_2$ is the union of porcupines of $\mathcal{P}_1$ and $\mathcal{P}_2$ with common tip.

Finally, we need terminology for recovering a prickle after applying projections and deletions. We say that a prickle $(R', \mathcal{P}')$ is a *retract* of a prickle $(R, \mathcal{P})$ if there is a bijection $\psi \colon \mathcal{P} \to \mathcal{P}'$ so that for each $P \in \mathcal{P}$, the porcupine $\psi(P)$ is a retract of P. Just as for porcupines, we will only work with retracts when $(R, \mathcal{P})$ is a prickle of a matroid M and $(R', \mathcal{P}')$ is a prickle of a minor N of M. Note that the collection of tips of porcupines in $\mathcal{P}'$ is equal to the collection of tips of porcupines in $\mathcal{P}$, and that $d(\mathbf{R}') = d(\mathbf{R})$; these properties allow us to think of $\mathbf{R}'$ as a copy of $\mathbf{R}$ which we recover in the minor N.

We now provide some properties of prickles which we will need to prove Theorem 4.0.1. Our first lemma deals with a maximal proper subprickle, which will be important for using inductive arguments with prickles.

Lemma 4.2.1. *Let $\mathbf{A}$ be a subprickle of a prickle $\mathbf{R}$ so that $d(\mathbf{A}) = d(\mathbf{R}) - 1$. Then*

(i) each $\mathbf{Z} \preceq \mathbf{R}$ with $\mathbf{Z} \not\preceq \mathbf{A}$ satisfies $\mathbf{A} \cup \mathbf{Z} = \mathbf{R}$, and

(ii) there is a prickle $\mathbf{C} \preceq \mathbf{R}$ so that $d(\mathbf{C}) = 1$ and $\mathbf{A} \cup \mathbf{C} = \mathbf{R}$.

Proof. Let $\mathbf{A} = (A, \mathcal{A})$ and $\mathbf{R} = (R, \mathcal{R})$. Since $d(\mathbf{A}) = d(\mathbf{R}) - 1$, there is a unique porcupine in $P \in \mathcal{R}$ so that $P \notin \mathcal{A}$. Since $\mathbf{A} \preceq \mathbf{R}$, there is some $P_1 \in \mathcal{A}$ so that P_1 is a subporcupine of P and $d(P_1) = d(P) - 1$. Let t denote the common tip of P and P_1.

We first prove (i). Let $\mathbf{Z} = (Z, \mathcal{P})$, and let P_2 denote the porcupine in $\mathcal{P}$ with tip t. Let $(P/t)|X$ be a simplification of P/t, and for each $i \in \{1, 2\}$ let $(P/t)|X_i$ be a simplification of P_i/t so that $X_i \subseteq X$. Since $(P/t|X_2)$ has no coloops and X_2 is not contained in X_1, we have $r^*((P/t)|(X_1 \cup X_2)) > r^*((P/t)|X_1)$. Since $(P/t)|X$ has no coloops and $r^*((P/t)|X) = r^*((P/t)|X_1) + 1$, it follows that $X_1 \cup X_2 = X$. Thus, $P_1 \cup P_2 = P$, and so $\mathbf{A} \cup \mathbf{Z} = \mathbf{R}$.

We now prove (ii). Since $\mathrm{si}(P/t)$ has no coloops, there is a subporcupine P' of P so that $d(P') = 1$ (P' is a spike) and P' is not a subporcupine of P_1. Let $\mathbf{C} = (C, \mathcal{C})$ be the unique subprickle of $\mathbf{R}$ so that $P' \in \mathcal{C}$ and each other porcupine $P'' \in \mathcal{C}$ satisfies $d(P'') = 0$. Then $d(\mathbf{C}) = 1$, and $\mathbf{C} \not\preceq \mathbf{A}$, and thus $\mathbf{A} \cup \mathbf{C} = \mathbf{R}$ by (i) □

Our second lemma finds a specific collection of maximal proper subprickles of a prickle, again for use in inductive arguments. It is perhaps more natural to write the lemma using set intersection instead of union; we use union because we will apply this lemma to bound $\kappa_M(J, E(\mathbf{R}) - E(\mathbf{Y}))$ for some set J.

Lemma 4.2.2. *Let M be a matroid with a prickle $\mathbf{R}$ such that $d(\mathbf{R}) > 0$, and let $\mathbf{Y} \preceq \mathbf{R}$. Then there is a collection $\mathcal{A}$ of subprickles of $\mathbf{R}$ so that each $\mathbf{A} \in \mathcal{A}$ satisfies $d(\mathbf{A}) = d(\mathbf{R}) - 1$ and $\mathbf{Y} \preceq \mathbf{A}$, while $|\mathcal{A}| \leq d(\mathbf{R}) - d(\mathbf{Y})$ and $E(\mathbf{R}) - E(\mathbf{Y}) \subseteq \cup_{\mathbf{A} \in \mathcal{A}}(E(\mathbf{R}) - E(\mathbf{A}))$.*

Proof. Let $\mathcal{A}$ be a minimal collection of subprickles of $\mathbf{R}$ such that each $\mathbf{A} \in \mathcal{A}$ satisfies $d(\mathbf{A}) = d(\mathbf{R}) - 1$ and $\mathbf{Y} \preceq \mathbf{A}$, while $E(\mathbf{R}) - E(\mathbf{Y}) \subseteq \cup_{\mathbf{A} \in \mathcal{A}}(E(\mathbf{R}) - E(\mathbf{A}))$. Some choice for $\mathcal{A}$ exists because for each $e \in E(\mathbf{R}) - E(\mathbf{Y})$ there is some $\mathbf{A} \preceq \mathbf{R}$ such that $d(\mathbf{A}) = d(\mathbf{R}) - 1$ and $\mathbf{Y} \preceq \mathbf{A}$ and $e \notin E(\mathbf{A})$, using that the porcupines of $\mathbf{R}$ are pairwise disjoint. For each

$\mathbf{A} \in \mathcal{A}$ there is some $\mathbf{C}_A \preceq \mathbf{R}$ such that $d(\mathbf{C}_A) = 1$ and $\mathbf{A} \cup \mathbf{C}_A = \mathbf{R}$, by Lemma 4.2.1 (ii). Since $\mathcal{A}$ is minimal, for each $\mathbf{A} \in \mathcal{A}$ there is an element of $E(\mathbf{R}) - E(\mathbf{Y})$ which is in $E(\mathbf{C}_A) - E((\cup_{\mathbf{A}' \in \mathcal{A} - \{\mathbf{A}\}} \mathbf{C}_{A'}))$. This implies that $d(\mathbf{Y} \cup (\cup_{\mathbf{A} \in \mathcal{A}} \mathbf{C}_A)) \geq d(\mathbf{Y}) + |\mathcal{A}|$, since $\mathrm{si}(P/t)$ has no coloops for any porcupine P with tip t. However, since $\mathbf{Y} \cup (\cup_{\mathbf{A} \in \mathcal{A}} \mathbf{C}_A) \preceq \mathbf{R}$ we have $d(\mathbf{Y} \cup (\cup_{\mathbf{A} \in \mathcal{A}} \mathbf{C}_A)) \leq d(\mathbf{R})$. Thus, $d(\mathbf{Y}) + |\mathcal{A}| \leq d(\mathbf{R})$, so $|\mathcal{A}| \leq d(\mathbf{R}) - d(\mathbf{Y})$. $\square$

Before proving our next lemma about prickles, we need a straightforward lemma about the corank of sets in a matroid. Again, we state this lemma using union instead of intersection for ease of application.

Lemma 4.2.3. *Let M be a matroid, and let $X \subseteq E(M)$ so that M and $M|X$ have no coloops. Then there is a collection $\mathcal{Y}$ of subsets of $E(M)$ so that $|\mathcal{Y}| \leq r^*(M) - r^*(M|X)$ and $E(M) - X \subseteq \cup_{Y \in \mathcal{Y}} (Y - X)$, while each $Y \in \mathcal{Y}$ satisfies $X \subseteq Y$ and $r^*(M|Y) = r^*(M|X) + 1$, and $M|Y$ has no coloops.*

Proof. Let $E = E(M)$. This statement is easier to prove in M^*. Since $(M|X)^* = M^*/(E - X)$, we have $r^*(M|X) = r^*(M) - r^*(E - X)$. Since $M|X$ has no coloops, the set $E - X$ is a flat of M^*. Let B be a basis of $M^*|(E - X)$, and let $\mathcal{H} = \{\mathrm{cl}_{M^*}(B - \{e\}) \colon e \in B\}$. Then each set $H \in \mathcal{H}$ is a hyperplane of $M^*|(E - X)$, and $|\mathcal{H}| = |B| = r^*(E - X) = r^*(M) - r^*(M|X)$. Moreover, for each $e \in E - X$ there is some $H \in \mathcal{H}$ so that $e \notin H$, which implies that $\cap \mathcal{H} = \varnothing$.

Let $\mathcal{Y} = \{E - H \colon H \in \mathcal{H}\}$, and note that $\cap \mathcal{Y} = X$, so $E - X \subseteq \cup_{Y \in \mathcal{Y}} (Y - X)$. Fix some $Y \in \mathcal{Y}$, and let $H \in \mathcal{H}$ so that $Y = E - H$. Since H is a hyperplane of $M^*|(E - X)$ and $E - X$ is a flat of M^*, it follows that H is a flat of M^*, so M^*/H has no loops. Thus, $M|Y$ has no coloops. Finally, since H is a hyperplane of $M^*|(E - X)$ we have $r(M^*/H) = r(M^*/(E - X)) + 1 = r^*(M|X) + 1$. Since $M^*/H = (M|(E - H))^*$, this implies that Y satisfies $r(M|Y)^* = r^*(M|X) + 1$. $\square$

The following lemma describes projections of a prickle from the perspective of a subprickle. It generalizes the fact that a projection of a spike by an element not parallel to the tip is the union of at most two spikes. Just as for Lemma 4.2.2, we will apply this lemma to bound $\kappa_M(J, E(\mathbf{A}) - E(\mathbf{A}_0))$ for some set J, so we write the statement using set union instead of intersection.

Lemma 4.2.4. *Let M be a matroid with prickles $\mathbf{A}_0$ and $\mathbf{A}$ such that $\mathbf{A}_0 \preceq \mathbf{A}$ and $d(\mathbf{A}_0) = d(\mathbf{A}) - 1$. Let $C \subseteq E(M)$ so that M/C has a retract $\mathbf{A}_0'$ of $\mathbf{A}_0$. Then there is a non-empty set C' and a collection $\mathcal{A}$ of prickles of M/C such that each $\mathbf{A}' \in \mathcal{A}$ is a retract of $\mathbf{A}$ with $\mathbf{A}_0' \preceq \mathbf{A}'$, while $|\mathcal{A}| + |C'| \leq r_M(C) + 2$ and $E(\mathbf{A}) - E(\mathbf{A}_0) \subseteq \mathrm{cl}_M(C \cup C' \cup (\cup_{\mathbf{A}' \in \mathcal{A}} (E(\mathbf{A}') - E(\mathbf{A}_0'))))$.*

Proof. We may assume that C is independent, since the lemma holds for C if and only if it holds for a basis of C. Since $d(\mathbf{A}_0) = d(\mathbf{A}) - 1$ and $\mathbf{A}_0 \preceq \mathbf{A}$, there are porcupines P of $\mathbf{A}$ and P_0 of $\mathbf{A}_0$ such that P_0 is a subporcupine of P and $d(P_0) = d(P) - 1$. Then M/C contains a retract P_0' of P_0. Let t denote the common tip of P, P_0, and P_0'. Note that $t \notin \mathrm{cl}_M(C)$ since P_0' is a retract of P_0, and thus each long line of P through t contains at most one element of $\mathrm{cl}_M(C)$.

Let $(P/t)|X$ be a simplification of P/t, let $(P/t)|X_0$ be a simplification of P_0/t, and let $M/(C \cup \{t\})|X_0'$ be a simplification of P_0'/t such that $X_0' \subseteq X_0 \subseteq X$ and $\mathrm{cl}_M(C) \cap X = \varnothing$. Such a set X exists because each long line of P through t contains at most one element of $\mathrm{cl}_M(C)$. Let $N = M/(C \cup \{t\})$. Then $N|X$ has no coloops, and

$$\begin{aligned} r^*(N|X) - r^*(N|X_0') &\le \big(r^*((M/t)|X) + |C|\big) - r^*(N|X_0') \\ &= |C| + d(P) - d(P_0') \\ &= |C| + 1. \end{aligned}$$

In particular, this implies that $N|X$ has at most $|C|+1$ nontrivial parallel classes since $N|X_0'$ is simple.

We first find a suitable simplification of $N|X$. Let $N|X'$ be a simplification of $N|X$ so that $X_0' \subseteq X'$, and let L_2 denote the set of coloops of $N|X'$. Since $N|X$ has no coloops, each element of L_2 is in a nontrivial parallel pair of $N|X$. Also, $L_2 \cap X_0' = \varnothing$ since $N|X_0'$ has no coloops. Let $X'' = X' - L_2$, and let T be a transversal of the nontrivial parallel classes of $N|X$ so that $L_2 \cap T = \varnothing$. Note that $|T| \le |C| + 1$. Then $X - X'' \subseteq \mathrm{cl}_M(C \cup \{t\} \cup T)$, since each element in $X - X''$ is a loop of N or is parallel in N to an element of T. We also see that $X_0' \subseteq X''$, the matroid $N|X''$ is simple and has no coloops, and

$$\begin{aligned} r^*(N|X'') &= |X''| - r(N|X'') \\ &= |X'| - |L_2| - (r(N|X') - |L_2|) \\ &= |X'| - r(N|X') \\ &= |X'| - r(N|X) \\ &\le |X| - |T| - r(N|X) = r^*(N|X) - |T|, \end{aligned}$$

where the inequality holds because we delete at least one element from each parallel class of $N|X$ to obtain $N|X'$. Then we have

$$\begin{aligned} r^*(N|X'') - r^*(N|X_0') &\le \big(r^*(N|X) - |T|\big) - r^*(N|X_0') \\ &\le |C| + 1 - |T|. \end{aligned}$$

Since $N|X''$ is simple and $r^*(N|X'') - r^*(N|X_0') \le |C| + 1 - |T|$, by Lemma 4.2.3 with $M = N|X''$ and $X = X_0'$, there is a collection $\mathcal{Y}$ of subsets of X'' such that $|\mathcal{Y}| \le |C|+1-|T|$

and $X''-X_0' \subseteq \cup_{Y\in\mathcal{Y}}(Y-X_0')$, and each $Y \in \mathcal{Y}$ satisfies $X_0' \subseteq Y$ and $r^*(N|Y) = r^*(N|X_0')+1$, and the matroid $N|Y$ is simple and has no coloops.

For each $Y \in \mathcal{Y}$, define $\mathbf{A}_Y$ to be the prickle of M/C obtained from $\mathbf{A}_0'$ by replacing P_0' by a porcupine P_Y with tip t such that $E(P_Y) \subseteq E(P)$, each line through t contains an element of Y, and each element of Y is on a line through t. Such a porcupine P_Y exists since $N|Y$ is simple. Since $X_0' \subseteq Y$ and $r^*(N|Y) = r^*(N|X_0') + 1$, we have $P_0' \preceq P_Y$ and $d(P_Y) = d(P_0') + 1 = d(P)$. Since $d(P_Y) = d(P)$ and P_Y and P have the same tip and $E(P_Y) \subseteq E(P)$ it follows that P_Y is a retract of P, and thus $\mathbf{A}_Y$ is a retract of $\mathbf{A}$. Since $X'' - X_0' \subseteq \cup_{Y\in\mathcal{Y}}(Y - X_0')$ and $X - X'' \subseteq \mathrm{cl}_M(C \cup \{t\} \cup T)$ we have

$$E(\mathbf{A}) - E(\mathbf{A}_0) \subseteq \mathrm{cl}_M(C \cup \{t\} \cup T \cup (\cup_{Y\in\mathcal{Y}}(E(\mathbf{A}_Y) - E(\mathbf{A}_0')))).$$

Thus, $\mathcal{A} = \{\mathbf{A}_Y : Y \in \mathcal{Y}\}$ and $C' = T \cup \{t\}$ satisfies the lemma statement. □

Given an g-porcupine P with tip t, it is often easier to work with a simplification of P/t, which is a matroid with girth at least g and no coloops. The following lemma shows that we can piece together two minors with large girth and corank to find a restriction with large girth and corank.

Lemma 4.2.5. *Let M be a matroid with sets $Z, K_1, K_3 \subseteq E(M)$ such that $Z \subseteq K_1 \subseteq E(M) - K_3$, and let $s \geq s' \geq 0$ and $g \geq 3$ be integers. If $M|Z$ has corank s' and girth at least g and $(M/K_1)|K_3$ has corank $s - s'$ and girth at least g, then $M|(K_1 \cup K_3)$ has a restriction with corank s and girth at least g.*

Proof. Let B be a basis of $(M/Z)|(K_1 - Z)$. If $M|(Z \cup B \cup K_3)$ has a circuit of size less than g, it is not contained in $Z \cup B$ since $M|Z$ has girth at least g, and B is independent in M/Z. But then $(M/K_1)|K_3$ has a circuit of size less than g, a contradiction. Also,

$$|Z{\cup}B{\cup}K_3|-r_M(Z{\cup}B{\cup}K_3) = \big(|Z{\cup}B|-r_M(Z{\cup}B)\big)+\big(|K_3|-r_{M/(Z\cup B)}(K_3)\big) = s'+(s-s') = s,$$

so $r^*(M|(Z \cup B \cup K_3)) = s$ and the lemma holds. □

The final lemma essentially shows that a projection of a matroid with corank s and girth at least g has a restriction with corank s and girth at least $g/2$.

Lemma 4.2.6. *For all integers $g \geq 3$, $s \geq 1$ and $m \geq 0$, if M is a matroid with $C, S \subseteq E(M)$ such that $M|S$ has corank s and girth at least $g2^m$ and $\sqcap_M(S, C) \leq m$, then $(M/C)|(S - C)$ has a restriction with corank s and girth at least g.*

Proof. Since $g \geq 3$ we may assume that M is simple. Let $|C|$ be minimal so that the lemma is false. Then $r_M(C) \geq 1$ and $C \subseteq \mathrm{cl}_M(S)$. If $e \in C \cap S$ then $(M/e)|(S - \{e\})$ has corank

s and girth at least $g2^m - 1$. Since $g2^m - 1 \geq g2^{m-1}$, the result holds by the minimality of $|C|$.

If $e \in C - S$ then $(M/e)|S$ has corank $s+1$, and has at most one circuit of size less than $g2^{m-1}$. To see this, if $(M/e)|S$ has distinct circuits C_1 and C_2 of size less than $g2^{m-1}$, then $C_1 \cup \{e\}$ and $C_2 \cup \{e\}$ are circuits of $M|(S \cup \{e\})$. But then $M|((C_1 \cup C_2) - \{e\})$ has a circuit of size at most $2(g2^{m-1}) - 1 < g2^m$, which contradicts that $M|S$ has girth at least $2g^m$. Thus, for each f in the smallest circuit of $(M/e)|S$, the matroid $(M/e)|(S - \{f\})$ has corank s and girth at least $g2^{m-1}$, and the result holds by the minimality of $|C|$. □

4.3 The Dowling-Geometry-Restriction Case

In this section we prove Theorem 4.0.1 in the case that G is a Dowling-geometry restriction of M. Since vertical connectivity is not preserved under taking minors, we strengthen Theorem 4.0.1 enough so that we no longer need vertical connectivity in the statement. To accomplish this, we identify a prickle $\mathbf{Q}$ which contains all prickles whose connectivity to $E(G)$ is 'too small', and we maintain the property that $\kappa_M(E(G), E(\mathbf{Q})) = r_M(E(\mathbf{Q}))$. The lemma below makes no reference to Dowling geometries, although we only invoke this lemma when $M|J$ is a Dowling geometry. This is the most technical result in this thesis, and uses the results from Sections 4.1 and 4.2.

Lemma 4.3.1. *Let $g \geq 3$ and $k, s \geq 1$ be integers, and define integers $n_0 = k+1$, and $n_i = 2kg2^{2in_{i-1}+1}$ for $i \geq 1$. Let M be a matroid with $J \subseteq E(M)$ such that $M \backslash J$ has a g-prickle $\mathbf{R} = (R, \mathcal{P})$ for which*

- *$|\mathcal{P}| \leq k$ and each $P \in \mathcal{P}$ satisfies $d(P) \leq s$, and*
- *there is a subprickle $\mathbf{Q}$ of $\mathbf{R}$ such that $\kappa_M(J, E(\mathbf{Q})) = r_M(E(\mathbf{Q})) < n_{d(\mathbf{Q})}$ and each $\mathbf{A} \preceq \mathbf{R}$ with $\kappa_M(J, E(\mathbf{A})) < n_{d(\mathbf{A})}$ satisfies $\mathbf{A} \preceq \mathbf{Q}$.*

Then M has a contract-minor M' which has $M|J$ as a spanning restriction, and has a g-retract $\mathbf{R}'$ of $\mathbf{R}$ so that $\mathbf{Q} \preceq \mathbf{R}'$.

Proof. Note that $n_i > 2kin_{i-1} + kg2^{2in_{i-1}+1}$ for each $i \geq 1$. Let M with prickle $\mathbf{R} = (R, \mathcal{P})$ and $\mathbf{Q} \preceq \mathbf{R}$ be a counterexample with $d(\mathbf{R})$ minimum. If $\mathbf{R} = \mathbf{Q}$, then by Theorem 4.1.3 applied to J and $E(Q)$, the result holds with $\mathbf{R}' = \mathbf{R}$. If $d(\mathbf{R}) = 0$, then $\mathbf{R} \preceq \mathbf{Q}$ and so $\mathbf{R} = \mathbf{Q}$ and the result holds. Thus, $\mathbf{R} \neq \mathbf{Q}$ and $d(\mathbf{R}) > 0$.

We will take a minimal minor M_1 of M with a retract $\mathbf{R}_1$ of $\mathbf{R}$ so that $\mathbf{R}_1$ satisfies a slightly weaker property than the second condition of the lemma statement, so that we can exploit the minimality of $d(\mathbf{R})$. We first develop some notation for the collection of all such minors and retracts. Let $\mathcal{M}$ denote the collection of pairs $(M', \mathbf{R}')$ where M' is a minor of M with $M'|J = M|J$, and $\mathbf{R}'$ is a retract of $\mathbf{R}$ contained in M' with $\mathbf{Q} \preceq \mathbf{R}'$ so that

(i) $\kappa_{M'}(J, E(\mathbf{Q})) = r_M(E(\mathbf{Q})) < n_{d(\mathbf{Q})}$, and

(ii) each $\mathbf{A} \prec \mathbf{R}'$ with $\kappa_{M'}(J, E(\mathbf{A})) < n_{d(\mathbf{A})}$ satisfies $\mathbf{A} \preceq \mathbf{Q}$.

Note that condition (ii) only applies for proper subprickles of $\mathbf{R}'$. Also, we do not require $\mathbf{R}'$ to be a g-prickle; we will show later that this follows from (i) and (ii). The idea behind conditions (i) and (ii) is that all prickles with connectivity 'too small' to J are subprickles of $\mathbf{Q}$ by (ii), and are thus 'safe' by (i). Our first claim shows that if $(M', \mathbf{R}') \in \mathcal{M}$, then proper subprickles of $\mathbf{R}'$ with 'small' connectivity to J in M' which are not subprickles of $\mathbf{Q}$ have a nested structure.

4.3.1.1. *Let $(M', \mathbf{R}') \in \mathcal{M}$. If $\mathbf{A}_1 \prec \mathbf{R}'$ and $\mathbf{A}_2 \prec \mathbf{R}'$ such that $\kappa_{M'}(J, E(\mathbf{A}_i)) \leq 2(d(\mathbf{A}_i) + 1)n_{d(\mathbf{A}_i)}$ for each $i \in \{1, 2\}$, then either $\mathbf{A}_1 \preceq \mathbf{A}_2$ or $\mathbf{A}_2 \preceq \mathbf{A}_1$, or $(\mathbf{A}_1 \cup \mathbf{A}_2) \preceq \mathbf{Q}$.*

Proof. If the claim is false, then $\mathbf{A}_1 \cup \mathbf{A}_2$ is a prickle such that $d(\mathbf{A}_1 \cup \mathbf{A}_2) > \max(d(\mathbf{A}_1), d(\mathbf{A}_2))$ and $(\mathbf{A}_1 \cup \mathbf{A}_2) \npreceq \mathbf{Q}$. Assume that $d(\mathbf{A}_1) \geq d(\mathbf{A}_2)$, without loss of generality. Then

$$\kappa_{M'}(J, E(\mathbf{A}_1 \cup \mathbf{A}_2)) \leq 2(d(\mathbf{A}_1)+1)n_{d(\mathbf{A}_1)} + 2(d(\mathbf{A}_2)+1)n_{d(\mathbf{A}_2)} \leq 4(d(\mathbf{A}_1)+1)n_{d(\mathbf{A}_1)} < n_{d(\mathbf{A}_1 \cup \mathbf{A}_2)},$$

so $(\mathbf{A}_1 \cup \mathbf{A}_2) \preceq \mathbf{Q}$, a contradiction. The last inequality holds since $4(i + 1)n_i < n_{i+1}$ for all $i \geq 1$. □

Our second claim shows that if $(M', \mathbf{R}') \in \mathcal{M}$, then each maximal proper subprickle of $\mathbf{R}'$ with 'small' connectivity to J in M' contains each other subprickle with 'small' connectivity to J in M'.

