

On K -theoretic polynomials and the chromatic symmetric function

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners. I understand that my thesis may be made electronically available to the public.

Abstract

This thesis explores various problems related to polynomials from *combinatorial K -theory* and/or to the *chromatic symmetric function*. We prove four main results:

1. Two alternating sum formulas involving K -theoretic polynomials, conjectured in [MPS21].
2. The fact that certain properties of a graph can be recovered from its Kromatic symmetric function.
3. A power sum expansion for the Kromatic symmetric function, which we show has integer coefficients.
4. Formulas for certain pieces of the chromatic symmetric homology for star graphs.

The organization is as follows:

- In Chapter 1, we introduce background on symmetric functions and K -theory. The basic idea of K -theory is to deform cohomology rings by introducing an extra parameter β , often set to -1 .
- In Chapter 2, we prove an alternating sum conjecture due to Monical, Pechenik, and Searles [MPS21]. The conjecture concerns four different bases for the ring $\text{Poly}_n[\beta] := \mathbb{Z}[x_1, \dots, x_n][\beta]$, each indexed by weak compositions a : the *Lascoux atoms* $\mathfrak{A}_a$, the *kaons* $\mathfrak{P}_a$, the *quasiLascoux polynomials* $\mathfrak{Q}_a$, and the *glide polynomials* $\mathfrak{F}_a$. The authors of [MPS21] proved two expansion formulas $\mathfrak{A}_a = \sum_b Q_b^a(\beta) \mathfrak{P}_b$ and $\mathfrak{Q}_a = \sum_b M_b^a(\beta) \mathfrak{F}_b$, where $Q_b^a(\beta)$ and $M_b^a(\beta)$ are monomials in β . We prove that for $\beta = -1$, we get the surprising alternating sum formulas $\sum_b Q_b^a(-1) = \sum_b M_b^a(-1) \in \{0, 1\}$.
- In Chapter 3, we give background on the *chromatic symmetric function* X_G (introduced by Stanley [Sta95]) and its K -analogue, the *Kromatic symmetric function* $\overline{X}_G$ (introduced by Crew, Pechenik and Spirkel [CPS23]). A *proper coloring* of a graph is a way of assigning a color to each vertex such that adjacent vertices always receive different colors. X_G is computed by assigning a variable to each color then summing the monomials for all proper colorings of G , where in each monomial, the exponent of each variable is the number of times that color is used. $\overline{X}_G$ is defined similarly except that the monomials correspond to *proper set colorings*, meaning each vertex is assigned a nonempty set of colors such that adjacent vertices have nonoverlapping color sets.
- In Chapter 4, we study a question posed in [CPS23] about how much information about a graph can be recovered from $\overline{X}_G$. It is known that $\overline{X}_G$ is a stronger graph invariant than X_G (i.e. some graphs have the same X_G but different $\overline{X}_G$). We conjecture that $\overline{X}_G$ is in fact a complete invariant that distinguishes all graphs. As evidence toward this conjecture, we prove that the number of copies in G of certain induced subgraphs can be recovered from $\overline{X}_G$.
- In Chapter 5, we give a formula for the expansion of $\overline{X}_G$ using a K -theoretic analogue $\overline{p}_\lambda$ for the *power sum basis* p_λ of the ring of symmetric functions, where the index λ ranges over all partitions. We show that this expansion has all integer coefficients, as conjectured in [CPS23].
- In Chapter 6, we study the *chromatic symmetric homology* $H_*(G) = \bigoplus_{i,j} H_{i,j}(G)$, which is a *categorification* of the chromatic symmetric function introduced by Sazdanović and Yip in [SY18] and further studied in [Cha+23] and [CM23]. We compute the multiplicity of certain Specht modules in $H_{1,0}(G)$ in the case where G is a star graph, confirming and extending some conjectures from [Cha+23].

Chapters 2, 4, 5, and 6 contain essentially the same content as the papers [Pie24d], [Pie24b], [Pie24a], and [Pie24c], respectively.

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Chapter 1

Background on symmetric functions

1.1 Symmetric functions

A **symmetric function** is a power series $f(x_1, x_2, \dots) \in \mathbb{C}[[x_1, x_2, x_3, \dots]]$ that stays the same when any two variables are swapped. More formally, it is a power series that satisfies

$$f(x_1, x_2, x_3, \dots) = f(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, \dots)$$

for any permutation σ of $\mathbb{N}$. Since the sum and product of any two symmetric functions is also symmetric, they form a ring, which we denote Sym . This ring has several important bases, which we will describe using partitions. A **partition of n** is a nonincreasing sequence

$$\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}) = \lambda_1 \lambda_2 \dots \lambda_{\ell(\lambda)}$$

of positive integers $\lambda_1 \geq \dots \geq \lambda_{\ell(\lambda)}$ with $\lambda_1 + \dots + \lambda_{\ell(\lambda)} = n$, and the numbers $\lambda_1, \dots, \lambda_{\ell(\lambda)}$ are its **parts**. We can equivalently write

$$\lambda = k^{i_k} (k-1)^{i_{k-1}} \dots 3^{i_3} 2^{i_2} 1^{i_1}$$

to denote the partition with i_j parts of size j for each $j = 1, 2, \dots, k$. The **length** $\ell(\lambda)$ is the number of parts in λ . The **size** $|\lambda|$ of λ is the sum $n = \lambda_1 + \lambda_2 + \dots + \lambda_{\ell(\lambda)}$, and if λ has size n , we also write $\lambda \vdash n$. A partition λ can be represented visually by a **Young diagram**, which is an upside-down staircase shape made up of left-justified rows of unit squares such that the i th row has λ_i squares. A **semistandard Young tableau (SSYT)** puts a positive integer in each square of the Young diagram such that the entries are weakly increasing across each row from left to right and strictly increasing down each column from top to bottom. We write $\text{SSYT}(\lambda)$ for the set of all SSYT of shape λ . The **content** of $T \in \text{SSYT}(\lambda)$ is the composition μ such that μ_i is the number of boxes with entry i , and we associate to T a corresponding monomial $x^T := x_1^{\mu_1} x_2^{\mu_2} \dots x_k^{\mu_k}$ encoding its content.

Example 1.1. The partition $\lambda = 651$ has length $\ell(\lambda) = 3$ since it has 3 parts, and size $|\lambda| = 6 + 5 + 1 = 12$. Below is a semistandard Young tableau T of shape λ :

1	1	1	2	2	3
2	2	3	4	5	
5					

The content of T is 34212, and the corresponding monomial is $x^T = x_1^3 x_2^4 x_3^2 x_4 x_5^2$, since T has three 1's, four 2's, two 3's, one 4, and two 5's.

For $\lambda \vdash n$, we say that T is a **standard Young tableau (SYT)** if $T \in \text{SSYT}(\lambda)$ and its entries are some permutation of $1, 2, \dots, n$, or equivalently, if the content of T is 1^n and $x^T = x_1 x_2 x_3 \dots x_n$. We write $\text{SYT}(\lambda)$ for the set of standard Young tableaux of shape λ .

We write $\mu \subseteq \lambda$ if all boxes of the Young diagram for μ are contained in the Young diagram for λ , or equivalently, if $\mu_i \leq \lambda_i$ for all i . In that case, we write λ/μ for the set of boxes contained in λ but not μ ,

which is called a *skew tableau*. A specific type of skew shape is the *tableau product* $\lambda * \mu$ formed by lining up the top right corner of λ with the bottom left corner of μ . Given a skew shape, an SSYT of that shape and its content and monomial are defined in the same way as above.

Example 1.2. Below is an SSYT of shape $874/43$, with content 44211 and monomial $x_1^4 x_2^4 x_3^2 x_4 x_5$:

				1	2	3	4
			1	1	3	5	
1	2	2	2				

Example 1.3. Below is an SSYT of shape $42 * 321$, with content 4432 and monomial $x_1^4 x_2^3 x_3^3 x_4^2$:

				1	1	2
				2	3	
				4		
1	1	2	4			
3	3					

We now list the common bases of Sym , in the variables $x_1, x_2, x_3, \dots$:

- The *monomial symmetric functions* are

$$m_\lambda := \sum_{\substack{i_1, i_2, \dots, i_{\ell(\lambda)} \in \mathbb{N} \\ \text{pairwise distinct}}} x_{i_1}^{\lambda_1} x_{i_2}^{\lambda_2} \dots x_{i_{\ell(\lambda)}}^{\lambda_{\ell(\lambda)}}.$$

- The *augmented monomial symmetric functions* are

$$\tilde{m}_{k^{i_k} \dots 2^{i_2} 1^{i_1}} := i_1! i_2! \dots i_k! \cdot m_{k^{i_k} \dots 2^{i_2} 1^{i_1}}.$$

- The *elementary symmetric functions* are

$$e_n : \sum_{\substack{i_1, i_2, \dots, i_n \in \mathbb{N} \\ \text{pairwise distinct}}} x_{i_1} x_{i_2} \dots x_{i_n}, \quad e_\lambda := e_{\lambda_1} e_{\lambda_2} \dots e_{\lambda_{\ell(\lambda)}}.$$

- The *homogeneous symmetric functions* are

$$h_n : \sum_{\substack{i_1, i_2, \dots, i_n \in \mathbb{N} \\ \text{not necessarily distinct}}} x_{i_1} x_{i_2} \dots x_{i_n}, \quad h_\lambda := h_{\lambda_1} h_{\lambda_2} \dots h_{\lambda_{\ell(\lambda)}}.$$

- The *power sum symmetric functions* are

$$p_n := \sum_{i \in \mathbb{N}} x_i^n, \quad p_\lambda := p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_{\ell(\lambda)}}.$$

- The *Schur functions* are

$$s_\lambda := \sum_{T \in \text{SSYT}(\lambda)} x^T.$$

Example 1.4. For $\lambda = 21$, if we restrict to three variables x_1, x_2, x_3 (i.e. set $x_4 = x_5 = \dots = 0$), we get

$$\begin{aligned} m_{21}(x_1, x_2, x_3) &= x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_2 + x_2 x_3^2, \\ e_{21}(x_1, x_2, x_3) &= (x_1 x_2 + x_1 x_3 + x_2 x_3)(x_1 + x_2 + x_3), \\ h_{21}(x_1, x_2, x_3) &= (x_1^2 + x_2^2 + x_3^2 + x_1 x_2 + x_1 x_3 + x_2 x_3)(x_1 + x_2 + x_3), \\ p_{21}(x_1, x_2, x_3) &= (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3), \\ s_{21}(x_1, x_2, x_3) &= x_1^2 x_2 + x_1 x_2^2 + x_1^2 x_3 + x_1 x_3^2 + x_2^2 x_3 + x_2 x_3^2 + 2x_1 x_2 x_3, \end{aligned}$$

where the Schur function is computed by listing all elements of $\text{SYT}(21)$ with entries in $\{1, 2, 3\}$:

1	1							
2								

1	2							
2								

1	1							
3								

1	3							
3								

2	2							
3								

2	3							
2								

1	2							
3								

1	3							
2								

Since Schur polynomials form a basis for Sym , any product $s_\lambda s_\mu$ of two Schur functions can be expanded in the Schur basis as $s_\lambda s_\mu = \sum_\nu c'_{\lambda\mu\nu} s_\nu$. Since s_λ is homogeneous of degree $|\lambda|$ and s_μ is homogeneous degree $|\mu|$, the product $s_\lambda s_\mu$ is homogeneous of degree $|\lambda| + |\mu|$, so $c'_{\lambda\mu\nu} = 0$ unless $|\nu| = |\lambda| + |\mu|$. These structure constants $c'_{\lambda\mu\nu}$ are called **Littlewood-Richardson coefficients**. They can be computed combinatorially via the **Littlewood-Richardson rule**. To state it, we first need a few more definitions. The **reading word** of a tableau is formed by reading the entries from left to right across each row, starting from the bottom row and moving up. If in every starting segment of the reading word, the number of i 's is at least the number of $(i+1)$'s for all i , then the reading word is called **ballot**, **Yamanouchi**, or a **reverse lattice word**. An SSYT whose reading word satisfies the above property is called a **Littlewood Richardson (LR) tableau**. Then $c'_{\lambda\mu\nu}$ counts the number of LR tableaux of shape $\lambda * \mu$ and content ν . (Alternatively, it also counts the number of LR tableaux of shape ν/λ and content μ .)

[illegible]

1.2 Schubert calculus

Question 1.6. *Given 4 generic lines in 3-dimensional space, how many lines intersect all of them?*

To answer this question, we use the *Grassmannian* $\text{Gr}(k, n)$, which is the set of k -dimensional subspaces of $\mathbb{C}^n$. If we fix an ordered basis of $\mathbb{C}^n$, we can encode a k -space $V \in \text{Gr}(k, n)$ as a $k \times n$ matrix whose rows form a basis for V . Performing row operations on the matrix does not change its row-span V , so we can assume the matrix is written in reduced row echelon form. We then associate a partition λ to V by looking at the shape formed by the zeros that are below and to the left of the pivot columns but not in pivot columns themselves, and then turning that shape upside down to get the Young diagram of λ . We illustrate this in the following example:

Example 1.7. Any matrix in $\text{Gr}(4, 10)$ with the row echelon form below corresponds to the partition $\lambda = 6321$, where the $*$'s can be any complex numbers. The boxed 0's are the ones used to determine λ :

$$\begin{pmatrix} \boxed{0} & 1 & * & 0 & * & 0 & * & * & * & 0 \\ \boxed{0} & 0 & \boxed{0} & 1 & * & 0 & * & * & * & 0 \\ \boxed{0} & 0 & \boxed{0} & 0 & \boxed{0} & 1 & * & * & * & 0 \\ \boxed{0} & 0 & \boxed{0} & 0 & \boxed{0} & 0 & \boxed{0} & \boxed{0} & \boxed{0} & 1 \end{pmatrix}.$$

The set of all k -spaces in $\text{Gr}(k, n)$ that correspond to λ in this way is called the **Schubert cell** X_λ° . Since k of the n columns are pivot columns and thus do not contain any boxed 0's, any λ that corresponds to an element of $\text{Gr}(k, n)$ must have at most k rows that all have length at most $n - k$, and thus λ fits inside the $k \times (n - k)$ rectangle, which we denote as $k \times (n - k)$. Since each k -space corresponds to a unique matrix in row echelon form, every element of $\text{Gr}(k, n)$ is in a unique Schubert cell, and thus

$$\text{Gr}(k, n) = \bigsqcup_{\lambda \subseteq k \times (n-k)} X_\lambda^\circ.$$

The **Schubert varieties** are the algebraic closures of the Schubert cells, and are given by

$$X_\lambda = \bigsqcup_{\mu \supseteq \lambda} X_\mu^\circ.$$

That is, X_λ consists of all matrices whose row echelon form is the one corresponding to λ , or anything that can be obtained from it by moving some of the pivot columns to the right.

Example 1.8. For $\text{Gr}(4, 2)$, the possible partitions are $\emptyset$, $\square$, $\square\square$, $\square$, $\square\square$, and $\square\square$, so

$$\text{Gr}(4, 2) = X_\emptyset^\circ \sqcup X_{\square}^\circ \sqcup X_{\square\square}^\circ \sqcup X_{\square}^\circ \sqcup X_{\square\square}^\circ \sqcup X_{\square\square}^\circ.$$

The Schubert cells are shown below, where the $*$'s can be arbitrary complex numbers, and the boxed 0's are used to read off the corresponding Young diagram shape:

$$\begin{aligned} X_\emptyset^\circ &= \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \end{pmatrix} \right\}, & X_{\square}^\circ &= \left\{ \begin{pmatrix} 1 & * & 0 & * \\ 0 & \boxed{0} & 1 & * \end{pmatrix} \right\}, & X_{\square\square}^\circ &= \left\{ \begin{pmatrix} 1 & * & * & 0 \\ 0 & \boxed{0} & \boxed{0} & 1 \end{pmatrix} \right\}, \\ X_{\square}^\circ &= \left\{ \begin{pmatrix} \boxed{0} & 1 & 0 & * \\ \boxed{0} & 0 & 1 & * \end{pmatrix} \right\}, & X_{\square\square}^\circ &= \left\{ \begin{pmatrix} \boxed{0} & 1 & * & 0 \\ \boxed{0} & 0 & \boxed{0} & 1 \end{pmatrix} \right\}, & X_{\square\square}^\circ &= \left\{ \begin{pmatrix} \boxed{0} & \boxed{0} & 1 & 0 \\ \boxed{0} & \boxed{0} & 0 & 1 \end{pmatrix} \right\}. \end{aligned}$$

The Schubert varieties are:

$$\begin{aligned} X_\emptyset &= \left\{ \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \end{pmatrix} \right\} = X_\emptyset^\circ \sqcup X_{\square}^\circ \sqcup X_{\square\square}^\circ \sqcup X_{\square}^\circ \sqcup X_{\square\square}^\circ \sqcup X_{\square\square}^\circ, \\ X_{\square} &= \left\{ \begin{pmatrix} * & * & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\} = X_{\square}^\circ \sqcup X_{\square\square}^\circ \sqcup X_{\square}^\circ \sqcup X_{\square\square}^\circ \sqcup X_{\square\square}^\circ, \\ X_{\square\square} &= \left\{ \begin{pmatrix} * & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} = X_{\square\square}^\circ \sqcup X_{\square\square}^\circ \sqcup X_{\square\square}^\circ, \\ X_{\square} &= \left\{ \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\} = X_{\square}^\circ \sqcup X_{\square\square}^\circ \sqcup X_{\square\square}^\circ, \\ X_{\square\square} &= \left\{ \begin{pmatrix} 0 & * & * & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} = X_{\square\square}^\circ \sqcup X_{\square\square}^\circ, \\ X_{\square\square} &= \left\{ \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} = X_{\square\square}^\circ. \end{aligned}$$

We can think of $\text{Gr}(k, n)$ as a topological space in which each k -space is a point (and in fact, it can be embedded as a variety in a certain high-dimensional projective space). The dimension of each Schubert cell

is the number of $*$'s, since each $*$ represents a degree of freedom. Thus, the Schubert cell X_λ has codimension $|\lambda|$, since each boxed 0 corresponds to one fewer $*$ and thus one fewer degree of freedom. The dimension of the Schubert variety X_λ is equal to the dimension of the corresponding cell X_λ° , since the “extra pieces” that get added to X_λ° to create X_λ are lower dimensional. The full space $\text{Gr}(k, n)$ is equal to $X_\emptyset$, so it has dimension $k(n - k)$. The Schubert varieties X_λ decrease in dimension as we add more boxes to λ , so the lowest dimensional Schubert variety is $X_{k \times (n-k)}$, which consists of a single point.

For a fixed λ , the set of Schubert varieties X_λ over all choices of the ordered basis forms an equivalence class $[X_\lambda]$ called a **Schubert cell**. We can put a ring structure on the Schubert cells where addition is given by unions and multiplication by intersections:

$$[X] + [Y] := [X \cup Y], \quad [X] \cdot [Y] := [X \cap Y].$$

This is **cohomology ring** $H^*(\text{Gr}(k, n))$ of the Grassmannian. We require the representatives X and Y of their respective equivalence classes to be “generic,” which essentially amounts to not allowing “coincidences” such as choosing the same element from $[X]$ and $[Y]$ if $[X] = [Y]$, or choosing an element of $[X]$ that is contained in the element of $[Y]$ when that would not normally be the case. With these operations, it is certainly the case that addition and multiplication are commutative and that multiplication distributes over addition. Also, the additive identity is $\emptyset$ and the multiplicative identity is the entire ambient space. Thus, to make this into a ring, we need only add in additive inverse, which do not have a physical meaning but can be constructed as formal $\mathbb{Z}$ -linear combinations of the equivalence classes. We will look at a few simpler examples of how this addition and multiplication structure works before describing how it works in the case of the Grassmannian:

Example 1.9. In 2-dimensional (projective) space,

$$2 \cdot [\text{line}] = [\text{line} \cup \text{line}], \quad [\text{line}]^2 = [\text{point}], \quad [\text{line}] \cdot [\text{point}] = 0$$

since two generic lines intersect in a point but a generic point and line do not intersect. In 3 dimensions,

$$2 \cdot [\text{plane}] = [\text{plane} \cup \text{plane}], \quad [\text{plane}]^2 = [\text{line}], \quad [\text{plane}] \cdot [\text{line}] = [\text{point}], \quad [\text{line}]^2 = 0,$$

since two planes intersect in a line, a plane and line intersect in a point, and two generic lines do not intersect.

Codimensions add under this product, that is,

$$\text{codim}([X] \cdot [Y]) = \text{codim}[X \cap Y] = \text{codim}[X] + \text{codim}[Y].$$

Intuitively, this comes from the fact we can think of a variety X of codimension m as being defined by m algebraic equations and a variety Y of codimension n as being defined by n algebraic equations, so their intersection $X \cap Y$ is defined by a total of $m + n$ equations and thus has codimension $m + n$.

The amazing connection is that there is a ring homomorphism $\phi : \text{Sym} \rightarrow H^*(\text{Gr}(k, n))$ given by

$$\phi(s_\lambda) = \begin{cases} [X_\lambda], & \text{if } \lambda \subseteq k \times (n - k); \\ 0, & \text{otherwise.} \end{cases}$$

Thus, the structure constants that arise in taking intersections of Schubert classes are precisely the Littlewood-Richardson coefficients that show up in products of Schur functions:

$$[X_\lambda][X_\mu] = \sum_{\nu \subseteq k \times (n-k)} c_{\lambda\mu}^\nu [X_\nu].$$

Example 1.10. We now return to Question 1.6. Since subspaces always pass through the origin while lines may not, we first go up a dimension and think of each line in 3-dimensional (projective) space as corresponding to a plane through the origin in 4-dimensional space, so we work in $\text{Gr}(2, 4)$. (We can visualize the original 3-space as being embedded in 4-space such that it does not contain the origin, and then we there is a unique plane through the origin in 4-space that contains each of the original lines.) Thus, we

want to count the planes P in 4-space that intersect four given planes P_1, P_2, P_3 , and P_4 . Saying that the P intersects P_i is equivalent to saying that P lies in a Schubert variety of the form

$$X_{\square} = \left\{ \begin{pmatrix} * & * & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\}$$

where the matrix is written with respect to a basis (v_1, v_2, v_3, v_4) such that $P_i = \text{span}(v_3, v_4)$. This is because the second row of the row echelon form is in $\text{span}(v_3, v_4)$ and thus corresponds to a point in P_i , so P intersects P_i if and only if it contains such a vector.

Thus, P must lie in the intersection of four different elements of the Schubert class $[X_{\square}]$ (one for each P_i). This intersection is the product $[X_{\square}]^4$, which corresponds to the product $s_{\square}^4$ in **Sym**. This product can be computed with the Littlewood-Richardson rule. The LR tableaux of shape $\square * \square * \square * \square$ are listed below, along with their reading words:

$$\begin{array}{cccccccccc}
 \begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} & \begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} & \begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} & \begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} & \begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} & \begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} & \begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} & \begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} & \begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} & \begin{array}{c} \square \\ \square \\ \square \\ \square \end{array} \\
 1111 & 2111 & 1211 & 1121 & 2211 & 2121 & 3211 & 3121 & 1321 & 4321
 \end{array}$$

Reading off their contents, we get

$$s_{\square}^4 = s_{\square\square\square\square} + 3s_{\square\square\square} + 2s_{\square\square\square} + 3s_{\square\square\square} + s_{\square\square\square}.$$

To compute the corresponding product $[X_{\square}]^4$, we ignore all terms for partitions that do not fit within the $k \times (n - k) = 2 \times 2$ rectangle, so we get

$$[X_{\square}]^4 = 2 \cdot [X_{\square\square}] = 2 \cdot [\text{point}].$$

Thus, there are two points of $\text{Gr}(2, 4)$ lying in the intersection $[X_{\square}]^4$, meaning there are two 2-spaces P intersecting the four 2-spaces P_1, P_2, P_3 , and P_4 , and thus in our original problem, there are exactly two lines in 3-dimensional spaces intersecting the four given lines.

1.3 K -theoretic Schubert calculus

We will explain the basic ideas of K -theory here to motivate the construction of the polynomials we will study in Chapters 2-5. In K -theory, we have a ring structure that is similar to that of cohomology rings, except that it “sees” the lower dimensional aspects of unions and intersections which cohomology ignores. The elements of the ring are **Grothendieck classes**, written $[\mathcal{O}_X]$ for a variety X . Products are again given by intersections, but for sums, instead of just taking the union, we take the union but count the overlap an extra time, since it shows up in both pieces being added. That is,

$$[\mathcal{O}_X][\mathcal{O}_Y] = [\mathcal{O}_{X \cap Y}], \quad [\mathcal{O}_{X \cup Y}] = [\mathcal{O}_X] + [\mathcal{O}_Y] - [\mathcal{O}_{X \cap Y}].$$

This ring is called the **Grothendieck ring** $K(X)$, where X is the ambient space.

More generally, we can choose an arbitrary parameter β and set

$$[\mathcal{O}_{X \cup Y}] = [\mathcal{O}_X] + [\mathcal{O}_Y] + \beta \cdot [\mathcal{O}_{X \cap Y}],$$

so the above is the special case $\beta = -1$. Thus, the Grothendieck ring is essentially a deformation of the cohomology ring that depends on β , and the cohomology ring is special case $\beta = 0$.

Example 1.11. In 2-dimensional space, in the cohomology ring we have $[\text{line} \cup \text{line}] = 2 \cdot [\text{line}]$, but in the Grothendieck ring,

$$[\mathcal{O}_{\text{line} \cup \text{line}}] = 2 \cdot [\mathcal{O}_{\text{line}}] - [\mathcal{O}_{\text{point}}],$$

since when we add the two lines, their point of intersection gets added twice and thus needs to be subtracted. Similarly, if we take the union of three lines, all three points of overlap get added twice and need to be subtracted, so

$$[\mathcal{O}_{\text{line} \cup \text{line} \cup \text{line}}] = 3 \cdot [\mathcal{O}_{\text{line}}] - 3 \cdot [\mathcal{O}_{\text{point}}].$$

Similarly, in 3-dimensional space, if we take the union of three planes, we need to add the three planes, then subtract the three lines where two of the planes intersect, then add back the point where all three planes intersect, since that point got added three times and then subtracted three times, so

$$[\mathcal{O}_{\text{plane} \cup \text{plane} \cup \text{plane}}] = 3 \cdot [\mathcal{O}_{\text{plane}}] - 3 \cdot [\mathcal{O}_{\text{line}}] + [\mathcal{O}_{\text{point}}].$$

If we replace -1 with β , these equalities become

$$\begin{aligned} [\mathcal{O}_{\text{line} \cup \text{line}}] &= 2 \cdot [\mathcal{O}_{\text{line}}] + \beta \cdot [\mathcal{O}_{\text{point}}], \\ [\mathcal{O}_{\text{line} \cup \text{line} \cup \text{line}}] &= 3 \cdot [\mathcal{O}_{\text{line}}] + 3\beta \cdot [\mathcal{O}_{\text{point}}], \\ [\mathcal{O}_{\text{plane} \cup \text{plane} \cup \text{plane}}] &= 3 \cdot [\mathcal{O}_{\text{plane}}] + 3\beta \cdot [\mathcal{O}_{\text{line}}] + \beta^2 \cdot [\mathcal{O}_{\text{point}}]. \end{aligned}$$

In $\text{Gr}(k, n)$, we write $\mathcal{O}_\lambda = [\mathcal{O}_{X_\lambda}]$ for the Grothendieck class corresponding to the Schubert class $[X_\lambda]$. Products of Grothendieck classes can be computed using a K -theoretic analogue of the Littlewood-Richardson rule, based on set-valued tableaux. A *set-valued tableau* of shape λ assigns to each box of λ a nonempty set of positive integers. The tableau is *semistandard* if the largest element in each box is weakly smaller than the smallest element in the box to its right and strictly smaller than the smallest element of the box below it. Equivalently, it is semistandard if any way of choosing one number from each of the boxes results in an SSYT. The *reading word* of a set-valued tableau is formed as before except that if a box contains multiple entries, we read them in increasing order. The *content* of a set-valued tableau T is the partition $k^{i_k} \dots 2^{i_2} 1^{i_1}$ such that i_j is the number of boxes containing j , and x_T is the corresponding monomial, and the *size* $|T|$ is the total number of entries, or equivalently, the size of the partition corresponding to its content. A semistandard set-valued tableau is an *LR tableau* if its reading word is ballot.

Example 1.12. Below is a set-valued tableau T of shape 431, with content $7^2 6^2 5^3 4^3 3^3 2^2 1$, monomial $x^T = x_1 x_2^2 x_3^3 x_4^3 x_5^3 x_6^2 x_7^2$, and reading word 5723456712334456:

1	23	34	456
23	4	567	
57			

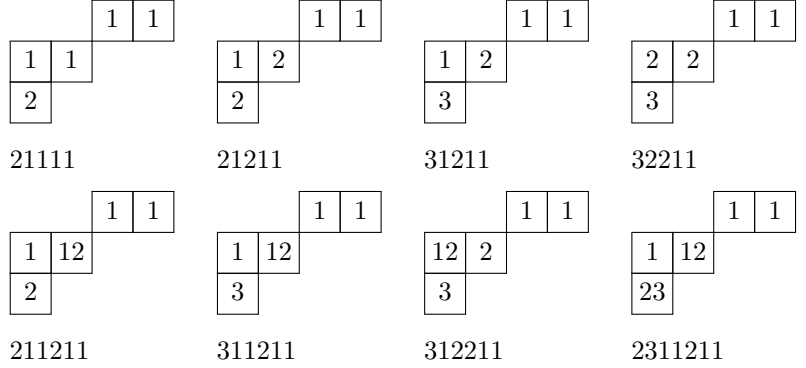
The K -theoretic Littlewood-Richardson rule says that

$$\mathcal{O}_\lambda \mathcal{O}_\mu = \sum_{\nu \subseteq k \times (n-k)} c_{\lambda\mu}^\nu \mathcal{O}_\nu,$$

where $c_{\lambda\mu}^\nu$ is $\beta^{|\nu| - |\mu| - |\lambda|}$ times the number of set-valued LR tableaux of shape $\lambda * \mu$ and content ν . Note that these coefficients agree with the ordinary Littlewood-Richardson coefficients when $|\nu| = |\mu| + |\lambda|$, and are 0 when $|\nu| < |\mu| + |\lambda|$.

