

# **Deterministic and Probabilistic Bijective Combinatorics for Macdonald Polynomials**

by

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## **Author's Declaration**

This thesis consists of material all of which I authored or co-authored: see **Statement of Contributions** included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

## Statement of Contributions

I am the sole author of **Chapters 1, 2 and 3**, which were written under the supervision of Olya Mandelshtam.

The research presented in **Chapter 4** is based on joint work with Olya Mandelshtam, to appear in an upcoming publication.

The research presented in **Chapter 5** is based on joint work with Olya Mandelshtam, in [DM25]: Guilherme Zeus Dantas e Moura and Olya Mandelshtam. *Probabilistic Entry Swapping Bijections for Non-Attacking Fillings*. 2025. arXiv: [2503.06051](#) [math.CO].

## Abstract

Permuted-basement Macdonald polynomials  $E_\alpha^\sigma(\mathbf{x}; q, t)$  are nonsymmetric generalizations of symmetric Macdonald polynomials indexed by a composition  $\alpha$  and a permutation  $\sigma$ . They form a basis for the polynomial ring  $\mathbb{Q}(q, t)[\mathbf{x}]$  for each fixed permutation  $\sigma$ . They can be described combinatorially as generating functions over augmented fillings of composition shape  $\alpha$  with a basement permutation  $\sigma$ .

We construct (deterministic) bijections and probabilistic bijections on fillings that prove identities relating  $E_\alpha^\sigma$ ,  $E_\alpha^{\sigma s_i}$ ,  $E_{s_i \cdot \alpha}^\sigma$ , and  $E_{s_i \cdot \alpha}^{\sigma s_i}$ . These identities correspond to two combinatorial operations on the shape and basement of the fillings: swapping adjacent parts in the shape, which expands  $E_\alpha^\sigma$  in terms of  $E_{s_i \cdot \alpha}^\sigma$  and  $E_{s_i \cdot \alpha}^{\sigma s_i}$ ; and swapping adjacent entries in the basement, which gives  $E_\alpha^\sigma = E_\alpha^{\sigma s_i}$  when  $\alpha_i = \alpha_{i+1}$ .

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# Chapter 1

## Introduction

Macdonald polynomials, in their many versions, form a family of polynomials studied in symmetric function theory that connect representation theory, algebra, combinatorics, statistical mechanics, and other areas of mathematics. A particularly fruitful approach to their study is through combinatorial formulas: many variants of Macdonald polynomials can be expressed as weight generating functions of tableaux, where the weights depend on classical tableaux statistics such as the inversion number and the major index. This perspective suggests that many identities and properties of Macdonald polynomials admit direct combinatorial proofs via correspondences between sets of tableaux.

We focus on the *permuted-basement Macdonald polynomials*  $E_{\alpha}^{\sigma}(\mathbf{x}; q, t)$  introduced by Ferreira [Fer11], a generalization of the nonsymmetric Macdonald polynomials  $E_{\alpha}(\mathbf{x}; q, t)$  first studied by Opdam [Opd95], Macdonald [Mac96], and Cherednik [Che95]. These polynomials depend on a composition  $\alpha$  and a permutation  $\sigma$ , and they specialize to several important families of polynomials in symmetric function theory, including nonsymmetric Macdonald polynomials, ASEP polynomials, key polynomials, Demazure atoms, and Schur polynomials. For a fixed permutation  $\sigma$ , the polynomials  $E_{\alpha}^{\sigma}(\mathbf{x}; q, t)$  form a basis for the polynomial ring  $\mathbb{Q}(q, t)[\mathbf{x}]$ . These polynomials admit a combinatorial formula as a weighted generating function over augmented non-attacking fillings with composition shape  $\alpha$  and basement permutation  $\sigma$ .

We employ two combinatorial tools for proving identities of such weight generating functions. The first, and most standard, is the method of weight-preserving bijections. These maps directly match fillings of one type with fillings of another while exactly preserving the combinatorial weight, proving the equality of the corresponding generating functions. The second tool is a generalization known as probabilistic bijections, where the bijection are

replaced by transition probabilities and the weight-preservation condition is replaced by a suitable balance condition between the weights and the probabilities. The existence of such a probabilistic bijection also implies the equality of the corresponding generating functions.

In the combinatorial study of Macdonald polynomials, certain descriptions involve summations over fillings subject to restrictive conditions. These restrictions reduce the number of terms to consider, which is computationally preferable, but the corresponding weights tend to be complicated expressions. In such cases, constructing a traditional weight-preserving bijection is often impossible, making probabilistic bijections a natural and powerful alternative. Other formulas allow summations over more general (and larger) sets of fillings, but with simpler weights. In these settings, weight-preserving bijections are often possible, and can be elegant and insightful.

We prove two identities for permuted-basement Macdonald polynomials using these two methods. The first is proved via a weight-preserving bijection, and shows that, for a composition  $\alpha$  with  $\alpha_i > \alpha_{i+1}$ ,

$$E_{\alpha}^{\sigma}(\mathbf{x}; q, t) = E_{s_i \cdot \alpha}^{\sigma s_i}(\mathbf{x}; q, t) + c_{\alpha, \sigma, i}(q, t) E_{s_i \cdot \alpha}^{\sigma}(\mathbf{x}; q, t),$$

with an explicit coefficient  $c_{\alpha, \sigma, i}(q, t) \in \mathbb{Q}(q, t)$ . The second identity is proved via a probabilistic bijection, and shows that, for a composition  $\alpha$  with  $\alpha_i = \alpha_{i+1}$ ,

$$E_{\alpha}^{\sigma}(\mathbf{x}; q, t) = E_{\alpha}^{\sigma s_i}(\mathbf{x}; q, t).$$

## Structure of the thesis

The remainder of this thesis is organized into four chapters.

**Chapter 2** introduces Macdonald polynomials from an algebraic perspective. We define several variations of Macdonald polynomials including the modified Macdonald polynomials  $\tilde{H}_{\mu}$ , the integral form symmetric Macdonald polynomials  $J_{\mu}$ , the symmetric Macdonald polynomials  $P_{\mu}$ , and the permuted-basement Macdonald polynomials  $E_{\alpha}^{\sigma}$ . Readers interested primarily in the combinatorial aspects may safely skip this chapter, as alternate combinatorial descriptions of these polynomials are provided in **Chapter 3**.

**Chapter 3** introduces a combinatorial framework for studying Macdonald polynomials. The central objects are fillings, considered in several variations. Each version of the Macdonald polynomials defined in **Chapter 2** is then realized as a weighted generating function over an appropriate class of fillings. The weights are expressed in terms of two key statistics on fillings: the major index (**maj**) and the inversion number (**inv**).

**Chapter 4** presents bijective results for Macdonald polynomials. On the first part of this chapter, following the work of [KLO22], we describe the bijection between fillings  $F(\alpha)$  and  $F(s_i \cdot \alpha)$  that preserves the statistics  $\text{inv}$  and  $\text{maj}$ . This bijection implies that  $\tilde{H}_\alpha(\mathbf{x}; q, t) = \tilde{H}_{s_i \cdot \alpha}(\mathbf{x}; q, t)$ , and by repeated application shows that the modified Macdonald polynomials are invariant under rearrangements of the partition. On the second part of this chapter, we adapt the bijection to the setting of permuted-basement Macdonald polynomials  $E_\alpha^\sigma(\mathbf{x}; q, t)$ , to obtain an identity relating  $E_\alpha^\sigma(\mathbf{x}; q, t)$ ,  $E_{s_i \cdot \alpha}^\sigma(\mathbf{x}; q, t)$ , and  $E_{s_i \cdot \alpha}^{\sigma s_i}(\mathbf{x}; q, t)$ .

**Chapter 5** extends the bijective methods to a probabilistic setting. We introduce the concept of probabilistic bijections as a generalization of weight-preserving bijections, which can also be used to prove identities between generating functions. We explore their applications to Macdonald polynomials by proving that  $E_\alpha^\sigma(\mathbf{x}; q, t) = E_\alpha^{\sigma s_i}(\mathbf{x}; q, t)$  when  $\alpha_i = \alpha_{i+1}$ .



# Chapter 2

## Macdonald polynomials

### 2.1 Permutations and compositions

The *symmetric group*  $\mathcal{S}_n$  is the group of all permutations of the set  $[n] = \{1, 2, \dots, n\}$ . We write permutations in one-line notation  $\sigma = [\sigma_1, \sigma_2, \dots, \sigma_n]$ . Given two permutations  $\sigma, \pi \in \mathcal{S}_n$ , their product is the permutation  $\sigma\pi \in \mathcal{S}_n$  defined by  $(\sigma\pi)_i = \sigma_{\pi_i}$ . The *reverse*  $\text{rev}(\sigma)$  of a permutation  $\sigma \in \mathcal{S}_n$  is the permutation  $\text{rev}(\sigma) = \sigma w_0 = [\sigma_n, \sigma_{n-1}, \dots, \sigma_1]$ , where  $w_0 = [n, n-1, \dots, 1]$ . The symmetric group  $\mathcal{S}_n$  is generated by the *simple transpositions*  $s_1, s_2, \dots, s_{n-1}$ , where  $s_i \in \mathcal{S}_n$  is the permutation that swaps the elements  $i$  and  $i+1$  and fixes all other elements. The *length*  $\ell(\sigma)$  of a permutation  $\sigma \in \mathcal{S}_n$  is the smallest number of simple transpositions whose product equals  $\sigma$ . A *reduced expression* of a permutation  $\sigma$  is a sequence  $s_{i_1}, s_{i_2}, \dots, s_{i_{\ell(\sigma)}}$  of simple transpositions whose product  $s_{i_1} s_{i_2} \cdots s_{i_{\ell(\sigma)}}$  equals  $\sigma$ .

A *composition of length  $n$*  is a sequence  $\alpha = \alpha_1 \alpha_2 \dots \alpha_n$  of nonnegative integers. A *partition* is a weakly decreasing composition, and an *antipartition* is a weakly increasing composition. Given a composition  $\alpha$ , let  $\text{dec}(\alpha)$  and  $\text{inc}(\alpha)$  denote the partition and antipartition obtained by sorting the parts of  $\alpha$  in weakly decreasing and weakly increasing order, respectively. Define  $\text{rev}(\alpha) = \alpha_n \alpha_{n-1} \dots \alpha_1$  to be the composition given by reversing the parts of  $\alpha$ .

