

Denormalized Lorentzian Laurent series

by

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A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Mathematics
in
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2025

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Lorentzian and denormalized Lorentzian (DL) polynomials are well studied classes of polynomials with several applications. Among them are lower bounds on the coefficients of DL polynomials. These bounds allow us to approximate the number of objects with given weights in a class whose generating series is a DL polynomial. However, if this generating series is a power series and not a polynomial, one would have to take extra steps to obtain an approximation using this method.

To address this problem, we introduce and study the class of denormalized Lorentzian (DL) Laurent series and prove lower bounds on their coefficients. We define the capacity function for Laurent series and utilize it in our lower bound. With this approach, we are able to use simpler arguments to re-prove an already known approximation on the number of contingency tables, which are lattice points of transportation polytopes; and tidy up and slightly improve an already known lower bound on the number of integral flows of a network.

Acknowledgments

First and foremost, I would like to thank my supervisor, Jonathan Leake, for his guidance and support during my degree. I truly appreciate the time and effort he has dedicated to helping me to become a better researcher. Having such an understanding and amazing supervisor has helped me become a more confident person, for which I am deeply grateful.

I would like to thank Stephen Melczer and Oliver Pechenik for agreeing to read thesis and giving me thoughtful comments, and the C&O staff and students for creating a warm and welcoming environment.

I am thankful to my friends in Ontario for making my time in Waterloo enjoyable, and to my friends all around the world for supporting me throughout these past two years and keeping in touch. I am especially grateful to my parents for always believing in me and, finally, to my sister, thank you for always being there for me.

Dedication

I would like to dedicate this thesis to my parents, Akram and Hassan, and to my sister Sara, for their love, trust, and support throughout my life.

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Chapter 1

Introduction

Suppose we are given a combinatorial problem about an object. We could encode this object into a polynomial, and then show that some combinatorial aspects of this object are reflected in the analytic or algebraic properties of the corresponding polynomial. For instance, [18] encodes the matchings of a graph into a polynomial, shows that this polynomial is real rooted, and concludes that the sequence $\{m_k\}_{k \in \mathbb{N}}$ is unimodal, where m_k is the numbers of matchings of this graph with k edges.

Real stable polynomials, as a generalization of univariate real rooted polynomials, have proven to be extremely useful tools for this purpose. Gurvits in [14] gives a short proof of the previously solved van der Waerden problem using real stable polynomials, [27] proves the famous Kadison-Singer problem posed in [23], and [20] gives a short proof of the sensitivity problem from [31].

The class of Lorentzian polynomials was developed in [6] to prove Mason's logarithmic concavity conjecture for matroids [28]. Simultaneously, completely log-concave polynomials were introduced in [1] to solve the same problem. These two classes of polynomials agree on the set of homogeneous polynomials, and are the same as the class of homogeneous strongly log-concave polynomials defined in [15]. Lorentzian polynomials generalize real stable polynomials, with a focus on log-concavity.

Lorentzian polynomials are well-studied and have several applications. Among them are lower bounds for the coefficients of *denormalized* Lorentzian polynomials. A polynomial $p(\mathbf{x}) = \sum_{\alpha} p_{\alpha} \mathbf{x}^{\alpha}$ is denormalized Lorentzian (DL) if $\sum_{\alpha} \frac{p_{\alpha}}{\alpha!} \mathbf{x}^{\alpha}$ is Lorentzian.

It is proven in [7] that for a DL polynomial $p \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ and any $\alpha \in \mathbb{N}^n$:

$$[\mathbf{x}^{\alpha}]p(\mathbf{x}) \geq \text{Cap}_{\alpha}(p) \cdot \prod_{i=2}^n \max \left\{ \frac{\alpha_i^{\alpha_i}}{(\alpha_i + 1)^{\alpha_i + 1}}, \frac{(d_i - \alpha_i)^{d_i - \alpha_i}}{(d_i - \alpha_i + 1)^{d_i - \alpha_i + 1}} \right\}, \quad (1)$$

where d_i is the degree of x_i in a polynomial derived from p , and $\text{Cap}_{\alpha}(p) := \inf_{\mathbf{x} > 0} \frac{p(\mathbf{x})}{\mathbf{x}^{\alpha}}$ is the *capacity* of p introduced in [13]. Inequality (1) is used in [7] to approximate the number

of *contingency tables* with given marginals and bounded entries. For $\alpha \in \mathbb{N}^n$, $\beta \in \mathbb{N}^m$ and $K \in (\mathbb{Z}_{\geq 0} \cup \{\infty\})^{n \times m}$, a K -contingency table with marginals (α, β) is a matrix $A \in \mathbb{Z}_{\geq 0}^{n \times m}$, with entries in the i -th row summing up to α_i , the entries in the j -th row summing up to β_j , and $A_{ij} \leq K_{ij}$ for all $i \in [n], j \in [m]$. The generating series for K -contingency tables with marginals (α, β) is a DL polynomial when K has finite entries, and inequality (1) gives us the desired bound.

The same argument cannot be used to bound the number of contingency tables with no restriction on their entries, as the generating series for these objects is not a polynomial. Instead, [7] takes the limit of some DL polynomials to converge to this generating series and prove a similar result. This bound on contingency tables is utilized in [25] to approximate the number of integral flows of networks.

In this thesis, we introduce denormalized Lorentzian (DL) Laurent series. Unlike the common definition of Laurent series, we do not impose a lower bound on the powers of monomials, and define $\mathbb{K}((x_1, \dots, x_n)) := \{p(\mathbf{x}) = \sum_{\alpha \in \mathbb{Z}^n} p_{\alpha} \mathbf{x}^{\alpha} : p_{\alpha} \in \mathbb{K} \ \forall \alpha \in \mathbb{Z}^n\}$. We will show that the class of DL Laurent series is closed under multiplication, whenever multiplication is possible, and that the generating series for both contingency tables and flows of a network are DL. We will study the capacity function for Laurent series. The capacity of a Laurent series $p \in \mathbb{R}_{\geq 0}((\mathbf{x}))$ is $\text{Cap}_{\alpha}(p) = \inf_{\mathbf{x} \in \Omega_p} \frac{p(\mathbf{x})}{\mathbf{x}^{\alpha}}$ where Ω_p , the *domain of convergence* of p , is the set of all $\mathbf{x} \in \mathbb{R}_{> 0}^n$ for which $p(\mathbf{x})$ converges. We will prove the following lower bound on the coefficients of DL Laurent series:

Theorem 1. *Let $p \in \mathbb{R}_{\geq 0}((x_1, \dots, x_n))$ be a DL Laurent series with $\Omega_p \neq \emptyset$. Assume further that the set of degrees of x_i in the Laurent series $[x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n}]p$ has either a lower bound or an upper bound (or both), and let b_i be a bound for this set. Then:*

$$[\mathbf{x}^{\alpha}]p(\mathbf{x}) \geq \text{Cap}_{\alpha}(p) \cdot \prod_{i=2}^n \frac{|b_i - \alpha_i|^{b_i - \alpha_i}}{(|b_i - \alpha_i| + 1)^{b_i - \alpha_i + 1}}.$$

We then use Theorem 1 to prove lower bounds on the number of contingency tables presented in [7] directly, and to tidy up and slightly improve the bounds for the number of integral flows of a network in [25]. For an acyclic directed graph $G = ([n], E)$ and $\mathbf{N} \in \mathbb{Z}^n$, an $\mathbf{N}$ -flow of G is an $|E|$ -tuple $\Phi = (\Phi_e)_{e \in E} \in \mathbb{N}^{|E|}$ satisfying

$$\sum_{e: v \rightarrow u \in E} \Phi_e - \sum_{e: u \rightarrow v \in E} \Phi_e = N_v,$$

for any $v \in [n]$. The number of $\mathbf{N}$ -flows of a complete graph is also called the *Kostant vector partition function* and appears in the representation theory of Lie algebras [22]. We will prove that:

Theorem 2. *Let $n \in \mathbb{N}$ and $\mathbf{N} \in \mathbb{Z}^{n+1}$ be a vector with $|\mathbf{N}|_1 = 0$. Let $K_n(\mathbf{N})$ be the number of $\mathbf{N}$ -flows of the acyclic complete directed graph with $n + 1$ vertices. Further, let*

$s_i = |\sum_{k=1}^i N_k|$ for $i \in [n]$. Then:

$$K_n(\mathbf{N}) \geq \text{Cap}_{\mathbf{N}}(p_{k_{n+1}}) \cdot \prod_{i=1}^n \frac{s_i^{s_i}}{(1+s_i)^{(1+s_i)}},$$

where:

$$p_{k_{n+1}}(x_1, \dots, x_n) = \prod_{1 \leq i < j \leq n} \frac{1}{1 - \frac{x_i}{x_j}}.$$

1.1 Basic definitions and notations

Throughout this thesis, we will be using basic definitions and notations from analysis and combinatorics. In order to have a consistent understanding of notation, some of them are listed here.

Integer, real and complex numbers. We will denote by $\mathbb{N}, \mathbb{Z}, \mathbb{R}$ and $\mathbb{C}$ the sets of positive integers, the set of integers, the set of real numbers and the set of complex numbers respectively. For any set $A \subseteq \mathbb{R}$, define $A_{\geq 0} = A \cap \{x \geq 0 : x \in \mathbb{R}\}$ and $A_{>0} = A \cap \{x > 0 : x \in \mathbb{R}\}$. Let us also define:

$$\begin{aligned} [n] &= \{1, \dots, n\} \quad \forall n \in \mathbb{N}, \\ \mathbb{R}^* &= \mathbb{R} \cup \{\infty\}, \\ \mathcal{H} &= \{z \in \mathbb{C} : \Im(z) > 0\}, \end{aligned}$$

where $\Im(z)$ is the imaginary part of $z \in \mathbb{C}$.

Vector short hands. For two vectors $\mathbf{x} = (x_1, \dots, x_n)$ and $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$, $\mathbf{x}^{\boldsymbol{\alpha}}$ is the product $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $\boldsymbol{\alpha}!$ is $\prod_{i=1}^n \alpha_i!$ whenever $\boldsymbol{\alpha} \in \mathbb{N}^n$. We say that $\mathbf{x} \geq \boldsymbol{\alpha}$ if $x_i \geq \alpha_i$ for all $i \in [n]$.

Polynomials and Laurent series. For a field $\mathbb{K}$, we will denote by $\mathbb{K}[x_1, \dots, x_n]$ the ring of polynomials in $x_1, \dots, x_n$ over $\mathbb{K}$. The degree of the monomial $\mathbf{x}^{\boldsymbol{\alpha}}$ is $|\boldsymbol{\alpha}|_1 := \sum_{i=1}^n \alpha_i$, and 0 is considered to be a monomial of any degree. For a polynomial $p \in \mathbb{K}[x_1, \dots, x_n]$, denote by $\deg_{x_i}(p)$ the degree of x_i in p . Given a vector $\boldsymbol{\mu} \in \mathbb{N}^n$, let $\mathbb{K}^{\boldsymbol{\mu}}[x_1, \dots, x_n]$ be the set of all polynomials in variables $x_1, \dots, x_n$ with $\deg_{x_i}(p) \leq \mu_i$ for all $i \in [n]$.

Unlike the common definition, the set of Laurent series in variables $x_1, \dots, x_n$ over a field $\mathbb{K}$ denoted by $\mathbb{K}((x_1, \dots, x_n))$, is the set of all infinite sums $p(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^n} p_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$ with $p_{\boldsymbol{\alpha}} \in \mathbb{K}$. $\mathbb{K}((x_1, \dots, x_n))$ has an addition operator defined as $(p+q)(\mathbf{x}) = \sum_{\boldsymbol{\alpha} \in \mathbb{Z}^n} (p_{\boldsymbol{\alpha}} + q_{\boldsymbol{\alpha}}) \mathbf{x}^{\boldsymbol{\alpha}}$, where $p_{\boldsymbol{\alpha}}$ and $q_{\boldsymbol{\alpha}}$ are the coefficient of $\mathbf{x}^{\boldsymbol{\alpha}}$ in p and q respectively. Assuming that $\mathbb{K}$ is equipped with a topology, two Laurent series $p(\mathbf{x})$ and $q(\mathbf{x})$ can be multiplied whenever for any $\boldsymbol{\gamma} \in \mathbb{Z}^n$, the infinite sum $\sum_{\boldsymbol{\alpha}+\boldsymbol{\beta}=\boldsymbol{\gamma}} p_{\boldsymbol{\alpha}} q_{\boldsymbol{\beta}}$ converges with respect to that topology. In

which case, we will define the Laurent series $p \cdot q$ as:

$$(p \cdot q)(\mathbf{x}) = \sum_{\gamma \in \mathbb{Z}^n} \left(\sum_{\alpha + \beta = \gamma} p_{\alpha} q_{\beta} \right) \mathbf{x}^{\gamma}.$$

For a polynomial or a Laurent series p , $[\mathbf{x}^{\alpha}]p$ is the coefficient of $\mathbf{x}^{\alpha}$ in p , and will sometimes be referred to as p_{α} for brevity. The support of p denoted by $\text{supp}(p)$ is the set of all vectors α for which $[\mathbf{x}^{\alpha}]p \neq 0$, and we say that p is n -homogeneous if $|\alpha|_1 = n$ whenever $\alpha \in \text{supp}(p)$.

Logarithmic concavity. We say that a sequence $\{a_n\}_{n \in \mathbb{Z}}$ does not have any internal zeros if for any index k with $a_k = 0$, all the terms a_m for either $m \geq k$ or $m \leq k$ are zero as well. Then a sequence $\{a_n\}_{n \in \mathbb{Z}}$ is said to be logarithmically concave (log-concave) if for any $n \in \mathbb{Z}$, we have $a_n^2 \geq a_{n-1}a_{n+1}$, and it has no internal zeros. A sequence $\{a_n\}_{n=0}^k$ is ultra logarithmically concave (ultra log-concave) if the sequence $\{\frac{a_n}{\binom{k}{n}}\}_{n=0}^k$ is log-concave.

A function $f : \mathbb{R}_{>0}^n \rightarrow \mathbb{R}_{>0}$ is said to be log-concave if for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}_{>0}^n$ and $\lambda \in [0, 1]$, we have:

$$f(\lambda \mathbf{x} + (1 - \lambda) \mathbf{y}) \geq f(\mathbf{x})^{\lambda} f(\mathbf{y})^{1-\lambda}.$$

1.2 Stable polynomials

$\mathcal{H}$ -Stable polynomials, or simply stable polynomials, are multivariate generalizations of real-rooted polynomials. Leonid Gurvits gave an amazingly short proof of the famous van der Waerden conjecture for permanents using stability [14], which we will glance at in this section. For a more thorough study on stable polynomials, we refer to [38].

The **permanent** of an $n \times n$ matrix $A = (a_{ij})_{i,j}$ denoted by $\text{per}(A)$ is the sum $\sum_{\sigma \in S_n} \prod_{i=1}^n a_{i\sigma(i)}$, where S_n is the set of all permutations $\sigma : [n] \rightarrow [n]$. Although the permanent's definition is similar to the determinant's, it is notoriously hard to compute. In fact, [35] proves that computing the permanent of a matrix is a **#P-complete** problem.

#P is the complexity class of the counting problems associated with the decision problems in the set NP. Loosely speaking, #P-complete problems are hard counting problems. A formal definition of #P-completeness is given in [35], which is not in the scope of this thesis. Counting the number of solutions of an NP-complete problem is usually #P-complete, for instance, counting the number of 3-colorings of a graph is #P-complete [26].

Still, the connection between #P-completeness and NP-completeness is not clear, and #P-complete problems are not always associated with an NP-complete problem. For example, computing the permanent of a matrix $A \in \mathbb{N}^{n \times n}$, while being a #P-complete problem, corresponds to the decision problem of finding a maximum matching in a bipartite multi-graph, which has a deterministic polynomial time algorithm by [19].

In 1926, van der Waerden proposed a conjecture regarding finding the minimal permanent of all non-negative doubly stochastic matrices (matrices with all row and column sums being 1) in [36, Page 117]. The following problem then became known as the van der Waerden conjecture, although van der Waerden told one of the authors of [37] that he had made no such conjecture ([37, Page 110]):

Theorem 1.2.1 (The van der Waerden problem, proven in [11], [12] and [14]). *Let $A \in \mathbb{R}_{\geq 0}^{n \times n}$ be a doubly stochastic matrix. Then the permanent of A is at least $\frac{n!}{n^n}$.*

The van der Waerden problem was first solved in 1981 in [11] and [12] using eigenvalue methods. Gurvits' proof on the other hand, uses log-concavity of real stable polynomials. He further proves that there is a polynomial time algorithm that computes $\text{per}(A)$ within multiplicative factor e^n for any $A \in \mathbb{R}_{\geq 0}^{n \times n}$ ([14, Theorem 4.7]).

Definition 1.2.2 ($\mathcal{H}$ -stable polynomials). A polynomial $p \in \mathbb{C}[x_1, \dots, x_n]$ is **$\mathcal{H}$ -stable** (stable) if either $p \equiv 0$ identically, or if $p(\mathbf{x}) \neq 0$ for any $\mathbf{x} \in \mathcal{H}^n$. We further say that p is **real stable** if it has real coefficients.

Remark 1.2.3. Let $p(x) \in \mathbb{R}[x]$ be a univariate real stable polynomial. Then p has to be real rooted. Otherwise, let $z, \bar{z} \in \mathbb{C} \setminus \mathbb{R}$ be a pair of conjugate roots of p . Then either z or $\bar{z}$ is in the upper half plane $\mathcal{H}$, yielding a contradiction. Thus, we can think of real-stable polynomials as a generalization of real-rooted univariate polynomials.

Lemma 1.2.4. *Let $A = (a_{ij})_{i,j} \in \mathbb{R}_{\geq 0}^{n \times n}$. Define:*

$$p_A(x_1, \dots, x_n) := \prod_{j=1}^n \sum_{i=1}^n a_{ij} x_i,$$

then p_A is a real-stable polynomial, and:

$$\text{per}(A) = \partial_1 \cdots \partial_{n-1} \partial_n p_A.$$

Proof. If A has an all zero column, $p_A \equiv 0$ and the statement is trivial. Otherwise, let $z_1, \dots, z_n \in \mathcal{H}$ be arbitrary. Then $\Im(\sum_{i=1}^n a_{ij} z_i) = \sum_{i=1}^n a_{ij} \Im(z_i) > 0$ for all $j \in [n]$, and therefore, $\sum_{i=1}^n a_{ij} z_i \neq 0$. Because $p_A(z_1, \dots, z_n)$ is the product of n non-zero sums, it is non-zero itself. So p_A has no roots in $\mathcal{H}^n$, and is real stable.

To prove the second statement of this lemma, look at the coefficient of $x_1 \cdots x_n$ in p_A . Choosing each x_k from exactly one of the sums $\sum_{i=1}^n a_{ij} x_i$ for all $k \in [n]$, corresponds to taking some permutation $\sigma \in S_n$, and choosing $a_{k, \sigma(k)} x_k$ from the $\sigma(k)$ -th sum, $\sum_{i=1}^n a_{i, \sigma(k)} x_i$. So:

$$[x_1 \cdots x_n] p_A = \sum_{\sigma \in S_n} \prod_{k=1}^n a_{k, \sigma(k)} = \text{per}(A).$$

Furthermore, since p is an n -homogeneous polynomial, $\partial_1 \cdots \partial_{n-1} \partial_n p_A$ is a constant polynomial, and is exactly the coefficient of $x_1 \cdots x_n$ in P_A . Thus:

$$\text{per}(A) = [x_1 \cdots x_n] p_A = \partial_1 \cdots \partial_{n-1} \partial_n p_A. \quad \square$$

Lemma 1.2.5 ([38, Lemma 2.4]). *Assuming that $p(x_1, \dots, x_n)$ is a stable polynomial, $\partial_i p$ is either a stable polynomial or a constant.*

The idea behind Gurvits' proof of the van der Waerden conjecture is to define a capacity for stable polynomials, whose behavior is traceable under differentiation. Let $\text{Cap}(p)$ be the **Gurvits' capacity** (or simply the capacity) of p , defined by:

$$\text{Cap}(p) = \inf_{\substack{\mathbf{x} > \mathbf{0} \\ \prod_{i=1}^n x_i = 1}} p(\mathbf{x}),$$

Gurvits shows that capacity cannot decay too fast under differentiation.

Theorem 1.2.6 ([14, Theorem 4.10]). *Let $p \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ be a homogeneous real stable polynomial of degree n . Then either $\partial_n p \equiv 0$, or:*

$$\text{Cap}(\partial_n p) \geq \text{Cap}(p) \cdot \left(\frac{k-1}{k} \right)^{k-1},$$

where k is the degree of x_n in p .

Proof of Theorem 1.2.1. As proven in Lemma 1.2.4, the polynomial $p_A(x_1, \dots, x_n)$ is an n -homogeneous real-stable polynomial, so by Theorem 1.2.6, we have:

$$\text{Cap}(\partial_n p_A) \geq \text{Cap}(p_A) \cdot \left(\frac{k-1}{k} \right)^{k-1}, \quad (1.1)$$

where $k = \deg_{x_n}(p) \leq n$. Note that $f(z) = (\frac{z-1}{z})^{z-1}$ is a decreasing function for $z \geq 1$, and therefore $\left(\frac{k-1}{k}\right)^{k-1} \geq \left(\frac{n-1}{n}\right)^{n-1}$. So we can rewrite Equation (1.1) as:

$$\text{Cap}(\partial_n p_A) \geq \text{Cap}(p_A) \cdot \left(\frac{n-1}{n} \right)^{n-1}.$$

$\partial_n p_A$ is an $(n-1)$ -homogeneous real stable polynomial by Lemma 1.2.5, and we can repeat this step. Using Theorem 1.2.6 multiple times, we obtain:

$$\text{Cap}(\partial_1 \cdots \partial_n p_A) \geq \text{Cap}(p_A) \cdot \prod_{i=1}^n \left(\frac{i-1}{i} \right)^{i-1}.$$

Since $\partial_1 \cdots \partial_n p_A = \text{per}(A)$ is a constant function, we have $\text{Cap}(\partial_1 \cdots \partial_n p_A) = \text{per}(A)$. Gurvits also shows that $\text{Cap}(p_A) = 1$ for any doubly stochastic matrix A , and therefore:

$$\begin{aligned} \text{per}(A) &\geq \prod_{i=1}^n \left(\frac{i-1}{i} \right)^{i-1} \\ &= \prod_{i=1}^n i \cdot \frac{(i-1)^{i-1}}{i^i} \\ &= \frac{n!}{n^n}. \end{aligned}$$

□

Later, Gurvits generalized Theorem 1.2.6 in [16] to the following:

Theorem 1.2.7 ([16, Theorem 5.1], see also [7, Theorem 5.11]). *Let $p \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ be a real stable polynomial of degree d , given by:*

$$p(x_1, \dots, x_n) = \sum_{\mu} p_{\mu} \mathbf{x}^{\mu},$$

and let d_i be the degree of x_i in $\partial_{i+1}^{\alpha_{i+1}} \dots \partial_n^{\alpha_n} p|_{x_{i+1}=\dots=x_n=0}$, and d_n , the degree of x_n in p . Then:

$$p_{\alpha} \geq \text{Cap}_{\alpha}(p) \cdot \prod_{i=2}^n \binom{d_i}{\alpha_i} \frac{\alpha_i^{\alpha_i} (d_i - \alpha_i)^{d_i - \alpha_i}}{d_i^{d_i}},$$

where $\text{Cap}_{\alpha}(p) = \inf_{\mathbf{x} > \mathbf{0}} \frac{p(\mathbf{x})}{\mathbf{x}^{\alpha}}$.

1.3 Lorentzian polynomials

To motivate the development of Lorentzian polynomials, we will first introduce matroids, and discuss a famous problem regarding these objects.

Definition 1.3.1 (Matroids). A **matroid** $\mathcal{M}$ is a pair $(\mathcal{X}, \mathcal{I})$, where $\mathcal{X}$ is a finite set, and $\mathcal{I} \subseteq 2^{\mathcal{X}}$, the collection of **independent sets** of $\mathcal{M}$, satisfies

1. $\emptyset \in \mathcal{I}$,
2. If $A \in \mathcal{I}$, then any subset of A is an element of $\mathcal{I}$,
3. If $A, B \in \mathcal{I}$ and $|A| > |B|$, then there exists some $x \in A \setminus B$ such that $B \cup \{x\}$ is an independent set.

The **rank** of $\mathcal{M}$ denoted by $r_{\mathcal{M}}$ is the size of its largest independent set. We say that $B \in \mathcal{I}$ is a **basis** for $\mathcal{M}$ if $|B| = r_{\mathcal{M}}$. Let $\mathcal{B}_{\mathcal{M}}$ be the collection of all bases of $\mathcal{M}$.

