

CONVECTIVE HEAT TRANSFER IN HYDRODYNAMICALLY-DEVELOPED LAMINAR FLOW IN ASYMMETRICALLY-HEATED ANNULI: A THREE-TEMPERATURE PROBLEM

Foroushani S.S.M.*, Naylor D., Wright J.L.

*Author for correspondence

Department of Mechanical & Mechatronics Engineering,
University of Waterloo,
200 University Ave West, Waterloo, ON, Canada
Email: ssmohadd@uwaterloo.ca

ABSTRACT

Heat transfer in hydrodynamically-developed flow in asymmetrically-heated channels and annuli has been studied extensively. This study is an extension of earlier work where heat transfer in an asymmetrically-heated parallel-plate channel was examined in a resistor-network framework. It was shown that the formulation of the problem in terms of a delta thermal-resistor network has several advantages. A delta network can also be used to represent heat transfer in asymmetrically-heated annuli. Nevertheless, the evaluation of the three paired convective resistances that characterize the network is not straightforward. In the present paper, a new technique based on solutions of the energy equation with perturbed boundary conditions is proposed. The proposed technique is first verified by comparison with the results previously obtained for the parallel-plate channel problem. A superposition solution to the energy equation is obtained for hydrodynamically-developed laminar flow in an asymmetrically-heated annulus. The developed technique is then applied to the annulus problem to obtain the corresponding resistances. Results are validated by examining limiting cases.

INTRODUCTION

The problem of calculating the forced-convective heat transfer in hydrodynamically developed flow in asymmetrically-heated channels and annuli has been studied extensively. Several analytical and numerical solutions have been published and many variations and extensions have been investigated. New developments in emerging areas such as fuel cells and micro flow devices have led to a renewed interest in heat transfer in asymmetrically-heated passages. Reviews of the classical treatments of the problem can be found in advanced heat transfer texts, *e.g.* [1,2]. In recent studies of the problem, the effects of viscous dissipation, porous media and non-Newtonian fluids have been examined. See [3-5], for example.

Nusselt numbers based on the difference between the wall temperatures and the bulk fluid temperature are traditionally used to characterize internal-flow convective heat transfer. For asymmetrically-heated passages, a temperature ratio is used to specify the ordering of the boundary temperatures, particularly the placement of the inlet fluid temperature with respect to the wall temperatures. As discussed in a recent paper [6], this approach has shortcomings when applied to asymmetrically-

NOMENCLATURE

A	[m ²]	surface area
C _n	[-]	series solution coefficient
f _n	[-]	eigenfunction
H	[m]	channel width
Q	[W]	heat transfer rate
q	[W/m ²]	heat flux
k	[W/mK]	thermal conductivity
L	[m]	length
Nu	[-]	Nusselt number
Pr	[-]	Prandtl number
Re	[-]	Reynolds number based on hydraulic diameter
R	[K/W]	thermal resistance
r	[m]	radius radial location
r	[-]	dimensionless radius: $r = r/r_1$
S	[m]	conduction shape factor
T	[K]	temperature
X	[-]	inverse Graetz number (Eq. 10)

Greek letters

dimensionless temperature: $\theta = (T - T_0)/(T_1 - T_0)$
eigenvalue
radius ratio: $\beta = r_2/r_1$

Subscripts

0 (inlet) fluid
1 upper/outer wall
2 lower/inner wall

Superscripts

(2T) two-temperature solution
* perturbed condition

-heated channels with isothermal walls. Most notably: *i*) a non-physical singularity occurs in the distribution of the wall Nusselt number at the point where the bulk fluid temperature reaches the temperature of either wall, and *ii*) the Nusselt numbers depend on temperature ratio, which is inconsistent with the physics of the problem. These features are apparent in results presented by Hatton and Turton [7] and Mitrovic *et al.* [8] for the parallel-plate channel, and by Mitrovic and Maletic [9] and Coelho and Pinho [5] for concentric annuli. It was shown [6] that the formulation of the parallel-plate channel problem in terms of a delta thermal-resistor network eliminates the non-physical singularities observed in the existing solutions. In the present paper, the earlier work on parallel-plate channels is extended to concentric annuli and the problem of forced convection in hydrodynamically-developed, laminar flow in an asymmetrically-heated annulus is examined in a

resistor-network framework. This approach reveals more details about the nature of the heat transfer phenomenon.