4.3.1.2. *Let $(M', \mathbf{R}') \in \mathcal{M}$, and let $\mathbf{Y}' \prec \mathbf{R}'$ with $d(\mathbf{Y}')$ maximal so that $\kappa_{M'}(J, E(\mathbf{Y}')) \leq 2(d(\mathbf{Y}') + 1)n_{d(\mathbf{Y}')}$. Then each $\mathbf{A} \prec \mathbf{R}'$ with $\kappa_{M'}(J, E(\mathbf{A})) \leq 2(d(\mathbf{A}) + 1)n_{d(\mathbf{A})}$ satisfies $\mathbf{A} \preceq \mathbf{Y}'$.*

Proof. Assume for a contradiction that there is a prickle $\mathbf{A} \prec \mathbf{R}'$ so that $\kappa_{M'}(J, E(\mathbf{A})) \leq 2(d(\mathbf{A}) + 1)n_{d(\mathbf{A})}$ and $\mathbf{A} \npreceq \mathbf{Y}'$. Then in particular $\mathbf{A} \neq \mathbf{Y}'$, so by the maximality of $d(\mathbf{Y}')$ we have $\mathbf{Y}' \npreceq \mathbf{A}$. By 4.3.1.1 with $\mathbf{A}_1 = \mathbf{A}$, and $\mathbf{A}_2 = \mathbf{Y}'$ we have $\mathbf{A} \cup \mathbf{Y}' \preceq \mathbf{Q}$. If $\mathbf{Y}' \neq \mathbf{Q}$ then $d(\mathbf{Y}') < d(\mathbf{Q})$, which contradicts the maximality of $d(\mathbf{Y}')$ since $\mathbf{Q}$ is a valid choice for $\mathbf{Y}'$ since $\kappa_{M'}(J, E(\mathbf{Q})) < n_{d(\mathbf{Q})} \leq 2(d(\mathbf{Q}) + 1)n_{d(\mathbf{Q})}$. Thus, $\mathbf{Y}' = \mathbf{Q}$, and so $\mathbf{A} \preceq \mathbf{Y}'$ since $\mathbf{A} \cup \mathbf{Y}' \preceq \mathbf{Q}$, which contradicts that $\mathbf{A} \npreceq \mathbf{Y}'$. □

We now take a minimal minor of M. Since $\mathbf{R} \neq \mathbf{Q}$ we have $\kappa_M(J, E(\mathbf{R})) \geq n_{d(\mathbf{R})}$, so there is some porcupine $P_k \in \mathcal{P}$ such that $\kappa_M(J, E(P_k)) \geq \frac{1}{k} n_{d(\mathbf{R})}$. Let M_1 be a minimal minor of M for which there is a retract $\mathbf{R}_1 = (R_1, \mathcal{P}_1)$ of $\mathbf{R}$ so that $(M_1, \mathbf{R}_1) \in \mathcal{M}$, and

(*) each $\mathbf{Y} \prec \mathbf{R}_1$ with $d(\mathbf{Y})$ maximal such that $\kappa_{M_1}(J, E(\mathbf{Y})) \leq 2(d(\mathbf{Y}) + 1)n_{d(\mathbf{Y})}$ satisfies $\kappa_{M_1}(J, E(\mathbf{Y}) \cup E(S_k)) \geq \frac{1}{k} n_{d(\mathbf{R}_1)}$, where $S_k \in \mathcal{P}_1$ has the same tip as $P_k \in \mathcal{P}$.

Note that $(M_1, \mathbf{R}_1)$ exists because $(M, \mathbf{R})$ is a valid choice. Also, note that (*) is a relaxation of the connectivity requirement that $\kappa_{M_1}(J, E(\mathbf{R}_1)) \geq n_{d(\mathbf{R}_1)}$. This allows us to find a proper subprickle of $\mathbf{R}_1$ for which we can exploit the minimality of $d(\mathbf{R}_1)$.

Let $\mathbf{Y}$ be a proper subprickle of $\mathbf{R}_1$ with $d(\mathbf{Y})$ maximal such that $\kappa_{M_1}(J, E(\mathbf{Y})) \leq 2(d(\mathbf{Y})+1)n_{d(\mathbf{Y})}$. Note that $\mathbf{Y}$ exists because the trivial subprickle of $\mathbf{R}_1$ is a choice for $\mathbf{Y}$ since $k < 2n_0$. One can show that $\mathbf{Y}$ is unique, although we will not use this fact.

The main idea of this proof is that there are three types of prickles $\mathbf{A} \preceq \mathbf{R}_1$, depending on the connectivity to J. If $\kappa_{M_1}(J, E(\mathbf{A}))$ is 'too small' to be useful, then $\mathbf{A} \preceq \mathbf{Q}$. If $\kappa_{M_1}(J, E(\mathbf{A}))$ is 'in danger' of becoming too small, then $\mathbf{A} \preceq \mathbf{Y}$. Finally, if $\mathbf{A} \not\preceq \mathbf{Y}$ then $\mathbf{A}$ is 'safe'. The following claim makes this idea more precise. When applied with $\mathbf{A} = \mathbf{R}_1$, it shows that for each $e \in E(M_1)$ which does not 'interact' with $\mathbf{Y}$, the matroid M_1/e has a retract $\mathbf{R}_1'$ so that $(M_1/e, \mathbf{R}_1') \in \mathcal{M}$ and $\mathbf{Y} \preceq \mathbf{R}_1'$. We only apply this claim with $\mathbf{A} = \mathbf{R}_1$, but we state it more generally so that we can prove it using induction.

4.3.1.3. *Let $e \in E(M_1) - (\mathrm{cl}_{M_1}(J) \cup \mathrm{cl}_{M_1}(E(\mathbf{Y})))$ such that each $\mathbf{Y}' \preceq \mathbf{Y}$ with $\kappa_{M_1/e}(J, E(\mathbf{Y}')) < n_{d(\mathbf{Y}')}$ satisfies $\mathbf{Y}' \preceq \mathbf{Q}$. Then for each prickle $\mathbf{A} \preceq \mathbf{R}_1$ with $\mathbf{Y} \preceq \mathbf{A}$, the matroid M_1/e has a retract $\mathbf{A}'$ of $\mathbf{A}$ so that $\mathbf{Y} \preceq \mathbf{A}'$, and each $\mathbf{Z} \preceq \mathbf{A}'$ with $\kappa_{M_1/e}(J, E(\mathbf{Z})) < n_{d(\mathbf{Z})}$ and $\mathbf{Z} \neq \mathbf{R}_1$ satisfies $\mathbf{Z} \preceq \mathbf{Q}$.*

Proof. Assume that the claim is false for some prickle $\mathbf{A}$ with $d(\mathbf{A})$ minimum and $e \in E(M_1)$. Then $d(\mathbf{A}) > d(\mathbf{Y})$ or else $\mathbf{A} = \mathbf{Y}$ and the claim holds with $\mathbf{A}' = \mathbf{Y}$, since $e \notin \mathrm{cl}_{M_1}(E(\mathbf{Y}))$ and each $\mathbf{Y}' \preceq \mathbf{Y}$ with $\kappa_{M_1/e}(J, E(\mathbf{Y}')) < n_{d(\mathbf{Y}')}$ satisfies $\mathbf{Y}' \preceq \mathbf{Q}$. By Lemma 4.2.2 applied to $\mathbf{A}$ and $\mathbf{Y}$, there is a collection $\mathcal{A}$ of subprickles of $\mathbf{A}$ such that each $\mathbf{A}_0 \in \mathcal{A}$ satisfies $d(\mathbf{A}_0) = d(\mathbf{A}) - 1$ and $\mathbf{Y} \preceq \mathbf{A}_0$, while $|\mathcal{A}| \leq d(\mathbf{A}) - d(\mathbf{Y})$ and $E(\mathbf{A}) - E(\mathbf{Y}) \subseteq \cup_{\mathbf{A}_0 \in \mathcal{A}}(E(\mathbf{A}) - E(\mathbf{A}_0))$.

Let $\mathbf{A}_0 \in \mathcal{A}$. Let $j = \min(d(\mathbf{A}), d(\mathbf{R}_1)-1)$; we will show that $\kappa_{M_1}(J, E(\mathbf{A}) - E(\mathbf{A}_0)) \leq 2n_j$. Since $d(\mathbf{A}_0) < d(\mathbf{A})$, by the minimality of $d(\mathbf{A})$ the matroid M_1/e has a retract $\mathbf{A}_0'$ of $\mathbf{A}_0$ so that $\mathbf{Y} \preceq \mathbf{A}_0'$, and each $\mathbf{Z} \preceq \mathbf{A}_0'$ with $\kappa_{M_1/e}(J, E(\mathbf{Z})) < n_{d(\mathbf{Z})}$ and $\mathbf{Z} \neq \mathbf{R}_1$ satisfies $\mathbf{Z} \preceq \mathbf{Q}$.

By Lemma 4.2.4 with $C = \{e\}$ there is a non-empty set $C' \subseteq E(M) - \{e\}$ and a collection $\mathcal{K}$ of prickles of M_1/e such that $|\mathcal{K}| + |C'| \leq 3$, each $\mathbf{K} \in \mathcal{K}$ is a retract of $\mathbf{A}$ with $\mathbf{A}_0' \preceq \mathbf{K}$, and

$$E(\mathbf{A}) - E(\mathbf{A}_0) \subseteq \mathrm{cl}_{M_1}\big(\big(\cup_{\mathbf{K} \in \mathcal{K}} (E(\mathbf{K}) - E(\mathbf{A}_0'))\big) \cup C' \cup \{e\}\big). \quad (1)$$

Note that for each $\mathbf{K} \in \mathcal{K}$ we have $\mathbf{Y} \preceq \mathbf{A}'_0 \preceq \mathbf{K}$ and $d(\mathbf{A}'_0) = d(\mathbf{A}) - 1 = d(\mathbf{K}) - 1$, by the definition of a retract.

Since the claim is false for $\mathbf{A}$, for each $\mathbf{K} \in \mathcal{K}$ there is some $\mathbf{Z} \preceq \mathbf{K}$ with $\mathbf{Z} \not\preceq \mathbf{Q}$ and $\mathbf{Z} \neq \mathbf{R}_1$ such that $\kappa_{M_1/e}(J, E(\mathbf{Z})) < n_{d(\mathbf{Z})} \leq n_j$. Since $d(\mathbf{A}'_0) = d(\mathbf{K}) - 1$ and $\mathbf{Z} \not\preceq \mathbf{A}'_0$, by Lemma 4.2.1 (i) we have $E(\mathbf{K}) - E(\mathbf{A}'_0) \subseteq E(\mathbf{Z})$, so $\kappa_{M_1/e}(J, E(\mathbf{K}) - E(\mathbf{A}'_0)) < n_j$. Thus, by (1) and the fact that $|\mathcal{K}| + |C'| \leq 3$, we have

$$\begin{aligned}\kappa_{M_1/e}(J, E(\mathbf{A}) - E(\mathbf{A}_0)) &\leq \kappa_{M_1/e}(J, \cup_{\mathbf{K}\in\mathcal{K}}(E(\mathbf{K}) - E(\mathbf{A}'_0))) + |C'| \\ &\leq |\mathcal{K}|(n_j - 1) + |C'| \\ &\leq (|\mathcal{K}| + |C'| - 1)(n_j - 1) + 1 \\ &\leq 2n_j - 1.\end{aligned}$$

Thus, $\kappa_{M_1}(J, E(\mathbf{A}) - E(\mathbf{A}_0)) \leq 2n_j$, as claimed.

We now use that $|\mathcal{A}| \leq d(\mathbf{A}) - d(\mathbf{Y})$, and $E(\mathbf{A}) - E(\mathbf{Y}) \subseteq \cup_{\mathbf{A}_0\in\mathcal{A}}(E(\mathbf{A}) - E(\mathbf{A}_0))$. We have

$$\begin{aligned}\kappa_{M_1}(J, E(\mathbf{A})) &\leq \kappa_{M_1}(J, E(\mathbf{Y})) + \kappa_{M_1}(J, E(\mathbf{A}) - E(\mathbf{Y})) \\ &\leq 2(d(\mathbf{Y}) + 1)n_{d(\mathbf{Y})} + \sum_{\mathbf{A}_0\in\mathcal{A}} \kappa_{M_1}(J, E(\mathbf{A}) - E(\mathbf{A}_0)) \\ &\leq 2(d(\mathbf{Y}) + 1)n_{d(\mathbf{Y})} + |\mathcal{A}|2n_j \\ &\leq 2(d(\mathbf{Y}) + 1)n_{d(\mathbf{Y})} + 2(d(\mathbf{A}) - d(\mathbf{Y}))n_j \\ &\leq 2(d(\mathbf{A}) + 1)n_j,\end{aligned}$$

since $d(\mathbf{Y}) \leq d(\mathbf{A}) - 1 \leq j$. If $\mathbf{A} \neq \mathbf{R}_1$, then $j = d(\mathbf{A})$ and $\kappa_{M_1}(J, E(\mathbf{A})) \leq 2(d(\mathbf{A}) + 1)n_{d(\mathbf{A})}$, and so $\mathbf{A} = \mathbf{Y}$ by 4.3.1.1 and the definition of $\mathbf{Y}$, so $\mathbf{A}$ is not a counterexample. If $\mathbf{A} = \mathbf{R}_1$, then $j = d(\mathbf{R}_1) - 1$ and

$$\kappa_{M_1}(J, E(\mathbf{R}_1)) \leq 2(d(\mathbf{R}_1) + 1)n_{d(\mathbf{R}_1)-1} < \frac{1}{k} n_{d(\mathbf{R}_1)},$$

where the last inequality holds since $2(i+2)n_i < \frac{1}{k} n_{i+1}$ for all $i \geq 0$. This contradicts (iii) since $E(\mathbf{Y}) \cup E(S_k) \subseteq E(\mathbf{R}_1)$. □

The following claim uses Lemma 4.1.2 to show that we can contract any suitable element in $E(M_1) - \mathrm{cl}_{M_1}(E(S_k))$, and contradict the minimality of M_1.

4.3.1.4. $E(M_1) = \mathrm{cl}_{M_1}(E(\mathbf{Y}) \cup E(S_k)) \cup \mathrm{cl}_{M_1}(J)$.

Proof. Assume for a contradiction that $E(M_1) - (\mathrm{cl}_{M_1}(E(\mathbf{Y}) \cup E(S_k)) \cup \mathrm{cl}_{M_1}(J)) \neq \varnothing$, and let

$$\mathcal{N} = \{E(\mathbf{Q})\} \cup \{E(\mathbf{A})\colon \mathbf{A} \prec \mathbf{R}_1, \mathbf{A} \not\preceq \mathbf{Q} \text{ and } \kappa_{M_1}(J, E(\mathbf{A})) = n_{d(\mathbf{A})}\} \cup \{E(\mathbf{Y}) \cup E(S_k)\}.$$

Other than $E(\mathbf{Q})$ and $E(\mathbf{Y}) \cup E(S_k)$, the set $\mathcal{N}$ contains the ground set of each prickle $\mathbf{A}$ with the minimum connectivity to J necessary to satisfy (ii). By applying 4.3.1.1 to each pair of sets in $\mathcal{N}$ we see that $\mathcal{N}$ is a nested collection of sets. In particular, if $\mathbf{A}_i \not\preceq \mathbf{Q}$ and $E(\mathbf{A}_i) \in \mathcal{N}$ for each $i \in \{1, 2\}$, then either $\mathbf{A}_1 \preceq \mathbf{A}_2$ or $\mathbf{A}_2 \preceq \mathbf{A}_1$. Then by Lemma 4.1.2 with $X = J$ there is some $e \in E(M_1) - (\mathrm{cl}_{M_1}(E(\mathbf{Y}) \cup E(S_k)) \cup \mathrm{cl}_{M_1}(J))$ so that $\kappa_{M_1/e}(J, Z) = \kappa_{M_1}(J, Z)$ for each $Z \in \mathcal{N}$. This means that we may apply 4.3.1.3 with the element e.

By 4.3.1.3 with $\mathbf{A} = \mathbf{R}_1$, the matroid M_1/e has a retract $\mathbf{R}'$ so that $(M_1/e, \mathbf{R}') \in \mathcal{M}$ and $\mathbf{Y} \preceq \mathbf{R}'$. Note that $(M_1/e, \mathbf{R}')$ satisfies (i) since $E(\mathbf{Q}) \in \mathcal{N}$. Since $\mathbf{R}'$ is a retract of $\mathbf{R}_1$ and $e \notin \mathrm{cl}_{M_1}(E(S_k))$, the porcupine of $\mathbf{R}'$ with the same tip as S_k is S_k itself. We will show that $(M_1/e, \mathbf{R}')$ satisfies (*). Let $\mathbf{Y}' \prec \mathbf{R}'$ be a prickle with $d(\mathbf{Y}')$ maximal such that $\kappa_{M_1/e}(J, E(\mathbf{Y}')) \leq 2(d(\mathbf{Y}') + 1)n_{d(\mathbf{Y}')}$. By 4.3.1.2 with $M' = M_1/e$ and $\mathbf{A} = \mathbf{Y}$ we have $\mathbf{Y} \preceq \mathbf{Y}'$. Then we have

$$\kappa_{M_1/e}(J, E(\mathbf{Y}') \cup E(S_k)) \geq \kappa_{M_1/e}(J, E(\mathbf{Y}) \cup E(S_k)) = \kappa_{M_1}(J, E(\mathbf{Y}) \cup E(S_k)) \geq \frac{1}{k} n_{d(\mathbf{R}_1)},$$

where the equality holds since $E(\mathbf{Y}) \cup E(S_k) \in \mathcal{N}$. Thus, $(M_1/e, \mathbf{R}')$ satisfies (*), so M_1/e contradicts the minimality of M_1. □

Define $D_Q = \{e \in E(M_1)\colon \kappa_{M_1/e}(J, E(\mathbf{Q})) < \kappa_{M_1}(J, E(\mathbf{Q}))\}$ and

$$D_Y = \{e \in E(M_1)\colon \kappa_{M_1/e}(J, E(\mathbf{A})) < n_{d(\mathbf{A})} \text{ for some } \mathbf{A} \preceq \mathbf{Y} \text{ with } \mathbf{A} \not\preceq \mathbf{Q}\},$$

and $D = \mathrm{cl}_{M_1}(E(\mathbf{Y})) \cup D_Q \cup D_Y$. This is essentially the set of elements e so that there is no retract $\mathbf{R}'$ of $\mathbf{R}$ in M_1/e so that $(M_1/e, \mathbf{R}') \in \mathcal{M}$ and $\mathbf{Y} \preceq \mathbf{R}'$. Then $\kappa_{M_1}(J, D) = \kappa_{M_1}(J, E(\mathbf{Y}))$, by Lemma 4.1.5 with $Y = E(\mathbf{Y})$. Let t denote the tip of S_k. The next claim partitions $E(S_k) - \{t\}$ into a subset of D and a subset of $\mathrm{cl}_{M_1}(J)$.

4.3.1.5. *$E(S_k) - \{t\} \subseteq D \cup \mathrm{cl}_{M_1}(J)$.*

Proof. If there is some $e \in E(S_k) - (D \cup \mathrm{cl}_{M_1}(J))$ with $e \neq t$, then by 4.3.1.3 and the definition of D, the matroid M_1/e has a retract $\mathbf{R}'$ so that $(M_1/e, \mathbf{R}') \in \mathcal{M}$ and $\mathbf{Y} \preceq \mathbf{R}'$. Note that $\kappa_{M_1/e}(J, E(\mathbf{Q})) = \kappa_{M_1}(J, E(\mathbf{Q}))$ by the definition of D_Q. Since $\mathbf{R}'$ is a retract of $\mathbf{R}_1$ and $e \in E(S_k) - \{t\}$, the porcupine S_k' of $\mathbf{R}'$ with tip t is a simplification of S_k/e.

We will show that $(M_1/e, \mathbf{R}')$ satisfies (*). Let $\mathbf{Y}' \prec \mathbf{R}'$ be a prickle with $d(\mathbf{Y}')$ maximal such that $\kappa_{M_1/e}(J, E(\mathbf{Y}')) \leq 2(d(\mathbf{Y}') + 1)n_{d(\mathbf{Y}')}$. By 4.3.1.2 with $M' = M_1/e$ and $\mathbf{A} = \mathbf{Y}$

we have $\mathbf{Y} \preceq \mathbf{Y}'$. Since $E(M_1) = \mathrm{cl}_{M_1}(E(\mathbf{Y}) \cup E(S_k)) \cup \mathrm{cl}_{M_1}(J)$ by 4.3.1.4, each set Z for which $J \subseteq Z \subseteq E(M_1) - (E(\mathbf{Y}) \cup E(S_k))$ and $\lambda_{M_1}(Z) = \kappa_{M_1}(J, E(\mathbf{Y}) \cup E(S_k))$ satisfies $Z \subseteq \mathrm{cl}_{M_1}(J)$. Then since $e \notin \mathrm{cl}_{M_1}(J)$, this implies that $\lambda_{M_1/e}(Z - \{e\}) = \lambda_{M_1}(Z)$ for each $Z \subseteq E(M_1)$ with $\lambda_{M_1}(Z) = \kappa_{M_1}(J, E(\mathbf{Y}) \cup E(S_k))$, and so

$$\kappa_{M_1/e}\big(J, (E(\mathbf{Y}) \cup E(S_k)) - \{e\}\big) = \kappa_{M_1}\big(J, E(\mathbf{Y}) \cup E(S_k)\big).$$

Combining this with the facts that $\mathbf{Y} \preceq \mathbf{Y}'$ and S_k' is a simplification of S_k/e gives

$$\kappa_{M_1/e}(J, E(\mathbf{Y}') \cup E(S_k')) \geq \kappa_{M_1/e}(J, E(\mathbf{Y}) \cup E(S_k')) = \kappa_{M_1}(J, E(\mathbf{Y}) \cup E(S_k)) \geq \frac{1}{k} n_{d(\mathbf{R}_1)},$$

so $(M_1/e, \mathbf{R}')$ satisfies (*). Thus, M_1/e contradicts the minimality of M_1. $\square$

We now use the minimality of $d(\mathbf{R}) = d(\mathbf{R}_1)$. Let Y_k denote the porcupine of $\mathbf{Y}$ which has the same tip as S_k, and let $\mathbf{R}_2$ denote the subprickle of $\mathbf{R}_1$ obtained from $(R_1, \mathcal{P}_1)$ by replacing $S_k \in \mathcal{P}_1$ with Y_k. If $d(Y_k) = d(S_k)$, then $Y_k = S_k$, and so $E(S_k) \subseteq E(\mathbf{Y})$. But then the facts that $\mathbf{Y} \neq \mathbf{R}_1$ and $2(i+1)n_i < \frac{1}{k} n_{i+1}$ for all $i \geq 0$ together imply that $\kappa_{M_1}(J, E(\mathbf{Y}) \cup E(S_k)) = \kappa_{M_1}(J, E(\mathbf{Y})) < \frac{1}{k} n_{d(\mathbf{R})}$, which contradicts (*). Thus, $d(Y_k) < d(S_k)$, so $d(\mathbf{R}_2) < d(\mathbf{R}_1)$. Also, $\mathbf{R}_2$ satisfies the conditions of the lemma statement using the same prickle $\mathbf{Q} \preceq \mathbf{R}_2$ and the matroid M_1.

By the minimality of $d(\mathbf{R}_1)$, the matroid M_1 with the prickle $\mathbf{R}_2$ is not a counterexample. Thus, there is some $X \subseteq E(M_1)$ such that M_1/X has $M_1|J$ as a spanning restriction and $\mathbf{Q}$ as a prickle, and has a g-retract $\mathbf{R}_3$ of $\mathbf{R}_2$. Let Y_k' denote the porcupine of $\mathbf{R}_3$ which is a retract of Y_k.

Let $(S_k/t)|K$ be a simplification of S_k/t. To finish the proof, it suffices to show that $(M_1/t/X)|K$ has a restriction with corank $d(S_k)$, girth at least g, and no coloops. This is because such a restriction implies that there is a g-porcupine S_k' of M_1/X which is a retract of S_k, and we can take $M' = M_1/X$ and obtain the desired prickle $\mathbf{R}'$ from the lemma statement by replacing Y_k' with S_k' in the prickle $\mathbf{R}_3$. Let $K_1 = K \cap D$ and $K_2 = K - D$, and note that $K_2 \subseteq \mathrm{cl}_{M_1}(J)$ by 4.3.1.5. Also note that there is some $K_Y \subseteq K_1$ so that $(S_k/t)|K_Y$ is a simplification of Y_k/t, since $E(\mathbf{Y}) \subseteq D$.

4.3.1.6. *There is some $K_3 \subseteq K_2$ such that $(M_1/t/(X \cup K_1))|K_3$ has corank $d(S_k) - d(Y_k)$ and girth at least g.*

Proof. Let $m = 2(d(\mathbf{Y}) + 1)n_{d(\mathbf{Y})}$, so $\kappa_{M_1}(J, D) = \kappa_{M_1}(J, E(\mathbf{Y})) \leq m$. We first show that $(M_1/t/K_1)|K_2$ has corank $d(S_k) - d(Y_k)$. Since $E(Y_k) \subseteq D$, the matroid $(M_1/t)|K_1$ has corank at least $d(Y_k)$. If $(M_1/t)|K_1$ has corank greater than $d(Y_k)$, then there is some $K_Y^+ \subseteq K_1$ so that $K_Y \subseteq K_Y^+$ and $(M_1/t)|K_Y^+$ has corank $d(Y_k) + 1$. Define Y_k^+ to be the subporcupine of

S_k so that each element of K_Y^+ is on a line of Y_k^+ through t and each line through t contains an element of K_Y^+, so $d(Y_k^+) = d(Y_k) + 1$. Define $\mathbf{A} \preceq \mathbf{R}_1$ to be the prickle obtained from $\mathbf{Y}$ by replacing Y_k with Y_k^+, so $d(\mathbf{A}) = d(\mathbf{Y}) + 1$. Since $E(\mathbf{Y}) \subseteq D$ and $K_Y^+ \subseteq K_1 \subseteq D$ we have $\kappa_{M_1}(J, E(\mathbf{A})) \leq \kappa_{M_1}(J, D) \leq m < \frac{1}{k} n_{d(\mathbf{A})}$, using that $2(i+1)n_i < \frac{1}{k} n_{i+1}$ for all $i \geq 0$. If $\mathbf{A} = \mathbf{R}_1$ this contradicts (*), so $\mathbf{A} \neq \mathbf{R}_1$. But then $\mathbf{A} \preceq \mathbf{Y}$ by 4.3.1.2 with $(M', \mathbf{R}') = (M_1, \mathbf{R}_1)$ and $\mathbf{Y}' = \mathbf{Y}$, which contradicts that $d(\mathbf{A}) = d(\mathbf{Y}) + 1$. Thus, $(M_1/t)|K_1$ has corank $d(Y_k)$. Then since $(M_1/t)|K$ has corank $d(S_k)$ and K is the disjoint union of K_1 and K_2, the matroid $(M_1/t/K_1)|K_2$ has corank $d(S_k) - d(Y_k)$.