The *exponent notation* for a composition  $\alpha = \alpha_1 \dots \alpha_n$  is given by a sequence of nonnegative integers  $p_1, p_2, \dots, p_k$  and positive integers  $m_1, m_2, \dots, m_k$  that sum to  $n$  such that  $p_i \neq p_{i+1}$  for  $i \in [k-1]$ , written as

$$\alpha = \underbrace{p_1 p_1 \dots p_1}_{m_1} \underbrace{p_2 p_2 \dots p_2}_{m_2} \cdots \underbrace{p_k p_k \dots p_k}_{m_k} = p_1^{m_1} p_2^{m_2} \cdots p_k^{m_k}.$$

The *conjugate* of a partition  $\lambda$  is the partition  $\lambda'$  defined by

$$\lambda'_i = |\{j : \lambda_j \geq i\}|.$$

For two partitions  $\mu$  and  $\lambda$  of the same integer, write  $\mu \leq \lambda$  if  $\mu_1 + \mu_2 + \cdots + \mu_k \leq \lambda_1 + \lambda_2 + \cdots + \lambda_k$  for all  $k \geq 1$ . This defines a partial order called the *dominance order*.

Given a permutation  $\pi \in \mathcal{S}_n$  and a composition  $\alpha = \alpha_1 \dots \alpha_n$ , the *left action* of  $\pi$  on  $\alpha$  is the composition  $\pi \cdot \alpha = \alpha_{\pi^{-1}(1)} \alpha_{\pi^{-1}(2)} \dots \alpha_{\pi^{-1}(n)}$ . In particular, this choice of notation guarantees that  $\sigma \cdot (\pi \cdot \alpha) = (\sigma\pi) \cdot \alpha$  for all  $\sigma, \pi \in \mathcal{S}_n$  and  $\alpha = \alpha_1 \dots \alpha_n$ . Note that  $\text{rev}(\alpha) = w_0 \cdot \alpha$ .

**Example 2.1.** For example,  $\alpha = 310220$  is a composition of length 6 with exponent notation  $3^1 1^1 0^1 2^2 0^1$ . It rearranges to the partition  $\text{dec}(\alpha) = 322100$ , the antipartition  $\text{inc}(\alpha) = 001223$ , and  $\text{rev}(\alpha) = 021023$ . The permutation  $\pi = [3, 1, 2, 4, 6, 5]$  has reduced expression  $s_2 s_1 s_5$ . The result of the left action of  $\pi$  on  $\alpha$  is the composition  $\pi \cdot \alpha = 103202$ , with a graphical representation shown in [Figure 2.1](#).

$$\begin{array}{cccccc} \alpha = & 3 & 1 & 0 & 2 & 2 & 0 \\ & \searrow \swarrow & \downarrow & \searrow \swarrow & \downarrow & \searrow \swarrow & \\ \pi \cdot \alpha = & 1 & 0 & 3 & 2 & 0 & 2 \end{array}$$

Figure 2.1: The left action of  $\pi = [3, 1, 2, 4, 6, 5]$  on  $\alpha = 310220$ .

*Remark 2.2.* Given permutations  $\sigma, \pi \in \mathcal{S}_n$  and a composition  $\alpha = \alpha_1 \dots \alpha_n$ , [\[Ale19\]](#) writes  $\pi\alpha$  to mean  $\pi \cdot \alpha$ , and writes  $\pi\sigma$  to mean  $\pi \cdot \sigma$ , interpreting  $\sigma$  as a composition, which we write as a product  $\pi\sigma^{-1}$ .

## 2.2 Symmetric functions and plethystic notation

We assume basic familiarity with the ring  $\Lambda(F)$  of symmetric functions in countably infinitely many variables  $\mathbf{x} = (x_1, x_2, \dots)$  with coefficients in a field  $F$  of characteristic zero. This includes important families of symmetric functions, such as the power sum symmetric functions  $p_k$ , and the Schur functions  $s_\lambda$ . We refer the reader to [\[Sta24, Chapter 7\]](#) for a comprehensive introduction to this topic. We take the coefficient field to be  $F = \mathbb{Q}(q, t)$ , the field of rational functions in the variables  $q$  and  $t$  with rational coefficients.

Recall that the *power sum symmetric functions*  $p_k$  for  $k \geq 1$  are defined by

$$p_k = \sum_{i \geq 1} x_i^k.$$

Any symmetric function can be uniquely expressed as a polynomial in the power sum symmetric functions. In other words,

$$\Lambda(F) = F[p_1, p_2, p_3, \dots].$$

We adopt plethystic notation as in [Hai99, Section 2], defined as follows.

Let  $A$  be a formal series (not necessarily symmetric, and not necessarily with  $x_1, x_2, \dots$  as indeterminates). We define  $p_k[A]$  to be the formal series obtained by replacing each indeterminate  $a$  in  $A$  by  $a^k$  (including the indeterminates  $q$  and  $t$ , if they appear in  $A$ ). For a symmetric function  $f \in \Lambda(F)$ , let

$$f = Q_f(p_1, p_2, \dots, p_m)$$

be the expression of  $f$  as a polynomial  $Q_f$  in the power sum symmetric functions. Then, the *plethystic substitution* of  $A$  into  $f$  is defined as

$$f[A] = Q_f(p_1[A], p_2[A], \dots, p_m[A]).$$

If  $A$  is a sum of indeterminates,  $A = a_1 + a_2 + \dots$ , then  $p_k[A] = a_1^k + a_2^k + \dots = p_k(a_1, a_2, \dots)$ , and thus  $f[A] = f(a_1, a_2, \dots)$ . This explains why we call  $f[A]$  a kind of substitution. Similarly, if  $A$  is a sum of monomials, then  $f[A]$  is  $f$  evaluated on these monomials. An important example is

$$f[1/(1-t)] = f[1 + t + t^2 + t^3 + \dots] = f(1, t, t^2, t^3, \dots).$$

We let  $X = x_1 + x_2 + x_3 + \dots$  denote the sum of all indeterminates in  $\mathbf{x}$ . Then,  $f[X] = f(\mathbf{x})$  is the usual evaluation of  $f$  at the variables  $\mathbf{x}$ , and

$$f[X/(1-t)] = f(x_1, x_2, x_3, \dots, tx_1, tx_2, tx_3, \dots, t^2x_1, t^2x_2, t^2x_3, \dots).$$

Note that some manipulations are easier in plethystic notation; for example, if  $g(\mathbf{x}) = f[X/(1-t)]$ , then  $f(\mathbf{x}) = g[X(1-t)]$ .

## 2.3 Algebraic definitions of Macdonald polynomials

In this section, we define multiple variations of Macdonald polynomials:

- $P_\mu$ , the symmetric Macdonald polynomials,
- $J_\mu$ , the integral form symmetric Macdonald polynomials,
- $\tilde{H}_\mu$ , the modified Macdonald polynomials,
- $E_\alpha^\sigma$ , the permuted-basement (nonsymmetric) Macdonald polynomials,
- $\mathcal{E}_\alpha^\sigma$ , the integral form permuted-basement (nonsymmetric) Macdonald polynomials.

### 2.3.1 Symmetric Macdonald polynomials $P_\mu$

We follow [Mac96, Chapter VI] for the definition of symmetric Macdonald polynomials.

Define the scalar product  $\langle \cdot, \cdot \rangle_{(q,t)}$  on  $\Lambda(\mathbb{Q}(q, t))$  by

$$\langle p_\lambda, p_\mu \rangle_{(q,t)} = \delta_{\lambda\mu} z_\lambda \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}},$$

where  $z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$ , with  $m_i$  being the multiplicity of the part  $i$  in the partition  $\lambda$ . When  $q = t$ , this scalar product reduces to the usual Hall inner product on symmetric functions.

**Theorem 2.3** ([Mac96, Theorem 4.7]). *There exists a unique family of symmetric functions (called **symmetric Macdonald polynomials**)  $P_\lambda = P_\lambda(\mathbf{x}; q, t) \in \Lambda(\mathbb{Q}(q, t))$  indexed by partitions  $\lambda$  such that*

- $P_\lambda = m_\lambda + \sum_{\mu < \lambda} u_{\lambda\mu}(q, t) m_\mu$  with coefficients  $u_{\lambda\mu}(q, t) \in \mathbb{Q}(q, t)$ , and;
- $\langle P_\lambda, P_\mu \rangle_{(q,t)} = 0$  if  $\lambda \neq \mu$ .

For any polynomial  $f(x_1, \dots, x_n)$  and  $1 \leq i \leq n$ , define the  $q$ -shift operator  $T_{q, x_i}$  by

$$(T_{q, x_i} f)(x_1, \dots, x_n) = f(x_1, \dots, qx_i, \dots, x_n).$$

Then, the operator  $D$  is defined as

$$D = \sum_{i=1}^n \left( \prod_{j \neq i} \frac{tx_i - x_j}{x_i - x_j} \right) T_{q, x_i}.$$

The operator  $D$  has distinct eigenfunctions, and these eigenfunctions are the symmetric Macdonald polynomials  $P_\lambda(x_1, \dots, x_n; q, t)$  where  $\lambda$  is a partition with at most  $n$  parts.

### 2.3.2 Integral form symmetric Macdonald polynomials $J_\mu$

We define the integral form symmetric Macdonald polynomials following [Mac95, Chapter VII, Section 8].

For a partition  $\mu$ , define the *normalization factor*  $c_\mu(q, t)$  by

$$c_\mu(q, t) = \prod_{u \in \text{dg}(\mu)} (1 - q^{\text{leg}(u)} t^{\text{arm}(u)+1}), \quad (2.1)$$

where the product is over all boxes  $u$  in the skyline diagram  $\text{dg}(\mu)$  of  $\mu$ , and  $\text{leg}(u)$  and  $\text{arm}(u)$  are the leg and arm lengths of the box  $u$  in  $\text{dg}(\mu)$ , respectively. The notions of skyline diagrams, leg lengths, and arm lengths are defined in Section 3.2.1.\* For now, it suffices to note  $c_\mu(q, t)$  is a polynomial in  $q$  and  $t$  that admits an explicit formula.

The *integral form symmetric Macdonald polynomial*  $J_\mu = J_\mu(\mathbf{x}; q, t)$  is defined by

$$J_\mu(\mathbf{x}; q, t) = c_\mu(q, t) P_\mu(\mathbf{x}; q, t). \quad (2.2)$$

The naming “integral form” is due to the fact that the coefficients of  $J_\mu(\mathbf{x}; q, t)$  are polynomials in  $q$  and  $t$  with integer coefficients. Several proofs of this integrality result appeared in the late 1990s; for example, see [Kno97].