THE DELTA NETWORK

In Figure 1, a schematic of flow in an asymmetrically-heated channel and the corresponding delta network are shown. A delta network is characterized by three “paired” convective resistances, each corresponding to a specific pair of nodes. Unlike the traditional formulation, the resistor-network formulation leads to convective resistances which are independent of temperature and/or temperature ratio and can be used to resolve the split of heat transfer between the channel walls and the flow [6].

It was shown [6] for channel flow that, given a symmetric velocity profile and constant fluid properties, the two wall-to-fluid resistances will be equal. Nodal energy balances were then used to obtain expressions for three convective resistances, $\{R_{ij}\}$, in terms of the three nodal temperatures, $\{T_i\}$, and the total heat transfer rates, $\{Q_i\}$. The analytical solution by Hatton and Turton [7] was used to derive expressions for $\{R_{ij}\}$ in dimensionless form, *i.e.* paired Nusselt numbers, $\{\overline{Nu}_{ij}\}$.

Given the advantages of the resistor-network approach, most notably significant simplification of the results and consistency with the physics of the problem, it is tempting to apply this approach to other three-temperature convection problems – convection problems where heat transfer occurs between more than two isothermal heat sources/sinks. Flow in asymmetrically-heated annuli, shown schematically in Figure 2, is in this category. While the delta network of Figure 1 can also be used to represent this configuration, the two wall-to-fluid resistances are not equal simply due to geometric asymmetry. Therefore, the nodal energy balances alone cannot be used to find the three convective resistances of the network.

In the present paper, a new technique is proposed to overcome this difficulty. This technique entails solutions of the energy equation with perturbed boundary conditions.

METHODOLOGY

Consider the three-resistor network shown in Figure 1. The set of paired resistances, $\{R_{ij}\}$, is sought.

The set of total heat transfer rates, $\{Q_i\}$, can be calculated for any given $\{T_i\}$. But heat transfer at a node, say Q_2 , is split between the two legs of the network connected to that node. See Equation 1.

$$Q_2 = Q_{20} + Q_{21} \quad (1)$$

Each component on the RHS of Equation 1 can be written in terms of the corresponding driving temperature difference and paired resistance. See Equation 2.

$$Q_{ij} = \frac{T_i - T_j}{R_{ij}} \quad (2)$$

Applying the nodal energy balance (Equation 1) at the three nodes, three algebraic equations are obtained with the three resistances, $\{R_{ij}\}$, unknown. But these equations are not independent; they are interconnected by the overall energy balance of the network: $Q_0 + Q_1 + Q_2 = 0$. The system of

equations is therefore under-defined with three unknowns and only two independent equations. For constant-property, hydrodynamically-developed flow in a parallel-plate channel, the two wall-to-fluid resistances are equal due to symmetry. The number of the unknowns is therefore reduced to two and the system of equations can be solved for $\{R_{ij}\}$ [6]. In the absence of symmetry, the more general case, an additional equation is needed to close the system.

If one of the temperatures, say T_1 , is perturbed by T_1^* , Q_2 will change as a result by some amount Q_2^* . These changes are shown in Equations 3 and 4, where asterisks are introduced to designate the perturbed condition.

$$T_1^* = T_1 + \Delta T_1 \quad (3)$$

$$Q_2^* = Q_2 + \Delta Q_2 \quad (4)$$

The energy balance (Equation 1) can also be applied to the perturbed condition, as shown in Equation 5.

$$Q_2^* = \frac{T_2 - T_0}{R_{20}^*} + \frac{T_2 - T_1^*}{R_{12}^*} \quad (5)$$

Note that a convective resistance is function of the velocity field and the fluid properties. Hence, a change in T_1 could change both R_{20} and R_{12} through temperature-dependent fluid properties or thermal effects on the velocity field, *e.g.* buoyancy. Nevertheless, in a constant-property forced-convection problem there is a one-way coupling between the velocity field and the temperature. The convective resistances are thus independent of the temperature field, *i.e.* $R_{ij}^* = R_{ij}$. Equation 4 can then be rewritten as shown in Equation 6.