Matroids are important and well-studied combinatorial objects. They have their roots in graph theory and linear algebra, and generalize the notion of linear independence. For a more detailed study of matroids and their geometry, we refer to [3]. Many combinatorial optimization problems are well understood for matroids, however, some counting problems seem to be harder to answer. For instance, counting the number of bases of a matroid is #P-complete, even for representable matroids [34]. Using connections between log-concave polynomials and matroids, [1] gives a deterministic polynomial time $2^{O(r_{\mathcal{M}})}$ -approximation algorithm for the number of bases of a matroid $\mathcal{M}$. More specifically, the authors of [1] prove that the multivariate basis generating polynomial of a matroid exhibits log-concavity properties, ultimately resulting in such an approximation algorithm, and proving Mason's conjecture, stated below:

Theorem 1.3.2 (Conjecture from [28], proven in [6] and [2]). *Let $\mathcal{M} = ([n], \mathcal{I})$ be a matroid and let I_k be the number of independent sets of size k in $\mathcal{M}$. Then the sequence $\{I_k\}_{k=0}^n$ is an ultra log-concave sequence.*

Authors in [6] prove Theorem 1.3.2 by introducing the class of **Lorentzian** polynomials. Similarly, [2] defines *completely log-concave* polynomials to prove this theorem. Lorentzian and completely log-concave polynomials agree on the set of homogeneous polynomials ([6, Theorem 2.30]), and are equivalent to the class of homogeneous *strongly log-concave* polynomials introduced in [15]. For the sake of consistency, we will refer to this class of homogeneous polynomials as Lorentzian polynomials.

Definition 1.3.3 ([6, Definition 2.1], Lorentzian polynomials). Let $H_n^d \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ be the set of d -homogeneous polynomials, and let $\mathring{L}_n^2 \subseteq H_n^d$ be the set of quadratic forms that have non-singular Hessian with exactly one positive eigenvalue. For any $d \geq 3$, define:

$$\mathring{L}_n^d = \{p \in H_n^d : \partial_i p \in \mathring{L}_n^{d-1} \quad \forall i \in [n]\}.$$

We say that the polynomials in $\mathring{L}_n^d$ are **strictly Lorentzian**. The closure of the set of strictly Lorentzian polynomials (in the Euclidean space of homogeneous polynomials of degree at most d with n variables and non-negative coefficients) is called the set of **Lorentzian** polynomials. Note that Lorentzian polynomials are always homogeneous.

Remark 1.3.4. Due to the recursive structure of Definition 1.3.3, $\partial_i p$ is Lorentzian if p is a Lorentzian polynomial.

Lorentzian polynomials have log-concavity properties. In fact, $p \in \mathbb{R}[x_1, \dots, x_n]$ is a Lorentzian polynomial if and only if for any $V = (V_{ij}) \in \mathbb{R}_{\geq 0}^{n \times k}$, the polynomial $D_V p$ is a log-concave function on the positive orthant [2, Definition 1.3], where D_V is the operator $\prod_{i=1}^k \sum_{j=1}^n V_{i,j} \partial_j$. When p is bivariate, log-concavity of $D_V p$ as a function translates to the log-concavity of p 's coefficient sequence:

Lemma 1.3.5 ([6, Example 2.26], see also [15, Proposition 2.7]). *A bivariate homogeneous polynomial $p(x, y) = \sum_{k=0}^d p_k x^{d-k} y^k$ is Lorentzian if and only if it has an ultra log-concave coefficient sequence, i.e., the sequence $\{p_k\}_{k=0}^d$ is ultra log-concave.*

Remark 1.3.6. Note that length of the sequence $\{p_k\}_{k=0}^d$ is a crucial part of its ultra log-concavity. For instance, the sequence $p_0 = 1, p_1 = 3, p_2 = 3$ is not ultra-log concave, since $\frac{p_0}{\binom{2}{0}}, \frac{p_1}{\binom{2}{1}}, \frac{p_2}{\binom{2}{2}}$ is not log-concave. However, we could increase the length of this sequence by adding a dummy zero term to it, and get the sequence $p_0 = 1, p_1 = 3, p_2 = 3, p_3 = 0$, which is ultra log-concave.

Lorentzian polynomials are closely related to matroids and M -convex sets. In fact, we see in [6, Theorem 3.10] that M -convex sets are characterized by Lorentzian polynomials:

Definition 1.3.7 (*M-convexity*). A nonempty set $A \subseteq \mathbb{Z}^n$ is **M-convex** if for any $\mathbf{x}, \mathbf{y} \in A$ and $i \in [n]$ such that $x_i > y_i$, there exists some $j \in [n]$ satisfying $x_j < y_j$ and $\mathbf{x} + \mathbf{e}_j - \mathbf{e}_i, \mathbf{y} - \mathbf{e}_j + \mathbf{e}_i \in A$ (The vector $\mathbf{e}_i \in \{0, 1\}^n$ has 1 in its i -th coordinate and is 0 everywhere else).

Theorem 1.3.8 ([6, Theorem 3.10]). *Let the generating function of a set $J \subseteq \mathbb{Z}_{\geq 0}^n$ be the polynomial*

$$f_J(x_1, \dots, x_n) = \sum_{\alpha \in J} \frac{\mathbf{x}^\alpha}{\alpha!}.$$

Then J is an M-convex set if and only if f_J is Lorentzian. Moreover, the support of any Lorentzian polynomial $p \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ is an M-convex set.

Theorem 1.3.9 ([2, Theorem 4.1]). *For any matroid $\mathcal{M} = ([n], \mathcal{I})$, the polynomial*

$$p_{\mathcal{M}}(x_1, \dots, x_n, y) = \sum_{I \in \mathcal{I}} y^{n-|I|} \mathbf{x}^I$$

is Lorentzian.

To prove Mason's log-concavity problem, we just need to prove that the polynomial $\sum_{k=0}^n I_k y^{n-k} x^k$ is Lorentzian. Then by Lemma 1.3.5, the sequence $\{I_k\}_{k=0}^n$ is ultra log-concave. In fact, we will show that $\sum_{k=0}^n I_k y^{n-k} x^k$ is obtained by applying a Lorentzian-preserving operator to $p_{\mathcal{M}}$.

Theorem 1.3.10 ([2, Theorem 2.2]). *If $p \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ is a Lorentzian polynomial and $T : \mathbb{R}_{\geq 0}^n \rightarrow \mathbb{R}_{\geq 0}^m$ is an affine transformation, then $p(T(x_1, \dots, x_n)) \in \mathbb{R}_{\geq 0}[y_1, \dots, y_m]$ is also Lorentzian.*

More generally, a linear operator $T : \mathbb{R}^\mu[x_1, \dots, x_n] \rightarrow \mathbb{R}[x_1, \dots, x_m]$ is said to be homogeneous if there exists some $k \in \mathbb{N}$ such that $T[\mathbf{x}^\alpha]$ is a homogeneous polynomial of degree $|\alpha|_1 + k$ for any $\alpha \in \mathbb{N}^n$. We say that T preserves the class of Lorentzian polynomials if $T[P]$ is Lorentzian for any Lorentzian polynomial p .

Theorem 1.3.11 ([6, Theorem 3.2]). *If $T : \mathbb{R}^\mu[x_1, \dots, x_n] \rightarrow \mathbb{R}[x_1, \dots, x_m]$ is a linear homogeneous operator, then T preserves the class of Lorentzian polynomials if $\text{Sym}_T(\mathbf{x}, \mathbf{z})$ is a Lorentzian polynomial, where:*

$$\text{Sym}_T(\mathbf{x}, \mathbf{z}) := T_{\mathbf{x}} \left[\prod_{i=1}^n (x_i + z_i)^{\mu_i} \right].$$

*Sym_T is called the **symbol** of T , and $T_{\mathbf{x}}$ treats all z variables as constants and only acts on x variables.*

Now we can prove Mason's log-concavity problem for matroids using Lorentzian-preserving operators as follows:

Proof of Theorem 1.3.2. Let $T : \mathbb{R}_{\geq 0}^{n+1} \rightarrow \mathbb{R}_{\geq 0}^{n+1}$ be an affine transformation defined by $T(x_1, \dots, x_n, y) = (x_1, \dots, x_1, y)$. Then by Theorem 1.3.9 and Theorem 1.3.10, we conclude that $p_{\mathcal{M}}(T(x_1, \dots, x_n, y)) = \sum_{k=0}^n I_k y^{n-k} x_1^k$ is Lorentzian, and by Lemma 1.3.5, $\{I_k\}_{k=0}^n$ is an ultra log-concave sequence. $\square$

1.4 Denormalized Lorentzian polynomials

In this section, we will discuss the class of denormalized Lorentzian polynomials, which is one of the main motivations behind this thesis. Depending heavily on Lorentzian polynomials, denormalized Lorentzian polynomials were introduced in [6] using a normalization operator $N : \mathbb{R}[x_1, \dots, x_n] \rightarrow \mathbb{R}[x_1, \dots, x_n]$, defined linearly by $N[\mathbf{x}^\alpha] = \frac{\mathbf{x}^\alpha}{\alpha!}$. A polynomial $p(\mathbf{x})$ is said to be **denormalized Lorentzian (DL)** if $N[p]$ is a Lorentzian polynomial. By [6, Corollary 3.7], all Lorentzian polynomials are denormalized Lorentzian since the normalization operator preserves the class of Lorentzian polynomials.

Remark 1.4.1. By Definition 1.3.3, the class of Lorentzian polynomials of degree at most d in n variables is closed by definition. Since normalization is a continuous operator, it follows that the class of denormalized Lorentzian polynomials of degree at most d in n variables is closed in the Euclidean space of homogeneous polynomials of degree at most d with n variables.

Example 1.4.1 (Schur polynomials). Schur polynomials, indexed by partitions, were first defined by Cauchy in [8], and their connections to the representations of $GL_m(\mathbb{C})$ was studied by Schur in [33]. We will define these polynomials and discuss their log-concavity.

A partition λ of some $d \in \mathbb{N}$ is a weakly decreasing sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k$ with $\lambda_1 + \dots + \lambda_k = d$. A Young tableau of shape λ is a tableaux with λ_i boxes in the i -th row. A representation of a Young tableau in English notation arranges the boxes in left-justified rows, with non-increasing row lengths. For instance, a Young tableau of shape $\lambda = (3, 1)$ in English notation is represented as follows:



Figure 1.1: Young tableau of shape $(3, 1)$

A semi-standard Young tableau T of shape λ filled with $[n]$ is a tableau of shape λ in English notation where each box is labeled with one of the numbers $1, \dots, n$, such that the numbers weakly increase in each row from left to right, and strictly increase in each column from top to bottom. The weight of a semi-standard Young tableau T is the vector $\text{wt}(T) = (w_1, \dots, w_n)$, where w_i is the number of boxes filled with i . The following tableaux are all of the possible semi-standard Young tableaux of shape $(3, 1)$:

1	1	1
2		

1	1	2
2		

1	2	2
2		

Figure 1.2: Semi-standard Young tableaux of shape $(3,1)$

For a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$, the **Schur** polynomial indexed by λ in n variable is:

$$s_\lambda(x_1, \dots, x_n) = \sum_T \mathbf{x}^{\text{wt}(T)},$$

where the sum is over all semi-standard Young tableaux of shape λ filled with $[n]$. For some $\boldsymbol{\mu} \in \mathbb{N}^n$, the coefficient of $\mathbf{x}^\mu$ in s_λ is the **Kostka number** $K_{\lambda, \boldsymbol{\mu}}$. Equivalently, the Kostka number $K_{\lambda, \boldsymbol{\mu}}$ is counting the number of semi-standard Young tableaux of shape λ and weight $\boldsymbol{\mu}$. Using the Bender-Knuth involutions from [4], one could show that $K_{\lambda, \boldsymbol{\mu}} = K_{\lambda, \boldsymbol{\gamma}}$ whenever $\boldsymbol{\gamma}$ is a permutation of $\boldsymbol{\mu}$, and thus, s_λ is a symmetric polynomial. In fact, Schur polynomials of degree d form a basis for the space of d -homogeneous polynomials in n variables. The problem of computing Kostka numbers $K_{\lambda, \boldsymbol{\mu}}$ is #P-complete by [30]. Schur polynomials are DL due to [21], but they are not generally Lorentzian. For instance, take the following Schur polynomial:

$$s_{(3,1)}(x_1, x_2) = x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3.$$

$s_{(3,1)}(x_1, x_2)$ is not Lorentzian because the sequence $(0, 1, 1, 1, 0)$ is not ultra log-concave.

Remark 1.4.2. Schubert polynomials are a generalization of Schur polynomials introduced by Lascoux and Schützenberger in [24]. Each Schubert polynomial in n variables is indexed by a permutation $w \in S_n$, and is denoted by $\mathfrak{S}_w(x_1, \dots, x_n)$. For a more detailed study of Schubert polynomials, see [32].

As a part of the proof of log-concavity of Schur polynomials, the authors of [21] prove that $x_1^{n-1} \dots x_n^{n-1} \mathfrak{S}_w(x_1^{-1}, \dots, x_n^{-1})$ is DL for any permutation w . They also conjecture that Schubert polynomials are DL themselves ([21, Conjecture 15]).

It is clear from Example 1.4.1 that the class of DL polynomials is not restricted to Lorentzian polynomials. DL polynomials still exhibit log-concavity properties, and are a generalization of Lorentzian polynomials.

Corollary 1.4.3. *A bivariate homogeneous polynomial is denormalized Lorentzian if and only if it has a log-concave coefficient sequence.*

Proof. Let $p(x, y) = \sum_{k=0}^d p_k x^k y^{d-k}$ be a denormalized Lorentzian polynomial. By definition, $N[p] = \sum_{k=0}^d \frac{p_k}{k!(d-k)!} x^k y^{d-k}$ is a Lorentzian polynomial and by Lemma 1.3.5, the sequence $\{\frac{p_k}{k!(d-k)!}\}_{k=0}^d$ is ultra log-concave, implying that $\{p_k\}_{k=0}^d$ is log-concave. The other direction of the proof is done similarly. $\square$

We are particularly interested in lower bounds on the coefficients of a DL polynomial. The authors of [7] use Gurvits' capacity function to prove such a lower bound, and then approximate the number of **contingency tables** of a given marginal. We will briefly mention their work here.

Theorem 1.4.4 ([7, Theorem 5.10]). *Let $\alpha \in \mathbb{Z}_{\geq 0}^n$ and $p(x_1, \dots, x_n) = \sum_{\mu \in \mathbb{N}^n} p_{\mu} \mathbf{x}^{\mu}$ be a DL polynomial of degree d . Let d_i be the degree of x_i in $\partial_{i+1}^{\alpha_{i+1}} \dots \partial_n^{\alpha_n} p|_{x_{i+1}=\dots=x_n=0}$, and let d_n be the degree of x_n in p . Then:*

$$[\mathbf{x}^{\alpha}]p \geq \text{Cap}_{\alpha}(p) \cdot \prod_{i=2}^n \max \left\{ \frac{\alpha_i^{\alpha_i}}{(\alpha_i + 1)^{\alpha_i+1}}, \frac{(d_i - \alpha_i)^{d_i - \alpha_i}}{(d_i - \alpha_i + 1)^{d_i - \alpha_i + 1}} \right\},$$

where $\text{Cap}_{\alpha}(p) = \inf_{\mathbf{x} > \mathbf{0}} \frac{f(\mathbf{x})}{\mathbf{x}^{\alpha}}$ is the **capacity** of p at the point α .

Remark 1.4.5 ([6, Remark 11.2]). For a polynomial $p \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$, we can interpret $\text{Cap}_{\alpha}(p)$ as a convex polynomial optimization problem:

$$-\log \text{Cap}_{\alpha}(p) = \inf_{\mathbf{y} \in \mathbb{R}^n} [-\log p(e^{\mathbf{y}}) + \langle \mathbf{y}, \alpha \rangle],$$

where $\log$ is the natural logarithm function and $\langle \cdot, \cdot \rangle$ is the usual dot product. Both $-\log$ and the dot product are convex functions, and $\mathbb{R}^n$ is a convex set. So $-\log \text{Cap}_{\alpha}(p)$ is a convex polynomial optimization problem, and can be solved using interior point methods.

For $\alpha \in \mathbb{Z}^n, \beta \in \mathbb{Z}^m$, a contingency table of marginals (α, β) is a matrix $A = (A_{ij}) \in \mathbb{Z}_{\geq 0}^{n \times m}$, with entries in the i -th row summing up to α_i and entries in the j -th row summing to β_j . Contingency tables are often studied as lattice points in transportation polytopes [9]. Let $K = (K_{ij})_{i,j}$ be an $n \times m$ matrix with entries in $\mathbb{Z}_{\geq 0} \cup \{\infty\}$. Then we say that A is a K -contingency table with marginals (α, β) if A is a contingency table with marginals (α, β) and $A_{ij} \leq K_{ij}$ for all $i, j \in [n]$. Let $CT_K(\alpha, \beta)$ be the number of K -contingency tables with marginals (α, β) . The problem of finding $CT_K(\alpha, \beta)$ is #p-complete, even if we restrict to the case $n = 2$ ([10, Theorem 1]).

For an $n \times m$ matrix K , the generating series for K -contingency tables with respect to their marginals is the following Laurent series:

$$P_K(x_1, \dots, x_n; y_1, \dots, y_m) = \prod_{i=1}^n \prod_{j=1}^m \sum_{l=0}^{K_{ij}} x_i^l y_j^l = \sum_{\alpha, \beta} CT_K(\alpha, \beta) \mathbf{x}^{\alpha} \mathbf{y}^{\beta}.$$

Note that even if P_K is a polynomial, it will not be denormalized Lorentzian, since it is not homogeneous. Instead, consider the following polynomial whenever K is a finite matrix with column marginals γ :

$$\tilde{P}_K(x_1, \dots, x_n; y_1, \dots, y_m) = \prod_{i=1}^n \prod_{j=1}^m \sum_{l=0}^{K_{ij}} x_i^l y_j^{K_{ij}-l} = \sum_{\alpha, \beta} CT_K(\alpha, \beta) \mathbf{x}^{\alpha} \mathbf{y}^{\gamma-\beta}.$$

All of the sums $\sum_{l=0}^{K_{ij}} x_i^l y_j^{K_{ij}-l}$ are bivariate homogeneous polynomials with log-concave coefficients. So $\tilde{P}_K$ is the product of some DL polynomials. The following lemma states that the class of DL polynomials is closed under multiplication, implying that $\tilde{p}_K$ is DL itself.

Theorem 1.4.6 ([6, Corollary 3.8]). *The class of DL polynomials is closed under multiplication.*

Now we can use Theorem 1.4.4 to approximate $CT_K(\alpha, \beta)$.

Theorem 1.4.7 ([7, Theorem 2.1]). *For $\alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^m$ and $K \in (\mathbb{Z}_{\geq 0} \cup \{\infty\})^{n \times m}$ we have:*

$$CT_K(\alpha, \beta) \geq \text{Cap}_{(\alpha, \beta)}(P_K) \cdot \prod_{i=2}^n \frac{\alpha_i^{\alpha_i}}{(1 + \alpha_i)^{1 + \alpha_i}} \prod_{j=1}^m \frac{\beta_j^{\beta_j}}{(1 + \beta_j)^{1 + \beta_j}}.$$

Proof. First, assume that K is a finite matrix. We can use Theorem 1.4.4 for $\tilde{P}_K$ to get:

$$\begin{aligned} CT_K(\alpha, \beta) &\geq \text{Cap}_{(\alpha, \gamma - \beta)}(\tilde{P}_K) \cdot \prod_{i=2}^n \frac{\alpha_i^{\alpha_i}}{(1 + \alpha_i)^{1 + \alpha_i}} \prod_{j=1}^m \frac{\beta_j^{\beta_j}}{(1 + \beta_j)^{1 + \beta_j}} \\ &= \text{Cap}_{(\alpha, \beta)}(P_K) \cdot \prod_{i=2}^n \frac{\alpha_i^{\alpha_i}}{(1 + \alpha_i)^{1 + \alpha_i}} \prod_{j=1}^m \frac{\beta_j^{\beta_j}}{(1 + \beta_j)^{1 + \beta_j}}. \end{aligned}$$

Now, suppose that K is not necessarily finite. Take a sequence $\{K_t\}_{t=0}^\infty$ of matrices in $\mathbb{Z}_{\geq 0}^{n \times m}$ that converge entry-wise to K . For each of these matrices K_t , we have a lower bound on the number of K_t -contingency tables with marginals (α, β) :

$$CT_{K_t}(\alpha, \beta) \geq \text{Cap}_{(\alpha, \beta)}(P_{K_t}) \cdot \prod_{i=2}^n \frac{\alpha_i^{\alpha_i}}{(1 + \alpha_i)^{1 + \alpha_i}} \prod_{j=1}^m \frac{\beta_j^{\beta_j}}{(1 + \beta_j)^{1 + \beta_j}}.$$

Note that as t goes to infinity, the number of K_t contingency tables of marginals (α, β) converges to $CT_K(\alpha, \beta)$. So we can write:

$$CT_K(\alpha, \beta) = \lim_{t \rightarrow \infty} CT_{K_t}(\alpha, \beta) \geq \lim_{t \rightarrow \infty} \text{Cap}_{(\alpha, \beta)}(P_{K_t}) \cdot \prod_{i=2}^n \frac{\alpha_i^{\alpha_i}}{(1 + \alpha_i)^{1 + \alpha_i}} \prod_{j=1}^m \frac{\beta_j^{\beta_j}}{(1 + \beta_j)^{1 + \beta_j}},$$

and by [17, Corollary 5.8], $\lim_{t \rightarrow \infty} \text{Cap}_{(\alpha, \beta)}(P_{K_t}) = \text{Cap}_{(\alpha, \beta)}(P_K)$. Therefore:

$$CT_K(\alpha, \beta) \geq \text{Cap}_{(\alpha, \beta)}(P_K) \cdot \prod_{i=2}^n \frac{\alpha_i^{\alpha_i}}{(1 + \alpha_i)^{1 + \alpha_i}} \prod_{j=1}^m \frac{\beta_j^{\beta_j}}{(1 + \beta_j)^{1 + \beta_j}}$$

and we are done. □

A part of this thesis is dedicated to bypassing the limit argument used in the proof of Theorem 1.4.7. We will develop a theory that works not only for denormalized Lorentzian polynomials, but for Laurent series that exhibit some log-concavity properties as well. Then we will use that theory to re-prove the lower bound for contingency tables in Theorem 1.4.7. We will also give a more natural proof of [25]’s bounds for complete acyclic graphs.

There exists a bijection between the integral flows of a complete acyclic directed graph, and K -contingency tables with certain marginals for some matrix K ([29, Section 1.3]). This bijection is employed in [25] to obtain a lower bound on the number of integral flows of such graphs. The number of integral flows of a complete graph is also known as the **Kostant partition number**. Kostant partition numbers are used to express other important quantities in representation theory, such as Kostka numbers and the Littlewood-Richardson coefficients [22].

Let $G = ([n], E)$ be an acyclic directed graph, and let $\mathbf{N} \in \mathbb{Z}_{\geq 0}^n$. An $\mathbf{N}$ -flow of G is a $|E|$ -tuple $\Phi = (\Phi_e)_{e \in E} \in \mathbb{N}^{|E|}$ satisfying:

$$\sum_{e: u \rightarrow v} \Phi_e - \sum_{e: v \rightarrow w} \Phi_e = N_v \quad \forall v \in [n].$$

Let K_{n+1} be the complete acyclic directed graph with vertices $0, 1, \dots, n$, and edges directed from j to i for $j > i$. Let $\mathbf{N} = (N_0, N_1, \dots, N_{n-1}, -\sum_{i=1}^{n-1} N_i) \in \mathbb{N}^{n+1}$. Denote by $K_n(\mathbf{N})$ the number of $\mathbf{N}$ -flows of K_{n+1} . We will introduce α, β, K such that the set of $\mathbf{N}$ -flows of K_{n+1} are in bijection with the set of K -contingency tables with marginals (α, β) .

