Acyclic List Colouring Locally Planar Graphs

by

Massimo Vicenzo

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2025

Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

The majority of what is presented is, and expands on, work done with Evelyne Smith-Roberge, under the supervision of Luke Postle [52]. Most sections have shared contributions, however the following are parts of which I am the sole author: Section 1.1, Section 1.2, Section 3.2, Section 4.1 and the whole of Chapter 5.

Abstract

A (vertex) colouring of a graph is *acyclic* if it contains no bicoloured cycle. In 1979, Borodin proved that planar graphs are acyclically 5-colourable. In 2010, Kawarabayashi and Mohar proved that locally planar graphs are acyclically 7-colourable. In 2002, Borodin, Fon-Der-Flaass, Kostochka, Raspaud, and Sopena proved that planar graphs are acyclically 7-list-colourable. We prove that locally planar graphs are acyclically 9-list-colourable—no bound for acyclic list colouring locally planar graphs for any fixed number of colours was previously known. We further show that triangle-free locally planar graphs are acyclically 8-list-colourable.

Acknowledgements

First and foremost I would like to thank my supervisor Luke Postle for his support and guidance, without his help, I don't think I could have gotten to this point, which is exactly where I want to be.

Thank you to my readers, Peter Nelson and Sophie Spirkl, both for reading this thesis, but also for allowing me to participate in other interesting projects outside of my comfort zone.

Of course thank you to all the friends I have made here at the University of Waterloo, I have changed so much since I have arrived here, mostly for the better, and it would not have been possible without any of you.

And lastly I would like to thank all of my friends and family back home. I am here because of you, and I want to thank you for always being there and supporting me, even though most of you don't understand what it is I actually do. I expect reading this thesis won't really help with that, but nonetheless, I appreciate you all.

Table of Contents

List of Figures

List of Tables

Chapter 1

Introduction

Given a graph G, a vertex colouring, or more simply, a colouring of G is an assignment of a set of colours to the vertices of G so that no two adjacent vertices are assigned the same colour. Formally, a *k-colouring* of a graph G is a function $\varphi : V(G) \to [k]$ such that for each edge $uv \in E(G)$, we require that $\varphi(u) \neq \varphi(v)$. A graph is *$k$-colourable* if it has a k-colouring. The *chromatic number* of G is denoted $\chi(G)$, and defined as the smallest number k such that G is k-colourable.

Given a class of graphs, the problem of determining the chromatic number of the class (i.e. the maximum number k such that there exists a graph in the class with chromatic number k) is perhaps the central question in the field of graph colouring. The Four Colour Theorem, due to Appel and Haken [5, 6], is one of the most celebrated results in the field, and does exactly this for the class of planar graphs, showing planar graphs are 4-colourable. Recall that a graph is *planar* if it can be embedded in the sphere in such a way that the edges meet only at common endpoints.

List colouring is a generalization of vertex colouring wherein the possible images of the colouring function are local to each vertex. More formally: given a graph G, a *k-list assignment* for G is a function L that assigns to each vertex $v \in V(G)$ a set $L(v)$ of size at least k. Given a list assignment L, an *L-colouring* is a colouring φ with the additional property that for each vertex v, we have that $\varphi(v) \in L(v)$. We say a graph is *k-list-colourable* if it admits an L-colouring for every k-list assignment L. Occasionally when comparing list colouring with the non-list colouring variant, we describe the latter as *ordinary colouring*.

In [65], Voigt showed that the Four Colour Theorem does not hold in the realm of list colouring by constructing a planar graph G and corresponding 4-list assignment L such that G is not L-colourable. Thomassen [61] proved via a remarkably short and elegant argument that lists of size four are *just* insufficient: every planar graph is 5-list-colourable.

This thesis investigates the related question of colouring graphs embedded in surfaces other than the sphere. For the more basic surface definitions not covered here (e.g. surface, handle, crosscap), we refer the reader to [47]. First recall the Classification of Surfaces Theorem (for one of many proofs see [59]), which states that every surface can be obtained from the sphere by either adding a set of handles, or by adding a nonempty set of crosscaps. If a surface Σ is obtained from the sphere by adding k handles, its *Euler genus* is defined as $g(\Sigma) := 2k$. If Σ is obtained by adding k crosscaps, its Euler genus is $g(\Sigma) := k$. For the remainder of the thesis, we refer to the Euler genus of a surface as simply its *genus*. A *non-contractible cycle* in a surface is a cycle that cannot be continuously deformed to a single point. We say a graph embedded in a surface is *ρ-locally planar* if every cycle in the graph that is non-contractible has length at least ρ. This is closely related to the concept of *edge-width*, which is the length of a shortest non-contractible cycle of an embedded graph: a ρ-locally planar has edge-width at least ρ. We use $\mathrm{ew}(G)$ to denote the edge-width of an embedded graph G. The "locally planar" nomenclature is of course very natural: given a ρ-locally planar graph G in a surface Σ, by taking any vertex v of G and the vertices X within distance at most $(\frac{\rho}{2} - 1)$ of v, the inherited embedding of the graph $G[X \cup \{v\}]$ is planar.

Locally planar embedded graphs often have colouring properties similar in spirit to planar graphs. To contextualize this in comparison to other embedded graphs: Heawood [38] famously proved in 1890 that if G is a graph embedded in a surface Σ other than the sphere, then $\chi(G) \leq \left\lfloor \frac{7+\sqrt{24g(\Sigma)+1}}{2} \right\rfloor$. In 1968, it was shown by Ringel and Youngs [54] that this is best possible unless Σ is the Klein bottle, which as proved by Franklin [34] requires exactly six colours. In comparison, Thomassen [60] showed that locally planar graphs are 5-colourable by showing that if G is embedded in a surface Σ and $\mathrm{ew}(G) \geq 2^{c(g(\Sigma))}$ for some constant c depending on the surface, then $\chi(G) \leq 5$. This is also best possible, as shown by Thomassen [62] (extending a construction of Fisk [33]).

When it comes to list colouring, Devos, Kawarabayashi, and Mohar [24] showed that locally planar graphs are 5-list-colourable: in particular, they showed that for every surface Σ there exists a constant c so that $\rho_0 = 2^{c(g(\Sigma))}$ such that every graph G embedded in Σ with edge-width $\rho \geq \rho_0$ is 5-list-colourable. This generalizes Thomassen's theorem ([60]) regarding the 5-colourability of such graphs; the edge-width bound for list colouring (and hence also for ordinary colouring) was later improved to the asymptotically best possible value of $\rho_0 = c \log(g(\Sigma))$ for some constant c depending on the surface, by Postle and Thomas [53].

Table 1.1 summarizes the best known bounds for the number of colours required for (list) colouring planar and locally planar graphs.

Graph Class \ Chromatic Numbers	Ordinary	List
Planar	$=4$ Four Colour Theorem (1977)	$=5$ Thomassen (1994)
Locally Planar	$=5$ Thomassen (1993)	$=5$ Devos et al. (2006)

Table 1.1: The best known bounds on the chromatic and list chromatic numbers of planar and locally planar graphs.

We now turn our attentions to *acyclic* colouring. First, note that a k-colouring can be thought of as a labelling where, for each colour $i \in [k]$, we impose a restriction on the set of vertices coloured i: in particular, we ask that these vertices form an independent set. We say a colouring φ of G is *acyclic* if for every cycle $C \subseteq G$, we have $|\{\varphi(v) : v \in V(C)\}| \geq 3$. A graph is *acyclically k-colourable* if it has an acyclic k-colouring. An acyclic k-colouring is therefore a k-colouring with, for each pair of distinct colours $i, j \in [k]$, an additional restriction on the set of vertices coloured i or j: in particular, we ask that these vertices induce a forest. The *acyclic chromatic number* of G is denoted $\chi_a(G)$, and defined as the least k such that G is acyclically k-colourable. Clearly, as acyclic k-colouring is a more restrictive notion than k-colouring, $\chi_a(G) \geq \chi(G)$ for every graph G.

Acyclic colouring was first introduced by Grünbaum [37] in 1973, whose original paper on the subject has accrued over 500 citations at the time of writing. While acyclic colouring is interesting in its own right, Grünbaum studies this concept in the setting of planar graphs to generalize the concept of "point-arboricity" of a graph—the decomposition of a graph into vertex-disjoint acyclic subgraphs.

Unlike ordinary colouring, the Four Colour Theorem does not hold for acyclic colouring: this was demonstrated by a construction of Grünbaum [37], who also conjectured in that $\chi_a(G) \leq 5$ if G is planar. In the same paper, Grünbaum showed that if G is planar, $\chi_a(G) \leq 9$. This bound was improved by Mitchem [45] to 8; and later, to 7 by Albertson and Berman [3]. Kostochka [42] showed $\chi_a(G) \leq 6$ for every planar graph G; and in [10], Borodin settled the question of the acyclic chromatic number of the class of planar graphs in 1979, showing every planar graph is acyclically 5-colourable. As mentioned prior, this is best possible.

Acyclic list colouring is defined analogously to its non-list counterpart: an L-colouring is acyclic if the vertices in every cycle together use at least three colours, and a graph is *acyclically-k-list-colourable* if it has an acyclic L-colouring for every k-list assignment L. Recall that all planar graphs are 5-list-colourable. This bound is also conjectured [13] to hold for acyclic list colouring:

Conjecture 1.1 (Borodin et al., [13]). *Every planar graph is acyclically* 5*-list-colourable.*

The best known bound on the number of colours needed is due to Borodin, Fon-Der-Flaass, Kostochka, Raspaud, and Sopena [13], the same group who provided Conjecture 1.1, who showed the following in 2002.

Theorem 1.2 (Borodin et al., [13]). *Every planar graph is acyclically 7-list-colourable.*

While progress for reducing the list sizes required has stagnated for some time, more progress has been made in showing that certain subsets of planar graphs do satisfy Conjecture 1.1. In 2006, planar graphs excluding 3- and 4-cycles were shown by Montassier, Ochem and Raspaud [49] and in the following year, those excluding 4- and 5-cycles or 4- and 6-cycles were shown by Montassier, Raspaud and Wang [50]. An improvement to the last result was made by Zhang and Xu [72], who showed that you need only exclude 4-cycles and chordal 6-cycles. Chen and Wang [21] showed that the Conjecture 1.1 holds for graphs excluding 4-cycles and two 3-cycles which are far apart, meanwhile Chen and Raspaud [20] showed the same for planar graphs excluding 4-cycles and intersecting 3-cycles. All these results are generalized by the what is currently the largest set of graphs to satisfy Conjecture 1.1 by Borodin and Ivanova [17] in 2011: the class of planar graphs which do not contain 4-cycles are acyclically 5-list-colourable.

For acyclic colouring graphs on surfaces, analogous to Heawood's theorem Alon, Mohar, and Sanders [4] proved that if G is a graph embedded in a surface Σ, then $\chi_a(G) \leq 100 \cdot g(\Sigma)^{\frac{4}{7}} + 10000$, and moreover there exists a constant c depending on the surface such that $\chi_a(G) \geq c\left(\frac{g(\Sigma)^{\frac{4}{7}}}{\log g(\Sigma)^{\frac{1}{7}}}\right)$. In 2005, Mohar [46] showed that locally planar graphs are acyclically 8-colourable. Kawarabayashi and Mohar [41] later proved that locally planar graphs are acyclically 7-colourable. (We note that it does not seem like the methods used in [41] can be used to prove an analogous result for list colouring. For instance, in certain parts of the proof, three colours are reserved for use in a specific subgraph H of the graph. In the list colouring framework, if our lists are big enough we can also avoid colouring parts of the graph using a reserved set of colours. However, there is no guarantee that the vertices in H contain the reserved set of colours in their lists, and so no guarantee that the colouring of the graph can be completed using this reserved colour set.)

With respect to acyclic list colouring graphs on surfaces, not much is known. This is the topic of this thesis. Our main result is the following theorem, which to our knowledge is the first that gives a bound on the acyclic list chromatic number of locally planar graphs.

Theorem 1.3. *For every surface Σ, there exists $\rho_0 := \rho_0(g(\Sigma))$ such that every ρ_0-locally planar graph embedded in Σ is acyclically 9-list-colourable.*

The bound in Theorem 1.3 is $\rho_0 = 516(g(\Sigma) - 2)$. Note that this implies for example that all graphs embeddable in at least one of the projective plane, torus, and the Klein bottle (which have genus 1,1, and 2, respectively) are acyclically 9-list-colourable.

Table 1.2 summarizes the best known bounds for the number of colours required for acyclic (list) colouring planar and locally planar graphs, giving context for the main result of this thesis.

Graph Class \ Acyclic Chromatic Numbers	Ordinary	List
Planar	$= 5$ Borodin (1979)	≤ 7 Borodin et al. (2002)
Locally Planar	≤ 7 Kawarabayashi Mohar (2010)	$\leq \mathbf{9}$ **Postle et al. (2024)**

Table 1.2: The best known bounds on the acyclic chromatic and acyclic list chromatic numbers of planar and locally planar graphs. No bound on the acyclic list chromatic number of locally planar graphs was previously known.

As ours is the first bound for acyclic list colouring locally planar graphs, we contrast it with a similar result for acyclic colouring; and since the edge-width bound for acyclic 7-colouring locally planar graphs in the paper of Kawarabayashi and Mohar [41] is not given, we compare the bound in Theorem 1.3 to that obtained by Mohar in [46]: Mohar showed an analogous result for acyclic 8-colouring locally planar graphs with $\rho_0 := O(g^3 2^g)$

(though as mentioned in [46], the author's priority was giving clear, elementary arguments perhaps at the cost of the edge-width bound obtained).

Given these previous results, we suspect that our list size is not optimal.

Conjecture 1.4. *Let Σ be a surface. There exists $k < 9$ and ρ depending on $g(\Sigma)$ such that every ρ-locally planar graph embedded in Σ is acyclically k-list-colourable.*

It seems likely that the methods used in this thesis may be used to attain $k = 8$ in Conjecture 1.4; however, the analysis would likely be much more lengthy and complex. It is not clear what the optimal value of k is: perhaps $k = 7$, which would match the previous results for acyclic list colouring planar graphs and acyclic colouring locally planar graphs, is achievable.

Moreover, despite the improvement from exponential to linear in the genus of the surface, we suspect the edge-width bound in Theorem 1.3 is not optimal.

Conjecture 1.5. *Let Σ be a surface. Let k be an integer such that there exists a value ρ_0 depending on $g(\Sigma)$ such that every ρ_0-locally planar graph embedded in Σ is acyclically k-list-colourable. There exists $\rho := O(\log g(\Sigma))$ such that every ρ-locally planar graph embedded in Σ is acyclically k-list-colourable.*

Since (see Theorem V.4.1 in [7]) this edge-width bound is asymptotically best possible for even ordinary colouring, it follows that it is also optimal for acyclic list colouring. Since there exist planar graphs that are not 4-acyclic colourable, we have moreover that $k \geq 5$.

1.1 Grötzsch's Theorem

Naturally one may wonder if instead of considering an entire class of graphs, if we consider only a subset of the class, can we reduce the number of colours needed in the case of ordinary colouring, or the size of the lists in the case of list colouring? Specifically, if we do not consider graphs with certain substructures, can we reduce the parameters previously mentioned? Indeed in 1959, Grötzsch [36] showed that planar graphs which are triangle-free are in fact 3-colourable. In the case of list colouring, one can show via a straightforward degeneracy argument that triangle-free planar graphs are 4-list-colourable. This upper bound on the sizes of the lists is tight, indeed in 1995, Voigt [66] provided an example of a triangle-free graph that is not 3-list-colourable. However if we also exclude larger cycles then we can show that the number of necessary colours decreases. Recall that the *girth* of a graph G is the shortest cycle contained in G. In the same year as Voigt's example, Thomassen [58] showed that planar graphs of girth at least five are 3-list-colourable.

In the setting of graphs on surfaces we may ask the same questions. In 2003, Thomassen [63] showed that locally planar graphs of girth at least five are 3-colourable. Forbidding both 3- and 4-cycles are necessary, as it was also shown by Youngs [71], and later expanded upon by Gimbel and Thomassen [35], that there are infinitely many triangle-free graphs which can be embedded in the projective plane that are not 3-colourable. Also of note is the series of papers by Dvořák, Král', and Thomas [25, 26, 27, 28, 29, 30, 31], first released on arXiv throughout the 2010's, which is now completely published as of 2024. They studied when triangle-free graphs on surfaces are indeed 3-colourable, and demonstrated a linear-time algorithm to decide the 3-colourability of a triangle-free graph embedded on a surface [30]. Further, they also provide a quadratic-time algorithm to output a valid 3-colouring. For list colouring, Postle [51] showed, as in the planar case, locally planar graphs that have girth at least five are also 3-list-colourable.

When considering acyclic colourings, the main result to consider is that of Borodin, Kostochka, and Woodall [9] who, in 1999, showed that planar graphs of girth at least five are acyclically 4-colourable and those of girth at least seven are acyclically 3-colourable. The girth of the former result cannot be reduced to four as Kostochka and Mel'nikov [43] in 1976 exhibited a planar graph of girth four which is not acyclically 4-colourable. For list colouring, recall in the acyclic setting that the best known bound for list sizes is that all planar graphs are acyclically 7-list-colourable [13]; Montassier, Ochem and Raspaud [49] in 2006 showed planar graphs of girth at least five are acyclically 5-list-colourable. In the same paper they also showed that planar graphs of girth at least six are acyclically 4-list-colourable, and planar graphs of girth at least eight are acyclically 3-list-colourable. This was then further improved by Montassier [48] later that year, who showed that planar graphs of girth at least five are in fact acyclically 4-list-colourable.

Despite the interest in determining the impact of girth on the colourability, acyclic or otherwise, there are very few results when considering graphs embedded in surfaces, or even locally planar graphs. We begin to fill this gap by looking at the first case to consider, locally planar graphs of girth at least four. The following theorem, using similar techniques used in the proof of Theorem 1.3, gives an upper bound for the sizes of lists needed to acyclically list colour.