$$Q_2^* = Q_{20}^* + Q_{21}^* = \frac{T_2 - T_0}{R_{20}} + \frac{T_2 - T_1^*}{R_{12}} \quad (6)$$

Equation 6 is the additional equation that closes the system; $\{R_{ij}\}$ can now be obtained. R_{12} , for example, is found as shown in Equation 7.

$$R_{12} = -\frac{T_1}{Q_2} \quad (7)$$

In a constant-property forced-convection problem, the energy equation is linear with respect to temperature. Hence Equation 7 can be written in the differential form shown in Equation 8.

$$\frac{1}{R_{12}} = -\frac{\partial Q_2}{\partial T_1} \quad (8)$$

For convenience, the proposed technique, which gives a paired resistance R_{ij} as the ratio between Q_j and T_i , is dubbed $dQdT$. A similar technique has been applied to evaluate the paired resistances of some other three-temperature convection problems [10,11,12].

VERIFICATION

For the purpose of verification, the $dQdT$ technique was applied to hydrodynamically-developed, laminar flow in an asymmetrically-heated parallel-plate channel (Figure 1). The

results obtained in [6] by equating the two wall-to-fluid resistances and solving the nodal energy balances is used here to validate the dQdT results. To apply dQdT, the solution by Hatton and Turton [7] was used to obtain expressions for $\{Q_{ij}\}$. These expressions were then differentiated with respect to $\{T_i\}$, according to Equation 8, to get $\{R_{ij}\}$. Results are presented in terms of paired Nusselt numbers, defined as shown in Equation 9. The two sets of results are plotted in Figure 3 versus the inverse Graetz number, X , which represents dimensionless position along the channel (Equation 10).

$$\overline{Nu}_{ij} = \left[\frac{Q_{ij}}{(T_i - T_j)X} \right] \frac{H}{k} = \left(\frac{1}{R_{ij}X} \right) \frac{H}{k} \quad (9)$$

$$X = \frac{x}{H/2} \cdot \frac{1}{Re \cdot Pr} \quad (10)$$

It can be seen from the curves shown in Figure 3 that the dQdT results are identical to the energy-balance results reported in [6]. The proposed technique for calculating the paired resistances is thus verified.

RESULTS

Lundberg *et al.* [13] solved the problem of constant-property, hydrodynamically-developed, laminar flow in asymmetrically-heated concentric annuli for the special case where one of the walls is maintained at the same temperature as the inlet flow and the other wall is heated to a second temperature. Similar to the solution by Hatton and Turton for channel flow, the solution by Lundberg *et al.* is expressed in terms of a one-dimensional solution for the non-homogeneous boundary conditions, *i.e.* the fully developed solution, $f_{d,i}$, and a series solution for the homogeneous boundary conditions. See Equation 11.

$$f_{d,i} = f_{d,i} - \sum_{n=0}^{\infty} (C_n)_i f_n \exp(-\frac{2}{n}X) \quad (11)$$

In Equation 11, $f_{d,i} = (T - T_0)/(T_i - T_0)$ is dimensionless temperature, and X is the inverse Graetz number defined similar to Equation 10 but with $H/2$ replaced with $2(r_o - r_i)$. C_n , f_n and $\frac{2}{n}$ are, respectively, the coefficients, eigenfunctions and eigenvalues of the series solution reported by Lundberg *et al.* [13] for the “fundamental solution of the first kind,” *i.e.* for Dirichlet conditions on all boundaries. The subscript i denotes the heated wall: $_2$, for example, is the fundamental solution of the first kind for the case where the outer wall is at the same temperature as the inlet flow ($T_1 = T_0$), while the inner wall is heated to a different temperature (T_2). The fully developed temperature profile, $f_{d,i}$, is given by Equation 12.