We now show that $(M_1/t/K_1)|K_2$ has girth at least $g2^{m+1}$. If $(M_1/t/K_1)|K_2$ has a circuit C of size less than $g2^{m+1}$, then $(M_1/t)|(K_1 \cup C)$ has a circuit C' which is not contained in $(M_1/t)|K_1$, and is thus not contained in $(M_1/t)|K_Y$. This implies that $(M_1/t)|(K_Y \cup C')$ has no coloops, and has corank greater than $d(Y_k)$, since $(M_1/t)|K_Y$ has corank $d(Y_k)$. Define Y_k^+ to be the subporcupine of S_k so that each element of $K_Y \cup C'$ is on a line of Y_k^+ through t and each line through t contains an element of $K_Y \cup C'$, so $d(Y_k^+) > d(Y_k)$. Define $\mathbf{A} \preceq \mathbf{R}_1$ to be the prickle obtained from $\mathbf{Y}$ by replacing Y_k with Y_k^+, so $d(\mathbf{A}) > d(\mathbf{Y})$. Then $E(\mathbf{A}) - D \subseteq \mathrm{cl}_{M_1}((C' - K_1) \cup \{t\}) \subseteq \mathrm{cl}_{M_1}(C \cup \{t\})$, so we have

$$\kappa_{M_1}(J, E(\mathbf{A})) \leq \kappa_{M_1}(J, D) + \kappa_{M_1}(J, E(\mathbf{A}) - D) \leq m + \kappa_{M_1}(J, C \cup \{t\}) < m + g2^{m+1} \leq \frac{1}{k} n_{d(\mathbf{A})},$$

since $|C| < g2^m$ and $2(i+1)n_i + g2^{2(i+1)n_i+1} \leq \frac{1}{k} n_{i+1}$ for all $i \geq 0$. If $\mathbf{A} = \mathbf{R}_1$ this contradicts (*), so $\mathbf{A} \neq \mathbf{R}_1$. But then $\mathbf{A} \preceq \mathbf{Y}$ by Lemma 4.3.1.2 with $(M', \mathbf{R}') = (M_1, \mathbf{R}_1)$ and $\mathbf{Y}' = \mathbf{Y}$, which contradicts that $d(\mathbf{A}) = d(\mathbf{Y}) + 1$. Thus, $(M_1/t/K_1)|K_2$ has girth at least $g2^{m+1}$.

Lastly, using $K_2 \subseteq \mathrm{cl}_{M_1}(J)$ and $K_1 \subseteq D$,

$$\begin{aligned}
\sqcap_{M_1/t/K_1}(K_2, X - K_1) &\leq \sqcap_{M_1/t}(K_2, X \cup K_1) && (2)\\
&\leq \sqcap_{M_1/t}(J, (X \cup D) - \{t\}) && (3)\\
&= \sqcap_{M_1/t}(J, X) + \sqcap_{M_1/t/X}(J, D - (\{t\} \cup X)) && (4)\\
&\leq 1 + \kappa_{M_1/t/X}(J, D - (\{t\} \cup X)) && (5)\\
&\leq 1 + \kappa_{M_1}(J, D) \leq 1 + m. && (6)
\end{aligned}$$

where $\sqcap_{M_1/t}(J, X) \leq 1$ since J and X are skew in M_1. Since $(M_1/t/K_1)|K_2$ has corank $d(S_k) - d(Y_k)$ and girth at least $g2^{m+1}$ and $\sqcap_{M_1/t/K_1}(X - K_1, K_2) \leq m + 1$, the claim holds by Lemma 4.2.6 applied with $M = M_1/t/K_1$ and $C = X - K_1$ and $S = K_2$. □

Since $\mathrm{cl}_{M_1}(E(\mathbf{Y})) \subseteq D$, there is some $Z \subseteq K_1$ such that $(M_1/t/X)|Z$ is a simplification of Y_k'/t. Then $(M_1/t/X)|Z$ has corank $d(Y_k)$ and girth at least g, so by Lemma 4.2.5 applied to $M_1/t/X$, Z, $K_1 - X$, and K_3, there is a restriction of $(M_1/t/X)|K$ with corank $d(S_k)$ and girth at least g. Then $(M_1/X)|E(S_k)$ has a g-porcupine S_k' with tip t so that $d(S_k') = d(S_k)$. Thus, the prickle obtained from $\mathbf{R}_3$ by replacing Y_k' with S_k' is a retract of $\mathbf{R}_1$, and thus a retract of $\mathbf{R}$, and is a g-prickle. This contradicts that $\mathbf{R}$ is a counterexample. □

4.4 Tangles

To prove Theorem 4.0.1 when the Dowling geometry is not a restriction, we use objects called 'tangles' to maintain connectivity between the prickle and the Dowling geometry as we take a minor. In this section we introduce tangles, and define the tangle which we will use for the remainder of this chapter. Tangles were first defined for graphs by Robertson and Seymour as part of the Graph Minors Project to describe the structure of graphs with no K_t-minor [44]. They were generalized explicitly to matroids in [3] and [6]. Most of the material in this section can be found in [40], which in turn is based on [7].

Roughly speaking, a tangle on a matroid M is the collection of the 'less interesting' sides of the small separations of M. Let M be a matroid and let $\theta \geq 2$ be an integer. We say that a set $Z \subseteq E(M)$ is *$(\theta - 1)$-separating in M* if $\lambda_M(Z) < \theta - 1$. A collection $\mathcal{T}$ of subsets of $E(M)$ is a *tangle of order θ* if

(1) each set in $\mathcal{T}$ is $(\theta - 1)$-separating in M, and for each $(\theta - 1)$-separating set Z, either $Z \in \mathcal{T}$ or $E(M) - Z \in \mathcal{T}$;

(2) if $A, B, C \in \mathcal{T}$ then $A \cup B \cup C \neq E(M)$; and

(3) $E(M) - \{e\} \notin \mathcal{T}$ for each $e \in E(M)$.

The idea is that if $\lambda_M(Z) < \theta - 1$, then either Z or $E - Z$ does not contain much information about M, and that is the set which goes into the tangle. If $Z \in \mathcal{T}$ then we say that Z is *$\mathcal{T}$-small*. Intuitively, if $Z \in \mathcal{T}$ then $\mathrm{cl}_M(Z) \in \mathcal{T}$, and this is indeed true.

Lemma 4.4.1. *If $\mathcal{T}$ is a tangle of order at least three on a matroid M and $Z \in \mathcal{T}$, then $\mathrm{cl}_M(Z) \in \mathcal{T}$.*

Proof. If the lemma is false, then there is some $Z \in \mathcal{T}$ and some $e \in \mathrm{cl}_M(Z) - Z$ so that $Z \cup \{e\} \notin \mathcal{T}$. Since $\lambda_M(Z \cup \{e\}) \leq \lambda_M(Z)$, tangle axiom (1) implies that $E - (Z \cup \{e\}) \in \mathcal{T}$. Since the order of $\mathcal{T}$ is at least three we have $\{e\} \in \mathcal{T}$, since $\lambda_M(\{e\}) \leq 1$ and $E - \{e\} \notin \mathcal{T}$ by axiom (3). But then $Z \cup (E - (Z \cup \{e\})) \cup \{e\} = E(M)$, which violates axiom (2). $\square$

A key property of tangles is that they induce another matroid with ground set $E(M)$. Given a tangle $\mathcal{T}$ of order θ on a matroid M and $X \subseteq E(M)$, define $r_{\mathcal{T}}(X) = \theta - 1$ if there is no set $Z \in \mathcal{T}$ so that $X \subseteq Z$, and define $r_{\mathcal{T}}(X) = \min\{\lambda_M(Z) \colon X \subseteq Z \in \mathcal{T}\}$ otherwise.

Lemma 4.4.2. *If $\mathcal{T}$ is a tangle of order θ on a matroid M, then $r_{\mathcal{T}}$ is the rank function of a rank-$(\theta - 1)$ matroid on $E(M)$.*

This matroid is called a *tangle matroid* of M, and is denoted $M(\mathcal{T})$. We denote the closure function $\mathrm{cl}_{M(\mathcal{T})}$ by $\mathrm{cl}_{\mathcal{T}}$ for readability. Note that Lemma 4.4.1 implies that $r_{\mathcal{T}}(X) =$

$r_{\mathcal{T}}(\mathrm{cl}_M(X))$ for each $X \subseteq E(M)$, and so $\mathrm{cl}_M(X) \subseteq \mathrm{cl}_{\mathcal{T}}(X)$. This also shows that $r_{\mathcal{T}}(X) \leq r_M(X)$, or else there is some $X' \subseteq X$ and $e \in X - X'$ so that $e \in \mathrm{cl}_M(X') - \mathrm{cl}_{\mathcal{T}}(X')$.

The next lemma shows the connection between tangles and minors of M.

Lemma 4.4.3. *If N is a minor of a matroid M and $\mathcal{T}_N$ is a tangle of order θ on N, then $\{X \subseteq E(M)\colon \lambda_M(X) < \theta - 1, X \cap E(N) \in \mathcal{T}_N\}$ is a tangle of order θ on M.*

This is the tangle on M *induced* by $\mathcal{T}_N$. In this thesis the minor N will always be a Dowling geometry, which allows us to use a special tangle which behaves very nicely with Dowling geometries, and in fact with round matroids in general.

For each matroid M and each integer $3 \leq k \leq r(M)$, let $\mathcal{T}_k(M)$ denote the collection of $(k-1)$-separating sets of M which are neither spanning nor cospanning in M. When M is round, $\mathcal{T}_k(M)$ is nearly a tangle.

Lemma 4.4.4. *If M is a round matroid and $3 \leq k \leq r(M)$, then $\mathcal{T}_k(M)$ is the collection of subsets of $E(M)$ of rank at most $k-2$. Moreover, $\mathcal{T}_k(M)$ satisfies tangle axioms (1) and (3).*

Proof. Since M is round, each nonspanning $(k-1)$-separating set has rank at most $k-2$, so each set in $\mathcal{T}_k(M)$ has rank at most $k-2$. If $r_M(X) \leq k-2$ and X is cospanning, then some subset of X is the complement of a basis of M and has rank at most $k-2$. But then $E(M)$ is the union of two hyperplanes, which contradicts that M is round. Thus, each set of rank at most $k-2$ is in $\mathcal{T}_k(M)$.

Clearly each set in $\mathcal{T}_k(M)$ is $(k-1)$-separating in M, and $\mathcal{T}_k(M)$ satisfies tangle axiom (3). Since M is round, if $\lambda_M(X) < k-1$, then either $r_M(X) \leq k-2$ or $r_M(E(M)-X) \leq k-2$. Thus, $\mathcal{T}_k(M)$ satisfies tangle axiom (1). □

This shows that when M is round, $\mathcal{T}_k(M)$ is a tangle if and only if $E(M)$ is not the union of three subsets of rank at most $k-2$. In the case that M is a rank-n Dowling geometry, it is not hard to see that $E(M)$ is the union of three subsets of rank m if and only if $m \geq \lceil 2n/3 \rceil$.

Lemma 4.4.5. *Let $n \geq 3$ be an integer, let $M \cong \mathrm{DG}(n, \Gamma)$, and let $3 \leq k \leq r(M)$. Then $\mathcal{T}_k(M)$ is a tangle of order k in M if and only if $3 \leq k \leq \lceil 2n/3 \rceil + 1$.*

If M is a matroid with a minor G so that $\mathcal{T}_k(G)$ is a tangle, then we write $\mathcal{T}_k(M, G)$ for the tangle of order k in M induced by $\mathcal{T}_k(G)$. We will work with this tangle for the remainder of this chapter, most often in the case that G is a Dowling geometry.

4.5 Tangles and Dowling Geometries

In this section we prove some properties of the tangle $\mathcal{T}_k(M, G)$ when G is round, although in this thesis we only apply these lemmas in the case that G is a Dowling geometry. We then use these lemmas to prove Theorem 4.5.7, a key ingredient in the proof of Theorem 1.7.2.

We first prove two lemmas which provide lower bounds on $r_{\mathcal{T}_k(M,G)}(X)$ for any $X \subseteq E(M)$.

Lemma 4.5.1. *Let M be a matroid with a round minor G, and let k be an integer so that $\mathcal{T}_k(G)$ is a tangle. Then each $X \subseteq E(M)$ satisfies $r_{\mathcal{T}_k(M,G)}(X) \geq \min(r_G(X \cap E(G)), k-1)$.*

Proof. Let $\mathcal{T} = \mathcal{T}_k(M, G)$, and $m = r_{\mathcal{T}}(X)$. If $m < k-1$ then there is some $Z \in \mathcal{T}$ so that $X \subseteq Z$ and $\lambda_M(Z) = m$. Since G is a minor of M, $\lambda_G(Z \cap E(G)) \leq \lambda_M(Z) \leq m$. Since G is round, either $r_G(Z \cap E(G)) \leq m$ or $r_G((E(M) - Z) \cap E(G)) \leq m < k - 1$. In the latter case, $(E(M) - Z) \cap E(G) \in \mathcal{T}_k(G)$ by Lemma 4.4.4. But then $E(M) - Z \in \mathcal{T}$ by the definition of $\mathcal{T}$, which contradicts that $\mathcal{T}$ is a tangle and $Z \in \mathcal{T}$. Thus, $r_G(Z \cap E(G)) \leq m$, so $r_G(X \cap E(G)) \leq m$ since $X \subseteq Z$. □

The second lemma shows that tangles behave nicely with vertical connectivity; the proof is similar to the proof of Lemma 4.5.1.

Lemma 4.5.2. *Let M be a vertically s-connected matroid with a round minor G, and let k be an integer so that $\mathcal{T}_k(G)$ is a tangle. If $X \subseteq E(M)$ and $r_{\mathcal{T}_k(M,G)}(X) < \min(k-1, s-1)$, then $r_{\mathcal{T}_k(M,G)}(X) \geq r_M(X)$.*

Proof. Let $\mathcal{T} = \mathcal{T}_k(M, G)$, and $m = r_{\mathcal{T}}(X)$. Since $m < k-1$ there is some $Z \in \mathcal{T}$ so that $X \subseteq Z$ and $\lambda_M(Z) = m$. Since M is vertically s-connected and $m < s-1$, either $r_M(Z) < m$ or $r_M(E(M) - Z) < m$. If $r_M(E(M) - Z) < m$, then $r_G((E(M) - Z) \cap E(G)) \leq k - 2$ since $m < k - 1$. Then $(E(M) - Z) \cap E(G) \in \mathcal{T}_k(G)$ by Lemma 4.4.4, so $E(M) - Z \in \mathcal{T}$ by the definition of $\mathcal{T}$. This contradicts that $Z \in \mathcal{T}$ and $\mathcal{T}$ is a tangle. Thus, $r_M(X) \leq r_M(Z) < m$, so the lemma holds. □

The following lemma shows the relationship between tangles and connectivity between a pair of sets. We apply this lemma in the case that G is a restriction of M.

Lemma 4.5.3. *Let M be a matroid with a round minor G, and let k be an integer so that $\mathcal{T}_k(G)$ is a tangle. If $J \subseteq E(G)$ and $X \subseteq E(M) - J$, then $\kappa_M(J, X) \geq \min(r_{\mathcal{T}_k(M,G)}(X), r_G(J))$.*

Proof. Let $\mathcal{T} = \mathcal{T}_k(M, G)$, and $E = E(M)$. Assume for a contradiction that $\kappa_M(J, X) < \min(r_{\mathcal{T}}(X), r_G(J))$ for some $J \subseteq E(G)$ and $X \subseteq E - J$. Let $(Z, E - Z)$ be a partition of E

such that $X \subseteq Z$ and $J \subseteq E - Z$, and $\lambda_M(Z) < \min(r_{\mathcal{T}}(X), r_G(J))$. Either Z or $E - Z$ is in $\mathcal{T}$, by tangle axiom (1). If $Z \in \mathcal{T}$, then

$$r_{\mathcal{T}}(X) \leq \lambda_M(Z) < \min(r_{\mathcal{T}}(X), r_G(J)) \leq r_{\mathcal{T}}(X),$$

a contradiction. If $E - Z \in \mathcal{T}$, then $E(G) \cap (E - Z) \in \mathcal{T}_k(G)$ by the definition of $\mathcal{T}_k(M, G)$. Then

$$\begin{aligned} r_G(J) &\leq r_G((E - Z) \cap E(G)) && (1)\\ &= \lambda_G((E - Z) \cap E(G)) && (2)\\ &\leq \lambda_M(E - Z) && (3)\\ &= \lambda_M(Z) < r_G(J), && (4) \end{aligned}$$

a contradiction. Line (1) holds because $J \subseteq (E - Z) \cap E(G)$ and line (2) holds by Lemma 4.4.4 and the fact that $(E - Z) \cap E(G) \in \mathcal{T}_k(G)$. □

Using tangles while taking a minor is tricky, because each time we contract an element we have a new tangle, and we must ensure that the rank of each set of interest does not decrease with respect to this new tangle. To deal with this, we prove two lemmas which provide sufficient conditions for maintaining tangle connectivity of a set as we contract towards a minor.

In the first case we contract an element which preserves the minor. Note that there are two different tangles in the statement of the lemma.

Lemma 4.5.4. *Let M be a matroid with a round minor G, and let k be an integer so that $\mathcal{T}_k(G)$ is a tangle. Let $e \in E(M)$ so that G is a minor of M/e, and let $X \subseteq E(M)$. Then $r_{\mathcal{T}_k(M/e,G)}(X - \{e\}) \geq r_{\mathcal{T}_k(M,G)}(X) - 1$. Moreover, if $e \notin \mathrm{cl}_{\mathcal{T}_k(M,G)}(X)$, then $r_{\mathcal{T}_k(M/e,G)}(X) = r_{\mathcal{T}_k(M,G)}(X)$ and $(M/e)|X = M|X$.*

Proof. Let $\mathcal{T} = \mathcal{T}_k(M, G)$. If $r_{\mathcal{T}_k(M/e,G)}(X - \{e\}) < r_{\mathcal{T}}(X) - 1$, then there is some $Z \in \mathcal{T}_k(M/e, G)$ so that $X - \{e\} \subseteq Z$ and $\lambda_{M/e}(Z) < r_{\mathcal{T}}(X) - 1$. Then $\lambda_M(Z \cup \{e\}) < r(M(\mathcal{T}))$, so either $Z \cup \{e\} \in \mathcal{T}$ or $E(M) - (Z \cup \{e\}) \in \mathcal{T}$ by tangle axiom (1).

If $E(M) - (Z \cup \{e\}) \in \mathcal{T}$, then $E(G) \cap (E(M) - (Z \cup \{e\}))$ is not spanning in G, by the definition of $\mathcal{T}_k(G)$. This means that $E(G) \cap (Z \cup \{e\})$ spans G, since G is round. But then $E(G) \cap Z$ spans G since $e \notin E(G)$, which contradicts that $Z \in \mathcal{T}_k(M/e, G)$, by the definition of $\mathcal{T}_k(M/e, G)$. Thus, $Z \cup \{e\} \in \mathcal{T}$. But $\lambda_M(Z \cup \{e\}) < r_{\mathcal{T}}(X)$ and $X \subseteq Z$, a contradiction. If $e \notin \mathrm{cl}_{\mathcal{T}}(X)$, then Lemma 4.4.1 implies that $e \notin \mathrm{cl}_M(X)$, so $(M/e)|X = M|X$. □

In the second case we contract a subset of the minor itself. Again, the lemma statement involves two different tangles. The proof is very similar to the proof of Lemma 4.5.4.

Lemma 4.5.5. *Let M be a matroid with a round minor G and $C \subseteq E(G)$, and let k be an integer so that $\mathcal{T}_k(G/C)$ is a tangle. Then each $X \subseteq E(M) - C$ satisfies $r_{\mathcal{T}_k(M/C,G/C)}(X - C) \geq r_{\mathcal{T}_k(M,G)}(X) - |C|$. Moreover, if $r_{\mathcal{T}_k(M,G)}(X \cup C) = r_{\mathcal{T}_k(M,G)}(X) + |C|$, then $r_{\mathcal{T}_k(M/C,G/C)}(X) = r_{\mathcal{T}_k(M,G)}(X)$ and $(M/C)|X = M|X$.*

Proof. Let $\mathcal{T} = \mathcal{T}_k(M, G)$. If $r_{\mathcal{T}_k(M/C,G/C)}(X) < r_{\mathcal{T}_k(M,G)}(X) - |C|$, then there is some $Z \in \mathcal{T}_k(M/C, G/C)$ such that $X - C \subseteq Z$ and $\lambda_{M/C}(Z) < r_{\mathcal{T}}(X) - |C|$. Then $\lambda_M(Z \cup C) < r_{\mathcal{T}}(X)$, so either $(Z \cup C) \in \mathcal{T}$ or $E(M) - (Z \cup C) \in \mathcal{T}$ by tangle axiom (1).

If $E(M) - (Z \cup C) \in \mathcal{T}$, then $E(G) \cap (E(M) - (Z \cup C))$ is not spanning in G, by the definition of $\mathcal{T}_k(G)$. This means that $E(G) \cap (Z \cup C)$ spans G, since G is round. But then $E(G) \cap Z$ spans G/C, which contradicts that $Z \in \mathcal{T}_k(M/C, G/C)$, by the definition of $\mathcal{T}_k(M/C, G/C)$. Thus, $(Z \cup C) \in \mathcal{T}$. But $X \subseteq Z \cup C$ and $\lambda_M(Z \cup C) < r_{\mathcal{T}}(X)$, a contradiction.

If $r_{\mathcal{T}}(X \cup C) = r_{\mathcal{T}}(X) + |C|$, then since $\mathrm{cl}_M(X \cup C') \subseteq \mathrm{cl}_{\mathcal{T}}(X \cup C')$ for all $C' \subseteq C$ by Lemma 4.4.1, we have

$$r_M(X \cup C) - r_M(X) \geq r_{\mathcal{T}}(X \cup C) - r_{\mathcal{T}}(X) = |C|,$$

which implies that $(M/C)|X = M|X$. □

Tangles can also tell us how close a minor is to being a restriction.

Lemma 4.5.6. *Let $\ell \geq 2$ and $m \geq 0$ be integers, and let $M \in \mathcal{U}(\ell)$ be a matroid so that M/C has a simple round restriction G. Let k be an integer so that $\mathcal{T}_k(G)$ is a tangle. If $r_{\mathcal{T}_k(M,G)}(C) \leq m$, then $r_M(E(G)) - r(G) \leq \ell^{m+1}$.*

Proof. Since $r_{\mathcal{T}_k(M,G)}(C) \leq m$, there is a set $Z \in \mathcal{T}_k(M, G)$ so that $C \subseteq Z$ and $\lambda_M(Z) \leq m$. Moreover, $r_G(Z \cap E(G)) \leq m$ by Lemma 4.5.1. Thus, $r_M(Z \cap E(G)) \leq |Z \cap E(G)| \leq \ell^m$ since G is simple and $G \in \mathcal{U}(\ell)$, by Theorem 1.3.2. We have

$$\begin{aligned} r_M(E(G)) - r(G) &= \sqcap_M(E(G), C) \\ &\leq \kappa_M(E(G), C) \\ &\leq \lambda_M(Z - E(G)) \\ &\leq \lambda_M(Z) + |Z \cap E(G)| = m + \ell^m \leq \ell^{m+1}, \end{aligned}$$

as desired. □

Finally, we prove a result which lets us move from a Dowling-geometry minor to a Dowling-geometry restriction in certain situations, while maintaining the connectivity of a set X to the Dowling geometry. This theorem uses Lemma 2.2.4, in addition to the lemmas from this section.

Theorem 4.5.7. *Let $\ell, s \geq 2$ and $n, m \geq 3$ be integers, and let Γ be a finite group. There is a function $r_{4.5.7}\colon \mathbb{Z}^3 \to \mathbb{Z}$ so that if $M \in \mathcal{U}(\ell)$ is a vertically $(s+2)$-connected matroid with no rank-n doubled-clique minor and with a $\mathrm{DG}(r_{4.5.7}(\ell, m, n), \Gamma)$-minor G, then for each $X \subseteq E(M)$ for which $r_M(X) \leq \min(s, m)$, there is minor N of M with a spanning $\mathrm{DG}(m, \Gamma)$-restriction so that $N|X = M|X$.*

Proof. Define $r_{4.5.7}(\ell, m, n) = r = \max(3m + f_{2.2.4}(\ell, n, \ell^{m+1}), 3(m+2))$. Let $k = m+2$, and note that $\mathcal{T}_k(G)$ is a tangle by Lemma 4.4.5 since $k \leq r/3 \leq \lceil 2r/3 \rceil$. Let M_1 be a minimal minor of M so that G is a minor of M_1, while $M_1|X = M|X$ and $r_{\mathcal{T}_k(M_1,G)}(X) = r_M(X)$. Since M is vertically $(s+2)$-connected and $r_M(X) \leq \min(s, k-2)$, the matroid M is a valid choice for M_1 by Lemma 4.5.2. Let $\mathcal{T} = \mathcal{T}_k(M_1, G)$.