Theorem 1.6. *For every surface Σ, there exists $\rho_0 := \rho_0(g(\Sigma))$ such that every triangle-free ρ_0-locally planar graph embedded in Σ is acyclically 8-list-colourable.*

The bound in Theorem 1.6 is $\rho_0 = 90(g(\Sigma) - 2)$. Note, as mentioned previously for Theorem 1.3, that this implies for example that all triangle-free graphs embeddable in at least one of the projective plane, torus, and the Klein bottle (which have genus 1,1, and 2, respectively) are acyclically 8-list-colourable.

As is the case with with Theorem 1.3, we expect that our value of 8 is not optimal. We suspect that a value of 7 is possible, however it would require substantially more work.

We prove both Theorem 1.3 and 1.6 via a *discharging* argument, the method used to prove many of the results regarding acyclic colouring planar graphs mentioned earlier (see e.g. [10, 13]). However, the standard discharging methods for proving colouring results are insufficient on their own in the context of acyclically list-colouring locally planar graphs, for reasons we elaborate on in Chapter 2. As such, we introduce a novel concept, *weighted edge-width*, which, when used in conjunction with discharging, provides a means to prove our result. This idea is also explained in Chapter 2.

For those unfamiliar with discharging, we provide a brief history and explanation of the technique. Along with the results mentioned previously, discharging was also the technique used to prove the Four Colour Theorem; in fact, the Four Colour Theorem immensely popularized this now famous technique, which was pioneered by Wernicke [70] in the early 1900s and systematized and publicized later by Heesch [39] in the 1960s. Broadly speaking, proving a theorem via discharging is a two-part process. We begin by assuming for a contradiction that a counterexample to our theorem exists. In one part of the process (the discharging phase), we assign a numerical quantity (called *charge*) to structures in our counterexample in such a way that the total sum of the charges is known. Next, we move the charge around neighbouring structures in the graph via a set of instructions (the *discharging rules*), preserving the total charge. Since the sum of the charges is known, we are able to gain insight into the sorts of structures that occur in the graph. We call these structures *unavoidable*. In the second part of the process, we show that every unavoidable configuration does not occur in a minimum counterexample (by showing the structures are *reducible*), thereby leading to a contradiction. For a more complete overview and further history on the method, we refer the reader to [23].

1.2 Steinberg's Conjecture

In this section we discuss some additional history concerning a problem known as Steinberg's Conjecture [2], which was first mentioned in 1978. It is related to our previous discussion in that it concerns colouring graphs with forbidden cycle lengths, however, unlike our previous discussion and our second result, we allow the graph to contain triangles.

Steinberg's Conjecture states that all planar graphs which do not contain 4- or 5-cycles are 3-colourable. Erdős [57] proposed the following question as a direction to prove the statement: What is the minimum integer $k \geq 4$ such that planar graphs without cycles of length 4 to k are 3-colorable? The bounds on k were steadily improved, starting from $k \leq 11$ by Abbott and Zhou [1] in 1991, followed by $k \leq 10$ by Borodin [11] in 1996, $k \leq 9$ by Borodin [8] and independently by Sanders and Zhao [55], and finally Borodin, Glebov, Raspaud, and Salavatipour [18] in 2005 showed that $k \leq 7$ which is where the upper bound remains today. In 2017, it was shown by Cohen-Addad, Hebdige, Král', Li, and Salgado [22] that in fact the conjecture is false. Therefore the only remaining question is whether planar graphs which exclude 4-, 5- and 6-cycles are 3-colourable, which remains open today.

In 2013, this question was asked in the context of list colouring by Borodin [12], whose proof that graphs without cycles of lengths 4 to 9 are 3-colourable [8] also holds for list colouring as well. Borodin specifically asked whether it was possible to reduce the lengths of cycles excluded to just those in the range of 4 to 8. Dvořák and Postle [32] the answered this positively in 2018 and proved this by first introducing and then using a generalized version of list colouring, known as correspondence colouring or DP-colouring. A construction of Voigt [64] given previously in 2007 showed that cycles of length 4 up to 6 necessarily need to be excluded by demonstrating a non-3-list-colourable planar graph without 4- and 5-cycles. This question remains open for lengths 4 to 6 and even 4 to 7.

While the question asks us to consider planar graphs not containing cycles of length 4 to some constant m, we could attempt to show something adjacent by only disallowing a subset of the lengths between 4 and m. Letting $x, y \in \{5, 6, 7, 8\}$ with $x \neq y$, it was shown that planar graphs not containing 4-cycles, 9-cycles, x-cycles and y-cycles are 3-list-colourable. Each of the six choices for x and y were done in separate articles the last of which was published in 2011, see [74, 73, 68, 67, 56, 69]. Notice that each of these articles forbid fewer lengths of cycles, unlike the Dvořák and Postle result mentioned previously. But on the other hand, they all forbid 9-cycles, so neither are generalizations of each other. Very recently, in 2021, an improvement was made to their paper however, Miao and Zhao [75] showed that planar graphs excluding 4-, 6- and 8-cycles are 3-list-colourable. Also, recent progress has been made toward proving that planar graphs without all cycles of length 4 to 7 are 3-list-colourable: Lv [44] in 2022 showed that on top of forbidding all cycles of length 4 to 7, if we also disallow triangles to intersect this is enough to ensure that planar graphs are 3-list-colourable.

In 2009, Hocquard and Montassier [40] and independently Borodin [15] introduced the Steinberg-style problem in the context of acyclic colouring, specifically acyclic list colouring. They show that planar graphs with no cycles of length 4 to 12 are acyclically 3-list-colourable. This was then improved by Borodin [16] a year later, who showed that planar graphs with no cycles of length 4 to 11 are acyclically 3-list-colourable. In terms of reducing the set of forbidden cycle lengths, this is the best result as of today.

If instead of asking which cycles are needed to forbid to only need lists of size three, we could instead ask this question for lists of size four. This question has been looked into more, and many results have shown various forbidden cycle lengths to ensure acyclic 4-list-colourability. The most general result, by Borodin and Ivanova [14] and independently by Chen and Raspaud [19] in 2013, is that planar graphs with no 4-cycles or 5-cycles are acyclically 4-list-colourable.

Outline of the Thesis. In Chapter 2, we explain why the existing techniques for arguing the reducibility of unavoidable configurations do not translate to the locally planar setup, and thus cannot be used to prove Theorem 1.3 or Theorem 1.6. We explain our new tools, and give a brief overview of the proof. In Chapter 3 we introduce definitions and general lemmas which we use throughout the proofs in order to avoid repeating the same arguments. In this chapter, Section 3.1 provides topological lemmas and Section 3.2 provides colouring lemmas. The proof of Theorem 1.3 is found in Chapter 4 and comprises two sections: In Section 4.1 we show each configuration in a set of configurations is reducible; and Section 4.2 contains the discharging portion of the proof, showing that at least one of the reducible configurations is unavoidable. Chapter 5 is nearly identical to Chapter 4 but for the proof of Theorem 1.6: Section 5.1 shows a set of configurations is reducible, and Section 5.2 shows that at least one of the reducible configurations is unavoidable.

Chapter 2

Proof Overview

First, we discuss why the discharging proof of Borodin, Fon-Der-Flaass, Kostochka, Raspaud, and Sopena [13] used in proving that planar graphs are acyclically list 7-colourable does not translate to the locally planar environment. For instance, in [13] the authors assume a vertex-minimum counterexample to their theorem is a triangulation. The authors also delete vertices and add edges between others. We cannot assume our graph is a near-triangulation or add edges between non-adjacent vertices, *as adding edges to our embedded graph can decrease its edge-width*. Moreover, in some of their reductions the authors of [13] use "Kempe chain" arguments, arguing by planarity that pairs of bicoloured paths whose colours are disjoint do not cross. We cannot replicate these arguments, as these bicoloured paths can have arbitrary lengths, and therefore no matter the edge-width bound there is still the possibility of these paths "crossing" for example via handles.

New ideas are needed to tackle these difficulties. In the reductions used in many acyclic colouring proofs, vertices are deleted and new edges added between others. As we will explain, adding edges seems crucial to proving that the reducible configurations are indeed acyclically colourable. We give a toy example: in the non-acyclic setting, when proving colouring results by induction, it suffices to delete a *low-degree vertex*, obtain a colouring of the rest of the graph via induction, and then extend the colouring by choosing a colour for the deleted vertex that is not used by its neighbours. As long as the number of colours available is more than the degree of the deleted vertex, the colouring will extend. For acyclic colouring, when extending the colouring to the deleted vertex we also need to avoid creating bicoloured cycles. Thus *even a vertex v of degree as low as two* does not permit an inductive argument via deletion alone: if two neighbours of the deleted vertex are identically coloured, then we risk creating a bicoloured cycle when extending the colouring to v no matter the choice of colour for the deleted vertex. Thus we would have to avoid not only the colours of neighbours of v, but also colours used in its second neighbourhood, which can be prohibitively large and rule out all colour options for v. In order to get around this issue, we add edges between neighbours of the deleted vertex to ensure they receive different colours, and therefore that no bicoloured cycle is created when extending the colouring to v. The innovations in our proof hinge on the following observation: often, these new added edges are added *only to ensure* their endpoints receive different colours (just like in an ordinary vertex colouring argument), rather than simultaneously to ensure endpoints receive different colours and no bicoloured cycles run through these edges.

This observation motivates partitioning the edge-set of our graph G into two types of edges, each enforcing distinct colouring restrictions. We define a set $E_1 \subseteq E(G)$ of edges whose endpoints receive distinct colours and which cannot be included in any bicoloured cycle. (These edges function as edges traditionally do in the acyclic colouring framework.) The non-E_1-edges enforce fewer restrictions, serving only to ensure that endpoints receive distinct colours. (These edges function as normal edges in ordinary colouring and list colouring.)

To get around the problem that adding edges judiciously still could result in a decrease of edge-width, we think of these edges as having different lengths: edges in E_1 have length 1, and edges not in E_1 have length $t > 1$ (for some suitably chosen fixed $t \in \mathbb{N}$ that arises naturally in the required reductions). For the right choice of t, we are able to argue using a set of useful topological lemmas (see Subsection 3.1) that carefully adding non-E_1-edges to our embedded graph does not decrease its edge-width (see Definition 2.2).

A summary of our main innovation follows.

➤ We define **two types of edges with different colouring restrictions and different lengths.** Throughout our reductions, we add new, long edges to a subgraph of our graph. The length of the new edges ensures the edge-width of the graph does not decrease, and the presence of the new edges allows us to restrict the colouring obtained via induction, thus ensuring the colouring extends to the whole graph.

➤ It is important to note that there is nothing in principle that prevents us from defining yet more edge-subsets with more distinct colouring restrictions and corresponding different edge-lengths: thus we expect this general idea to be useful in other colouring problems.

We prove Theorem 1.3 via a more technical (equivalent) theorem (Theorem 2.3) that speaks of edge-width in terms of these edge-lengths. To state this technical theorem, we require the following definitions.

Definition 2.1. Let G be a graph, k a positive integer, and L, a k-list assignment for G. Let $E_1 \subseteq E(G)$. An E_1-*cycle* in G is a cycle $C \subseteq G$ with $E(C) \subseteq E_1$. An E_1-*acyclic* L-*colouring* of G is a proper L-colouring φ with the property that G does not contain an E_1-cycle C with $|\{\varphi(v) : v \in V(C)\}| = 2$. We say G is E_1-*acyclic* k-*list-colourable* if it has an E_1-acyclic L'-colouring for every k-list assignment L'.

We highlight again that this differs from the standard definition of acyclic colouring, which does not allow for *any* bichromatic cycles: in our setup, we allow bichromatic cycles in G so long as they contain at least one edge from from $E(G) \setminus E_1$.

Definition 2.2. Let (G, Σ) be an embedded graph, let $\mathcal{C}$ be the set of non-contractible cycles of G, let t be a positive integer, and let $E_1 \subseteq E(G)$. We define

$$\mathrm{ew}_t(G, E_1) := \min_{C \in \mathcal{C}} \left(|E_1 \cap E(C)| + t \times |E(C) \setminus E_1|\right).$$

If Σ is the sphere, then $\mathcal{C}$ is empty and we define $\mathrm{ew}_t(G, E_1)$ as being infinite.

Note that when $t = 1$, we recover the standard edge-width definition. For our analysis, it suffices to fix $t = 2$.

The more technical version of our main theorem is given below. Note that the value for ρ is given by considering the maximum value for ε in Lemma 4.10.

Theorem 2.3. *Let Σ be a surface with genus g, and let $\rho := 516 \cdot (g - 2)$. Let (G, Σ) be an embedded graph, and $E_1 \subseteq E(G)$. If $\mathrm{ew}_2(G, E_1) > \rho$, then G is E_1-acyclic 9-list-colourable.*

We recover the statement of Theorem 1.3 by taking $E_1 = E(G)$. (Note the theorems are nearly equivalent: with ρ_0 as in Theorem 1.3, we recover Theorem 2.3 (with less specific edge-width bound) from Theorem 1.3 by setting $\rho = 2 \cdot \rho_0$.)

For Theorem 1.6, we prove the statement on triangle-free graphs, which would normally inhibit our ability to add edges as we may create unwanted triangles. Here, we may use our distinguished E_1-edges to our advantage. On top of enforcing the set E_1 to not contain any bicoloured cycle, we additionally place the condition that $G[E_1]$ is triangle-free. Note that the remaining edges do not have have this restriction. This allows us to add non-E_1-edges to ensure the endpoints receive different colours without creating a triangle in $G[E_1]$.

As with Theorem 1.3, we provide a more technical version of Theorem 1.6 below. Again, note that the value of ρ is given by considering the maximum value for ε in Lemma 5.7.

Theorem 2.4. *Let Σ be a surface of genus g, and let $\rho := 90(g-2)$. Let (G, Σ) be an embedded graph, and $E_1 \subseteq E(G)$ such that $G[E_1]$ is triangle-free. If $\mathrm{ew}_2(G, E_1) \geq \rho$, then G is E_1-acyclic 8-list-colourable.*

As with Theorem 2.3, we recover Theorem 1.6 by setting $E_1 = E(G)$.

Chapter 3

Topological and Colouring Lemmas

In this chapter we discuss definitions and lemmas commonly used in Chapter 4 and Chapter 5. In Section 3.1 we prove lemmas which allow us to add non-E_1-edges without increasing edge-width. This will be used in the proofs of every reduction in both of the following chapters. Section 3.2 outlines terminology which will be used frequently and provides lemmas which we use as general tools to modify or extend acyclic colourings.

3.1 Topological Lemmas

In this section, we prove two lemmas regarding the addition of non-E_1-edges to an embedded graph.

Our first lemma will be used to show that in a minimum counterexample to either Theorem 2.3 or 2.4, the faces surrounding low-degree vertices are mostly triangles. This will be helpful in performing colouring reductions throughout, as it ensures adjacent low-degree vertices have neighbours in common, and therefore that not too many colours are used when colouring their neighbourhoods.

Lemma 3.1. *Let (G, Σ) be an embedded graph, and let $E_1 \subseteq E(G)$. If uv and wv are edges in E_1 that are cofacial in (G, Σ) and $uw \notin E(G)$, then there exists an embedding of $G + uw$ in Σ where uw is embedded in a common face of uv and wv and such that $\mathrm{ew}_2(G + uw, E_1) \geq \mathrm{ew}_2(G, E_1)$.*

Proof. Let $(G + uw, \Sigma)$ be the embedded graph obtained from (G, Σ) by embedding the edge uw in a face whose boundary contains both uv and vw in such a way that $uvwu$

bounds an open disk. We claim $\mathrm{ew}_2(G+uw, E_1) = \mathrm{ew}_2(G, E_1)$. To see this, suppose not, and let C be a non-contractible cycle in $(G+uw, \Sigma)$ with $\mathrm{ew}_2(C, E_1 \cap E(C)) < \mathrm{ew}_2(G, E_1)$. Since $\mathrm{ew}_2(G, E_1) > \mathrm{ew}_2(G+uw, E_1)$, it follows that $uw \in E(C)$. Note that $C \neq uvwu$, since by our choice of embedding of uw, the cycle $uvwu$ is contractible in Σ. In what follows, let Δ be the closed disk bounded by the cycle $uwvu$ in $(G+uw, \Sigma)$.

Let W be the closed walk obtained by listing the vertices of C in cyclic order beginning with uw, and let W' be a closed walk obtained from W by replacing uw with the walk uvw. Note that since C is non-contractible and the regions bounded by C and the walk W' differ only in Δ, it follows that W' is also non-contractible; and since every non-contractible walk contains a non-contractible cycle, the graph C' whose edge- and vertex-sets are precisely those of W' contains a non-contractible cycle. Moreover, $uw \in V(C)$ and every cycle in C' contains a subset of the edges of $C \setminus \{uw\}$ along with a subset of $\{uv, vw\}$. Since $\{uv, vw\}$ is a subset of E_1 of size two and $uw \notin E_1$, it follows that $\mathrm{ew}_2(C', E_1 \cap E(C')) \leq \mathrm{ew}_2(C, E_1 \cap E(C))$. This is a contradiction, since $C' \subseteq G$, and hence $\mathrm{ew}_2(G, E_1) \leq \mathrm{ew}_2(C, E_1 \cap E(C))$.

□

The following lemma allows us to add specific non-E_1-edges when performing our reductions. We require two definitions.

Definition 3.2. Let H be a graph, and $S \subseteq V(H)$. We denote by $\delta(S)$ the set of edges in H with exactly one endpoint in S. When $|S| = 1$ (say $S = \{v\}$), we write $\delta(v)$ instead of $\delta(\{v\})$.

Definition 3.3. Let G be a graph and $v \in V(G)$. We define the *E_1-neighbourhood of v*, $N_{E_1}(v)$, as the set $\{u \in N(v) \ : \ vu \in E_1\}$.