$$f_{d,i} = \frac{\ln(r/r_j)}{\ln(r_i/r_j)} \quad (12)$$

Given the linearity of the energy equation, a solution to the generic case where neither wall is at the same temperature as the inlet flow – the three-temperature problem – can be constructed using superposition. See Equation 13, where the solution to the three-temperature case, T , is expressed as the

sum of the solution to a two-temperature case where the outer wall is heated; $T_1^{(2T)}$, and the solution to a two-temperature case where the inner wall is heated; $T_2^{(2T)}$. This is illustrated schematically in Figure 4.

$$T = T_1^{(2T)} + T_2^{(2T)} \quad (13)$$

With $T_i^{(2T)}$ known (in dimensionless form, $f_{d,i}$) from the work of Lundberg *et al.*, and noting that superposition requires $T_1 + T_1'' = T_i$; the three-temperature solution is obtained as shown in Equation 14.

$$T = (T_1 - T_0) \cdot f_{d,1} + (T_2 - T_0) \cdot f_{d,2} + T_0 \quad (14)$$

Equation 14 is differentiated to find the heat flux at a wall. The wall heat fluxes are then integrated along the annulus to obtain total heat transfer rates, $\{Q_{ij}\}$. Finally, Equation 8 is applied to obtain the convective resistances. Results are reported in terms of paired Nusselt numbers, defined in Equation 15. In this definition, a characteristic length of $(r_1 - r_2)$ is used to match the parallel-plate channel results (based on the channel width, H) in the $r_2/r_1 \rightarrow 1$ limit.

$$\overline{Nu}_{ij} = \left(\frac{1}{R_{ij}A_i} \right) \left(\frac{r_1 - r_2}{k} \right) \quad (15)$$

The resulting expressions for paired Nusselt numbers are shown in Equations 16, 17 and 18. In these equations, $r = r/r_1$ is dimensionless radial location and r_2/r_1 denotes radius ratio.

$$\overline{Nu}_{12} = \frac{(1 - r_2/r_1)}{X} \left[\frac{1}{\ln(1/r_2/r_1)} - \sum_{n=0}^{\infty} \left\{ \frac{(C_n)_1}{-\frac{2}{n}} \left(\frac{df_n}{dr} \right)_{r_2} \exp(-\frac{2}{n}X) \right\} \right] \quad (16)$$

$$\overline{Nu}_{10} = \frac{(1 - r_2/r_1)}{X} \sum_{n=0}^{\infty} \left\{ \left[\frac{(C_n)_1 + (C_n)_2}{-\frac{2}{n}} \right] \left(\frac{df_n}{dr} \right)_{r_1} \exp(-\frac{2}{n}X) \right\} \quad (17)$$

$$\overline{Nu}_{20} = \frac{(1 - r_2/r_1)}{X} \sum_{n=0}^{\infty} \left\{ \left[\frac{(C_n)_1 + (C_n)_2}{-\frac{2}{n}} \right] \left(\frac{df_n}{dr} \right)_{r_2} \exp(-\frac{2}{n}X) \right\} \quad (18)$$

These (average) paired Nusselt numbers are plotted in Figure 5 for a sample radius ratio of $r_2/r_1 = 0.5$. The coefficients, eigenvalues and eigenfunction derivatives given by Lundberg *et al.* [13] were used to evaluate $\{\overline{Nu}_{ij}\}$. Wall-to-fluid heat transfer is infinitely large at the annulus inlet (similar to the leading-edge singularity point in flow over a flat plate). The asymmetry in geometry, *i.e.* the different curvature of the inner and outer walls, leads to a difference between the two wall-to-fluid Nusselt numbers, \overline{Nu}_{10} and \overline{Nu}_{20} . As expected, the surface with the lower curvature has a lower wall-to-fluid Nusselt number: $\overline{Nu}_{10} < \overline{Nu}_{20}$. As the flow develops thermally, with the temperature profile approaching the fully developed profile of Equation 12, wall-to-fluid heat transfer decays, approaching the limiting value of zero at $X \rightarrow \infty$. Wall-to-wall heat transfer, on the other hand, increases from zero at the inlet and to the pure-conduction limit. The slight departure

of \overline{Nu}_{12} from 0 at $X=0$ is because only the first four eigenvalues were used.

