

Edge-Coloring Planar Graphs and the Cycling Conjecture

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

An r -graph is defined to be a graph where each vertex has degree r and any odd subset has at least r edges leaving it. This thesis focuses on the conjecture that any planar r -graph can be edge-colored with r colors. Past work has shown that the conjecture holds for $r \leq 8$, but it becomes more difficult with each increase of r . We consider what occurs when r is very large. The main ideas of the thesis work with a *minimal counterexample* graph that is one of the smallest graphs to contradict the conjecture for a given r . To make a minimal counterexample easier to work with, we generalize to grafts, working with T -joins and T -cuts. We go through various directions to approach the problem and show properties of a minimal counterexample as well as questions that stand in the way of proving it does not exist.

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Chapter 1

Introduction

Consider coloring a graph by assigning a color to each edge. The coloring is valid if any two edges that are incident to the same vertex are not assigned the same color. The minimum number of colors needed to make a valid edge-coloring of a graph G is the edge chromatic number of G and is written as $\chi'(G)$. A classic theorem for edge-coloring is Vizing's Theorem. The theorem states that for graphs with no parallel edges or loops, called simple graphs, $\chi'(G) \in \{\Delta(G), \Delta(G) + 1\}$, where $\Delta(G)$ is the maximum degree of G .

We will focus on multigraphs that allow parallel edges but no loops. For multigraphs, Vizing showed that $\chi'(G) \leq \Delta(G) + \mu$, where μ is the multiplicity of the graph - the maximum number of parallel edges between two vertices [23]. In the 1970s, it was conjectured by Goldberg [9] and Seymour [19, Conjecture 3.2] independently that $\chi'(G) \leq \max\{\Delta(G) + 1, \lceil \omega(G) \rceil\}$. Here, $\omega(G)$ is the density of G , defined as the maximum value of $\frac{2|E(U)|}{|U|-1}$ for all $U \subset V(G)$ where $|U|$ is odd and at least 3, and $E(U)$ is the set of edges with both vertices in U . Proofs of the Goldberg-Seymour Conjecture have been proposed, most recently in 2024 by Chen et al. [2]

If the maximum degree of our graph is r , the edge chromatic number is at least r , so a natural case to look at is when all the vertices have degree r . A graph where every vertex has degree r is called **r -regular**. Clearly, $\chi'(G) \geq r$ for r -regular graphs, but when is $\chi'(G) = r$?

We can show that for $\chi'(G) = r$ to hold, it is necessary that for every odd subset $U \subseteq V(G)$, the size of the set of edges with exactly one endpoint in U , $|\delta(U)|$, is at least r . If $\chi'(G) = r$, then the edges can be partitioned into r perfect matchings. Consider an odd subset U (so $|U|$ is odd). Every perfect matching must appear in $\delta(U)$, so there must

be at least r edges in $\delta(U)$. Therefore, if $\chi'(G) = r$, then $|\delta(U)| \geq r$ for every odd subset $U \subseteq V(G)$. With this property, we define the following:

Definition 1. An r -**graph** is an r -regular graph where for any odd subset $U \subseteq V(G)$, it holds that $|\delta(U)| \geq r$.

Seymour showed in 1979 that if the Goldberg-Seymour Conjecture was true for an r -graph G , then $\chi'(G) \leq r + 1$ [19]. This can be quickly shown: Since G is an r -graph, for every odd subset U , $2|E(U)| \leq r|U| - r$. In the original conjecture, $\omega(G)$ was defined as the maximum value of $\frac{2|E(U)|}{|U|-1}$. So $\omega(G) \leq \frac{r|U|-r}{|U|-1} = r$. Then the conjecture gives that $\chi'(G) \leq \max\{r+1, \lceil \omega(G) \rceil\} = \max\{r+1, r\} = r+1$. Consider the smallest case of interest for r -graphs: the 3-graphs.

Proposition 1 (Folklore). *A graph is a 3-graph if and only if it is cubic and bridgeless.*

Proof. 3-graphs are cubic, and it can be shown easily that they are bridgeless. For a contradiction, assume that there exists a 3-graph with a bridge e , so there is a cut $\delta(U)$ with exactly one edge leaving it. By definition of r -graph, $|U|$ cannot be odd since then $|\delta(U)| \geq 3$. Then $|U|$ must be even. Let x be the vertex in U that is the endpoint of e . Since the graph is cubic, there are two edges between $U - \{x\}$ and $\{x\}$. $U - \{x\}$ is odd so $|\delta(U - \{x\})| \geq 3$. So the remaining edge(s) that are not incident to x must have endpoints in $\bar{U} = V(G) - U$, making e no longer a bridge.

In the other direction, consider a cubic bridgeless graph G . We want to show that for any odd subset U , $|\delta(U)| \geq 3$. Let $U \subseteq V(G)$ be an odd subset. Since G is bridgeless, $|\delta(G)| \geq 2$. For a contradiction, assume that $|\delta(U)| = 2$. Then $E(U)$, the number of edges within U , is $\frac{3|U|-2}{2}$. However, since $|U|$ is odd, this is not an integer, so $|\delta(U)| \geq 3$. $\square$

Therefore, when trying to edge-color a 3-graph with 3 colors, it is natural to try to color a bridgeless cubic graph with 3 colors. It has been shown by Tait that the Four Color Theorem is equivalent to the statement that all planar bridgeless cubic graphs are 3-edge-colorable [21]. To show that the Four Color Theorem implies that all planar bridgeless cubic graphs are 3-edge-colorable, start with a planar bridgeless cubic graph. Take the dual of the graph, and since it is planar, the faces can be colored with four colors. Let the colors be the binary numbers 00, 01, 10, and 11. To edge color the dual graph, color the edge between two faces as the sum mod 2 of the colors of the two faces. Then all edges around a face have different colors. Color each edge's corresponding color in the original graph with the same color. This 3-edge-colors the planar bridgeless cubic graph.

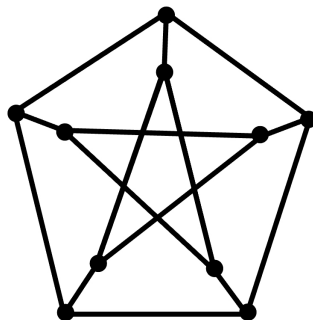


Figure 1.1: Petersen Graph

Now assume that it holds that all planar bridgeless cubic graphs are 3-edge-colorable. Begin with a planar graph where we want to color the vertices with four colors. Triangulate the graph so that it is maximally planar and then take the dual. Because the graph is a triangulation, the dual is cubic and bridgeless. 3-edge-color the dual. Pick two colors a, b . Remove all edges from the graph that are colored a . The remaining edges form a disjoint union of cycles. Color with 01 any faces that in the plane are contained in an odd number of cycles. Separately, remove all edges from the graph that are colored b and color with 10 any faces that in the plane are contained in an odd number of cycles. Faces that are not colored from this get the color 00. Now add the colors in the two graphs to get a coloring of the faces with the colors 00, 01, 10, and 11. Therefore, the vertices of the original planar graph can be validly colored with 4 colors.

The Four Color Theorem was proven by Appel and Haken [1], so it is possible to 3-edge-color any planar cubic bridgeless graph. Can the planar condition be removed? It cannot because the Petersen graph $\mathcal{P}$, shown in Figure 1.1, is a cubic bridgeless graph where $\chi'(\mathcal{P}) = 4$. This obstacle lead to Tutte's conjecture in 1966:

Conjecture 1. [22] *If G is a bridgeless cubic graph that does not contain the Petersen graph as a minor, then G is 3-edge-colorable.*

Work following this conjecture explored whether different graphs could be excluded instead of the Petersen graph in order to prove the conjecture. A graph G is **apex** if for some vertex v , $G - v$ is planar. A graph is **doublecross** if it can be drawn in the plane where there are only two crossings and they are both on the infinite region. Neither apex nor doublecross graphs contain the Petersen graph as a minor. A graph is **theta-connected** if it is cubic, has girth at least 5, and $|\delta(U)| \geq 6$ for all $U \subset V(G)$ where

$|U|, |V(G) - U| \geq 6$. It was shown by Robertson, Seymour, and Thomas that for every theta-connected graph G that has no Petersen minor, G is either apex or doublecross, with the exception of one graph called Starfish [15, Theorem 1.1]. The Starfish graph is a cubic graph on 20 vertices that is 3-edge-colorable. Then it was shown that every minimal counterexample to Tutte’s conjecture above is either apex or theta-connected, and therefore it is either apex or doublecross [14, Theorem 1.3]. It was first shown by Sanders and Thomas that bridgeless apex cubic graphs are 3-edge-colorable [16] and later by Edwards, Sanders, Seymour, and Thomas that bridgeless doublecross cubic graphs are 3-edge-colorable [8, Theorem 1.1], proving Tutte’s conjecture.

Returning the planarity condition for the general case, there is the conjecture by Seymour [17]:

Conjecture 2 (Edge-Coloring Conjecture). *A planar r -graph is r -edge-colorable.*

We have already stated that the conjecture holds for $r = 3$ since it is implied by the Four Color Theorem. The $r = 4, 5$ cases were proven by Guenin in 2003 [10, Corollary 1.4], $r = 6$ was proven by Dvorak, Kawarabayashi and Kral in 2010 [6, Theorem 2], $r = 7$ by Edwards and Kawarabayashi in 2011 [7] and later by Chudnovsky, Edwards, Kawarabayashi, and Seymour [3, Theorem 1.2], and $r = 8$ by Chudnovsky, Edwards, and Seymour in 2012 [4, Theorem 1.2]. In each of these papers, the proofs became more involved when just increasing r by one each time, so they were not simple induction proofs. This thesis considers the question of whether it is sufficient to prove the conjecture for a finite number of values of r . Namely, does there exist an $\bar{r}$ such that if the Edge-Coloring Conjecture holds for all values up to $\bar{r}$, it holds for all larger values?

Conjecture 3 (Edge-Coloring for Large r Conjecture). *There exists a nonnegative integer $\bar{r}$ such that if it holds for all $r < \bar{r}$ that a planar r -graph is r -edge-colorable, then for all r , planar r -graphs are r -edge-colorable.*

1.1 Organization of the Thesis

In the thesis, we take the approach in graph theory of considering a minimal counterexample to the conjecture. In order to make it easier to make the r -graph smaller, we restate the problem in terms of grafts, T -joins, and T -cuts. Chapter 2 provides the definitions and background that lead to this restatement as well as a proof of the equivalence of the two conjectures. The next chapters use set theoretic and graph theoretic arguments to study minimal counterexample grafts.

Chapters 3 and 4 examine a counterexample through the set theoretic lens by rewriting the problem through clutters. Chapter 3 deals with the basics of clutters and defines Eulerian clutters in order to restate the conjecture. In Chapter 4, we consider a certificate of when a clutter does not pack, called a *mate*, and observe what occurs when a minimal counterexample packs except for one edge. Then transitioning back to thinking of the problem through grafts, Chapter 5 explores when a graft comes even closer to packing. It also characterizes mates in the environment of a graft and explores ways to count the number of mates. Chapter 6 considers what properties occur in a minimal counterexample graft when r is very large, and Chapter 7 looks into a recoloring argument.

Chapter 2

Packing T -joins

2.1 An Introduction to T -joins

First, let's consider the Edge-Coloring Conjecture of edge-coloring a planar r -graph with r colors. A classic strategy in graph theory proofs is to consider a minimal counterexample to the conjecture. For this problem, that would look like starting with an r -graph for a given r such that it is the smallest one (with respect to the number of edges or vertices) that cannot be edge-colored with r colors. Then make a change to the graph to make a smaller r -graph. By the assumption, this smaller r -graph can be edge-colored with r colors. Then extend the coloring to the original graph. It is difficult to follow this strategy when working with r -graphs because it is not easy to make the graph smaller while maintaining r . Instead, we work with a generalization of matchings called T -joins.

T -joins are a structure found in grafts. A **graft** (G, T) is a pair made up of an undirected graph G that may have parallel edges and a subset of vertices $T \subseteq V(G)$ where $|T|$ is even. This set T is used to define the following structures:

Definition 2. A **T -join** of a graft (G, T) is a subset of edges $B \subseteq E(G)$ where T is the set of vertices with odd degree in $G[B]$, the subgraph induced by the edges in B .

A **minimal T -join** is inclusion-wise minimal. Note that there are no parallel edges in minimal T -joins.

Definition 3. A **T -cut** of a graft (G, T) is a cut $\delta(U)$ for $U \subseteq V(G)$ where $|U \cap T|$ is odd.

Familiar structures in graph theory show up for certain sets T . For example, here are two of the extreme cases. If T is just a set of two vertices $\{s, t\}$, then the minimal T -joins are the st -paths of G , since s and t are the only two vertices of the T -joins that can have odd degree. The T -cuts for $T = \{s, t\}$ are the st -cuts since they must include exactly one of s or t . If T is the maximum it can be, which is the entire set $T = V(G)$, then the T -joins of size $|V(G)|/2$ are perfect matchings. Additionally, when $T = V(G)$, any cut formed by an odd number of vertices (called an *odd cut*) is a T -cut.

An r -edge-coloring of an r -regular graph is a set of r disjoint perfect matchings. A similar concept appears with T -joins, where a set of pairwise disjoint T -joins is called a **packing** of T -joins. When we refer to a packing of T -joins, we may assume that we are using minimal T -joins. The cardinality of the largest packing of T -joins in a graft (G, T) is denoted as $\nu(G, T)$. The cardinality of the smallest T -cut of (G, T) is denoted as $\tau(G, T)$. The following remark gives a relationship between T -joins and T -cuts.

Remark 1. For a cut $\delta(U)$ and a T -join B , $|\delta(U) \cap B|$ is odd if and only if $\delta(U)$ is a T -cut.

Since any T -cut must intersect each T -join at least once and there is a set of $\nu(G, T)$ disjoint T -joins, then $\tau(G, T) \geq \nu(G, T)$ for any graft (G, T) . When equality holds, we define the following property:

Definition 4. A graft (G, T) **packs** if $\tau(G, T) = \nu(G, T)$.

This property appears as Menger's Theorem [11] when T is two vertices. Namely, the largest packing of T -joins is the maximum number of disjoint paths between two vertices. The cardinality of the smallest T -cut is the size of the smallest cut-set. For a graft (G, T) where G is an r -graph and T is $V(G)$, $\tau(G, V(G)) = r$ and $\nu(G, V(G)) = \chi'(G)$, so our goal is for the graft to pack. In generalizing our planar r -graph problem, we try to restate the problem without specifying that G is a planar r -graph.

First, we check if the planarity condition is sufficient; namely does every planar graft pack? There is a simple example shown in Figure 2.1 to prove that there exists a planar graft that does not pack. In this graft, G is the bipartite graph $K_{2,3}$ and T is the set of vertices that are in boxes: v_2, v_3, v_4 , and v_5 . The smallest T -cut has cardinality 2, so $\tau(G, T) = 2$. The T -joins of the graft are either the set of edges $\{v_2v_5, v_3v_5, v_4v_5\}$ or of the following form: two of the edges incident to v_1 and the edge incident to v_5 that does not share an endpoint with those two edges. For example, $\{v_1v_2, v_1v_3, v_4v_5\}$ is a T -join. Every pair of T -joins shares an edge, so the size of the largest packing, $\nu(G, T)$, is 1. Then $\tau(G, T) > \nu(G, T)$, so the graft does not pack. Let's call this graft $\tilde{K}_{2,3}$. In fact, Seymour's

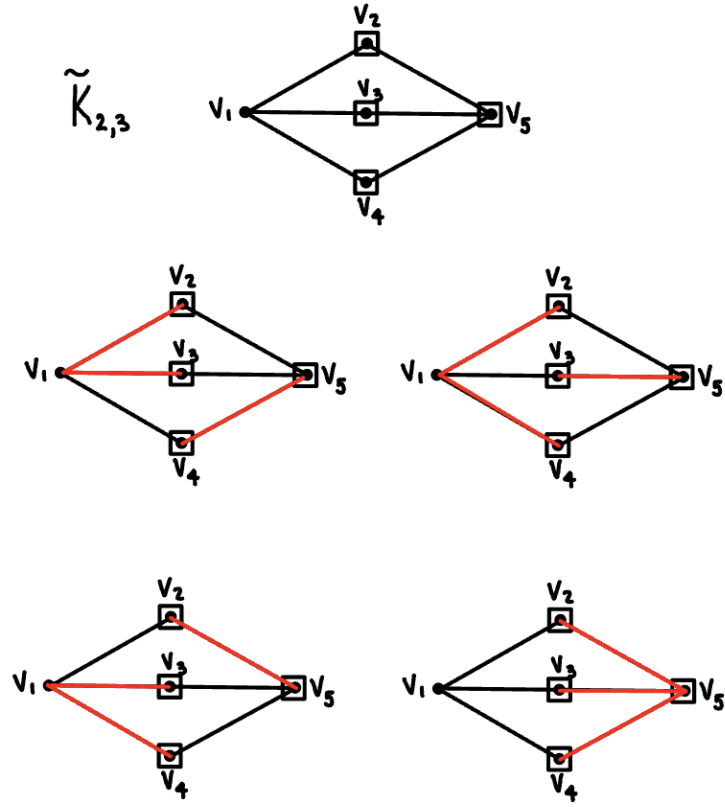


Figure 2.1: Graft $\tilde{K}_{2,3} = (K_{2,3}, \{v_2, v_3, v_4, v_5\})$ and its T -joins below it.