As explained in the previous chapter, given a graph G embedded in Σ, we would like to be able to delete a vertex v and add edges between its neighbors without increasing the edge-width of the resultant graph. We do this by using the non-E_1-edges. The intuition of how we go about adding these edges to the existing embedding is as follows: consider only small portion of the surface, a closed disk Δ which contains v and the edges incident with it in the interior, and the vertices of $N(v)$ on the boundary of Δ. Notice that if the vertex v is deleted, all that remains in Δ is the vertices in $N(v)$. So, if we embed all of the edges we would like to add wholly within Δ, then we obtain our new embedded graph by 'cutting out' Δ from the embedding of G and replacing it with a new Δ containing the non-E_1-edges.

This idea is formalized in the following lemma.

Lemma 3.4. *Let (G, Σ) be an embedded graph, and $E_1 \subseteq E(G)$. Let $v \in V(G)$, and let Δ be a closed disk in (G, Σ) whose boundary intersects G only in vertices in $N(v)$ and whose interior contains only v and the edges of $\delta(v)$. Suppose H is a graph on the vertex-set $N_{E_1}(v)$ embedded in $\Delta - v$ where $E(H) \cap E(G) = \emptyset$. If (G', Σ) is the graph obtained from $(G - v, \Sigma)$ by embedding H in $\Delta - v$ then $\mathrm{ew}_2(G', E_1 \setminus \delta(v)) \geq \mathrm{ew}_2(G, E_1)$.*

Proof. Let (G, H) be a counterexample chosen to minimize $e(H)$ and subject to that, to minimize $\deg_G(v)$. Note that $\deg(v) \neq 0$, as otherwise the statement trivially holds since deleting vertices does not decrease edge-width. We claim $V(H) = N(v)$ and the minimum degree of H is at least one. To see this, suppose not, and let u be either a vertex in $N(v) \setminus V(H)$, or a vertex in H with $\deg_H(u) = 0$. By the minimality of (G, H), we have that $(G - uv, H)$ is not a counterexample. Since deleting edges does not decrease edge-width, this implies (G, H) is not a counterexample—a contradiction. Hence H has minimum degree at least one, and so $e(H) \geq 1$.

We claim $E(H)$ contains an edge between two consecutive vertices u, w in the boundary of Δ. To see this, let C be the cycle on the vertex-set $V(H) = N(v)$ formed by adding edges between vertices that are consecutive in the cyclic ordering around v, and let H' be the graph obtained from C by adding the edges $E(H) \setminus E(C)$. Note that H' is outerplanar, and hence since outerplanar graphs are 2-degenerate, H' has minimum degree at most 2. Since $\delta(C) = 2$, it follows that there exists a vertex $u \in V(H')$ with degree exactly 2. Since $\delta(H) \geq 1$, this implies u is incident with an edge $uw \in E(H) \cap E(C)$. Hence uw is an edge in $E(H)$ between two consecutive vertices in C, the boundary of Δ, and so u and w are cofacial in G. By Lemma 3.1, there exists an embedding of $G + uw$ with uw in the common face of uv and wv such that $\mathrm{ew}_2(G + uw, E_1) \geq \mathrm{ew}_2(G, E_1)$. Let Δ^+ be a closed disk in $(G + uw, \Sigma)$ whose boundary intersects G only in vertices of $N(v)$ and whose interior contains only v and the edges of $\delta(v)$. Let $\Delta' = \Delta^+ - v$ and let $H - uw$ be embedded in Δ'. By the minimality of $e(H)$, the graph (G^+, Σ) obtained from $(G + uw, \Sigma)$ by replacing Δ^+ by Δ' satisfies $\mathrm{ew}_2(G^+, E_1 \setminus \delta(v)) \geq \mathrm{ew}_2(G + uw, E_1)$, a contradiction since G^+ is an embedded graph obtained from $(G - v, \Sigma)$ by embedding H in $\Delta - v$.

□

3.2 Colouring Lemmas

Throughout this thesis, we will use the following definition and notation.

Definition 3.5. Let G be a graph and $v \in V(G)$. We define the *second neighbourhood of* v as the set of vertices at distance exactly two from v in G.

Definition 3.6. Let H be a graph. Given a (not necessarily proper) colouring φ of H and subset S of $V(H)$, we define $\varphi(S) := \{\varphi(x) : x \in S\}$. We say a subgraph $H' \subseteq H$ is *bicoloured with respect to* φ if $|\varphi(V(H'))| = 2$.

Throughout, given an L-colouring φ of G, we are mainly concerned with cycles $C \subseteq G$ that are bicoloured with respect to φ and have $E(C) \subseteq E_1$.

Definition 3.7. Let G be a graph, $G' \subseteq G$, L be a list assignment for G, $E_1' \subseteq E(G')$, and $E_1 \subseteq E(G)$ such that $E_1' \subseteq E_1$. Suppose that φ' is an E_1'-acyclic L-colouring of G'. We say φ' *extends to* G if there exists an E_1-acyclic L-colouring φ of G that agrees with φ' on $V(G')$. We call φ an *extension of* φ'.

Note that when we extend a colouring φ to a larger domain since the extension agrees with φ on its domain, we use the symbol φ to represent both the original and the extended colouring.

To prove both of the main theorems, we contradict the existence of a minimum counterexample (G, E_1, L). Given $v \in V(G)$, the primary way we contradict its existence is by extending an $(E_1 \setminus \delta(v))$-acyclic L-colouring φ of $G - v$ to G. To ensure that the extension of φ to G is indeed a proper colouring we choose $\varphi(v) \notin \varphi(N(v))$. To ensure that G contains no E_1-cycle that is bicoloured with respect to φ, it is sufficient to ensure $\varphi(v)$ does not appear in the set of colours used in the second neighbourhood of v. However, in our reductions, the set of colours that appear in the first and second neighbourhoods of v often exceeds the size of $L(v)$. To get around this, we argue that it suffices to colour v to avoid $\varphi(N(v))$ and only *a subset* of the colours that appear in the second neighbourhood of v. In particular: let X be the set of pairs $\{u_1, u_2\} \subseteq N_{E_1}(v)$ with $\varphi(u_1) = \varphi(u_2)$. If G contains an E_1-cycle C that is bicoloured with respect to φ and where $v \in V(C)$, then C also contains a pair of vertices in X. Therefore, instead of avoiding all colours in the second neighbourhood of v, it suffices to avoid the colours of the neighbours of at least one vertex from each pair of vertices in X. This notion is formalized in the following lemma.

Lemma 3.8. *Let G be a graph, $E_1 \subseteq E(G)$ and L be a list assignment for G. Let $v \in V(G)$ and $E_1' = E_1 \setminus \delta(v)$. Suppose that φ is an E_1'-acyclic L-colouring of $G - v$. Let $S \subseteq N_{E_1}(v)$ be such that for all $\{u_1, u_2\} \subseteq N_{E_1}(v)$ such that $\varphi(u_1) = \varphi(u_2)$, we have that $S \cap \{u_1, u_2\} \neq \emptyset$. Let $N_S := \bigcup_{u \in S} (N_{E_1}(u) \setminus \{v\})$. If $L(v) \setminus (\varphi(N(v)) \cup \varphi(N_S))$ is non-empty, then φ extends to an E_1-acyclic L-colouring of G.*

Proof. We have that $L(v) \setminus (\varphi(N(v)) \cup \varphi(N_S))$ is non-empty therefore we extend φ to G by choosing $\varphi(v) \in L(v) \setminus (\varphi(N(v)) \cup \varphi(N_S))$. Suppose that the extension of φ to G is not a valid E_1-acyclic L-colouring of G. Since $\varphi(v) \notin \varphi(N(v))$, we have that φ is a proper colouring and therefore G contains a cycle C which is bicoloured with respect to φ.

If $v \notin V(C)$ then $C \subseteq G - v$, contradicting that φ is an E_1'-acyclic L-colouring of $G - v$. Thus $v \in V(C)$ and therefore there exist two neighbours u_1 and u_2 of v such that $\{u_1, u_2\} \subseteq V(C)$. By definition of S, at least one of u_1 and u_2 is in S; we may assume that $u_1 \in S$. Let $u \neq v$ be a vertex adjacent to u_1 in C. Note that since C is an E_1-cycle, we have that $u_1 u \in E_1$. Since C is bicoloured with respect to φ, $\varphi(v) = \varphi(u)$. However this contradicts the fact that $\varphi(v) \neq \varphi(u)$ since $u \in N_S$. □

Often in the reductions, we divide our proofs into cases based on the quantity $|\varphi(N(v))|$, where $v \in V(G)$ and φ is an $(E_1 \setminus \delta(v))$-acyclic L-colouring of $G - v$. We assume that φ is chosen to maximize $|\varphi(N(v))|$, and therefore if we create an $(E_1 \setminus \delta(v))$-acyclic L-colouring φ' with $|\varphi'(N(v))| > |\varphi(N(v))|$ we arrive at a contradiction. Let $u \in N(v)$, and assume that all colours in $\varphi(N_{E_1}(u))$ are distinct. If $L(u) \setminus \varphi(N(v) \cup N(u))$ is non-empty, then there exists an alternate colour for u which gives us an $(E_1 \setminus \delta(v))$-acyclic L-colouring with $|\varphi'(N(v))| > |\varphi(N(v))|$. We then can conclude that $|\varphi(N_{E_1}(u))| < |N_{E_1}(u)|$, which will help us in applying Lemma 3.8, since it will reduce the size of N_S. A more general version of this recolouring argument is captured in the following lemma.

Lemma 3.9. *Let G be a graph, $E_1 \subseteq E(G)$, L be a list assignment for G and $v \in V(G)$. Suppose that φ is an E_1-acyclic L-colouring of G. Let C be a set of colours such that $\varphi(N(v) \cup \{v\}) \subseteq C$. If $|\varphi(N_{E_1}(v))| = |N_{E_1}(v)|$ and $L(v) \setminus C$ is non-empty, then there exists an E_1-acyclic L-colouring $\varphi' \neq \varphi$ such that $\varphi'(x) = \varphi(x)$ for all $x \in V(G) \setminus \{v\}$ and $\varphi'(v) \notin C$.*

Proof. Let φ' be a colouring of G' obtained by defining $\varphi'(x) = \varphi(x)$ for all $x \in V(G) \setminus \{v\}$ and $\varphi'(v) \in L(v) \setminus C$. Note that φ' is a proper colouring as $\varphi'(v) \notin \varphi(N(v)) \subseteq C$. In addition, G contains no bicoloured cycle with respect to φ' since every E_1-cycle containing v also contains two vertices $\{v_1, v_2\} \subseteq N_{E_1}(v)$ which have distinct colours different from $\varphi'(v)$ since $|\varphi(N_{E_1}(v))| = |N_{E_1}(v)|$.

□

Chapter 4

Proof of Theorem 2.3

This chapter contains the proof of Theorem 2.3. The chapter is divided into three sections which contain required tools for the proof. The proof itself is found below. The reductions are found in Subsection 4.1, wherein we show that a minimum counterexample to our main theorem does not contain certain substructures. Finally, the discharging portion of the proof (showing at least one of these these substructures must occur, thereby disproving the existence of a minimum counterexample) is found in Section 4.2.

Throughout this chapter we use the following notation to more efficiently talk about the degrees of vertices and faces.

Definition 4.1. Given a positive integer k, we refer to a vertex of degree equal to k as a k-vertex, a vertex of degree at most k as a k^--vertex, and a vertex of degree at least k as a k^+-vertex. Analogously, given an embedded graph G, we refer to a face with boundary of length k, at most k, and at least k as a k-face, a k^--face, and a k^+-face, respectively.

4.1 Reducible Configurations

In what follows in this chapter, a *counterexample* is a graph, edge-set, and list assignment triple (G, E_1, L) that satisfy the hypotheses but not the conclusion of Theorem 2.3. A *minimum* counterexample is a counterexample chosen to minimize $\sum_{v \in V(G)} |L(v)|$, and subject to that, to maximize $e(G)$. In particular, this minimizes $v(G)$ and makes $|L(v)| = 9$ for all $v \in V(G)$. Let (G, E_1, L) be a minimum counterexample, in all of the following lemmas in this section, G is assumed to be this minimum counterexample. We will use the following definition to more easily describe one of our reductions, and later to use in the discharging.

Definition 4.2. A *triangular vertex* in an embedded graph is a vertex incident only with 3-faces.

To facilitate the reductions in this section, we will use the following.

Lemma 4.3. *If $v \in V(G)$ is an 8^--vertex, then there are at least four E_1-edges incident with v.*

Proof. Suppose towards a contradiction that there exists $v \in V(G)$ with $\deg(v) \leq 8$ and $|N_{E_1}(v)| \leq 3$.

Let H be the complete graph with vertex-set $N_{E_1}(v)$. Let G' be the graph with vertex-set $V(G)\setminus\{v\}$ and edge-set $(E(G)\cup E(H))\setminus\delta(v)$. Let $E_1' := E_1\setminus\delta(v)$. By Lemma 3.4, G' has an embedding such that $\mathrm{ew}_2(G', E_1') \geq \mathrm{ew}_2(G, E_1)$. Since G is a minimum counterexample to Theorem 2.3 and $v(G') < v(G)$, the graph G' has an E_1'-acyclic L-colouring φ.

Since $\deg_G(v) \leq 8$, we have that $|L(v) \setminus \varphi(N(v))| \geq |L(v)| - |\varphi(N(v))| \geq 9 - 8 = 1$ and therefore $L(v) \setminus \varphi(N(v))$ is non-empty. By Lemma 3.8 with $S := \emptyset$, φ extends to an E_1-acyclic L-colouring of G, a contradiction.

□

As an easy consequence of Lemma 3.1 and the edge-maximality of G, we have the following.

Lemma 4.4. *G does not contain vertices u, v, w where $\{uv, vw\} \subset E_1$, where uv and vw are cofacial, and where $uw \notin E(G)$.*

Proof. Let $\{uv, vw\} \subset E_1$ and suppose by way of contradiction that $uw \notin E(G)$. By Lemma 3.1 the graph G' obtained from G by adding the non E_1-edge uw to G satisfies $\mathrm{ew}_2(G', E_1) \geq \mathrm{ew}_2(G, E_1)$. This contradicts the fact that G is edge-maximal.

□

The lemmas below establish the reducibility of the unavoidable configurations listed in Lemma 4.10.

Lemma 4.5. *G does not contain a 3^--vertex.*

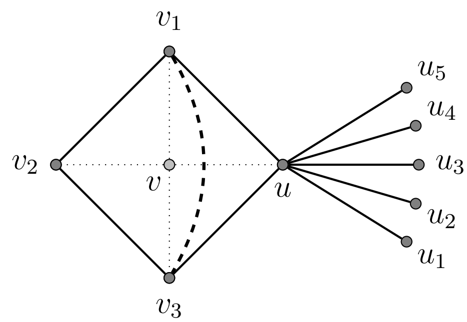

Figure 4.1: The structure of the graph in Lemma 4.6. A 4-vertex v is adjacent to a 8^--vertex u. The set N_u is labelled as $\{u_1, \ldots, u_5\}$.

Proof. By Lemma 4.3, every 8^--vertex is incident to four E_1-edges. Therefore, if $v \in V(G)$ has $\deg(v) \leq 8$, it also has $\deg(v) \geq 4$, so no 3^--vertex exists in $V(G)$. $\square$

Our second reducible configuration (referred to in Lemma 4.6) is illustrated in Figure 4.1.

Lemma 4.6. *G does not contain a 4-vertex adjacent to an 8^--vertex.*

Proof. Suppose not: that is, suppose there exists a 4-vertex v adjacent to an 8^--vertex u. By Lemma 4.5, G does not contain a 3^--vertex and hence $\deg(u) \geq 4$. By Lemma 4.3 every 8^--vertex is incident with at least four E_1-edges and hence since $\deg(v) = 4$, every edge incident with v is in E_1. Let u, v_1, v_2, v_3 be the neighbours of v listed in cyclic order. Since $\{vu, vv_1, vv_3\} \subseteq E_1$, by Lemma 4.4 we find that $\{v_1, v_3\} \subseteq N(u)$. Let $N_u := N_{E_1}(u) \setminus (N(v) \cup \{v\})$; note that since $\deg(u) \leq 8$, we have that $|N_u| \leq 5$.

Let H be the graph with vertex-set $N_{E_1}(v)$ and edge-set $\{v_1v_3\}$. Let G' be the graph with vertex-set $V(G) \setminus \{v\}$ and edge-set $(E(G) \cup E(H)) \setminus \delta(v)$. Let $E_1' := E_1 \setminus \delta(v)$. By Lemma 3.4, G' has an embedding such that $\mathrm{ew}_2(G', E_1') \geq \mathrm{ew}_2(G, E_1)$. Since G is a minimum counterexample to Theorem 2.3 and $v(G') < v(G)$, the graph G' has an E_1'-acyclic L-colouring φ.

First suppose that $|\varphi(N(v))| = 4$. Since $|L(v) \setminus \varphi(N(v))| \geq |L(v)| - |\varphi(N(v))| = 9 - 4 = 5$, we have that $L(v) \setminus \varphi(N(v))$ is non-empty. By Lemma 3.8 with $S := \emptyset$, we have that φ extends to a E_1-acyclic L-colouring of G, contradicting that G is a counterexample to Theorem 2.3. Thus we may assume $|\varphi(N(v))| \leq 3$. Since $\{v_1v_3, v_1v_2, v_2v_3, uv_1, uv_3\} \subseteq E(G')$, it follows that $\varphi(v_2) = \varphi(u)$ and $|\varphi(N(v))| = 3$.