Let $C_0 \subseteq E(M_1)$ so that G is a restriction of M_1/C_0. Then $C_0 \subseteq \mathrm{cl}_{\mathcal{T}}(X)$ by the minimality of M_1 and Lemma 4.5.4, so $r_{\mathcal{T}}(C_0) \leq r_{\mathcal{T}}(X) \leq r_M(X) \leq m$. Then by Lemma 4.5.6 we have $r_{M_1}(E(G)) - r(G) \leq \ell^{m+1}$. By Lemma 2.2.4 applied to M_1 with $d = \ell^{m+1}$, the matroid $M_1|E(G)$ has a $\mathrm{DG}^-(3m, \Gamma)$-restriction G_1. Then G_1 has a $\mathrm{DG}^-(m+2, \Gamma)$-restriction G_2 so that $X \cap E(G_2) = \varnothing$. By Lemma 4.5.3 we have

$$\kappa_{M_1}(E(G_2), X) \geq \min(r_{\mathcal{T}}(X), r_G(E(G_2))) = r_{\mathcal{T}}(X) = r_M(X).$$

By Theorem 4.1.3, the matroid M_1 has a minor N so that $N|E(G_2) = M_1|E(G_2)$ and $N|X = M_1|X$, while $E(N) = E(G_2) \cup X$ and $\lambda_N(E(G_2)) = \kappa_{M_1}(E(G_2), X) = r_M(X)$. Since $\lambda_N(X) = r_M(X) = r_N(X)$ and $E(N) = X \cup E(G_2)$, we have $r_N(X) + r_N(E(G_2)) - r(N) = r_N(X)$, so $E(G_2)$ spans N. By Lemma 2.2.3 and the fact that $r_M(X) \leq m$, there is a set C of N of size at most two so that N/C has a $\mathrm{DG}(r(N/C), \Gamma)$-restriction, and $(N/C)|X = N|X$. □

When we apply this theorem X will either be a stack, or a collection of nearly skew spikes of rank at most four.

4.6 The Proof

We now prove a theorem which easily implies Theorem 4.0.1, the main result of this chapter. Recall that if P is a porcupine with tip f, then we write $d(P)$ for the corank of $\mathrm{si}(P/f)$.

Theorem 4.6.1. *Let $\ell \geq 2$ and $k, s \geq 1$ and $g, m, n \geq 3$ be integers, and let Γ be a finite group. There are functions $s_{4.6.1}\colon \mathbb{Z}^3 \to \mathbb{Z}$ and $r_{4.6.1}\colon \mathbb{Z}^6 \to \mathbb{Z}$ so that if $M \in \mathcal{U}(\ell)$ is a vertically $s_{4.6.1}(k, s, g)$-connected matroid with no rank-n doubled-clique minor, and with a $\mathrm{DG}(r_{4.6.1}(\ell, m, k, n, s, g), \Gamma)$-minor G and sets $T' \subseteq T$ so that $|T| \leq k$ and each element of T' is the tip of a g-porcupine restriction P of M with $d(P) = s$, then M has a minor N of rank at least m so that*

- *N has a* $\mathrm{DG}(r(N), \Gamma)$*-restriction,*
- *$N|T = M|T$, and*
- *each element of T' is the tip of a g-porcupine restriction P of N with $d(P) = s$.*

Proof. Define integers $n_0 = k+1$, and $n_i = 2kg2^{2in_{i-1}+1}$ for $i \geq 1$, which appear in the statement of Lemma 4.3.1. Let $m_1 = \max(3(ks+1)n_{ks}, m)$. Define $s_{4.6.1}(k, g, s) = n_{ks} + 1$ and $r_{4.6.1}(\ell, m, k, n, s, g) = r = \max(m_1 + 3 + f_{2.2.4}(\ell, n, \ell^{m_1+1}), 3m_1)$. Let $M \in \mathcal{U}(\ell)$ be a vertically $s_{4.6.1}(k, s, g)$-connected matroid with a $\mathrm{DG}(r_{4.6.1}(\ell, m, k, n, s, g), \Gamma)$-minor G, a set $T \subseteq E(M)$ with $|T| \leq k$, and $T' \subseteq T$ so that each element of T' is the tip of a g-porcupine restriction P of M with $d(P) = s$. Let $t = |\Gamma|$. We may assume that T contains no loop of M.

Let M_0 be obtained from M by performing parallel extensions so that these porcupine restrictions are pairwise disjoint, their union is disjoint from $E(G)$, and T is disjoint from $E(G)$. Say that $E(M_0) = E(M) \cup X$, where each element of X is parallel to an element of $M_0|(E(M))$. Note that M_0 is vertically $s_{4.6.1}(k, s, g)$-connected. Let $\mathbf{R}_0 = (R_0, \mathcal{P}_0)$ be a g-prickle of M_0 so that

- the tip of each porcupine of $\mathcal{P}_0$ is in T,
- each element of T is the tip of a porcupine in $\mathcal{P}_0$,
- each $P \in \mathcal{P}_0$ with tip in T' satisfies $d(P) = s$, and
- each $P \in \mathcal{P}_0$ with tip in $T - T'$ satisfies $d(P) = 0$ (so P is simply a tip).

We first show that there exists a subprickle $\mathbf{Q}$ of $\mathbf{R}_0$ with 'very small' rank which contains all other subprickles with 'very small' rank.

4.6.1.1. *There is a subprickle $\mathbf{Q}$ of $\mathbf{R}_0$ so that $r_{M_0}(E(\mathbf{Q})) < n_{d(\mathbf{Q})}$, and each $\mathbf{A} \preceq \mathbf{R}_0$ with $r_{M_0}(E(\mathbf{A})) < n_{d(\mathbf{A})}$ satisfies $\mathbf{A} \preceq \mathbf{Q}$.*

Proof. Let $\mathbf{Q}$ be a subprickle of $\mathbf{R}_0$ with $d(\mathbf{Q})$ maximal such that $r_{M_0}(E(\mathbf{Q})) < n_{d(\mathbf{Q})}$. The trivial subprickle of $\mathbf{R}_0$ is a candidate for $\mathbf{Q}$ since $k < n_0$, so $\mathbf{Q}$ exists. If there is some $\mathbf{A} \not\preceq \mathbf{Q}$ such that $r_{M_0}(E(\mathbf{A})) < n_{d(\mathbf{A})}$, then $(\mathbf{Q} \cup \mathbf{A}) \preceq \mathbf{R}_0$ and $d(\mathbf{Q} \cup \mathbf{A}) > d(\mathbf{Q})$ and $r_{M_0}(E(\mathbf{Q} \cup \mathbf{A})) < n_{d(\mathbf{Q})} + n_{d(\mathbf{A})} < n_{d(\mathbf{Q} \cup \mathbf{A})}$. We use that $2n_i < n_{i+1}$ for all $i \geq 1$. This contradicts the maximality of $d(\mathbf{Q})$. □

The main idea of this proof is that we will use Lemmas 2.2.4 and 2.2.3 to find a $\mathrm{DG}(m, \Gamma)$-restriction, and then apply Lemma 4.3.1. In order to apply Lemma 2.2.4, we need to find a minor of M_0 for which we can contract a set of bounded size and obtain G as a restriction. We also need to preserve the tangle rank of prickles so that we can apply Lemma 4.3.1.

Let M_1 be a minimal minor of M_0 such that G is a minor of M_1 and $M_1|T = M_0|T$, while M_1 has a retract $\mathbf{R}_1$ of $\mathbf{R}$ with $\mathbf{Q} \preceq \mathbf{R}_1$ so that

(i) $r_{\mathcal{T}_{\lceil 2r/3\rceil-2}(M_1,G)}(E(\mathbf{Q})) = r_{M_0}(E(\mathbf{Q}))$, and

(ii) each $\mathbf{A} \preceq \mathbf{R}_1$ with $r_{\mathcal{T}_{\lceil 2r/3\rceil-2}(M_1,G)}(E(\mathbf{A})) < n_{d(\mathbf{A})}$ satisfies $\mathbf{A} \preceq \mathbf{Q}$.

Since M_0 is vertically $(n_{d(\mathbf{R})}+1)$-connected, each prickle $\mathbf{A} \preceq \mathbf{R}$ with $r_{\mathcal{T}_{\lceil 2r/3\rceil-2}(M_0,G)}(E(\mathbf{A})) < n_{d(\mathbf{A})}$ satisfies $r_{\mathcal{T}_{\lceil 2r/3\rceil-2}(M_0,G)}(E(\mathbf{A})) = r_{M_0}(E(\mathbf{A}))$ by Lemma 4.5.2. This implies that M_0 is a valid choice for M_1, using 4.6.1.1.

Let $\mathcal{T} = \mathcal{T}_{\lceil 2r/3\rceil-2}(M_1, G)$, and let $C_0 \subseteq E(M_1)$ such that G is a restriction of M_1/C_0. Note that $\mathcal{T}_{\lceil 2r/3\rceil-2}(M_1/C, G/C)$ is a tangle for each $C \subseteq E(G)$ of rank at most two, by Lemma 4.4.5; this is why we work with $\mathcal{T}$ instead of $\mathcal{T}_{\lceil 2r/3\rceil}(M_1, G)$. We will show that there is a prickle $\mathbf{Y} \preceq \mathbf{R}_1$ with 'small' tangle rank so that $C_0 \subseteq \mathrm{cl}_{\mathcal{T}}(E(\mathbf{Y}))$. We first prove the following claim, which shows that the union of two subprickles of $\mathbf{R}_1$ with 'small' tangle rank also has 'small' tangle rank.

4.6.1.2. *If $\mathbf{A}_1 \preceq \mathbf{R}_1$ and $\mathbf{A}_2 \preceq \mathbf{R}_1$ such that $r_{\mathcal{T}}(E(\mathbf{A}_i)) \le 3(d(\mathbf{A}_i)+1)n_{d(\mathbf{A}_i)}$ for $i \in \{1,2\}$, then $r_{\mathcal{T}}(E(\mathbf{A}_1 \cup \mathbf{A}_2)) \le 3(d(\mathbf{A}_1 \cup \mathbf{A}_2)+1)n_{d(\mathbf{A}_1\cup\mathbf{A}_2)}$.*

Proof. This is clearly true if $\mathbf{A}_1 = \mathbf{A}_2$ or $\mathbf{A}_i = \mathbf{R}_1$, so assume that $\mathbf{A}_1 \neq \mathbf{A}_2$ and $\mathbf{A}_i \neq \mathbf{R}_1$ for each $i \in \{1,2\}$. Then $\mathbf{A}_1 \cup \mathbf{A}_2$ is a prickle such that $d(\mathbf{A}_1 \cup \mathbf{A}_2) > \max(d(\mathbf{A}_1), d(\mathbf{A}_2))$. Assume that $d(\mathbf{A}_1) \ge d(\mathbf{A}_2)$, without loss of generality. Then

$$\begin{aligned} r_{\mathcal{T}}(E(\mathbf{A}_1 \cup \mathbf{A}_2)) &\le r_{\mathcal{T}}(E(\mathbf{A}_1)) + r_{\mathcal{T}}(E(\mathbf{A}_2)) \\ &\le 6(d(\mathbf{A}_1)+1)n_{d(\mathbf{A}_1)} \\ &\le 3(d(\mathbf{A}_1 \cup \mathbf{A}_2)+1)n_{d(\mathbf{A}_1\cup\mathbf{A}_2)}, \end{aligned}$$

as desired. The last inequality holds because $2(i+1)n_i \le (i+1)n_{i+1}$ for all $i \ge 0$. □

Let $\mathbf{Y}$ be a subprickle of $\mathbf{R}_1$ with $d(\mathbf{Y})$ maximal such that $r_{\mathcal{T}}(E(\mathbf{Y})) \le 3(d(\mathbf{Y})+1)n_{d(\mathbf{Y})}$. Note that $\mathbf{Y}$ exists because the trivial subprickle of $\mathbf{R}_1$ is a choice for $\mathbf{Y}$. One can show that $\mathbf{Y}$ is unique by 4.6.1.2, but we only need the existence of $\mathbf{Y}$. Also note that $\mathbf{Q} \preceq \mathbf{Y}$, by applying 4.6.1.2 with $\mathbf{A}_1 = \mathbf{Q}$ and $\mathbf{A}_2 = \mathbf{Y}$.

The following claim shows that we can contract certain elements of $E(M_1) - \mathrm{cl}_{\mathcal{T}}(E(\mathbf{Y}))$ and recover a prickle which satisfies (i) and (ii) and has $\mathbf{Y}$ as a subprickle. We apply this claim with $\mathbf{A} = \mathbf{R}_1$ to show that $C_0 \subseteq \mathrm{cl}_{\mathcal{T}}(E(\mathbf{Y}))$ so that we can apply Theorem 2.2.4 and reduce to the case with a $\mathrm{DG}^-(m+2,\Gamma)$-restriction. We then apply this claim to show that we can contract two elements and recover a $\mathrm{DG}(m,\Gamma)$-restriction, and still have a prickle with connectivity properties which satisfy the hypotheses of Lemma 4.3.1. While we only apply this claim with $\mathbf{A} = \mathbf{R}_1$, we state it more generally so that we can prove it using induction on $d(\mathbf{A})$.

4.6.1.3. *Let $C \subseteq C_0$ or $C \subseteq E(G)$ so that $|C| \leq 2$ and $r_{\mathcal{T}}(E(\mathbf{Y}) \cup C) = r_{\mathcal{T}}(E(\mathbf{Y})) + |C|$. Let $\mathcal{T}/C = \mathcal{T}_{\lceil 2r/3 \rceil - 2}(M_1/C, G)$ or $\mathcal{T}/C = \mathcal{T}_{\lceil 2r/3 \rceil - 2}(M_1/C, G/C)$ if $C \subseteq C_0$ or $C \subseteq E(G)$, respectively. Then for each prickle $\mathbf{A} \preceq \mathbf{R}_1$ with $\mathbf{Y} \preceq \mathbf{A}$, the matroid M_1/C has a retract $\mathbf{A}'$ of $\mathbf{A}$ so that $\mathbf{Y} \preceq \mathbf{A}'$, and each $\mathbf{Z} \preceq \mathbf{A}'$ with $r_{\mathcal{T}/C}(E(\mathbf{Z})) < n_{d(\mathbf{Z})}$ satisfies $\mathbf{Z} \preceq \mathbf{Q}$.*

Proof. Assume that the claim is false for some prickle $\mathbf{A}$ with $d(\mathbf{A})$ minimum. Then $d(\mathbf{A}) > d(\mathbf{Y})$ or else $\mathbf{A} = \mathbf{Y}$ and the claim holds, since Lemma 4.5.5 applied with $X = E(\mathbf{Y})$ implies that $(M/C)|E(\mathbf{Y}) = M|E(\mathbf{Y})$ and that $r_{\mathcal{T}/C}(Y') = r_{\mathcal{T}}(Y')$ for all $Y' \subseteq E(\mathbf{Y})$. Then by Lemma 4.2.2 applied to $\mathbf{A}$ and $\mathbf{Y}$, there is a collection $\mathcal{A}$ of subprickles of $\mathbf{A}$ such that each $\mathbf{A}_0 \in \mathcal{A}$ satisfies $d(\mathbf{A}_0) = d(\mathbf{A}) - 1$ and $\mathbf{Y} \preceq \mathbf{A}_0$, while $|\mathcal{A}| \leq d(\mathbf{A}) - d(\mathbf{Y})$ and $E(\mathbf{A}) - E(\mathbf{Y}) \subseteq \cup_{\mathbf{A}_0 \in \mathcal{A}}(E(\mathbf{A}) - E(\mathbf{A}_0))$.

Let $\mathbf{A}_0 \in \mathcal{A}$. We will show that $r_{\mathcal{T}}(E(\mathbf{A}) - E(\mathbf{A}_0)) \leq 3n_{d(\mathbf{A})}$. Since $d(\mathbf{A}_0) < d(\mathbf{A})$ and $d(\mathbf{A})$ is minimum, the matroid M_1/C has a retract $\mathbf{A}_0'$ of $\mathbf{A}_0$ so that $\mathbf{Y} \preceq \mathbf{A}_0'$, and each $\mathbf{Z} \preceq \mathbf{A}_0'$ with $r_{\mathcal{T}/C}(E(\mathbf{Z})) < d(\mathbf{Z})$ satisfies $\mathbf{Z} \preceq \mathbf{Q}$.

By Lemma 4.2.4 there is a non-empty set $C' \subseteq E(M) - C$ and a collection $\mathcal{K}$ of prickles of M_1/C such that $|\mathcal{K}| + |C'| \leq 4$, each $\mathbf{K} \in \mathcal{K}$ is a retract of $\mathbf{A}$ with $\mathbf{A}_0' \preceq \mathbf{K}$, and

$$E(\mathbf{A}) - E(\mathbf{A}_0) \subseteq \mathrm{cl}_{M_1}\Big(\big(\cup_{\mathbf{K} \in \mathcal{K}}\,(E(\mathbf{K}) - E(\mathbf{A}_0'))\big) \cup C' \cup C\Big). \tag{1}$$

Note that for each $\mathbf{K} \in \mathcal{K}$ we have $\mathbf{Y} \preceq \mathbf{A}_0' \preceq \mathbf{K}$ and $d(\mathbf{A}_0') = d(\mathbf{K}) - 1$, by the definition of a retract. Since the claim is false for $\mathbf{A}$, for each $\mathbf{K} \in \mathcal{K}$ there is some $\mathbf{Z} \preceq \mathbf{K}$ with $\mathbf{Z} \not\preceq \mathbf{Q}$ such that $r_{\mathcal{T}/C}(E(\mathbf{Z})) < n_{d(\mathbf{Z})} \leq n_{d(\mathbf{A})}$. Since $d(\mathbf{A}_0') = d(\mathbf{K}) - 1$ and $\mathbf{Z} \not\preceq \mathbf{A}_0'$, by Lemma 4.2.1 (i) we have $E(\mathbf{K}) - E(\mathbf{A}_0') \subseteq E(\mathbf{Z})$, so $r_{\mathcal{T}/C}(E(\mathbf{K}) - E(\mathbf{A}_0')) < n_{d(\mathbf{A})}$. Thus, by (1) and the fact that $|\mathcal{K}| + |C'| \leq 4$, we have

$$\begin{aligned} r_{\mathcal{T}/C}(E(\mathbf{A}) - E(\mathbf{A}_0)) &\leq r_{\mathcal{T}/C}(\cup_{\mathbf{K} \in \mathcal{K}}(E(\mathbf{K}) - E(\mathbf{A}_0'))) + |C'| \\ &\leq |\mathcal{K}|(n_{d(\mathbf{A})} - 1) + |C'| \\ &\leq (|\mathcal{K}| + |C'| - 1)(n_{d(\mathbf{A})} - 1) + 1 \\ &\leq 3n_{d(\mathbf{A})} - 2. \end{aligned}$$

Then since $|C| \leq 2$ we have $r_{\mathcal{T}}(E(\mathbf{A}) - E(\mathbf{A}_0)) \leq 3n_{d(\mathbf{A})}$, using Lemma 4.5.4 if $C \subseteq C_0$ and Lemma 4.5.5 if $C \subseteq E(G)$.

We now use that $|\mathcal{A}| \leq d(\mathbf{A}) - d(\mathbf{Y})$ and $E(\mathbf{A}) - E(\mathbf{Y}) \subseteq \cup_{\mathbf{A}_0 \in \mathcal{A}}(E(\mathbf{A}) - E(\mathbf{A}_0))$. We have

$$\begin{aligned} r_{\mathcal{T}}(E(\mathbf{A})) &\le r_{\mathcal{T}}(E(\mathbf{Y})) + r_{\mathcal{T}}(E(\mathbf{A}) - E(\mathbf{Y})) \\ &\le 3(d(\mathbf{Y})+1)n_{d(\mathbf{Y})} + \sum_{\mathbf{A}_0 \in \mathcal{A}} r_{\mathcal{T}}(E(\mathbf{A}) - E(\mathbf{A}_0)) \\ &\le 3(d(\mathbf{Y})+1)n_{d(\mathbf{Y})} + |\mathcal{A}| 3 n_{d(\mathbf{A})} \\ &\le 3(d(\mathbf{Y})+1)n_{d(\mathbf{Y})} + 3(d(\mathbf{A}) - d(\mathbf{Y}))n_{d(\mathbf{A})} \\ &\le 3(d(\mathbf{A})+1)n_{d(\mathbf{A})}. \end{aligned}$$

But then $\mathbf{A} \preceq \mathbf{Y}$ by 4.6.1.2 and the definition of $\mathbf{Y}$, so $\mathbf{A}$ is not a counterexample. □

If there is some $e \in C_0 - \mathrm{cl}_{\mathcal{T}}(E(\mathbf{Y}))$, then $r_{\mathcal{T}_{\lceil 2r/3 \rceil - 2}(M_1/e, G)}(E(\mathbf{Q})) = r_{M_0}(E(\mathbf{Q}))$ by Lemma 4.5.4 since $E(\mathbf{Q}) \subseteq E(\mathbf{Y})$. Then by 4.6.1.3 applied with $C = \{e\}$ and $\mathbf{A} = \mathbf{R}_1$, the matroid M_1/e contradicts the minor-minimality of M_1. Thus, $C_0 \subseteq \mathrm{cl}_{\mathcal{T}}(E(\mathbf{Y}))$ and so $r_{\mathcal{T}}(C_0) \le 3(ks+1)n_{ks} \le m_1$. So by Lemma 4.5.6 we have $r_{M_1}(E(G)) - r(G) \le \ell^{m_1+1}$. Then since $r(G) \ge m_1 + 3 + f_{2.2.4}(\ell, n, \ell^{m_1+1})$, by Lemma 2.2.4 with $m = m_1 + 3$ and $d = \ell^{m_1+1}$ there is a $\mathrm{DG}^-(m_1+3, \Gamma)$-restriction G_1 of $M_1|E(G)$.

Since $r_{\mathcal{T}}(E(G_1)) \ge r_G(E(G_1)) \ge m_1 + 2$ by Lemma 4.5.1, by Lemma 2.2.3 there is a set $C \subseteq E(G_1)$ of size at most two so that $r_{\mathcal{T}}(E(\mathbf{Y}) \cup C) = r_{\mathcal{T}}(E(\mathbf{Y})) + |C|$ and G_1/C has a $\mathrm{DG}(m_1, \Gamma)$-restriction G_2. Let $\mathcal{T}/C = \mathcal{T}_{\lceil 2r/3 \rceil - 2}(M_1/C, G/C)$. Since $r_{\mathcal{T}}(E(\mathbf{Y}) \cup C) = r_{\mathcal{T}}(E(\mathbf{Y})) + |C|$, by 4.6.1.3 applied to $\mathbf{R}_1$ and C, the matroid M_1/C has a retract $\mathbf{R}_2$ of $\mathbf{R}_1$ so that $\mathbf{Y} \preceq \mathbf{R}_2$, and each $\mathbf{A} \preceq \mathbf{R}_2$ with $r_{\mathcal{T}/C}(E(\mathbf{A})) < n_{d(\mathbf{A})}$ satisfies $\mathbf{A} \preceq \mathbf{Q}$. This last condition implies that $\mathbf{R}_2$ is a g-prickle.

4.6.1.4. $\mathbf{R}_2$ *is a g-prickle.*

Proof. If not, then there is some porcupine P of $\mathbf{R}_2$ which satisfies $d(P) = 1$ and $r_{M_1/C}(P) \le g$ (so P is a spike of rank at most g). Then the subprickle $\mathbf{A}$ of $\mathbf{R}_2$ consisting of P and otherwise only trivial porcupines is not a g-prickle, and satisfies $d(\mathbf{A}) = 1$. But then $r_{M_1/C}(E(\mathbf{A})) < |T| + g < n_1$, and so $r_{\mathcal{T}/C}(E(\mathbf{A})) \le r_{M_1/C}(E(\mathbf{A})) < n_{d(\mathbf{A})}$, so $\mathbf{A} \preceq \mathbf{Q}$. But then $\mathbf{A} \preceq \mathbf{R}_0$ since $\mathbf{Q} \preceq \mathbf{R}_0$, which contradicts that $\mathbf{R}_0$ is a g-prickle, since every subprickle of a g-prickle is also a g-prickle. □

We now show that M_1/C and $\mathbf{R}_2$ satisfy the connectivity conditions of Lemma 4.3.1 with $J = E(G_2)$. Since $r_{\mathcal{T}}(E(\mathbf{Y}) \cup C) = r_{\mathcal{T}}(E(\mathbf{Y})) + |C|$ we have $r_{\mathcal{T}/C}(E(\mathbf{Q})) = r_{\mathcal{T}}(E(\mathbf{Q}))$ by Lemma 4.5.5. If $\mathbf{A} \preceq \mathbf{R}_2$ with $\mathbf{A} \not\preceq \mathbf{Q}$, then

$$\kappa_{M_1/C}(E(G_2), E(\mathbf{A})) \ge \min\left(r_{\mathcal{T}/C}(E(\mathbf{A})), r_{G/C}(E(G_2))\right) \ge \min(n_{d(\mathbf{A})}, m_1) \ge n_{d(\mathbf{A})},$$

where the first inequality holds by applying Lemma 4.5.3 to $\mathcal{T}/C$ with $J = E(G_2)$, and the last inequality holds since $m_1 \geq n_{ks}$. Since $T \subseteq E(\mathbf{Q})$, by Lemma 4.3.1 with $J = E(G_2)$ the matroid M_1/C has a minor N with G_2 a spanning restriction, $N|T = (M_1/C)|T = M_0|T$, and a g-retract $\mathbf{R}_3$ of $\mathbf{R}_2$. Since $\mathbf{R}_3$ is a retract of $\mathbf{R}_2$ and is thus a retract of $\mathbf{R}_0$, each element of T' is the tip of a g-porcupine restriction P of N with $d(P) = s$.