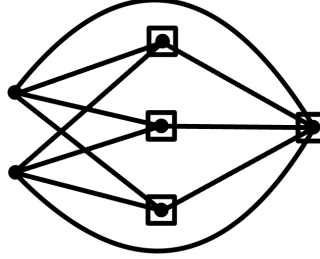


Figure 2.2: Example of a Eulerian graft (G, T) , where T is the vertices in the boxes, that contains $\tilde{K}_{2,3}$ as a minor

work [18] implies that any graft will pack if it does not contain $\tilde{K}_{2,3}$ as a minor. A minor of a graft is obtained by performing edge deletions and contractions. When an edge is contracted, the resulting vertex is in T if exactly one of the endpoints of the original edge was in T . Is there a way to add a property to our grafts so that $\tilde{K}_{2,3}$ is excluded? We define the following:

Definition 5. A graft (G, T) is **Eulerian** if all T -cuts have the same parity.

The parity of a T -cut $\delta(U)$ is the parity of $|\delta(U)|$. $\tilde{K}_{2,3}$ is not Eulerian since it can be seen in Figure 2.1 that the T -cut made with the set $\{v_2\}$ has size 2 and the one made with the set $\{v_5\}$ has size 3. Therefore, requiring the graft to be Eulerian will remove the obstruction of $\tilde{K}_{2,3}$. Note that a graft that contains $\tilde{K}_{2,3}$ as a minor can be Eulerian. For example, consider the graft in Figure 2.2.

A graft (G, T) being Eulerian is different from G being a Eulerian graph. There is the case where the graft (G, T) could be a **postman set**, where T is the set of vertices of G with odd degree. Note that this means that $E(G)$ is a T -join. These two cases are shown in the following proposition presented in the $r = 4, 5$ paper by Guenin [10, Proposition 1.2].

Proposition 2. For a graft (G, T) where T is nonempty, (G, T) is Eulerian if and only if G is Eulerian or (G, T) is a postman set.

Proof. First consider the easier direction where it is assumed that G is Eulerian or (G, T) is a postman set. If G is Eulerian, then each vertex has even degree. Consider any cut $\delta(U)$. $|\delta(U)|$ is the sum of the degrees of each vertex in U subtracted by twice the number of edges with both endpoints in U . In a Eulerian graph, the degree of each vertex is even,

so $|\delta(U)|$ is even. Therefore, all cuts are even, so all T -cuts are even. If (G, T) is a postman set, then $E(G)$ is a T -join. By Remark 1, for every T -cut $\delta(U)$, $|\delta(U) \cap E(G)| = |\delta(U)|$ is odd. Then all T -cuts are odd. This completes the direction that if G is Eulerian or (G, T) is a postman set, then all T -cuts have the same parity.

Now let's assume that all T -cuts have the same parity. Some useful facts for this direction are that $\delta(A \triangle B) = \delta(A) \triangle \delta(B)$ and that if X, Y have the same parity, then $|X \triangle Y|$ is even. If $T = V(G)$, then $\delta(v)$ for any vertex in G is a T -cut, and therefore all vertices have degrees of the same parity. Then G is Eulerian or (G, T) is a postman set.

Assume that $T \neq V(G)$. Take the T -cut $\delta(U)$ and vertex $v \notin T$. $\delta(U \triangle \{v\})$ is a T -cut since v won't affect the size of $|(U \triangle \{v\}) \cap T|$. Since $U \triangle U = \emptyset$, $\{v\} = U \triangle U \triangle \{v\}$. Then $\delta(v) = \delta(U \triangle U \triangle \{v\}) = \delta(U) \triangle \delta(U \triangle \{v\})$. Because it is assumed that all T -cuts have the same parity, $\delta(U)$ and $\delta(U \triangle \{v\})$ have the same parity, so $|\delta(v)| = |\delta(U \triangle U \triangle \{v\})|$ is even. Therefore, any vertex not in T has even degree.

If there does not exist a vertex $v \in V(G)$ of odd degree, then G is Eulerian. Assume such a v exists, so $v \in T$. To show that (G, T) is a postman set, we must show that all vertices in T have odd degree. Take $w \in T$. Since $|\delta(v)|$ is odd and all T -cuts have the same parity, then $|\delta(w)|$ must be odd. Therefore, all vertices in T have odd degree, so (G, T) is a postman set. $\square$

With these new terms, we restate the Edge-Coloring Conjecture in terms of grafts:

Conjecture 4 (T -join Packing Conjecture). *Eulerian planar grafts pack.*

2.2 Equivalence Between the Edge-Coloring and T -join Packing Conjectures

In order to continue in the thesis with packing T -joins, we must show that for a given r the statement that all planar r -graphs are r -edge-colorable is equivalent to the statement that every Eulerian planar graft where $\tau(G, T) = r$ packs. We begin with the easier direction, that the T -join Packing Conjecture implies the Edge-Coloring Conjecture.

Proposition 3. *For positive integer r , if every Eulerian planar graft where $\tau(G, T) = r$ packs, then all planar r -graphs are r -edge-colorable.*

Proof. Fix an r . All planar r -graphs are r -edge-colorable if for any planar r -graph G , there exists a set of r pairwise disjoint perfect matchings. Assume that for the fixed r , every

Eulerian planar graft where $\tau(G, T) = r$ packs. Let G be a planar r -graph. $|V(G)|$ is even since for an r -graph, all odd subsets $U \subseteq V(G)$ have $|\delta(U)| \geq r$. Set $T = V(G)$. Then every odd cut is a T -cut, making $\tau(G, T) \geq r$. Since G is r -regular, $|\delta(v)| = r$ for any vertex $v \in V(G)$. $\delta(v)$ is an odd cut and therefore a T -cut, so $\tau(G, T) = r$. If r is even, then G is Eulerian. If r is odd, then (G, T) is a postman set because $T = V(G)$ and all vertices have odd degree. Therefore, by Proposition 2, (G, T) is a Eulerian graft. So with the given G and T , (G, T) is a planar Eulerian graft where $\tau(G, T) = r$. Then by assumption, it packs.

Therefore, there exists a packing of r T -joins, $F_1, \dots, F_r$. Since $\delta(v)$ is an odd cut, it is a T -cut. The intersection between any T -cut and T -join must be odd and the T -joins in the packing are disjoint, so $\delta(v)$ must contain an edge from each of the r T -joins. $|\delta(v)| = r$, so it contains exactly one edge from each T -join. Since this holds for every vertex, the T -joins are perfect matchings. Assign a different color to each perfect matching, and you have an edge-coloring of the graph G . Therefore, all planar r -graphs are r -edge-colorable, so the T -join Packing Conjecture implies the Edge-Coloring Conjecture. $\square$

In order to prove that the Edge-Coloring Conjecture implies the T -join Packing Conjecture, we first need a useful fact about T -cuts and the two ways that they can be thought of as “the smallest”.

Definition 6. A *minimum T -cut* is a T -cut $\delta(U)$ in the graft that has $|\delta(U)| = \tau(G, T)$.

Definition 7. A *trivial T -cut* is a T -cut $\delta(U)$ in the graft where $|U| = 1$ or $|\bar{U}| = 1$.

We want to show that for a Eulerian planar graft, the following holds:

Proposition 4. Let $\delta(U)$ be a minimum T -cut of (G, T) . Let G' be the graph obtained by contracting edges so that U is just one vertex, u , and let $T' = (T - U) \cup \{u\}$. Then let G'' and T'' be the same but this time contracting edges so that $\bar{U}$ is just one vertex, $\bar{u}$. If (G', T') and (G'', T'') pack, then (G, T) packs.

Proof. These structures can be seen in Figure 2.3. Assume that (G', T') and (G'', T'') pack. Note that $|\delta(u)|$ and $|\delta(\bar{u})|$ are both $\tau(G, T)$ since $\delta(U)$ is a minimum cut. Let $\tau = \tau(G, T)$. Then there is a set of disjoint T' -joins $F'_1, \dots, F'_\tau$ so that (G', T') packs. Since $\delta(U)$ is a T -cut, then $\delta(u)$ is a T' -cut. Any T' -cut and T' -join have odd intersection, so each of the τ edges incident with u belongs to a different T' -join. Now consider the set of disjoint T'' -joins $F''_1, \dots, F''_\tau$ which shows that (G'', T'') packs. Similarly, each of the τ edges incident to $\bar{u}$ belongs to a different T'' -join. Renumber the T'' -joins so that F''_i uses the

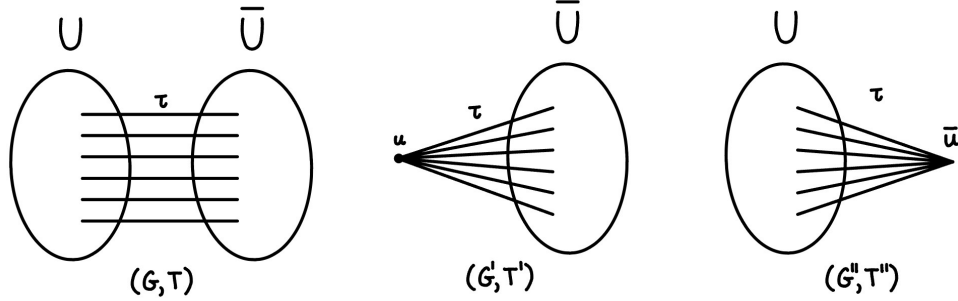


Figure 2.3: The structures (G, T) , (G', T') , and (G'', T'') described in Proposition 4

same edge of $\delta(U)$ as F'_i . Form τ T -joins by setting $F_i = F'_i \cup F''_i$. These τ T -joins are edge disjoint, so it is a packing of (G, T) . Therefore, if (G', T') and (G'', T'') pack, then (G, T) packs. $\square$

Now we can prove that the Edge-Coloring Conjecture implies the T -join Packing Conjecture.

Proposition 5. *If all planar r -graphs are r -edge-colorable, then every Eulerian planar graft where $\tau(G, T) = r$ packs.*

Proof. Fix r . We begin by assuming it is not true that every Eulerian planar graft where $\tau(G, T) = r$ packs. Therefore, there exists at least one Eulerian planar graft where $\tau(G, T) = r$ that does not pack. We pick (G, T) to be one these Eulerian planar grafts where $\tau(G, T) = r$ that do not pack with the number of edges minimized. We call this a **minimal counterexample graft**. Note that it is possible that there are multiple choices for a minimal counterexample graft. With this graft we have the following claim:

Claim 1. *In a minimal counterexample graft (G, T) , a minimum T -cut is a trivial T -cut.*

Proof. In a minimal counterexample graft (G, T) , let $\delta(U)$ be a minimum T -cut, so $|\delta(U)| = \tau(G, T)$. If $|U| = 1$ or $|\bar{U}| = 1$, then it is a trivial T -cut. Assume that it is not. First we show that $G[U]$ and $G[\bar{U}]$ must both be connected. Assume that $G[U]$ is not connected, so U can be partitioned into sets U_1 and U_2 with no edges between them. Since $\delta(U)$ is a T -cut, then $|U \cap T|$ is odd. Without loss of generality, assume $|U_1 \cap T|$ is odd, so $|U_2 \cap T|$ is even. Then $\delta(U_1)$ is a T -cut. In order for G to be connected, there must exist at least one edge between U_2 and $\bar{U}$. Then $|\delta(U_1)| < |\delta(U)|$. This gives a contradiction since by assumption, $\delta(U)$ is a minimum T -cut. Therefore, $G[U]$ is connected. Since $|T|$ is even,

$|\overline{U} \cap T|$ is odd. Then the same can be done to show that $G[\overline{U}]$ is connected. Contract the edges in $G[U]$ so that it is one vertex, u , to make new graft (G', T') . This is a Eulerian planar graft with $\tau(G', T') = r$ and $|E(G')| < |E(G)|$. Then (G', T') packs. Similarly, (G'', T'') is formed from (G, T) by contracting the edges in $G[\overline{U}]$ so that it is one vertex, $\bar{u}$, and (G'', T'') packs. By Proposition 4, since (G', T') and (G'', T'') pack, (G, T) packs. This is a contradiction since (G, T) is a minimal counterexample graft. Therefore, $\delta(U)$ is a trivial T -cut. $\diamond$

More properties of a minimal counterexample graft (G, T) were shown by Guenin [10, Lemma 2.2]:

Claim 2. *T consists of all the vertices of G , and G is r -regular for $r = \tau(G, T)$.*

Proof. For a contradiction, assume that there exists a vertex v in G such that $v \notin T$ or $d(v) > r$. First we show that not all edges incident to v are parallel. Assume that all edges incident to v are parallel. We work with the two cases, $v \in T$ or $v \notin T$.

In this first case, $v \in T$, so $d(v) > r$. Let $e_1, e_2 \in \delta(v)$ and $G' = G - \{e_1, e_2\}$. Since (G, T) is Eulerian, all T -cuts of G have the same parity. The degree of v decreases by 2, so all T -cuts of G' have the same parity. Therefore, $d(v) \geq r + 2$ to maintain parity. Since the edges incident to v are all parallel, $\tau(G', T) \geq r$. By minimality of (G, T) , (G', T) has a packing of r T -joins. The T -joins of G' are T -joins of G , so (G, T) packs, a contradiction.

In the second case, $v \notin T$. This time, let $G' = G - \delta(v)$. Since $\delta(v)$ is made up of parallel edges, no minimum T -cut uses an edge of $\delta(v)$. Therefore, $\tau(G', T) \geq r$, as it is unaffected. Then by the minimality of (G, T) , (G', T) packs, but then so does (G, T) , a contradiction. From these two cases, we have shown that the edges of $\delta(v)$ are not all parallel to each other or that v has a degree of 0. If $d(v) = 0$, then you can delete v .

Now consider a planar embedding of G , and let's look at the edges of $\delta(v)$ in a clockwise order. From what we have just shown, there must exist $u_1 \neq u_2 \in V(G)$ such that u_1v and u_2v are consecutive edges in $\delta(v)$. We create the planar graph G' by deleting the edges u_1v and u_2v and adding the edge u_1u_2 . Note that since G has no loops, G' has no loops. Also, the degree of any vertex other than v is unchanged. The degree of v has decreased by 2.

Let $\delta(U)$ be a T -cut such that if it is a trivial T -cut, $U = \{v\}$. If $U = \{v\}$, then v is in T , so $d(U) = d(v) > r$. If $\delta(U)$ is not a trivial T -cut, then $d(U) > r$ because a minimum T -cut is a trivial T -cut for a minimal counterexample. To maintain parity, $d(U) \geq r + 2$. Since $\tau(G', T) = r$ and all T -cuts of G' have the same parity, the minimality of (G, T) gives that (G', T) has a packing of T -joins $B'_1, \dots, B'_r$ of G' . If none of the B'_i use u_1u_2 , then clearly this is also a packing for (G, T) . Otherwise, without loss of generality,

say that $u_1u_2 \in B'_1$. Then $B'_1 - \{u_1u_2\} \cup \{u_1v, u_2v\}, B'_2, \dots, B'_r$ is packing of r T -joins of G . In either case, this gives a contradiction, since (G, T) is a minimal counterexample. Therefore, $T = V(G)$ and $d(v) = r$ for all $v \in V(G)$. $\diamond$

Since $V(G) = T$, all odd cuts are T -cuts. Because all degrees are r and the minimum T -cut is a trivial T -cut, all odd cuts have size at least r . So G is an r -graph. Since (G, T) does not pack, then G is not r -edge-colorable. Therefore, the Edge-Coloring Conjecture does not hold for this r . $\square$

With these propositions, it has been shown that the Edge-Coloring Conjecture is equivalent to the T -join Packing Conjecture. In the introduction, we gave another conjecture for edge-coloring for large r . We now do the same for packing T -joins.