Let $s_j = \sum_{k=0}^j N_k$ and define $\alpha = (s_0, \dots, s_{n-1})$, and $\beta = (s_{n-1}, \dots, s_0)$. Further, let K be an $n \times n$ matrix with $K_{ij} = 0$ for $i + j \geq n + 2$, and $K_{ij} = \infty$ otherwise (this matrix has zero entries in its bottom right corner). There is a bijection between the $\mathbf{N}$ -flows of K_{n+1} and K -contingency tables with marginals (α, β) . See [29, Section 1.3] and [25, Section 2.1] for more details and proof of this bijection. Theorem 1.4.7 for K -contingency tables with marginals (α, β) implies:

Theorem 1.4.8 ([25, Theorem 2.28]). *Let $\mathbf{N} = (N_0, \dots, N_n) \in \mathbb{Z}^{n+1}$, and $s_i = \sum_{k=0}^i N_k$ for $0 \leq i \leq n - 1$, and let $\alpha = (s_0, \dots, s_{n-1})$ and $\beta = (s_{n-1}, \dots, s_0)$. We have:*

$$K_n(\mathbf{N}) \geq \text{Cap}_{\alpha, \beta}(\Phi) \cdot \max_{0 \leq i \leq n-1} \left\{ \frac{(s_i + 1)^{s_i+1}}{s_i^{s_i}} \right\} \prod_{i=0}^{n-1} \left[\frac{s_i^{s_i}}{(s_i + 1)^{s_i+1}} \right]^2,$$

where

$$\Phi(\mathbf{x}, \mathbf{y}) = \prod_{\substack{0 \leq i, j \leq n-1 \\ i+j \leq n}} \frac{1}{1 - x_i y_j}.$$

Note that Theorem 1.4.8 uses a bijection between the integral flows of a complete graph and K -contingency tables of a given marginal. A similar bijection can be constructed for

any arbitrary acyclic graph by setting the entries of K corresponding to the missing edges in our graph to be zero. So with some work, one could prove a result similar to Theorem 1.4.8 for any acyclic directed graph.

Throughout the rest of this section, we will review some useful operations that preserve the class of DL polynomials.

Theorem 1.4.9 ([7, Lemma 4.8]). *If $p(x_1, \dots, x_n)$ is a DL polynomial and $y_1, \dots, y_n \in \mathbb{R}_{>0}$, then $p(y_1 x_1, \dots, y_n x_n)$ is a DL polynomial as well.*

Theorem 1.4.10 ([7, Lemma 4.8]). *Let $T_{x_1=x_2}^{\text{Poly}} : \mathbb{R}[x_1, \dots, x_n] \rightarrow \mathbb{R}[x_1, \dots, x_n]$ be the linear operator defined as $T_{x_1=x_2}^{\text{Poly}}[p(x_1, x_2, x_3, \dots, x_n)] = p(x_1, x_1, x_3, \dots, x_n)$. Then $T_{x_1=x_2}^{\text{Poly}}$ preserves the class of DL polynomials.*

Theorem 1.4.11. *Let $t \in \mathbb{N}$ and $p, q \in \mathbb{R}[x_1, \dots, x_n]$ be polynomials satisfying $p(\mathbf{x}) = x_1^t q(\mathbf{x})$. Then $p(\mathbf{x})$ is DL if and only if $q(\mathbf{x})$ is DL.*

Proof. If $q(\mathbf{x})$ is a DL polynomial, then $p(\mathbf{x})$ is also DL using Theorem 1.4.6. Conversely, if $p(\mathbf{x})$ is DL, then:

$$\begin{aligned} N[q(\mathbf{x})] &= N[x_1^{-t} p(\mathbf{x})] \\ &= N\left[\sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^n \\ \alpha_1 \geq t}} p_{\alpha} x_1^{\alpha_1 - t} x_2^{\alpha_2} \dots x_n^{\alpha_n}\right] \\ &= \sum_{\substack{\alpha \in \mathbb{Z}_{\geq 0}^n \\ \alpha_1 \geq t}} p_{\alpha} \frac{x_1^{\alpha_1 - t}}{(\alpha_1 - t)!} \cdot \frac{x_2^{\alpha_2}}{\alpha_2!} \dots \frac{x_n^{\alpha_n}}{\alpha_n!} \\ &= \partial_1^t \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} p_{\alpha} \frac{x_1^{\alpha_1}}{\alpha_1!} \cdot \frac{x_2^{\alpha_2}}{\alpha_2!} \dots \frac{x_n^{\alpha_n}}{\alpha_n!} \\ &= \partial_1^t N[p(\mathbf{x})], \end{aligned}$$

and $N[q(\mathbf{x})]$ is Lorentzian by Remark 1.3.4. □

Theorem 1.4.12. *For any $t \in \mathbb{Z}, i \in [n]$, let $\text{Trunc}_{x_i}^t : \mathbb{R}[x_1, \dots, x_n] \rightarrow \mathbb{R}[x_1, \dots, x_n]$ be a linear operator defined on monomials by:*

$$\text{Trunc}_{x_i}^t[\mathbf{x}^{\alpha}] = \begin{cases} x_i^{-t} \mathbf{x}^{\alpha} & \text{if } \alpha_i \geq t \\ 0 & \text{otherwise} \end{cases}.$$

Then Trunc_k^i preserves the class of DL polynomials.

Proof. Without loss of generality, assume that $i = 1$. Let $S := N \circ \text{Trunc}_{x_1}^t \circ N^{-1}$. We just need to prove that S preserves the class of Lorentzian polynomials. It suffices to prove that

$S|_{\mathbb{R}^\mu[x]}$ preserves the class of DL polynomials for any μ . Note that S is a homogeneous operator, and we only need to show that the symbol of $S|_{\mathbb{R}^\mu[x]}$ is Lorentzian.

$$\begin{aligned} \text{Sym}_{S|_{\mathbb{R}^\mu[x]}}(\mathbf{x}, \mathbf{z}) &= \sum_{0 \leq \lambda \leq \mu} \binom{\mu}{\lambda} z^{\mu-\lambda} S[\mathbf{x}^\lambda] \\ &= (\mu!)^n \sum_{\substack{0 \leq \lambda \leq \mu \\ \lambda_1 \geq t}} \frac{z^{\mu-\lambda}}{(\mu-\lambda)!} \cdot \frac{\mathbf{x}^{(\lambda_1-t, \lambda_2, \dots, \lambda_n)}}{(\lambda)!}, \end{aligned}$$

The set $\{(\mu - \lambda; \lambda_1 - t, \lambda_2, \dots, \lambda_n) : \mathbf{0} \leq \lambda \leq \mu, t \leq \lambda_1\}$ is an M-convex set, and $\text{Sym}_{S|_{\mathbb{R}^\mu[x]}}(\mathbf{x}, \mathbf{z})$ is the generating polynomial of an M-convex set up to a scalar, and therefore, is Lorentzian by Theorem 1.3.8. So S preserves the class of Lorentzian polynomials, and we are done. $\square$

Theorem 1.4.13. *For any $t \in \mathbb{Z}, i \in [n]$, let $\text{trunc}_{x_i}^t : \mathbb{R}[x_1, \dots, x_n] \rightarrow \mathbb{R}[x_1, \dots, x_n]$ be a linear operator defined on monomials as follows:*

$$\text{trunc}_{x_i}^t[\mathbf{x}^\alpha] = \begin{cases} \mathbf{x}^\alpha & \text{if } \alpha_i \leq t \\ 0 & \text{otherwise} \end{cases}.$$

Then trunc_k^i preserves the class of DL polynomials.

Proof. Similar to Theorem 1.4.12, we only need to show that $\text{Sym}_{S|_{\mathbb{R}^\mu[x]}}(\mathbf{x}, \mathbf{z})$ is Lorentzian for $S = N \circ \text{trunc}_{x_i}^t \circ N^{-1}$:

$$\begin{aligned} \text{Sym}_{S|_{\mathbb{R}^\mu[x]}}(\mathbf{x}, \mathbf{z}) &= \sum_{0 \leq \lambda \leq \mu} \binom{\mu}{\lambda} z^{\mu-\lambda} S[\mathbf{x}^\lambda] \\ &= (\mu!)^n \sum_{\substack{0 \leq \lambda \leq \mu \\ \lambda_1 \leq t}} \frac{z^{\mu-\lambda}}{(\mu-\lambda)!} \cdot \frac{\mathbf{x}^\lambda}{(\lambda)!}, \end{aligned}$$

which is the generating polynomial of an M-convex set up to a scalar, and therefore, is Lorentzian by Theorem 1.3.8. So S preserves the class of Lorentzian polynomials, and we are done. $\square$

1.5 Thesis outline

In Section 2.1, we will define denormalized Lorentzian Laurent series, and explore some basic properties of these objects. We prove that these Laurent series are closed under

multiplication, whenever multiplication is possible, in Section 2.2. We then proceed to define the domain of convergence for these Laurent series and study their behavior in Section 2.3, and use them to prove capacity bounds for DL Laurent series in Section 2.4. Lastly, we will apply these bounds in Section 2.5 to approximate the number of contingency tables and integral flows of a network.

Chapter 2

Denormalized Lorentzian Laurent series

2.1 Definition and basic properties

In this section, we define the class of denormalized Lorentzian Laurent series. We explore some of their basic properties, and try to draw analogs between this class of Laurent series and DL polynomials. Recall that in the context of this thesis, a Laurent series is an infinite sum $\sum_{\alpha \in \mathbb{Z}^n} p_{\alpha} \mathbf{x}^{\alpha}$.

Let $\text{Poly} : \mathbb{R}((x_1, \dots, x_n)) \rightarrow \mathbb{R}((x_1, \dots, x_n))$ be the following operator:

$$\text{Poly}\left[\sum_{\alpha \in \mathbb{Z}^n} p_{\alpha} \mathbf{x}^{\alpha}\right] = \sum_{\alpha \in \mathbb{Z}_{\geq 0}^n} p_{\alpha} \mathbf{x}^{\alpha}.$$

$\text{Poly}[p]$ is not necessarily a polynomial, however, it will be if p is a d -homogeneous Laurent series, since there are only finitely many $\alpha \in \mathbb{Z}_{\geq 0}^n$ satisfying $|\alpha|_1 = d$.

Definition 2.1.1 (denormalized Lorentzian Laurent series). For a homogeneous Laurent series $p \in \mathbb{R}_{\geq 0}((x_1, \dots, x_n))$, say that p is a **denormalized Lorentzian (DL) Laurent series** if, for any $\alpha \in \mathbb{Z}_{\geq 0}^n$, the polynomial $\text{Poly}[\mathbf{x}^{\alpha} p(\mathbf{x})]$ is denormalized Lorentzian.

Remark 2.1.2. The intersection of the class of DL Laurent series and polynomials is the class of DL polynomials. Let $p \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ be a DL polynomial. Theorem 1.4.11 gives us that $\text{Poly}[\mathbf{x}^{\alpha} p(\mathbf{x})] = \mathbf{x}^{\alpha} p(\mathbf{x})$ is a DL polynomial. Hence, p is a DL Laurent series by definition. Conversely, if a DL Laurent series p is a polynomial, then $\text{Poly}[\mathbf{x}^0 p(\mathbf{x})] = p(\mathbf{x})$, and $p(\mathbf{x})$ is a DL polynomial by definition.

So there is no ambiguity when we say that a polynomial p is DL, without specifying whether it is a DL Laurent series or a DL polynomial.

Example 2.1.3. Let $p(x, y) = \sum_{n \geq 0} x^n y^{-n-1}$. Then p is a (-1) -homogeneous Laurent series. For any $\alpha, \beta \in \mathbb{Z}_{\geq 0}$:

$$\text{Poly}[x^\alpha y^\beta p(x, y)] = \sum_{n=0}^{\beta-1} x^{n+\alpha} y^{\beta-1-n}.$$

$\text{Poly}[x^\alpha y^\beta p(x, y)]$ has a log-concave coefficient sequence, and is DL by Corollary 1.4.3, implying that p is a DL Laurent series by definition. Note that this Laurent series can be viewed as $\frac{1}{x-y}$ in one of its domains of convergence. We will discuss the domains of convergence of Laurent series more in Section 2.3.

Lemma 2.1.4. Let $p(x, y) = \sum_{n \in \mathbb{Z}} p_n x^{d-n} y^n$. Then p is DL if and only if the sequence $\{p_n\}_{n \in \mathbb{Z}}$ is log-concave, that is, it has no internal zeroes and for all $n \in \mathbb{Z}$, we have $p_n^2 \geq p_{n-1} p_{n+1}$.

Proof. Let us first do the forward direction of the proof. If p is a DL Laurent series, then the polynomial

$$\text{Poly}[x^m y^m p(x, y)] = \sum_{-m \leq k \leq d+m} p_k x^{d-k+m} y^{k+m}$$

is DL for all $m \in \mathbb{Z}$, and therefore the sequence $\{p_k\}_{k=-m}^{d+m}$ is log-concave by Corollary 1.4.3. Now given n , we just need to take a large enough m to see that $p_n^2 \geq p_{n-1} p_{n+1}$. Furthermore, $\{p_n\}_{n \in \mathbb{Z}}$ has no internal zeros since $\{p_k\}_{k=-m}^{d+m}$ does not have any internal zeros for all $m \in \mathbb{Z}$.

For the other direction of the proof, suppose $\{p_n\}_{n \in \mathbb{Z}}$ is a log concave sequence. We need to prove that the polynomial

$$\text{Poly}[x^\alpha y^\beta p(x, y)] = \sum_{d+\alpha \geq k \geq -\beta} p_k x^{d-k+\alpha} y^{k+\beta}$$

is DL. The sequence $\{p_k\}_{k=-\beta}^{d+\alpha}$ inherits log-concavity from $\{p_n\}_{n \in \mathbb{Z}}$, and $\text{Poly}[x^\alpha y^\beta p(x, y)]$ is DL by Corollary 1.4.3. $\square$

DL Laurent series maintain an intimate connection with M-convex sets.

Lemma 2.1.5. The support of any DL Laurent series $p \in \mathbb{R}_{\geq 0}((x_1, \dots, x_n))$ is M-convex.

Proof. Let $\boldsymbol{\mu}, \boldsymbol{\nu} \in \text{supp}(p)$, and let $\mu_i > \nu_i$ for some $i \in [n]$. Take a vector $\boldsymbol{\gamma} \in \mathbb{N}^n$ with large enough coordinates $\gamma_1, \dots, \gamma_n$ so that $\boldsymbol{\mu} + \boldsymbol{\gamma}, \boldsymbol{\nu} + \boldsymbol{\gamma} \in \mathbb{Z}_{>0}^n$.

Note that if a vector $\boldsymbol{\alpha}$ satisfies $\boldsymbol{\alpha} + \boldsymbol{\gamma} \geq \mathbf{0}$, $\boldsymbol{\alpha} \in \text{supp}(p)$ if and only if $\boldsymbol{\alpha} + \boldsymbol{\gamma}$ is in the support of $N[\text{Poly}[\boldsymbol{x}^\gamma p]]$ because:

$$[\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\gamma}}] (N[\text{Poly}[\boldsymbol{x}^\gamma p]]) = [\boldsymbol{x}^{\boldsymbol{\alpha}+\boldsymbol{\gamma}}] \left(\sum_{\boldsymbol{d}+\boldsymbol{\gamma} \geq \mathbf{0}} \frac{p_{\boldsymbol{d}}}{(\boldsymbol{d}+\boldsymbol{\gamma})!} \boldsymbol{x}^{\boldsymbol{d}+\boldsymbol{\gamma}} \right) = \frac{[\boldsymbol{x}^{\boldsymbol{\alpha}}] p(\boldsymbol{x})}{(\boldsymbol{\alpha}+\boldsymbol{\gamma})!}.$$

By definition, $N[\text{Poly}[\mathbf{x}^\gamma p]]$ is a Lorentzian polynomial, and its support is M-convex by Theorem 1.3.8. Note that both $\boldsymbol{\mu} + \boldsymbol{\gamma}$ and $\boldsymbol{\nu} + \boldsymbol{\gamma}$ are elements of the support of $N[\text{Poly}[\mathbf{x}^\gamma p]]$, and we have $(\boldsymbol{\mu} + \boldsymbol{\gamma})_i = \mu_i + \gamma_i > (\boldsymbol{\nu} + \boldsymbol{\gamma})_i = \nu_i + \gamma_i$. Therefore, there should exist some $j \in [n] - \{i\}$ such that $(\boldsymbol{\mu} + \boldsymbol{\gamma})_j < (\boldsymbol{\nu} + \boldsymbol{\gamma})_j$ and $\boldsymbol{\mu} + \boldsymbol{\gamma} - \mathbf{e}_i + \mathbf{e}_j, \boldsymbol{\nu} + \boldsymbol{\gamma} + \mathbf{e}_i - \mathbf{e}_j$ are both elements of $\text{supp}(N[\text{Poly}[\mathbf{x}^\gamma p]])$. So $\mu_j < \nu_j$ and $\boldsymbol{\mu} - \mathbf{e}_i + \mathbf{e}_j, \boldsymbol{\nu} + \mathbf{e}_i - \mathbf{e}_j \in \text{supp}(p)$ by what we discussed above, and $\text{supp}(p)$ is M-convex by definition. $\square$

The following is an analog of Theorem 1.4.11.

Lemma 2.1.6. *Let $p \in \mathbb{R}_{\geq 0}((x_1, \dots, x_n))$ be a DL Laurent series. Then $x_1^t p(\mathbf{x})$ is DL for any $t \in \mathbb{Z}$.*

Proof. It suffices to prove that for any vector $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^n$, the polynomial $\text{Poly}[x_1^t \mathbf{x}^\alpha p(\mathbf{x})]$ is DL.

Let $M > \alpha_1 + t$ be an integer, and let $\boldsymbol{\beta} := (M, \alpha_2, \dots, \alpha_n)$. Then $\text{Poly}[\mathbf{x}^\beta p(\mathbf{x})]$ is a DL polynomial by definition, and Theorem 1.4.12 implies that $\text{Trunc}_{x_1}^{-(\alpha_1+t)}[\text{Poly}[\mathbf{x}^\beta p(\mathbf{x})]]$ is also a DL polynomial. Now observe that:

$$\text{Poly}[\mathbf{x}^\beta p(\mathbf{x})] = \sum_{\substack{d_1 \geq -M \\ d_i \geq -\alpha_i \ \forall i \geq 2}} p_{\mathbf{d}} x_1^{d_1+M} x_2^{d_2+\alpha_2} \dots x_n^{d_n+\alpha_n},$$

and:

$$\begin{aligned} \text{Trunc}_{x_1}^{-(\alpha_1+t)} \text{Poly}[\mathbf{x}^\beta p(\mathbf{x})] &= \sum_{\substack{d_1 \geq -M \\ d_1 \geq -(\alpha_1+t) \\ d_i \geq -\alpha_i \ \forall i \geq 2}} p_{\mathbf{d}} x_1^{d_1+M+\alpha_1+t} x_2^{d_2+\alpha_2} \dots x_n^{d_n+\alpha_n} \\ &= x_1^M \sum_{\substack{d_1 \geq -(\alpha_1+t) \\ d_i \geq -\alpha_i \ \forall i \geq 2}} p_{\mathbf{d}} x_1^{d_1+\alpha_1+t} x_2^{d_2+\alpha_2} \dots x_n^{d_n+\alpha_n} \\ &= x_1^M \text{Poly}[x_1^t \mathbf{x}^\alpha p(\mathbf{x})]. \end{aligned}$$

We conclude that $x_1^M \text{Poly}[x_1^t \mathbf{x}^\alpha p(\mathbf{x})]$ is DL, and by Theorem 1.4.11, so is $\text{Poly}[x_1^t \mathbf{x}^\alpha p(\mathbf{x})]$. $\square$

Corollary 2.1.7. *A homogeneous Laurent series $p \in \mathbb{R}_{\geq 0}((x_1, \dots, x_n))$ is DL if and only if $\text{Poly}[\mathbf{x}^\alpha p(\mathbf{x})]$ is DL for any $\boldsymbol{\alpha} \in \mathbb{Z}^n$.*

Proof. The backward direction of the proof is trivial. To prove the forward direction, assume that p is a DL Laurent series, and let $\boldsymbol{\alpha} \in \mathbb{Z}^n$. By Lemma 2.1.6, $\mathbf{x}^\alpha p(\mathbf{x})$ is a DL Laurent series, and therefore $\text{Poly}[\mathbf{x}^\alpha p(\mathbf{x})]$ is a DL polynomial. $\square$

We will be using Corollary 2.1.7 as an alternate definition for DL Laurent series. The rest of this section is dedicated to exploring some operators on Laurent series that preserve the class of denormalized Lorentzian Laurent series.

Lemma 2.1.8. *The class of DL Laurent series is closed in the Euclidean space of homogeneous Laurent series of degree at most d in n variables.*

Proof. Let $\{p_i\}_{i \in \mathbb{N}}$ be a sequence of homogeneous DL Laurent series in variables $x_1, \dots, x_n$, converging coefficient-wise to a Laurent series p . Then for any $\alpha \in \mathbb{Z}^n$, the sequence $\{\text{Poly}[\mathbf{x}^\alpha p_i(\mathbf{x})]\}_{i \in \mathbb{N}}$ also converges coefficient-wise to $\text{Poly}[\mathbf{x}^\alpha p(\mathbf{x})]$. So $\text{Poly}[\mathbf{x}^\alpha p(\mathbf{x})]$ is the limit point of a sequence of DL polynomials, and is DL itself by Remark 1.4.1. $\square$

Lemma 2.1.9. *If $p \in \mathbb{R}_{\geq 0}((x_1, \dots, x_n))$ is a DL Laurent series. Then, for any $y_1, \dots, y_n > 0$, the Laurent series $p(y_1 x_1, \dots, y_n x_n)$ is also DL.*

Proof. Let $\alpha \in \mathbb{Z}^n$. Rewrite $\text{Poly}[\mathbf{x}^\alpha p(y_1 x_1, \dots, y_n x_n)]$ as:

$$\begin{aligned} \text{Poly}[\mathbf{x}^\alpha p(y_1 x_1, \dots, y_n x_n)] &= \sum_{\substack{d+\alpha \geq 0 \\ d \in \mathbb{Z}^n}} p_d \mathbf{x}^{d+\alpha} \mathbf{y}^d \\ &= \mathbf{y}^{-\alpha} \text{Poly}[\mathbf{x}^\alpha p](y_1 x_1, \dots, y_n x_n). \end{aligned}$$

The statement follows from Theorem 1.4.9. $\square$

Lemma 2.1.10. *Let $p \in \mathbb{R}_{\geq 0}((x_1, \dots, x_n))$ be a DL Laurent series. For any $d \in \mathbb{Z}$, the Laurent series $[x_1^d]p \in \mathbb{R}((x_2, \dots, x_n))$ defined as:*

$$[x_1^d]p = \sum_{d_2, \dots, d_n \in \mathbb{Z}} p_{d, d_2, \dots, d_n} x_2^{d_2} \cdots x_n^{d_n}$$

is a DL Laurent series as well.

Proof. Let $q(x_2, \dots, x_n) = [x_1^d]p$. We need to prove that $\text{Poly}[x_2^{\alpha_2} \cdots x_n^{\alpha_n} q]$ is DL for any choice of $\alpha_2, \dots, \alpha_n \in \mathbb{Z}$. Observe that:

$$\begin{aligned} \text{Poly}[x_2^{\alpha_2} \cdots x_n^{\alpha_n} q] &= \sum_{d_i + \alpha_i \geq 0, \forall i \geq 2} p_{(d, d_2, \dots, d_n)} x_2^{d_2 + \alpha_2} \cdots x_n^{d_n + \alpha_n} \\ &= \text{Poly}[x_1^{-d} x_2^{\alpha_2} \cdots x_n^{\alpha_n} p](0, x_2, \dots, x_n) \end{aligned}$$

We know that $\text{Poly}[x_1^{-d} x_2^{\alpha_2} \cdots x_n^{\alpha_n} p](x_1, x_2, \dots, x_n)$ is a DL polynomial, and for any $c > 0$, the polynomial $\text{Poly}[x_1^{-d} x_2^{\alpha_2} \cdots x_n^{\alpha_n} p](cx_1, x_2, \dots, x_n)$ is still DL by Theorem 1.4.9. Note that the class of DL polynomials is closed by Remark 1.4.1, so $\text{Poly}[x_1^d x_2^{\alpha_2} \cdots x_n^{\alpha_n} p](0, x_2, \dots, x_n)$ as the limit point of $\{\text{Poly}[x_1^{-d} x_2^{\alpha_2} \cdots x_n^{\alpha_n} p](\frac{1}{k} x_1, x_2, \dots, x_n)\}_{k \in \mathbb{N}}$ is DL and we are done. $\square$

Applying Lemma 2.1.10 repeatedly to a Laurent series p , and using the fact that $[x_i^{d_i} x_j^{d_j}]p = [x_i^{d_i}]([x_j^{d_j}]p)$ give us the following corollary:

Corollary 2.1.11. *Let $p \in \mathbb{R}_{\geq 0}((x_1, \dots, x_n))$ be a DL Laurent series. Given a subset $I \subseteq [n]$, and $d_i \in \mathbb{Z}$ for all $i \in I$, the Laurent series $[\prod_{i \in I} x_i^{d_i}]p$ is a DL Laurent series in variables x_j for $j \in [n] \setminus I$, where:*

$$[\prod_{i \in I} x_i^{d_i}]p = \sum_{d_j \in \mathbb{Z}: \forall j \in [n] \setminus I} p_d \prod_{j \in [n] \setminus I} x_j^{d_j}.$$

Corollary 2.1.12. *Generalize $\text{Trunc}_{x_1}^t$, $\text{trunc}_{x_1}^t$ from Theorem 1.4.12 and Theorem 1.4.13 to operators on Laurent series as follows:*

$$\text{Trunc}_{x_1}^t \left[\sum_{\alpha \in \mathbb{Z}^n} p_{\alpha} \mathbf{x}^{\alpha} \right] = x_i^{-t} \sum_{\substack{\alpha \in \mathbb{Z}^n \\ \alpha_1 \geq t}} p_{\alpha} \mathbf{x}^{\alpha},$$

and

$$\text{trunc}_{x_1}^t \left[\sum_{\alpha \in \mathbb{Z}^n} p_{\alpha} \mathbf{x}^{\alpha} \right] = \sum_{\substack{\alpha \in \mathbb{Z}^n \\ \alpha_1 \leq t}} p_{\alpha} \mathbf{x}^{\alpha}.$$

These new operators both preserve the class of DL Laurent series, that is, $\text{Trunc}_{x_1}^t[p]$ and $\text{trunc}_{x_1}^t[p]$ are both DL whenever p is a DL Laurent series.