Therefore, M_0 has a minor N with a $\mathrm{DG}(r(N), \Gamma)$-restriction so that $N|T = M_0|T$ and each element of T' is the tip of a g-porcupine restriction P of N with $d(P) = s$. Since porcupines and Dowling geometries are simple, we may assume that $N|(E(N) - T)$ is simple. Let $C_1, D \subseteq E(M_0)$ be disjoint sets so that $N = M_0/C_1 \backslash D$. Since $N|(E(N) - T)$ is simple, we may assume that $X \subseteq D$. Thus, N is a minor of M, and the theorem holds. □

Chapter 5

The Main Result

In this chapter we combine the main results of Chapters 3 and 4 to prove Theorem 1.7.2, which we restate below for convenience.

Theorem 1.7.2. *There is a function $f_{1.7.2} : \mathbb{Z}^4 \to \mathbb{Z}$ so that for all integers $\ell \geq 2$, $t \geq 1$ and $n, k \geq 3$, if $\mathcal{M}$ is a minor-closed class of matroids such that $U_{2,\ell+2} \notin \mathcal{M}$, then either*

- *$\mathcal{M}$ contains a rank-n doubled clique,*
- *$\mathcal{M}$ contains $\mathrm{DG}(k, \Gamma)$ for some group Γ with $|\Gamma| \geq t$, or*
- *each $M \in \mathcal{M}$ satisfies $\varepsilon(M) \leq (t-1)\binom{r(M)}{2} + f_{1.7.2}(\ell, n, k, t) \cdot r(M)$.*

We first prove Theorem 1.7.2 for matroids with a spanning clique restriction in Section 5.1; we must treat this case separately as it is one of the two outcomes of Theorem 3.1.3 and does not promise any additional structure. Then in Sections 5.2-5.4 we show that each of the three structures found in Theorem 3.6.1 leads to either a doubled-clique minor or a Dowling-geometry minor with group size at least t. This allows us to complete step (III) of the general growth-rates-proof sketch from Section 1.6.

5.1 The Spanning Clique Case

We combine Corollary 2.1.5, Theorem 1.6.2, and Theorem 1.3.5 to prove Theorem 1.7.2 for matroids with a spanning clique restriction.

Proposition 5.1.1. *There is a function $f_{5.1.1} : \mathbb{Z}^3 \to \mathbb{Z}$ so that for all integers $\ell \geq 2$, $t \geq 1$ and $k, n \geq 3$, if $\mathcal{M}$ is a minor-closed class of matroids such that $U_{2,\ell+2} \notin \mathcal{M}$, then either*

- *$\mathcal{M}$ contains a rank-n doubled clique,*
- *$\mathcal{M}$ contains $\mathrm{DG}(k, \Gamma)$ for some group Γ with $|\Gamma| \geq t$, or*
- *each $M \in \mathcal{M}$ with a spanning clique restriction satisfies $\varepsilon(M) \leq (t-1)\binom{r(M)}{2} + f_{5.1.1}(\ell, n, k) \cdot r(M)$.*

Proof. Let $h = h_{1.6.2}(\ell, n)$ and $\alpha = \alpha_{1.3.5}(\ell, n)$, and define $f_{5.1.1}(\ell, n, k) = 2\ell k^{2^\ell} + h\ell\alpha$. Suppose that the first two outcomes do not hold for this value of $f_{5.1.1}$, and let $M \in \mathcal{M}$ be simple with a spanning clique restriction. By Theorem 1.6.2 there are disjoint sets $C_1, C_2 \subseteq E(M)$ with $r_M(C_1 \cup C_2) \leq h$ and a B-clique $\hat{M}$ such that $\varepsilon(\hat{M}) \geq \varepsilon(M/(C_1 \cup C_2))$ and $\mathrm{si}(\hat{M})$ is isomorphic to a restriction of M/C_1.

5.1.1.1. $\varepsilon(M) \leq \varepsilon(\hat{M}) + (h\ell\alpha)r(M)$.

Proof. Since $\varepsilon(M/(C_1 \cup C_2)) \leq \varepsilon(\hat{M})$ it suffices to show that $\varepsilon(M) \leq \varepsilon(M/(C_1 \cup C_2)) + (h\ell\alpha)r(M)$. Assume for a contradiction that $\varepsilon(M) - \varepsilon(M/(C_1 \cup C_2)) > (h\ell\alpha)r(M)$. Let

$C \subseteq C_1 \cup C_2$ be a maximal independent set so that $\varepsilon(M) - \varepsilon(M/C) \leq r_M(C)(\ell\alpha)r(M)$, and let $e \in (C_1 \cup C_2) - \mathrm{cl}_M(C)$. Then

$$\begin{aligned}\varepsilon(M/C) - \varepsilon(M/C/e) &= \big(\varepsilon(M) - \varepsilon(M/C/e)\big) - \big(\varepsilon(M) - \varepsilon(M/C)\big) \\ &> r_M(C \cup \{e\})(\ell\alpha)r(M) - r_M(C)(\ell\alpha)r(M) \\ &\geq (\ell\alpha)r(M).\end{aligned}$$

Since $M \in \mathcal{U}(\ell)$, Lemma 1.1.4 implies that there are greater than $\alpha \cdot r(M)$ long lines of M/C through e. Let $T_0 \subseteq E(M/C)$ be the set of elements of M/C on long lines through e, and let $N = (M/C)|T_0$.

Let $(N/e)|T$ be a simplification of N/e, so $\varepsilon((N/e)|T) > \alpha \cdot r(N/e)$. By Theorem 1.3.5, $(N/e)|T$ has an $M(K_n)$-minor. Say $(N/X/e)|T_1 \cong M(K_n)$ for disjoint sets $X, T_1 \subseteq T$. Let $J \subseteq E(N)$ denote the set of elements of N on a line through e and an element of T_1. Each line of N through e and an element of T_1 has rank two in N/X, since $(N/X/e)|T_1$ is simple. Thus, $(N/X)|J$ has a rank-n doubled-clique restriction with tip e, a contradiction. □

Since $\mathrm{si}(\hat{M})$ is isomorphic to a restriction of M/C_1, the B-clique $\hat{M}$ has no $U_{2,\ell+2}$-minor and no $\mathrm{DG}(k,\Gamma)$-minor with $|\Gamma| \geq t$. Then by Lemma 2.1.5 we have $\varepsilon(\hat{M}) \leq (t-1)\binom{r(\hat{M})}{2} + 2\ell k^{2^\ell} \cdot r(\hat{M})$. By 5.1.1.1,

$$\varepsilon(M) \leq (t-1)\binom{r(M)}{2} + \Big(2\ell k^{2^\ell} + h\ell\alpha\Big)r(M),$$

as desired. □

5.2 Stacks

We now prove that a matroid with a spanning clique restriction and a huge $\mathcal{F} \cap \mathcal{U}(t)$-stack restriction has either a projective-geometry minor or a Dowling-geometry minor. Recall that a matroid M is an $(\mathcal{O}, b, h)$-*stack* if there are disjoint sets $P_1, P_2, \ldots, P_h \subseteq E(M)$ such that $\cup_i P_i$ spans M, and for each $i \in [h]$ the matroid $(M/(P_1 \cup \cdots \cup P_{i-1}))|P_i$ has rank at most b and is not in $\mathcal{O}$. Note that for each $j \in [h]$, the matroid $M/(P_1 \cup \cdots \cup P_j)$ is an $(\mathcal{O}, b, h-j)$-stack. We say that a matroid N is a *good* $(\mathcal{O}, b, h-j)$-stack-minor of M if there is some $j \in [h]$ so that $N = M/(P_1 \cup \cdots \cup P_j)$. It is clear from this definition that if M_1 is a good $(\mathcal{O}, b, h-j)$-stack-minor of M, and M_2 is a good $(\mathcal{O}, b, h-j-i)$-stack-minor of M_1 for $i \in [h-j]$, then M_2 is a good $(\mathcal{O}, b, h-j-i)$-stack-minor of M.

We first prove a general lemma which shows that stacks are somewhat robust under lifts and projections.

Lemma 5.2.1. *Let $d, b, m \geq 0$ be integers, and let M and N be matroids. If $M|S$ is an $(\mathcal{O}, b, m2^d)$-stack and $\text{dist}(M, N) \leq d$, then there are sets C and S' so that $(M/C)|S' = (N/C)|S'$ is a good $(\mathcal{O}, b, m)$-stack-minor of $M|S$.*

Proof. Let d be minimal so that the claim is false, so $d > 0$. Let M_0 be a matroid so that $\text{dist}(M, M_0) \leq d-1$ and $\text{dist}(M_0, N) = 1$. By the minimality of d, there are sets C_0 and S_0 so that $(M/C_0)|S_0 = (M_0/C_0)|S_0$ is a good $(\mathcal{O}, b, 2m)$-stack-minor of $M|S$, since $\frac{m2^d}{2^{d-1}} = 2m$. Let $M' = M_0/C_0$ and $N' = N/C_0$, and note that $\text{dist}(M', N') \leq \text{dist}(M_0, N) = 1$. Let (S_1, S_2) be a partition of S_0 so that $M'|S_1$ and $(M'/S_1)|S_2$ are $(\mathcal{O}, b, m)$-stacks.

It suffices to show that there are sets C_1 and S_3 so that $(M'/C_1)|S_3 = (N'/C_1)|S_3$ is a good $(\mathcal{O}, b, m)$-stack-minor of $M'|S_0$, because then $(M/(C_0 \cup C_1))|S_3 = (N/(C_0 \cup C_1))|S_3$ is a good $(\mathcal{O}, b, m)$-stack-minor of $M|S$, and the lemma holds with $C = C_0 \cup C_1$ and $S' = S_3$. Let K be a matroid so that $\{K/f, K\backslash f\} = \{M', N'\}$ for some $f \in E(K)$. First assume that $(K\backslash f, K/f) = (M', N')$. If $f \notin \text{cl}_K(S_1)$, then $N'|S_1$ is an $(\mathcal{O}, b, m)$-stack and we take $C_1 = \varnothing$. If $f \in \text{cl}_K(S_1)$, then

$$(N'/S_1)|S_2 = (K/S_1\backslash f)|S_2 = (M'/S_1)|S_2,$$

so we take $C_1 = S_1$. Now assume that $(K/f, K\backslash f) = (M', N')$. If $f \notin \text{cl}_K(S_1)$, then $N'|S_1 = K|S_1 = (K/f)|S_1 = M'|S_1$, so we take $C = \varnothing$. If $f \in \text{cl}_K(S_1)$, then $(N'/S_1)|S_2 = (K/S_1/f)|S_2 = (M'/S_1)|S_2$, so we take $C_1 = S_1$. □

We will need the following corollary of Lemma 5.2.1 in the next section.

Corollary 5.2.2. *Let $d, b, m \geq 0$ be integers, and let M be a matroid. If $M|S$ is an $(\mathcal{O}, b, m2^d)$-stack and $r_M(X) \leq d$, then there are sets C and S' so that $X \subseteq C$ and $(M/C)|S'$ is a good $(\mathcal{O}, b, m)$-stack-minor of $M|S$.*

Proof. Let N be the matroid with ground set $E(M)$ so that $N\backslash X = M/X$ and $r_N(X) = 0$. Then $\text{dist}(M, N) \leq d$, so the statement holds by Lemma 5.2.1. □

We now combine Theorem 1.6.2 and Lemma 5.2.1 to prove the main result of this section.

Proposition 5.2.3. *There is a function $h_{5.2.3} \colon \mathbb{Z}^3 \to \mathbb{Z}$ so that for all integers $\ell, t, b \geq 2$ and $n, k \geq 3$, if $M \in \mathcal{U}(\ell)$ has a spanning clique restriction and an $(\mathcal{F} \cap \mathcal{U}(t), b, h_{5.2.3}(\ell, k, n))$-stack restriction S, then M has either a rank-n projective-geometry minor or a $\text{DG}(k, \Gamma)$-minor with $|\Gamma| \geq t$.*

Proof. Let $h = h_{1.6.2}(\ell, n)$, let $m = k^{2^\ell}$, and define $h_{5.2.3}(\ell, k, n) = m2^h$. Let B_0 be a frame for the spanning clique restriction of M. Assume that M has no rank-n projective-geometry minor. By Theorem 1.6.2 there is some $B_1 \subseteq B_0$ and a B_1-clique N so that $\text{dist}(M, N) \leq h$

and $\mathrm{si}(N)$ is isomorphic to a minor of M. By Lemma 5.2.1, N has a contract-minor N_1 with a set S_1 so that $N_1|S_1$ is an $(\mathcal{F} \cap \mathcal{U}(t), b, m)$-stack. Since N_1 is a frame matroid, the matroid $N_1|S_1$ is an $(\mathcal{U}(t), b, m)$-stack. Then each part of this stack has a $U_{2,t+2}$-minor, so by contracting elements of each part of this stack we see that N_1 has a contract-minor N_2 with $S_2 \subseteq S_1$ so that $N_2|S_2$ is a spanning $(\mathcal{U}(t), 2, m)$-stack restriction of N_2. Note that N_2 is a B-clique for some $B \subseteq B_1$, since N_2 is a contract-minor of N.

Since $N_2|S_2$ is a spanning $(\mathcal{U}(t), 2, m)$-stack restriction of N_2, there are disjoint sets $P_1, \dots, P_m \subseteq E(N_2)$ so that $\cup_i P_i$ spans N_2 and for each $i \in [m]$ the matroid $(N_2/(P_1 \cup \dots \cup P_{i-1}))|P_i$ is isomorphic to $U_{2,t+2}$. Since $t \geq 2$ and each line of length at least four spans two elements of B, this gives a partition $(B_1, \dots, B_m)$ of B so that $|B_i| = 2$ and $P_i \subseteq \mathrm{cl}_{N_2/(P_1 \cup \dots \cup P_{i-1})}(B_i)$ for each $i \in [m]$. Then $P_1 \cup \dots \cup P_{i-1}$ and $B_1 \cup \dots \cup B_{i-1}$ span the same flat of N_2, so

$$(N_2/(P_1 \cup \dots \cup P_{i-1}))|P_i = (N_2/(B_1 \cup \dots \cup B_{i-1}))|P_i.$$

Since $P_i \subseteq \mathrm{cl}_{N_2/(B_1 \cup \dots \cup B_{i-1})}(B_i)$ and $|P_i| \geq 4$, this implies that there are two elements $e \in P_i$ for which the unique circuit of $N_2|(B \cup \{e\})$ contains B_i. Since N_2 is a frame matroid and $|B_i| = 2$, this circuit contains no other elements of B. Thus, two elements of P_i are spanned in N_2 by B_i and are not parallel to either element of B_i, which implies that $P_i \subseteq \mathrm{cl}_{N_2}(B_i)$ for each $i \in [m]$. Since $\varepsilon(M|(B_i \cup P_i)) \geq t + 2$ for each $i \in [m]$, the matroid N_2 has a $\mathrm{DG}(k, \Gamma)$-minor with $|\Gamma| \geq t$ by Lemma 2.1.4. Since $\mathrm{DG}(k, \Gamma)$ is simple and $\mathrm{si}(N)$ is isomorphic to a minor of M, the matroid M has a $\mathrm{DG}(k, \Gamma)$-minor. $\square$

5.3 Small Spikes with Common Tip

In this section we prove that a matroid with a spanning clique restriction and lots of nearly skew small spikes with common tip has a doubled-clique minor. We first prove a continuation of Lemma 5.2.1 in the case that N is a B-clique.

Lemma 5.3.1. *Let $d, b, m \geq 0$ be integers. There is a function $f_{5.3.1} \colon \mathbb{Z}^3 \to \mathbb{Z}$ so that if M and N are matroids with $\mathrm{dist}(M, N) \leq d$ so that N is a B-clique and $M|S = N|S$ is an $(\mathcal{O}, b, f_{5.3.1}(d, b, m))$-stack, then there are sets C, D, B', S' so that*

- *$M/C \backslash D = N/C \backslash D$,*
- *$N/C \backslash D$ is a B'-clique, and*
- *$(M/C \backslash D)|S' = (N/C \backslash D)|S'$ is a good $(\mathcal{O}, b, m)$-stack-minor of $M|S = N|S$.*

Proof. Define $f_{5.3.1}(0, b, m) = 1$, and inductively define

$$f_{5.3.1}(d, b, m) = f_{5.3.1}(d-1, b, m)\big(1 + 2^{2b \cdot f_{5.3.1}(d-1, b, m)}\big),$$

for $d > 0$. Let d be minimal so that the claim is false, so $d > 0$, and let $m_1 = f_{5.3.1}(d-1, b, m)$. Let M_1 be a matroid so that $\text{dist}(N, M_1) = 1$ and $\text{dist}(M_1, M) \leq d - 1$. Let (S_1, S_2) be a partition of S so that $N|S_1$ is an $(\mathcal{O}, b, m_1)$-stack and $(N/S_1)|S_2$ is an $(\mathcal{O}, b, m_1 2^{2bm_1})$-stack. We consider two cases, depending on whether M_1 is a projection or lift of N. Recall that any contract-minor of N is a B_1-clique for some $B_1 \subseteq B$.

First assume that there is a matroid K with an element f so that $K\backslash f = N$ and $K/f = M_1$.

5.3.1.1. *There are sets C_1, D_1, B_1, S_1' so that $N/C_1\backslash D_1 = M/C_1\backslash D_1$ is a B_1'-clique, and $(N/C_1\backslash D_1)|S_1' = (M_1/C_1\backslash D_1)|S_1'$ is a good $(\mathcal{O}, b, m_1)$-stack-minor of $N|S$.*

Proof. Let B_1' be a minimal subset of B so that $S_1 \subseteq \text{cl}_N(B_1')$, and note that $|B_1'| \leq 2r_N(S_1) \leq 2bm_1$. If $f \notin \text{cl}_N(B_1')$, then let $D_1 = E(N) - \text{cl}_N(B_1')$. Then $N\backslash D_1$ is a B_1'-clique, and $N\backslash D_1 = M_1\backslash D_1$ since $f \notin \text{cl}_K(E(N) - D_1)$, so the claim holds with $(C_1, D_1, B_1, S_1') = (\varnothing, D_1, B_1', S_1)$.

If $f \in \text{cl}_N(B_1')$, then by Corollary 5.2.2 applied with $(M, X, d, m) = (N, B_1', 2bm_1, m_1)$, there are sets C' and S_1' so that $B_1' \subseteq C'$ and $(N/C')|S_1'$ is a good $(\mathcal{O}, b, m_1)$-stack-minor of $N|S$. Thus, the claim holds with $(C_1, D_1, S_1') = (C', \varnothing, S')$, since $f \in \text{cl}_N(C')$ and N/C' is a contract-minor of N. $\square$

Now assume that there is a matroid K with an element f so that $K/f = N$ and $K\backslash f = M_1$.

5.3.1.2. *There is a set $C_3 \subseteq E(N)$ so that $|C_3| \leq 3$ and $f \in \text{cl}_K(C_3)$.*

Proof. If B is a basis of M_1, then $r(M_1) = r(N)$. But then f is a coloop of K, which implies that $N = M_1$, and this contradicts that $d(N, M_1) > 0$. Thus, there is some element $e \in M_1 - \text{cl}_{M_1}(B)$. Since N is a B-clique there are elements $b, b' \in B$ so that $e \in \text{cl}_N(\{b, b'\})$. Since $r_{M_1}(\{b, b', e\}) = 3$ and $r_N(\{b, b', e\}) = 2$ it follows that $f \in \text{cl}_K(\{b, b', e\})$. $\square$

If $f \in \text{cl}_K(S_1)$, then let $C_1 = S_1$. Then there are sets $B_1 \subseteq B$ and $S_2' \subseteq S_2$ so that $N/C_1 = M/C_1$ is a B_1-clique, and $(N/C_1)|S_2' = (M_1/C_1)|S_2'$ is a good $(\mathcal{O}, b, m_1)$-stack-minor of $N|S$. If $f \notin \text{cl}_K(S_1)$, then let C_3 be the set given by 5.3.1.2. By Corollary 5.2.2 applied with $(M, X, d, m) = (N, C_3, 3, m_1)$, there are sets C' and S_1' so that $C_3 \subseteq C'$ and $(N/C')|S_1'$ is a good $(\mathcal{O}, b, m_1)$-stack-minor of $N|S$. Let $C_1 = C'$. Then $N/C_1 = M_1/C_1$ since $f \in \text{cl}_N(C_1)$, and $(N/C_1)|S_1' = (M_1/C_1)|S_1'$ is a good $(\mathcal{O}, b, m_1)$-stack-minor of $N|S$.

In all cases, we have shown that there are sets C_1, D_1, B_1, S_1 so that $N/C_1\backslash D_1 = M_1/C_1\backslash D_1$ is a B_1-clique, and $(N/C_1\backslash D_1)|S_1 = (M_1/C_1\backslash D_1)|S_1$ is a good $(\mathcal{O}, b, m_1)$-stack-minor of $N|S$. Let $N' = N/C_1\backslash D_1$, and $M' = M/C_1\backslash D_1$. Then N' is a B_1-clique, and $\text{dist}(N'/C_1\backslash D_1, M/C_1\backslash D_1) \leq \text{dist}(M_1, M) \leq d-1$. By the minimality of d, the definition of m_1, and the fact that N' is a B_1-clique, there are sets C_2, D_2, B_1', S_1' so that

- $M'/C_2\backslash D_2 = N'/C_2\backslash D_2$,
- $N'/C_2\backslash D_2$ is a B_1'-clique, and
- $(M'/C_2\backslash D_2)|S_1' = (N'/C_2\backslash D_2)|S_1'$ is a good $(\mathcal{O}, b, m)$-stack-minor of $M'|S_1 = N'|S_1$.

Let $C = C_1 \cup C_2$ and $D = D_1 \cup D_2$. Then $M/C\backslash D = N/C\backslash D$ is a B_1'-clique, and $(M/C\backslash D)|S_1' = (N/C\backslash D)|S_1'$ is a good $(\mathcal{O}, b, m)$-stack-minor of $M|S = N|S$. □

To prove the following proposition we apply Theorem 1.6.2 and Lemma 5.3.1, and then apply Lemma 2.2.2 to find a doubled-clique minor. Given a collection $\mathcal{Z}$ of sets, we will write $\cup\mathcal{Z}$ for $\cup_{Z\in\mathcal{Z}} Z$, for convenience.

Proposition 5.3.2. *There is a function $m_{5.3.2}\colon \mathbb{Z}^2 \to \mathbb{Z}$ so that for all integers $\ell \geq 2$ and $n \geq 3$, if $M \in \mathcal{U}(\ell)$ has a spanning clique restriction and there is some $e \in E(M)$ and a collection $\mathcal{S}$ of $m_{5.3.2}(\ell, n)$ mutually skew sets in M/e so that for each $S \in \mathcal{S}$, the matroid $M|(S \cup \{e\})$ is a spike of rank at most four with tip e, then M has a rank-n doubled-clique minor.*

Proof. Let $h = h_{1.6.2}(\ell, n)$ and $m_1 = n^{2^{\ell+1}}$, and define $m_{5.3.2}(\ell, n) = f_{5.3.1}(h, 2, 2m_1)2^h$. Assume that M has no rank-n doubled-clique minor; this implies that M has no rank-n projective-geometry minor. Let M with $\mathcal{S}$ be a counterexample so that M is minor-minimal. Then each $S \in \mathcal{S}$ satisfies $r_{M/e}(S) = 2$, since every rank-4 spike has a rank-3 spike as a minor. For each $S \in \mathcal{S}$, let X_S be a transversal of the parallel classes of $(M/e)|S$, and let $\mathcal{X} = \{X_S\colon S \in \mathcal{S}\}$. Let $\mathcal{O} = \mathcal{M} - \{(M/e)|X\colon X \in \mathcal{X}\}$, where $\mathcal{M}$ is the class of all matroids. Then $(M/e)|\cup\mathcal{X}$ is an $(\mathcal{O}, 2, m_{5.3.2}(\ell, n))$-stack. Note that if M' is a good $(\mathcal{O}, 2, j)$-stack-minor of $(M/e)|\cup\mathcal{X}$, then there is collection of j sets in $\mathcal{X}$ which are mutually skew in M', by the definition of a good $(\mathcal{O}, 2, j)$-stack-minor.

Since M/e has a spanning clique restriction, by Theorem 1.6.2 there is a B-clique N so that $\mathrm{dist}(M/e, N) \leq h$. By Lemma 5.2.1 with $m = f_{5.3.1}(h, 2, 2m_1)$ and $d = h$ there are sets C_1 and S_1 so that $(M/e/C_1)|S_1 = (N/C_1)|S_1$ is a good $(\mathcal{O}, 2, f_{5.3.1}(h, 2, 2m_1))$-stack-minor of $(M/e)|\cup\mathcal{X}$. Note that $\mathrm{dist}(M/e/C_1, N/C_1) \leq \mathrm{dist}(M/e, N) \leq h$ and that N/C_1 is a B_1-clique for some $B_1 \subseteq B$. Then by Lemma 5.3.1 with $M = M/e/C_1$ and $N = N/C_1$ and $S = S_1$, there are sets C, D, B', S' so that $M/e/(C_1 \cup C)\backslash D = N/(C_1 \cup C)\backslash D$ is a B'-clique, and $(N/(C_1 \cup C)\backslash D)|S'$ is a good $(\mathcal{O}, 2, 2m_1)$-stack-minor of $(N/C_1)|S_1$.