Conjecture 5 (*T -Join Packing for Large r Conjecture*). *There exists a nonnegative integer $\bar{r}$ such that if every Eulerian planar graft (G, T) with $\tau(G, T) \leq \bar{r}$ packs, then every Eulerian planar graft packs.*

The following chapters study a minimal counterexample to this conjecture. This uses set theoretic as well as graph theoretic arguments. We split these up, and for set theoretic ones, we generalize to clutters. Afterwards, we return to thinking about the problem through grafts.

Chapter 3

Basics of Clutters

We now turn to a more general structure to view our problem through a set theoretical lens. We define a clutter.

Definition 8. A **clutter** $\mathcal{C}$ is defined by a ground set $E(\mathcal{C})$ and a family of subsets of $E(\mathcal{C})$ so that for any two subsets $S, S' \in \mathcal{C}$, if $S \subseteq S'$, then $S = S'$.

So $\mathcal{C}$ is a set of subsets of its ground set $E(\mathcal{C})$. As an example, we can take a look at minimal T -joins and T -cuts. For a graft (G, T) , consider a clutter where the ground set is the edges of G and the family of subsets is the set of the minimal T -joins of the graft. We call this the **clutter of T -joins**, so any time we reference the clutter of T -joins, we are referring to only the minimal T -joins. In order to tie in T -cuts, we define a useful structure of clutters.

Definition 9. A **cover** B of a clutter is a set of elements $B \subseteq E(\mathcal{C})$ where for any set $S \in \mathcal{C}$, $B \cap S \neq \emptyset$.

A cover is **minimal** if removing any edge would make it no longer a cover. It is **minimum** if there does not exist a cover with a smaller cardinality.

Definition 10. A **blocker** $b(\mathcal{C})$ of the clutter $\mathcal{C}$ is a clutter with the same ground set as $\mathcal{C}$ where $b(\mathcal{C})$ is the set of minimal covers of $\mathcal{C}$.

A nice property of blockers is that for a clutter $\mathcal{C}$, $b(b(\mathcal{C})) = \mathcal{C}$ [5, Theorem 1.15]. The T -cuts of a graft are covers of the minimal T -joins, and in fact, the blocker of the clutter of T -joins is the clutter of T -cuts. Similarly to grafts, we denote the minimum size of a cover of $\mathcal{C}$ as $\tau(\mathcal{C})$ and the maximum size of a set of disjoint subsets in the clutter as $\nu(\mathcal{C})$. Note that $\tau(\mathcal{C}) \geq \nu(\mathcal{C})$.

Definition 11. For a clutter $\mathcal{C}$, if $\tau(\mathcal{C}) = \nu(\mathcal{C})$, then $\mathcal{C}$ *packs*.

In the clutter of T -joins, $\mathcal{C}$, the smallest T -cut corresponds to the minimum cover, so $\tau(\mathcal{C})$ equals $\tau(G, T)$. The family of subsets of the clutter are T -joins, so the maximum size of a set of disjoint subsets in the clutter is the size of the largest packing of T -joins in the graft. Then $\nu(\mathcal{C})$ is equivalent to $\nu(G, T)$. Therefore, when the clutter of T -joins packs, the graft packs.

Recall that grafts have the property that any T -join and T -cut have an odd intersection. A similar property describes some clutters.

Definition 12. A clutter is *binary* if for any $S \in \mathcal{C}$ and $B \in b(\mathcal{C})$, $|S \cap B|$ is odd.

From the definition, we can see that if a clutter is binary, then its blocker is also binary. Due to the previously mentioned property of grafts, the clutter of T -joins is binary, and naturally the clutter of T -cuts is also binary. Now we move towards writing a more general version of the T -join Packing Conjecture with clutters. We want to also include something equivalent to the Eulerian property of a graft.

Definition 13. A clutter $\mathcal{C}$ is *Eulerian* if it is binary and the sizes of the sets of the blocker all have the same parity.

With this definition, the clutter of T -joins of a Eulerian graft is Eulerian since all T -cuts have the same parity. Now we want to explore when Eulerian clutters don't pack. First we look at $\mathcal{P}_{10}$, the clutter of the postman sets of the Petersen graph. A postman set is a T -join where T is the odd degree vertices of the graph. Since the Petersen graph is cubic, T will be all the vertices of the graph. Figure 3.1 shows types of minimal postman sets of the Petersen graph. Note that there are no two postman sets that are disjoint. Therefore, $\nu(\mathcal{P}_{10}) = 1$. The minimum T -cut has 3 edges so $\tau(\mathcal{P}_{10}) = 3$. Therefore, $\mathcal{P}_{10}$ does not pack.

Next, we consider $\mathcal{F}_7$, the clutter corresponding to the Fano plane. The Fano plane, shown in Figure 3.2, is the projective plane on 7 points and 7 lines where there are 3 points on every line and 3 lines through every point. The ground set of the clutter is the points of the plane and the subsets are the lines. Any two lines in the Fano plane intersect at a point, so the maximum size of a set of disjoint subsets is one, so $\nu(\mathcal{F}_7) = 1$. There is no cover of size one since only three lines go through a point, so the clutter does not pack.

$\mathcal{O}_5$ is defined to be the clutter where the ground set is the edges of K_5 and the subsets are the odd cycles of K_5 . K_5 has 10 cycles of length 3 and 12 cycles of length 5. There are 10 edges in K_5 , so $\nu(\mathcal{O}_5) < 4$. The sets in the blocker are complements of cuts of K_5 . Then $\tau(\mathcal{O}_5) \geq 4$, so the clutter $\mathcal{O}_5$ does not pack.

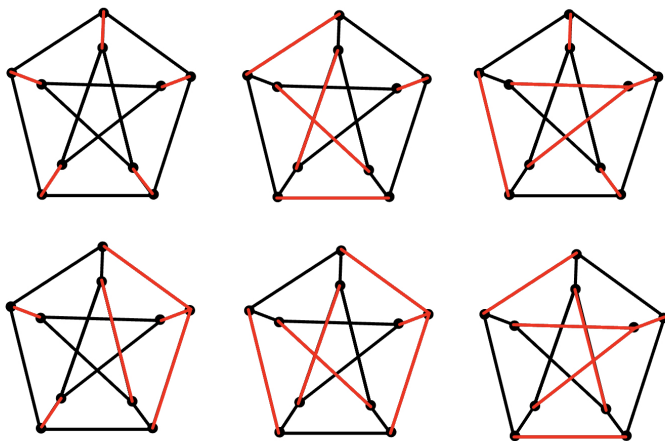


Figure 3.1: Minimal postman sets of the Petersen graph

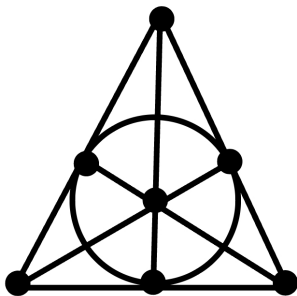


Figure 3.2: The Fano Plane

Now let's consider the blocker of this clutter. The covers of $\mathcal{O}_5$ have size at least 4. So $\nu(b(\mathcal{O}_5)) \leq 2$. The covers of $b(\mathcal{O}_5)$ are the odd cycles, so $\tau(b(\mathcal{O}_5)) = 3$. So $b(\mathcal{O}_5)$ does not pack.

Since these clutters do not pack, we want to exclude them, so we define a **minor** of a clutter. For clutter $\mathcal{C}$ and $j \in E(\mathcal{C})$, a **contraction** $\mathcal{C}/j$ forms the clutter where $E(\mathcal{C}/j) = E(\mathcal{C}) - \{j\}$ and $\mathcal{C}/j$ is the set of minimal elements of $\{S - \{j\} : S \in \mathcal{C}\}$. A **deletion** $\mathcal{C} \setminus j$ forms the clutter where $E(\mathcal{C} \setminus j) = E(\mathcal{C}) - \{j\}$ and $\mathcal{C} \setminus j = \{S : j \notin S \in \mathcal{C}\}$. A minor of $\mathcal{C}$ is formed by performing contractions and deletions. In order to avoid confusion in notation with minors, we let $A - B$ denote $\{a \in A : a \notin B\}$ and use this notation when we simply want to “remove” a set. A useful property of minors of clutters that we will use in the next chapter is the following:

Remark 2. [5, Proposition 1.35] For clutter $\mathcal{C}$ and $U \subset E(\mathcal{C})$,

$$b(\mathcal{C}/U) = b(\mathcal{C}) \setminus U$$

and

$$b(\mathcal{C} \setminus U) = b(\mathcal{C})/U.$$

Note that the clutter of T -joins does not contain $\mathcal{P}_{10}$, $\mathcal{F}_7$, $\mathcal{O}_5$, or $b(\mathcal{O}_5)$ as minors [5]. This leads to a generalization of the T -join Packing Conjecture in terms of clutters:

Conjecture 6 (Cycling Conjecture). [20] A Eulerian clutter packs if it does not contain $\mathcal{P}_{10}$, $\mathcal{F}_7$, $\mathcal{O}_5$, or $b(\mathcal{O}_5)$ as minors.

Note that we do not list the blockers of $\mathcal{P}_{10}$ and $\mathcal{F}_7$ because $b(\mathcal{P}_{10})$ is not Eulerian and $b(\mathcal{F}_7) = \mathcal{F}_7$. Since the clutter of T -joins does not contain any of $\mathcal{P}_{10}$, $\mathcal{F}_7$, $\mathcal{O}_5$, or $b(\mathcal{O}_5)$, this conjecture implies the T -join Packing Conjecture. The following chapter will continue our study of clutters and consider what occurs in a minimal counterexample.

Chapter 4

Minimal Counterexample Clutter

We now think about what occurs when a clutter $\mathcal{C}$ doesn't pack. Is there a “certificate” to show that a certain set is not in a maximum packing? In a clutter that packs, we observe what occurs when a set $S_1 \in \mathcal{C}$ is in a maximum packing. If S_1 is in a maximum packing, then there exist $S_2, \dots, S_\tau \in \mathcal{C}$ such that $S_1, S_2, \dots, S_\tau$ are pairwise disjoint where $\tau = \tau(\mathcal{C})$. This means that $S_2, \dots, S_\tau$ are all in $\mathcal{C} \setminus S_1$. So there exists a packing of $\tau - 1$ disjoint sets, making $\tau(\mathcal{C} \setminus S_1) \geq \tau - 1$. Then every cover $B' \in b(\mathcal{C} \setminus S_1)$ has size at least $\tau - 1$. Because $b(\mathcal{C} \setminus S_1) = b(\mathcal{C})/S_1$, $B' \in b(\mathcal{C})/S_1$ is formed by removing the elements in S_1 . So $B' = B - S_1$ for some $B \in b(\mathcal{C})$. If S_1 is in a maximum packing, then for every cover $B \in b(\mathcal{C})$, $|B - S_1| \geq \tau - 1$. Therefore, if there exists a $B \in b(\mathcal{C})$ such that $|B - S_1| < \tau - 1$, then S_1 is not in a maximum packing. We call this cover a mate of S_1 :

Definition 14. A *mate* of $S \in \mathcal{C}$ is a $B \in b(\mathcal{C})$ such that $|B - S| < \tau(\mathcal{C}) - 1$.

By this definition, if every $S \in \mathcal{C}$ has a mate, then $\mathcal{C}$ does not pack. Note that the converse is not necessarily true. A clutter not packing does not guarantee that there is a mate for each set of the clutter.

Similar to how we defined a minimal counterexample graft in Chapter 2, we define a **minimal counterexample clutter**. Let τ be the smallest value of $\tau(\mathcal{C})$ where the Cycling Conjecture does not hold. Consider the set of Eulerian clutters that do not contain $\mathcal{P}_{10}$, $\mathcal{F}_7$, $\mathcal{O}_5$, or $b(\mathcal{O}_5)$ as minors, have $\tau(\mathcal{C}) = \tau$, and do not pack. Let a minimal counterexample clutter be one of these clutters that minimizes the number of elements of the ground set. Note that if $\mathcal{C}$ is a minimal counterexample clutter, then any Eulerian clutter $\mathcal{C}'$ with $\tau(\mathcal{C}') < \tau(\mathcal{C})$ packs.

To use the minimality in our proofs for the following results, we often delete a set or contract an element of the ground set. In order for these strategies to be used, we now show that a Eulerian clutter will remain Eulerian if a set is deleted or if an element of the ground set is contracted. First we show that a minor of a binary clutter is always binary.

Proposition 6. *A minor of a binary clutter is a binary clutter.*

Proof. A minor of a clutter is formed by performing contractions and/or deletions. We show that after either operation, the clutter remains binary. Let $\mathcal{C}$ be a binary clutter and take $e \in E(\mathcal{C})$.

First we perform a contraction. The sets of $\mathcal{C}/e$ are now of the form $S - e$ for $S \in \mathcal{C}$. The blocker $b(\mathcal{C}/e)$ is now $b(\mathcal{C}) - \{B: e \in B\}$. If $B' \in b(\mathcal{C}/e)$, then $B' \in b(\mathcal{C})$. Since $\mathcal{C}$ is binary, $|B' \cap S|$ is odd for any $B' \in b(\mathcal{C}/e)$ and $S \in \mathcal{C}$. Any $B' \in b(\mathcal{C}/e)$ does not contain e , so for any $S \in \mathcal{C}$, $B' \cap S = B' \cap (S - e)$. Then for any $S' \in \mathcal{C}/e$ and $B' \in b(\mathcal{C}/e)$, $|B' \cap S'|$ is odd, making $\mathcal{C}/e$ binary.

Now we perform a deletion, $\mathcal{C} \setminus e$. $\mathcal{C} \setminus e = \{S: e \notin S \in \mathcal{C}\}$, so if $S' \in \mathcal{C} \setminus e$, then $S' \in \mathcal{C}$. Since $\mathcal{C}$ is binary, for any $B \in b(\mathcal{C})$ and $S' \in \mathcal{C} \setminus e$, $|B \cap S'| = |(B - e) \cap S'|$ is odd. Since $b(\mathcal{C} \setminus e) = b(\mathcal{C})/e = \{B - e: B \in b(\mathcal{C})\}$, for any $S' \in \mathcal{C} \setminus e$ and $B' \in b(\mathcal{C} \setminus e)$, $|B' \cap S'|$ is odd, making $\mathcal{C} \setminus e$ binary. Therefore, any minor of a binary clutter is binary. $\square$

Proposition 7. *For Eulerian clutter $\mathcal{C}$ and $S \in \mathcal{C}$, the clutter $\mathcal{C} \setminus S$ is Eulerian.*

Proof. Due to the previous proposition, $\mathcal{C} \setminus S$ is binary, so it remains to show that the sets in the blocker all have the same parity. Since $\mathcal{C}$ is Eulerian, $|B \cap S|$ is odd for any $B \in b(\mathcal{C})$ and all $B \in b(\mathcal{C})$ have the same parity. Then $|B - S| = |B| - |B \cap S|$ has the same parity for all B . Since $b(\mathcal{C} \setminus S) = \{B - S: B \in b(\mathcal{C})\}$, this means that all $B' \in b(\mathcal{C} \setminus S)$ have the same parity. Therefore, the clutter is Eulerian. $\square$

Proposition 8. *For Eulerian clutter $\mathcal{C}$ and $e \in E(\mathcal{C})$, the clutter $\mathcal{C}/e$ is Eulerian.*

Proof. Proposition 6 showed that $\mathcal{C}/e$ is binary. So it remains to show that the sets in the blocker have the same parity. Since the blocker $b(\mathcal{C}/e)$ is now $b(\mathcal{C}) - \{B: e \in B\}$, covers are only removed from the blocker and not changed. Therefore, their parity remains the same, making $\mathcal{C}/e$ Eulerian. $\square$

Due to these propositions, if a set of a minimal counterexample clutter is deleted or an element of the ground set is contracted, all that is needed to show that this new clutter packs is that either the minimum size of a cover is the same and the size of the ground set

has decreased or the minimum size of a cover has decreased. With these results to use in our proofs, we move back to mates, focusing on their properties in a minimal counterexample clutter.