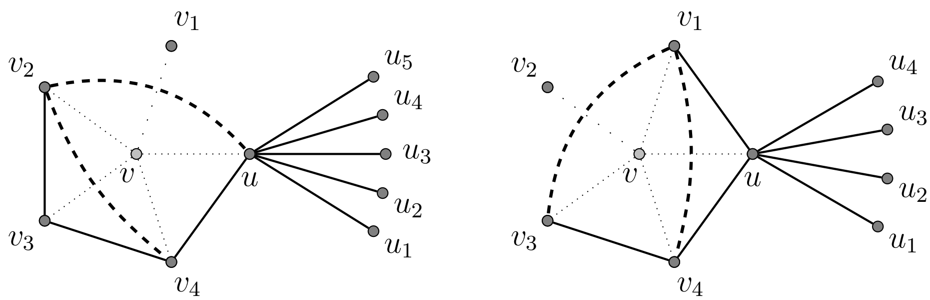

Figure 4.2: The structure of the graph in Lemma 4.7. A 5-vertex v is adjacent to a 7^{-}-vertex u such that $vu \in E_1$ and $|E_1 \cap \delta(v)| = 4$. The case where $vv_1 \notin E_1$ is on the left, and the case where $vv_2 \notin E_1$ is on the right. The set N_u is labelled as $\{u_1, \ldots, u_5\}$ (left) or $\{u_1, \ldots, u_4\}$ (right).

Let $S := \{u\}$. We next argue that there exists a colour for v avoiding both $\varphi(N(v))$ and $\varphi(N(u))$, which will imply G is E_1-acyclically L-colourable. Since $|\varphi(N(v))| = 3$ and $|N_u| \leq 5$, we have $|L(v) \setminus (N(v) \cup N_u)| \geq |L(v)| - |\varphi(N(v))| - |N_u| \geq 9 - 3 - 5 = 1$ and therefore $L(v) \setminus \varphi(N(v) \cup N_u)$ is non-empty. By Lemma 3.8 with S as previously defined, we have that φ extends to an E_1-acyclic L-colouring of G, a contradiction.

$\square$

Lemma 4.7. *G does not contain a 5-vertex v adjacent to a 7^{-}-vertex u, such that $vu \in E_1$ and $|\delta(v) \cap E_1| = 4$.*

Proof. Suppose not. By Lemma 4.5, G does not contain a 3^{-}-vertex and hence $\deg(u) \geq 4$. Let u, v_1, v_2, v_3, v_4 be the neighbours of v listed in cyclic order and set $v_0 := u$. Let $i \in \{1, 2, 3, 4\}$ be the index with $v_i \in N(v) \setminus N_{E_1}(v)$. By symmetry, we may assume that $i \in \{1, 2\}$. Since vv_4 and vu are cofacial and $\{vv_4, vu\} \subseteq E_1$, by Lemma 4.4, we have that $v_4 \in N(u)$. Let $N_u := N_{E_1}(u) \setminus (N(v) \cup \{v\})$; note that since $\deg(u) \leq 7$, we have $|N_u| \leq 5$. Also by Lemma 4.4, we have that consecutive vertices in $N_{E_1}(v)$, with respect to the cyclic ordering of $N(v)$, are adjacent. See Figure 4.2 for an illustration for when $i = 1$ (left) and $i = 2$ (right).

Let H be the graph with vertex-set $N_{E_1}(v)$ and edge-set $\{v_{3-i}v_4, v_{i-1}v_{i+1}\}$. Let G' be the graph with vertex-set $V(G) \setminus \{v\}$ and edge-set $(E(G) \cup E(H)) \setminus \delta(v)$. Let $E_1' := E_1 \setminus \delta(v)$. By Lemma 3.4, G' has an embedding such that $\mathrm{ew}_2(G', E_1') \geq \mathrm{ew}_2(G, E_1)$. Since G is a minimum counterexample to Theorem 2.3 and $v(G') < v(G)$, the graph G' has an E_1'-acyclic L-colouring. Let φ be an E_1'-acyclic L-colouring of G' that maximizes $|\varphi(N_{E_1}(v))|$.

First suppose that $|\varphi(N_{E_1}(v))| = 4$. Since $|L(v) \setminus \varphi(N(v))| \geq |L(v)| - |\varphi(N(v))| \geq 9 - 5 = 4$, we have that $L(v) \setminus \varphi(N(v))$ is non-empty. By Lemma 3.8 with $S := \emptyset$, we have that φ extends to an E_1-acyclic L-colouring of G, a contradiction. Thus we assume that $|\varphi(N_{E_1}(v))| \leq 3$. Since $\{uv_{3-i}, v_{3-i}v_3, v_3v_4, uv_4, v_{3-i}v_4\} \subseteq E(G')$, it follows that $\varphi(v_3) = \varphi(u)$ and $\{\varphi(v_{3-i}), \varphi(v_3), \varphi(v_4)\}$ are all distinct colours. Note then that $|\varphi(N_{E_1}(v))| = 3$, and hence $|\varphi(N(v))| \leq 4$.

Now suppose that $|\varphi(N_u) \setminus \varphi(N(v))| = 5$. Note that $|N_u| \leq 5$ since u is adjacent to v_4. Moreover this implies that no vertex of $N(v) \setminus \{v_4\}$ is adjacent to u. Furthermore all vertices in $N(u) \setminus \{v\}$ receive different colours. Now notice that $|L(u) \setminus \varphi(N_{E_1}(v) \cup N(u))| = |L(u) \setminus \varphi(N_{E_1}(v) \cup N_u)| \geq |L(u)| - |\varphi(N_{E_1}(v))| - |\varphi(N_u) \setminus \varphi(N(v))| \geq 9 - 3 - 5 = 1$. Therefore we have that $L(u) \setminus \varphi(N_{E_1}(v) \cup N(u))$ is non-empty. By Lemma 3.9 we obtain a colouring φ' of G' that agrees with φ everywhere except for u, and where $\varphi'(u) \notin \varphi(N_{E_1}(v))$. Since $|\varphi'(N_{E_1}(v))| = 4 > |\varphi(N_{E_1}(v))|$, we have that φ' contradicts the our choice of φ. We therefore conclude that $|\varphi(N_u) \setminus \varphi(N(v))| \leq 4$.

Let $S := \{u\}$. We next argue that there exists a colour for v avoiding both $\varphi(N(v))$ and $\varphi(N(u))$, which will imply G is E_1-acyclically L-colourable. Since $|L(v) \setminus \varphi(N(v) \cup N_u)| \geq |L(v)| - |\varphi(N(v))| - |\varphi(N_u) \setminus \varphi(N(v))| \geq 9 - 4 - 4 = 1$, we have that $L(v) \setminus \varphi(N(v) \cup N_u)$ is non-empty. By Lemma 3.8 (with $S = \{u\}$) we have that φ extends to an E_1-acyclic L-colouring of G, a contradiction.

□

Lemma 4.8. *G does not contain a 5-vertex adjacent to a both 7^--vertex u and a 6^--vertex w distinct from u.*

Proof. Suppose not, and let v be a 5-vertex adjacent to a 7^--vertex u and a 6^--vertex w distinct from u. Since v is an 8^--vertex, by Lemma 4.3 we find that $|\delta(v) \cap E_1| \geq 4$. At least one of u and w is in $N_{E_1}(v)$ and hence by Lemma 4.7 we have $|\delta(v) \cap E_1| \neq 4$, and therefore $|\delta(v) \cap E_1| = 5$.

We split into cases based on whether or not u and w are consecutive in the cyclic ordering of $N(v)$. The two cases considered are illustrated in Figure 4.3. When u and w are consecutive, let u, w, v_1, v_2, v_3 be a cyclic ordering of $N(v)$. When u and w are not consecutive, let u, x_1, w, x_2, x_3 be a cyclic ordering of $N(v)$. By Lemma 4.4 it follows that any two consecutive vertices are adjacent in G. Let $N_u := N_{E_1}(u) \setminus (N(v) \cup \{v\})$ and $N_w := N_{E_1}(w) \setminus (N(v) \cup \{v\})$. Since $\deg(u) \leq 7$ and $\deg(w) \leq 6$, it follows that $|N_u| \leq 4$ and $|N_w| \leq 3$.

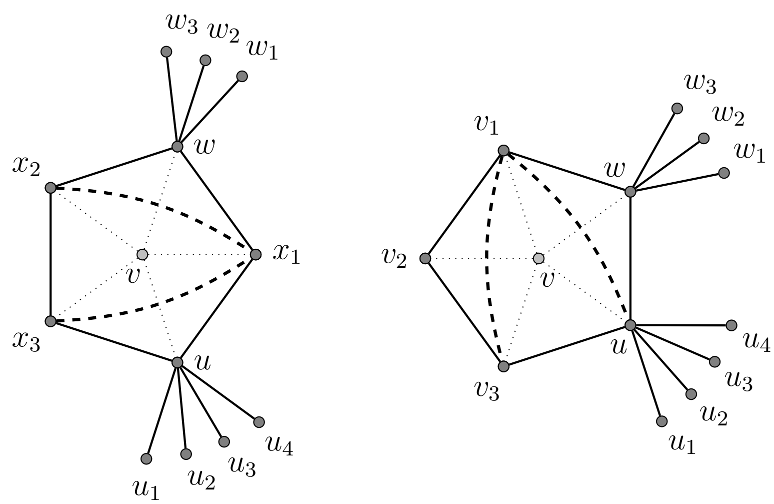

Figure 4.3: The structure of the graph in Lemma 4.8. A 5-vertex v is adjacent to a 7^--vertex u and 6^--vertex w distinct from u. The case where u and w are not consecutive in the cyclic order of $N(v)$ is on the left, and the case where u and w are consecutive is on the right. The sets N_u and N_w are labelled as $\{u_1, \dots, u_4\}$ and $\{w_1, \dots, w_3\}$, respectively.

If u and w are consecutive, let H be the graph with vertex-set $N_{E_1}(v)$ and edge-set $\{v_1v_3, uv_1\}$. If u and w are not consecutive, let H be the graph with vertex-set $N_{E_1}(v)$ and edge-set $\{x_1x_2, x_1x_3\}$. Let G' be the graph with vertex-set $V(G) \setminus \{v\}$ and edge-set $(E(G) \cup E(H)) \setminus \delta(v)$. Let $E_1' := E_1 \setminus \delta(v)$. By Lemma 3.4, G' has an embedding such that $\mathrm{ew}_2(G', E_1') \geq \mathrm{ew}_2(G, E_1)$. Since G is a minimum counterexample to Theorem 2.3 and $v(G') < v(G)$, the graph G' has an E_1'-acyclic L-colouring. Let φ be an E_1'-acyclic L-colouring which maximizes $|\varphi(N(v))|$.

First suppose that $|\varphi(N(v))| = 5$. Since $|L(v) \setminus \varphi(N(v))| \geq |L(v)| - |\varphi(N(v))| = 9 - 5 = 4$, the set $L(v) \setminus \varphi(N(v))$ is non-empty. By Lemma 3.8 with $S := \emptyset$, we have that φ extends to an E_1-acyclic L-colouring of G, a contradiction. Thus we assume that $|\varphi(N(v))| \leq 4$.

Next suppose that $|\varphi(N(v))| = 4$. Since $|N(v)| = 5$, there is exactly one pair of vertices in $N(v)$ with the same colour. Since $G'[N(v) \setminus \{u, w\}]$ is isomorphic to K_3, at least one of u or w is in this pair. Let $S := \{z\}$ where $z \in \{u, w\}$ is said vertex. We next argue that there exists a colour for v avoiding both $\varphi(N(v))$ and $\varphi(N(z))$, which will imply G is E_1-acyclically L-colourable. Since $|L(v) \setminus \varphi(N(v) \cup N_z)| \geq |L(v)| - |\varphi(N(v))| - |N_z| \geq 9 - 4 - 4 = 1$, we have that $L(v) \setminus \varphi(N(v) \cup N_z)$ is non-empty. By Lemma 3.8 with S as defined previously, we have that φ extends to an E_1-acyclic L-colouring of G, a contradiction. Thus we assume that $|\varphi(N(v))| = 3$.

As before, $G'[N(v) \setminus \{u, w\}]$ is isomorphic to K_3 and therefore any pair of vertices in $N(v)$ with the same colour includes at least one of $\{u, w\}$. Let $S := \{u, w\}$.

Recall the cyclic order of $N(v)$ when u and w are not consecutive: u, x_1, w, x_2, x_3. Since $|\varphi(N(v))| = 3$, we have $\varphi(u) = \varphi(x_2) \neq \varphi(x_3) = \varphi(w)$. Suppose that $|\varphi(N_w) \setminus \varphi(N(v))| = 3$. It follows that all vertices in $N(w)$ receive different colours. Since $|L(w) \setminus \varphi(N(v) \cup N_w)| \geq |L(w)| - |\varphi(N(v))| - |\varphi(N_w) \setminus \varphi(N(v))| \geq 9 - 3 - 3 = 3$, the set $L(w) \setminus \varphi(N(v) \cup N_w)$ is non-empty. By Lemma 3.9 we obtain a colouring φ' of G' that agrees with φ everywhere except for w, and where $\varphi'(w) \notin \varphi(N(v))$. Since $|\varphi'(N_{E_1}(v))| = 4 > |\varphi(N_{E_1}(v))|$, we have that φ' contradicts our choice of φ. We therefore conclude that $|\varphi(N_w) \setminus \varphi(N(v))| \leq 2$. A symmetrical argument shows that $|\varphi(N_u) \setminus \varphi(N(v))| \leq 3$.

Since $|L(v) \setminus \varphi(N(v) \cup N_u \cup N_w)| \geq |L(v)| - |\varphi(N(v))| - |\varphi(N_u) \setminus \varphi(N(v))| - |\varphi(N_w) \setminus \varphi(N(v))| \geq 9 - 3 - 3 - 2 = 1$, we have that $L(v) \setminus (\varphi(N(v) \cup N_u \cup N_w))$ is non-empty. By Lemma 3.8 (with $S := \{u, w\}$), φ extends to an E_1-acyclic L-colouring of G, a contradiction. We therefore conclude that u and w are consecutive in the cyclic order of $N(v)$.

Recall the cyclic order of $N(v)$ when u and w are consecutive: u, w, v_1, v_2, v_3. Since $|\varphi(N(v))| = 3$ and both $\{v_1, v_2, v_3\}$ and $\{v_1, v_3, u\}$ induce triangles, we have $\varphi(v_2) = \varphi(u) \neq \varphi(w) = \varphi(v_3)$. Let $S := \{u, w\}$. Suppose that $|\varphi(N_w) \setminus \varphi(N(v))| = 3$. It follows that all vertices in $N(w)$ receive different colours. Since $|L(w) \setminus \varphi(N(v) \cup N_w)| \geq |L(w)| - |\varphi(N(v))| - |\varphi(N_w) \setminus \varphi(N(v))| \geq 9 - 3 - 3 = 3$, the set $L(w) \setminus \varphi(N(v) \cup N_w)$ is non-empty. By Lemma 3.9 we obtain a colouring φ' of G' that agrees with φ everywhere except for w, and where $\varphi'(w) \notin \varphi(N(v))$. Since $|\varphi'(N_{E_1}(v))| = 4 > |\varphi(N_{E_1}(v))|$, we have that φ' contradicts our choice of φ. We therefore conclude that $|\varphi(N_w) \setminus \varphi(N(v))| \leq 2$.

Note that we may not use a symmetrical recolouring argument for u in general since $\varphi(w) = \varphi(v_3)$ and $\{w, v_3\} \subseteq N(u)$. Suppose that $uw \notin E_1$. We first consider the case that $|\varphi(N_u) \setminus \varphi(N(v))| = 4$. It follows that all vertices in $N(u)$, except for w and v_3, receive different colours. Since $|L(u) \setminus \varphi(N(v) \cup N_u)| \geq |L(u)| - |\varphi(N(v))| - |\varphi(N_w) \setminus \varphi(N(v))| \geq 9 - 3 - 4 = 2$, the set $L(u) \setminus \varphi(N(v) \cup N_u)$ is non-empty. Since $w \notin N_{E_1}(u)$, by Lemma 3.9 we obtain a colouring φ' of G' that agrees with φ everywhere except for u, and where $\varphi'(u) \notin \varphi(N(v))$. Since $|\varphi'(N_{E_1}(v))| = 4 > |\varphi(N_{E_1}(v))|$, we have that φ' contradicts our choice of φ. We therefore conclude that $|\varphi(N_u) \setminus \varphi(N(v))| \leq 3$.

Since $|L(v) \setminus \varphi(N(v) \cup N_u \cup N_w)| \geq |L(v)| - |\varphi(N(v))| - |\varphi(N_u) \setminus \varphi(N(v))| - |\varphi(N_w) \setminus \varphi(N(v))| \geq 9 - 3 - 3 - 2 = 1$, we have that $L(v) \setminus \varphi(N(v) \cup N_u \cup N_w)$ is non-empty. By Lemma 3.8 (with $S := \{u, w\}$), φ extends to E_1-acyclic L-colouring of G, a contradiction. We may then assume that $uw \in E_1$.