To demonstrate the validity of these results, two limiting cases may be considered. First: in the thermally developed limit ($X \rightarrow \infty$), there is zero heat transfer between the annulus walls and the flow, hence: $\lim_{X \rightarrow \infty} \overline{Nu}_{10} = \lim_{X \rightarrow \infty} \overline{Nu}_{20} = 0$. Wall-to-wall heat transfer, on the other hand, approaches the pure-conduction limit with $\lim_{X \rightarrow \infty} \overline{Nu}_{12} = 0.72$. Using this limiting value, obtained by dQdT, the wall-to-wall heat transfer rate at $X \rightarrow \infty$ can then be calculated as shown in Equation 19.

$$\begin{aligned} \lim_{X \rightarrow \infty} Q_{12} &= \left(\lim_{X \rightarrow \infty} \overline{Nu}_{12} \right) \left(\frac{k}{r_1 - r_2} \right) A_1 (T_1 - T_2) \\ &= 9.064 \text{ kL}(T_1 - T_2) \end{aligned} \quad (19)$$

Alternatively, the wall-to-wall heat transfer rate in the pure-conduction limit can be calculated using the conduction shape factor of the annulus. The conduction shape factor of a concentric annulus is $S = 2L/\ln(r_1/r_2)$, which yields $S = 9.065L$ for $r_1/r_2 = 0.5$. The fully-developed wall-to-wall heat transfer rate is then obtained as shown in Equation 20. Comparing Equations 19 and 20, the accuracy of the dQdT results is obvious.

$$\lim_{X \rightarrow \infty} Q_{12} = Sk(T_1 - T_2) = 9.065 \text{ kL}(T_1 - T_2) \quad (20)$$

The second case is the limit where the curvature of both annulus walls approaches zero and $r_1/r_2 \rightarrow 1$. In this limit, the solution to the annular problem must approach the solution to the parallel-plate channel problem. In Figure 6, the distribution of the inner-to-fluid paired Nusselt number, \overline{Nu}_{20} , is plotted for various radius ratios. It can be seen that as $r_1/r_2 \rightarrow 1$, inner-to-fluid Nusselt number approaches the wall-to-fluid Nusselt number of the channel problem reported in [6]. Note that the $r_1/r_2 = 1$ curve and the channel-flow curve (dashed) were generated using two different sets of eigenvalues which were obtained using two different numerical schemes. The slight discrepancy between the two is, therefore, not unexpected.

CONCLUSIONS

This study is part of an ongoing research project on multi-temperature convection problems, and an extension of earlier work on resistor-network modeling and characterization of heat transfer in an asymmetrically-heated parallel-plate channel. Since the two wall-to-fluid convective resistances of an annulus are not equal, the nodal energy balances alone cannot be used to obtain the three resistances that characterize the delta thermal-resistor network of the problem. A new technique, dQdT, was proposed to generate an additional equation which, in combination with the nodal energy balances, can be solved for all three resistances. The validity of the proposed technique was demonstrated by application to the parallel-plate channel geometry: dQdT successfully reproduces the energy-balance results.

A superposition solution to the energy equation for hydrodynamically-developed, laminar flow in asymmetrically-

heated annuli was obtained using a classical solution from the literature. dQdT was then applied to obtain the three paired Nusselt numbers that characterize the delta network representing the problem. The results were shown to be in agreement with the physics of the problem. Two limiting cases were examined: 1) in the thermally developed limit, where there is zero wall-to-fluid heat transfer, the dQdT results for wall-to-wall heat transfer match calculations based on the conduction shape factor of a concentric annulus. 2) In the limit where the curvature of the walls approaches zero, the wall-to-fluid Nusselt number of the annulus was shown to approach the wall-to-fluid Nusselt number of the parallel-plate channel.

Although the dQdT technique is developed for and applied to a specific case with hydrodynamically-developed, laminar flow; there are no assumptions or restrictions precluding its application to other multi-temperature convection problems. Extension of the resistor-network approach and application of the dQdT technique to turbulent and simultaneously developing flows will be examined in future work.

