

Disturbance Vulnerability Analysis and Reduction for Nonlinear Systems using Modes of Instability

by

Jinghan Wang

A thesis
presented to the University of Waterloo
in fulfillment of the
thesis requirement for the degree of
Master of Applied Science
in
Electrical and Computer Engineering

Waterloo, Ontario, Canada, 2025

© Jinghan Wang 2025

Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Engineered systems naturally experience large disturbances which have the potential to disrupt desired operation because the system may fail to recover to a desired exponentially [stable equilibrium point \(SEP\)](#). It is valuable to determine the mechanism of instability when the system is subject to a particular finite-time disturbance, because this information can be used to improve vulnerability detection, and to design controllers capable of mitigating these disturbances to enhance system resilience and ensure reliable performance, thereby reducing vulnerability.

For a large class of nonlinear systems there exists a particular [unstable equilibrium point \(UEP\)](#) on the [region of attraction \(RoA\)](#) boundary of the desired [SEP](#) such that the unstable eigenvector of the Jacobian at this [UEP](#) represents the mode of instability for the disturbance. Unfortunately, it is challenging to find this mode of instability, especially in high dimensional systems, because it is often computationally intractable to compute this particular [UEP](#) for a given disturbance. Consider a particular finite-time disturbance applied to a nonlinear system which possesses a [SEP](#) representing desired behavior. The system is able to recover from the disturbance if its post-disturbance [initial condition \(IC\)](#) lies inside the [RoA](#) of the desired [SEP](#). In cases where the system fails to recover, the nonlinear mode of instability for the disturbance represents the subset of system dynamics most responsible for this failure to recover.

This thesis develops a novel algorithm for numerically computing the mode of instability for parameter-dependent nonlinear systems without prior knowledge of the particular [UEP](#), resulting in a computationally efficient method. The key idea is to first consider the setting where the system recovers, and to average the Jacobian along the system trajectory from the post-disturbance state up until the Jacobian becomes stable. As the system approaches inability to recover, the averaged Jacobians converge to the Jacobian at the particular [UEP](#), and can be used to extract the unstable eigenvector representing the mode of instability. Convergence guarantees are provided for computing the mode of instability, both for the theoretical setting in continuous time, and for the proposed algorithm which relies on numerical integration. Numerical examples illustrate the successful application of the method to identify the mechanism of instability in power systems subject to temporary short circuits.

Then a novel approach to control design for reducing disturbance vulnerability of nonlinear systems using knowledge of the mode of instability is developed. The main idea is to tune controller parameter values so as to drive the post-disturbance [IC](#) further inwards away from the [RoA](#) boundary by driving it in the direction opposite to the mode of instability. To achieve this, the problem is formulated as a nonconvex optimization problem, and

an efficient algorithm is developed to solve it. Local convergence guarantees are provided for this method. Numerical examples illustrate the successful application of the method to reduce the vulnerability of power systems subject to temporary short circuits.

Acknowledgements

First and foremost, I would like to express my deepest gratitude to my supervisor, Professor Michael W. Fisher, for giving me the opportunity to work in the DOCS group and for his invaluable support throughout my MASc journey. At the beginning of my graduate studies, I faced challenges in transitioning to the expectations of graduate-level research. Professor Fisher guided me through this period with patience and encouragement. Our many fruitful discussions shaped the direction of my research and deepened my understanding of the subject. He encouraged me to first apply the proposed algorithm for computing the mode of instability to a simple nonlinear pendulum example subject to a disturbance before moving on to theoretical proofs. This hands-on approach greatly helped me grasp the practical aspects of the algorithm.

I am especially grateful for Professor Fisher’s detailed guidance in proving the convergence guarantees, both for the theoretical continuous-time setting and for the proposed algorithm based on numerical integration. He walked me through each step of the proof, offering insight into foundational results from his prior work, including those published in his SIAM paper. His introduction to essential theoretical tools such as the Implicit Function Theorem and the Inclination Lemma, as well as his intuitive breakdown of the system trajectory into three segments, left a lasting impact on my understanding. For the more complex case study involving the IEEE 9-bus power system, I deeply appreciate his willingness to explain the role of each MATLAB file and his help in debugging simulations when issues arose. Our discussions about the results were not only insightful but also instrumental in shaping the conclusions of this thesis. During times when progress was slow, he was supportive and flexible, always suggesting promising directions to explore further.

I am also sincerely thankful for Professor Fisher’s detailed feedback on the journal and conference papers I submitted. His advice on identifying relevant historical work, highlighting the contributions of our research, and presenting clear and coherent proofs has been invaluable in developing my academic writing skills and preparing me for future research endeavors. This thesis would not have been possible without Professor Fisher’s guidance, encouragement, and unwavering support, qualities that go above and beyond the role of a supervisor. I feel incredibly fortunate to have been under his mentorship.

I would also like to thank my readers, Professor Christopher Nielsen and Professor Jun Liu. I sincerely appreciate your time in reviewing my thesis and attending my seminar. The insightful questions you raised during the seminar not only deepened my understanding of the research setting but also offered valuable perspectives and promising directions for future work.

I would also like to extend my sincere thanks to all my lab mates: Zhong Fang, Murphy Wu, Penny Kitaev, and Elin Liu, for making my time in the DOCS group both intellectually enriching and personally meaningful.

To Zhong Fang, I feel incredibly fortunate to have had you as my lab mate. I will truly miss our conversations and interactions in the office. You were always willing to help whenever I faced difficulties with assignments, and I greatly appreciated your support. Beyond academics, we shared many unforgettable moments, from going to the gym together, to our trips to Niagara Falls where we saw the rainbow, and to High Park during sakura season. I remember seeing baby geese at the University of Waterloo and visiting Waterloo Park and Victoria Park to enjoy the autumn maple leaves. We captured so many wonderful memories in photos, and each experience holds a special place in my heart. I also thank you for your valuable advice on the importance of independence in research, something I will strive to improve upon in my future endeavors. I am proud to have you as one of my closest friends in Waterloo.

To Murphy Wu, I am grateful for the opportunity to share courses and thoughtful conversations with you in the office. Your presence made my academic journey more engaging and enjoyable. To Penny Kitaev, thank you for always being there when I encountered issues with MATLAB coding. I deeply appreciate the time and effort you took to support me, it made a big difference. To Elin Liu, it was a pleasure to have you as the lively and extroverted spirit of our lab. You consistently brought warmth and a sense of community to the DOCS group, making it feel like a supportive family. I especially enjoyed the Thanksgiving event you organized, which provided a wonderful opportunity for us to connect and build friendships.

Furthermore, I would like to thank all my other friends in Waterloo who made my time in Waterloo so memorable. Thank you to Wincy Huang, Wenkai Zhao, Olivia Ouyang, Patrick Luo, and Andy Meng for the wonderful moments we shared: playing badminton and card games during weekends, visiting Wonderland during reading week, skiing during my winter break, and enjoying the delicious meals you prepared. To Tianyi Zhang, Manuel Wang, and Taiqian Zhang, thank you for inviting me to join the University of Waterloo Intramural Soccer Team. It was a pleasure to be your teammate and to experience the spirit of sportsmanship together. Overall, These experiences played a crucial role in helping me unwind and manage the stresses of academic life. I am truly grateful for your friendship and support. You made my time in Waterloo truly memorable, and I will undoubtedly look back on this chapter of my life with fondness and gratitude.

Finally, I am deeply grateful to my parents, Dongbai Wang and Yan Wang, to whom this thesis is dedicated. You have always reminded me not to be too hard on myself, and that

it is just as important to eat well, sleep well, and take breaks when feeling overwhelmed. Your words have consistently helped me regain perspective: as long as I try my best, there is nothing to regret, regardless of the outcome. I will miss our weekly calls that kept me grounded and connected. Without your unwavering love, support, and encouragement, I could not have completed this journey.

Dedication

I dedicate this thesis to my parents, Dongbai Wang and Yan Wang. Their unwavering support, wholehearted care, comforting reassurance, and constant encouragement have made this journey possible.

Table of Contents

Author’s Declaration	ii
Abstract	iii
Acknowledgements	v
Dedication	viii
List of Figures	xii
List of Tables	xv
List of Abbreviations	xvi
List of Symbols	xvii
1 Introduction	1
1.1 Background	1
1.2 Literature Review	2
1.2.1 Historical Work	2
1.2.2 Limitations	3
1.3 Motivation	3
1.4 Contribution	4

1.4.1	Computing the Nonlinear Mode of Instability	4
1.4.2	Control Design for Reducing Disturbance Vulnerability	5
1.5	Organization of the Thesis	6
2	Preliminaries	7
2.1	Introduction	7
2.2	Dynamical System Model	7
2.3	Notations and Definitions	8
2.4	Key Results from Previous Work	10
2.5	Conclusion	11
3	Main Results for Computing the Nonlinear Mode of Instability	12
3.1	Introduction	12
3.2	Defining Mode of Instability for Parameterized Systems	12
3.3	Obtaining Mode of Instability via Jacobian Averaging	13
3.3.1	Theorem 3.3.1	14
3.3.2	Theorem 3.3.2	15
3.4	Algorithm for Numerically Computing Mode of Instability	15
3.4.1	Theorem 3.4.1	16
3.4.2	Theorem 3.4.2	16
3.5	Conclusion	17
4	Algorithm for Reducing Vulnerability	18
4.1	Introduction	18
4.2	Finding the Desired Direction to Move the IC	18
4.3	Driving the IC in the Desired Direction	20
4.4	Conclusion	23

5	Proofs	24
5.1	Proof of Theorem 3.3.1	24
5.2	Proof of Theorem 3.3.2	30
5.3	Proof of Theorem 3.4.1	30
5.4	Proof of Theorem 3.4.2	34
5.5	Proof of Lemma 4.2.1	37
5.6	Proof of Theorem 4.2.1	37
5.7	Proof of Corollary 4.2.1	38
5.8	Proof of Theorem 4.3.1	38
6	Application to Power Systems	39
6.1	Damped, Driven Nonlinear Pendulum	39
6.1.1	Introduction	39
6.1.2	Numerical Experiments	40
6.1.3	Results	41
6.1.4	Conclusion	42
6.2	IEEE 9-Bus Power System	44
6.2.1	Introduction	44
6.2.2	Numerical Experiments	46
6.2.3	Results	49
6.2.4	Conclusion	54
7	Conclusion and Future Work	59
	References	61

List of Figures

4.1	The system trajectory (cyan dashed line) for parameter values near the recovery boundary in parameter space (recovery boundary) from the IC $x_0(p)$ (green circle) and converges to the SEP $x^s(p)$ (dark blue circle). The stable manifold $W^s(x^u(p))$ (blue solid line) and the unstable manifold of $W^u(x^u(p))$ (red solid line) of the controlling UEP $x^u(p)$ (red circle), are shown. The illustration of mapping $\hat{v}(p)$ (black arrow) and $\hat{U}(p)$ (red arrow) backward in time until the IC, which gives $\tilde{v}(p)$ (brown arrow) and $\tilde{U}(p)$ (blue arrow), is shown.	21
5.1	The system trajectory (cyan dashed line) for parameter values near the recovery boundary from the IC $x_0(p)$ (green circle) and converges to the SEP $x^s(p)$ (dark blue circle). The stable manifold $W^s(x^u(p))$ (blue solid line) and the unstable manifold of $W^u(x^u(p))$ (red solid line) of the controlling UEP $x^u(p)$ (red circle), are shown.	25
5.2	The system trajectory in discrete time (cyan dotted line) for parameter values near the recovery boundary. This figure is similar to Fig. 5.1, except we are considering the state obtained from the continuous time system (2.1)-(2.2) with new labels and definitions.	31
6.1	The RoA boundary $\partial W^s(x_{0.02}^s(\hat{p}))$ (solid black line) of the SEP $x_{0.02}^s(\hat{p})$ (blue star) of equations (6.3)-(6.4) is shown. It is equal to $W^s(x_{0.02}^u(\hat{p}))$ where $x_{0.02}^u(\hat{p})$ (red triangle) is the UEP that can be shown to be the controlling UEP for the given disturbance. The IC $x_0(\hat{p})$ (green circle) is shown. The unstable eigenvector $v(\hat{p})$ of $\frac{\partial f}{\partial x}(x_{0.02}^u(\hat{p}))$ (solid magenta line with an arrow), which is the mode of instability for the disturbance, is shown. . . .	42
6.2	The closest boundary parameter value $p^*(h)$ to p_0 as a function of h	43

6.3	The norm of the difference between $\tilde{F}(p^*(h))$ and $A(p^*(0.02))$ as a function of h	43
6.4	The absolute value of the difference between the unstable eigenvalues $\tilde{\lambda}(p^*(h))$ of $\tilde{F}(p^*(h))$ and $\lambda(p^*(0.02))$ of $A(p^*(0.02))$ as a function of h	43
6.5	The norm of the difference between the unstable eigenvectors $\tilde{v}(p^*(h))$ of $\tilde{F}(p^*(h))$ and $v(p^*(0.02))$ of $A(p^*(0.02))$ as a function of h	43
6.6	The RoA boundaries $\partial W^s(x^s(p))$ of the SEPs $x^s(p)$ (stars) for parameter values $p = \bar{p}$ (red dashed) and $p = p_0$ (solid blue), are shown. Each boundary is equal to $W^s(x^*(p))$ where $x^*(p)$ (triangles) is the controlling UEP. The IC $x_0(p)$ (circles) are shown for each parameter value.	44
6.7	IEEE 9-Bus Power System (SG = synchronous generator).	45
6.8	The frequency dynamics of all three generators before reaching synchronism in the one-dimensional parameter space, where the moment of inertia scaling factor is the parameter of interest, at $p = p^*(\frac{1}{60})$	50
6.9	The angle of generator two and generator three relative to generator one in the one-dimensional parameter space, where the moment of inertia scaling factor is the parameter of interest, at $p = p^*(\frac{1}{60})$	51
6.10	The internal state dynamics of the Active Voltage Regulator (AVR) of all three synchronous generators in the one-dimensional parameter space, where the moment of inertia scaling factor is the parameter of interest, at $p = p^*(\frac{1}{60})$	52
6.11	The frequency dynamics of all three generators before reaching synchronism in the 15-dimensional parameter space, where the parameters of interest are the moment of inertia, real and reactive load voltage exponents, AVR and Power System Stabilizer (PSS) controller gains, at $p = p^*(\frac{1}{60})$	53
6.12	The angle of generator two and generator three relative to generator one in the 15-dimensional parameter space, where the parameters of interest are the moment of inertia, real and reactive load voltage exponents, AVR and PSS controller gains, at $p = p^*(\frac{1}{60})$	54
6.13	The internal state dynamics of the AVR of all three synchronous generators in the 15-dimensional parameter space, where the parameters of interest are the moment of inertia, real and reactive load voltage exponents, AVR and PSS controller gains, at $p = p^*(\frac{1}{60})$	55

6.14	Frequency dynamics of generator one for the parameter value p_0 and $\bar{p}$ in the 9-dimensional parameter space, where the parameters of interest are the AVR, PSS, and governor controller gains.	56
6.15	Frequency dynamics of generator two for the parameter value p_0 and $\bar{p}$ in the 9-dimensional parameter space, where the parameters of interest are the AVR, PSS, and governor controller gains.	57
6.16	Frequency dynamics of generator three for the parameter value p_0 and $\bar{p}$ in the 9-dimensional parameter space, where the parameters of interest are the AVR, PSS, and governor controller gains.	58

List of Tables

6.1	Components of $\tilde{v}(\hat{p})$ representing the frequencies ω , angles θ , and the internal state of the AVR x for the three synchronous generators in the case of one-dimensional parameter space.	46
6.2	Test case parameter values.	47
6.3	Closest boundary parameter value for moment of inertia, real and reactive load voltage exponents, AVR and PSS controller gains.	48
6.4	Components of $\tilde{v}(\hat{p})$ representing the frequencies ω , angles θ , and the internal state of the AVR x for the three synchronous generators in the case of 15-dimensional parameter space.	48
6.5	Test case parameter values.	49
6.6	Closest boundary parameter value for the AVR, PSS, and governor controller gains.	49
6.7	Local minimum of the control design optimization problem.	55
6.8	Closest boundary parameter value for the parameter value in Table 6.7. . .	56

List of Abbreviations

AVR Active Voltage Regulator [xiii–xv](#), [44–58](#)

IC initial condition [iii](#), [xii](#), [xiii](#), [xviii](#), [2–10](#), [13–15](#), [18–25](#), [27](#), [37](#), [40–42](#), [44](#), [59](#)

ODEs ordinary differential equations [7](#)

PSS Power System Stabilizer [xiii–xv](#), [5](#), [44](#), [45](#), [47–50](#), [52–58](#)

recovery boundary recovery boundary in parameter space [xii](#), [3–5](#), [9–11](#), [16](#), [21](#), [25](#), [31](#), [46](#), [47](#)

recovery region recovery region in parameter space [9](#), [59](#)

RoA region of attraction [iii](#), [xii](#), [xiii](#), [2–6](#), [9](#), [10](#), [12](#), [18–20](#), [22](#), [23](#), [41](#), [42](#), [44](#), [59](#)

SEP stable equilibrium point [iii](#), [xii](#), [xiii](#), [1](#), [2](#), [4](#), [9](#), [16](#), [21](#), [24](#), [25](#), [35](#), [39](#), [40](#), [42](#), [44](#), [45](#)

UEP unstable equilibrium point [iii](#), [xii](#), [xiii](#), [2–6](#), [13](#), [14](#), [16](#), [19–21](#), [24](#), [25](#), [34](#), [39–42](#), [44–47](#), [59](#), [60](#)

List of Symbols

- $B_r(x)$ The open ball of radius r centered at x . 8
- ∂E The topological boundary of E . 8
- ∂R The recovery parameter boundary. 9–12, 14, 20, 36, 40
- C^1 Continuously differentiable. 7–11, 24–27, 35, 36
- $x(p)$ The critical element corresponding to parameter value p . 8
- $\overline{E}$ The topological closure of E . 8
- $\mathbb{R}^n \times J$ The Cartesian product of $\mathbb{R}^n$ and J . 7, 8
- d_{C_1} The C_1 distance. 9
- x The dynamic states. 7, 8
- $\mathbb{R}^n$ The n -dimensional Euclidean space. 7–9, 11
- $\phi(x, p, t)$ The flow from x corresponding to time t and parameter value p . 8
- $\|\cdot\|$ The Frobenius norm. 11
- I_n The $n \times n$ identity matrix. 11
- int** E The topological interior of E . 8
- $W_{\text{loc}}^s(x(p))$ The local stable manifold of $x(p)$. 8
- $W_{\text{loc}}^u(x(p))$ The local unstable manifold of $x(p)$. 8
- $\mathbb{N}$ The set of natural numbers. 11

$N_x A$ The normal space to A at x . 8
 $\Omega(f)$ The nonwandering set under f . 8
 S The Poincaré cross section. 8
 τ The Poincaré first return map. 8
 $\frac{\partial f}{\partial x}(x(p))$ The partial derivative of f with respect to x at $x(p)$. 8
 $\phi(x, p, T) \sqcup_{t \in T} \phi(x, p, t)$. 8
 $\phi(x_0(J'), J', T) \sqcup_{p \in J'} \phi(x_0(p), p, T)$. 8
 $\mathbf{proj}_V(x)$ The projection of x onto the image of V . 9
 J The connected smooth manifold which represents parameter values. 7–11
 R The set of recovery parameter values. 9, 11, 14–16, 20, 24, 26, 27, 29–32, 34–36, 40, 48, 53
 $\mathbb{R}^{n \times n}$ The set of real matrices of dimension $n \times n$. 11
 $\mathbb{R}$ The set of real numbers. 8
 $W^s(x(p))$ The stable manifold of $x(p)$. 8, 9
 $W^s(x(J')) \sqcup_{p \in J'} W^s(x(p))$. 8
 $T_x A$ The tangent space to A at x . 8, 9
 $\dot{x}$ The time derivative of the dynamic states. 7
 $x_0(p)$ The IC corresponding to parameter value p . 7, 9
 $\|\cdot\|_2$ The Euclidean norm. 11
 $W^u(x(p))$ The unstable manifold of $x(p)$. 8
 $W^u(x(J')) \sqcup_{p \in J'} W^u(x(p))$. 8
 f_p The vector field corresponding to parameter value p . 7, 8
 f The vector field. 7, 8, 24, 27
 $x(J') \sqcup_{p \in J'} x(p)$. 8

Chapter 1

Introduction

1.1 Background

Disturbances naturally occur in engineered systems, and may disrupt desired system behavior. For example, temporary short circuits in a power system can lead to blackout conditions, and other common disturbances in power systems include sudden load changes, equipment failures, and fluctuations in renewable energy generation, all of which may result in instability or blackout conditions. When the system is unable to recover from a given finite-time disturbance to a [SEP](#) representing desired behavior, often there is a particular subset of system states whose coupled dynamics is responsible for this failure to recover. We refer to these lower order dynamics that result in failure to recover as the mechanism of instability of the disturbance [\[20\]](#). In networked or complex systems, the mechanism of instability typically also reveals the components of the system most responsible for this inability to recover. Determining the mechanism of instability is valuable because focusing on the most important subset of system dynamics for a given disturbance facilitates detecting proximity to vulnerability to the disturbance and designing controllers to reduce this vulnerability [\[10, 16\]](#).

To ensure safe and reliable operation, it is important to assess the system's vulnerability to credible disturbances. If the likelihood of recovery from a particular potential disturbance is too low, the system is considered to be vulnerable, and preventive measures should be implemented. Ideally, these measures would keep the system at the same operating point in order to maintain desired behavior, while tuning controllers so as to more effectively mitigate the particular nonlinear disturbance, thereby reducing vulnerability.

Fix a particular finite-time disturbance, such as a particular short circuit at a particular location in the system, and denote the system state immediately after the disturbance as the post-disturbance **IC**, or **IC** for short. Whether or not the system recovers from the disturbance to the desired **SEP** depends on whether the **IC** lies within the **RoA** of that **SEP**. When the system is just marginally unable to recover, its **IC** lies on the **RoA** boundary of the **SEP** and, for a large class of nonlinear systems, within the stable manifold of some **UEP**, which is known as the controlling **UEP** [8]. Generically, the Jacobian at the controlling **UEP** has a unique unstable eigenvalue [10] and, as in [16], we define the nonlinear mode of instability to be the corresponding eigenvector of that unstable eigenvalue. In the situations where the system fails to recover, this mode of instability is responsible for driving the system state away from the **RoA** after the disturbance [16], and thus represents the mechanism of instability.

Whether the system can recover from the given disturbance to the desired **SEP** depends on the system parameter values, including controller parameters such as controller gains. For certain parameter values, the **IC** lies outside the **RoA** of the **SEP**, and the system is not able to recover from the disturbance. For other parameter values, the **IC** lies inside the **RoA** of the **SEP**, in which case the system should be able to recover from the disturbance.