### 2.3.3 Modified Macdonald polynomials $\tilde{H}_\mu$

We define the modified Macdonald polynomials following [Hai99, Section 2]. The *modified Macdonald polynomial*  $\tilde{H}_\mu = \tilde{H}_\mu(\mathbf{x}; q, t)$  is defined by the plethystic substitution

$$\tilde{H}_\mu(\mathbf{x}; q, t) = t^{n(\mu)} J_\mu \left[ \frac{X}{1 - t^{-1}}; q, t^{-1} \right], \quad (2.3)$$

where  $n(\mu) = \sum_{i \geq 1} (i - 1) \mu_i$ .

**Proposition 2.4** ([Hai99, Proposition 2.6]). *The modified Macdonald polynomials  $\tilde{H}_\mu$  satisfy and are uniquely determined by the following properties:  $\tilde{H}_\mu[X(1 - q); q, t]$  is a linear combination of Schur functions  $s_\lambda$  with  $\lambda \geq \mu$  in dominance order;  $\tilde{H}_\mu[X(1 - t); q, t]$  is a linear combination of Schur functions  $s_\lambda$  with  $\lambda \geq \mu'$  in dominance order; and  $\langle \tilde{H}_\mu, s_{(n)} \rangle = 1$ .*

---

\*The exponents of  $q$  and  $t$  in the formula in [Mac95, Chapter VII, Section 8] are  $a(u)$  and  $l(u) + 1$ , where  $a(u)$  and  $l(u)$  denote the arm and leg lengths of the box  $u$  in the Young diagram of  $\mu$ . Our formula measures arm and leg lengths in the skyline diagram, which effectively swaps the roles of arms and legs compared to the Young diagram. Hence the apparent swap of arms and legs is due to the different diagram conventions, and our formula is consistent with that in [Mac95].

### 2.3.4 Permuted-basement Macdonald polynomials $E_\alpha^\sigma$

We follow the exposition of [Fer11] and the notation of [Ale19] for the definition of non-symmetric Macdonald polynomials and permuted-basement Macdonald polynomials, which are polynomials in the ring  $\mathbb{Q}(q, t)[x_1, x_2, \dots, x_n]$ . Now, we have finitely many variables  $\mathbf{x} = (x_1, x_2, \dots, x_n)$ . For more background on this topic, see [GR22], where these polynomials are referred to as the *relative Macdonald polynomials*.

Given a composition  $\alpha = \alpha_1 \dots \alpha_n$ , we write  $\mathbf{x}^\alpha$  to denote the monomial  $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$ . A permutation  $\sigma \in \mathbb{S}_n$  acts on a polynomial  $f(\mathbf{x})$  by permuting the variables. In other words,  $\sigma(\mathbf{x}^\alpha) = \mathbf{x}^{\sigma \cdot \alpha}$  for a composition  $\alpha$ , and this action extends linearly to all polynomials in  $\mathbb{Q}(q, t)[\mathbf{x}]$ . For example, the permutation  $\sigma = [2, 3, 1]$  sends the monomial  $x_1^2 x_2 = \mathbf{x}^{210}$  to the monomial  $[2, 3, 1](x_1^2 x_2) = [2, 3, 1](\mathbf{x}^{210}) = \mathbf{x}^{[2,3,1] \cdot 210} = \mathbf{x}^{021} = x_2^2 x_3$ .

The *Demazure–Lusztig operators*  $T_1, \dots, T_{n-1}$  on  $\mathbb{Q}(q, t)[\mathbf{x}]$  are defined by

$$T_i(f) = ts_i(f) + (1 - t)x_{i+1} \frac{f - s_i(f)}{x_i - x_{i+1}},$$

for  $i \in [n - 1]$ , and the operator  $g$  on  $\mathbb{Q}(q, t)[\mathbf{x}]$  is defined by

$$g(f(x_1, \dots, x_n)) = f(x_2, x_3, \dots, x_n, q^{-1}x_1).$$

For a reduced expression  $\sigma = s_{i_1} \dots s_{i_k}$ , we write  $T_\sigma$  to denote the composition  $T_{i_1} \circ \dots \circ T_{i_k}$ . The *Cherednik–Dunkl operators*  $Y_1, \dots, Y_n$  on  $\mathbb{Q}(q, t)[\mathbf{x}]$  are defined by, for each  $i \in [n]$ ,

$$Y_i = t^{i-1} T_{i-1}^{-1} \dots T_1^{-1} g T_{n-1} \dots T_i.$$

**Definition 2.5** ([Fer11, Theorem 4.1.4]). For a composition  $\alpha = \alpha_1 \dots \alpha_n$ , the *nonsymmetric Macdonald polynomial*  $E_\alpha = E_\alpha(\mathbf{x}; q, t)$  is the unique element of  $\mathbb{Q}(q, t)[\mathbf{x}]$  such that the coefficient of  $\mathbf{x}^\alpha$  is 1, and for all  $i \in [n]$ ,

$$Y_i E_\alpha = q^{-\alpha_i} t^{k_i} E_\alpha,$$

where  $k_i = |\{j \in [i - 1] : \alpha_j > \alpha_i\}| + |\{j \in [i + 1, n] : \alpha_j \geq \alpha_i\}|$ .

*Remark 2.6.* We note that there is a typo in [Fer11]: the corresponding formula for  $k_i$  has a minus sign instead of a plus sign. Our formulation aligns with the definition in [Kno97], although the notation there differs.

**Example 2.7.** Let  $n = 4$  and  $\alpha = 1011$ . Then,  $k_1 = 2$ ,  $k_2 = 3$ ,  $k_3 = 1$  and  $k_4 = 0$ . Therefore, the nonsymmetric Macdonald polynomial  $E_{1011}(\mathbf{x}; q, t)$  is the unique element of  $\mathbb{Q}(q, t)[x_1, x_2, x_3, x_4]$  such that the coefficient of  $x_1 x_3 x_4$  is 1, and

$$\begin{aligned} Y_1 E_{1011} &= q^{-1} t^2 E_{1011}, & Y_2 E_{1011} &= q^0 t^3 E_{1011}, \\ Y_3 E_{1011} &= q^{-1} t^1 E_{1011}, & Y_4 E_{1011} &= q^{-1} t^0 E_{1011}. \end{aligned}$$

The unique polynomial satisfying the above is

$$E_{1011}(x_1, x_2, x_3, x_4; q, t) = x_1 x_2 x_3 \frac{1-t}{1-qt^2} + x_1 x_2 x_4 \frac{1-t}{1-qt^2} + x_1 x_3 x_4.$$

The following definition is a reformulation based on [Ale19, Corollary 16] and [Fer11, Remark 4.4.5].

**Definition 2.8** (Permuted-basement Macdonald polynomials). For a composition  $\alpha = \alpha_1 \dots \alpha_n$  and a permutation  $\sigma \in \mathfrak{S}_n$ , the *permuted-basement Macdonald polynomial*  $E_\alpha^\sigma = E_\alpha^\sigma(\mathbf{x}; q, t)$  is given by

$$E_\alpha^\sigma = t^{-\text{twinv}(\alpha, \sigma)} T_{\text{rev}(\sigma)} E_{\text{rev}(\alpha)},$$

where  $\text{twinv}(\alpha, \sigma) = |\{(i, j) : i < j \text{ and } \alpha_i \geq \alpha_j \text{ and } \sigma_i < \sigma_j\}|$ .

Note that, for  $\sigma = w_0 = [n, n-1, \dots, 1]$ , the permuted-basement Macdonald polynomial  $E_\alpha^{w_0}(\mathbf{x}; q, t)$  is the nonsymmetric Macdonald polynomial  $E_{\text{rev}(\alpha)}(\mathbf{x}; q, t)$ .

*Remark 2.9.* Our notation for the permuted-basement Macdonald polynomial agrees with the notation in [Ale19; CMW22], which is different from the notation in [Fer11]. Notably, the permuted-basement Macdonald polynomial  $E_\alpha^\sigma(\mathbf{x}; q, t)$  is denoted by  $E_{\text{rev}(\alpha)}^{\text{rev}(\sigma)}(\mathbf{x}; q, t)$  in [Fer11].

*Remark 2.10.* We note a typo in [Ale19] concerning the definition of  $\text{twinv}(\alpha, \sigma)$ : the inequality involving  $\sigma$  is reversed. This error becomes apparent by considering the special case  $E_\alpha^{w_0}$ . Our formulation aligns with the definition provided in [Fer11], although we use different notation.

**Example 2.11.** Let  $n = 4$ ,  $\alpha = 1101$ , and  $\sigma = [2, 4, 1, 3]$ . Then,  $\text{rev}(\sigma) = [3, 1, 4, 2]$  has reduced expression  $s_2 s_1 s_3$ , so the Demazure–Lusztig operator  $T_{\text{rev}(\sigma)}$  is given by  $T_2 T_1 T_3$ . We compute  $\text{twinv}(\alpha, \sigma) = \text{twinv}(1101, [2, 4, 1, 3]) = |\{(1, 2), (1, 4)\}| = 2$ . Therefore, the permuted-basement Macdonald polynomial  $E_{1101}^{[2, 4, 1, 3]}(\mathbf{x}; q, t)$  is given by

$$E_{1101}^{[2, 4, 1, 3]}(\mathbf{x}; q, t) = t^{-2} T_2 T_1 T_3 E_{1011}(\mathbf{x}; q, t),$$

which can be computed from [Example 2.7](#) as

$$E_{1101}^{[2,4,1,3]} = x_1 x_2 x_3 \frac{t(1-t)}{1-qt^2} + x_1 x_3 x_4 \frac{1-t}{1-qt^2} + x_2 x_3 x_4.$$

**Example 2.12.** Let  $n = 4$ ,  $\alpha = 1101$ , and  $\tau = [4, 2, 1, 3]$ . Then,  $\text{rev}(\tau) = [3, 1, 4, 2]$  has a reduced expression  $\text{rev}(\tau) = s_2 s_1$ , so the Demazure–Lusztig operator  $T_{\text{rev}(\tau)}$  is given by  $T_2 T_1$ . We compute  $\text{twinv}(\alpha, \tau) = \text{twinv}(1101, [4, 2, 1, 3]) = |\{(1, 4)\}| = 1$ . Therefore, the permuted-basement Macdonald polynomial  $E_{1101}^{[4,2,1,3]}(\mathbf{x}; q, t)$  is given by

$$E_{1101}^{[4,2,1,3]}(\mathbf{x}; q, t) = t^{-1} T_2 T_1 E_{1011}(\mathbf{x}; q, t),$$

which can be computed as

$$E_{1101}^{[4,2,1,3]} = x_1 x_2 x_3 \frac{t(1-t)}{1-qt^2} + x_1 x_3 x_4 \frac{1-t}{1-qt^2} + x_2 x_3 x_4.$$

Note that the computations in [Example 2.11](#) and [Example 2.12](#) yield the same result, where the permutations  $\sigma = [2, 4, 1, 3]$  and  $\tau = [4, 2, 1, 3]$  differ by an application of the simple transposition  $s_1$ . This equality exemplifies a property that will be explored in [Chapter 5](#).