Proof. Given that $p(\mathbf{x})$ is DL, we need to prove that $\text{Poly}[\mathbf{x}^{\beta} \text{Trunc}_{x_i}^t[p]]$ and $\text{Poly}[\mathbf{x}^{\beta} \text{trunc}_{x_i}^t[p]]$ are DL polynomials for any $\beta \in \mathbb{Z}^n$. Let $\gamma = (-\max\{-\beta_1, t\}, \beta_2, \dots, \beta_n)$, then:

$$\text{Poly}[\mathbf{x}^{\beta} \text{Trunc}_{x_i}^t[p]] = x_i^{-t} \sum_{\substack{\alpha_i + \beta_i \geq 0 \forall i \\ \alpha_1 - t \geq 0}} p_{\alpha} \mathbf{x}^{\alpha} = x_i^{-t} \text{Poly}[\mathbf{x}^{\gamma} p(\mathbf{x})],$$

and:

$$\text{Poly}[\mathbf{x}^{\beta} \text{trunc}_{x_1}^t[p]] = \sum_{\substack{\alpha_i + \beta_i \geq 0 \forall i \\ \alpha_1 \leq t}} p_{\alpha} \mathbf{x}^{\alpha} = \text{trunc}_{x_1}^t[\text{Poly}[\mathbf{x}^{\beta} p(\mathbf{x})]].$$

$x_i^{-t} \text{Poly}[\mathbf{x}^{\gamma} p(\mathbf{x})]$ is DL by Theorem 1.4.11 and Theorem 1.4.12, and $\text{trunc}_{x_1}^t[\text{Poly}[\mathbf{x}^{\beta} p(\mathbf{x})]]$ is DL by Theorem 1.4.13. Therefore both $\text{Poly}[\mathbf{x}^{\beta} \text{Trunc}_{x_i}^t[p]]$ and $\text{Poly}[\mathbf{x}^{\beta} \text{trunc}_{x_i}^t[p]]$ are DL polynomials, and we are done. $\square$

2.2 Product of DL Laurent series

We know from Theorem 1.4.6 that the class of DL polynomials is closed under multiplication. In [7], this fact is used to prove that the generating series for contingency tables with bounded entries (see Section 1.4 for definition and details) is DL, so we naturally search for an analog of Theorem 1.4.6 for DL Laurent series. But first, we need to address the problem that $\mathbb{R}((x_1, \dots, x_n))$ itself is not closed under multiplication, i.e., the product of two Laurent series may not be well defined. Recall that for two Laurent series

$p, q \in \mathbb{R}((x_1, \dots, x_n))$, their product is well defined if the sums $\sum_{\alpha+\beta=\gamma} p_\alpha q_\beta$ converge in $\mathbb{R}$ for all $\gamma \in \mathbb{Z}^n$. Then the product $p.q(\mathbf{x})$ is defined as:

$$(p \cdot q)(\mathbf{x}) = \sum_{\gamma \in \mathbb{Z}^n} \left(\sum_{\alpha+\beta=\gamma} p_\alpha q_\beta \right) \mathbf{x}^\gamma.$$

For instance, the product of $\sum_{k \in \mathbb{Z}} x^k y^{-k}$ and $\sum_{k \in \mathbb{Z}} y^k x^{-k}$ is not a well defined Laurent series, since the sum corresponding to the constant coefficient of $\sum_{k \in \mathbb{Z}} x^k y^{-k} \cdot \sum_{k \in \mathbb{Z}} y^k x^{-k}$ does not converge.

As a special case, assume that $p, q \in \mathbb{R}((x_1, \dots, x_n; y_1, \dots, y_m))$ are two Laurent series in disjoint sets of variables. Say p only uses the x variables and $[\mathbf{x}^\mu \mathbf{y}^\nu]p = 0$ for all $\nu \neq \mathbf{0}$, and q only uses the y variables and $[\mathbf{x}^\mu \mathbf{y}^\nu]q = 0$ for all $\mu \neq \mathbf{0}$. Then $\sum_{\alpha+\beta=\gamma} p_\alpha q_\beta$ is always a finite sum for any $\gamma \in \mathbb{Z}^{n+m}$. So $p.q$ is a well defined Laurent series. It is also easy to see that in this case, if p, q are both DL, their product will be DL as well:

Remark 2.2.1. If $p(\mathbf{x}), q(\mathbf{y})$ are two DL Laurent series in disjoint sets of variables $\mathbf{x}$ and $\mathbf{y}$, then $p(\mathbf{x})q(\mathbf{y})$ is DL since:

$$\begin{aligned} \text{Poly}[\mathbf{x}^\alpha \mathbf{y}^\beta p(\mathbf{x})q(\mathbf{y})] &= \sum_{\substack{\mu+\alpha \geq \mathbf{0} \\ \nu+\beta \geq \mathbf{0}}} p_\mu q_\nu \mathbf{x}^{\mu+\alpha} \mathbf{y}^{\nu+\beta} \\ &= \sum_{\mu+\alpha \geq \mathbf{0}} p_\mu \mathbf{x}^{\mu+\alpha} \cdot \sum_{\nu+\beta \geq \mathbf{0}} q_\nu \mathbf{y}^{\nu+\beta} \\ &= \text{Poly}[\mathbf{x}^\alpha p(\mathbf{x})] \text{Poly}[\mathbf{y}^\beta q(\mathbf{y})] \end{aligned}$$

$\text{Poly}[\mathbf{x}^\alpha p(\mathbf{x})] \text{Poly}[\mathbf{y}^\beta q(\mathbf{y})]$ is the product of two DL polynomials and is itself DL by Theorem 1.4.6. So $\text{Poly}[\mathbf{x}^\alpha \mathbf{y}^\beta p(\mathbf{x})q(\mathbf{y})]$ is DL for all α, β , and $p.q$ is DL by definition.

Suppose we have two DL Laurent series p and q , and we want to see whether their product is DL. Consider two disjoint sets of variables $\mathbf{x}, \mathbf{y}$, and multiply $p(\mathbf{x})$ and $q(\mathbf{y})$ to get a well-defined Laurent series $p(\mathbf{x})q(\mathbf{y})$. Now in this Laurent series, set $y_1 = x_1$, then $y_2 = x_2$, and so on. We will prove that as long as the final result is a well defined Laurent series, it will be denormalized Lorentzian.

Definition 2.2.2 (Admissible pairs). Let $T_{x_1=x_2}^{\text{PS}} : \mathbb{R}((x_1, \dots, x_n)) \rightarrow \mathbb{R}^*((x_1, \dots, x_n))$ be an operator that sends $p(x_1, \dots, x_n)$ to $p(x_1, x_1, x_3, \dots, x_n)$. This operator is a generalization of $T_{x_1=x_2}^{\text{Poly}}$ in Theorem 1.4.10.

We say that (x_1, x_2) is an **admissible pair** of p if $T_{x_1=x_2}^{\text{PS}}[p]$ is a well defined Laurent series. In other words, if p is given by its coefficients as:

$$p(x_1, \dots, x_n) = \sum_{\alpha \in \mathbb{Z}^n} p_\alpha \mathbf{x}^\alpha,$$

then (x_1, x_2) is an admissible pair of p if for any $\alpha, \alpha_3, \dots, \alpha_n \in \mathbb{Z}$, the sum $\sum_{\alpha_1+\alpha_2=\alpha} p_{(\alpha_1, \alpha_2, \dots, \alpha_n)}$ converges in $\mathbb{R}$. Similarly, one could define $T_{x_i=x_j}^{\text{PS}}$ and admissibility for any pair (x_i, x_j) .

Example 2.2.3. Let $p(x, y) = \sum_{n \geq 0} x^n y^{-n-1}$ be the Laurent series introduced in Example 2.1.3. Then (x, y) is not an admissible pair of p .

We will later see in Lemma 2.3.5 that (x_1, x_2) is an admissible pair of $p(x_1, x_2)$ if and only if $(1, 1)$ is in the domain of convergence of p .

Now let us prove that the operator $T_{x_1=x_2}^{\text{PS}}$ “almost” preserves the class of DL Laurent series.

Theorem 2.2.4. Assume that (x_1, x_2) is an admissible pair of a t -homogeneous DL polynomial p . Then $T_{x_1=x_2}^{\text{PS}}[p]$ is also a DL Laurent series.

To be more specific, for any given vector $\alpha = (\alpha_1, \alpha_3, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^{n-1}$, we will show that:

$$\text{Poly}[x_1^{\alpha_1} x_3^{\alpha_3} \dots x_n^{\alpha_n} T_{x_1=x_2}^{\text{PS}}[p]] = \lim_{k \rightarrow \infty} \text{Poly}[x_1^{\alpha_1-2k} T_{x_1=x_2}^{\text{PS}}[\text{Poly}[\mathbf{x}^{\beta_k} p]]],$$

where $\beta_k = (k, k, \alpha_3, \dots, \alpha_n)$, and the limit is taken in the Euclidean space of homogeneous polynomials of degree at most $|\alpha|_1 + t$ in $n-1$ variables.

Note that a similar theorem can be proven for $T_{x_i=x_j}^{\text{PS}}[p]$ when (x_i, x_j) is an admissible pair of p .

Proof. Let $p(\mathbf{x}) = \sum_{d \in \mathbb{Z}^n} p_d \mathbf{x}^d$ be a DL Laurent series and assume that (x_1, x_2) is an admissible pair of p . Then :

$$T_{x_1=x_2}^{\text{PS}}[p] = p(x_1, x_1, x_3, \dots, x_n) = \sum_{d, d_3, \dots, d_n \in \mathbb{Z}} \left(\sum_{d_1+d_2=d} p_{d_1, d_2, d_3, \dots, d_n} \right) x_1^d x_3^{d_3} \dots x_n^{d_n}.$$

We need to show that for any $\alpha = (\alpha_1, \alpha_3, \dots, \alpha_n) \in \mathbb{Z}_{\geq 0}^{n-1}$, the polynomial $\text{Poly}[\mathbf{x}^\alpha T_{x_1=x_2}^{\text{PS}}[p]]$ is DL. We have:

$$\text{Poly}[\mathbf{x}^\alpha T_{x_1=x_2}^{\text{PS}}[p]] = \sum_{\substack{d+\alpha_1 \geq 0 \\ d_i+\alpha_i \geq 0 \forall i \geq 3}} \left(\sum_{d_1+d_2=d} p_{d_1, d_2, d_3, \dots, d_n} \right) x_1^{d+\alpha_1} x_3^{d_3+\alpha_3} \dots x_n^{d_n+\alpha_n}. \quad (2.1)$$

So we want to show that the RHS of Equation (2.1) is a DL polynomial.

Let $\beta_k = (k, k, \alpha_3, \dots, \alpha_n)$. Since p itself is a DL Laurent series, the polynomial $T_{x_1=x_2}^{\text{Poly}}[\text{Poly}[\mathbf{x}^{\beta_k} p]]$ is also DL for all $k \in \mathbb{Z}$. Observe that:

$$\text{Poly}[\mathbf{x}^{\beta_k} p] = \text{Poly}[x_1^k x_2^k x_3^{\alpha_3} \dots x_n^{\alpha_n} p] = \sum_{\substack{d_1+k \geq 0 \\ d_2+k \geq 0 \\ d_i+\alpha_i \geq 0 \forall i \geq 3}} p_d x_1^{d_1+k} x_2^{d_2+k} x_3^{d_3+\alpha_3} \dots x_n^{d_n+\alpha_n}.$$

$T_{x_1=x_2}^{\text{Poly}}[\text{Poly}[\mathbf{x}^{\beta_k} p]]$ is also a DL polynomial by Theorem 1.4.10:

$$T_{x_1=x_2}^{\text{Poly}}[\text{Poly}[\mathbf{x}^{\beta_k} p]] = \sum_{\substack{d+2k \geq 0 \\ d_i + \alpha_i \geq 0 \forall i \geq 3}} \left(\sum_{\substack{d_1+d_2=d \\ d_1+k \geq 0 \\ d_2+k \geq 0}} p_{d_1, d_2, \dots, d_n} \right) x_1^{d+2k} x_3^{d_3+\alpha_3} \dots x_n^{d_n+\alpha_n}. \quad (2.2)$$

Note that as we take k to ∞ , the coefficients of Equation (2.2) converge to their corresponding coefficients in Equation (2.1). So let us apply some changes to Equation (2.2) to get closer to what Equation (2.1) looks like.

$$\begin{aligned} \text{Poly}[x_1^{\alpha_1-2k} T_{x_1=x_2}^{\text{Poly}}[\text{Poly}[\mathbf{x}^{\beta_k} p]]] &= \text{Poly} \left[\sum_{\substack{d+2k \geq 0 \\ d_i + \alpha_i \geq 0 \forall i \geq 3}} \left(\sum_{\substack{d_1+d_2=d \\ d_1+k \geq 0 \\ d_2+k \geq 0}} p_{d_1, d_2, \dots, d_n} \right) x_1^{d+\alpha_1} x_3^{d_3+\alpha_3} \dots x_n^{d_n+\alpha_n} \right] \\ &= \sum_{\substack{d+\alpha_1 \geq 0 \\ d+2k \geq 0 \\ d_i + \alpha_i \geq 0 \forall i \geq 3}} \left(\sum_{\substack{d_1+d_2=d \\ d_1+k \geq 0 \\ d_2+k \geq 0}} p_{d_1, d_2, \dots, d_n} \right) x_1^{d+\alpha_1} x_3^{d_3+\alpha_3} \dots x_n^{d_n+\alpha_n}. \end{aligned}$$

For any $k \geq \frac{\alpha_1}{2}$, the first summation would only impose the constraints $d + \alpha_1 \geq 0$ and $d_i + \alpha_i \geq 0$ for $i \geq 3$. Moreover:

$$\lim_{k \rightarrow \infty} \sum_{\substack{d_1+d_2=d \\ d_1+k \geq 0 \\ d_2+k \geq 0}} p_{d_1, d_2, \dots, d_n} = \sum_{d_1+d_2=d} p_{d_1, d_2, d_3, \dots, d_n},$$

and therefore:

$$\begin{aligned} \lim_{k \rightarrow \infty} \text{Poly}[x_1^{\alpha_1-2k} T_{x_1=x_2}^{\text{Poly}}[\text{Poly}[\mathbf{x}^{\beta_k} p]]] &= \lim_{k \rightarrow \infty} \sum_{\substack{d+\alpha_1 \geq 0 \\ d+2k \geq 0 \\ d_i + \alpha_i \geq 0 \forall i \geq 3}} \left(\sum_{\substack{d_1+d_2=d \\ d_1+k \geq 0 \\ d_2+k \geq 0}} p_{d_1, d_2, \dots, d_n} \right) x_1^{d+\alpha_1} x_3^{d_3+\alpha_3} \dots x_n^{d_n+\alpha_n} \\ &= \sum_{\substack{d+\alpha_1 \geq 0 \\ d_i + \alpha_i \geq 0 \forall i \geq 3}} \left(\sum_{d_1+d_2=k} p_{d_1, d_2, \dots, d_n} \right) x_1^{d+\alpha_1} x_3^{d_3+\alpha_3} \dots x_n^{d_n+\alpha_n} \\ &= \text{Poly}[x_1^{\alpha_1} x_3^{\alpha_3} \dots x_n^{\alpha_n} T_{x_1=x_2}^{\text{PS}}[p]]. \end{aligned}$$

Observe that LHS of the above equality is DL since the set of DL polynomials is closed by Remark 1.4.1, and therefore $\text{Poly}[x_1^{\alpha_1} x_3^{\alpha_3} \dots x_n^{\alpha_n} T_{x_1=x_2}^{\text{PS}}[p]]$ is DL for any $\alpha_1, \alpha_3, \dots, \alpha_n \in \mathbb{Z}$. $\square$

For the sake of brevity, we will denote by $T_{x_{i_k}=y_{i_k}, x_{i_{k-1}}=y_{i_{k-1}}, \dots, x_{i_1}=y_{i_1}}^{\text{PS}}$ the operator $T_{x_{i_k}=y_{i_k}}^{\text{PS}} \circ \dots \circ T_{x_{i_1}=y_{i_1}}^{\text{PS}}$.

Corollary 2.2.5. *Suppose $p(\mathbf{x}), q(\mathbf{x})$ are two homogeneous DL Laurent series. Further assume that (x_i, y_i) is an admissible pair of $T_{x_{i-1}=y_{i-1}, \dots, x_1=y_1}^{\text{PS}}[p(\mathbf{x})q(\mathbf{y})]$ for any $i \in [n]$ (i.e., $p(\mathbf{x})q(\mathbf{x})$ is a well defined Laurent series). Then $p(\mathbf{x})q(\mathbf{x})$ is also a DL Laurent series.*

Remark 2.2.6. Assuming that (x_1, y_1) and (x_2, y_2) are both admissible pairs of $p(\mathbf{x}, \mathbf{y})$, (x_2, y_2) is not necessarily an admissible pair of $T_{x_1=y_1}^{\text{PS}}[p]$. For instance, (x_1, y_1) and (x_2, y_2) are both admissible pairs of the Laurent series $p(x_1, x_2, y_1, y_2) = \sum_{n \geq 0} x_1^n y_1^{-n} x_2^n y_2^{-n}$, but (x_2, y_2) is not an admissible pair of $T_{x_1=y_1}^{\text{PS}}[p] = \sum_{n \geq 0} x_2^n y_2^{-n}$. So it is essential in the statement of Corollary 2.2.5 to assume that (x_i, y_i) is an admissible pair of $T_{x_{i-1}=y_{i-1}, \dots, x_1=y_1}^{\text{PS}}[p(\mathbf{x})q(\mathbf{y})]$, and not an admissible pair of $p(\mathbf{x})q(\mathbf{y})$.

2.3 Domains of convergence

The goal of this section is to study the domain of convergence of denormalized Lorentzian Laurent series, to ultimately define the capacity function for Laurent series in Section 2.4 and use it to prove lower bounds on the coefficients of denormalized Lorentzian Laurent series. We also examine the relationship between the domain of convergence of a DL Laurent series $p \in \mathbb{R}_{\geq 0}((x_1, \dots, x_n))$ and the Laurent series $[x_n^k]p$ for any $k \in \mathbb{Z}$.

Definition 2.3.1 (Domain of convergence). Let $p \in \mathbb{R}_{\geq 0}((x_1, \dots, x_n))$. For some $\mathbf{x} \in \mathbb{R}_{>0}^n$, say that $p(\mathbf{x})$ converges absolutely to $L \in \mathbb{R}$ if the following limit converges to L :

$$\lim_{k \rightarrow \infty} \sum_{\forall i: |\alpha_i| \leq k} p_{\alpha} \mathbf{x}^{\alpha}.$$

Then the **domain of convergence** of p , denoted by $\Omega_p \subseteq \mathbb{R}_{>0}^n$, is the set of all points $\mathbf{x} \in \mathbb{R}_{>0}^n$ for which $p(\mathbf{x})$ converges. If $p \equiv 0$ identically, let $\Omega_p = \mathbb{R}_{>0}^n$.

We are purposefully excluding the points $\mathbf{x} \in \mathbb{R}_{\geq 0}^n \setminus \mathbb{R}_{>0}^n$ because our definition of Laurent series allows negative powers, and working with such points brings up unnecessary complications.

Example 2.3.2. Let $p(z) = \sum_{n \in \mathbb{Z}} z^n$. Ω_p is empty since either $\sum_{n \geq 0} z^n$ or $\sum_{n \leq 0} z^n$ is divergent for any $z \in \mathbb{R}_{>0}$.

Example 2.3.3. Let $p(x, y) = \sum_{n \geq 0} y^n x^{-n-1}$ be the Laurent series from Example 2.1.3, then $\Omega_p = \{(x, y) : x > y > 0\}$. Since for any $x > y > 0$, $\frac{1}{x-y}$ can be expanded as $\sum_{n \geq 0} y^n x^{1-n}$, we have $p(x, y) = \frac{1}{x-y}$ for any $(x, y) \in \Omega_p$. Note that $\frac{1}{x-y}$ can also be expanded for any $y > x > 0$ as $\sum_{n \geq 0} (-1)^n x^n y^{1-n}$.

Remark 2.3.4. Assume that $p \in \mathbb{R}_{\geq 0}((x_1, \dots, x_n))$ is d -homogeneous. If $\mathbf{x} \in \Omega_p$, then $\lambda \mathbf{x} \in \Omega_p$ for any $\lambda > 0$ since $p(\lambda \mathbf{x}) = \lambda^d p(\mathbf{x})$.

Lemma 2.3.5. Let $p \in \mathbb{R}_{\geq 0}((x_1, \dots, x_n))$ be a d -homogeneous Laurent series. Then (x_1, x_2) is an admissible pair of p if and only if $(1, 1) \in \Omega_{[x_3^{\alpha_3} \dots x_n^{\alpha_n}]_p}$ for any $\alpha_3, \dots, \alpha_n \in \mathbb{Z}$.

Proof. Write:

$$[x_3^{\alpha_3} \dots x_n^{\alpha_n}]p = \sum_{\alpha_1 + \alpha_2 = d - (\alpha_3 + \dots + \alpha_n)} p_{\alpha} x_1^{\alpha_1} x_2^{\alpha_2}.$$

$(1, 1) \in \Omega_{[x_3^{\alpha_3} \dots x_n^{\alpha_n}]_p}$ for all $\alpha_3, \dots, \alpha_n \in \mathbb{Z}$ if and only if $\sum_{\alpha_1 + \alpha_2 = d - (\alpha_3 + \dots + \alpha_n)} p_{\alpha}$ is convergent for all $\alpha_3, \dots, \alpha_n \in \mathbb{Z}$, which is exactly the condition we need for admissibility of (x_1, x_2) from Definition 2.2.2. $\square$

We would like to determine the circumstances under which the domain of convergence of a bivariate homogeneous DL Laurent series is nonempty. We will answer the same question for n -variate homogeneous DL Laurent series later in this section.

Definition 2.3.6 (Trivially DL). As previously discussed in Lemma 2.1.4, $p(x, y) = \sum_{m \in \mathbb{Z}} p_m x^{d-m} y^m$ is a DL Laurent series if and only if $\{p_m\}_{m \in \mathbb{Z}}$ is a log-concave sequence. Further say that $p(x, y)$ is **trivially DL** if $\{p_m\}_{m \in \mathbb{Z}}$ is a geometric progression, that is, if there exists $a, c > 0$ such that $p_m = ac^m$ for all $m \in \mathbb{Z}$.

Lemma 2.3.7. Let $p(x, y) = \sum_{m \in \mathbb{Z}} p_m x^{d-m} y^m$ be a DL Laurent series. Ω_p is empty if and only if p is trivially DL. Moreover, for any index $m \in \mathbb{Z}$ with $p_m \neq 0$, we have:

$$\{1\} \times \left(\frac{p_{m-1}}{p_m}, \frac{p_m}{p_{m+1}} \right) \subseteq \Omega_p.$$

Proof. First observe that $\{p_m\}_{m \in \mathbb{Z}}$ is a log-concave sequence by Lemma 2.1.4, so $p_m^2 \geq p_{m-1}p_{m+1}$ for all $m \in \mathbb{Z}$, and assuming that $p_m \neq 0$, $(\frac{p_{m-1}}{p_m}, \frac{p_m}{p_{m+1}}) \subseteq \mathbb{R}_{>0}$ is a (possibly empty) interval.