1.2 Literature Review

1.2.1 Historical Work

There is a long history on identifying the nonlinear mode of instability. In [24] the mode of instability is computed and then used to identify the coupled oscillators in the system that are the first to lose synchronization in response to a disturbance. More recent work includes using the mode of instability to quantify the relative contributions of each state to transient dynamics using controlling **UEP**-based participation factor analysis with its corresponding unstable eigenvector [23]. The main method in the literature for computing the mode of instability is to first determine the controlling **UEP**, and then directly find the unstable eigenvector of its Jacobian [16]. Much of the historical work on disturbance vulnerability has focused on **RoA** estimation using sublevel sets of Lyapunov functions [8, 31], and the solutions of related partial differential equations [35, 33]. Nevertheless, there has been some work on enlarging the **RoA**, and on making it more robust to parameter uncertainty, in order to reduce vulnerability [6, 7, 3]. These methods require the construction of a Lyapunov function and the maximization of the largest estimate of the **RoA** it can provide.

1.2.2 Limitations

However, determining the controlling UEP is itself a challenging problem and can often be intractable in practice [4]. Although several methods exist in the literature for finding the controlling UEP, including the mode of disturbance method [15], the exit point method (also known as the BCU method) [9], and the shadowing method [32], these methods typically lack rigorous convergence guarantees, may converge to an incorrect equilibrium point or fail to converge at all, and often require many time-consuming simulations, which can lead to prohibitively large computational cost, especially for high dimensional systems. Moreover, [8, 31, 35, 33, 6, 7, 3] are often either overly conservative in practice or computationally inefficient for high dimensional systems with uncertain parameters.

1.3 Motivation

The first goal of this thesis is to develop a computationally efficient method with rigorous convergence guarantees for computing the nonlinear mode of instability without requiring knowledge of the controlling UEP. This provides guarantees of reliability for correctly computing the mode of instability while avoiding the challenge of identifying the controlling UEP. While much of the historical work focused on finding the mode of instability for a single vector field, this thesis considers the more general and more challenging case of finding this mode for a parameter-dependent vector field with parameter-dependent IC. This more general setting is adopted here because disturbance recovery often depends on system parameters, which may be uncertain and time-varying. As in [14], we define the *recovery boundary* to be the boundary in parameter space between the parameter values for which the system recovers from the disturbance and the parameter values for which it fails to recover. In the special case where the parameter is chosen to be the IC, the *recovery boundary* becomes equal to the RoA boundary, so it can be interpreted as a generalization of the RoA boundary to assess vulnerability in parameter space.

In practice, due to uncertainty and modeling errors, in order to improve the likelihood that the system recovers it is valuable for the IC to lie within the RoA and a significant distance away from the RoA boundary. Then the goal is to develop a novel and efficient method for reducing disturbance vulnerability in nonlinear systems by varying parameters so as to drive the IC further inwards and away from the RoA boundary. This is very challenging to accomplish because identifying the RoA boundary in high dimensional systems is generally intractable. Rather than explicitly computing the RoA boundary, the approach employed here involves first identifying a direction in which to move the IC so

that it moves away from the [RoA](#) boundary, and then varying parameter values so as to drive the [IC](#) along this direction.

1.4 Contribution

1.4.1 Computing the Nonlinear Mode of Instability

The method presented in this thesis exploits the properties that as parameter values approach the [recovery boundary](#) in parameter space, the [IC](#) approaches the [RoA](#) boundary in state space and the amount of time the system trajectory spends near the controlling [UEP](#) diverges towards infinity [13]. For a parameter value for which the system recovers from the disturbance, the key idea of the proposed approach is to average the Jacobian along the system trajectory from the [IC](#) until the final time at which the Jacobian transitions from unstable to stable. The intuition is that as parameter values approach the [recovery boundary](#), since the system trajectory spends increasing time near the controlling [UEP](#), the Jacobians along the system trajectory will spend increasing time close to the Jacobian at the controlling [UEP](#), and their average will approach the Jacobian at the controlling [UEP](#). However, it is important to only average along the trajectory until the final time at which the Jacobian transitions to becoming stable, since the remaining portion of the trajectory converges to the [SEP](#) and, thus, averaging over the entire trajectory would result in the average approaching the Jacobian at the [SEP](#) rather than the Jacobian at the controlling [UEP](#).

The method presented in this thesis relies on prior work [12], [11] to find parameter values on the [recovery boundary](#), and numerically computes this average of the Jacobians as parameter values approach the [recovery boundary](#). We prove that as the parameter values approach the [recovery boundary](#), this average of the Jacobians converges to the true Jacobian at the controlling [UEP](#). Furthermore, under the same conditions we prove that the average of the Jacobians will have a unique unstable eigenvalue whose corresponding eigenvector converges to the mode of instability. The average of the Jacobians is evaluated in practice using numerical integration of the underlying system trajectory, which introduces approximation errors. Therefore, for the proposed algorithm which relies on numerical integration, we further prove that the approximation of the average of the Jacobians obtained from numerical integration, and its eigenvector corresponding to its unstable eigenvalue, will converge to the true Jacobian at the controlling [UEP](#) and the mode of instability, respectively, in the limit as the time step of the numerical integration approaches zero. Ultimately, the proposed approach is computationally efficient, does not

require knowledge of the controlling UEP, and these convergence guarantees ensure it will reliably compute the mode of instability accurately.

The method is first validated on the simple example of a nonlinear pendulum subject to a disturbance. In this case, since the system is low dimensional the controlling UEP and mode of instability can be easily obtained analytically. It is shown that the proposed method accurately computes the mode of instability. Then, the method is applied to determine the mode of instability of a power system containing multiple generators subject to a temporary short circuit at a particular location in the network without attempting to identify the controlling UEP. The points on the recovery boundary are considered both in one and higher dimensional parameter spaces. The identified modes of instability provide non-intuitive insights into the mechanism of instability, including a case where the generator most responsible for failure to recover is distant from the location of the disturbance, a situation where one generator plays a minimal role for the inability to recover, and that the PSS controllers, which are often associated with the failure to recover, do not exhibit much impact in these cases. This unexpected dynamic behavior is challenging to predict or identify without applying the proposed algorithm.

1.4.2 Control Design for Reducing Disturbance Vulnerability

Since in the situations where the system fails to recover, this nonlinear mode of instability is responsible for driving the system state away from the RoA after the disturbance. Therefore, the direction in which to move the IC is chosen to be in the opposite direction to the mode of instability, thereby helping mitigate the impact of the disturbance.

The problem of varying parameter values so as to drive the IC in the opposite direction to the mode of instability is then formulated as a nonconvex optimization problem, and an efficient algorithm is developed to solve it. Convergence guarantees are provided that show that the algorithm will converge to a local minimum if the initial parameter value is sufficiently close to its local minimizer. The algorithm is validated first on a simple test case in two dimensions in which the RoA boundary and the mode of instability can be directly computed and visualized. Then, it is demonstrated in a higher dimensional setting of a power system subject to a temporary short circuit where it is intractable to efficiently compute the RoA boundary and the mode of instability directly, but it can still successfully reduce vulnerability to the disturbance.

1.5 Organization of the Thesis

The remainder of the thesis is organized as follows. Chapter 1 provides background information, a literature review, research motivation, and an overview of the contributions of each chapter. We introduce the algorithm for computing the nonlinear mode of instability without requiring identification of the controlling UEP, as well as the concept of varying controller parameter values to reduce disturbance vulnerability using the identified mode of instability.

Chapter 2 presents preliminary material, including the dynamical system model that forms the foundation of this thesis. It also introduces the notations and definitions necessary for the theoretical development. In addition, key results from previous work that support the convergence guarantees of the proposed algorithm are reviewed.

Chapter 3 presents the main results for computing the nonlinear mode of instability. It begins with a formal definition of the mode of instability for parameterized systems. The chapter then introduces the concept of obtaining the mode of instability through Jacobian averaging in a continuous-time theoretical setting. Finally, this idea is extended to a practical algorithm for computing the mode of instability using numerical integration.

Chapter 4 addresses the control design for reducing disturbance vulnerability using the identified mode of instability. It first establishes a direction in which to shift the IC to increase its distance from the RoA boundary. This direction is then used to guide the variation of parameter values, effectively steering the IC along the desired path to enhance system robustness.

Chapter 5 provides proofs that establish convergence guarantees for computing the mode of instability, both in the continuous-time theoretical setting, and for the proposed algorithm based on numerical integration. The chapter also proves the theoretical foundation of the control design for reducing vulnerability, including local convergence results.

Chapter 6 shows illustrative examples demonstrating the application of the proposed method. It demonstrates the successful application of the algorithm in computing the mode of instability and reducing disturbance vulnerability for the damped, driven nonlinear pendulum and IEEE 9-bus power system.

Finally, Chapter 7 presents a comprehensive conclusion of the thesis and outlines future research directions. These include extending the proposed method to controlling periodic orbits and other controlling sets beyond equilibrium points, as well as addressing piecewise smooth vector fields with switching and saturation limits.

Chapter 2

Preliminaries

2.1 Introduction

In this chapter, we first present preliminary material, including the dynamical system model that forms the foundation of this thesis in Section 2.2. Then we introduce the notations and definitions necessary for the theoretical development in Section 2.3. In addition, key results from previous work that support the convergence guarantees of the proposed algorithm are reviewed in Section 2.4.

2.2 Dynamical System Model

Let J be a connected smooth manifold that is a subset of $\mathbb{R}^m$ for some integer $m > 0$ representing the parameter space, where $\mathbb{R}^m$ is the m -dimensional Euclidean space. Consider a family of vector fields $f : \mathbb{R}^n \times J \rightarrow \mathbb{R}^n$, parameterized by J , such that f is C^1 and $n > 0$ is an integer. Then for any $p \in J$, we have that $f_p := f(\cdot, p)$ is a vector field on $\mathbb{R}^n$. We use $\dot{x}$ to denote the time derivative of the dynamic states $x \in \mathbb{R}^n$. We model a particular finite-time parameter-dependent disturbance as a parameter-dependent post-disturbance IC, which we simply refer to as the IC $x_0 : J \rightarrow \mathbb{R}^n$ given by $x(0) := x_0(p)$. Hence, we consider the family of ordinary differential equations (ODEs) given for $p \in J$ by

$$\dot{x} = f(x, p), \tag{2.1}$$

$$x(0) = x_0(p). \tag{2.2}$$

Let $\phi(x, p, t)$ denote the flow of the vector field f from the IC $x \in \mathbb{R}^n$ at time $t \in \mathbb{R}$ for parameter value $p \in J$. For any subset $T \subset \mathbb{R}$, let $\phi(x, p, T) = \bigcup_{t \in T} \phi(x, p, t)$. For any subsets $J' \subset J$ and $T \subset \mathbb{R}$, let $\phi(x_0(J'), J', T)$ denote the disjoint union $\sqcup_{p \in J'} \phi(x_0(p), p, T) = \bigcup_{p \in J'} \phi(x_0(p), p, T) \times \{p\} \subset \mathbb{R}^n \times J$.

2.3 Notations and Definitions

Consider a fixed $p \in J$. For any $x \in \mathbb{R}^n$ we say that x is nonwandering under f_p if for every open neighborhood U of x and any $T \in \mathbb{R}$, there exists $t \in \mathbb{R}$ of the same sign as T with $|t| > T$ such that $\phi(U, p, t) \cap U \neq \emptyset$. Let $\Omega(f)$ denote the set of all nonwandering points of f_p , and note that this includes all equilibria and periodic orbits of f_p , as well as chaotic dynamics and other forms of recurrent behavior. Let a *critical element* $x(p)$ be either an equilibrium point or a periodic orbit of f_p . An equilibrium point $x(p)$ of f_p is hyperbolic if its linearization $\frac{\partial f}{\partial x}(x(p))$ has no purely imaginary eigenvalues. A periodic orbit $x(p)$ of f_p is hyperbolic if there exists $x \in x(p)$ and a cross section S containing x such that the Poincaré first return map $\tau : S \rightarrow S$ is well-defined and its linearization $\frac{d\tau(x)}{dx}$ has no eigenvalues of norm one. Every hyperbolic critical element $x(p)$ of f_p possesses a stable manifold $W^s(x(p))$ and an unstable manifold $W^u(x(p))$ where $W^s(x(p))$ ($W^u(x(p))$) consists of all IC that converge to $x(p)$ in forwards (backwards) time, respectively. Furthermore, there exist local stable and unstable manifolds, denoted by $W_{\text{loc}}^s(x(p))$ and $W_{\text{loc}}^u(x(p))$, which are invariant in forwards and backwards time, respectively. Furthermore, for any hyperbolic critical element $x(\tilde{p})$ of $f_{\tilde{p}}$, for J sufficiently small and any $p \in J$, there exists a unique hyperbolic critical element $x(p)$ that is equal to $x(\tilde{p})$ at $p = \tilde{p}$, and such that $x(p)$, $W_{\text{loc}}^s(x(p))$, and $W_{\text{loc}}^u(x(p))$ all vary C^1 with p . For any $J' \subset J$, define $x(J') = \sqcup_{p \in J'} x(p) = \bigcup_{p \in J'} x(p) \times \{p\} \subset \mathbb{R}^n \times J$. Define $W^s(x(J'))$ and $W^u(x(J'))$ analogously.

The notion of a generic C^1 vector field is used to describe typical behavior, similar to the concept of probability one in a probability space. A property that holds for a generic class of C^1 vector fields is therefore considered to represent typical or expected behavior. Since there exists many pathological C^1 vector fields, it is often beneficial to restrict analysis to certain classes of generic C^1 vector fields, which facilitates establishing the theoretical framework for generic vector fields that often would not hold for arbitrary vector fields. For any set E in a topological space, let ∂E denote its topological boundary, $\overline{E}$ its topological closure, and $\text{int } E$ its topological interior. For any point $x \in \mathbb{R}^n$ in state space and any $r > 0$, define $B_r(x)$ to be the open ball of radius r centered at x . For any manifold A and $x \in A$, let $T_x A$ denote the tangent space to A at x and $N_x A$ denote the normal space to A at x . Two manifolds $A, B \subset \mathbb{R}^n$ are transverse if for every $x \in A \cap B$,

$T_x A + T_x B = T_x \mathbb{R}^n$. Given two differentiable manifolds M and N , a differentiable map $g : M \rightarrow N$ is a diffeomorphism if it is a bijection and its inverse $g^{-1} : N \rightarrow M$ is differentiable as well. Let $V \in \mathbb{R}^{n \times k}$ be the matrix whose columns are the orthonormal basis vectors. For any $x \in \mathbb{R}^n$, the projection of x onto the image of V is $\text{proj}_V(x) = VV^\top x$. For additional background in differential topology, including the strong and weak C^1 topologies, we refer the reader to [17, Chapter 2]. For more information regarding d_{C^1} and ϵ C^1 -close, we refer the reader to the background section in [13].

Suppose $p_0 \in J$ such that f_{p_0} possesses a hyperbolic SEP, denoted by $x^s(p_0)$. Then for J sufficiently small and any $p \in J$, there exists a unique hyperbolic SEP $x^s(p)$ near $x^s(p_0)$. For $p \in J$, we define the RoA of $x^s(p)$ to be its stable manifold $W^s(x(p))$, and $\partial W^s(x^s(p))$ denotes the RoA boundary. For a parameter value $p \in J$, the system recovers from the disturbance if and only if $x_0(p) \in W^s(x(p))$. Let the *recovery region in parameter space* (*recovery region*) R be the set of parameter values in J for which the system recovers, and define the *recovery boundary* to be its topological boundary ∂R in J . We define parameter values that lie on the *recovery boundary* as *boundary parameter values*. For an initial parameter value $p_0 \in R$, its *safety margin* is defined to be the distance from p_0 to the closest boundary parameter value, and provides a measure of proximity to vulnerability. Algorithms were developed in [11] to efficiently compute the closest boundary parameter value for a given initial value p_0 .

Trajectory sensitivities are defined as partial derivatives of the system state with respect to ICs and/or parameter values. They measure the sensitivity of the system's states along a trajectory to small changes in ICs and/or parameter values. The trajectory sensitivities of (2.1) are obtained by differentiating both sides of the equation with respect to the parameter value p :

$$\frac{\partial \dot{x}}{\partial p} = \frac{\partial f}{\partial x}(x, p) \frac{\partial x}{\partial p}(p) + \frac{\partial f}{\partial p}(x, p). \quad (2.3)$$

In the special case where the parameter is chosen to be the IC (i.e., $p = x_0$, (2.4) can be reduced to

$$\frac{\partial \dot{x}}{\partial x_0} = \frac{\partial f}{\partial x}(x_0) \frac{\partial x}{\partial x_0}(x_0). \quad (2.4)$$

Trajectory sensitivities can be efficiently computed numerically (i.e., integrated) as a byproduct of numerical integration of the underlying dynamical system (2.1)-(2.2) [18].

2.4 Key Results from Previous Work

Propositions 2.4.1 and 2.4.2 will be useful for establishing rigorous guarantees for our method to compute the nonlinear mode of instability for a large class of parameterized nonlinear systems. Proposition 2.4.1 was proved in [13] and [14], and states that under general assumptions satisfied by a large class of dynamical systems, and for sufficiently small J , the RoA boundary in state space varies continuously with respect to parameter, and equals the union of the stable manifolds of the critical elements it contains. Since each parameter value on the recovery boundary has corresponding IC lying on the RoA boundary in state space, and therefore in the stable manifold of some critical element, which we call the *controlling critical element*, and whose unstable manifold intersects the RoA. This proposition establishes a direct connection between the recovery boundary in parameter space and the RoA boundary in state space, which will be used for proving the theoretical guarantees of the proposed algorithm for computing the mode of instability for parameterized nonlinear systems.

Proposition 2.4.1. [14, Theorem 4.17, Corollary 4.18, Theorem 4.21, Corollary 4.23] Assume there exists $p_0 \in J$ such that:

- (i) Every critical element in $\partial W^s(x^s(p_0))$ is hyperbolic.
- (ii) The intersections of the stable and unstable manifolds of the critical elements in $\partial W^s(x^s(p_0))$ are transverse.
- (iii) The intersection of $\partial W^s(x^s(p_0))$ with $\Omega(f_{p_0})$ consists of a finite union of critical elements $\{x^i(p_0)\}_{i \in I}$, where the set I indexes the critical elements that lie on $\partial W^s(x^s(p_0))$.
- (iv) There exists a neighborhood of infinity which contains no nonwandering points of f_{p_0} and no orbits which diverge to infinity in both forwards and backwards time.
- (v) The vector field family $\{f_p\}_{p \in J}$ is strong C^1 continuous and x_0 is C^1 .
- (vi) Certain lower semicontinuous functions over J are continuous at p_0 , see [14, Theorem 4.17] for more details.

Then for J sufficiently small, $\partial W^s(x^s(p))$ varies continuously with p and $\partial W^s(x^s(J)) = \bigcup_{i \in I} W^s(x^i(J))$. For any $p^* \in \partial R$, $x_0(p^*) \in \partial W^s(x^s(p^*))$. Then $x_0(p^*) \in W^s(x^j(p^*))$ for a unique $j \in I$, $x^j(p^*)$ is called the controlling critical element for p^* , and $W^s(x^s(p^*)) \cap W^u(x^j(p^*)) \neq \emptyset$.

Remark 2.4.1. Note that Assumptions (i), (ii), and (vi) are generic, and it is generically true that $\Omega(f_{p_0})$ is equal to the closure of the union of the critical elements of f_{p_0} [28] (compare to Assumption (iii)), so (iii) is equivalent to a finite number of critical elements, along with this generic condition. Thus, the conditions of Proposition 2.4.1 are very general and hold for a large class of realistic engineering system models.

Assumption 2.4.1. By Proposition 2.4.1, $\partial W^s(x^s(J)) = \bigcup_{i \in I} W^s(x^i(J))$. Assume that $x_0(J)$ is transverse to $W^s(x^i(J))$ for all $i \in I$ [11, Assumption 1].

Remark 2.4.2. Assumption 2.4.1 is generic because J submanifolds are generically transverse [19, Theorem A.3.20].

Proposition 2.4.2. Assume the conditions of Proposition 2.4.1 and Assumption 2.4.1. Then there exists an open neighborhood N of ∂R such that for generic $p_0 \in N$, there exists a unique boundary parameter value $p^* \in \partial R$ that is closest to p_0 , and that p^* varies C^1 continuously with initial parameter value p_0 [11, Theorem 4].

Remark 2.4.3. Proposition 2.4.2 shows that for generic initial parameter value p_0 sufficiently close to the recovery boundary, p_0 has a unique closest point on the recovery boundary that varies smoothly with respect to p_0 .

Recent algorithms have been developed for efficiently computing the closest boundary parameter value $p^* \in \partial R$ to an initial value $p_0 \in R$ in high dimensional parameter and state space [11]. Throughout this thesis, for $p^* \in \partial R$, let $\lim_{p \rightarrow p^*}$ denote the limit as p approaches p^* from within R , i.e., $\lim_{p \rightarrow p^*, p \in R}$. For any $A \in \mathbb{R}^{n \times n}$, let $\|\cdot\|$ denote the Frobenius norm defined as $\|A\| = \sqrt{\sum_{i=1}^n \sum_{j=1}^n |A_{ij}|^2}$. For any $y \in \mathbb{R}^n$, let $\|\cdot\|_2$ denote the two norm defined as $\|y\|_2 = \sqrt{\sum_{i=1}^n y_i^2}$. Let $\mathbb{N}$ and I_n denote the set of natural numbers and $n \times n$ identity matrix, respectively.

2.5 Conclusion

Therefore, we have presented preliminary material, including the foundational dynamical system model in Section 2.2, followed by the introduction of key notations and definitions required for the theoretical development in Section 2.3. Additionally, we have reviewed essential results from prior work that support the convergence guarantees of the proposed algorithm in Section 2.4.

Chapter 3

Main Results for Computing the Nonlinear Mode of Instability

3.1 Introduction

In this chapter, we present the main results for computing the nonlinear mode of instability. Section 3.2 introduces the formal definition of the nonlinear mode of instability for parameterized systems. In Section 3.3, we propose a method for obtaining the mode of instability with convergence guarantees using Jacobian averaging. These theoretical results are then extended in Section 3.4 to a practical setting, where the system trajectory is approximated through numerical integration.

3.2 Defining Mode of Instability for Parameterized Systems

We begin by generalizing the definition of the nonlinear mode of instability of a disturbance to parameter-dependent vector fields. Let N be the neighborhood of ∂R from Proposition 2.4.2, and fix any initial parameter value $p_0 \in N$ belonging to the generic set described in Proposition 2.4.2. Then there exists a unique closest boundary parameter value $p^* \in \partial R$ to p_0 . By Proposition 2.4.1, there exists a critical element $x^*(p^*)$ in the RoA boundary of $x^s(p^*)$ such that $x_0(p^*)$ lies in the stable manifold of $x^*(p^*)$.