Example 1.13. To compute $\mathcal{O}_{\square\square} \mathcal{O}_{\square\square}$ in $\text{Gr}(k, n)$, we list the set-valued LR tableaux of shape $\square\square * \square\square$,

along with their reading words:



Reading off the contents, we find that (assuming $k \geq 3$ and $n \geq 7$),

$$\mathcal{O}_{\square\square}\mathcal{O}_{\square\square} = \mathcal{O}_{\square\square\square\square} + \mathcal{O}_{\square\square\square} + \mathcal{O}_{\square\square\square} + \mathcal{O}_{\square\square\square} - \mathcal{O}_{\square\square\square\square} - \mathcal{O}_{\square\square\square\square} - \mathcal{O}_{\square\square\square\square} + \mathcal{O}_{\square\square\square\square}.$$

Just as there is a ring homomorphism $\phi : \text{Sym} \rightarrow H^*(\text{Gr}(k, n))$ taking Schur polynomials to Schubert classes, there is also a ring homomorphism $\psi : \text{Sym} \rightarrow K(\text{Gr}(k, n))$ given by

$$\psi(\bar{s}_\lambda) = \begin{cases} \mathcal{O}_\lambda, & \text{if } \lambda \subseteq k \times (n - k); \\ 0, & \text{otherwise,} \end{cases}$$

where the $\bar{s}_\lambda$'s are the *stable Grothendieck polynomials*

$$\bar{s}_\lambda := \sum_{\substack{T \text{ a semistandard} \\ \text{set-tableau of shape } \lambda}} \beta^{|T| - |\lambda|} x^T.$$

Thus, the stable Grothendieck polynomials are the natural K -theoretic analogue of the Schur polynomials, and the lowest degree terms of $\bar{s}_\lambda$ match s_λ .

Many other polynomials can similarly be generalized to a K -theoretic setting by introducing a parameter β such that the original polynomial is the special case $\beta = 0$. Often the case $\beta = -1$ is of particular interest and is the main one studied. In particular, in Chapter 2, we will prove a conjecture about the $\beta = -1$ case for the K -theoretic analogues of four different families of polynomials. Then in Chapters 3, 4, and 5, we will study the Kromatic symmetric function, which is the $\beta = -1$ case for the K -theoretic analogue of the chromatic symmetric function.

1.4 Representations of symmetric groups

The idea of representation theory is to encode an arbitrary finite group as a group of matrices. More formally, a *representation* of a finite group G is a group homomorphism $\phi : G \rightarrow \text{GL}(V)$ to the group $\text{GL}(V)$ of invertible linear maps from some finite dimensional vector space V to itself (i.e. the *general linear group* of V , which can equivalently be thought of as the group of invertible matrices of size $\dim V \times \dim V$). The *dimension* of the representation is the dimension of V . A *subrepresentation* is a subspace $W \subseteq V$ such that $\phi(g)W \subseteq W$ for all $g \in G$, i.e. the action of G on V fixes W . A representation is *irreducible* if it has no proper subrepresentations. Every representation $\phi : G \rightarrow \text{GL}(V)$ can be uniquely broken down into a direct sum of irreducible representations $\phi_i : G \rightarrow \text{GL}(V_i)$, meaning that $V = \bigoplus_i V_i$ and $\phi(g) = \bigoplus_i \phi_i(g)$ for each $g \in G$. Thus, irreducible representations form the building blocks for all representations.

If V is a vector space over the field k , then a representation V of G can equivalently be thought of as a *$k[G]$ -module*, where $k[G] := \{\sum_{g \in G} a_g g : a_g \in k\}$ is the set of formal linear combinations of elements of G with coefficients in k . This is essentially just saying that the action of G on V can be linearly extended to

an action of $k[G]$ on V . Then given a representation $\phi : H \rightarrow \mathrm{GL}(V)$ of a subgroup H of G , we can extend it to a corresponding **induced representation** of G , defined by

$$\mathrm{Ind}_H^G(V) := k[G] \otimes_{k[H]} V.$$

The **symmetric group** $\mathfrak{S}_n$ is the group of permutations of the set $[n] = \{1, 2, \dots, n\}$. We can encode a permutation $\sigma : [n] \rightarrow [n]$ using **cycle notation** $\sigma = \sigma_1 \sigma_2 \dots \sigma_k$ where each cycle σ_i is of the form $(a \ \sigma(a) \ \sigma^2(a) \ \dots \ \sigma^{m-1}(a))$, where $m > 1$ is the smallest value such that $\sigma^m(a) = a$. The surprising connection to symmetric functions stems from the fact that the irreducible representations of symmetric groups over $\mathbb{C}$ have essentially the same ring structure as Sym , with irreducible representations corresponding to Schur polynomials. Like Schur polynomials, the irreducible representations of the symmetric group $\mathfrak{S}_n$, called the **Specht modules** $\mathcal{S}_\lambda$, are indexed by the partitions $\lambda \vdash n$. We defer the construction of Specht modules to Chapter 6, but we will say a bit more here about their connection to Sym .

The **tensor product** of two representations $\phi : G \rightarrow \mathrm{GL}(V)$ and $\rho : H \rightarrow \mathrm{GL}(W)$ is the map $\phi \otimes \rho : G \times H \rightarrow \mathrm{GL}(V \otimes W)$ given by

$$(\phi \otimes \rho)(g, h) : v \otimes w = \phi(g)v \otimes \rho(h)w.$$

Given two irreducible representation $\mathcal{S}_\lambda$ of $\mathfrak{S}_{|\lambda|}$ and $\mathcal{S}_\mu$ of $\mathfrak{S}_{|\mu|}$, their tensor product $\mathcal{S}_\lambda \otimes \mathcal{S}_\mu$ gives a representation of $\mathfrak{S}_{|\lambda|} \times \mathfrak{S}_{|\mu|}$. Since $\mathfrak{S}_{|\lambda|} \times \mathfrak{S}_{|\mu|}$ is a subgroup of $\mathfrak{S}_{|\lambda|+|\mu|}$, we can construct a corresponding induced representation of $\mathfrak{S}_{|\lambda|+|\mu|}$. This induced representation can then be broken down into a direct sum of irreducible representations of the form $\mathcal{S}_\nu$ where $|\nu| = |\lambda| + |\mu|$. The key connection is that the number of copies of $\mathcal{S}_\nu$ in this decomposition is given by the **Littlewood-Richardson coefficient** $c_{\lambda\mu}^\nu$, where the Littlewood-Richardson coefficients are the same structure constants that show up when taking the product $s_\lambda s_\mu$ of two Schur functions and then expanding it as a sum of Schur functions:

$$\mathrm{Ind}_{\mathfrak{S}_{|\lambda|} \times \mathfrak{S}_{|\mu|}}^{\mathfrak{S}_{|\lambda|+|\mu|}}(\mathcal{S}_\lambda \otimes \mathcal{S}_\mu) = \bigoplus_{\nu} \mathcal{S}_\nu^{\oplus c_{\lambda\mu}^\nu} \quad \longleftrightarrow \quad s_\lambda s_\mu = \sum_{\nu} c_{\lambda\mu}^\nu s_\nu.$$

There is a structure-preserving map called the **Frobenius characteristic** that takes the Specht module $\mathcal{S}_\lambda$ to the Schur polynomial s_λ , and in some cases one can also construct maps that go in the opposite direction, turning Schur polynomials into Specht modules. This is the basic idea of **categorification**, which is the process used to construct the **chromatic symmetric homology** of a graph, our focus in Chapter 6.

Chapter 2

Proof of a K -theoretic polynomial conjecture of Monical, Pechenik, and Searles

2.1 Introduction

Our goal here is to prove a conjecture of Monical, Pechenik, and Searles from [MPS21] stating that two alternating sums involving polynomials arising from *combinatorial K -theory* are always equal to either 0 or 1. These polynomials were defined as part of a project to extend the combinatorics associated to the ring Sym_n of symmetric polynomials in n variables to the larger ring $\text{Poly}_n := \mathbb{Z}[x_1, \dots, x_n]$ of arbitrary polynomials in n variables, and then further to the ring $\text{Poly}_n[\beta]$ of K -theoretic deformations of these polynomials. The motivation for defining these K -theoretic analogues arises from the connections these polynomials have to algebraic geometry, specifically to *Schubert calculus*, which studies the cohomology rings of *flag varieties*. As explained in §1.3, the basic idea of K -theory is to deform the structure of cohomology rings by introducing an additional parameter β . Polynomials coming from cohomology rings often also have corresponding K -analogues, and conversely, when polynomials arising from combinatorics have nice K -analogues, that indicates that they may have geometric interpretations.

Recall from Chapter 1 that a classic basis for Sym_n is the *Schur polynomials* s_λ , which have rich connections to representation theory and algebraic geometry. Schur polynomials are a special case of *Schubert polynomials* $\mathfrak{S}_a$, which were introduced by Lascoux and Schützenberger in [LS82] as a basis for Poly_n that is closely connected to Schubert calculus. The Schubert polynomials are indexed by *weak compositions* $a = (a_1, \dots, a_n)$, i.e. finite sequences of nonnegative integers. Various related bases for Poly_n have also been studied, each also indexed by weak compositions a , including the *Demazure characters* $\mathfrak{D}_a$ [Dem74], the *Demazure atoms* $\mathfrak{A}_a$ [LS88], the *(fundamental) slide polynomials* $\mathfrak{F}_a$ [AS17], the *(fundamental) particles* $\mathfrak{P}_a$ [Sea20], and the *quasikey polynomials* $\mathfrak{Q}_a$ [AS18]. Hicks and Niese showed in [HN24] that all these polynomials arise naturally using a quasisymmetric analogue of the *divided difference operator*, and some of these polynomials also arise naturally from representation theory. These bases all have K -theoretic lifts to corresponding bases for $\text{Poly}_n[\beta]$: the *Grothendieck polynomials* $\overline{s}_\lambda$ and $\overline{\mathfrak{S}}_a$ [FK94], the *Lascoux polynomials* $\overline{\mathfrak{D}}_a$, the *Lascoux atoms* $\overline{\mathfrak{A}}_a$ [Mon16], the *glide polynomials* $\overline{\mathfrak{F}}_a$ [PS19], the *kaons* $\overline{\mathfrak{P}}_a$ [MPS21], and the *quasiLascoux polynomials* $\overline{\mathfrak{Q}}_a$ [MPS21]. An integrable vertex model for Lascoux polynomials and Lascoux atoms has been developed in [BSW20], using ideas from statistical mechanics.

The authors of [MPS21] proved two expansion formulas involving these K -theoretic polynomials:

$$\overline{\mathfrak{A}}_a = \sum_b Q_b^a(\beta) \overline{\mathfrak{P}}_b, \quad \overline{\mathfrak{Q}}_a = \sum_b M_b^a(\beta) \overline{\mathfrak{F}}_b,$$

where each $Q_b^a(\beta)$ and $M_b^a(\beta)$ is a monomial of the form $m\beta^n$ for nonnegative integers m and n ([MPS21], Theorems 3.12 and 4.12). They then conjectured the following formulas, which we now prove:

Theorem 2.1 ([MPS21], Conjecture 1.4). *If the nonzero parts of a are weakly decreasing, then*

$$\sum_b Q_b^a(-1) = \sum_b M_b^a(-1) = 1,$$

and otherwise

$$\sum_b Q_b^a(-1) = \sum_b M_b^a(-1) = 0,$$

where the sums are taken over all weak compositions b .

Theorem 2.1 is suggestive of an Euler characteristic calculation and may provide some insight into the geometric context of these polynomials.

We will explain the relevant definitions and prior results in §2.2, and then in §2.3, we will prove Theorem 2.1 and give some examples to illustrate it.

2.2 Background

In this section, we will define the four families of polynomials studied here, in the following order:

1. The **kaons** $\overline{\mathfrak{P}}_a$, which are generating functions for **mesonic glides**.
2. The **Lascoux atoms** $\overline{\mathfrak{A}}_a$, which are generating functions for **set-valued skyline fillings**.
3. The **glide polynomials** $\overline{\mathfrak{F}}_a$, which are sums of kaons.
4. The **quasiLascoux polynomials** $\overline{\mathfrak{Q}}_a$, which are sums of Lascoux atoms.

We will also explain the expansion formulas from [MPS21] that express Lascoux atoms in terms of kaons and quasiLascoux atoms in terms of glide polynomials, which will be the main tools for our proof.

Remark 2.2. The overlines indicate that these are K -theoretic polynomials (i.e. they involve β). The corresponding versions without the overlines are obtained by setting $\beta = 0$. The letter $\mathfrak{A}$ is an “A” for “atom.” The letter $\mathfrak{Q}$ is a “Q” for “quasi.” The letter $\mathfrak{P}$ is a “P” for “particle,” because the kaons $\overline{\mathfrak{P}}_a$ are a K -analogue of the particles $\mathfrak{P}_a$. The letter $\mathfrak{F}$ is an “F” for “fundamental,” because the glides $\overline{\mathfrak{F}}_a$ are a K -analogue of the fundamental slides $\mathfrak{F}_a$, which lift the fundamental basis F_a of the ring QSym of quasisymmetric functions. The name “atoms” makes sense because the Lascoux atoms form a basis of $\text{Poly}_n[\beta]$ and hence a set of fundamental building blocks. “Kaons” are named after a type of subatomic particle, which fits because they are “subatomic” in the sense that Lascoux atoms can be built out of kaons, and also because in particle physics, the symbol for kaons is “K.” The name “meson” comes from the name in particle physics of the family of particles that includes kaons (kaons are also called “K mesons”). The name “skyline fillings” comes from the fact that the diagrams look like city skylines turned sideways.

Kaons $\overline{\mathfrak{P}}_a$

First we introduce the definitions needed for the kaons. A **weak komposition** $b = (b_1, \dots, b_n)$ is a weak composition whose entries are colored red or black, and its **excess** $\text{ex}(b)$ is the number of red entries. Let $a = (a_1, \dots, a_n)$ be a weak composition with nonzero entries at positions $n_1 \leq n_2 \leq \dots \leq n_\ell$. We also set $n_0 = 0$. A weak komposition b is a **mesonic glide** of a if the following three properties are satisfied for every $1 \leq j \leq \ell$:

$$(G.1') \quad a_{n_j} = b_{n_{j-1}+1} + \dots + b_{n_j} - \text{ex}(b_{n_{j-1}+1}, \dots, b_{n_j}).$$

$$(G.3') \quad \text{The leftmost nonzero entry of } (b_{n_{j-1}+1}, \dots, b_{n_j}) \text{ is black.}$$

$$(G.4') \quad b_{n_j} \neq 0.$$

In our examples below, the red entries are also shown in bold to help them stand out.

Example 2.3. The weak komposition $b = (2, \mathbf{2}, 1, \mathbf{3}, \mathbf{2})$ is a mesonic glide of $a = (0, 3, 0, 0, 4)$, since

$$\begin{aligned} a_{n_1} &= a_2 = 3 = 2 + \mathbf{2} - \text{ex}(2, \mathbf{2}), \\ a_{n_2} &= a_5 = 4 = 1 + \mathbf{3} + \mathbf{2} - \text{ex}(1, \mathbf{3}, \mathbf{2}), \end{aligned}$$

the leftmost entries of $(2, \mathbf{2})$ and $(1, \mathbf{3}, \mathbf{2})$ are black, $b_{n_1} = b_2 = \mathbf{2} \neq 0$, and $b_{n_2} = b_5 = \mathbf{2} \neq 0$.

Write $\mathbf{x}^b := x_1^{b_1} \dots x_n^{b_n}$. The *kaon* $\overline{\mathfrak{P}}_a$ is the generating function

$$\overline{\mathfrak{P}}_a := \sum_{b \text{ a mesonic glide of } a} \beta^{\text{ex}(b)} \mathbf{x}^b.$$

In the example below, we show the x variables corresponding to bolded red entries as also being bolded and red to help illustrate the correspondence. This coloring of the x variables is not actually part of the definition of the polynomials.

Example 2.4. For $a = (0, 2, 0, 1)$, the possible mesonic slides are

$$\begin{array}{cccc} (0, 2, 0, 1) & (1, 1, 0, 1) & (1, \mathbf{2}, 0, 1) & (2, \mathbf{1}, 0, 1) \\ (0, 2, 1, \mathbf{1}) & (1, 1, 1, \mathbf{1}) & (1, \mathbf{2}, 1, \mathbf{1}) & (2, \mathbf{1}, 1, \mathbf{1}) \end{array}$$

so the corresponding kaon is

$$\overline{\mathfrak{P}}_{(0,2,0,1)} = x_2^2 x_4 + x_1 x_2 x_4 + \beta x_1 \mathbf{x}_2^2 x_4 + \beta x_1^2 \mathbf{x}_2 x_4 + \beta x_2^2 x_3 \mathbf{x}_4 + \beta x_1 x_2 x_3 \mathbf{x}_4 + \beta^2 x_1 \mathbf{x}_2^2 x_3 \mathbf{x}_4 + \beta^2 x_1^2 \mathbf{x}_2 x_3 \mathbf{x}_4.$$

Remark 2.5. We follow the seemingly odd numbering (G.1'), (G.3'), and (G.4') to match the notation in [MPS21]. They use this notation because the conditions (G.1'), (G.3'), and (G.4') are a modification of the conditions (G.1), (G.2), and (G.3) used to define the glide polynomials $\overline{\mathfrak{F}}_a$, which were originally defined before the kaons $\overline{\mathfrak{P}}_a$. We choose here to define the kaons $\overline{\mathfrak{P}}_a$ first and then the glides $\overline{\mathfrak{F}}_a$, since it is simpler to define the glides $\overline{\mathfrak{F}}_a$ in terms of the kaons $\overline{\mathfrak{P}}_a$.

Remark 2.6. Both sets of glide conditions (G.1), (G.2), (G.3) and (G.1'), (G.3'), (G.4') are themselves modifications of a simpler set of *slide conditions*, which are used to define the fundamental particles $\mathfrak{P}_a$ and the fundamental slide polynomials $\mathfrak{F}_a$. The particles $\mathfrak{P}_a$ and slides $\mathfrak{F}_a$ are the respective $\beta = 0$ cases of $\overline{\mathfrak{P}}_a$ and $\overline{\mathfrak{F}}_a$ and hence do not involve the red entries. These sorts of slide and glide moves that allow entries to be moved in only one direction arise naturally from the study of the ring of *quasisymmetric functions* QSym_n , which sits between Sym_n and Poly_n (i.e. $\text{Sym}_n \subset \text{QSym}_n \subset \text{Poly}_n$). Much of the combinatorics of Sym_n was originally extended to QSym_n before it was extended to Poly_n . Adding the red entries to the “slides” to create the “glides” turns out to be the natural way to introduce the extra parameter β .

Lascoux atoms $\overline{\mathfrak{A}}_a$

Next, we introduce the definitions needed for the Lascoux atoms. A *skyline diagram* of shape $a = (a_1, \dots, a_n)$ consists of n left justified rows of boxes such that the i th row from the bottom has a_i boxes. A *set-valued skyline filling* T of shape a is a filling of each box of the skyline diagram of a with a nonempty set of positive integers. The largest entry of each box is called the *anchor*, and the other entries are *free*. We say T is *semistandard* if it satisfies the following properties:

- (S.1) There are no repeat entries in any column.
- (S.2) The entries weakly decrease across rows, in the sense that the smallest entry in each box is at least the largest entry in the box to its right.
- (S.3) Suppose the anchors α, β , and γ satisfy all of the following conditions:
 - α is immediately to the right of β .
 - The row containing α and β is at least as long as the row containing γ if it is lower than the row containing γ , or is strictly longer if it is higher than the row containing γ .

- γ is either above α in the same column or below β in the same column.

Then we must have either $\gamma < \alpha$ or $\gamma > \beta$.

Equivalently, this says that if α, β , and γ are arranged in either an “upside down L” or “backwards L” as shown below such that the row containing the base of the L is longer (strictly if it is the higher row, and weakly if it is the lower row), then γ cannot be equal to or between α and β :

	γ		β	α
	$\vdots$		$\vdots$	
β	α		γ	

(Note that since α is to the right of β , we must have $\alpha \leq \beta$ by (S.2)).

(S.4) Each free entry is assigned to the box in its column with the smallest possible anchor such that (S.2) is satisfied (so choosing the column of a free entry uniquely determines its row as well).

(S.5) Each anchor in the leftmost column equals the index of its row.

We write $\overline{\mathfrak{A}}\text{SSF}(a)$ for the set of all semistandard skyline tableaux of shape a . The *weight* $\text{wt}(T)$ of a tableau T is the weak composition whose i th part is the number of times i appears as an entry in T .

The *Lascoux atom* $\overline{\mathfrak{A}}_a$ is the generating function

$$\overline{\mathfrak{A}}_a := \sum_{T \in \overline{\mathfrak{A}}\text{SSF}(a)} \beta^{\text{ex}(T)} \mathbf{x}^{\text{wt}(T)}.$$

Remark 2.7. The Lascoux atoms are the K -analogue of the Demazure atoms $\mathfrak{A}_a$, which are the $\beta = 0$ case and hence involve only the anchors and no free entries. The Demazure atoms are a modification of the *Demazure characters* (also called *key polynomials*) from [Dem74], which were originally defined algebraically in terms of divided difference operators and were later shown in [LS88] to also have a combinatorial description in terms of *key tableaux*. We can think of these key tableaux and skyline tableaux as nonsymmetric analogues of SSYT. Although the rules for defining these tableaux are more complicated, they turn out to have some nice associated combinatorics that is analogous to the combinatorics of SSYT, such as analogues of the Littlewood-Richardson rule for multiplying Schur polynomials ([Bru+21]).

Transition formula from Lascoux atoms $\overline{\mathfrak{A}}_a$ to kaons $\overline{\mathfrak{P}}_a$

We now give the definitions needed to state the $\overline{\mathfrak{A}}_a$ to $\overline{\mathfrak{P}}_a$ transition formula. A skyline tableau T has the *meson-highest* property if for every i that shows up as an entry in T , at least one of the following holds:

- There is an i that is an anchor in row i (i.e. row i is nonempty).
- There is an occurrence of the next smallest entry (which we denote $i^\uparrow$) that is weakly to the right of and in a different box from the leftmost i .

The set $\overline{\mathfrak{A}}2\overline{\mathfrak{P}}(a)$ consists of all semistandard meson-highest tableaux of shape a . We write $|a|$ as the sum of the entries of a and $|T|$ for the total number of entries of T . We can now state the first expansion formula:

Theorem 2.8 (Monical, Pechenik, and Searles [MPS21], Theorem 3.12). *The kaon expansion of the Lascoux atoms is*

$$\overline{\mathfrak{A}}_a = \sum_{T \in \overline{\mathfrak{A}}2\overline{\mathfrak{P}}(a)} \beta^{|T| - |a|} \overline{\mathfrak{P}}_{\text{wt}(T)}.$$

Example 2.9. The tableaux in $\overline{\mathfrak{A}}2\overline{\mathfrak{P}}(0, 0, 1, 2)$ are below, grouped into columns by number $|T| - |a|$ of free entries (i.e. the exponent on β), with the anchors shown bolded and the other entries unbolded. The corresponding term in the kaon expansion of $\overline{\mathfrak{A}}_{(0,0,2,2)}$ is listed below each tableau. Note that there are

actually an extra two rows numbered 1 and 2 at the bottom of each of these tableaux, which we omit because they are empty.

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		$\beta^2 \cdot \overline{\mathfrak{P}}_{(1,0,2,2)}$	$\beta^3 \cdot \overline{\mathfrak{P}}_{(2,0,2,2)}$																										

Example 2.10. The tableaux in $\overline{\mathfrak{A}2\mathfrak{P}}(0, 0, 2, 2)$ are shown below, again grouped in columns by the power of β , and with the corresponding monomial listed below each tableau. As in Example 2.9, each tableau actually has an extra two rows numbered 1 and 2 at the bottom that are not drawn here.

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Glide polynomials $\overline{\mathfrak{F}}_a$ and quasiLascoux polynomials $\overline{\mathfrak{Q}}_a$

We will now define the last two types of polynomials and explain the transition formula between them. For weak compositions a and b , a^+ denotes the nonzero elements of a . We say that b *dominates* a and write $b \geq a$ if $b_1 + \cdots + b_i \geq a_1 + \cdots + a_i$ for every i . Then the *glide polynomial* $\overline{\mathfrak{F}}_a$ is

$$\overline{\mathfrak{F}}_a := \sum_{\substack{b \geq a \\ b^+ = a^+}} \overline{\mathfrak{P}}_b,$$

and analogously, the *quasiLascoux polynomial* $\bar{\Omega}_a$ is

$$\bar{\Omega}_a := \sum_{\substack{b \geq a \\ b^+ = a^+}} \bar{\Omega}_b.$$

A *set-valued quasi-skyline filling* of shape a is the same as a semistandard set-valued filling, except that (S.5) is replaced with the following property:

- (S.5') The leftmost anchor in row i is at most i , and the anchors in the left column decrease from top to bottom.

We write $\bar{\Omega}\text{SSF}(a)$ for the set of set-valued quasi-skyline fillings of shape a . Then $\bar{\Omega}\text{SSF}(a)$ is a superset of $\bar{\Omega}\text{SSF}(a)$ of a , and in fact

$$\bar{\Omega}\text{SSF}(a) = \bigcup_{\substack{b \geq a \\ b^+ = a^+}} \bar{\Omega}\text{SSF}(b).$$

It follows that we can alternately define $\bar{\Omega}_a$ as

$$\bar{\Omega}_a = \sum_{T \in \bar{\Omega}\text{SSF}(a)} \beta^{|T| - |a|} \mathbf{x}^{\text{wt}(T)}.$$

A tableau $T \in \bar{\Omega}\text{SSF}(a)$ is *quasiYamanouchi* if for every i in T , at least one of the following holds:

- The leftmost i is the anchor in the leftmost column of row i .
- There is an $i + 1$ weakly to the right of the leftmost i and in a different box.

We write $\bar{\Omega}2\bar{\mathfrak{F}}(a)$ for the set of quasiYamanouchi tableau of shape a . We can now state the second expansion formula that we will need:

Theorem 2.11 (Monical, Pechenik, and Searles [MPS21], Theorem 4.12). *The glide expansion for the quasiLascoux polynomials is*

$$\bar{\Omega}_a = \sum_{T \in \bar{\Omega}2\bar{\mathfrak{F}}(a)} \beta^{|T| - |a|} \bar{\mathfrak{F}}_{\text{wt}(T)}.$$

Example 2.12. Returning to $a = (0, 0, 1, 2)$ as in Example 2.9, the two tableaux from the second row in Example 2.9 are in $\bar{\Omega}2\bar{\mathfrak{P}}(0, 0, 1, 2)$ but not in $\bar{\Omega}2\bar{\mathfrak{F}}(0, 0, 1, 2)$ because they have a 1 that is not an anchor in the left column its own row but do not have any 2's, violating the quasiYamanouchi property. Thus, the tableaux in the glide expansion of $\bar{\Omega}_{(0,0,1,2)}$ are the ones shown below, with the corresponding terms of the expansion listed below them. Again, we omit the bottom two empty rows of each tableau.

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$\bar{\mathfrak{F}}_{(0,0,1,2)}$		$\beta \cdot \bar{\mathfrak{F}}_{(0,0,2,2)}$		$\beta^2 \cdot \bar{\mathfrak{F}}_{(0,1,2,2)}$		$\beta^3 \cdot \bar{\mathfrak{F}}_{(0,2,2,2)}$		$\beta^4 \cdot \bar{\mathfrak{F}}_{(1,2,2,2)}$		$\beta^5 \cdot \bar{\mathfrak{F}}_{(2,2,2,2)}$	

Example 2.13. Using $a = (0, 0, 2, 2)$ as in Example 2.10, the four tableaux with a 1 but not a 2 are no longer valid, as they violate the quasiYamanouchi property. Thus, the tableaux in $\bar{\Omega}2\bar{\mathfrak{F}}(0, 0, 2, 2)$ and their corresponding monomials in the glide expansion of $\bar{\Omega}_{(0,0,2,2)}$ are the ones shown below. As in our other

examples, the two empty rows at the bottom are omitted.

<table><tr><td>4</td><td>4</td></tr><tr><td>3</td><td>3</td></tr></table>	4	4	3	3	<table><tr><td>4</td><td>43</td></tr><tr><td>3</td><td>2</td></tr></table>	4	43	3	2	<table><tr><td>4</td><td>43</td></tr><tr><td>32</td><td>2</td></tr></table>	4	43	32	2	<table><tr><td>4</td><td>432</td></tr><tr><td>32</td><td>1</td></tr></table>	4	432	32	1	<table><tr><td>4</td><td>432</td></tr><tr><td>321</td><td>1</td></tr></table>	4	432	321	1
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$\widetilde{\mathfrak{F}}_{(0,1,2,1)}$	$\beta \cdot \widetilde{\mathfrak{F}}_{(0,2,2,1)}$																							

2.3 Proof of Theorem 2.1 and examples of the involution

We first consider $\sum_b Q_b^a(-1)$. By Theorem 2.8,

$$\sum_b Q_b^a(-1) = \sum_{T \in \overline{\mathfrak{A}2\overline{\mathfrak{P}}}(a)} (-1)^{|T|-|a|}.$$

To prove that this equals 0 or 1, construct the following sign reversing involution ι , defined on either the full set of tableaux $T \in \overline{\mathfrak{A}2\overline{\mathfrak{P}}}(a)$ or the set $\overline{\mathfrak{A}2\overline{\mathfrak{P}}}(a) \setminus \{T_0\}$ for one particular tableau T_0 :

1. Set m to be the smallest entry appearing in T .
2. Find the rightmost column of T where a free m can be either added or removed to get a new tableau that is still in $\overline{\mathfrak{A}2\overline{\mathfrak{P}}}(a)$, and add or remove m from that column.
3. If no such column exists, reset m to be the next smallest entry $m^\uparrow$ and repeat step 2. Continue until either step 2 succeeds or m is equal to the maximum entry of T , in which case we take $\iota(T)$ to be undefined.

We can see that ι is well-defined (except in the case it is undefined) because removing an m in a given column can only be done in one way, and (S.5) guarantees that adding a free m can also be done in only one way. Also, it is sign reversing since $|\iota(T)| = |T| \pm 1$, as we are either adding or removing one entry of T to get $\iota(T)$. Thus, it remains to check that ι is an involution and is either defined for all $T \in \overline{\mathfrak{A}2\overline{\mathfrak{P}}}(a)$ or undefined for exactly one $T \in \overline{\mathfrak{A}2\overline{\mathfrak{P}}}(a)$. To show these facts, we use the following lemma:

Lemma 2.14. *The entry m that is added or removed to get from T to $\iota(T)$ is the largest entry such that for every $m' < m$ in T , row m' has all entries equal to m' and is weakly longer than all rows above it.*

Proof. First we note that for the choice of m described in Lemma 2.14, no $m' < m$ can possibly be added to or removed from a column of T . We cannot remove an m' because every m' is an anchor. We also cannot add a free m' because the only columns that do not already contain an m' are those in rows longer than row m' , meaning rows below row m' , and those rows have anchors less than m' , meaning adding a free m' would violate (S.2). This shows that the m chosen for ι must be at least the m described in Lemma 2.14.