ACKNOWLEDGMENTS

This work was funded by the Smart Net-Zero Energy Buildings Strategic Research Network (SNEBRN) of the Natural Sciences and Engineering Research Council of Canada (NSERC) and the University of Waterloo.

REFERENCES

- [1] Shah, R.K., and London, A.L., 1978, Laminar forced convection in ducts, Supplement to advances in heat transfer, Academic Press, NY
- [2] Kakac, S., Shah, R.K., and Aung, W., 1987, Handbook of single-phase convective heat transfer, John Wiley & Sons
- [3] Coelho P.M., and Pinho F.T., Fully-developed heat transfer in annuli with viscous dissipation, *Int. J. Heat Mass Transfer*, 49, 2006, pp. 3349-3359
- [4] Satyamurty, V.V., Bhargavi, D. Forced convection in thermally developing region of a channel partially filled with a porous material and optimal porous fraction, *Int. J. Thermal Sciences*, Vol. 49(2), 2010, pp. 319-332
- [5] Oliveira, P.J., Coelho, P.M., Pinho, F.T. The Graetz problem with viscous dissipation for FENE-P fluids, *J Non-Newton Fluid*, Vol. 121(1), 2004, pp. 69-72
- [6] Foroushani S.S.M., Naylor D. and Wright J.L., The asymmetric Graetz problem: The analytical solution revisited, *AIAA J. Thermophysics Heat Transfer*, 2016 [In review]
- [7] Hatton A.P. and Turton J.S., Heat transfer in the thermal entry length with laminar flow between parallel walls at unequal temperatures, *Int. J. Heat Mass Transfer*, Vol. 5, 1962, pp. 673-679
- [8] Mitrovi J., Maleti B., and Ba li B.S., Some peculiarities of the asymmetric Graetz problem, *Int. J. Eng. Science*, Vol. 44(7), 2006, pp. 436-455
- [9] Mitrovi J., and Maleti B., Effect of thermal asymmetry on heat transfer in a laminar annular flow, *Chem. Eng Technology*, Vol. 28(10), 2005, pp. 1144-1150
- [10] Foroushani S.S.M., Naylor D., and Wright J.L., Laminar free convection from a pair of horizontal cylinders: A three-temperature problem, *23rd Annual Conf CFD Society of Canada*, Waterloo, Canada, June 2015
- [11] Foroushani S.S.M., Wright J.L., and Naylor D., Forced convection from a pair of spheres: A three-temperature problem, *8th Int. Conf Computational Heat & Mass Transfer*, Istanbul, Turkey, May 2015
- [12] Foroushani S.S.M., Wright J.L., Naylor D., and Collins M.R., Assessing convective heat transfer coefficients associated with

indoor shading attachments using a new technique based on Computational Fluid Dynamics, *ASHRAE Winter Conf*, Chicago, USA, January 2015

[13] Lundberg R.E., McCuen P.A., and Reynolds W.C., Heat transfer in annular passages – Hydrodynamically developed laminar flow with arbitrarily prescribed wall temperatures or heat fluxes, *Int. J. Heat Mass Transfer*, Vol. 6, 1963, pp. 495-529

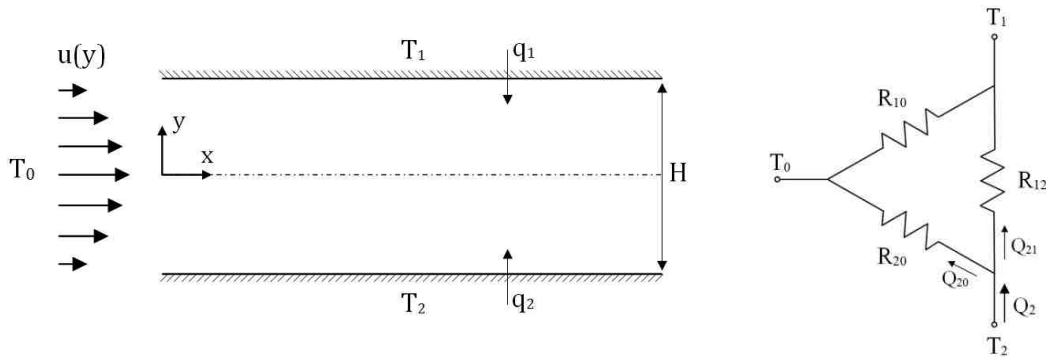


Figure 1 Schematic and resistor network of hydrodynamically developed flow in asymmetrically-heated channel