Assumption 3.2.1. $W^s(x^*(p^*))$ has codimension one.

Remark 3.2.1. Assumption 3.2.1 is generic [11, Lemma 3].

Assumption 3.2.2. $x^*(p^*)$ is an equilibrium point.

Remark 3.2.2. For many practical nonlinear systems including power systems, Assumption 3.2.2 is ubiquitous and has been observed through numerical experiments on a wide range of realistic models [32].

Definition 3.2.1. Under Assumptions 3.2.1-3.2.2, $x^*(p^*)$ is known as the controlling UEP, and we denote it by $x^u(p^*)$. Define $A(p^*)$ to be the Jacobian at $x^u(p^*)$:

$$A(p^*) := \frac{\partial f}{\partial x}(x^u(p^*)). \quad (3.1)$$

By Assumption 3.2.1 and Proposition 2.4.1(i), $A(p^*)$ has a unique unstable eigenvalue which we denote by $\lambda(p^*)$. The mode of instability is defined to be the eigenvector of $A(p^*)$ associated with the eigenvalue $\lambda(p^*)$, and is denoted $v(p^*)$.

3.3 Obtaining Mode of Instability via Jacobian Averaging

In order to avoid the complexity of determining the controlling UEP directly, we propose a method that can indirectly calculate the mode of instability without requiring prior knowledge of the controlling UEP. At the closest boundary parameter value p^* to p_0 , the IC lies in the stable manifold of the controlling UEP $x^u(p^*)$, and thus the system trajectory converges to $x^u(p^*)$. As p approaches p^* from within the set of recovery values, the system trajectory spends a very large amount of time around the controlling UEP $x^u(p)$ before converging to $x^s(p)$. Thus, since the Jacobian depends continuously on state and parameter values, it seems natural to approximate $A(p^*)$ (the Jacobian at $x^u(p^*)$) by averaging the Jacobian along the system trajectory for p near p^* .

Unfortunately, averaging the Jacobian along the system trajectory for all time does not approximate $A(p^*)$ well since as time approaches infinity, the system converges to $x^s(p)$. This implies that the Jacobian along the trajectory converges to the Jacobian at $x^s(p)$, and the average over time of these Jacobians would also converge to the Jacobian at $x^s(p)$ rather than $A(p^*)$. To avoid this, we instead only average the Jacobians along the system trajectory for a finite length of time $t(p)$ given by the supremum of the times at which

these Jacobians are unstable. More precisely, we define

$$t(p) := \sup_{\substack{t > 0 \\ \frac{\partial f}{\partial x}(\phi(x_0(p), p, t)) \text{ is unstable}}} t, \quad (3.2)$$

where $\frac{\partial f}{\partial x}(\phi(x_0(p), p, t))$ is the Jacobian along the system trajectory at time t for parameter value p .

Note that for any p within the set of recovery values, $t(p)$ is finite since the system trajectory converges to $x^s(p)$, and the Jacobian is stable in a neighborhood of $x^s(p)$. The introduction of this $t(p)$ ensures that the values of the Jacobian near the controlling UEP dominate the average of the Jacobians along the system trajectory, and that this average will converge to $A(p^*)$ as p approaches p^* . To make this precise, define the piecewise function $F : R \cup \partial R \rightarrow \mathbb{R}^{n \times n}$ such that for any $p \in R$, F averages the Jacobians along the system trajectory from the IC up until $t(p)$, i.e.,

$$F(p) := \frac{1}{t(p)} \int_0^{t(p)} \frac{\partial f}{\partial x}(\phi(x_0(p), p, t)) dt. \quad (3.3)$$

For any $p^* \in \partial R$, $t(p^*) = \infty$ since $x_0(p^*) \in W^s(x^u(p^*))$. Thus, (3.2) is not defined at p^* , and so for any $p^* \in \partial R$ we define

$$F(p^*) := A(p^*). \quad (3.4)$$

3.3.1 Theorem 3.3.1

Theorem 3.3.1 justifies the proposed method for computing $A(p^*)$ indirectly. It shows that $F(p)$ will converge to $F(p^*)$ as p approaches p^* from within R , which implies that the average of the Jacobians along the system trajectory computed in (3.3) converges to $A(p^*)$.

Theorem 3.3.1. *Assume that p_0 satisfies the conditions of Proposition 2.4.1, and that Assumptions 2.4.1-3.2.1 hold. Then the piecewise function F defined in (3.3)-(3.4) is continuous at p^* :*

$$\lim_{p \rightarrow p^*} F(p) = F(p^*) = A(p^*). \quad (3.5)$$

3.3.2 Theorem 3.3.2

Theorem 3.3.2 shows that as p approaches p^* from within R , $F(p)$ has a unique unstable eigenvalue that converges to the unique unstable eigenvalue of $A(p^*)$, and its corresponding unstable eigenvector converges to the mode of instability.

Theorem 3.3.2. *Assume the conditions of Theorem 3.3.1. Then for p sufficiently close to p^* in R , there exists a unique unstable eigenvalue $\lambda(p)$ of $F(p)$, we denote its corresponding unstable eigenvector as $v(p)$, and*

$$\lim_{p \rightarrow p^*} \lambda(p) = \lambda(p^*), \quad (3.6a)$$

$$\lim_{p \rightarrow p^*} v(p) = v(p^*). \quad (3.6b)$$

3.4 Algorithm for Numerically Computing Mode of Instability

The results of Section 3.3, and in particular the computation of the average of the Jacobians in (3.5), requires exact evaluation of an integral along the system trajectory. However, determining the exact system trajectory in continuous time is in general intractable, and so this section extends the results of Section 3.3 to the practical setting where the system trajectory is approximated via numerical integration. To do so, we will first require a discrete time analog to Theorems 3.3.1-3.3.2. Let $h > 0$ represent a constant time step. For any integer n , let $T^n(x_0(p), p, h) := \phi(x_0(p), p, nh)$ be the state obtained from the continuous time system (3.1)-(3.2) after n time steps starting from IC $x_0(p)$. Analogously to the definition of $t(p)$ in continuous time, define $j(p)$ in discrete time as

$$j(p) := \sup_{n \in \mathbb{N}, n > 0} n. \quad (3.7)$$

$\frac{\partial f}{\partial x}(T^n(x_0(p), p, h)) \text{ is unstable}$

Finally, analogously to the definition of $F(p)$ in continuous time, we define its discrete approximation $\hat{F}(p)$ by

$$\hat{F}(p) := \frac{1}{j(p)} \sum_{n=0}^{j(p)} \frac{\partial f}{\partial x}(T^n(x_0(p), p, h)). \quad (3.8)$$

To derive practical guarantees for our approach, fix any autonomous constant step size numerical integration scheme with the property that for any IC x_0 and any time

$t > 0$, the system state obtained from numerical integration after n time steps (where $n = \inf_{\substack{n \in \mathbb{N} \\ nh \geq t}} n$) converges to $\phi(x_0, p, t)$ as the time step h approaches 0. The approximation error resulting from numerical integration can change the [recovery boundary](#) and, thereby, change the value of p^* , so let $p^*(h)$ denote the unique closest boundary parameter value on the [recovery boundary](#) to p_0 for the chosen integration scheme and the step size h . Define $\tilde{j}(p)$ to be the value of $j(p)$ obtained from (3.7) when T^n is approximated using numerical integration. Similarly, define $\tilde{F}(p)$ to be the value of $\hat{F}(p)$ obtained from (3.8) when T^n is approximated using numerical integration. For h sufficiently small, there exist $x_h^s(p)$ near $x^s(p)$ which is the hyperbolic [SEP](#), and $x_h^u(p)$ near $x^u(p)$ which is the hyperbolic [UEP](#), when T^n is approximated using numerical integration.

3.4.1 Theorem 3.4.1

Theorem 3.4.1 shows that as p approaches p^* from within [R](#), $\hat{F}(p)$ converges to $A(p^*)$, and it has a unique unstable eigenvalue that converges to the unique unstable eigenvalue of $A(p^*)$, and its corresponding unstable eigenvector converges to the mode of instability.

Theorem 3.4.1. *Assume the conditions of Theorem 3.3.1, and fix any time step $h > 0$. Then for p sufficiently close to p^* in [R](#), there exists a unique unstable eigenvalue $\hat{\lambda}(p)$ of $\hat{F}(p)$, let $\hat{v}(p)$ denote its corresponding unstable eigenvector, and*

$$\lim_{p \rightarrow p^*} \hat{F}(p) = A(p^*), \quad (3.9a)$$

$$\lim_{p \rightarrow p^*} \hat{\lambda}(p) = \lambda(p^*), \quad (3.9b)$$

$$\lim_{p \rightarrow p^*} \hat{v}(p) = v(p^*). \quad (3.9c)$$

3.4.2 Theorem 3.4.2

Theorem 3.4.2 shows that as h approaches 0, $p^*(h)$ converges to p^* . As p approaches $p^*(h)$ from within the set of recovery values, $\tilde{F}(p)$ converges to the Jacobian at $x_h^u(p^*(h))$. As h approaches zero, $\tilde{F}(p^*(h))$ converges to $A(p^*)$, and it has a unique unstable eigenvalue that converges to the unique unstable eigenvalue of $A(p^*)$, and its corresponding unstable eigenvector converges to the mode of instability.

Theorem 3.4.2. *Assume the conditions of Theorem 3.3.1. Then for $h > 0$ sufficiently small,*

$$\lim_{h \rightarrow 0} p^*(h) = p^*, \quad (3.10a)$$

$$\lim_{p \rightarrow p^*(h)} \tilde{F}(p) = \frac{\partial f}{\partial x}(x_h^u(p^*(h))), \quad (3.10b)$$

$$\lim_{h \rightarrow 0} \lim_{p \rightarrow p^*(h)} \tilde{F}(p) = A(p^*), \quad (3.10c)$$

and there exists a unique unstable eigenvalue $\tilde{\lambda}(p)$ of $\tilde{F}(p)$, with corresponding unstable eigenvector $\tilde{v}(p)$, and

$$\lim_{h \rightarrow 0} \lim_{p \rightarrow p^*(h)} \tilde{\lambda}(p) = \lambda(p^*), \quad (3.11a)$$

$$\lim_{h \rightarrow 0} \lim_{p \rightarrow p^*(h)} \tilde{v}(p) = v(p^*). \quad (3.11b)$$

3.5 Conclusion

In this chapter, we presented the main results for computing the mode of instability in parameterized nonlinear systems. We began by formally defining the mode of instability in Section 3.2. In Section 3.3, we introduced a method for computing the mode of instability with convergence guarantees based on Jacobian averaging. Finally, in Section 3.4, we extended these results to a practical setting by applying the approach to system trajectories approximated through numerical integration.

Chapter 4

Algorithm for Reducing Vulnerability

4.1 Introduction

This chapter introduces the algorithm for reducing vulnerability to the disturbance using the mode of instability, which will require two main steps: (1) finding the desired direction in which to move the IC in order to go further inwards and away from the RoA boundary, as introduced in Section 4.2, and (2) varying parameter values so as to drive the IC in this direction, as introduced in Section 4.3.

4.2 Finding the Desired Direction to Move the IC

For finding the desired direction to move the IC, the goal will be to find a vector $\tilde{v}$ pointing inwards from the RoA boundary at $x_0(p^*)$, and a basis $\tilde{U}$ for the tangent space to the RoA boundary at $x_0(p^*)$.¹ Then, the IC will be moved in a direction along $\tilde{v}$ after first subtracting the component along the tangent space spanned by $\tilde{U}$. In this way, distance will be measured inwards from the RoA boundary and avoid measuring the component tangent to the RoA boundary.

In order to find the desired $\tilde{v}$ and $\tilde{U}$, the key idea of the proposed approach is to first find a point $\hat{x}$ in the forward orbit of $x_0(p^*)$ such that the Jacobian at that point $\frac{\partial f}{\partial x}(\hat{x}, p^*)$

¹In general, the RoA boundary is not a smooth manifold, so it does not have a well-defined tangent space everywhere. However, as $x_0(p^*) \in W^s(x^u(p^*))$ in the RoA boundary, this tangent space is well-defined at $x_0(p^*)$.

is close to $A(p^*)$. Such a point $\hat{x}$ always exists since the forward orbit of $x_0(p^*)$ converges to $x^u(p^*)$ where the Jacobian is $A(p^*)$, and the Jacobian is continuous. Next, we make use of the following results. Lemma 4.2.1 relates the eigenspaces of the Jacobian at the controlling UEP to the tangent and normal spaces of the RoA boundary at the controlling UEP.

Lemma 4.2.1. *Let E^s and E^u denote the stable and unstable eigenspaces of $A(p^*)$, respectively. Then $T_{x^u(p^*)}W^s(x^u(p^*)) = E^s$ and $N_{x^u(p^*)}W^s(x^u(p^*)) = E^u$. Furthermore, E^u is the span of the mode of instability.*

Theorem 4.2.1 shows that we can use the trajectory sensitivity equation (2.4) to propagate vectors in the tangent and normal spaces of the RoA boundary near the controlling UEP backwards in time to the tangent and normal spaces of the RoA boundary near the IC, respectively. This will also send a vector pointing inwards toward the RoA near the controlling UEP to a vector pointing inwards toward the RoA near the IC.

Theorem 4.2.1. *Let $\hat{x}$ lie on the forward orbit of $x_0(p^*)$, say $\hat{x} = \phi(x_0(p^*), p^*, \hat{t})$, with both in $W^s(x^u(p^*))$. Let $\hat{U}$ be a basis for $T_{\hat{x}}W^s(x^u(p^*))$ and let $\hat{v}$ be a basis for $N_{\hat{x}}W^s(x^u(p^*))$ that points inwards towards the RoA. Let v^* be the result of numerically integrating the trajectory sensitivities in backwards time using (2.4) with IC $\frac{\partial x}{\partial x_0}(t=0) = \hat{v}$. Similarly, let U^* be the collection of vectors resulting from numerically integrating the trajectory sensitivities in backwards time using (2.4) with IC set to each vector of $\hat{U}$. Then U^* is a basis for $T_{x_0(p^*)}W^s(x^u(p^*))$, v^* is a basis for $N_{x_0(p^*)}W^s(x^u(p^*))$, and v^* points inwards towards the RoA.*

Corollary 4.2.1 shows that we can perform a similar propagation of vectors backwards in time using a parameter value $\hat{p}$ that is sufficiently close to the true closest boundary parameter value p^* to p_0 . In particular, it shows that the resulting vectors $\tilde{v}$ and $\tilde{U}$ will be arbitrarily close to v^* and U^* of Theorem 4.2.1. Then, $\tilde{v}$ and $\tilde{U}$ can be used to determine the desired direction to move the IC in order to move inwards and away from the RoA boundary.

When $p = \hat{p}$, in order to find a suitable $\hat{x}$ will require solving the following optimization problem to find a point on the forward orbit of $x_0(\hat{p})$ whose Jacobian is close to $A(p^*)$:

$$\hat{t} := \arg \min_{t \geq 0} \left\| \frac{\partial f}{\partial x}(\phi(x_0(\hat{p}), \hat{p}, t), \hat{p}) - A(p^*) \right\|. \quad (4.1)$$

After solving (4.1), we will set $\hat{x} = \phi(x_0(\hat{p}), \hat{p}, \hat{t})$. In practice, we efficiently solve (4.1) after obtaining the system trajectory from numerical integration by evaluating the norm in (4.1) at each time step and taking the minimum.

Corollary 4.2.1. *Let $\epsilon > 0$. Then, making an additional generic assumption,² there exists $r > 0$ such that $\|\hat{p} - p^*\| < r$ with $\hat{p} \in R$ implies that the following holds. Let $\hat{t}$ be the solution to (4.1), and let $\hat{x} = \phi(x_0(\hat{p}), \hat{p}, \hat{t})$. Then $\frac{\partial f}{\partial x}(\hat{x}, \hat{p})$ is hyperbolic with a one-dimensional unstable eigenspace. Let $\hat{v}$ be a vector in that unstable eigenspace that points inwards towards the RoA. Then $\hat{v}$ is arbitrarily close to the negative of the mode of instability. Let $\hat{U}$ be a basis for the stable eigenspace of $\frac{\partial f}{\partial x}(\hat{x}, \hat{p})$. Let $\tilde{v}$ and $\tilde{U}$ be the results of numerically integrating the trajectory sensitivities in backwards time using (2.4) with IC set to each vector in $\hat{v}$ and $\hat{U}$, respectively. Then $\|\tilde{v} - v^*\| < \epsilon$ and for each vector $u^* \in U^*$ and corresponding vector $\tilde{u} \in \tilde{U}$, $\|\tilde{u} - u^*\| < \epsilon$. Furthermore, $\tilde{v}$ points inwards towards the RoA.*

Therefore, based on Corollary 4.2.1, we start from the initial parameter value p_0 and follow the steps shown in Algorithm 1 and illustrated in Fig. 4.1. This procedure yields $\tilde{v}$ and $\tilde{U}$, where the span of $\tilde{U}$ approximates the tangent space to the RoA boundary near the IC, and $\tilde{v}$ points inwards towards the RoA near the IC. Thus, after computing $\tilde{U}$ and $\tilde{v}$, these will be used to drive the IC to move in a direction along $\tilde{v}$ (towards the RoA) after first subtracting the component along the span of $\tilde{U}$ (tangent to the RoA boundary).

Algorithm 1 Algorithm for Finding Desired Direction

- 1) Find the closest boundary parameter value $\hat{p} \in \partial R$ to p_0 using the algorithm from [11].³
 - 2) Find the Jacobian at the controlling UEP $A(p^*)$ using the algorithm from [34].
 - 3) Find the point $\hat{x} = \phi(x_0(\hat{p}), \hat{p}, \hat{t})$ by finding the minimum of (4.1).
 - 4) For the Jacobian at $(\hat{x}, \hat{p})$, find $\hat{v}$ its unique unstable eigenvector, and find $\hat{U}$ a basis for its stable eigenspace.
 - 5) Letting $\hat{v}$ and $\hat{U}$ be ICs, use (2.4) to numerically integrate these vectors backwards by time $\hat{t}$, and let $\tilde{v}$ and $\tilde{U}$ denote the results.
 - 6) Choose to move the IC $x_0(p)$ in a direction that excludes the component tangent to $\tilde{U}$, and in the direction of $\tilde{v}$.
-

4.3 Driving the IC in the Desired Direction

Once the direction in which to move the IC has been found by computing $\tilde{v}$ and $\tilde{U}$, the next step is to vary parameter values so as to drive the IC in that desired direction. To

²The generic assumption is that the eigenvalues of $A(p^*)$ are distinct.

³Since ∂R is a set of measure zero, in general we cannot numerically compute p^* which lies exactly on ∂R . Instead, we numerically approximate p^* to arbitrary accuracy with a value $\hat{p}$ that lies just inside R .

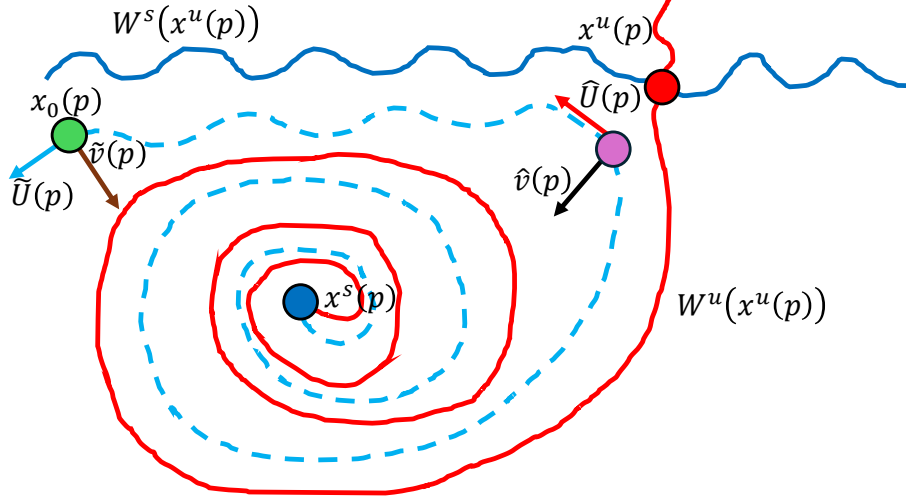


Figure 4.1: The system trajectory (cyan dashed line) for parameter values near the [recovery boundary](#) from the [IC](#) $x_0(p)$ (green circle) and converges to the [SEP](#) $x^s(p)$ (dark blue circle). The stable manifold $W^s(x^u(p))$ (blue solid line) and the unstable manifold of $W^u(x^u(p))$ (red solid line) of the controlling [UEP](#) $x^u(p)$ (red circle), are shown. The illustration of mapping $\hat{v}(p)$ (black arrow) and $\hat{U}(p)$ (red arrow) backward in time until the [IC](#), which gives $\tilde{v}(p)$ (brown arrow) and $\tilde{U}(p)$ (blue arrow), is shown.

achieve this, we first note that the movement of the [IC](#) relative to its value at the closest boundary parameter value is equal to $x_0(p) - x_0(\hat{p}) =: C(p)$. Therefore, $C(p)$ is the direction of movement of the [IC](#), so we want $C(p)$ to move in the direction of $\tilde{v}$ after subtracting the component of $C(p)$ that lies in the span of $\tilde{U}$. At the same time, we want to achieve this with as little change in parameter values as possible, since these typically represent controller parameter values that have been tuned to provide local stability and performance. Towards that end, define the function $H : J \rightarrow [0, \infty)$ as

$$H(p) := \lambda \|p - p_0\|_2^2 - \left\| \underbrace{\text{proj}_{\tilde{v}} [I_n - \text{proj}_{\tilde{U}}]}_B \underbrace{[x_0(p) - x_0(\hat{p})]}_{C(p)} \right\|_2^2. \quad (4.2)$$

The first term in H penalizes the deviation of parameter values from their initial value, where $\lambda > 0$ is a fixed weight controlling the trade-off between the two terms. The second term first subtracts the component of $C(p) = x_0(p) - x_0(\hat{p})$ that lies in the span of $\tilde{U}$,

then projects the result onto $\tilde{v}$ and takes the norm. This serves to measure the movement of the IC inwards from the RoA boundary without including the motion tangent to the RoA boundary. The second term is subtracted from the first so that the second term is minimized when the distance from the RoA boundary is maximized. Thus, the goal is to solve the following minimization problem:

$$\min_{p \in R} H(p). \quad (4.3)$$

Any solution to (4.3) will achieve the desired goal of driving the IC inwards and away from the RoA boundary with the minimal change in controller parameter values. However, (4.3) is challenging to solve because H is nonconvex and since $x_0(p)$ is an implicit function requiring numerical integration to evaluate. To solve (4.3) we will require the gradient of H , which is given by:

$$\nabla H(p) = 2\lambda(p - p_0) - 2 \frac{\partial x_0(p)^\top}{\partial p} B^\top B [x_0(p) - x_0(\hat{p})]. \quad (4.4)$$

To solve (4.3), we use gradient descent:

$$p^{k+1} = p^k - \alpha \nabla H(p^k), \quad (4.5)$$

where k denotes the iteration and $\alpha > 0$ is a constant step size. It is an iterative algorithm where in each iteration we numerically compute the derivative of x_0 with respect to p . If $x_0(p)$ is defined by a static map, we can solve it using Newton-Raphson and the implicit function theorem, and if it is defined by a dynamic map, we can find it through numerical integration including integration of the trajectory sensitivities as a byproduct.