Suppose that the face F incident with uw distinct from the triangle given by u, v, w

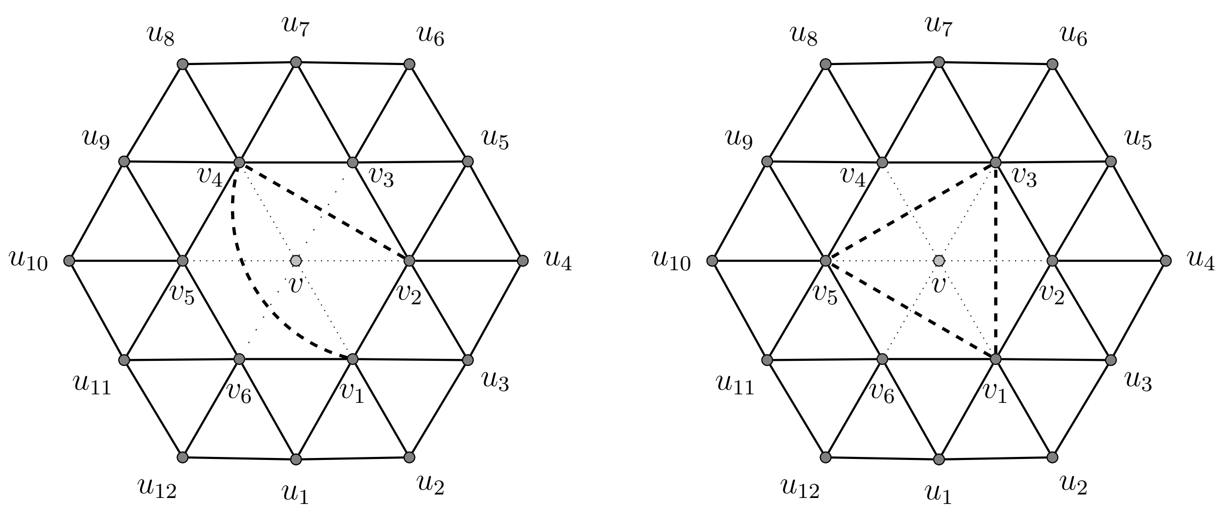

Figure 4.4: The structure of the graph in Lemma 4.9. A triangular 6-vertex v is adjacent to only triangular 6-vertices. On the left we demonstrate one of the cases where $\{vv_1, vv_3, vv_5\} \not\subseteq E_1$ and $\{vv_2, vv_4, vv_6\} \not\subseteq E_1$. The case where $\{vv_1, vv_3, vv_5\} \subseteq E_1$ is on the right. The sets N_4, N_5, and N_6 are labelled as $\{u_7, u_8, u_9\}$, $\{u_9, u_{10}, u_{11}\}$, and $\{u_{11}, u_{12}, u_1\}$, respectively. We use the convention that sparse dotted lines are non-E_1-edges which are deleted with the removal of v.

is also a triangle. This implies that $|N_u \cap N_w| \geq 1$. Since $|L(v) \setminus \varphi(N(v) \cup N_u \cup N_w)| \geq |L(v)| - |\varphi(N(v))| - |\varphi(N_w) \setminus \varphi(N(v))| - |N_u \setminus N_w| \geq 9 - 3 - 2 - 3 = 1$, we have that $L(v) \setminus \varphi(N(v) \cup N_u \cup N_w)$ is non-empty. By Lemma 3.8 with S as previously defined, φ extends to E_1-acyclic L-colouring of G, a contradiction. We then conclude that the face F is not a triangle.

Since F is not a triangle and $uw \in E_1$, by Lemma 4.4 there exists a vertex $u_F \in N(u) \setminus N_{E_1}(u)$ in the boundary of F. Symmetrically, there exists a vertex $w_F \in N(w) \setminus N_{E_1}(w)$ in the boundary of F. Therefore, $|N_u| \leq 3$ and $|N_w| \leq 2$. Since $|L(v) \setminus \varphi(N(v) \cup N_u \cup N_w)| \geq |L(v)| - |\varphi(N(v))| - |N_u| - |N_w| \geq 9 - 3 - 3 - 2 = 1$, we have that $L(v) \setminus \varphi(N(v) \cup N_u \cup N_w)$ is non-empty. By Lemma 3.8 (with $S := \{u, w\}$), φ extends to an E_1-acyclic L-colouring of G, a contradiction.

□

Recall Definition 4.2, that is, a vertex is *triangular* if it is incident with only 3-faces.

Our final reducible configuration follows.

Lemma 4.9. *G does not contain a triangular 6-vertex whose neighbourhood contains only triangular 6-vertices.*

Proof. Suppose not, and let v be a triangular 6-vertex whose neighbourhood contains only triangular 6-vertices. Let $v_1, \dots, v_6$ be the neighbours of v listed in cyclic order. By Lemma 4.3, every 8^--vertex is incident with at least four E_1-edges and therefore $|\delta(v) \cap E_1| \geq 4$.

First suppose that $\{vv_1, vv_3, vv_5\} \not\subseteq E_1$ and $\{vv_2, vv_4, vv_6\} \not\subseteq E_1$. We may assume that $vv_1 \in E_1$ and $vv_6 \notin E_1$. We conclude that $\{vv_2, vv_4\} \subseteq E_1$ and exactly one of vv_3 and vv_5 is in E_1. Let u be the vertex in $\{v_3, v_5\} \cap N_{E_1}(v)$.

Let H be the graph with vertex-set $N_{E_1}(v)$ and edge-set $\{v_1v_4, v_2v_4\}$. See Figure 4.4 (left) for an illustration of this case. Let G' be the graph with vertex-set $V(G) \setminus \{v\}$ and edge-set $(E(G) \cup E(H)) \setminus \delta(v)$. Let $E_1' := E_1 \setminus \delta(v)$. By Lemma 3.4, G' has an embedding such that $\mathrm{ew}_2(G', E_1') \geq \mathrm{ew}_2(G, E_1)$. Since G is a minimum counterexample to Theorem 2.3 and $v(G') < v(G)$, the graph G' has an E_1'-acyclic L-colouring. Let φ be an E_1'-acyclic L-colouring of G'.

First suppose that $|\varphi(\{v_1, v_2, v_4, u\})| = 4$. Since $|L(v) \setminus \varphi(N(v))| \geq |L(v)| - |N(v)| = 9 - 6 = 3$, we have that $L(v) \setminus \varphi(N(v))$ is non-empty. By Lemma 3.8 with $S := \emptyset$, we have that φ extends to an E_1-acyclic L-colouring of G, a contradiction. Thus $|\varphi(\{v_1, v_2, v_4, u\})| \leq 3$. Since v is a triangular vertex, we have $v_1v_2 \in E(G)$. As well, since $\{v_1v_2, v_1v_4, v_2v_4\} \subseteq E(G')$, we have that $\{v_1, v_2, v_4\}$ induces a triangle in G' and therefore $|\varphi(\{v_1, v_2, v_4, u\})| = 3$.

Since v is a triangular vertex, $\{v_3v_4, v_4v_5\} \subseteq E(G)$. Since $\{v_1v_2, v_1v_4, v_2v_4\} \subseteq E(G')$, it follows that $\varphi(u)$ and exactly one of $\varphi(v_1)$ and $\varphi(v_2)$ are equal. Let $S := \{u\}$. Since $|\varphi(\{v_1, v_2, v_4, u\})| \leq 3$, we have that $|\varphi(N(v))| \leq 5$. Let $N_u := N_{E_1}(u) \setminus (N(v) \cup \{v\})$. Since u is a triangular 6-vertex by assumption, $|\varphi(N_u)| \leq |N_u| \leq 3$. Since $|L(v) \setminus \varphi(N(v) \cup N_u)| \geq |L(v)| - |\varphi(N(v))| - |\varphi(N_u)| \geq 9 - 5 - 3 = 1$, the set $L(v) \setminus \varphi(N(v) \cup N_u)$ is non-empty. By Lemma 3.8 with S as previously defined, we have that φ extends to an E_1-acyclic L-colouring of G, a contradiction.

Thus we may assume without loss of generality that $\{vv_1, vv_3, vv_5\} \subseteq E_1$. See Figure 4.4 (right) for an illustration of this case. Let H be the graph with vertex-set $N_{E_1}(v)$ and edge-set $\{v_1v_3, v_1v_5, v_3v_5\}$. Let G' be the graph with vertex-set $V(G) \setminus \{v\}$ and edge-set $(E(G) \cup E(H)) \setminus \delta(v)$. Let $E_1' := E_1 \setminus \delta(v)$. By Lemma 3.4, G' has an embedding such that $\mathrm{ew}_2(G', E_1') \geq \mathrm{ew}_2(G, E_1)$. Since G is a minimum counterexample to Theorem 2.3 and $v(G') < v(G)$, the graph G' has an E_1'-acyclic L-colouring. Let φ be an E_1'-acyclic L-colouring which maximizes $|\varphi(N(v))|$.

First suppose that $|\varphi(N(v))| = 6$. Since $|L(v) \setminus \varphi(N(v))| \geq |L(v)| - |\varphi(N(v))| = 9 - 6 = 3$, we have that $L(v) \setminus \varphi(N(v))$ is non-empty. By Lemma 3.8 with $S := \emptyset$, we have that φ extends to an E_1-acyclic L-colouring of G, a contradiction. Thus we may assume that $|\varphi(N(v))| \leq 5$.

Suppose that $|\varphi(N(v))| = 5$. Then there are exactly two vertices $\{u, w\} \subseteq N(v)$ such that $\varphi(u) = \varphi(w)$. Let $S := \{u\}$, and let $N_u := N_{E_1}(u) \setminus (N(v) \cup \{v\})$. Since u is a triangular 6-vertex, $|\varphi(N_u)| \leq |N_u| \leq 3$. Since $|L(v) \setminus \varphi(N(v) \cup N_u)| \geq |L(v)| - |\varphi(N(v))| - |\varphi(N_u)| = 9 - 5 - 3 = 1$, the set $L(v) \setminus \varphi(N(v) \cup N_u)$ is non-empty. By Lemma 3.8 with S as previously defined, we have that φ extends to an E_1-acyclic L-colouring of G, a contradiction. Thus we may assume that $|\varphi(N(v))| \leq 4$.

For the remainder of the proof we use the notation N_i to denote $N_{E_1}(v_i) \setminus (N(v) \cup \{v\})$ for $i \in [6]$. Next suppose that $|\varphi(N(v))| = 4$. Since $\{v_1v_3, v_1v_5, v_3v_5\} \subseteq E(G')$, there exists a vertex $u \in \{v_2, v_4, v_6\}$ such that $|\varphi(\{v_1, v_3, v_5, u\})| = 4$. Up to relabelling the vertices, we may assume that $u = v_2$. Let $S := \{v_4, v_6\}$.

Recall that v_4 is a triangular 6-vertex by assumption. First suppose that$|\varphi(N_4) \setminus \varphi(N(v))| = 3$. Since $|L(v_4) \setminus (\varphi(N(v) \cup N_4))| \geq |L(v_4)| - |\varphi(N(v))| - |\varphi(N_4) \setminus \varphi(N(v))| \geq 9 - 4 - 3 = 2$, the set $L(v_4) \setminus \varphi(N(v) \cup N_4)$ is non-empty. By Lemma 3.9 we obtain a colouring φ' of G' that agrees with φ everywhere except for v_4, and where $\varphi'(v_4) \notin \varphi(N(v))$. Since $|\varphi'(N_{E_1}(v))| = 5 > |\varphi(N_{E_1}(v))|$, we have that φ' contradicts our choice of φ. Therefore we conclude that $|\varphi(N_4) \setminus \varphi(N(v))| \leq 2$. A symmetrical argument shows that $|\varphi(N_6) \setminus \varphi(N(v))| \leq 2$.

Since $|L(v) \setminus (\varphi(N(v) \cup N_4 \cup N_6))| \geq |L(v)| - |\varphi(N(v))| - |\varphi(N_4) \setminus \varphi(N(v))| - |\varphi(N_6) \setminus \varphi(N(v))| \geq 9 - 4 - 2 - 2 = 1$, we have that $L(v) \setminus \varphi(N(v) \cup N_4 \cup N_6)$ is non-empty. By Lemma 3.8 (with $S := \{v_4, v_6\}$), we have that φ extends to an E_1-acyclic L-colouring of G, a contradiction. We conclude that $|\varphi(N(v))| = 3$.

Since $\{v_1v_3, v_1v_5, v_3v_5\} \subseteq E(G')$, we have $\varphi(v_i) = \varphi(v_{i+3})$ for $i \in \{1, 2, 3\}$. Let $S := \{v_4, v_5, v_6\}$. As before, using Lemma 3.9 we conclude that for $i \in \{4, 6\}$ we have that $|\varphi(N_i) \setminus \varphi(N(v))| \leq 2$.

Since v_5 is a triangular vertex, $\{v_4v_5, v_5v_6\} \subseteq E(G)$ and so $|N_5 \setminus (N_4 \cup N_6)| \leq 1$. Thus $|\varphi(N(v) \cup N_4 \cup N_5 \cup N_6)| \leq |\varphi(N(v))| + |\varphi(N_4) \setminus \varphi(N(v))| + |\varphi(N_6) \setminus \varphi(N(v))| + |N_5 \setminus (N_4 \cup N_6)| \leq 3 + 2 + 2 + 1 = 8$. Since $|L(v)| = 9$, we have that $L(v) \setminus (\varphi(N(v) \cup N_4 \cup N_5 \cup N_6))$ is non-empty. By Lemma 3.8 (with $S := \{v_4, v_5, v_6\}$), we have that φ extends to an E_1-acyclic L-colouring of G, a contradiction.

□

4.2 Discharging

In this section, we prove our discharging lemma, which is used to show that every locally planar embedded graph contains at least one member of a set of configurations (listed in Lemma 4.10). As shown in Subsection 4.1, each of these configurations is reducible, and therefore does not appear in a minimum counterexample to Theorem 2.3.

Lemma 4.10. *Let ε be a real number with $0 < \varepsilon \leq \frac{1}{43}$. If G is a graph embedded in a surface of genus g with $v(G) > \frac{6(g-2)}{\varepsilon}$, then G contains at least one of the following configurations:*

(i) *a 3^- vertex,*

(ii) *a 4-vertex adjacent to an 8^--vertex,*

(iii) *a 5-vertex adjacent to both a 6^--vertex v_1 and a 7^--vertex v_2 distinct from v_1, or*

(iv) *a triangular 6-vertex adjacent only to triangular 6-vertices.*

Proof. Suppose not. We assign an initial charge of $ch_0(v) = \deg(v) - 6$ to each $v \in V(G)$, and $ch_0(f) = 2(\deg(f) - 3)$ to each face f in the set F of faces of the embedding of G. Using Euler's formula for graphs embedded in surfaces,

$$\begin{aligned}\sum_{v \in V(G)} ch_0(v) + \sum_{f \in F} ch_0(f) &= 6(e(G) - v(G) - |F|) \\ &\leq 6(g-2). \qquad (4.2.1)\end{aligned}$$

We discharge via three rules. For $i \in [3]$, we denote by ch_i the charge after applying rule Ri. We aim to show that $ch_3(v) \geq \varepsilon$ and $ch_3(f) \geq 0$ for each vertex $v \in V(G)$ and face $f \in F$. This will imply that the sum of all charges is at least $\varepsilon \cdot v(G)$; and since $v(G) > \frac{6(g-2)}{\varepsilon}$ by assumption, this will imply further that the sum of all charges is greater than $6(g-2)$, contradicting Equation 4.2.1. Note that $ch_0(f) \geq 0$ for each face $f \in F$, and $ch_0(v) \geq \varepsilon$ for each 7^+-vertex. Our discharging rules (which follow below) therefore revolve around sending extra charge to 6^--vertices.

Discharging Rules

R1: Every 4^+-face f splits its charge $ch_0(f)$ evenly between its incident 6^--vertices.

R2: Every 7^+-vertex v sends $\frac{ch_0(v)-\varepsilon}{\deg(v)}$ charge to each of its neighbouring vertices. Charge sent via this rule to 7^+-vertices is instead redirected: $\frac{1}{2}\left(\frac{ch_0(v)-\varepsilon}{\deg(v)}\right)$ is sent to each of the next 6^--vertex clockwise and anticlockwise from u in the cyclic ordering of $N(v)$.

R3: Every 6-vertex v that is incident with a 4^+-face or adjacent to a 7^+-vertex splits $(ch_2(v)-\varepsilon)$ evenly between its neighbouring triangular 6-vertices that are not adjacent to a 7^+-vertex.

In R2, if charge sent from u to w is instead redirected to v, we say u *sends* v *charge redirected from* w.

We state a useful observation regarding R2.

Observation 4.11. *Let $v \in V(G)$, and let u and w be consecutive 7^+-vertices in the cyclic order of $N(v)$. Let $c_u, c_w > 0$ be the amount of charge sent directly (not via redirection) to v from u and w, respectively. If there exists a triangular face incident with both vu and vw, then u sends v at least $\frac{c_u}{2}$ charge redirected from w, and w sends v at least $\frac{c_w}{2}$ charge redirected from u.*

Note that if vu and vw are incident with the same triangular face, then uw is an edge.

First note that since $ch_0(f) \geq 0$ for each face $f \in F$ and faces only send charge via R1, it follows from R1 that $ch_3(f) \geq 0$ for each face f. Next, we claim every 7^+-vertex $v \in V(G)$ has $ch_3(v) \geq \varepsilon$: this follows easily from the facts that $ch_0(v) = \deg(v) - 6$, and v sends charge to at most $\deg(v)$ neighbours via R2. Hence $ch_3(v) \geq (\deg(v) - 6) - \deg(v) \cdot \left(\frac{ch_0(v)-\varepsilon}{\deg(v)}\right) = \varepsilon$, as desired.

To show all vertices in G have final charge at least ε, it remains only to consider the 6^--vertices. Since G contains no 3^--vertex by assumption, we limit our attentions to the 4-, 5-, and 6-vertices. For the remainder of the analysis, we break into cases.

Case 1: v is a 4-vertex. Since v is a 4-vertex, $ch_0(v) = -2$. By assumption, v is not adjacent to an 8^--vertex, and hence all neighbours of v are 9^+-vertices. Note that if f is a 4^+-face incident with v, then two of the 9^+-neighbours of v are in the boundary walk of f, and hence we have that f sends v at least $\frac{ch_0(f)}{\deg(f)-2} = \frac{2(\deg(f)-3)}{\deg(f)-2}$ via R1. This is at least 1, since $\deg(f) \geq 4$. We break into further cases depending on the number of 4^+-faces incident with v.

First suppose v is incident with at least two 4^+-faces. Each incident 4^+-face sends v at least 1 charge via R1, and each neighbouring 9^+-vertex sends at least $\frac{3-\varepsilon}{9}$ charge via R2, and hence $ch_3(v) \geq ch_0(v) + 2(1) + 4\left(\frac{3-\varepsilon}{9}\right) = 4\left(\frac{3-\varepsilon}{9}\right) > \varepsilon$, where the last inequality follows since $\varepsilon < \frac{12}{13}$.

