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Robust-stochastic models for profit maximizing hub location problems

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This paper introduces robust-stochastic models for profit maximizing capacitated hub location problems in which two different types of uncertainty including stochastic demand and uncertain revenue are simultaneously incorporated into the problem. First, a two-stage stochastic program is presented where demand and revenue are jointly stochastic. Next, robust-stochastic models are developed to better model uncertainty in the revenue while keeping the demand stochastic. Two particular cases are studied based on the dependency between demand and revenue. In the first case, a robust-stochastic model with a min-max regret objective is developed assuming a finite set of scenarios that describe uncertainty associated with the revenue under a revenue-elastic demand setting. For the case when demand and revenue are independent, robust-stochastic models with a max-min criterion and a min-max regret objective are formulated considering both interval uncertainty and discrete scenarios, respectively. It is proved that the robust-stochastic version with max-min criterion can be viewed as a special case of the min-max regret stochastic model. Exact algorithms based on Benders decomposition coupled with sample average approximation scheme are proposed. Exploiting the repetitive nature of sample average approximation, generic acceleration methodologies are developed to enhance the performance of the algorithms enabling them to solve large-scale intractable instances. Extensive computational experiments are performed to consider the efficiency of the proposed algorithms and also to analyze the effects of uncertainty under different settings. The qualities of the solutions obtained from different modeling approaches are compared under various parameter settings. Computational results justify the need to solve robust-stochastic models to embed uncertainty in decision making to design resilient hub networks.

Key words: Hub location, robust optimization, stochastic demand, Benders decomposition, sample average approximation.

1. Introduction

Hubs are intermediate facilities serving as consolidation, sorting, and transshipment points in many-to-many distribution networks. In a hub network, demand of commodities are routed through hubs instead of using point-to-point connections. The aim is to exploit economies of scale by consolidating flows at hubs and also to save on network costs by establishing a network connecting many origins to many destinations with fewer number of links. Hub location problems determine the location of hubs, the allocation of demand nodes to these hubs, and the routes of flows through the network while optimizing a given objective. Hub location problems have widespread applications in the design of transportation and telecommunication networks including, but not limited to, airline passenger and freight transportation, express shipment and postal delivery, truckload and less-than-truckload transportation, and computer network design. The interested reader can refer to Campbell et al. (2002), Alumur and Kara (2008), Campbell and O'Kelly (2012), and Contreras (2015) for reviews on hub location problems.

In this paper, we consider profit maximizing capacitated hub location problems with multiple demand classes. In this problem, the demand of commodities are segmented into different classes and there is available capacity at hubs which is to be allocated to these different demand segments. The decision maker needs to determine the proportion of each class of demand to serve between origin-destination pairs based on the profit to be obtained from satisfying this demand. The decisions to be made are to find the optimal number and locations of hubs, allocation of demand nodes to these hubs, and the optimal routes of flow of different classes of commodities that are to be served while maximizing profit. We consider a multiple allocation setting, where the demand nodes can be allocated to as many hubs as necessary.

The main applications of this problem arise in the design of freight transportation and parcel delivery networks in which the amount of demand of different commodities to serve is very dependent on hub locations. To provide the big picture, in 2019 alone, US department of shipment reported 17.8 billion tones of shipments across the United States using freight transportation; the world's largest package delivery company served 5.5 billion parcels and documents in the global delivery volume (UPS 2020), which were both made possible through the employment of hub networks. Different classes of demand in freight transportation may include different types of services, for example, standard, temperature controlled shipments, and commodities requiring special equipment. In postal delivery networks, on the other hand, there is demand for services such as priority, express, and standard mail. Moreover, hub terminals have limited capacity due to the availability and capacity of the material handling equipment or the available number of docks. Hence, the decision on the proportion of each class of demand to serve between origin-destination pairs based on these available capacities needs to be incorporated into the problem.

Profit maximizing hub location problems are network design problems involving strategic decisions. In strategic planning, decisions need to be held for a considerable time frame. During this time, in real world, many unpredictable causes may lead to changes in operating conditions. For example, the amount of demand may be greater or less than its expected value. Changes may also occur in the amount of revenue obtained from the satisfied demand due to some unpredictable variations in a competitive environment. In these conditions, solving a deterministic model may result in wrong and costly decisions. Hence, taking uncertainty into account in the decision process is a necessity. To provide more reliable models, we consider two sources of uncertainty in our problem. We assume that demand of commodities and revenue are not precisely known and the optimal decisions have to be anticipated under uncertainty.

While uncertainty in demand is a natural extension, one may argue that the decision maker might have control over the prices and, hence, the revenues in practice. Although this is a plausible assumption in pricing problems, note that the main focus in hub location problems is on the strategic level location decisions. Over the planning horizon for these long-term decisions, there might be unpredictable forces that can affect the revenue (e.g., economic instability, competition), which are beyond a company's control. Hence, to produce robust and reliable strategic decisions, one needs to simultaneously consider uncertainties in demand and revenue.

Optimization under uncertainty generally consists of two streams of approaches: stochastic and robust optimization. In stochastic optimization, there are some known probability distributions describing the behavior of uncertain parameters and these distributions can be used to optimize the expected value of the objective function. In robust optimization, on the other hand, no probabilistic information is available for the uncertain parameters. In this case, uncertainty can be described by using a finite set of scenarios or can be modeled assuming that the values of the uncertain parameters can change within predefined intervals (for more information on robust optimization see, e.g., Bertsimas and Sim 2003, Ben-Tal et al. 2004, Bertsimas et al. 2011, Gabrel et al. 2014, and Correia and Saldanha-da Gama 2015).

In this paper, we consider two sources of uncertainty: demand and revenue. We first model the case where both of these parameters are stochastic. Next, we model uncertainty in revenue using robust optimization techniques while keeping the demand stochastic. Because of the availability of historical data, it makes sense to assume that demand is described by a known probability distribution. On the other hand, since the decision maker has control over the revenue parameter, it may not be realistic to take revenue as stochastic. Hence, we consider interval representation and discrete scenarios to incorporate uncertain revenue into the problem and use robust optimization techniques. In interval uncertainty, revenues are only known in the form of a range of possible values without any probabilistic information. Modeling profit maximizing hub location problems by concurrently using robust and stochastic optimization techniques surely brings on extra computational challenges, yet we believe this is a much more realistic problem setting with respect to information availability.

For both the stochastic and robust-stochastic settings, we study two cases based on the dependency between demand and revenue. In the first case, we assume that the distribution of demand depends on the realizations of revenue, referred as the revenue-elastic demand. This is the case in many practical applications where demand is intimately related to the prices set by a company. On the other hand, over the planning horizon for long-term strategic decisions, there might be unforeseeable situations where demand and revenue can no longer be dependent. For example, during the COVID-19 pandemic in 2020, many businesses and shopping centers were closed and the demand for freight transportation was drastically reduced. In such a situation, any reduction in prices would not have had any influence on the demand. Moreover, online shopping, particularly in grocery, became a mainstream part of the North American life during this pandemic. No matter how high the delivery costs were, people deliberately purchased their requirements online. Such circumstances highlight the situations where revenue might be independent from the demand.

We first present a two-stage stochastic program by assuming that the demand and revenue are jointly stochastic. We consider three separate cases of this model depending on the relation between demand and revenue. Next, we develop a robust-stochastic model with revenue-elastic demand. We assume a finite set of scenarios that describe uncertainty associated with the revenue to capture the control of the decision maker on revenue and develop a min-max regret stochastic model. With the min-max regret criterion, the decision maker decides based on the regret (or opportunity loss) from not selecting the best strategy. We then propose robust-stochastic models with independent demand. Under this setting, we first take interval uncertainty into account for revenue using the max-min criterion with a budget of uncertainty. The max-min criterion maximizes profit under the worst case scenario. We then propose a min-max regret stochastic model by considering a finite set of scenarios for the revenue. Both max-min and min-max regret criteria fit a conservative decision maker approach (Aissi et al. 2009). In this study, we model both approaches to observe the level of robustness and conservatism of each metric in addressing the uncertainty associated with revenue.

We develop exact algorithms based on Benders decomposition coupled with a sample average approximation (SAA) scheme to solve large-scale instances of the problem. We also propose novel techniques for accelerating Benders decomposition coupled with SAA capable of reusing the cuts generated during the SAA process. We perform extensive computational analysis and are able to solve instances with up to 75 nodes and 16,875 commodities of multiple demand classes. Furthermore, we investigate the effects of uncertainty under different settings on optimal hub networks and empirically evaluate the quality of the solutions obtained from different modeling approaches under various parameter settings.

The main contribution of this paper from a modeling perspective is that we incorporate uncertainty in both demand and revenue into the profit maximizing hub location problems and develop novel robust-stochastic programming approaches to simultaneously model two different types of uncertainty. We present stochastic and robust-stochastic models for the case where demand is dependent on the realizations of the revenue as well as when the revenue is independent from the demand. When demand is independent, we prove that the robust-stochastic version with max-min criterion can be viewed as a special case of the min-max regret stochastic model. To the best of our knowledge, this is the first study that compares two robust approaches under a stochastic setting. We additionally compare the quality of the solutions obtained from the stochastic and robust-stochastic models and provide several important insights in the design of optimal hub networks to maximize profit. The proposed modeling techniques can be used to formulate uncertainty in other types of discrete location and network design problems.

From a methodological point of view, our contribution is to propose Benders-based algorithms that generate strong cuts to optimally solve robust-stochastic versions of the profit maximizing capacitated hub location problems. Furthermore, we develop generic acceleration techniques to enhance the convergence of the algorithms. The acceleration techniques we propose in this study represent a contribution to the development of reoptimization capabilities of Benders cuts for two-stage programming. In particular, in our computational experimentation, the developed acceleration methodologies speed up the computation time up to 15 times on average and enables to solve large-scale intractable instances of the problem. These proposed techniques are applicable for any implementation of Benders decomposition coupled with SAA.

The remainder of the paper is organized as follows. In Section 2, we review the related literature. We define the problem setting and present the mathematical formulations in Section 3. Sections 4 and 5 provide the solution scheme for the stochastic and robust-stochastic models, respectively. Section 6 proposes acceleration techniques for reducing the computational time of the algorithms. Computational experiments are presented in Section 7. Finally, Section 8 provides concluding remarks.

2. Literature Review

Most of hub location research focused on minimizing total cost. In comparison with min-cost models, relatively less work has been devoted to maximize profits. Alibeyg et al. (2016) study hub network design problems with a profit-oriented objective by including the decisions on the nodes to be served and the commodities to be routed. They proposed alternative models of this problem and use CPLEX solver to test the performance of their models. Alibeyg et al. (2018) propose a Lagrangian relaxation as a solution algorithm for the profit-oriented models introduced in Alibeyg et al. (2016). They incorporate Lagrangian relaxation within a branch and-bound algorithm and also apply reduction tests to reduce the computational effort. Taherkhani and Alumur (2019) study profit maximizing hub location problems and propose new formulations with profit-oriented objectives in which the demand of commodities are allowed to be shipped through any number of hubs and network connections. Alternative allocation strategies including multiple, single, and r-allocation, as well as allowing for the possibility of direct connections between non-hub nodes are modeled.

Lin and Lee (2018) study a hub network design problem with price elasticity of demand for time definite less-than-truckload freight transportation. They compare the hub network configurations under cost minimization and profit maximization behaviors and numerically show that profit maximization leads to a denser hub network compared to cost minimization.

More recently, Taherkhani et al. (2020) incorporate revenue management decisions within hub location problems and determine how to allocate available capacities of hubs to demand of commodities from different market segments to maximize profit. They present a deterministic formulation of the problem and further extend it to a two-stage stochastic program considering stochastic demand. They devise two exact algorithms based on Benders decomposition and develop a new methodology to strengthen the Benders optimality cuts by decomposing the subproblem in a two-phase fashion. They are able to solve the largest deterministic hub location instances with 500 nodes and 750,000 commodities of different demand segments. In this paper, we adapt the two-phase solution methodology proposed by Taherkhani et al. (2020) to solve our Benders subproblems.

The stochastic model studied in Taherkhani et al. (2020) and the acceleration techniques therein is a special case of the stochastic model introduced in this paper that do not consider any uncertainty in revenue. Because of the structural differences between the stochastic model presented in Taherkhani et al. (2020) and the models proposed in this paper, the acceleration methodologies developed in this paper are much more comprehensive and complicated as will be detailed in Section 6. The contribution of this current paper on top of Taherkhani et al. (2020) is to simultaneously model uncertainties in demand and revenue through stochastic and robust-stochastic models and extend the methodology in order to optimally solve several versions of the problem with revenue-elastic and independent demand.

A related line of research is competitive hub location problems, where there are a number of firms competing to serve the demand instead of having a single firm. Hence, competitors' decisions might affect the profit of a firm. Different objective functions are considered in competitive hub location problems, such as maximizing the demand captured, maximizing total revenue or profit (e.g., Aykin 1995, Eiselt and Marianov 2009, Lüer-Villagra and Marianov 2013, and O'Kelly et al. 2015). Even though objective functions of some competitive hub location studies also aim at maximizing profit, in this study, we do not consider a competitive environment; that is, there is only a single firm that wants to design its hub network in the most profitable way. Moreover, we do not force all demand to be served.

There are a few studies incorporating uncertainty into hub location problems. Marianov and Serra (2003) study a problem within the context of airline transportation. They develop a formula for the probability of the number of customers in the system which is later employed to propose a capacity constraint restricting the number of airplanes. Yang (2009) studies hub location problem within airline transportation under seasonal demand variations. He develops a two-stage stochastic programming model with finite set of scenarios and uses data from the air freight market in Taiwan and China to test the proposed model. Sim et al. (2009) consider a stochastic p-hub center problem with normally distributed travel times. They employ chance constraints to model service level considerations.

Contreras et al. (2011b) study the stochastic uncapacitated multiple allocation hub location problem with uncertain demands and transportation costs. They show that the stochastic problem with uncertain demand is equivalent to its associated deterministic expected value problem where random variables are replaced by their expected value. However, when uncertainty is associated with transportation costs, this equivalence does not hold and an SAA method is developed to solve the corresponding stochastic problems. Numerical results on a set of instances with up to 50 nodes are reported.

Alumur et al. (2012) address single and multiple allocation hub location problems in which two sources of uncertainty, set-up costs for the location of hubs and the demands to be transported between the nodes, are incorporated. They assume that no probabilistic information can be associated with uncertain setup costs and propose a min-max regret formulation. For the problem with uncertain demand, they consider a stochastic programming model. They then merge these two models and propose a min-max regret stochastic formulation to model both sources of uncertainty. They generate a finite set of scenarios for the uncertain parameters (i.e., five scenarios for each parameter) and use CPLEX to solve instances with up to 25 nodes. To the best of our knowledge, this is the only paper that integrates a stochastic hub location model within a min-max regret objective. This model maximizes the expected regret, whereas, in this paper, we model the expectation of the maximum regret, which is more realistic, but also more challenging to solve.

Meraklı and Yaman (2016) model the robust uncapacitated multiple allocation p-hub median problem under polyhedral demand uncertainty with two different uncertainty sets; hose and hybrid. The hose model assumes that the only available information is the upper limit on the total flow adjacent to each node, while the hybrid model additionally imposes lower and upper bounds on each pairwise demand. They adopt a min-max robustness criterion for a cost-minimization objective function and develop two exact algorithms based on Benders decomposition. Meraklı and Yaman (2017) extend this study by incorporating capacity constraints for hubs and devise two different Benders decomposition algorithms capable of solving instances with up to 50 nodes. To the best of our knowledge, Meraklı and Yaman (2017) is the first paper that incorporates robustness into capacitated hub location problems.

Zetina et al. (2017) present robust counterparts for uncapacitated hub location problems considering uncertain demands and transportation costs. They employ a budget of uncertainty to control the level of conservatism in their mathematical models. They implement a branch-and-cut algorithm and are able to solve instances with up to 50 nodes.

de Sá et al. (2018) focus on a robust multiple allocation incomplete hub location problem in which a hub network can be partially interconnected by hub arcs, and where both demand and transportation costs are subject to uncertainty. They develop a Benders decomposition algorithm to solve their problem with up to 50 nodes. Ghaffarinasab (2018) considers robust multiple allocation p-hub median problem under polyhedral demand uncertainty. Three variants of polyhedral uncertainty models are used in the problem and a tabu search based matheuristic algorithm is developed to solve the presented models. More recently, Peiro et al. (2019) investigate a stochastic uncapacitated r-allocation p-hub median problem with direct connections. Uncertainty is associated with the demands and the transportation costs. A heuristic approach is developed to solve the proposed model.

Another related line of research is the distributionally robust optimization technique, where a parameter assumes a partially known random distribution (see e.g., Wiesemann et al. 2014, Chen et al. 2019, and Wang et al. 2020). We remark here that although we consider both types of uncertainty in our models, we depart from the distributionally robust optimization approach by assuming that one set of parameters (i.e., demand) have a fully known distribution, while there is no known distribution for the other set of parameters (i.e., revenue).

To the best of our knowledge, this is the first study in hub location that models both revenue and demand as uncertain parameters. We present several models that take into account all the plausible cases depending on the relation between these two uncertain parameters. Solving models that reflect these relationships provide several important insights in the design of optimal hub networks to maximize profit. In the next section, we present the mathematical formulations for these different versions of the problem.

3. Mathematical Formulations

We consider a directed complete graph $G = (N, A)$, where N is the set of nodes and A is the set of arcs representing possible direct links between the nodes. We allow the arc set A to have both an ordered set and a self-loop by defining $\{a_1, a_2\}$ if $|a| = 2$, and $\{a_1\}$ if $|a| = 1$, respectively. We denote the set of potential hub locations by $H \subseteq N$ and assume that there is an installation cost as well as an available capacity for a hub located at node $i \in H$ denoted by f_i and Γ_i, respectively.

There is demand for a set of commodities denoted by $K \subseteq N \times N$. Each $k \in K$ indicates a unique O-D pair whose origin and destination points belong to N. The demand for commodities are segmented into M classes. Let w_k^m be the demand for commodity $k \in K$ of class $m \in M$ to be routed from origin $o(k) \in N$ to destination $d(k) \in N$. A per unit revenue, denoted by r_k^m, is obtained from satisfying a unit commodity $k \in K$ of class $m \in M$.

Each arc has a transportation cost defined as $c_{ij} = \zeta d_{ij}$, where d_{ij} represents the distance between nodes $i \in N$ and $j \in N$ and ζ is the resource cost per unit distance. Direct transportation between two non-hub nodes are not allowed and accordingly, any satisfied demand has to be routed via at least one hub. As a direct consequence of not having any fixed costs for inter-hub links while assuming that distances satisfy the triangle inequality, every path between an origin $o(k)$ and a destination $d(k)$ will contain at least one and at most two hubs represented by $(o(k), i, j, d(k))$, where $(i, j) \in H \times H$ is the ordered pair of hubs. Accordingly, the unit transportation cost of routing commodity k along path $(o(k), i, j, d(k))$ is expressed as $C_{ijk} = \chi c_{o(k)i} + \alpha c_{ij} + \delta c_{jd(k)}$, where χ, α, δ are the collection, transfer, and distribution cost factors along the path. To reflect economies of scale between hubs, we assume that $\alpha < \chi$ and $\alpha < \delta$.

Note that in any optimal solution, every commodity $k \in K$ uses at most one of the paths $o(k), i, j, d(k)$ and $o(k), j, i, d(k)$; the one with the lower transportation cost. This property can be used to reduce the size of the mathematical formulations. Accordingly, we replace C_{ijk} with $\hat{C}_{ijk} = \min\{C_{ijk}, C_{jik}\}$, and reduce the set of candidate hub arcs for commodity $k \in K$ to A_k by defining $A_k = \{(i, j) \in A : i \leq j, \hat{C}_{ijk} \leq \min\{C_{iik}, C_{jjk}\}\}$.

3.1. Stochastic Model

We first present a two-stage stochastic program for the profit maximizing capacitated hub location problem assuming that the demand and revenue are jointly stochastic. Let $w_k^m(\psi)$ and $r_k^m(\psi)$ be the random variables representing the future demand and revenue for commodity $k \in K$ of class $m \in M$, respectively. The strategic location decisions are considered in the first stage, while the tactical decisions, including the allocations, the optimal routes of flows through the network, and the decision on how much of total capacity to allocate to the demand from different classes are determined in the second stage depending on the particular realization of the random vector $\psi \in \Psi$, where Ψ is the support of ψ.

Define $y_i = 1$ if a hub is located at node $i \in H$, and 0 otherwise, let $x_{ak}^m(\psi)$ be the fraction of commodity $k \in K$ of class $m \in M$ that is satisfied through a path with hub arc $a \in A_k$ for realization $\psi \in \Psi$, and also let $\mathbb{E}_\psi$ denote the expectation with respect to ψ. Using the aforementioned settings, we model the *profit maximizing capacitated hub location problem with stochastic demand and revenue* as follows:

$$\max \quad \mathbb{E}_\psi \left[\sum_{k \in K} \sum_{m \in M} \sum_{a \in A_k} (r_k^m(\psi) - \hat{C}_{ak}) w_k^m(\psi) x_{ak}^m(\psi) \right] - \sum_{i \in H} f_i y_i \tag{1}$$

$$\text{s.t.} \quad \sum_{a\in A_k} x^m_{ak}(\psi) \leq 1 \qquad k\in K, m\in M, \psi\in\Psi \tag{2}$$

$$\sum_{a\in A_k: i\in a} x^m_{ak}(\psi) \leq y_i \qquad i\in H, k\in K, m\in M, \psi\in\Psi \tag{3}$$

$$\sum_{k\in K}\sum_{m\in M}\sum_{a\in A_k: i\in a} w^m_k(\psi) x^m_{ak}(\psi) \leq \Gamma_i y_i \qquad i\in H, \psi\in\Psi \tag{4}$$

$$x^m_{ak}(\psi) \geq 0 \qquad k\in K, m\in M, a\in A_k, \psi\in\Psi \tag{5}$$

$$y_i \in \{0,1\} \qquad i\in H. \tag{6}$$

The objective function (1) maximizes the total expected profit by determining the first stage decisions, which calculate the hub installation costs, and the expectation of the second stage decisions, which calculate the expected values for revenue and transportation cost. Constraints (2) ensure that if the demand of commodity is to be satisfied, the flow should be routed via hubs. Constraints (3) prevent the demand of commodities to be satisfied through non-hub nodes. Constraints (4) model capacity restriction on the total incoming flow at a hub. Note that constraints (2) and (4) imply constraints (3); however, we keep these in the model to have a stronger formulation. Finally, Constraints (5)–(6) represent the non-negative and binary variables. In the sequel, we consider three special cases of this model.