Theorem 4.3.1 shows that the gradient descent algorithm (4.5) can guarantee convergence to a local minimum of (4.3) with probability one if the step size is sufficiently small.

Theorem 4.3.1. *There exists $r > 0$ such that for any $\alpha \in (0, r)$, the sequence $\{p^k\}_{k=1}^\infty$ starting from $p^1 = p_0$ and defined by the algorithm (4.5) with ∇H given by (4.4) converges to a local minimum of $H(p)$ with probability one.*

Thus, we arrive at the final optimization problem given by (4.3), which is solved using (4.5) with (4.4). Convergence to a local minimum is guaranteed by Theorem 4.3.1, which moves the IC further from the RoA boundary, thereby reducing the system's vulnerability to the disturbance.

4.4 Conclusion

Overall, we introduced an algorithm for reducing vulnerability to disturbances using the mode of instability in this chapter. As detailed in Section 4.2, the first step involves identifying a direction in which to move the IC further inwards and away from the RoA boundary. The second step, as discussed in Section 4.3, involves varying system parameter values to steer the IC along this direction, thereby reducing vulnerability.

Chapter 5

Proofs

5.1 Proof of Theorem 3.3.1

Proof of Theorem 3.3.1. This proof proceeds by splitting the system trajectory for $p \in R$ with p near p^* into three segments as illustrated in Fig. 5.1: (i) a segment that starts from the IC $x_0(p)$ and enters a ball N around the controlling UEP, (ii) another segment that travels within N , and (iii) a final segment that leaves N and enters the local stable manifold of the SEP. The key idea is that as p approaches p^* , the system trajectory spends only a finite amount of time on segments (i) and (iii), but the amount of time it spends in segment (ii) diverges towards infinity. Thus, the value of the Jacobian along segment (ii), which is very close to the Jacobian at the controlling UEP, will dominate in the calculation of the average $F(p)$ as p approaches p^* .

Let $\epsilon > 0$. To show that $\lim_{p \rightarrow p^*} F(p) = A(p^*)$, it suffices to show that there exists $\delta > 0$ such that $\|p - p^*\| < \delta$ with $p \in R$ implies that $\|F(p) - A(p^*)\| < \epsilon$. Let p^* be the unique closest boundary parameter value to p_0 as in Proposition 2.4.1, and let $x^u(p^*)$ be the controlling UEP. Then, by [13, Corollary 4.28], $\lim_{t \rightarrow \infty} \phi(x_0(p^*), p^*, t) = x^u(p^*)$. Since f is C^1 its first derivatives are continuous, so this implies that $\lim_{t \rightarrow \infty} \frac{\partial f}{\partial x}(\phi(x_0(p^*), p^*, t)) = \frac{\partial f}{\partial x}(x^u(p^*)) = A(p^*)$. As f is C^1 at $(x^u(p^*), p^*)$, there exist $\delta > 0$ and $r > 0$ such that whenever $\|p - p^*\| < \delta$ and $\|x - x^u(p^*)\| < r$,

$$\left\| \frac{\partial f}{\partial x}(x, p) - A(p^*) \right\| < \frac{\epsilon}{3}. \quad (5.1)$$

Let $N = B_r(x^u(p^*))$ denote the closed ball of radius r centered at $x^u(p^*)$. Consider the system trajectory for $p = p^*$. As $\lim_{t \rightarrow \infty} \phi(x_0(p^*), p^*, t) = x^u(p^*) \in N$, there exists

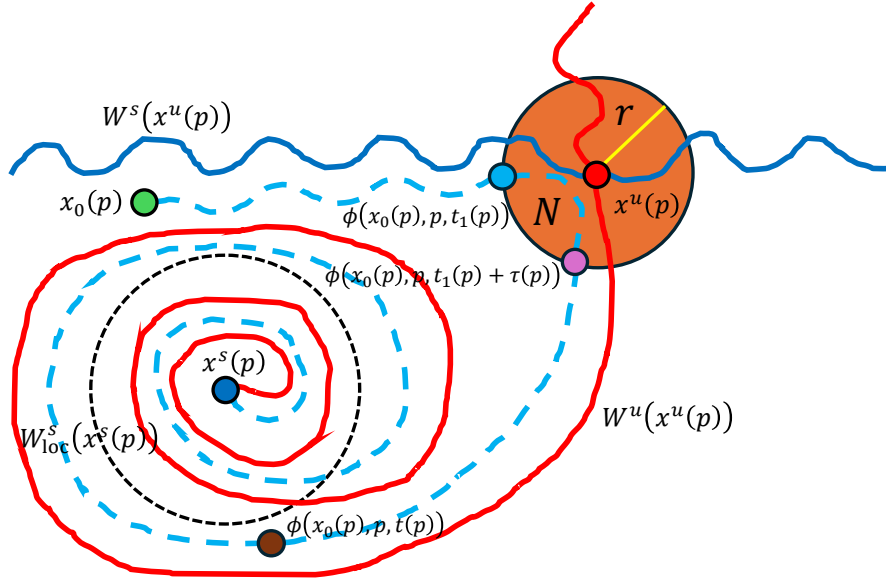


Figure 5.1: The system trajectory (cyan dashed line) for parameter values near the **recovery boundary** from the **IC** $x_0(p)$ (green circle) and converges to the **SEP** $x^s(p)$ (dark blue circle). The stable manifold $W^s(x^u(p))$ (blue solid line) and the unstable manifold of $W^u(x^u(p))$ (red solid line) of the controlling **UEP** $x^u(p)$ (red circle), are shown.

some finite time at which $\phi(x_0(p^*), p^*, t) \in \partial N$. Shrink r if necessary such that for $p = p^*$ the system trajectory intersects ∂N transversely at the final time $\hat{t}_1 > 0$ at which it passes inwards through ∂N , and such that ∂N is transverse to the local unstable manifold $W_{loc}^u(x^u(p^*))$ with a single point of intersection inside $W^s(x^s(p^*))$.

Define the C^1 function

$$\Phi(p, t) := [\phi(x_0(p), p, t) - x^u(p)]^\top [\phi(x_0(p), p, t) - x^u(p)] - r^2.$$

As $\phi(x_0(p^*), p^*, \hat{t}_1) \in \partial N = \partial B_r(x^u(p^*))$, $\Phi(p^*, \hat{t}_1) = 0$. Furthermore, $\frac{\partial \Phi}{\partial t}(p^*, \hat{t}_1) \neq 0$ (i.e., is full rank) because of the transversal intersection between the system trajectory at $p = p^*$ and ∂N . By the implicit function theorem [22, Theorem C.40], there exist open neighborhoods $V_0 \subseteq \mathbb{R}^m$ of p^* , $W_0 \subseteq \mathbb{R}$ of $\hat{t}_1$, and a C^1 function $t_1 : V_0 \rightarrow W_0$ such that $\Phi(p, t) = 0$ for $(p, t) \in V_0 \times W_0$ if and only if $t = t_1(p)$. This gives $\hat{t}_1 = t_1(p^*)$ so that $\lim_{p \rightarrow p^*} t_1(p) = t_1(p^*) = \hat{t}_1$. Shrink δ if necessary such that whenever $\|p - p^*\| < \delta$,

$$|t_1(p) - \hat{t}_1| < \epsilon. \quad (5.2)$$

For $p \in R$ with $\|p - p^*\| < \delta$, define $\tau(p)$ such that $t_1(p) + \tau(p)$ is the final time at which the system trajectory passes outwards through ∂N . Note that this is finite and well-defined since $\lim_{t \rightarrow \infty} \phi(x_0(p), p, t) = x^s(p) \notin \overline{N}$ and there exists at least one inward crossing of ∂N at $t_1(p)$. Let $x_2(p) = \phi(x_0(p), p, t_1(p) + \tau(p))$. By [13, Theorem 4.32], $\lim_{p \rightarrow p^*} \tau(p) = \infty$.

Let $\hat{x}_2$ be the single point of transversal intersection between $W_{\text{loc}}^u(x^u(p^*))$ and ∂N in $W^s(x^s(p^*))$. We next show that $\lim_{p \rightarrow p^*} x_2(p) = \hat{x}_2$. So, let $\tilde{\epsilon} > 0$. As $W_{\text{loc}}^u(x^u(p^*))$ and ∂N are compact and have a single point of transversal intersection in $W^s(x^s(p^*))$, there exists $\hat{\epsilon} > 0$ such that for any manifold M that satisfies $d_{C_1}(M, W_{\text{loc}}^u(x^u(p^*))) < \hat{\epsilon}$, M intersects N transversely with a single point of intersection in $W^s(x^s(p^*))$ and $d_{C_1}(M \cap \partial N \cap W^s(x^s(p^*)), \hat{x}_2) < \tilde{\epsilon}$ [19, Corollary A.3.18]. Note that $M \cap \partial N \cap W^s(x^s(p^*))$ is this single point of intersection.

Consider $B_\delta(p^*)$, the open ball of radius δ centered at p^* . Let $\phi(x_0(B_\delta(p^*)), p, t) := \bigcup_{p \in B_\delta(p^*)} \phi(x_0(p), p, t)$, and note that $x_0(B_\delta(p^*))$ is a C^1 manifold that is transverse to $W^s(x^u(p^*))$ by Proposition 2.4.1(ii). Thus, by the Inclination Lemma [26], shrinking δ further if necessary, there exists $T > 0$ such that whenever $p \in B_\delta(p^*)$ and $t > T$,

$$d_{C_1}(\phi(x_0(B_\delta(p^*)), p, t) \cap N, W_{\text{loc}}^u(x^u(p))) < \frac{\hat{\epsilon}}{2}, \quad (5.3)$$

where $\phi(x_0(B_\delta(p^*)), p, t) \cap N$ refers to the connected component of this intersection that contains $\phi(x_0(p^*), p^*, t)$. Since $W_{\text{loc}}^u(x^u(p))$ is continuous in p , $\lim_{p \rightarrow p^*} W_{\text{loc}}^u(x^u(p)) = W_{\text{loc}}^u(x^u(p^*))$. Shrink δ if necessary such that whenever $\|p - p^*\| < \delta$,

$$d_{C_1}(W_{\text{loc}}^u(x^u(p)), W_{\text{loc}}^u(x^u(p^*))) < \frac{\hat{\epsilon}}{2}. \quad (5.4)$$

Thus, we have

$$\begin{aligned} & d_{C_1}(\phi(x_0(B_\delta(p^*)), p, t) \cap N, W_{\text{loc}}^u(x^u(p^*))) \\ & \stackrel{\text{triangle inequality}}{\leq} d_{C_1}(\phi(x_0(B_\delta(p^*)), p, t) \cap N, W_{\text{loc}}^u(x^u(p))) + d_{C_1}(W_{\text{loc}}^u(x^u(p)), W_{\text{loc}}^u(x^u(p^*))) \\ & \stackrel{(5.3), (5.4)}{<} \frac{\hat{\epsilon}}{2} + \frac{\hat{\epsilon}}{2} = \hat{\epsilon}. \end{aligned}$$

By the above, this implies that for any $\|p - p^*\| < \delta$ and $t > T$, $\phi(x_0(B_\delta(p^*)), p, t)$ is transverse to ∂N with a single point of intersection in $W^s(x^s(p^*))$, and that

$$d_{C_1}(\phi(x_0(B_\delta(p^*)), p, t) \cap \partial N \cap W^s(x^s(p^*)), \hat{x}_2) < \tilde{\epsilon}.$$

Since $\lim_{p \rightarrow p^*} t_1(p) + \tau(p) = \infty$, shrink δ if necessary such that whenever $\|p - p^*\| < \delta$, $t_1(p) + \tau(p) > T$. As $x_2(p) = \phi(x_0(p), p, t_1(p) + \tau(p)) \in \partial N$ and $\phi(x_0(B_\delta(p^*)), p, t_1(p) + \tau(p))$ has a single point of intersection with ∂N in $W^s(x^s(p^*))$, it must hold that $x_2(p)$ is this single point of intersection. Thus, we have

$$d_{C_1}(x_2(p), \hat{x}_2) = d_{C_1}(\phi(x_0(B_\delta(p^*)), p, t_1(p) + \tau(p)) \cap \partial N \cap W^s(x^s(p^*)), \hat{x}_2) < \tilde{\epsilon}.$$

Since $\tilde{\epsilon} > 0$ was arbitrary, this proves $\lim_{p \rightarrow p^*} x_2(p) = \hat{x}_2$.

Next, consider the system trajectory at time t starting from the IC $\hat{x}_2$, denoted by $\phi(\hat{x}_2, p^*, t)$. As $\hat{x}_2 \in W^s(x^s(p^*))$, there exists $\hat{t}_2 > 0$ such that $\phi(\hat{x}_2, p^*, \hat{t}_2) \in \text{int } W_{\text{loc}}^s(x^s(p^*))$. As $W_{\text{loc}}^s(x^s(p))$ varies continuously with p , $\lim_{p \rightarrow p^*} x_2(p) = \hat{x}_2$, and the flow is continuous, shrinking δ further if necessary implies that for $\|p - p^*\| < \delta$, $\phi(x_2(p), p, \hat{t}_2) \in \text{int } W_{\text{loc}}^s(x^s(p))$. Recall that for all $\|p - p^*\| < \delta$, $W_{\text{loc}}^s(x^s(p))$ is forward invariant and has $\frac{\partial f}{\partial x}(x)$ stable for all $x \in W_{\text{loc}}^s(x^s(p))$. Thus, for all $\|p - p^*\| < \delta$ and $t \geq \hat{t}_2$, $\frac{\partial f}{\partial x}(\phi(x_2(p), p, t))$ is stable. For $p \in R$, as $t(p)$ is the supremum over time at which $\frac{\partial f}{\partial x}(\phi(x_0(p), p, t))$ is unstable, this implies that

$$t(p) < t_1(p) + \tau(p) + \hat{t}_2 \quad (5.5)$$

for all $\|p - p^*\| < \delta$.

For $\|p - p^*\| < \delta$ with $p \in R$, define the functions

$$\Psi(p) := \sup_{t \in [0, t_1(p)]} \left\| \frac{\partial f}{\partial x}(\phi(x_0(p), p, t)) - A(p^*) \right\|, \quad (5.6)$$

$$\hat{\Psi}(p) := \sup_{t \in [0, \hat{t}_2]} \left\| \frac{\partial f}{\partial x}(\phi(x_2(p), p, t)) - A(p^*) \right\|. \quad (5.7)$$

As f and the flow are C^1 , $x_0(p)$ and $x_2(p)$ are continuous at p^* , the norm is continuous, so $\left\| \frac{\partial f}{\partial x}(\phi(x_0(p), p, t)) - A(p^*) \right\|$ and $\left\| \frac{\partial f}{\partial x}(\phi(x_2(p), p, t)) - A(p^*) \right\|$ are both continuous in (p, t) . Thus, as $[0, t_1(p)]$ and $[0, \hat{t}_2]$ are compact, nonempty, and continuous in p (since $t_1(p)$ is continuous), by the maximum theorem [5, Page 116], Ψ and $\hat{\Psi}$ are continuous at p^* . Thus,

$$\begin{aligned} \lim_{p \rightarrow p^*} \Psi(p) &= \lim_{p \rightarrow p^*} \sup_{t \in [0, t_1(p)]} \left\| \frac{\partial f}{\partial x}(\phi(x_0(p), p, t)) - A(p^*) \right\| \\ &= \sup_{t \in [0, \hat{t}_1]} \left\| \frac{\partial f}{\partial x}(\phi(x_0(p^*), p^*, t)) - A(p^*) \right\| \\ &=: \Psi(p^*) \end{aligned} \quad (5.8)$$

and

$$\begin{aligned}
\lim_{p \rightarrow p^*} \hat{\Psi}(p) &= \lim_{p \rightarrow p^*} \sup_{t \in [0, \hat{t}_2]} \left\| \frac{\partial f}{\partial x}(\phi(x_2(p), p, t)) - A(p^*) \right\| \\
&= \sup_{t \in [0, \hat{t}_2]} \left\| \frac{\partial f}{\partial x}(\phi(\hat{x}_2, p^*, t)) - A(p^*) \right\| \\
&=: \hat{\Psi}(p^*).
\end{aligned} \tag{5.9}$$

Since $t(p) > \tau(p) + t_1(p)$ and $\lim_{p \rightarrow p^*} \tau(p) = \infty$, $\lim_{p \rightarrow p^*} t(p) = \infty$. Thus, shrink δ further if necessary such that whenever $\|p - p^*\| < \delta$,

$$t(p) > \frac{3(\hat{t}_1 + \epsilon)(\Psi(p^*) + \epsilon)}{\epsilon}, \tag{5.10}$$

$$t(p) > \frac{3\hat{t}_2(\hat{\Psi}(p^*) + \epsilon)}{\epsilon}, \tag{5.11}$$

$$|\Psi(p) - \Psi(p^*)| < \epsilon, \tag{5.12}$$

$$|\hat{\Psi}(p) - \hat{\Psi}(p^*)| < \epsilon, \tag{5.13}$$

where the final two equations follow from (5.8)-(5.9).

By the definitions of Ψ and $\hat{\Psi}$ in (5.6) and (5.7), we also have that for $t \in [0, t_1(p)]$, $\left\| \frac{\partial f}{\partial x}(\phi(x_0(p), p, t)) - A(p^*) \right\| \leq \Psi(p)$, and for $t \in [0, \hat{t}_2]$, $\left\| \frac{\partial f}{\partial x}(\phi(x_2(p), p, t)) - A(p^*) \right\| \leq \hat{\Psi}(p)$. Thus, for any $\|p - p^*\| < \delta$, we have

$$\begin{aligned}
&\|F(p) - A(p^*)\| \\
&= \left\| \frac{1}{t(p)} \int_0^{t(p)} \frac{\partial f}{\partial x}(\phi(x_0(p), p, t)) dt - A(p^*) \right\| \\
&= \left\| \frac{1}{t(p)} \int_0^{t(p)} \left[\frac{\partial f}{\partial x}(\phi(x_0(p), p, t)) - A(p^*) \right] dt \right\| \\
&\leq \frac{1}{t(p)} \int_0^{t(p)} \left\| \frac{\partial f}{\partial x}(\phi(x_0(p), p, t)) - A(p^*) \right\| dt \\
&= \frac{1}{t(p)} \left[\int_0^{t_1(p)} \left\| \frac{\partial f}{\partial x}(\phi(x_0(p), p, t)) - A(p^*) \right\| dt \right.
\end{aligned}$$

$$\begin{aligned}
& + \int_{t_1(p)}^{t_1(p)+\tau(p)} \left\| \frac{\partial f}{\partial x} (\phi(x_0(p), p, t)) - A(p^*) \right\| dt \\
& + \int_{t_1(p)+\tau(p)}^{t(p)} \left\| \frac{\partial f}{\partial x} (\phi(x_0(p), p, t)) - A(p^*) \right\| dt \Bigg] \\
& \stackrel{(5.5)}{<} \frac{1}{t(p)} \left[\int_0^{t_1(p)} \left\| \frac{\partial f}{\partial x} (\phi(x_0(p), p, t)) - A(p^*) \right\| dt \right. \\
& + \int_{t_1(p)}^{t_1(p)+\tau(p)} \left\| \frac{\partial f}{\partial x} (\phi(x_0(p), p, t)) - A(p^*) \right\| dt \\
& + \left. \int_{t_1(p)+\tau(p)}^{t_1(p)+\tau(p)+\hat{t}_2} \left\| \frac{\partial f}{\partial x} (\phi(x_0(p), p, t)) - A(p^*) \right\| dt \right] \\
& = \frac{1}{t(p)} \left[\int_0^{t_1(p)} \left\| \frac{\partial f}{\partial x} (\phi(x_0(p), p, t)) - A(p^*) \right\| dt \right. \\
& + \int_{t_1(p)}^{t_1(p)+\tau(p)} \left\| \frac{\partial f}{\partial x} (\phi(x_0(p), p, t)) - A(p^*) \right\| dt \\
& + \left. \int_0^{\hat{t}_2} \left\| \frac{\partial f}{\partial x} (\phi(x_2(p), p, t)) - A(p^*) \right\| dt \right] \\
& \stackrel{(5.1), (5.6), (5.7)}{<} \frac{1}{t(p)} \left[\int_0^{t_1(p)} \Psi(p) dt + \int_{t_1(p)}^{t_1(p)+\tau(p)} \frac{\epsilon}{3} dt + \int_0^{\hat{t}_2} \hat{\Psi}(p) dt \right] \\
& = \frac{t_1(p) \Psi(p)}{t(p)} + \frac{\tau(p) \epsilon}{t(p) 3} + \frac{\hat{t}_2 \hat{\Psi}(p)}{t(p)} \\
& \stackrel{(5.2), (5.12), (5.13)}{<} \frac{(\hat{t}_1 + \epsilon) (\Psi(p^*) + \epsilon)}{t(p)} + \frac{\epsilon}{3} + \frac{\hat{t}_2 (\hat{\Psi}(p^*) + \epsilon)}{t(p)} \\
& \stackrel{(5.10), (5.11)}{<} \frac{(\hat{t}_1 + \epsilon) (\Psi(p^*) + \epsilon)}{\frac{3(\hat{t}_1 + \epsilon)(\Psi(p^*) + \epsilon)}{\epsilon}} + \frac{\epsilon}{3} + \frac{\hat{t}_2 (\hat{\Psi}(p^*) + \epsilon)}{\frac{3\hat{t}_2(\hat{\Psi}(p^*) + \epsilon)}{\epsilon}} \\
& = \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,
\end{aligned}$$

so $\|F(p) - A(p^*)\| < \epsilon$ for all $\|p - p^*\| < \delta$ with $p \in \textcolor{blue}{R}$, which completes the proof of Theorem 3.3.1. $\square$

5.2 Proof of Theorem 3.3.2

Proof of Theorem 3.3.2. Recall that $\lambda(p^*)$ is the simple unstable eigenvalue of $A(p^*)$ and $v(p^*)$ is its corresponding eigenvector. Furthermore, $A(p^*)$ is hyperbolic by Proposition 2.4.1 and its proof, so all eigenvalues of $A(p^*)$ other than $\lambda(p^*)$ are stable (i.e., no eigenvalues of $A(p^*)$ have zero real part). Since the eigenvalues of a matrix are continuous functions of the elements of that matrix [25, Theorem 3.1.2], and by Theorem 3.3.1 $\lim_{p \rightarrow p^*} F(p) = A(p^*)$, this implies that the eigenvalues of $F(p)$ converge to the eigenvalues of $A(p^*)$ in the limit as $p \rightarrow p^*$. Thus, there exists $\delta > 0$ sufficiently small such that $\|p - p^*\| < \delta$ implies that $F(p)$ is hyperbolic and has exactly one unstable eigenvalue, call it $\lambda(p)$. Let $v(p)$ denote the eigenvector associated with $\lambda(p)$. As the eigenvalues of $F(p)$ converge to those of $A(p^*)$ in the limit as $p \rightarrow p^*$, $\lambda(p)$ is the unique unstable eigenvalue of $F(p)$, and $\lambda(p^*)$ is the unique unstable eigenvalue of $A(p^*)$, this implies that $\lim_{p \rightarrow p^*} \lambda(p) = \lambda(p^*)$. Since $\lambda(p)$ is a simple (i.e., multiplicity one) eigenvalue of $F(p)$ and $v(p)$ is its associated eigenvector, and since $\lim_{p \rightarrow p^*} F(p) = A(p^*)$, by [25, Theorem 3.1.3] $\lim_{p \rightarrow p^*} v(p) = v(p^*)$. This completes the proof of Theorem 3.3.2. $\square$

5.3 Proof of Theorem 3.4.1

Proof of Theorem 3.4.1. The goal of this proof is to follow a similar argument as in the proof of Theorem 3.3.1, which involves splitting the system trajectory for $p \in R$ with p close to p^* into three segments as illustrated in Fig. 5.2. Recall the definition of $j(p)$ from (3.7). We begin by showing that $\lim_{p \rightarrow p^*} j(p) = \infty$. Let $K > 0$. As $\lim_{n \rightarrow \infty} T^n(x_0(p^*), p^*, h) = x^u(p^*)$, $\frac{\partial f}{\partial x}(x^u(p^*)) = A(p^*)$ is unstable and hyperbolic, and $\frac{\partial f}{\partial x}$ is continuous, there exists $i \in \mathbb{N}$ such that $ih > K$ and $\frac{\partial f}{\partial x}(T^i(x_0(p^*), p^*, h))$ is unstable. Since $\frac{\partial f}{\partial x}(T^i(x_0(p), p, h))$ is continuous in p , there exists $\delta > 0$ such that whenever $\|p - p^*\| < \delta$, $\frac{\partial f}{\partial x}(T^i(x_0(p), p, h))$ is unstable. Hence, by the definition of $j(p)$ in (3.7), $j(p) \geq ih > K$. As K was arbitrary, this implies that $\lim_{p \rightarrow p^*} j(p) = \infty$.