It remains to show that this choice of m actually can be added to or removed from some column of T . We can remove all rows of T below row m without changing which entry of T will be added or removed, since no entry from one of those rows will be the entry chosen or impact which entry is chosen. Thus, we may assume that m is the smallest entry of T . We consider several cases:

- If T contains multiple m 's and at least one of them is free, then we can remove the rightmost free m without violating any conditions. Thus, we can assume that either T contains only one m , or all m 's in T are anchors.
- Suppose T contains only one m , the m is free, and it is not in the rightmost column. Then we can add a free m to the rightmost column without violating any conditions.

- Suppose T contains only one m , the m is free, and it is in the rightmost column. Then by the meson-highest condition, $m^\uparrow$ must also appear in that same rightmost column in a different box from m . However, if $m^\uparrow$ is an anchor in that column, then m should instead be in the same box of the column as $m^\uparrow$ by (S.5), since $m^\uparrow$ is the smallest anchor in that column. Similarly, if $m^\uparrow$ is not an anchor, then both m and $m^\uparrow$ must be in whichever box of the last column has the smallest anchor, again by (S.5). This contradicts m and $m^\uparrow$ being in different boxes, so this case is impossible.
- Suppose now that every m in T is an anchor, but that not every column of T contains an m . Consider the rightmost column not containing an m , say column c . Then column $c + 1$ contains an m as an anchor. This means we can add a free m to column c , since adding an m to the box immediately to the left of the one containing the m in column $c + 1$ will not violate (S.2) or any other conditions.
- The only remaining case is that every column of T contains an m and all the m 's are anchors. The m in the leftmost column must then be in row m , so row m must be nonempty, and it must be the lowest row of T since m is the smallest entry. By (S.2) and minimality of m , every other box of row m must then contain an m and no free entries. The only thing left to check is that no m appears in a higher row, or equivalent, row m is at least as long as every other row. If not, let c be the rightmost column of row m . Then column $c + 1$ contains an m in some higher row. Taking that m together with the box to its left and the m in column c then creates an upside-down L shape violating (S.3), since both m 's are anchors and thus $\beta = \gamma = m$. This is a contradiction, so we conclude that row m has maximal length. But then m is not the maximal entry satisfying the condition of Lemma 2.14, since $m^\uparrow$ also satisfies the condition. Thus, this case is not possible.

Thus, in each possible case, some m can be added or removed from T , which shows that the m chosen is indeed the one described in the lemma statement. $\square$

Now to show that ι is an involution on the subset of $\overline{\mathfrak{A}2\overline{\mathfrak{P}}}(a)$ on which it is defined, note that as long as the same m and the same column is chosen when applying ι to $\iota(T)$ as when applying it to T , we will get $\iota^2(T) = T$, since the operations of adding and removing an m from the same column cancel out. Thus, we consider the possible ways that a different m or column could be chosen when computing $\iota(\iota(T))$:

- The same m is chosen, but in a different column. Whichever column was used when finding $\iota(T)$ is certainly also a valid column to modify to find $\iota(\iota(T))$, so the only potential issue is if there is a column of $\iota(T)$ further to the right in which we can add or remove an m . But then that same column would also have been modified when finding $\iota(T)$ instead of the one that was modified, because the existence or nonexistence of an extra m in some column further to the left will not impact whether or not (S.1)-(S.5) and the meson-highest condition hold if an m is added or removed from a column further to the right, so modifying the further right column is valid for $\iota(T)$ if and only if it is valid for T . This is a contradiction, so provided the same m is chosen, the same column will also be chosen.
- A different m is chosen. One potential issue is that after adding or removing an m , a smaller value m' can now be either added to or removed from T , but this cannot happen because m' is the value chosen if and only if the condition in Lemma 2.14 applies to m' , and that condition is not impacted by adding or removing an entry of T that is larger than m' . The only other potential issue is that there is only one m in T and it gets removed when going from T to $\iota(T)$, so $\iota(T)$ no longer contains an m . For this to happen, the lone m must be in the rightmost column of T , because otherwise it can be added to the rightmost column without violating any of the conditions. It must also be free, since otherwise it could not be removed. But by the argument in the proof of Lemma 2.14, it is not possible that there is only one m in T and that the lone m is a free entry in the far right column. Thus, this case cannot actually happen.

Thus, $\iota(T)$ is an involution. It remains to show that $\iota(T)$ is undefined for exactly one $T \in \overline{\mathfrak{A}2\overline{\mathfrak{P}}}(a)$ if the parts of a are nonincreasing, and is defined for all $T \in \overline{\mathfrak{A}2\overline{\mathfrak{P}}}(a)$ otherwise. But $\iota(T)$ is undefined if and only if the condition in Lemma 2.14 applies to every entry of T , which is equivalent to saying that all nonempty rows of T are weakly decreasing in length and that every box of T is filled with a single entry equal to the corresponding row number. No such tableau T exists if the nonzero parts of a are not weakly decreasing, and exactly one such T exists if they are weakly decreasing. Thus, the sum is 0 if the parts of a are not

weakly decreasing, and if they are, the sum is 1 because the single unpaired T has no free entries and thus has positive sign since $|T| - |a| = 0$.

The proof for $\sum_b M_b^a(-1)$ is essentially identical. Instead of Theorem 2.8, we use Theorem 2.11 to get

$$\sum_b M_b^a(-1) = \sum_{T \in \overline{\Omega}2\overline{\mathfrak{F}}(a)} \beta^{|T| - |a|} \overline{\mathfrak{F}}_{\text{wt}(T)}.$$

We then define ι in the same manner as before, except that it is an involution on either the full set of tableau $T \in \overline{\Omega}2\overline{\mathfrak{F}}$ or on the set $\overline{\Omega}2\overline{\mathfrak{F}} \setminus \{T_0\}$ for one particular tableau T_0 . The only difference in the proof is that if m is the entry chosen by ι and $m' < m$ is another entry of T , it could be the case that instead of all the occurrences of m' appearing as anchors in row m' , they all appear as anchors in some higher row. Thus, m will instead be the smallest entry such that for every $m' < m$, all occurrences of m' appear as anchors in the same row, and that row is weakly longer than all rows above it.

So, $\iota(T)$ will be undefined if and only if the nonzero elements of a are weakly decreasing, T has no free entries, and every row of T contains only repeats of the same entry. We claim that in fact the repeated entry in each row must equal the row index. If not, there is some m that is the anchor of a row higher than row m . Then by the quasiYamanouchi condition, $m^\uparrow = m + 1$, since m is not an anchor in its own row. Then $m + 1$ must also be the anchor of some row higher than its own row. By the same logic, we must have $(m + 1)^\uparrow = m + 2$, and $m + 2$ must then be the anchor of a row higher than row $m + 2$. Continuing this logic inductively, we find that the largest entry M appearing in T is also an anchor of some row higher than row M . But then the quasiYamanouchi condition requires T to contain an entry equal to $M + 1$, which is a contradiction. So in fact, every entry m of a tableau T for which $\iota(T)$ is undefined must be an anchor in its own row m rather than in another row.

This shows that again, there is exactly one unpaired tableau T in the case where the parts of a are weakly decreasing and no unpaired tableaux otherwise. As with $\sum_b Q_b^a(-1)$, in the case where such an unpaired tableau exists, it appears with positive sign since it has no free entries. Thus, we get that $\sum_b M_b^a(-1) = 1$ if the parts of a are weakly decreasing and $\sum_b M_b^a(-1) = 0$ otherwise, as desired. This completes the proof of Theorem 2.1. $\square$

We end with two examples to illustrate the involution.

Example where $\sum_b Q_b^a(-1) = \sum_b M_b^a(-1) = 0$

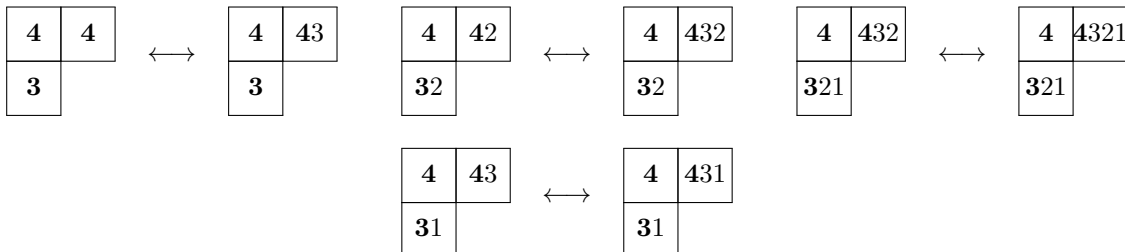
Example 2.15. Revisiting $a = (0, 0, 1, 2)$ as in Examples 2.9 and 2.12, we get

$$\begin{aligned} \sum_b Q_b^a(\beta) &= 1 + \beta + 2\beta^2 + 2\beta^3 + \beta^4 + \beta^5, \\ \sum_b M_b^a(\beta) &= 1 + \beta + \beta^2 + \beta^3 + \beta^4 + \beta^5. \end{aligned}$$

Plugging in $\beta = -1$ gives

$$\begin{aligned} \sum_b Q_b^a(\beta) &= 1 - 1 + 2 - 2 + 1 - 1 = 0, \\ \sum_b M_b^a(\beta) &= 1 - 1 + 1 - 1 + 1 - 1 = 0. \end{aligned}$$

This matches Theorem 2.1 since the nonzero parts of $a = (0, 0, 1, 2)$ are not weakly decreasing, so we expect the sums to be 0. The pairing induced by ι for the tableaux in $\overline{\mathfrak{A}}_2\overline{\mathfrak{P}}$ is shown below:



The pairing of the tableaux in $\overline{\Omega 2 \mathfrak{F}}$ is the same except with the two tableaux in the bottom row omitted.

Example where $\sum_b M_b^a(-1) = \sum_b Q_b^a(-1) = 1$

Example 2.16. Revisiting $a = (0, 0, 2, 2)$ as in Examples 2.10 and 2.13, we get

$$\sum_b Q_b^a(\beta) = 3 + 4\beta + 2\beta^2 + \beta^3 + \beta^4,$$

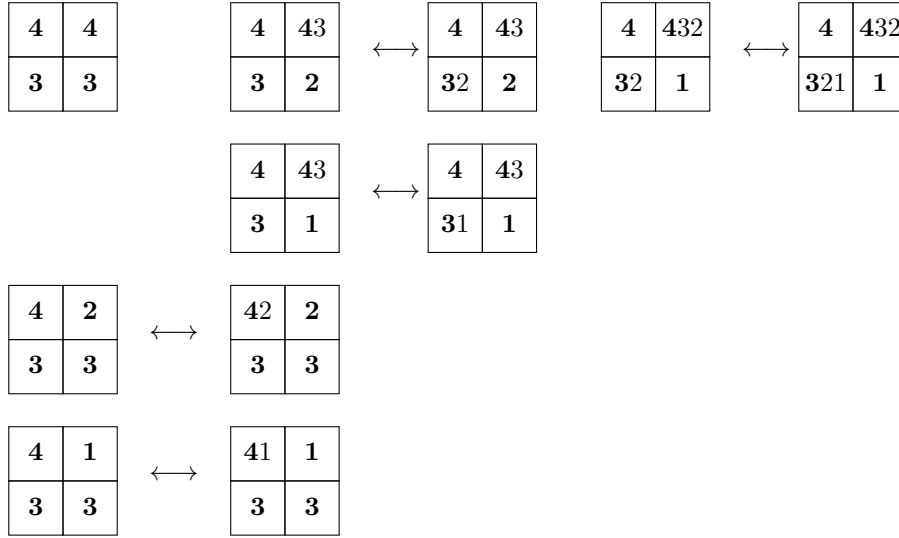
$$\sum_b M_b^a(\beta) = 2 + 2\beta + \beta^2 + \beta^3 + \beta^4.$$

When $\beta = -1$, these sums become

$$\sum_b Q_b^a(-1) = 3 - 4 + 2 - 1 + 1 = 1,$$

$$\sum_b M_b^a(-1) = 2 - 2 + 1 - 1 + 1 = 1.$$

This again matches Theorem 2.1 because the nonzero parts of $a = (0, 0, 2, 2)$ are in weakly decreasing order, so we expect both sums to be 1. The pairing of the tableaux in $\overline{\mathfrak{A} 2 \mathfrak{P}}(0, 0, 2, 2)$ induced by ι is shown below. Note that the top left tableau is the unique one that is unpaired.



The pairing of tableaux in $\overline{\Omega 2 \mathfrak{F}}$ is the same except without the four tableaux containing 1's but not 2's (i.e. the tableaux in the second and fourth rows from the top).

Chapter 3

Background on the chromatic and Kromatic symmetric functions

3.1 Introduction

The remaining chapters of this thesis will focus on variations of the *chromatic symmetric function* X_G . Its history begins with a famous and deceptively simple question, first posed by Guthrie in 1852:

Question 3.1. *How many colors are needed to color the countries on a map so that no two bordering countries are the same color?*

The answer is four, and this result, known as the *four-color theorem*, is famous for being surprisingly difficult to prove: it remained open for over a century, and after many failed attempts, the only known proofs, due to Appel and Haken ([AHK77]) and Robertson, Sanders, Seymour, and Thomas ([Rob+96]), rely on extensive computer searches. It was the first major theorem whose proof relied on computers, prompting philosophical questions about the role of computers in mathematics and what constitutes a proof ([App84; DL80; Tym79]).

Work on the four-color theorem also led to a great deal of mathematical theory being developed, much of which continues to be actively studied. In the language of graph theory, the *chromatic number* $\chi(G)$ is the minimum number of colors required to color the vertices of a graph so that no two adjacent vertices receive the same color (called a *proper coloring*). A strengthening of the four-color theorem that remains a major open question is the *Hadwiger conjecture* ([Had43]), which says that any graph with no K_t minor has chromatic number less than t . Another surprisingly difficult open problem in coloring theory is the *Hadwiger-Nelson problem* ([Had45]), which asks for the number of colors needed to color all points in the plane so that no two points one unit apart are the same color, also called the *chromatic number of the plane*. It can be shown fairly easily that the answer is between 4 and 7 inclusive ([Soi16]), but that was all that was known until 2018, when De Grey used a computer-aided construction of a graph on 1581 vertices to raise the lower bound to 5 ([De 18]). Various other graph coloring problems have and continue to be studied, such as colorings of graphs embedded in surfaces besides the plane ([LZZ04]), *list colorings* (where each vertex has a different list of possible colors [Bab09]), *edge colorings* (where edges sharing an endpoint may not receive the same color [Cao+19]), *Ramsey theory* (where edges are colored to avoid large monochromatic cliques [Rad11]), and, most relevantly for us, *multicolorings*, where each vertex may be assigned more than one color ([HK04]), which has applications to job scheduling problems ([TZP18]).

Of particular interest to us is the *chromatic polynomial* $\chi_G(k)$, which equals the number of proper colorings of G with k colors. It was introduced by Birkhoff in [Bir13] in an attempt to prove the four-color theorem. Although it did not prove helpful for that, it turned out to be interesting in its own right, and has been generalized in various ways. Whitney ([Whi31]) generalized it to arbitrary graphs, as Birkhoff had initially defined it only for planar graphs. An important bivariate generalization is the *Tutte polynomial*, introduced by Tutte in [Tut54] and generalized to arbitrary matroids by Crapo in [Cra69], which has connections to diverse areas such as knot theory ([Kwu+16]) and statistical physics ([WM00]).

The *chromatic symmetric function* $X_G(x_1, x_2, x_3, \dots)$ was introduced by Stanley in [Sta95] as a symmetric function generalization of the chromatic polynomial. It is a sum of monomials such that each monomial corresponds to a coloring and each variable to a color, and the exponent of a variable in a monomial represents the number of vertices receiving that color in the corresponding coloring. Stanley’s paper proves a number of results about X_G , including an expansion formula in the basis of *power sum symmetric functions*, which we will see in Chapter 5. He also makes several conjectures, which have led to two big open questions in the study of the chromatic symmetric function:

1. The first is the *Stanley-Stembridge conjecture* ([SS93; Sta95]), which says that X_G is *e-positive* whenever G is a claw-free incomparability graph of a poset. The *incomparability graph* of a poset P has a vertex for each element of P and an edge between each pair of elements that are incomparable in P , and being *e-positive* means that when X_G is expanded in the basis of *elementary symmetric functions* e_λ , all the coefficients are nonnegative. Stanley ([Sta95]) proved this for the special cases of cobipartite graphs (i.e. graphs whose complement is bipartite), paths, and cycles. Gasharov [Gas96] proved the weaker result that for the same set of graphs G (claw-free incomparability graphs), X_G is always *s-positive* (meaning it has nonnegative coefficients when expanded in the basis of *Schur functions* s_λ), by showing that the coefficients count combinatorial objects called *P-tableaux*. The *e-positivity* conjecture has since been proven in the special cases of *lollipop graphs* ([DW18]) and graphs that are claw-free and also avoid certain other induced subgraphs ([HHT19]). There have also been *e-positivity* results proven for various analogues of X_G , including a *quasisymmetric* version ([SW16]), a *nonsymmetric* version ([HW17]), and a version with noncommuting variables ([GS01]).
2. The second major open question about X_G is whether it always *distinguishes trees*, which originated because of an offhanded remark in Stanley’s original paper ([Sta95]) that he did not know of any two nonisomorphic trees with the same chromatic symmetric function. Much work has since been dedicated to trying to prove that no such pair of trees can exist. The chromatic symmetric function is known to distinguish all trees on up to 29 vertices ([SST15; HJ19]), various infinite families of trees including *caterpillars* and *spiders* ([Mor05; MMW08; AZ14; Ger17; Ali+23]), and several other infinite families of graphs that are not trees including *squids* ([MMW08]) and *trivially perfect graphs* ([Tsu18]). Various generalizations of X_G have also been shown to distinguish many or all trees, including a *rooted* version of X_G ([LW24]), a *group algebra* version ([Fol+21]), a *noncommutative* version ([GS01]), and a *quasisymmetric* version ([ADM23]). Various properties of a tree are known to be computable from X_G , including the *subtree polynomial* ([MMW08]), the path and degree sequence ([MMW08]), the number of vertices of degree at least 3 ([Cre22]), and counts of certain subtrees [Lyd16]. For general graphs (not necessarily trees), X_G determines the girth and the number of vertices, edges, connected components, matchings, and triangles ([OS14]). However, it does not distinguish all graphs: [OS14] and [Ali+21] construct infinite families of pairs of nonisomorphic graphs with the same X_G , and [CS] lists 1000 pairs of small graphs with equal chromatic symmetric functions.

Various generalizations of X_G have been studied, including several that involve multicolorings instead of colorings. The function Y_G (which we will revisit in Chapter 5), was studied by Stanley in [Sta98], and is defined as a sum of monomials corresponding to set colorings where each vertex receives a set of colors (possibly empty) such that adjacent vertices have nonoverlapping color sets. As a tool for his *s-positivity* proof, Gasharov [Gas96] also introduced a similar function $\tilde{X}_G^{\mathbf{m}}$ that tracks colorings where vertex i is assigned m_i colors for a fixed sequence of nonnegative integers $\mathbf{m} = (m_1, m_2, \dots)$.

Our focus will be on two other analogues of X_G . The first is the *Kromatic symmetric function* $\overline{X}_G$, which we will study in Chapters 4 and 5, and the second is the *chromatic symmetric homology* $H_*(G)$, which we will study in Chapter 6. $\overline{X}_G$ was introduced by Crew, Pechenik, and Spirkel [CPS23] as a *K-theoretic* analogue of X_G and was further studied by Marberg in [Mar23]. It is defined in the same way as Y_G except that each vertex must receive a nonempty set of colors. One of their main motivations for studying $\overline{X}_G$ was to test whether the Stanley-Stembridge conjecture is likely to have a topological interpretation, that is, whether X_G might correspond to some subvariety of the Grassmannian in the case where G is a claw-free incomparability graph, since X_G is known to be *s-positive* in that case and the cohomology classes of the Grassmannian correspond to Schur polynomials. They were able to lift Gasharov’s *s-positivity* result to the *K-theoretic* setting by showing that X_G expands positively in the basis of *Grothendieck functions* $\overline{s}_\lambda$.

(the K -theoretic analogue of Schur polynomials), but it does not seem to expand positively for any obvious K -theoretic lifting of the e -basis. This suggests that X_G may indeed have a topological interpretation, but likely not one that is useful for proving e -positivity.

In the remainder of this chapter, we will introduce background on graphs and colorings in §3.2 and define X_G and $\overline{X}_G$ in §3.3. Then we will explain three of the main results proven about $\overline{X}_G$ in [CPS23]: an expansion formula using a basis they call the *augmented monomial symmetric functions* (§3.4), a recursive *contraction-deletion relation* similar to the one satisfied by the chromatic polynomial (§3.5), and the Grothendieck expansion formula (§3.6). Finally, in §3.7, we will explain the Hopf algebra construction of $\overline{X}_G$ from [Mar23].

In the next two chapters, we will prove two new results about $\overline{X}_G$: in Chapter 4 we will show that the number of copies in G of certain induced subgraphs can be computed from $\overline{X}_G$, and in Chapter 5 we will give an expansion formula for $\overline{X}_G$ using a K -theoretic analogue of the *power sum symmetric functions* p_λ , which we will show has all integer coefficients. Then in Chapter 6 we will define the chromatic symmetric homology $H_*(G)$ and prove several results about it.

3.2 Graphs and colorings

A *graph* $G = (V, E)$ is a finite set $V(G)$ of *vertices* together with a set $E(G)$ of *edges* such that each edge is an unordered pair vw of vertices $v, w \in V(G)$. Two vertices v and w are *adjacent* or *neighbors* if $vw \in E(G)$, and $v \sim w$ means that v and w are adjacent. An *isolated vertex* is a vertex with no neighbors. The *order* of G is $|V(G)|$. A *tree* is a connected graph with no cycles. The *complete graph* K_n is the graph on n vertices with every two vertices adjacent. A *star graph* is a graph where one vertex is adjacent to all others but no other pairs of vertices are adjacent. Two graphs G and H are *isomorphic* if there is a bijection $\phi : V(G) \rightarrow V(H)$ such that

$$E(H) = \{\phi(v)\phi(w) : vw \in E(G)\}.$$

A *subgraph* of G is a graph H with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. An *induced subgraph* is a subgraph H with $V(H) \subseteq V(G)$ and

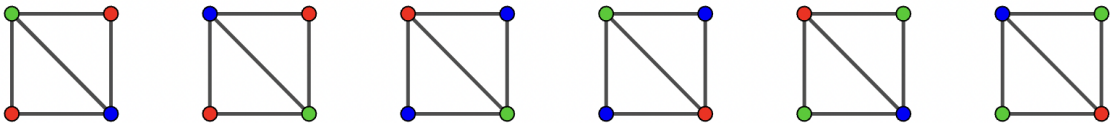
$$E(H) = \{vw : v, w \in V(H), vw \in E(G)\}.$$

For $W \subseteq V$, we write $G|_W$ for the induced subgraph of G with vertex set W . An induced subgraph is a *clique* if every two vertices are adjacent and a *stable set* or *independent set* if no two vertices are adjacent. The *complement graph* of G is the graph $\overline{G}$ given by

$$V(\overline{G}) = V(G), \quad E(\overline{G}) = \{vw : v, w \in V(G), vw \notin E(G)\}.$$

A *proper coloring* κ of a graph G is a function $\kappa : V(G) \rightarrow \mathbb{N}$ such that if $vw \in E(G)$, then $\kappa(v) \neq \kappa(w)$, where $\mathbb{N} = \{1, 2, 3, \dots\}$ denotes the set of positive integers. The *chromatic polynomial* $\chi_G(k)$ is number of proper colorings of G using only the colors $1, 2, \dots, k$. The fact that $\chi_G(k)$ is always a polynomial is surprising, but can be proven recursively through a *contraction-deletion relation* which we will describe shortly. First, however, we will give examples of what $\chi_G(k)$ is in a few special cases:

Example 3.2. For the graph G below, $\chi_G(3) = 6$ since there are 6 proper colorings of $V(G)$ with 3 colors:



Example 3.3. If G is any tree on n vertices, then $\chi_G(k) = k(k-1)^{n-1}$. To see this, note first that there are k ways to color the first vertex. Then we can color each new vertex one at a time in such a way that each vertex has a unique colored neighbor at the time it is being colored. We can thus assign the new vertex any color except the color of its colored parent, so it has $k-1$ options. Thus, the chromatic polynomial is the same for all trees on n vertices.

Example 3.4. If $G = \overline{K_n}$ is an edgeless graph, then $\chi_G(k) = k^n$ since we can independently assign any one of the k colors to each vertex. If $G = K_n$ is a complete graph, then $\chi_G(k) = k(k-1)\dots(k-n+1)$, since there are k choices for the first vertex, then $k-1$ for the second (since it must be a different color from the first), then $k-2$ for the third (since it must be a different color than the first two), and so on.

Now the reason why $\chi_G(k)$ is a polynomial follows from the fact that $\chi_{\overline{K_n}}(k) = k^n$ and the *contraction-deletion relation*

$$\chi_G(k) = \chi_{G \setminus e}(k) - \chi_{G/e}(k), \quad (3.1)$$

where $e \in E(G)$ is any edge, $G \setminus e$ denotes G with the edge e removed, and G/e denotes G with the edge e *contracted*, which means e is removed and its two endpoints v and w are replaced with a single vertex connected to all vertices that were originally neighbors of v or w or both.

Proof of the contraction-deletion relation. Every proper coloring of G is also a proper coloring of $G \setminus e$, since removing an edge can only make the graph easier to color (as there are now fewer restrictions). However, some proper colorings of $G \setminus e$ are not actually proper colorings of G , namely, those in which v and w get the same color. But proper colorings of G where v and w get the same color correspond precisely to proper colorings of G/e (as we can give the new contracted vertex is given the shared color of v and w). Thus, the number of colorings of $G \setminus e$ with k colors that are not also proper colorings of G is equal to the number of proper colorings of G/e with k colors. It follows that the number $\chi_G(k)$ of proper colorings of G with k colors equals the number $\chi_{G \setminus e}(k)$ of proper colorings of $G \setminus e$ with k colors minus the number $\chi_{G/e}$ of proper colorings of G/e with k colors. $\square$

The fact that $\chi_G(k)$ is a polynomial then follows by induction on the number of edges of G , with the base case being the edgeless graph $\overline{K_n}$.

3.3 The chromatic and Kromatic symmetric functions

The chromatic symmetric function was introduced by Stanley in 1995 ([Sta95]) as a symmetric function generalization of the chromatic polynomial:

Definition 3.5 (Stanley [Sta95]). The *chromatic symmetric function* is

$$X_G(x_1, x_2, x_3, \dots) := \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)},$$

where κ ranges over all proper colorings of G , where a *proper coloring* means that no two adjacent vertices receive the same color. For a weighted graph (G, ω) , we can analogously define

$$X_{(G, \omega)} := \sum_{\kappa} \prod_{v \in V(G)} x_{\kappa(v)}^{\omega(v)}.$$

We can see that $\chi_G(k)$ is a specialization of X_G because when we set the first k variables to 1 and all others to 0, we get

$$X_G(\underbrace{1, 1, \dots, 1}_k, 0, 0, \dots) = \chi_G(k),$$

since monomials corresponding to colorings using only the first k colors become 1 while all others become 0.

In [CPS23], Crew, Pechenik, and Spirkel introduced the following K -theoretic analogue of X_G :

Definition 3.6 (Crew, Pechenik, and Spirkel [CPS23]). A *proper set coloring* of G is a function $\kappa : V(G) \rightarrow 2^{\mathbb{N} \setminus \{\emptyset\}}$ such that $\kappa(v) \cap \kappa(w) = \emptyset$ whenever $vw \in E(G)$, so each vertex receives a nonempty set of colors such that adjacent vertices have nonoverlapping color sets. For a vertex-weighted graph G with weight function $\omega : V(G) \rightarrow \mathbb{N}$, the *Kromatic symmetric function* is

$$\overline{X}_{(G, \omega)} := \sum_{\kappa} \prod_{v \in V(G)} \left(\prod_{i \in \kappa(v)} x_i \right)^{\omega(v)},$$

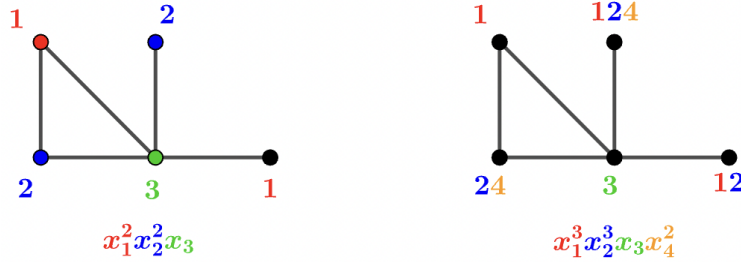
where the sum ranges over all proper set colorings κ . If no weight function is specified, we will assume all vertices have weight 1.

Example 3.7. For the graph from Example 3.2, if we restrict to 3 variables $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$, then there are no other proper colorings or proper set colorings besides the ones shown, so

$$X_G(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = \overline{X}_G(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3) = 2\mathbf{x}_1^2\mathbf{x}_2\mathbf{x}_3 + 2\mathbf{x}_1\mathbf{x}_2^2\mathbf{x}_3 + 2\mathbf{x}_1\mathbf{x}_2\mathbf{x}_3^2.$$

This evaluates to $\chi_G(3) = 6$ when $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}_3 = 1$.

Example 3.8. Below is a proper coloring of a graph and the corresponding monomial in X_G , followed by a proper set coloring of the same graph and the corresponding monomial in $\overline{X}_G$:



Example 3.9. For the complete graph K_n , we have

$$X_{K_n} = e_n, \quad \overline{X}_{K_n} = \sum_{m \geq n} S(m, n) e_m,$$

where $S(m, n)$ is the *Stirling number* that counts set partitions of a set of size m into n nonempty parts. The first formula follows because for a proper coloring of K_n , all the vertices must be assigned different colors. The second follows because for a proper set coloring of K_n , again no color may be used twice, but we must also choose how to partition the colors among the n vertices such that each vertex receives at least one color.

Example 3.10. For the edgeless graph $\overline{K_n}$, we have

$$X_{\overline{K_n}} = (x_1 + x_2 + x_3 + \dots)^n, \quad \overline{X}_{\overline{K_n}} = (e_1 + e_2 + e_3 + \dots)^n.$$

The first formula follows because for a proper coloring of $\overline{K_n}$, we can independently choose any color for each of the n vertices, and the second follows because for a proper set coloring of $\overline{K_n}$, we can independently choose any nonempty set of colors for each vertex.

Note that the $X_{(G, \omega)}$ is the homogeneous part of $\overline{X}_{(G, \omega)}$ of lowest degree, since it represents the colorings where we assign just one color to each vertex.