We will prove that $\{1\} \times (\frac{p_{m-1}}{p_m}, \frac{p_m}{p_{m+1}}) \subseteq \Omega_p$ for any DL Laurent series $p(x, y)$. Then for non-trivially DL Laurent series, this immediately implies that the domain of convergence is non-empty since $p_m^2 > p_{m-1}p_{m+1}$ for some $m \in \mathbb{Z}$. we will also have to show that the domain of convergence of any trivially DL Laurent series is empty.

Fix $m \in \mathbb{Z}$ and $z \in (\frac{p_{m-1}}{p_m}, \frac{p_m}{p_{m+1}})$. Write $p(1, z) = \sum_{n \in \mathbb{Z}} p_n z^n$, and define $q_1(z) := \sum_{n \geq 0} p_{m+n} z^n$ and $q_2(z) := \sum_{n \geq 0} p_{m-n} z^{-n}$. $p(1, z)$ converges if and only if $q_1(z)$ and $q_2(z)$ both converge.

To show that $q_1(z)$ is convergent, we can reduce to the case where q_1 is an infinite sum. Otherwise, $q_1(z)$ is a finite sum and converges without any further issues. Additionally,

since $\{p_n\}_{n \in \mathbb{Z}}$ has no internal zeroes, all the coefficients p_n for $n \geq m$ are non-zero. We obtain:

$$\frac{p_{m+n+1}}{p_{m+n}} \leq \frac{p_{m+n}}{p_{m+n-1}} \leq \dots \leq \frac{p_{m+1}}{p_m},$$

and therefore,

$$\lim_{n \rightarrow \infty} \frac{p_{m+n+1} z^{m+n+1}}{p_{m+n} z^{m+n}} \leq \frac{p_{m+1}}{p_m} z < 1,$$

The ratio test from Lemma A.1.1 for $\{p_{m+n}\}_{n \geq 0}$ implies the convergence of q_1 .

Similarly, if $q_2(z)$ is a finite power sum, q_2 is convergent, so we can assume that p_{m-n} is nonzero for any $n \geq 0$, and we can write:

$$\frac{p_{m-(n+1)}}{p_{m-n}} \leq \frac{p_{m-n}}{p_{m-(n-1)}} \leq \dots \leq \frac{p_{m-1}}{p_m},$$

and therefore,

$$\lim_{n \rightarrow \infty} \frac{p_{m-(n+1)} z^{m-(n+1)}}{p_{m-n} z^{m-n}} \leq \frac{p_{m-1}}{p_m} z^{-1} < 1,$$

and q_2 is convergent as well by Lemma A.1.1. So far, we have proven that:

$$\{1\} \times \left(\frac{p_{m-1}}{p_m}, \frac{p_m}{p_{m+1}} \right) \subseteq \Omega_p,$$

Implying that $\Omega_p \neq \emptyset$ for non-trivially DL Laurent series. Finally, suppose $\{p_m\}_{m \in \mathbb{Z}}$ is a geometric progression, and $p_m = p_0 a^m$ for some $a \in \mathbb{R}_{>0}$. Rewrite $p(x, y)$ as:

$$p(x, y) = \sum_{m \in \mathbb{Z}} p_m x^{d-m} y^m = p_0 x^d \sum_{m \in \mathbb{Z}} \left(\frac{ay}{x} \right)^m.$$

By Example 2.3.2, the sum $\sum_{m \in \mathbb{Z}} \left(\frac{ay}{x} \right)^m$ does not converge for any $x, y \in \mathbb{R}_{>0}$, and Ω_p is empty. $\square$

Now we will derive a result similar to Lemma 2.3.7 for general DL Laurent series.

Lemma 2.3.8. *Let $n \geq 2$ and assume that $p \in \mathbb{R}_{\geq 0}((x_1, \dots, x_n))$ is a d -homogeneous DL Laurent series linear in x_n , given by:*

$$p(x_1, \dots, x_n) = x_n s(x_1, \dots, x_{n-1}) + r(x_1, \dots, x_{n-1}).$$

Then $\Omega_r = \Omega_s$.

Proof. We just need to prove that some vector $\mathbf{a} = (a_1, \dots, a_{n-1})$ is an element of Ω_r if and only if it is an element of Ω_s . Note that we can further assume $\mathbf{a} = \mathbf{1}_{n-1} = (1, \dots, 1) \in \mathbb{R}^{n-1}$, since we can scale the variables of p , and work with $p(x_1/a_1, \dots, x_n/a_n)$ instead (which is still a DL Laurent series by Lemma 2.1.9). So we want to prove that $\mathbf{1}_{n-1} \in \Omega_r$ if and only if $\mathbf{1}_{n-1} \in \Omega_s$.

The proof is done by induction over n . For the induction base, let $n = 2$. p is a homogeneous bivariate Laurent series of degree d and linear in x_2 , so we should have:

$$p(x_1, x_2) = ax_2x_1^{d-1} + bx_1^d$$

for some $a, b \in \mathbb{R}_{\geq 0}$, and the statement is trivial. For $n = 3$, the statement is proven in Lemma A.2.1.

There are two facts at the core of this proof for $n = 3$. First, there is a correspondence between the elements of $\text{supp}(s)$ and the elements of $\text{supp}(r)$ due to the M-convexity of $\text{supp}(p)$. Second, the growth of the coefficients of s and r is roughly similar because p is log-concave.

Now let $n > 3$. For any $k_3, \dots, k_{n-1} \in \mathbb{Z}$, write:

$$[x_3^{k_3} \cdots x_{n-1}^{k_{n-1}}]p = x_n[x_3^{k_3} \cdots x_{n-1}^{k_{n-1}}]s + [x_3^{k_3} \cdots x_{n-1}^{k_{n-1}}]r.$$

By Corollary 2.1.11, all of the Laurent series $s, r, [x_3^{k_3} \cdots x_{n-1}^{k_{n-1}}]p, [x_3^{k_3} \cdots x_{n-1}^{k_{n-1}}]s$ and $[x_3^{k_3} \cdots x_{n-1}^{k_{n-1}}]r$ are DL. Note that $[x_3^{k_3} \cdots x_{n-1}^{k_{n-1}}]p$ is a Laurent series with 3 variables x_1, x_2, x_n and is linear in x_n . So by the induction hypothesis, $(1, 1) \in \Omega_{[x_3^{k_3} \cdots x_{n-1}^{k_{n-1}}]s}$ if and only if $(1, 1) \in \Omega_{[x_3^{k_3} \cdots x_{n-1}^{k_{n-1}}]r}$.

Suppose $\mathbf{1}_{n-1} \in \Omega_s$, we need to show that $\mathbf{1}_{n-1} \in \Omega_r$. Since $([x_3^{k_3} \cdots x_{n-1}^{k_{n-1}}]s)(1, 1) < s(\mathbf{1}_{n-1})$, we have that $(1, 1) \in \Omega_{[x_3^{k_3} \cdots x_{n-1}^{k_{n-1}}]s}$, and using the induction hypothesis, $(1, 1) \in \Omega_{[x_3^{k_3} \cdots x_{n-1}^{k_{n-1}}]r}$. This holds for any choice of $k_3, \dots, k_{n-1}$, which by Lemma 2.3.5, means that (x_1, x_2) is an admissible pair of both s and r , implying that $(1, 1)$ is an admissible pair of p as well.

Let $p_1 = T_{x_1=x_2}^{\text{PS}}[p]$. p_1 is DL by Theorem 2.2.4, and:

$$p_1(x_1, x_3, \dots, x_n) = x_n s_1(x_1, x_3, \dots, x_{n-1}) + r_1(x_1, x_3, \dots, x_{n-1}),$$

where $s_1 = T_{x_1=x_2}^{\text{PS}}[s]$ and $r_1 = T_{x_1=x_2}^{\text{PS}}[r]$. Going back to our initial assumption, we have $\mathbf{1}_{n-1} \in \Omega_s$, giving us $\mathbf{1}_{n-2} \in \Omega_{s_1}$. By the induction hypothesis for s_1 and r_1 , $\mathbf{1}_{n-2} \in \Omega_{r_1}$, which means that $r_1(\mathbf{1}_{n-2})$ is convergent, and so is $r(\mathbf{1}_{n-1}) = r_1(\mathbf{1}_{n-2})$. Thus, we have proven $\mathbf{1}_{n-1} \in \Omega_r$, and we are done. The other direction follows similarly. $\square$

Lemma 2.3.9. Assume that p is a d -homogeneous DL Laurent series in $x_1, \dots, x_n$. Further assume that:

$$p(x_1, \dots, x_n) = \sum_{k \in \mathbb{Z}} x_n^k p_k(x_1, \dots, x_{n-1}).$$

The following statements hold for p :

1. Let $(y_1, \dots, y_{n-1}) \in \bigcap_k \Omega_{p_k}$. Then $\{p_k(y_1, \dots, y_{n-1})\}_k$ is a log-concave sequence with no internal zeros,

2. $\Omega_{p_m} = \Omega_{p_k}$ whenever p_m, p_k are both non-zero Laurent series,
3. Let $(z_1, \dots, z_{n-1}) \in \Omega_{p_k}$ for some nonzero p_k . Then $(z_1, \dots, z_{n-1}, y) \in \Omega_p$ for any y in the following interval for some $m \in \mathbb{Z}$:

$$\left(\frac{p_{m-1}(z_1, \dots, z_{n-1})}{p_m(z_1, \dots, z_{n-1})}, \frac{p_m(z_1, \dots, z_{n-1})}{p_{m+1}(z_1, \dots, z_{n-1})} \right).$$

Proof. To prove the first statement, assume that $(y_1, \dots, y_{n-1}) \in \Omega_k$ for any k . Ideally, if $T_{x_1=x_2, x_2=x_3, \dots, x_{n-2}=x_{n-1}}^{\text{PS}}[p](s, t) = p(s, \dots, s, t)$ was still a well defined DL Laurent series, we could have simply used Lemma 2.1.9 to prove that the following Laurent series is DL:

$$p(y_1 s, \dots, y_{n-1} s, t) = \sum_{k \in \mathbb{Z}} t^k p_k(y_1 s, \dots, y_{n-1} s) = \sum_{k \in \mathbb{Z}} t^k s^{d-k} p_k(y_1, \dots, y_{n-1}),$$

and by Lemma 2.1.4, $\{p_k(y_1, \dots, y_{n-1})\}_{k \in \mathbb{Z}}$ would have been a log-concave sequence. But we are not assuming anything about the admissibility of (x_i, x_j) . Instead, we will utilize the Poly operator and a similar argument for $\text{Poly}[\mathbf{x}^\alpha p]$.

First, note that it suffices to prove that for any given $l \leq u$, the sequence $\{p_k(y_1, \dots, y_{n-1})\}_{k=l}^u$ is log-concave. For any $\alpha = (\alpha_1, \dots, \alpha_{n-1}) \in \mathbb{Z}^{n-1}$, we have:

$$\text{Poly}[x_1^{\alpha_1} \dots x_{n-1}^{\alpha_{n-1}} x_n^{-l} p](x_1, \dots, x_n) = \sum_{k \geq l} x_n^{k-l} \text{Poly}[x_1^{\alpha_1} \dots x_{n-1}^{\alpha_{n-1}} p_k](x_1, \dots, x_{n-1}).$$

For the sake of brevity, define:

$$\begin{aligned} q_\alpha(x_1, \dots, x_n) &:= \text{Poly}[x_1^{\alpha_1} \dots x_{n-1}^{\alpha_{n-1}} x_n^{-l} p](x_1, \dots, x_n), \\ q_{k,\alpha}(x_1, \dots, x_{n-1}) &:= \text{Poly}[x_1^{\alpha_1} \dots x_{n-1}^{\alpha_{n-1}} p_k](x_1, \dots, x_{n-1}). \end{aligned}$$

So $q_\alpha(x_1, \dots, x_n) = \sum_{k \geq l} x_n^{k-l} q_{k,\alpha}(x_1, \dots, x_{n-1})$. q_α is a DL polynomial by definition, giving us that $q_\alpha(y_1 t, \dots, y_{n-1} t, s)$ is a $(d + |\alpha|_1 - l)$ -homogeneous bivariate DL polynomial in t and s by Theorem 1.4.9 and Theorem 1.4.10. Rewrite $q_\alpha(y_1 t, \dots, y_{n-1} t, s)$ as:

$$q_\alpha(y_1 t, \dots, y_{n-1} t, s) = \sum_{k \geq l} s^{k-l} t^{d+|\alpha|_1-k} q_{k,\alpha}(y_1, \dots, y_{n-1}).$$

We conclude that the sequence of coefficients of $q_\alpha(y_1 t, \dots, y_{n-1} t, s)$, $\{q_{k,\alpha}(y_1, \dots, y_{n-1})\}_{k \geq l}$ is log-concave by Corollary 1.4.3 (note that this sequence is log-concave regardless of the fact that $(y_1, \dots, y_{n-1}) \in \cap_k \Omega_{p_k}$. We will use this in the proof of the second statement of this theorem).

Assume further that $(y_1, \dots, y_{n-1}) \in \Omega_{p_k}$ for all $k \in \mathbb{Z}$. Using Corollary 1.4.3 again, for any $u \geq l$, the polynomial $\sum_{k=l}^u s^{k-l} t^{u-k} q_{k,\alpha}(y_1, \dots, y_{n-1})$ is DL, and since the class of DL

polynomials is closed by Remark 1.4.1, the polynomial $\lim_{\alpha \rightarrow \infty} \sum_{k=l}^u s^{k-l} t^{u-k} q_{k,\alpha}(y_1, \dots, y_{n-1})$ is also DL. We have:

$$\begin{aligned} \lim_{\alpha \rightarrow \infty} \sum_{k=l}^u s^{k-l} t^{u-k} q_{k,\alpha}(y_1, \dots, y_{n-1}) &= \sum_{k=l}^u s^{k-l} t^{u-k} \lim_{\alpha \rightarrow \infty} q_{k,\alpha}(y_1, \dots, y_{n-1}) \\ &= \sum_{k=l}^u s^{k-l} t^{u-k} p_k(y_1, \dots, y_{n-1}). \end{aligned}$$

So the coefficient sequence of this polynomial, $\{p_k(y_1, \dots, y_{n-1})\}_{k=l}^u$, is log concave, and we are done.

Let us move on to the proof of the second statement. We have proven in the first part of this theorem that for any arbitrary $\alpha \in \mathbb{Z}^n$, $(y_1, \dots, y_{n-1}) \in \mathbb{R}_{>0}^{n-1}$ and $l \leq u$, the sequence $\{q_{k,\alpha}\}_{k=l}^u$ is log-concave, and has no internal zeroes. $q_{k,\alpha}$ is a "part" of the Laurent series p_k , and $\{p_k\}_{k \in \mathbb{Z}}$ cannot have any internal zeros either, in the sense that if $p_k \equiv 0$, then $p_m \equiv 0$ either for all $m \geq k$ or for all $m \leq k$. So now, it suffices to show that $\Omega_{p_k} = \Omega_{p_{k+1}}$ whenever p_k, p_{k+1} are non-zero. Take the Laurent series:

$$\text{Trunc}_{x_n}^k \circ \text{trunc}_{x_n}^{k+1}[p] = p_k(x_1, \dots, x_{n-1}) + x_n p_{k+1}(x_1, \dots, x_{n-1}).$$

$\text{Trunc}_{x_n}^k \circ \text{trunc}_{x_n}^{k+1}[p]$ is DL by Corollary 2.1.12, and $\Omega_{p_k} = \Omega_{p_{k+1}}$ by Lemma 2.3.8. We are done.

To prove the third statement, take some $(z_1, \dots, z_{n-1}) \in \Omega_{p_k}$. We have that $(z_1, \dots, z_{n-1}) \in \Omega_{p_j}$ for any nonzero p_j by statement 2, and therefore by statement 1, $\{p_j(z_1, \dots, z_{n-1})\}_{j \in \mathbb{Z}}$ is a log-concave sequence, and the Laurent series

$$q(s, t) = \sum_{j \in \mathbb{Z}} p_j(z_1, \dots, z_{n-1}) s^j t^{-j}$$

is a DL bivariate Laurent series. Apply Lemma 2.3.7 to get that $q(s, t)$ for any $\frac{t}{s} \in (\frac{p_{m-1}(z_1, \dots, z_{n-1})}{p_m(z_1, \dots, z_{n-1})}, \frac{p_m(z_1, \dots, z_{n-1})}{p_{m+1}(z_1, \dots, z_{n-1})})$. Equivalently, for any y in the following interval:

$$\left(\frac{p_{m-1}(z_1, \dots, z_{n-1})}{p_m(z_1, \dots, z_{n-1})}, \frac{p_m(z_1, \dots, z_{n-1})}{p_{m+1}(z_1, \dots, z_{n-1})} \right),$$

we have that the Laurent series $q(y, 1) = p(z_1, \dots, z_{n-1}, y)$ converges. $\square$

It follows from the proof of the third statement of Lemma 2.3.9 and Lemma 2.3.7 that if the sequence $\{p_j(z_1, \dots, z_{n-1})\}_{j \in \mathbb{Z}}$ is not geometric, then $(z_1, \dots, z_{n-1})$ can be extended to a point $(z_1, \dots, z_{n-1}, y)$ in Ω_p . So for instance if $p_m \equiv 0$ for some index m (or in other words if the set of powers of x_n in p is bounded), then $\{p_j(z_1, \dots, z_{n-1})\}_{j \in \mathbb{Z}}$ is not a geometric sequence, and $(z_1, \dots, z_{n-1})$ can be extended to a point $(z_1, \dots, z_{n-1}, y) \in \Omega_p$. Also any point $(z_1, \dots, z_{n-1}, y) \in \Omega_p$ can be traced back to a point $(z_1, \dots, z_{n-1}) \in \Omega_{p_k}$.

There are more inclusive conditions we could impose on p to obtain that p is a sense non-trivially DL, and that $\{p_j(z_1, \dots, z_{n-1})\}_{j \in \mathbb{Z}}$ is not geometric. But boundedness of the set of powers of x_n is relatively easy to check and is sufficient for the purpose of this thesis.

Definition 2.3.10 (The set $\text{Deg}_{x_i}^\alpha$). Let $p(x_1, \dots, x_n)$ be a Laurent series and $\alpha \in \mathbb{Z}^n$ be a given vector. For any $2 \leq i \leq n$, let $\text{Deg}_{x_i}^\alpha(p)$ denote the set

$$\{k_i : (k_1, \dots, k_i, \alpha_{i+1}, \dots, \alpha_n) \in \text{supp}(p) \text{ for some } k_1, \dots, k_{i-1} \in \mathbb{Z}\},$$

and let $\text{Deg}_{x_i}(p)$ denote the set

$$\{k_i : (k_1, \dots, k_i, \dots, k_n) \in \text{supp}(p) \text{ for some } k_1, \dots, k_{i-1}, k_{i+1}, \dots, k_n \in \mathbb{Z}\}.$$

Remark 2.3.11. If $\text{Deg}_{x_n}(p)$ is bounded below or above, then there exists some $m \in \mathbb{Z}$ such that the interval

$$\left(\frac{p_{m-1}(z_1, \dots, z_{n-1})}{p_m(z_1, \dots, z_{n-1})}, \frac{p_m(z_1, \dots, z_{n-1})}{p_{m+1}(z_1, \dots, z_{n-1})} \right)$$

is nonempty for any $(z_1, \dots, z_{n-1}) \in \Omega_{p_k}$. If $\text{Deg}_{x_n}(p)$ is bounded below, let m be the minimum element of $\text{Deg}_{x_n}(p)$, then we have $\frac{p_{m-1}(z_1, \dots, z_{n-1})}{p_m(z_1, \dots, z_{n-1})} = 0$ and $\frac{p_m(z_1, \dots, z_{n-1})}{p_{m+1}(z_1, \dots, z_{n-1})} > 0$. If $\text{Deg}_{x_n}(p)$ is bounded above, let m be the maximum element of $\text{Deg}_{x_n}(p)$, then $\frac{p_m(z_1, \dots, z_{n-1})}{p_{m+1}(z_1, \dots, z_{n-1})} = \infty$ and $\frac{p_{m-1}(z_1, \dots, z_{n-1})}{p_m(z_1, \dots, z_{n-1})} < \infty$. Therefore, a point $(z_1, \dots, z_{n-1}) \in \Omega_{p_k}$ can always be extended to some $(z_1, \dots, z_{n-1}, y) \in \Omega_p$ for $p_k \neq 0$ by the third statement of Lemma 2.3.9.

Remark 2.3.12. If $(y_1, \dots, y_n) \in \Omega_p$ then $(y_1, \dots, y_{n-1}) \in \Omega_{p_k}$ for any $k \in \mathbb{Z}$ since:

$$y_n^k p_k(y_1, \dots, y_{n-1}) \leq p(y_1, \dots, y_n) < \infty$$

and therefore $p_k(y_1, \dots, y_n)$ converges as well.

Corollary 2.3.13. If $\text{Deg}_{x_n}(p)$ is bounded below or above, then by Remark 2.3.11 and Remark 2.3.12:

$$\pi_n(\Omega_p) = \Omega_{p_k},$$

where $\pi_n : \mathbb{R}^n \rightarrow \mathbb{R}^{n-1}$ is defined as $\pi_n(x_1, \dots, x_n) = (x_1, \dots, x_{n-1})$.

Now we are ready to define and use capacity to prove lower bounds on the coefficients of DL Laurent series.

2.4 Capacity bounds

In this section, we will study the capacity function for Laurent series, and prove a lower bound similar to Theorem 1.4.4 for the coefficients of denormalized Lorentzian Laurent series.

Definition 2.4.1. Recall that the capacity of a polynomial $p \in \mathbb{R}_{\geq 0}[x_1, \dots, x_n]$ at the point $\alpha \in \mathbb{Z}^n$ is the function $\text{Cap}_\alpha(p) = \inf_{\mathbf{x}^\alpha} \frac{p(\mathbf{x})}{\mathbf{x}^\alpha}$. For a Laurent series $p \in \mathbb{R}_{\geq 0}((x_1, \dots, x_n))$, capacity is defined as:

$$\text{Cap}_\alpha(p) = \begin{cases} \inf_{\mathbf{x} \in \Omega_p} \frac{p(\mathbf{x})}{\mathbf{x}^\alpha} & \text{if } \Omega_p \neq \emptyset \\ \infty & \text{otherwise} \end{cases},$$

and can also be written as $\text{Cap}_\alpha(p) = \inf_{\mathbf{x} > \mathbf{0}} \frac{p(\mathbf{x})}{\mathbf{x}^\alpha}$. Capacity of a Laurent series is always nonnegative, and is finite if and only if $\Omega_p \neq \emptyset$.

Example 2.4.2. Let $p(x, y) = \sum_{m \in \mathbb{Z}} y^m x^{-d-m}$, then $\text{Cap}_\alpha(p) = \infty$ since the domain of convergence of p is empty by Example 2.3.2.

Lemma 2.4.3. Assume that $p \in \mathbb{R}_{\geq 0}((x_1, \dots, x_n))$ is a nonzero Laurent series with a non-empty domain of convergence. Then $\text{Cap}_\alpha(p) > 0$ if and only if $\alpha \in \text{Newt}(p)$, where $\text{Newt}(p)$ is the convex hull of $\text{supp}(p) \subseteq \mathbb{R}^n$.

Proof. Let $\alpha \in \text{Newt}(p)$. Then there exists some $k \in \mathbb{N}$ and $\mathbf{d}_1, \dots, \mathbf{d}_k \in \text{supp}(p)$ satisfying $\alpha = \lambda_1 \mathbf{d}_1 + \dots + \lambda_k \mathbf{d}_k$ for some $\lambda_1 + \dots + \lambda_k = 1$ and $\lambda_i > 0$. For any $\mathbf{x} > \mathbf{0}$ we have:

$$p(\mathbf{x}) \geq \sum_{i=1}^k p_{\mathbf{d}_i} \mathbf{x}^{\mathbf{d}_i} = \sum_{i=1}^k \lambda_i \left(\frac{p_{\mathbf{d}_i}}{\lambda_i} \mathbf{x}^{\mathbf{d}_i} \right) \geq \prod_{i=1}^k \left(\frac{p_{\mathbf{d}_i}}{\lambda_i} \mathbf{x}^{\mathbf{d}_i} \right)^{\lambda_i} = \prod_{i=1}^k \left(\frac{p_{\mathbf{d}_i}}{\lambda_i} \right)^{\lambda_i} \mathbf{x}^\alpha,$$

and $\inf_{\mathbf{x} > \mathbf{0}} \frac{p(\mathbf{x})}{\mathbf{x}^\alpha} \geq \prod_{i=1}^n \left(\frac{p_{\mathbf{d}_i}}{\lambda_i} \right)^{\lambda_i} > 0$.