**Theorem 2.13** ([[Fer11](#)]). *Let  $\sigma$  be a permutation in  $S_n$ . The set of permuted-basement Macdonald polynomials*

$$\{E_{\alpha}^{\sigma}(x_1, \dots, x_n; q, t) : \alpha \text{ is a composition of length } n\}$$

*is a basis of  $\mathbb{Q}(q, t)[x_1, \dots, x_n]$ .*

### 2.3.5 Integral form permuted-basement Macdonald polynomials $\mathcal{E}_{\alpha}^{\sigma}$

We define the integral form permuted-basement Macdonald polynomials following [[Kno97](#), Section 5]. While [[Kno97](#)] only defined the integral form nonsymmetric Macdonald polynomials (i.e., the case  $\sigma = w_0$ ), the extension to the permuted-basement case is straightforward by using the same normalization factor.

For a composition  $\alpha = \alpha_1 \dots \alpha_n$ , define the *normalization factor*  $d_{\alpha}(q, t)$  by

$$d_{\alpha}(q, t) = \prod_{u \in \text{dg}(\alpha)} (1 - q^{\text{leg}(u)+1} t^{\text{arm}(u)+1}), \quad (2.4)$$

where the product is over all boxes  $u$  in the skyline diagram  $\text{dg}(\alpha)$  of  $\alpha$ , and  $\text{leg}(u)$  and  $\text{arm}(u)$  are the leg and arm lengths of the box  $u$  in  $\text{dg}(\alpha)$ , respectively. The notions of skyline diagrams, leg lengths, and arm lengths are defined in [Section 3.2.1](#).<sup>\*</sup> The *integral form permuted-basement Macdonald polynomial*  $\mathcal{E}_\alpha^\sigma = \mathcal{E}_\alpha^\sigma(\mathbf{x}; q, t)$  is defined by

$$\mathcal{E}_\alpha^\sigma(\mathbf{x}; q, t) = d_\alpha(q, t) E_\alpha^\sigma(\mathbf{x}; q, t). \quad (2.5)$$

Just like the integral form symmetric Macdonald polynomials, the “integral form” in the name refers to the fact that the coefficients of  $\mathcal{E}_\alpha^\sigma(\mathbf{x}; q, t)$  are polynomials in  $q$  and  $t$  with integer coefficients.

**Example 2.14.** Let  $n = 4$ ,  $\alpha = 1011$ , and  $\sigma = [2, 4, 1, 3]$ , as in [Example 2.11](#). We compute the normalization factor  $d_{1011}(q, t)$  as

$$d_{1011}(q, t) = (1 - qt^3)(1 - qt^2)(1 - qt).$$

Multiplying the permuted-basement Macdonald polynomial  $E_{1011}^{[2,4,1,3]}$  computed in [Example 2.11](#) by this normalization factor, we obtain the integral form permuted-basement Macdonald polynomial

$$\begin{aligned} \mathcal{E}_{1011}^{[2,4,1,3]} &= x_1 x_2 x_3 t(1 - t)(1 - qt^3)(1 - qt) + \\ &\quad + x_1 x_3 x_4(1 - t)(1 - qt^3)(1 - qt) + x_2 x_3 x_4(1 - qt^3)(1 - qt^2)(1 - qt), \end{aligned}$$

which has coefficients in  $\mathbb{Z}[q, t]$ .

---

<sup>\*</sup>The exponents of  $q$  and  $t$  in the formula in [\[Kno97, Section 5\]](#) are  $a(u) + 1$  and  $l(u) + 1$ . We call  $\text{leg}(u)$  the quantity that [\[Kno97\]](#) denotes by  $a(u)$ , and  $\text{arm}(u)$  the quantity that [\[Kno97\]](#) denotes by  $l(u)$ . Hence our formula is consistent with that in [\[Kno97\]](#).



# Chapter 3

## Combinatorial formulas for Macdonald polynomials

### 3.1 Unsigned and signed alphabets

An *unsigned alphabet*  $\mathcal{A}$  is a totally ordered set. We write  $\chi_{ab}$  to denote the indicator function of  $a > b$ . Some unsigned alphabets we use are  $\mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$  and its subalphabet  $[n] = \{1, 2, \dots, n\}$ .

A *signed alphabet*  $\mathcal{A}_{\pm}$  is a total order on the disjoint union of two copies of a set  $\mathcal{A}$ , a *positive* copy  $\mathcal{A}_+$  and a *negative* copy  $\mathcal{A}_-$ . We write  $\chi_{ab}$  to denote the indicator function of  $a > b$  or  $a = b \in \mathcal{A}_-$ . Given an element  $a \in \mathcal{A}$ , we denote its positive copy by  $a \in \mathcal{A}_+$  and its negative copy by  $\bar{a} \in \mathcal{A}_-$ , and we say that  $a \in \mathcal{A}$  is the *absolute value* of  $a \in \mathcal{A}_+$  and of  $\bar{a} \in \mathcal{A}_-$ , denoted by  $|a| = |\bar{a}| = a$ . Some signed alphabets we use are  $\mathbb{Z}_{\pm}$ , with the total order

$$1 < \bar{1} < 2 < \bar{2} < 3 < \bar{3} < \dots,$$

and its subalphabet  $[n]_{\pm}$ .

For example, in the signed alphabet  $\mathbb{Z}_{\pm}$ , we have

$$\chi_{53} = \chi_{\bar{5}\bar{3}} = \chi_{5\bar{3}} = \chi_{\bar{5}3} = 1, \quad \chi_{22} = \chi_{2\bar{2}} = 0, \quad \text{and} \quad \chi_{\bar{2}2} = \chi_{2\bar{2}} = 1.$$

We also use the symbol  $\infty$  to denote an element larger than any element in  $\mathcal{A}$ , so that  $\chi_{a\infty} = 0$  and  $\chi_{\infty a} = 1$  for all  $a \in \mathcal{A}$ . Note that  $\infty$  is not an element of  $\mathcal{A}$ .

For both unsigned and signed alphabets, define

$$\chi_{abc} = \chi_{ab} + \chi_{bc} - \chi_{ac}.$$

We have  $\chi_{abc} \in \{0, 1\}$ , since it is impossible for  $\chi_{ab} = \chi_{bc} = 1 - \chi_{ac}$  by transitivity.

Here's one way to think about  $\chi_{abc}$ . For an unsigned alphabet,  $\chi_{abc}$  indicates whether  $a, b, c$  is cyclically weakly decreasing, where the earlier of two equal entries is considered smaller than the later one. That is, in an unsigned alphabet,  $\chi_{abc} = 1$  if and only if

$$a > b > c \quad \text{or} \quad b > c \geq a \quad \text{or} \quad c \geq a > b.$$

Figure 3.1 provides examples of the indicator function  $\chi_{abc}$ .

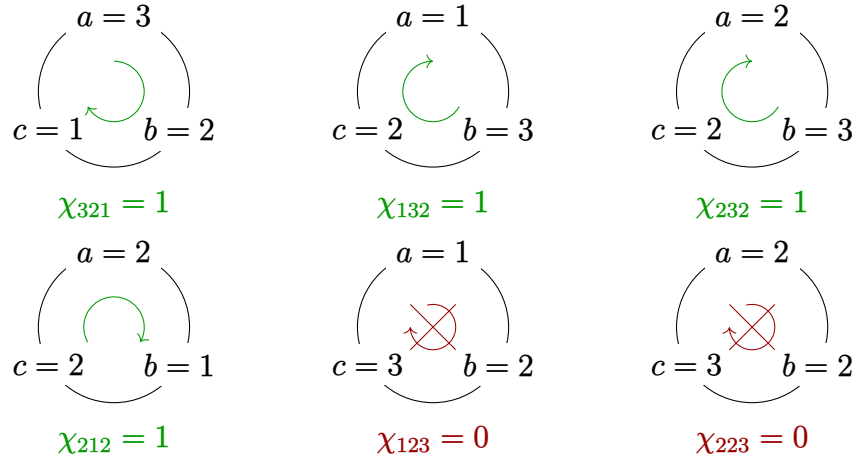


Figure 3.1: Examples of the indicator function  $\chi_{abc}$  for an unsigned alphabet.

In signed alphabets,  $\chi_{abc}$  indicates whether  $a, b, c$  is cyclically decreasing, where the earlier of two equal positive entries is considered smaller than the later one, and the earlier of two equal negative entries is considered larger than the later one.

## 3.2 Fillings

### 3.2.1 Non-augmented fillings

The *skyline diagram* of a composition  $\alpha = \alpha_1 \dots \alpha_n$  is the set

$$\text{dg}(\alpha) = \{(i, r) : i \in [n] \text{ and } r \in [\alpha_i]\},$$

where the first coordinate indexes the columns and the second coordinate indexes the rows. In [Section 3.2.2](#), we define the *augmented skyline diagram* of a composition. The optional term *non-augmented* is used when the distinction is necessary. The elements of  $\mathbf{dg}(\alpha)$  are called *boxes*. We draw the box  $(i, r)$  as a square in column  $i$  and row  $r$ . [Figure 3.2](#) provides an example of a skyline diagram.

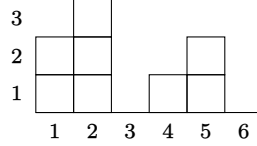


Figure 3.2: The skyline diagram of the composition  $\alpha = 230120$ .

A box  $u = (i_u, r_u)$  is *after*  $v = (i_v, r_v)$  in the *reading order* of  $\mathbf{dg}(\alpha)$  if  $r_u > r_v$ , or  $r_u = r_v$  and  $i_u < i_v$ . This defines a total order on the boxes of  $\mathbf{dg}(\alpha)$ .

A box  $u = (i_u, r_u)$  *attacks*  $v = (i_v, r_v)$  in  $\mathbf{dg}(\alpha)$  if  $r_u = r_v$  and  $i_u < i_v$ , that is,  $u$  and  $v$  are in the same row and  $u$  is to the left of  $v$ ; or  $r_v = r_u - 1$  and  $i_v < i_u$ , that is,  $u$  is in the row directly above  $v$  and  $u$  is to the right of  $v$ . [Figure 3.3](#) illustrates the two configurations in which a box  $u$  attacks a box  $v$ . Let  $\mathbf{Attack}_{\mathbf{dg}(\alpha)}(u) \subset \mathbf{dg}(\alpha)$  be the set of boxes attacked by  $u$ . When there is no ambiguity, we may simply write  $\mathbf{Attack}(u)$ .