We assume a linear interdependence between the revenue and demand of different segments of commodities characterized by a revenue-demand function of the following form (O'Kelly et al. 2015):

$$\boldsymbol{w}_k(\psi) = \bar{\boldsymbol{w}}_k - W(\boldsymbol{r}_k(\psi) - \bar{\boldsymbol{r}}_k) + \boldsymbol{\epsilon}_k(\psi), \tag{7}$$

where $\boldsymbol{w}_k(\psi) = (w^m_k(\psi))_{m\in M}$ and $\boldsymbol{r}_k(\psi)$ denote the vectors of stochastic demand and the revenue of different segments of commodity k, respectively. Moreover, $\bar{\boldsymbol{w}}_k$ and $\bar{\boldsymbol{r}}_k$ are the vectors of nominal demand and revenue of commodity k, respectively, $\boldsymbol{\epsilon}_k(\psi)$ is an error term that captures the fluctuations in demand of commodity k independent from the revenue, and W is a symmetric positive definite matrix that represents the dependency between demand and revenue.

We consider three special cases under different settings of the parameters W, $\boldsymbol{\epsilon}_k(\psi)$, and the marginal distribution of $\boldsymbol{r}_k(\psi)$:

(i) $\boldsymbol{\epsilon}_k(\psi) = \mathbf{0}$: All fluctuations in demand are due to revenue.
(ii) $W = [0]_{|M|\times|M|}$: Demand and revenue are independent.
(iii) $\boldsymbol{r}_k(\psi) = \bar{\boldsymbol{r}}_k$: Marginal probability distribution of the revenue is constant.

The third case yields to a stochastic model with deterministic revenue and stochastic demand studied by Taherkhani et al. (2020). Following their notation, we represent the revenue of commodity k of class m by r^m_k, and denote the random vector describing the demand by ξ and its domain by Ξ, such that $\xi\in\Xi$.

3.2. Robust-Stochastic Model with Revenue-Elastic Demand

When the decision maker has control over the revenue, it may not be realistic to take the revenue as a stochastic parameter. However, forces beyond a firm's control may exert pressures and constraints on those revenues, and those forces might be uncertain. Unlike the stochastic case, revenue and demand may no longer jointly follow a distribution. Instead, in this section, we assume that demand of commodities is on average linearly dependent on the revenue scenarios. More specifically, let $\boldsymbol{r}_k$ be a realization of the $|M|$-dimensional vector of revenue of different segments, and let $\boldsymbol{w}_k(\xi)$ be the vector of demand of commodity k, with $\bar{\boldsymbol{w}}_k$ being its nominal value. We establish the dependence of $\boldsymbol{w}_k(\xi)$ to $\boldsymbol{r}_k$ via

$$\boldsymbol{w}_k(\xi|\boldsymbol{r}_k) = \bar{\boldsymbol{w}}_k - W(\boldsymbol{r}_k - \bar{\boldsymbol{r}}_k) + \boldsymbol{\epsilon}_k(\xi), \tag{8}$$

where $\bar{\boldsymbol{w}}_k$, $\bar{\boldsymbol{r}}_k$, and W are defined as in (7) and $\boldsymbol{\epsilon}_k(\xi)$ is the error term. In other words, as depicted in Figure 1, the mean demand values for given revenue realizations are linearly dependent to the revenue realizations, and the demand realizations fluctuate around these mean values with some error $\boldsymbol{\epsilon}_k(\xi)$. For simplicity, demands and revenues are depicted one-dimensional in Figure 1.

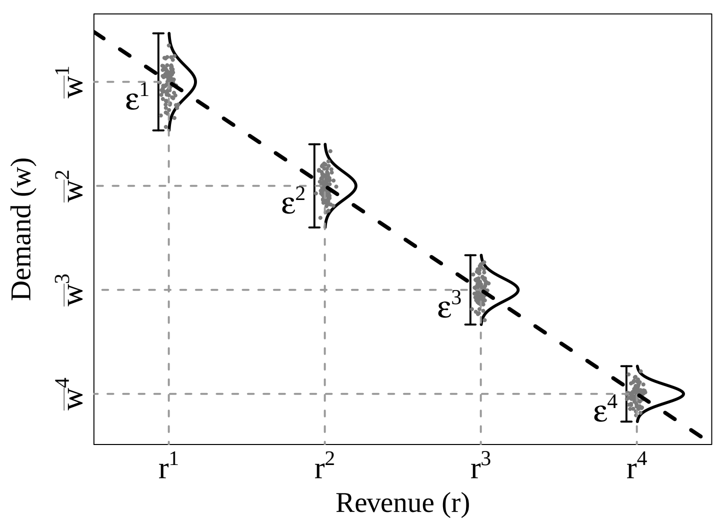


Figure 1 **Revenue-elastic demand.**

To capture the control of the decision maker on revenue, we assume that a discrete set of scenarios are available for the revenue (e.g., low, medium, and high revenue). To make long-term robust decisions with discrete scenarios, one may aim at minimizing the regret incurred by the lack of perfect information (Alumur et al. 2012, Correia and Saldanha-da Gama 2015). This approach bodes well to a conservative decision maker, who wishes to mitigate the opportunity loss by not making the best decision. Let S_r define the set of scenarios for uncertain revenue and r_k^{ms} denote the amount of revenue obtained from satisfying a unit commodity $k \in K$ of class $m \in M$ under scenario $s \in S_r$. Additionally, let ξ^s be the random variable describing demand realizations under revenue scenario $s \in S_r$ with Ξ^s its domain. Defining $\boldsymbol{r}_k^s = (r_k^{ms})_{m \in M}$ as the vector of revenue of commodity k under revenue scenario s, we may reformulate equation (8) as

$$\boldsymbol{w}_k(\xi^s) = \bar{\boldsymbol{w}}_k - W(\boldsymbol{r}_k^s - \bar{\boldsymbol{r}}_k^s) + \boldsymbol{\epsilon}_k(\xi^s) \qquad \forall s \in S_r. \tag{9}$$

Note that the decisions that need to be made under uncertainty are the first stage location decisions described by $\boldsymbol{y}$, while the second stage routing decisions defined by $\boldsymbol{x}$ are to be made after both uncertain demand and revenue are realized. For a given solution $\boldsymbol{y} = \hat{\boldsymbol{y}}$, let $Z^s(\hat{\boldsymbol{y}}, \xi^s)$ be the total profit that can be generated under revenue scenario $s \in S_r$ and a respective demand scenario $\xi^s \in \Xi^s$, defined by

$$Z^s(\hat{\boldsymbol{y}}, \xi^s) = \max \sum_{k \in K} \sum_{m \in M} \sum_{a \in A_k} (r_k^{ms} - \hat{C}_{ak}) w_k^m(\xi^s) x_{ak}^m(\xi^s) - \sum_{i \in H} f_i \hat{y}_i \tag{10}$$

$$\text{s.t.} \sum_{a \in A_k} x_{ak}^m(\xi^s) \leq 1 \qquad k \in K, m \in M \tag{11}$$

$$\sum_{a \in A_k : i \in a} x_{ak}^m(\xi^s) \leq \hat{y}_i \qquad i \in H, k \in K, m \in M \tag{12}$$

$$\sum_{k \in K} \sum_{m \in M} \sum_{a \in A_k : i \in a} w_k^m(\xi^s) x_{ak}^m(\xi^s) \leq \Gamma_i \hat{y}_i \qquad i \in H \tag{13}$$

$$x_{ak}^m(\xi^s) \geq 0 \qquad k \in K, m \in M, a \in A_k. \tag{14}$$

Moreover, let $Z^s(\hat{\boldsymbol{y}}) = \mathbb{E}_{\xi^s}[Z^s(\hat{\boldsymbol{y}}, \xi^s)]$ be the expected total profit, and $Z^s = \max_{\boldsymbol{y}} \{Z^s(\boldsymbol{y})\}$ denote the highest revenue one can hope for achieving under revenue scenario s. For a given revenue scenario $s \in S_r$, the regret of a solution $\boldsymbol{y}$ is defined as the difference between the optimal profit that can be achieved under that scenario (i.e., Z^s) and the total profit associated with $\boldsymbol{y}$ (i.e., $Z^s(\boldsymbol{y})$). Thus, the maximum regret is attained by the revenue scenario that maximizes $Z^s - Z^s(\boldsymbol{y})$, and the goal of the *min-max regret robust-stochastic optimization model* is to find $\boldsymbol{y}$ that minimizes this regret as formulated in the following:

$$\text{RS-I} \quad \min_{\boldsymbol{y}} \max_{s \in S_r} \{Z^s - Z^s(\boldsymbol{y})\}. \tag{15}$$

We remark that the decision variables and uncertain parameters take their values in the following order. First, decisions on hub locations (y_i) are made. Next, revenue scenario s and the corresponding parameters (r_k^{ms}) are realized. Then, the demand parameters ($w_k^m(\xi^s)$) are realized according to (9). Finally, the routing decisions ($x_{ak}^m(\xi^s)$) are made after all parameters are known. The regret formulated in (15) captures this sequence of events.

For notational convenience, we define $\mho(\boldsymbol{y},\xi)$ as the set of all feasible routing decision variables $x_{ak}^m(\xi)$ under demand realization ξ that comply with the given location decisions $\boldsymbol{y}$. That is,

$$\mho(\boldsymbol{y},\xi) = \{x_{ak}^m(\xi) : (11) \text{ –}(14) \text{ are satisfied}\}. \tag{16}$$

Replacing the maximum regret term in (15) with a variable V and using the set $\mho(\boldsymbol{y},\xi)$, we can reformulate problem (15) as

$$\min \quad V \tag{17}$$

$$\text{s.t.} \quad V \geq Z^s - \mathbb{E}_{\xi^s}\left[\sum_{k\in K}\sum_{m\in M}\sum_{a\in A_k}(r_k^{ms} - \hat{C}_{ak})w_k^m(\xi^s)x_{ak}^m(\xi^s)\right] + \sum_{i\in H} f_i y_i \tag{18}$$

$$\boldsymbol{x}(\xi^s) \in \mho(\boldsymbol{y},\xi^s) \qquad s\in S_r, \xi^s \in \Xi^s \tag{19}$$

$$y_i \in \{0,1\} \qquad i\in H, \tag{20}$$

where $\boldsymbol{x}(\xi^s) = (x_{ak}^m(\xi^s))$ is the vector of routing variables replicated for revenue scenario $s\in S_r$ and the corresponding demand realization $\xi^s \in \Xi^s$.

3.3. Robust-Stochastic Models with Independent Demand

We now study the case where demand and revenue are independent. This setting is advocated by situations where a change in revenue does not impact the demand (e.g., during a pandemic). Under this setting, in addition to considering discrete scenarios, we incorporate uncertain revenue into the problem by considering an interval representation, as the revenue is no longer dependent on the demand. In this section, unless otherwise stated, with robust-stochastic models we refer to the cases where demand is considered independently from the revenue.

3.3.1. Max-min criterion. We first use interval uncertainty for revenue in which each parameter r_k^m for $k\in K, m\in M$ takes values in $[\bar{r}_k^m - \hat{r}_k^m, \bar{r}_k^m]$, where $\bar{r}_k^m$ is the nominal value of revenue and $\hat{r}_k^m \geq 0$ represents the deviation from the nominal value. Note that we may equivalently define the intervals of uncertainty as two-sided intervals; however, the one-sided definition is preferable as the worst case scenarios correspond to the cases where revenue decreases. Let $\gamma_r \in [0, |K|\times|M|]$ be an integer value controlling the level of conservatism in the objective and denote the uncertainty budget on the maximum number of revenue parameters r_k^m whose value is allowed to differ from its nominal value. We are interested in finding an optimal solution that optimizes against all realizations under which a number γ_r of the revenue coefficients can vary in such a way as to maximally influence the objective (Bertsimas and Sim 2003). The *robust-stochastic model with max-min criterion* for the profit maximizing hub location problem with capacity allocation is then modeled as:

$$\text{RS-II} \quad \max_{(\boldsymbol{x},\boldsymbol{y})\in\mho} \quad \mathbb{E}_\xi\left[\sum_{k\in K}\sum_{m\in M}\sum_{a\in A_k}(\bar{r}_k^m - \hat{C}_{ak})w_k^m(\xi)x_{ak}^m(\xi) - \nu_\xi(\boldsymbol{x})\right] - \sum_{i\in H} f_i y_i, \tag{21}$$

where $\mho = \{(\boldsymbol{y},\boldsymbol{x}) : \boldsymbol{x}(\xi) \in \mho(\boldsymbol{y},\xi)\ \forall \xi \in \Xi, y_i \in \{0,1\}\ \forall i \in H\}$ is the set of solutions $(\boldsymbol{x},\boldsymbol{y})$ that are feasible under all realizations of ξ, and $\nu_\xi(\boldsymbol{x})$ is defined as follows:

$$\nu_\xi(\boldsymbol{x}) = \max_{U_r \subseteq K\times M : |U_r| \leq \gamma_r} \sum_{(k,m)\in U_r} \sum_{a \in A_k} \hat{r}_k^m w_k^m(\xi) x_{ak}^m(\xi). \tag{22}$$

The goal of $\nu_\xi(\boldsymbol{x})$ is to determine the worst case deviation from the total revenue over all possible revenue realizations for a given solution $\boldsymbol{x}$. Note that in extreme cases when $\gamma_r = 0$ or $\gamma_r = |K| \times |M|$ (alternatively, when $U_r = \emptyset$ or $U_r = K \times M$, respectively), the problem can be reduced to the stochastic model and it has trivial solutions such that for all commodities (k,m), in the former case, $r_k^m = \bar{r}_k^m$, whereas in the latter case, $r_k^m = \bar{r}_k^m - \hat{r}_k^m$, where these cases represent the least and highest levels of conservatism, respectively. In general, a higher value of γ_r leads to a more conservative solution in the expense of a possibly lower profit.

We can reformulate $\nu_\xi(\boldsymbol{x})$ by introducing a binary variable z_k^m which determines whether or not class $m \in M$ of commodity $k \in K$ is subject to uncertainty; i.e., $z_k^m = 1$ if $(k,m) \in U_r$, and 0 otherwise.

$$\nu_\xi(\boldsymbol{x}) = \max \sum_{k\in K}\sum_{m\in M} \left(\hat{r}_k^m w_k^m(\xi) \sum_{a\in A_k} x_{ak}^m(\xi) \right) z_k^m \tag{23}$$

$$\text{s.t.} \quad \sum_{k\in K}\sum_{m\in M} z_k^m \leq \gamma_r \tag{24}$$

$$z_k^m \in \{0,1\} \qquad k \in K, m \in M. \tag{25}$$

Note that since γ_r is integer, $\nu_\xi(\boldsymbol{x})$ simply sorts the commodities (k,m) in the non-increasing order of $\hat{r}_k^m w_k^m(\xi) \sum_{a\in A_k} x_{ak}^m(\xi)$ and selects the first γ_r of them. Hence, constraint (25) can be replaced with its linear relaxation counterpart without losing integrality. Let $\mu(\xi)$ and $\lambda_k^m(\xi)$ be the dual variables associated with constraints (24) and the linear relaxation of (25), respectively. The dual of problem (23)–(25) can be obtained as:

$$\nu_\xi(\boldsymbol{x}) = \min\ \gamma_r \mu(\xi) + \sum_{k\in K}\sum_{m\in M} \lambda_k^m(\xi)$$

$$\text{s.t.} \quad \mu(\xi) + \lambda_k^m(\xi) \geq \hat{r}_k^m w_k^m(\xi) \sum_{a\in A_k} x_{ak}^m(\xi) \qquad k \in K, m \in M \tag{26}$$

$$\lambda_k^m(\xi), \mu(\xi) \geq 0 \qquad k \in K, m \in M. \tag{27}$$

With this formulation of $\nu_\xi(\boldsymbol{x})$, mathematical program (21) can be reformulated as the following stochastic MILP:

$$\max_{(\boldsymbol{x},\boldsymbol{y})\in\mho} \quad \mathbb{E}_\xi \left[\sum_{k\in K}\sum_{m\in M}\sum_{a\in A_k} (\bar{r}_k^m - \hat{C}_{ak}) w_k^m(\xi) x_{ak}^m(\xi) - \gamma_r \mu(\xi) - \sum_{k\in K}\sum_{m\in M} \lambda_k^m(\xi) \right] - \sum_{i\in H} f_i y_i \tag{28}$$

$$\text{s.t.} \quad (26), (27).$$

3.3.2. Min-max regret. Provided a set of scenarios describing uncertainty associated with the revenue, we use the min-max regret type objective function to model the problem. For a given demand realization $\xi \in \Xi$, the maximum profit that can be achieved under revenue scenario $s \in S_r$, denoted by $Z^s(\xi)$, can be calculated by

$$Z^s(\xi) = \max_{(\boldsymbol{x}(\xi),\boldsymbol{y}) \in \mho(\xi)} \sum_{k\in K}\sum_{m\in M}\sum_{a\in A_k}(r_k^{ms} - \hat{C}_{ak})w_k^m(\xi)x_{ak}^m(\xi) - \sum_{i\in H} f_i y_i, \tag{29}$$

where $\mho(\xi) = \{(\boldsymbol{x}(\xi), \boldsymbol{y}) : \boldsymbol{x}(\xi) \in \mho(\boldsymbol{y}, \xi), y_i \in \{0,1\}\ \forall i \in H\}$. For a given demand realization $\xi \in \Xi$, the regret of a solution $(\boldsymbol{x}(\xi), \boldsymbol{y})$ under revenue scenario $s \in S_r$ is defined as the difference between the optimal profit that can be achieved under that scenario (i.e., $Z^s(\xi)$) and the total profit associated with $(\boldsymbol{x}(\xi), \boldsymbol{y})$. With this definition, the *min-max regret stochastic model* can be formulated as follows:

$$\text{RS-III} \quad \min_{(\boldsymbol{x},\boldsymbol{y})\in\mho} \quad \mathbb{E}_\xi \left[\max_{s\in S_r} \left\{ Z^s(\xi) - (\sum_{k\in K}\sum_{m\in M}\sum_{a\in A_k}(r_k^{ms} - \hat{C}_{ak})w_k^m(\xi)x_{ak}^m(\xi) - \sum_{i\in H} f_i y_i) \right\} \right], \tag{30}$$

where $\mho$ is the set of feasible solutions defined by $\mho(\xi)$ for each $\xi \in \Xi$. The inner maximization in (30) calculates the maximum regret among all revenue scenarios. Replacing the inner maximization with a continuous variable $V(\xi)$, the above formulation can be linearized as follows:

$$\min_{(\boldsymbol{x},\boldsymbol{y})\in\mho} \quad \mathbb{E}_\xi[V(\xi)] \tag{31}$$

$$\text{s.t.} \quad V(\xi) \geq Z^s(\xi) - \left(\sum_{k\in K}\sum_{m\in M}\sum_{a\in A_k}(r_k^{ms} - \hat{C}_{ak})w_k^m(\xi)x_{ak}^m(\xi) - \sum_{i\in H} f_i y_i \right) \qquad \xi \in \Xi, s \in S_r. \tag{32}$$

We now like to compare the min-max regret stochastic model (RS-III) with the robust-stochastic model with max-min criterion (RS-II). Let's first assume that the set of revenue scenarios considered in the min-max regret model (i.e., S_r) complies with the requirements of the uncertainty sets considered in the robust-stochastic model with max-min criterion (i.e., U_r). In other words, let S_r consists of all revenue scenarios involving at most γ_r commodities with an uncertain revenue. For a given solution $(\boldsymbol{x}, \boldsymbol{y})$, the robust-stochastic model with max-min criterion selects from S_r the scenario that minimizes the total revenue, and maximizes the expectation of this minimal revenue over all possible solutions $(\boldsymbol{x}, \boldsymbol{y})$. The min-max regret stochastic model, on the other hand, selects from S_r the scenario that maximizes the regret, and minimizes the expectation of this maximal regret over all possible solutions $(\boldsymbol{x}, \boldsymbol{y})$. Interestingly, as shown in Theorem 1 below, the robust-stochastic version with max-min criterion is actually a special case of the min-max regret stochastic model in which $Z^s(\xi)$ is replaced by a fixed value $\hat{Z}$ (e.g. 0) for each revenue scenario $s \in S_r$ and demand realization $\xi \in \Xi$.

THEOREM 1. *Let S_r be the set of revenue scenarios where at most γ_r commodities are subject to revenue uncertainty. Then, the min-max regret stochastic model* (30)*, in which regrets are calculated with respect to a fixed reference point $\hat{Z}$, is equivalent to the robust-stochastic model with max-min criterion* (21)*.*

Proof of Theorem 1 is given in the supplementary Appendix EC.1. As a consequence of this theorem, the robust-stochastic model with max-min criterion (RS-II) is computationally less challenging than the min-max regret stochastic version (RS-III) as there is no need to compute $Z^s(\xi)$ for each scenario. We empirically analyze the outcome and the level of robustness with both of the models through our computational experiments. In the sequel, we present a solution scheme to solve each of these large mixed-integer stochastic programs.

4. Solution Scheme for the Stochastic Model

The main challenge in solving the stochastic problems is the evaluation of the expected value of the objective function. To overcome this issue, we suggest using a Monte Carlo simulation based method known as sample average approximation scheme (Kleywegt et al. 2002, Contreras et al. 2011b, Adulyasak et al. 2015, and Taherkhani et al. 2020). In SAA, a random sample of realizations with size $|\mathcal{N}|$ of the random vector ψ (or ξ) is generated, and the second-stage expectation is approximated by the sample average function. This procedure is then replicated $\mathcal{M}$ times, and the overall average value is considered as the approximation of the optimal value of the stochastic problem.

At each replication of the sample average optimization problem, we solve the SAA counterpart of the stochastic problem (1)–(6) using a Benders decomposition (BD) algorithm. BD is based on the premise that for fixed values of integer variables, the resulting problem, known as the primal subproblem, is relatively easier to solve compared to the original problem. In BD, the problem is reformulated based on the information inferred from the dual space of the continuous variables. The equivalent reformulation, known as the master problem (MP), contains the integer variables and exponentially many constraints corresponding to the dual variables. Therefore, MP is usually solved using a cutting-plane method, where relaxations of MP are iteratively solved until the optimal solution is obtained.