As in the proof of Theorem 3.3.1, there exist $r > 0$ and $\delta > 0$ such that whenever $\|p - p^*\| < \delta$ and $\|x - x^u(p^*)\| < r$,

$$\left\| \frac{\partial f}{\partial x}(x, p) - A(p^*) \right\| < \frac{\epsilon}{3}. \quad (5.14)$$

Shrink r further so that (5.14) holds for some $r' > r$. As in the proof of Theorem 3.3.1, shrink r further if necessary such that $\partial B_r(x^u(p^*))$ is transverse to the local unstable

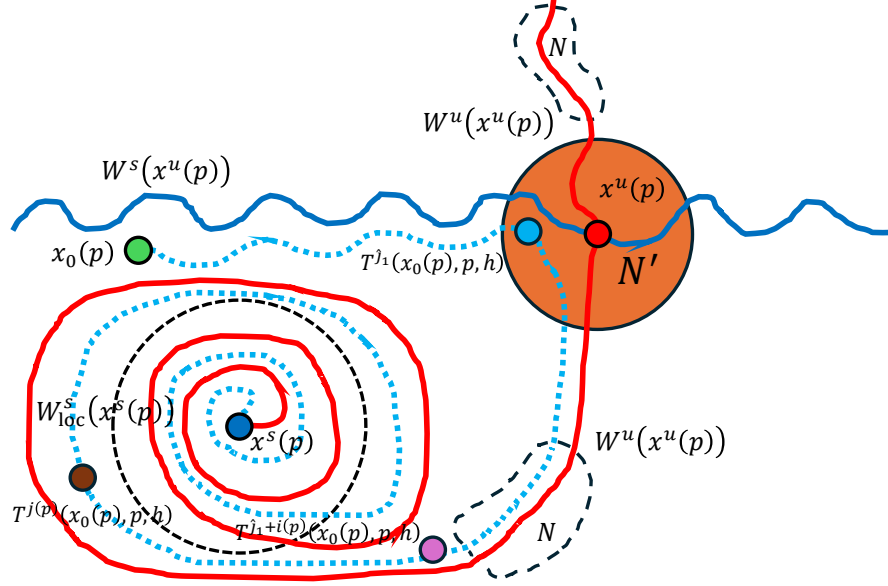


Figure 5.2: The system trajectory in discrete time (cyan dotted line) for parameter values near the [recovery boundary](#). This figure is similar to Fig. 5.1, except we are considering the state obtained from the continuous time system (2.1)-(2.2) with new labels and definitions.

manifold $W^u_{loc}(x^u(p^*))$ with a single point of intersection $\hat{x}_2$ inside $W^s(x^s(p^*))$ and a single point of intersection $\tilde{x}_2$ outside $W^s(x^s(p^*))$. Let $\hat{x}_1 = T^{-1}(\hat{x}_2, p^*, h)$ and $\tilde{x}_1 = T^{-1}(\tilde{x}_2, p^*, h)$. Since $W^u(x^u(p^*))$ is one-dimensional, let $\hat{D}$ be the subset of $W^u(x^u(p^*))$ from $\hat{x}_1$ to $\hat{x}_2$, $\tilde{D}$ be the subset of $W^u(x^u(p^*))$ from $\tilde{x}_1$ to $\tilde{x}_2$, and $D = \hat{D} \cup \tilde{D}$. Then by [14, Lemma 5.4] (or, similarly, from the proof of [26, Theorem 1.9]), shrinking δ further if necessary implies that there exist open neighborhoods N and N' such that $D \subset N$, $\bar{N}$ is disjoint from $W^s(x^u(p^*))$ and $\partial W^s(x^s(p^*))$, $\bar{N} \subset B_{r'}(x^u(p^*))$, $\bar{N}$ is compact, and for $\|p - p^*\| < \delta$, N' contains $x^u(p)$ and $\bigcup_{i=0}^{\infty} T^{-i}(N, p, h) \cup W^s(x^u(p))$ contains N' [13, Lemma 5.1]. Let $\hat{N} = (N' \cup N) \cap B_{r'}(x^u(p^*))$, and note that $N \subset \hat{N}$.

Let $\hat{j}_1$ be the final time index at which $T^n(x_0(p^*), p^*, h)$ enters the interior of $\hat{N}$. Since $T^{\hat{j}_1}(x_0(p), p, h)$ is continuous in p , shrink δ if necessary such that whenever $\|p - p^*\| < \delta$, $T^{\hat{j}_1}(x_0(p), p, h)$ enters the interior of $\hat{N}$. For $p \in \mathcal{R}$ with $\|p - p^*\| < \delta$, define $i(p)$ such that $\hat{j}_1 + i(p)$ is the final time index at which $T^n(x_0(p), p, h)$ exits N , and let $x_2(p) = T^{\hat{j}_1+i(p)}(x_0(p), p, h)$. Note that $i(p)$ is well-defined since for $p \in \mathcal{R}$ with $\|p - p^*\| < \delta$, $T^{\hat{j}_1}(x_0(p), p, h) \in \hat{N}$, so since $T^{\hat{j}_1}(x_0(p), p, h)$ is not in $W^s(x^u(p))$, it must be in $T^{-i}(N, p, h)$ for some positive integer i . This implies that $T^{\hat{j}_1+n}(x_0(p), p, h)$ must pass

through N after a finite number of time steps, before ultimately converging to $x^s(p)$. Let $S = \overline{N} \cap \overline{W}^s(x^s(p^*))$, which is compact. Then $S \subset W^s(x^s(p^*))$ since $\overline{N}$ and $\partial W^s(x^s(p^*))$ are disjoint. Then as S is compact and T^n is continuous, there exists a time index $\hat{j}_2$ sufficiently large such that $T^{\hat{j}_2}(S, p^*, h) \subset \text{int } W_{\text{loc}}^s(x^s(p^*))$. As $W_{\text{loc}}^s(x^s(p))$ is continuous in p , S is compact, and $T^{\hat{j}_2}$ is continuous, shrink δ further if necessary such that $\|p - p^*\| < \delta$ with $p \in R$ implies that $T^{\hat{j}_2}(S, p, h) \subset \text{int } W_{\text{loc}}^s(x^s(p))$. As $\text{int } W_{\text{loc}}^s(x^s(p))$ is forward invariant and the Jacobian is stable for each $x \in W_{\text{loc}}^s(x^s(p))$, this implies that for $p \in R$ with $\|p - p^*\| < \delta$,

$$j(p) < \hat{j}_1 + i(p) + \hat{j}_2. \quad (5.15)$$

For $p \in R$ with $\|p - p^*\| < \delta$, define the functions

$$\Psi(p) := \max_{n \in \{0, \dots, \hat{j}_1 - 1\}} \left\| \frac{\partial f}{\partial x}(T^n(x_0(p), p, h)) - A(p^*) \right\|, \quad (5.16)$$

$$\hat{\Psi}(p) := \max_{n \in \{0, \dots, \hat{j}_2\}} \left\| \frac{\partial f}{\partial x}(T^n(x_2(p), p, h)) - A(p^*) \right\|. \quad (5.17)$$

Note that $\Psi(p)$ and $\hat{\Psi}(p)$, as the maximums of a finite number of continuous functions, are themselves continuous. We have

$$\lim_{p \rightarrow p^*} \Psi(p) = \Psi(p^*) := \max_{n \in \{0, \dots, \hat{j}_1 - 1\}} \left\| \frac{\partial f}{\partial x}(T^n(x_0(p^*), p^*, h)) - A(p^*) \right\|.$$

As $\max_{n \in \{0, \dots, \hat{j}_2\}} \left\| \frac{\partial f}{\partial x}(T^n(x_2(p), p, h)) - A(p^*) \right\|$ is continuous, $x_2(p) \in N$, and $\overline{N}$ is compact,

$$\hat{\Psi}(p) \leq \sup_{x \in \overline{N}} \max_{n \in \{0, \dots, \hat{j}_2\}} \left\| \frac{\partial f}{\partial x}(T^n(x, p, h)) - A(p^*) \right\| =: \hat{\Psi}(p^*) < \infty. \quad (5.18)$$

Since $\lim_{p \rightarrow p^*} j(p) = \infty$, shrink δ if necessary such that whenever $p \in R$ with $\|p - p^*\| < \delta$,

$$j(p) > \frac{3\hat{j}_1(\Psi(p^*) + \epsilon)}{\epsilon}, \quad (5.19)$$

$$j(p) > \frac{3(\hat{j}_2 + 1)\hat{\Psi}(p^*)}{\epsilon}, \quad (5.20)$$

$$|\Psi(p) - \Psi(p^*)| < \epsilon. \quad (5.21)$$

For $n \in \{\hat{j}_1, \dots, \hat{j}_1 + i(p) - 1\}$, since $T^n(x_0(p), p, h) \in B_{r'}(x^u(p^*))$, by (5.14) we have

$$\left\| \frac{\partial f}{\partial x}(T^n(x_0(p), p, h)) - A(p^*) \right\| < \frac{\epsilon}{3}. \quad (5.22)$$

Thus, we have whenever $\|p - p^*\| < \delta$,

$$\begin{aligned} & \left\| \hat{F}(p) - A(p^*) \right\| \\ &= \left\| \frac{1}{j(p)} \sum_{n=0}^{j(p)} \frac{\partial f}{\partial x}(T^n(x_0(p), p, h)) - A(p^*) \right\| \\ &= \left\| \frac{1}{j(p)} \sum_{n=0}^{j(p)} \left[\frac{\partial f}{\partial x}(T^n(x_0(p), p, h)) - A(p^*) \right] \right\| \\ &\leq \frac{1}{j(p)} \sum_{n=0}^{j(p)} \left\| \frac{\partial f}{\partial x}(T^n(x_0(p), p, h)) - A(p^*) \right\| \\ &= \frac{1}{j(p)} \left[\sum_{n=0}^{\hat{j}_1-1} \left\| \frac{\partial f}{\partial x}(T^n(x_0(p), p, h)) - A(p^*) \right\| \right. \\ &\quad + \sum_{n=\hat{j}_1}^{\hat{j}_1+i(p)-1} \left\| \frac{\partial f}{\partial x}(T^n(x_0(p), p, h)) - A(p^*) \right\| \\ &\quad \left. + \sum_{n=\hat{j}_1+i(p)}^{j(p)} \left\| \frac{\partial f}{\partial x}(T^n(x_0(p), p, h)) - A(p^*) \right\| \right] \\ &\stackrel{(5.15)}{<} \frac{1}{j(p)} \left[\sum_{n=0}^{\hat{j}_1-1} \left\| \frac{\partial f}{\partial x}(T^n(x_0(p), p, h)) - A(p^*) \right\| \right. \\ &\quad + \sum_{n=\hat{j}_1}^{\hat{j}_1+i(p)-1} \left\| \frac{\partial f}{\partial x}(T^n(x_0(p), p, h)) - A(p^*) \right\| \\ &\quad \left. + \sum_{n=\hat{j}_1+i(p)}^{\hat{j}_1+i(p)+\hat{j}_2} \left\| \frac{\partial f}{\partial x}(T^n(x_0(p), p, h)) - A(p^*) \right\| \right] \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{j(p)} \left[\sum_{n=0}^{\hat{j}_1-1} \left\| \frac{\partial f}{\partial x} (T^n(x_0(p), p, h)) - A(p^*) \right\| \right. \\
&\quad + \sum_{n=\hat{j}_1}^{\hat{j}_1+i(p)-1} \left\| \frac{\partial f}{\partial x} (T^n(x_0(p), p, h)) - A(p^*) \right\| \\
&\quad \left. + \sum_{n=0}^{\hat{j}_2} \left\| \frac{\partial f}{\partial x} (T^n(x_2(p), p, h)) - A(p^*) \right\| \right] \\
&\stackrel{(5.16), (5.17), (5.22)}{<} \frac{1}{j(p)} \left[\sum_{n=0}^{\hat{j}_1-1} \Psi(p) + \sum_{n=\hat{j}_1}^{\hat{j}_1+i(p)-1} \frac{\epsilon}{3} + \sum_{n=0}^{\hat{j}_2} \hat{\Psi}(p) \right] \\
&\stackrel{(5.18), (5.21)}{<} \frac{1}{j(p)} \left[\sum_{n=0}^{\hat{j}_1-1} (\Psi(p^*) + \epsilon) + \sum_{n=\hat{j}_1}^{\hat{j}_1+i(p)-1} \frac{\epsilon}{3} + \sum_{n=0}^{\hat{j}_2} \hat{\Psi}(p^*) \right] \\
&= \frac{\hat{j}_1(\Psi(p^*) + \epsilon)}{j(p)} + \frac{i(p)\epsilon}{j(p)3} + \frac{(\hat{j}_2 + 1)\hat{\Psi}(p^*)}{j(p)} \\
&\stackrel{(5.19), (5.20)}{<} \frac{\hat{j}_1(\Psi(p^*) + \epsilon)}{\frac{3\hat{j}_1(\Psi(p^*) + \epsilon)}{\epsilon}} + \frac{\epsilon}{3} + \frac{(\hat{j}_2 + 1)\hat{\Psi}(p^*)}{\frac{3(\hat{j}_2 + 1)\hat{\Psi}(p^*)}{\epsilon}} \\
&= \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon,
\end{aligned}$$

so $\|\hat{F}(p) - A(p^*)\| < \epsilon$ for all $\|p - p^*\| < \delta$ with $p \in R$, which completes the proof of (3.9a). (3.9b) and (3.9c) can be proved by following analogous reasoning as in the proof of Theorem (3.3.2). This completes the proof of Theorem (3.4.1). $\square$

5.4 Proof of Theorem 3.4.2

Proof of Theorem 3.4.2. We first prove $\lim_{h \rightarrow 0} p^*(h) = p^*$. Recall that $x_0(p^*) \in W^s(x^u(p^*))$, where $x^u(p^*)$ is the controlling UEP. Thus, there exists $T > 0$ such that

$$x_0(p^*) \in \phi(W_{\text{loc}}^s(x^u(p^*)), p^*, -T) \subset \phi(W_{\text{loc}}^s(x^u(J)), J, -T).$$

By Assumption 2.4.1, $W^s(x^u(J))$ and $x_0(J)$ are transverse, which implies that $x_0(J)$ and $\phi(W_{\text{loc}}^s(x^u(J)), J, -T)$ are transverse, and that $x_0(p^*)$ is a point of their transver-

sal intersection. Then for any $\epsilon > 0$, by [19, Proposition A.3.16] (i.e., the stability of transversal intersections), there exists a $\hat{\delta} > 0$ such that for any manifold M that satisfies $d_{C^1}(M, \phi(W_{\text{loc}}^s(x^u(J)), J, -T)) < \hat{\delta}$, there exists a point $(x, p) \in M \cap x_0(J)$ with $\|(x, p) - (x_0(p^*), p^*)\| < \epsilon$. As $\phi(\cdot, \cdot, -T)$ is C^1 , there exists $\delta > 0$ such that for any manifold M that satisfies $d_{C^1}(M, W_{\text{loc}}^s(x^u(J))) < \delta$, $d_{C^1}(\phi(M, -T), \phi(W_{\text{loc}}^s(x^u(J)), -T)) < \hat{\delta}$, so $\phi(M, -T)$ intersects $x_0(J)$ at a point within an ϵ distance of $(x_0(p^*), p^*)$.

Recall that $T^n(x, p, h) = \phi(x, p, nh)$ for any state x , parameter value p , and time step n . We refer to T as the exact discrete time map with time step h . Let T_d denote the approximation of T that is obtained using numerical integration. Fix some time $\hat{t} > 0$. For any $h > 0$, let $\hat{n}(h)$ be the minimum integer n such that $nh \geq \hat{t}$. Over any compact set $K \times J' \subset \mathbb{R}^n \times J$, define the maps $\hat{T}$ and $\hat{T}_d$ by $\hat{T}(x, p, h) = T^{\hat{n}(h)}(x, p, h)$ and $\hat{T}_d(x, p, h) = T_d^{\hat{n}(h)}(x, p, h)$, respectively, for all $(x, p) \in K \times J'$ and $h > 0$. Then, since the approximation error of the numerical integration approaches zero as $h \rightarrow 0$, we have that $\lim_{h \rightarrow 0} d_{C^1}(\hat{T}_d(\cdot, \cdot, h), \hat{T}(\cdot, \cdot, h)) = 0$. Furthermore, as $\lim_{h \rightarrow 0} h\hat{n}(h) = \hat{t}$, and since $\hat{T}$ represents the exact discrete time map, we also have that $\lim_{h \rightarrow 0} d_{C^1}(\hat{T}(\cdot, \cdot, h), \phi(\cdot, \cdot, \hat{t})) = 0$, where $\phi(\cdot, \cdot, \hat{t})$ is the time- $\hat{t}$ map for the flow ϕ , and is itself a discrete time map. Thus, combining these implies that

$$\lim_{h \rightarrow 0} d_{C^1}(\hat{T}_d(\cdot, \cdot, h), \phi(\cdot, \cdot, \hat{t})) = 0. \quad (5.23)$$

Note that $x^u(p)$ is a hyperbolic fixed point of $\phi(\cdot, \cdot, \hat{t})$ for all $p \in J$, and that $W_{\text{loc}}^s(x^u(J))$ is its local stable manifold under $\phi(\cdot, \cdot, \hat{t})$, since $\phi(\cdot, \cdot, \hat{t})$ is the exact time- $\hat{t}$ map for the flow ϕ . Thus, there exists $\tilde{\delta} > 0$ such that for any map $\tilde{T}$ with $d_{C^1}(\tilde{T}, \phi(\cdot, \cdot, \hat{t})) < \tilde{\delta}$, $\tilde{T}$ possesses a hyperbolic fixed point $\tilde{x}^u$ near x^u and $d_{C^1}(W_{\text{loc}}^s(\tilde{x}^u(J)), W_{\text{loc}}^s(x^u(J))) < \delta$. By (5.23), this implies that there exists $\hat{h} > 0$ such that for any $h \in (0, \hat{h})$, $\hat{T}_d(\cdot, \cdot, h)$ possesses a hyperbolic fixed point x_h^u near x^u and $d_{C^1}(W_{\text{loc}}^s(x_h^u(J)), W_{\text{loc}}^s(x^u(J))) < \delta$. Thus, by the above there exists a point $(x_h, p_h) \in W_{\text{loc}}^s(x_h^u(J)) \cap x_0(J)$ with $\|(x_h, p_h) - (x_0(p^*), p^*)\| < \epsilon$. As $\epsilon > 0$ was arbitrary, this implies that the points (x_h, p_h) can be selected such that $\lim_{h \rightarrow 0} p_h = p^*$. Furthermore, as $x_0(p_h) = (x_h, p_h) \in W_{\text{loc}}^s(x_h^u(J))$ for the map $\hat{T}_d(\cdot, \cdot, h)$, $x_0(p_h) \notin W_{\text{loc}}^s(x_h^s(J))$ for this map, where x_h^s represents the SEP of $\hat{T}_d(\cdot, \cdot, h)$ near x^s , so $p_h \notin \text{R}$ for this map. Also, x_h^u is a hyperbolic fixed point for both $\hat{T}_d(\cdot, \cdot, h)$ and $T_d(\cdot, \cdot, h)$, and $W_{\text{loc}}^s(x_h^u(J))$ is its local stable manifold under both maps, so $p_h \notin \text{R}$ for the map $T_d(\cdot, \cdot, h)$ as well.

Assume towards a contradiction that $\lim_{h \rightarrow 0} p^*(h) \neq p^*$. Then there must exist a monotonically decreasing sequence $\{h_n\}_{n=1}^\infty$ with $\lim_{n \rightarrow \infty} h_n = 0$ such that $\lim_{n \rightarrow \infty} p^*(h_n) \neq p^*$.