3.4 Monomial expansion

The Kromatic symmetric function $\overline{X}_{(G, \omega)}$ seems a bit unwieldy, since it is a sum of infinitely many terms. However, it can be expressed as a finite sum of certain basis vectors $\overline{m}_\lambda$, which are a K -theoretic analogue of the augmented monomial symmetric functions, and there is an efficient way to recursively compute this $\overline{m}$ -expansion, which we will see in §3.5. The $\overline{m}$ -expansion also turns out to be a useful tool for proving that certain properties of a graph G can be recovered from $\overline{X}_G$, which will be our main focus in Chapter 4.

The authors of [CPS23] define the *K -theoretic augmented monomial symmetric functions* $\overline{m}_\lambda$ as

$$\overline{m}_\lambda := \overline{X}_{K_\lambda},$$

where K_λ is the complete graph on $\ell(\lambda)$ vertices whose weights $\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}$ are the parts of λ . A *stable set cover* is a set $C = \{V_1, \dots, V_k\}$ where $V_1, \dots, V_k \subseteq V(G)$ are distinct nonempty independent sets with

$V_1 \cup \dots \cup V_k = V(G)$. We write $\text{SSC}(G)$ for the collection of all stable set covers of G . To each $C \in \text{SSC}(G)$, we associate a partition $\lambda(C)$ whose parts are the weights $\omega(V_1), \dots, \omega(V_k)$ of the stable sets, where the weight of a subset is $\omega(V_i) := \sum_{v \in V_i} \omega(v)$, i.e. the sum of the weights of all the vertices in S_i . They then prove the following $\widetilde{m}$ -expansion formula for $\overline{X}_G$:

Theorem 3.11 (Crew, Pechenik, and Spirkl [CPS23], Proposition 3.4). $\overline{X}_G$ satisfies

$$\overline{X}_{(G, \omega)} = \sum_{C \in \text{SSC}(G)} \widetilde{m}_{\lambda(C)}.$$

Proof idea. For each color used by a coloring κ , the vertices receiving that color form a stable set, and the collection of all stable sets that correspond to some color in this manner is a stable set cover C . The monomial corresponding to κ will then show up in the $\widetilde{m}_{\lambda(C)}$ term of the $\widetilde{m}$ -expansion, because it corresponds to a coloring of $K_{\lambda(C)}$ with the same associated monomial. This coloring of $K_{\lambda(C)}$ is formed by labeling the vertices of $K_{\lambda(C)}$ with the stable sets $V_1, \dots, V_k \in C$ such that the vertex labeled V_i has weight $\omega(V_i)$. Then, in the coloring of $K_{\lambda(C)}$, the vertex labeled V_i gets assigned precisely the colors that were assigned by κ to the vertices in V_i and to no other vertices. $\square$

Example 3.12. Let $G = \{u, v, w\}$ be the unweighted path graph on 3 vertices, with $u \sim v, v \sim w$, and $u \nsim w$. Then the nonempty stable sets are $\{u\}, \{v\}, \{w\}$, and $\{u, w\}$. The stable set covers and their corresponding partitions are listed below:

Stable set cover	Partition
$\{\{u\}, \{v\}, \{w\}\}$	1^3
$\{\{u, w\}, \{v\}\}$	21
$\{\{u, w\}, \{u\}, \{v\}\}$	21^2
$\{\{u, w\}, \{v\}, \{w\}\}$	21^2
$\{\{u, w\}, \{u\}, \{v\}, \{w\}\}$	21^3

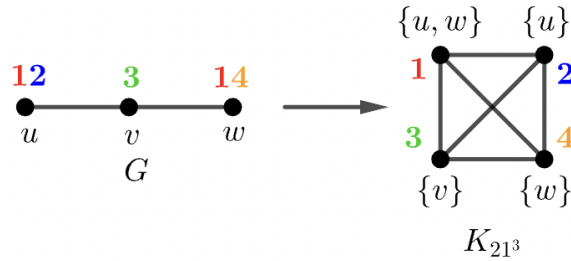
Thus, the $\widetilde{m}$ -expansion of $\overline{X}_G$ is

$$\overline{X}_G = \widetilde{m}_{1^3} + \widetilde{m}_{21} + 2\widetilde{m}_{21^2} + \widetilde{m}_{21^3}.$$

Now consider the coloring

$$\kappa(u) = \{\mathbf{1}, \mathbf{2}\}, \quad \kappa(v) = \{\mathbf{3}\}, \quad \kappa(w) = \{\mathbf{1}, \mathbf{4}\},$$

shown below:



We can see that $\kappa(u) \cap \kappa(v) = \kappa(v) \cap \kappa(w) = \emptyset$, so this is a proper set coloring. The corresponding monomial in $\overline{X}_G$ is $\mathbf{x}_1^2 \mathbf{x}_2 \mathbf{x}_3 \mathbf{x}_4$. To find the corresponding stable set cover, we record for each stable set the colors assigned to the vertices in that stable set and to no other vertices:

Stable set	Colors
$\{u, w\}$	$\mathbf{1}$
$\{u\}$	$\mathbf{2}$
$\{v\}$	$\mathbf{3}$
$\{w\}$	$\mathbf{4}$

Since there is a nonempty set of colors assigned to all four stable sets, the corresponding stable set cover is $\{\{u, w\}, \{u\}, \{v\}, \{w\}\}$, and the associated partition is 21^3 . Thus, this coloring should show up as a monomial in the $\widetilde{m}_{21^3}$ term of the $\widetilde{m}$ -expansion, meaning it should correspond to a coloring of the complete graph K_{21^3} with four vertices of weights 2, 1, 1, and 1. To find this coloring, we can label the vertices of K_{21^3} with the stable sets $\{u, w\}$, $\{u\}$, $\{v\}$, and $\{w\}$, and then assign to each vertex the colors assigned by κ to its corresponding stable set, as shown above.

3.5 Contraction-deletion relation

The contraction-deletion relation gives us an efficient way to recursively compute $\overline{X}_{(G, \omega)}$:

Theorem 3.13 (Crew, Pechenik, and Spirkel [CPS23], Proposition 3.6). *For any two vertices $v, w \in V(G)$ with $vw = e \notin E(G)$,*

$$\overline{X}_{(G, \omega)} = \overline{X}_{(G/e, \omega/e)} + \overline{X}_{(G \cup e, \omega)} + \overline{X}_{(G^1, \omega^1)} + \overline{X}_{(G^2, \omega^2)} + \overline{X}_{(G^*, \omega^*)},$$

where:

- In $(G/e, \omega/e)$, v and w are combined into one vertex z_{vw} with weight $\omega(v) + \omega(w)$.
- In $(G \cup e, \omega)$, the edge $e = vw$ is added.
- In (G^1, ω^1) , e is added, v is connected to all neighbors of w , and $\omega(w)$ is added to $\omega(v)$.
- In (G^2, ω^2) , e is added, w is connected to all neighbors of v , and $\omega(v)$ is added to $\omega(w)$.
- In (G^*, ω^*) , e is added, as is a new weight $\omega(v) + \omega(w)$ vertex z^* connected to v, w , and their neighbors.

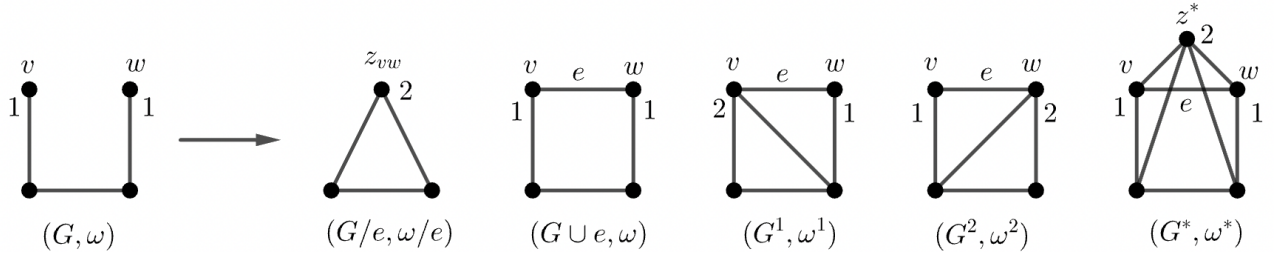
Proof idea. A proper set coloring of G corresponds to a proper set coloring of one of the 5 graphs on the right of the expansion formula, based on how $\kappa(v)$ and $\kappa(w)$ overlap. Specifically, it corresponds to:

- A coloring of G/e if $\kappa(v) = \kappa(w)$.
- A coloring of $G \cup e$ if $\kappa(v) \cap \kappa(w) = \emptyset$.
- A coloring of G^1 if $\kappa(v) \subsetneq \kappa(w)$, where in G^1 , v gets the colors assigned by κ to both v , and w gets the colors assigned by κ to only w .
- A coloring of G^2 if $\kappa(v) \supsetneq \kappa(w)$, where in G^2 , v gets the colors assigned by κ to only v and w gets the colors assigned to both.
- A coloring of G^* if $\kappa(v)$ and $\kappa(w)$ overlap, but neither is a subset of the other, where in G^* , z^* gets the colors assigned by κ to v and w , v gets the colors assigned just to v , and w gets the colors assigned just to w .

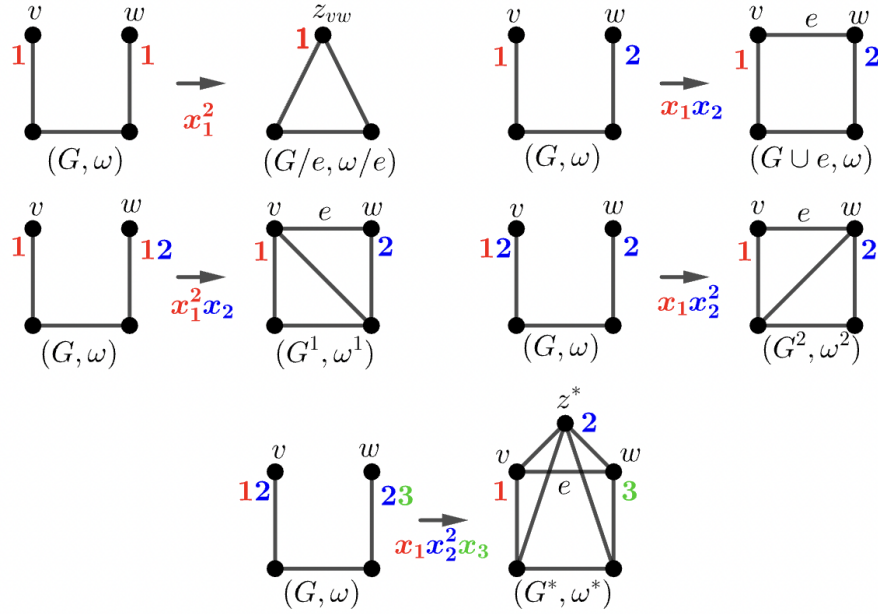
In each case, the corresponding monomial for the two colorings is the same. $\square$

In contrast to the contraction-deletion relation for $\chi_G(k)$, the graphs on the right side of the contraction-deletion relation have *more* edges and vertices than the starting graph G , so they do not seem to be getting smaller and thus it is not obvious that this process will terminate. However, the idea is that the graphs on the right side are closer to being weighted complete graphs than the starting graph, since they generally have fewer missing edges. Thus, eventually we will get a sum of Kromatic symmetric functions of weighted complete graphs, which will precisely give us the $\widetilde{m}$ -expansion of $\overline{X}_{(G, \omega)}$.

Example 3.14. Below is a weighted graph (G, ω) , along with the weighted graphs we get from the five terms of the contraction-deletion relation. The numerical labels represent the vertex weights. We ignore the weights and labels of the bottom two vertices, since they do not change.



The figure below illustrates the correspondence between colorings of G and colorings of the other graphs, by showing a coloring of v and w that would fall under each case, along with the corresponding coloring of one of the other graphs. We ignore the colors of the bottom two vertices, since they do not change.



3.6 Grothendieck positivity

The other main result that the authors of [CPS23] prove about $\overline{X}_G$ is an $\bar{s}$ -positivity result lifting Gasharov's s -positivity result about X_G from [Gas96], in the case where G is a claw-free incomparability graph. Since the Grothendieck polynomials $\bar{s}_\lambda$ are the analogue of the Schur polynomials s_λ in K -theoretic Schubert calculus, this result supports the possibility that X_G and $\overline{X}_G$ may have some topological interpretation in terms of Schubert calculus when G is a claw-free incomparability graph.

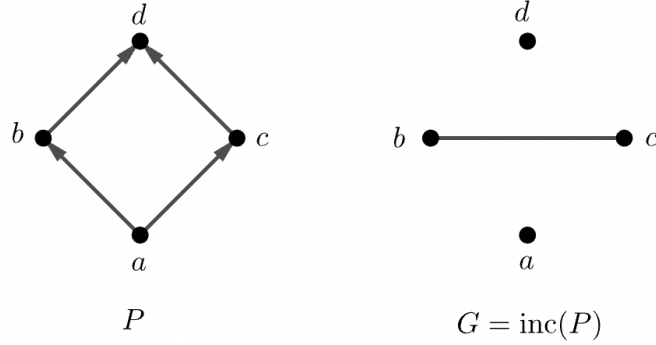
We will need a few more definitions to state these results. A **partially ordered set (poset)** is a set P together with a binary relation $\leq$ that is **transitive** (meaning if $a \leq b$ and $b \leq c$, then $a \leq c$), **reflexive** (meaning $a \leq a$ for all a), and **weakly antisymmetric** (meaning if $a \leq b$ and $b \leq a$, then $a = b$). Two elements $a, b \in P$ are **incomparable** if $a \not\leq b$ and $b \not\leq a$. The **incomparability graph** $\text{inc}(P)$ is the graph with vertex set P such that $a \sim b$ if and only if a and b are incomparable in P . A graph is **claw-free** if it does not contain the claw $\begin{smallmatrix} & \bullet & \\ & / \backslash & \\ \bullet & & \bullet \end{smallmatrix}$ as an induced subgraph. The graph $\text{inc}(P)$ is claw-free if and only if P is **(3+1)-free**, meaning it does not contain 4 elements a, b, c , and d with $a < b < c$ and d incomparable to a, b , and c (More generally, P is **(m + n)-free** if it does not contain two chains $a_1 < a_2 < \dots < a_m$ and $b_1 < b_2 < \dots < b_n$ such that a_i is incomparable to b_j for all i and j .)

Gasharov ([Gas96]) defines a **P -tableau** of shape λ to be a filling of the boxes of λ with elements of P , such that every element of P is used exactly once, and each entry is strictly less than the entry to its right and not greater than the entry below it. He uses P -tableaux to prove his s -positivity result:

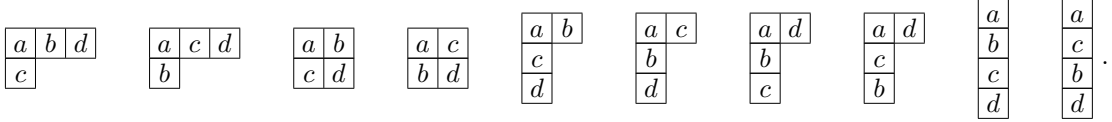
Theorem 3.15 (Gasharov [Gas96]). *If $G = \text{inc}(P)$ is claw-free, the coefficient $[s_\lambda]X_G$ is equal to the number of P -tableaux of shape λ , and is thus nonnegative.*

A poset can be represented visually using a **Hasse diagram**, in which we draw an arrow between all pairs of elements a and b such that $a < b$ and there is no c with $a < c < b$. In such a case, we say that b **covers** a , and the arrow from a to b is called a **cover relation**. By transitivity, if $a < b$ and $b < c$, it automatically follows that $a < c$. More generally, $a < c$ in P if and only if there is a directed path in the Hasse diagram from a to c , so the full structure of a poset can be recovered from its Hasse diagram. Thus, we generally draw the Hasse diagram to represent a poset instead of drawing an edge between every pair of comparable elements, as the Hasse diagram contains the same information but gives a cleaner and less cluttered picture.

Example 3.16. Below is the Hasse diagram for a poset P , along and its incomparability graph $G = \text{inc}(P)$:



There are ten possible P -tableaux for this poset, as shown below:



Reading off the shapes of the P -tableaux, we get

$$X_G = 2s_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}} + 2s_{\begin{smallmatrix} \square & \square \\ \square \end{smallmatrix}} + 4s_{\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}} + 2s_{\begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix}}.$$

The authors of [CPS23] define a **Grothendieck P -array** of shape λ to be a filling of the boxes of λ with either entries of P or positive integers, such that:

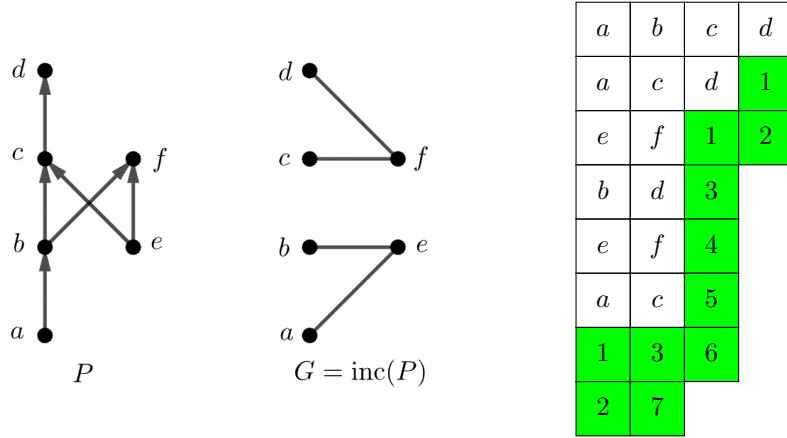
- The boxes filled with elements of P form a Young diagram $\mu \subseteq \lambda$, each element of P is used at least once, the entries in P increase strictly across rows, and no entry in P is greater than the one below it.
- The boxes filled with positive integers form an SSYT of the remaining skew shape λ/μ and satisfy the additional property that all entries in the i th row are at most $i - 1$.

They then prove:

Theorem 3.17 (Crew, Pechenik, and Spirkl [CPS23]). *For $G = \text{inc}(P)$ claw-free, $[\bar{s}_\lambda]\bar{X}_G$ counts the number of Grothendieck P -arrays of shape λ , and is thus nonnegative.*

Example 3.18. Below is the Hasse diagram for a poset P along with its incomparability graph $G = \text{inc}(P)$

and a possible Grothendieck P -array:



The P -tableau has shape $\lambda = 4^3 3^4 2$, and the part consisting of elements of P has shape $\mu = 4 3 2^4$. The boxes that contain positive integers (which form the skew shape λ/μ) are colored green.

3.7 Hopf algebra construction

The last thing we will describe about $\overline{X}_G$ in this chapter is an alternative algebraic construction of it in terms of Hopf algebras, given by Marberg in [Mar23]. The existence of this construction supports the idea that $\overline{X}_G$ is a natural and algebraically meaningful analogue of X_G to be studying. The idea is that one can put Hopf algebra structures $(\text{Sym}, \zeta_{\text{Sym}})$ and $(\text{Graphs}, \zeta_{\text{Graphs}})$ on the ring of symmetric functions and the set of graphs such that there is a homomorphism of Hopf algebras

$$\phi : (\text{Graphs}, \zeta_{\text{Graphs}}) \rightarrow (\text{Sym}, \zeta_{\text{Sym}}) \quad \text{such that} \quad \phi(G) = X_G.$$

Then, both these Hopf algebras can be completed into corresponding structures called LC-Hopf algebras, denoted $(\mathbf{mGraphs}, \overline{\zeta}_{\text{Graphs}})$ and $(\mathbf{mSym}, \overline{\zeta}_{\text{Sym}})$, such that there is a corresponding homomorphism

$$\psi : (\mathbf{mGraphs}, \overline{\zeta}_{\text{Graphs}}) \rightarrow (\mathbf{mSym}, \overline{\zeta}_{\text{Sym}}) \quad \text{such that} \quad \psi(G) = \overline{X}_G.$$

To explain how this works, we first need some definitions. A **combinatorial Hopf algebra** (A, ζ) over an integral domain $\mathbb{K}$ is a $\mathbb{K}$ -module H equipped with:

- A **product** $\nabla : A \otimes A \rightarrow A$, with a **unit** $\iota : \mathbb{K} \rightarrow A$ which is a morphism of algebras.
- A **coproduct** $\Delta : A \rightarrow A \otimes A$, with a **counit** $\varepsilon : A \rightarrow \mathbb{K}$ which is a morphism of algebras.
- A **grading** $A = \bigoplus_{n \geq 0} A_n$ with $A_0 = \mathbb{K}$, $\nabla(A_m \otimes A_n) \subseteq A_{m+n}$, and $\Delta(A_n) \subseteq \bigoplus_{k \geq 0} A_k \otimes A_{n-k}$.
- An **algebra homomorphism** $\zeta : A \rightarrow \mathbb{K}$.

There is also a compatibility requirement between the product and coproduct: the **convolution** operator $*$ on $\text{End}(A)$ given by $f * g = \nabla \circ (f \otimes g) \circ \Delta$ must have $\iota \circ \varepsilon$ as the unit, and there must be a function $S : A \rightarrow A$ called the **antipode** such that $S * \text{id} = \text{id} * S = \iota \circ \varepsilon$, where id is the identity function on A .

Given a combinatorial Hopf algebra (A, ζ) , one can take the **completion** to get a corresponding **linearly compact (LC)-Hopf algebra** $(\overline{A}, \overline{\zeta})$, which has essentially the same structures associated to it except that we allow infinite linear combinations of the basis vectors, the homomorphism $\zeta : A \rightarrow \mathbb{K}$ is replaced with a homomorphism $\overline{\zeta} : \overline{A} \rightarrow \mathbb{K}[[t]]$, and the product and coproduct need not respect the grading.

Recall that for a graph G and $S \subseteq V(G)$, $G|_S$ denotes the restriction of G to the vertex set S . We also write $\emptyset$ for the empty graph. Then **(Graphs, ζ_{Graphs})** consists of all finite $\mathbb{K}$ -linear combinations of isomorphism classes of graphs, with:

- **Product:** $\nabla(G \otimes H) := G \sqcup H$, with unit $\iota(a) := a \cdot \emptyset$ for $a \in \mathbb{K}$.

- **Coproduct:** $\Delta(G) := \sum_{S \sqcup T = V(G)} G|_S \otimes G|_T$, with counit $\varepsilon(G) := \begin{cases} 1, & \text{if } G = \emptyset; \\ 0, & \text{otherwise.} \end{cases}$
- **Grading:** Graphs_n is spanned by the isomorphism classes of graphs on n vertices.
- **Homomorphism:** $\zeta_{\text{Graphs}}(G) := \begin{cases} 1, & \text{if } G \text{ is edgeless;} \\ 0, & \text{otherwise.} \end{cases}$

Write $\lambda = \mu \sqcup \nu$ to mean that the partition λ is formed by interleaving the parts of μ with the parts of ν , i.e. taking the multiset of parts of either μ or ν and arranging them in nonincreasing order. Then **(Sym, ζ_{Sym})** is the ring of all symmetric functions of finite degree, with:

- **Product:** $\nabla(f \otimes g) := fg$ is the usual multiplication, with unit $\iota(a) := a$ for $a \in \mathbb{K}$.
- **Coproduct:** $\Delta(m_\lambda) := \sum_{S \sqcup T = \{1, 2, \dots, \ell(\lambda)\}} m_{\lambda|_S} \otimes m_{\lambda|_T}$, with counit $\varepsilon(1) := 1$, $\varepsilon(x_i) := 0$ for all i .
- **Grading:** Sym_n is spanned by the polynomials m_λ with $|\lambda| = n$.
- **Homomorphism:** $\zeta_{\text{Sym}}(m_\lambda) := \begin{cases} 1, & \text{if } \ell(\lambda) \leq 1; \\ 0, & \text{otherwise.} \end{cases}$

We can now check that the map $\phi(G) := X_G$ respects each aspect of the Hopf algebra structures:

- **Product:** Note that $X_{G \sqcup H} = X_G X_H$, since a proper coloring of $G \sqcup H$ corresponds to independently choosing a proper coloring of G and a proper coloring of H , and the monomial of $X_{G \sqcup H}$ corresponding to that coloring is the product of the monomials from X_G and X_H corresponding to the two original colorings. Thus,

$$\phi(\nabla(G \otimes H)) = \phi(G \sqcup H) = X_{G \sqcup H} = X_G X_H = \nabla(X_G \otimes X_H) = \nabla(\phi(G) \otimes \phi(H)).$$

- **Coproduct:** Consider $\Delta(\phi(G)) = \Delta(X_G)$. Note that $[m_\lambda]X_G$ counts the number of colorings κ of $V(G)$ that use color i exactly λ_i times for $i = 1, 2, \dots, \ell(\lambda)$. Thus, each term of $\Delta(X_G)$ is of the form $m_\mu \otimes m_\nu$ where $\mu \sqcup \nu = \lambda$ for such a λ and κ . We may think of the parts of μ and ν are interleaved to form λ in such a way that when $\mu_i = \nu_j$, the part μ_i comes before the part ν_j . Thus, we can uniquely determine which of μ or ν each part of λ belongs to, meaning for each color $1, 2, \dots, \ell(\lambda)$, we can uniquely assign it to one of μ or ν . Now let S be the set of vertices receiving colors in μ and T the set of vertices receiving colors in ν . Then if we relabel the colors of μ to be $1, 2, \dots, \ell(\mu)$ and the colors of ν to be $1, 2, \dots, \ell(\nu)$, then $\kappa|_S$ gives a proper coloring of $G|_S$ in which color i is used μ_i times, and $\kappa|_T$ gives a proper coloring of $G|_T$ in which color i is used ν_i times. Such a pair of proper colorings corresponds precisely to an occurrence of $m_\mu \otimes m_\nu$ in $X_{G|_S} \otimes X_{G|_T}$. It follows that

$$\Delta(\phi(G)) = \Delta(X_G) = \sum_{S \sqcup T = V(G)} X_{G|_S} \otimes X_{G|_T} = \sum_{S \sqcup T = V(G)} \phi(G|_S) \otimes \phi(G|_T) = \phi(\Delta(G)).$$

- **Grading:** $\phi(\text{Graphs}_n) \subseteq \text{Sym}_n$ since X_G has degree equal to $|V(G)|$.
- **Homomorphism:** $\phi(G) = X_G$ contains a monomial m_n if and only if $|V(G)| = n$ and G is edgeless (in which case $[m_n]X_G = 1$), since m_n corresponds to assigning the same color to all vertices, which can only happen if there are no edges. Thus,

$$\zeta_{\text{Graphs}}(G) = \zeta_{\text{Sym}}(X_G) = \zeta_{\text{Sym}}(\phi(G)) = \begin{cases} 1, & \text{if } G \text{ is edgeless;} \\ 0, & \text{otherwise.} \end{cases}$$

Example 3.19. Note that $X_{\bullet \rightarrow \bullet} = 6m_{13} + m_{21}$, since there are $3! = 6$ proper colorings that use the colors 1, 2, and 3 each once, and one that uses color 1 twice and color 2 once. Also, $X_{\bullet \rightarrow \bullet} = 2m_{12}$, since the two vertices must receive different colors but we can choose which vertex gets which color, and $X_{(\bullet \rightarrow \bullet)} = 2m_{12} + m_2$, since we could assign a different color to each vertex or the same color to both. Thus,

$$\begin{aligned}
\phi(\Delta(\bullet \rightarrow \bullet)) &= \sum_{S \sqcup T = V(\bullet \rightarrow \bullet)} X_{\bullet \rightarrow \bullet|_S} \otimes X_{\bullet \rightarrow \bullet|_T} \\
&= X_{\bullet \rightarrow \bullet} \otimes X_{\emptyset} + 2 \cdot X_{\bullet \rightarrow \bullet} \otimes X_{\bullet} + X_{(\bullet \rightarrow \bullet)} \otimes X_{\bullet} \\
&\quad + X_{\bullet} \otimes X_{(\bullet \rightarrow \bullet)} + 2 \cdot X_{\bullet} \otimes X_{\bullet \rightarrow \bullet} + X_{\emptyset} \otimes X_{\bullet \rightarrow \bullet} \\
&= (6m_{13} + m_{21}) \otimes 1 + 2(2m_{12} \otimes m_1) + (2m_{12} + m_2) \otimes m_1 \\
&\quad + m_1 \otimes (2m_{12} + m_2) + 2(m_1 \otimes 2m_{12}) + 1 \otimes (6m_{13} + m_{21}) \\
&= 6(m_{13} \otimes 1 + m_{12} \otimes m_1 + m_1 \otimes m_{12} + 1 \otimes m_{13}) \\
&\quad + (m_{21} \otimes 1 + m_2 \otimes m_1 + m_1 \otimes m_2 + 1 \otimes m_{21}) \\
&= \Delta(6m_{13} + m_{21}) = \Delta(X_{\bullet \rightarrow \bullet}) = \Delta(\phi(\bullet \rightarrow \bullet)).
\end{aligned}$$

So, we can see that ϕ respects the coproduct Δ in this example.

The LC-Hopf algebra $(\mathbf{mGraphs}, \bar{\zeta}_{\mathbf{Graphs}})$ is spanned by all possibly infinite linear combinations of isomorphism classes of graphs. It uses the same product as $(\mathbf{Graphs}, \zeta_{\mathbf{Graphs}})$, but a different coproduct,

$$\blacktriangle(G) := \sum_{S \cup T = V(G)} G|_S \otimes G|_T,$$

and a different homomorphism $\bar{\zeta}_{\mathbf{Graphs}} : \mathbf{mGraphs} \rightarrow \mathbb{K}[[t]]$, given by

$$\bar{\zeta}_{\mathbf{Graphs}}(G) := \begin{cases} t^{|V(G)|}, & \text{if } G \text{ is edgeless;} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\blacktriangle$ differs from the coproduct Δ from $(\mathbf{Graphs}, \zeta_{\mathbf{Graphs}})$ in that S and T need not be disjoint, and it does not respect the grading of $\mathbf{mGraphs}$: $\blacktriangle(\mathbf{mGraphs}_n)$ contains tensors in $\mathbf{mGraphs}_k \otimes \mathbf{mGraphs}_m$ with $k + m > n$, since S and T can overlap and thus we can have $|S| + |T| > |V(G)|$.