In the SAA counterpart of the stochastic model (1)–(6), we approximate Ψ with the sample $\mathcal{N}$, and for each $n \in \mathcal{N}$, we denote the revenue and demand realizations of commodity k of class m with r_k^{mn} and w_k^{mn}, respectively. Note that this notation covers all special cases of the stochastic model, where the relationship between revenue and demand is captured through scenarios contained in $\mathcal{N}$ (e.g., r_k^{mn} is the same for each scenario $n \in \mathcal{N}$ in the third case, when revenue is deterministic). Under a sample $\mathcal{N}$, for given hub locations $\boldsymbol{y} := \bar{\boldsymbol{y}}$ the *primal subproblem* S-PS($\mathcal{N}$) of the SAA counterpart of the stochastic model (1)–(6) can be formulated as:

$$\text{S-PS}(\mathcal{N}) \quad \max \ \frac{1}{|\mathcal{N}|}\Big[\sum_{n\in\mathcal{N}}\sum_{k\in K}\sum_{m\in M}\sum_{a\in A_k}(r_k^{mn} - \hat{C}_{ak})w_k^{mn}x_{ak}^{mn}\Big]$$

$$\text{s.t.} \quad \sum_{a\in A_k} x_{ak}^{mn} \leq 1 \qquad k \in K, m \in M, n \in \mathcal{N} \qquad (33)$$

$$\sum_{a\in A_k: i\in a} x_{ak}^{mn} \leq \bar{y}_i \qquad i\in H, k\in K, m\in M, n\in \mathcal{N} \tag{34}$$

$$\sum_{k\in K}\sum_{m\in M}\sum_{a\in A_k: i\in a} w_k^{mn} x_{ak}^{mn} \leq \Gamma_i \bar{y}_i \qquad i\in H, n\in \mathcal{N} \tag{35}$$

$$x_{ak}^{mn} \geq 0 \qquad k\in K, m\in M, a\in A_k, n\in \mathcal{N}. \tag{36}$$

Observe that S-PS($\mathcal{N}$) can be decomposed into $|\mathcal{N}|$ independent subproblems, one for each $n\in\mathcal{N}$. Let α_k^{mn}, u_{ik}^{mn}, and b_i^n be the dual variables associated with constraints (33)–(35), respectively. The *dual subproblem* associated with scenario $n\in\mathcal{N}$ can then be formulated as:

$$\text{S-DS}(\mathcal{N},n) \quad \min \sum_{k\in K}\sum_{m\in M} \alpha_k^{mn} + \sum_{i\in H} \bar{y}_i (\sum_{k\in K}\sum_{m\in M} u_{ik}^{mn} + \Gamma_i b_i^n) \tag{37}$$

$$\text{s.t.} \quad \alpha_k^{mn} + u_{ik}^{mn} + u_{jk}^{mn} + w_k^{mn}(b_i^n + b_j^n) \geq (r_k^{mn} - \hat{C}_{ijk}) w_k^{mn} \qquad k\in K, m\in M, (i,j)\in A_k : i\neq j \tag{38}$$

$$\alpha_k^{mn} + u_{ik}^{mn} + w_k^{mn} b_i^n \geq (r_k^{mn} - \hat{C}_{iik}) w_k^{mn} \qquad k\in K, m\in M, i\in H \tag{39}$$

$$\alpha_k^{mn}, u_{ik}^{mn}, b_i^n \geq 0 \qquad k\in K, m\in M, i\in H. \tag{40}$$

Note that the dual subproblem is always feasible and bounded, hence, it attains its optimum at one of its extreme points. Let $P_{\mathcal{N}}^n$ be the set of extreme points of the polyhedron defined by (38)–(40) for $n\in\mathcal{N}$. Since the subproblem can be decomposed by each scenario $n\in\mathcal{N}$, the Benders optimality cuts can be separated by each $n\in\mathcal{N}$. Hence, the Benders *master problem* S-MP($\mathcal{N}$) can be reformulated as:

$$\text{S-MP}(\mathcal{N}) \quad \max \quad \frac{1}{|\mathcal{N}|}\sum_{n\in\mathcal{N}} \eta^n - \sum_{i\in H} f_i y_i$$

$$\text{s.t.} \quad \eta^n \leq \sum_{k\in K}\sum_{m\in M} \alpha_k^{mn} + \sum_{i\in H} y_i (\Gamma_i b_i^n + \sum_{k\in K}\sum_{m\in M} u_{ik}^{mn}) \qquad n\in\mathcal{N}, (\boldsymbol{\alpha}^n, \boldsymbol{u}^n, \boldsymbol{b}^n)\in P_{\mathcal{N}}^n \tag{41}$$

$$y_i \in \{0,1\} \qquad i\in H. \tag{42}$$

We solve S-MP($\mathcal{N}$) by relaxing the Benders optimality cuts (41) and producing them as needed by solving the dual subproblems, until the upper and lower bounds on the optimal value of S-MP($\mathcal{N}$) are sufficiently close. An overview of this procedure is described in Algorithm 1, in which UB and LB denote the upper and lower bounds on the optimal value, while Z_{MP}^e and Z_{DS}^{en} stand for the optimal values obtained from the master problem and dual subproblems for sample n at iteration e, respectively.

The computational efficiency of the BD algorithm generally depends on the number of iterations it takes and the computational effort required to solve the master problem as well as the subproblems at each iteration. In the sequel, we describe how the subproblem can be solved efficiently.

Producing effective cuts is the core ingredient to enhance the convergence of the BD algorithm. Note that, for each revenue-demand scenario $n\in\mathcal{N}$, the subproblem consists of $O(|M||N|^4)$ variables and $O(|M||N|^3)$ constraints. Hence, generating effective cuts by solving these huge linear programming (LP) problems is the most challenging part in the BD algorithm. We solve S-DS($\mathcal{N},n$)

Algorithm 1 Benders decomposition for the stochastic model

$UB \leftarrow +\infty, LB \leftarrow -\infty, e \leftarrow 1$
$P^n_{\mathcal{N}} \leftarrow \emptyset \quad \forall n \in \mathcal{N}$
while $LB < UB$ **do**
  SOLVE S-MP($\mathcal{N}$) and obtain $\boldsymbol{y}^e$ and Z^e_{MP}
  $UB \leftarrow Z^e_{\text{MP}}$
  for n in $\mathcal{N}$ **do**
    SOLVE S-DS($\mathcal{N}, n$) with $\bar{\boldsymbol{y}} = \boldsymbol{y}^e$ and obtain $(\boldsymbol{\alpha}^n, \boldsymbol{u}^n, \boldsymbol{b}^n)^e$ and Z^{en}_{DS}
    $P^n_{\mathcal{N}} \leftarrow P^n_{\mathcal{N}} \cup \{(\boldsymbol{\alpha}^n, \boldsymbol{u}^n, \boldsymbol{b}^n)^e\}$
  end for
  $LB \leftarrow \max\{LB, \frac{1}{|\mathcal{N}|}\sum_{n\in\mathcal{N}} Z^{en}_{\text{DS}} - \sum\limits_{i\in H} f_i y^e_i\}$
  $e \leftarrow e + 1$
end while

in two sequential phases based on the set of open/closed hubs as proposed by Taherkhani et al. (2020).

Observe that for $\bar{y}_i = 0$ (i.e., if hub i is closed) any feasible value of u^{mn}_{ik} and b^n_i are optimal, since their coefficients in the objective function of S-DS($\mathcal{N}, n$) are equal to zero. Therefore, the dual subproblems have trivial optimal solutions, in which one can set b^n_i and u^{mn}_{ik} to a very large number M for closed hubs (i.e., $\bar{y}_i = 0$) and find optimal values for the remaining variables. This is in fact the solution that a solver would produce. To produce stronger cuts than these trivial cuts, we initially remove the variables and constraints associated with closed hubs from S-DS($\mathcal{N}, n$) to solve the Phase I subproblem. Then, in Phase II, we produce the smallest feasible values for the dual variables associated with the closed hubs. In this manner, the optimality of the subproblem is guaranteed in Phase I, while in Phase II, the feasible values of the remaining variables are calculated so as to strengthen the cut.

Let H^1_e and H^0_e denote the set of open and closed hubs at iteration e, respectively. In Phase I, we remove the variables u^{mn}_{ik} and b^n_i associated with $i \in H^0_e$ and calculate the values of the remaining variables. Note that when $i \in H^1_e$, constraints (33) and (35) imply constraints (34). Consequently, there exists an optimal solution where the dual values associated with constraints (34) (i.e., u^{mn}_{ik}) are equal to 0 for $i \in H^1_e$. Thus, the Phase I subproblem can be formulated as:

$$\text{S-DS-I}(\mathcal{N}, n) \quad \min \quad \sum_{k\in K}\sum_{m\in M} \alpha^{mn}_k + \sum_{i\in H^1_e} \Gamma_i b^n_i$$

$$\text{s.t.} \quad \alpha^{mn}_k + w^{mn}_k (b^n_i + b^n_j) \geq (r^{mn}_k - \hat{C}_{ijk}) w^{mn}_k \qquad k \in K, m \in M, (i,j) \in A^1_{ke}$$

$$\alpha_k^{mn} + w_k^{mn} b_i^n \geq (r_k^{mn} - \hat{C}_{iik}) w_k^{mn} \qquad k \in K, m \in M, i \in H_e^1$$

$$\alpha_k^{mn}, b_i^n \geq 0 \qquad k \in K, m \in M, i \in H_e^1,$$

where $A_{ke}^1 = \{(i,j) \in A_k \cap H_e^1 \times H_e^1 : i \neq j\}$. Given that the cardinality of H_e^1 is almost certainly small, the Phase I subproblem can be solved using LP solvers. Once the optimal values of all the variables in the Phase I subproblem are obtained, the optimal value of the rest of variables will be computed in Phase II formulated as:

$$\text{S-DS-II}(\mathcal{N}, n) \quad \min \sum_{i \in H_e^0} (\sum_{k \in K} \sum_{m \in M} u_{ik}^{mn} + \Gamma_i b_i^n) \tag{43}$$

$$\text{s.t.} \quad u_{ik}^{mn} + u_{jk}^{mn} + w_k^{mn}(b_i^n + b_j^n) \geq \rho_{ijk}^{mn} \qquad k \in K, m \in M, (i,j) \in A_{ke}^0 \tag{44}$$

$$u_{ik}^{mn} + w_k^{mn} b_i^n \geq \rho_{iik}^{mn} \qquad k \in K, m \in M, i \in H_e^0 \tag{45}$$

$$u_{ik}^{mn}, b_i^n \geq 0 \qquad k \in K, m \in M, i \in H_e^0, \tag{46}$$

where $A_{ke}^0 = \{(i,j) \in A_k \cap H_e^0 \times H_e^0 : i \neq j\}$, $\rho_{ijk}^{mn} = (r_k^{mn} - \hat{C}_{ijk}) w_k^{mn} - \alpha_k^{mn}$ for $(i,j) \in A_{ke}^0$ and $\rho_{iik}^{mn} = \max\{\max_{j \in H_{ie}^1} \{(r_k^{mn} - \hat{C}_{ijk}) w_k^{mn} - u_{jk}^{mn} - w_k^{mn} b_j^n\}, (r_k^m - \hat{C}_{iik}) w_k^{mn}\} - \alpha_k^{mn}$ for $i \in H_e^0$, in which $H_{ie}^1 = \{j \in H_e^1 : (i,j) \in A_k \text{ or } (j,i) \in A_k\}$. To efficiently generate the cuts, we solve S-DS-II($\mathcal{N}, n$) as a series of continuous knapsack problems as proposed in Taherkhani et al. (2020).

5. Solution Scheme for the Robust-Stochastic Models

In this section, we present exact algorithms based on BD coupled with SAA to solve the robust-stochastic versions of the problem both with revenue-elastic and independent demand.

5.1. Revenue-Elastic Demand

For the min-max regret robust-stochastic model with revenue-dependent demand, for each revenue scenario $s \in S_r$, we approximate the corresponding random demand variable ξ^s with a sample $\mathcal{N}^s$, which captures the linear dependency of demand to revenue. With this approximation, we may linearize the stochastic problem (15) as the following

$$\min \quad V \tag{47}$$

$$\text{s.t.} \quad V \geq Z^s - \frac{1}{|\mathcal{N}^s|} [\sum_{n \in \mathcal{N}^s} \sum_{k \in K} \sum_{m \in M} \sum_{a \in A_k} (r_k^{ms} - \hat{C}_{ak}) w_k^{msn} x_{ak}^{msn}] + \sum_{i \in H} f_i y_i \qquad s \in S_r \tag{48}$$

$$\boldsymbol{x}^{sn} \in \mho^{sn}(\boldsymbol{y}) \qquad s \in S_r, n \in \mathcal{N}^s \tag{49}$$

$$y_i \in \{0,1\} \qquad i \in H, \tag{50}$$

in which $\bar{Z}^s$ is an estimation of Z^s defined in Section 3.2 and $\mho^{sn}(\boldsymbol{y})$ is the set of feasible routing solutions $\boldsymbol{x}^{sn} = (x_{ak}^{msn})$ under revenue scenario $s \in S_r$ and demand scenario $n \in \mathcal{N}^s$ for a given solution $\boldsymbol{y}$ as defined by (33)–(36).

We assume that the $(\boldsymbol{y}, V)$ variables are handled in the master problem and the rest is left to the subproblem. Note that for a given $(\bar{\boldsymbol{y}}, \bar{V})$, the remaining problem is a feasibility problem in the space of the x-variables for each revenue s. Let RS-I-DS(n,s) denote the dual subproblem equivalent to problem (37)–(40) for a given demand scenario $n \in \mathcal{N}_s$, in which r_k^{mn} is replaced with r_k^{ms}, and let $P^s_{\mathcal{N}^s}$ be the set of extreme points of the feasible region of the aggregated subproblems. As shown in Appendix EC.2 and following the procedure presented in Section 4, we reformulate (47)–(50) as the following master problem

$$\begin{aligned}
\text{RS-I-MP} \quad & \max \quad \eta - \sum_{i\in H} f_i y_i \\
& \text{s.t.} \quad \eta \le \frac{1}{|\mathcal{N}^s|} \sum_{n\in\mathcal{N}^s} \left(\sum_{k\in K}\sum_{m\in M} \alpha_k^{mn} + \sum_{i\in H} y_i \big(\sum_{k\in K}\sum_{m\in M} u_{ik}^{mn} + \Gamma_i b_i^n\big)\right) - \bar{Z}^s \qquad s\in S_r, (\boldsymbol{\alpha}, \boldsymbol{u}, \boldsymbol{b}) \in P^s_{\mathcal{N}^s} \quad (51)\\
& \qquad y_i \in \{0,1\} \qquad i\in H,
\end{aligned}$$

where $\eta = -V + \sum_{i\in H} f_i y_i$. Consequently, upon solving RS-I-MP at iteration e and obtaining the MP solution $(\eta^e, \boldsymbol{y}^e)$, $V^e := -\eta^e + \sum_{i\in H} f_i y_i^e$ provides a lower bound on the minimum regret. Provided the optimal values of the subproblems RS-I-DS(n,s) for a given master solution $\boldsymbol{y}^e$ denoted by Z^{esn}_{DS}, we can obtain an upper bound on the minimum regret using (15) via

$$\max_{s\in S_r} \left\{ \bar{Z}^s - \left(\frac{1}{|\mathcal{N}^s|} \sum_{n\in\mathcal{N}^s} Z^{esn}_{\text{DS}} - \sum_{i\in H} f_i y_i^e \right) \right\}.$$

Thus, we solve RS-I-MP using the BD algorithm described in Algorithm 2. Note that, as shown in EC.2, we can use the cut generation algorithm described for S-MP in Section 4 to produce cuts of the form (51).

5.2. Independent Demand with Max-Min Criterion

To solve the robust-stochastic model with max-min criterion, we again approximate the demand realizations (i.e., ξ) with a sample $\mathcal{N}$, and take the y-variables as the master problem solutions. As demonstrated in Appendix EC.3, we may restate the SAA counterpart of problem (28) as the following Benders master problem

$$\begin{aligned}
\text{RS-II-MP}(\mathcal{N}) \quad & \max \quad \frac{1}{|\mathcal{N}|} \sum_{n\in\mathcal{N}} \eta^n - \sum_{i\in H} f_i y_i \\
& \text{s.t.} \quad \eta^n \le \sum_{k\in K}\sum_{m\in M} \alpha_k^{mn} + \sum_{i\in H} y_i \big(\Gamma_i b_i^n + \sum_{k\in K}\sum_{m\in M} u_{ik}^{mn}\big) \qquad n\in\mathcal{N}, (\boldsymbol{\beta}^n, \boldsymbol{\alpha}^n, \boldsymbol{u}^n, \boldsymbol{b}^n) \in \hat{P}^n_{\mathcal{N}} \\
& \qquad y_i \in \{0,1\} \qquad i\in H,
\end{aligned}$$

where $\hat{P}^n_{\mathcal{N}}$ is the set of extreme points of the dual subproblem associated with scenario $n \in \mathcal{N}$ formulated as:

$$\begin{aligned}
\text{RS-II-DS}(\mathcal{N}, n) \quad & \min \sum_{k\in K}\sum_{m\in M} \alpha_k^{mn} + \sum_{i\in H} \bar{y}_i \big(\sum_{k\in K}\sum_{m\in M} u_{ik}^{mn} + \Gamma_i b_i^n\big) \\
\text{s.t.} \quad & \hat{r}_k^m w_k^{mn} \beta_k^{mn} + \alpha_k^{mn} + u_{ik}^{mn} + u_{jk}^{mn} + w_k^{mn} (b_i^n + b_j^n) \ge (\bar{r}_k^m - \hat{C}_{ijk}) w_k^{mn}
\end{aligned}$$

Algorithm 2 Benders decomposition for robust-stochastic model with revenue-elastic demand

$UB \leftarrow +\infty, LB \leftarrow -\infty, e \leftarrow 1$
$P^s_{\mathcal{N}^s} \leftarrow \emptyset \quad \forall s \in S_r$
while $LB < UB$ **do**
  SOLVE RS-I-MP to obtain $\boldsymbol{y}^e$ and η^e
  $LB \leftarrow -\eta^e + \sum_{i\in H} f_i y_i^e$
  for s in S_r **do**
    for n in $\mathcal{N}^s$ **do**
      SOLVE RS-I-DS(n,s) with $\bar{\boldsymbol{y}} = \boldsymbol{y}^e$ to obtain $(\boldsymbol{\alpha}^{esn}, \boldsymbol{u}^{esn}, \boldsymbol{b}^{esn})$ and Z^{esn}_{DS}
    end for
    $P^s_{\mathcal{N}^s} \leftarrow P^s_{\mathcal{N}^s} \cup \{(\boldsymbol{\alpha}^{esn}, \boldsymbol{u}^{esn}, \boldsymbol{b}^{esn})_n\}$
  end for
  $UB \leftarrow \min\left\{UB, \max_{s\in S_r}\left\{\bar{Z}^s - \left(\frac{1}{|\mathcal{N}^s|}\sum_{n\in\mathcal{N}^s} Z^{esn}_{\mathrm{DS}} - \sum_{i\in H} f_i y_i^e\right)\right\}\right\}$
  $e \leftarrow e+1$
end while

$$k\in K, m\in M, (i,j)\in A_k : i\neq j \qquad (52)$$

$$\hat{r}^m_k w^{mn}_k \beta^{mn}_k + \alpha^{mn}_k + u^{mn}_{ik} + w^{mn}_k b^n_i \geq (\bar{r}^m_k - \hat{C}_{iik}) w^{mn}_k \qquad k\in K, m\in M, i\in H \qquad (53)$$

$$\sum_{k\in K}\sum_{m\in M}\beta^{mn}_k \leq \gamma_r$$

$$\beta^{mn}_k \leq 1 \qquad k\in K, m\in M$$

$$\beta^{mn}_k, \alpha^{mn}_k, u^{mn}_{ik}, b^n_i \geq 0 \qquad k\in K, m\in M, i\in H.$$

A pseudo-code of the basic BD algorithm to solve the robust-stochastic model with max-min criterion is presented in Algorithm 3. As described in the previous sections, we solve the dual subproblem with a two-phase procedure by removing the variables u^{mn}_{ik} and b^n_i associated with $i\in H^0_e$, and calculating the values of the remaining variables via the following Phase I subproblem

$$\text{RS-II-DS-I}(\mathcal{N},n) \quad \min \sum_{k\in K}\sum_{m\in M}\alpha^{mn}_k + \sum_{i\in H^1_e}\Gamma_i b^n_i$$

$$\text{s.t.} \quad \hat{r}^m_k w^{mn}_k \beta^{mn}_k + \alpha^{mn}_k + w^{mn}_k (b^n_i + b^n_j) \geq (\bar{r}^m_k - \hat{C}_{ijk}) w^{mn}_k \qquad k\in K, m\in M, (i,j)\in A^1_{ke}$$

$$\hat{r}^m_k w^{mn}_k \beta^{mn}_k + \alpha^{mn}_k + w^{mn}_k b^n_i \geq (\bar{r}^m_k - \hat{C}_{iik}) w^{mn}_k \qquad k\in K, m\in M, i\in H^1_e$$

$$\sum_{k\in K}\sum_{m\in M}\beta^{mn}_k \leq \gamma_r$$

$$\beta^{mn}_k \leq 1 \qquad k\in K, m\in M$$

$$\beta^{mn}_k, \alpha^{mn}_k, b^n_i \geq 0 \qquad k\in K, m\in M, i\in H^1_e.$$

Upon solving the Phase I subproblem, we compute the optimal value of the remaining variables in Phase II. As the β-variables are computed in Phase I, constraints (52) and (53) associated with each commodity can respectively be rewritten as:

$$\alpha_k^{mn} + u_{ik}^{mn} + u_{jk}^{mn} + w_k^{mn}(b_i^n + b_j^n) \geq (\bar{r}_k^m - \hat{r}_k^m \beta_k^{mn} - \hat{C}_{ijk})w_k^{mn} \qquad (i,j) \in A_k : i \neq j$$

$$\alpha_k^{mn} + u_{ik}^{mn} + w_k^{mn} b_i^n \geq (\bar{r}_k^m - \hat{r}_k^m \beta_k^{mn} - \hat{C}_{iik})w_k^{mn} \qquad i \in H.$$

Note that β_k^{mn} expresses the extent to which revenue of commodity (k,m) is subject to uncertainty. Consequently, obtaining the optimal value of the β-variables implies resolving the data uncertainty in Phase I. Hence, in Phase II, we work with the realized revenue r_k^m (i.e., $r_k^m = \bar{r}_k^m - \hat{r}_k^m \beta_k^{mn}$) for each commodity (k,m). Therefore, the Phase II subproblem can be formulated as the LP (43)–(46), which can be solved again as a series of continuous knapsack problems.