Since $p^*(h_n)$ is the unique closest parameter value in ∂R to p_0 for the map $T_d(\cdot, \cdot, h_n)$, and since $p_{h_n} \notin R$ for the map $T_d(\cdot, \cdot, h_n)$, for each integer n we have

$$\|p^*(h_n) - p_0\| \leq \|p_{h_n} - p_0\| \leq \sup_{n \geq 1} \|p_{h_n} - p_0\| =: r < \infty,$$

where the supremum is finite since $\lim_{n \rightarrow \infty} p_{h_n} = p^*$. Therefore, the sequence $\{p^*(h_n)\}_{n=1}^\infty$ is contained in the closed ball $\overline{B}_r(p_0)$, which is compact. Thus, $\{p^*(h_n)\}_{n=1}^\infty$ must have a convergent subsequence. If every convergent subsequence of $\{p^*(h_n)\}_{n=1}^\infty$ converges to p^* , then (since the sequence is contained in a compact set) $\lim_{n \rightarrow \infty} p^*(h_n) = p^*$ [29], and we obtain a contradiction, so suppose there exists a convergent subsequence $\{p^*(h_{n_m})\}_{m=1}^\infty$ with $\lim_{m \rightarrow \infty} p^*(h_{n_m}) = \hat{p} \neq p^*$. Since $\|p^*(h_{n_m}) - p_0\| \leq \|p_{h_{n_m}} - p_0\|$ for all m , taking the limit as $m \rightarrow \infty$ implies that $\|\hat{p} - p_0\| \leq \|p^* - p_0\|$. By Proposition 2.4.2, p^* is the unique closest parameter value in ∂R to p_0 under the map $\phi(\cdot, \cdot, \tilde{t})$. Therefore, since $\hat{p} \neq p^*$ and is at least as close to p_0 as p^* , we must have $\hat{p} \in R$ under the map $\phi(\cdot, \cdot, \tilde{t})$. Thus, $x_0(\hat{p}) \in W^s(x^s(\hat{p}))$, so there exists $\hat{T} > 0$ such that $x_0(\hat{p}) \in \text{int } \phi(W_{\text{loc}}^s(x^s(\hat{p})), \hat{p}, -\hat{T})$.

As ϕ and x_0 are C^1 , and by (5.23), there exist $h' > 0$ and $\delta' > 0$ such that $h \in (0, h')$ and $\|p - \hat{p}\| < \delta'$ implies that $W_{\text{loc}}^s(x_h^s(p))$ under $\hat{T}_d(\cdot, p, h)$ is sufficiently C^1 close to $W_{\text{loc}}^s(x^s(\hat{p}))$ under $\phi(\cdot, \hat{p}, \tilde{t})$ such that $x_0(p) \in \phi(W_{\text{loc}}^s(x_h^s(p)), p, -\hat{T})$. Thus, for m sufficiently large, $x_0(p^*(h_{n_m})) \in \phi(W_{\text{loc}}^s(x_{h_{n_m}}^s(p^*(h_{n_m}))), p^*(h_{n_m}), -\hat{T})$. Thus, for m sufficiently large, $p^*(h_{n_m}) \in R$ under $T_d(\cdot, \cdot, h_{n_m})$, which contradicts the definition of $p^*(h)$ as the closest point in ∂R to p_0 under $T_d(\cdot, \cdot, h)$. Therefore, we must have that $\lim_{h \rightarrow 0} p^*(h) = p^*$, which proves (3.10a).

Then, (3.10b) can be proved by following analogous reasoning as in the proof of Theorem 3.4.1, with the following changes:

- All instances of T should be replaced with $T_d(\cdot, \cdot, h)$, including in the definition of $j(p)$ from (3.7) and the definition of $\hat{F}(p)$ from (3.8).
- All instances of p^* should be replaced with $p^*(h)$.
- For $h > 0$ sufficiently small, there exists $x_h^s(p)$ near $x^s(p)$ stable, and $x_h^u(p)$ near $x^u(p)$ unstable, which are hyperbolic fixed points of $T_d(\cdot, \cdot, h)$ such that

$$\lim_{n \rightarrow \infty} T_d^n(x_0(p^*(h)), p^*(h), h) = x_h^u(p^*(h)).$$

- Since $W^u(x_h^u(p^*(h)))$ is one-dimensional, define $\hat{D}$ and $\tilde{D}$ to be the subsets of $W^u(x_h^u(p^*(h)))$ from $\hat{x}_1$ to $T_d(\hat{x}_1, p^*(h), h)$, and from $\tilde{x}_1$ to $T_d(\tilde{x}_1, p^*(h), h)$, respectively.

By continuity of $\frac{\partial f}{\partial x}$ and $x_h^u(p)$, and by (3.10a) and (3.10b) $\lim_{h \rightarrow 0} \lim_{p \rightarrow p^*(h)} \tilde{F}(p) = \lim_{h \rightarrow 0} \frac{\partial f}{\partial x}(x_h^u(p^*(h))) = \frac{\partial f}{\partial x}(x^u(p^*)) = A(p^*)$, and this proves (3.10c). (3.11a) and (3.11b) can be proved by following analogous reasoning as in the proof of Theorem 3.3.2. This completes the proof of Theorem 3.4.2. $\square$

5.5 Proof of Lemma 4.2.1

Proof of Lemma 4.2.1. Since $x^u(p^*)$ is a hyperbolic equilibrium point, by [27, Theorem 2.7.1] it follows that $T_{x^u(p^*)}W^s(x^u(p^*)) = E^s$ and $N_{x^u(p^*)}W^s(x^u(p^*)) = E^u$. As the mode of instability is the single unstable eigenvector of the Jacobian at $x^u(p^*)$, this implies that E^u is its span. $\square$

5.6 Proof of Theorem 4.2.1

Proof of Theorem 4.2.1. We have $x_0(p^*) = \phi(\hat{x}, p^*, -\hat{t})$. So, flowing backwards in time from $\hat{x}$, we have $\frac{\partial \phi(\hat{x}, p^*, -\hat{t})}{\partial \hat{x}} = \frac{\partial x}{\partial x_0}(-\hat{t})$. The solution to (2.4) integrated backwards for time $\hat{t}$ with IC $\frac{\partial x}{\partial x_0}(0) = \hat{v}$ is $\frac{\partial x}{\partial x_0}(-\hat{t})\hat{v}$. Thus, $v^* = \frac{\partial \phi(\hat{x}, p^*, -\hat{t})}{\partial \hat{x}}\hat{v}$ and $U^* = \frac{\partial \phi(\hat{x}, p^*, -\hat{t})}{\partial \hat{x}}\hat{U}$. As ϕ is a diffeomorphism, its derivative is full rank. As $\hat{U}$ consists of linearly independent vectors, this implies that U^* consists of linearly independent vectors. Since $\hat{x} \in W^s(x^u(p^*))$ and $W^s(x^u(p^*))$ is invariant under the flow, the derivative $\frac{\partial \phi(\hat{x}, p^*, -\hat{t})}{\partial \hat{x}}$ maps $T_{\hat{x}}W^s(x^u(p^*))$ to $T_{x_0(p^*)}W^s(x^u(p^*))$. As $\hat{U}$ is a basis for $T_{\hat{x}}W^s(x^u(p^*))$, and since the tangent space has constant dimension over $W^s(x^u(p^*))$, U^* is a basis for $T_{x_0(p^*)}W^s(x^u(p^*))$. Note that $\frac{\partial \phi(\hat{x}, p^*, -\hat{t})}{\partial \hat{x}}$ must map $N_{\hat{x}}W^s(x^u(p^*))$ to $N_{x_0(p^*)}W^s(x^u(p^*))$. Otherwise, if there exists $w \in N_{\hat{x}}W^s(x^u(p^*))$ which maps to $v \in T_{x_0(p^*)}W^s(x^u(p^*))$, then this would contradict that $\left(\frac{\partial \phi(\hat{x}, p^*, -\hat{t})}{\partial \hat{x}}\right)^{-1} = \frac{\partial \phi(\hat{x}, p^*, \hat{t})}{\partial \hat{x}}$ must map $T_{x_0(p^*)}W^s(x^u(p^*))$ to $T_{\hat{x}}W^s(x^u(p^*))$. As $\hat{v} \in N_{\hat{x}}W^s(x^u(p^*))$, this implies that $v^* \in N_{x_0(p^*)}W^s(x^u(p^*))$. As $N_{x_0(p^*)}W^s(x^u(p^*))$ has constant dimension over $W^s(x^u(p^*))$, v^* is a basis for it. Finally, assume towards a contradiction that v^* points outwards. For $t \in [0, \hat{t}]$, let $v^*(t)$ be the solution to (2.4) integrated backwards for time t with IC $\hat{v}$. As $v^*(0) = \hat{v}$ points inwards and $v^*(\hat{t}) = v^*$ points outwards, there must exist $\bar{t} \in [0, \hat{t}]$ such that $v^*(\bar{t}) \in T_{\phi(\hat{x}, p^*, -\bar{t})}W^s(x^u(p^*))$, which contradicts that $v^*(\bar{t}) \in N_{\phi(\hat{x}, p^*, -\bar{t})}W^s(x^u(p^*))$. Thus, v^* must point inwards. $\square$

5.7 Proof of Corollary 4.2.1

Proof of Corollary 4.2.1. As $A(p^*) = \frac{\partial f}{\partial x}(x^u(p^*), p^*)$ has distinct eigenvalues (so that each eigenspace is one dimensional) and $\frac{\partial f}{\partial x}(x, p)$ is continuous, by [25, Theorem 3.1.3] there exists a ball B around $(x^u(p^*), p^*)$ over which the eigenvectors of $\frac{\partial f}{\partial x}(x, p)$ are continuous. Thus, shrinking B if necessary implies that all eigenvectors of $\frac{\partial f}{\partial x}(x, p)$ are arbitrarily close to the eigenvectors at $A(p^*)$. As distance from the orbit of $x_0(p)$ to $x^u(p^*)$ converges to zero as $p \rightarrow p^*$, there exists $r > 0$ such that $\|\hat{p} - p^*\| < r$ implies that the solution $\hat{x}(\hat{p})$ to (4.1) (with time $\hat{t}(\hat{p})$) lies in B . Let $\hat{v}(\hat{p})$ and $\hat{U}(\hat{p})$ be the unstable and stable eigenspaces of $\frac{\partial f}{\partial x}(\hat{x}(\hat{p}), \hat{p})$. Let $\bar{x}(\hat{p})$ be the closest point in $W^s(x^u(\hat{p}))$ to $\hat{x}(\hat{p})$, and let $\bar{v}(\hat{p})$ and $\bar{U}(\hat{p})$ be the unstable and stable eigenspaces of $\frac{\partial f}{\partial x}(\bar{x}(\hat{p}), \hat{p})$. Thus, we may shrink r and B if necessary so that $\|\hat{p} - p^*\| < r$ implies that $\frac{\partial \phi(\hat{x}(\hat{p}), \hat{p}, -\hat{t}(\hat{p}))}{\partial \hat{x}(\hat{p})}$ and $\frac{\partial \phi(\bar{x}(\hat{p}), \hat{p}, -\hat{t}(\hat{p}))}{\partial \bar{x}(\hat{p})}$ are sufficiently close in norm that they map $\hat{v}(\hat{p})$ and $\bar{v}(\hat{p})$ to within ϵ norm of each other, and similar for $\hat{U}(\hat{p})$ and $\bar{U}(\hat{p})$. The result then follows by Theorem 4.2.1. $\square$

5.8 Proof of Theorem 4.3.1

Proof of Theorem 4.3.1. The result follows from [21, Theorem 4]. $\square$

Chapter 6

Application to Power Systems

6.1 Damped, Driven Nonlinear Pendulum

6.1.1 Introduction

To illustrate the algorithm of Section 3.4 and the results of Theorem 3.4.2, we first consider the simple example of a damped, driven nonlinear pendulum with constant driving force. For this simple example, it is straightforward to compute the mode of instability analytically by directly finding the controlling UEP, so this example can be used as a benchmark to validate the proposed algorithm. The dynamics are given by

$$\dot{x}_1 = x_2, \quad (6.1)$$

$$\dot{x}_2 = -c_1 \sin(x_1) - c_2 x_2 + c_3, \quad (6.2)$$

where $c_1, c_2, c_3 > 0$ are real parameters and $x = (x_1, x_2) \in \mathbb{R}^2$. Physically, x_1 represents the angle of the pendulum, x_2 its angular velocity, c_1 the square of the natural frequency of the pendulum (under the small angle approximation), c_2 a damping coefficient due to air drag, and c_3 the constant driving torque. Using the simple power system model of a single machine infinite bus [30], (6.1)-(6.2) can also be interpreted as an electrical generator, with (x_1, x_2) the angle and angular velocity of the turbine, c_1 a constant determining the electrical torque supplied by the generator, c_2 a damping coefficient, and c_3 the constant driving mechanical torque.

For the demonstration below, we set $c = (c_1, c_2, c_3) = (2, 0.5, 1.5)$. This system possesses one SEP at $(0.848, 0)$ and one UEP at $(2.294, 0)$. As we vary the parameter values c_1 , c_2 , and/or c_3 , the SEP and UEP will also vary.

We establish an **IC** to (6.1)-(6.2) as the output of the following related system which models a finite-time disturbance:

$$\dot{z}_1 = z_2, \tag{6.3}$$

$$\dot{z}_2 = -c_2 z_2 + c_3, \tag{6.4}$$

starting from the **SEP** and integrating for time $t = 0.8$ seconds, which is the length of time the disturbance is active. This yields the post-disturbance **IC** for (6.1)-(6.2). If (6.1)-(6.2) are interpreted as an electrical generator, then (6.3)-(6.4) represent a temporary short circuit across the terminals of the generator, which is modeled by setting the electrical torque term in (6.1)-(6.2) to zero temporarily, resulting in (6.3)-(6.4).

6.1.2 Numerical Experiments

For simplicity, consider the one-dimensional parameter space consisting of just the single parameter $p = c_3$ which we will subsequently treat as a free parameter, with all other parameters constant, and let $p_0 = 1.5$. We set the time step $h = 0.02$ seconds, use trapezoidal integration for the numerical integration, and solve each trapezoidal integration step using the Newton-Raphson method with a tolerance of 10^{-15} . Then the numerical integration of (6.3)-(6.4) yields the **IC**, and the numerical integration of (6.1)-(6.2) provides the full system trajectory, which is used to calculate $\tilde{F}(p)$ using (3.8).

Since the parameter space is one-dimensional in this case, the closest boundary parameter value $p^*(h)$ to p_0 can be found using a backtracking line search performed by bisection. Due to finite numerical precision, and since ∂R is a set of measure zero, in general we cannot numerically compute $p^*(h)$ which lies exactly on ∂R . Instead, we numerically approximate $p^*(h)$ to arbitrary accuracy with a value that lies just inside R . For the remainder of this chapter, we therefore abuse notation and let $p^*(h)$ denote this numerical approximation, which lies slightly inside R , rather than the exact theoretical boundary value. Using the bisection method, we determine that $p^*(0.02) = 1.5686$. In this case, the system with $p = p^*(0.02)$ has two equilibrium points (where the angle is restricted to a single interval of length 2π since the angle is defined modulo 2π), one of which is the **SEP** where the system begins prior to the disturbance, while the other is a **UEP** that can be shown to be the controlling **UEP** for this disturbance (see Fig. 6.1), consistent with Assumption 3.2.2.

6.1.3 Results

Fig. 6.2 shows the value of $p^*(h)$ for $h \in \{0.02, 0.04, 0.08, 0.1, 0.2, 0.4, 0.8\}$. The graph of $p^*(h)$ versus h becomes a flat line as h gets sufficiently small, which shows that $p^*(h)$ converges to the constant p^* as h approaches zero, consistent with (3.10a). Let $p^*(0.02)$ be the approximation of the exact theoretical boundary parameter value. For the same values of h , $\left\| \tilde{F}(p^*(h)) - \frac{\partial f}{\partial x}(x_h^u(p^*(h))) \right\| < 6.7\%$, which is consistent with (3.10b). Fig. 6.3 shows the value of $\left\| \tilde{F}(p^*(h)) - A(p^*(0.02)) \right\|$ for the same values of h , which shows that $\tilde{F}(p^*(h))$ converges to the true Jacobian as h approaches zero, consistent with (3.10c). Fig. 6.4 and Fig. 6.5 show $\left| \tilde{\lambda}(p^*(h)) - \lambda(p^*(0.02)) \right|$ and $\|\tilde{v}(p^*(h)) - v(p^*(0.02))\|_2$ for the same values of h , which shows that the unstable eigenvalue and its associated eigenvector of $\tilde{F}(p^*(h))$ converge to the unstable eigenvalue of the true Jacobian and the mode of instability, respectively, as h approaches zero, consistent with (3.11a) and (3.11b). As the controlling UEP $x_{0.02}^u(p^*(0.02))$ is known and available analytically, the Jacobian at the controlling UEP, $A(p^*(0.02))$, can therefore be computed directly and exactly. Consistent with Assumption 3.2.1, it has a single unstable eigenvalue, and its corresponding unstable eigenvector is $v(p^*(0.02)) = \begin{bmatrix} 0.7464 \\ 0.6655 \end{bmatrix}$. Let $\hat{p} = p^*(0.02)$ in this case, and note that $\hat{p} \in R$ and is very close to the exact theoretical boundary value. Numerically computing $\tilde{F}(\hat{p})$ from the system trajectory, consistent with Theorem 3.4.2, it is shown to have a single unstable eigenvalue with associated unstable eigenvector $\tilde{v}(\hat{p}) = \begin{bmatrix} 0.7569 \\ 0.6536 \end{bmatrix}$. Thus, $\|\tilde{v}(\hat{p}) - v(p^*(0.02))\|_2 < 0.016$, so $\tilde{v}(\hat{p})$ and $v(p^*(0.02))$ have less than 1.6% difference since both are normalized to unit length. Fig. 6.1 also shows the RoA boundary and the mode of instability at $\hat{p}$.

We follow the steps shown in Algorithm 1, and find that $A(p^*(0.02)) = \begin{bmatrix} 0 & 1 \\ 1.2407 & -0.5 \end{bmatrix}$, $\hat{x} = \begin{bmatrix} 2.2004 \\ 0.0547 \end{bmatrix}$, $\hat{v} = \begin{bmatrix} -0.7464 \\ -0.6655 \end{bmatrix}$, $\hat{U} = \begin{bmatrix} -0.5836 \\ 0.8121 \end{bmatrix}$, $\tilde{v} = \begin{bmatrix} -0.9509 \\ -0.3094 \end{bmatrix}$, $\tilde{U} = \begin{bmatrix} -0.7556 \\ 0.6550 \end{bmatrix}$. Next, we formulate the optimization problem (4.3) and solve it using the gradient descent algorithm given by (4.5) with the starting parameter value $p^0 = 1.5$, the step size $\alpha = 0.01$, solution tolerance set to $\epsilon = 10^{-5}$, and $\lambda = 0.4$. The method converges to $\bar{p} = 1.2083$, which is consistent with Theorem 4.3.1. Note that the safety margin increases by over five times from its initial value of 0.0686 at p^0 to 0.3603 at $\bar{p}$, indicating that the method successfully reduced vulnerability to the disturbance.

Since this example is two-dimensional, we can visualize the IC and the RoA boundary of

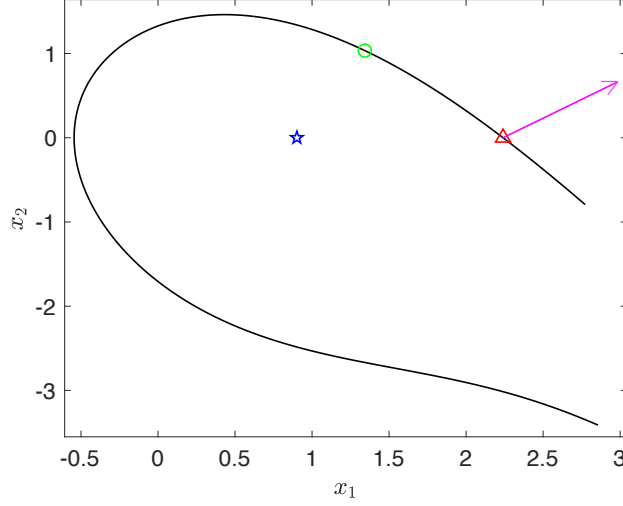


Figure 6.1: The **RoA** boundary $\partial W^s(x_{0.02}^s(\hat{p}))$ (solid black line) of the **SEP** $x_{0.02}^s(\hat{p})$ (blue star) of equations (6.3)-(6.4) is shown. It is equal to $W^s(x_{0.02}^u(\hat{p}))$ where $x_{0.02}^u(\hat{p})$ (red triangle) is the **UEP** that can be shown to be the controlling **UEP** for the given disturbance. The **IC** $x_0(\hat{p})$ (green circle) is shown. The unstable eigenvector $v(\hat{p})$ of $\frac{\partial f}{\partial x}(x_{0.02}^u(\hat{p}))$ (solid magenta line with an arrow), which is the mode of instability for the disturbance, is shown.

the **SEP** at p_0 and $\bar{p}$. As illustrated in Fig. 6.6, we can see that the **IC** at $\bar{p}$ is substantially further from the **RoA** boundary than the **IC** at p_0 .

6.1.4 Conclusion

Therefore, this illustrates that the nonlinear mode of instability of this low dimensional system can be accurately computed and the vulnerability of this system subject to the given disturbance can be reduced by the proposed algorithm.

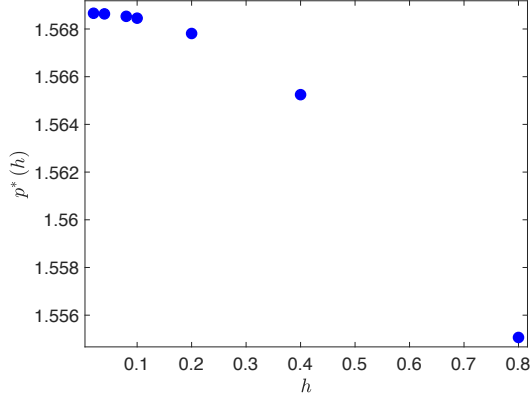


Figure 6.2: The closest boundary parameter value $p^*(h)$ to p_0 as a function of h .

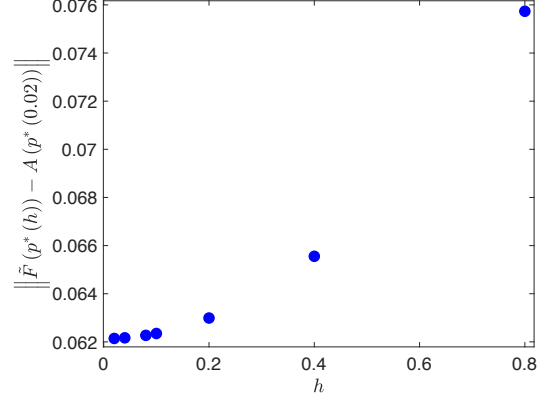


Figure 6.3: The norm of the difference between $\tilde{F}(p^*(h))$ and $A(p^*(0.02))$ as a function of h .

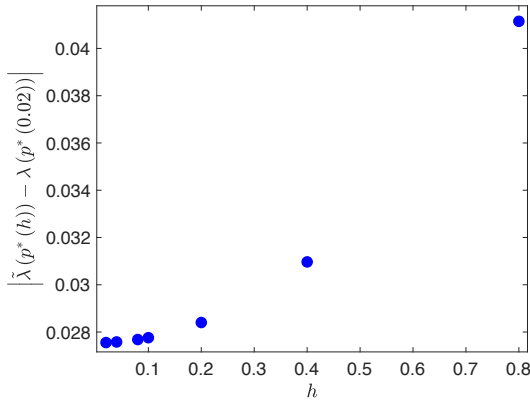


Figure 6.4: The absolute value of the difference between the unstable eigenvalues $\tilde{\lambda}(p^*(h))$ of $\tilde{F}(p^*(h))$ and $\lambda(p^*(0.02))$ of $A(p^*(0.02))$ as a function of h .