The LC-Hopf algebra $(\mathbf{mSym}, \bar{\zeta}_{\mathbf{Sym}})$ is simply the linear extension of $(\mathbf{Sym}, \zeta_{\mathbf{Sym}})$ to symmetric functions of unbounded degree, with the homomorphism $\bar{\zeta}_{\mathbf{Sym}} : \mathbf{mSym} \rightarrow \mathbb{K}[[t]]$ given by

$$\bar{\zeta}_{\mathbf{Sym}}(m_\lambda) := \begin{cases} t^n, & \text{if } \lambda = n \text{ for some } n \in \{0, 1, 2, 3, \dots\}, \\ 0, & \text{otherwise.} \end{cases}$$

The reasoning for why the map $\psi(G) := \bar{X}_G$ is a homomorphism is similar to the reasoning for ϕ :

- **Product:** By the same computation as for X_G , it suffices to show that $\bar{X}_{G \sqcup H} = \bar{X}_G \bar{X}_H$, which holds because a proper set coloring of $G \sqcup H$ is formed by independently taking proper set colorings of G and H , and the monomial in $\bar{X}_{G \sqcup H}$ corresponding to the resulting set coloring is the product of the corresponding monomials from $\bar{X}_G$ and $\bar{X}_H$.
- **Coproduct:** Again by a similar computation to the X_G case, it suffices to show that

$$\Delta(\bar{X}_G) = \sum_{S \cup T = V(G)} \bar{X}_{G|_S} \otimes \bar{X}_{G|_T}.$$

To see this, note that a term m_λ in $\overline{X}_G$ corresponds to a proper set-coloring of $V(G)$ that uses color i exactly λ_i times for $i = 1, 2, \dots, \ell(\lambda)$. Then $\Delta(m_\lambda) = \sum_{\mu \sqcup \nu = \lambda} m_\mu \otimes m_\nu$. As before, we can think of each part of λ as belonging to μ or ν , and use that to assign each color $1, 2, \dots, \ell(\lambda)$ to either μ or ν . We then S to be the set of vertices receiving colors in μ and T to be the set of vertices receiving colors in ν , which may overlap since vertices may receive multiple colors. Then relabeling the colors in μ as $1, 2, \dots, \ell(\mu)$ and the colors in ν as $1, 2, \dots, \ell(\nu)$, we get a proper set-coloring of $G|_S$ that uses color i μ_i times and a proper set-coloring of $G|_T$ that uses color i ν_i times, which corresponds to an occurrence of m_μ in $\overline{X}_{G|_S}$ together with an occurrence of m_ν in $\overline{X}_{G|_T}$, or equivalently, an occurrence of $m_\mu \otimes m_\nu$ in $\overline{X}_{G|_S} \otimes \overline{X}_{G|_T}$.

- **Homomorphism:** The only way G has a coloring using only one color is if G is edgeless, in which case that color gets used $|V(G)|$ times since no vertex can receive the same color twice. Thus, $\overline{X}_G$ has a single $m_{|V(G)|}$ term when G is edgeless, and no m_k terms with $k \in \mathbb{N}$ otherwise, so $\overline{\zeta}_{\text{Graphs}}(G) = \overline{\zeta}_{\text{Sym}}(\overline{X}_G)$.

Note that ψ does not respect the grading, since it maps an n -vertex graph to a sum of monomials that may have degree greater than n , as some vertices may get multiple colors and thus some monomials in $\overline{X}_G$ have degree greater than n .

Similar reasoning can be used to extend this map to a homomorphism

$$\psi : (\mathbf{mWGraphs}, \overline{\zeta}_{\text{WGraphs}}) \rightarrow (\mathbf{mSym}, \overline{\zeta}_{\text{Sym}}) \quad \text{such that} \quad \psi(G, \omega) = \overline{X}_{(G, \omega)},$$

where $(\mathbf{mWGraphs}, \overline{\zeta}_{\text{WGraphs}})$ is the LC-Hopf algebra of weighted graphs (G, ω) , defined analogously to $(\mathbf{mGraphs}, \overline{\zeta}_{\text{Graphs}})$. In particular, in the case of $\overline{X}_{K_\lambda} = \widetilde{m}_\lambda$, we get

$$\Delta(\widetilde{m}_\lambda) = \sum_{S \cup T = \{1, 2, \dots, \ell(\lambda)\}} \widetilde{m}_{\lambda|_S} \otimes \widetilde{m}_{\lambda|_T},$$

where $\lambda|_S$ represents taking only the parts of λ with indices in S , and again, S and T may overlap.

Example 3.20. For $G = \bullet \rightarrow \bullet$, the fact that ψ is a homomorphism implies that

$$\begin{aligned} \Delta(\overline{X}_{\bullet \rightarrow \bullet}) &= \sum_{S \cup T = V(\bullet \rightarrow \bullet)} \overline{X}_{\bullet \rightarrow \bullet|_S} \otimes \overline{X}_{\bullet \rightarrow \bullet|_T} \\ &= \overline{X}_{\bullet \rightarrow \bullet} \otimes \overline{X}_\emptyset + 2\overline{X}_\bullet \otimes \overline{X}_\bullet + \overline{X}_\emptyset \otimes \overline{X}_{\bullet \rightarrow \bullet} \\ &\quad + 2\overline{X}_{\bullet \rightarrow \bullet} \otimes \overline{X}_\bullet + 2\overline{X}_\bullet \otimes \overline{X}_{\bullet \rightarrow \bullet} \\ &\quad + \overline{X}_{\bullet \rightarrow \bullet} \otimes \overline{X}_{\bullet \rightarrow \bullet}. \end{aligned}$$

Since $\overline{X}_{\bullet \rightarrow \bullet} = \widetilde{m}_{1^2}$, $\overline{X}_\bullet = \widetilde{m}_1$, and $\overline{X}_\emptyset = 1$, this can alternatively be written in terms of $\widetilde{m}_\lambda$'s as

$$\begin{aligned} \Delta(\widetilde{m}_{1^2}) &= \widetilde{m}_{1^2} \otimes 1 + 2\widetilde{m}_1 \otimes \widetilde{m}_1 + 1 \otimes \widetilde{m}_{1^2} \\ &\quad + 2\widetilde{m}_{1^2} \otimes \widetilde{m}_1 + 2\widetilde{m}_1 \otimes \widetilde{m}_{1^2} \\ &\quad + \widetilde{m}_{1^2} \otimes \widetilde{m}_{1^2}. \end{aligned}$$

Marberg ([Mar23]) goes on to define a quasisymmetric analogue of $\overline{X}_G$ via a similar Hopf algebra construction, but we will not recall it here since our focus is on $\overline{X}_G$.

We will now move on to proving some new results about $\overline{X}_G$. To start, in the next chapter we will make use of the $\widetilde{m}$ -expansion formula from Theorem 3.11 to explore what properties of a graph G can be recovered given its Kromatic symmetric function.

Chapter 4

Results on distinguishing graphs by their Kromatic symmetric functions

4.1 Introduction

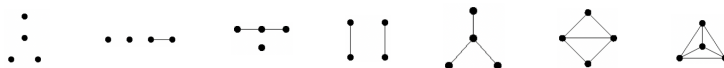
This chapter is based around the following conjecture:

Conjecture 4.1. *There do not exist nonisomorphic graphs G and H with $\overline{X}_G = \overline{X}_H$.*

As evidence for Conjecture 4.1, we show that $\overline{X}_G$ can be used to count certain induced subgraphs of G . As an application of our results, we give an alternative proof that $\overline{X}_G$ distinguishes the graphs in each of three examples from [CPS23] of pairs of graphs with the same X_G but different $\overline{X}_G$ (see Example 4.4 below).

We prove three main results towards Conjecture 4.1. Our first result counts certain order 4 subgraphs:

Theorem 4.2. *The number of induced copies of the following order 4 graphs can be computed from $\overline{X}_G$:*



and the counts of the remaining 4 order 4 graphs satisfy a system of 3 linear equations determined by $\overline{X}_G$. Also, for T a tree, the multiplicity of all induced subgraphs of order 4 can be determined from $\overline{X}_T$.

In particular, Theorem 4.2 implies that $\overline{X}_G$ can determine whether or not G contains an induced claw , which is of interest because X_G cannot tell whether G is claw-free and much is known about the structure of claw-free graphs ([CS05]). Claw-free graphs are of particular interest in the study of the chromatic symmetric function because another major open question about X_G is the Stanley-Stembridge conjecture (mentioned in Chapter 3), which says that X_G is e -positive for G a claw-free incomparability graph of a partially ordered set.

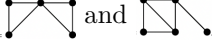
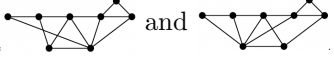
Our second result counts certain subgraphs of order 5:

Theorem 4.3. *The number of induced copies of the following order 5 graphs can be computed from $\overline{X}_G$:*



and the counts of the remaining 23 order 5 graphs satisfy a system of 14 linear equations determined by $\overline{X}_G$. Also, for T a tree, the multiplicity of all induced subgraphs of order 5 can be determined from $\overline{X}_T$.

Example 4.4. The authors of [CPS23] give examples of three pairs of graphs with the same X_G but different $\overline{X}_G$. We can use Theorems 4.2 and 4.3 to distinguish all three pairs:

- For , the second graph has an induced  while the first does not.
- For , the first graph has an induced  while the second does not.
- For , the second graph has an induced  while the first does not.

Our third result concerns an infinite family of subgraphs that can be counted using $\overline{X}_G$:

Theorem 4.5. *For each pair $h, k \geq 0$, the number of induced copies in G of the disjoint union of an h -vertex star and k isolated vertices can be recovered from $\overline{X}_G$.*

In [MMW08], Martin, Morin, and Wagner proved the stronger fact that the degree sequence of a tree can be recovered from the ordinary symmetric function X_G . Our proof of Theorem 4.5 is shorter and uses different methods, which suggests that proving that $\overline{X}_G$ distinguishes all trees may be easier than proving that X_G does.

In §4.2 of this chapter, we prove a key lemma that will be used in several of our proofs. Then in §4.3, §4.4, and §4.5, we give the proofs of our three main results.

4.2 Key lemma

Our proofs will make use of the following lemma, which we will prove using Theorem 3.11:

Lemma 4.6. *For any $i_1, \dots, i_\ell$, the number $\#(i_1, i_2, \dots, i_\ell)$ of ways to cover exactly i_1 of the vertices of $\overline{G}$ using i_2 edges of $\overline{G}$, i_3 K_3 's in $\overline{G}$, $\dots$, and i_ℓ K_ℓ 's in $\overline{G}$ can be determined from $\overline{X}_G$.*

Proof. By Theorem 3.11, $[\widetilde{m}_\lambda]$ counts the number of ways to cover $V(G)$ with stable sets whose sizes are the parts of λ . Note first that the number of vertices $n := |V(G)|$ can be determined from $\overline{X}_G$ because $[\widetilde{m}_{1^k}] = 0$ for all $k \neq n$ while $[\widetilde{m}_{1^n}] = 1$.

To compute $\#(i_1, i_2, \dots, i_\ell)$, note that by Theorem 3.11, $[\widetilde{m}_{\ell^{i_\ell}(\ell-1)^{i_{\ell-1}} \dots 3^{i_3} 2^{i_2} 1^{n-i_1}}]$ counts the number of ways to cover $V(G)$ using $n - i_1$ singletons and i_j stable sets of size j for each $j = i_2, \dots, i_\ell$. Translating this to $\overline{G}$, $[\widetilde{m}_{\ell^{i_\ell}(\ell-1)^{i_{\ell-1}} \dots 3^{i_3} 2^{i_2} 1^{n-i_1}}]$ counts the number of ways to cover $\overline{G}$ with $n - i_1$ singletons together with i_j cliques K_j for each j . Now we split this count into cases based on how many of the vertices covered by one of the $n - i_1$ singletons are also covered by one of the cliques K_j . The number of ways to choose one of the covers counted by $[\widetilde{m}_{\ell^{i_\ell}(\ell-1)^{i_{\ell-1}} \dots 3^{i_3} 2^{i_2} 1^{n-i_1}}]$ that includes at least k vertices that are each covered by at least one stable set besides a singleton is $\#(k, i_2, \dots, i_\ell) \binom{k}{i_1}$. To see this, note first that by the definition of our notation, there are $\#(k, i_2, \dots, i_\ell)$ ways to cover k vertices using i_j cliques K_j for each j . Then, $n - k$ of the singletons are needed to cover the remaining $n - k$ uncovered vertices, so $(n - i_1) - (n - k) = k - i_1$ singletons will cover one of the k already covered vertices. So, there are $\binom{k}{k-i_1} = \binom{k}{i_1}$ ways to choose which $k - i_1$ of those k vertices are covered by a singleton in addition to another stable set. Putting this together and then rearranging to isolate the value $\#(i_1, i_2, \dots, i_\ell)$ that we want to solve for, we get

$$\#(i_1, i_2, \dots, i_\ell) = [\widetilde{m}_{\ell^{i_\ell}(\ell-1)^{i_{\ell-1}} \dots 3^{i_3} 2^{i_2} 1^{n-i_1}}] - \sum_{k=i_1+1}^n \#(k, i_2, \dots, i_\ell) \binom{k}{i_1}.$$

Thus, we can recursively compute $\#(i_1, i_2, \dots, i_\ell)$ by fixing $i_2, \dots, i_\ell$ and letting i_1 range from n down to 1, so the values $\#(i_1, i_2, \dots, i_\ell)$ are all computable from $\overline{X}_G$ by induction. $\square$

In particular, it follows that the number $\#(j, i)$ of (not necessarily induced) subgraphs of the complement graph $\overline{G}$ with j vertices, i edges, and no isolated vertices can be determined from $\overline{X}_G$.

4.3 Proof of Theorem 4.2

Finding induced subgraphs of G isomorphic to H is equivalent to finding induced subgraphs of $\overline{G}$ isomorphic to $\overline{H}$, so we will instead focus on induced subgraphs of $\overline{G}$, as Lemma 4.6 makes them easier to think about. We will use a picture of H to represent the number of induced subgraphs of $\overline{G}$ isomorphic to H , and unless stated otherwise, all references to subgraphs and edges will be for $\overline{G}$ rather than for G . The pictures of the graphs are taken from [Rid23].

From [MMW08], the number of vertices, edges, triangles, and induced copies of $\bullet\text{---}\bullet\text{---}\bullet$ in G can be computed from X_G , and thus also from $\overline{X}_G$ since the lowest degree terms of $\overline{X}_G$ determine X_G . For $|V(H)| = 4$, there are 11 total choices of H . We can directly compute 5 of the values:

$$\begin{array}{c} \triangle \\ \bullet \end{array} = [\widetilde{m}_{41^{n-4}}] \quad (4.1)$$

$$\begin{array}{c} \triangle \\ \diagup \diagdown \\ \bullet \end{array} = \#(4, 5) - 6 \cdot \begin{array}{c} \triangle \\ \bullet \end{array} \quad (4.2)$$

$$\begin{array}{c} \bullet \\ | \\ \triangle \\ \bullet \end{array} = (\bullet\text{---}\bullet - 3) \cdot \begin{array}{c} \triangle \\ \bullet \end{array} - [\widetilde{m}_{321^{n-5}}] - 12 \cdot \begin{array}{c} \triangle \\ \diagup \diagdown \\ \bullet \end{array} - 4 \cdot \begin{array}{c} \triangle \\ \diagup \diagdown \\ \diagup \diagdown \\ \bullet \end{array} \quad (4.3)$$

$$\begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array} = \#(4, 4) - 15 \cdot \begin{array}{c} \triangle \\ \bullet \end{array} - 5 \cdot \begin{array}{c} \triangle \\ \diagup \diagdown \\ \bullet \end{array} - \begin{array}{c} \bullet \\ | \\ \triangle \\ \bullet \end{array} \quad (4.4)$$

$$\begin{array}{c} \triangle \\ \bullet \end{array} \bullet = (n-3) \cdot \begin{array}{c} \triangle \\ \bullet \end{array} - 4 \cdot \begin{array}{c} \triangle \\ \diagup \diagdown \\ \bullet \end{array} - 2 \cdot \begin{array}{c} \triangle \\ \diagup \diagdown \\ \diagup \diagdown \\ \bullet \end{array} - \begin{array}{c} \bullet \\ | \\ \triangle \\ \bullet \end{array}. \quad (4.5)$$

(4.1) is immediate, and the reasoning for the other equations is as follows:

- (4.2): We count subgraphs of $\overline{G}$ with 5 edges and 4 vertices, of which there are $\#(4, 5)$ total. For each $\begin{array}{c} \triangle \\ \bullet \end{array}$, there are 6 such subgraphs since we can choose any of the 6 edges to omit, and for each $\begin{array}{c} \triangle \\ \diagup \diagdown \\ \bullet \end{array}$, there is one.
- (4.3): We count ways to choose a triangle plus an edge not contained in the triangle, of which there are $(\bullet\text{---}\bullet - 3) \cdot \begin{array}{c} \triangle \\ \bullet \end{array}$ total. Then $[\widetilde{m}_{321^{n-5}}]$ counts the ways where the triangle and the edge do not overlap, for each $\begin{array}{c} \triangle \\ \bullet \end{array}$ there are 12 ways (4 options for the triangle, then 3 for the edge), for each $\begin{array}{c} \triangle \\ \diagup \diagdown \\ \bullet \end{array}$ there are 4 ways (2 options for the triangle, then 2 for the edge), and for each $\begin{array}{c} \bullet \\ | \\ \triangle \\ \bullet \end{array}$ there is one way.
- (4.4): We count ways to cover 4 vertices with 4 edges. There are $\#(4, 4)$ ways total. For each $\begin{array}{c} \triangle \\ \bullet \end{array}$, there are $\binom{6}{4} = 15$ ways since we choose which 4 edges to include, for each $\begin{array}{c} \triangle \\ \diagup \diagdown \\ \bullet \end{array}$, there are $\binom{5}{4} = 5$ ways, and for each $\begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \bullet \end{array}$ or $\begin{array}{c} \bullet \\ | \\ \triangle \\ \bullet \end{array}$, there is just one way since we need all 4 edges.
- (4.5): We count ways to choose a triangle plus a vertex not contained in the triangle. There are $(n-3) \cdot \begin{array}{c} \triangle \\ \bullet \end{array}$ ways total. For each $\begin{array}{c} \triangle \\ \bullet \end{array}$, there is one way since $\begin{array}{c} \triangle \\ \bullet \end{array}$ has 4 triangles, for each $\begin{array}{c} \triangle \\ \diagup \diagdown \\ \bullet \end{array}$, there are 2 ways since it has 2 triangles, and for each $\begin{array}{c} \bullet \\ | \\ \triangle \\ \bullet \end{array}$ or $\begin{array}{c} \bullet \\ | \\ \bullet \end{array}$, there is one way.

For the remaining 6 cases where $|V(H)| = 4$, we set up a system of 5 equations in 6 variables:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & 3 & 2 & 2 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \text{---} \cdot \cdot \cdot \text{---} \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \\ \cdot \cdot \cdot \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{pmatrix},$$

where

$$\begin{aligned} b_1 &= \binom{n}{4} - \text{triangle} - \text{diamond} - \text{star} - \text{square} - \text{path} \cdot \\ b_2 &= \binom{n-2}{2} \cdot \text{---} \cdot \cdot - 6 \cdot \text{triangle} - 5 \cdot \text{diamond} - 4 \cdot \text{star} - 4 \cdot \text{square} - 3 \cdot \text{path} \cdot \\ b_3 &= \#(4,3) - 16 \cdot \text{triangle} - 8 \cdot \text{diamond} - 3 \cdot \text{star} - 4 \cdot \text{square} \\ b_4 &= [\widetilde{m}_{221^{n-4}}] - 3 \cdot \text{triangle} - 2 \cdot \text{diamond} - \text{star} - 2 \cdot \text{square} \\ b_5 &= (n-3) \cdot \text{---} \cdot \cdot - 2 \cdot \text{diamond} - 2 \cdot \text{star} - 4 \cdot \text{square} \end{aligned}$$

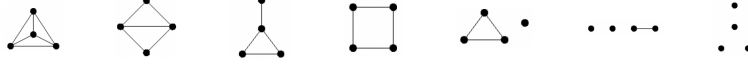
The explanations for these equations are as follows:

- **Row 1:** We count all ways to choose four vertices, which on the one hand is $\binom{n}{4}$, and on the other is the sum over all H with $|V(H)| = 4$ of the number of induced subgraphs isomorphic to H .
- **Row 2:** We count ways to choose an edge plus two other vertices, which on the one hand is $\binom{n-2}{2}$ times the number of edges, and on the other is the sum over all H of the number of induced copies of H times the number of edges in H .
- **Row 3:** We count ways to cover 4 vertices with 3 edges. The total number of ways is $\#(4,3)$, and for each H , the number of ways is $\binom{|E(H)|}{3}$ minus the number of triangles in H , since to cover all 4 vertices of $V(H)$ with 3 edges from $E(H)$, we can choose any 3 edges that do not form a triangle.
- **Row 4:** We count ways to choose two nonoverlapping edges. The total number of ways is $[\widetilde{m}_{221^{n-4}}]$, and for each H , the number of ways equals the number of pairs of nonoverlapping edges in H .
- **Row 5:** We count ways to choose an induced $\text{---} \cdot \cdot$ plus an extra vertex. The total number of ways is $(n-3) \cdot \text{---} \cdot \cdot$, and the number of ways for each H is the number of induced copies of $\text{---} \cdot \cdot$ in H .

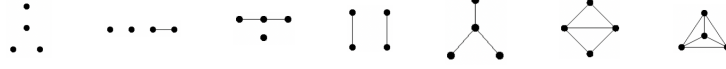
The reduced row echelon form of the above matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since the last two rows each have a single 1 and the rest 0's, the number of copies in $\overline{G}$ of the corresponding graphs can be computed from $\overline{X}_G$, so the number of copies in $\overline{G}$ of the following 7 graphs can be determined:



Taking complements, the number of copies in G of the following 7 graphs can be determined:



Finally, note that for T a tree, this lets us determine the induced number of copies in T of all order 4 forests except $\bullet - \bullet - \bullet - \bullet$. But since T a tree, the multiplicity of $\bullet - \bullet - \bullet - \bullet$ can then also be computed since it is the only remaining possible 4 vertex subgraph of T . $\square$

4.4 Proof of Theorem 4.3

There are 34 graphs with $|V(H)| = 5$. We first directly compute 6 of the values:

$$\begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} = [\widetilde{m}_{51^{n-5}}] \quad (4.6)$$

$$\begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} = \#(5, 9) - 10 \cdot \begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} \quad (4.7)$$

$$\begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} = \#(5, 0, 1, 1) - 30 \cdot \begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} - 6 \cdot \begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} \quad (4.8)$$

$$\begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} = \#(5, 8) - 45 \cdot \begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} - 9 \cdot \begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} - \begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} \quad (4.9)$$

$$\begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} = (\bullet - \bullet - 6) \cdot \begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} - [\widetilde{m}_{421^{n-6}}] - 20 \cdot \begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} - 6 \cdot \begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} - 2 \cdot \begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} \quad (4.10)$$

$$\begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} \bullet = (n-4) \cdot \begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} - 5 \cdot \begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} - 2 \cdot \begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} - \begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} - \begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array} \quad (4.11)$$

(4.6) is immediate, and the other equations are obtained as follows:

- (4.7): We count ways to cover 5 vertices with 9 edges. The total is $\#(5, 9)$, for each , there are 10 ways since any edge can be removed, and for each , there is one way.
- (4.8): We count ways to cover 5 vertices with a triangle and a K_4 . For each , there are 5 ways to choose the K_4 and then $\binom{4}{2} = 6$ ways to choose the triangle, for $5 \cdot 6 = 30$ total, for each , there are 2 ways to choose the K_4 and then 3 ways to choose the triangle, for $2 \cdot 3 = 6$ total, and for each , there is one way.
- (4.9): We count ways to cover 5 vertices with 8 edges. The total is $\#(5, 8)$. For each , there are $\binom{10}{2} = 45$ ways since we can choose any 2 edges to remove, for each , there are 9 ways since we can choose any one edge to remove, and for each  or , there is one way.
- (4.10): We count ways to choose a K_4 plus an extra edge. There are $(\bullet - \bullet - 6) \cdot \begin{array}{c} \text{pentagon} \\ \text{pentagon} \end{array}$ ways total. There are $[\widetilde{m}_{421^{n-6}}]$ ways where the edge and triangle do not overlap, 20 for each , 6 for each , 2 for each , and one for each .

- (4.11): We count ways to choose a K_4 and another vertex. There are $(n-4) \cdot \triangle$ total, and for each 5 vertex graph H , the number of ways is the number of K_4 subgraphs of H .

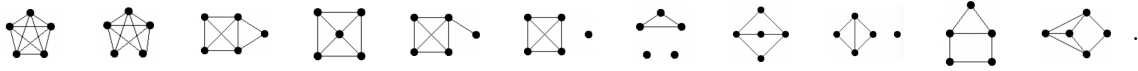
For the remaining 28 values, we set up the system of 19 equations shown in Figure 4.7, with the aid of Sage ([Ste+24]), where $\mathbf{b}$ is computable from $\overline{X}_G$. The rows of the matrix are obtained as follows:

- **Rows 1-9:** We count induced copies of the order 4 subgraphs in the order 5 subgraphs.
- **Rows 10-13:** We count connected subgraphs with each number of edges from 4 to 7.
- **Rows 14-17:** We count ways to cover 5 vertices with a triangle plus 1, 2, 3, or 4 edges.
- **Rows 18-19:** We count ways to cover 5 vertices with 2 triangles, or 2 triangles plus an edge.

The reduced row echelon form of the matrix is:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 2 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & -6 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & -2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 3 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

There are 5 rows with a single 1 and the rest 0's, so combining those with our initial 6 graphs, we find that $\overline{X}_G$ determines the number of induced copies in $\overline{G}$ of the following graphs:



Taking complements, $\overline{X}_G$ determines the counts in G of the following 11 graphs:



For T a tree, this gives the counts of all order 5 forests except $\bullet \bullet \bullet \bullet \bullet$ and $\bullet \bullet \bullet \bullet \bullet$. We can then compute the multiplicity of $\bullet \bullet \bullet \bullet \bullet$ because we know the multiplicity of each induced subgraph on 4 vertices from the proof of Theorem 4.2, and each $\bullet \bullet \bullet \bullet \bullet$ has an induced $\bullet \bullet \bullet \bullet$ while each $\bullet \bullet \bullet \bullet \bullet$ does not. Then all remaining sets of 5 vertices would form an induced $\bullet \bullet \bullet \bullet \bullet$, so that number can be determined as well. $\square$

4.5 Proof of Theorem 4.5

An induced copy of a j -vertex star in G is equivalent to an induced copy in $\overline{G}$ of the order j graph consisting of a K_{j-1} together with an isolated vertex, which we will denote as $K_{j-1} \sqcup v$. For $1 \leq i \leq j-1$, let $K_{j-1} \sqcup i \cdot e$

Figure 4.7: System of linear equations to count induced subgraphs of order 5

[illegible]

denote the order j graph consisting of a K_{j-1} plus an extra vertex v connected to exactly i of the vertices in the clique. Thus, the complement $\overline{K_{j-1} \sqcup i \cdot e}$ is an $(j-i)$ -vertex star together with i isolated vertices, since when we take the complement, the vertex v not in the K_{j-1} becomes the center of the star and the vertices of the K_{j-1} adjacent to v become the isolated vertices.

For a graph H , write $\#(H)$ for the number of induced copies of H in $\overline{G}$. By Lemma 4.6, the number

$$\#(\underbrace{j, i, 0, \dots, 0, 1}_{j-1})$$

of ways to cover j vertices in $\overline{G}$ with i edges and one K_{j-1} can be computed from $\overline{X}_G$ for each i . Thus, our strategy will be to set up a system of linear equations by expressing these known values in terms of the numbers $\#(K_{j-1} \sqcup v)$ and $\#(K_{j-1} \sqcup i \cdot e)$ that we are interested in computing.

Every set of j vertices in H that can be covered by i edges and a K_{j-1} is contained $K_{j-1} \sqcup k \cdot e$ for some k , since those are the only induced subgraphs on j vertices that contain a K_{j-1} . Given such an induced subgraph of H isomorphic to $K_{j-1} \sqcup k \cdot e$, if we want to cover it with a K_{j-1} and i edges, we must first choose the K_{j-1} . Thus, let a_k denote the number of induced copies of K_{j-1} in $K_{j-1} \sqcup k \cdot e$. If $k = j-1$, then all vertices of the K_{j-1} are connected to the extra vertex v , so $K_{j-1} \sqcup (j-1) \cdot e = K_j$. There are j induced copies of K_{j-1} in K_j is j , so $a_{j-1} = j$. If $k = j-2$, we get $a_{j-2} = 2$ induced copies of K_{j-1} in $K_{j-1} \sqcup (j-2) \cdot e$, since the vertex not in the K_{j-1} can be either of the two nonadjacent vertices. For $k < j-2$, there is just one copy of K_{j-1} in $K_{j-1} \sqcup k \cdot e$, since the vertex with only $j-2$ neighbors must be the one left out. Thus,

$$a_k = \begin{cases} j, & \text{if } k = j-1; \\ 2, & \text{if } k = j-2; \\ 1, & \text{if } k < j-2. \end{cases}$$

Now that we have the K_{j-1} , let v be the remaining vertex not in the K_{j-1} . We need to choose i of the $\binom{j-1}{2} + k$ total edges in the $K_{j-1} \sqcup k \cdot e$ such that at least one of them has v as an endpoint (in order to cover v). The number of ways to do this is

$$\binom{\binom{j-1}{2} + k}{i} - \binom{\binom{j-1}{2}}{i},$$

since the first term represents the number of ways to choose any i edges of $K_{j-1} \sqcup k \cdot e$ and the second represents the number of ways to choose all k edges from the K_{j-1} (which we do not want since then v is not covered).

Since i can take any value from 1 to $j-1$, we have a system of $j-1$ equations

$$\sum_{k=1}^{j-1} a_k \left(\binom{\binom{j-1}{2} + k}{i} - \binom{\binom{j-1}{2}}{i} \right) \cdot \#(K_{j-1} \sqcup k \cdot e) = \#(\underbrace{j, i, 0, \dots, 0, 1}_{j-1})$$

for $i = 1, 2, \dots, j-1$. This gives $j-1$ linearly independent equations in the $j-1$ variables $\#(K_{j-1} \sqcup e), \#(K_{j-1} \sqcup 2 \cdot e), \dots, \#(K_{j-1} \sqcup (j-1) \cdot e)$, so we can solve for all the variables to compute the values $\#(K_{j-1} \sqcup j \cdot e)$ for $k = 1, \dots, j-1$ given $\overline{X}_G$.