Now assume that $\alpha \notin \text{Newt}(p)$. Since $\text{Newt}(p)$ is a closed convex set and α is a point outside $\text{Newt}(p)$, there exists a separating hyperplane $\mathbf{c} \in \mathbb{R}^n$ satisfying $\inf_{\mathbf{d} \in \text{supp}(p)} \mathbf{c}^T(\alpha - \mathbf{d}) = \epsilon > 0$. Let $\mathbf{x} \in \Omega_p$ and $\mathbf{y}_t = (x_1 e^{t c_1}, \dots, x_n e^{t c_n})$ for all $t \in \mathbb{R}$. Then $\mathbf{y}_t \in \Omega_p$ by Lemma 2.1.9, and:

$$\begin{aligned} 0 \leq \text{Cap}_\alpha(p) &\leq \inf_{t \in \mathbb{R}} \frac{p(\mathbf{y}_t)}{\mathbf{y}_t^\alpha} \\ &= \inf_{t \in \mathbb{R}} \frac{\sum_{\mathbf{d} \in \text{supp}(p)} p_{\mathbf{d}} \mathbf{y}_t^{\mathbf{d}}}{\mathbf{y}_t^\alpha} \\ &= \inf_{t \in \mathbb{R}} \sum_{\mathbf{d} \in \text{supp}(p)} p_{\mathbf{d}} \mathbf{y}_t^{\mathbf{d} - \alpha} \\ &= \inf_{t \in \mathbb{R}} \sum_{\mathbf{d} \in \text{supp}(p)} p_{\mathbf{d}} e^{t \mathbf{c}^T(\mathbf{d} - \alpha)} \mathbf{x}^{\mathbf{d} - \alpha} \\ &\leq \inf_{t \in \mathbb{R}} e^{\epsilon t} \sum_{\mathbf{d} \in \text{supp}(p)} p_{\mathbf{d}} \mathbf{x}^{\mathbf{d} - \alpha} \\ &= \frac{p(\mathbf{x})}{\mathbf{x}^\alpha} \inf_{t \in \mathbb{R}} e^{\epsilon t} = 0, \end{aligned}$$

and therefore $\text{Cap}_\alpha(p) = 0$. □

Example 2.4.4. Let $n, d \in \mathbb{Z}$ be arbitrary. Similar to Example 2.3.3, the domain of convergence of the Laurent series $p(x, y) = \sum_{m \geq n} y^m x^{d-m}$ is the set $\Omega_p = \{(x, y) : x > y > 0\}$, and for any $(x_0, y_0) \in \Omega_p$, $p(x_0, y_0) = y_0^n x_0^{-n-d} (1 - \frac{y_0}{x_0})^{-1}$. Now we can compute the capacity of p at point (α, β) as follows:

$$\begin{aligned} \text{Cap}_{(\alpha, \beta)}(p) &= \inf_{0 < y < x} \frac{y^n x^{-n-d} (1 - \frac{y}{x})^{-1}}{x^\alpha y^\beta} \\ &= \inf_{0 < y < x} \frac{1}{x^{\alpha+\beta+d}} \cdot \frac{(1 - \frac{y}{x})^{-1}}{(\frac{y}{x})^{\beta-n}} \\ &= \inf_{0 < x} \frac{1}{x^{\alpha+\beta+d}} \cdot \inf_{0 < z < 1} \frac{(1-z)^{-1}}{z^{\beta-n}} \\ &= \begin{cases} 0 & \text{if } \alpha + \beta + d \neq 0 \\ \inf_{0 < z < 1} \frac{(1-z)^{-1}}{z^{\beta-n}} & \text{if } \alpha + \beta + d = 0 \end{cases} \end{aligned}$$

For any $k \in \mathbb{R}$, $\frac{d}{dz} \left(\frac{(1-z)^{-1}}{z^k} \right)$ is only zero at point $z_0 = \frac{k}{1+k}$. If $\frac{k}{1+k} \in (0, 1)$, then it is easy to check that $\frac{k}{1+k} = \arg \min \left(\frac{(1-z)^{-1}}{z^k} \right)_{0 < z < 1}$. If $\frac{k}{1+k} \notin (0, 1)$ however, $\frac{(1-z)^{-1}}{z^k}$ is a monotone function in the interval $(0, 1)$, and:

$$\inf_{0 < z < 1} \frac{(1-z)^{-1}}{z^k} = \min \left\{ \lim_{z \rightarrow 0^+} \frac{(1-z)^{-1}}{z^k}, \lim_{z \rightarrow 1^-} \frac{(1-z)^{-1}}{z^k} \right\} = 0.$$

Therefore:

$$\begin{aligned} \text{Cap}_{(\alpha, \beta)}(p) &= \begin{cases} 0 & \text{if } \alpha + \beta + d \neq 0 \\ \frac{(1+\beta-n)^{1+\beta-n}}{(\beta-n)^{\beta-n}} & \text{if } \alpha + \beta + d = 0 \text{ and } 0 \leq \frac{\beta-n}{1+\beta-n} < 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 0 & \text{if } \alpha + \beta + d \neq 0 \text{ or } \beta < n \\ \frac{(1+\beta-n)^{1+\beta-n}}{(\beta-n)^{\beta-n}} & \text{if } \alpha + \beta + d = 0 \text{ and } \beta \geq n \end{cases} \\ &= \begin{cases} 0 & \text{if } (\alpha, \beta) \notin \text{Newt}(p) \\ \frac{(1+\beta-n)^{1+\beta-n}}{(\beta-n)^{\beta-n}} & \text{if } (\alpha, \beta) \in \text{Newt}(p) \end{cases} \end{aligned}$$

So for any $k \geq n$, $\text{Cap}_{(t-k, k)}(\sum_{m \geq n} y^m x^{t-m}) = \frac{(1+k-n)^{1+k-n}}{(k-n)^{k-n}}$. By a similar argument, for any $k \leq n$, one can prove that $\text{Cap}_{(t-k, k)}(\sum_{m \leq n} y^m x^{t-m}) = \frac{(1+n-k)^{(1+n-k)}}{(n-k)^{n-k}}$.

Definition 2.4.5 (weighted log-concavity). Let $p(x, y) = \sum_{m \in \mathbb{Z}} p_m x^{d-m} y^m$ and $w(x, y) = \sum_{m \in \mathbb{Z}} w_m x^{d-m} y^m$. Assume further that w 's support does not have any holes (i.e, the sequence $\{w_m\}_{m \in \mathbb{Z}}$ does not have any internal zeroes). Say that p is a **w -log-concave** Laurent series if $\text{supp}(p) \subseteq \text{supp}(w)$ and the sequence $\{\frac{p_m}{w_m}\}_{m \in \mathbb{Z}, w_m \neq 0}$ is a log-concave sequence.

Remark 2.4.6. It follows by definition that the support of p cannot have any internal zeros if p is w -log-concave.

Lemma 2.4.7. Assume that p is w -log-concave and $\Omega_w \neq \emptyset$. Then Ω_p is also non-empty.

Proof. Let $p(x, y) = \sum_{m \in \mathbb{Z}} p_m x^{d-m} y^m$ and $w(x, y) = \sum_{m \in \mathbb{Z}} w_m x^{d-m} y^m$, and let $(x_0, y_0) \in \Omega_w$. we will find $s, r \in \mathbb{R}_{>0}$ such that $(sx_0, ry_0) \in \Omega_p$.

Suppose $\{\frac{p_m}{w_m}\}_{m \in \mathbb{Z}, p_m \neq 0}$ is a geometric sequence. So there exists some $c, a \in \mathbb{R}_{>0}$ satisfying $p_m = cw_m a^m$ for all non zero p_m , and:

$$p(x_0, \frac{1}{a}y_0) = \sum_{m \in \mathbb{Z}} \frac{p_m}{a^m} x_0^{d-m} y_0^m = cw(x_0, y_0).$$

Therefore, $(x_0, \frac{1}{a}y_0) \in \Omega_p$.

Now assume that $\{\frac{p_m}{w_m}\}_{m \in \mathbb{Z}, p_m \neq 0}$ is not a geometric sequence. So the Laurent series $q(x, y) := \sum_{m \in \mathbb{Z}, w_m \neq 0} \frac{p_m}{w_m} x^{d-m} y^m$ is non-trivially DL, and by Lemma 2.3.7, Ω_q is non-empty. Take an arbitrary point $(x_1, y_1) \in \Omega_q$ and let $L := q(x_1, y_1)$. Then for any $m \in \mathbb{Z}$, each monomial $\frac{p_m}{w_m} x_1^{d-m} y_1^m$ is less than L , and $p_m(x_0 x_1)^{d-m} (y_0 y_1)^m$ which is a monomial of $p(x_0 x_1, y_0 y_1)$, is less than its corresponding monomial $L w_m x_0^{d-m} y_0^m$ in $Lw(x_0, y_0)$. We can now use the comparison test to show that $p(x_0 x_1, y_0 y_1)$ is a converging sequence since $p(x_0 x_1, y_0 y_1) \leq Lw(x_0, y_0)$. $\square$

Lemma 2.4.8. Let p be w -log-concave Laurent series given by $p(x, y) = \sum_{m \in \mathbb{Z}} p_m x^{d-m} y^m$ and $w(x, y) = \sum_{m \in \mathbb{Z}} w_m x^{d-m} y^m$. For any $k \in \mathbb{Z}$ we have:

$$\frac{p_k}{\text{Cap}_{(d-k, k)}(p)} \geq \frac{w_k}{\text{Cap}_{(d-k, k)}(w)}.$$

Proof. The right hand side of this inequality will be zero if $\Omega_w = \emptyset$, and by Lemma 2.4.3, the left hand side of the inequality is $+\infty$ if $(d-k, k) \notin \text{Newt}(p)$. Since the support of p does not have internal zeros by Remark 2.4.6, $(d-k, k)$ is a lattice point of $\text{Newt}(p)$ if and only if $(d-k, k) \in \text{supp}(p)$. So the statement of this lemma is trivial if $\Omega_w = \emptyset$, or if $(d-k, k) \notin \text{supp}(p)$, and we will assume that neither are the case for the rest of this proof.

We have that $(d-k, k) \in \text{supp}(p) \subseteq \text{supp}(w)$. Again, the statement is trivial if $\text{supp}(w) = \{(d-k, k)\}$, since we would have $\text{Cap}_{(d-k, k)}(p) = p_k$ and $\text{Cap}_{(d-k, k)}(w) = w_k$. So we can assume that $(d-k, k)$ is not the only element of $\text{supp}(w)$, and either w_{k+1} or w_{k-1} is nonzero as well. Moreover, since $\Omega_w \neq \emptyset$, by Lemma 2.4.7, Ω_p is non-empty as well.

Let:

$$C_{d-k, k} := \sup_{l.c. \mathbf{a}} \inf_{x, y > 0} \left(\frac{1}{a_k x^{d-k} y^k} \sum_{(d-j, j) \in \text{supp}(w)} w_j a_j x^{d-j} y^j \right).$$

Where the sup is over all positive log concave sequences $\mathbf{a} = \{a_m\}_{m \in \mathbb{Z}, w_m \neq 0}$. In other words, $C_{(d-k, k)}$ is the supremum of the capacity of all possible w -log-concave Laurent

series at point $(d-k, k)$. Now the statement of the lemma is equivalent to proving that $C_{d-k,k} = \text{Cap}_{(d-k,k)}(w)$.

We can rewrite $C_{(d-k,k)}$ as follows:

$$\begin{aligned} C_{d-k,k} &= \sup_{l.c.\mathbf{a}} \inf_{x,y>0} \left(\frac{1}{a_k x^{d-k} y^k} \sum_{(d-j,j) \in \text{supp}(w)} w_j a_j x^{d-j} y^j \right) \\ &= \sup_{l.c.\mathbf{a}} \inf_{z>0} \left(\sum_{(d-j,j) \in \text{supp}(w)} w_j \frac{a_j}{a_k} z^{j-k} \right) \\ &= \sup_{\substack{l.c.\mathbf{a} \\ a_k=1}} \inf_{z>0} \left(\sum_{(d-j,j) \in \text{supp}(w)} w_j a_j z^{j-k} \right). \end{aligned}$$

For any log-concave sequence $\{a_m\}_{m \in \mathbb{Z}}$ with $a_k = 1$, we have the following inequalities for any valid index $j \geq 1$:

$$a_{k-j} \leq a_{k-1}^j \quad \text{and} \quad a_{k+j} \leq a_{k+1}^j.$$

So the supremum of $\inf_{z>0} \left(\sum_{(d-j,j) \in \text{supp}(w)} w_j a_j z^{j-k} \right)$ is attained for a log-concave sequence $\mathbf{a}$ that is almost geometric, and satisfies $a_{k-j} = a_{k-1}^j$, $a_{k+j} = a_{k+1}^j$ and $a_k = 1$:

$$C_{d-k,k} = \sup_{a_{k-1}, a_{k+1} \leq 1} \inf_{z>0} \left[\left(\sum_{\substack{j \leq k-1 \\ (d-j,j) \in \text{supp}(w)}} w_j a_{k-1}^{k-j} z^{j-k} \right) + w_k + \left(\sum_{\substack{j \leq k+1 \\ (d-j,j) \in \text{supp}(w)}} w_j a_{k+1}^{j-k} z^{j-k} \right) \right].$$

As discussed before, either w_{k+1} or w_{k-1} is nonzero. We can impose a restraint that either a_{k+1} or a_{k-1} has to be non-zero as well, or the supremum will not be attained for the sequence $\mathbf{a}$. Without loss of generality, let $a_{k+1} \neq 0$. So $a_{k-1} \leq a_{k+1}^{-1}$, and we can further restrict to the case $a_{k-1} = a_{k+1}^{-1}$ in the supremum, which implies:

$$C_{d-k,k} = \sup_{a_{k+1}>0} \inf_{z>0} \left(\sum_{(d-j,j) \in \text{supp}(w)} w_j (a_{k+1} z)^{j-k} \right) = \inf_{z>0} \left(\sum_{(d-j,j) \in \text{supp}(w)} w_j z^{j-k} \right).$$

Therefore $C_{d-k,k} = \text{Cap}_{(d-k,k)}(w)$, proving the result. $\square$

Corollary 2.4.9. *Let $p(x, y) = \sum_{m \geq n} p_m x^{d-m} y^m$ be a DL Laurent series. Then:*

$$p_k \geq \text{Cap}_{(d-k,k)}(p) \cdot \frac{(k-n)^{k-n}}{(1+k-n)^{1+k-n}},$$

for any $k \geq n$. Similarly if $p(x, y) = \sum_{m \leq n} p_m x^{d-m} y^m$ is DL, then for any $k \leq n$:

$$p_k \geq \text{Cap}_{(d-k,k)}(p) \cdot \frac{(n-k)^{n-k}}{(1+n-k)^{1+n-k}}.$$

Proof. if $p(x, y) = \sum_{m \geq n} p_m x^{d-m} y^m$ is DL, then p is $(\sum_{m \geq n} x^{d-m} y^m)$ -log-concave. Using Example 2.4.4 and Lemma 2.4.8, we have:

$$\frac{p_k}{\text{Cap}_{(d-k, k)}(p)} \geq \frac{(k-n)^{k-n}}{(1+k-n)^{1+k-n}},$$

for any $k \geq n$. p is also $(\sum_{m \leq d-n} x^m y^{d-m})$ -log-concave, but Lemma 2.4.8 gives the same bound as above.

The other statement can be proven similarly. $\square$

Remark 2.4.10. We cannot use the same method to bound the coefficients of a general DL Laurent series $p(x, y) = \sum_{m \in \mathbb{Z}} p_m x^{d-m} y^m$. The best thing we can say about this Laurent series is that it is w -log-concave for $w(x, y) = \sum_{m \in \mathbb{Z}} x^{d-m} y^m$, and using Corollary 2.4.9 for this p and w gives us a trivial bound. We can try and fix this by swapping w for a Laurent series that captures the “growth” of p ’s coefficients, but that would beat the purpose of bounding the coefficients of a Laurent series without knowing much about the coefficients themselves.

We can rewrite Corollary 2.4.9 using $\text{Deg}_{x_i}^\alpha$ from Definition 2.3.10 as the following corollary, which will be the base case for an inductive argument to prove lower bounds on the coefficients of n -variate DL Laurent series:

Corollary 2.4.11. *Let $p \in \mathbb{R}((x_1, x_2))$ be a DL Laurent series and let $(\alpha_1, \alpha_2) \in \text{supp}(p)$. Further assume that b_2 is either a lower or an upper bound for $\text{Deg}_{x_2}^{(\alpha_1, \alpha_2)}(p)$, then we have:*

$$p_{(\alpha_1, \alpha_2)} \geq \text{Cap}_{(\alpha_1, \alpha_2)}(p) \cdot \frac{(|\alpha_2 - b_2|)^{|\alpha_2 - b_2|}}{(1 + |\alpha_2 - b_2|)^{1 + |\alpha_2 - b_2|}}.$$

Proof. Assume that $p(x_1, x_2) = \sum_{m \geq n} p_{(d-m, m)} x_1^{d-m} x_2^m$ and $p_{(d-n, n)} \neq 0$, so that n is a tight lower bound for the set of powers of x_2 in p , $\text{Deg}_{x_2}(p)$. We proved in Corollary 2.4.9 that:

$$p_{(\alpha_1, \alpha_2)} \geq \text{Cap}_{(\alpha_1, \alpha_2)} \cdot \frac{(|n - \alpha_2|)^{|n - \alpha_2|}}{(1 + |n - \alpha_2|)^{1 + |n - \alpha_2|}}.$$

Since $f(z) = \frac{z^z}{(1+z)^{1+z}}$ is a decreasing function on $\mathbb{R}_{>0}$, for any (looser) bound b_2 for the set $\text{Deg}_{x_2}(p) = \text{Deg}_{x_2}^{(\alpha_1, \alpha_2)}(p)$, we have:

$$p_{(\alpha_1, \alpha_2)} \geq \text{Cap}_{(\alpha_1, \alpha_2)}(p) \cdot \frac{(|\alpha_2 - b_2|)^{|\alpha_2 - b_2|}}{(1 + |\alpha_2 - b_2|)^{1 + |\alpha_2 - b_2|}}.$$

A similar argument works for $p(x_1, x_2) = \sum_{m \leq n} p_{(d-m, m)} x_1^{d-m} x_2^m$. $\square$

Theorem 2.4.12. *Let $p \in \mathbb{R}_{\geq 0}((x_1, \dots, x_n))$ be a d -homogeneous DL Laurent series and $\alpha \in \text{supp}(p)$. Assume further that Ω_p is nonempty, and let b_i be an upper or lower bound for $\text{Deg}_{x_i}^\alpha(p)$ for all $i \geq 2$. Then:*

$$\frac{p_\alpha}{\text{Cap}_\alpha(p)} \geq \prod_{i=2}^n \frac{|b_i - \alpha_i|^{b_i - \alpha_i}}{(1 + |b_i - \alpha_i|)^{1 + |b_i - \alpha_i|}}.$$

Proof. The proof is by induction over $n \geq 2$. The case $n = 2$ is Corollary 2.4.11. Let $n > 2$. First, assume that $\text{Deg}_{x_n}^\alpha(p)$ is bounded below by b_n , and write $p(x_1, \dots, x_n) = \sum_{k=b_n}^\infty x_n^k p_k(x_1, \dots, x_{n-1})$.

For any $(z_1, \dots, z_{n-1}, z_n) \in \Omega_p$, we know that $(y_1 t, \dots, y_{n-1} t, t) \in \Omega_p$ by Lemma 2.1.9, where $y_i = \frac{z_i}{z_n}$ and $t > 0$ is arbitrary, besides, for any $s \in (0, t)$:

$$\begin{aligned} p(y_1 t, \dots, y_{n-1} t, s) &= \sum_{k \geq b_n} s^k t^{d-k} p_k(y_1, \dots, y_{n-1}) \\ &= s^{b_n} \sum_{k \geq b_n} s^{k-b_n} t^{d-k} p_k(y_1, \dots, y_{n-1}) \\ &\leq s^{b_n} \sum_{k \geq b_n} t^{d-b_n} p_k(y_1, \dots, y_{n-1}) \\ &= \left(\frac{s}{t}\right)^{b_n} p(y_1 t, \dots, y_{n-1} t, t) < \infty. \end{aligned}$$

Note that each p_k is a $(d-k)$ -homogeneous Laurent series with $(y_1, \dots, y_{n-1})$ in its domain of convergence by Remark 2.3.12. So $p(y_1 t, \dots, y_{n-1} t, s)$ also converges for any $0 < s < t$. Observe that the capacity of p at point α is at most $\frac{p(y_1 t, \dots, y_{n-1} t, s)}{(y_1 t, \dots, y_{n-1} t, s)^\alpha}$, and we can write:

$$\begin{aligned} \text{Cap}_\alpha(p) &\leq \inf_{0 < s < t} \frac{\sum_{k \geq b_n} s^k t^{d-k} p_k(y_1, \dots, y_{n-1})}{y_1^{\alpha_1} \dots y_{n-1}^{\alpha_{n-1}} s^{\alpha_n} t^{d-\alpha_n}} \\ &= \frac{1}{y_1^{\alpha_1} \dots y_{n-1}^{\alpha_{n-1}}} \inf_{0 < s < t} \frac{\sum_{k \geq b_n} s^k t^{d-k} p_k(y_1, \dots, y_{n-1})}{s^{\alpha_n} t^{d-\alpha_n}}. \end{aligned}$$

The sequence $\{p_k(y_1, \dots, y_{n-1})\}_{k \geq b_n}$ is log-concave by Lemma 2.3.9, so the Laurent series $\sum_{k \geq b_n} s^k t^{d-k} p_k(y_1, \dots, y_{n-1})$ is a bivariate DL Laurent series in s and t . Using the induction hypothesis for $n = 2$, we obtain:

$$\begin{aligned} \text{Cap}_\alpha(p) &\leq \frac{1}{y_1^{\alpha_1} \dots y_{n-1}^{\alpha_{n-1}}} \inf_{0 < s < t} \frac{\sum_{k \geq b_n} s^k t^{d-k} p_k(y_1, \dots, y_{n-1})}{s^{\alpha_n} t^{d-\alpha_n}} \\ &\leq \frac{p_{\alpha_n}(y_1, \dots, y_{n-1})}{y_1^{\alpha_1} \dots y_{n-1}^{\alpha_{n-1}}} \cdot \frac{(1 + \alpha_n - b_n)^{1+\alpha_n-b_n}}{(\alpha_n - b_n)^{\alpha_n-b_n}} \\ &= \frac{p_{\alpha_n}(z_1, \dots, z_{n-1})}{z_1^{\alpha_1} \dots z_{n-1}^{\alpha_{n-1}}} \cdot \frac{(1 + \alpha_n - b_n)^{1+\alpha_n-b_n}}{(\alpha_n - b_n)^{\alpha_n-b_n}}. \end{aligned}$$

Since this inequality holds for any $(z_1, \dots, z_n) \in \Omega_p$, we conclude:

$$\begin{aligned} \text{Cap}_\alpha(p) &\leq \inf_{(z_1, \dots, z_{n-1}, z_n) \in \Omega_p} \frac{p_{\alpha_n}(z_1, \dots, z_{n-1})}{z_1^{\alpha_1} \dots z_{n-1}^{\alpha_{n-1}}} \cdot \frac{(1 + \alpha_n - b_n)^{1+\alpha_n-b_n}}{(\alpha_n - b_n)^{\alpha_n-b_n}} \\ &= \inf_{(z_1, \dots, z_{n-1}) \in \Omega_{p_{\alpha_n}}} \frac{p_{\alpha_n}(z_1, \dots, z_{n-1})}{z_1^{\alpha_1} \dots z_{n-1}^{\alpha_{n-1}}} \cdot \frac{(1 + \alpha_n - b_n)^{1+\alpha_n-b_n}}{(\alpha_n - b_n)^{\alpha_n-b_n}} \\ &= \text{Cap}_{(\alpha_1, \dots, \alpha_{n-1})}(p_{\alpha_n}) \cdot \frac{(1 + \alpha_n - b_n)^{1+\alpha_n-b_n}}{(\alpha_n - b_n)^{\alpha_n-b_n}}. \end{aligned}$$

The second to the last step is due to Corollary 2.3.13. The results follows from the induction hypothesis for p_{α_n} .