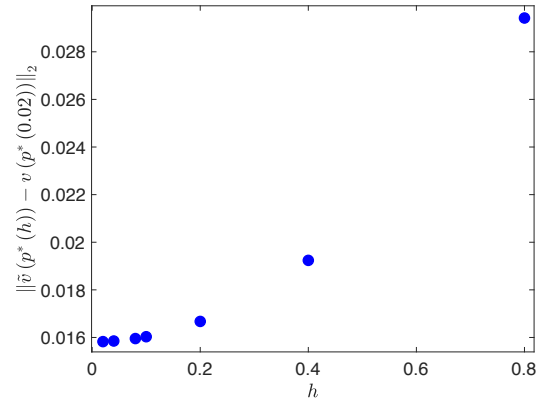


Figure 6.5: The norm of the difference between the unstable eigenvectors $\tilde{v}(p^*(h))$ of $\tilde{F}(p^*(h))$ and $v(p^*(0.02))$ of $A(p^*(0.02))$ as a function of h .

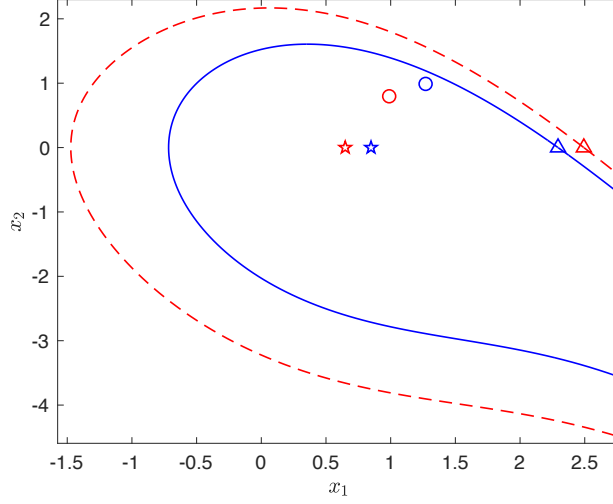


Figure 6.6: The RoA boundaries $\partial W^s(x^s(p))$ of the SEPs $x^s(p)$ (stars) for parameter values $p = \bar{p}$ (red dashed) and $p = p_0$ (solid blue), are shown. Each boundary is equal to $W^s(x^*(p))$ where $x^*(p)$ (triangles) is the controlling UEP. The IC $x_0(p)$ (circles) are shown for each parameter value.

6.2 IEEE 9-Bus Power System

6.2.1 Introduction

Next, the algorithm described in Section 3.4 and 4 is applied to the more complex example of the IEEE 9-bus power system [2], where the mode of instability is not so straightforward to obtain analytically. As shown in Fig. 6.7, this is a network consisting of nine nodes that contains three synchronous generators, three loads, six transmission lines, and three transformers. Each generator is modeled using a 4th order synchronous machine model, along with an AVR and PSS for control, which are modeled according to the IEEE standard ST1C and PSS1A models [1], respectively. The loads are modeled as constant power loads. The dimension of the dynamic states for this system is 60. The disturbance considered is a

temporary short circuit, known as a fault, at the terminals of generator three. The system is initially at a [SEP](#), then the fault occurs at 0.02 seconds. It lasts for 0.2 seconds and is cleared at 0.22 seconds. For this nonlinear power system model with higher dimensions, the mode of instability is determined without direct identification of the controlling [UEP](#) using the algorithm from Section [3.4](#).

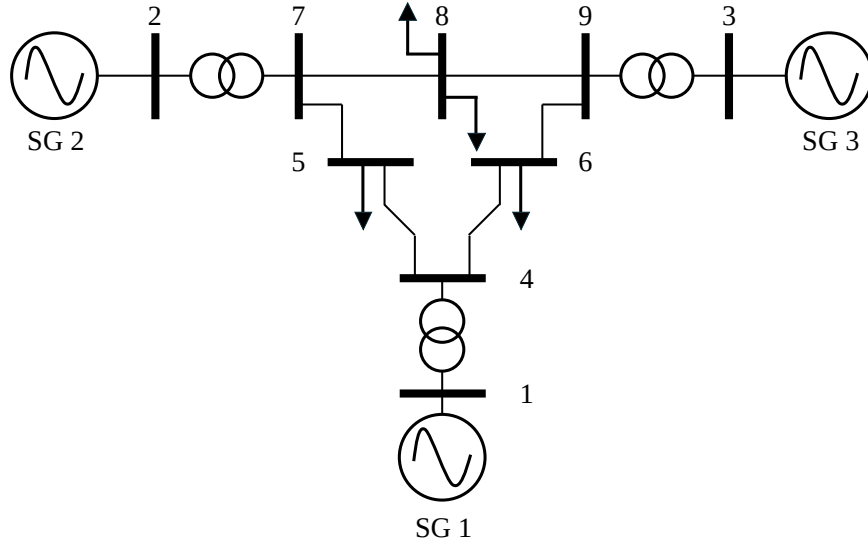


Figure 6.7: IEEE 9-Bus Power System (SG = synchronous generator).

Many model parameters of the system are of interest for recovery considerations. The moment of inertia scaling factor multiplies the moment of inertia of all three synchronous generators simultaneously. The gains of the [AVR](#) controllers can help to capture the impact of controller turning on system stability, the gains of the [PSS](#) controllers are crucial for enhancing system stability by improving the damping of low-frequency oscillations in power systems, and the gains of the governor controllers are critical in frequency control for enhancing system stability. The active and reactive power loads are modeled using the standard exponential form of a voltage-dependent load model. The voltage exponents are set uniformly for all active power loads and similarly for all reactive power loads. As load is often uncertain, given the complexity of modeling load dynamics, these parameters can help to capture the impact of uncertain load behavior on system recoverability.

6.2.2 Numerical Experiments

To demonstrate the effectiveness of the proposed algorithm, we first consider the one-dimensional parameter space with the parameter of interest being the moment of inertia scaling factor. We use trapezoidal integration with the Newton-Raphson method for the numerical integration with a fixed time step of $h = \frac{1}{60}$ seconds and a tolerance of 10^{-10} . This choice of h represents a single cycle of the 60 Hz AC voltage and, thus, is faster than all other dynamics explicitly included in the system model, so it represents a choice of h that is close to zero for this application. Using the backtracking line search performed by bisection, or using the method from [11], which finds the closest point on the [recovery boundary](#) in one-dimensional parameter space, we compute the boundary value of the moment of inertia scaling factor to be $p^* \left(\frac{1}{60} \right) = 0.8724$, which is very close to the exact theoretical boundary value. For this example, we do not know the location of the controlling [UEP](#), so we are unable to compute $v \left(p^* \left(\frac{1}{60} \right) \right)$ directly from $A \left(p^* \left(\frac{1}{60} \right) \right)$. Instead, we set $\hat{p} = p^* \left(\frac{1}{60} \right)$, and numerically compute $\tilde{F}(\hat{p})$.

Consistent with Theorem 3.4.2, $\tilde{F}(\hat{p})$ has exactly one unstable eigenvalue $\tilde{\lambda}(\hat{p})$. As the choice of h is close to zero for this application, by Theorem 3.4.2 its corresponding eigenvector $\tilde{v}(\hat{p})$ is a close approximation to the true mode of instability. Table 6.1 shows the components of $\tilde{v}(\hat{p})$ other than those which are close to zero. The components of ω_1 , ω_2 , and ω_3 of $\tilde{v}(\hat{p})$ correspond to the frequency components of generator one, generator two, and generator three, respectively. The components of θ_1 , θ_2 , and θ_3 of $\tilde{v}(\hat{p})$ correspond to the angle of generator one, generator two, and generator three, respectively. The components of x_1 , x_2 , and x_3 correspond to the internal state of the [AVR](#) of generator one, generator two, and generator three, respectively.

ω_1	ω_2	ω_3
-0.0112	0.7594	0.4227
θ_1	θ_2	θ_3
-0.0026	0.1754	0.0977
x_1	x_2	x_3
0.0863	0.3473	0.2673

Table 6.1: Components of $\tilde{v}(\hat{p})$ representing the frequencies ω , angles θ , and the internal state of the [AVR](#) x for the three synchronous generators in the case of one-dimensional parameter space.

For the multi-dimensional parameter space, we consider the parameter set consisting of

the moment of inertia of each of the three generators, the real and reactive load voltage exponents for each of the three loads, the AVR and PSS controller gains for each of the three generators (15 parameters). This set captures both the uncertainty in load dynamic characteristics and the influence of controller response on system recoverability. Let H_1 , H_2 , and H_3 be the moment of inertia of generator one, generator two, and generator three, respectively. Let e_1 , e_2 , and e_3 be the real load voltage exponents for each of the three loads. Let e_4 , e_5 , and e_6 be the reactive load voltage exponents for each of the three loads. Let K_1 , K_2 , and K_3 be the AVR controller gains of generator one, generator two, and generator three, respectively. Let L_1 , L_2 , and L_3 be the PSS controller gains of generator one, generator two, and generator three, respectively. Let $H = [H_1 \ H_2 \ H_3]$, $e = [e_1 \ e_2 \ e_3 \ e_4 \ e_5 \ e_6]$, $K = [K_1 \ K_2 \ K_3]$, and $L = [L_1 \ L_2 \ L_3]$. Let $p = [H \ e \ K \ L]^\top$ be the vector of chosen parameters. Let p_0 be the initial parameter values, as given in Table 6.2. We apply the optimization approach [11] to find the closest point on the recovery boundary to the nominal value of this 15-dimensional parameter space, with the solution tolerance set to $\epsilon = 10^{-5}$.

H_1	H_2	H_3	e_1	e_2
$\frac{23.64}{60\pi}$	$\frac{6.4}{60\pi}$	$\frac{3.01}{60\pi}$	2	2
e_3	e_4	e_5	e_6	K_1
2	2	2	2	20
K_2	K_3	L_1	L_2	L_3
20	20	$\frac{0.5}{120\pi}$	$\frac{0.5}{120\pi}$	$\frac{0.5}{120\pi}$

Table 6.2: Test case parameter values.

The parameter value on the recovery boundary $p^*(\frac{1}{60}) = [H^* \ e^* \ K^* \ L^*]^\top$ for this 15-dimensional parameter space is shown in Table 6.3, which closely approximates the exact theoretical boundary value, and is arbitrarily close to the initial parameter value. Since the location of the controlling UEP is unknown, we are unable to determine $A(p^*(\frac{1}{60}))$ and, consequently, $v(p^*(\frac{1}{60}))$ directly. Instead, we set $\hat{p} = p^*(\frac{1}{60})$ once again and numerically compute $\tilde{F}(\hat{p})$.

Consistent with Theorem 3.4.2, $\tilde{F}(\hat{p})$ possesses a unique unstable eigenvalue and, by Theorem 3.4.2, its associated unstable eigenvector is a close approximation to the mode of instability. Table 6.4 presents the components of $\tilde{v}(p^*(\frac{1}{60}))$ apart from those that are close to zero. The components of ω_1 , ω_2 , ω_3 , θ_1 , θ_2 , θ_3 , x_1 , x_2 , x_3 of $\tilde{v}(\hat{p})$ carry the same interpretation as those in the one-dimensional parameter space.

For the algorithm described in Chapter 4, we consider the parameter set consisting of

H_1^*	H_2^*	H_3^*	e_1^*	e_2^*
0.1254	0.0340	0.0160	2	2
e_3^*	e_4^*	e_5^*	e_6^*	K_1^*
2	2	2	2	20
K_2^*	K_3^*	L_1^*	L_2^*	L_3^*
20	20	0.0013	0.0013	0.0013

Table 6.3: Closest boundary parameter value for moment of inertia, real and reactive load voltage exponents, AVR and PSS controller gains.

ω_1	ω_2	ω_3
0.1106	0.4388	0.6850
θ_1	θ_2	θ_3
0.0300	0.1192	0.1861
x_1	x_2	x_3
0.1150	0.2648	0.4267

Table 6.4: Components of $\tilde{v}(\hat{p})$ representing the frequencies ω , angles θ , and the internal state of the AVR x for the three synchronous generators in the case of 15-dimensional parameter space.

the AVR, PSS, and governor controller gains for each of the three generators (9 parameters). These parameters are tunable, allowing them to be varied when applying the algorithm of Chapter 4 to reduce vulnerability. Let K_1 , K_2 , and K_3 be the AVR controller gains of generator one, generator two, and generator three, respectively. Let L_1 , L_2 , and L_3 be the PSS controller gains of generator one, generator two, and generator three, respectively. Let M_1 , M_2 , and M_3 be the governor controller gains of generator one, generator two, and generator three, respectively. Let $K = [K_1 \ K_2 \ K_3]$, $L = [L_1 \ L_2 \ L_3]$, and $M = [M_1 \ M_2 \ M_3]$. Let $p = [K \ L \ M]^\top$ be the vector of chosen parameters, and note that this results in a 9-dimensional parameter space. Let p_0 be the initial parameter values, as given in Table 6.5. We apply the algorithm from prior work [11] to find the closest boundary parameter value $\hat{p} = [K^* \ L^* \ M^*]^\top$ to p_0 , which is shown in Table 6.6.¹ The safety margin at p_0 is 0.6338.

¹As before, $\hat{p}$ lies slightly inside R , rather than the exact theoretical boundary value.

K_1	K_2	K_3
2.6909	2.6837	2.6650
L_1	L_2	L_3
0.0013	0.0013	0.0013
M_1	M_2	M_3
0.0502	0.0497	0.0497

Table 6.5: Test case parameter values.

K_1^*	K_2^*	K_3^*
2.3233	2.3173	2.3011
L_1^*	L_2^*	L_3^*
0.0013	0.0013	0.0013
M_1^*	M_2^*	M_3^*
0.0503	0.0496	0.0496

Table 6.6: Closest boundary parameter value for the [AVR](#), [PSS](#), and governor controller gains.

6.2.3 Results

For the one-dimensional parameter space, the mode of instability, as shown in Table 6.1, reveals the subset of dynamics responsible for failure to recover from the fault. In particular, among the three generators, the frequency component, angle component, and the [AVR](#) internal state component of generator two are the largest in magnitude. This leads to the non-intuitive information that generator two is most responsible for the failure to recover from the disturbance, even though the disturbance originates at a distant location from generator two (near generator three). To further validate this observation, we plot the frequency dynamics of all three generators before they achieve synchronism, as shown in Fig. 6.8. The figure shows that the frequency of generator two has larger amplitude oscillations after the disturbance has occurred than the frequencies of the other two generators. Fig. 6.9 shows the angle of generator three relative to generator one and the angle of generator two relative to generator one as a function of time. We observe that the relative angle of generator two has larger amplitude oscillations after the disturbance has occurred than the relative angle of generator three, indicative of generator two's role as the generator most important to the mechanism of instability for the disturbance. Fig. 6.10 shows the dy-

namics for the internal state of the AVR of all three generators. It shows that the internal state of the AVR of generator two has a larger initial transient than the internal state of the AVR of generator one and generator three, which indicates that generator two is most important for determining whether the system will fail to recover from the disturbance. In addition, Table 6.1 shows that the frequency component, angle component, and the AVR internal state component of generator one are very small, indicating that generator one plays a more minimal role in the onset of failure to recover than generators two and three. Furthermore, the components of the internal states of the PSS are close to zero, indicating the nontrivial fact that, unlike for many other faults, for this particular disturbance the PSS controllers do not have a significant contribution to the mechanism of instability.

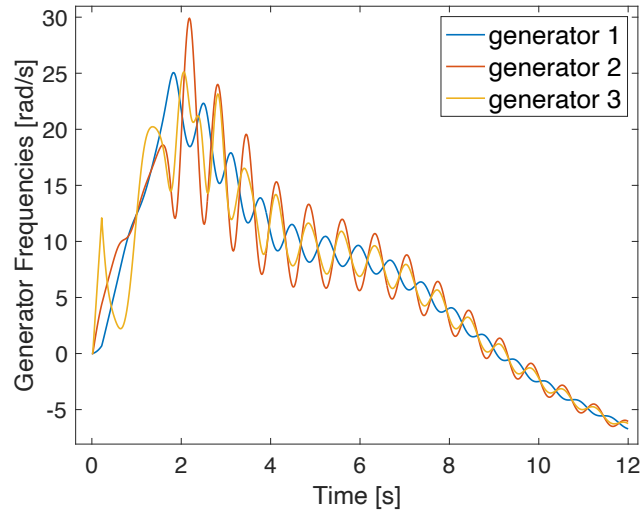


Figure 6.8: The frequency dynamics of all three generators before reaching synchronism in the one-dimensional parameter space, where the moment of inertia scaling factor is the parameter of interest, at $p = p^* \left(\frac{1}{60} \right)$.

Unlike the one-dimensional case with the moment of inertia scaling factor, for the given higher dimensional (15-dimensional) parameter space, generator three has the largest frequency, angle, and the AVR internal state components in magnitude, followed by generator

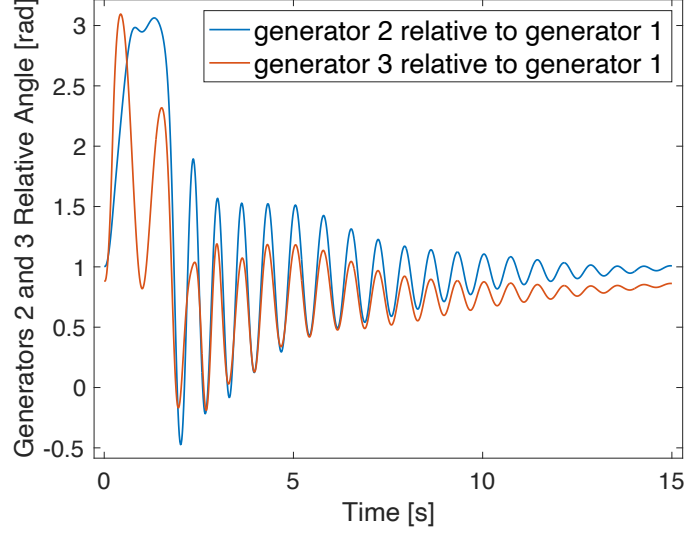


Figure 6.9: The angle of generator two and generator three relative to generator one in the one-dimensional parameter space, where the moment of inertia scaling factor is the parameter of interest, at $p = p^* \left(\frac{1}{60} \right)$.

two and then generator one. This provides a clear indication that generator three plays the most significant role in the failure to recover from the disturbance, which is intuitive since the disturbance originates near generator three, followed by generator two and then generator one. To further validate this observation, the frequency dynamics of all three generators before they reach synchronism, and the angle of generator three and generator two relative to generator one as a function of time, are shown in Fig. 6.11 and Fig. 6.12, respectively. We observe that after the fault is cleared, the frequency of generator three exhibits a long transient, with large oscillations starting earlier and ending later than for the other generators, with generator two having the second largest initial frequency transient, and finally generator one having the smallest. Furthermore, the relative angle of generator three has greater amplitude oscillations than those of generator two. Fig. 6.13 illustrates the AVR internal state dynamics for all three generators. We can see that the internal state of the AVR of generator three has a larger initial transient than the internal

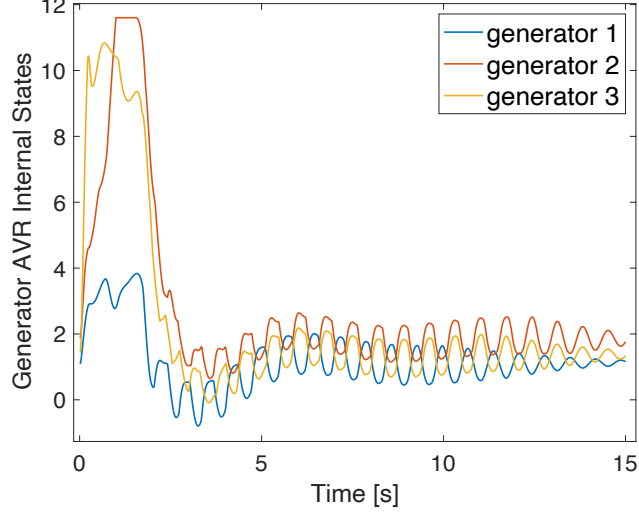


Figure 6.10: The internal state dynamics of the AVR of all three synchronous generators in the one-dimensional parameter space, where the moment of inertia scaling factor is the parameter of interest, at $p = p^* \left(\frac{1}{60} \right)$.

state of the AVR of generator two, and generator one has the smallest initial transient in its AVR internal state. All of these observations highlight that the frequency and angle dynamics of generator three, along with the operation of its AVR, are the primary factors influencing whether the system can recover from the disturbance, followed by those of generator two, and then of generator one. Notably and non-intuitively, although both the frequency and voltage dynamics play a major role in the mechanism of instability in this case, the dynamics of the PSS, which are designed to limit the coupling between frequency and voltage dynamics, do not play a significant role in the failure to recover.

Next, we apply the algorithm described in Chapter 4 to solve the optimization problem (4.3) with initial parameter value $p^0 = p_0$, step size $\alpha = 0.01$, solution tolerance of $\epsilon = 10^{-5}$, and $\lambda = 0.6$. The method converges to a local minimum $\bar{p} = [\bar{K} \quad \bar{L} \quad \bar{M}]^\top$, consistent with Theorem 4.3.1, which is shown in Table 6.7. To assess whether the method was successful at reducing vulnerability, we again apply the algorithm from prior work [11] to find the

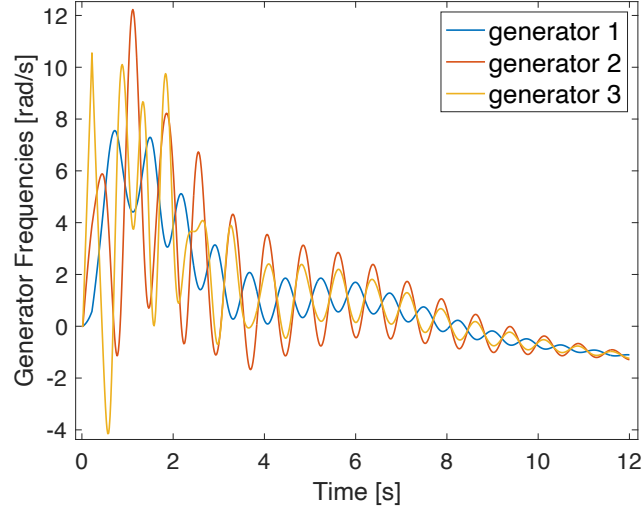


Figure 6.11: The frequency dynamics of all three generators before reaching synchronism in the 15-dimensional parameter space, where the parameters of interest are the moment of inertia, real and reactive load voltage exponents, AVR and PSS controller gains, at $p = p^* \left(\frac{1}{60} \right)$.

closest boundary parameter value to $\bar{p}$, which is shown in Table 6.8². From this value, the safety margin can be found to be 1.5398, which is more than 2.4 times larger than the safety margin was for p_0 . This illustrates that the proposed method successfully tuned controller parameter values so as to significantly reduce the vulnerability of the system to the given disturbance. To further validate this observation, we plot the frequency dynamics of generator one, generator two, and generator three for the parameter value p_0 and $\bar{p}$, as shown in Fig. 6.14, Fig. 6.15, and Fig. 6.16, respectively. The figures show that at p_0 , the frequencies of generator one, generator two, and generator three have larger amplitude oscillations after the disturbance than those at $\bar{p}$, which indicates that the system is less vulnerable to the disturbance for parameter value $\bar{p}$ than for p_0 .