Let C' be a maximal subset of $C_1 \cup C$ so that $e \notin \mathrm{cl}_M(C_1 \cup C)$, and let $M' = M/C'\backslash D$. Then M'/e is a B'-clique, and there is a collection $\mathcal{X}' \subseteq \mathcal{X}$ of $2m_1$ mutually skew subsets of M'/e. Let $\mathcal{S}' = \{S \in \mathcal{S}\colon X_S \in \mathcal{X}'\}$. For each $S \in \mathcal{S}'$ we have $M'|(S \cup \{e\}) = M|(S \cup \{e\})$, since e is a nonloop of M' and $(M'/e)|X = (M/e)|X$. Thus, $\mathcal{S}'$ is a collection of mutually skew subsets of M'/e so that for each $S \in \mathcal{S}'$, the matroid $M'|(S \cup \{e\})$ is a rank-3 spike with tip e. Let $C_2 \subseteq E(M')$ have minimum size so that $C_2 \cup (\cup\mathcal{S}')$ spans M', and let M_2

be a simplification of M'/C_2. Then $M_2|(\{e\} \cup (\cup \mathcal{S}')) = M'|(\{e\} \cup (\cup \mathcal{S}'))$ and M_2/e is a B_2-clique for some $B_2 \subseteq B'$, while $\cup \mathcal{S}'$ spans M_2.

5.3.2.1. *There is a collection $\mathcal{Y}$ of m_1 pairwise-disjoint 2-subsets of B_2 so that each $Y \in \mathcal{Y}$ spans a nontrivial parallel class of M_2/e which contains neither element of Y.*

Proof. Let $\mathcal{Y}$ be a maximal such collection of subsets of B_2, and assume for a contradiction that $|\mathcal{Y}| < m_1$. Since each element of $\cup \mathcal{S}'$ is in a nontrivial parallel class of M_2/e, the maximality of $|\mathcal{Y}|$ implies that each nonloop element of $(\cup \mathcal{S}') - (\cup \mathcal{Y})$ is parallel in $M_2/e/(\cup \mathcal{Y})$ to an element of $B_2 - (\cup \mathcal{Y})$. Since $r_{M_2/e}(\cup \mathcal{Y}) \leq 2(m_1 - 1) < |\mathcal{S}'|$ and the sets in $\mathcal{S}'$ are mutually skew in M_2/e, there is some set $S \in \mathcal{S}'$ so that $(M_2/e/(\cup \mathcal{Y}))|S = (M_2/e)|S$. But since $(M_2/e)|S$ contains a rank-2 circuit, this implies that three elements of $B_2 - (\cup \mathcal{Y})$ are in a circuit in $(M_2/e/(\cup \mathcal{Y}))$, which contradicts that B_2 is independent in M_2/e. □

Lemma 2.2.2 and 5.3.2.1 and the definition of m_1 imply that M_2 has a rank-n doubled-clique minor, a contradiction. □

5.4 Porcupines

The following lemma shows that any matroid with a large independent set so that each element is the tip of a g-porcupine is not a bounded distance from a frame matroid. We will apply this with $h = h_{1.6.2}(\ell, n)$ to find a rank-n projective-geometry minor. Recall that if P is a porcupine with tip f, then we write $d(P)$ for the corank of $\mathrm{si}(P/f)$.

Proposition 5.4.1. *For each integer $h \geq 0$, if M is a matroid with a size-$(h+1)$ independent set S so that each element is the tip of a $(5 \cdot 2^h)$-porcupine P with $d(P) = h + 1$, then there are no sets $C_1, C_2 \subseteq E(M)$ with $r_M(C_1 \cup C_2) \leq h$ and a frame matroid N on ground set $E(M)$ such that*

(*) for all $X \subseteq E(M) - (C_1 \cup C_2)$, if $(M/(C_1 \cup C_2))|X$ is simple, then $N|X = (M/C_1)|X$.

Proof. Let $e \in S - \mathrm{cl}_M(C_1 \cup C_2)$, and note that e is a nonloop of N by (*). We will show that e is the tip of a spike of rank at least five in N. Let P be a $(5 \cdot 2^h)$-porcupine restriction of M with tip e and $d(P) = h + 1$, and let $(P/e)|T_0$ be a simplification of P/e. By Lemma 4.2.6 there is some $T_1 \subseteq T_0$ so that $(M/e/(C_1 \cup C_2))|T_1$ has corank $h + 1$ and girth at least five. In particular, $(M/e/(C_1 \cup C_2))|T_1$ is simple.

Then $(M/e/C_1)|T_1$ has corank at least one and girth at least five, since $r_M(C_2) \leq h$. So there is some $T_2 \subseteq T_1$ so that $(M/C_1/e)|T_2$ is a circuit of size at least five. Let $S \subseteq E(P)$ be the union of lines of P through e and an element of T_2.

5.4.1.1. *$(M/C_1)|S$ is a spike of rank at least five with tip e, and $(M/C_1/e)|T_2$ is a simplification of $(M/C_1/e)|(S-\{e\})$.*

Proof. First note that each line of $P|S$ through e has rank two in M/C_1, or else $(M/C_1/e)|T_2$ contains a loop. Also, each line of $P|S$ through e is a line of $(M/C_1)|S$, or else two elements of $(M/C_1/e)|T_2$ are parallel. Since $(M/C_1/e)|T_2$ is a circuit of size at least five, the claim holds. □

Since $(M/e/(C_1 \cup C_2))|T_2$ is simple, no two elements of S are parallel in $M/(C_1 \cup C_2)$, and S is disjoint from $\mathrm{cl}_{M/C_1}(C_2)$. Thus, $(M/(C_1 \cup C_2))|S$ is simple. By (*) we have $N|S = (M/C_1)|S$, so N has a spike restriction of rank at least five, which contradicts that N is a frame matroid. □

Theorems 1.6.2 and Proposition 5.4.1 combine to give the following corollary, which we will use in the proof of Theorem 1.7.2 to find a rank-n doubled-clique minor.

Corollary 5.4.2. *Let $\ell, n \geq 2$ be integers, and let $h = h_{1.6.2}(\ell, n)$. If $M \in \mathcal{U}(\ell)$ is a matroid with a spanning B-clique restriction, and a size-$(h+1)$ independent set S so that each element is the tip of a $(5 \cdot 2^h)$-porcupine P with $d(P) = h+1$, then M has a rank-n projective-geometry minor.*

5.5 The Main Proof

We are now ready to prove Theorem 1.7.2. The proof applies Theorems 3.1.3, 3.6.1 and 4.0.1 in that order, and then uses the results of Sections 5.2-5.4.

Theorem 1.7.2. *There is a function $f_{1.7.2} : \mathbb{Z}^4 \to \mathbb{Z}$ so that for all integers $\ell \geq 2$, $t \geq 1$ and $n, k \geq 3$, if $\mathcal{M}$ is a minor-closed class of matroids such that $U_{2,\ell+2} \notin \mathcal{M}$, then either*

- *$\mathcal{M}$ contains a rank-n doubled clique, or*
- *$\mathcal{M}$ contains $\mathrm{DG}(k, \Gamma)$ for some group Γ with $|\Gamma| \geq t$, or*
- *each $M \in \mathcal{M}$ satisfies $\varepsilon(M) \leq (t-1)\binom{r(M)}{2} + f_{1.7.2}(\ell, n, k, t) \cdot r(M)$.*

Proof. Let $h_0 = h_{1.6.2}(\ell, n)$, let $m_0 = m_{5.3.2}(\ell, n)$, and let $h_1 = h_{5.2.3}(\ell, k, n)$. Then let $h = \max(h_0, m_0, h_1)$; we will use h to apply Theorem 3.6.1. Let $m_1 = \max(4m_0, 15h_1 2^h)$, which will be an upper bound on the rank of a stack or collection of small spikes which we find. Let

$$n_0 = \max\Big(k, r_{4.0.1}(\ell, 3, h_0+1, n, h_0+1, 5 \cdot 2^{h_0}), r_{4.5.7}(\ell, m_1, n)\Big),$$

which is the rank of a Dowling-geometry minor that we will find, and let

$$s_1 = \max\Big(s_{4.0.1}(h_0+1, h_0+1, 5\cdot 2^{h_0}) + h2^{h+7}, m_1 + 2 + h2^{h+7}, 2^{15h}\Big).$$

Define

$$f_1 = \max\Big(f_{5.1.1}(\ell, n, k), \ell^{28h}, \alpha_{1.3.5}(\ell, n_0 + h2^{h+7})\Big),$$

to handle matroids with a spanning clique restriction and the case $t = 1$, and

$$r_1 = r_{3.1.3}\Big(\frac{t-1}{2}, f_1, 0, \ell, 1, s_1\Big),$$

and finally

$$f_{1.7.2}(\ell, t, n, k) = \max(\ell^{r_1}, f_1).$$

Let $p(r) = (t-1)\binom{r}{2} + f_{1.7.2}(\ell, t, n, k)\cdot r$. Assume that the third outcome of the theorem statement does not hold, and that $\mathcal{M}$ contains no rank-n doubled clique.

Let $M \in \mathcal{M}$ be a matroid so that $\varepsilon(M) > p(r(M))$. Then $r(M) \geq r_1$, by the definition of $f_{1.7.2}(\ell, t, n, k)$ and the fact that $M \in \mathcal{U}(\ell)$. We will show that M has a $\mathrm{DG}(k, \Gamma)$-minor with $|\Gamma| \geq t$. By Theorem 3.1.3 with $r = 1$ and $s = s_1$, the matroid M has a minor N so that $r(N) \geq 1$ and $\varepsilon(N) > p(r(N))$, while N either has a spanning clique restriction, or is vertically s_1-connected and has an s_1-element independent set S so that each $e \in S$ satisfies $\varepsilon(N) - \varepsilon(N/e) > p(r(N)) - p(r(N)-1)$. Since $\varepsilon(N) > p(r(N))$ and $f_1 \geq f_{5.1.1}(\ell, n, k)$, the matroid N does not have a spanning clique restriction by Theorem 5.1.1. Since $\varepsilon(N) > \alpha_{1.3.5}(\ell, n_0 + h2^{h+7})\cdot r(N)$ by the definition of f_1, the matroid N has an $M(K_{n_0+1+h2^{h+7}})$-minor G by Theorem 1.3.5. Since $n_0 \geq k$, this proves the result in the case that $t = 1$, so we may assume that $t \geq 2$.

By Theorem 3.6.1 with $h = \max(h_0, m_0, h_1)$ and the facts that $f_1 \geq \ell^{28h}$ and $s_1 \geq 2^{15h}$, there is some $C \subseteq E(N)$ with $r_N(C) \leq h2^{h+7}$ so that N/C has either

(i) an $(\mathcal{F} \cap \mathcal{U}(t), 15\cdot 2^h, h_1)$-stack restriction,

(ii) an element e and a collection $\mathcal{S}$ of m_0 mutually skew sets in $N/(C \cup \{e\})$ such that for each $R \in \mathcal{S}$, the matroid $(N/C)|(R \cup \{e\})$ is a spike of rank at most four with tip e, or

(iii) a size-(h_0+1) independent set so that each element is the tip of a $(5\cdot 2^{h_0})$-porcupine restriction P of N/C with $d(P) = h_0 + 1$.

Note that N/C has an $M(K_{n_0+1})$-minor since $r_N(C) \leq h2^{h+7}$ and N has an $M(K_{n_0+1+h2^{h+7}})$-minor. Also, N/C is vertically $(s_1 - h2^{h+7})$-connected since N is vertically s_1-connected and $r_M(C) \leq h2^{h+7}$. Note that $s_1 - h2^{h+7} \geq m_1 + 2$.

If (i) holds, then N/C has a rank-n projective-geometry minor or a $\mathrm{DG}(k,\Gamma)$-minor with $|\Gamma| \geq t$ by Theorem 4.5.7 with $m = m_1$ and Proposition 5.2.3. If (ii) holds, then N/C has a rank-n doubled-clique minor by Theorem 4.5.7 with $m = m_1$ and Proposition 5.3.2.

If (iii) holds, then by Theorem 4.0.1 with $s = k = h_0 + 1$ and $g = 5 \cdot 2^{h_0}$ and $m = 3$, the matroid N/C has a minor with a spanning clique restriction and a size-(h_0+1) independent set so that each element is the tip of an $5 \cdot 2^{h_0}$-porcupine P with $d(P) = h_0+1$. By Corollary 5.4.2 and the definition of h_0, the matroid N/C has a rank-n projective-geometry minor and thus a rank-n doubled-clique minor. □

We conclude this chapter by proving Theorem 1.7.3, which shows that there are dichotomies among minor-closed classes which exclude a line and all rank-n doubled cliques. Recall that a Γ-*frame matroid* is a frame matroid associated with a directed graph whose edges are labeled by elements of Γ.

Theorem 1.7.3. *For all integers $\ell \geq 2$ and $n \geq 3$, if $\mathcal{M}$ is a minor-closed class of matroids so that $U_{2,\ell+2} \notin \mathcal{M}$ and $\mathcal{M}$ contains no rank-n doubled clique, then there is a constant $c_{\mathcal{M}}$ so that either*

(1) $h_{\mathcal{M}}(r) \leq c_{\mathcal{M}} \cdot r$ for all $r \geq 0$, or

(2) there is a finite group Γ so that $|\Gamma|\binom{r}{2} + r \leq h_{\mathcal{M}}(r) \leq |\Gamma|\binom{r}{2} + c_{\mathcal{M}} \cdot r$ for all $r \geq 0$, and $\mathcal{M}$ contains all Γ-frame matroids.

Proof. We may assume that $\mathcal{M}$ contains all graphic matroids, or else (1) holds by Theorem 1.3.5. Let $t \in \mathbb{Z}$ be maximal so that $\mathcal{M}$ contains a rank-k Dowling geometry with group size t for infinitely many integers k. Since there are finitely many groups of size t, there is some group Γ such that $|\Gamma| = t$ and $\mathcal{M}$ contains $\mathrm{DG}(k,\Gamma)$ for infinitely many integers k. Since $\mathrm{DG}(n,\Gamma)$ is a minor of $\mathrm{DG}(n+1,\Gamma)$ for all $n \geq 3$, this shows that M contains all Dowling geometries over Γ. By Theorem 1.7.2 and maximality of t, there is a constant $c_{\mathcal{M}}$ so that $h_{\mathcal{M}}(r) \leq |\Gamma|\binom{r}{2} + c_{\mathcal{M}} \cdot r$ for all $r \geq 0$.

Using the fact that if $M \cong \mathrm{DG}(r(M),\Gamma)$, then $\mathrm{si}(M/e) \cong \mathrm{DG}(r(M)-1,\Gamma)$, it is not hard to show that $\mathrm{DG}(2n+1,\Gamma)$ has a minor N so that $\mathrm{si}(N) \cong \mathrm{DG}(n,\Gamma)$, and each parallel class of N has size at least two. This can be done by contracting one element b of a frame of $\mathrm{DG}(2n+1,\Gamma)$, and a set of elements which forms a maximum matching of the Γ-labeled graph associated with $\mathrm{DG}(2n+1,\Gamma)/b$. Combined with the fact that each simple rank-n Γ-frame matroid is a restriction of $\mathrm{DG}(n,\Gamma)$, this shows that each Γ-frame matroid is a minor of a sufficiently large Dowling geometry over Γ. Thus, $\mathcal{M}$ contains all Γ-frame matroids. □

Chapter 6

An Exact Theorem

The remainder of this thesis deals with applications of Theorem 1.7.2. In this chapter we use Theorem 1.7.2 to prove Theorem 1.7.6, which we restate below for convenience. Recall that a matroid M is a nontrivial extension of a Dowling geometry if M is simple, and has no coloops.

Theorem 1.7.6. *For all integers $t \geq 1$, $\ell \geq 2$, and $k, n \geq 3$, if $\mathcal{M}$ is a minor-closed class of matroids so that $U_{2,\ell+2} \notin \mathcal{M}$, then either*

- *$\mathcal{M}$ contains a rank-n doubled clique, or*
- *$\mathcal{M}$ contains a nontrivial extension of $\mathrm{DG}(k, \Gamma)$ with $|\Gamma| \geq t$, or*
- *each $M \in \mathcal{M}$ with sufficiently large rank satisfies $\varepsilon(M) \leq t\binom{r(M)}{2} + r(M)$. Moreover, if $r(M)$ is sufficiently large and $\varepsilon(M) = t\binom{r(M)}{2} + r(M)$, then $\mathrm{si}(M)$ is isomorphic to a Dowling geometry.*

We first use Theorem 1.7.2 to prove an analogue of Theorem 3.1.3 for classes which exclude a doubled-clique minor, in Section 6.1. This is enough to prove the upper bound in Theorem 1.7.6, but to prove uniqueness of Dowling geometries we need some properties of matroids for which each bounded-rank restriction is a frame matroid, and we prove these in Section 6.2.

6.1 A New Connectivity Reduction

In this section we prove an analogue of Theorem 3.1.3 for matroids with no doubled-clique minor, where the spanning minor we find is a Dowling geometry, not a clique. The proof is essentially identical to the proof of Theorem 3.1.3, except we apply Theorem 1.7.2 instead of Theorem 1.3.5.

Theorem 6.1.1. *There is a function $r_{6.1.1}\colon \mathbb{R}^8 \to \mathbb{Z}$ so that for all integers $\ell, k, t \geq 2$ and $r, s \geq 1$ and any real polynomial $p(x) = ax^2 + bx + c$ with $a > \frac{t-1}{2}$, if $M \in \mathcal{U}(\ell)$ has no rank-k doubled-clique minor and satisfies $r(M) \geq r_{6.1.1}(a, b, c, \ell, t, k, r, s)$ and $\varepsilon(M) > p(r(M))$, then M has a minor N with $\varepsilon(N) > p(r(N))$ and $r(N) \geq r$ such that either*

(1) N has a $\mathrm{DG}(r(N), \Gamma)$-restriction with $|\Gamma| \geq t$, or

(2) N is vertically s-connected and has an s-element independent set S so that each $e \in S$ satisfies $\varepsilon(N) - \varepsilon(N/e) > p(r(N)) - p(r(N) - 1)$.

Proof. We first define the function $r_{6.1.1}$. Let $\nu = \nu_{3.1.1}(a, b, c, \ell, k, r, s)$, and define $\hat{r}_1$ to be an integer so that

$$(2s + 1)a(x + y) + s(\nu + b) + c - as^2 \leq 2axy$$

and $p(x-s) \le p(x-s+1)$ for all real $x, y \ge \hat{r}_1$. Let f be a function which takes in an integer m and outputs an integer $f(m) \ge \max(r, 2m, 2\hat{r}_1)$ such that $p(x) - p(x-1) \ge ax + \ell^{\max(m,\hat{r}_1)}$ for all real $x \ge f(m)$. Define $r_{\lceil \nu/a \rceil} = 1$, and for each $i \in \{0, 1, 2, \dots, \lceil \nu/a \rceil - 1\}$ recursively define r_i to be an integer so that

$$p(x) > (t-1)\binom{x}{2} + f_{1.7.2}\Big(\ell, k, n_{3.1.2}\big(a, b, c, \ell, f(r_{i+1}), s\big), t\Big) \cdot x$$

for all $x \ge r_i$. Such an integer r_i exists because $a > \frac{t-1}{2}$. Finally, define $r_{6.1.1}(a, b, c, \ell, t, k, r, s) = r_0$.

Let $M \in \mathcal{U}(\ell)$ with no rank-k doubled-clique minor such that $r(M) \ge r_0$ and $\varepsilon(M) > p(r(M))$. Let $\mathcal{M}$ denote the class of minors of M. We may assume that $h_{\mathcal{M}}(n) \le p(n) + \nu n$ for all $n \ge 1$, or else (2) holds by Lemma 3.1.1. The following claim essentially finds some ν' so that the coefficient of the linear term of $h_{\mathcal{M}}(n)$ is in the interval $[\nu' + b - a, \nu' + b + a]$.

6.1.1.1. *There is some $0 \le \nu' < \nu$ and $i \ge 0$ so that $h_{\mathcal{M}}(n) > p(n) + \nu' n$ for some $n \ge r_i$, and $h_{\mathcal{M}}(n) \le p(n) + (\nu' + a)n$ for all $n \ge r_{i+1}$.*

Proof. We will break up the real interval $[0, \nu]$ into subintervals of size a. Define $\nu_i = ai$ for $i \in \{0, 1, 2, \dots, \lceil \frac{\nu}{a} \rceil\}$. Let $i \ge 0$ be minimal so that $h_{\mathcal{M}}(n) \le p(n) + \nu_{i+1} n$ for all $n \ge r_{i+1}$. This choice of i is well-defined, because $i = \lceil \nu/a \rceil - 1$ is a valid choice since $\nu_{\lceil \nu/a \rceil} \ge \nu$ and $h_{\mathcal{M}}(n) \le p(n) + \nu n$ for all $n \ge 1 = r_{\lceil \nu/a \rceil}$.

If $i > 0$, then $h_{\mathcal{M}}(n) > p(n) + \nu_i$ for some $n \ge r_i$ by the minimality of i. If $i = 0$, then M certifies that $h_{\mathcal{M}}(n) > p(n)$ for some $n \ge r_0$. Thus, there is some $i \ge 0$ so that $h_{\mathcal{M}}(n) > p(n) + \nu_i n$ for some $n \ge r_i$, and $h_{\mathcal{M}}(n) \le p(n) + \nu_{i+1} n$ for all $n \ge r_{i+1}$. Since $\nu_i + a = \nu_{i+1}$, we may choose $\nu' = \nu_i$. Note that $\nu_i = ai < \nu$ since $i \le \lceil \frac{\nu}{a} \rceil - 1$. □

By 6.1.1.1, M has a minor M_1 such that $r(M_1) \ge r_i$ and $\varepsilon(M_1) > p(r(M_1)) + \nu' r(M_1)$. By Theorem 1.7.2 and definition of r_i, the matroid M_1 has a $\mathrm{DG}(n_{3.1.2}(a, b, c, \ell, f(r_{i+1}), s), \Gamma)$-minor with $|\Gamma| \ge t$. Then by Lemma 3.1.2 with $r = f(r_{i+1})$ and $q = p + \nu'$, the matroid M_1 has a minor N such that $r(N) \ge f(r_{i+1})$ and $\varepsilon(N) > p(r(N)) + \nu' r(N)$, and N either has a $\mathrm{DG}(r(N), \Gamma)$-restriction or an s-element independent set S so that each $e \in S$ satisfies $\varepsilon(N) - \varepsilon(N/e) > p(r(N)) - p(r(N) - 1) + \nu'$. We may assume that N is simple. Since $f(r_{i+1}) \ge r$ and $\nu' \ge 0$ we may assume that N is not vertically s-connected, or else either (1) or (2) holds.

Let (A, B) be a partition of $E(N)$ so that $r_N(A) \le r_N(B) < r(N)$ and $r_N(A) + r_N(B) - r(N) < s - 1$. Let $r_N = r(N)$ and $r_A = r_N(A)$ and $r_B = r_N(B)$. We first show that $r_A \ge \max(\hat{r}_1, r_{i+1})$. If not, then $r_B \ge r_N - r_A \ge \max(r_{i+1}, \hat{r}_1)$, using that $r_N \ge f(r_{i+1}) \ge$

$\max(2r_{i+1}, 2\hat{r}_1)$. Also,

$$|B| = |N| - |A| > p(r_N) + \nu' r_N - \ell^{\max(\hat{r}_1, r_{i+1})} \quad (1)$$
$$\geq p(r_N - 1) + (\nu' + a)r_N \quad (2)$$
$$\geq p(r_B) + (\nu' + a)r_B. \quad (3)$$

Line (1) holds because $r_A < \max(\hat{r}_1, r_{i+1})$ and $N \in \mathcal{U}(\ell)$, and line (2) holds because $r_N \geq f(r_{i+1})$. Line (3) holds because $r_B \geq \hat{r}_1$, so $p(r_B) \leq p(r_N - 1)$ since $r_B \leq r_N - 1$. But then $r_B \geq r_{i+1}$ and $|B| > p(r_B) + (\nu' + a)r_B$, which contradicts 6.1.1.1 and the choice of ν'. Thus, $r_B \geq r_A \geq \max(\hat{r}_1, r_{i+1})$. Then

$$p(r_A + r_B - s) + \nu'(r_A + r_B - s) \leq p(r_N) + \nu' r_N < |A| + |B| \leq p(r_A) + p(r_B) + (\nu' + a)(r_A + r_B),$$

where the first inequality holds because $r_A + r_B - s \leq r_N$ and $p(x - s) \leq p(x - s + 1)$ for all $x \geq \hat{r}_1$, and the last inequality holds by 6.1.1.1 because $r_B \geq r_A \geq r_{i+1}$. Expanding $p(x) = ax^2 + bx + c$ and simplifying, we have

$$(2s + 1)a(r_A + r_B) + s(\nu' + b) + c - as^2 > 2ar_A r_B,$$

which contradicts that $r_A \geq \hat{r}_1$, since $\nu' < \nu$. □

6.2 Locally Frame Matroids

In this section we prove a lemma which will help prove the uniqueness of Dowling geometries in Theorem 1.7.6. We first need a straightforward lemma about frame matroids.

Lemma 6.2.1. *If M is a simple frame matroid and $e \in E(M)$, then no element on a line through e of length at least four is the tip of a spike in M/e.*

Proof. If f is on a line of M through e of length at least four in M, then f is parallel to a frame element in M/e, by Lemma 1.2.3 (i). Thus, f is not the tip of a spike in M/e by Lemma 1.2.3 (iii). □

The following proposition is the main result of this section, and relies on Proposition 3.4.5. We freely use the fact that if M is framed by B and $e \in E(M)$ is on at least two lines of M of length at least four, then e is parallel to an element of B.

Proposition 6.2.2. *Let $t \geq 1$ be an integer, and let M be a simple matroid of rank at least seven so that $|M| = t\binom{r(M)}{2} + r(M)$ and there is some $e \in E(M)$ so that $\mathrm{si}(M/e) \cong \mathrm{DG}(r(M) - 1, \Gamma)$ for some group Γ with $|\Gamma| = t$. Then either*

(1) there is a set $X \subseteq E(M)$ for which $r_M(X) \leq 15$ and $M|X$ is either not a frame matroid, or has a $U_{2,t+3}$-minor, or

(2) e is the tip of a spike of rank at least five, or

(3) $M \cong \mathrm{DG}(r(M), \Gamma)$.

Proof. Assume for a contradiction that (1), (2), and (3) do not hold for M.