Then to compute $\#(K_{j-1} \sqcup v)$, we count the total number of ways to choose a K_{j-1} in $\overline{G}$ plus an extra vertex. The total number of K_{j-1} 's in $\overline{G}$ is just $[\widetilde{m}_{(j-1)1^{n-j+1}}]$, and the remaining vertex can be any of the other $n-j+1$ vertices in $\overline{G}$, so the total number of ways is $(n-j+1) \cdot [\widetilde{m}_{(j-1)1^{n-j+1}}]$. The j chosen vertices will then form either an induced $K_{j-1} \sqcup v$ or an induced $K_{j-1} \sqcup k \cdot e$ in $\overline{G}$ for some i , since those are the only induced subgraphs of order j that contain a K_{j-1} . For each induced $K_{j-1} \sqcup k \cdot e$ in $\overline{G}$, there are a_k ways to choose the K_{j-1} , and the extra vertex would have to be the remaining vertex. For each induced $K_{j-1} \sqcup v$, there is one choice for the clique and the vertex. Thus, we get

$$\#(K_{j-1} \sqcup v) = (n-j+1) \cdot [\widetilde{m}_{(j-1)1^{n-j+1}}] - \sum_{k=1}^{j-1} a_k \cdot \#(K_{j-1} \sqcup k \cdot e).$$

Since all values on the right side can be computed from $\overline{X}_G$, so can $\#(K_{j-1} \sqcup v)$.

Since we were counting induced subgraphs of $\overline{G}$, taking complements tells us that for each j and k , we can compute of copies in G of $\overline{K_{j-1} \sqcup k \cdot e}$, which is an a star on $h = j - k$ vertices together with k isolated vertices, as well as the number of copies in G of $K_{j-1} \sqcup v$, which is a star on $h = j$ vertices. This completes the proof of Theorem 4.5. $\square$

Chapter 5

A power sum expansion for the Kromatic symmetric function

5.1 Introduction

Recall from Chapter 1 that for a partition λ , the corresponding *power sum symmetric function* is

$$p_\lambda := \prod_{i=1}^{\ell(\lambda)} \left(\sum_{j \in \mathbb{N}} x_j^{\lambda_i} \right) = (x_1^{\lambda_1} + x_2^{\lambda_1} + \dots)(x_1^{\lambda_2} + x_2^{\lambda_2} + \dots) \dots (x_1^{\lambda_{\ell(\lambda)}} + x_2^{\lambda_{\ell(\lambda)}} + \dots).$$

Stanley ([Sta95]) gave a p -expansion formula for X_G as an alternating sum over subsets of $E(G)$, and Bernardi and Nadeau ([BN20]) give an alternate interpretation of the p -expansion in terms of acyclic orientations. A natural question is thus whether there is some corresponding p -expansion formula for $\overline{X}_G$, in terms of an appropriate analogue of the p -basis. The authors of [CPS23] defined a K -theoretic analogue of the power sum symmetric functions as follows:

Definition 5.1 (Crew, Pechenik, and Spirkl [CPS23]). Let $\overline{K}_\lambda$ be the edgeless graph on $\ell(\lambda)$ vertices with weights $\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}$ of λ . Then the *K-theoretic power sum symmetric function* $\overline{p}_\lambda$ is

$$\overline{p}_\lambda := \overline{X}_{\overline{K}_\lambda}.$$

The authors of [CPS23] gave a table listing the first few coefficients of the $\overline{p}$ -expansions of $\overline{X}_G$ for several small graphs G , which they computed using Sage [Ste+24] based on their contraction-deletion relation (see §3.5). The leading terms (i.e. the terms where $|\lambda|$ is minimal), must always match the corresponding terms from ordinary p -expansion of the ordinary chromatic symmetric function X_G . (The leading terms are those where $|\lambda| = |V(G)|$ for an unweighted graph, or where $|\lambda| = \omega(G) := \sum_{v \in V(G)} \omega(v)$ for a weighted graph with weight function $\omega : V(G) \rightarrow \mathbb{N}$.)

Unlike the p -expansion of X_G , they found that the $\overline{p}$ -expansions for $\overline{X}_G$ appeared to have infinitely many terms, but the coefficients still seemed to always be integers. For instance, below are first few terms (grouped by $|\lambda|$) of the $\overline{p}$ -expansions for the single edge $\bullet \text{---} \bullet$, and for the graph K_{21} consisting of a single edge connecting a vertex of weight 2 and a vertex of weight 1:

$$\begin{aligned} \overline{X}_{K_{21}} = & (-\overline{p}_3 + \overline{p}_{21}) + (\overline{p}_4 - \overline{p}_{31}) + (\overline{p}_{41} - \overline{p}_{32}) + (-\overline{p}_6 + \overline{p}_{42} + \overline{p}_{33} - \overline{p}_{321}) \\ & + (-\overline{p}_{61} + \overline{p}_{421} + \overline{p}_{331}) + (\overline{p}_8 - \overline{p}_{62} + \overline{p}_{332}) + (\overline{p}_{81} + \overline{p}_{63} - \overline{p}_{621} - \overline{p}_{333} + \overline{p}_{3321}) + \dots \end{aligned}$$

$$\begin{aligned} \overline{X}_{\bullet \text{---} \bullet} = & (-\overline{p}_2 + \overline{p}_{11}) + (2\overline{p}_3 - 2\overline{p}_{21}) + (-4\overline{p}_4 + 4\overline{p}_{31} + \overline{p}_{22} - \overline{p}_{211}) \\ & + (6\overline{p}_5 - 8\overline{p}_{41} - 3\overline{p}_{32} + 2\overline{p}_{311} + 2\overline{p}_{221}) \\ & + (-9\overline{p}_6 + 12\overline{p}_{51} + 4\overline{p}_{42} - 4\overline{p}_{411} + \overline{p}_{33} - 4\overline{p}_{321} - \overline{p}_{222} + \overline{p}_{2211}) \dots \end{aligned}$$

Since $\bullet \text{---} \bullet$ has $|V(G)| = 2$, the leading terms are $-\bar{p}_2 + \bar{p}_{11}$, matching the ordinary p -expansion $X_{\bullet \text{---} \bullet} = -p_2 + p_{11}$. Similarly, K_{21} has weight $\omega(K_{21}) = 2 + 1 = 3$, so the leading terms are $-\bar{p}_3 + \bar{p}_{21}$, which matches the ordinary p -expansion $X_{K_{21}} = -p_3 + p_{21}$. We will explain where the rest of the coefficients in these expansions come from when we revisit these examples in §5.3.

The authors of [CPS23] asked whether there is an explicit formula for the $\bar{p}$ -expansion of $\bar{X}_G$. Here, we give such a formula that applies to any weighted graph G with a vertex weight function $\omega : V \rightarrow \mathbb{N}$, in terms of its *independence polynomial*

$$I_{(G,\omega)}(t) := \sum_{S \subseteq V(G) \text{ an independent set}} t^{\omega(S)},$$

where the weight of a subset S is $\omega(S) := \sum_{v \in S} \omega(v)$. We write $G|_W$ to denote the induced subgraph of G with vertex set W .

Theorem 5.2. $\bar{X}_{(G,\omega)}$ satisfies the formula

$$\bar{X}_{(G,\omega)} = \sum_{W \subseteq V(G)} (-1)^{|V(G)|-|W|} \prod_{k \geq 1} (1 + \bar{p}_k)^{a_W(k)},$$

where the exponents $a_W(k)$ are defined to satisfy

$$(1+t)^{a_W(1)}(1+t^2)^{a_W(2)}(1+t^3)^{a_W(3)} \dots = I_{(G|_W,\omega)}(t).$$

This idea for where the $(1+t^k)$ terms come from is the factorization $1 + \bar{p}_k = \prod_{i \geq 1} (1 + x_i^k)$. As a consequence of Theorem 5.2, we can write down an explicit formula for the coefficients $[\bar{p}_\lambda] \bar{X}_{(G,\omega)}$:

Corollary 5.3. The coefficient of $\bar{p}_\lambda$ in $\bar{X}_{(G,\omega)}$ is given by

$$[\bar{p}_\lambda] \bar{X}_{(G,\omega)} = \sum_{W \subseteq V(G)} (-1)^{|V(G)|-|W|} \prod_{k=1}^{\ell} \binom{a_W(k)}{i_k},$$

where $\lambda = \ell^{i_\ell} \dots 2^{i_2} 1^{i_1}$.

The numbers $a_W(k)$ above can be computed recursively as follows:

Proposition 5.4. The exponents $a_W(k)$ satisfy the recurrence

$$a_W(k) = [t^k] \log(I_{(G|_W,\omega)}(t)) + \sum_{d|k} \frac{(-1)^{k/d} d \cdot a_W(d)}{k}.$$

We also show that all coefficients in the above expansion are integers:

Theorem 5.5. For every partition λ , the coefficient $[\bar{p}_\lambda] \bar{X}_G$ is an integer.

Another question one can ask is a question of $\bar{p}$ -positivity. For G an unweighted graph, it can be shown that the sign of $[p_\lambda] X_G$ is $(-1)^{|\lambda|-\ell(\lambda)}$. From checking some examples, the coefficients $[\bar{p}_\lambda] \bar{X}_G$ seem to follow the same pattern of signs, which suggests the following conjecture:

Conjecture 5.6. For G unweighted, the sign of $[\bar{p}_\lambda] \bar{X}_G$ is $(-1)^{|\lambda|-\ell(\lambda)}$.

Note that Conjecture 5.6 does not apply to weighted graphs. For instance, in the $\bar{p}$ -expansion of K_{21} above, the coefficient for $\lambda = 3$ is -1 while $|\lambda| - \ell(\lambda) = 3 - 1 = 2$ is even.

The remainder of this chapter is organized as follows: In §5.2, we explain the p -expansion formulas for X_G from [Sta95] and [BN20]. In §5.3, we illustrate our $\bar{p}$ -expansion formula for $\bar{X}_G$ using the examples of $\bullet \text{---} \bullet$ and K_{21} from above. Finally, in §5.4, we prove Theorem 5.2, Corollary 5.3, Proposition 5.4, and Theorem 5.5.

5.2 Prior power sum expansion formulas for X_G

Expansion in terms of subsets of the edges

Stanley [Sta95] gives the following p -expansion formula for the chromatic symmetric function:

Theorem 5.7 (Stanley [Sta95], Theorem 1.1). X_G satisfies

$$X_G = \sum_{S \subseteq E(G)} (-1)^{|S|} p_{\lambda(S)},$$

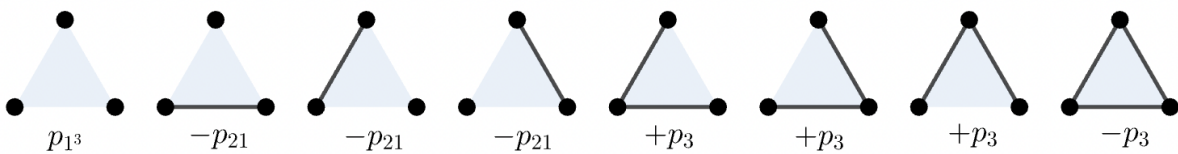
where $\lambda(S)$ is the partition whose parts are the sizes of the connected components of the subgraph of G with vertex set $V(G)$ and edge set S .

Proof. This follows from the principle of inclusion-exclusion. For $S = \emptyset$, we get $p_{1^n} = (x_1 + x_2 + \dots)^n$, which represents independently assigning a color to each vertex. However, we then have to subtract the colorings where some pair of adjacent vertices v and w receive the same color, and then add back the colorings where two pairs of adjacent vertices receive the same color, and subtract the ones where three pairs receive the same color, and so on. A coloring in which k pairs of adjacent vertices $v_1w_1, v_2w_2, \dots, v_kw_k$ get the same color as its partner can equivalently be thought of as a coloring that assigns the same color to all vertices in the same connected component of the subgraph of G with edge set $S = \{v_1w_1, \dots, v_kw_k\}$. If the sizes of the connected components of this subgraph are $\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}$, we thus have to choose $\ell(\lambda)$ colors $i_1, i_2, \dots, i_{\ell(\lambda)}$ such that the j th color i_j is assigned to λ_j vertices, which gives a monomial $x_{i_1}^{\lambda_1} \dots x_{i_{\ell(\lambda)}}^{\lambda_{\ell(\lambda)}}$. The set of all such monomials is precisely the set of monomials that appear in the expansion

$$p_{\lambda(S)} = p_{\lambda_1} \dots p_{\lambda_{\ell(\lambda)}} = (x_1^{\lambda_1} + x_2^{\lambda_2} + \dots) \dots (x_1^{\lambda_{\ell(\lambda)}} + x_2^{\lambda_{\ell(\lambda)}} + \dots).$$

Thus, we need to add or subtract a $p_{\lambda(S)}$ term for each subset $S \subseteq E(G)$, with the sign $(-1)^{|S|}$ given by the parity of $|S|$, since the signs must alternate. $\square$

Example 5.8. For $G = \triangle$, the 8 possible subsets $S \subseteq E(G)$ are listed below, along with their corresponding power sum terms:



It follows that

$$X_{\triangle} = p_{1^3} - 3p_{21} + 2p_3.$$

Expansion in terms of subsets without broken circuits

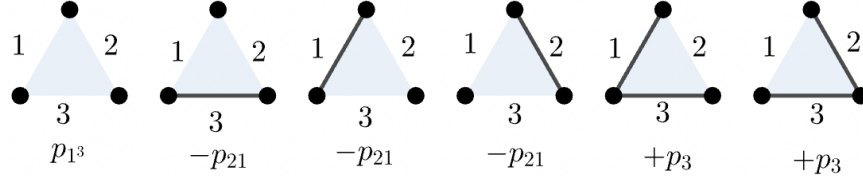
Stanley ([Sta95]) also gives an alternative version without cancellation, given an ordering on $E(G)$:

Theorem 5.9 (Stanley [Sta95], Theorem 1.3).

$$X_G = \sum_{\substack{S \subseteq E(G) \text{ not containing} \\ \text{a broken circuit}}} (-1)^{|S|} p_{\lambda(S)},$$

where a **broken circuit** is a cycle in G with its largest edge removed.

Example 5.10. Let $G = \triangle$ again, and label the edges 1, 2, and 3, as shown below. Then the last two subsets from Example 5.8 are not valid since they contain the broken circuit consisting of edges 1 and 2, so we instead have a sum of just the first 6 terms:



This gives the same result for $X_{\triangle}$, since the final two terms from the expansion in Example 5.8 cancel.

Expansion in terms of acyclic orientations

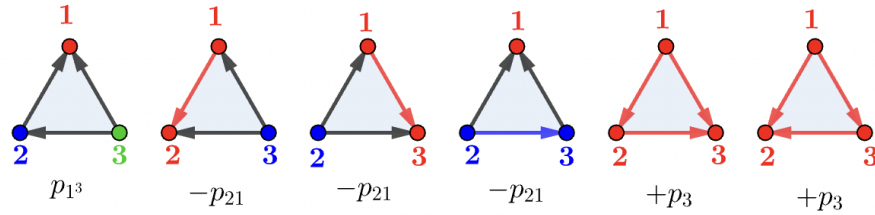
Bernardi and Nadeau [BN20] give an alternative p -expansion X_G in terms of acyclic orientations, given an ordering on $V(G)$. An **acyclic orientation** γ is a way of assigning an orientation to each edge of G without forming any directed cycles. The partition $\lambda(\gamma)$ has parts equal to the sizes of the **sources components** of γ arranged in nonincreasing order, which are constructed as follows:

1. The first source component consists of the smallest vertex in $V(G)$ and all vertices that can be reached from it by following directed paths.
2. At each subsequent step until all vertices are assigned to a source component, choose the smallest vertex v not yet in a source component as the new source. The next source component consists of all vertices not yet assigned to a source component that are reachable from v by following directed paths.

Theorem 5.11 (Bernardi-Nadeau [BN20], Proposition 5.3). X_G has the power sum expansion

$$X_G = (-1)^{|V(G)|} \sum_{\substack{\gamma \text{ an acyclic} \\ \text{orientation of } G}} (-1)^{\ell(\lambda(\gamma))} p_{\lambda(\gamma)}.$$

Example 5.12. Again let $G = \triangle$, this time with the vertices labeled 1, 2, and 3, as shown. The 6 possible acyclic orientations of G are shown below, along with their corresponding monomials. In each case, we use a different color for each source component.



The above formulas can be extended to weighted graphs with a vertex weight function $\omega : V \rightarrow \mathbb{N}$, by defining the parts of $\lambda(S)$ to be the sum of the weights of the vertices in the connected components formed by S in Stanley's formulas, and $\lambda(\gamma)$ to be the sum of weights of the vertices in the source components of γ in Bernardi and Nadeau's formula.

5.3 Examples of Theorem 5.2

We will now revisit the two example $\bar{p}$ -expansions of $\bar{X}_{K_{21}}$ and $\bar{X}_{\bullet\bullet}$ from §5.1 and show how to compute their coefficients using Theorem 5.2.

Example 5.13. Let $G = K_{21}$ consist of two connected vertices v and w , with $\omega(v) = 1$ and $\omega(w) = 2$. Then

$$I_{(G,\omega)}(t) = 1 + t + t^2,$$

since G has three independent sets: $\emptyset$ has weight 0, $\{v\}$ has weight 1, and $\{w\}$ has weight 2. By difference of cubes,

$$I_{(G,\omega)}(t) = \frac{1 - t^3}{1 - t}.$$

Then by repeated difference of squares,

$$\frac{1}{1 - t} = (1 + t)(1 + t^2)(1 + t^4)(1 + t^8) \cdots = \prod_{j \geq 0} (1 + t^{2^j}),$$

and similarly,

$$1 - t^3 = \frac{1}{1 + t^3} \cdot \frac{1}{1 + t^6} \cdot \frac{1}{1 + t^{12}} \cdot \frac{1}{1 + t^{24}} \cdots = \prod_{j \geq 0} \frac{1}{1 + t^{3 \cdot 2^j}},$$

which gives

$$I_{(G,\omega)}(t) = \prod_{j \geq 0} (1 + t^{2^j}) \cdot \prod_{j \geq 0} \frac{1}{1 + t^{3 \cdot 2^j}}.$$

The other subsets of $V(G)$ are $\{v\}$, $\{w\}$, and $\emptyset$. In these cases, we get:

$$\begin{aligned} I_{(G|\{v\},\omega)}(t) &= 1 + t, \\ I_{(G|\{w\},\omega)}(t) &= 1 + t^2, \\ I_{(G|\emptyset,\omega)}(t) &= 1. \end{aligned}$$

Putting this together, Theorem 5.2 gives

$$\overline{X}_{(G,\omega)}(t) = \prod_{j \geq 0} (1 + \overline{p}_{2^j}) \cdot \prod_{j \geq 0} \frac{1}{1 + \overline{p}_{3 \cdot 2^j}} - (1 + \overline{p}_1) - (1 + \overline{p}_2) + 1.$$

Note that in this case, the superscript 2^j refers to the partition with one part whose size is 2^j rather than the partition with j parts of size 2. It follows that all $\overline{p}$ -coefficients are ± 1 or 0 in this case. In particular,

$$[\overline{p}_\lambda] \overline{X}_{(G,\omega)} = (-1)^{\# \text{ of parts of } \lambda \text{ that are multiples of } 3}$$

provided that λ satisfies all of the following properties:

- $|\lambda| \geq 3$.
- All parts are of the form 2^j or $3 \cdot 2^j$ for some $j \geq 0$.
- No parts of the form 2^j are repeated.

All other coefficients are 0. To illustrate this, we can expand out the leading terms to get

$$\begin{aligned} \overline{X}_{(G,\omega)}(t) &= (1 + \overline{p}_1)(1 + \overline{p}_2)(1 + \overline{p}_4) \cdots \frac{1}{1 + \overline{p}_3} \cdot \frac{1}{1 + \overline{p}_6} \cdot \frac{1}{1 + \overline{p}_{12}} \cdots - (1 + \overline{p}_1 + \overline{p}_2) \\ &= (1 + \overline{p}_1)(1 + \overline{p}_2)(1 + \overline{p}_4) \cdots (1 - \overline{p}_3 + \overline{p}_{33} - \overline{p}_{333} + \cdots)(1 - \overline{p}_6 + \overline{p}_{66} - \cdots) \cdots - (1 + \overline{p}_1 + \overline{p}_2) \\ &= (-\overline{p}_3 + \overline{p}_{21}) + (\overline{p}_4 - \overline{p}_{31}) + (\overline{p}_{41} - \overline{p}_{32}) + (-\overline{p}_6 + \overline{p}_{42} + \overline{p}_{33} - \overline{p}_{321}) \\ &\quad + (-\overline{p}_{61} + \overline{p}_{421} + \overline{p}_{331}) + (\overline{p}_8 - \overline{p}_{62} + \overline{p}_{332}) + (\overline{p}_{81} + \overline{p}_{63} - \overline{p}_{621} - \overline{p}_{333} + \overline{p}_{3321}) + \cdots \end{aligned}$$

We can see that these terms match the ones from §5.1.

For the seemingly simpler case of a single *unweighted* edge $\bullet \text{---} \bullet$, there does not seem to be such a simple formula for the coefficients:

Example 5.14. Let $G = \bullet \text{---} \bullet = \{v, w\}$, where v and w both have weight 1. Then we get

$$I_G(t) = 1 + 2t,$$

since $\{v\}$ and $\{w\}$ are stable sets of size 1 and $\emptyset$ is a stable set of size 2. In this case the factorization begins

$$1 + 2t = (1 + t)^2 \cdot \frac{1}{1 + t^2} \cdot (1 + t^3)^2 \cdot \frac{1}{(1 + t^4)^4} \cdot (1 + t^5)^6 \dots$$

The other stable sets are $\{v\}$, $\{w\}$, and $\emptyset$, which have independence polynomials

$$I_{G|\{v\}}(t) = I_{G|\{w\}}(t) = 1 + t, \quad I_{G|\emptyset}(t) = 1.$$

Putting this together, Theorem 5.2 gives

$$\begin{aligned} \bar{X}_G(t) &= (1 + \bar{p}_1)^2 \cdot \frac{1}{1 + \bar{p}_2} \cdot (1 + \bar{p}_3)^2 \cdot \frac{1}{(1 + \bar{p}_4)^4} \cdot (1 + \bar{p}_5)^6 \dots - (1 + \bar{p}_1) - (1 + \bar{p}_1) + 1 \\ &= (1 + 2\bar{p}_1 + \bar{p}_{11})(1 - \bar{p}_2 + \bar{p}_{22} - \dots)(1 + 2\bar{p}_3 + \bar{p}_{33})(1 - 4\bar{p}_4 + 10\bar{p}_{44} - 20\bar{p}_{444} + \dots) \dots - (1 + 2\bar{p}_1) \\ &= (-\bar{p}_2 + \bar{p}_{11}) + (2\bar{p}_3 - 2\bar{p}_{21}) + (-4\bar{p}_4 + 4\bar{p}_{31} + \bar{p}_{22} - \bar{p}_{211}) \\ &\quad + (6\bar{p}_5 - 8\bar{p}_{41} - 3\bar{p}_{32} + 2\bar{p}_{311} + 2\bar{p}_{221}) \\ &\quad + (-9\bar{p}_6 + 12\bar{p}_{51} + 4\bar{p}_{42} - 4\bar{p}_{411} + \bar{p}_{33} - 4\bar{p}_{321} - \bar{p}_{222} + \bar{p}_{2211}) \dots, \end{aligned}$$

again matching the expansion from §5.1. The sequence of exponents 2, -1, 2, -4, 6, ... is sequence A038067 in the Online Encyclopedia of Integer Sequences (OEIS) [OEI24].

5.4 Proofs

In this section we will prove the Theorem 5.2, Corollary 5.3, Proposition 5.4, and Theorem 5.5.

Proof of Theorem 5.2. Our key lemma is an expansion formula for Stanley's Y_G function (introduced in [Sta98]), where $Y_{(G,\omega)}$ is defined to be the same as $\bar{X}_{(G,\omega)}$ except that it includes monomials corresponding to colorings where some vertices are not assigned any colors:

$$Y_{(G,\omega)} := \sum_{\kappa} \prod_{v \in V(G)} \left(\prod_{i \in \kappa(v)} x_i \right)^{\omega(v)},$$

where the sum ranges over all functions $\kappa : V(G) \rightarrow 2^n$ such that $\kappa(v) \cap \kappa(w) = \emptyset$ whenever $vw \in E(G)$.

Lemma 5.15. $Y_{(G,\omega)}$ factors as

$$Y_{(G,\omega)} = \prod_{k \geq 1} (1 + \bar{p}_k)^{a_{V(G)}(k)}.$$

Proof. Since $Y_{(G,\omega)}$ allows vertices to not receive any colors, we can think of assigning each color to a subset of the vertices independently of all other colors. Specifically, for each color, we can independently choose a subset of $V(G)$ which will receive that color, and the only restriction is that the subset be an independent set. Thus, the generating series for the ways to choose which vertices receive color i is given by the independence polynomial $I_{(G,\omega)}(x_i)$. Since different colors can be assigned independently, we can compute $Y_{(G,\omega)}$ by simply multiplying over all colors, so

$$Y_{(G,\omega)} = \prod_{i \geq 1} I_{(G,\omega)}(x_i).$$

Then by definition of $a_{V(G)}(k)$, we have

$$I_{(G,\omega)}(x_i) = \prod_{k \geq 1} (1 + x_i^k)^{a_{V(G)}(k)}.$$

Now note that $\bar{p}_k$ represents the sum of all monomials corresponding to ways to assign a nonempty set of colors to a single weight k vertex, so $1 + \bar{p}_k$ represents the sum of monomials corresponding to all ways to assign a set of colors to the weight k vertex, including the empty set. To choose such a coloring, we can independently choose whether or not to use each color. If color i is used, then we get an x_i^k in the monomial, and otherwise the power of x_i is 0. Since we can independently choose an exponent of 0 or k for each variable,

$$1 + \bar{p}_k = \prod_{i \geq 1} (1 + x_i^k).$$

It follows that

$$Y_{(G,\omega)} = \prod_{i \geq 1} I_{(G,\omega)}(x_i) = \prod_{i,k \geq 1} (1 + x_i^k)^{a_{V(G)}(k)} = \prod_{k \geq 1} (1 + \bar{p}_k)^{a_{V(G)}(k)},$$

as claimed. $\square$

Theorem 5.2 then follows from the principle of inclusion-exclusion. The difference between $\bar{X}_{(G,\omega)}$ and $Y_{(G,\omega)}$ is that $\bar{X}_{(G,\omega)}$ requires all vertices to receive a nonempty set of colors. Thus, by inclusion-exclusion, we can compute $\bar{X}_{(G,\omega)}$ by adding the monomials all possible colorings without that restriction, then subtracting the monomials where a particular vertex receives no colors, then adding back the ones where a particular pair of vertices receive no colors, and so on. It follows that

$$\bar{X}_{(G,\omega)} = \sum_{W \subseteq V(G)} Y_{(G,\omega)}.$$

Plugging in the formula from Lemma 5.15 proves Theorem 5.2. $\square$

The formula in Corollary 5.3 is now fairly immediate:

Proof of Corollary 5.3. Let $\lambda = \ell^{i_\ell} \dots 2^{i_2} 1^{i_1}$. First, we note that

$$\bar{p}_\lambda = \bar{p}_{\lambda_1} \dots \bar{p}_{\lambda_{\ell(\lambda)}} = \bar{p}_\ell^{i_\ell} \dots \bar{p}_2^{i_2} \bar{p}_1^{i_1}.$$

To see this, recall that $\bar{p}_\lambda$ is the sum of all monomials corresponding to colorings of the vertices of the edgeless graph $\overline{K_\lambda}$ with vertex weights $\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}$, or equivalently, with i_k vertices of weight k for each $k = 1, 2, \dots, \ell$. Since the graph has no edges, the set of colors assigned to a vertex has no bearing on how we can color the other vertices, so we can choose the coloring of each vertex independently. Thus, $\bar{p}_\lambda$ is the product over all vertices of the sum of all possible monomials for ways to color that vertex, which is just $\bar{p}_k$ for a vertex of weight k .

Thus, get a $\bar{p}_\lambda$ term from $\prod_{k \geq 1} (1 + \bar{p}_k)^{a_W(k)}$, we need to choose a $\bar{p}_k^{i_k}$ term from $(1 + \bar{p}_k)^{a_W(k)}$ for each $k = 1, 2, \dots, \ell$. By the binomial theorem,

$$(1 + \bar{p}_k)^{a_W(k)} = \sum_{i \geq 0} \binom{a_W(k)}{i} \bar{p}_k^i,$$

so the coefficient of $\bar{p}_\lambda$ in the product $\prod_{k \geq 1} (1 + \bar{p}_k)^{a_W(k)}$ is just the product of these coefficients:

$$[\bar{p}_\lambda] \prod_{k \geq 1} (1 + \bar{p}_k)^{a_W(k)} = \prod_{k=1}^{\ell} \binom{a_W(k)}{i_k} \bar{p}_k^{i_k}.$$

Summing over all $W \subseteq V(G)$ completes the proof. $\square$

Next, we prove the recursive formula in Proposition 5.4:

Proof of Proposition 5.4. Taking logarithms of both sides of the equation

$$(1+t)^{a_W(1)}(1+t^2)^{a_W(2)}(1+t^3)^{a_W(3)} \dots = I_{(G|_W, \omega)}(t)$$

gives

$$\sum_{j \geq 1} a_W(j) \log(1 + t^j) = \log(I_{G|W, \omega}(t)).$$

Expanding the left side using the power series $\log(1 + x) = \sum_{i \geq 1} \frac{(-1)^{i+1} x^i}{i}$, we get

$$\sum_{j \geq 1} a_W(j) \sum_{i \geq 1} \frac{(-1)^{i+1} t^{ij}}{i} = \log(I_{G|W, \omega}(t)).$$

Since $j = k$ is the largest value of j on the left side that contributes a t^k term, we can compute $a_W(k)$ recursively from the values $a_W(j)$ for $j < k$ by making the coefficient of t^k on the left side match the coefficient $[t^k] \log(I_{G|W, \omega}(t))$ on the right side. The values $j < k$ that contribute a t^k term on the left are precisely the values where j is some divisor d of k , in which case $i = k/d$. The contribution from the $j = d$ term is then

$$a_W(d) \frac{(-1)^{k/d+1}}{k/d} = \frac{(-1)^{k/d} \cdot a_W(d)}{k}.$$

Summing over all values $d \mid k$ and then subtracting the resulting sum from the right side coefficient $[t^k] \log(I_{G|W, \omega}(t))$ gives the formula in Proposition 5.4. $\square$

Finally, we prove that these coefficients are all integers:

Proof of Theorem 5.5. Recall from §3.4 that the *K -theoretic augmented monomial symmetric functions* $\widetilde{m}_\lambda$ are defined in [CPS23] as

$$\widetilde{m}_\lambda := \overline{X}_{K_\lambda},$$

where K_λ is a complete graph on $\ell(\lambda)$ whose vertex weights are the parts of λ , and that $\overline{X}_G$ has the $\widetilde{m}$ -expansion formula

$$\overline{X}_{(G, \omega)} = \sum_{C \in \text{SSC}(G)} \widetilde{m}_{\lambda(C)},$$

where $\text{SSC}(G)$ is the set of *stable set covers* of G , meaning ways to cover $V(G)$ with some set of independent sets $C = \{S_1, \dots, S_k\}$, and the associated partition $\lambda(C)$ has parts $\omega(S_1), \dots, \omega(S_k)$ arranged in nonincreasing order. Now consider the $\widetilde{m}$ -expansion of $\overline{p}_\lambda = \overline{X}_{\overline{K}_\lambda}$. Since $\overline{K}_\lambda$ has no edges, all subsets of its vertices are independent sets. One possible stable set cover is the one where each vertex is in a stable set by itself, in which case $\lambda(C) = \lambda$, so the expansion of $\overline{p}_\lambda$ has an $\widetilde{m}_\lambda$ term. For every other term $\widetilde{m}_\mu$, we must have either $|\mu| > |\lambda|$ (if some vertices are covered by multiple stable sets in C) or else μ is formed by taking λ and combining some of its parts (if each vertex is only in one stable set but some stable sets contain multiple vertices), which is equivalent to saying that λ is a *refinement* of μ . Thus, we have $[\widetilde{m}_\lambda] \overline{p}_\lambda = 1$ and $[\widetilde{m}_\mu] \overline{p}_\lambda = 0$ whenever $|\mu| < |\lambda|$ or $|\mu| = |\lambda|$ but λ is not a refinement of μ . Now, order the set of all partitions in such a way that partitions of smaller numbers come first and λ comes before μ whenever λ is a refinement of μ , which can be done by ordering the partitions of each size lexicographically. Then the transition matrix from the $\overline{p}$ -basis to the $\widetilde{m}$ -basis is upper triangular with 1's along the diagonal and all integer entries. It follows that the inverse matrix which transitions from the $\widetilde{m}$ -basis to the $\overline{p}$ -basis is also upper triangular with 1's on the diagonal and all integer entries. Thus, each $\widetilde{m}_\mu$ is a linear combination of $\overline{p}_\lambda$ terms with integer coefficients. Since $\overline{X}_{(G, \omega)}$ is an integral linear combination of $\widetilde{m}_\mu$ terms for every weighted graph (G, ω) , it follows that $\overline{X}_{(G, \omega)}$ is an integral linear combination of $\overline{p}_\lambda$ terms, as claimed. $\square$

Chapter 6

The chromatic symmetric homology for star graphs

6.1 Introduction

In this chapter, we study a different enrichment of X_G , the *chromatic symmetric homology* $H_*(G)$, which was introduced by Sazdanović and Yip in [SY18] as a *categorification* of X_G . The idea of categorification is to take a “set-like” object and replace it with a “category-like” object that has additional structure. Specifically, this often involves taking a polynomial and replacing it with a collection of modules over some ring, such as replacing *Schur polynomials* s_λ with *Specht modules* S_λ , which are irreducible representations of the symmetric group $\mathfrak{S}_n$, or equivalently, modules over the ring $\mathbb{C}[\mathfrak{S}_n]$ (see §1.4). The way these categorifications are constructed is generally analogous to the construction of the homology groups of a topological space.