Proposition 9. *For a minimal counterexample $\mathcal{C}$, each $S \in \mathcal{C}$ has a mate.*

Proof. Assume for a contradiction that there exists an $S \in \mathcal{C}$ for minimal counterexample $\mathcal{C}$ such that S does not have a mate. Let $\tau(\mathcal{C}) = \tau$. Delete S to form the clutter $\mathcal{C} \setminus S$. Since $b(\mathcal{C} \setminus S) = \{B - S : B \in b(\mathcal{C})\}$ and S has no mate, for any $B' \in b(\mathcal{C} \setminus S)$, $|B'| \geq \tau - 1$. By Proposition 7, $\mathcal{C} \setminus S$ is Eulerian. Since S and every $B \in b(\mathcal{C})$ must intersect, every minimal cover decreases in size when deleting S , so $\tau(\mathcal{C} \setminus S) < \tau$. Then by minimality, $\mathcal{C} \setminus S$ packs. Since $\tau(\mathcal{C} \setminus S) \geq \tau - 1$, there exists a packing of $\mathcal{C} \setminus S$ made of $\tau - 1$ disjoint sets, all of which do not intersect S . Adding S back in forms a set of τ disjoint sets, so $\mathcal{C}$ packs - a contradiction. Therefore, S must have a mate. $\square$

Note that in a minimal counterexample, since the clutter is Eulerian, all mates have the same parity. Also, it is binary, so the intersection between a set and a cover is odd. Therefore, $|B - S| = |B| - |B \cap S| = \tau + 2k - (2l + 1)$ for nonnegative integers k and l . Then $|B - S|$ is τ minus some odd integer. By the definition of a mate, $|B - S| \leq \tau - 2$. Then for any mate B of S , $|B - S| \leq \tau(\mathcal{C}) - 3$.

Next, we consider what happens when a clutter almost packs. Say there is just one element of the ground set that keeps the clutter from packing.

Definition 15. *For $e \in E(\mathcal{C})$, an e -near-packing of $\mathcal{C}$ is a collection of sets $S_1, \dots, S_\tau \in \mathcal{C}$ where $\tau = \tau(\mathcal{C})$ and*

1. $|\{S_i : e \in S_i, i \in [\tau]\}| > 1$ and
2. for any $f \in E(\mathcal{C}) - \{e\}$, $|\{S_i : f \in S_i, i \in [\tau]\}| \leq 1$.

so that there does not exist a collection of sets $S'_1, \dots, S'_\tau \in \mathcal{C}$ that satisfies the properties above and $|\{S'_i : e \in S'_i, i \in [\tau]\}| < |\{S_i : e \in S_i, i \in [\tau]\}|$.

So an e -near-packing behaves like a packing except at the element e , where multiple sets can intersect. By the definition, an e -near-packing has the smallest possible amount of overlap at e . Generally, this overlap must have size at least 2. For example, consider the clutter $\mathcal{C}$ where $E(\mathcal{C}) = V(C_5)$ and $\{S : S \in \mathcal{C}\} = E(C_5)$. The C_5 is shown in Figure 4.1 with a v_1 -near-packing consisting of the edges v_1v_2, v_3v_4, v_1v_5 . There is only overlap at v_1 ,

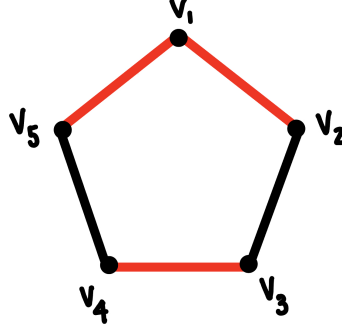


Figure 4.1: Clutter with $E(\mathcal{C}) = V(C_5)$ and $\{S : S \in \mathcal{C}\} = E(C_5)$ and a v_1 -near-packing that has overlap of size 2

which is contained by both v_1v_2 and v_1v_5 . So generally in clutters, there are cases where it is possible to have an overlap of 2 in a near-packing. We see what occurs for Eulerian clutters in the following propositions.

Proposition 10. *Let $\mathcal{C}$ be a minimal counterexample clutter and let $e \in E(\mathcal{C})$ where there exists a $B' \in b(\mathcal{C})$ such that $e \notin B'$ and $|B'| = \tau(\mathcal{C})$. If the sets $S_1, \dots, S_\tau$ form an e -near-packing then for any set S_i for $i \in [\tau]$ and a mate B of S_i ,*

- (1) $e \in B$
- (2) $B \subseteq S_1 \cup \dots \cup S_\tau$
- (3) $|B \cap S_i| \geq 3$
- (4) $|B \cap S_j| = 1$ for $j \in [\tau], j \neq i$.

Proof. Let $\tau = \tau(\mathcal{C})$. Since B is in $b(\mathcal{C})$, $|B| \geq \tau$. Since B is a mate of S_i ,

$$\tau - |B \cap S_i| \leq |B| - |B \cap S_i| \leq |B - S_i| \leq \tau - 3,$$

giving that $|B \cap S_i| \geq 3$. This proves (3).

Let $|\{S_i : e \in S_i, i \in [\tau]\}| = p$. By definition of a near-packing, there does not exist an e -near-packing with overlap less than p at e . We “replicate” e k times, for an even number k , so that the number of sets that contain e is 2 if p is even and 3 if p is odd. This forms a new clutter $\mathcal{C}'$. To do this replication, we add k new elements $e_1, \dots, e_k$ to the ground set $E(\mathcal{C})$ so that $E(\mathcal{C}') = E(\mathcal{C}) \cup \{e_1, \dots, e_k\}$. For any set of $\mathcal{C}$ that contains e , make another

k sets so that each contains a different copy of e in place of e . This means that $b(\mathcal{C}')$ is made up of all sets in $b(\mathcal{C})$ that do not contain e as well as $B' = B \cup \{e_1, \dots, e_k\}$ for any $B \in b(\mathcal{C})$ that contains e . Since k is even, an even amount of elements are added to each set of the blocker (either 0 or k). Therefore, all sets of the blocker have the same parity. Furthermore, $|S \cap B|$ is odd for any $S \in \mathcal{C}'$ and $B \in b(\mathcal{C}')$. Then $\mathcal{C}'$ is Eulerian.

Claim 3. $\tau(\mathcal{C}') = \tau(\mathcal{C})$.

Proof. By replicating e , none of the sets in the blocker decreased in size, so $\tau(\mathcal{C}') \geq \tau(\mathcal{C})$. For a contradiction we assume that $\tau(\mathcal{C}') > \tau(\mathcal{C})$. This means that all minimum sets in $b(\mathcal{C})$ increased in size. Only minimal covers that contained e changed, so every minimum set in $b(\mathcal{C})$ must contain e . However, we assumed there exists a $B' \in b(\mathcal{C})$ such that $e \notin B'$ and $|B'| = \tau(\mathcal{C})$. This gives a contradiction, so $\tau(\mathcal{C}') = \tau(\mathcal{C})$. $\diamond$

By adding these k new elements, there now exists an e -near-packing of $\mathcal{C}'$, $S'_1, \dots, S'_\tau$, where only two or three of these sets contain e .

Claim 4. Let S' be one of these sets of the e -near-packing of $\mathcal{C}$. Then there exists a $B \in b(\mathcal{C}')$ such that $|B - S'| < \tau - 1$.

Proof. Assume for a contradiction that for any $B \in b(\mathcal{C}')$, $|B - S'| \geq \tau - 1$. Delete S' to make $\mathcal{C}' \setminus S'$. Then $\tau(\mathcal{C}' \setminus S') \geq \tau - 1$. Since S' and every $B \in b(\mathcal{C}')$ must intersect, every minimal cover decreases in size when S' is deleted, so $\tau(\mathcal{C}' \setminus S') < \tau$. By the definition of a minimal counterexample, $\mathcal{C}' \setminus S'$ packs. Then there is a packing of $\tau - 1$ disjoint sets. Add S' back in to get τ disjoint sets. Then e and each of the k new elements are each in different sets. Then in $\mathcal{C}$ there is an e -near-packing with overlap of $k + 1$ at e . This gives a contradiction because $p > k + 1$ is the smallest possible overlap of an e -near-packing of $\mathcal{C}$. Therefore, there exists a $B \in b(\mathcal{C}')$ such that $|B - S'| < \tau - 1$. $\diamond$

So in $\mathcal{C}'$, each set S' of the e -near-packing has a mate B , meaning that $\mathcal{C}'$ does not pack.

Claim 5. For mate B of S'_i in the e -near-packing, $S'_1, \dots, S'_\tau$, of $\mathcal{C}'$, $B \subseteq S'_1 \cup \dots \cup S'_\tau$ for $\tau = \tau(\mathcal{C}')$.

Proof. Since B is a mate of S'_i , $|B - S'_i| \leq \tau - 2$. If $p - k = 2$, then $\mathcal{C} \setminus S'_i$ is made up of $\tau - 2$ disjoint sets. Each must have a nonempty intersection with $B - S'_i$, so there do not exist any elements in $E(\mathcal{C}') - (S'_1 \cup \dots \cup S'_\tau)$ that are in B . Therefore, $B \subseteq S'_1 \cup \dots \cup S'_\tau$. Now let's consider the case where $p - k = 3$. Since $\mathcal{C}'$ is Eulerian, $|B - S'_i| \leq \tau - 3$. Then

$\mathcal{C} \setminus S'_i$ is made up of $\tau - 3$ disjoint sets. So again we have that there are no elements in B that are outside of $S'_1 \cup \dots \cup S'_\tau$. So we have shown that $B \subseteq S'_1 \cup \dots \cup S'_\tau$. $\diamond$

Due to this claim, $B \subseteq S_1 \cup \dots \cup S_\tau$ since the only change is the copies of e . Therefore, (2) holds.

Claim 6. *Any mate B of a set S' in the e -near-packing of $\mathcal{C}'$ contains e .*

Proof. Assume that there is a mate B of S' where $e \notin B$. Contract e to make $\mathcal{C}'/e$. Then there is a packing of τ disjoint sets, one of which is $S' - \{e\}$. Deleting S' from $\mathcal{C}'/e$ gives a packing of $\tau - 1$ disjoint sets. So any minimal cover has size at least $\tau - 1$. $B \cap S' = B \cap (S' - \{e\})$, so $|B - (S' - \{e\})| \leq \tau - 3$. Then there is a minimal cover after deleting S' with size less than $\tau - 1$, giving a contradiction. So all mates of sets in the e -near-packing contain e . $\diamond$

If a mate of S' in the e -near-packing of $\mathcal{C}'$ contains e , then a mate of S in the e -near-packing of $\mathcal{C}$ contains e , proving (1).

We want to show that for any S'_i in the e -near-packing of $\mathcal{C}'$ and a mate B of S'_i , $|B \cap S'_j| = 1$ for $j \in [\tau]$ and $j \neq i$. There are $p - k$ sets containing e , where $p - k$ is either 2 or 3. We look at two cases for S'_i : when $i \in [p - k]$ and when $i > p - k$.

First, say $i \in [p - k]$, so $e \in S'_i$. Let B be a mate of S'_i . Delete S'_i to make $\mathcal{C}' \setminus S'_i$. There are $\tau - (p - k)$ disjoint sets $S'_{p-k+1}, \dots, S'_\tau$. $B - S'_i$ is a cover of $\mathcal{C}' \setminus S'_i$ so it must intersect each of these $\tau - (p - k)$ disjoint sets at least once. Since $|B - S'_i| \leq \tau - 3 \leq \tau - (p - k)$, B intersects each S'_j for $j > p - k$ at most once. Due to the constraint on its size, $B - S'_i$ does not intersect S'_j for $j \in [p - k] - \{i\}$. Therefore, $B \cap S_j = \{e\}$ for $j \in [p - k] - \{i\}$. Then for any $j \in [\tau]$, $j \neq i$, $|B \cap S_j| = 1$.

Now for the second case, let $i > p - k$, so $e \notin S'_i$. Let B be a mate of S'_i . Delete S'_i to make $\mathcal{C}' \setminus S'_i$. There are $\tau - (p - k + 1)$ disjoint subsets, $S'_{p-k+1}, \dots, S'_{i-1}, S'_{i+1}, \dots, S'_\tau$. Since $B - S'_i$ is a cover of $\mathcal{C}' \setminus S'_i$, it must intersect each of these $\tau - (p - k + 1)$ disjoint sets at least once. Because B is a mate, $|B - S'_i| \leq \tau - 3 \leq \tau - (p - k)$. Since $e \notin S'_i$, $|(B - S'_i) - \{e\}| \leq \tau - (p - k + 1)$. Therefore, B intersects each S'_j for $j > p - k$, $j \neq i$ at most once. Due to the constraint on the size of B , $B \cap S_j = \{e\}$ for $j \in [p - k]$. Then for any $j \in [\tau]$, $j \neq i$, $|B \cap S_j| = 1$. This proves (4). $\square$

Due to this proposition, there cannot exist a $B \in b(\mathcal{C})$ where it is a mate of two sets that appear in the same near-packing. So given a near-packing, each set has a distinct mate. Note that we assumed that for the edge e , there is a minimum set of the blocker that does

not contain e . We see that since we will be applying these results to grafts, this assumption will hold for all e . Earlier in the thesis, we showed that in a minimal counterexample graft, each vertex has degree τ and T is all the vertices. Then a cut formed by just one vertex is a minimum T -cut. Since we have more than two vertices, for any edge e , the cut formed by a vertex that is not an endpoint of v is a minimum T -cut that does not contain e . Therefore, the assumption will always be satisfied for every edge of a minimal counterexample graft. Next, we show that the overlap for an e -near-packing must in fact be odd. To prove this, we use the following remark that follows from the definition of a binary clutter:

Remark 3. *Let $\mathcal{C}$ be a binary clutter. For $S_1, \dots, S_k \in \mathcal{C}$ and $B \in b(\mathcal{C})$,*

$$|(S_1 \triangle S_2 \triangle \dots \triangle S_k) \cap B|$$

is odd if and only if k is odd.

Proposition 11. *Let $\mathcal{C}$ be a minimal counterexample clutter and let $S_1, \dots, S_\tau$ be an e -near-packing for $\tau = \tau(\mathcal{C})$. Then $|\{S_i : e \in S_i, i \in [\tau]\}|$ is odd.*

Proof. Let B_i be a mate of S_i in the e -near-packing where $B_i \subseteq S_1 \cup \dots \cup S_\tau$. Let $W = S_1 \triangle \dots \triangle S_\tau$. The previous remark gives that $|B_i \cap W|$ has the same parity as τ . Since the clutter is Eulerian, $|B_i|$ and τ have the same parity. Therefore, $|B_i \cap W|$ and $|B_i|$ have the same parity. If W contains e , then an odd number of sets in $S_1 \triangle \dots \triangle S_\tau$ contain e . So it remains to show that $e \in W$. For a contradiction, assume that $e \notin W$. By Proposition 10, $e \in B_i$. Since $B_i \subseteq S_1 \cup \dots \cup S_\tau = W \cup \{e\}$, then $|B_i \cap W| = |B_i - \{e\}|$. This means that $|B_i \cap W|$ and $|B_i|$ have distinct parities - a contradiction. Therefore, $|\{S_i : e \in S_i, i \in [\tau]\}|$ is odd. $\square$

Note that this proof did not use the assumption that the near-packing had the smallest overlap possible, so the result holds for any near-packing of a minimal counterexample clutter. Some of the following results will also apply to any near-packing if they do not use the assumption of it having the smallest overlap. The following proposition shows where a near-packing can appear in a minimal counterexample.