Algorithm 3 Benders decomposition for the robust-stochastic model with max-min criterion

1: $UB \leftarrow +\infty, LB \leftarrow -\infty, e \leftarrow 1$
2: $\hat{P}_{\mathcal{N}}^n \leftarrow \emptyset \quad \forall n \in \mathcal{N}$
3: **while** $LB < UB$ **do**
4:     **SOLVE** RS-II-MP($\mathcal{N}$) to obtain $\boldsymbol{y}^e$ and Z_{MP}^e
5:     $UB \leftarrow Z_{\text{MP}}^e$
6:     **for** n in $\mathcal{N}$ **do**
7:         **SOLVE** RS-II-DS($\mathcal{N}, n$) with $\bar{\boldsymbol{y}} = \boldsymbol{y}^e$ to obtain $(\boldsymbol{\beta}^n, \boldsymbol{\alpha}^n, \boldsymbol{u}^n, \boldsymbol{b}^n)^e$ and Z_{DS}^{en}
8:         $\hat{P}_{\mathcal{N}}^n \leftarrow \hat{P}_{\mathcal{N}}^n \cup \{(\boldsymbol{\beta}^n, \boldsymbol{\alpha}^n, \boldsymbol{u}^n, \boldsymbol{b}^n)^e\}$
9:     **end for**
10:     $LB \leftarrow \max\{LB, \frac{1}{|\mathcal{N}|}\sum_{n \in \mathcal{N}} Z_{\text{DS}}^{en} - \sum_{i \in H} f_i y_i^e\}$
11:     $e \leftarrow e + 1$
12: **end while**

5.3. Independent Demand with Min-Max Regret

For the robust-stochastic model with independent demand and min-max regret objective, we approximate the demand realizations (i.e., ξ) with a sample $\mathcal{N}$. Note that $\mathcal{N}$ is defined independently of revenue scenarios. For a given demand scenario $n \in \mathcal{N}$ and revenue scenario $s \in S_r$, let $\bar{Z}^{ns}$ be an estimation of the optimal value of (29). To solve the SAA counterpart of (30), we again assume that the hub location decisions are handled in the master problem and the rest is left to the subproblem. As shown in Appendix EC.4, we may reformulate (30) as the following Benders master problem

$$\text{RS-III-MP}(\mathcal{N}) \quad \max \quad \frac{1}{|\mathcal{N}|}\sum_{n \in \mathcal{N}} \eta^n - \sum_{i \in H} f_i y_i$$

$$\text{s.t.} \quad \eta^n \leq \sum_{k\in K}\sum_{m\in M}\alpha_k^{mn} + \sum_{i\in H} y_i(\Gamma_i b_i^n + \sum_{k\in K}\sum_{m\in M} u_{ik}^{mn}) - \sum_{s\in S_r}\bar{Z}^{ns}\omega^{ns} \qquad n\in\mathcal{N}, (\boldsymbol{\alpha}^n, \boldsymbol{u}^n, \boldsymbol{b}^n, \boldsymbol{\omega}^n) \in \bar{P}_{\mathcal{N}}^n$$

$$y_i \in \{0,1\} \qquad i\in H,$$

in which $\bar{P}_{\mathcal{N}}^n$ is the set of extreme points of the feasible region of RS-III-DS($\mathcal{N},n$) formulated as

$$\text{RS-III-DS}(\mathcal{N},n) \quad \min \sum_{k\in K}\sum_{m\in M}\alpha_k^{mn} + \sum_{i\in H}\bar{y}_i(\Gamma_i b_i^n + \sum_{k\in K}\sum_{m\in M}u_{ik}^{mn}) - \sum_{s\in S_r}\bar{Z}^{ns}\omega^{ns}$$

$$\text{s.t.} \quad \sum_{s\in S_r}\omega^{ns} = 1$$

$$\alpha_k^{mn} + u_{ik}^{mn} + u_{jk}^{mn} + w_k^{mn}(b_i^n + b_j^n) \geq \sum_{s\in S_r}\omega^{ns}(r_k^{ms} - \hat{C}_{ijk})w_k^{mn} \qquad k\in K, m\in M, (i,j)\in A_k : i\neq j$$

$$\alpha_k^{mn} + u_{ik}^{mn} + w_k^{mn}b_i^n \geq \sum_{s\in S_r}\omega^{ns}(r_k^{ms} - \hat{C}_{iik})w_k^{mn} \qquad k\in K, m\in M, i\in H$$

$$\alpha_k^{mn}, u_{ik}^{mn}, b_i^n, \omega^{ns} \geq 0 \qquad k\in K, m\in M, i\in H, s\in S_r.$$

Observe that for $\omega^{n\hat{s}} = 1$ and for arbitrary $\hat{s}\in S_r$, the feasible region of RS-III-DS($\mathcal{N},n$) is equivalent to the feasible region of S-DS($\mathcal{N},n$) under revenue scenario $\hat{s}$.

For a given demand scenario $n\in\mathcal{N}$, we solve RS-III-DS($\mathcal{N},n$) in two sequential phases based on the set of open (H_e^1) and closed (H_e^0) hubs. In Phase I, the optimal value of the α- and ω-variables, along with the value of u_{ik}^{mn} and b_i^n for $i\in H_e^1$ are calculated. Similar to the stochastic case, it can be shown that the optimal value of u_{ik}^{mn} for $i\in H_e^1$ is equal to 0. Hence, the Phase I subproblem can be formulated as

$$\text{RS-III-DS-I}(\mathcal{N},n) \quad \min \sum_{k\in K}\sum_{m\in M}\alpha_k^{mn} + \sum_{i\in H_e^1}\Gamma_i b_i^n - \sum_{s\in S_r}\bar{Z}_s^n\omega^{ns}$$

$$\text{s.t.} \quad \sum_{s\in S_r}\omega^{ns} = 1$$

$$\alpha_k^{mn} + w_k^{mn}(b_i^n + b_j^n) \geq \sum_{s\in S_r}\omega^{ns}(r_k^{ms} - \hat{C}_{ijk})w_k^{mn} \qquad k\in K, m\in M, (i,j)\in A_{ke}^1$$

$$\alpha_k^{mn} + w_k^{mn}b_i^n \geq \sum_{s\in S_r}\omega^{ns}(r_k^{ms} - \hat{C}_{iik})w_k^{mn} \qquad k\in K, m\in M, i\in H_e^1$$

$$\alpha_k^{mn}, b_i^n, \omega^{ns} \geq 0 \qquad k\in K, m\in M, i\in H_e^1, s\in S_r.$$

Upon computing the optimal value of the Phase I variables, we obtain the optimal value of the rest of variables (i.e., u_{ik}^{mn} and b_i^n for $i\in H_e^0$) in Phase II. Since ω-variables take their values in Phase I, one can formulate and solve the Phase II subproblem using the adjusted revenue $r_k^m = \sum_{s\in S_r}\omega^{ns}r_k^{ms}$ for each commodity (k,m), resulting in the LP (43)–(46) given for the Phase II subproblem of the stochastic version. An overview of the BD algorithm for the min-max regret stochastic model is presented in Algorithm 4.

To improve the efficiency of all of the proposed algorithms, we apply variable fixing techniques presented in Contreras et al. (2011a) and later on improved by Taherkhani et al. (2020). In the following section we develop novel acceleration techniques specific to Benders decomposition coupled with SAA.

Algorithm 4 Benders decomposition for the min-max regret stochastic model with independent demand

$UB \leftarrow +\infty, LB \leftarrow -\infty, e \leftarrow 1$
$\bar{P}^n_{\mathcal{N}} \leftarrow \emptyset \quad \forall n \in \mathcal{N}$
while $LB < UB$ **do**
  SOLVE RS-III-MP($\mathcal{N}$) and obtain $\boldsymbol{y}^e$ and Z^e_{MP}
  $UB \leftarrow Z^e_{\text{MP}}$
  for n in $\mathcal{N}$ **do**
    SOLVE RS-III-DS($\mathcal{N}, n$) with $\bar{\boldsymbol{y}} = \boldsymbol{y}^e$ and obtain $(\boldsymbol{\alpha}^n, \boldsymbol{u}^n, \boldsymbol{b}^n, \boldsymbol{\omega}^n)^e$ and Z^{en}_{DS}
    $\bar{P}^n_{\mathcal{N}} \leftarrow \bar{P}^n_{\mathcal{N}} \cup \{(\boldsymbol{\alpha}^n, \boldsymbol{u}^n, \boldsymbol{b}^n, \boldsymbol{\omega}^n)^e\}$
  end for
  $LB \leftarrow \max\{LB, \frac{1}{|\mathcal{N}|}\sum_{n\in\mathcal{N}} Z^{en}_{\text{DS}} - \sum_{i\in H} f_i y^e_i\}$
  $e \leftarrow e + 1$
end while

6. Acceleration Techniques

We now develop acceleration techniques to reduce the computational effort to solve the SAA algorithms. We start with the general idea of transforming dual solutions from one sample to another, which layouts the statement of the acceleration methodology for each case. Our methodology is based on the observation that the cuts generated in solving sample $\hat{\mathcal{N}}$ can be transformed into valid cuts for sample $\mathcal{N}$. From scenario $\hat{n} \in \hat{\mathcal{N}}$ to scenario $n \in \mathcal{N}$, revenue, demand, or both may change. For simplicity of exposition, we first consider the individual changes, and then show how simultaneous changes can be handled. Proposition 1 shows how feasible dual solutions can be obtained by assuming that revenue under both scenarios are the same.

Proposition 1. *Let $(\boldsymbol{\alpha}^{\hat{n}}, \boldsymbol{u}^{\hat{n}}, \boldsymbol{b}^{\hat{n}})$ be a feasible solution for S-DS($\hat{\mathcal{N}}, \hat{n}$), and $w^{m\hat{n}}_k$ be the demand for commodity $k \in K$ of class $m \in M$ under scenario $\hat{n} \in \hat{\mathcal{N}}$. Then, assuming that revenue under scenarios n and $\hat{n}$ are the same, $(\boldsymbol{\alpha}^n, \boldsymbol{u}^n, \boldsymbol{b}^n)$ defined by* (54)–(56) *is feasible for S-DS($\mathcal{N}, n$):*

$$b^n_i = b^{\hat{n}}_i \qquad i \in H \tag{54}$$

$$\alpha^{mn}_k = \frac{w^{mn}_k}{w^{m\hat{n}}_k}\alpha^{m\hat{n}}_k \qquad k \in K, m \in M \tag{55}$$

$$u^{mn}_{ik} = \frac{w^{mn}_k}{w^{m\hat{n}}_k}u^{m\hat{n}}_{ik} \qquad k \in K, m \in M, i \in H. \tag{56}$$

Proof. From (55) and (56), we obtain $\alpha^{m\hat{n}}_k = \frac{w^{m\hat{n}}_k}{w^{mn}_k}\alpha^{mn}_k$ and $u^{m\hat{n}}_{ik} = \frac{w^{m\hat{n}}_k}{w^{mn}_k}u^{mn}_{ik}$, respectively. Feasibility of $(\boldsymbol{\alpha}^n, \boldsymbol{u}^n, \boldsymbol{b}^n)$ for S-DS($\mathcal{N}, n$) can easily be verified by replacing $b^{\hat{n}}_i$, $\alpha^{m\hat{n}}_k$, and $u^{m\hat{n}}_{ik}$ respectively with b^n_i, $\frac{w^{m\hat{n}}_k}{w^{mn}_k}\alpha^{mn}_k$, and $\frac{w^{m\hat{n}}_k}{w^{mn}_k}u^{mn}_{ik}$, in constraints (38)–(40) associated with S-DS($\hat{\mathcal{N}}, \hat{n}$) and noting that revenues are the same under both scenarios. □

Similarly, Proposition 2 shows how feasible dual solutions can be obtained by assuming that demands under both scenarios are the same.

PROPOSITION 2. *Let* $(\boldsymbol{\alpha}^{\hat{n}}, \boldsymbol{u}^{\hat{n}}, \boldsymbol{b}^{\hat{n}})$ *be a feasible solution for S-DS*$(\hat{\mathcal{N}}, \hat{n})$*, and* $r_k^{m\hat{n}}$ *be the revenue for commodity* $k \in K$ *of class* $m \in M$ *under scenario* $\hat{n} \in \hat{\mathcal{N}}$*. Then, assuming that demands under scenarios* n *and* $\hat{n}$ *are the same,* $(\boldsymbol{\alpha}^{n}, \boldsymbol{u}^{n}, \boldsymbol{b}^{n})$ *defined by* (57)–(59) *is feasible for S-DS*$(\mathcal{N}, n)$*:*

$$b_i^n = b_i^{\hat{n}} \qquad i \in H \tag{57}$$

$$\alpha_k^{mn} = \max\{0, \alpha_k^{m\hat{n}} + w_k^{mn}(r_k^{mn} - r_k^{m\hat{n}})\} \qquad k \in K, m \in M \tag{58}$$

$$u_{ik}^{mn} = u_{ik}^{m\hat{n}} \qquad k \in K, m \in M, i \in H. \tag{59}$$

Proof. Replacing $b_i^{\hat{n}}$ and $u_{ik}^{m\hat{n}}$ respectively with b_i^n and u_{ik}^{mn} in constraints (38) and (39) of S-DS$(\hat{\mathcal{N}}, \hat{n})$, and adding $w_k^{mn}(r_k^{mn} - r_k^{m\hat{n}})$ to both sides of these constraints yield

$$\alpha_k^{m\hat{n}} + w_k^{mn}(r_k^{mn} - r_k^{m\hat{n}}) + u_{ik}^{mn} + u_{jk}^{mn} + w_k^{mn}(b_i^n + b_j^n) \geq (r_k^{mn} - \hat{C}_{ijk})w_k^{mn}$$
$$\alpha_k^{m\hat{n}} + w_k^{mn}(r_k^{mn} - r_k^{m\hat{n}}) + u_{ik}^{mn} + w_k^{mn} b_i^n \geq (r_k^{mn} - \hat{C}_{iik})w_k^{mn}.$$

Hence, any $\alpha_k^{mn} \geq 0$ satisfying $\alpha_k^{mn} \geq \alpha_k^{m\hat{n}} + w_k^{mn}(r_k^{mn} - r_k^{m\hat{n}})$ provides a feasible solution to S-DS$(\mathcal{N}, n)$. □

Using Propositions 1 and 2, we may obtain feasible dual solutions when both revenue and demand are different via Proposition 3.

PROPOSITION 3. *Let* $(\boldsymbol{\alpha}^{\hat{n}}, \boldsymbol{u}^{\hat{n}}, \boldsymbol{b}^{\hat{n}})$ *be a feasible solution for S-DS*$(\hat{\mathcal{N}}, \hat{n})$*, and* $w_k^{m\hat{n}}$ *and* $r_k^{m\hat{n}}$ *be the demand and revenue, respectively, for commodity* $k \in K$ *of class* $m \in M$ *under scenario* $\hat{n} \in \hat{\mathcal{N}}$*. Then,* $(\boldsymbol{\alpha}^{n}, \boldsymbol{u}^{n}, \boldsymbol{b}^{n})$ *defined by* (60)–(62) *is feasible for S-DS*$(\mathcal{N}, n)$*:*

$$b_i^n = b_i^{\hat{n}} \qquad i \in H \tag{60}$$

$$\alpha_k^{mn} = \max\{0, \frac{w_k^{mn}}{w_k^{m\hat{n}}}\alpha_k^{m\hat{n}} + w_k^{mn}(r_k^{mn} - r_k^{m\hat{n}})\} \qquad k \in K, m \in M \tag{61}$$

$$u_{ik}^{mn} = \frac{w_k^{mn}}{w_k^{m\hat{n}}} u_{ik}^{m\hat{n}} \qquad k \in K, m \in M, i \in H. \tag{62}$$

Proof. The equations follow from first applying Proposition 1 and then Proposition 2. □

Consequently, the solution obtained by (60)–(62) yields a valid cut for S-MP$(\mathcal{N})$. To avoid overloading the master problem with too many cuts, we restrict the algorithm to selecting the best potential cut that is associated with scenario $\tilde{n} \in \hat{\mathcal{N}}$, which accounts for the least relative deviation from scenario $n \in \mathcal{N}$, using the following equation:

$$\tilde{n} = \arg\min_{\hat{n} \in \hat{\mathcal{N}}} \left\{ \sum_{k \in K} \sum_{m \in M} \left(\frac{|w_k^{m\hat{n}} - w_k^{mn}|}{w_k^{mn}} + \frac{|r_k^{m\hat{n}} - r_k^{mn}|}{r_k^{mn}} \right) \right\}. \tag{63}$$

We remark that if we eliminate a set of hubs through any variable fixing techniques (Contreras et al. 2011a, Taherkhani et al. 2020), the dual variables associated with those hubs will not be computed in the subproblem. Therefore, we solve the first replication of SAA without using any variable fixing and obtain the complete dual solutions. We then use these solutions for producing initial cuts in the subsequent iterations, and continue with performing variable fixing in those replications.

For the robust-stochastic models, in particular with the min-max regret objective (i.e., RS-I and RS-III), the SAA counterparts of the models exhibit a more repetitive structure, which can be exploited further as discussed in the following subsections.

6.1. Revenue-Elastic Demand

The SAA algorithm has several replications, we exploit the fact that each replication has the capability of passing information to the other. Recall that each replication of the SAA algorithm for solving the robust stochastic model with revenue-elastic demand (RS-I) involves two steps. In the first step (SAA-I), one needs to compute the $\bar{Z}^s$ values for each revenue scenario $s \in S_r$, which amounts to performing $|S_r|$ replications of Algorithm 1 with $\mathcal{N}_s$ demand scenarios, one for each $s \in S_r$. In the second step (SAA-II), with the provided $\bar{Z}^s$ values, one should solve a min-max regret SAA problem to find the optimal hub locations. Although the additional step SAA-I imposes extra computational effort, if treated carefully, its repetitive structure can be exploited for speeding up the overall SAA algorithm.

At the first replication of the SAA algorithm, we compute $\bar{Z}^s$ for $s = 1$ using Algorithm 1, which as a byproduct, produces several dual solutions $(\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{u}}, \hat{\boldsymbol{b}})$. We can transform these solutions to feasible cuts using Proposition 3 and speed up Algorithm 1 for computing $\bar{Z}^2$. In general, we accelerate SAA-I for computing $\bar{Z}^s$ by generating initial cuts from the dual solutions produced in SAA-I for $\hat{s} < s$. On the other hand, the generated dual solutions can be directly employed for generating initial cuts for the second step SAA-II. Hence, we can reduce the computation time for performing the first replication of the SAA algorithm. We repeat this procedure in the subsequent replications of the SAA algorithm, with the difference that we can also transform the dual solutions produced in the first replication of SAA (either in SAA-I or SAA-II) and generate initial cuts for each component of these replications.

Although producing initial cuts can have a profound impact on accelerating the overall SAA algorithm, computing the optimal $\bar{Z}^s$ values in SAA-I is still computationally burdensome and also unnecessary. Note that, in the min-max regret criterion, we use the $\bar{Z}^s$ as reference values for comparing objective values under different revenue realizations. Hence, one can use near-optimal values instead of the optimal $\bar{Z}^s$ to reduce the computational time. Our approach for approximating these values is based on the observation that, for sufficiently large demand samples, the optimal hub locations under each specific revenue scenario remain the same from one replication to another. Hence, in the first replication of the SAA algorithm, we perform a complete round of Algorithm 1 for each revenue scenario $s \in S_r$ to obtain the optimal hub locations $\hat{\boldsymbol{y}}^s$. In the subsequent replications, we fix the hub locations at $\hat{\boldsymbol{y}}^s$ and perform a single iteration of Algorithm 1 and set $\bar{Z}^s$ to the objective value corresponding to $\hat{\boldsymbol{y}}^s$. Our computational results show that this approximation scheme reduces the computation time by orders of magnitude. Additional implementation details are provided in Appendix EC.5.

6.2. Independent Demand

6.2.1. Max-min criterion. For the robust-stochastic model with max-min criterion (RS-II), the SAA algorithm can be enhanced just by adapting Proposition 1 as only demand changes from one sample to another. Let $(\boldsymbol{\beta}^{\hat{n}}, \boldsymbol{\alpha}^{\hat{n}}, \boldsymbol{u}^{\hat{n}}, \boldsymbol{b}^{\hat{n}})$ be a feasible solution for RS-II-DS$(\hat{\mathcal{N}}, \hat{n})$. It can easily be shown that $(\boldsymbol{\beta}^{n}, \boldsymbol{\alpha}^{n}, \boldsymbol{u}^{n}, \boldsymbol{b}^{n})$ with $(\boldsymbol{\alpha}^{n}, \boldsymbol{u}^{n}, \boldsymbol{b}^{n})$ defined by (54)–(56) and $\boldsymbol{\beta}^{n} = \boldsymbol{\beta}^{\hat{n}}$ is feasible for RS-II-DS$(\mathcal{N}, n)$. Hence, we can use these cuts to warm start the subsequent replications of the SAA algorithm.

6.2.2. Min-max regret. The SAA counterpart of the min-max regret stochastic model with independent demand (RS-III) is in structure and nature similar to that of RS-I. Hence, we devise similar acceleration strategies by exploiting this repetitive structure. As per RS-I, the main computational challenge in RS-III is computing the $\bar{Z}^{ns}$ value for each pair of demand and revenue scenario (n, s), which amounts to solving the deterministic counterparts of (29) $|\mathcal{N}| \times |S_r|$ times. Hence, cardinality of the sets $\mathcal{N}$ and S_r drastically affect the computational efficiency of solving the RS-III problem, which warrants an efficient method for estimating these $\bar{Z}^{ns}$ values. In the following, we provide an overview of the proposed acceleration technique.

Our proposed acceleration technique for the SAA counterpart of RS-III is four-fold: (i) generating valid cuts for solving scenario pair (n, s) from the cuts generated for solving scenario pair $(\hat{n}, s)$, (ii) generating valid cuts for solving scenario pair (n, s) from the cuts generated for solving scenario pair $(n, \hat{s})$, (iii) approximating the $\bar{Z}^{ns}$ values, and (iv) generating valid cuts for solving the min-max regret stochastic problem from the cuts generated for obtaining the $\bar{Z}^{ns}$ values. Steps (i) and (ii) can be achieved by applying Propositions 1 and 2, respectively. We elaborate steps (iii) and (iv) in the following.