Next suppose v is incident with exactly one 4^+-face, and therefore is incident with three triangles. The incident 4^+-face sends v at least 1 charge via R1. By Observation 4.11 we conclude that $ch_3(v) \geq ch_0(v) + 1 + 4\left(\frac{3-\varepsilon}{9}\right) + 3\left(\frac{1}{2} \cdot \frac{3-\varepsilon}{9} + \frac{1}{2} \cdot \frac{3-\varepsilon}{9}\right) = -1 + 7\left(\frac{3-\varepsilon}{9}\right) > \varepsilon$, where the last inequality follows since $\varepsilon < \frac{3}{4}$.

Thus we may assume v is a triangular vertex. By Observation 4.11 we conclude that $ch_3(v) \geq ch_0(v) + 4\left(\frac{3-\varepsilon}{9}\right) + 4\left(\frac{1}{2} \cdot \frac{3-\varepsilon}{9} + \frac{1}{2} \cdot \frac{3-\varepsilon}{9}\right) = -2 + 8\left(\frac{3-\varepsilon}{9}\right) > \varepsilon$ where the last inequality follows since $\varepsilon < \frac{6}{17}$.

Case 2: v is a 5-vertex. Since v is a 5-vertex, $ch_0(v) = -1$. By assumption, v is not adjacent to a 7^--vertex u and a 6^- vertex distinct from u, and hence v is adjacent to at most one 6^--vertex. Moreover, if v is adjacent to exactly one 6^--vertex, all other neighbours of v are 8^+-vertices. Since each face incident with v is also incident with at least one 7^+-vertex, each 4^+-face f incident with v sends v at least $\frac{ch_0(f)}{\deg(f)-1} = \frac{2(\deg(f)-3)}{\deg(f)-1} \geq \frac{2}{3}$ charge via R1. Hence if v is incident with at least two 4^+-faces, then $ch_2(v) \geq -1 + 2 \cdot \frac{2}{3} > \varepsilon$ since $\varepsilon < \frac{1}{3}$. Thus we may assume v is incident with at most one 4^+-face.

First suppose that v is incident with exactly one 4^+-face f (and hence exactly four triangles). Then v receives $\frac{2}{3}$ charge from f via R1. If v is adjacent to a 6^--vertex, then since the remaining neighbours of v are 8^+-vertices, at least two faces satisfy the premises of Observation 4.11 and so $ch_3(v) \geq -1 + \frac{2}{3} + 4\left(\frac{2-\varepsilon}{8}\right) + 2\left(\frac{1}{2} \cdot \frac{2-\varepsilon}{8} + \frac{1}{2} \cdot \frac{2-\varepsilon}{8}\right) = -\frac{1}{3} + 6\left(\frac{2-\varepsilon}{8}\right)$. Note this is greater than ε since $\varepsilon < \frac{2}{3}$. Meanwhile if every vertex in $N(v)$ is a 7^+-vertex, all four triangular faces satisfy the premises of Observation 4.11, and therefore we have $ch_3(v) \geq -1 + \frac{2}{3} + 5\left(\frac{1-\varepsilon}{7}\right) + 4\left(\frac{1}{2} \cdot \frac{1-\varepsilon}{7} + \frac{1}{2} \cdot \frac{1-\varepsilon}{7}\right) = -\frac{1}{3} + 9\left(\frac{1-\varepsilon}{7}\right) > \varepsilon$ since $\varepsilon < \frac{5}{12}$.

Lastly suppose that v is a triangular vertex. If v is adjacent to a 6^--vertex, then since it is adjacent to at most one 6^--vertex by assumption, exactly three faces satisfy the premises of Observation 4.11 and so $ch_3(v) \geq -1 + 4\left(\frac{2-\varepsilon}{8}\right) + 3\left(\frac{1}{2} \cdot \frac{2-\varepsilon}{8} + \frac{1}{2} \cdot \frac{2-\varepsilon}{8}\right) = -1 + 7\left(\frac{2-\varepsilon}{8}\right)$. Note this is greater than ε since $\varepsilon < \frac{2}{5}$. Meanwhile if every vertex in $N(v)$ is a 7^+-vertex, then every face incident with v satisfies Observation 4.11 and therefore we have $ch_3(v) \geq -1 + 5\left(\frac{1-\varepsilon}{7}\right) + 5\left(\frac{1}{2} \cdot \frac{1-\varepsilon}{7} + \frac{1}{2} \cdot \frac{1-\varepsilon}{7}\right) = -1 + 10\left(\frac{1-\varepsilon}{7}\right) > \varepsilon$ since $\varepsilon < \frac{3}{17}$.

Case 3: v is a 6-vertex. If v receives charge via R1, then it is incident with a 4^+-face, and hence v receives at least $\frac{2(\deg(f)-3)}{\deg(f)} \geq \frac{2(4-3)}{4} = \frac{1}{2} > \varepsilon$ charge via R1. Next, v gives $ch_2(v) - \varepsilon$ to its neighbours via R3, leaving ε charge for v, as desired. We may assume then that v does not receive charge via R1, and hence v is triangular.

Suppose then that v receives charge via R2. Then v is adjacent to a 7^+-vertex, and hence v receives at least $\frac{1-\varepsilon}{7} > \varepsilon$ charge via R2. Note this is greater than ε since $\varepsilon < \frac{1}{8}$. Once again, v gives $ch_2(v) - \varepsilon$ to its neighbours via R3, leaving ε charge for v, as desired. We may assume then that v does not receive charge via R2, and hence is not adjacent to any 7^+-vertex.

Suppose then that v receives charge via R3. Then it is adjacent to a 6-vertex w which is adjacent to a 7^+-vertex or incident with a 4^+-face. Hence w is a 6-vertex that received charge via either R1 or R2, and so as argued above, we see that $ch_2(w) \geq \frac{1-\varepsilon}{7}$ if w is adjacent to a 7^+-vertex or $ch_2(w) \geq \frac{1}{2}$ if w is incident with a 4^+-face. In the former case, since w is adjacent to at least one 7^+-vertex, it sends charge to at most five neighbours via R3. Therefore v receives at least $\frac{1}{5}\left(\frac{1-\varepsilon}{7} - \varepsilon\right) \geq \varepsilon$ from w via R3, where the last inequality follows since $\varepsilon \leq \frac{1}{43}$. Meanwhile, in the latter case we have that v receives $\frac{1}{6}\left(\frac{1}{2} - \varepsilon\right) > \varepsilon$ from w, where the last inequality follows since $\varepsilon < \frac{1}{14}$.

Thus we may assume v is adjacent to only triangular 6-vertices and 5^--vertices. Since G does not contain a 4-vertex adjacent to an 8^--vertex, all vertices in $N(v)$ are 5-vertices or triangular 6-vertices. Since G does not contain a 5-vertex adjacent to two 6^--vertices, it follows further that v is a triangular 6-vertex adjacent only to triangular 6-vertices. This is a contradiction.

□

We conclude this chapter with the proof of Theorem 2.3 which almost immediately follows from applying the previously shown lemmas.

Proof of Theorem 2.3. Suppose not, and let G be a counterexample that minimizes $\sum_{v \in V(G)} |L(v)|$ and subject to that, maximizes $e(G)$. If G does not contain a non-contractible cycle, then G is planar and hence G is acyclically 9-list colourable by Theorem 1.2. Thus G contains a non-contractible cycle, and hence $v(G) > \rho = \frac{6(g-2)}{\varepsilon}$, where $\varepsilon := \frac{1}{43}$.

By Lemma 4.5, G does not contain a 3^--vertex. By Lemma 4.6, G does not contain a 4-vertex adjacent to an 8^--vertex. By Lemma 4.8, G does not contain 5-vertex adjacent to both a 6^--vertex v_1 and a 7^--vertex v_2 distinct from v_1. Finally, by Lemma 4.9, G does not contain a 6-vertex incident with only 3-faces adjacent only to 6-vertices that are themselves only incident with 3-faces. Yet by Lemma 4.10, if Σ is a surface of genus g and G is a graph embedded in Σ where $v(G) > \frac{6(g-2)}{\varepsilon}$, then G contains one of the configurations listed above. This is a contradiction, and hence a counterexample to Theorem 2.3 does not exist. □

Chapter 5

Proof of Theorem 2.4

This chapter is structured very similarly to the previous chapter. In Section 5.1 we present a series of reductions, in which we claim that a minimum counterexample to Theorem 2.4 does not contain certain substructures. We then present the discharging portion of the proof in Section 5.2, where we show that a minimum counterexample contains at least one of these substructures, and hence arrive at a contradiction, disproving the existence of a counterexample.

Since we will be considering both the degrees of a vertex v in both the graph with edge set E_1 and the edge set $E := E(G)$, we use the following notation:

Definition 5.1. Given positive integers k, ℓ such that $k \leq \ell$, we refer to a vertex v with $\deg_{E_1}(v) = k$ and $\deg_E(v) = \ell$ as a (k, ℓ)-vertex. If $\deg_{E_1}(v) \leq k$ and $\deg_E(v) \leq \ell$, or $\deg_{E_1}(v) \geq k$ and $\deg_E(v) \geq \ell$, we write that v is a (k^-, ℓ^-)-vertex or (k^+, ℓ^+)-vertex respectively. Furthermore, in the cases where we have no restriction on ℓ, we write that v is a $(k^-, \cdot)$-vertex or a $(k^+, \cdot)$-vertex.

Analogously, given an embedded graph G, we refer to a face with boundary of length k, at most k, and at least k as a k-face, a k^--face, and a k^+-face, respectively.

5.1 Reducible Configurations

In what follows in this chapter, a *counterexample* is a graph, edge-set, and list assignment triple (G, E_1, L) that satisfy the hypotheses but not the conclusion of Theorem 2.4. A *minimum* counterexample is a counterexample chosen to minimize $\sum_{v \in V(G)} |L(v)|$, and subject to that, to maximize $e(G)$. In particular, this minimizes $v(G)$ and makes $|L(v)| = 8$ for all $v \in V(G)$. Let (G, E_1, L) be a minimum counterexample, in all of the following lemmas in this section, G is assumed to be this minimum counterexample.

Lemma 5.2. *G does not contain a $(3^-, 7^-)$-vertex.*

Proof. Suppose not: that is, there exists a $(4, 4+k)$-vertex v adjacent to a $((5-k)^-, \cdot)$-vertex u. Let u, v_1, v_2, v_3 be the E_1-neighbours of v listed in cyclic order. Let $N_u := N_{E_1}(u) \setminus (N(v) \cup \{v\})$; note that since $\deg_{E_1}(u) \leq 5-k$, we have that $|N_u| \leq 4-k$.

Let H be the graph with vertex-set $N_{E_1}(v)$ and edge-set $\{v_1v_3\} \cup \{uv_1, v_1v_2, v_2v_3, v_3u\}$. Let G' be the graph with vertex-set $V(G) \setminus \{v\}$ and edge-set $(E(G) \cup E(H)) \setminus \delta(v)$. Let $E_1' := E_1 \setminus \delta(v)$. By Lemma 3.4, G' has an embedding such that $\mathrm{ew}_2(G', E_1') \geq \mathrm{ew}_2(G, E_1)$. Since G is a minimum counterexample to Theorem 2.4 and $v(G') < v(G)$, the graph G' has an E_1'-acyclic L-colouring φ. □

Similar to the previous chapter, as an easy consequence of Lemma 3.1 and the edge-maximality of G, we have the following.

Lemma 5.3. *G does not contain vertices u, v, w where $\{uv, vw\} \subset E_1$, where uv and vw are cofacial, and where $uw \notin E \setminus E_1$.*

Proof. Let $\{uv, vw\} \subset E_1$ and suppose by way of contradiction that $uw \notin E \setminus E_1$. First note that $uw \notin E_1$, since then u, v, w forms a triangle, but $G[E_1]$ is triangle-free. Therefore we may assume that $uw \notin E(G)$. By Lemma 3.1 the graph G' obtained from G by adding the non E_1-edge uw to G satisfies $\mathrm{ew}_2(G', E_1) \geq \mathrm{ew}_2(G, E_1)$. This contradicts the fact that G is edge-maximal.

□

Lemma 5.4. *G does not contain a $(4, 4+k)$-vertex v adjacent to a $((5-k)^-, \cdot)$-vertex u, for $k \in \{0, 1, 2\}$.*

Proof. Suppose not: that is, there exists a $(4, 4+k)$-vertex v adjacent to a $((5-k)^-, \cdot)$-vertex u. Let u, v_1, v_2, v_3 be the E_1-neighbours of v listed in cyclic order. Let $N_u := N_{E_1}(u) \setminus (N(v) \cup \{v\})$; note that since $\deg_{E_1}(u) \leq 5-k$, we have that $|N_u| \leq 4-k$.

Let H be the graph with vertex-set $N_{E_1}(v)$ and edge-set $\{v_1v_3\} \cup \{uv_1, v_1v_2, v_2v_3, v_3u\}$. Let G' be the graph with vertex-set $V(G) \setminus \{v\}$ and edge-set $(E(G) \cup E(H)) \setminus \delta(v)$. Let $E_1' := E_1 \setminus \delta(v)$. By Lemma 3.4, G' has an embedding such that $\mathrm{ew}_2(G', E_1') \geq \mathrm{ew}_2(G, E_1)$.

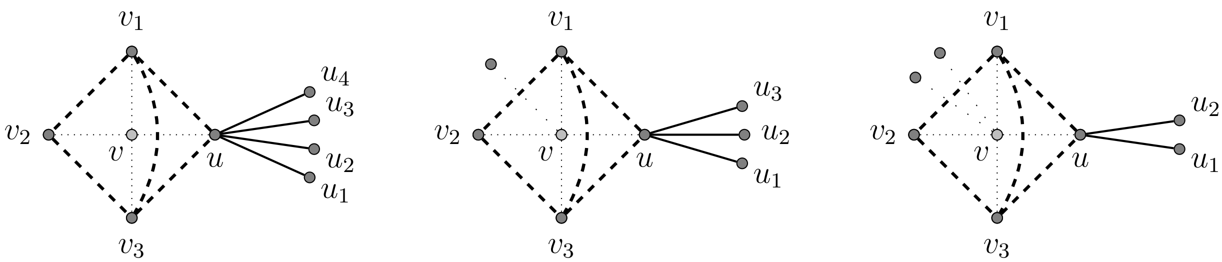

Figure 5.1: The structure of the graphs in Lemma 5.4. A $(4, 4+k)$-vertex v is adjacent to a $((5-k)^-, \cdot)$-vertex u. The set N_u is labelled as $\{u_1, \dots, u_{k-1}\}$. From left to right, the cases shown are when $k = 0$, $k = 1$ and $k = 2$.

Since G is a minimum counterexample to Theorem 2.4 and $v(G') < v(G)$, the graph G' has an E_1'-acyclic L-colouring φ.

First suppose that $|\varphi(N_{E_1}(v))| = 4$. Since $|L(v) \setminus \varphi(N(v))| \geq |L(v)| - |\varphi(N(v))| \geq 8 - (4+k) \geq 2$, $L(v) \setminus \varphi(N(v))$ is non-empty, by Lemma 3.8, with $S := \emptyset$ we have that φ is an E_1-acyclic L-colouring of G, contradicting that G is a counterexample. We thus assume that $|\varphi(N_{E_1}(v))| \leq 3$. Note that since v_1, v_2, v_3 form a triangle, that in fact $|\varphi(N_{E_1}(v))| = 3$.

Let $S := \{u\}$. Since $|\varphi(N_{E_1}(v))| = 3$ and $|N_u| \leq 4 - k$, we have that $|L(v) \setminus \varphi(N(v) \cup N_u)| \geq |L(v)| - |\varphi(N(v))| - |\varphi(N_u)| \geq 8 - (3+k) - (4-k) = 1$. Therefore $L(v) \setminus \varphi(N(v) \cup N_u)$ is non-empty. By Lemma 3.8 with S as previously defined, we have that φ extends to an E_1-acyclic L-colouring of G, a contradiction. $\square$

Lemma 5.5. *G does not have a $(4, 6)$-vertex v adjacent to either*

- *a $(4^-, 5^-)$-vertex u_1, or*
- *a $(4^-, 6^-)$-vertex u_2, where, in the consecutive ordering of $N_{E_1}(v)$, there exists a vertex w either immediately before or after u_2 such that $u_2 w \in E \setminus E_1$.*

Proof. Suppose not: that is, there exists a $(4, 6)$-vertex v adjacent to a vertex u_i with $i \in \{1, 2\}$. As denoted in the statement of the lemma, u_1 is a $(4^-, 5^-)$-vertex and u_2 is a $(4^-, 6^-)$-vertex, where, in the consecutive ordering of $N_{E_1}(v)$, there exists a vertex w either immediately before or after u_2 such that $u_2 w \in E \setminus E_1$. Let u_i, v_1, v_2, v_3 be the neighbours of v listed in cyclic order. Note that if $i = 2$, then w is either v_1 or v_3. Let $N_{u_i} := N_{E_1}(u_i) \setminus (N(v) \cup \{v\})$; note that since $\deg_{E_1}(u_i) \leq 4$, we have that $|N_{u_i}| \leq 3$.

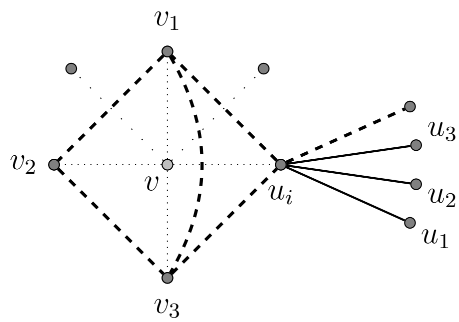

Figure 5.2: The structure of the graphs in Lemma 5.5. A $(4,6)$-vertex v is adjacent to a $(4^-,5^-)$-vertex u_1 or a $(4^-,6^-)$-vertex u_2, where, in the consecutive ordering of $N_{E_1}(v)$, there exists a vertex w either immediately before or after u_2 such that $u_2w \in E \setminus E_1$. In the case presented above, if $i = 2$ then $w = v_3$. The set N_u is labelled as $\{u_1, u_2, u_3\}$.