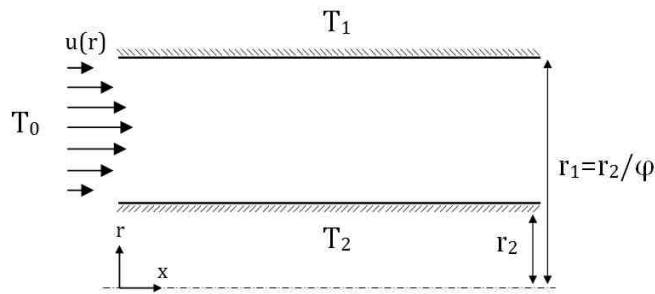


Figure 2 Schematic of hydrodynamically developed flow in asymmetrically-heated annulus

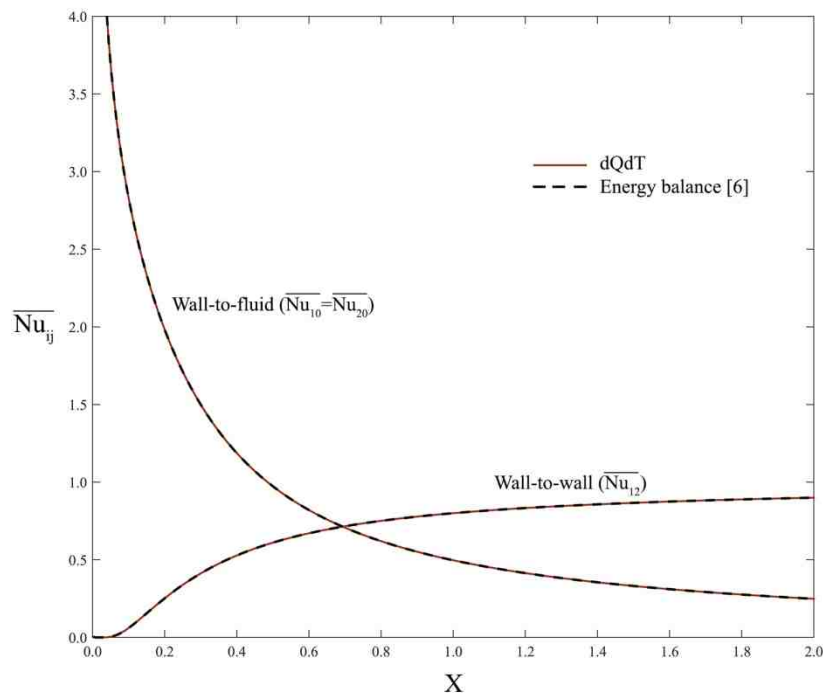


Figure 3 Average paired Nusselt numbers obtained by dQdT and from nodal energy balances

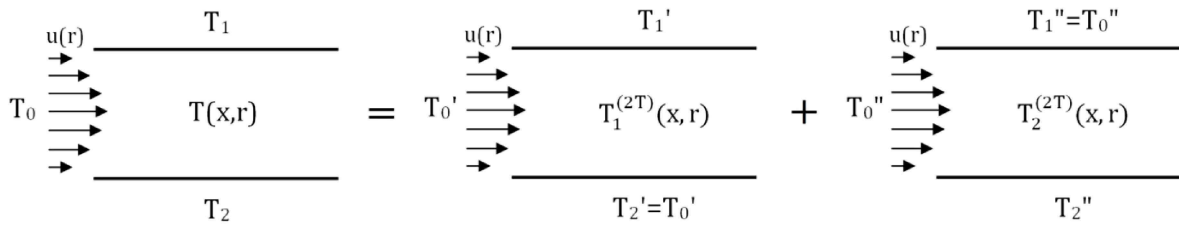


Figure 4 Generic three-temperature solution as superposition of fundamental two-temperature solutions