To obtain near-optimal values for the $\bar{Z}^{ns}$ values, we observe that similar demand scenarios are likely to result in the same optimal hub locations. Therefore, similar to RS-I, we only calculate the optimal $\bar{Z}^{ns}$ values for the first replication of the SAA algorithm, and approximate the $\bar{Z}^{ns}$ values (step iii) for the subsequent replications as follows. Let $\mathcal{N}_t$ denote the realized demand sample $\mathcal{N}$ at replication t of the SAA algorithm, for $t = 1, \ldots, \mathcal{M}$. Upon completing the first replication of the SAA algorithm, we obtain $|\mathcal{N}_1| \times |S_r|$ hub locations $\hat{\boldsymbol{y}}^{ns}$, one for each $n \in \mathcal{N}_1$ and $s \in S_r$. At replication $t > 1$, for a given demand scenario $n \in \mathcal{N}_t$, let $\hat{n}$ be the closest demand scenario to n among the demand scenarios in the first replication, as selected via (63). We estimate $\bar{Z}^{ns}$ by fixing $\boldsymbol{y}$ at $\hat{\boldsymbol{y}}^{\hat{n}s}$, where $\hat{\boldsymbol{y}}^{\hat{n}s}$ is the optimal location of the hubs under the scenario pair $(\hat{n}, s)$ for $\hat{n} \in \mathcal{N}_1$ and $s \in S_r$.

Finally, in step (iv), once the $\bar{Z}^{ns}$ values are obtained, we generate valid cuts for RS-III using Proposition 4 below.

Proposition 4. *For a given $\hat{s} \in S_r$, let $(\boldsymbol{\alpha}^{n\hat{s}}, \boldsymbol{u}^{n\hat{s}}, \boldsymbol{b}^{n\hat{s}})$ be a feasible solution for S-DS$(\mathcal{N}, n)$, then $(\boldsymbol{\alpha}^n, \boldsymbol{u}^n, \boldsymbol{b}^n, \boldsymbol{\omega}^n)$ is feasible for RS-III-DS$(\mathcal{N}, n)$, where $(\boldsymbol{\alpha}^n, \boldsymbol{u}^n, \boldsymbol{b}^n) = (\boldsymbol{\alpha}^{n\hat{s}}, \boldsymbol{u}^{n\hat{s}}, \boldsymbol{b}^{n\hat{s}})$ and*

$$\omega^{ns} = 1 \text{ if } s = \hat{s}, \text{ and } \omega^{ns} = 0 \text{ if } s \neq \hat{s} \qquad s \in S_r.$$

Proof. The proof follows from the fact that the feasible region of S-DS$(\mathcal{N}, n)$ under revenue scenario $\hat{s}$ is equivalent to the feasible region of RS-III-DS$(\mathcal{N}, n)$ when $\omega^{n\hat{s}}$ is set to 1. □

In Appendix EC.6, we provide the pseudocode of the acceleration algorithm that can be adopted for any implementation of Benders decomposition coupled with SAA.

7. Computational Experiments

We performed extensive computational experiments to test the performance of our models and algorithms. We use the well-known Australia Post (AP) dataset, the most commonly used dataset in hub location literature (Ernst and Krishnamoorthy 1996). The distances and the postal flow between pairs of nodes are provided in OR Library (Beasley 1990). Two different sets for installation costs and capacities of hubs are available on the AP dataset referred to as loose (L) and tight (T). In our experiments, we assume that all nodes can potentially become hubs (i.e., $H = N$) and each pair of nodes generate a separate commodity (i.e., $K = N \times N$).

The demand of commodities are segmented into three classes ($|M| = 3$). For each class, the nominal demand is set as $\bar{w}_k^m = \rho_m^k w_k$, where ρ_m^k is the proportion of demand of segment m of commodity k. For the revenue from commodity $k \in K$ of class $m \in M$, we generate the nominal values using $\bar{r}_k^m = \upsilon^m \frac{c_k}{w_k^m}$, where $\upsilon^1 \sim U[50, 60]$, $\upsilon^2 \sim U[40, 50]$, and $\upsilon^3 \sim U[30, 40]$. Collection, transfer, and distribution costs per unit are taken as $\chi = 2$, $\alpha = 0.75$, and $\delta = 3$ as defined in the AP dataset (Beasley 1990). We tested instances with $|N| \in \{10, 20, 25, 40, 50, 75\}$ and denote each instance as $nf\Gamma$ where n is the instance size, f is the installation cost, and Γ is the capacity.

For the SAA, we use the sample size and the number of replications as $|\mathcal{N}| = 50$ and $|\mathcal{M}| = 60$, respectively, since this combination provides the best trade-off between solution quality and computational time.

Computational experiments were carried out on a workstation that contains: Intel Core i7-3930K 2.61GHz CPU, and 39 GB of RAM. The algorithms were coded in $C\#$ and the time limit was set to 24 hours. The master problems of all versions of the Benders decomposition algorithms as well as the Phase I subproblems were solved using the callable library of CPLEX 12.7.

7.1. Stochastic Model

This section presents the results with the stochastic model on the AP dataset. In our analysis, we first consider the case where all fluctuations in demand are due to revenue. Accordingly, in Equation (7), we set $\boldsymbol{\epsilon}_k = 0$ and $W = [W_{ij}]$ with $W_{ij} = 0.3\bar{\rho}$ for $i = j$, $W_{ij} = -0.1\bar{\rho}$ for $|i-j| = 1$, and $W_{ij} = 0$ otherwise, where $\bar{\rho} = \frac{\sum_{k\in K}\sum_{m\in M}\bar{w}_k^m}{\sum_{k\in K}\sum_{m\in M}\bar{r}_k^m}$ is the ratio of total nominal demand to total nominal revenue. We then consider the case where demand and revenue are independent (i.e., $W = 0$). To generate samples for revenue and demand, we assume that the total revenue and total demand of each commodity k (i.e., r_k, and w_k) are drawn from normal distributions in which the mean revenue and mean demand are set to be the nominal revenue and nominal demand (i.e., $\bar{r}_k$ and $\bar{w}_k$) and their standard deviations are equal to $\sigma_k^r = \nu\bar{r}_k$ and $\sigma_k^w = \nu\bar{w}_k$, respectively, where ν is set to 0.5. Note that in the first case, since demand is dependent to revenue, we need to generate samples only for the revenue.

We first evaluate the performance of the acceleration techniques proposed for the stochastic model. Computational times on the 10–20 node instances from the AP dataset without and with the implementation of the acceleration techniques for revenue-elastic demand and independent demand are reported in Table 1. As observed from the table, the algorithm runs around two times faster on average with the implementation of the acceleration techniques. Accordingly, the rest of the computational results with the stochastic model are performed using these techniques.

Table 1 Computational times for the stochastic model without and with the implementation of acceleration techniques.

$\|N\|$	Time (sec)			
	Revenue-elastic demand		Independent demand	
	Without acceleration	With acceleration	Without acceleration	With acceleration
10LL	92	59	113	41
10LT	85	32	108	49
10TL	108	76	96	43
10TT	98	42	104	44
20LL	2,053	1,262	3,964	2,524
20LT	2,831	1,779	2,928	1,848
20TL	1,065	568	898	477
20TT	518	256	779	298
Average	856	509	1,124	666

The computational results are summarized in Table 2. The first column presents the instance size and name. The columns labeled “Profit”, “Avg. iter.”, and “Time (sec)” indicate the net profit, the average number of iterations required for the convergence of the BD algorithm at each replication of SAA, and the computation time of the instances (in seconds) obtained from solving the model, respectively. The columns labeled "Open hubs" show the locations of the hub nodes. The table is split into two parts to present the results for the revenue-elastic demand and independent demand cases.

Table 2 **Computational results for the stochastic model with revenue-elastic demand and independent demand.**

Instance	Revenue-elastic demand				Independent demand			
$\|N\|$	Profit	Avg. iter.	Time (sec)	Open hubs	Profit	Avg. iter.	Time (sec)	Open hubs
10LL	18,779	3.00	59	5,9,10	26,132	2.06	41	4,5,10
10LT	8,642	2.03	32	5,10	9,546	3.03	49	4,5,10
10TL	13,717	6.80	76	5,9	18,595	4.33	43	5,9
10TT	7,977	5.55	42	5,9	11,782	7.88	44	5,9
20LL	104,818	7.48	1,262	7,10,19	118,968	13.37	2,524	7,10,18,19
20LT	71,668	7.93	1,779	5,10,12,14	83,529	7.13	1,848	5,10,12,14,19
20TL	65,472	6.55	568	7,10	75,942	5.37	477	7,10
20TT	29,217	3.97	256	5,10	36,095	4.00	298	5,10
25LL	125,416	13.13	4,031	9,17,19	142,163	21.13	7,247	7,14,17,23
25LT	98,762	6.07	2,586	6,9,12,14,25	113,087	6.23	2,748	6,9,12,14,25
25TL	89,340	5.30	1,433	6,9,14	103,484	6.20	1,676	6,9,14
25TT	56,154	10.90	3,026	6,9,10,14	67,351	9.20	2,700	6,10,14,25
40LL	75,097	31.67	25,796	12,22,26,29	87,734	33.96	29,014	12,22,26,29
40LT	67,142	28.93	31,968	12,14,26,30,38	79,012	34.63	43,685	12,14,26,30,38
40TL	61,709	11.43	4,475	14,29	73,360	12.76	6,279	14,19,29
40TT	52,342	8.23	5,600	14,19,35,38	63,200	9.00	7,262	14,19,25,38
50LL	70,797	38.53	45,455	15,27,33,35	83,486	34.66	48,182	15,28,33,35
50LT	67,353	19.43	27,597	6,26,32,46	79,933	15.90	25,114	6,26,32,46
50TL	54,095	9.13	5,504	26,45	65,643	10.96	7,218	3,26,45
50TT	50,253	2.07	1,810	26,48	59,562	5.10	4,150	26,48
75LL	197,518	42.95	69,562	14,23,35,37,56	214,874	44.63	73,318	5,14,23,35,37,56
75LT	153,764	36.68	71,529	14,25,32,35,46,59	166,818	38.09	74,971	14,26,32,35,46,59
75TL	92,776	9.90	34,299	14,35,37	111,644	9.30	35,330	14,35,37
75TT	81,871	15.00	29,400	25,32,38	98,900	14.20	34,789	25,32,38,59

All instances in Table 2 are solved to optimality. The average computational times for revenue-elastic demand and independent demand cases are 4.26 and 4.73 hours, respectively. The number of open hubs increases or remains the same when revenue and demand become independent. Locations of the hubs, on the other hand, do not vary significantly except for the instance 25LL, where for the revenue-elastic demand case, hubs are located at nodes 9, 17, 19, while for the independent demand case, hubs are located at nodes 7, 14, 17, 23.

Observe from Table 2 that for each instance, the net profit value obtained from the revenue-elastic demand case is lower than that of the independent demand case. We use the following toy example to explain why this happens. Assume that we have a single commodity between two hubs of sufficiently large capacities, and that the demand is fully satisfied. We first consider the case where revenue (r) and demand (w) of the commodity are independent random variables with expected values $\bar{r}$ and $\bar{w}$, respectively. Trivially, the total profit for satisfying the demand of this commodity is $\mathbb{E}[w(r-C)] = \bar{w}(\bar{r}-C)$, where C is the transportation cost. Next, we consider the revenue-elastic demand case in which demand and revenue are dependent random variables through the demand-revenue function $w = \bar{w} - W(r-\bar{r})$, with $W > 0$. It is not difficult to show that under this case $\mathbb{E}[w(r-C)] = \bar{w}(\bar{r}-C) - W(\sigma^r)^2$ which is strictly smaller than that of the independent demand case.

Table 3 **Percentage of demand satisfied for each demand class with the stochastic model.**

Instance type	Revenue-elastic demand				Independent demand			
	Demand class (%)			Avg.	Demand class (%)			Avg.
	$m=1$	$m=2$	$m=3$		$m=1$	$m=2$	$m=3$	
LL	98.56	88.47	68.87	85.30	99.83	91.33	73.62	88.26
LT	93.83	78.17	47.68	73.23	95.02	81.25	50.97	75.75
TL	94.14	79.33	50.47	74.65	96.07	82.51	56.57	78.38
TT	82.98	57.38	32.96	57.78	86.87	62.72	38.75	62.78

Table 3 presents the percentages of satisfied demand from different market segments. The averages for each demand class are calculated over instances from Table 2 with the same type of installation costs and capacities. The average percentages of total satisfied demand are provided in the last columns corresponding to revenue-elastic demand and independent demand cases. The percentage of satisfied demand for all three market segments for the revenue-elastic demand case are lower than that of the independent demand case. In the instances with the same configuration of hub installation costs and capacities, the first class is the one with the highest percentages of satisfied demand, while the third demand class has the lowest. This is because serving the first and third classes result the highest and lowest revenue, respectively. On average, instances with loose capacities (LL and TL) yield higher percentages compared to the instances with tight capacities (LT and TT).

7.2. Robust-Stochastic Model with Revenue-Elastic Demand

In this section, we analyze the results obtained from the min-max regret stochastic model with revenue-elastic demand. For each instance, we perform two sets of experiments each involving five different scenarios with uncertain revenue (i.e., $|S_r| = 5$). In the first set, revenue scenarios are randomly generated from the interval $[0.75\bar{r}_k^m, \bar{r}_k^m]$, while in the second set, revenue scenarios are drawn from the interval $[0.5\bar{r}_k^m, \bar{r}_k^m]$, where $\bar{r}_k^m$ is the nominal revenue of commodity k of class m. To present the dependency between revenue and demand in Equation (9), we use the same W matrix detailed in Section 7.1. We assume that the error term that captures the fluctuations in demand of commodity k (i.e., $\boldsymbol{\epsilon}_k$) is drawn from a multivariate normal distribution in which the mean is zero and the standard deviation is equal to $\sigma_{km} = \nu \ddot{w}_{km}$, where $\ddot{\boldsymbol{w}}_k = \bar{\boldsymbol{w}}_k - W(\boldsymbol{r}_k - \bar{\boldsymbol{r}}_k)$ is the expected demand and ν is the coefficient of variation that is set to 0.5.

First, the performance of the acceleration techniques proposed for this model are evaluated by taking runs on instances with 10–20 nodes. Computational times of the instances without and with the implementation of the acceleration techniques for $r_k^{ms} \in [0.75\bar{r}_k^m, \bar{r}_k^m]$ and $r_k^{ms} \in [0.5\bar{r}_k^m, \bar{r}_k^m]$ are reported in Table 4.

Table 4 Computational times for the robust-stochastic model with revenue-elastic demand without and with the implementation of acceleration techniques.

$\|N\|$	Time (sec)			
	$r_k^{ms} \in [0.75\bar{r}_k^m, \bar{r}_k^m]$		$r_k^{ms} \in [0.5\bar{r}_k^m, \bar{r}_k^m]$	
	Without acceleration	With acceleration	Without acceleration	With acceleration
10LL	195	72	117	57
10LT	152	52	97	53
10TL	93	23	71	22
10TT	101	24	82	23
20LL	14,687	1,765	12,624	1,239
20LT	8,371	1,819	5,849	1,289
20TL	3,344	223	2,833	223
20TT	1,631	166	1,272	182
Average	3,572	518	2,868	386

The results in Table 4 highlight the effectiveness of the acceleration techniques, where the algorithm performs more than six times faster on average with the implementation of the acceleration techniques even in these small-sized instances. Accordingly, the rest of the computational experiments with this model are performed using the acceleration techniques, enabling our algorithm to solve large-scale intractable instances of this problem.

We now use instances with up to 75 nodes from the AP dataset. The computational results are summarized in Table 5, which is split into two parts to represent the results for $r_k^{ms} \in [0.75\bar{r}_k^m, \bar{r}_k^m]$ and $r_k^{ms} \in [0.5\bar{r}_k^m, \bar{r}_k^m]$, respectively. In this table, the column "Regret" indicates the optimal regret of the problem and "Avg. profit" represents the average anticipated profits, which are computed by taking the average profits over 50 demand and 5 revenue scenarios in 60 replications.

All of the instances in Table 5 are solved to optimality. The CPU times and average number of iterations required for solving the instances to optimality indicate the efficiency and robustness of the algorithm and the acceleration techniques. As can be observed from Table 5, different revenue intervals have no significant impact on the performance of the algorithm. In particular, the averages of the computational time for $r_k^{ms} \in [0.75\bar{r}_k^m, \bar{r}_k^m]$ and $r_k^{ms} \in [0.5\bar{r}_k^m, \bar{r}_k^m]$ are 3.1 and 2.8 hours, respectively. These results clearly confirm the efficiency of the proposed algorithm for the robust-stochastic model with revenue-elastic demand.

Table 5 Computational results for the robust-stochastic model with revenue-elastic demand.

Instance	$r_k^{ms} \in [0.75\bar{r}_k^m, \bar{r}_k^m]$					$r_k^{ms} \in [0.5\bar{r}_k^m, \bar{r}_k^m]$				
$\|N\|$	Regret	Avg. profit	Avg. iter.	Time (sec)	Open hubs	Regret	Avg. profit	Avg. iter.	Time (sec)	Open hubs
10LL	0	14,085	1.98	72	5,9,10	41	11,204	1.93	57	5,9
10LT	73	5,920	1.98	52	5,10	119	5,195	1.93	53	5,9
10TL	39	12,695	1.98	23	5	94	8,622	1.93	22	5
10TT	68	5,173	1.98	24	5	184	4,571	1.93	23	5
20LL	108	78,731	2.02	1,765	7,9,10,19	261	53,458	1.98	1,239	7,10,19
20LT	215	40,773	2.65	1,819	5,10,12,19	603	24,549	2.95	1,289	5,10,14
20TL	259	33,605	1.98	223	7,10	420	18,307	1.97	223	7,10
20TT	283	8,270	1.98	166	10	507	4,344	1.93	182	10
25LL	28	102,272	1.87	5,030	7,14,17,23	98	73,593	1.93	3,603	7,14,17,23
25LT	54	64,924	1.87	3,306	6,9,10,12,14,25	334	42,841	2.02	2,558	9,12,14,25
25TL	136	53,007	2.10	1,638	6,9,14	483	32,530	1.93	1,311	6,9,14
25TT	218	20,637	1.87	1,473	6,10,14	519	12,174	1.93	1,035	10,14
40LL	54	66,191	1.87	9,310	12,22,26,29	397	50,825	2.21	8,042	12,22,26,29
40LT	59	56,940	1.87	11,979	12,14,26,29,30,38	421	42,184	3.20	9,174	12,14,26,30,38
40TL	0	51,157	1.87	4,349	14,19,29	217	37,213	1.97	3,601	14,29
40TT	61	40,211	1.87	4,686	14,19,25,38	479	28,045	1.97	3,782	14,19,38
50LL	101	62,462	2.00	10,562	15,28,33,35	259	47,117	2.10	8,432	15,28,33,35
50LT	95	58,016	2.00	7,339	6,26,32,46	213	43,390	1.97	5,727	6,26,32,46
50TL	132	43,414	3.00	5,739	26,45	309	30,796	1.97	3,135	26,45
50TT	252	36,771	2.00	4,638	26,48	527	27,355	1.97	3,848	26,48
75LL	117	154,614	4.21	64,517	14,23,37,38,59	374	112,834	4.00	61,114	14,23,38,59
75LT	163	114,095	4.01	58,193	14,25,32,37,59	469	78,231	3.93	53,907	14,25,32,59
75TL	47	70,136	3.00	39,568	14,35,37	171	53,197	3.01	37,613	14,35,37
75TT	79	59,624	4.00	32,741	25,32,38,59	219	41,930	3.00	33,194	25,32,59

The regret values reported in Table 5 indicate the maximum amount of profit that can be lost under this data uncertainty; implying that if the decision maker employs the obtained solution, the anticipated loss in profit is guaranteed to be less than this value. Moreover, the average profit values provide an insight on the expected profit.

We now analyze the effect of the lower bound of the interval from which revenue scenarios are generated on the optimal solutions. For a given instance, when the lower bound decreases from 0.75 to 0.5, that is, when the range of fluctuations in revenue increases, the regret of the solution increases, while the average profit and the optimal number of open hubs decrease. A wider range of fluctuations in the data may result in lower expected revenues, which in turn results in opening fewer hubs due to installation costs.

Table 6 Percentage of demand satisfied for each demand class with the robust-stochastic model with revenue-elastic demand.

Instance type	$r_k^{ms} \in [0.75\bar{r}_k^m, \bar{r}_k^m]$				$r_k^{ms} \in [0.5\bar{r}_k^m, \bar{r}_k^m]$			
	Demand class (%)			Avg.	Demand class (%)			Avg.
	$m=1$	$m=2$	$m=3$		$m=1$	$m=2$	$m=3$	
LL	97.85	85.13	53.83	78.94	95.13	79.85	48.67	74.55
LT	91.37	73.79	42.87	69.34	89.46	68.67	37.85	65.33
TL	93.67	74.67	45.84	71.39	90.54	69.74	39.83	66.70
TT	79.94	50.83	24.64	51.81	70.34	39.62	20.78	43.58

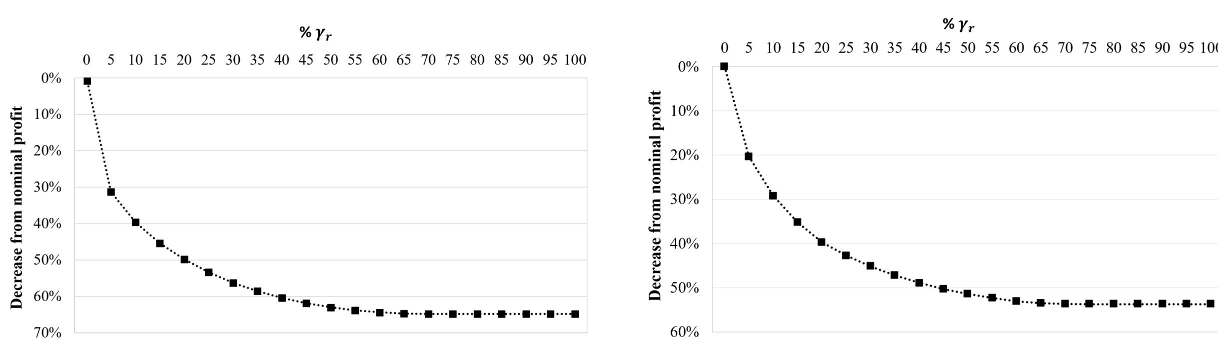
Table 6 presents the percentages of satisfied demand from different market segments for $r_k^{ms} \in [0.75\bar{r}_k^m, \bar{r}_k^m]$ and $[0.5\bar{r}_k^m, \bar{r}_k^m]$. When the lower bound of the interval decreases, the percentage of satisfied demand for all three market segments also decrease. In line with our observations from Table 3, in the instances with the same configuration, the first and third classes result in the highest and lowest percentages of satisfied demand, respectively. For a given revenue interval, the percentage of satisfied demand in the instances with loose capacities, on average, is higher than that of the instances with tight capacities.

7.3. Robust-Stochastic Models with Independent Demand

In this section, we present computational results for the robust-stochastic models with independent demand. We first present the results with the max-min criterion and then with the min-max regret objective.