If $\text{Deg}_{x_n}^\alpha$ is bounded above, then a similar argument works. $\square$

All DL polynomials $p \in \mathbb{R}[\mathbf{x}]$ and “shifted” DL polynomials $\mathbf{x}^\alpha p(\mathbf{x})$ satisfy Theorem 2.4.12’s conditions. It might seem that our assumptions for this theorem are too restrictive, and that if all of the sets $\text{Deg}_{x_i}^\alpha(p)$ are bounded below or above for some Laurent series p , then p is either a DL polynomial or a shifted DL polynomial. But we will see in the next section that there are other DL Laurent series that satisfy this condition. In fact, any DL Laurent series whose support is contained in a proper cone satisfies Theorem 2.4.12’s assumptions.

2.5 Applications of capacity bounds for DL Laurent series

In this section, we first introduce the flow Laurent series of an acyclic directed graph. We then look at the flow Laurent series for directed complete graphs and directed complete bipartite graphs. In the former case, we will be able to bound the number of directed flows of a complete graph, and in the latter case we will be able to bound the number of contingency tables of a given marginal.

Definition 2.5.1 (flow Laurent series). Let $G = ([n], E)$ be a simple directed graph. For any edge $e : v \rightarrow u \in E$, define:

$$p_e(x_v, x_u) = \sum_{k \geq 0} x_u^k x_v^{-k}$$

and:

$$p_G(\mathbf{x}) = \prod_{e=v \rightarrow u \in E} p_e(x_v, x_u)$$

We will first check when p_G is a DL Laurent series.

Lemma 2.5.2. p_G is a well-defined Laurent series if and only if G is acyclic. Furthermore, p_G is a DL Laurent series in this case, and we will refer to p_G as the **flow Laurent series of G** .

Proof. First, assume that G has a directed cycle C . Without loss of generality, let C be a cycle with vertices $1, 2, \dots, i$ and edges $1 \rightarrow 2, 2 \rightarrow 3, \dots, i-1 \rightarrow i, i \rightarrow 1$. Then there are infinite ways to obtain the monomial $x_1^0 x_2^0 \dots x_i^0$ in the following product of Laurent series:

$$\begin{aligned} p_C(x_1, \dots, x_i) &= p_{1 \rightarrow 2}(x_1, x_2) p_{2 \rightarrow 3}(x_2, x_3) \dots p_{i-1 \rightarrow i}(x_{i-1}, x_i) p_{i \rightarrow 1}(x_i, x_1) \\ &= \left(\sum_{k \geq 0} x_2^k x_1^{-k} \right) \left(\sum_{k \geq 0} x_3^k x_2^{-k} \right) \dots \left(\sum_{k \geq 0} x_i^k x_{i-1}^{-k} \right) \left(\sum_{k \geq 0} x_1^k x_i^{-k} \right). \end{aligned}$$

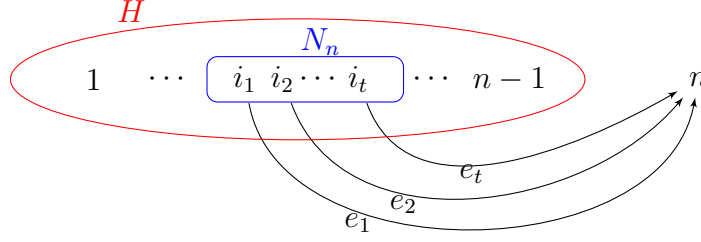


Figure 2.1: The set of neighbors of the vertex n in H ($i_1, \dots, i_t$ are not necessarily consecutive, unlike what this figure might suggest).

One would just have to fix $k \in \mathbb{Z}_{\geq 0}$, and choose $x_{j+1}^k x_j^{-k}$ from $p_{j \rightarrow j+1}$ for $j \in [i]$ (with $x_{i+1} = x_1$). Therefore, p_C is not a well-defined Laurent series. Because $p_G(\mathbf{x})$ has a factor of p_C , it is not a well-defined Laurent series either.

Now assume that G is an acyclic directed graph. We want to prove that p_G is a DL Laurent series. We will proceed by induction on n , the number of vertices. If G has only two vertices and one directed edge, then $p_G(x_1, x_2)$ is a DL Laurent series by Example 2.1.3.

Since G is acyclic, one can relabel the vertices of G so that each edge of G is directed from the vertex with the smaller label to the vertex with the larger label. Let H be the induced subgraph of G on the vertex set $\{1, \dots, n-1\}$, and let $N_n = \{i_1, \dots, i_t\}$ be the set of neighbors of the vertex n , and $e_1, \dots, e_t$ be the edges $i_1 \rightarrow n, \dots, i_t \rightarrow n$ respectively:

By the induction hypothesis, $p_H(x_1, \dots, x_{n-1})$ is a well-defined DL Laurent series, and by Remark 2.2.1 and Example 2.1.3,

$$f_G(x_1, \dots, x_n, y_{i_1}, \dots, y_{i_t}) := p_H(x_1, \dots, x_{n-1}) p_{e_1}(y_{i_1}, x_n) \cdots p_{e_t}(y_{i_t}, x_n)$$

is also a DL Laurent series. We will show that (x_{i_1}, y_{i_1}) is an admissible pair of f_G , (x_{i_2}, y_{i_2}) is an admissible pair of $T_{x_{i_1}=y_{i_1}}^{\text{PS}}[f_G]$ and so on. Then by Corollary 2.2.5, $p_G(\mathbf{x})$ will be a well-defined DL Laurent series. First, let us prove that (x_{i_1}, y_{i_1}) is an admissible pair of f_G .

Fix $k, \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in \mathbb{Z}$. We will show that the number of pairs (d_x, d_y) such that $d_x + d_y = k$ and $x_{i_1}^{d_x} y_{i_1}^{d_y} \prod_{t \neq i_1} x_t^{\alpha_t} \prod_{t \neq i_1} y_{i_t}^{\beta_{i_t}}$ is a monomial of f_G with nonzero coefficient, is finite. The only way to obtain a monomial from f_G with the degree of x_n being d_n , is to choose a positive integer partition of n such as $0 \leq \lambda_{i_1}, \lambda_{i_2}, \dots, \lambda_{i_t}$, and choose the monomial $x_n^{\lambda_{i_j}} y_{i_j}^{-\lambda_{i_j}}$ from p_{e_j} . Therefore, the degree of y_1 in such monomial would be $-\lambda_{i_1}$. Now since d_n is a fixed integer, there are finitely many positive integer partitions of d_n , and therefore, finitely many possible choices of d_y :

$$f_G(x_1, \dots, x_n, y_{i_1}, \dots, y_{i_t}) = p_H \cdot \underbrace{\left(\sum_{m \geq 0} x_n^m y_{i_1}^{-m} \right)}_{x_n^{\lambda_{i_1}} y_{i_1}^{-\lambda_{i_1}}} \underbrace{\left(\sum_{m \geq 0} x_n^m y_{i_2}^{-m} \right)}_{x_n^{\lambda_{i_2}} y_{i_2}^{-\lambda_{i_2}}} \cdots \underbrace{\left(\sum_{m \geq 0} x_n^m y_{i_t}^{-m} \right)}_{x_n^{\lambda_{i_t}} y_{i_t}^{-\lambda_{i_t}}}.$$

Hence, (x_{i_1}, y_{i_1}) is an admissible pair of f_G .

Now note that:

$$T_{x_{i_1}=y_{i_1}}^{\text{PS}}[f_G] = p_H \left(\sum_{m \geq 0} x_n^m x_{i_1}^{-m} \right) \left(\sum_{m \geq 0} x_n^m y_{i_2}^{-m} \right) \cdots \left(\sum_{m \geq 0} x_n^m y_{i_t}^{-m} \right).$$

Since we can only choose positive powers for x_n from each of these parentheses, (x_{i_2}, y_{i_2}) is an admissible pair of $T_{x_{i_1}=y_{i_1}}^{\text{PS}}[f_G]$. Similarly, (x_{i_j}, y_{i_j}) is an admissible pair of the Laurent series $T_{x_{i_{j-1}=y_{i_{j-1}}}, \dots, x_{i_1}=y_{i_1}}^{\text{PS}}[f_G]$ and we are done. $\square$

Remark 2.5.3. Let G be an acyclic directed graph. Then as proven in Lemma 2.5.2, $p_G(x_1, \dots, x_n)$ is a Laurent series and:

$$\Omega_{p_G} = \{(x_1, \dots, x_n) \in \mathbb{R}_{\geq 0} : x_v > x_u > 0 \text{ for all } e = u \rightarrow v \in E(G)\}$$

Further for any $\mathbf{x} \in \Omega_{p_G}$:

$$p_G(\mathbf{x}) = \prod_{v \rightarrow u \in E(G)} \frac{1}{1 - \frac{x_v}{x_u}}.$$

For a flow $\Phi \in \mathbb{Z}_{\geq 0}^{|E|}$ of the graph $G = ([n], E)$, the in flow of a vertex $v \in [n]$ is the sum $\Phi^+(v) = \sum_{e: u \rightarrow v} \Phi_e$, the out flow of v is the sum $\Phi^-(v) = \sum_{e: v \rightarrow u} \Phi_e$, and the net flow of v is $\Phi^+(v) - \Phi^-(v)$. Note that if Φ is an $\mathbf{N}$ -flow, then N_v is the net flow of v .

Remark 2.5.4. For a vector $\mathbf{N} \in \mathbb{Z}^n$, there is a one to one correspondence between the set of $\mathbf{N}$ -flows of $G = ([n], E)$, and the ways of obtaining the monomial $\mathbf{x}^{\mathbf{N}}$ from the Laurent series $p_G = \prod_{e: u \rightarrow v \in E} p_e$. For an $\mathbf{N}$ -flow Φ of G , choose $x_u^{-\Phi_e} x_v^{\Phi_e}$ from $p_{u \rightarrow v}$ to finally obtain $\prod_{v \in [n]} x_v^{\Phi^+(v) - \Phi^-(v)} = \mathbf{x}^{\mathbf{N}}$. So $[\mathbf{x}^{\mathbf{N}}]_{p_G}$ is the number of $\mathbf{N}$ -flows of G .

Theorem 2.5.5 (Theorem 2). *Let $G = ([n+1], E)$ be a directed complete graph with edges directed from the larger vertex to the smaller ones, and $\mathbf{N} \in \mathbb{Z}_{\geq 0}^{n+1}$. Recall from Section 1.4 that $K_{\mathbf{N}}(n)$ is the number of $\mathbf{N}$ -flows of G . We have:*

$$K_n(\mathbf{N}) \geq \text{Cap}_{\mathbf{N}}(p_G) \cdot \prod_{i=1}^n \frac{|s_i|^{s_i}}{(|s_i| + 1)^{|s_i| + 1}},$$

where:

$$p_G(x_1, \dots, x_{n+1}) = \prod_{1 \leq i < j \leq n+1} \frac{1}{1 - \frac{x_i}{x_j}},$$

is the flow Laurent series of G .

Proof. First, note that the statement is trivial if $\mathbf{N} \notin \text{supp}(p_G)$, since both $[\mathbf{x}^{\mathbf{N}}]_{p_G}$ and $\text{Cap}_{\mathbf{N}}(p_G)$ are zero. So let us assume that $\mathbf{N} \in \text{supp}(p_G)$.

We want to find a bound for the set $\text{Deg}_{x_i}^N(p_G)$. $\text{Deg}_{x_i}^N(p_G)$ is the set of all possible net flows that vertex i can have with respect to a flow Φ , with the constraint that the net flow of any other vertex $j > i$ is N_j .

Let H be the induced subgraph of G on vertices $i + 1, \dots, n$. Since edges are directed from the larger vertices to the smaller vertices, there are no edges going into H , and all the vertices of H send $\sum_{j>i} N_j$ units of flow to the rest of the graph:

$$\begin{aligned} \sum_{\substack{e:j \rightarrow u \\ u \leq i \\ j > i}} \Phi_e &= \sum_{\substack{e:j_2 \rightarrow j_1 \\ j_2 > j_1 > i}} \Phi_e - \sum_{\substack{e:j_2 \rightarrow j_1 \\ j_2 > j_1 > i}} \Phi_e + \sum_{\substack{e:j \rightarrow u \\ u \leq i \\ j > i}} \Phi_e \\ &= \sum_{j>i} \left(\sum_{\substack{e:u \rightarrow j \\ u \in [n]}} \Phi_e - \sum_{\substack{e:u \rightarrow j \\ u \in [n]}} \Phi_e \right) \\ &= \sum_{j>i} \Phi^+(j) - \Phi^-(j) \\ &= \sum_{j>i} N_j, \end{aligned}$$

and the vertices $1, \dots, i$ receive $-\sum_{j>i} N_j$ units of flow from H .

The most amount of in flow that i can have is achieved by absorbing all of this $-\sum_{j>i} N_j$ units of flow that is coming from H , and the least amount of out flow that i can have is 0, obtained by not sending any flows to vertices $1, \dots, i - 1$. So the maximum net flow of i is $-\sum_{j>i} N_j$, which is an upper bound for the set $\text{Deg}_{x_i}^N(p_G)$.

Note that Ω_{p_G} is nonempty by Remark 2.5.3, and we can use Theorem 2.4.12, to conclude:

$$[\mathbf{x}^N]p_G \geq \text{Cap}_N(p_G) \cdot \prod_{i=2}^{n+1} \frac{|N_i + \sum_{j>i} N_j|^{N_i + \sum_{j>i} N_j}}{(|N_i + \sum_{j>i} N_j| + 1)^{|N_i + \sum_{j>i} N_j| + 1}}.$$

Additionally, for a valid $\mathbf{N}$ -flow Φ , the sum of net flows of all vertices is zero, and $s_{i-1} + \sum_{j \geq i} N_j = 0$. Finally, we could use Remark 2.5.4 to rewrite:

$$K_n(\mathbf{N}) \geq \text{Cap}_N(p_G) \cdot \prod_{i=1}^n \frac{|s_i|^{s_i}}{(|s_i| + 1)^{|s_i| + 1}}. \quad \square$$

We will derive a similar bound for the number of contingency tables. Recall that for some $\boldsymbol{\alpha} \in \mathbb{Z}_{\geq 0}^n$ and $\boldsymbol{\beta} \in \mathbb{Z}_{\geq 0}^m$, $\text{CT}_K(\boldsymbol{\alpha}, \boldsymbol{\beta})$ is the number of K -contingency tables with marginals $(\boldsymbol{\alpha}, \boldsymbol{\beta})$. Let $\text{CT}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ be the number of contingency tables with marginals $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ with no restrictions on their entries.

Remark 2.5.6. Define:

$$C(x_1, \dots, x_n; y_1, \dots, y_m) = \prod_{i=1}^n \prod_{j=1}^m \left(\sum_{k \geq 0} x_i^k y_j^{-k} \right)$$

C is the flow Laurent series of a complete bipartite graph with two parts of size n and m . For a contingency tables $A \in \mathbb{Z}_{\geq 0}^{n \times m}$ of marginals (α, β) , choose the monomial $x_i^{A_{ij}} y_j^{-A_{ij}}$ from the sum $\sum_{k \geq 0} x_i^k y_j^{-k}$, to ultimately obtain $\prod_{i=1}^n x_i^{\sum_{k=1}^m A_{ik}} \prod_{j=1}^m y_j^{\sum_{k=1}^n A_{kj}} = \mathbf{x}^\alpha \mathbf{y}^{-\beta}$. This gives us a bijection between the ways of getting $\mathbf{x}^\alpha \mathbf{y}^{-\beta}$ in C and the set of contingency tables of marginals (α, β) .

Theorem 2.5.7. Let $\alpha \in \mathbb{Z}_{\geq 0}^n, \beta \in \mathbb{Z}_{\geq 0}^m$, and let $s_i = \sum_{k=1}^i \alpha_k$ for $i \in [n]$. Then:

$$\text{CT}(\alpha, \beta) \geq \text{Cap}_{(\alpha, -\beta)}(C) \cdot \prod_{i=1}^m \frac{\beta_i^{\beta_i}}{(1 + \beta_i)^{1 + \beta_i}} \cdot \prod_{i=2}^n \max \left\{ \frac{\alpha_i^{\alpha_i}}{(1 + \alpha_i)^{1 + \alpha_i}}, \frac{s_{i-1}^{s_{i-1}}}{(1 + s_{i-1})^{1 + s_{i-1}}} \right\},$$

where:

$$C(\mathbf{x}, \mathbf{y}) = \prod_{i=1}^n \prod_{j=1}^m \frac{1}{1 - \frac{x_i}{y_j}}.$$

Proof. Similar to the proof of Theorem 2.5.5, the statement is trivial if $(\alpha, -\beta) \notin \text{supp}(C)$. So assume that there exists at least one contingency table with marginals (α, β) . This would imply that $|\alpha|_1$ and $|\beta|_1$ are equal, and both count the sum of entries of this contingency table.

To use Theorem 2.4.12, we need to find a bound for the sets $\text{Deg}_{x_i}^{(\alpha, -\beta)}$ and $\text{Deg}_{y_j}^{(\alpha, -\beta)}$. We have:

$$\text{Deg}_{y_j}^{(\alpha; -\beta)} = \{-t_i : \mathbf{x}^{\mathbf{d}} y_1^{-t_1} \cdots y_j^{-t_j} y_{j+1}^{-\beta_{j+1}} \cdots y_m^{-\beta_m} \in \text{supp}(C) \text{ for some } \mathbf{d} \in \mathbb{N}^n, t_1, \dots, t_{i-1} \in \mathbb{N}\},$$

and a bound for this set is a really a bound for the sum of entries in the j -th column of a contingency table with the constraint that the sum of columns $j + 1, \dots, m$ is predetermined. In this circumstances, the sum of the entries of the j -th column can be any nonnegative integer, so an upper bound for $\text{Deg}_{y_j}^{(\alpha, -\beta)}$ is 0, and we cannot find any other lower or upper bound for this set.

Now to find a bound for $\text{Deg}_{x_i}^{(\alpha, -\beta)}$. Write:

$$\text{Deg}_{x_i}^{(\alpha; -\beta)} = \{d_i : x_1^{d_1} \cdots x_i^{d_i} x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n} \mathbf{y}^{-\beta} \in \text{supp}(C) \text{ for some } d_1, \dots, d_{i-1} \in \mathbb{N}, \beta \in \mathbb{N}^m\}.$$

Same as before, a bound for this set is a bound for the sum of entries in the i -th row of a matrix $A \in \mathbb{Z}_{\geq 0}^{n \times m}$, with fixed column sums and row sums for rows $i + 1, \dots, n$. For this

matrix, we have:

$$\begin{aligned}
\sum_{k=1}^n (\text{sum of the entries of the } k\text{-th row}) &= \sum_{k=1}^m (\text{sum of the entries of the } k\text{-th column}) \\
\implies (\text{sum of the entries of the } i\text{-th row}) &= |\beta|_1 - \sum_{k=1}^{i-1} (\text{sum of the entries of the } k\text{-th row}) - \sum_{k=i+1}^n (\text{sum of the entries of the } k\text{-th row}) \\
&\leq |\beta|_1 - \sum_{k \geq i+1} \alpha_k.
\end{aligned}$$

We conclude that $|\beta|_1 - \sum_{k \geq i+1} \alpha_k$ is an upper bound for the set $\text{Deg}_{x_i}^{(\alpha; -\beta)}$. We can simplify this bound further since $|\alpha|_1 = |\beta|_1$. So $s_i = |\alpha|_1 - \sum_{k \geq i+1} \alpha_k$ is an upper bound for this set. We could also always use the fact that the sum of entries of a row is nonnegative to get a lower bound of 0 for this set. Now while implying Theorem 2.4.12, we could use either of this bounds on $\text{Deg}_{x_i}^{(\alpha; -\beta)}$ that gives us a better estimation of $[x^\alpha y^{-\beta}]C$:

$$[x^\alpha y^{-\beta}]C \geq \text{Cap}_{(\alpha, -\beta)}(C) \cdot \prod_{j=1}^m \frac{\beta_j^{\beta_j}}{(1 + \beta_j)^{1+\beta_j}} \cdot \prod_{i=2}^n \max \left\{ \frac{\alpha_i^{\alpha_i}}{(1 + \alpha_i)^{1+\alpha_i}}, \frac{(s_i - \alpha_i)^{s_i - \alpha_i}}{(1 + s_i - \alpha_i)^{1+s_i - \alpha_i}} \right\}.$$

For the final step of the proof, use Remark 2.5.6 to see:

$$\text{CT}(\alpha, \beta) \geq \text{Cap}_{(\alpha, -\beta)}(C) \cdot \prod_{j=1}^m \frac{\beta_j^{\beta_j}}{(1 + \beta_j)^{1+\beta_j}} \cdot \prod_{i=2}^n \max \left\{ \frac{\alpha_i^{\alpha_i}}{(1 + \alpha_i)^{1+\alpha_i}}, \frac{(s_{i-1})^{s_{i-1}}}{(1 + s_{i-1})^{1+s_{i-1}}} \right\}.$$

□

2.6 Future work

DL preserving operators on Laurent series and symbol of an operator. We saw in Theorem 1.3.11 that there is a sufficient condition that implies an operator $T : \mathbb{R}^\mu[x_1, \dots, x_n] \rightarrow \mathbb{R}[x_1, \dots, x_m]$ preserves the class of Lorentzian polynomials. The authors of [5, Theorem 1.2] introduce algebraic and transcendental symbols of an operator $T : \mathbb{R}[x_1, \dots, x_n] \rightarrow \mathbb{R}[x_1, \dots, x_n]$ to check if it preserves real stability:

Theorem 2.6.1 ([5, Theorem 1.1]). *For a linear operator $T : \mathbb{R}^\mu[x_1, \dots, x_n] \rightarrow \mathbb{R}[x_1, \dots, x_n]$, define the **algebraic symbol** of T as:*

$$G_T(\mathbf{x}, \mathbf{z}) = T_{\mathbf{x}} \left[\prod_{i=1}^n (x_i + z_i)^{\mu_i} \right] = \sum_{\alpha \leq \mu} \binom{\mu}{\alpha} T(\mathbf{x}^\alpha) \mathbf{z}^{\mu-\alpha},$$

where $T_{\mathbf{x}}$ treats all z variables as constants and only acts on x variables, and $\binom{\mu}{\alpha} = \frac{\mu!}{\alpha!(\mu-\alpha)!}$. Then T preserves the class of real stable polynomials if $G_T(\mathbf{x}, \mathbf{z})$ is real stable.

Theorem 2.6.2 ([5, Theorem 1.3]). *For a linear operator $T : \mathbb{R}[x_1, \dots, x_n] \rightarrow \mathbb{R}[x_1, \dots, x_n]$, define the **transcendental symbol** of T as:*

$$\bar{G}_T(\mathbf{x}, \mathbf{y}) = \sum_{\alpha \in \mathbb{N}^n} (-1)^{|\alpha|_1} T[\mathbf{x}^\alpha] \frac{\mathbf{y}^\alpha}{\alpha!}.$$

By abuse of notation, we can write $\bar{G}_T(\mathbf{x}, \mathbf{z}) = T_{\mathbf{x}}[e^{-\mathbf{x} \cdot \mathbf{z}}]$. Then T preserves the class of real stable polynomials if $\bar{G}_T$ is the limit (uniformly on compact sets) of real stable polynomials in $2n$ variables.

Suppose we have a linear operator $T : \mathbb{R}[x_1, \dots, x_n] \rightarrow \mathbb{R}[x_1, \dots, x_n]$, and we want to check whether T preserves real stability. We could either check if $\bar{G}_T$ is the limit of real stable polynomials, or check if for all $\mu \in \mathbb{N}^n$, the polynomial G_{T_μ} is real stable, where $T_\mu = T|_{\mathbb{R}^\mu[x_1, \dots, x_n]}$. Can we utilize the methods from [5] to find an algebraic and a transcendental symbol for operators on Laurent series that preserve the class of DL Laurent series?

Connections to symmetric functions. The idea of DL Laurent series was developed originally to attempt prove that Schur polynomials are DL using a more combinatorial method compared to [21]. Consider the following equality, known as the Cauchy identity:

$$\prod_{i=1}^n \prod_{j=1}^m \frac{1}{1 - \frac{x_i}{y_j}} = \sum_{\lambda \text{ a partition}} s_\lambda(x_1, \dots, x_n) s_\lambda(y_1^{-1}, \dots, y_m^{-1}).$$

The left hand side of this identity is a DL Laurent series, in fact it is the Laurent series $C(x_1, \dots, x_n; y_1, \dots, y_m)$ introduced in Remark 2.5.6. Would it be possible to apply an operator to the right hand side of this equality to obtain $s_\lambda(x_1, \dots, x_n)$, and then show that this operator preserves the class of DL Laurent series?