Proposition 12. *Let $\mathcal{C}$ be a minimal counterexample clutter. For any $e \in E(\mathcal{C})$ where there exists a $B \in b(\mathcal{C})$ such that $e \notin B$ and $|B| = \tau(\mathcal{C})$, there exists an e -near-packing.*

Proof. For $e \in E(\mathcal{C})$, we contract e . So e is removed from the ground set and from any S that contained it. We use the idea that a contraction in a clutter affects the blocker in the same way as a deletion in the blocker [5]. Namely, $b(\mathcal{C}/e) = b(\mathcal{C}) \setminus e$. Therefore, after the

contraction of e in $\mathcal{C}$, the blocker behaves in the same way as if we deleted e from $b(\mathcal{C})$. Therefore, some of the covers of $\mathcal{C}$ are removed but none have decreased in size. Therefore, $\tau(\mathcal{C}/e) \geq \tau(\mathcal{C})$. In fact, $\tau(\mathcal{C}/e) = \tau(\mathcal{C})$ because we assumed that there exists a minimum cover that does not contain e . Since $|E(\mathcal{C}/e)| < |E(\mathcal{C})|$, by the minimality of $\mathcal{C}$, $\mathcal{C}/e$ packs. So there exists a packing $S'_1, \dots, S'_\tau$ of $\mathcal{C}/e$. Let $S_1, \dots, S_\tau$ be sets where S_i contains S'_i and adds e if it originally had it. Then $S_1, \dots, S_\tau$ is an e -near-packing of $\mathcal{C}$. $\square$

In Proposition 10, we found that any mate B of a set in an e -near-packing contains e when there is a minimum cover that does not contain e . The proof did not use the assumption that there is a minimum cover that does not contain e . So for a minimal counterexample clutter $\mathcal{C}$ and any $e \in E(\mathcal{C})$, a mate of any set in an e -near-packing contains e . Using this result, we can bound the number of elements of the ground set where near-packings at each element all contain a given set.

Proposition 13. *Let $\mathcal{C}$ be a minimal counterexample clutter and let $L \subseteq E(\mathcal{C})$. Fix an $S \in \mathcal{C}$. If, for any $e \in L$, S is in an e -near-packing, then $|L| < \tau(\mathcal{C}) + |L \cap S| - 2$.*

Proof. Let B be a mate of S . We have assumed that for any element $e \in L$, S is in an e -near-packing. By the previous proposition, B contains all elements of L , so $L \subseteq B$. Therefore, $|B| \geq |L|$ and $|B - S| \geq |L - S|$. Since B is a mate of S , $|B - S| \leq \tau(\mathcal{C}) - 3$. Note that $|L - S| = |L| - |L \cap S|$. Then

$$|L| = |L - S| + |L \cap S| \leq |B - S| + |L \cap S| \leq \tau(\mathcal{C}) - 3 + |L \cap S|.$$

Therefore,

$$|L| < \tau(\mathcal{C}) + |L \cap S| - 2.$$

$\square$

We find a similar result for multiple sets of the clutter.

Corollary 1. *Let $\mathcal{C}$ be a minimal counterexample clutter and let $L \subseteq E(\mathcal{C})$. Let $S_1, S_2, \dots, S_m \in \mathcal{C}$. If, for any $e \in L$, S_i is in an e -near-packing for all $i \in [m]$, then*

$$|L| < \tau(\mathcal{C}) - 2 + \min_{i \in [m]} \{|L \cap S_i|\}.$$

Proof. By the previous proposition, $|L| < \tau(\mathcal{C}) + |L \cap S_i| - 2$ for each $i \in [m]$. Therefore,

$$|L| < \min_{i \in [m]} \{\tau(\mathcal{C}) + |L \cap S_i| - 2\}.$$

The only difference for each is the $|L \cap S_i|$ term. So this can be rewritten as

$$|L| < \tau(\mathcal{C}) - 2 + \min_{i \in [m]} \{|L \cap S_i|\}.$$

□

In the previous propositions, we assumed that a given set S of the clutter is contained in a near-packing. We now characterize which sets are contained in near-packings.

Definition 16. For clutter $\mathcal{C}$, **core**($\mathcal{C}$) is the set

$$\{S \in \mathcal{C} : S \text{ appears in some } e\text{-near-packing for } e \in E(\mathcal{C})\}.$$

Proposition 14. Let $\mathcal{C}$ be a minimal counterexample clutter where for every $e \in E(\mathcal{C})$, there exists a $B \in b(\mathcal{C})$ such that $e \notin B$ and $|B| = \tau(\mathcal{C})$. Let $S \in \mathcal{C}$. The following are equivalent:

1. $S \in \text{core}(\mathcal{C}, w)$
2. $\bigcap \{B : \text{mates of } S\} \neq \emptyset$.

Proof. First, we show that 1 implies 2. This is equivalent to showing that if $\bigcap \{B : \text{mates of } S\}$ is empty, then S is not in $\text{core}(\mathcal{C})$. We found that for an S in an e -near-packing, all mates of S must contain e . If the mates of S have no intersection, then S cannot be in an e -near-packing since there is no e that all mates of S contain.

Now we show that 2 implies 1. If 2 holds, then there exists an element e such that it appears in all mates of S . Contract e . As in previous proofs, the elements of $b(\mathcal{C}/e)$ do not decrease in size. So the smallest cover of $\mathcal{C}/e$ has size at least τ . Since there exists a minimum cover of $\mathcal{C}$ that does not contain e , $\tau(\mathcal{C}/e) = \tau$. Note that the mates of S all contain e so they are not minimal covers of $\mathcal{C}/e$. By minimality, $\mathcal{C}/e$ packs. Since the mates of S are not minimal covers of $\mathcal{C}/e$, then there does not exist a minimal cover B of $\mathcal{C}/e$ where $|B - S'| < \tau - 1$ for $S' = S - \{e\}$. Therefore, S' is in a maximum packing of $\mathcal{C}/e$. Add e back in so that it is returned to its original sets. This creates an e -near-packing that contains S . Therefore, $S \in \text{core}(\mathcal{C})$. □

Any set in the core of a clutter must have a mate. Considering some overlap, we want to lower bound the number of mates of sets that are in the core. We found in Proposition 10 that in a near-packing there is a distinct mate of each set of the near-packing. This result leads to the following proposition.

Proposition 15. *Let $\mathcal{C}$ be a minimal counterexample clutter where for every $e \in E(\mathcal{C})$, there exists a $B \in b(\mathcal{C})$ such that $e \notin B$ and $|B| = \tau(\mathcal{C})$. Then there are at least $\frac{|E(\mathcal{C})|\tau}{\lambda}$ distinct mates of sets in $\text{core}(\mathcal{C})$ where λ is the maximum size of a mate of a set in the core.*

Proof. Let $e \in E(\mathcal{C})$. Let $S_1, \dots, S_\tau$ be an e -near-packing. Each S_i has a distinct mate, so there are at least τ mates for this near-packing. Since there is a near-packing at every element of the ground set, there are at least $|E(\mathcal{C})|\tau$ mates - but this includes overcounting. To account for this, let λ be the maximum size of a mate of a set in the core. If B is a mate of a set S that appears in an e -near-packing, then B must contain e . Then mate B of S is counted for at most $|B|$ near-packings. So there are at least $\frac{|E(\mathcal{C})|\tau}{\lambda}$ mates of sets in the core. $\square$

We assumed that for any edge e , there is a minimum set of the blocker that does not contain e . As mentioned earlier, this assumption holds for a minimal counterexample graft. Later in this chapter we bound λ . Note that while we specifically lower bounded the number of mates in the core of a clutter, this is also a lower bound on the number of mates in the clutter. Next, we will consider these results when a clutter is paired with a set of weights.

4.1 Replication

Up to this point we have thought of our graph as unweighted with parallel edges. We can instead think of it as a simple weighted graph. For clutters, this can be represented by a pairing of a clutter $\mathcal{C}$ and set of weights $w \in \mathbb{Z}_+^{E(\mathcal{C})}$ so that each element in $E(\mathcal{C})$ is assigned a positive integer weight. We call this the **weighted clutter**. The weight of $e \in E(\mathcal{C})$ is denoted as $w(e)$. For the weighted clutter, $\tau(\mathcal{C}, w)$ represents the minimum sum of the weights of a cover. A set of the weighted clutter consists of elements each with weight one, so an element can only appear once in a set. $\nu(\mathcal{C}, w)$ is again the maximum size of a packing of sets where any element $e \in E(\mathcal{C})$ is in at most $w(e)$ sets of the packing. These definitions are more complicated, so instead of redefining all the previous definitions and propositions for this pairing, we use a given $\mathcal{C}$ and w to define a **replication clutter**. The results of basic clutters will apply to the replication clutter.

In order to define the replication clutter, we first describe how to replicate a single element. Consider a clutter $\mathcal{C}$ and $e \in E(\mathcal{C})$. Say the clutter $\mathcal{D}$ is obtained from $\mathcal{C}$ by *replicating* e . Then

$$E(\mathcal{D}) = E(\mathcal{C}) \cup \{e'\}$$

$$\mathcal{D} = \mathcal{C} \cup \{S \cup \{e'\} - \{e\} : e \in S \in \mathcal{C}\}.$$

So a copy of e , labeled as e' , is added to the ground set, and for any set S that contains e , another set is made that contains e' instead of e .

The blocker of $\mathcal{D}$ is defined so that any cover that contained e now also contains e' :

$$b(\mathcal{D}) = \{B : B \in b(\mathcal{C}), e \notin B\} \cup \{B \cup \{e'\} : e \in B \in b(\mathcal{C})\}.$$

Consider the weighted clutter $(\mathcal{C}, w)$ where all weights are 1 except for at $e \in E(\mathcal{C})$ where $w(e) = 2$. If a minimum cover contains e , its weight for $(\mathcal{C}, w)$ will be one more than in $\mathcal{C}$. The same would occur if we instead think of the clutter that replicates e . We see this in the following remark:

Remark 4. Let $\mathcal{D}$ be the clutter obtained from $\mathcal{C}$ by replicating $e \in E(\mathcal{C})$. Define $w \in \mathbb{Z}_+^{E(\mathcal{C})}$ so that for $f \in E(\mathcal{C})$

$$w(f) = \begin{cases} 2 & \text{if } f = e \\ 1 & \text{otherwise} \end{cases}.$$

Then $\tau(\mathcal{C}, w) = \tau(\mathcal{D})$ and $\nu(\mathcal{C}, w) = \nu(\mathcal{D})$.

We use the idea of replicating an element to formally define the replication clutter.

Definition 17. For clutter $\mathcal{C}$ and $w \in \mathbb{Z}_+^{E(\mathcal{C})}$, the **replication clutter** $\mathcal{C}^w$ is formed as follows. For every $e \in E(\mathcal{C})$,

- if $w(e) = 0$, delete e
- if $w(e) \geq 2$, replicate e $w(e) - 1$ times.

While the weighted clutter can be viewed through graphs as a weighted simple graph, the replication clutter corresponds to a graph that has parallel edges instead of weighted edges. $\tau(\mathcal{C}^w)$ is the size of a minimum cover of $\mathcal{C}^w$. This will be the same as $\tau(\mathcal{C}, w)$ which is the minimum sum of weights of a cover. Since any cover of $\mathcal{C}^w$ is made up of a cover of $\mathcal{C}$ and the copies of all its elements,

$$\tau(\mathcal{C}, w) = \tau(\mathcal{C}^w).$$

$\nu(\mathcal{C}^w)$ is the maximum size of a set of disjoint sets in $\mathcal{C}^w$. In the weighted clutter $(\mathcal{C}, w)$, a set of disjoint sets can include $e \in \mathcal{C}$ as many as $w(e)$ times. Therefore,

$$\nu(\mathcal{C}, w) = \nu(\mathcal{C}^w).$$

Since $\tau(\mathcal{C}, w) = \tau(\mathcal{C}^w)$ and $\nu(\mathcal{C}, w) = \nu(\mathcal{C}^w)$, if $\mathcal{C}^w$ packs, then $(\mathcal{C}, w)$ packs. Then it is preferable to consider $\mathcal{C}^w$ as it is just a clutter with no weights attached to the elements, so the definitions and propositions about near-packings do not have to be redefined. A minimal counterexample clutter is defined in the same way and there is a near-packing at every $e \in \mathcal{C}^w$ where there is a minimum cover that does not contain e .

Note that for a cover $B \in b(\mathcal{C})$, $w(B)$ in the weighted clutter is the same as $|B'|$ of a cover $B' \in b(\mathcal{C}^w)$ corresponding to B since B' contains all copies of elements of B . A mate of $S' \in \mathcal{C}^w$ is a $B' \in b(\mathcal{C}^w)$ with $|B' - S'| \leq \tau(\mathcal{C}^w) - 3$. As before, in a minimal counterexample $\mathcal{C}^w$, any mate of a set in an e -near-packing of $\mathcal{C}^w$ must contain e and its copies. For S in minimal counterexample $\mathcal{C}^w$ and set $L \subseteq E(\mathcal{C}^w)$ where S is in an e -near-packing for every $e \in L$, $|L| < \tau(\mathcal{C}^w) + |L \cap S| - 2$. In the weighted clutter, this is equivalent to $w(L) < \tau(\mathcal{C}, w) + |L \cap S| - 2$ for $L \subseteq E(\mathcal{C})$ and $S \in \mathcal{C}$.

Due to the equivalence between the weighted clutter and the replication clutter, we can move back and forth between them, depending on what is more useful for our arguments. To think of the size of mates, we use the weighted clutter. The definition of a mate can be rewritten so that for a mate B of $S \in \mathcal{C}$ in $(\mathcal{C}, w)$, $w(B) - |B \cap S| \leq \tau - 3$. Since $|B| \geq |B \cap S|$, a useful property of a mate is that

$$w(B) - |B| \leq \tau - 3.$$

This can assist with upper-bounding the size of mates in a clutter. When elements of the ground set have weight 1, not much is clear, but when all elements have weight at least 2, then for any mate B , $w(B) \geq 2|B|$. So

$$\tau - 3 \geq w(B) - |B| \geq 2|B| - |B| = |B|,$$

making $|B| \leq \tau - 3$.

Earlier we showed that for a minimal counterexample clutter $\mathcal{C}$ where for any edge e , there exists a minimum set $B \in b(\mathcal{C})$ where $e \notin B$, the number of mates in the clutter is at least $\frac{|E(\mathcal{C})|\tau}{\lambda}$, where λ is the maximum size of a mate of a set in the core of the clutter. So $\lambda \leq \max\{|B| : B \text{ is a mate}\}$. Therefore, the clutter has at least $\frac{|E(\mathcal{C})|\tau}{\max\{|B| : B \text{ is a mate}\}}$ mates. So when the clutter has all weights at least 2, the number of mates is at least $\frac{|E(\mathcal{C})|\tau}{\tau-3}$. Generalizing, we find that when all weights are at least k , then

$$\tau - 3 \geq w(B) - |B| \geq k|B| - |B| = (k - 1)|B|$$

so $|B| \leq \frac{\tau-3}{k-1}$. Then $\lambda \leq \frac{\tau-3}{k-1}$. Plugging into $\frac{|E(\mathcal{C})|\tau}{\lambda}$ gives the following remark:

Remark 5. *Let $\mathcal{C}$ be a minimal counterexample clutter where for any edge e , there exists a minimum set $B \in b(\mathcal{C})$ where $e \notin B$. If $w(e) \geq k$ for any $e \in E(\mathcal{C})$, there are at least $|E(\mathcal{C})|^{\frac{(k-1)\tau}{\tau-3}}$ distinct mates.*

Note that k is the minimum weight and cannot be the average weight of an element in the clutter since we do not know what occurs within the mates. It is possible that they only use elements of small weight. The result with the minimum weight shows that, when the weights are all greater than 1, the number of mates increases with $|E(\mathcal{C})|$. If k is an increasing function of τ , then the number of mates increases with τ . The question still remains of whether this occurs in the general case.

Question 1. *Can it be shown that the number of distinct mates of a minimal counterexample clutter is at least $|E(\mathcal{C})|f(\tau)$ where f is a strictly increasing monotone function?*

This function would be useful, as perhaps it can be shown that there are too many mates in a minimal counterexample when τ is large. This concludes our set theoretic arguments, and in the next chapter, we return to our discussion of the T -join Packing Conjecture.

Chapter 5

Minimal Counterexample Weighted Graft

In this chapter, we return to thinking of our problem through grafts. Similarly to the previous chapter, our graft can be thought of as (G^w, T) , the clutter of T -joins C^w where it is a multigraph with no weights. We can also think of it as a weighted graft (G, T, w) where G is a simple graph with corresponding weights w , so the weighted clutter of T -joins (C, w) . In this chapter, we use the weighted graft.

As with the clutters, for $e \in E(G)$, $w(e)$ is the weight of edge e , and for $A \subseteq E(G)$, $w(A) = \sum_{e \in A} w(e)$. Since the graft is now simple, $|e|$ for any edge is 1. Now we refer to the degree of vertex v as $|\delta(v)| = \sum_{e \in \delta(v)} |e|$. If we want to use the sum of the weights around v , we write $w(\delta(v)) = \sum_{e \in \delta(v)} w(e)$. Recall that we are working with minimal T -joins, so there are no parallel edges. Therefore, there is only a weight of one for each edge in a T -join.