The idea of categorifying polynomials in this way began with the *Khovanov homology*, which was introduced by Khovanov in [Kho00] as a categorification of the *Jones polynomial* from knot theory. Sazdanović and Yip constructed the chromatic symmetric homology in [SY18] as a categorification of the chromatic symmetric function. A corresponding vertex-weighted analogue of the chromatic symmetric homology was introduced in [Cil24]. For a ring R , the chromatic symmetric homology $H_*(G; R)$ is constructed as

$$H_*(G; R) = \bigoplus_{0 \leq i \leq j \leq |E(G)|} H_{i,j}(G; R),$$

where each $H_{i,j}(G; R)$ is an $R[\mathfrak{S}_n]$ -module. The chromatic symmetric homology $H_*(G)$ is a strictly stronger graph invariant than X_G in the sense that there exist pairs of graphs with $X_{G_1} = X_{G_2}$ but $H_*(G_1) \neq H_*(G_2)$, as proven by Chandler, Sazdanović, Stella, and Yip in [Cha+23]. In particular, the authors of [Cha+23] conjectured that $H_*(G; \mathbb{Z})$ can detect planarity based on whether $H_{1,0}(G; \mathbb{Z})$ has $\mathbb{Z}_2$ -torsion (i.e. contains a subgroup isomorphic to $\mathbb{Z}_2$). One direction of this conjecture was proven in [CM23], namely, that all nonplanar graphs contain $\mathbb{Z}_2$ -torsion, but the other direction remains open.

The authors of [Cha+23] also present a number of computations and conjectures about special cases of $H_*(G)$. In this chapter, we focus on the special case of *star graphs* (graphs where one central vertex is connected to all others and no other pairs of vertices are connected). The authors of [Cha+23] conjectured the following formulas involving the chromatic symmetric homology for star graphs:

Conjecture 6.1. *For G the n -vertex star, the multiplicity of $\mathcal{S}_{(n-2,2)}$ in $H_{1,0}(G)$ is 1.*

Conjecture 6.2. *For G the n -vertex star, the multiplicity of $\mathcal{S}_{2^{2^{1^{n-4}}}}$ in $H_{1,0}(G)$ is $\binom{n-2}{2}$.*

They also gave the following table ([Cha+23], §4.3) with their conjectures for the full values of $H_{1,0}(G)$ where G is a star on 4, 5, 6, or 7 vertices:

G	$H_{1,0}(G; \mathbb{C})$
	$\mathcal{S}_{2^2}$
	$\mathcal{S}_{2^2 1}^{\oplus 3} \oplus \mathcal{S}_{32}$
	$\mathcal{S}_{2^2 1^2}^{\oplus 6} \oplus \mathcal{S}_{2^3}^{\oplus 5} \oplus \mathcal{S}_{321}^{\oplus 4} \oplus \mathcal{S}_{42}$
	$\mathcal{S}_{2^2 1^3}^{\oplus 10} \oplus \mathcal{S}_{2^3 1}^{\oplus 16} \oplus \mathcal{S}_{321^2}^{\oplus 10} \oplus \mathcal{S}_{32^2}^{\oplus 9} \oplus \mathcal{S}_{421}^{\oplus 5} \oplus \mathcal{S}_{52}$

Table 6.1: The degree 0 chromatic symmetric homology for stars on 4 through 7 vertices.

We verify Conjectures 6.1 and 6.2 as well as the conjectured values from Table 6.1 as special cases of the following result:

Theorem 6.3. *For G the n -vertex star, the multiplicity of $\mathcal{S}_{\ell 2^k 1^{n-\ell-2k}}$ in $H_{1,0}(G)$ is*

$$\binom{n-1}{\ell-1} f^{2^k 1^{n-\ell-2k}} - f^{\ell 2^k 1^{n-\ell-2k}},$$

where f^λ is the number of standard Young tableaux of shape λ .

In particular, Conjectures 6.1 and 6.2 follow immediately from the following special case of Theorem 6.3:

Corollary 6.4. *For G the n -vertex star multiplicity of $\mathcal{S}_{\ell 21^{n-\ell-2}}$ in $H_{1,0}(G)$ is $\binom{n-2}{\ell}$.*

We also give a more explicit formula for the special case where all parts of λ have size 1 or 2, which we use to verify some of the values in Table 6.1:

Corollary 6.5. *For G the n -vertex star, the multiplicity of $\mathcal{S}_{2^k 1^{n-2k}}$ in $H_{1,0}(G)$ is*

$$(n-2k+1) \left(\binom{n-1}{k-1} - \frac{1}{k} \binom{n}{k-1} \right).$$

Finally, we conjecture that Theorem 6.3 gives essentially a full description of the degree 0 chromatic symmetric homology for star graphs, in the sense that all other multiplicities are 0 (except $H_{0,0}(G) = \mathcal{S}_{1^n}$):

Conjecture 6.6. *Whenever $i \geq 2$ or $\lambda_2 \geq 3$, the multiplicity of $\mathcal{S}_\lambda$ in $H_{i,0}(G)$ is 0.*

The remainder of this chapter is organized as follows. In §6.2 we give the general construction of $H_*(G)$. In §6.3, we explain how to concretely compute the degree $j = 0$ case $H_{i,0}(G; \mathbb{C})$ and its decomposition into Specht modules using the combinatorics of Young tableaux. In §6.4 we prove Theorem 6.3. In §6.5, we prove Corollary 6.4, and in §6.6, we prove Corollary 6.5. Finally, in §6.7, we use our results to verify the conjectured values from Table 6.1.

6.2 General construction of $H_*(G)$

Let $n := |V(G)|$ and $m := |E(G)|$. For a ring R (generally $\mathbb{C}$ or $\mathbb{Z}$), the chromatic symmetric homology $H_*(G; R)$ is defined by building a sequence of **chain modules** $C_i(G)$ with associated **differentials** or **chain maps** $d_i : C_i(G) \rightarrow C_{i-1}(G)$, where $0 \leq i \leq m$:

$$0 \xrightarrow{d_{m+1}} C_m(G) \xrightarrow{d_m} C_{m-1}(G) \xrightarrow{d_{m-1}} \dots \xrightarrow{d_2} C_1(G) \xrightarrow{d_1} C_0(G) \xrightarrow{d_0} 0.$$

Each $C_i(G)$ is a representation of $\mathfrak{S}_n$, or equivalently, an $R[\mathfrak{S}_n]$ -module, and each d_i is a homomorphism of $R[\mathfrak{S}_n]$ -modules. The **homology groups** associated to G are then defined by

$$H_i(G; R) := \ker(d_i) / \operatorname{im}(d_{i+1}).$$

Each chain module has an associated **grading** $C_i(G; R) = \bigoplus_{j=0}^i C_{i,j}(G; R)$, and the chain maps and homology groups inherit this grading, so we can write $d_i = \bigoplus_{j=0}^i d_{i,j}$ and $H_i(G; R) = \bigoplus_{j=0}^i H_{i,j}(G; R)$. The key connection to X_G is that when $R = \mathbb{C}$, if we break each $H_{i,j}$ into a direct sum of Specht modules, replace each Specht module with the associated Schur polynomial, and sum the resulting Schur polynomials with the appropriate signs, we recover the chromatic symmetric function:

$$H_{i,j}(G; \mathbb{C}) = \bigoplus_{\lambda} \mathcal{S}_{\lambda}^{\oplus n_{i,j}^{\lambda}} \longrightarrow X_G = \sum_{i,j} (-1)^{i+j} n_{i,j}^{\lambda} s_{\lambda},$$

where $n_{i,j}^{\lambda}$ is the multiplicity of $\mathcal{S}_{\lambda}$ in $H_{i,j}(G; \mathbb{C})$. Assume the vertices of G are labeled 1 through n . The chain module $C_i(G)$ can be constructed as a direct sum of $\mathbb{C}[\mathfrak{S}_n]$ -modules $\mathcal{M}_F$, one for each subset $F \subseteq E$ with $|F| = i$:

$$C_i(G) := \bigoplus_{F \subseteq E, |F|=i} \mathcal{M}_F.$$

Each such subset F corresponds to a **spanning subgraph** of G , where “spanning” means we take the vertex set of the subgraph to be the full vertex set $V(G)$. If $b_1, \dots, b_r$ are the sizes of the connected components of F , then the definition of $\mathcal{M}_F$ is

$$\mathcal{M}_F := \text{Ind}_{\mathfrak{S}_{b_1} \times \dots \times \mathfrak{S}_{b_r}}^{\mathfrak{S}_n} (\mathcal{L}_{b_1} \otimes \dots \otimes \mathcal{L}_{b_r}),$$

where $\text{Ind}_H^G(V)$ denotes the **induced representation** of G corresponding to a representation V of H ,

$$\mathcal{L}_b := \bigoplus_{j=0}^{b-1} \mathcal{S}_{(b-j, 1^j)},$$

and $\mathcal{S}_{\lambda}$ is the **Specht module** corresponding to the partition λ . The **degree (or grading)** is defined such that each $\mathcal{S}_{(b-j, 1^j)}$ piece has degree j , and the degree of a tensor product of two representations is the sum of their degrees. In particular, the degree 0 parts are

$$(\mathcal{M}_F)_0 = \text{Ind}_{\mathfrak{S}_{b_1} \times \dots \times \mathfrak{S}_{b_r}}^{\mathfrak{S}_n} (\mathcal{S}_{b_1} \otimes \dots \otimes \mathcal{S}_{b_r}).$$

Then for each pair F and F' such that F' is formed by removing one edge from F , the differential map $d_{F,F'}$ is the natural embedding of $\mathcal{M}_F$ into $\mathcal{M}_{F'}$. (The precise construction of this unique “natural embedding” is a bit technical and is given in [SY18], §2.3.)

Finally, given an ordering on $|E(G)|$, the differential maps d_i are defined by

$$d_i := \bigoplus_{|F|=i, |F'|=i-1} (-1)^{\# \text{ of edges in } F \text{ that come after the removed edge}} \cdot d_{F,F'}.$$

The name “differentials” is justified by the following property:

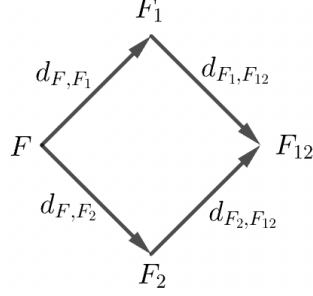
Theorem 6.7 (Sazdanović and Yip [SY18], Proposition 2.10). *These maps are in fact differentials in the sense that $d_i \circ d_{i+1} = 0$, or equivalently, $\text{im}(d_{i+1}) \subseteq \ker(d_i)$.*

The full proof of this fact is a bit technical, but we will sketch the idea:

Proof idea. The key is that the choice of signs above makes everything cancel. For each subgraph F and pair of edges $e_1, e_2 \in F$, write

$$F_1 := F - e_1, \quad F_2 := F - e_2, \quad F_{12} := F - e_1 - e_2.$$

Consider the differential maps between these four subgraphs, as shown below:



We claim that in all cases, three of the edges must have the same sign while the fourth has the opposite sign. To see this, suppose e_1 comes before e_2 and that there are k edges in F between e_1 and e_2 and ℓ edges after e_2 . Then d_{F,F_1} has sign $(-1)^{k+\ell+1}$, $d_{F_2,F_{12}}$ has sign $(-1)^{k+\ell}$, and d_{F,F_2} and $d_{F_1,F_{12}}$ both have signs $(-1)^\ell$. It follows that the signs of d_{F,F_2} and $d_{F_1,F_{12}}$ are always the same while the signs of d_{F,F_1} and $d_{F_2,F_{12}}$ are always opposites, so in total there must be three edges with one sign and one edge with the opposite sign.

It can then be shown that

$$d_{F_1,F_{12}} \circ d_{F,F_1} = d_{F_2,F_{12}} \circ d_{F,F_2},$$

so the signs of the edges make these terms cancel. Since all terms contributing to $d_i \circ d_{i+1}$ can be paired up in this way, it follows that $d_i \circ d_{i+1} = 0$. $\square$

6.3 $H_{i,0}(G)$ and its decomposition into Specht modules

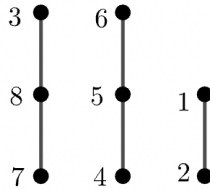
Our focus from now on will be on the case $j = 0$ and $R = \mathbb{C}$, so will now describe how $H_{i,0}(G)$ can be explicitly computed using the combinatorics of Young tableaux.

In that case, the $\mathbb{C}[\mathfrak{S}_n]$ -modules $\mathcal{M}_F$ (which we henceforth use as shorthand for $(\mathcal{M}_F)_0$) can alternatively be constructed as

$$\mathcal{M}_F := \mathbb{C}[\mathfrak{S}_n] \cdot a_{T(F)},$$

where $\mathbb{C}[\mathfrak{S}_n] \cdot a_{T(F)} := \{\sigma \cdot a_{T(F)} : \sigma \in \mathbb{C}[\mathfrak{S}_n]\}$, and $a_{T(F)}$ is a particular element of $\mathbb{C}[\mathfrak{S}_n]$ that we will define momentarily. To define $a_{T(F)}$, let $\lambda(F)$ be *partition type* of F , meaning the partition whose parts are the sizes of the connected components of F , arranged in decreasing order. We construct a Young tableau $T(F)$ of shape $\lambda(F)$ as follows. For each connected component of F , list the vertices of F in increasing order. Order these lists first in decreasing order by length, and then within that in increasing order by the smallest element. Then use the i th list to fill the i th row of the Young diagram. Note that the resulting Young tableau need not be standard, as the numbers increase across each row but may not increase down each column.

Example 6.8. Let F be the spanning subgraph shown below:



Then $\lambda(F) = 332$, and

$$T(F) = \begin{array}{|c|c|c|} \hline 3 & 7 & 8 \\ \hline 4 & 5 & 6 \\ \hline 1 & 2 & \\ \hline \end{array}.$$

The *row symmetrizer* $a_T \in \mathbb{C}[\mathfrak{S}_n]$ for a tableau T is the sum of all $\sigma \in \mathfrak{S}_n$ such that σ permutes the elements in each row of $T(F)$ among themselves, and the associated *permutation module* is

$$\mathcal{M}_T := \mathbb{C}[\mathfrak{S}_n] \cdot a_T.$$

We can factor a_T as the product over rows of the sum of all permutations of elements in that row.

Example 6.9. Write e for the identity permutation and $(i_1 i_2 \dots i_k)$ for the k cycle that maps i_j to i_{j+1} for $1 \leq j \leq k-1$ and i_k to i_1 . Then for the F from Example 6.8,

$$a_{T(F)} = (e + (37) + (38) + (78) + (378) + (387)) \cdot (e + (45) + (46) + (56) + (456) + (465)) \cdot (e + (12)).$$

The **column symmetrizer** $b_T \in \mathbb{C}[\mathfrak{S}_n]$ is the sum of $\text{sign}(\sigma) \cdot \sigma$ over all permutations σ that permute the elements of each column among themselves.

Example 6.10. For the F from Example 6.8,

$$b_{T(F)} = (e - (13) - (14) - (34) + (134) + (143)) \cdot (e - (25) - (27) - (57) + (257) + (275)) \cdot (e - (68)).$$

The **Young symmetrizer** for T is $c_T := b_T a_T$. The corresponding **Specht module** is

$$\mathcal{S}_T := \mathbb{C}[\mathfrak{S}_n] \cdot c_T.$$

For any tableaux S and T of the same shape λ , $\mathcal{S}_T$ and $\mathcal{S}_{T'}$ are isomorphic, so we can write $\mathcal{S}_\lambda$ for this isomorphism class of Specht modules. Similarly, $\mathcal{M}_T$ and $\mathcal{M}_{T'}$ are isomorphic when T and T' have the same shape μ , so we can write $\mathcal{M}_\mu$ for this isomorphism class.

It turns out that the Specht modules $\mathcal{S}_\lambda$ are precisely the **irreducible representations** of $\mathfrak{S}_n$ (see §1.4), which means any $\mathbb{C}[\mathfrak{S}_n]$ module can be written as a direct sum of Specht modules in a way that is unique up to isomorphism. In the case of $\mathcal{M}_F$, this decomposition turns out to be

$$\mathcal{M}_F \cong \bigoplus_{\lambda} \mathcal{S}_\lambda^{\oplus K_{\lambda, \lambda(F)}},$$

where $K_{\lambda\mu}$ is the **Kostka number** that counts the number of SSYT of shape λ and content μ .

We can realize this decomposition for $\mathcal{M}_F \cong \mathcal{M}_{\lambda(F)}$ as

$$\mathcal{M}_F = \bigoplus_{\lambda} \bigoplus_{j=1}^{K_{\lambda, \lambda(F)}} \mathbb{C}[\mathfrak{S}_n] \cdot v_{X_F^j}^{T(\lambda)}.$$

It remains to explain what x_F^j , $T(\lambda)$, and $v_{X_F^j}^{T(\lambda)}$ are. For $S, T \in \text{SYT}(\lambda)$, write $\sigma_{S,T} \in \mathfrak{S}_n$ for the permutation such that $\sigma_{S,T} \cdot S = T$. Then

$$v_S^T := c_T \cdot \sigma_{S,T} = \sigma_{S,T} \cdot c_S.$$

The tableau $T(\lambda) \in \text{SYT}(\lambda)$ is constructed by filling in the numbers $1, 2, \dots, n$ in order row by row. For instance, for $\lambda = 432$,

$$T(\lambda) = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 5 & 6 & 7 & \\ \hline 8 & 9 & & \\ \hline \end{array}.$$

Finally, the tableaux X_F^j are constructed as follows. Let $T_1, \dots, T_{K_{\lambda, \lambda(F)}}$ be the SSYT of shape λ and content $\lambda(F)$. Then we turn T_j into the corresponding tableau X_F^j of shape λ by replacing the 1's in T_j with the entries from the first row of $T(F)$ in order, the 2's with the entries from the second row of $T(F)$ in order, and so on, and then reordering so the entries in each row are increasing. The resulting tableau X_F^j will have entries 1 through n each showing up exactly once, and the entries will increase across each row, but it will not necessarily be standard as the entries need not increase going down the columns.

Example 6.11. Let F be the spanning subgraph from Example 6.8, so $\lambda(F) = 332$, and let $\lambda = 431$. Then $K_{\lambda, \lambda(F)} = 2$, so there are two copies of $\mathcal{S}_\lambda$ in $\mathcal{M}_T$. The two the SSYT of shape λ and content $\lambda(F)$ are

$$T_1 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 2 \\ \hline 2 & 2 & 3 & \\ \hline 3 & & & \\ \hline \end{array}, \quad T_2 = \begin{array}{|c|c|c|c|} \hline 1 & 1 & 1 & 3 \\ \hline 2 & 2 & 2 & \\ \hline 3 & & & \\ \hline \end{array}.$$

To get the tableaux X_F^1 and X_F^2 , we replace the three 1's in each tableau by the 3, 7, and 8 from the first row of $\lambda(F)$, we replace the three 2's by 4, 5, and 6 from the second row, and we replace the two 3's by 1 and 2 from the third row. This gives:

$$\begin{array}{|c|c|c|c|} \hline 3 & 7 & 8 & 4 \\ \hline 5 & 6 & 1 & \\ \hline 2 & & & \\ \hline \end{array}, \quad \begin{array}{|c|c|c|c|} \hline 3 & 7 & 8 & 1 \\ \hline 4 & 5 & 6 & \\ \hline 2 & & & \\ \hline \end{array}.$$

Then we reorder each row so the entries are in increasing order to get

$$X_F^1 = \begin{array}{|c|c|c|c|} \hline 3 & 4 & 7 & 8 \\ \hline 1 & 5 & 6 & \\ \hline 2 & & & \\ \hline \end{array}, \quad X_F^2 = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 7 & 8 \\ \hline 4 & 5 & 6 & \\ \hline 2 & & & \\ \hline \end{array}.$$

Thus, the two copies of $\mathcal{S}_\lambda$ in $\mathcal{M}_T$ are given by

$$\mathcal{M}_T|_{\mathcal{S}_{431}} = \mathbb{C}[\mathfrak{S}_n] \cdot v^{T(\lambda)} \begin{array}{|c|c|c|c|} \hline 3 & 4 & 7 & 8 \\ \hline 1 & 5 & 6 & \\ \hline 2 & & & \\ \hline \end{array} \oplus \mathbb{C}[\mathfrak{S}_n] \cdot v^{T(\lambda)} \begin{array}{|c|c|c|c|} \hline 3 & 7 & 8 & 1 \\ \hline 4 & 5 & 6 & \\ \hline 2 & & & \\ \hline \end{array},$$

where

$$T(\lambda) = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 7 & 8 \\ \hline 4 & 5 & 6 & \\ \hline 2 & & & \\ \hline \end{array}.$$

Note that neither X_F^1 nor X_F^2 is an SYT, since the entries do not always increase down the columns.

There is one other formula we will need for our dimension counting arguments. Write

$$f^\lambda := K_{\lambda, 1^n} = |\text{SYT}(\lambda)|$$

for the number of SYT of shape λ . Thus, f^λ is the multiplicity of $\mathcal{S}_\lambda$ in $\mathcal{M}_{1^n}$, and it can also be shown that

$$\dim \mathcal{S}_\lambda = f^\lambda.$$

Example 6.12. We have $f^{32} = 5$, with the 5 elements of $\text{SYT}(32)$ shown below:

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 4 \\ \hline 3 & 5 & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 3 & 4 \\ \hline 2 & 5 & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array}.$$

The number of SYT of shape λ can be calculated using the [hook length formula](#). For each box $b \in \lambda$ in the Young diagram of shape λ , the [hook](#) $\text{hook}(b)$ consists of b itself together with all the boxes below it in its column or to the right of it in its row (forming a Γ shape). Then the formula is

$$f^\lambda = \frac{|\lambda|!}{\prod_{b \in \lambda} |\text{hook}(b)|},$$

where $|\text{hook}(b)|$ denotes the number of boxes in $\text{hook}(b)$.

Example 6.13. For $\lambda = 32$, the hooks are shown below:

$$\begin{array}{|c|c|c|} \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline & \blacksquare & \blacksquare \\ \hline & \blacksquare & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline & & \blacksquare \\ \hline & & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline & & \\ \hline \blacksquare & \blacksquare & \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline & & \\ \hline & \blacksquare & \\ \hline \end{array}.$$

Their lengths are 4, 3, 1, 2, and 1, respectively, so the hook length formula gives

$$f^{32} = \frac{5!}{4 \cdot 3 \cdot 1 \cdot 2 \cdot 1} = 5,$$

which matches what we found in Example 6.12.

6.4 Proof of Theorem 6.3

Let $\lambda = \ell 2^k 1^{n-\ell-2k}$. The restriction of $C_0(G)$ to $\mathcal{S}_\lambda$ is

$$C_0(G)|_{\mathcal{S}_\lambda} = \bigoplus_{Y_i \in \text{SYT}(\lambda)} \mathbb{C}[\mathfrak{S}_n] \cdot v_{Y_i}^{T(\lambda)} \subseteq \mathbb{C}[\mathfrak{S}_n],$$

so the multiplicity of $\mathcal{S}_\lambda$ in $C_0(G)$ is $|\text{SYT}(\lambda)| = f^\lambda$.

To find $C_1(G)|_{\mathcal{S}_\lambda}$, we need to consider the spanning subgraphs of G with a single edge. Label the vertices of G as $1, \dots, n$, and without loss of generality assume vertex 1 is the central vertex. Then there are $n-1$ spanning subgraphs of G with one edge, each of the form F_{1i} for some $i \neq 1$. Since $\lambda(F_{1i}) = 21^{n-2}$ for all i , the restriction of $C_1(G)$ to $\mathcal{S}_\lambda$ is

$$C_1(G)|_{\mathcal{S}_\lambda} = \bigoplus_{i=2}^n \mathcal{M}_{F_{1i}}|_{\mathcal{S}_\lambda} = \bigoplus_{i=2}^n \bigoplus_{r=1}^{K_{\lambda, 21^{n-2}}} \mathbb{C}[\mathfrak{S}_n] \cdot v_{X_i^r}^{T(\lambda)},$$

where each tableau X_i^r is formed by taking an SSYT of shape λ and content $\lambda(F_{1i}) = 21^{n-2}$, then relabeling it so the second 1 (which is necessarily the second entry of the top row) becomes an i and the entries greater than $i-1$ each get increased by 1, then reordering the first row so the entries are in increasing order. Every Y_k is equal to X_i^r for some choice of i and r (e.g. by taking i to be the second entry in the top row of Y_k), so $d_1|_{\mathcal{S}_\lambda}$ is surjective. Thus, by rank-nullity, the multiplicity of $\mathcal{S}_\lambda$ in $\ker(d_1)$ is equal to its multiplicity in $C_1(G)$ minus its multiplicity in $C_0(G)$. The multiplicity in $C_0(G)$ is simply f^λ . We claim that the multiplicity in $C_1(G)$ is

$$(\ell-1) \cdot \binom{n-1}{\ell-1} f^{2^k 1^{n-\ell-k}}.$$

To see this, note that to form an X_i^r , we can choose any $\ell-1$ numbers besides 1 to fill the first row (in $\binom{n-1}{\ell-1}$ ways), choose any one of them to be i (in $\ell-1$ ways), and then fill the remaining rows by taking any SSYT of shape $2^k 1^{n-\ell-k}$ (in $f^{2^k 1^{n-\ell-k}}$ ways) and relabeling it so the 1 becomes the smallest remaining entry, the 2 becomes the second smallest, and so on.

For $C_2(G)|_{\mathcal{S}_\lambda}$, we consider the $\binom{n-1}{2}$ spanning subgraphs of G with two edges, which are each of the form $F_{1i,1j}$ for i and j . Every one of these spanning subgraphs has partition type $\lambda(F_{1i,1j}) = 31^{n-3}$, so

$$C_2(G)|_{\mathcal{S}_\lambda} = \bigoplus_{2 \leq i < j \leq n} \mathcal{M}_{F_{1i,1j}}|_{\mathcal{S}_\lambda} = \bigoplus_{2 \leq i < j \leq n} \bigoplus_{k=1}^{K_{\lambda, 31^{n-3}}} \mathbb{C}[\mathfrak{S}_n] \cdot v_{W_{ij}^k}^{T(\lambda)},$$

where each W_{ij}^k is formed from an SSYT of shape λ and content 31^{n-3} by replacing two of the 1's with i and j , adding 1 to each entry between i and $j-2$, adding 2 to each entry between $j-1$ and $n-2$, and reordering the first row so its entries are increasing. Every W_{ij}^k formed in this manner is also of the form X_i^r and an X_j^s for some r and s . Thus, the map $d_2|_{\mathcal{S}_\lambda}$ directly embeds each $\mathbb{C}[\mathfrak{S}_n] \cdot v_{W_{ij}^k}^{T(\lambda)} \subseteq \mathcal{M}_{1i,1j}$ into

$$\mathbb{C}[\mathfrak{S}_n] \cdot v_{X_i^r}^{T(\lambda)} \oplus \mathbb{C}[\mathfrak{S}_n] \cdot v_{X_j^s}^{T(\lambda)} \subseteq \mathcal{M}_{1i} \oplus \mathcal{M}_{1j},$$

via a map that looks like

$$d_2|_{\mathcal{S}_\lambda}(\tau) = (-\tau) \oplus \tau$$

for each $\tau \in \mathbb{C}[\mathfrak{S}_n] \cdot v_{W_{ij}^k}^{T(\lambda)}$. We write the first component with a negative sign because we can assume without loss of generality that the edge $1i$ comes before the edge $1j$ in the edge ordering, and hence the map $\mathcal{M}_{1i,1j} \rightarrow \mathcal{M}_{1i}$ has negative sign (since there is one edge in F , namely $1j$, that comes after $1i$ in the ordering) while the map $\mathcal{M}_{1i,1j} \rightarrow \mathcal{M}_{1j}$ has positive sign (since there are no edges in F coming after $1j$ in the ordering).