6.2.2.1. *There is no $B \subseteq E(M)$ so that M is framed by B.*

Proof. If the claim is false, then, since M has no $U_{2,t+3}$-restriction and $\varepsilon(M) = t\binom{r(M)}{2}+r(M)$, each pair of elements of B spans a $U_{2,t+2}$-restriction of M. Since no rank-4 subset of $E(M)$ has a $U_{2,t+3}$-minor, Corollary 2.1.2 implies that there is a group Γ' so that $M \cong \mathrm{DG}(r(M), \Gamma')$. Then $\mathrm{si}(M/e) \cong \mathrm{DG}(r(M) - 1, \Gamma')$, which implies that $\Gamma' \cong \Gamma$, since Dowling geometries are isomorphic if and only if their groups are isomorphic. Thus, $M \cong \mathrm{DG}(r(M), \Gamma)$ and (3) holds, a contradiction. □

We now reduce to the case that $t \geq 2$. Note that $\varepsilon(M) - \varepsilon(M/e) = t(r(M) - 1) + 1$.

6.2.2.2. *$t \geq 2$.*

Proof. Assume for a contradiction that $t = 1$. Since (1) does not hold, each restriction of M of rank at most 15 is binary, by Theorem 1.1.1. Since every binary spike has an F_7-minor and F_7 is not a frame matroid by Proposition 1.2.1, binary spikes are not frame matroids. This implies that e is not the tip of a spike, since (1) and (2) do not hold. Since $\varepsilon(M) - \varepsilon(M/e) = r(M)$ and M has no $U_{2,4}$-restriction, there are $r(M) - 1$ lines of length three of M through e. We claim that each transversal B of the long lines of M through e is a frame for M/e. Since e is not the tip of a spike in M and $\mathrm{si}(M/e) \cong M(K_n)$, the set B corresponds to a spanning tree of M/e. If this tree has a path P with three edges, then there is some f so that $P \cup \{f\}$ is a size-4 circuit. But then e is the tip of a rank-3 spike in M/f, and (1) holds since binary spikes are not frame matroids. Thus, B corresponds to a spanning star of K_n and is a thus a frame for M/e. In particular, this implies that each pair F, F' of long lines of M through e satisfies $|\operatorname{cl}_M(F \cup F')| \geq \varepsilon_{M/e}(\operatorname{cl}_{M/e}(F \cup F')) + 3 = 6$, since $\mathrm{si}(M/e) \cong M(K_n)$.

We will choose a specific transversal B' of the long lines of M through e, and show that M is framed by $B' \cup \{e\}$. Let F_1 be a long line of M through e, and let $b_1 \in F_1 - \{e\}$. Let F be a long line through e other than F_1. Since $|\operatorname{cl}_M(F_1 \cup F)| = 6$ and (1) does not hold, we have $M|\operatorname{cl}_M(F_1 \cup F) \cong M(K_4)$. Since e and b_1 are in a triangle, the matroid $M|\operatorname{cl}_M(F_1 \cup F)$ has a unique frame containing e and b_1, and the third frame element b_F is on $F - \{e\}$. Then $\{b_1\} \cup \{b_F \colon F \neq F_1$ is a long line through $e\}$ is a transversal B' of the long lines of M through e.

We claim that $B' \cup \{e\}$ is a frame for M. Clearly $B' \cup \{e\}$ is a basis for M since B' is a basis for M/e. Let $x \in E(M)$. Since M/e is framed by B', there are elements $b, b' \in B'$ so that $x \subseteq \mathrm{cl}_M(\{e, b, b'\})$. If $b_1 \in \{b, b'\}$, then $B' \cup \{e\}$ contains a frame for $M|\,\mathrm{cl}_M(\{e, b, b'\})$ and x is spanned by two elements of $\{e, b, b'\}$, so we may assume that $b_1 \notin \{b, b'\}$. Then $r_M(\{e, b_1, b, b'\}) = 4$, and $|\,\mathrm{cl}_M(\{e, b_1, b, b'\})| = \varepsilon_{M/e}(\mathrm{cl}_{M/e}(\{b_1, b, b'\})) + 4 = 10$, since B' frames M/e. Since (1) does not hold, this implies that $M|\,\mathrm{cl}_M(\{e, b_1, b, b'\}) \cong M(K_5)$. Since e and b_1 are in a triangle, this matroid has a unique frame B_1 containing e and b_1. This set B_1 contains the unique frame for $M|\,\mathrm{cl}_M(\{e, b_1, b\})$ and $M|\,\mathrm{cl}_M(\{e, b_1, b'\})$, and so $b, b' \in B_1$. Since $r_M(B_1) = 4$ and $e, b_1, b, b' \in B_1$, we have $B_1 = \{e, b_1, b, b'\} \subseteq B' \cup \{e\}$. Thus, x is spanned by two elements of $B' \cup \{e\}$. Therefore, M is framed by $B' \cup \{e\}$, which contradicts 6.2.2.1. □

Since $t \geq 2$, we can exploit that fact that any rank-3 frame matroid with $3t + 3$ elements and no $U_{2,t+3}$-restriction has three elements each on two lines of length at least four, and thus has a unique frame. Let $(M/e)|T_1$ be a simplification of M, and let $T \subseteq T_1$ be a transversal of the nontrivial parallel classes of M/e. Let B be a frame for $(M/e)|T_1$. There are two distinct cases for the structure of the long lines of M through e.

6.2.2.3. *If e is not the tip of a spike, then e is on $r(M) - 1$ lines of length $t + 2$ so that each contains an element of B.*

Proof. Recall that $\varepsilon(M) - \varepsilon(M/e) = t(r(M) - 1) + 1$. If e is not the tip of a spike, then e is on $r(M) - 1$ lines of length $t + 2$ since M has no $U_{2,t+3}$-restriction. Let F be a line of M through e. If $F \cap B = \varnothing$, then there is an element of F which is the tip of a spike S of rank at most four in M/e, since each element of $(M/e)|(T_1 - B)$ is the tip of a spike of rank at most four in M since $(M/e)|T_1 \cong \mathrm{DG}(r(M/e), \Gamma)$. But then $M|(F \cup S)$ is not a frame matroid by Lemma 6.2.1 and (1) holds, a contradiction. □

The structure is a bit more complex when e is the tip of a spike.

6.2.2.4. *If e is the tip of a spike, then $(M/e)|T$ has a star-partition $(\mathcal{L}, \{x\})$ so that $|\,\mathrm{cl}_M(\{e, x\})| = t+2$, and $|L| = t$ for each $L \in \mathcal{L}$. Moreover, $x \in B$ and for each $b \in B - \{x\}$ there some $L \in \mathcal{L}$ so that $L \subseteq \mathrm{cl}_M(\{e, x, b\})$.*

Proof. Since (1) and (2) do not hold, by Proposition 3.4.5 with $g = 5$ there is a star-partition $(X, \mathcal{L})$ of $(M/e)|T$ so that each line of M through e and an element of $T - X$ has length three. Note that each $L \in \mathcal{L}$ satisfies $|L| \leq t + 2$, or else (1) holds. Then $|\mathcal{L}| \leq r(M) - 1$, or else (2) holds by taking the union of lines through e and each element of a transversal of $\mathcal{L}$. Then there is some $L \in \mathcal{L}$ so that $|L| \geq 2$, or else $\varepsilon(M) - \varepsilon(M/e) \leq (r(M) - 1) + 2t + 2 <$

$t(r(M)-1)+1$, using that $r(M) \geq 7$. This implies that $|L| \leq t$ for all $L \in \mathcal{L}$, or else (1) holds by Lemma 3.5.1. Letting $m = |\operatorname{cl}_M(\{e\} \cup X)|$, we have

$$\begin{aligned} t(r(M)-1)+1 &= \varepsilon(M) - \varepsilon(M/e) \\ &= \sum_{L \in \mathcal{L}} |L| + m - 1 \\ &\leq t(r(M)-2) + m - 1, \end{aligned}$$

which implies that $m \geq t+2$ and each $L \in \mathcal{L}$ satisfies $|L| = t$.

By the same reasoning as in 6.2.2.3, $\operatorname{cl}_M(\{e,x\}) \cap B \neq \varnothing$. Since $B \subseteq T_1$ and $\operatorname{cl}_M(\{e,x\}) \cap T_1 = \{x\}$ we have $x \in B$. Fix some $L \in \mathcal{L}$. By the definition of a star-partition, the set $\{x\} \cup L$ is a line of M/e of length $t+1 \geq 3$, and thus spans some element $b_L \in B - \{x\}$ in M/e since $x \in B$. Thus, $L \subseteq \operatorname{cl}_M(\{e,x,b_L\})$, so for each $L \in \mathcal{L}$ there is some $b_L \in B - \{x\}$ so that $\{e,x,b_L\}$ spans L in M. If $L \neq L'$ then $b_L \neq b_{L'}$, or else $L \cup X$ is not a flat of $(M/e)|T$, which contradicts the definition of a star-partition. Since $|\mathcal{L}| = |B - \{x\}| = r(M) - 2$, for each $b \in B - \{x\}$ there is a unique $L \in \mathcal{L}$ so that $L \subseteq \operatorname{cl}_M(\{e,x,b\})$. □

Let $b_1 \in B$ be on a line of length $t+2$ through e; such an element exists by 6.2.2.3 and 6.2.2.4. We will show that for each $b \in B - \{b_1\}$, the matroid $N = M|\operatorname{cl}_M(\{e,b_1,b\}))$ is a frame matroid with a unique frame. Note that $\varepsilon_{M/e}(\operatorname{cl}_{M/e}(\{b_1,b\})) = t+2$ since $\operatorname{si}(M/e)$ is isomorphic to a Dowling geometry. If e is not the tip of a spike of M, then since B is a transversal of the long lines of M through e by 6.2.2.3 we have $\varepsilon(N) = (t+2)+2t+1 = 3t+3$, and thus N has a unique frame since it has no $U_{2,t+3}$-restriction. If e is the tip of a spike of M, then by 6.2.2.4 the point e is on t lines of length three and a line of length $t+2$ in N, so $\varepsilon(N) = (t+2)+2t+1 = 3t+3$ and again N has a unique frame.

We now define a frame for M. Fix some element $b_2 \in B - \{b_1\}$, and let $\{e', b_1', b_2'\}$ be the unique frame for $M|\operatorname{cl}_M(\{e,b_1,b_2\})$, where $\{e',b_1'\}$ spans $\{e,b_1\}$. Define a function f from $B - \{b_1\}$ to $E(M)$ which maps b to the unique element $f(b)$ so that $\{e', b_1', f(b)\}$ is a frame for $M|\operatorname{cl}_M(\{e,b_1,b\})$. We will show that $\hat{B} = \{e',b_1'\} \cup \{f(b) \colon b \in B - \{b_1\}\}$ is a frame for M. Clearly $|\hat{B}| \leq r(M)$ and $B \cup \{e\} \subseteq \operatorname{cl}_M(\hat{B})$ by the definition of f, so $\hat{B}$ is a basis of M. Let $x \in E(M)$, so $x \in \operatorname{cl}_M(\{e,b_i,b_j\})$ for some $b_i, b_j \in B$, since B is a frame for M/e. Then $x \in \operatorname{cl}_M(\{e', b_1', f(b_i), f(b_j)\})$, since $\{e,b_1\}$ spans $\{e',b_1'\}$. The matroid $M|\operatorname{cl}_M(\{e',b_1',f(b_i),f(b_j)\})$ is a frame matroid of rank at most four, and the frame contains $\{e',b_1',f(b_i),f(b_j)\}$, as each of these elements is in the unique frame for $M|\operatorname{cl}_M(\{e,b_1,b_i\})$ or $M|\operatorname{cl}_M(\{e,b_1,b_j\})$. Since $\{e',b_1',f(b_i),f(b_j)\}$ spans $\{e,b_1,b_i,b_j\}$ and is contained in a frame, the set $\{e',b_1',f(b_i),f(b_j)\}$ is a frame for $M|\operatorname{cl}_M(\{e,b_1,b_i,b_j\})$. Therefore, x is spanned by two elements of $\hat{B}$. But then M is framed by $\hat{B}$, which contradicts 6.2.2.1. □

6.3 The Proof

In this section we prove Theorem 1.7.6, which we state more precisely below. The proof of the upper bound on the extremal function follows the same outline as the proof of Theorem 1.7.2.

Theorem 6.3.1. *There is a function $r_{6.3.1}\colon \mathbb{Z}^4 \to \mathbb{Z}$ so that for all integers $t \geq 1$, $\ell \geq 2$, and $k, n \geq 3$, if $\mathcal{M}$ is a minor-closed class of matroids so that $U_{2,\ell+2} \notin \mathcal{M}$, then either*

- *$\mathcal{M}$ contains a rank-n doubled clique, or*
- *$\mathcal{M}$ contains a nontrivial extension of $\mathrm{DG}(k, \Gamma)$ with $|\Gamma| \geq t$, or*
- *each $M \in \mathcal{M}$ with $r(M) \geq r_{6.3.1}(\ell, t, k, n)$ satisfies $\varepsilon(M) \leq t\binom{r(M)}{2} + r(M)$. Moreover, if $r(M) \geq r_{6.3.1}(\ell, t, k, n)$ and $\varepsilon(M) = t\binom{r(M)}{2} + r(M)$, then $\mathrm{si}(M)$ is isomorphic to a Dowling geometry.*

Proof. We first define a sequence of large integers, ending with $r_{6.3.1}(\ell, t, k, n)$. Define $n_0 = \max(r_{4.5.7}(\ell, k, n), r_{4.0.1}(\ell, k, 1, n, 1, 5))$, and define r_0 to be an integer so that $t\binom{r}{2} + r > (t-1)\binom{r}{2} + f_{1.7.2}(\ell, t, n_0, n)r$ for all $r \geq r_0$. This will allow us to find a $\mathrm{DG}(n_0, \Gamma)$-minor with $|\Gamma| \geq t$. Define $r_1 = \max(k, r_0)$, and $r_2 = r_{6.1.1}(\frac{t}{2}, 1, 0, \ell, t, k, r_1, 200)$.

This integer r_2 is large enough to show that $h_{\mathcal{M}}(r) \leq t\binom{r}{2} + r$ for all $r \geq r_2$, but we must define a larger integer to prove that each extremal matroid is a Dowling geometry. Define $n_1 \geq r_2$ to be an integer so that

$$txy \geq 400(x + y) + 200(1 - t/2) - 200^2$$

for all real $x, y \geq n_1$. Define n_2 to be an integer so that

$$\frac{t}{2}(2x - 1) > (1 - t/2) + \ell^{r_2} + \ell^{n_1}$$

for all real $x \geq n_2$. Finally, define $r_{6.3.1}(\ell, t, k, n)$ to be an integer so that $t\binom{r}{2} + r > (t-1)\binom{r}{2} + f_{1.7.2}(\ell, t, n_2, n) \cdot r$ for all $r \geq r_{6.3.1}(\ell, t, k, n)$. We will write $\mathrm{DG}(k, \Gamma) + e$ to denote any nontrivial extension of $\mathrm{DG}(k, \Gamma)$. Assume that $\mathcal{M}$ contains no rank-n doubled clique and no $\mathrm{DG}(k, \Gamma) + e$ with $|\Gamma| \geq t$. The following claim provides sufficient conditions for finding a $(\mathrm{DG}(k, \Gamma) + e)$-minor with $|\Gamma| \geq t$.

6.3.1.1. *Let $N \in \mathcal{M}$ be a vertically 200-connected matroid so that N has a $\mathrm{DG}(n_0, \Gamma)$-minor with $|\Gamma| \geq t$. If N has either a restriction of rank at most 15 which is not in $\mathcal{F} \cap \mathcal{U}(t+1)$ or a spike restriction of rank at least five, then N has a $(\mathrm{DG}(k, \Gamma) + e)$-minor with $|\Gamma| \geq t$.*

Proof. If there is some $X \subseteq E(N)$ so that $r(N|X) \leq 15$ and $N|X \notin \mathcal{F} \cap \mathcal{U}(t+1)$, then by Theorem 4.5.7 with $m = k$ and $s = 15$, the matroid N has a minor N_1 with a spanning $\mathrm{DG}(k, \Gamma)$-restriction so that $N_1|X = N|X$. Since $N_1|X$ is not a restriction of a Dowling geometry with group size at most t, the matroid N_1 has a $(\mathrm{DG}(k, \Gamma) + e)$-restriction.

If N has a spike restriction of rank at least five, then by Theorem 4.0.1 with $m = k$, $s = k = 1$ and $g = 5$, the matroid N has a minor N_2 of rank at least k with a $\mathrm{DG}(r(N_2), \Gamma)$-restriction and a spike restriction of rank at least five, using that $s_{4.0.1}(1, 1, 5) < 200$. Since $r(N_2) \geq k$ and spikes of rank at least five are not frame matroids, N_2 has a $(\mathrm{DG}(k, \Gamma) + e)$-minor. □

We first prove the upper bound on the extremal function for $\mathcal{M}$.

6.3.1.2. *$h_{\mathcal{M}}(r) \leq t\binom{r}{2} + r$ for all $r \geq r_2$.*

Proof. If this is false, let $M \in \mathcal{M}$ so that $r(M) \geq r_2$ and $\varepsilon(M) > t\binom{r(M)}{2} + r(M)$. By Theorem 6.1.1 with $r = r_1$ and $s = 200$, M has a minor N so that $r(N) \geq r_1$ and $\varepsilon(N) > t\binom{r(N)}{2} + r(N)$ and N has either a $\mathrm{DG}(r(N), \Gamma)$-restriction with $|\Gamma| \geq t$, or is vertically 200-connected and has an element e so that $\varepsilon(N) - \varepsilon(N/e) > t(r(N) - 1) + 1$. We may assume that the first outcome does not hold or else N has a $(\mathrm{DG}(r(N), \Gamma) + e)$-restriction, and thus a $(\mathrm{DG}(k, \Gamma) + e)$-minor since $r(N) \geq k$. Since $r(N) \geq r_0$ and $\varepsilon(N) > t\binom{r(N)}{2} + r(N)$, we have $\varepsilon(N) > (t-1)\binom{r(N)}{2} + f_{1.7.2}(\ell, t, n_0, n) \cdot r(N)$, so N has a $\mathrm{DG}(n_0, \Gamma)$-minor with $|\Gamma| \geq t$ by Theorem 1.7.2.

Since N has an element e so that $\varepsilon(N) - \varepsilon(N/e) > t(r(N) - 1) + 1$, by Theorem 3.6.1 with $h = 0$, N either has a restriction of rank at most 15 which is not in $\mathcal{F} \cap \mathcal{U}(t+1)$, or a spike restriction of rank at least five. Here we use the fact that the union of two spikes with tip e which are skew in M/e is not a frame matroid, which follows from Lemma 3.4.1. Then N has a $(\mathrm{DG}(k, \Gamma) + e)$-minor by 6.3.1.1, a contradiction. □

We now prove the uniqueness of Dowling geometries as extremal matroids. Let $M \in \mathcal{M}$ be a simple matroid so that $r(M) \geq r_{6.3.1}(\ell, t, k, n)$ and $|M| = t\binom{r(M)}{2} + r(M)$. Assume for a contradiction that M is not isomorphic to a Dowling geometry. Since $r(M) \geq r_{1.7.2}(\ell, t, n_2, n)$, the matroid M has a $\mathrm{DG}(n_2, \Gamma)$-minor G with $|\Gamma| \geq t$, by Theorem 1.7.2. Let $C \subseteq E(M)$ so that G is a restriction of M/C. Let C_1 be a maximal subset of C so that $\varepsilon(M/C_1) = t\binom{r(M/C_1)}{2} + r(M/C_1)$ and $\mathrm{si}(M/C_1)$ is not isomorphic to a Dowling geometry. Let M_1 be a simplification of M/C_1, and note that $C_1 \neq C$.

6.3.1.3. *M_1 is vertically 200-connected.*

Proof. If not, then there is a partition (A, B) of $E(M_1)$ with $r_{M_1}(A) \leq r_{M_1}(B) < r(M_1)$ so that $r_{M_1}(A) + r_{M_1}(B) < r(M_1) + 200$. Let $r_A = r_{M_1}(A)$ and $r_B = r_{M_1}(B)$. We first show that $r_A \geq n_1$. If not, then

$$\begin{aligned}|M_1| = |A| + |B| &< \ell^{n_1} + \max\left(\ell^{r_2}, t\binom{r_B}{2} + r_B\right)\\ &\leq \ell^{n_1} + \ell^{r_2} + t\binom{r_B}{2} + r_B\\ &< t\binom{r(M_1)}{2} + r(M_1),\end{aligned}$$

a contradiction. The last line holds since $r(M_1) \geq r(G) \geq n_2$ and $r_B < r(M_1)$, and by the definition of n_2. Thus, $r_B \geq r_A \geq n_1 \geq r_2$. Then using that $r_A + r_B - 200 < r(M_1)$, we have

$$\begin{aligned}t\binom{r_A + r_B - 200}{2} + r_A + r_B - 200 &< t\binom{r(M_1)}{2} + r(M_1)\\ &= |M_1| = |A| + |B|\\ &\leq t\binom{r_A}{2} + r_A + t\binom{r_B}{2} + r_B.\end{aligned}$$

After expanding these polynomials and rearranging, this contradicts that $r_B \geq r_A \geq n_1$. □

Let $e \in C - C_1$. By the maximality of C_1, either $\varepsilon(M_1/e) < t\binom{r(M_1/e)}{2} + r(M_1/e)$, or $\mathrm{si}(M_1/e)$ is isomorphic to a Dowling geometry. If $\varepsilon(M_1/e) < t\binom{r(M_1/e)}{2} + r(M_1/e)$, then $\varepsilon(M_1) - \varepsilon(M_1/e) > 1 + t(r(M_1) - 1)$. Then by Theorem 3.6.1 with $h = 0$, the matroid M_1 either has a restriction of rank at most 15 which is not in $\mathcal{F} \cap \mathcal{U}(t+1)$, or a spike restriction of rank at least five. By 6.3.1.1 and 6.3.1.3, this implies that M_1 has a $(\mathrm{DG}(k, \Gamma) + e)$-minor. This is a contradiction, so M_1/e is isomorphic to a Dowling geometry. Then by Proposition 6.2.2, the matroid M_1 either has a restriction of rank at most 15 which is not in $\mathcal{F} \cap \mathcal{U}(t+1)$, or a spike restriction of rank at least five. Again, 6.3.1.1 and 6.3.1.3 imply that M_1 has a $(\mathrm{DG}(k, \Gamma) + e)$-minor with $|\Gamma| \geq t$, a contradiction. Therefore, M is isomorphic to a Dowling geometry. □

We conclude this chapter by stating a consequence of Theorem 6.3.1, which is interesting in light of Theorems 1.5.8 and 1.5.9.

Corollary 6.3.2. *If $\mathcal{M}$ is a minor-closed class of matroids such that $h_{\mathcal{M}}(r) = \binom{r+1}{2}$ for sufficiently large r, then each simple extremal matroid of sufficiently large rank is isomorphic to a clique.*

Proof. There is some integer $n \geq 3$ so that $\mathcal{M}$ contains no rank-n doubled clique, or else $h_{\mathcal{M}}(r) \geq 2\binom{r}{2}+1$ for all $r \geq 3$. There is some integer $k \geq 3$ so that $\mathcal{M}$ contains no nontrivial extension of $M(K_k)$, or else $h_{\mathcal{M}}(r) \geq \binom{r+1}{2}+1$ for all $r \geq 3$. Thus, Theorem 6.3.1 applies. $\square$

Chapter 7

Representable Matroids

In this chapter we show that Theorems 1.7.2 and 1.7.6 apply to many interesting classes of representable matroids. A key fact is that doubled cliques of large enough rank are representable over fields of at most one characteristic, and are not representable over fields of characteristic zero. In Section 7.1 we show that the connection between doubled cliques and representable matroids is due to a class of rank-3 matroids called Reid geometries. Then in Sections 7.2 and 7.3 we use this connection to state some corollaries of Theorems 1.7.2 and 1.7.6 for representable matroids.

7.1 Reid Geometries

A *Reid geometry* is a simple rank-3 matroid R consisting of long lines L_1, L_2, L_3 with a common intersection point x so that $L_3 = \{x, y, z\}$ has length three. The *incidence graph* $I(R)$ of R is the bipartite graph with bipartition $(L_1 - \{x\}, L_2 - \{x\})$ such that $a_1 \in L_1 - \{x\}$ is adjacent to $a_2 \in L_2 - \{x\}$ if and only if $\{a_1, a_2, y\}$ or $\{a_1, a_2, z\}$ is a line of R. Note that $I(R)$ has maximum degree at most two, since each $a_1 \in L_1 - \{x\}$ is on at most one long line of R with each of y and z.

We say that a Reid geometry R is *proper* if $I(R)$ contains a cycle; these are the most interesting Reid geometries. The following lemma is due to Joseph Kung [30], but we include a proof for completeness. We remark that a partial converse is true: if $I(R)$ is a cycle of length $2p$ for a prime p, then R is representable over every field of characteristic p.