7.3.1. Max-min criterion. For the analysis with the max-min model, we take $\hat{r}_k^m \sim U[0, \varphi\bar{r}_k^m]$ to generate intervals of uncertainty, where φ is the maximum possible deviation from the nominal value of revenue. We first evaluate the effect of the uncertainty budget (γ_r) on total profit. We select two instances of the AP dataset on 20 and 25 nodes with $\varphi = 0.5$, and test the model using $\gamma_r \in \{0, 5\%, 10\%, \ldots, 100\%\}$. For simplicity, we use percentage to represent the budget of uncertainty which corresponds to the percentage of the revenue parameters under uncertainty.

Figures 2(a) and 2(b) plot the percentage of decrease from the nominal profit for different values of γ_r for the AP20TL and AP25LT instances, respectively. Let Z_{γ_r} denote the optimal profit obtained from the robust-stochastic model with max-min criterion when budget of uncertainty is γ_r and Z_0 denote the objective function value with the nominal profit that can be obtained from the stochastic model. The percentage of decrease from the nominal profit can then be calculated as $\frac{Z_0 - Z_{\gamma_r}}{Z_0}$ for any γ_r.



(a) AP20TL (b) AP25LT

Figure 2 Effect of the uncertainty budget.

It is clear that a higher value of γ_r leads to a more conservative solution with a lower Z_{γ_r}. Moreover, as shown in Bertsimas and Sim (2003), Z_{γ_r} is a concave function of γ_r. Consequently, as noted from the Figures 2(a) and 2(b), for smaller values of γ_r, percentage of decrease from the nominal profit drops faster compared to higher values of the budget of uncertainty. In particular, when we select our budget of uncertainty with $\gamma_r \geq 55\%$, then observe from both of the figures that, there is not much deviation in the optimal profits. This observation indicates that small and moderate values of γ_r provide better insights for evaluating the effects of the uncertainty budget on the solutions. For this reason, we use $\gamma_r \in \{15\%, 25\%, 50\%\}$ during the rest of our computational analysis.

Next, we evaluate the performance of the acceleration techniques proposed for the max-min model. The computational results with the 10–20 node instances are reported in Table 7. The column labeled "φ" presents the amount of deviation from the nominal revenue. The next six columns show the computational time without and with the implementation of the acceleration techniques for $\gamma_r = 15\%$, 25%, and 50%, respectively. The results in Table 7 show that the algorithm runs around three times faster on average with the implementation of the proposed acceleration methodologies. Hence, the rest of the computational experiments with the max-min model are performed using the acceleration techniques.

Table 7 Computational times for the robust-stochastic model with the max-min criterion without and with the implementation of acceleration techniques.

$\|N\|$	φ	Time (sec)					
		$\gamma_r = 15\%$		$\gamma_r = 25\%$		$\gamma_r = 50\%$	
		Without acceleration	With acceleration	Without acceleration	With acceleration	Without acceleration	With acceleration
10LL	0.5	56	16	37	16	65	19
	1	53	21	86	19	107	78
10LT	0.5	32	14	41	13	53	13
	1	30	18	43	14	26	12
10TL	0.5	37	15	24	15	50	19
	1	35	21	38	17	112	77
10TT	0.5	36	13	23	13	50	13
	1	33	17	52	14	28	13
20LL	0.5	4,373	2,342	5,484	1,948	6,761	2,402
	1	4,006	1,526	5,659	2,257	5,670	2,130
20LT	0.5	3,789	2,265	4,588	1,099	2,739	1,383
	1	3,557	1,802	4,534	2,022	1,752	285
20TL	0.5	812	347	519	154	347	143
	1	977	311	724	194	273	105
20TT	0.5	271	78	446	71	170	57
	1	506	139	405	79	458	60
Average	0.5	1,176	636	1,395	310	1,279	506
	1	1,150	482	1,443	577	1,053	345

We now analyze the results obtained from the max-min model for larger size instances with up to 75 nodes and report the computational results in Table 8. The table is split into three parts to represent the results with $\gamma_r = 15\%$, 25%, 50%. The columns report the same as previously explained for Tables 2 and 5.

Table 8 Computational results for the robust-stochastic model with the max-min criterion

Instance		$\gamma_r = 15\%$				$\gamma_r = 25\%$				$\gamma_r = 50\%$			
$\|N\|$	φ	Profit	Avg. iter.	Time (sec)	Open hubs	Profit	Avg. iter.	Time (sec)	Open hubs	Profit	Avg. iter.	Time (sec)	Open hubs
10LL	0.5	11,051	2.01	16	5,9	10,629	2.01	16	5,9	10,129	2.01	19	5,9
	1	10,057	2.01	21	5,9	9,536	2.01	19	5,9	9,336	2.01	78	5,9
10LT	0.5	2,799	2.01	14	5	2,522	2.01	13	5	2,222	2.01	13	5
	1	2,370	2.01	18	5	1,973	2.01	14	5	1,773	2.01	12	5
10TL	0.5	10,805	2.01	15	5,9	10,384	2.01	15	5,9	9,784	2.01	19	5,9
	1	9,720	2.01	21	5,9	9,098	2.01	17	5,9	9,088	2.01	77	5,9
10TT	0.5	2,409	2.01	13	5	2,349	2.01	13	5	2,049	2.01	13	5
	1	2,378	2.01	17	5	2,185	2.01	14	5	1,685	2.01	13	5
20LL	0.5	71,687	13.26	2,342	7,9,10,19	67,929	13.05	1,948	7,9,10,19	55,471	13.36	2,402	7,9,10,19
	1	49,394	9.71	1,526	7,10,19	43,784	9.5	2,257	7,10,19	26,496	7.83	2,130	10,19
20LT	0.5	33,992	8.03	2,265	5,10,12,14	31,514	7.82	1,099	5,10,12,14	24,129	6.10	1,383	5,10,12,14
	1	20,639	6.09	1,802	5,10,12,14	16,736	5.81	2,022	10,12,14	12,166	4.06	285	10,14
20TL	0.5	28,963	3.02	347	7,10	26,602	2.74	154	7,10	19,612	3.07	143	7,10
	1	16,662	3.01	311	7,10	12,667	3.17	194	7,10	9,168	3.08	105	10
20TT	0.5	5,416	2.01	78	10	4,911	2.01	71	10	4,565	2.01	57	10
	1	1,558	2.01	139	10	527	2.01	79	10	436	2.01	60	10
25LL	0.5	95,864	18.53	6,567	7,14,17,23	91,273	18.32	1,783	7,14,17,23	76,665	18.73	5,300	7,14,17,23
	1	67,639	12.63	2,028	7,14,17	60,669	12.42	3,848	7,14,17	38,708	17.63	2,035	9,17
25LT	0.5	59,640	8.02	2,413	6,9,10,12,14,25	55,493	7.91	1,555	6,9,10,12,25	44,722	9.64	2,607	6,9,10,12,25
	1	41,798	6.60	1,886	6,9,12,14,25	36,685	6.39	2,186	9,12,14,25	23,167	6.60	1,762	12,14,25
25TL	0.5	49,639	8.07	1,966	6,9,14	46,153	7.86	1,586	6,9,14	35,093	7.20	1,638	6,9,14
	1	33,417	7.20	1,118	6,9,14	27,273	6.99	2,078	6,9,14	16,152	6.06	860	14
25TT	0.5	17,306	8.03	1,933	6,10,14	15,481	7.82	1,067	10,14	12,515	8.03	1,219	10,14
	1	10,282	7.02	1,329	10,14	8,632	6.81	1,632	14	8,205	2.01	321	14
40LL	0.5	61,336	26.03	21,631	12,22,26,29	58,575	25.82	16,324	12,22,26,29	53,575	24.01	19,324	12,22,26,29
	1	42,520	25.54	20,749	17,26,35	38,203	25.33	19,017	17,26,35	31,203	23.18	18,017	17,26,35
40LT	0.5	48,425	22.13	21,852	10,14,26,30,38	46,080	21.92	18,448	10,14,26,30,38	42,080	17.42	18,448	10,14,26,38
	1	29,070	19.09	20,318	10,17,26, 38	27,736	18.88	21,547	10,26, 38	22,736	15.75	16,547	10,26, 38
40TL	0.5	42,569	16.83	8,116	14,19,29	40,477	16.12	6,903	14,19,29	35,185	10.09	71,032	14,29
	1	25,900	10.96	6,401	14,29	22,730	10.75	9,165	14,29	14,752	8.16	3,367	14,29
40TT	0.5	32,427	10.03	10,331	14,19,25,38	30,583	9.82	7,523	14,19,25,38	26,691	7.55	3,027	14,19,38
	1	20,168	8.43	8,252	14,19,38	17,523	8.22	10,048	14,19,38	11,809	4.24	948	14,38
50LL	0.5	59,282	22.71	14,393	15,28,33,35	57,541	22.5	12,143	15,28,33,35	54,541	16.94	12,143	15,28,33,35
	1	39,066	20.08	16,141	15,28,33,35	36,174	17.27	16,732	15,28,35	31,174	14.18	10,732	15,28,35
50LT	0.5	55,565	19.23	15,057	26,32,46	54,186	16.02	10,436	26,32,46	52,186	13.07	13,436	26,32,46
	1	39,119	17.08	13,326	26,32,46	35,992	9.87	14,681	26,46	30,992	8.74	8,681	26,46
50TL	0.5	34,324	9.13	9,645	26,45	32,431	8.92	5,912	26,45	27,449	8.23	6,113	26,45
	1	17,729	7.73	7,912	24	15,340	4.52	9,448	24	9,882	3.23	2,436	24
50TT	0.5	29,183	4.06	7,299	26,48	27,727	3.85	4,479	26,48	24,434	3.01	4,125	26,48
	1	16,113	3.81	6,729	26,48	13,639	3.6	8,759	26	9,800	3.02	2,403	26
75LL	0.5	93,115	47.37	68,749	14,26,35,38,56	81,343	47.16	65,231	14,26,35,38,56	55,318	41.82	63,194	14,26,38,56
	1	64,054	45.29	64,184	14,26,35,38,56	53,172	41.23	57,418	14,26,38,56	32,719	23.08	56,319	14,38,56
75LT	0.5	74,926	43.16	59,317	25,32,35,38,59	57,343	42.95	53,568	25,35,38,59	41,663	24.76	43,717	25,38,59
	1	40,963	45.83	61,732	25,32,35,38,59	31,568	40.26	55,619	25,35,38,59	24,428	19.80	21,368	25,38
75TL	0.5	52,219	11.76	47,273	14,35,37	39,618	11.21	45,236	14,35,37	25,763	9.96	20,813	14,37
	1	27,165	10.74	45,613	14,35,37	21,308	5.91	34,192	14,37	19,233	2.21	12,341	37
75TT	0.5	42,708	11.63	45,763	26,32,38,59	30,784	11.42	30,118	26,38	20,619	9.59	21,679	26,38
	1	28,816	8.89	31,573	26,38,59	17,193	8.28	29,918	26,38	14,672	2.20	11,972	38

All of the instances presented in Table 8 are solved to optimality. We observe that the computation times and average number of iterations do not vary significantly by varying γ_r values. This can be attributed to the fact that the number of dual variables associated with the intervals of uncertainty is independent from the value of γ_r. This characteristic enables the algorithm to solve instances with up to 16,875 commodities containing stochastic demand and uncertain revenue. The averages of the computational times reported in Table 8 for the $\gamma_r = 15\%$, 25%, and 50% instances are 3.8, 3.4, and 2.8 hours, respectively.

The profits obtained from the max-min model represent the lowest profit that can be expected, as long as the revenues comply with the model of uncertainty. This profit provides a valuable information, in particular to a conservative decision maker, since the profit associated with this solution will never fall below the obtained value.

Next, we analyze the effects of variability in uncertain revenue on the optimal solutions presented in Table 8. When the level of uncertainty (i.e., φ and γ_r) increases, the net profit value and the number of open hubs in the optimal solutions decrease. It can also be observed that the set of open hubs with a high level of uncertainty (e.g., $\gamma_r = 50\%$ and $\varphi = 1$) is a subset of the open hubs, when the level of uncertainty is low (e.g., $\gamma_r = 15\%$ and $\varphi = 0.5$). For example, in the optimal solution of 20LL with $\gamma_r = 50\%$ and $\varphi = 1$, hubs are located at nodes 10 and 19. While, by decreasing γ_r to 15% and φ to 0.5, hubs are located at nodes 7, 9, 10, and 19.

Table 9 Percentage of demand satisfied for each demand class with the max-min model.

Instance type	φ	$\gamma_r = 15\%$				$\gamma_r = 25\%$				$\gamma_r = 50\%$			
		Demand Class (%)			Avg.	Demand Class (%)			Avg.	Demand Class (%)			Avg.
		$m=1$	$m=2$	$m=3$		$m=1$	$m=2$	$m=3$		$m=1$	$m=2$	$m=3$	
LL	0.5	97.56	84.19	69.38	83.71	87.50	74.75	58.42	73.56	74.89	62.81	44.81	60.84
	1	87.80	75.94	50.83	71.52	78.42	67.10	41.85	62.45	66.58	55.96	31.07	51.20
LT	0.5	91.13	78.44	57.69	75.75	84.29	69.84	48.88	67.67	72.92	57.68	37.97	56.19
	1	83.40	68.46	39.14	63.67	73.65	59.34	30.87	54.62	62.90	49.21	19.89	44.00
TL	0.5	93.69	82.34	61.56	79.20	86.66	71.72	51.08	69.82	75.66	58.29	38.05	57.33
	1	85.69	71.10	40.62	65.80	77.27	66.41	32.07	58.58	65.54	56.78	20.81	47.71
TT	0.5	83.62	72.19	42.38	66.06	76.43	65.68	32.75	58.29	65.53	54.48	21.89	47.30
	1	73.73	63.00	26.98	54.57	64.45	53.98	18.14	45.52	53.02	41.70	8.62	34.45

The percentages of satisfied demand from different market segments are reported in Table 9. For a given (φ, γ_r) pair, the averages for each demand class are calculated over instances from Table 8 with the same type of installation costs and capacities. As expected, when φ or γ_r value increases, the percentage of satisfied demand for all three market segments decreases.

7.3.2. Min-max regret. We now analyze the results obtained with the min-max regret stochastic model with independent demand. For uncertain revenues, we generate five scenarios using the two intervals considered in Section 7.2. First, the performance of the acceleration techniques proposed for this model are evaluated by taking runs on instances with 10–20 nodes. Table 10 presents the computation times of the instances without and with the implementation of the acceleration techniques.

Similar to our observations in Tables 4 and 7, the results in Table 10 underline the efficiency of the acceleration techniques. The algorithm significantly speeds up by performing fifteen times

Table 10 Computational times for the min-max regret stochastic model without and with the implementation of acceleration techniques.

$\lvert N\rvert$	Time (sec)			
	$r_k^{ms} \in [0.75\bar{r}_k^m, \bar{r}_k^m]$		$r_k^{ms} \in [0.5\bar{r}_k^m, \bar{r}_k^m]$	
	Without acceleration	With acceleration	Without acceleration	With acceleration
10LL	1,076	39	895	40
10LT	474	31	494	39
10TL	351	32	478	31
10TT	532	26	567	31
20LL	28,606	2,237	24,317	1,820
20LT	24,439	1,946	14,043	1,241
20TL	8,460	104	7,605	144
20TT	6,330	91	6,936	102
Average	8,783	563	6,917	431

faster on average with the implementation of the acceleration techniques for the min-max regret stochastic model with independent demand. Accordingly, the rest of the computational experiments on this model are performed using the acceleration techniques.

Table 11 Computational results for the min-max regret stochastic model.

Instance	$r_k^{ms} \in [0.75\bar{r}_k^m, \bar{r}_k^m]$					$r_k^{ms} \in [0.5\bar{r}_k^m, \bar{r}_k^m]$				
$\lvert N\rvert$	Regret	Avg. profit	Avg. iter.	Time (sec)	Open hubs	Regret	Avg. profit	Avg. iter.	Time (sec)	Open hubs
10LL	139	13,642	2.00	39	5,9	678	8,379	2.00	40	5,9
10LT	565	4,700	2.00	31	5	1,296	1,692	2.00	39	5
10TL	137	11,023	2.00	32	5,9	975	7,942	2.00	31	5,9
10TT	561	3,560	2.00	26	5	1,368	1,799	2.00	31	5
20LL	550	73,309	5.70	2,237	7,9,10,19	2,323	40,839	7.20	1,820	7,10,19
20LT	902	35,684	3.04	1,946	5,10,12,14	1,994	23,234	6.00	1,241	5,10,14
20TL	1,027	29,181	2.03	104	7,10	2,137	21,532	3.00	144	7,10
20TT	1,074	7,231	2.00	91	10	1,799	5,712	2.00	102	10
25LL	145	94,318	2.10	4,563	7,14,17,23	781	70,435	3.84	4,548	7,14,17,23
25LT	383	59,317	2.00	5,394	6,9,10,12,14,25	1,203	41,059	5.03	5,637	6,9,12,14,25
25TL	760	48,941	2.90	2,572	6,9,14	1,094	31,520	4.00	2,776	6,9,14
25TT	979	17,593	4.00	1,774	10,14	1,872	11,451	3.00	457	14
40LL	219	65,561	6.07	22,437	12,22,26,29	935	49,338	6.79	18,717	12,26,29
40LT	189	56,402	4.70	20,214	12,14,26,29,30,38	869	37,804	4.03	15,749	14,26,29,38
40TL	127	48,637	2.00	4,426	14,19,29	929	34,389	2.21	2,616	14,29
40TT	397	37,614	4.01	9,180	14,19,25,38	1,170	25,789	2.16	4,736	14,19,38
50LL	126	64,073	4.09	14,718	15,28,33,35	489	35,848	5.12	15,134	15,28,33,35
50LT	148	60,578	2.00	7,625	6,26,32,46	577	35,812	4.07	7,526	6,26,32,46
50TL	233	37,516	3.00	4,712	26,45	780	15,456	4.40	4,495	26,45
50TT	363	31,974	2.00	5,021	26,48	1,083	13,384	2.00	5,549	26,48
75LL	368	100,106	5.86	72,163	14,23,35,38,56	954	59,933	6.14	74,318	14,23,35,38,56
75LT	464	80,243	4.91	67,335	14,25,32,38,59	1,063	37,002	5.08	67,660	14,25,32,38,59
75TL	534	55,236	4.79	43,307	14,35,37	1,187	24,537	4.36	47,208	14,35,37
75TT	599	46,406	4.15	37,639	26,32,38	1,608	25,810	4.09	41,314	26,32,38

The computational results for the instances with up to 75 nodes are summarized in Table 11. All of the instances are solved to optimality as before. The averages of the computational times for $r_k^{ms} \in [0.75\bar{r}_k^m, \bar{r}_k^m]$ and $r_k^{ms} \in [0.5\bar{r}_k^m, \bar{r}_k^m]$ are 3.8 and 3.7 hours, respectively.

Analogous to the observations that are drawn from Table 5, by decreasing the lower bound from 0.75 to 0.5, the regret of the solution increases, while the average profit and the optimal number of open hubs decrease. It is also worthwhile to note that the set of open hubs when $r_k^{ms} \in [0.5\bar{r}_k^m, \bar{r}_k^m]$, turned out to be a subset of the open hubs when $r_k^{ms} \in [0.75\bar{r}_k^m, \bar{r}_k^m]$, in all the instances presented in Table 11.

Table 12 **Percentage of demand satisfied for each demand class with the min-max regret model.**

Instance type	$r_k^{ms} \in [0.75\bar{r}_k^m, \bar{r}_k^m]$				$r_k^{ms} \in [0.5\bar{r}_k^m, \bar{r}_k^m]$			
	Demand class (%)			Avg.	Demand class (%)			Avg.
	$m=1$	$m=2$	$m=3$		$m=1$	$m=2$	$m=3$	
LL	97.19	83.34	48.67	76.40	93.21	74.66	37.56	68.48
LT	89.90	70.80	36.68	65.79	84.02	58.25	26.97	56.41
TL	92.21	71.38	38.35	67.31	88.07	62.51	28.05	59.55
TT	76.90	47.51	17.96	47.46	66.09	32.70	12.56	37.12

The percentages of satisfied demand from different market segments for $r_k^{ms} \in [0.75\bar{r}_k^m, \bar{r}_k^m]$ and $[0.5\bar{r}_k^m, \bar{r}_k^m]$ are presented in Table 12. As in Table 6, we observe the general trend that loose capacity instances result in higher levels of satisfied demand, and the level of satisfied demand decreases as the class level increases.

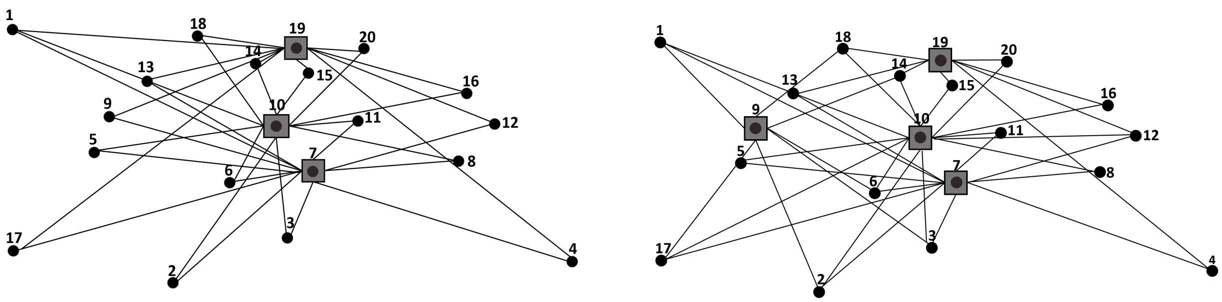
7.4. Comparison of Stochastic and Robust-Stochastic Solutions

To analyze the effect of modeling uncertainty with stochastic and robust-stochastic methodologies on the total profit and the resulting hub networks, in this section, we conduct two sets of comparisons based on the dependency between the revenue and demand parameters.