² $\bar{p}^*$ lies slightly inside R , rather than the exact theoretical boundary value.

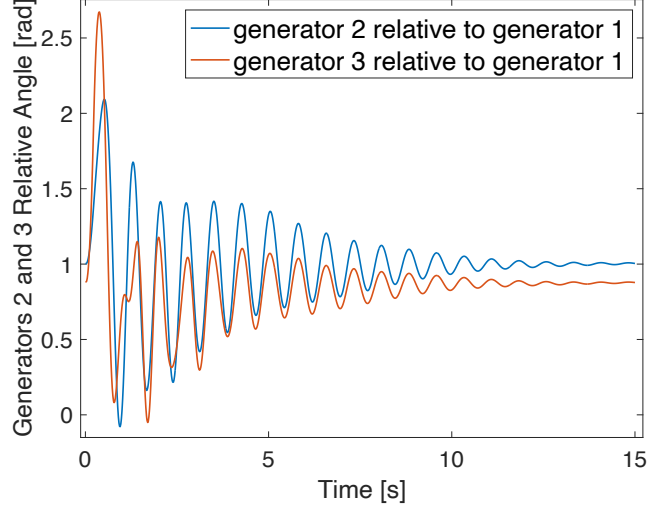


Figure 6.12: The angle of generator two and generator three relative to generator one in the 15-dimensional parameter space, where the parameters of interest are the moment of inertia, real and reactive load voltage exponents, [AVR](#) and [PSS](#) controller gains, at $p = p^* \left(\frac{1}{60} \right)$.

6.2.4 Conclusion

Overall, the proposed algorithm was successfully applied to the IEEE 9-bus benchmark power system to compute the nonlinear mode of instability for both one-dimensional and higher dimensional parameter spaces, and revealed unexpected dynamic behavior responsible for the mechanism of instability that would not have been straightforward to identify otherwise. The control design was successfully applied to the IEEE 9-bus power system to reduce vulnerability to a temporary short circuit.

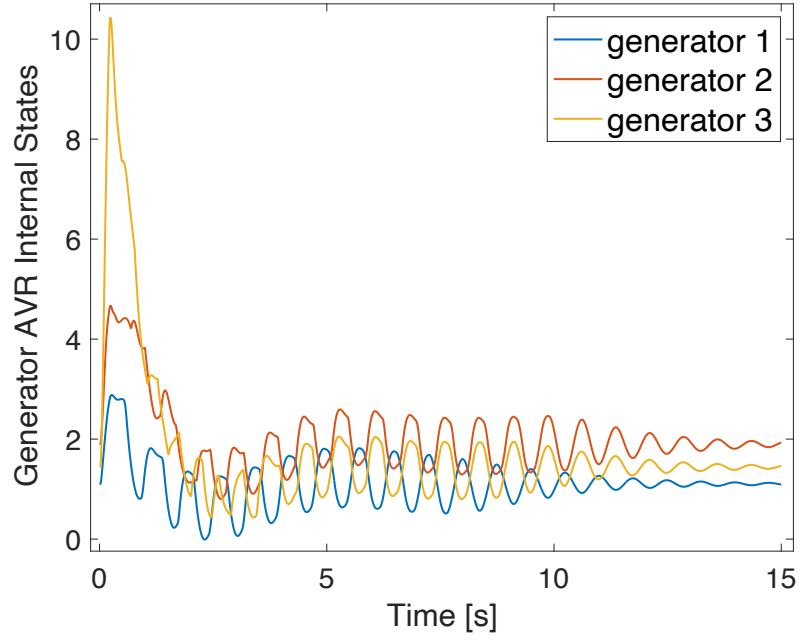


Figure 6.13: The internal state dynamics of the AVR of all three synchronous generators in the 15-dimensional parameter space, where the parameters of interest are the moment of inertia, real and reactive load voltage exponents, AVR and PSS controller gains, at $p = p^* \left(\frac{1}{60} \right)$.

$\bar{K}_1$	$\bar{K}_2$	$\bar{K}_3$
3.0415	5.4341	4.6327
$\bar{L}_1$	$\bar{L}_2$	$\bar{L}_3$
0.0013	0.0013	0.0013
$\bar{M}_1$	$\bar{M}_2$	$\bar{M}_3$
0.0502	0.0497	0.0397

Table 6.7: Local minimum of the control design optimization problem.

$\bar{K}_1^*$	$\bar{K}_2^*$	$\bar{K}_3^*$
2.6652	4.2275	3.7534
$\bar{L}_1^*$	$\bar{L}_2^*$	$\bar{L}_3^*$
0.0013	0.0013	0.0013
$\bar{M}_1^*$	$\bar{M}_2^*$	$\bar{M}_3^*$
0.0503	0.0496	0.0397

Table 6.8: Closest boundary parameter value for the parameter value in Table 6.7.

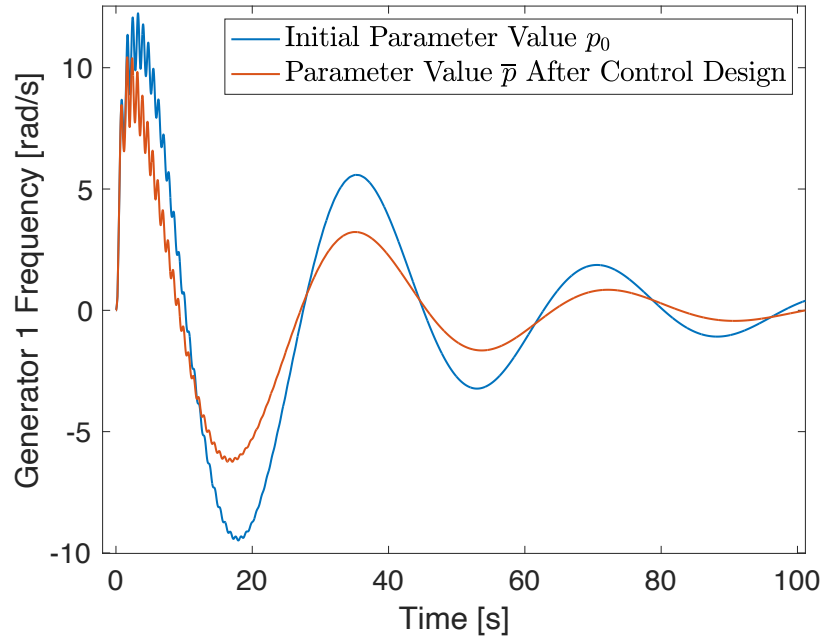


Figure 6.14: Frequency dynamics of generator one for the parameter value p_0 and $\bar{p}$ in the 9-dimensional parameter space, where the parameters of interest are the AVR, PSS, and governor controller gains.

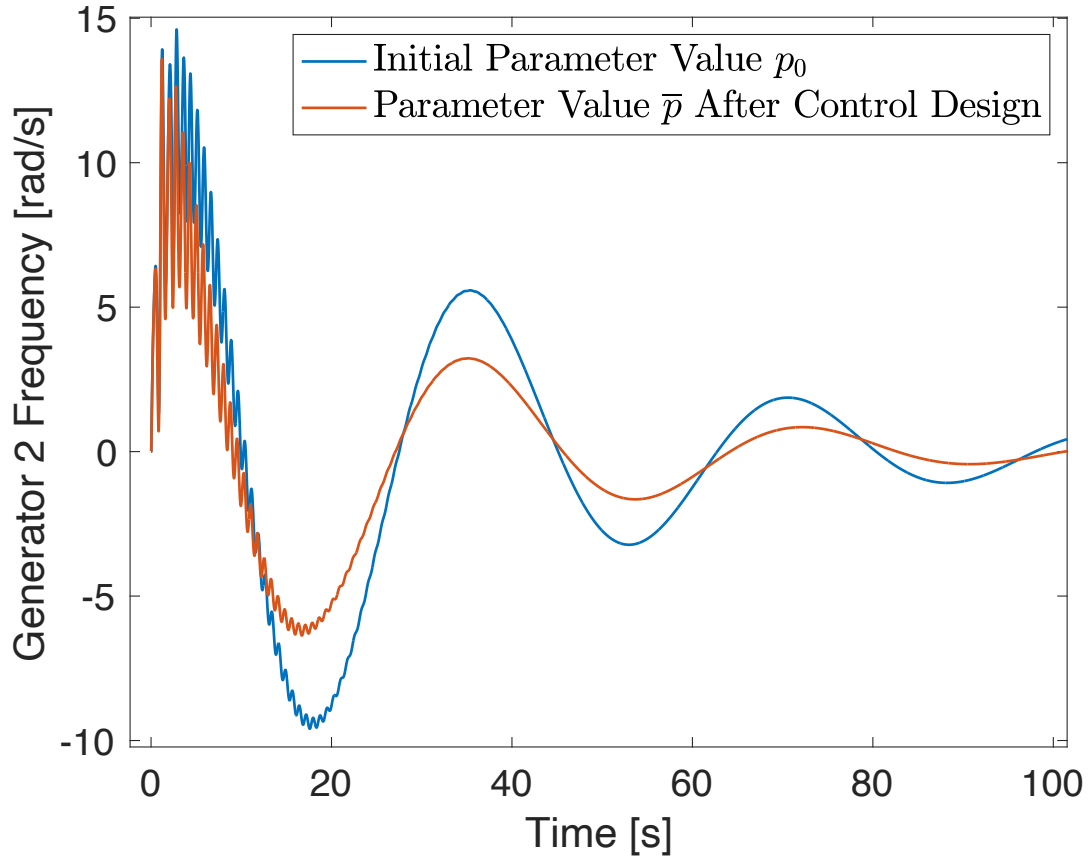


Figure 6.15: Frequency dynamics of generator two for the parameter value p_0 and $\bar{p}$ in the 9-dimensional parameter space, where the parameters of interest are the [AVR](#), [PSS](#), and governor controller gains.

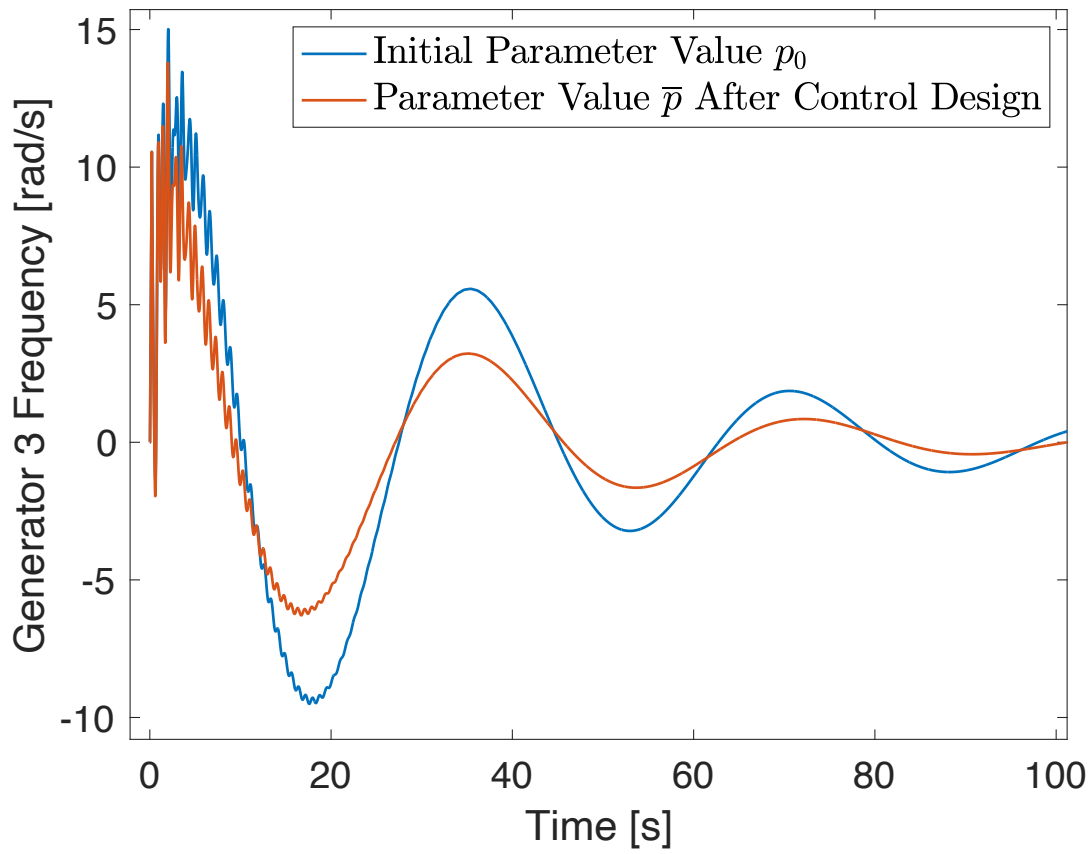


Figure 6.16: Frequency dynamics of generator three for the parameter value p_0 and $\bar{p}$ in the 9-dimensional parameter space, where the parameters of interest are the [AVR](#), [PSS](#), and governor controller gains.

Chapter 7

Conclusion and Future Work

This thesis develops a computationally efficient method with rigorous convergence guarantees for numerically computing the mode of instability for parameterized nonlinear systems while avoiding the challenge of identifying the controlling UEP. For parameter values where the system recovers, it averages the Jacobian along the system trajectory from the IC until the final time at which the Jacobian transitions from unstable to stable. It is shown that as boundary parameter values are approached from within the recovery region, this average of the Jacobians converges to the true Jacobian at the controlling UEP, and it has a unique unstable eigenvalue whose corresponding eigenvector converges to the mode of instability. Under the same conditions, the approximation of the average of the Jacobians obtained from numerical integration, and its eigenvector corresponding to its unique unstable eigenvalue, are shown to converge to the true Jacobian at the controlling UEP and the mode of instability, respectively, as the step size approaches zero.

A novel method to reduce vulnerability of nonlinear systems to large disturbances using the mode of instability was developed. Since the mode of instability is responsible for driving the system out of the RoA after the disturbance, the key idea of the method was to vary controller parameters so as to drive the IC in the opposite direction to the mode of instability, thereby driving it further inwards and away from the RoA boundary. This was accomplished by formulating and solving a nonconvex optimization problem to directly drive the IC in the desired direction. When the initial parameter value was sufficiently close to a local minimum of the optimization problem, and the step size was sufficiently small, it was shown that the solution method would converge to this local minimum.

The method was first validated on the damped, driven, nonlinear pendulum example subject to a disturbance. By directly finding the controlling UEP and analytically comput-

ing the mode of instability for this low dimensional system, it was shown that the mode of instability could be accurately computed using the proposed algorithm in this case, thereby validating the proposed approach. Then the algorithm was applied to compute the mode of instability for the IEEE 9-bus power system subject to a temporary short circuit. For this higher dimensional power system model, it was no longer straightforward to identify the controlling UEP and the mode of instability analytically. The method was used to identify the mode of instability in high dimensional parameter space for this system. The identified modes of instability revealed non-intuitive information about the mechanism of instability, and would have been difficult to predict or identify without the use of the proposed method. The proposed method was successfully validated on power system models subject to temporary short circuits with high dimensional state and parameter spaces, where it was shown to increase the safety margins and, thus, reduce disturbance vulnerability.

Future work will involve extending the method to controlling periodic orbits and other controlling sets beyond equilibrium points, and considering piecewise smooth vector fields with switching and saturation limits.

References

- [1] *IEEE Std. 421.5-2016, IEEE Recommended Practice for Excitation System Models for Power System Stability Studies*. New York: Institute of Electrical and Electronics Engineers, Inc., 2016.
- [2] P M Anderson and A A Fouad. *The Elementary Mathematical Model*. Hoboken, NJ, USA:Wiley-IEEE Press, 2003, pp. 13-52.
- [3] Jan Awrejcewicz, Dmytro Bilichenko, Akram Khalil Cheib, Nataliya Losyeva, and Volodymyr Puzyrov. Estimating the region of attraction based on a polynomial lyapunov function. *Applied Mathematical Modelling*, 90:1143–1152, 2021.
- [4] Anup K Behera, MA Pai, and PW Sauer. Analytical approaches to determine critical clearing time in multi-machine power systems. *24th Conference on Decision and Control (CDC)*, pages 818–823, 1985.
- [5] Claude Berge. *Topological Spaces (Oliver and Boyd, Edinburgh)*. 1963.
- [6] Graziano Chesi. Computing output feedback controllers to enlarge the domain of attraction in polynomial systems. *IEEE Transactions on Automatic Control*, 49(10):1846–1853, 2004.
- [7] Graziano Chesi. Rational lyapunov functions for estimating and controlling the robust domain of attraction. *Automatica*, 49(4):1051–1057, 2013.
- [8] H-D Chiang, Morris W Hirsch, and Felix Fulih Wu. Stability regions of nonlinear autonomous dynamical systems. *IEEE Transactions on Automatic Control*, 33(1):16–27, 1988.
- [9] H-D Chiang, Felix F Wu, and Pravin P Varaiya. Foundations of the potential energy boundary surface method for power system transient stability analysis. *IEEE Transactions on Circuits and Systems*, 35(6):712–728, 1988.

- [10] Hsiao-Dong Chiang, Jianzhong Tong, and Karen Nan Miu. Predicting unstable modes in power systems: Theory and computations. *IEEE Transactions on Power Systems*, 8(4):1429–1437, 1993.
- [11] Michael W Fisher. Computing safety margins of parameterized nonlinear systems for vulnerability assessment via trajectory sensitivities. 2025, under review. Preprint available at <https://arxiv.org/abs/2501.07498>.
- [12] Michael W Fisher and Ian A Hiskens. Numerical computation of critical system recovery parameter values by trajectory sensitivity maximization. *58th Conference on Decision and Control (CDC)*, pages 8000–8006, 2019.
- [13] Michael W Fisher and Ian A Hiskens. Hausdorff continuity of region of attraction boundary under parameter variation with application to disturbance recovery. *SIAM Journal on Applied Dynamical Systems*, 21(1):327–365, 2022.
- [14] Michael W Fisher and Ian A Hiskens. Stability of the nonwandering set in the region of attraction boundary under perturbations with application to vulnerability assessment. *SIAM Journal on Applied Dynamical Systems*, 22(4):3390–3430, 2023.
- [15] AA Fouad, Vijay Vittal, and Tae Kyoo Oh. Critical energy for direct transient stability assessment of a multimachine power system. *IEEE Transactions on Power Apparatus and Systems*, 103(8):2199–2206, 1984.
- [16] Abdel Azia Fouad and Vijay Vittal. *Power System Transient Stability Analysis Using the Transient Energy Function Method*. Englewood Cliffs, NJ, USA: Prentice-Hall, 1992.
- [17] Morris W Hirsch. *Differential Topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, 1976.
- [18] Ian A Hiskens and M Anantha Pai. Trajectory sensitivity analysis of hybrid systems. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 47(2):204–220, 2000.
- [19] A Katok and B Hasselblatt. *Introduction to the Modern Theory of Dynamical Systems*, volume 54 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, 1999.
- [20] Prabha Kundur, John Paserba, Venkat Ajjarapu, Göran Andersson, Anjan Bose, Claudio Canizares, Nikos Hatziargyriou, David Hill, Alex Stankovic, Carson Taylor,

- T Van Cutsem, and V Vittal. Definition and classification of power system stability ieeecigre joint task force on stability terms and definitions. *IEEE Transactions on Power Systems*, 19(3):1387–1401, 2004.
- [21] Jason D Lee, Max Simchowitz, Michael I Jordan, and Benjamin Recht. Gradient descent only converges to minimizers. In *Conference on learning theory*, pages 1246–1257. PMLR, 2016.
 - [22] John M Lee. *Introduction to Smooth Manifolds*. Graduate Texts in Mathematics, Springer, 2 ed., 2013.
 - [23] Rui Ma, Yayao Zhang, Meng Zhan, Kan Cao, Dan Liu, Kezheng Jiang, and Shijie Cheng. Dominant transient equations of grid-following and grid-forming converters by controlling-unstable-equilibrium-point-based participation factor analysis. *IEEE Transactions on Power Systems*, 39(3):4818–4834, 2023.
 - [24] AN Michel and V Vittal. On the mechanism of transient instability of power systems. *Circuits, Systems and Signal Processing*, 4(3):413–434, 1985.
 - [25] James M Ortega. *Numerical Analysis: A Second Course*. Philadelphia, PA, USA: SIAM, 1990.
 - [26] Jacob Palis. On morse-smale dynamical systems. *Topology*, 8(4):385–405, 1969.
 - [27] Lawrence Perko. *Differential equations and dynamical systems*, volume 7. Springer Science & Business Media, 2013.
 - [28] Charles C Pugh and Clark Robinson. The c1 closing lemma, including hamiltonians. *Ergodic Theory and Dynamical Systems*, 3(2):261–313, 1983.
 - [29] Ricardo G Sanfelice. Asymptotic properties of solutions to set dynamical systems. *56th Conference on Decision and Control (CDC)*, pages 2287–2292, 2014.
 - [30] P W Sauer and M A Pai. *Power System Dynamics and Stability*. 1997.
 - [31] Weehong Tan and Andrew Packard. Stability region analysis using polynomial and composite polynomial lyapunov functions and sum-of-squares programming. *IEEE Transactions on Automatic Control*, 53(2):565–571, 2008.
 - [32] Roger Thomas Treinen, Vijay Vittal, and Wolfgang Kliemann. An improved technique to determine the controlling unstable equilibrium point in a power system. *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, 43(4):313–323, 1996.

- [33] A. Vannelli and M. Vidyasagar. Maximal lyapunov functions and domains of attraction for autonomous nonlinear systems. *Automatica*, 21(1):69–80, 1985.
- [34] Jinghan Wang and Michael W Fisher. Computing modes of instability of parameterized nonlinear systems for vulnerability assessment. 2025, under review. Preprint available at <https://arxiv.org/abs/2503.12298>.
- [35] Vladimir Ivanovich Zubov. *Methods of A.M. Lyapunov and their Application*. 1964.