Each tableau that shows up as an X_i^r for some i actually shows up as an X_i^r for $\ell-1$ different values of i , since we can take i to be any entry in the top row except the 1. It also shows up as W_{ij}^k for $\binom{\ell-1}{2}$ pairs (i, j) , since we can take i and j to be any two entries in the top row except the two 1's. Furthermore, a

tableau of shape λ is one of the X_i^T 's if and only if it is one of the W_{ij}^k 's, since the only requirement in both cases is that the first row be in increasing order and the portion below the first row be standard.

To find the multiplicity of $\mathcal{S}_\lambda$ in d_2 , consider the restriction of d_2 to submodules of the form $\mathbb{C}[\mathfrak{S}_n] \cdot v_S^{T(\lambda)}$ for a particular tableau S . There are $\ell - 1$ such modules in $C_1(G)$, one for each entry in the top row of S besides the 1, each isomorphic to a copy of $\mathcal{S}_\lambda$. From the description of d_2 above, an element $\sigma_1 \oplus \cdots \oplus \sigma_{\ell-1}$ in the direct sum of these $\ell - 1$ submodules is in $\text{im}(d_2)$ if and only if $\sigma_1 + \cdots + \sigma_{\ell-1} = 0$. Thus, $\sigma_1, \dots, \sigma_{\ell-1}$ can be chosen independently, and $\sigma_{\ell-1}$ is then determined. Thus, for every $\ell - 1$ copies of $\mathcal{S}_\lambda$ in $C_1(G)$, there are $\ell - 2$ copies in $\text{im}(d_2)$, so the multiplicity of $\mathcal{S}_\lambda$ in $\text{im}(d_2)$ is $\frac{\ell-1}{\ell-2}$ times its multiplicity in $C_1(G)$, or

$$(\ell - 2) \cdot \binom{n-1}{\ell-1} f^{2^k 1^{n-\ell-2k}}.$$

Then since $H_{1,0}(G) := \ker(d_1)/\text{im}(d_2)$, the multiplicity of $\mathcal{S}_\lambda$ in $H_{1,0}(G)$ is equal to its multiplicity in $\ker(d_1)$ minus its multiplicity in $\text{im}(d_2)$, or

$$(\ell - 1) \cdot \binom{n-1}{\ell-1} f^{2^k 1^{n-\ell-2k}} - f^\lambda - (\ell - 2) \cdot \binom{n-1}{\ell-1} f^{2^k 1^{n-\ell-2k}} = \binom{n-1}{\ell-1} f^{2^k 1^{n-\ell-2k}} - f^\lambda,$$

as claimed. $\square$

6.5 Proof of Corollary 6.4

Let $\lambda = \ell 21^{n-\ell-2}$. By Theorem 6.3, the multiplicity of $\mathcal{S}_{\ell 21^{n-\ell-2}}$ in $H_{1,0}(G)$ is

$$\binom{n-1}{\ell-1} f^{21^{n-\ell-2}} - f^\lambda.$$

We note that

$$f^{21^{n-\ell-2}} = n - \ell - 1,$$

since to get an SYT of shape $21^{n-\ell-2}$, we can choose any of the $n - \ell - 1$ numbers from $2, \dots, n - \ell$ to be the second entry in the first row, and then the rest is determined since the entries in the first column must be in increasing order. To compute f^λ , we draw out the Young diagram of shape λ and label each box with its corresponding hook length:

$n-1$	ℓ	$\ell-2$	$\dots$	2	1
$n-\ell$	1				
$n-\ell-2$					
$\vdots$					
2					
1					

Thus, the hook length formula gives

$$f^\lambda = \frac{n!}{(n-1)(n-\ell)\ell \cdot (n-\ell-2)! \cdot (\ell-2)!} = \frac{n(\ell-1)}{n-\ell} \binom{n-2}{\ell}.$$

Putting this together, our multiplicity is

$$(n - \ell - 1) \binom{n-1}{\ell-1} - \frac{n(\ell-1)}{n-\ell} \binom{n-2}{\ell} = \frac{(n-1)\ell - n(\ell-1)}{n-\ell} \binom{n-2}{\ell} = \binom{n-2}{\ell},$$

as claimed. Conjectures 6.1 and 6.2 then follow immediately as the special cases $\ell = n - 2$ and $\ell = 2$.

6.6 Proof of Corollary 6.5

Let $\lambda = 2^k 1^{n-2k}$. Since $\ell = 2$, Theorem 6.3 implies that the multiplicity of $\mathcal{S}_\lambda$ in $H_{1,0}(G)$ is

$$(n-1)K_{\lambda, 21^{n-2}} - f^\lambda.$$

By the hook-length formula, we get

$$f^\lambda = \frac{n!}{(n-k+1) \dots (n-2k+2) \cdot k! \cdot (n-2k)!} = \frac{n-2k+1}{k} \binom{n}{k-1},$$

since we can label each box with its hook length as shown below:

$n-k+1$	k
$n-k$	$k-1$
$\vdots$	$\vdots$
$n-2k+2$	1
$n-2k$	
$\vdots$	
1	

For $K_{\lambda, 21^{n-2}}$, note that to get an SSYT of shape λ and content 21^{n-2} , the two 1's must be in the top row, and then the number of ways to fill the remaining part is equal to the number of SYT of shape $2^{k-2} 1^{n-2k}$. Using the formula above for $f^{2^{k-2} 1^{n-2k}}$ but replacing n with $n-2$ and k with $k-1$ gives

$$f^{2^{k-2} 1^{n-2k}} = \frac{(n-2) - 2(k-1) + 1}{k-1} \binom{n-2}{k-2} = \frac{n-2k+1}{n-1} \binom{n-1}{k-1}.$$

Putting this together, the multiplicity of $\mathcal{S}^\lambda$ in $H_{1,0}(G)$ is

$$(n-1) \cdot \frac{n-2k+1}{n-1} \binom{n-1}{k-1} - \frac{n-2k+1}{k} \binom{n}{k-1} = (n-2k+1) \left(\binom{n-1}{k-1} - \frac{1}{k} \binom{n}{k-1} \right),$$

as claimed. □

6.7 Proof of the conjectured values from Table 6.1

We reproduce Table 6.1 here for convenience:

G	$H_{1,0}(G; \mathbb{C})$
	$\mathcal{S}_{2^2}$
	$\mathcal{S}_{2^2 1}^{\oplus 3} \oplus \mathcal{S}_{32}$
	$\mathcal{S}_{2^2 1^2}^{\oplus 6} \oplus \mathcal{S}_{2^3}^{\oplus 5} \oplus \mathcal{S}_{321}^{\oplus 4} \oplus \mathcal{S}_{42}$
	$\mathcal{S}_{2^2 1^3}^{\oplus 10} \oplus \mathcal{S}_{2^3 1}^{\oplus 16} \oplus \mathcal{S}_{321^2}^{\oplus 10} \oplus \mathcal{S}_{32^2}^{\oplus 9} \oplus \mathcal{S}_{421}^{\oplus 5} \oplus \mathcal{S}_{52}$

Corollary 6.4 immediately implies the following multiplicities from Table 6.1:

Specht module	Multiplicity
$\mathcal{S}_{2^2}$ in $H_{1,0} \left(\begin{smallmatrix} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{smallmatrix} \right)$	$\binom{4-2}{2} = \binom{2}{2} = 1$
$\mathcal{S}_{2^2 1}$ in $H_{1,0} \left(\begin{smallmatrix} \bullet \\ \vdots \\ \bullet \end{smallmatrix} \right)$	$\binom{5-2}{2} = \binom{3}{2} = 3$
$\mathcal{S}_{32}$ in $H_{1,0} \left(\begin{smallmatrix} \bullet \\ \vdots \\ \bullet \end{smallmatrix} \right)$	$\binom{5-2}{3} = \binom{3}{3} = 1$
$\mathcal{S}_{2^2 1^2}$ in $H_{1,0} \left(\begin{smallmatrix} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{smallmatrix} \right)$	$\binom{6-2}{2} = \binom{4}{2} = 6$
$\mathcal{S}_{321}$ in $H_{1,0} \left(\begin{smallmatrix} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{smallmatrix} \right)$	$\binom{6-2}{3} = \binom{4}{3} = 4$
$\mathcal{S}_{42}$ in $H_{1,0} \left(\begin{smallmatrix} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{smallmatrix} \right)$	$\binom{6-2}{4} = \binom{4}{4} = 1$
$\mathcal{S}_{2^2 1^3}$ in $H_{1,0} \left(\begin{smallmatrix} \bullet \\ \vdots \\ \bullet \end{smallmatrix} \right)$	$\binom{7-2}{2} = \binom{5}{2} = 10$
$\mathcal{S}_{321^2}$ in $H_{1,0} \left(\begin{smallmatrix} \bullet \\ \vdots \\ \bullet \end{smallmatrix} \right)$	$\binom{7-2}{3} = \binom{5}{3} = 10$
$\mathcal{S}_{421}$ in $H_{1,0} \left(\begin{smallmatrix} \bullet \\ \vdots \\ \bullet \end{smallmatrix} \right)$	$\binom{7-2}{4} = \binom{5}{4} = 5$
$\mathcal{S}_{52}$ in $H_{1,0} \left(\begin{smallmatrix} \bullet \\ \vdots \\ \bullet \end{smallmatrix} \right)$	$\binom{7-2}{5} = \binom{5}{5} = 1$

Corollary 6.5 then implies two of the others:

Specht module	Multiplicity
$\mathcal{S}_{2^3}$ in $H_{1,0} \left(\begin{smallmatrix} \bullet \\ \vdots \\ \bullet \end{smallmatrix} \right)$	$(6 - 2 \cdot 3 + 1) \left(\binom{6-1}{3-1} - \frac{1}{3} \binom{6}{3-1} \right) = \binom{5}{2} - \frac{1}{3} \binom{6}{2} = 5$
$\mathcal{S}_{2^3 1}$ in $H_{1,0} \left(\begin{smallmatrix} \bullet \\ \vdots \\ \bullet \end{smallmatrix} \right)$	$(7 - 2 \cdot 3 + 1) \left(\binom{7-1}{3-1} - \frac{1}{3} \binom{7}{3-1} \right) = 2 \left(\binom{6}{2} - \frac{1}{3} \binom{7}{2} \right) = 16$

The only other multiplicity to check is $\mathcal{S}_{32^2}^9$ in $H_{1,0} \left(\begin{smallmatrix} \bullet \\ \vdots \\ \bullet \end{smallmatrix} \right)$. For that one, we directly use the formula

$$\binom{n-1}{\ell-1} f^{2^k 1^{n-\ell-2k}} - f^\lambda$$

from Theorem 6.3, which in this case becomes

$$\binom{7-1}{3-1} f^{2^2} - f^{3 \cdot 2^2} = \binom{6}{2} f^{2^2} - f^{3 \cdot 2^2} = 15 f^{2^2} - f^{3 \cdot 2^2}.$$

We have $f^{2^2} = 2$, since the two SYT of shape 2^2 are

$$\begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}.$$

For $f^{3 \cdot 2^2}$, we use the hook length formula. Below, we label each of the 7 boxes with the length of its hook:

$$\begin{array}{|c|c|c|} \hline 5 & 4 & 1 \\ \hline 3 & 2 & \\ \hline 2 & 1 & \\ \hline \end{array}.$$

Thus, the hook length formula gives

$$f^{3 \cdot 2^2} = \frac{7!}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 2 \cdot 1 \cdot 1} = 21.$$

Plugging these values in gives a multiplicity of

$$15 \cdot 2 - 21 = 9,$$

as claimed in the table.

For the stars on 4 through 7 vertices, it follows by dimension counting that the full decompositions of $H_{1,0}(G)$ are as claimed in Table 6.1, because the authors of [Cha+23] computed what the dimension of $H_{1,0}(G)$ is in each case. $\square$

Bibliography

- [ADM23] Jean-Christophe Aval, Karimatou Djenabou, and Peter RW McNamara. “Quasisymmetric functions distinguishing trees”. In: *Algebraic Combinatorics* 6.3 (2023), pp. 595–614. URL: <https://alco.centre-mersenne.org/articles/10.5802/alco.273/>.
- [AHK77] Kenneth Appel, Wolfgang Haken, and John Koch. “Every planar map is four colorable”. In: *Illinois Journal of Mathematics* 21 (1977), pp. 439–657.
- [Ali+21] José Aliste-Prieto, Logan Crew, Sophie Spirkl, and José Zamora. “A vertex-weighted Tutte symmetric function, and constructing graphs with equal chromatic symmetric function”. In: *The Electronic Journal of Combinatorics* 28 (2 2021). URL: <https://www.combinatorics.org/ojs/index.php/eljc/article/view/v28i2p1/pdf>.
- [Ali+23] José Aliste-Prieto, Anna De Mier, Rosa Orellana, and José Zamora. “Marked graphs and the chromatic symmetric function”. In: *SIAM Journal on Discrete Mathematics* 37.3 (2023), pp. 1881–1919. URL: <https://epubs.siam.org/doi/10.1137/22M148046X>.
- [App84] Kenneth I Appel. “The use of the computer in the proof of the four color theorem”. In: *Proceedings of the American Philosophical Society* 128.1 (1984), pp. 35–39. URL: <https://www.jstor.org/stable/986491>.
- [AS17] Sami Assaf and Dominic Searles. “Schubert polynomials, slide polynomials, Stanley symmetric functions and quasi-Yamanouchi pipe dreams”. In: *Advances in Mathematics* 306 (2017), pp. 89–122. URL: <https://doi.org/10.1016/j.aim.2016.10.015>.
- [AS18] Sami Assaf and Dominic Searles. “Kohnert tableaux and a lifting of quasi-Schur functions”. In: *Journal of Combinatorial Theory, Series A* 156 (2018), pp. 85–118. URL: <https://doi.org/10.1016/j.jcta.2018.01.001>.
- [AZ14] José Aliste-Prieto and José Zamora. “Proper caterpillars are distinguished by their chromatic symmetric function”. In: *Discrete Mathematics* 315 (2014), pp. 158–164. URL: <https://www.sciencedirect.com/science/article/pii/S0012365X13004433>.
- [Bab09] Courtney Leigh Baber. “An introduction to list colorings of graphs”. PhD thesis. Virginia Tech, 2009. URL: <https://vtechworks.lib.vt.edu/items/4d192f82-e162-4d14-bb6c-f9cd338da07c>.
- [Bir13] George D Birkhoff. “The reducibility of maps”. In: *American Journal of Mathematics* 35.2 (1913), pp. 115–128. URL: <https://doi.org/10.2307/2370276>.
- [BN20] Olivier Bernardi and Philippe Nadeau. “Combinatorial reciprocity for the chromatic polynomial and the chromatic symmetric function”. In: *Discrete mathematics* 343.10 (2020), p. 111989. URL: <https://doi.org/10.1016/j.disc.2020.111989>.
- [Bru+21] Ben Brubaker, Valentin Buciumas, Daniel Bump, and Henrik PA Gustafsson. “Colored five-vertex models and Demazure atoms”. In: *Journal of Combinatorial Theory, Series A* 178 (2021), p. 105354. URL: <https://doi.org/10.1016/j.jcta.2020.105354>.
- [BSW20] Valentin Buciumas, Travis Scrimshaw, and Katherine Weber. “Colored five-vertex models and Lascoux polynomials and atoms”. In: *J. Lond. Math. Soc. (2)* 102.3 (2020), pp. 1047–1066. ISSN: 0024-6107. URL: <https://doi.org/10.1112/jlms.12347>.

- [Buc05] Anders Skovsted Buch. “Combinatorial K-theory”. In: *Topics in Cohomological Studies of Algebraic Varieties: Impanga Lecture Notes* (2005), pp. 87–103. URL: https://link.springer.com/chapter/10.1007/3-7643-7342-3_3.
- [Cao+19] Yan Cao, Guantao Chen, Guangming Jing, Michael Stiebitz, and Bjarne Toft. “Graph edge coloring: a survey”. In: *Graphs and Combinatorics* 35 (2019), pp. 33–66. URL: <https://link.springer.com/article/10.1007/s00373-018-1986-5>.
- [Cha+23] Alex Chandler, Radmila Sazdanovic, Salvatore Stella, and Martha Yip. “On the strength of chromatic symmetric homology for graphs”. In: *Advances in Applied Mathematics* 150 (2023), p. 102559. URL: <https://doi.org/10.1016/j.aam.2023.102559>.
- [Cil24] Azzurra Ciliberti. “A deletion–contraction long exact sequence for chromatic symmetric homology”. In: *European Journal of Combinatorics* 115 (2024), p. 103788. URL: <https://doi.org/10.1016/j.ejc.2023.103788>.
- [CM23] Azzurra Ciliberti and Luca Moci. “On chromatic symmetric homology and planarity of graphs”. 2023. URL: <https://doi.org/10.37236/11397>.
- [CPS23] Logan Crew, Oliver Pechenik, and Sophie Spirkl. “The Kromatic Symmetric Function: A K-theoretic Analogue of X_G ”. 2023. URL: <https://arxiv.org/abs/2301.02177>.
- [Cra69] Henry H Crapo. “The tutte polynomial”. In: *aequationes mathematicae* 3 (1969), pp. 211–229. URL: <https://link.springer.com/article/10.1007/BF01817442>.
- [Cre22] Logan Crew. “A note on distinguishing trees with the chromatic symmetric function”. In: *Discrete Mathematics* 345.2 (2022), p. 112682. URL: <https://www.sciencedirect.com/science/article/abs/pii/S0012365X21003952>.
- [CS] Logan Crew and Sophie Spirkl. “Pairs of small graphs with equal chromatic symmetric function”. URL: https://drive.google.com/file/d/1zrA1hlmer_NEco95CiznCe9QjhfxhroY/view.
- [CS05] Maria Chudnovsky and Paul D Seymour. “The structure of claw-free graphs.” In: *BCC*. 2005, pp. 153–171. URL: https://web.math.princeton.edu/~mchudnov/claws_survey.pdf.
- [De 18] Aubrey DNJ De Grey. “The chromatic number of the plane is at least 5”. 2018. URL: <https://doi.org/10.48550/arXiv.1804.02385>.
- [Dem74] Michel Demazure. “Une nouvelle formule des caracteres”. In: *Bull. Sci. Math* 2.98 (1974), pp. 163–172.
- [DL80] Michael Detlefsen and Mark Luker. “The four-color theorem and mathematical proof”. In: *The Journal of Philosophy* 77.12 (1980), pp. 803–820. URL: <https://doi.org/10.2307/2025806>.
- [DW18] Samantha Dahlberg and Stephanie van Willigenburg. “Lollipop and lariat symmetric functions”. In: *SIAM Journal on Discrete Mathematics* 32.2 (2018), pp. 1029–1039. URL: <https://doi.org/10.1137/17M1144805>.
- [FK94] Sergey Fomin and Anatol N Kirillov. “Grothendieck polynomials and the Yang-Baxter equation”. In: *Proceedings on formal power series and algebraic comb.* 1994, pp. 183–190. URL: https://www.researchgate.net/publication/248559296_Grothendieck_polynomials_and_the_Yang_-_Baxter_equation.
- [Fol+21] Angele M Foley, Joshua Kazdan, Larissa Kröll, Sofía Martínez Alberga, Oleksii Melnyk, and Alexander Tenenbaum. “Transplanting Trees: Chromatic Symmetric Function Results through the Group Algebra of S_n ”. 2021. URL: <https://doi.org/10.48550/arXiv.2112.09937>.
- [Gas96] Vesselin Gasharov. “Incomparability graphs of $(3+1)$ -free posets are s-positive”. In: *Discrete Mathematics* 157.1-3 (1996), pp. 193–197. URL: [https://doi.org/10.1016/S0012-365X\(96\)83014-7](https://doi.org/10.1016/S0012-365X(96)83014-7).
- [Ger17] Melanie Gerling. “On distinguishing special trees by their chromatic symmetric functions”. 2017. URL: https://www.researchgate.net/publication/313879095_On_distinguishing_special_trees_by_their_chromatic_symmetric_functions.

- [GS01] David D Gebhard and Bruce E Sagan. “A chromatic symmetric function in noncommuting variables”. In: *Journal of Algebraic Combinatorics* 13.3 (2001), pp. 227–255. URL: <https://doi.org/10.1023/A:1011258714032>.
- [Had43] Hugo Hadwiger. “Über eine klassifikation der streckenkomplexe”. In: *Vierteljschr. Naturforsch. Ges. Zürich* 88.2 (1943), pp. 133–142. URL: https://www.ngzh.ch/archiv/1943_88/88_2/88_17.pdf.
- [Had45] Hugo Hadwiger. “Überdeckung des euklidischen Raumes durch kongruente Mengen”. In: *Portugaliae mathematica* 4 (1945), pp. 238–242. URL: <http://dml.mathdoc.fr/item/2142/>.
- [HHT19] Angele M Hamel, Chinh T Hoàng, and Jake E Tuero. “Chromatic symmetric functions and H-free graphs”. In: *Graphs and Combinatorics* 35 (2019), pp. 815–825. URL: <https://link.springer.com/article/10.1007/s00373-019-02034-1>.
- [HJ19] Sam Heil and Caleb Ji. “On an algorithm for comparing the chromatic symmetric functions of trees”. In: *Australian Journal of Combinatorics* 72 (5 2019), pp. 210–222. URL: https://www.researchgate.net/publication/322755095_On_an_Algorithm_for_Comparing_the_Chromatic_Symmetric_Functions_of_Trees.
- [HK04] Magnús M Halldórsson and Guy Kortsarz. “Multicoloring: Problems and techniques”. In: *International Symposium on Mathematical Foundations of Computer Science*. Springer, 2004, pp. 25–41. URL: https://link.springer.com/chapter/10.1007/978-3-540-28629-5_2.
- [HN24] Angela Hicks and Elizabeth Niese. “Quasisymmetric divided difference operators and polynomial bases”. 2024. URL: <https://arxiv.org/abs/2406.02420>.
- [HW17] James Haglund and Andrew Timothy Wilson. “Macdonald polynomials and chromatic quasisymmetric functions”. 2017. URL: <https://doi.org/10.48550/arXiv.1701.05622>.
- [Kho00] Mikhail Khovanov. “A categorification of the Jones polynomial”. In: *Duke Mathematics Journal* 101 (3 2000). URL: <https://doi.org/10.1215/S0012-7094-00-10131-7>.
- [Kwu+16] Young Chel Kwun, Abdul Rauf Nizami, Mobeen Munir, Waqas Nazeer, and Shin Min Kang. “The Tutte and the Jones polynomials”. In: *Global Journal of Pure and Applied Mathematics* 12.6 (2016), pp. 4717–4740. URL: https://www.researchgate.net/publication/311202041_The_Tutte_and_the_Jones_Polynomials.
- [LS82] Alain Lascoux and Marcel-Paul Schützenberger. “Structure de Hopf de l’anneau de cohomologie et de l’anneau de Grothendieck d’une variété de drapeaux”. In: *CR Acad. Sci. Paris Sér. I Math* 295.11 (1982), pp. 629–633.
- [LS88] A. Lascoux and M.-P. Schützenberger. “Invariant theory and tableaux”. In: Minneapolis, MN, 1988. Chap. Keys and standard bases.
- [LW24] Nicholas A Loehr and Gregory S Warrington. “A rooted variant of Stanley’s chromatic symmetric function”. In: *Discrete Mathematics* 347.3 (2024), p. 113805. URL: <https://www.sciencedirect.com/science/article/abs/pii/S0012365X23004910>.
- [Lyd16] Mark Thomas Lydon. *On the chromatic symmetric function of graphs*. University of California, Davis, 2016. URL: <https://www.proquest.com/openview/a5119f3c1fcc69f6115ba0373b65c712/1?pq-origsite=gscholar&cbl=18750&diss=y>.
- [LZZ04] Sergei K Lando, Alexander K Zvonkin, and Don Bernard Zagier. *Graphs on surfaces and their applications*. Vol. 141. Springer, 2004. URL: <https://link.springer.com/book/10.1007/978-3-540-38361-1>.
- [Mar23] Eric Marberg. “Kromatic quasisymmetric functions”. 2023. URL: <https://doi.org/10.48550/arXiv.2312.16474>.
- [MMW08] Jeremy L Martin, Matthew Morin, and Jennifer D Wagner. “On distinguishing trees by their chromatic symmetric functions”. In: *Journal of Combinatorial Theory, Series A* 115.2 (2008), pp. 237–253. URL: <https://www.sciencedirect.com/science/article/pii/S0097316507000775>.
- [Mon16] Cara Monical. “Set-valued skyline fillings”. 2016. URL: <https://doi.org/10.48550/arXiv.1611.08777>.

- [Mor05] Matthew Morin. “Caterpillars, ribbons, and the chromatic symmetric function”. PhD thesis. University of British Columbia, 2005. URL: <https://open.library.ubc.ca/soa/cIRcle/collections/ubctheses/831/items/1.0079438>.
- [MPS21] Cara Monical, Oliver Pechenik, and Dominic Searles. “Polynomials from Combinatorial K -theory”. In: *Canadian Journal of Mathematics* 73.1 (2021), pp. 29–62. URL: <https://doi.org/10.4153/S0008414X19000464>.
- [OEI24] OEIS. *The Online Encyclopedia of Integer Sequences Entry A038067*. 2024. URL: <https://oeis.org/A038067>.
- [OS14] Rosa Orellana and Geoffrey Scott. “Graphs with equal chromatic symmetric functions”. In: *Discrete Mathematics* 320 (2014), pp. 1–14. URL: <https://www.sciencedirect.com/science/article/pii/S0012365X13004974>.
- [Pie24a] Laura Pierson. “A power sum expansion for the Kromatic symmetric function”. 2024. URL: <https://arxiv.org/abs/2408.01395>.
- [Pie24b] Laura Pierson. “Counting induced subgraphs with the Kromatic symmetric function”. 2024. URL: <https://doi.org/10.48550/arXiv.2403.15929>.
- [Pie24c] Laura Pierson. “On the chromatic symmetric homology for star graphs”. 2024. URL: <https://arxiv.org/abs/2408.01396>.
- [Pie24d] Laura Pierson. “Proof of a K -theoretic polynomial conjecture of Monical, Pechenik, and Searles”. 2024. URL: <https://arxiv.org/abs/2408.01390>.
- [PS19] Oliver Pechenik and Dominic Searles. “Decompositions of Grothendieck polynomials”. In: *International Mathematics Research Notices* 2019.10 (2019), pp. 3214–3241. URL: <https://doi.org/10.1093/imrn/rnx207>.
- [Rad11] Stanislaw Radziszowski. “Small ramsey numbers”. In: *The electronic journal of combinatorics* 1000 (2011), DS1–Aug. URL: <https://www.emis.de/journals/EJC/ojs/index.php/eljc/article/view/ds1>.
- [Rid23] H.N. de Ridder et al. “List of small graphs”. In: *Information System on Graph Classes and their Inclusions (ISGCI)* (updated 24 December 2023). URL: <https://www.graphclasses.org/smallgraphs.html>.
- [Rob+96] Neil Robertson, Daniel Sanders, Paul Seymour, and Robin Thomas. “A new proof of the four-colour theorem”. In: *Electronic research announcements of the American Mathematical Society* 2.1 (1996), pp. 17–25. URL: <https://doi.org/10.1090/S1079-6762-96-00003-0>.
- [Sea20] Dominic Searles. “Polynomial bases: positivity and Schur multiplication”. In: *Transactions of the American Mathematical Society* 373.2 (2020), pp. 819–847. URL: <https://www.ams.org/journals/tran/2020-373-02/S0002-9947-2019-07670-5/>.
- [Soi16] Alexander Soifer. “The Hadwiger–Nelson problem”. In: *Open problems in mathematics* (2016), pp. 439–457. URL: https://link.springer.com/chapter/10.1007/978-3-319-32162-2_14.
- [SS93] Richard P Stanley and John R Stembridge. “On immanants of Jacobi-Trudi matrices and permutations with restricted position”. In: *Journal of Combinatorial Theory, Series A* 62.2 (1993), pp. 261–279. URL: [https://doi.org/10.1016/0097-3165\(93\)90048-D](https://doi.org/10.1016/0097-3165(93)90048-D).
- [SST15] Isaac Smith, Zane Smith, and Peter Tian. “Symmetric chromatic polynomial of trees”. 2015. URL: <https://arxiv.org/abs/1505.01889>.
- [Sta95] Richard P Stanley. “A symmetric function generalization of the chromatic polynomial of a graph”. In: *Advances in Mathematics* 111.1 (1995), pp. 166–194. URL: <https://www.sciencedirect.com/science/article/pii/S0001870885710201>.
- [Sta98] Richard P Stanley. “Graph colorings and related symmetric functions: ideas and applications a description of results, interesting applications, & notable open problems”. In: *Discrete Mathematics* 193.1-3 (1998), pp. 267–286. URL: [https://doi.org/10.1016/S0012-365X\(98\)00146-0](https://doi.org/10.1016/S0012-365X(98)00146-0).

- [Ste+24] W. A. Stein et al. *Sage Mathematics Software (Version 10.3)*. The Sage Development Team. 2024. URL: <http://www.sagemath.org>.
- [SW16] John Shareshian and Michelle L Wachs. “Chromatic quasisymmetric functions”. In: *Advances in Mathematics* 295 (2016), pp. 497–551. URL: <https://doi.org/10.1016/j.aim.2015.12.018>.
- [SY18] Radmila Sazdanovic and Martha Yip. “A categorification of the chromatic symmetric function”. In: *Journal of Combinatorial Theory, Series A* 154 (2018), pp. 218–246. URL: <https://doi.org/10.1016/j.jcta.2017.08.01>.
- [Tsu18] Shuhei Tsujie. “The chromatic symmetric functions of trivially perfect graphs and cographs”. In: *Graphs and Combinatorics* 34 (2018), pp. 1037–1048. URL: <https://link.springer.com/article/10.1007/s00373-018-1928-2>.
- [Tut54] William Thomas Tutte. “A contribution to the theory of chromatic polynomials”. In: *Canadian journal of mathematics* 6 (1954), pp. 80–91. URL: <https://doi.org/10.4153/CJM-1954-010-9>.
- [Tym79] Thomas Tymoczko. “The four-color problem and its philosophical significance”. In: *The journal of philosophy* 76.2 (1979), pp. 57–83. URL: <https://doi.org/10.2307/2025976>.
- [TZP18] Simon Thevenin, Nicolas Zufferey, and Jean-Yves Potvin. “Graph multi-coloring for a job scheduling application”. In: *Discrete Applied Mathematics* 234 (2018), pp. 218–235. URL: <https://doi.org/10.1016/j.dam.2016.05.023>.
- [Whi31] Hassler Whitney. “The coloring of graphs”. In: *Proceedings of the National Academy of Sciences* 17.2 (1931), pp. 122–125. URL: <https://doi.org/10.2307/1968214>.
- [WM00] Dominic JA Welsh and Criel Merino. “The Potts model and the Tutte polynomial”. In: *Journal of Mathematical Physics* 41.3 (2000), pp. 1127–1152. URL: <https://doi.org/10.1063/1.533181>.