7.4.1. Revenue-elastic demand. We first compare the solutions obtained from the stochastic and robust-stochastic models under revenue-elastic demand. To make a fair comparison, for the stochastic model, we generate revenues from uniform distributions using the same intervals as in the robust-stochastic model, i.e., $r_k^m \sim U[0.75\bar{r}_k^m, \bar{r}_k^m]$ and $r_k^m \sim U[0.5\bar{r}_k^m, \bar{r}_k^m]$, respectively. Since the stochastic model optimizes the expected profit while the robust-stochastic model optimizes the expected regret, their optimal values cannot be compared directly. Let y^* and R^* be the optimal hub locations and the minimum regret obtained from the robust-stochastic model, respectively. It follows from (15) that $R^* = \max_{s \in S_r}\{Z^s - Z^s(y^*)\}$, yielding $\frac{1}{|S_r|}\sum_{s \in S_r} Z^s(y^*) \geq \frac{1}{|S_r|}\sum_{s \in S_r} Z^s - R^*$. Hence, $\frac{1}{|S_r|}\sum_{s \in S_r} Z^s - R^*$ provides a lower bound on the expected profit from y^*, which can be compared with the expected profit from the stochastic model. The expected profits and hub locations for 20–40 node instances for both stochastic and robust-stochastic models are presented in Table 13. Note that in all of the instances in Table 13, the robust-stochastic model resulted in higher profit values, except for one instance (20TT), where only a single hub is located. Also note

Table 13 Optimal solutions from stochastic and robust-stochastic models with uniform distributions.

Instance	$r_k^{ms} \in [0.75\bar{r}_k^m, \bar{r}_k^m]$				$r_k^{ms} \in [0.5\bar{r}_k^m, \bar{r}_k^m]$			
	Stochastic model		Robust-stochastic model		Stochastic model		Robust-stochastic model	
$\|N\|$	Profit	Open hubs	Profit	Open hubs	Profit	Open hubs	Profit	Open hubs
20LL	75,253	7,10,19	78,623	7,9,10,19	51,966	7,10,19	53,197	7,10,19
20LT	38,291	5,10,12,14	40,558	5,10,12,19	23,239	10,12,14	23,946	5,10,14
20TL	32,542	7,10	33,346	7,10	17,188	7,10	17,887	7,10
20TT	8,252	10	7,987	10	4,109	10	3,837	10
25LL	96,834	9,17,19	102,244	7,14,17,23	67,696	7,14,19,23	73,495	7,14,17,23
25LT	61,575	6,9,12,14,25	64,870	6,9,10,12,14,25	40,351	9,12,14,25	42,507	9,12,14,25
25TL	51,108	6,9,14	52,871	6,9,14	30,546	6,9,14	32,047	6,9,14
25TT	19,410	6,10,14	20,419	6,10,14	11,256	10,14	11,655	10,14
40LL	62,686	12,22,26,29	66,137	12,22,26,29	48,260	12,22,26,29	50,428	12,22,26,29
40LT	53,776	10,14,26,30,38	56,881	12,14,26,29,30,38	40,279	10,14,26,30,38	41,763	12,14,26,30,38
40TL	48,232	14,29	51,157	14,19,29	35,419	14,29	36,996	14,29
40TT	38,083	14,19,35,38	40,150	14,19,25,38	27,235	14,19,38	27,566	14,19,38



(a) Stochastic solution (b) Robust-stochastic solution

Figure 3 Hub networks of stochastic and robust-stochastic models with revenue-elastic demand for 20LL where $r_k^{ms} \in [0.75\bar{r}_k^m, \bar{r}_k^m]$.

that more hubs are opened in the robust-stochastic solutions compared with the pure stochastic model.

Figure 3 demonstrates the optimal hub networks obtained from stochastic (Figure 3a) and robust-stochastic (Figure 3b) models with revenue elastic demand for 20LL where $r_k^{ms} \in [0.75\bar{r}_k^m, \bar{r}_k^m]$. In these figures, squares represent the established hubs and the thin lines the allocation connections. To provide a better representation, we omitted the inter-hub links in these figures. Comparing the networks, note that the number of open hubs and allocation links in the stochastic model (Figure 3a) is less than that of the robust-stochastic model (Figure 3b). We observe in general that the robust-stochastic model results in building denser hub networks.

7.4.2. Independent demand. We now compare the solutions obtained from the stochastic and the two robust-stochastic models with independent demand. For consistency, the set of revenue scenarios considered in the stochastic (i.e., $\mathcal{N}$) and the min-max regret models (i.e., S_r) should comply with the requirements of the uncertainty sets considered in the max-min model (i.e., φ and γ_r). Recall that in the max-min version, we use φ to determine the variability in uncertain revenue, such that $\hat{r}_k^m \sim U[0, \varphi \bar{r}_k^m]$ and the interval of uncertainty for revenue is $[\bar{r}_k^m - \hat{r}_k^m, \bar{r}_k^m]$. These intervals are used to satisfy the budget of uncertainty constraint (24). For the stochastic model, we generate $|\mathcal{N}| = 50$ scenarios using the same intervals as in the max-min version (i.e., $r_k^{mn} \in [\bar{r}_k^m - \hat{r}_k^m, \bar{r}_k^m]$ for each $n \in \mathcal{N}$) and implicitly satisfy the budget of uncertainty constraint (24) by ensuring that $\sum_k \sum_m \frac{\bar{r}_k^m - r_k^{mn}}{\hat{r}_k^m} \leq \gamma_r$, for each scenario $n \in \mathcal{N}$. In the same manner, we generate $|S_r| = 10$ revenue scenarios for the min-max regret stochastic model.

Note that each model optimizes a different metric, hence, we cannot compare the quality of the solutions based on the individual objective function values. However, we can compare the hub networks obtained from each of the models. We suggest evaluating the quality of the solutions under two metrics: the profit that can be expected from each solution, and the frequency at which each solution attains the highest profit among other solutions.

For the first metric, let $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ denote the optimal solution obtained from any of the stochastic or robust-stochastic models. For revenue $r_k^m \in [\bar{r}_k^m - \hat{r}_k^m, \bar{r}_k^m]$, the total profit associated with a solution $(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ is calculated as:

$$\begin{aligned} Z(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}}) &= \mathbb{E}_\xi[\sum_{k \in K} \sum_{m \in M} \sum_{a \in A_k} (r_k^m - \hat{C}_{ak}) w_k^m(\xi) \tilde{x}_{ak}^m(\xi)] - \sum_{i \in H} \tilde{y}_i f_i \\ &= \sum_{k \in K} \sum_{m \in M} r_k^m \tilde{X}_k^m - \tilde{C} \end{aligned}$$

where

$$\tilde{X}_k^m = \mathbb{E}_\xi[\sum_{a \in A_k} w_k^m(\xi) \tilde{x}_{ak}^m(\xi)] \text{ and } \tilde{C} = \mathbb{E}_\xi[\sum_{k \in K} \sum_{m \in M} \sum_{a \in A_k} \hat{C}_{ak} w_k^m(\xi) \tilde{x}_{ak}^m(\xi)] + \sum_{i \in H} \tilde{y}_i f_i.$$

Recall that r_k^m is a random variable (with unknown distribution); therefore, $Z(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})$ is also a random variable with the expected value

$$\mathbb{E}_r[Z(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})] = \sum_{k \in K} \sum_{m \in M} \mathbb{E}[r_m^k] \tilde{X}_k^m - \tilde{C}.$$

In our experiments, we adapt the SAA scheme and use $|\mathcal{N}| = 50$ demand scenarios drawn from normal distributions as explained in Section 7.1. Moreover, we assume that $r_k^m \sim U[\bar{r}_k^m - \hat{r}_k^m, \bar{r}_k^m]$; therefore, $\mathbb{E}_r[Z(\tilde{\boldsymbol{x}}, \tilde{\boldsymbol{y}})]$ can easily be computed by setting $\mathbb{E}[r_m^k] = \bar{r}_k^m - 0.5\hat{r}_k^m$.

The second metric estimates the percentage that a solution outperforms the other two solutions in terms of the profit that can be obtained under different revenue realizations. To compute these percentages, we generate a large sample of revenue scenarios of size 1,000 and estimate the percentages by counting the realizations under which each model outperforms the others.

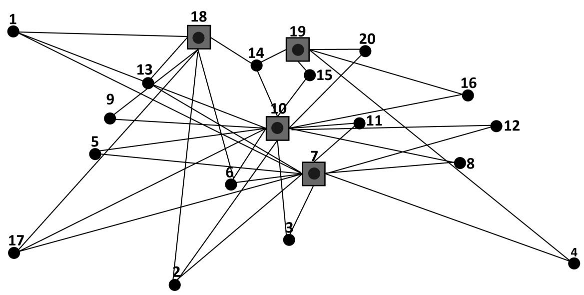
We performed the above analysis using four selected instances from the AP dataset with a variety of types and sizes: 20LL, 25LT, 40TL, and 50LL. For each instance, we consider five levels of variability for the revenue intervals, $\varphi \in \{0.2, 0.4, 0.6, 0.8, 1\}$, and two values for the budget of uncertainty with $\gamma_r \in \{25\%, 50\%\}$. Computational results are reported in Table 14. Columns under the heading of "Average expected profit" represent the average expected profit obtained from the stochastic, robust-stochastic model with max-min criterion, and min-max regret stochastic model, respectively. Columns under the heading "Frequency of attaining the highest expected profit (%)", on the other hand, provide the percentage that each model yields the highest profit among the three models under 1000 scenarios. The bold entries in Table 14 highlight the highest expected profit for each instance.

Table 14 Profit comparison with stochastic and robust-stochastic models.

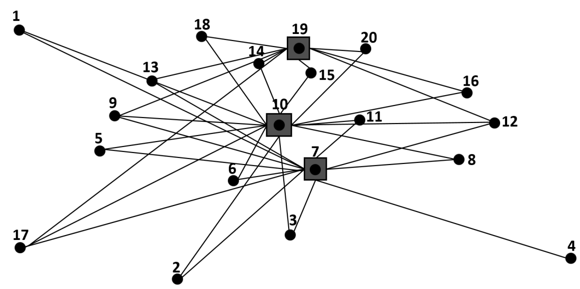
Instance		$\gamma_r = 25\%$					$\gamma_r = 50\%$				
		Average expected profit			Frequency of highest expected profit (%)		Average expected profit			Frequency of highest expected profit (%)	
$\|N\|$	φ	Stochastic revenue	Max-min criterion	Min-max regret	Max-min criterion	Min-max regret	Stochastic revenue	Max-min criterion	Min-max regret	Max-min criterion	Min-max regret
20LL	0.2	54,426	54,375	**54,498**	0.4	99.6	54,426	54,374	**59,769**	0.2	99.8
	0.4	42,981	42,618	**43,518**	0.0	100.0	42,981	42,759	**43,538**	0.2	99.8
	0.6	30,881	29,751	**32,529**	0.0	100.0	30,881	29,934	**32,529**	0.2	99.8
	0.8	16,542	13,799	**21,555**	0.0	100.0	16,542	15,319	**21,545**	0.2	99.8
	1	7,155	6,383	**11,748**	0.2	99.8	7,155	6,997	**11,748**	0.2	99.8
25LT	0.2	80,655	80,545	**80,869**	0.1	99.9	80,655	80,621	**80,840**	0.1	99.9
	0.4	67,711	68,097	**69,584**	0.2	99.8	67,711	68,630	**69,584**	0.2	99.8
	0.6	55,399	54,070	**58,309**	0.2	99.8	55,399	55,351	**58,309**	0.2	99.8
	0.8	45,243	44,261	**49,346**	0.2	99.8	45,243	45,890	**49,315**	0.2	99.8
	1	35,784	34,272	**41,554**	0.1	99.9	35,784	34,661	**41,554**	0.2	99.8
40TL	0.2	51,917	51,832	**52,025**	0.2	99.8	51,917	51,854	**52,098**	0.2	99.8
	0.4	45,075	44,771	**45,299**	0.2	99.8	45,075	44,772	**45,220**	0.2	99.8
	0.6	37,665	**39,684**	38,890	99.8	0.2	37,665	**39,486**	38,835	98.3	1.7
	0.8	33,549	33,123	**35,962**	0.2	99.8	33,549	32,717	**35,962**	0.2	99.8
	1	27,157	26,298	**30,051**	0.2	99.8	26,157	26,081	**30,051**	0.2	99.8
50TT	0.2	54,697	57,477	**55,806**	84.7	15.3	54,697	55,842	**56,060**	0.2	99.8
	0.4	47,431	47,550	**47,897**	5.8	94.2	47,431	47,620	**47,933**	18.7	81.3
	0.6	39,995	40,332	**40,666**	81.3	18.7	39,995	**41,626**	40,811	100.0	0.0
	0.8	30,671	**34,614**	32,712	100.0	0.0	30,671	31,982	**32,836**	0.0	100.0
	1	20,214	**25,405**	23,390	100.0	0.0	20,214	22,979	**23,606**	0.0	100.0
Avg.		41,257	41,463	**43,311**	23.7	76.3	41,207	41,475	**43,607**	11.0	89.0

The solutions obtained from both of the robust-stochastic models outperform the solutions obtained from the stochastic model, in general. On average, the robust-stochastic models yield significantly higher expected profits compared to the stochastic model. Moreover, when we look at the frequencies of attaining the highest expected profit, in none of the 1000 scenarios the stochastic model resulted in a better solution in terms of profit. More specifically, in our experiments, there is a 100% chance that at least one of the robust-stochastic models dominates the stochastic case. The robustness of the solutions obtained from the robust-stochastic models increases as the uncertainty in the revenue (i.e., φ) increases. These results highlight the superior performance of the robust-stochastic models on the out-of-sample observations, and underline the robustness of the solutions obtained from these models over the stochastic model.

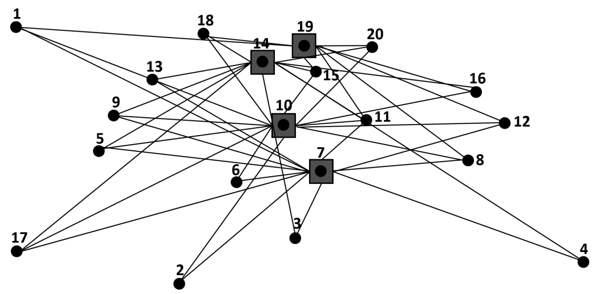
When the two robust-stochastic models are compared with each other, the min-max regret model turned out to be likely to yield the highest profit. In particular, the min-max regret model attained the highest profit in 76% of the instances when $\gamma_r = 25\%$ and 89% of the instances when $\gamma_r = 50\%$, whereas the max-min model attained the highest profit only in 24% of the instances when $\gamma_r = 25\%$ and 11% when $\gamma_r = 50\%$. It can also be observed from the results reported in Table 14 that increasing the budget of uncertainty from $\gamma_r = 25\%$ to 50% results in a higher expected profit for the robust-stochastic models, on average. This suggests a positive effect of increasing the level of conservatism on the quality of the solutions obtained from the robust-stochastic models. Increasing the level of conservatism plays a role in favour, in particular, of the min-max regret model, as it not only increases the expected profit, but also increases the percentage that this model yields the highest profit among the three models.



(a) Stochastic solution



(b) Robust-stochastic solution with max-min criterion



(c) Min-max regret stochastic solution

Figure 4 Hub networks of stochastic and robust-stochastic models with independent demand for 20LL with $\varphi = 0.6$ and $\gamma_r = 0.5$.

We now depict the differences between the optimal hub networks obtained with stochastic and robust-stochastic models on the 20LL instance with $\varphi = 0.6$ and $\gamma_r = 0.5$. Figures 4(a), 4(b), and 4(c) show the optimal hub networks from stochastic, max-min, and min-max regret models, respectively. Note that each model selects a different set of hub locations. The least number of hubs are opened in Figure 4(b), which corresponds to the conservative attitude of the max-min criterion.

Apart from the locations of the hubs, we observe that the network induced by the solution of the min-max regret stochastic model (Figure 4c) is considerably denser than that of the both stochastic (Figure 4a) and max-min stochastic models (Figure 4b). This supports our observation from the revenue-elastic demand case (Figure 3) in that the min-max regret criterion results in locating more hubs and allocation links. We believe this is in the spirit of the observation made by Kleywegt and Shapiro (2001) in that the min-max regret criterion tends to produce *less extreme* solutions compared with the ones obtained with the max-min criterion. Within our context, less extreme solutions correspond to diversifying the demand of satisfied commodities by locating more hubs and allocation links.

8. Conclusion

In this paper, we modeled uncertainty in demand and revenue within profit maximizing capacitated hub location problems with multiple demand classes. We first proposed a two-stage stochastic program by taking demand and revenue jointly stochastic and considered three separate cases of this model depending on the relation between these two parameters. We then developed novel robust-stochastic models in which two different types of uncertainty were simultaneously incorporated into the problem. We studied two particular cases based on the dependency between revenue and demand: revenue-elastic demand and independent demand. For the robust-stochastic model with revenue-elastic demand, we developed a min-max regret stochastic program with discrete scenarios. For independent demand, on the other hand, we formulated robust-stochastic models with interval representation and discrete scenarios using a max-min criterion and a min-max regret objective, respectively. For the latter case, we showed that the robust-stochastic version with max-min criterion is a special case of the min-max regret stochastic model.

We developed exact algorithms based on Benders decomposition coupled with sample average approximation scheme to solve the proposed models. We additionally developed novel acceleration techniques to enhance the performance of the algorithms. We performed extensive computational experiments on the well-known AP dataset to evaluate the efficiency of the algorithms and also to analyze the effects of uncertainty under different settings. The results show that our algorithms were able to optimally solve instances involving up to 75 nodes and 16,875 commodities of different demand classes. The developed acceleration methodologies speeded up the computation time up to 15 times on average confirming their efficiency.

We compared the quality of the solutions obtained from the stochastic and robust-stochastic models. The results provide several important insights in the design of optimal hub networks to maximize profit. The expected profits obtained from the robust-stochastic models with both revenue-elastic demand and independent demand cases turned out to be significantly higher than that of the pure stochastic model. In general, when the decision maker does not have any information on the distribution of revenue, our empirical results show that it is better to use a robust-stochastic modeling approach rather than solving a pure stochastic model.

The computational results showed that the net profits and the number of open hubs in the optimal solutions decrease significantly by increasing level of uncertainty. We also compared the resulting hub networks obtained under the pure stochastic model and the two robust-stochastic models. We observed that the min-max regret stochastic model for both revenue-elastic demand and independent demand cases resulted in building denser hub networks with a higher number of hubs and allocation connections yielding a higher expected profit by diversifying the demand of satisfied commodities. Our observations imply that the min-max regret stochastic model provides the most robust solutions among the three modeling approaches.

For future research, the mathematical formulations can be extended to take into account uncertainty in the cost parameters, in addition to revenue and demand. In our problem setting, there are two types of costs including transportation and hub installation costs where both directly affect the objective function value (i.e., net profit) and thus the optimal hub network. Another future direction would be to consider the single allocation variant of the problem. The mathematical models can easily be extended to the single allocation setting by defining new decision variables and modifying the formulations. Regarding the proposed algorithms, the single allocation setting requires enforcing integrality on the routing decision variables, which poses computational complexities on the Benders decomposition algorithm, as the subproblems can no longer be handled efficiently through LP duality.

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Technical Extensions

EC.1. Proof of Theorem 1

Proof. Let $Z^s(\xi) = \hat{Z}$ for each revenue scenario $s \in S_r$ and demand realization $\xi \in \Xi$. Then, the min-max regret stochastic model (30) reads as:

$$\min_{(\boldsymbol{x},\boldsymbol{y})\in\mho} \quad \mathbb{E}_\xi \left[\max_{s\in S_r} \left\{ \hat{Z} - (\sum_{k\in K} \sum_{m\in M} \sum_{a\in A_k} (r_k^{ms} - \hat{C}_{ak}) w_k^m(\xi) x_{ak}^m(\xi) - \sum_{i\in H} f_i y_i) \right\} \right]. \tag{EC.1}$$

We now prove that (21) is equivalent to (EC.1). Since $\hat{Z}$ is constant, it can be taken out from the inner maximization, the expectation, and the minimization, respectively. Hence, (EC.1) can be reformulated as:

$$\hat{Z} - \max_{(\boldsymbol{x},\boldsymbol{y})\in\mho} \quad \mathbb{E}_\xi \left[\min_{s\in S_r} \left\{ \sum_{k\in K} \sum_{m\in M} \sum_{a\in A_k} (r_k^{ms} - \hat{C}_{ak}) w_k^m(\xi) x_{ak}^m(\xi) - \sum_{i\in H} f_i y_i \right\} \right]. \tag{EC.2}$$

For a given $(\boldsymbol{x},\boldsymbol{y}) \in \mho$ and for each $\xi \in \Xi$, the inner minimization in (EC.2) calculates the worst possible profit associated with $(\boldsymbol{x},\boldsymbol{y})$ over all revenue scenarios. Given that each revenue scenario $s \in S_r$ involves at most γ_r commodities with uncertain revenue, we can map each revenue scenario s to a subset of commodities $U_r(s)$ including the commodities with uncertain revenue. That is, $r_k^{ms} = \bar{r}_k^m - \hat{r}_k^m$ if $(k,m) \in U_r(s)$, and $r_k^{ms} = \bar{r}_k^m$, otherwise. Therefore, the inner minimization in (EC.2) can be rewritten as:

$$\min_{s\in S_r} \left\{ \sum_{k\in K} \sum_{m\in M} \sum_{a\in A_k} (\bar{r}_k^m - \hat{C}_{ak}) w_k^m(\xi) x_{ak}^m(\xi) - \sum_{i\in H} f_i y_i - \sum_{(k,m)\in U_r(s)} \sum_{a\in A_k} (\hat{r}_k^m) w_k^m(\xi) x_{ak}^m(\xi) \right\}. \tag{EC.3}$$

Since the first two terms of (EC.3) are constant with respect to s, we can reformulate (EC.3) as:

$$\sum_{k\in K} \sum_{m\in M} \sum_{a\in A_k} (\bar{r}_k^m - \hat{C}_{ak}) w_k^m(\xi) x_{ak}^m(\xi) - \sum_{i\in H} f_i y_i - \max_{s\in S_r} \left\{ \sum_{(k,m)\in U_r(s)} \sum_{a\in A_k} (\hat{r}_k^m) w_k^m(\xi) x_{ak}^m(\xi) \right\}. \tag{EC.4}$$

Note that the maximization in (EC.4) is equivalent to $\nu_\xi(\boldsymbol{x})$ as defined in (22), therefore (EC.1) is equivalent to

$$\hat{Z} - \max_{(\boldsymbol{x},\boldsymbol{y})\in\mho} \quad \mathbb{E}_\xi \left[\sum_{k\in K} \sum_{m\in M} \sum_{a\in A_k} (\bar{r}_k^m - \hat{C}_{ak}) w_k^m(\xi) x_{ak}^m(\xi) - \nu_\xi(\boldsymbol{x}) \right] - \sum_{i\in H} f_i y_i,$$

which is equivalent to the robust-stochastic model with max-min criterion (21). □

EC.2. Deriving the Benders Reformulation for the Robust-Stochastic Model with Revenue-Elastic Demand