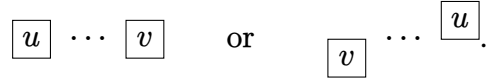


Figure 3.3: Configurations in which a box  $u$  attacks a box  $v$ : either in the same row (left), or in consecutive rows (right).

Let  $u = (i, r) \in \mathbf{dg}(\alpha)$ . The box *south* of  $u = (i, r)$  is the box  $d(u) = (i, r - 1)$ , that is, the box directly below  $u$ . If  $r > 1$ , then  $d(u) \in \mathbf{dg}(\alpha)$ ; while if  $r = 1$ , then  $d(u)$  is still defined (in row 0), but it is not in  $\mathbf{dg}(\alpha)$ . The *leg set* of  $u$  is the set  $\mathbf{Leg}_{\mathbf{dg}(\alpha)}(u) \subset \mathbf{dg}(\alpha)$  of boxes above  $u$  in the same column, that is,  $\mathbf{Leg}_{\mathbf{dg}(\alpha)}(u) = \{(i, r') \in \mathbf{dg}(\alpha) : r' > r\}$ . The *leg statistic* of  $u$  is defined as  $\mathbf{leg}_{\mathbf{dg}(\alpha)}(u) = |\mathbf{Leg}_{\mathbf{dg}(\alpha)}(u)| = \alpha_i - r$ . The *arm set* of  $u$  is the set  $\mathbf{Arm}_{\mathbf{dg}(\alpha)}(u) \subset \mathbf{dg}(\alpha)$  of boxes  $v$  attacked by  $u$  such that  $\mathbf{leg}_{\mathbf{dg}(\alpha)}(v) \leq \mathbf{leg}_{\mathbf{dg}(\alpha)}(u)$ , that is,

$$\mathbf{Arm}_{\mathbf{dg}(\alpha)}(u) = \left\{ v \in \mathbf{Attack}_{\mathbf{dg}(\alpha)}(u) : \mathbf{leg}_{\mathbf{dg}(\alpha)}(v) \leq \mathbf{leg}_{\mathbf{dg}(\alpha)}(u) \right\}.$$

The *arm statistic* of  $u$  is defined as  $\mathbf{arm}_{\mathbf{dg}(\alpha)}(u) = |\mathbf{Arm}_{\mathbf{dg}(\alpha)}(u)|$ . When there is no ambiguity, we may simply write  $\mathbf{Leg}(u)$ ,  $\mathbf{leg}(u)$ ,  $\mathbf{Arm}(u)$ , and  $\mathbf{arm}(u)$ . The arm set  $\mathbf{Arm}(u)$  of a box

$u = (i, r)$  can be equivalently described as

$$\{(j, r-1) \in \mathbf{dg}(\alpha) : j < i \text{ and } \alpha_j \in [r-1, \alpha_i - 1]\} \cup \{(j, r) \in \mathbf{dg}(\alpha) : j > i \text{ and } \alpha_j \in [r, \alpha_i]\}$$

and consequently, the arm statistic  $\mathbf{arm}(u)$  can be computed as

$$\mathbf{arm}(u) = |\{j < i : \alpha_j \in [r-1, \alpha_i - 1]\}| + |\{j > i : \alpha_j \in [r, \alpha_i]\}|.$$

See Figure 3.4 for an example of the south, leg set, arm set, and attacked set of a box  $u$ .

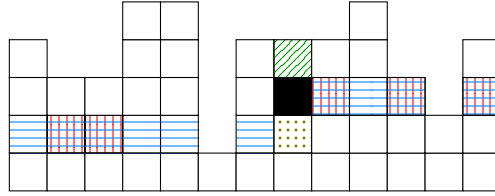


Figure 3.4: The leg set () , the arm set () , the attacked set () , and the south () of the box  $u = (8, 3)$  () in the skyline diagram of  $\alpha = 4335514445324$  with  $\mathbf{leg}(u) = 1$  and  $\mathbf{arm}(u) = 5$ .

A *(non-augmented) filling* of shape  $\alpha = \alpha_1 \dots \alpha_n$  with labels in an alphabet  $\mathcal{A}$  is a function  $T : \mathbf{dg}(\alpha) \rightarrow \mathcal{A}$ . Let  $\mathbf{F}_{\mathcal{A}}(\alpha)$  denote the set of (non-augmented) fillings of shape  $\alpha$  with labels in  $\mathcal{A}$ .

For a signed alphabet  $\mathcal{A}_{\pm}$  and a filling  $T \in \mathbf{F}_{\mathcal{A}_{\pm}}(\alpha)$ , the *absolute value of  $T$*  is the filling  $|T| \in \mathbf{F}_{\mathcal{A}}(\alpha)$  defined by  $|T|(u) = |T(u)|$  for all  $u \in \mathbf{dg}(\alpha)$ .

For an unsigned alphabet  $\mathcal{A}$ , a filling  $T \in \mathbf{F}_{\mathcal{A}}(\alpha)$  is *non-attacking* if  $T(u) \neq T(v)$  whenever  $u$  attacks  $v$  in  $\mathbf{dg}(\alpha)$ . For a signed alphabet  $\mathcal{A}_{\pm}$ , a filling  $T \in \mathbf{F}_{\mathcal{A}_{\pm}}(\alpha)$  is *non-attacking* if  $|T|$  is a non-attacking filling in  $\mathbf{F}_{\mathcal{A}}(\alpha)$ . Let  $\mathbf{NAF}_{\mathcal{A}}(\alpha) \subseteq \mathbf{F}_{\mathcal{A}}(\alpha)$  denote the set of non-attacking fillings of shape  $\alpha$  with labels in  $\mathcal{A}$ . See Figure 3.5 for an example of a non-attacking filling.

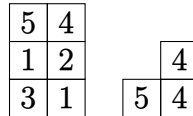


Figure 3.5: A non-attacking filling of shape  $\alpha = 33012$  with labels in  $[5]$ .

The *content* of a filling  $T$  with labels in an alphabet  $\mathcal{A}$  is the multiset  $\mathbf{content}(T)$  whose elements are the labels  $a \in \mathcal{A}$ , with multiplicity equal to the number of boxes  $u \in \mathbf{dg}(\alpha)$  such that  $T(u) = a$ .

When the alphabet is  $\mathbb{Z}_{>0}$  or  $[n]$ , the content is identified with the composition  $\beta = \text{content}(T)$  where  $\beta_i$  is the multiplicity of  $i$  in  $\text{content}(T)$ . We define

$$\mathbf{x}^T = \mathbf{x}^{\text{content}(T)} = \prod_i x_i^{\beta_i}.$$

For example, the filling  $T$  in [Figure 3.5](#) has content  $\text{content}(T) = 21132$  and  $\mathbf{x}^T = \mathbf{x}^{\text{content}(T)} = x_1^2 x_2^1 x_3^1 x_4^3 x_5^2$ .

Recall that boxes in row 0, which are not in  $\text{dg}(\alpha)$ , are obtained as the south  $d(u)$  of boxes  $u$  in row 1. By convention, if  $w$  is a box in row 0, then we define  $T(w) = \infty$ .

Given a filling  $T \in \mathbf{F}(\alpha)$ , we say that a box  $u \in \text{dg}(\alpha)$  is a *descent* in  $T$  if  $\chi_{T(u)T(d(u))} = 1$ . Note that boxes in row 1 are never descents, since the label of the south of a box in row 1 is always  $\infty$ , by convention. Let  $\text{Des}(T) \subseteq \text{dg}(\alpha)$  denote the set of descents in  $T$ . The *major index* of  $T$  is

$$\text{maj}(T) = \sum_{u \in \text{Des}(T)} (\text{leg}(u) + 1) = \sum_{u \in \text{dg}(\alpha)} \chi_{T(u)T(d(u))} (\text{leg}(u) + 1).$$

See [Figure 3.6](#) for an example.

|   |   |   |   |
|---|---|---|---|
| 2 | 4 |   |   |
| 1 | 2 |   | 4 |
| 3 | 1 | 2 | 4 |

Figure 3.6: The filling  $T$  above has three descents ( ). The leg statistics of the descents are  $\text{leg}(1, 3) = \text{leg}(2, 3) = 0$  and  $\text{leg}(2, 2) = 1$ . The major index of  $T$  is  $\text{maj}(T) = 1 + 1 + 2 = 4$ .

A *triple* is an ordered tuple  $(u, v, w)$  of boxes such that  $u \in \text{dg}(\alpha)$ ,  $v \in \text{Arm}(u)$ , and  $w = d(u)$ . The total number of triples in  $\text{dg}(\alpha)$  is

$$\sum_{u \in \text{dg}(\alpha)} \text{arm}(u). \tag{3.1}$$

If  $v$  is to the right of  $u$  in the same row as  $u$ , then the triple is of *type A*, and if  $v$  is to the left of  $u$  in the row directly below  $u$ , then the triple is of *type B*.<sup>\*</sup> The configurations of the boxes  $u$ ,  $v$ , and  $w$  when the tuple  $(u, v, w)$  is of type A or type B are shown in [Figure 3.7](#). Note that if  $\alpha$  is a partition, there are no triples of type B.

---

<sup>\*</sup>This is not related to Cartan types  $A_n$  and  $B_n$ . In [\[HHL08\]](#) and [\[Fer11\]](#), they are called *type II* and *type I*, respectively.



(TYPE A) The column containing  $u$  and  $w$  is at least as high as the column containing  $v$ .

(TYPE B) The column containing  $u$  and  $w$  is strictly higher than the column containing  $v$ .

Figure 3.7: Configurations for triples of type A (left) and type B (right).

Note that the definition of a triple  $(u, v, w)$  allows for  $u$  to be in row 1, and consequently for  $w$  to be in row 0. A triple that has  $u$  in row 1 is called a *degenerate triple*. A degenerate triple is always of type A, since  $v \in \text{Arm}(u) \subset \text{dg}(\alpha)$ , and therefore cannot be in row 0.

Given a filling  $T \in \mathbf{F}(\alpha)$ , a triple  $(u, v, w)$  in  $\text{dg}(\alpha)$  is an *inversion triple* if  $\chi_{T(u)T(v)T(w)} = 1$ , and is a *coinversion triple* otherwise. Let  $\text{inv}(T)$  and  $\text{coinv}(T)$  denote the number of inversion triples and coinversion triples in  $T$ , respectively. We have

$$\text{inv}(T) + \text{coinv}(T) = \sum_{u \in \text{dg}(\alpha)} \text{arm}(u), \quad (3.2)$$

the total number of triples in  $\text{dg}(\alpha)$ . See Figure 3.8 for an example.