Comparing bonds from Theorem 2 and [25]. Although our bound for $K_n(\mathbf{N})$ from Theorem 2 is obtained more naturally compared to Theorem 1.4.8, how do these two bounds compare numerically and asymptotically? Moreover, the authors of [25] use convex analysis to find a dual formulation of $\text{Cap}_\alpha(\Phi)$:

Theorem 2.6.3 ([25, Proposition 3.1]). *Let $\mathbf{N}$, α and β be the vectors from Theorem 1.4.8. Then:*

$$\text{Cap}_{(\alpha, \beta)} \left(\prod_{i+j \leq n} \frac{1}{1 - x_i y_j} \right) = \sup_{A \in \varphi(\mathcal{F}_n(\mathbf{N}))} \prod_{i+j \leq n} \frac{(a_{ij} + 1)^{a_{ij}+1}}{a_{ij}^{a_{ij}}},$$

where $\varphi(\mathcal{F}_n(\mathbf{N}))$ is the image of a bijection φ from the set of $\mathbf{N}$ -flows of complete acyclic graphs with $n+1$ vertices, to the set of K -contingency tables with marginals α, β for some K . See [29, Section 1.3] and [25, Section 2.1] for more details about this bijection.

So any matrix $A \in \varphi(\mathcal{F}_n(\mathbf{N}))$ will give us a lower bound for $\text{Cap}_{(\alpha, \beta)}(\Phi)$ using Theorem 2.6.3 that is easy to compute from the computational complexity point of view. Can

we apply their analysis to $\text{Cap}_{\mathbf{N}}(p_{k_n+1})$, and replace the capacity factor in our lower bounds in Theorem 2 with its dual formulation, to obtain a bound that is easier to compute?

Other applications of lower bounds for DL Laurent series' coefficients. Finally, what other classes of combinatorial objects could we find that Theorem 2.4.12 would give a meaningful bound for?

References

- [1] Nima Anari, Shayan Oveis Gharan, and Cynthia Vinzant. Log-concave polynomials, I: entropy and a deterministic approximation algorithm for counting bases of matroids. *Duke Math. J.*, 170(16):3459–3504, 2021.
- [2] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. Log-concave polynomials III: Mason’s ultra-log-concavity conjecture for independent sets of matroids. *Proc. Amer. Math. Soc.*, 152(5):1969–1981, 2024.
- [3] Fredrico Ardila. The geometry of matroids. *Notices Amer. Math. Soc.* 65.8, page 902–908, 2018.
- [4] Edward A. Bender and Donald E. Knuth. Enumeration of plane partitions. *J. Combinatorial Theory Ser. A*, 13:40–54, 1972.
- [5] Julius Borcea and Petter Brändén. The Lee-Yang and Pólya-Schur programs. I. Linear operators preserving stability. *Invent. Math.*, 177(3):541–569, 2009.
- [6] Petter Brändén and June Huh. Lorentzian polynomials. *Ann. of Math. (2)*, 192(3):821–891, 2020.
- [7] Petter Brändén, Jonathan Leake, and Igor Pak. Lower bounds for contingency tables via Lorentzian polynomials. *Israel J. Math.*, 253(1):43–90, 2023.
- [8] Augustin Louis Cauchy. Oeuvres complètes: Series 1 (in French). *Cambridge University Press*, 2009.
- [9] Jesús A. De Loera and Edward D. Kim. Combinatorics and geometry of transportation polytopes: an update. 625:37–76, 2014.
- [10] Martin Dyer, Ravi Kannan, and John Mount. Sampling contingency tables. *Random Structures Algorithms*, 10(4):487–506, 1997.
- [11] G. P. Egoryčev. yr Reshenie problemy van-der-Vardena dlya permanentov (Russian)[Solution of the van der Waerden problem for permanents] . page 12, 1980. Preprint IFSO-13 M.

- [12] D. I. Falikman. Proof of the van der Waerden conjecture on the permanent of a doubly stochastic matrix. *Mat. Zametki*, 29(6):931–938, 957, 1981.
- [13] Leonid Gurvits. The van der Waerden conjecture for mixed discriminants. *Adv. Math.*, 200(2):435–454, 2006.
- [14] Leonid Gurvits. Van der Waerden/Schrijver-Valiant like conjectures and stable (aka hyperbolic) homogeneous polynomials: one theorem for all. *Electron. J. Combin.*, 15(1):Research Paper 66, 26, 2008. With a corrigendum.
- [15] Leonid Gurvits. On multivariate Newton-like inequalities. pages 61–78, 2009.
- [16] Leonid Gurvits. Boolean matrices with prescribed row/column sums and stable homogeneous polynomials: combinatorial and algorithmic applications. *Inform. and Comput.*, 240:42–55, 2015.
- [17] Leonid Gurvits and Jonathan Leake. Counting matchings via capacity-preserving operators. *Combin. Probab. Comput.*, 30(6):956–981, 2021.
- [18] Ole J. Heilmann and Elliott H. Lieb. Theory of monomer-dimer systems. *Comm. Math. Phys.*, 25:190–232, 1972.
- [19] John E. Hopcroft and Richard M. Karp. An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs. *SIAM J. Comput.*, 2:225–231, 1973.
- [20] Hao Huang. Induced subgraphs of hypercubes and a proof of the sensitivity conjecture. *Ann. of Math. (2)*, 190(3):949–955, 2019.
- [21] June Huh, Jacob P. Matherne, Karola Mészáros, and Avery St. Dizier. Logarithmic concavity of Schur and related polynomials. *Trans. Amer. Math. Soc.*, 375(6):4411–4427, 2022.
- [22] James E. Humphreys. *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$* , volume 94 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.
- [23] Richard V. Kadison and I. M. Singer. Extensions of pure states. *Amer. J. Math.*, 81:383–400, 1959.
- [24] Alain Lascoux and Marcel-Paul Schützenberger. Polynômes de Schubert. *Comptes Rendus*, 294:447–450, 1982.
- [25] Jonathan Leake and Alejandro H. Morales. Capacity bounds on integral flows and the Kostant partition function. *arXiv*, 2024, arXiv:2406.07838.
- [26] Nathan Linial. Hard enumeration problems in geometry and combinatorics. *SIAM J. Algebraic Discrete Methods*, 7(2):331–335, 1986.

- [27] Adam W. Marcus, Daniel A. Spielman, and Nikhil Srivastava. Interlacing families II: Mixed characteristic polynomials and the Kadison-Singer problem. *Ann. of Math. (2)*, 182(1):327–350, 2015.
- [28] John Mason. Matroids: unimodal conjectures and motzkin’s theorem. *Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst.) Inst. Math. Appl.*, page 207–220, 3 1972.
- [29] Karola Mészáros, Alejandro H. Morales, and Brendon Rhoades. The polytope of Tesler matrices. *Selecta Math. (N.S.)*, 23(1):425–454, 2017.
- [30] Hariharan Narayanan. On the complexity of computing Kostka numbers and Littlewood-Richardson coefficients. *J. Algebraic Combin.*, 24(3):347–354, 2006.
- [31] Noam Nisan and Mária Szegedy. On the degree of Boolean functions as real polynomials. *Comput. Complexity*, 4(4):301–313, 1994. Special issue on circuit complexity (Barbados, 1992).
- [32] Oliver Pechenik and Dominic Searles. Asymmetric function theory. 332:73–112, [2020] ©2020.
- [33] Issai Schur. Über eine klasse von matrizen, die sich einer gegebenen matrix zuordnen lassen. *Dietrich, Berlin; Göttingen*, 1901.
- [34] Michael Snook. Counting bases of representable matroids. *Electron. J. Combin.*, 19(4):Paper 41, 11, 2012.
- [35] L. G. Valiant. The complexity of computing the permanent. *Theoret. Comput. Sci.*, 8(2):189–201, 1979.
- [36] B.L. van der Waerden. Aufgabe 45. *Jber. Deutsch. Math. -Verein*, 35:117, 1926.
- [37] J. H. van Lint and R. M. Wilson. *A course in combinatorics*. Cambridge University Press, Cambridge, 1992.
- [38] David G. Wagner. Multivariate stable polynomials: theory and applications. *Bull. Amer. Math. Soc. (N.S.)*, 48(1):53–84, 2011.

APPENDICES

Appendix A

Missing proofs

For completeness, the proofs of results used without proof in previous sections are included in this appendix.

A.1 Ratio test

We state and prove the ratio test for general positive sequences, and log-concave sequences.

Lemma A.1.1. *Let $\{a_n\}_{n \geq 0}$ be a sequence with $a_i \in \mathbb{R}_{>0}$. The sum $\sum_{n \geq 0} a_n$ converges if $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1$, and diverges if $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} > 1$.*

Proof. Let $L = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$. If $L < 1$, take some $r \in \mathbb{R}_{\geq 0}$ satisfying $L < r < 1$. Then there exists some large enough $n_0 \in \mathbb{N}$ such that $\frac{a_{n+1}}{a_n} \leq r$ for all $n \geq n_0$, and therefore, $a_n \leq a_{n_0} \cdot r^{n-n_0}$. Write:

$$\begin{aligned} \sum_{n \geq 0} a_n &\leq \sum_{n < n_0} a_n + \sum_{n \geq n_0} a_{n_0} r^{n-n_0} \\ &= \sum_{n < n_0} a_n + a_{n_0} \frac{1}{1-r} < \infty, \end{aligned}$$

so $\sum_{n \geq 0} a_n$ converges.

If $L > 1$ however, take some $s \in \mathbb{R}$ such that $L > s > 1$, and there exists some $n_1 \in \mathbb{N}$ satisfying $\frac{a_{n+1}}{a_n} \geq s \geq 1$ for all $n \geq n_1$, and therefore $a_n \geq a_{n_1}$. So we have:

$$\sum_{n \geq 0} a_n \geq \sum_{n \leq n_1} a_n + \sum_{n \geq n_1} a_{n_1},$$

this sum diverges. □

Lemma A.1.2. *Let $\{a_n\}_{n \geq 0}$ be a log-concave sequence with $a_i \in \mathbb{R}_{>0}$. Then the sequence $\{\frac{a_{n+1}}{a_n}\}_{n \in \mathbb{N}}$ is decreasing and convergent, and the sum $\sum_{n \geq 0} a_n$ converges if and only if $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} < 1$.*

Proof. Because $\frac{a_{n+1}}{a_n} \leq \frac{a_n}{a_{n-1}}$ for a log-concave sequence $\{a_n\}_{n \in \mathbb{N}}$, the sequence $\{\frac{a_{n+1}}{a_n}\}_{n \in \mathbb{N}}$ is decreasing. So $\{\frac{a_{n+1}}{a_n}\}_{n \in \mathbb{N}}$ is a decreasing positive sequence, and converges to some $L \in \mathbb{R}_{>0}$.

By Lemma A.1.1, $\sum_{n \geq 0} a_n$ converges if $L < 1$, and diverges if $L > 1$. It remains to show that this sum diverges in case $L = 1$ as well.

Note that $\{\frac{a_{n+1}}{a_n}\}_{n \in \mathbb{N}}$ is a decreasing sequence, and if $L = 1$, none of the terms $\frac{a_{n+1}}{a_n}$ can be less than 1, or we will end up with a sequence that is eventually strictly less 1. So we should have $\frac{a_{n+1}}{a_n} \geq 1$ for all $n \in \mathbb{N}$, and $a_n \geq a_1$. Now we have:

$$\sum_{n \in \mathbb{N}} a_n \geq \sum_{n \in \mathbb{N}} a_1$$

and this sum diverges. □

A.2 Proof of Lemma 2.3.8 for $n = 3$

Lemma A.2.1. *Let*

$$p(x_1, x_2, x_3) = x_3 r(x_1, x_2) + s(x_1, x_2)$$

be a $(d+1)$ -homogeneous DL Laurent series. Assuming that r and s are both non-zero Laurent series, we have $(1, 1) \in \Omega_r$ if and only if $(1, 1) \in \Omega_s$.

Proof. Write:

$$\begin{aligned} r(x_1, x_2) &= \sum_{n \in \mathbb{Z}} r_n x_1^n x_2^{-n+d}, \\ s(x_1, x_2) &= \sum_{n \in \mathbb{Z}} s_n x_1^n x_2^{-n+d+1}, \end{aligned}$$

and let $D_s := \text{Deg}_{x_1}(s) = \{n : s_n \neq 0\}$, $D_r := \text{Deg}_{x_1}(r) = \{n : r_n \neq 0\}$, and $D := D_r \cap D_s$. We need to show that $\sum_{n \in D_r} r_n$ converges if and only if $\sum_{n \in D_s} s_n$ converges. s and r are two bivariate DL Laurent series by Corollary 2.1.11, and therefore, the sequences of their coefficients are log-concave by Lemma 2.1.4. Particularly, $\{r_n\}_{n \in \mathbb{Z}}$ and $\{s_n\}_{n \in \mathbb{Z}}$ cannot have internal zeros, implying that D_r and D_s are two sets of consecutive integers. We will prove that $D_r \Delta D_s$ has finitely many elements.

Take an arbitrary $n \in D_s$ and an arbitrary element $m \in D_r$. So we have $\nu := (n, -n + d + 1, 0) \in \text{supp}(p)$ and $\mu := (m, -m + d, 1) \in \text{supp}(p)$. Note that the support of p is M-convex by Lemma 2.1.5, and we have two elements of p 's support differing in their

third coordinate, implying that there should exist some $i \in \{1, 2\}$ satisfying $\boldsymbol{\mu} - \mathbf{e}_3 + \mathbf{e}_i, \boldsymbol{\nu} + \mathbf{e}_3 - \mathbf{e}_i \in \text{supp}(p)$. If $i = 1$, we obtain $(m+1, -m+d, 0), (n-1, -n+d+1, 1) \in \text{supp}(p)$, and therefore $m+1 \in D_s$ and $n-1 \in D_r$. In case $i = 2$, we get $(m, -m+d+1, 0), (n, -n+d, 1) \in \text{supp}(p)$, so $m \in D_s$ and $n \in D_r$. Thus, we have proven so far that for any element $n \in D_s$, either n itself or $n-1$ is an element of D_r as well, and conversely, for any $m \in D_r$, either m or $m+1$ is in D_s . So we have:

$$\min(D_r) \geq \min(D_s) - 1 \quad \text{and} \quad \max(D_s) \leq \max(D_r) + 1,$$

where $\min(A), \max(A)$ are the minimum and the maximum elements of a finite set A respectively. Taking into consideration that both D_r and D_s are sets of consecutive integers, we conclude $|D \Delta D_s|, |D \Delta D_r| \leq 1$. We now just have to prove that the sum $\sum_{n \in D} s_n$ converges if and only if $\sum_{n \in D} r_n$ converges.

For any $n \in D$, define:

$$\epsilon_n = \frac{r_{n+1}}{r_n},$$

$$\delta_n = \frac{s_{n+1}}{s_n},$$

The idea is to prove that $\epsilon_n = o(\delta_n)$, and that the coefficients of neither s nor r grow faster than those of the other. Since p is a DL Laurent series, for any $k \in \mathbb{Z}$, the polynomial:

$$\text{Poly}[x_1^{-k} x_2^{k-d+1} p] = s_k x_2^2 + s_{k+1} x_1 x_2 + s_{k+2} x_1^2 + r_k x_3 x_2 + r_{k+1} x_3 x_1$$

is DL, and the matrix:

$$\nabla^2 N[\text{Poly}[x_1^{-k} x_2^{k-d+1} p]] = \begin{bmatrix} 0 & r_k & r_{k+1} \\ r_k & s_k & s_{k+1} \\ r_{k+1} & s_{k+1} & s_{k+2} \end{bmatrix}$$

has at most one positive eigen-value by Definition 1.3.3 (this hessian is written in the reverse order of variables). So its determinant must be nonnegative. If $k+1 \in D$, we can divide the first row and the first column of this hessian by r_{k+1} , to get a matrix whose determinant is still nonnegative, giving us:

$$0 \leq \begin{vmatrix} 0 & 1 & \epsilon_k \\ 1 & s_k & \delta_k s_k \\ \epsilon_k & \delta_k s_k & \delta_k \delta_{k+1} s_k \end{vmatrix}$$

$$= -\epsilon_k^2 s_k + 2\epsilon_k \delta_k s_k - \delta_k \delta_{k+1} s_k,$$

which is a quadratic form in ϵ_k , and as long as $s_k > 0$ (so for instance if $k \in D$), we will get:

$$\delta_k - \sqrt{\delta_k^2 - \delta_k \delta_{k+1}} \leq \epsilon_k \leq \delta_k + \sqrt{\delta_k^2 - \delta_k \delta_{k+1}}.$$

First, suppose $(1, 1) \in \Omega_s$, and let us prove that $\sum_{n \in D_{\geq 0}} r_n$ and $\sum_{n \in D_{< 0}} r_n$ both converge, where $D_{\geq 0} = D \cap \mathbb{Z}_{\geq 0}$ and $D_{< 0} = D \cap \mathbb{Z}_{< 0}$.

If $D_{\geq 0}$ is a finite set, $\sum_{n \in D_{\geq 0}} r_n$ converges without any further issues. So suppose $D_{\geq 0}$ is an infinite set. Recall that D is a set of consecutive integers, and therefore, $D_{\geq 0}$ contains all nonnegative integers greater than some $M \in \mathbb{Z}_{\geq 0}$. We conclude that δ_n is a well defined ratio for all $n \geq M$. Apply Lemma A.1.2 to $\sum_{n \geq M} s_n$ to obtain $\delta^* := \lim_{n \rightarrow \infty} \delta_n < 1$.

Let $k_0 \in \mathbb{N}$ be a large enough integer satisfying $k_0, k_0 + 1 \in D$ and $\delta_{k_0} < 1$, and let $c = \frac{(1 - \delta_{k_0})^2}{2\delta_{k_0}}$. Now, there exists some large enough $k \geq k_0$ satisfying $\delta_k - \delta_{k+1} \leq c$, therefore:

$$\epsilon_k \leq \delta_k + \sqrt{\delta_k} \cdot \sqrt{\delta_k - \delta_{k+1}} \leq \delta_{k_0} + \sqrt{\delta_{k_0}} \cdot \sqrt{c} < 1$$

So $\lim_{n \rightarrow \infty} \epsilon_n < 1$, and $\sum_{n \in D_{\geq 0}} r_n$ converges by Lemma A.1.2.

If $D_{< 0}$ is finite, we don't have anything to prove. So we can assume that $D_{< 0}$ is the set of all negative integers smaller than some $M \in \mathbb{Z}$, and that δ_{-n} is a well defined ratio for all $n > |M|$. Apply Lemma A.1.2 to get $\delta_*^{-1} := \lim_{n \rightarrow \infty} \delta_{-n}^{-1} < 1$.

If $\delta_* \neq \infty$, we have that $\lim_{n \rightarrow \infty} \delta_{-n} = \delta_* > 1$. Let k_1 be a large enough integer satisfying $\delta_{-k_1} > 1$, and let $c = \frac{(\delta_{-k_1} - 1)^2}{2\delta_*}$. There exists some large enough $k \geq k_1$ satisfying $\delta_{-k} - \delta_* \leq c$ and $\delta_{-k} - \delta_{-k+1} \leq c$. Note that $\{\delta_{-n}\}_{n \in \mathbb{N}}$ is an increasing sequence and we have:

$$\epsilon_{-k} \geq \delta_{-k} - \sqrt{\delta_{-k}} \cdot \sqrt{(\delta_{-k} - \delta_{-k+1})} \geq \delta_{-k_1} - \sqrt{\delta_*} \cdot \sqrt{c} > 1$$

So $\{\delta_{-n}\}_{n \in \mathbb{N}}$ is an increasing sequence with $\delta_k > 1$, and therefore $\lim_{n \rightarrow \infty} \delta_{-n}^{-1} < 1$.

On the other hand, let $\delta_* = \infty$, and assume for the sake of contradiction that ϵ_{-k} is always less than 1 for $k \in \mathbb{N}$. We know that $\delta_{-k} - \sqrt{\delta_{-k}^2 - \delta_{-k}\delta_{-k+1}} \leq \epsilon_{-k} \leq 1$, and we can rewrite this inequality to get:

$$\begin{aligned} \delta_{-k}^2 - \delta_{-k}\delta_{-k+1} &\geq (\delta_{-k} - 1)^2 \\ &= \delta_{-k}^2 - 2\delta_{-k} + 1 \\ \implies \delta_{-k}(2 - \delta_{-k+1}) &\geq 1 \end{aligned}$$

and therefore, $\delta_{-k+1} \leq 2$, a contradiction since we know $\lim_{n \rightarrow \infty} \delta_{-n} = \infty$. Note that we could assume that the sequence $\{\epsilon_{-k}\}_{k \in \mathbb{N}}$ is bounded above by any number and derive a similar contradiction. So in this case, not only $\lim_{n \rightarrow \infty} \epsilon_{-n}$ is strictly larger than 1, but it is infinite as well.

Overall, we have proven that $\lim_{n \rightarrow \infty} \epsilon_{-n}^{-1} < 1$, and $\sum_{n \in D_{< 0}} r_n$ converges by Lemma A.1.2.

For the second part of the proof, assume that $(1, 1) \in \Omega_r$. We need to show that both of the sums $\sum_{n \in D_{\geq 0}} s_n$ and $\sum_{n \in D_{< 0}} s_n$ converge.

Suppose $D_{\geq 0}$ is the set of all nonnegative integers greater than M , or we have nothing to prove. By Lemma A.1.2, the limit of ϵ_n as n goes to infinity be some $\epsilon^* < 1$.

Let m_0 be a large enough integer satisfying $\epsilon_{m_0} < 1$, and let $c = \frac{(1-\epsilon_{m_0})^2}{2\delta_{m_0}}$. Note that the sequence $\{\delta_k\}_{k>M}$ converges by Lemma A.1.2, and therefore, there exists some large enough m satisfying $\delta_m - \delta_{m+1} \leq c$, therefore:

$$\begin{aligned}\delta_m &\leq \epsilon_m + \sqrt{\delta_m} \sqrt{\delta_m - \delta_{m+1}} \\ &\leq \epsilon_{m_0} + \sqrt{\delta_m} \sqrt{\delta_m - \delta_{m+1}} \\ &\leq \epsilon_{m_0} + \sqrt{\delta_{m_0}} \sqrt{c} < 1\end{aligned}$$

For the last part of this proof, assume that $D_{<0} = \mathbb{Z}_{<0}$. The ratio test for the convergence of $\sum_{n \in \mathbb{N}} r_{-n}$ tells as that $\lim_{n \rightarrow \infty} \epsilon_{-n}^{-1} \leq 1$. If this limit is exactly 1, we will get a contradiction, so let $\epsilon_*^{-1} = \lim_{n \rightarrow \infty} \epsilon_{-n}^{-1} < 1$.

For the last part of this proof, suppose $D_{<0}$ is an infinite set and contains all negative integers smaller than some $M \in \mathbb{Z}$. Then ϵ_{-n} is a well-defined ratio for all $n > |M|$, and by Lemma A.1.2, $\epsilon_*^{-1} = \lim_{n \rightarrow \infty} \epsilon_{-n}^{-1} < 1$.

If $\epsilon_* = \infty$, we have:

$$\epsilon_{-k} \leq \delta_{-k} + \sqrt{\delta_{-k}^2 - \delta_{-k} \delta_{-k+1}} \leq 2\delta_{-k},$$

and $\{\delta_{-n}\}_{n>|M|}$ goes to infinity, in which case $\lim_{n \rightarrow \infty} \delta_{-n}^{-1} < 1$ and $\sum_{n \in D_{<0}} s_n$ converges by Lemma A.1.2.

So assume that $1 < \epsilon_* < \infty$. Moreover, if $\delta_*^{-1} := \lim_{n \rightarrow \infty} \delta_{-n}^{-1}$, we can assume $\delta_* < \infty$ as well (note that $\{\delta_{-n}^{-1}\}_{n>|M|}$ converges by Lemma A.1.2). Let m_1 be a large enough integer satisfying $\epsilon_{-m_1} > 1$, and let $c = \frac{(\epsilon_{-m_1}-1)^2}{2\delta_*}$. There exists some large enough m such that $\delta_{-m} - \delta_{-m+1} \leq c$, and we will get:

$$\begin{aligned}\delta_{-m} &\geq \epsilon_{-m} - \sqrt{\delta_{-m}} \sqrt{\delta_{-m} - \delta_{-m+1}} \\ &\geq \epsilon_{-m_1} - \sqrt{\delta_*} \sqrt{c} \\ &> 1\end{aligned}$$

So $\delta_{-m} > 1$ for some $m > |M|$, and $\lim_{n \rightarrow \infty} \delta_{-n}^{-1} < 1$. So $\sum_{n \in D_{<0}} s_n$ converges by Lemma A.1.2, completing the proof. $\square$