For given master solutions $(\bar{V}, \bar{\boldsymbol{y}})$ under revenue scenario $s \in S_r$, we formulate the primal subproblem (RS-I-PS($\mathcal{N}^s$)) as

$$\text{RS-I-PS}(\mathcal{N}^s) \quad \max 0$$

$$\begin{aligned}
\text{s.t.}\quad & \frac{1}{|\mathcal{N}^s|}[\sum_{n\in\mathcal{N}^s}\sum_{k\in K}\sum_{m\in M}\sum_{a\in A_k}(r_k^{mn}-\hat{C}_{ak})w_k^{mn}x_{ak}^{mn}] \geq Z^s-\bar{V}+\sum_{i\in H}f_i\bar{y}_i \\
& \sum_{a\in A_k} x_{ak}^{mn} \leq 1 && k\in K, m\in M, n\in\mathcal{N}^s \\
& \sum_{a\in A_k: i\in a} x_{ak}^{mn} \leq \bar{y}_i && i\in H, k\in K, m\in M, n\in\mathcal{N}^s \\
& \sum_{k\in K}\sum_{m\in M}\sum_{a\in A_k: i\in a} w_k^{mn}x_{ak}^{mn} \leq \Gamma_i\bar{y}_i && i\in H, n\in\mathcal{N}^s \\
& x_{ak}^{mn} \geq 0 && k\in K, m\in M, a\in A_k, n\in\mathcal{N}^s.
\end{aligned}$$

Note that RS-I-PS($\mathcal{N}^s$) is a feasibility problem in the space of the x-variables. Dualizing the constraints with variables β, α_k^{mn}, u_{ik}^{mn}, and b_i^n, we derive the following dual subproblem

$$\begin{aligned}
\text{RS-I-DS}(\mathcal{N}^s)\quad & \min -|\mathcal{N}^s|(Z^s-\bar{V}+\sum_{i\in H}f_i\bar{y}_i)\beta+\sum_{n\in\mathcal{N}^s}\left(\sum_{k\in K}\sum_{m\in M}\alpha_k^{mn}+\sum_{i\in H}\bar{y}_i(\sum_{k\in K}\sum_{m\in M}u_{ik}^{mn}+\Gamma_i b_i^n)\right) \\
\text{s.t.}\quad & \alpha_k^{mn}+u_{ik}^{mn}+u_{jk}^{mn}+w_k^{mn}(b_i^n+b_j^n) \geq \beta(r_k^{mn}-\hat{C}_{ijk})w_k^{mn} && k\in K, m\in M, (i,j)\in A_k: i\neq j, n\in\mathcal{N}^s \\
& \alpha_k^{mn}+u_{ik}^{mn}+w_k^{mn}b_i^n \geq \beta(r_k^{mn}-\hat{C}_{iik})w_k^{mn} && k\in K, m\in M, i\in H, n\in\mathcal{N}^s \\
& \alpha_k^{mn}, u_{ik}^{mn}, b_i^n \geq 0 && k\in K, m\in M, i\in H, n\in\mathcal{N}^s.
\end{aligned}$$

Since RS-I-PS($\mathcal{N}^s$) is a feasibility problem, the dual subproblem provides feasibility cuts of the form

$$-|\mathcal{N}^s|(Z^s-V+\sum_{i\in H}f_iy_i)\beta+\sum_{n\in\mathcal{N}^s}\left(\sum_{k\in K}\sum_{m\in M}\alpha_k^{mn}+\sum_{i\in H}y_i(\sum_{k\in K}\sum_{m\in M}u_{ik}^{mn}+\Gamma_i b_i^n)\right) \geq 0. \tag{EC.5}$$

Based on the value of β, we may consider two cases: For $\beta=0$, the cut is trivially redundant. For $\beta>0$, dividing both sides of (EC.5) by β yields:

$$\frac{1}{\beta}\sum_{n\in\mathcal{N}^s}\left(\sum_{k\in K}\sum_{m\in M}\alpha_k^{mn}+\sum_{i\in H}y_i(\sum_{k\in K}\sum_{m\in M}u_{ik}^{mn}+\Gamma_i b_i^n)\right) \geq |\mathcal{N}^s|(Z^s-V+\sum_{i\in H}f_iy_i).$$

Since the feasible region of RS-I-DS($\mathcal{N}^s$) is a cone, the magnitude of the dual variables does not affect the cut. Hence, without loss of generality, we may normalize the variables by setting $\beta=1$, which yields

$$\sum_{n\in\mathcal{N}^s}\left(\sum_{k\in K}\sum_{m\in M}\alpha_k^{mn}+\sum_{i\in H}y_i(\sum_{k\in K}\sum_{m\in M}u_{ik}^{mn}+\Gamma_i b_i^n)\right) \geq |\mathcal{N}^s|(Z^s-V+\sum_{i\in H}f_iy_i). \tag{EC.6}$$

Setting $\eta=-V+\sum_{i\in H}f_iy_i$ and rearranging (EC.6) yields the desired cut

$$\eta \leq \frac{1}{|\mathcal{N}^s|}\sum_{n\in\mathcal{N}^s}\left(\sum_{k\in K}\sum_{m\in M}\alpha_k^{mn}+\sum_{i\in H}y_i(\sum_{k\in K}\sum_{m\in M}u_{ik}^{mn}+\Gamma_i b_i^n)\right)-\bar{Z}^s. \tag{EC.7}$$

Consequently, to derive the Benders cut (EC.7), it suffices to solve the dual subproblems proposed in Section 4 (i.e., S-DS($\mathcal{N}^s, n$)) for computing the values of the dual variables $(\boldsymbol{\alpha}, \boldsymbol{u}, \boldsymbol{b})$ for each revenue scenario $s\in S_r$ and demand scenario $n\in\mathcal{N}^s$. We remark that although the cut is not separable by demand scenarios, we may still derive the cut by solving the subproblems S-DS($\mathcal{N}^s, n$) independently, one for each pair (s,n).

EC.3. Deriving the Benders Reformulation for the Robust-Stochastic Model with Max-Min Criterion

For a given solution $\boldsymbol{y} := \bar{\boldsymbol{y}}$ and a demand sample $\mathcal{N}$, the *primal subproblem* RS-II-PS($\mathcal{N}$) of the SAA counterpart of RS-II can be formulated as:

$$\text{RS-II-PS}(\mathcal{N}) \quad \max \frac{1}{|\mathcal{N}|}[\sum_{n\in\mathcal{N}}\sum_{k\in K}\sum_{m\in M}\sum_{a\in A_k}(\bar{r}_k^m - \hat{C}_{ak})w_k^{mn}x_{ak}^{mn} - (\gamma_r\mu^n + \sum_{n\in\mathcal{N}}\sum_{k\in K}\sum_{m\in M}\lambda_k^{mn})]$$

$$\text{s.t.} \quad \hat{r}_k^m w_k^{mn}\sum_{a\in A_k}x_{ak}^{mn} - \mu^n - \lambda_k^{mn} \le 0 \qquad k\in K, m\in M, n\in\mathcal{N} \tag{EC.8}$$

$$\sum_{a\in A_k}x_{ak}^{mn} \le 1 \qquad k\in K, m\in M, n\in\mathcal{N} \tag{EC.9}$$

$$\sum_{a\in A_k:i\in a}x_{ak}^{mn} \le \bar{y}_i \qquad i\in H, k\in K, m\in M, n\in\mathcal{N} \tag{EC.10}$$

$$\sum_{k\in K}\sum_{m\in M}\sum_{a\in A_k:i\in a}w_k^{mn}x_{ak}^{mn} \le \Gamma_i\bar{y}_i \qquad i\in H, n\in\mathcal{N} \tag{EC.11}$$

$$x_{ak}^{mn}, \lambda_k^{mn}, \mu^n \ge 0 \qquad k\in K, m\in M, a\in A_k, n\in\mathcal{N}.$$

Observe that RS-II-PS($\mathcal{N}$) can be decomposed into $|\mathcal{N}|$ independent subproblems, one for each $n\in\mathcal{N}$, which means the Benders optimality cuts can be separated by each $n\in\mathcal{N}$. To produce the cuts, let β_k^{mn}, α_k^{mn}, u_{ik}^{mn}, and b_i^n be the dual variables associated with constraints (EC.8)–(EC.11), and formulate the dual subproblem for each $n\in\mathcal{N}$ as:

$$\text{RS-II-DS}(\mathcal{N},n) \ \min \sum_{k\in K}\sum_{m\in M}\alpha_k^{mn} + \sum_{i\in H}\bar{y}_i(\sum_{k\in K}\sum_{m\in M}u_{ik}^{mn} + \Gamma_i b_i^n)$$

$$\text{s.t.} \ \hat{r}_k^m w_k^{mn}\beta_k^{mn} + \alpha_k^{mn} + u_{ik}^{mn} + u_{jk}^{mn} + w_k^{mn}(b_i^n + b_j^n) \ge (\bar{r}_k^m - \hat{C}_{ijk})w_k^{mn} \qquad k\in K, m\in M, (i,j)\in A_k : i\ne j$$

$$\hat{r}_k^m w_k^{mn}\beta_k^{mn} + \alpha_k^{mn} + u_{ik}^{mn} + w_k^{mn}b_i^n \ge (\bar{r}_k^m - \hat{C}_{iik})w_k^{mn} \qquad k\in K, m\in M, i\in H$$

$$\sum_{k\in K}\sum_{m\in M}\beta_k^{mn} \le \gamma_r$$

$$\beta_k^{mn} \le 1 \qquad k\in K, m\in M$$

$$\beta_k^{mn}, \alpha_k^{mn}, u_{ik}^{mn}, b_i^n \ge 0 \qquad k\in K, m\in M, i\in H.$$

Thus, the Benders *master problem* RS-II-MP($\mathcal{N}$) is formulated as:

$$\text{RS-II-MP}(\mathcal{N}) \quad \max \frac{1}{|\mathcal{N}|}\sum_{n\in\mathcal{N}}\eta^n - \sum_{i\in H}f_i y_i$$

$$\text{s.t.} \quad \eta^n \le \sum_{k\in K}\sum_{m\in M}\alpha_k^{mn} + \sum_{i\in H}y_i(\Gamma_i b_i^n + \sum_{k\in K}\sum_{m\in M}u_{ik}^{mn}) \qquad n\in\mathcal{N}, (\boldsymbol{\beta}^n, \boldsymbol{\alpha}^n, \boldsymbol{u}^n, \boldsymbol{b}^n)\in P_{\mathcal{N}}^n$$

$$y_i\in\{0,1\} \qquad i\in H.$$

EC.4. Deriving the Benders Reformulation for the Min-Max Regret Stochastic Model with Independent Demand

Let V^n be the regret incurred under demand scenario $n\in\mathcal{N}$. For ease of exposition, let's define $\bar{V}^n := -(V^n - \sum_{i\in H} f_i y_i)$ and rewrite $V^n = -(\bar{V}^n - \sum_{i\in H} f_i y_i)$. Additionally, for a given demand scenario $n\in\mathcal{N}$ and revenue scenario $s\in S_r$, let $\bar{Z}^{ns}$ be an estimation of the optimal value of (29).

Consequently, we may reformulate the SAA counterpart of (31)–(32) as the following maximization program:

$$\begin{aligned}
\max \quad & \frac{1}{|\mathcal{N}|}\sum_{n\in\mathcal{N}} \bar{V}^n - \sum_{i\in H} f_i y_i \\
\text{s.t.} \quad & \bar{V}^n - \sum_{k\in K}\sum_{m\in M}\sum_{a\in A_k}(r_k^{ms} - \hat{C}_{ak}) w_k^{mn} x_{ak}^{mn} \le -\bar{Z}^{ns} && n\in\mathcal{N}, s\in S_r, \\
& \boldsymbol{x}^n \in \mho^n(\boldsymbol{y}) && n\in\mathcal{N},
\end{aligned}$$

where $\boldsymbol{x}^n = (x_{ak}^{mn})$ and $\mho^n(\boldsymbol{y})$ is the set of feasible routing solutions under demand scenario $n\in\mathcal{N}$ for a given solution $\boldsymbol{y}$ as defined by (33)–(36). With $\boldsymbol{y}$ set to a specific vector $\boldsymbol{y} := \bar{\boldsymbol{y}}$, the *primal subproblem* RS-III-PS($\mathcal{N}$) reads as:

$$\begin{aligned}
\text{RS-III-PS}(\mathcal{N}) \quad \max \quad & \frac{1}{|\mathcal{N}|}\sum_{n\in\mathcal{N}} \bar{V}^n \\
\text{s.t.} \quad & \bar{V}^n - \sum_{n\in\mathcal{N}}\sum_{k\in K}\sum_{m\in M}\sum_{a\in A_k}(r_k^{ms} - \hat{C}_{ak}) w_k^{mn} x_{ak}^{mn} \le -\bar{Z}^{ns} && n\in\mathcal{N}, s\in S_r && \text{(EC.12)} \\
& \boldsymbol{x}^n \in \mho^n(\bar{\boldsymbol{y}}) && n\in\mathcal{N}. && \text{(EC.13)}
\end{aligned}$$

Observe that RS-III-PS($\mathcal{N}$) can be decomposed into $|\mathcal{N}|$ independent subproblems, one for each $n\in\mathcal{N}$. Dualizing (EC.12) with ω^{ns} and (EC.13) with dual variables $(\boldsymbol{\alpha}^n, \boldsymbol{u}^n, \boldsymbol{b}^n)$, the *dual subproblem* RS-III-DS($\mathcal{N}, n$) can be stated as:

$$\begin{aligned}
\text{RS-III-DS}(\mathcal{N}, n) \ \min \ & \sum_{k\in K}\sum_{m\in M}\alpha_k^{mn} + \sum_{i\in H}\bar{y}_i(\Gamma_i b_i^n + \sum_{k\in K}\sum_{m\in M} u_{ik}^{mn}) - \sum_{s\in S_r}\bar{Z}^{ns}\omega^{ns} \\
\text{s.t.} \quad & \sum_{s\in S_r}\omega^{ns} = 1 \\
& \alpha_k^{mn} + u_{ik}^{mn} + u_{jk}^{mn} + w_k^{mn}(b_i^n + b_j^n) \ge \sum_{s\in S_r}\omega^{ns}(r_k^{ms} - \hat{C}_{ijk}) w_k^{mn} && k\in K, m\in M, (i,j)\in A_k : i\ne j \\
& \alpha_k^{mn} + u_{ik}^{mn} + w_k^{mn} b_i^n \ge \sum_{s\in S_r}\omega^{ns}(r_k^{ms} - \hat{C}_{iik}) w_k^{mn} && k\in K, m\in M, i\in H \\
& \alpha_k^{mn}, u_{ik}^{mn}, b_i^n, \omega^{ns} \ge 0 && k\in K, m\in M, i\in H, s\in S_r.
\end{aligned}$$

Define $\bar{P}_{\mathcal{N}}^n$ as the set of extreme points of the feasible region of RS-III-DS($\mathcal{N}, n$) for $n\in\mathcal{N}$. Each demand scenario $n\in\mathcal{N}$ can provide a Benders cut; hence, the Benders *master problem* RS-III-MP($\mathcal{N}$) can be reformulated as below:

$$\begin{aligned}
\text{RS-III-MP}(\mathcal{N}) \ \max \ & \frac{1}{|\mathcal{N}|}\sum_{n\in\mathcal{N}}\eta^n - \sum_{i\in H} f_i y_i \\
\text{s.t.} \quad & \eta^n \le \sum_{k\in K}\sum_{m\in M}\alpha_k^{mn} + \sum_{i\in H} y_i(\Gamma_i b_i^n + \sum_{k\in K}\sum_{m\in M} u_{ik}^{mn}) - \sum_{s\in S_r}\bar{Z}^{ns}\omega^{ns} && n\in\mathcal{N}, (\boldsymbol{\alpha}^n, \boldsymbol{u}^n, \boldsymbol{b}^n, \boldsymbol{\omega}^n)\in\bar{P}_{\mathcal{N}}^n, \\
& y_i\in\{0,1\} && i\in H.
\end{aligned}$$

EC.5. Technical Details of the Acceleration Techniques for Robust-Stochastic Model with Revenue-Elastic Demand (RS-I).

The proposed accelerated SAA algorithm for the robust-stochastic model with revenue-elastic demand is detailed in Algorithm 1. At replication t of the SAA algorithm, we denote the set of demand scenarios associated with revenue scenario s by $\mathcal{N}_t^s$. Moreover, P_1^s denotes the set of dual solutions obtained in solving SAA-I in the first replication under the revenue scenario $s \in S_r$. At replication t, $\bar{P}_t^s$ consists of feasible solutions for scenario $s \in S_r$ obtained by converting the solutions contained in P_1^s. To convert the solutions, we use Equation (63) to select the closest scenarios to $\mathcal{N}_t^s$ among the scenarios contained in $\mathcal{N}_1^s$.

Algorithm 1 Accelerated SAA for the robust-stochastic model with revenue-elastic demand

1: $t \leftarrow 1$, $P_1^s \leftarrow \emptyset$ for each $s \in S_r$
2: **for** $s \in S_r$ **do**
3:     **if** $s > 1$ **then**
4:         Convert the solutions contained in $P_1^{\hat{s}}$ for each $\hat{s} < s$ to feasible solutions for revenue scenario s using Proposition 3 and Equation (63), and store them in P_1^s.
5:     **end if**
6:     **Calculate** $\bar{Z}_1^s$ (without variable fixing) using Algorithm 1 by generating initial cuts from the solutions contained in P_1^s. Store the dual solutions obtained by Algorithm 1 in P_1^s.
7:     $\bar{P}_1^s \leftarrow P_1^s$
8: **end for**
9: **SOLVE** RS-I-MP($\{\mathcal{N}_t^s\}_{s\in S_r}$) via Algorithm 2 by generating initial cuts from the solutions in $\{\bar{P}_t^s\}_{s\in S_r}$.
10: **if** $t < \mathcal{M}$ **then**
11:     $t \leftarrow t+1$
12:     **for** $s \in S_r$ **do**
13:         Let $\hat{\boldsymbol{y}}^s$ be the optimal solution obtained for computing $\bar{Z}_1^s$.
14:         **Approximate** $\bar{Z}_t^s$ by fixing $\boldsymbol{y}$ at $\boldsymbol{y} = \hat{\boldsymbol{y}}^s$: for each $n \in \mathcal{N}_t^s$ compute $\hat{Z}^s(\hat{\boldsymbol{y}}^s, n)$ using (10)–(14); set $\bar{Z}_t^s \leftarrow \frac{1}{|\mathcal{N}_t^s|}\hat{Z}^s(\hat{\boldsymbol{y}}^s, n)$
15:         Convert the solutions contained in P_1^s using Proposition 1 and Equation (63), and store them in $\bar{P}_t^s$.
16:     **end for**
17:     Go to Step 9.
18: **else**
19:     Terminate
20: **end if**

EC.6. Technical Details of the Acceleration Techniques for Min-Max Regret Stochastic Model (RS-III).

The proposed accelerated SAA algorithm for the min-max regret stochastic model with independent demand is detailed in Algorithm 2. We refer to each demand scenario by an integer n and each revenue scenario by an integer s. Moreover, P_1^s denotes the set of dual solutions obtained in solving the first demand scenario (i.e., $n=1$) of the first replication under the revenue scenario $s \in S_r$. At replication t, the set demand scenarios is denoted by $\mathcal{N}_t$, and $\bar{P}_t^{ns}$ consists of feasible solutions for scenario pair $(n,s) \in \mathcal{N}_t \times S_r$ obtained by converting the solutions contained in P_1^s.

Algorithm 2 Accelerated SAA for the min-max regret stochastic model

1: $t \leftarrow 1$, $P_1^s \leftarrow \emptyset$ for each $s \in S_r$.
2: **for** each $s \in S_r$ and each $n \in \mathcal{N}_1$ **do**
3: $\quad \bar{P}_1^{ns} \leftarrow \emptyset$
4: $\quad$ **if** $n = 1$ **then**
5: $\quad\quad$ **if** $s > 1$ **then**
6: $\quad\quad\quad$ Convert the solutions contained in $P_1^{\hat{s}}$ for each $\hat{s} < s$ to feasible solutions for revenue scenario s using Proposition 2 and add them to P_1^s.
7: $\quad\quad$ **end if**
8: $\quad\quad$ **Calculate** $\bar{Z}^{ns}$ (without variable fixing) by generating initial cuts from the solutions contained in P_1^s. Store the obtained dual solutions in P_1^s.
9: $\quad\quad \bar{P}_1^{ns} \leftarrow P_1^s$
10: $\quad$ **else**
11: $\quad\quad$ Convert the solutions contained in P_1^s to feasible solutions for demand scenario n using Proposition 1 and store the obtained solutions in $\bar{P}_1^{ns}$.
12: $\quad\quad$ **Calculate** $\bar{Z}^{ns}$ (with variable fixing) using the initial cuts contained in $\bar{P}_1^{ns}$.
13: $\quad$ **end if**
14: **end for**
15: **SOLVE** RS-III-MP($\mathcal{N}_t$) via Algorithm 4 by generating initial cuts from the solutions contained in $\{\bar{P}_t^{ns}\}_{s \in S_r, n \in \mathcal{N}_t}$ using Proposition 4.
16: **if** $t < \mathcal{M}$ **then**
17: $\quad t \leftarrow t + 1$
18: $\quad$ **for** $n \in \mathcal{N}_t$ **do**
19: $\quad\quad$ Let $\tilde{n} = \arg\min_{\hat{n} \in \mathcal{N}_1} \left\{ \sum_{k \in K} \sum_{m \in M} \left(\frac{|w_k^{m\hat{n}} - w_k^{mn}|}{w_k^{mn}} \right) \right\}$ be the closest demand scenario in $\mathcal{N}_1$ to n.
20: $\quad\quad$ **for** $s \in S_r$ **do**
21: $\quad\quad\quad$ **Approximate** $\bar{Z}^{ns}$ by fixing $\boldsymbol{y}$ at $\boldsymbol{y} = \hat{\boldsymbol{y}}^{\tilde{n}s}$: compute $\hat{Z}^s(\hat{\boldsymbol{y}}^{\tilde{n}s}, n)$ using (10)–(14); set $\bar{Z}^{ns} \leftarrow \hat{Z}^s(\hat{\boldsymbol{y}}^{\tilde{n}s}, n)$
22: $\quad\quad\quad$ Convert the solutions contained in P_1^s to feasible solutions for demand scenario n using Proposition 1 and store the obtained solutions in $\bar{P}_t^{ns}$.
23: $\quad\quad$ **end for**
24: $\quad$ **end for**
25: $\quad$ Go to Step 15.
26: **else**
27: $\quad$ Terminate
28: **end if**