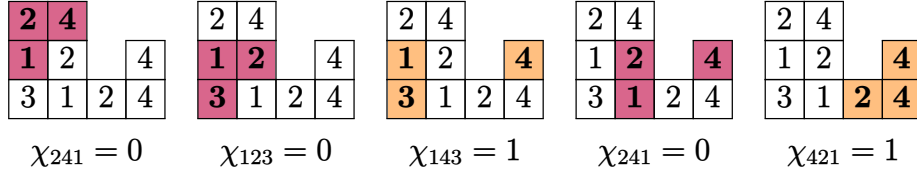


Figure 3.8: We show all five triples of the filling  $T$  in Figure 3.5. The first four are of type A, and the last one is of type B. There are two inversion triples (orange) and three coinversion triples (pink), which gives  $\text{inv}(T) = 2$  and  $\text{coinv}(T) = 3$ .

There is another interpretation of  $\text{inv}(T)$  and  $\text{coinv}(T)$ , in terms of inversion pairs. An *inversion pair* in a filling  $T$  is a pair  $(u, v)$  of boxes in  $\text{dg}(\alpha)$  such that  $u$  attacks  $v$  and  $\chi_{T(u)T(v)} = 1$ . Let  $\text{IP}(T)$  denote the set of inversion pairs in  $T$ .

**Proposition 3.1** ([HHL05, p. 739]). *Let  $T \in \mathbf{F}(\alpha)$ . Then*

$$\text{inv}(T) = |\text{IP}(T)| - \sum_{u \in \text{Des}(T)} \text{arm}(u).$$

*Proof.* Recall that  $(u, v, w)$  forms a triple if  $v \in \text{Arm}(u)$  and  $w = d(u)$ , and such triple is an inversion triple if and only if  $\chi_{T(u)T(v)} + \chi_{T(v)T(w)} - \chi_{T(u)T(w)} = 1$ . Finally, note that  $u$

attacking  $v$  is equivalent to  $v$  attacking  $d(u)$ . Thus, the total number of inversion triples in  $T$  is

$$\begin{aligned}
& \sum_{u \in \mathbf{dg}(\alpha)} \sum_{v \in \mathbf{Arm}(u)} \left( \chi_{T(u)T(v)} + \chi_{T(v)T(d(u))} - \chi_{T(u)T(d(u))} \right) = \\
&= \sum_{\substack{u, v \in \mathbf{dg}(\alpha) \\ v \in \mathbf{Arm}(u)}} \chi_{T(u)T(v)} + \sum_{\substack{u, v \in \mathbf{dg}(\alpha) \\ v \in \mathbf{Arm}(u)}} \chi_{T(v)T(d(u))} - \sum_{\substack{u, v \in \mathbf{dg}(\alpha) \\ v \in \mathbf{Arm}(u)}} \chi_{T(u)T(d(u))} \\
&= \sum_{\substack{u, v \in \mathbf{dg}(\alpha) \\ u \text{ attacks } v \\ \mathbf{leg}(v) \leq \mathbf{leg}(u)}} \chi_{T(u)T(v)} + \sum_{\substack{v, w \in \mathbf{dg}(\alpha) \\ v \text{ attacks } w \\ \mathbf{leg}(w) > \mathbf{leg}(v)}} \chi_{T(v)T(w)} - \sum_{u \in \mathbf{dg}(\alpha)} \chi_{T(u)T(d(u))} \mathbf{arm}(u) \\
&= |\mathbf{IP}(T)| - \sum_{u \in \mathbf{Des}(T)} \mathbf{arm}(u),
\end{aligned}$$

as desired. There are two subtleties in the computation. First, if  $u$  is in row 1, then  $d(u)$  is in row 0 and thus not in  $\mathbf{dg}(\alpha)$ ; nevertheless, since  $T(d(u)) = \infty$  by convention, we have  $\chi_{T(v)T(d(u))} = 0$  and  $\chi_{T(u)T(d(u))} = 0$ , so the formula still holds. Second, not all  $w \in \mathbf{dg}(w)$  are attained as the south of some  $u \in \mathbf{dg}(\alpha)$ , so changing the variable from  $u$  to  $w$  in the second sum would incorrectly account for such boxes; nevertheless, those boxes  $w \in \mathbf{dg}(\alpha)$  that are not attained as the south of some  $u \in \mathbf{dg}(\alpha)$  have  $\mathbf{leg}(w) = 0$ , and therefore are not included in the sum since  $\mathbf{leg}(w) > \mathbf{leg}(v) \geq 0$  cannot hold.  $\square$

**Proposition 3.1** gives an alternative description of  $\mathbf{inv}(T)$ . Together with (3.2), we also obtain an alternative description of  $\mathbf{coinv}(T)$  by

$$\mathbf{coinv}(T) = \sum_{u \in \mathbf{dg}(\alpha)} \mathbf{arm}(u) - \mathbf{inv}(T) = \sum_{u \in \mathbf{dg}(\alpha) \setminus \mathbf{Des}(T)} \mathbf{arm}(u) - |\mathbf{IP}(T)|.$$

### 3.2.2 Augmented fillings

In the context of augmented fillings, we either take the unsigned alphabet  $\mathcal{A} = [n]$ , or the signed alphabet  $\mathcal{A}_{\pm} = [n]_{\pm}$ .

The *augmented skyline diagram* of a composition  $\alpha = \alpha_1 \dots \alpha_n$  is the set

$$\widehat{\mathbf{dg}}(\alpha) = \mathbf{dg}(\alpha) \cup \{(i, 0) : i \in [n]\},$$

which is obtained from  $\mathbf{dg}(\alpha)$  by adding an extra row of  $n$  boxes below row 1, called the *row 0* or the *basement*. Figure 3.9 provides an example of an augmented skyline

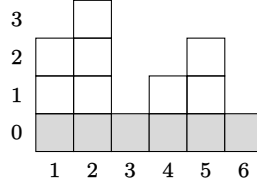


Figure 3.9: The augmented skyline diagram of the composition  $\alpha = 230120$ . The basement (row 0) is shaded in gray ( $\blacksquare$ ).

diagram. Augmented skyline diagrams are always drawn with the basement shaded in gray to distinguish them from non-augmented skyline diagrams.

The attacking relation, leg set, arm set, leg statistic, and arm statistic in  $\widehat{\mathbf{dg}}(\alpha)$  are defined as in Section 3.2.1, with boxes in row 0 treated like any other boxes. However, since row 0 is now part of the diagram, there are subtle differences: for  $u = (i, r) \in \widehat{\mathbf{dg}}(\alpha)$ , if  $r \geq 2$ , then  $\text{Attack}_{\widehat{\mathbf{dg}}(\alpha)}(u) = \text{Attack}_{\mathbf{dg}(\alpha)}(u)$  and  $\text{arm}_{\widehat{\mathbf{dg}}(\alpha)}(u) = \text{arm}_{\mathbf{dg}(\alpha)}(u)$ ; but if  $r = 1$ ,  $\text{Attack}_{\widehat{\mathbf{dg}}(\alpha)}(u)$  and  $\text{Arm}_{\widehat{\mathbf{dg}}(\alpha)}(u)$  may include boxes in row 0 not present in the non-augmented case. Thus, extra attention is needed when computing these sets and statistics for boxes in row 1, as they may differ depending on whether  $\mathbf{dg}(\alpha)$  or  $\widehat{\mathbf{dg}}(\alpha)$  is used. If we need to distinguish between the two settings, we use the subscripts  $\mathbf{dg}(\alpha)$  or  $\widehat{\mathbf{dg}}(\alpha)$  to indicate which diagram is being used.

The definition of the south of a box is slightly different in the augmented setting. Given a box  $u = (i, r)$ , if  $r \geq 1$ , the *south* of  $u$  is the box  $d(u) = (i, r - 1)$ , while if  $r = 0$ , then  $d(u)$  is not defined. Note that the south of a box in row 1 of  $\widehat{\mathbf{dg}}(\alpha)$  is also a box in  $\widehat{\mathbf{dg}}(\alpha)$ , unlike in Section 3.2.1 where the south of a box in row 1 is not in  $\mathbf{dg}(\alpha)$ . We also remark that the south of a box in row 0 is not a box in row  $-1$  outside  $\widehat{\mathbf{dg}}(\alpha)$ , but rather is simply undefined.

An *augmented filling* of shape  $\alpha = \alpha_1 \dots \alpha_n$ , basement  $\sigma = [\sigma_1, \dots, \sigma_n]$ , and labels in an alphabet  $\mathcal{A}$  is a function  $T: \widehat{\mathbf{dg}}(\alpha) \rightarrow \mathcal{A}$  such that  $T(i, 0) = \sigma_i$  for all  $i \in [n]$ . Let  $F_{\mathcal{A}}(\alpha, \sigma)$  denote the set of augmented fillings of shape  $\alpha$ , basement  $\sigma$ , and labels in  $\mathcal{A}$ . The absolute value and the non-attacking condition for augmented fillings are defined as in Section 3.2.1. Let  $\text{NAF}_{\mathcal{A}}(\alpha, \sigma) \subseteq F_{\mathcal{A}}(\alpha, \sigma)$  denote the set of non-attacking augmented fillings of shape  $\alpha$ , basement  $\sigma$ , and labels in  $\mathcal{A}$ .

See Figure 3.10 for an example of a non-attacking augmented filling. Augmented fillings are always drawn with the basement shaded in gray to distinguish them from non-augmented fillings.

Note that the non-attacking condition implies that, for a non-attacking augmented

|   |   |   |   |   |
|---|---|---|---|---|
| 5 | 4 |   |   |   |
| 1 | 2 |   |   | 4 |
| 3 | 1 |   | 5 | 4 |
| 2 | 1 | 3 | 5 | 4 |

Figure 3.10: A non-attacking augmented filling of shape  $\alpha = 33012$  and basement  $\sigma = [2, 1, 3, 5, 4]$ , with labels in  $[5]$ . The basement is shaded in gray ()

filling with labels in  $[n]$ , the entries  $\sigma_1, \dots, \sigma_n$  in the basement are distinct. Therefore, the basement  $\sigma$  is identified with a permutation of  $[n]$ . For (possibly attacking) augmented fillings with labels in  $[n]$  or  $[n]_{\pm}$ , the definition allows for repeated or negative entries in the basement. However, the applications in this work only require the case of a permutation basement, that is, positive and distinct entries in the basement.