Let H be the graph with vertex-set $N_{E_1}(v)$ and edge-set $\{v_1v_3\} \cup \{uv_1, v_1v_2, v_2v_3, v_3u\}$. Let G' be the graph with vertex-set $V(G) \setminus \{v\}$ and edge-set $(E(G) \cup E(H)) \setminus \delta(v)$. Let $E_1' := E_1 \setminus \delta(v)$. By Lemma 3.4, G' has an embedding such that $\mathrm{ew}_2(G', E_1') \geq \mathrm{ew}_2(G, E_1)$. Since G is a minimum counterexample to Theorem 2.4 and $v(G') < v(G)$, the graph G' has an E_1'-acyclic L-colouring φ. Let φ be a colouring chosen so that $|\varphi(N_{E_1}(v))|$ is minimized.

First suppose that $|\varphi(N_{E_1}(v))| = 4$. Since $|L(v) \setminus \varphi(N(v))| \geq |L(v)| - |\varphi(N(v))| \geq 8 - 6 \geq 2$, $L(v) \setminus \varphi(N(v))$ is non-empty, by Lemma 3.8, with $S := \emptyset$ we have that φ is an E_1-acyclic L-colouring of G, contradicting that G is a counterexample. We thus assume that $|\varphi(N_{E_1}(v))| \leq 3$. Note that since v_1, v_2, v_3 form a triangle, that in fact $|\varphi(N_{E_1}(v))| = 3$.

Let $S = \{u_i\}$. First suppose that $|\varphi(N_{u_i}) \setminus \varphi(N(v))| = 3$, and therefore all vertices in $N_{E_1}(u_i)$ are given distinct colours. Now since $w \in N_{E_1}(v)$ we have that for both $i = 1$ and $i = 2$, $|L(u_1) \setminus \varphi(N_{E_1}(v) \cup (N(u_1) \setminus \{v\}))| = |L(u_1)| - |\varphi(N_{E_1}(v))| - |\varphi(N(u_1) \setminus (N(v) \cup \{v\}))| \geq 8 - 3 - 4 = 1$. Therefore we have that $L(u_i) \setminus \varphi(N_{E_1}(v) \cup N(u) \setminus \{v\})$ is non-empty. By Lemma 3.8 we obtain a colouring φ' of G' that agrees with φ everywhere except for u_i, and where $\varphi'(u_i) \notin \varphi(N_{E_1}(v))$. Since $|\varphi'(N_{E_1}(v))| > 3 = |\varphi(N_{E_1}(v))|$, we have that φ' contradicts our choice of φ. Therefore we conclude that $|\varphi(N_{u_i}) \setminus \varphi(N(v))| \leq 2$.

Since $|\varphi(N_{E_1}(v))| = 3$ and $|\varphi(N_{u_i}) \setminus \varphi(N(v))| \leq 2$, we have that $|L(v) \setminus \varphi(N(v) \cup N_{u_i})| \geq |L(v)| - |\varphi(N(v))| - |\varphi(N_{u_i}) \setminus \varphi(N(v))| \geq 8 - 5 - 2 = 1$. Therefore $L(v) \setminus \varphi(N(v) \cup N_{u_i})$ is non-empty. By Lemma 3.8 (with $S = \{u_i\}$), we have that φ extends to an E_1-acyclic L-colouring of G, a contradiction. $\square$

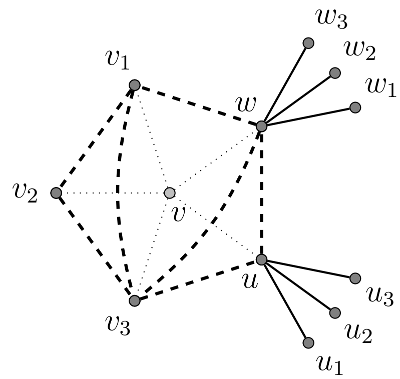

Figure 5.3: The structure of the graph in Lemma 5.6. A $(5,5)$-vertex v is adjacent to two consecutive $(4,6)$-vertices u and w. The sets N_u and N_w are labelled as $\{u_1, u_2, u_3\}$ and $\{w_1, w_2, w_3\}$, respectively.

Lemma 5.6. *G does not have a $(5,5)$-vertex v adjacent to two $(4^-, 6^-)$-vertices u and w, which are consecutive in the cyclic ordering of $N(v)$.*

Proof. Suppose not: that is, there exists a $(5,5)$-vertex v adjacent to two $(4^-, 6^-)$-vertices u and w, which are consecutive in the cyclic ordering of $N(v)$. Let u, w, v_1, v_2, v_3 be the neighbours of v listed in cyclic order. Since G is chosen to maximize $e(G)$ and v has no non-E_1-edges incident with it, by Lemma 5.3, we have that any two consecutive vertices in the cyclic ordering of $N(v)$ are adjacent via a non-E_1-edge. Therefore we have that both u and v are incident with at least two non-E_1-edges. By Lemma 5.2, we have that $\deg_{E_1}(u) = \deg_{E_1}(w) = 4$, and therefore we conclude that both u and v are exactly $(4,6)$-vertices. Let $N_u := N_{E_1}(u) \setminus (N(v) \cup \{v\})$ and $N_w := N_{E_1}(w) \setminus (N(v) \cup \{v\})$. Since $\deg_{E_1}(u) = \deg_{E_1}(w) = 4$, we have that $|N_u| \leq 3$ and $|N_w| \leq 3$.

Let H be the graph with vertex-set $N(v)$ and edge-set $\{v_1v_3, wv_3\}$. Let G' be the graph with vertex-set $V(G) \setminus \{v\}$ and edge-set $(E(G) \cup E(H)) \setminus \delta(v)$. Let $E_1' := E_1 \setminus \delta(v)$. By Lemma 3.4, G' has an embedding such that $\mathrm{ew}_2(G', E_1') \geq \mathrm{ew}_2(G, E_1)$. Since G is a minimum counterexample to Theorem 2.4 and $v(G') < v(G)$, the graph G' has an E_1'-acyclic L-colouring φ. Let φ be a colouring chosen so that $|\varphi(N(v))|$ is minimized.

First suppose that $|\varphi(N(v))| = 5$. Since $|L(v) \setminus \varphi(N(v))| \geq |L(v)| - |\varphi(N(v))| = 8 - 5 = 3$, we have that $L(v) \setminus \varphi(v)$ is non-empty. Therefore by Lemma 3.8 with $S := \emptyset$ we have that φ is an E_1-acyclic L-colouring of G, contradicting that G is a counterexample. We thus assume that $|\varphi(N(v))| \leq 4$.

Now suppose that $|\varphi(N(v))| = 4$. Since $|N(v)| = 5$, there exist exactly one pair of vertices in $N(v)$ with the same colour. Since $G'[N(v) \setminus \{u, w\}]$ is isomorphic to K_3, at least one of u or w is in this pair. Let $S := \{z\}$ where $z \in \{u, w\}$ is said vertex. We next argue that there exists a colour for v avoiding both $\varphi(N(v))$ and $\varphi(N_{E_1}(z))$, which will imply G is E_1-acyclically L-colourable. Since $|L(v) \setminus \varphi(N(v) \cup N_z)| \geq |L(v)| - |\varphi(N(v))| - |N_z| \geq 8 - 4 - 3 = 1$, we have that $L(v) \setminus \varphi(N(v) \cup N_z)$ is non-empty. By Lemma 3.8 (with $S = \{z\}$) we have that φ extends to an E_1-acyclic L-colouring of G, a contradiction. We may therefore conclude that $|\varphi(N(v))| = 3$.

As before, $G'[N(v) \setminus \{u, w\}]$ is isomorphic to K_3 and therefore any pair of vertices in $N(v)$ with the same colour includes at least one of $\{u, w\}$. In particular we have that $\varphi(v_2) = \varphi(u) \neq \varphi(w) = \varphi(v_3)$. Let $S := \{u, w\}$.

Suppose that $|\varphi(N_u) \setminus \varphi(N(v))| = 3$. It follows that all vertices in $N_{E_1}(u)$ receive different colours. Since $|L(u) \setminus \varphi(N(v) \cup N_u)| \geq |L(u)| - |\varphi(N(v))| - |\varphi(N_u) \setminus \varphi(N(v))| \geq 8 - 3 - 3 = 2$, we have that the set $L(u) \setminus \varphi(N(v) \cup N_u)$ is nonempty. By Lemma 3.9 we obtain a colouring φ' of G' that agrees with φ everywhere except for u, and where $\varphi'(u) \notin \varphi(N(v))$. Since $|\varphi'(N_{E_1}(v))| = 4 > |\varphi(N_{E_1}(v))|$, we have that φ' contradicts our choice of φ. We therefore conclude that $|\varphi(N_u) \setminus \varphi(N(v))| \leq 2$. A symmetrical argument shows that $|\varphi(N_w) \setminus \varphi(N(v))| \leq 2$.

Since $|L(v) \setminus \varphi(N(v) \cup N_u \cup N_w)| \geq |L(v)| - |\varphi(N(v))| - |\varphi(N_u) \setminus \varphi(N(v))| - |\varphi(N_w) \setminus \varphi(N(v))| \geq 8 - 3 - 2 - 2 = 1$, we have that $L(v) \setminus \varphi(N(v) \cup N_u \cup N_w)$ is non-empty. By Lemma 3.8 (with $S = \{u, w\}$), we have that φ extends to a E_1-acyclic L-colouring of G, a contradiction.

□

5.2 Discharging

In this section, we prove our discharging lemma, which is used to show that every locally planar embedded graph contains at least one member of a set of configurations (listed in Lemma 5.7). As shown in Subsection 5.1, each of these configurations is reducible, and therefore does not appear in a minimum counterexample to Theorem 2.4.

Differently from Lemma 4.10, we use the fact that $G[E_1]$ is triangle-free and G is not by using different discharging formulas for both of these graphs separately. The formula for G is as it was in Lemma 4.10, however since there are no 3-faces in $G[E_1]$ using the same formula would be inefficient as the formula was chosen to force all faces to have non-negative charge. Using a different formula which assigns negative charge to 3-faces allows us to give positive charge to more vertices in $G[E_1]$. This reduces the number of cases we need to consider.

Let F_E be the set of faces in G and F_{E_1} be the set of faces of $G[E_1]$ with the same embedding as G. In practice, we give faces the amount of charge they would normally receive from the formulas of their respective graphs. However the vertices receive the sum of the two initial charges for both graphs. This gives some vertices a smaller deficit of charge, which would normally have a large deficit. For example, we will find that vertex v with $\deg_E(v) = 5$ would normally have a charge of -1. However if $\deg_{E_1}(v) = 5$ then v also receives 1 charge from the discharging formula on $G[E_1]$, resulting in a net charge of 0.

Our discharging lemma is presented and subsequently proved below.

Lemma 5.7. *Let ε be a real number with $0 < \varepsilon \leq \frac{1}{9}$. If G is a graph embedded in a surface of genus g with $v(G) > \frac{10(g-2)}{\varepsilon}$, then G contains at least one of the following configurations:*

(i) *a $(3^-, 7^-)$-vertex,*

(ii) *a $(4, 4)$-vertex adjacent to a $(5^-, \cdot)$-vertex,*

(iii) *a $(4, 5)$-vertex adjacent to a $(4^-, \cdot)$-vertex, or*

(iv) *a $(4, 6)$-vertex adjacent to a $(3^-, \cdot)$- or a $(4^-, 5^-)$-vertex, or*

(v) *a $(4, 6)$-vertex v adjacent to a $(4^-, 6^-)$-vertex u, where, in the consecutive ordering of $N_{E_1}(v)$, there exists a vertex w either immediately before or after u_2 such that $u_2w \in E \setminus E_1$, or*

(vi) *a $(5, 5)$-vertex v adjacent to two $(4^-, 6^-)$-vertices, which are consecutive in the cyclic ordering of $N(v)$.*

Proof. Suppose not. We assign an initial charge of $ch_0(v) = (\deg_{E_1}(v) - 4) + (\deg_E(v) - 6)$ to each $v \in V(G)$, an initial charge of $ch_0(f) = (\deg_{E_1}(f) - 4)$ for each $f \in F_{E_1}$ and $ch_0(f) = 2(\deg_E(f) - 3)$ for each face $f \in F_E$. Using Euler's formula for graphs embedded in surfaces,

$$\begin{aligned}
\sum_{v\in V(G)} ch_0(v) + &\sum_{f\in F_{E_1}} ch_0(f) + \sum_{f\in F_E} ch_0(f) \\
&= \left(\sum_{v\in V(G)} (\deg_{E_1}(v) - 4) + \sum_{f\in F_{E_1}} ch_0(f)\right) + \left(\sum_{v\in V(G)} (\deg_E(v) - 6) + \sum_{f\in F_{E_1}} ch_0(f)\right) \\
&= 4(|E_1| - v(G) - |F_{E_1}|) + 6(|E| - v(G) - |F_E|) \\
&\leq 10(g-2). \qquad (5.2.1)
\end{aligned}$$

We discharge via two rules. For $i \in [2]$, we denote ch_i the charge after applying rule Ri. We aim to show that $ch_2(v) \geq \varepsilon$ and $ch_2(v) \geq 0$ for each vertex $v \in V(G)$ and face $f \in F_{E_1} \cup F_E$. This will imply that the sum of all charges is at least $\varepsilon \cdot v(G)$; and since $v(G) > \frac{10(g-2)}{\varepsilon}$ by assumption, this will imply further that the sum of all charges is greater than $10(g-2)$, contradicting Equation 5.2.1.

Discharging Rules

R1: Every 5^+-face $f \in F_{E_1}$ splits its charge $ch_0(f)$ evenly between its incident $(k^-, (10-k)^-)$-vertices for $k \in \{2, 3, 4, 5\}$.

R2: Every $(k^+, (11-k)^+)$-vertex v, for $k \in \{2, 3, 4, 5\}$, sends $\frac{ch_0(v)-\varepsilon}{\deg(v)}$ charge to each $(k^-, (10-k)^-)$-vertex in $N_{E_1}(v)$.

First we note a short observation concerning R2:

Observation 5.8. *If a vertex v gives charge to a vertex u via R2, then it sends at least $\frac{1-\varepsilon}{5}$ to u. Furthermore, if v is not a $(5,6)$-vertex, then v sends at least $\frac{1-\varepsilon}{4}$ to u, which is tight if v is a $(4,7)$-vertex.*

First note that since $ch_0(f) \geq 0$ for each face $f \in F_{E_1} \cup F_E$ and faces only send charge via R1, it follows from R1 that $ch_2(f) \geq 0$ for each face f. Next, we claim every $(k^+, (11-k)^+)$-vertex $v \in V(G)$ has $ch_2(v) \geq \varepsilon$: this follows easily from the facts that $ch_0(v) = (\deg_{E_1}(v) - 4) + (\deg_E(v) - 6) \geq 1$, and v sends charge to at most $\deg(v)$ neighbours via R2. Hence $ch_2(v) \geq ch_0(v) - \deg(v) \cdot \left(\frac{ch_0(v)-\varepsilon}{\deg(v)}\right) = \varepsilon$, as desired.

To show all vertices in G have final charge at least ε, it remains only to consider the $(k^-, (10-k)^-)$-vertices. For the remainder of the analysis, we break into cases.

Case 1: v is a $(2,\cdot)$-vertex. First note that G does not contain a $(3^-,7^-)$-vertex, therefore we may assume that v is a $(2,8^+)$-vertex. Since we need only consider $(k^-,(10-k)^-)$-vertices, the only case that remains is when v is exactly a $(2,8)$-vertex.

At least one of the faces f in F_{E_1} which is incident with v contains at least three non-E_1-edges incident with v. This face therefore contains at least six vertices, and therefore $ch_2(v) \geq 0 + \frac{\deg_{E_1}(f)-4}{\deg_{E_1}(f)} \geq \frac{1}{3} > \varepsilon$ since $\varepsilon < \frac{1}{3}$.

Case 2: v is a $(3,\cdot)$-vertex. Again, note that G does not contain a $(3^-,7^-)$-vertex, therefore we may assume that v is a $(3,8^+)$-vertex. However, as we have shown, $(k^+,(11-k)^+)$-vertices have positive charge after discharging is complete.

Case 3: v is a $(4,\cdot)$-vertex. Since we need only consider $(k^-,(10-k)^-)$-vertices, we need only consider when v is a $(4,4)$- $(4,5)$- or $(4,6)$-vertex. We handle each of these cases separately.

If v is a $(4,4)$-vertex, then since G does not contain a $(4,4)$-vertex adjacent to a $(5^-,\cdot)$-vertex, we may assume that each neighbour of v is a $(6^+,\cdot)$-vertex. By Lemma 5.3, each vertex in $N(v)$ is incident to two non-E_1-edges, and therefore each neighbour of v is in fact $(6^+,8^+)$-vertex. From R2 each of these vertices give at least $\frac{4-\varepsilon}{6}$ charge. Therefore, $ch_2(v) \geq -2 + 4\cdot\frac{4-\varepsilon}{6} > \varepsilon$ since $\varepsilon < \frac{2}{5}$.

If v is a $(4,5)$-vertex, then since G does not contain a $(4,5)$-vertex adjacent to a $(4^-,\cdot)$-vertex, we may assume that each neighbour of v is a $(5^+,\cdot)$-vertex. By Lemma 5.3, two vertices in $N_{E_1}(v)$ is incident to two non-E_1-edges and the remaining two are incident to at least one non-E_1-edge each. Therefore two neighbours v are in fact $(5^+,7^+)$-vertices and the other two are $(5^+,6^+)$-vertices. From R2 each of these vertices give at least $\frac{2-\varepsilon}{5}$ and $\frac{1-\varepsilon}{5}$ charge. Therefore, $ch_2(v) \geq -1 + 2\cdot\frac{2-\varepsilon}{5} + 2\cdot\frac{1-\varepsilon}{5} \geq \varepsilon$ since $\varepsilon \leq \frac{1}{9}$.