Similarly to clutters, $\tau(G, T, w) = \tau(G^w, T)$ and $\nu(G, T, w) = \nu(G^w, T)$. Therefore, (G, T, w) packs when (G^w, T) packs. Let $\tau = \tau(G, T, w) = \tau(G^w, T)$. So when (G, T, w) packs there is a set of τ minimal T -joins where for any edge $e \in E(G)$, it is in $w(e)$ T -joins. We now define a **minimal counterexample weighted graft** (G, T, w) . Note that it should be the same as for (G^w, T) . We want to minimize the number of edges in G^w , so we want to minimize $\sum_{e \in E(G)} w(e)$ in G . Let τ be the smallest value of $\tau(G, T, w)$ where the T -join Packing Conjecture does not hold. A minimal counterexample to the T -join Packing Conjecture is a Eulerian planar weighted graft with $\tau(G, T, w) = \tau$ that does not pack and minimizes $\sum_{e \in E(G)} w(e)$.

5.1 Tight Near-Packings

Earlier we showed that for a Eulerian clutter that minimally does not pack, an e -near-packing must have odd overlap. Therefore, there must be at least 3 sets containing e . Since our minimal counterexample is a Eulerian graft, this still holds. Consider a near-packing where the overlap is exactly 3. We call this way of getting as close as possible to a packing a tight near-packing:

Definition 18. For graft (G^w, T) and $e \in E(G^w)$, an e -near-packing, $S_1, \dots, S_\tau$, is **tight** if $|\{i: e \in S_i, i \in [\tau]\}| = 3$.

Generally, for a mate B of T -join S of (G^w, T) in an e -near-packing, $|B - S|$ can be very small if many T -joins of the e -near-packing contain e . However, for a tight e -near-packing, exactly 3 T -joins contain e . Because $B - S$ must intersect each T -join of the near-packing other than S , we have the following remark:

Remark 6. In a tight e -near-packing of (G^w, T) for $e \in E(G^w)$, for a mate B of T -join S in the near-packing, $|B - S| = \tau - 3$.

Now we redefine for weighted graft (G, T, w) . A tight near-packing of (G^w, T) has two extra T -joins containing e , so in (G, T, w) , the weight covering e is $w(e) + 2$.

Definition 19. For weighted graft (G, T, w) and $e \in E(G)$, an e -near-packing $S_1, \dots, S_\tau$ is **tight** if $|\{i: e \in S_i, i \in [\tau]\}| = w(e) + 2$.

We do not know if in any minimal counterexample weighted graft there is a tight near-packing, but we find that after choosing a vertex, there will always exist a minimal counterexample that has a tight near-packing at an edge incident to that vertex. When τ is fixed, there is a set of grafts (G, T, w) that are all minimal counterexamples where they are Eulerian planar weighted grafts that have $\tau(G, T, w) = \tau$, do not pack, and the total weight, $w(E(G)) = \sum_{e \in E(G)} w(e)$, is minimized. Since the total weight is minimized, then all the options for minimal counterexample have the same total weight.

In (G^w, T) , $|E(G^w)| = \frac{1}{2} \sum_{v \in v(G^w)} \deg(v)$, which can be rewritten for (G, T, w) as $w(E(G)) = \frac{1}{2} \sum_{v \in V(G)} w(\delta(v))$. Since $w(\delta(v)) = \tau$ for any vertex in these grafts, $w(E(G)) = \frac{1}{2}|V(G)|\tau$, so

$$|V(G)| = \frac{2w(E(G))}{\tau}.$$

All minimal counterexample grafts have the same total weight, so we have the remark:

Remark 7. For given τ , all minimal counterexample weighted grafts have the same number of vertices.

So you can think of a base set of vertices, $V(G)$, that all minimal counterexamples use where each one has a different assignment of weights, w .

Proposition 16. Consider a vertex v in the base $V(G)$ for all possible minimal counterexample weighted grafts (G, T, w) for given τ . There exists a minimal counterexample weighted graft where there is a tight near-packing at an edge incident to v .

Proof. Consider all the possible weight assignments w for a minimal counterexample. Each weight assignment w has a weight multiplicity set of the edges incident to v , $\{m_0, \dots, m_{\tau-1}\}$, where $m_i = |\{e \in \delta(v) : w(e) = i\}|$. We define a partial ordering of the weight assignments so that w' is **smaller** than w if for w and w' with weight multiplicity sets around v $\{m_0, \dots, m_{\tau-1}\}$ and $\{m'_0, \dots, m'_{\tau-1}\}$, respectively, there exists an $i \in [\tau-1]$ such that $m'_i > m_i$ while $m'_j = m_j$ for all $j < i$. Therefore, if a weight set changes so that only the minimum weight edge around v decreases in weight, it becomes smaller.

Let (G, T, w) be a minimal counterexample weighted graft that is smallest according to this partial ordering. Choose $x \in N(v)$ so that xv is an edge incident to v with minimum weight. The goal is to decrease the weight of xv while creating another minimal counterexample weight graft. This means it must hold that

- $w(\delta(a)) = \tau$ for any vertex a
- the total weight of the graft remains the same
- the parity of the T -cuts remains the same.

Choose another two edges $yv, zv \in \delta(v)$. Define w' so that

$$w'(xv) = w(xv) - 1,$$

$$w'(yv) = w(yv) - 1,$$

$$w'(xy) = w(xy) + 1,$$

and

$$w'(zv) = w(zv) + 2,$$

and $w'(e) = w(e)$ for all $e \in E(G) \setminus \{xv, yv, xy, zv\}$.

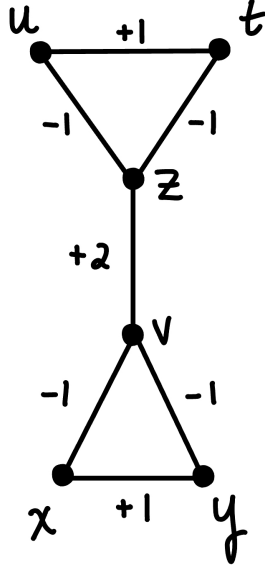


Figure 5.1: Changes in weights to make w' to create a tight near-packing

Note that $w'(\delta(z)) = \tau + 2$ and $w(E(G)) \neq w(E(G'))$. To fix this, we choose two edges $zt, zu \in \delta(z) \setminus \{zv\}$ and make changes to weights so that

$$w'(zt) = w(zt) - 1,$$

$$w'(zu) = w(zu) - 1,$$

and

$$w'(tu) = w(tu) + 1.$$

We see all the changes in Figure 5.1. Now the sum of weights around any vertex is τ , keeping the size of the trivial T -cut the same, and the total sum of weights is the same. Note that it is possible that $w(xy)$ and $w(tu)$ are 0, so this operation could be creating edges xy, tu . The size of every T -cut increases by 2, decreases by 2, or remains the same, so the parity is maintained. All three properties of a minimal counterexample are satisfied, so it remains to show that this created a smaller multiplicity set. The minimum edge weight around v is decreased by one and the only other changes around v are one weight decreased by one and one increased by two. Therefore, the weight multiplicity set around v has $m'_i > m_i$ for $i = w'(xv)$ and $m'_j = m_j$ for all $j < i$, so w' is smaller than w . We chose the weight assignment w to be the smallest from all minimal counterexample options with respect to the weight multiplicity set around v . Then the weighted graft (G, T, w') packs.

Let $S_1, \dots, S_i, \dots, S_j, \dots, S_\tau$ be the packing of T -joins of (G, T, w') . We create a tight near-packing of (G, T, w) at zv , made up of $S_1, \dots, S'_i, \dots, S'_j, \dots, S_\tau$. Let T -join S_i contain edge xy and let S_j contain tu . Since xv and yv are not in S_i , in our near-packing, we let

$$S'_i = (S_i \setminus \{xy\}) \cup \{xv, yv\}.$$

Similarly, since $zt, zu \notin S_j$, let

$$S'_j = (S_j \setminus \{tu\}) \cup \{zt, zu\}.$$

S'_i is a T -join because the parity of the number of times it hits each vertex does not change. The edges of S'_i have x as an endpoint the same number of times as in S_i . The same is true for y . At v there are two more edges, but that does not change the parity. Similarly, S'_j is a T -join. Therefore, $S_1, \dots, S'_i, \dots, S'_j, \dots, S_\tau$ is a set of T -joins where each edge e is contained in $w(e)$ T -joins except for zv . We did not change anything at zv , so it is contained in $w(zv) + 2$ T -joins. So this is a tight zv -near-packing of (G, T, w) . $\square$

A natural question that follows is whether this can be done when a set of vertices is chosen instead of just one.

Proposition 17. *Let $V(G)$ be the base set of vertices for all possible minimal counterexample weighted grafts (G, T, w) for given τ . Consider a set of vertices $A \subseteq V(G)$ where for any two vertices $u, v \in A$, there is a distance of at least 3 between them. There exists a minimal counterexample weighted graft where there is a tight near-packing at an edge incident to v for every $v \in A$.*

Proof. As in the previous proof, consider all the possible weight assignments w for a minimal counterexample. This time, let the weight multiplicity set $\{m_0, \dots, m_{\tau-1}\}$ for weight assignment w consider all edges incident to a vertex in A . Note that for any two vertices $u, v \in A$, $\delta(u) \cap \delta(v) = \emptyset$ because they are a distance of at least 3 apart. So the set of all edges incident to a vertex in A is $\delta(A)$. Then $m_i = |\{e \in \delta(A) : w(e) = i\}|$. Define a partial ordering of the weight assignments for a minimal counterexample so that w' is smaller than w if there exists an $i \in [\tau - 1]$ such that $m'_i > m_i$ while $m'_j = m_j$ for all $j < i$. Let (G, T, w) be a minimal counterexample weighted graft that is smallest according to this partial order around the vertices of A .

We can create a tight near-packing of (G, T, w) at an edge incident to any vertex in A . Pick a vertex $v \in A$ around which to create a tight near-packing. We use the same steps as in the previous proof to make changes around v and create a new weight assignment

w' . This doesn't affect any edge in $\delta(u)$ for $u \in A - \{v\}$ since u and v have distance at least 3. As the previous proof showed, the weight assignment w' keeps $w(\delta(x)) = \tau$ for any vertex $x \in V(G)$, maintains the parity of cuts, and does not change the total weight. Only two edges of $\delta(A)$ decrease in weight, both by 1. One of these is the minimum weight edge, so w' is smaller than w . The weight assignment w was chosen so that (G, T, w) is a minimal counterexample weight graft that is smallest according to the partial order around A . Therefore, (G, T, w') packs. Then using the same process as before, make a tight near-packing around an edge in $\delta(v)$. $\square$

5.2 Crossing Mates

We now turn to searching for mates in a minimal counterexample weighted graft. Later on, we tie in the idea that τ is very large in the graft. A possible direction to finding that a minimal counterexample does not exist could be to have too many mates that they cannot fit in the graft. Inspired by this idea, we look to see how mates can appear due to the existence of other mates. First, we consider what occurs when two mates “cross”.

Definition 20. *Mate $B = \delta(U)$ **crosses** mate $C = \delta(V)$ if $U \cap V, U \cap \bar{V}, \bar{U} \cap V$, and $\bar{U} \cap \bar{V}$ are all nonempty.*

The goal is to show that for a given mate, there will always exist another mate that crosses it, and if that is the case, to find how many mates cross it. To do this, we define a relaxation of the term mate.

Definition 21. *An odd cut $\delta(U)$ is a **mate cut** if $w(\delta(U)) - |\delta(U)| \leq \tau - 3$.*

We mentioned in an earlier chapter that for any mate B of T -join S in the weighted graft, $w(B) - |B \cap S| \leq \tau - 3$, so $w(B) - |B| \leq \tau - 3$. In Chapter 2, it was shown that for a minimal counterexample graft, $T = V(G)$, so odd cuts are T -cuts. Then, all mates are mate cuts. We will work only with nontrivial mates, so there must be at least two vertices on each side of the cut. The goal is to show that for a given nontrivial mate, there exist many mates that cross it. In the proof we actually show a stronger result, that for a given nontrivial mate cut, there exist many mates that cross it, when the minimum weight of the mate cut is at least 3.

Theorem 1. *For a minimal counterexample weighted graft (G, T, w) and a given nontrivial mate with minimum weight k at least 3, there are at least $\frac{k-3}{k-1}\tau + \frac{6k}{k-1}$ distinct mates that cross it.*

Proof. Let $C = \delta(U)$ be a nontrivial mate cut in (G, T, w) with minimum weight $k \geq 3$, so $w(C) - |C| \leq \tau - 3$. Since C is nontrivial, $|U| \geq 2$ and $|\bar{U}| \geq 2$. We first show that there exists a mate that crosses C , and then we will count how many such mates there are.

Since the minimum weight is k , $w(C) \geq k|C|$, so

$$\tau - 3 \geq w(C) - |C| \geq k|C| - |C| = (k - 1)|C|.$$

And therefore,

$$|C| \leq \frac{\tau - 3}{k - 1}.$$

Because $T = V(G)$ in a minimal counterexample, every odd cut is a T -cut, making C a T -cut. Since every T -join and T -cut have an odd intersection, every T -join of (G, T, w) must intersect C at least once. A T -join of (G^w, T) will not contain parallel edges, so in (G, T, w) it must overlap with C at distinct edges. Then a T -join intersects C no more than $|C|$ times.

Since C is a mate cut, we can bound $w(C)$ using the bound on $|C|$:

$$w(C) \leq \tau - 3 + |C| \leq \tau - 3 + \frac{\tau - 3}{k - 1} = \frac{k}{k - 1}(\tau - 3).$$

Since C is nontrivial, there exist at least two vertices in U . In fact, there must exist an edge with both endpoints in U since $w(\delta(v)) = \tau$ for any $v \in U$ and $w(C) \leq \frac{k}{k-1}(\tau - 3) \leq \frac{3}{2}(\tau - 3)$. By the same argument, there must exist an edge with both endpoints in $\bar{U}$. Let A be the set of edges of G with both endpoints in U , and let B be the set of edges of G with both endpoints in $\bar{U}$. Then A, B, C are disjoint sets with $E(G) = A \cup B \cup C$. Let $e \in A$ and $f \in B$. Let $S_1, \dots, S_\tau$ be an e -near-packing of (G, T, w) and $T_1, \dots, T_\tau$ be an f -near-packing of (G, T, w) .

Our first goal in the proof is to find T -joins S_i and T_j of these near-packings such that they both intersect C once, and at the same edge. To do this, first we show that there exist T -joins that intersect C exactly once and then that some of the T -joins of the near-packings intersect the same edges.

Claim 7. *There must exist a T -join in each near-packing that intersects C at exactly one edge.*

Proof. Assume for a contradiction that no such T -join exists. Then all τ T -joins in a near-packing have at least three edges of C , since they must have an odd intersection. So the total weight taken up is 3τ , but $w(C) \leq \frac{k}{k-1}(\tau - 3) < 3\tau$, a contradiction. Therefore, there must exist at least one T -join that intersects C exactly once. $\diamond$

Claim 8. *There exists an edge $g \in C$ such that there are T -joins S_i and T_j of the near-packings where $g \in S_i$, $|S_i \cap C| = 1$, $g \in T_j$, and $|T_j \cap C| = 1$.*

Proof. Each of the τ T -joins of the e -near-packing, $S_1, \dots, S_\tau$, intersects C at least once. We count how many of them intersect C exactly once. Some of these intersections can (and will) be at the same edge. Since every T -join will have at least one edge of C , assign an edge to each T -join. This gives a total weight of τ . The remaining weight is

$$w(C) - \tau \leq \frac{k}{k-1}(\tau - 3) - \tau = \frac{1}{k-1}\tau - \frac{3k}{k-1}.$$

The number of T -joins with intersection of size one with C is minimized when the rest of them have intersection of 3. Then the remaining weight found is split in half to give that at most $\frac{1}{2}\left(\frac{1}{k-1}\tau - \frac{3k}{k-1}\right)$ T -joins have intersection of size 3 with C . So the number of T -joins S_i where $|S_i \cap C| = 1$ is at least

$$\tau - \frac{1}{2}\left(\frac{1}{k-1}\tau - \frac{3k}{k-1}\right) = \frac{2k-3}{2(k-1)}\tau + \frac{3k}{2(k-1)}.$$

The same is true for the near-packing at f .