The *content* of an augmented filling  $T$  with labels in an alphabet  $\mathcal{A}$  is the multiset  $\text{content}(T)$  whose elements are the labels  $a \in \mathcal{A}$ , with multiplicity equal to the number of boxes  $u \in \text{dg}(\alpha)$  such that  $T(u) = a$ . Note that boxes in row 0 are not counted in the content. When the alphabet is  $\mathbb{Z}_{>0}$  or  $[n]$ , the content is identified with the composition  $\beta = \text{content}(T)$  where  $\beta_i$  is the multiplicity of  $i$  in  $\text{content}(T)$ . We define

$$\mathbf{x}^T = \mathbf{x}^{\text{content}(T)} = \prod_i x_i^{\beta_i}.$$

For example, the augmented filling  $T$  in **Figure 3.10** has content  $\text{content}(T) = 21132$  and  $\mathbf{x}^T = \mathbf{x}^{\text{content}(T)} = x_1^2 x_2^1 x_3^1 x_4^3 x_5^2$ .

The definitions of descents, and major index for augmented fillings are as in **Section 3.2.1**. Note that a box  $u$  in row 1 can be a descent, since the label of its south  $d(u)$  is now a basement entry (rather than  $\infty$  by convention). However, a box  $u$  in row 0 is never a descent, since its south  $d(u)$  is undefined.

A triple  $(u, v, w)$  is defined as in **Section 3.2.1**. Since boxes in row 0 do not have a south,  $u$  cannot be in row 0. There is no concept of a degenerate triple in augmented fillings. The definitions of inversion triples, coinversion triples, and the statistics  $\text{inv}(T)$  and  $\text{coinv}(T)$  for augmented fillings are as in **Section 3.2.1**. In the augmented setting, the total number of triples in  $\widehat{\text{dg}}(\alpha)$  is

$$\sum_{u \in \text{dg}(\alpha)} \text{arm}_{\widehat{\text{dg}}(\alpha)}(u) \tag{3.3}$$

Note that this sum does not include boxes  $u$  in row 0. We remark that this sum may be different from  $\sum_{u \in \text{dg}(\alpha)} \text{arm}_{\text{dg}(\alpha)}(u)$  in **(3.1)**, since the arm statistic for boxes in row 1 may be different in the two settings.

Similarly to [Section 3.2.1](#), there is another interpretation of  $\text{inv}(T)$  and  $\text{coinv}(T)$ , in terms of inversion pairs. An *inversion pair* in an augmented filling  $T$  is a pair  $(u, v)$  of boxes with  $u \in \text{dg}(\alpha)$  and  $v \in \widehat{\text{dg}}(\alpha)$  such that  $u$  attacks  $v$  and  $\chi_{T(u)T(v)} = 1$ . Let  $\text{IP}(T)$  denote the set of inversion pairs in  $T$ .

**Proposition 3.2** ([[HHL05](#), p. 739]). *Let  $T \in \mathbf{F}(\alpha)$ . Then*

$$\text{inv}(T) = |\text{IP}(T)| - \sum_{u \in \text{Des}(T)} \text{arm}(u) - |\{(i, j) : i < j \text{ and } \alpha_i < \alpha_j \text{ and } \sigma_i > \sigma_j\}|.$$

*Proof.* Recall that  $(u, v, w)$  forms a triple if  $u \in \text{dg}(\alpha)$ ,  $v \in \text{Arm}(u)$  and  $w = d(u)$ . Such triple is an inversion triple if and only if  $\chi_{T(u)T(v)} + \chi_{T(v)T(w)} - \chi_{T(u)T(w)} = 1$ . Finally, note that  $u$  attacking  $v$  is equivalent to  $v$  attacking  $d(u)$ . Thus, the total number of inversion triples in  $T$  is

$$\begin{aligned} & \sum_{u \in \text{dg}(\alpha)} \sum_{v \in \text{Arm}(u)} \left( \chi_{T(u)T(v)} + \chi_{T(v)T(d(u))} - \chi_{T(u)T(d(u))} \right) = \\ &= \sum_{\substack{u \in \text{dg}(\alpha), v \in \widehat{\text{dg}}(\alpha) \\ v \in \text{Arm}(u)}} \chi_{T(u)T(v)} + \sum_{\substack{u \in \text{dg}(\alpha), v \in \widehat{\text{dg}}(\alpha) \\ v \in \text{Arm}(u)}} \chi_{T(v)T(d(u))} - \sum_{\substack{u \in \text{dg}(\alpha), v \in \widehat{\text{dg}}(\alpha) \\ v \in \text{Arm}(u)}} \chi_{T(u)T(d(u))} \\ &= \sum_{\substack{u \in \text{dg}(\alpha), v \in \widehat{\text{dg}}(\alpha) \\ u \text{ attacks } v \\ \text{leg}(v) \leq \text{leg}(u)}} \chi_{T(u)T(v)} + \sum_{\substack{v, w \in \widehat{\text{dg}}(\alpha) \\ v \text{ attacks } w \\ \text{leg}(w) > \text{leg}(v)}} \chi_{T(v)T(w)} - \sum_{u \in \text{dg}(\alpha)} \chi_{T(u)T(d(u))} \text{arm}(u) \\ &= |\text{IP}(T)| - \sum_{u \in \text{Des}(T)} \text{arm}(u) - \sum_{\substack{v, w \text{ in row } 0 \\ v \text{ attacks } w \\ \text{leg}(w) > \text{leg}(v)}} \chi_{T(v)T(w)} \\ &= |\text{IP}(T)| - \sum_{u \in \text{Des}(T)} \text{arm}(u) - |\{(i, j) : i < j \text{ and } \alpha_i < \alpha_j \text{ and } \sigma_i > \sigma_j\}|, \end{aligned}$$

as desired. □

[Proposition 3.2](#) gives an alternative description of  $\text{inv}(T)$ . Together with [\(3.3\)](#), we also obtain an alternative description of  $\text{coinv}(T)$  by

$$\begin{aligned} \text{coinv}(T) &= \sum_{u \in \text{dg}(\alpha)} \text{arm}(u) - \text{inv}(T) \\ &= \sum_{u \in \text{dg}(\alpha) \setminus \text{Des}(T)} \text{arm}(u) - |\text{IP}(T)| + |\{(i, j) : i < j \text{ and } \alpha_i < \alpha_j \text{ and } \sigma_i > \sigma_j\}|. \end{aligned}$$

### 3.2.3 Mistaking augmented fillings for non-augmented fillings

If we didn't shade the basement in gray, it would be easy to mistake an augmented filling for a non-augmented filling. This would lead to mistakes in computing the statistics  $\text{inv}$  and  $\text{coinv}$ . However, the mistakes are systematic, and we quantify them below. In [Section 4.2](#), this difference is relevant when we define a bijection for non-attacking fillings from a bijection for augmented fillings.

Let  $T \in F_{\mathcal{A}}(\alpha, \sigma)$  be an augmented filling of shape  $\alpha = \alpha_1 \dots \alpha_n$ , basement  $\sigma = [\sigma_1, \dots, \sigma_n]$ , and labels in an alphabet  $\mathcal{A}$ . Let  $T' \in F_{\mathcal{A}}(\alpha')$  be the non-augmented filling of shape  $\alpha' = (\alpha_1 + 1) \dots (\alpha_n + 1)$  defined by  $T'(i, r) = T(i, r - 1)$  for all  $(i, r) \in \text{dg}(\alpha')$ . See [Figure 3.11](#) for an example.

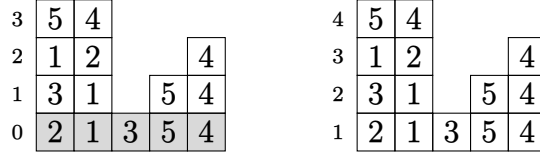


Figure 3.11: An augmented filling  $T$  of shape  $\alpha = 33012$  and basement  $\sigma = [2, 1, 3, 5, 4]$ , with labels in  $[5]$  (left); and the corresponding non-augmented filling  $T'$  of shape  $\alpha' = 44123$  (with labels in  $[5]$ ) obtained by ignoring the basement (right).

It's straightforward to check that  $\text{maj}(T) = \text{maj}(T')$ .

However,  $\text{inv}(T)$  and  $\text{coinv}(T)$  may differ from  $\text{inv}(T')$  and  $\text{coinv}(T')$ , respectively. We define new statistics  $\text{syminv}$  and  $\text{symcoinv}$  on augmented fillings by

$$\text{syminv}(T) = \text{inv}(T') \quad \text{and} \quad \text{symcoinv}(T) = \text{coinv}(T').$$

**Proposition 3.3.** *Let  $T \in F_{\mathcal{A}}(\alpha, \sigma)$  be an augmented filling of shape  $\alpha = \alpha_1 \dots \alpha_n$  and basement  $\sigma = [\sigma_1, \dots, \sigma_n]$ . Then,*

$$\text{syminv}(T) = \text{inv}(T) + |\{(i, j) : 1 \leq i < j \leq n \text{ and } \alpha_i \geq \alpha_j \text{ and } \sigma_i > \sigma_j = 1\}| \quad (3.4)$$

and

$$\text{symcoinv}(T) = \text{coinv}(T) + |\{(i, j) : 1 \leq i < j \leq n \text{ and } \alpha_i \geq \alpha_j \text{ and } \sigma_i > \sigma_j = 0\}|. \quad (3.5)$$

Note that the correction term in (3.5) is  $\text{twinv}(\sigma, \alpha)$  as defined in [Definition 2.8](#).