Lemma 7.1.1. *For each integer $k \geq 2$, if R is a Reid geometry so that $I(R)$ has a cycle of length $2k$, then R is representable over a field $\mathbb{F}$ only if k is prime and $\mathbb{F}$ has characteristic k.*

Proof. Let $\{u_1, \dots, u_k\} \subseteq L_1 - \{x\}$ and $\{v_1, \dots, v_k\} \subseteq L_2 - \{x\}$ be the elements in a length-$2k$ cycle of $I(R)$ so that $\{u_i, v_i, y\}$ and $\{v_i, u_{i+1}, z\}$ are lines of R for each $i \in [k]$, taking indices modulo k. We may assume that $E(R) = \{x, y, z\} \cup \{u_1, \dots, u_k\} \cup \{v_1, \dots, v_k\}$. Let $\mathbb{F}$ be a field, and consider a representation of R over $\mathbb{F}$ so that $x = [1, 0, 0]^T$, $y = [0, 1, 1]^T$, $z = [a, 1, 1]^T$, $u_i = [c_i, 1, 0]^T$ and $v_i = [d_i, 0, 1]^T$, where a, c_i, and d_i are nonzero elements of $\mathbb{F}$ for each $i \geq 2$, and $c_1 = d_1 = 0$. Since $r_R(\{u_i, v_i, y\}) = 2$ we have $c_i + d_i = 0$ over $\mathbb{F}$ for each $i \in [k]$. Since $r_R(\{v_i, u_{i+1}, z\}) = 2$ we have $c_{i+1} + d_i - a = 0$ over $\mathbb{F}$ for each $i \in [k]$, taking indices modulo k. This tells us that $c_{i+1} = c_i + a$ for each $i \in [k]$. By induction on k it follows that $c_k = (k-1)a$. Then $0 = c_1 = c_k + a = ka$. Since $a \neq 0$ we conclude that $k = 0$ over $\mathbb{F}$, so the characteristic of $\mathbb{F}$ divides k. If the characteristic of $\mathbb{F}$ is some proper factor p of k, then $c_{p+1} = pa = 0$. But then u_1 and u_{p+1} are parallel in R, a contradiction since $p < k$. Thus, k is prime and $\mathbb{F}$ has characteristic k. □

The following lemma shows the connection between Dowling geometries and Reid geometries. Specifically, every nontrivial extension of a Dowling geometry with lines as long as possible has a proper Reid-geometry minor. Recall that a matroid M is a nontrivial extension of a Dowling geometry if M is simple, and has no coloops.

Proposition 7.1.2. *For all integers $\ell \geq 2$ and $k \geq 3$, if $M \in \mathcal{U}(\ell)$ is isomorphic to a nontrivial extension of* $\mathrm{DG}(k, \Gamma)$ *with $|\Gamma| = \ell - 1$, then M has a proper Reid-geometry minor.*

Proof. Let e be an element of M so that $M \backslash e \cong \mathrm{DG}(k, \Gamma)$ with $|\Gamma| = \ell - 1$. Let B be a frame for the spanning $\mathrm{DG}(k, \Gamma)$-restriction of M, and let $B_e \subseteq B$ so that $B_e \cup \{e\}$ is the unique circuit contained in $B \cup \{e\}$ which contains e. Then $|B_e| \geq 3$, or else $\mathrm{cl}_M(B_e)$ has a $U_{2,\ell+2}$-restriction. Let b_1, b_2, b_3 be distinct elements of B_e, and let $N = M/(B - \{b_1, b_2, b_3\})$. Let $L_1 = \mathrm{cl}_N(\{b_1, b_2\})$ and $L_2 = \mathrm{cl}_N(\{b_1, b_3\})$. There is some element y spanned by $\{b_2, b_3\}$ such that $\{b_1, y, e\}$ is a line, or else N/b_1 has a $U_{2,\ell+2}$-restriction. Let $L_3 = \{b_1, y, e\}$, and $R = N|(L_1 \cup L_2 \cup L_3)$, so R is a Reid geometry. Note that each long line of R through e or y other than L_3 contains precisely one element from both L_1 and L_2. Then each element of $(L_1 \cup L_2) - \{b_1\}$ is on a long line with y, or else $\varepsilon(N/y) \geq \varepsilon(R) - (\ell - 1) = \ell + 2$. Similarly, each element of $(L_1 \cup L_2) - \{b_1\}$ is on a long line with e. Thus, each vertex of $I(R)$ has degree at least two in $I(R)$, so $I(R)$ contains a cycle. □

The following lemma shows that every large-rank doubled clique in $\mathcal{U}(\ell)$ has a proper Reid-geometry minor. Together with Lemma 7.1.1, this shows the connection between doubled cliques and representable matroids.

Proposition 7.1.3. *For each integer $\ell \geq 2$, if $M \in \mathcal{U}(\ell)$ is a rank-3^{2^ℓ} doubled clique, then M has a proper Reid-geometry minor.*

Proof. Define a function $g \colon \{2, 3, \ldots, \ell\} \to \mathbb{Z}$ by $g(\ell) = 3$, and $g(s) = 3^{\binom{g(s+1)}{2}}$ for $s < \ell$. It is an easy induction proof to show that $g(s) \leq 3^{(2^{\ell-s+1}-1)}$ for all $s \geq 2$. Let $k = g(2)$, and note that $k \leq 3^{2^\ell}$. For all integers $d \geq 2$ and $r \geq 3$, a *rank-r d-doubled clique* is a simple matroid M with an element t so that $\mathrm{si}(M/t) \cong M(K_r)$, and each parallel class of M/t has size d. Note that a 2-doubled clique is a doubled clique.

Let $s \in \{2, 3, \ldots, \ell\}$ be maximal so that there exists a rank-$g(s)$ s-doubled clique $M \in \mathcal{U}(\ell)$ with no proper Reid-geometry minor. If $s = \ell$, let L_1 and L_2 be lines of M through e of length $\ell + 1$, and let $L_3 = \{e, x, y\}$ be a line of M through e so that L_1, L_2 and $\{x, y\}$ are pairwise disjoint. Then $R = M|(L_1 \cup L_2 \cup L_3)$ is a Reid geometry, and each vertex of $I(R)$ has degree two or else R has a $U_{2,\ell+2}$-minor. Thus, $I(R)$ contains a cycle, so $s < \ell$.

Let $(M/t)|X$ be a simplification of M/t, and note that $(M/t)|X \cong M(K_{g(s)})$. Let $\mathcal{X}$ be a collection of $\binom{g(s+1)}{2}$ size-three subsets of X which correspond to pairwise vertex-disjoint triangles of $(M/t)|X$.

7.1.3.1. *For each $F \in \mathcal{X}$ there is some $x_F \in F$ so that $\mathrm{cl}_{M/x_F}(\{t\} \cup F)$ has a $U_{2,s+2}$-restriction.*

Proof. Let $F \in \mathcal{X}$. Let L_1 and L_2 be distinct lines of M of length $s+1$ through e and elements of F, and let $L_3 = \{t, x, y\}$ be a line of M with $x \in F$ so that $x, y \notin L_1 \cup L_2$. Then $L_1 \cup L_2 \cup L_3$ is a Reid geometry, and either $(L_1 \cup L_2 \cup L_3)/x$ or $(L_1 \cup L_2 \cup L_3)/y$ has a $U_{2,s+2}$-restriction, or else the incidence graph has minimum degree two and thus has a cycle. □

Let $C = \{x_F \colon F \in \mathcal{X}\}$, and let X' be a transversal of $\{F - \{x_F\} \colon F \in \mathcal{X}\}$. Then $\mathrm{si}((M/t/C)|X)$ is isomorphic to a clique, and X' corresponds to a size-$\binom{g(s+1)}{2}$ matching of $(M/t/C)|X$. Thus, $(M/t/C)|X$ has a minor with ground set X' which is isomorphic to $M(K_{g(s+1)})$. So there is some $C' \subseteq X - X'$ so that $(M/t/C/C')|X' \cong M(K_{g(s+1)})$ and $t \notin \mathrm{cl}_M(C \cup C')$.

For each $x \in X'$, the matroid $(M/C)|\,\mathrm{cl}_{M/C}(\{t, x\})$ has a $U_{2,s+2}$-restriction, by 7.1.3.1 and the definition of C. In $M/C/C'$ there is no line through t which contains two elements of X', since $(M/t/C/C')|X'$ is simple. Thus, for each $x \in X'$, the set $\{t, x\}$ spans a distinct $U_{2,s+2}$-restriction of $M/C/C'$. Since $(M/t/C/C')|X' \cong M(K_{g(s+1)})$, this implies that $(M/C/C')|(\{t\} \cup X')$ has a $(s+1)$-doubled-clique restriction of rank $g(s+1)$. By the maximality of s, this matroid has a proper Reid-geometry minor. □

7.2 Approximate Results

For classes of representable matroids, Theorem 1.7.2 has the following corollary, which determines the correct leading coefficient of the extremal function for a huge family of well-studied classes.

Theorem 7.2.1. *Let $\ell \geq 2$ and $\alpha \geq 1$ be integers so that $\ell > \alpha$. Let $\mathcal{F}$ be a family of fields having no common subfield so that α is the size of the largest common subgroup (up to isomorphism) of size less than ℓ, of the multiplicative groups of the fields in $\mathcal{F}$. Then the class $\mathcal{M}$ of matroids representable over all fields in $\mathcal{F}$ and with no $U_{2,\ell+2}$-minor has extremal function $h_{\mathcal{M}}(n) = \alpha\binom{n}{2} + O(n)$.*

Proof. By Proposition 7.1.3 and Lemma 7.1.1, $\mathcal{M}$ contains no rank-3^{2^ℓ} doubled clique. By Theorem 1.2.4, $\mathcal{M}$ contains no Dowling geometry with group size greater than α. Thus,

$$h_{\mathcal{M}}(n) \leq \alpha\binom{n}{2} + f_{1.7.2}(\ell, \alpha+1, 3^{2^\ell}, 3) \cdot n$$

for all $n \geq 0$. By Theorem 1.2.4, $\mathcal{M}$ contains all Dowling geometries over the cyclic group of size α, so $h_{\mathcal{M}}(n) \geq \alpha\binom{n}{2} + n$ for all $n \geq 0$. □

We highlight two special cases which are particularly interesting. The first was conjectured in [10].

Theorem 7.2.2. *Let $\mathbb{F}_1$ and $\mathbb{F}_2$ be finite fields with different characteristic, and let α be the size of the largest common subgroup, up to isomorphism, of the groups $\mathbb{F}_1^\times$ and $\mathbb{F}_2^\times$. Then the class of matroids representable over both $\mathbb{F}_1$ and $\mathbb{F}_2$ has extremal function $h(n) = \alpha\binom{n}{2} + O(n)$ for all sufficiently large n.*

The previous best upper bound on the leading coefficient was $q^{2^{q-1}-1} - q^{2^{q-1}-2}$, where $q = \min(|\mathbb{F}_1|, |\mathbb{F}_2|)$, which was proved by Kung [30].

The second special case follows from the fact that the largest finite subgroup of the multiplicative group of the real numbers has size two.

Theorem 7.2.3. *For each integer $\ell \geq 3$, the class of $\mathbb{R}$-representable matroids with no $U_{2,\ell+2}$-minor has extremal function $h(n) = 2\binom{n}{2} + O(n)$ for all sufficiently large n.*

This is the current best upper bound for $\ell \geq 4$. The class of $\mathbb{R}$-representable matroids with no $U_{2,4}$-minor is equal to the class of regular matroids, so in this case Theorem 1.5.4 says that $h(n) = \binom{n+1}{2}$ for all $n \geq 0$. When $\ell = 3$, Theorem 1.5.7 tells us that $h(n) = n^2$ for all $n \geq 0$. However, for $\ell \geq 4$ the previous best upper bound on the leading coefficient is $\ell^{2^\ell-1} - \ell^{2^\ell-2}$, proved by Kung in [30].

7.3 Exact Results

Theorem 1.7.6 has the following consequence for quadratically dense classes of representable matroids.

Theorem 7.3.1. *Let $\mathcal{F}$ be a family of fields having no common subfield, and let $t \geq 1$ be an integer. Then the class $\mathcal{M}$ of matroids representable over all fields in $\mathcal{F}$ and with no $U_{2,t+3}$-minor satisfies $h_{\mathcal{M}}(n) \leq t\binom{n}{2} + n$ for all sufficiently large n. Moreover, if $\varepsilon(M) = t\binom{r(M)}{2} + r(M)$ and $r(M)$ is sufficiently large, then $\mathrm{si}(M)$ is isomorphic to a Dowling geometry.*

Proof. By Proposition 7.1.3 and Lemma 7.1.1, $\mathcal{M}$ contains no rank-$3^{2^{t+1}}$ doubled clique. By Lemma 7.1.2 and Lemma 7.1.1, $\mathcal{M}$ contains no nontrivial extension of $\mathrm{DG}(3, \Gamma)$ with $|\Gamma| \geq t$. Thus, Theorem 6.3.1 applies. $\square$

Whenever the size of the largest common subgroup (up to isomorphism) of the multiplicative groups of the fields in $\mathcal{F}$ has size t, Dowling geometries over that group give a matching lower bound for the extremal function. There are two notable cases for which this occurs. The first was conjectured independently by Nelson [38] and Kapadia [25].

Theorem 7.3.2. *For each integer $t \geq 1$, the class of $\mathbb{C}$-representable matroids with no $U_{2,t+3}$-minor has extremal function $h(n) = t\binom{n}{2} + n$ for sufficiently large n. Moreover, if n is sufficiently large and equality holds for M, then $\mathrm{si}(M)$ is isomorphic to a Dowling geometry over the cyclic group of size t.*

The second was conjectured by Geelen, Gerards and Whittle in [10].

Theorem 7.3.3. *If $\mathbb{F}_1$ and $\mathbb{F}_2$ are finite fields with different characteristic such that $|\mathbb{F}_1| - 1$ divides $|\mathbb{F}_2| - 1$, then the class of matroids representable over $\mathbb{F}_1$ and $\mathbb{F}_2$ has extremal function $h(n) = (|\mathbb{F}_1| - 1)\binom{n}{2} + n$ for sufficiently large n. Moreover, if n is sufficiently large and equality holds for M, then $\mathrm{si}(M)$ is isomorphic to a Dowling geometry over $\mathbb{F}_1^{\times}$.*

In fact, we can prove a result which implies Theorem 7.3.3, and was also conjectured in [10]. We first need a result about Reid geometries. We remark that if R is a Reid geometry so that $I(R)$ is connected, then $I(R)$ is either a cycle or a path, since $I(R)$ has maximum degree at most two.

Lemma 7.3.4. *If R and R' are Reid geometries so that $I(R)$ and $I(R')$ are connected isomorphic graphs, then R and R' are isomorphic matroids.*

Proof. Since $I(R)$ is connected and has maximum degree at most two, either $I(R)$ is a cycle or a path. We prove the case that $I(R)$ is a cycle of length $2k$ for some $k \geq 2$; the proof when $I(R)$ is a path on $2k$ vertices is nearly identical. Let L_1, L_2, L_3 be long lines of R with common point x so that $L_3 = \{x, y, z\}$, and $L_1 - \{x\} = \{u_1, \dots, u_k\}$ and $L_2 - \{x\} = \{v_1, \dots, v_k\}$. By reordering $\{u_1, \dots, u_k\}$ and $\{v_1, \dots, v_k\}$, we may assume that $\{u_i, v_i, y\}$ and $\{v_i, u_{i+1}, z\}$ are lines of R for each $i \in [k]$, taking indices modulo k. Note that these lines and L_1, L_2, L_3 are the only long lines of R, since for each $i \in [k]$, each long line through u_i other than L_1 contains y or z and a unique element of $L_2 - \{x\}$.

Let L_1', L_2', L_3' be long lines of R' with common point x' so that $L_3' = \{x', y', z'\}$, and $L_1' - \{x'\} = \{u_1', \dots, u_k'\}$ and $L_2' - \{x'\} = \{v_1', \dots, v_k'\}$. Again, by reordering $\{u_1', \dots, u_k'\}$ and $\{v_1', \dots, v_k'\}$, we may assume that $\{u_i', v_i', y'\}$ and $\{v_i', u_{i+1}', z'\}$ are lines of R' for each $i \in [k]$, taking indices modulo k. Define a function $f \colon E(R) \to E(R')$ by $f(e) = e'$ for each $e \in E(R)$. Then each set $Z \subseteq E(R)$ is a long line of R if and only if $f(Z)$ is a long line of R'. Since $r(R) = r(R') = 3$, this implies that f is an isomorphism from R to R'. □

For each prime p, define $R(p)$ to be the unique Reid geometry so that $I(R(p))$ is a cycle of length $2p$.

Theorem 7.3.5. *Let $\mathbb{F}$ be a finite field with characteristic p, and let $\mathcal{M}$ be the class of $\mathbb{F}$-representable matroids with no $R(p)$-minor. Then $h_{\mathcal{M}}(n) = (|\mathbb{F}| - 1)\binom{n}{2} + n$ for all sufficiently large n, and if equality holds for M and $r(M)$ is sufficiently large, then $\mathrm{si}(M)$ is isomorphic to a Dowling geometry.*

Proof. Assume for a contradiction that there is a rank-$3^{(2^{|\mathbb{F}|})}$ doubled clique $M \in \mathcal{M}$. By Proposition 7.1.3, M has a proper Reid-geometry minor R. Since $R \in \mathcal{M}$ and $\mathbb{F}$ has characteristic p, this cycle has length $2p$ by Lemma 7.1.1. Thus, R has an $R(p)$-restriction, a contradiction, so $\mathcal{M}$ contains no rank-$3^{(2^{|\mathbb{F}|})}$ doubled clique. By Lemma 7.1.2 and similar reasoning, $\mathcal{M}$ does not contain any nontrivial extension of $\mathrm{DG}(3, \Gamma)$ with $|\Gamma| \geq |\mathbb{F}^{\times}|$. Thus, Theorem 6.3.1 applies. □

Chapter 8

Future Work

In this chapter we discuss some natural questions which arise from results and conjectures in this thesis.

8.1 Projections of Dowling Geometries

The results of this thesis indicate that minor-closed classes which contain no rank-n doubled clique warrant further study. Perhaps the most interesting avenue of further research is Conjecture 1.7.12, which we restate for convenience.

Conjecture 1.7.12. *Let $\ell \geq 2$ and $n \geq 3$ be integers, and let $\mathcal{M}$ be a minor-closed class of matroids so that $U_{2,\ell+2} \notin \mathcal{M}$, and $\mathcal{M}$ contains no rank-n doubled clique. Then there is an integer $k \geq 0$ so that each simple extremal matroid of $\mathcal{M}$ with sufficiently large rank is a k-element projection of a Dowling geometry.*

Theorem 1.7.6 shows that this conjecture is true (with $k = 0$) for classes which do not contain any nontrivial extension of $\mathrm{DG}(m, \Gamma)$ with $|\Gamma| \geq t$ for some fixed $m \geq 3$. The extremal matroids in Theorem 1.5.5 are projections of cliques, so this conjecture also holds for the the class of sixth-root-of-unity matroids and the class of near-regular matroids. As a final piece of evidence, the extremal matroids representable over GF(4) and GF(5) are expected to be rank-2 projections of cliques. This is known as Archer's conjecture [1], and has been verified by Grace [22] subject to the Matroid Minors Structure Theorem.

Conjecture 1.7.12 may be difficult to prove in general, but there are some cases which may not be so hard. In general, it should be easier when the maximum group size of a Dowling geometry in $\mathcal{M}$ is close to ℓ, because in this case the integer k should be smaller. For example, the following conjecture might be approachable using the techniques of this thesis.

Conjecture 8.1.1. *The simple extremal $\mathbb{R}$-representable matroids with no $U_{2,6}$-minor and sufficiently large rank are projections of Dowling geometries with group* $\mathrm{GF}(3)^{\times}$.

A more general approach would be to try to prove Conjecture 1.7.12 for classes which contain all Γ-frame matroids, but do not contain $U_{2,|\Gamma|+4}$.

In another direction, if Conjecture 1.7.12 is true, then it is worthwhile to study k-element projections of Dowling geometries. From the perspective of Problem 1.4.1, we would like to determine all possibilities for the number of points of a k-element projection of a Dowling geometry. These matroids are not well-studied at all, so perhaps a first step would be to find a lower bound on the number of points of a k-element projection of a Dowling geometry. In the case that the Dowling geometry is a clique we make the following conjecture.

Conjecture 8.1.2. *For each integer $k \geq 0$, if M is a matroid with a spanning clique restriction G and $X \subseteq E(M) - E(G)$ is a rank-k flat of M, then $\varepsilon(M/X) \geq \binom{r(M)+1}{2} -$*

$(3\binom{k}{2} + 2k)$.

This bound is achieved when M is a specific binary matroid for which each column indexed by X has support of size four. We may assume that X is a flat, or else the problem reduces to an instance with a smaller value of k. In [16] Geelen and Nelson proved that this conjecture is true when $k = 1$. It is difficult to even make an analogous conjecture for Dowling geometries in general. Since each Dowling geometry over a non-cyclic group is not representable over any field, it is hard to imagine what the 'worst-case' k-element projection might be.

Lastly, perhaps there is some analogue of Theorem 1.6.2 for classes with no rank-n doubled clique.

Problem 8.1.3. *What is the structure of matroids with no $U_{2,\ell+2}$-minor, no rank-n doubled-clique minor, and a spanning clique restriction?*

In the proof of Theorem 1.7.2, Theorem 1.6.2 finds a projective-geometry minor when all we need is a doubled-clique minor. Since the constant $h_{1.6.2}(\ell, n)$ is enormous and plays such an important role in the proof of Theorem 1.7.2, it would be nice to at least answer Problem 8.1.3 in a way that allows us to use a much smaller constant.

Since it should be much easier to find a doubled-clique minor than a projective-geometry minor, it might be the case that the matroids in Problem 8.1.3 have a strong structural description. This is particularly interesting since Problem 8.1.3 applies to many quadratically dense classes of representable matroids, such as the ones considered in Chapter 7.

8.2 Lifts of Dowling Geometries

Another clear avenue for further research is to try to use the techniques of this thesis to prove Conjecture 1.4.4, which we restate for convenience.

Conjecture 1.4.4. *If $\mathcal{M}$ is a quadratically dense minor-closed class of matroids, then there are integers $\alpha \geq 1$ and $t \geq 0$ so that*

- $h_{\mathcal{M}}(n) = \alpha\binom{n}{2} + O(n)$, *and*
- *for each integer $n > t$, $\mathcal{M}$ contains an (α, t)-frame matroid of rank n.*

In order to adapt the proof of Theorem 1.7.2 for Conjecture 1.4.4, we would have to make two improvements. The first would be a result analogous to Theorem 4.0.1, in which we do not exclude rank-n doubled-clique minors. For such a result to be true, we would have to reduce both the number of porcupines and the constant g during the proof, but this is not an issue since Conjecture 1.4.4 has a linear error term.

The second improvement would be replacing Theorem 3.6.1 by a similar result for which each outcome leads to either a projective-geometry minor or an (α, t)-frame matroid minor. More specifically, we would have to replace outcome (2) of Theorem 3.6.1, since a collection of nearly skew spikes is not helpful for finding either of these matroids as a minor. In order to prove such a strengthening of Theorem 3.6.1 we would have to improve the results in Section 3.2. In particular, we would need to investigate which structures arise when $\delta(M, f) - \delta(M/e, f)$ is large, as in Lemma 3.3.2. This would likely be easier for classes which contain no $\mathrm{DG}(k, \Gamma)$ with $|\Gamma| = 2$, because any $U_{2,4}$-restriction is helpful for finding a $\mathrm{DG}(k, \Gamma)$-minor with $|\Gamma| = 2$.

If Conjecture 1.4.4 is true, then it is worthwhile to study (α, t)-frame matroids. For example, it may not be difficult to prove that every (α, t)-frame matroid with huge rank has an (α, t)-frame matroid minor M with big rank such that $\mathrm{si}(M/T)$ is isomorphic to a Dowling geometry. This would be good first step to refine the structure of (α, t)-frame matroids.

As a second step, since we are interested in matroids with large density, we would like to solve the following problem concerning (α, t)-frame matroids which are at least as dense as all of their minors.

Problem 8.2.1. *What is the structure of an (α, t)-frame matroid M so that $\varepsilon(M) = \alpha\binom{r(M)}{2} + b \cdot r(M) + c$ for some constants b and c, and each minor N of M satisfies $\varepsilon(N) \leq \alpha\binom{r(N)}{2} + b \cdot r(N) + c$?*

Theorem 2.1.1 shows that when $(t, b, c) = (0, 1, 0)$, the answer is given by Dowling geometries. It seems likely that when $t \geq 1$ these maximally dense (α, t)-frame matroids are also described by a group, and the proof may be similar to the proof of Theorem 2.1.1.

Finally, if Conjecture 1.4.4 is true then we would like to use it to find exact extremal functions, just as we used Theorem 1.7.2 to prove Theorem 1.7.6. To do this we would have to prove an analogue of Theorem 6.1.1 for (α, t)-frame matroids, where we find a spanning (α, t)-frame restriction instead of a Dowling-geometry restriction. It is unclear how difficult this would be, but it is worth investigating since there are several natural minor-closed classes for which the extremal matroids are conjectured to be lifts of Dowling geometries. Most notably, lifts of Dowling geometries are likely the extremal matroids for the classes of $\mathrm{GF}(p^k)$-matroids with no $\mathrm{PG}(n, p)$-minor, where p is a prime number.

8.3 Linearly Dense Classes

As discussed in Chapter 1, linearly dense minor-closed classes can be a bit wild. However, there are some enticing conjectures in [10] which would go some way towards classifying the extremal functions of linearly dense classes.

Conjecture 8.3.1. *If $\mathcal{M}$ is a linearly dense minor-closed class of matroids, then*

$\lim_{n\to\infty} h_{\mathcal{M}}(n)/n$ *exists and is rational.*

Conjecture 8.3.2. *If* $\mathcal{M}$ *is a linearly dense minor-closed class of matroids, then there exists a sequence* $(a, b_0, b_1, \ldots, b_{t-1})$ *of rational numbers so that, for all sufficiently large* n, $h_{\mathcal{M}}(n) = an + b_i$ *where* $i \in \{0, \ldots, t-1\}$ *and* $i \equiv n \mod t$.

These conjectures are not even known to be true for minor-closed classes of graphs, so they are certainly difficult. On the other hand, Kapadia [25] proved both conjectures in the case that $\mathcal{M}$ is a class of $\mathbb{F}$-representable matroids of bounded branch-width for a finite field $\mathbb{F}$. Hill [24] was able to generalize this to all classes of bounded branch-width, although his results have not yet been published. If these conjectures are true, they would indicate that even the 'wildest' minor-closed classes of matroids which exclude a line have some structure.

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