To show that there is overlap in C between the T -joins of the near-packing at e and the one at f that have intersection of size 1 with C , it suffices to show that each takes up more than half of the edges of C . The total weight taken up by T -joins of one of the near-packings with intersection of 1 with C is at least $\frac{2k-3}{2(k-1)}\tau + \frac{3k}{2(k-1)}$. So the fraction of the edges of C taken up is at least

$$\frac{\frac{2k-3}{2(k-1)}\tau + \frac{3k}{2(k-1)}}{w(C)} \geq \frac{\frac{2k-3}{2(k-1)}\tau + \frac{3k}{2(k-1)}}{\frac{k}{k-1}\tau} = \frac{(2k-3)\tau + 3k}{2k\tau} = \frac{2k-3}{2k} + \frac{3}{2\tau}.$$

For $k \geq 3$, $\frac{2k-3}{2k} + \frac{3}{2\tau} > \frac{1}{2}$. Thus, when $k \geq 3$, there exists an edge $g \in C$ such that there are T -joins S_i, T_j that both contain g and $|S_i \cap C| = 1$ and $|T_j \cap C| = 1$. This is shown in Figure 5.2. $\diamond$

Let $g = xy$ be an edge as described in the claim where $x \in U$ and $y \in \bar{U}$. Let S_i, T_j be the T -joins described in the claim. Then $(S_i \cap B) \cup \{g\}$ is a T -join of $\bar{U}$. Similarly, $(T_j \cap A) \cup \{g\}$ is a T -join of U . So $\{g\} \cup (S_i \cap B) \cup (T_j \cap A)$ is a T -join of G . It is shown circled in red in Figure 5.2. Since (G, T, w) is a minimal counterexample graft, this T -join has a mate, D . Finally, we need to show that D crosses C . It suffices to show that D has edges in both A and B .

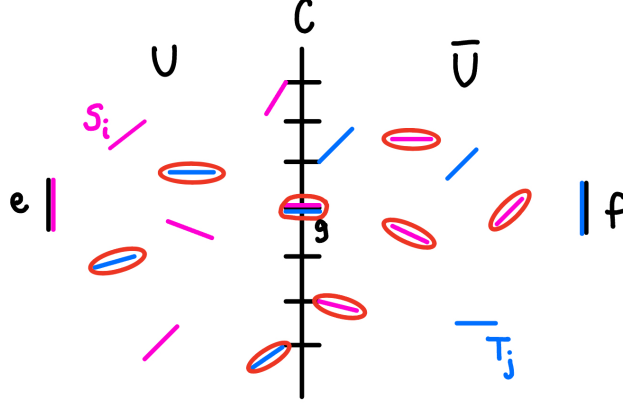


Figure 5.2: The mate cut $C = \delta(U)$ with $e \in U$ and $f \in \bar{U}$. S_i is a T -join of a near-packing at e . T_j is a T -join of a near-packing at f . They both hit C only at edge g . Circled in red is a T -join of the graph using the two T -joins.

Claim 9. D must have edges in both A and B .

Proof. Assume for a contradiction that D has no edges in B . Then it is a mate of T_j , so it must contain f , a contradiction since $f \in B$. Similarly, if D has no edges in A , then it is a mate of S_i . So it must contain e , a contradiction since $e \in A$. Therefore, D must contain edges in A and B . $\diamond$

So when there is an edge $g \in C$ such that there exist T -joins S_i, T_j in the near-packings where $g \in S_i, T_j$, $|S_i \cap C| = 1$, and $|T_j \cap C| = 1$, there exists a mate D that crosses mate cut C . Thus, for any mate cut where the minimum weight is at least 3, there is a mate that crosses it. Next, we look to how many mates cross C . To try to find this, we show how much weight is taken up by the overlap of T -joins that have intersection of one with C .

Claim 10. *There is a total weight of $\frac{k-3}{k-1}\tau + \frac{6k}{k-1}$ taken up by edges of type $g \in C$ where there exist T -joins S_i, T_j in the near-packings as before where $g \in S_i$, $|S_i \cap C| = 1$, $g \in T_j$, and $|T_j \cap C| = 1$.*

Proof. The total weight taken up by T -joins S_i of the e -near-packing with $|S_i \cap C| = 1$ is at least $\frac{2k-3}{2(k-1)}\tau + \frac{3k}{2(k-1)}$. The same is true for T -joins T_j of the f -near-packing where $|T_j \cap C| = 1$.

Their sum is larger than $\frac{k}{k-1}(\tau - 3)$, so there must exist overlap. So there is weight of at least

$$\begin{aligned} 2\left(\frac{2k-3}{2(k-1)}\tau + \frac{3k}{2(k-1)}\right) - w(C) &\geq \frac{(2k-3)\tau + 3k}{k-1} - \frac{k}{k-1}(\tau - 3) \\ &= \frac{k\tau - 3\tau + 6k}{k-1} = \frac{k-3}{k-1}\tau + \frac{6k}{k-1} \end{aligned}$$

that is taken up by both types of T -joins. $\diamond$

It is a bit trickier to get the number of edges that this takes up when the weight of any edge is at least k . If all edges have weight k , then there are $\frac{k-3}{k(k-1)}\tau + \frac{6}{k-1}$ edges $g \in C$ taken up by the weight described in the previous proposition. In our graft, weights vary, but some information can still be obtained from the weight. We can find a lower bound for how many mates cross a given mate cut.

We showed that the mates that cross C came from joining parts of two T -joins and their overlapping edge to make a T -join of (G, T, w) . The mate D must contain at least 3 edges of that T -join and exactly one for disjoint T -joins. The T -joins of the near-packing $S_1, \dots, S_\tau$ only overlap at e , so in B there is no overlap. The same is true in A for the T -joins of the f -near-packing. Because the crossing mate will contain edges in A and B , T -joins made from joining different T -joins of the e -near-packing and f -near-packing will lead to different mates.

Since there is a total weight of at least $\frac{k-3}{k-1}\tau + \frac{6k}{k-1}$ of edges in C that are intersected once by T -joins S_i and T_j , there are at least $\frac{k-3}{k-1}\tau + \frac{6k}{k-1}$ pairings of T -joins (one from each near-packing) so that no T -join is used more than once. Then given a nontrivial mate cut, there are at least $\frac{k-3}{k-1}\tau + \frac{6k}{k-1}$ distinct mates that cross it. So the result holds with a given nontrivial mate. $\square$

Since the proof actually used a mate cut instead of a mate, we have the following, stronger, corollary:

Corollary 2. *For a minimal counterexample weighted graft (G, T, w) and a given nontrivial mate cut with minimum weight k at least 3, there are at least $\frac{k-3}{k-1}\tau + \frac{6k}{k-1}$ distinct mates that cross it.*

A way to find more mates that cross C is to choose different starting edges e and f so that there are different near-packings. However, we do not know if any of the mates that were already counted will reappear. Therefore, this is not part of the count at the end of the proof. This gives us a direction to continue work to answer the question:

Question 2. *What is the maximum number of mates that cross a nontrivial mate cut in a minimal counterexample?*

While the result of Theorem 1 is interesting, it heavily relies on there being no edges of weight 1 or 2 in the mate cut since it specifies that k , the minimum weight, must be at least 3. We can make a slight change that while still having a restriction related to the weights, allows some small weights. Instead of bounding the minimum weight, we bound the average weight of the mate cut.

Corollary 3. *For a minimal counterexample weighted graft (G, T, w) and a given nontrivial mate cut with average weight $l \geq 3$, there are at least $\frac{l-3}{l-1}\tau + \frac{6l}{l-1}$ distinct mates that cross it.*

Proof. Let C be a nontrivial mate cut with average weight l . The proof follows exactly as the previous one since by definition $w(C) = l|C|$, so $w(C) \geq l|C|$. This is the only assumption that used the minimum degree and it still applies for average degree. Then using $w(C) - |C| \leq \tau - 3$, it is still true that $|C| \leq \frac{\tau-3}{k-1}$. Therefore, the results above still hold, so there are at least $\frac{l-3}{l-1}\tau + \frac{6l}{l-1}$ mates that cross C . Now C is allowed to have edges of weight 1 or 2 as long as the average is at least 3. $\square$

To put the results of Theorem 1 and Corollary 3 to use, it must be shown that there exist such mate cuts. The condition of the average weight is less restrictive than the minimum weight since if the minimum weight is at least 3, then the average weight is at least 3. Showing that there exists at least one nontrivial mate cut with average weight at least 3 is enough to show that the crossing mates will appear.

For a minimal counterexample weighted graft (G, T, w) , $w(\delta(v)) = \tau$ for all $v \in V(G)$. The degree can vary, but since this is a simple planar graph, the average degree is less than 6. Therefore, there have to be large weights in the graft. In fact,

Remark 8. *The average weight of the edges of a minimal counterexample graft is greater than $\frac{\tau}{6}$.*

This unfortunately does not immediately give a result about the mate cut, so we are still left with the question that perhaps can be solved with a counting argument:

Question 3. *Does a minimal counterexample weighted graft contain a nontrivial mate cut with average weight at least 3?*

The count for crossing mates suggests that perhaps the number of mates could be quadratic in the number of edges. What might a graft with quadratic number of mates look like? We consider a G that we call the **donut graph** where there are two large faces (one of which is the infinite face) and all edges are “near” one of these large faces. It is a “thin” donut so that a cut across it will be a mate cut. Then given a mate cut the number of crossing mate cuts is quadratic in the number of edges of the graft. Inspired by this graft with large faces, we continue to explore this in the next chapter.

5.3 Finding New Cuts

The discussion of crossing mate cuts leads us to ask whether crossing mate cuts cause the existence of other cuts. Let $C = \delta(W)$ and $D = \delta(U)$ be two crossing odd cuts. In a minimal counterexample graft, $T = V(G)$, so there are an even number of vertices. Then $|W|, |\overline{W}|, |U|$, and $|\overline{U}|$ are all odd.

Remark 9. *Assume $|U|$ and $|W|$ are odd, the total number of vertices is even, and $|W \cap U|, |W \cap \overline{U}|, |\overline{W} \cap U|$, and $|\overline{W} \cap \overline{U}|$ are all nonempty. Then either $|W \cap U|$ and $|\overline{W} \cap \overline{U}|$ are odd or $|W \cap \overline{U}|$ and $|\overline{W} \cap U|$ are odd, where both cannot occur at the same time.*

Without loss of generality, let $|W \cap U|$ and $|\overline{W} \cap \overline{U}|$ be odd while $|W \cap \overline{U}|$ and $|\overline{W} \cap U|$ are even. We will use the following remark to help our results:

Remark 10. *For sets of vertices U, W ,*

$$\begin{aligned}\delta(W \cap U) &= \delta(\overline{W} \cup \overline{U}) \\ \delta(\overline{W} \cap U) &= \delta(W \cup \overline{U}) \\ \delta(W \cap \overline{U}) &= \delta(\overline{W} \cup U) \\ \delta(\overline{W} \cap \overline{U}) &= \delta(W \cup U)\end{aligned}$$

We use submodular functions to find some results about what occurs when cuts cross. The key property of submodular functions is that $f(W) + f(U) \geq f(W \cap U) + f(W \cup U)$. From the remark above, we can say that

$$f(W) + f(U) \geq f(W \cap U) + f(W \cup U) = f(W \cap U) + f(\overline{W} \cap \overline{U}).$$

First, we consider the function $f(U) = w(\delta(U))$, which counts the total weight of the edges in a cut.

Remark 11. *The function $f(U) = w(\delta(U))$ is submodular.*

Note that since $T = V(G)$, for any odd cut $\delta(U)$ in a minimal counterexample graft, $f(U) \geq \tau$. If equality holds, then it is a **tight cut**. We show that two crossing tight cuts create two more tight cuts.

Proposition 18. *If $\delta(W)$ and $\delta(U)$ are crossing tight odd cuts, the cuts created by the sets of $W \cap U, W \cap \bar{U}, \bar{W} \cap U$, and $\bar{W} \cap \bar{U}$ that are odd are also tight.*

Proof. Without loss of generality, let $|W \cap U|$ and $|\bar{W} \cap \bar{U}|$ be odd while $|W \cap \bar{U}|$ and $|\bar{W} \cap U|$ are even. Since $\delta(W)$ and $\delta(U)$ are tight, $f(W) = f(U) = \tau$. Then

$$f(W) + f(U) = 2\tau \geq f(W \cap U) + f(\bar{W} \cap \bar{U}).$$

Since $W \cap U$ and $\bar{W} \cap \bar{U}$ are odd cuts, $f(W \cap U) \geq \tau$ and $f(\bar{W} \cap \bar{U}) \geq \tau$. Then $f(W \cap U) = f(\bar{W} \cap \bar{U}) = \tau$, making the cuts tight. $\square$

Now we consider what occurs when mate cuts cross. This time define $f(U) = w(\delta(U)) - |\delta(U)|$. So $\delta(U)$ is a mate cut if and only if $f(U) \leq \tau - 3$. This is submodular because of Remark 11 since the edges are still counted the same way but the weight of each one is just one less.

Proposition 19. *If $\delta(W)$ and $\delta(U)$ are crossing mate cuts, at least one of the cuts created by the sets of $W \cap U, W \cap \bar{U}, \bar{W} \cap U$, and $\bar{W} \cap \bar{U}$ that are odd is also a mate cut.*

Proof. As before, let $|W \cap U|$ and $|\bar{W} \cap \bar{U}|$ be odd while $|W \cap \bar{U}|$ and $|\bar{W} \cap U|$ are even. Since $\delta(W)$ and $\delta(U)$ are mate cuts, $f(W) \leq \tau - 3$ and $f(U) \leq \tau - 3$. Therefore,

$$2(\tau - 3) \geq f(W) + f(U) \geq f(W \cap U) + f(\bar{W} \cap \bar{U}).$$

Then at least one of $f(W \cap U)$ and $f(\bar{W} \cap \bar{U})$ must be at most $\tau - 3$. So at least one of them is a mate cut. $\square$

Two crossing tight cuts created two tight cuts, but when two mate cuts cross, we can only guarantee one mate cut in this way. While the results of this section do not depend on the weights of the cuts, the set up of two mates crossing does depend on this. Moving forward from here should use some counting idea to show that since the average weight of the edges in the graft is greater than $\frac{\tau}{6}$, then there must exist a mate with average weight at least three. The challenge here arises from unknown overlap of mates. From there, showing that a minimal counterexample graft does not exist can come from too many distinct mates that cannot fit in the graft. The large τ could assist here. In the following chapter, we further explore what comes from the graft with large τ .

Chapter 6

Minimal Counterexample Grafts for Large τ

In this chapter we return to the T -join Packing for Large r Conjecture, so we focus on large τ . Earlier in the thesis we have tried at times to remove the planarity assumption from our conjectures, often finding minors to omit. In the previous chapters, we rarely use the planarity of the graft. For large τ , perhaps the sparsity of the graph is doing most of the work instead of the planarity.

In a 1999 paper [13], Rizzi explored **indecomposable** r -graphs, which are r -graphs that cannot be expressed as a sum of an r_1 -graph and an r_2 -graph. A sum of graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ is the graph $G_1 + G_2 = (V, E_1 \cup E_2)$. Note that the sum of an r_1 -graph and an r_2 -graph is an $(r_1 + r_2)$ -graph. Rizzi constructs an indecomposable r -graph for every r that serves as a counterexample to the following question:

Question 4. *Is there an $\bar{r}$ such that all indecomposable r -graphs have $r < \bar{r}$?*

Unsliceable r -graphs cannot be expressed as the sum of an $r - 1$ -graph and a 1-factor, which is a spanning subgraph that is a 1-graph. A 1-graph is a perfect matching, so in our terms of grafts, if a graph is unsliceable, you cannot remove a T -join to make τ reduce to $\tau - 1$. Note that if a graph is indecomposable then it is unsliceable. Then Rizzi's counterexample also serves as a counterexample to this question posed by Seymour:

Question 5. [19] *Is there an $\bar{r}$ such that all unsliceable r -graphs have $r < \bar{r}$?*

This counterexample shows that we cannot simply remove the planarity constraint from the T -join Packing Conjecture for Large r but perhaps we can replace it with other conditions, leading us to this conjecture:

Conjecture 7. *There exists a function f such that for any Eulerian weighted graft (G, T, w) where*

1. *G has average degree at most γ*
2. *There are no nontrivial T -cuts of minimum weight*

and where $\tau(G, T, w) \geq f(\gamma)$, there exists a removable T -join.