Finally if v is a $(4,6)$-vertex, then none of the incident faces in F_{E_1} can be 5^+-faces, since otherwise $ch_2(v) \geq \frac{\deg_{E_1}(f)-4}{\deg_{E_1}(f)} \geq \frac{1}{5} > \varepsilon$. Similarly, v is not incident with any 4^+-faces in F_E, since otherwise $ch_3(v) \geq \frac{2(\deg_E(f)-3)}{\deg_E(f)} \geq \frac{1}{2} > \varepsilon$. Therefore we may assume that all faces incident with v in F_{E_1} are 4-faces, and in F_E are 3-faces.

Since G does not contain a $(4,6)$-vertex adjacent to a $(4^-,5^-)$-vertex, we may assume that each neighbour of v is a $(4^+,6^+)$- or a $(5^+,5^+)$-vertex. Since there are only two non-E_1-edges incident with v, by Lemma 5.3, at least two consecutive neighbours v_1, v_2 in the cyclic order of $N_{E_1}(v)$ are adjacent via an non-E_1-edge. Notice that v_1 and v_2 cannot be $(5,5)$-vertices as they are incident to an non-E_1-edge. Also, if either v_1 or v_2 is a $(4,6)$-vertex, say v_1, then we notice that v is a $(4,6)$-vertex adjacent to a $(4^-,6^-)$-vertex v_1, where $v_2 \in N_{E_1}(v) \setminus \{v_1\}$ such that $v_1v_2 \in E \setminus E_1$. However this is exactly one of our configurations which we assumed did not appear, therefore we then conclude that at least one of v_1 and v_2 is a $(k^+,(11-k)^+)$-vertex. Therefore v receives charge via R2. In particular, by Observation 5.8 we have $ch_2(v) \geq \frac{1-\varepsilon}{5} > \varepsilon$ since $\varepsilon < \frac{1}{6}$.

Case 4: v is a $(5,\cdot)$-vertex. Since we need only consider $(k^-,(10-k)^-)$-vertices, we need only consider the case where v is a $(5,5)$-vertex. Notice that since v is not incident to any non-E_1-edge, by Lemma 5.3, each pair of consecutive vertices in the cyclic order of $N_{E_1}(v)$ are adjacent via a non-E_1-edge. Moreover, none of these pairs of vertices can both be $(4^-,6^-)$-vertices, as this is one of the configurations that G does not contain. We can then conclude that at least one of the neighbours of v is a $(4^+,7^+)$-vertex.

Therefore v receives charge from this vertex via R2. By Observation 5.8, the final charge of v is then $ch_2(v) \geq \frac{1-\varepsilon}{4} > \varepsilon$ since $\varepsilon < \frac{1}{5}$.

□

We conclude this chapter with the proof of Theorem 2.4 which almost immediately follows from applying the previously shown lemmas.

Proof of Theorem 2.4. Suppose not, and let G be a counterexample that minimizes $\sum_{v\in V(G)} |L(v)|$ and subject to that, maximizes $e(G)$. If G does not contain a non-contractible cycle, then G is planar and hence G is acyclically 8-list-colourable by Theorem 1.2. Thus G contains a non-contractible cycle, and hence $v(G) > \rho = \frac{10(g-2)}{\varepsilon}$, where $\varepsilon := \frac{1}{9}$.

By Lemma 5.2, G does not contain a $(3^-,7^-)$-vertex. By Lemma 5.4, G does not contain a $(4,4+k)$-vertex adjacent to a $((5-k)^-,\cdot)$-vertex, for $k \in \{0,1,2\}$. By Lemma 5.5, G does not contain a $(4,6)$-vertex v adjacent to a $(4^-,5^-)$-vertex u_1. Additionally, v is not adjacent to a $(4^-,6^-)$-vertex u_2, where, in the consecutive ordering of $N_{E_1}(v)$, there exists a vertex w either immediately before or after u_2 such that $u_2w \in E \setminus E_1$. By Lemma 5.6, G does not contain a $(5,5)$-vertex v adjacent to two $(4^-,6^-)$-vertices which are consecutive in the cyclic ordering of $N(v)$. Yet by Lemma 5.7, if Σ is a surface of genus g and G is embedded in Σ where $v(G) > \frac{10(g-2)}{\varepsilon}$, then G contains one of the configurations listed above. This is a contradiction, and hence a counterexample to Theorem 2.4 does not exist. □

References

[1] H.L. Abbott and Bing Zhou. On small faces in 4-critical graphs. *Ars Combinatoria*, 32, 01 1991.

[2] VA Aksionov and LS Mel'nikov. Essay on the theme: the three-color problem. In *Colloq. Math. Soc. Janos Bolyai*, volume 18, pages 23–34, 1978.

[3] Michael O Albertson and David M Berman. Every planar graph has an acyclic 7-coloring. *Israel Journal of Mathematics*, 28:169–174, 1977.

[4] Noga Alon, Bojan Mohar, and Daniel P Sanders. On acyclic colorings of graphs on surfaces. *Israel Journal of Mathematics*, 94(1):273–283, 1996.

[5] Kenneth Appel and Wolfgang Haken. Every planar map is four colorable. Part I: Discharging. *Illinois Journal of Mathematics*, 21:429–490, 1977.

[6] Kenneth Appel, Wolfgang Haken, and John Koch. Every planar map is four colorable. Part II: Reducibility. *Illinois Journal of Mathematics*, 21(3):491–567, 1977.

[7] Béla Bollobás. Extremal graph theory. *Academic Press, New York*, 4, 1978.

[8] O. V. Borodin. Structural properties of plane graphs without adjacent triangles and an application to 3-colorings. *Journal of Graph Theory*, 21(2):183–186, 1996.

[9] O. V. BORODIN, A. V. KOSTOCHKA, and D. R. WOODALL. Acyclic colourings of planar graphs with large girth. *Journal of the London Mathematical Society*, 60(2):344–352, 1999.

[10] Oleg V Borodin. On acyclic colorings of planar graphs. *Discrete Mathematics*, 25(3):211–236, 1979.

[11] Oleg V. Borodin. To the paper of h.l. abbott and b. zhou on 4-critical planar graphs. *Ars Comb.*, 43, 1996.

[12] Oleg V Borodin. Colorings of plane graphs: a survey. *Discrete Mathematics*, 313(4):517–539, 2013.

[13] Oleg V Borodin, Dmitry G Fon-Der-Flaass, Alexandr V Kostochka, André Raspaud, and Éric Sopena. Acyclic list 7-coloring of planar graphs. *Journal of Graph Theory*, 40(2):83–90, 2002.

[14] Oleg V Borodin and Anna O Ivanova. Acyclic 4-choosability of planar graphs with no 4-and 5-cycles. *Journal of Graph Theory*, 72(4):374–397, 2013.

[15] Oleg Veniaminovich Borodin. Acyclic 3-choosability of plane graphs without cycles of length from 4 to 12. *Diskretnyi Analiz i Issledovanie Operatsii*, 16(5):26–33, 2009.

[16] Oleg Veniaminovich Borodin and Anna Olegovna Ivanova. Acyclic 3-choosability of planar graphs with no cycles of length from 4 to 11. , 7(0):275–283, 2010.

[17] Oleg Veniaminovich Borodin and Anna Olegovna Ivanova. Acyclic 5-choosability of planar graphs without 4-cycles. *Siberian mathematical journal*, 52(3):411–425, 2011.

[18] O.V. Borodin, A.N. Glebov, A. Raspaud, and M.R. Salavatipour. Planar graphs without cycles of length from 4 to 7 are 3-colorable. *Journal of Combinatorial Theory, Series B*, 93(2):303–311, 2005.

[19] Min Chen and André Raspaud. Planar graphs without 4-and 5-cycles are acyclically 4-choosable. *Discrete Applied Mathematics*, 161(7-8):921–931, 2013.

[20] Min Chen and André Raspaud. A sufficient condition for planar graphs to be acyclically 5-choosable. *Journal of Graph Theory*, 70(2):135–151, 2012.

[21] Min Chen and Weifan Wang. Acyclic 5-choosability of planar graphs without 4-cycles. *Discrete Mathematics*, 308(24):6216–6225, 2008.

[22] Vincent Cohen-Addad, Michael Hebdige, Daniel Král', Zhentao Li, and Esteban Salgado. Steinberg's conjecture is false. *Journal of Combinatorial Theory, Series B*, 122:452–456, 2017.

[23] Daniel W Cranston and Douglas B West. An introduction to the discharging method via graph coloring. *Discrete Mathematics*, 340(4):766–793, 2017.

[24] Matt DeVos, Ken-ichi Kawarabayashi, and Bojan Mohar. Locally planar graphs are 5-choosable. *Journal of Combinatorial Theory, Series B*, 6(98):1215–1232, 2008.

[25] Zdenek Dvorak, Dan Kral, and Robin Thomas. Three-coloring triangle-free graphs on surfaces i. extending a coloring to a disk with one triangle. *Journal of Combinatorial Theory, Series B*, 120, 10 2010.

[26] Zdeněk Dvořák, Daniel Král', and Robin Thomas. Three-coloring triangle-free graphs on surfaces ii. 4-critical graphs in a disk. *Journal of Combinatorial Theory, Series B*, 132:1–46, 2018.

[27] Zdeněk Dvořák, Daniel Král', and Robin Thomas. Three-coloring triangle-free graphs on surfaces iii. graphs of girth five. *Journal of Combinatorial Theory, Series B*, 145:376–432, 2020.

[28] Zdeněk Dvořák, Daniel Král', and Robin Thomas. Three-coloring triangle-free graphs on surfaces iv. bounding face sizes of 4-critical graphs. *Journal of Combinatorial Theory, Series B*, 150:270–304, 2021.

[29] Zdeněk Dvořák, Daniel Král', and Robin Thomas. Three-coloring triangle-free graphs on surfaces v. coloring planar graphs with distant anomalies. *Journal of Combinatorial Theory, Series B*, 150:244–269, 2021.

[30] Zdeněk Dvořák, Daniel Král', and Robin Thomas. Three-coloring triangle-free graphs on surfaces vii. a linear-time algorithm. *Journal of Combinatorial Theory, Series B*, 152:483–504, 2022.

[31] Zdeněk Dvořák, Daniel Král', and Robin Thomas. Three-coloring triangle-free graphs on surfaces vi. 3-colorability of quadrangulations. *Journal of Combinatorial Theory, Series B*, 164:517–548, 2024.

[32] Zdeněk Dvořák and Luke Postle. Correspondence coloring and its application to list-coloring planar graphs without cycles of lengths 4 to 8. *Journal of Combinatorial Theory, Series B*, 129:38–54, 2018.

[33] Steve Fisk. Geometric coloring theory. *Advances in Mathematics*, 24(3):298–340, 1977.

[34] Philip Franklin. A six color problem. *Journal of Mathematics and Physics*, 13(1-4):363–369, 1934.

[35] John Gimbel and Carsten Thomassen. Coloring graphs with fixed genus and girth. *Transactions of the American Mathematical Society*, 349(11):4555–4564, 1997.

[36] H Grotzsch. Ein dreifarbensatz fur dreikreisfreie netze auf der kugel. *Wiss. Z. Martin Luther Univ. Halle-Wittenberg, Math. Nat. Reihe*, 8:109–120, 1959.

[37] Branko Grünbaum. Acyclic colorings of planar graphs. *Israel journal of mathematics*, 14(4):390–408, 1973.

[38] PJ Heawood. Map colour theorem', quart. d. *J. Math*, 1890.

[39] Heinrich Heesch. Untersuchungen zum Vierfarbenproblem. *Hochschulskriptum 810ab*, 1969.

[40] Hervé Hocquard and Mickaël Montassier. Acyclic choosability of planar graphs: a steinberg like approach. *Electronic Notes in Discrete Mathematics*, 34:199–205, 2009. European Conference on Combinatorics, Graph Theory and Applications (EuroComb 2009).

[41] Ken-ichi Kawarabayashi and Bojan Mohar. Star coloring and acyclic coloring of locally planar graphs. *SIAM Journal on Discrete Mathematics*, 24(1):56–71, 2010.

[42] A. V. Kostochka. Acyclic 6-coloring of planar graphs. *Discretny analys*, 28:40–56, 1976.

[43] A.V. Kostochka and L.S. Mel'nikov. Note to the paper of grünbaum on acyclic colorings. *Discrete Mathematics*, 14(4):403–406, 1976.

[44] Jian-Bo Lv. Planar graphs without cycles of length from 4 to 7 and intersecting triangles are dp-3-colorable. *Graphs and Combinatorics*, 38:1–12, 2022.

[45] J. Mitchem. Every planar graph has an acyclic 8-coloring. *Duke Math. J.*, 41:177–181, 1974.

[46] Bojan Mohar. Acyclic colorings of locally planar graphs. *European Journal of Combinatorics*, 26(3-4):491–503, 2005.

[47] Bojan Mohar and Carsten Thomassen. *Graphs on surfaces*, volume 10. JHU press, 2001.

[48] Mickaël Montassier. Acyclic 4-choosability of planar graphs with girth at least 5. In *Graph Theory in Paris: Proceedings of a Conference in Memory of Claude Berge*, pages 299–310. Springer, 2006.

[49] Mickaël Montassier, Pascal Ochem, and André Raspaud. On the acyclic choosability of graphs. *Journal of Graph Theory*, 51:281–300, 04 2006.

[50] Mickaël Montassier, André Raspaud, and Weifan Wang. Acyclic 5-choosability of planar graphs without small cycles. *Journal of Graph Theory*, 54(3):245–260, 2007.

[51] Luke Postle. 3-list-coloring graphs of girth at least five on surfaces. *Journal of Combinatorial Theory, Series B*, 147:1–36, 2021.

[52] Luke Postle, Evelyne Smith-Roberge, and Massimo Vicenzo. Acyclic list colouring locally planar graphs, 2024.

[53] Luke Postle and Robin Thomas. Five-list-coloring graphs on surfaces II. A linear bound for critical graphs in a disk. *Journal of Combinatorial Theory, Series B*, 119:42–65, 2016.

[54] Gerhard Ringel and John WT Youngs. Solution of the Heawood map-coloring problem. *Proceedings of the National Academy of Sciences*, 60(2):438–445, 1968.

[55] Daniel P Sanders and Yue Zhao. A note on the three color problem. *Graphs and Combinatorics*, 11(1):91–94, 1995.

[56] Liang Shen and Yingqian Wang. A sufficient condition for a planar graph to be 3-choosable. *Inf. Process. Lett.*, 104(4):146–151, November 2007.

[57] Richard Steinberg. The state of the three color problem. In John Gimbel, John W. Kennedy, and Louis V. Quintas, editors, *Quo Vadis, Graph Theory?*, volume 55 of *Annals of Discrete Mathematics*, pages 211–248. Elsevier, 1993.

[58] C. Thomassen. 3-list-coloring planar graphs of girth 5. *Journal of Combinatorial Theory, Series B*, 64(1):101–107, 1995.

[59] Carsten Thomassen. The jordan-schönflies theorem and the classification of surfaces. *The American Mathematical Monthly*, 99(2):116–130, 1992.

[60] Carsten Thomassen. Five-coloring maps on surfaces. *Journal of Combinatorial Theory, Series B*, 59(1):89–105, 1993.

[61] Carsten Thomassen. Every planar graph is 5-choosable. *Journal of Combinatorial Theory, Series B*, 62(1):180–181, 1994.

[62] Carsten Thomassen. Color-critical graphs on a fixed surface. *Journal of Combinatorial Theory, Series B*, 70(1):67–100, 1997.

[63] Carsten Thomassen. The chromatic number of a graph of girth 5 on a fixed surface. *Journal of Combinatorial Theory, Series B*, 87(1):38–71, 2003.

[64] M. Voigt. A non-3-choosable planar graph without cycles of length 4 and 5. *Discrete Mathematics*, 307(7):1013–1015, 2007. Cycles and Colourings 2003.

[65] Margit Voigt. List colourings of planar graphs. *Discrete Mathematics*, 120(1-3):215–219, 1993.

[66] Margit Voigt. A not 3-choosable planar graph without 3-cycles. *Discrete Mathematics*, 146(1):325–328, 1995.

[67] Yingqian Wang, Huajing Lu, and Ming Chen. A note on 3-choosability of planar graphs. *Information Processing Letters*, 105(5):206–211, 2008.

[68] Yingqian Wang, Huajing Lu, and Ming Chen. Planar graphs without cycles of length 4, 5, 8, or 9 are 3-choosable. *Discrete Mathematics*, 310(1):147–158, 2010.

[69] Yingqian Wang, Qian Wu, and Liang Shen. Planar graphs without cycles of length 4, 7, 8, or 9 are 3-choosable. *Discrete Applied Mathematics*, 159(4):232–239, 2011.

[70] Paul Wernicke. Über den kartographischen Vierfarbenproblemsatz. *Mathematische Annalen*, 58(3):413–426, 1904.

[71] D. A. Youngs. 4-chromatic projective graphs. *Journal of Graph Theory*, 21(2):219–227, 1996.

[72] Haihui Zhang and Baogang Xu. Acyclic 5-choosability of planar graphs with neither 4-cycles nor chordal 6-cycles. *Discrete Mathematics*, 309(20):6087–6091, 2009.

[73] Li Zhang and Baoyindureng Wu. Three-coloring planar graphs without certain small cycles. *Graph Theory Notes of New York*, 46:27–30, 2004.

[74] Li Zhang and Baoyindureng Wu. A note on 3-choosability of planar graphs without certain cycles. *Discrete Mathematics*, 297(1):206–209, 2005.

[75] Yueying Zhao and Lianying Miao. Planar graphs without $\{4,6,8\}$-cycles are 3-choosable. *AIMS Mathematics*, 6(9):10253–10265, 2021.