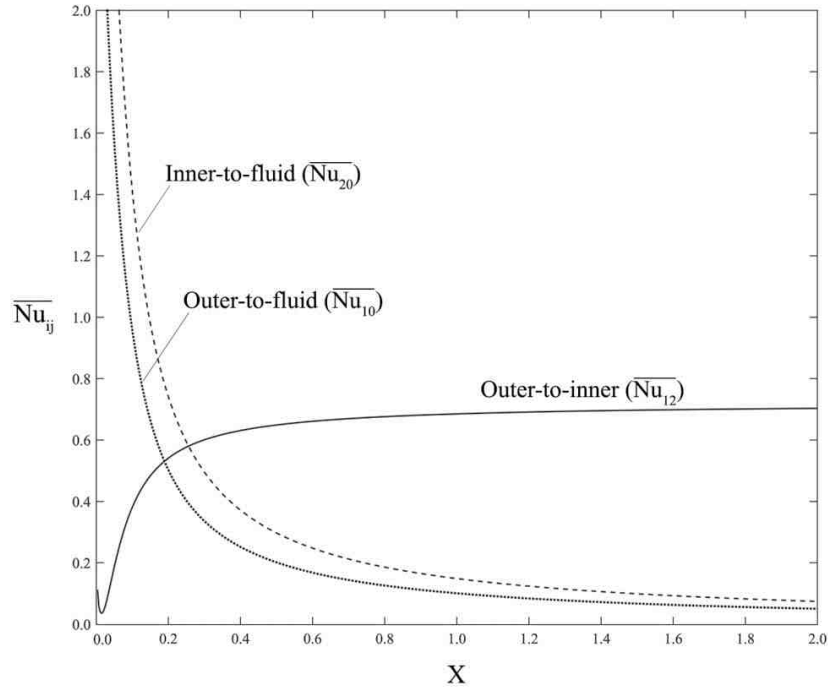


Figure 5 Average paired Nusselt numbers calculated using dQdT ($\phi=0.5$)

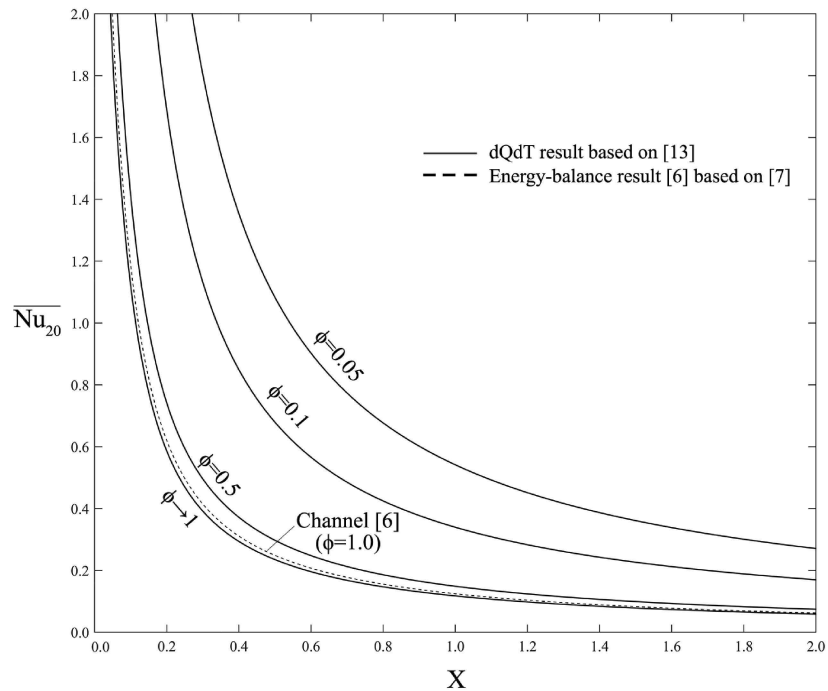


Figure 6 Average inner-wall-to-fluid Nusselt number for different radius ratios