The paper by Rizzi [13] shows that due to the counterexample, both conditions cannot be removed from this conjecture. Since our minimal counterexample graft is planar and has that all minimum cuts are trivial cuts, we have the following remark.

Remark 12. *The previous conjecture implies the T -join Packing for Large r Conjecture.*

So far we have not considered what the structure of a minimal counterexample will be like. Could it be “average” or “regular” where the structure is the same all throughout it? We can picture something like a grid where both the degree and face size are bounded. An upper bound on the degree lower bounds the average weight. Since a minimal counterexample (G, T, w) is a planar simple weighted graft, the average degree is less than 6, and therefore, the average weight is greater than $\frac{\tau}{6}$. Note that there can exist small weights that are not a function of τ as long as all weights average to more than $\frac{\tau}{6}$. We consider what occurs when there are no small weights, so all weights are linear in τ :

Definition 22. *The weights are α -**bounded** for $0 < \alpha < 1$ if for any edge e , $w(e) \geq \alpha\tau$ where $\alpha\tau > 1$.*

Note that in order to consider a situation where weights are at all restricted, $\alpha\tau > 1$. Otherwise, they can be any weight. In an α -bounded minimal counterexample, can the size of the faces still be upper bounded by a constant? Namely, will there exist faces whose sizes increase with τ ? We observe what occurs around a given edge. First, we define what it means to be “near” an edge.

Definition 23. *The β -**neighborhood** of an edge e is the set of all edges within β faces of e where you can only move from edge to edge if they are part of the same face.*

In the dual of the graph, the set of edges corresponding to those in the β -neighborhood of e includes anything that has a path of length at most β to e . We want to show that in our α -bounded graft, there is a large face in the β neighborhood of every edge for β that

is a function of α . We define large to be lower bounded by a monotone increasing function of τ .

In order to assist with finding a large face, we use mates. Recall that for a mate B of T -join S , $w(B) - |B \cap S| \leq \tau - 3$, and we defined a mate cut to be an odd cut where $w(\delta(U)) - |\delta(U)| \leq \tau - 3$. Then all mates are mate cuts. Note that in this chapter, the odd condition on mate cuts is not necessary and the results still hold without it.

Proposition 20. *Let B be a mate cut of a minimal counterexample graft with α -bounded weights for some $\alpha \in (0, 1)$. Then*

$$|B| \leq \frac{1}{\alpha^2}.$$

Proof. Let B be a mate cut of an α -bounded minimal counterexample graft for some $\alpha \in (0, 1)$. Then

$$\tau - 3 \geq w(B) - |B| \geq \alpha\tau|B| - |B|$$

so

$$|B| \leq \frac{\tau - 3}{\alpha\tau - 1}.$$

We want to show that $|B| \leq \frac{1}{\alpha^2}$ for α -bounded minimal counterexample graft. We first show that if $\tau \geq \frac{3\alpha^2 - 1}{\alpha^2 - \alpha}$, then $\frac{\tau - 3}{\alpha\tau - 1} \leq \frac{1}{\alpha^2}$:

If

$$\tau \geq \frac{3\alpha^2 - 1}{\alpha^2 - \alpha}$$

then since $0 < \alpha < 1$, $\alpha^2 < \alpha$ so

$$(\alpha^2 - \alpha)\tau \leq 3\alpha^2 - 1$$

$$\alpha^2\tau - \alpha\tau \leq 3\alpha^2 - 1$$

$$\alpha^2(\tau - 3) \leq \alpha\tau - 1$$

and since $\alpha^2 > 0$ and $\alpha\tau > 1$,

$$\frac{\tau - 3}{\alpha\tau - 1} \leq \frac{1}{\alpha^2}.$$

We would like the bound we are using to hold for all τ that apply to our situation. Since $\alpha\tau > 1$, we consider all τ where $\tau > \frac{1}{\alpha}$. So how much does the bound $\tau \geq \frac{3\alpha^2 - 1}{\alpha^2 - \alpha}$ not account for?

$$\frac{3\alpha^2 - 1}{\alpha^2 - \alpha} - \frac{1}{\alpha} = \frac{3\alpha^2 - \alpha}{\alpha^2 - \alpha} = \frac{2\alpha^2}{\alpha^2 - \alpha} + 1$$

Since $\alpha^2 - \alpha < 0$, $\frac{2\alpha^2}{\alpha^2 - \alpha} + 1 < 1$. So $\tau \geq \frac{1}{\alpha} + 1$ includes all τ that satisfy $\tau \geq \frac{3\alpha^2 - 1}{\alpha^2 - \alpha}$. If $\tau \geq \frac{1}{\alpha} + 1$, then $\alpha\tau \geq 1 + \alpha$. Note that $2 > 1 + \alpha$, so this only gives the restriction that weights cannot be 1. This is already done by setting $\alpha\tau > 1$. The α -bounded weights are used so that all weights are a function of τ , so it is okay to not allow edges of weight one. With this work, we have shown that for α -bounded weights,

$$|B| \leq \frac{1}{\alpha^2}.$$

□

Proposition 21. *Let (G, T, w) be a minimal counterexample weighted graft and let $\beta = \frac{1}{2\alpha^2}$. If G has α -bounded weights, then for every edge e , the size of the largest face in the β -neighborhood of e is lower bounded by a monotone increasing function of τ .*

Proof. Let $e \in E(G)$. Let vertices s, t of the dual graph correspond to the faces of G that e is incident to. A large face in G corresponds to a vertex of high degree in the dual. Every face in the β -neighborhood of e corresponds to a vertex that is distance β to s or t . Let m be the maximum degree of the vertices in the β -neighborhood of e in the dual. Therefore, it is the size of the largest face in G in the β -neighborhood of e .

As this is a minimal counterexample graft, there exists an e -near-packing. Each T -join of that near-packing has a distinct mate, and every mate contains e . Therefore, there are τ mate cuts that contain e . In the dual graph, this corresponds to τ distinct cycles that include the edge st . So there are τ distinct paths in the dual from s to t . These paths have length at most $\frac{1}{\alpha^2}$ because $|B| \leq \frac{1}{\alpha^2}$. Since $\beta = \frac{1}{2\alpha^2}$, they have length at most 2β , so they are entirely in the β -neighborhood. We want to bound τ from above in terms of m and β .

Consider a path from s to t with length 2β . $(2\beta)!$ paths can be made from just the vertices in that path in different arrangements by adding edges. Each vertex in the path can have degree at most m . Then there are at most $(2\beta)!m^{2\beta}$ paths from s to t . So

$$(2\beta)!m^{2\beta} \geq \tau$$

and therefore

$$m \geq \left(\frac{\tau}{(2\beta)!} \right)^{1/2\beta}.$$

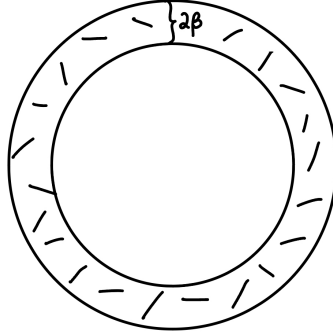


Figure 6.1: Donut graph with width of 2β

So the size of the largest face in the β neighborhood of an edge e is lower bounded by a monotone increasing function of τ . $\square$

Note that this is a rough bound, and there is room to improve here. This shows that a minimal counterexample graph cannot both have α -bounded weights and small faces.

An extreme case of having a large face in every β -neighborhood is if every edge is within β distance from one large face. Notice that this actually means there would be two large faces - one inside and one outside. In the previous chapter, we defined this to be the donut graph. Notice the “width” of the donut graph can be as large as 2β , as seen in Figure 6.1. Say $|E(G)|$ and β are given. Then any constraint on the size of the large face is determined by these two values, regardless of what τ may be. Then,

Remark 13. *In the donut graph, the size of the largest face is arbitrarily large and does not grow with τ .*

Therefore, the donut graph does not work as a counterexample to the T -join Packing for Large r Conjecture.

Chapter 7

Recoloring

In this chapter, we use the idea of recoloring to consider a bit of a different direction in which to approach the packing conjectures. A recoloring operation will start with a coloring, perform some changes, and end with a different coloring with the same number of colors. A popular recoloring operation is the Kempe change, which works with vertex colorings. As we are working with edge-colorings, we define a recoloring operation similarly.

For a graph G and an integer $k \geq 1$, a k -edge-coloring is represented by a mapping $c: E(G) \rightarrow \{1, \dots, k\}$. Take a k -edge-coloring of G . This is a set of k edge-disjoint matchings. Choose two distinct colors $a, b \in [k]$. Let $G(a, b)$ be the subgraph of G induced on the edges of color a or b . Let K be a component of $G(a, b)$. For the edges in K , switch the colors a and b to create a new edge-coloring with k colors.

We apply this recoloring operation to our minimal counterexample Eulerian graft. Our goal is to recolor one near-packing to create another near-packing. To do this, we first must define different changes we can make to the graft. To begin, we define a **recoloring operation** from one packing of T -joins to another. Consider a packing of T -joins and choose two T -joins S_i, S_j . Let G' be the graph induced on the edges of the T -joins. Then G' is a set of cycles. Choose a cycle and make T -joins S'_i, S'_j so that edges in this cycle belonging to S_i are now in S'_j and edges in the cycle belonging to S_j are now in S'_i .

Up to this point of the thesis, we have been working with minimal T -joins. Now we will slightly deviate from this. Whenever we state that we have a near-packing of a minimal counterexample, it will be made of minimal T -joins. However, in the process, we will make changes to the T -joins so that they are not at all times minimal. Namely, we will add parallel edges to them. If a T -join contains a cycle, then it is not minimal. So if it contains

two edges that are parallel to each other, then they can be removed and it will still be a T -join.

Next we define a **swap** between edges e and f . This can be performed when e has two edges, e', e'' , parallel to it and there exists a packing of T -joins where e and its parallel edges are each in different T -joins. Begin by adding two edges f', f'' to be parallel to f and add them to the T -join of the packing that contains f . Note that the T -join is no longer minimal. Now perform recoloring operations as defined before so that e, e' , and e'' are all in the same T -join and f, f' , and f'' are each in different T -joins. Finally, remove e' and e'' so that the T -join that contained them is minimal.

Now we can use the swap operation to recolor an e -near-packing to an f -near-packing of a minimal counterexample graft (G^w, T) for $e \neq f$. Begin with an e -near-packing where $S_1, \dots, S_m$ are the T -joins that contain e . Note that $m \geq w(e) + 2$ since this is a near-packing. Add $m - w(e)$ parallel edges so that each T -join contains exactly one of the parallel edges. *Swap* between edges e and f so that there are only $w(e)$ parallel edges in the place of e , all in different T -joins, and there are at least $w(f) + 2$ parallel edges in place of f that are all contained in different T -joins. This is now an f -near-packing. We call two near-packings at different edges that share a T -join **equivalent** if there exists a recoloring as just described from one to the other.

Given a T -join, **equivalence classes** can be created of near-packings that contain the T -join where they can be recolored to form each other. Note that a T -join can have multiple equivalence classes and a near-packing is in τ equivalence classes. This idea of equivalence classes based on recoloring has been studied by others, particularly for Kempe changes for vertex colorings [12].

The following proposition found that learning more about our recoloring equivalence classes can be quite useful to the T -join Packing Conjecture.

Proposition 22. *Let (G^w, T) be a minimal counterexample graft. If there exists a T -join S and set of edges L such that*

- *there is an equivalence class for S with an e -near-packing for each $e \in L$*
- $w(L) \geq \tau + |L \cap S| - 2$,

then the T -join Packing Conjecture holds.

Proof. Say that for a minimal counterexample graft (G^w, T) , there exists a T -join S and set of edges L where there is an equivalence class for S with an e -near-packing for each

$e \in L$ and $w(L) \geq \tau + |L \cap S| - 2$. Then each $e \in L$ has an e -near-packing that contains S . Recall that due to Proposition 13, for a set L of edges and a given T -join where the T -join appears in an e -near-packing, for each $e \in L$, $w(L) < \tau + |L \cap S| - 2$. This contradicts our assumption, so a minimal counterexample graft cannot exist. Therefore, the T -join Packing Conjecture holds. $\square$

So this leaves us with the question,

Question 6. *In a minimal counterexample graft, is there a T -join S where S has an equivalence class such that if M is the set of edges where for all $e \in M$ there is an e -near-packing in the equivalence class, there is a set $L \subseteq M$ such that $w(L) \geq \tau + |L \cap S| - 2$?*

Finally, we note that recoloring to match a T -join will likely not be an easy achievement due to the example of C_4 given in the Chudnovsky et al. paper [4] that proved the Edge-Coloring Conjecture for $r = 8$. Consider a C_4 with the vertices u, v, x, y in a minimal counterexample weighted graft, and note that there may be edges between u and x or v and y . We make a change to the weights so that the weights of uv and xy both decrease by one and the weights of vx and yu both increase by one. Say there is a packing of the graft on these new weights. We would be able to make the original graft pack if one of the T -joins of the packing contains both vx and yu . However, the work by Chudnovsky et al. shows that this is difficult to do and requires studying specific configurations [4]. Therefore, it must not be simple to recolor a packing so that vx and yu are in the same T -join.

Chapter 8

Conclusion

This thesis aimed to get closer to answering whether a planar r -graph is r -edge-colorable when r is large. Particularly, what does a minimal counterexample look like? To make the problem easier to work with, we redefined in terms of grafts, working with packing T -joins. We used clutters to characterize a certificate of not packing, which we called a mate.

Since we are working with planar grafts, the sparsity of the graph means that there will be large weights. If the number of mates grows with τ , then perhaps for large τ , there will be too many mates. We found that for mates of sets in near-packings, there are at least $|E(G)|\frac{(k-1)\tau}{(\tau-3)}$ distinct mates where k is the minimum weight. A possible direction to improve in is to see how mates overlap, perhaps accounting for the overcount of $|E(G)|\tau$ in a better way. In fact, an open question that remains is whether the number of distinct mates is lower bounded by $|E(G)|f(\tau)$ for strictly increasing monotone function f . Note also that the current bound depends on the minimum weight. Is there a situation where the average weight can help? Since the simple weighted graph is planar, the average degree is less than 6, making the average weight greater than $\frac{\tau}{6}$. While the average weight generally in the graph is large, we do not know what occurs in mates. It is possible that they use only edges of small weight. Therefore, for any number of vertices n , there is an r that is a function of n such that the Edge-Coloring Conjecture is true for r -graphs on n vertices.

We saw that in tight near-packings, for a mate B of T -join S of the near-packing, $|B-S| = \tau-3$. We also showed that for any vertex, there exists a minimal counterexample that has a tight near-packing at an edge incident to that vertex. Further work could learn more about the mates in this situation.

The thesis also explored a way to find more mates from a given one in the section about crossing mates. We found that in a minimal counterexample weighted graft with

a nontrivial mate cut with minimum weight k at least 3, there are at least $\frac{k-3}{k-1}\tau + \frac{6k}{k-1}$ distinct mates that cross it. The same is true if the mate cut has average weight $k \geq 3$. But in order for this result to help, there needs to exist a mate cut with average weight 3. This seems to be likely for large τ , but not much is known about the mate cuts. Could they only be using edges with small weights? There may be a counting argument to show that this is not possible.

As mentioned with the mates, not much is known about the structure of a minimal counterexample. An initial question to start with is whether it is “average” like a grid. This means that the degrees as well as the face sizes are bounded. Upper bounding the average degree is essentially lower bounding the average weight. This could be by a constant. We consider what occurs if the weight is lower bounded by a function of τ . If the weights are α -bounded, then we found that for every edge, there must exist a large face near it. So we know that it cannot look “average” when the weights are α -bounded. The large face result lead us to think about the donut graph where there are only two large faces and every edge is near them. This also raises the question of whether the number of mates could be quadratic in the number of edges.

With these results about tight near-packings, counting mates, and large faces, this thesis explored different approaches to the problem of r -edge-coloring planar r -graphs. We found properties of a minimal counterexample that moved the progress forward in each direction and showed ways that research could be continued.

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