*Proof.* Consider a triple  $(u, v, w)$  in  $\widehat{\mathbf{dg}}(\alpha)$ . Define  $u' = u + (0, 1)$ ,  $v' = v + (0, 1)$ , and  $w' = w + (0, 1)$ . Then,  $(u', v', w')$  is a triple in  $\mathbf{dg}(\alpha')$ . Note that all triples  $(u', v', w')$  such that  $u'$  is not in row 1 are obtained in this way from triples  $(u, v, w)$  in  $\widehat{\mathbf{dg}}(\alpha)$ . Moreover,  $(u, v, w)$  is an inversion triple in  $T$  if and only if  $(u', v', w')$  is an inversion triple in  $T'$ . Therefore,

$$|\{\text{inversion triples } (u', v', w') \text{ in } T' : u' \text{ is not in row 1}\}| = \text{inv}(T), \quad (3.6)$$

and similarly,

$$|\{\text{coinversion triples } (u', v', w') \text{ in } T' : u' \text{ is not in row 1}\}| = \text{coinv}(T). \quad (3.7)$$

Now, consider triples  $(u', v', w')$  in  $\mathbf{dg}(\alpha')$  such that  $u'$  is in row 1. This is equivalent to  $u' = (i, 1)$  for some  $i \in [n]$ , and  $v' = (j, 1)$  for some  $j \in [n]$  such that  $j > i$  and  $\alpha'_i \geq \alpha'_j$ . This is an inversion triple in  $T'$  if and only if  $\chi_{\sigma_i \sigma_j \infty} = 1$ , which is equivalent to  $\sigma_i > \sigma_j$ . Therefore,

$$\begin{aligned} |\{\text{inversion triples } (u', v', w') \text{ in } T' : u' \text{ is in row 1}\}| &= \\ &= |\{(i, j) : 1 \leq i < j \leq n \text{ and } \alpha_i \geq \alpha_j \text{ and } \sigma_i > \sigma_j\}|, \end{aligned} \quad (3.8)$$

and similarly,

$$\begin{aligned} |\{\text{coinversion triples } (u', v', w') \text{ in } T' : u' \text{ is in row 1}\}| &= \\ &= |\{(i, j) : 1 \leq i < j \leq n \text{ and } \alpha_i \geq \alpha_j \text{ and } \sigma_i < \sigma_j\}|. \end{aligned} \quad (3.9)$$

The results follow by adding (3.6) and (3.8), and adding (3.7) and (3.9).  $\square$

## 3.3 Tableaux formulas for Macdonald polynomials

### 3.3.1 Modified Macdonald polynomials $\tilde{H}_\mu$

Recall that the Modified Macdonald polynomials  $\tilde{H}_\mu$  are a family of polynomials indexed by partitions  $\mu$ . Haglund, Haiman, and Loehr [HHL05, Theorem 2.2] give a combinatorial formula for  $\tilde{H}_\mu$  as\*

$$\tilde{H}_\mu(\mathbf{x}; q, t) = \sum_{T \in \mathcal{F}_{\geq 0}(\mu)} q^{\text{maj}(T)} t^{\text{inv}(T)} \mathbf{x}^T, \quad (3.10)$$

---

\*We remark that (3.10) uses the skyline diagram of  $\mu$ , while [HHL05, (14)] use the Ferrers diagram of  $\mu$  that corresponds to the skyline diagram of the conjugate partition  $\mu'$ . Moreover, (3.10) interchanges the roles of  $q$  and  $t$  compared to [HHL05, (14)]. Nevertheless, these choices yield the same polynomial  $\tilde{H}_\mu(\mathbf{x}; q, t)$  as in [HHL05] due to the symmetry  $\tilde{H}_\mu(\mathbf{x}; q, t) = \tilde{H}_{\mu'}(\mathbf{x}; t, q)$  [Mac95, Chapter VI, (8.6)].

where  $F_{\mathbb{Z}_{>0}}(\mu)$  is the set of fillings of shape  $\mu$  with labels in the positive integers  $\mathbb{Z}_{>0}$ .

In their subsequent paper [HHL08], Haglund, Haiman, and Loehr show that the set may be changed to  $F_{\mathbb{Z}_{>0}}(\alpha)$  for any rearrangement  $\alpha$  of  $\mu$ , and the formula still holds. In Section 4.1, we give a bijective proof of this rearrangement property.

### 3.3.2 Integral form symmetric Macdonald polynomials $J_\mu$

Recall that the integral form Macdonald polynomials  $J_\mu$  are a family of polynomials indexed by partitions  $\mu$ . Haglund, Haiman, and Loehr [HHL05, Section 7] derive a tableaux formula for the integral form Macdonald polynomials  $J_\mu$  from the combinatorial formula for the modified Macdonald polynomials  $\tilde{H}_\mu$  in (3.10). We briefly recall the main steps.

From the plethystic formula in (2.3), we have

$$J_\mu(\mathbf{x}; q, t) = t^{n(\mu)} \tilde{H}_\mu [X(1-t); q, t^{-1}] = t^{n(\mu)+|\mu|} \tilde{H}_\mu [X(t^{-1}-1); q, t^{-1}], \quad (3.11)$$

where  $n(\mu) = \sum_i (i-1)\mu_i = \sum_{u \in \text{dg}(\mu)} \text{arm}(u)$  and  $|\mu| = \sum_i \mu_i$ . Applying a standard technique called *superization*, [HHL05] derive from (3.10) and (3.11) the following combinatorial formula for  $J_\mu$  as a sum over all fillings using the signed alphabet  $[n]_\pm$ :

$$J_\mu(\mathbf{x}; q, t) = \sum_{T \in F_{[n]_\pm}(\mu)} \mathbf{x}^{|T|} q^{\text{maj}(T)} t^{\text{coinv}(T)} (-t)^{\text{neg}(T)}, \quad (3.12)$$

where  $\text{neg}(T)$  denotes the number of negative entries in  $T$ . We refer the reader to [HHL05, Section 7] for a complete derivation.

Two other combinatorial formulas for  $J_\mu$  in terms of non-attacking fillings are also given in [HHL05], below in Propositions 3.4 and 3.5. We reproduce their proofs because their ideas are useful in Section 3.3.5.

**Proposition 3.4.** *For a partition  $\mu$  with at most  $n$  parts, the integral form Macdonald polynomial  $J_\mu(\mathbf{x}; q, t)$  can be expressed as a sum over all non-attacking fillings using the signed alphabet  $[n]_\pm$ ,*

$$J_\mu(\mathbf{x}; q, t) = \sum_{T \in \text{NAF}_{[n]_\pm}(\mu)} \mathbf{x}^{|T|} q^{\text{maj}(T)} t^{\text{coinv}(T)} (-t)^{\text{neg}(T)}. \quad (3.13)$$

*Proof ([HHL05, p. 744]).* We define a sign-reversing involution on the set of attacking fillings in  $F_{[n]_\pm}(\mu) \setminus \text{NAF}_{[n]_\pm}(\mu)$ . Recall that, in  $[n]_\pm$ , we have

$$1 < \bar{1} < 2 < \bar{2} < \cdots < n < \bar{n}.$$

This ordering has two useful properties:

$$\chi_{ak} = \chi_{\bar{a}k} \quad \text{for any } a \in [n] \text{ and any } k \in [n]_{\pm}, \text{ and} \quad (3.14)$$

$$\chi_{ka} = \chi_{k\bar{a}} \quad \text{for any } a \in [n] \text{ and any } k \in [n]_{\pm} \text{ with } |k| \neq a. \quad (3.15)$$

Given an attacking filling  $T \in F_{[n]_{\pm}}(\mu) \setminus \text{NAF}_{[n]_{\pm}}(\mu)$ , let  $a$  be the smallest label which is the absolute value of two attacking entries in  $T$ , which exists since  $T$  is attacking. Let  $v$  be the earlier box (in reading order) with entry of absolute value  $a$  that is attacked by another box with entry of absolute value  $a$ . Let  $u$  be the earlier box (in reading order) which attacks  $v$  and has absolute value  $a$ . Define  $T'$  to be the filling obtained from  $T$  by switching the sign of the entry in box  $u$ . Then,  $T'$  is also an attacking filling in  $F_{[n]_{\pm}}(\mu) \setminus \text{NAF}_{[n]_{\pm}}(\mu)$ .

Since the definitions of  $a$ ,  $v$  and  $u$  only depend on the absolute values of the entries in  $T$ , we have that applying the same procedure to  $T'$  returns  $T$ . Hence, the map  $T \mapsto T'$  is an involution.

Without loss of generality, assume that  $T(u) = a$  and  $T'(u) = \bar{a}$ . It is straightforward to check that  $\mathbf{x}^{|T'|} = \mathbf{x}^{|T|}$  and  $\text{neg}(T') = \text{neg}(T) + 1$ .

I claim that  $\text{Des}(T') = \text{Des}(T)$ , which implies that  $\text{maj}(T') = \text{maj}(T)$ . Let  $w$  be the box directly above  $u$ , if it exists. For any box  $b \notin \{w, u\}$ , the fillings  $T$  and  $T'$  agree on the entries in boxes  $b$  and  $d(b)$ , so  $b$  is a descent in  $T$  if and only if it is a descent in  $T'$ . For  $w$ , if it exists, we can apply (3.14) to conclude that  $w$  is a descent in  $T$  if and only if it is a descent in  $T'$ . For  $u$ , note that  $d(u)$  is attacked by  $v$  and is earlier than  $v$  in reading order. Therefore, by minimality of  $v$ , we have  $|T(d(u))| \neq a$ , so we can apply (3.15) to conclude that  $u$  is a descent in  $T$  if and only if it is a descent in  $T'$ . Thus, we have shown that  $\text{Des}(T') = \text{Des}(T)$ , which implies that  $\text{maj}(T') = \text{maj}(T)$ .

I claim that  $\text{inv}(T') = \text{inv}(T) + 1$ . Using Proposition 3.1, since  $\text{Des}(T') = \text{Des}(T)$ , it suffices to show that  $|\text{IP}(T')| = |\text{IP}(T)| + 1$ . Any pair of attacking boxes that does not involve box  $u$  is an inversion pair in  $T$  if and only if it is an inversion pair in  $T'$ . Moreover, by (3.14), any box  $b$  that attacks  $u$  forms an inversion pair  $(b, u)$  in  $T$  if and only if it forms an inversion pair  $(b, u)$  in  $T'$ . Consider a box  $b$  that is attacked by  $u$ . If  $b = v$ , then  $(u, v)$  is an inversion pair in  $T'$  but not in  $T$ . If  $b \neq v$ , then by minimality of  $u$ , we have  $|T(b)| \neq a$ , so by (3.15),  $(u, b)$  is an inversion pair in  $T$  if and only if it is an inversion pair in  $T'$ . Therefore,  $\text{IP}(T')$  has exactly one more inversion pair than  $\text{IP}(T)$ , namely  $(u, v)$ , which proves the claim. Consequently,  $\text{coinv}(T') = \text{coinv}(T) - 1$ .

Putting this together, we have

$$\begin{aligned} \mathbf{x}^{|T'|} q^{\text{maj}(T')} t^{\text{coinv}(T')} (-t)^{\text{neg}(T')} &= \mathbf{x}^{|T|} q^{\text{maj}(T)} t^{\text{coinv}(T)-1} (-t)^{\text{neg}(T)+1} \\ &= -\mathbf{x}^{|T|} q^{\text{maj}(T)} t^{\text{coinv}(T)} (-t)^{\text{neg}(T)}, \end{aligned}$$

which implies that the contributions of fillings in  $F_{[n]_{\pm}}(\mu) \setminus \text{NAF}_{[n]_{\pm}}(\mu)$  to the sum in (3.12) cancel out, and the result follows.  $\square$

**Proposition 3.5.** *For a partition  $\mu$  with at most  $n$  parts, the integral form Macdonald polynomial  $J_{\mu}(\mathbf{x}; q, t)$  can be expressed as a sum over all non-attacking fillings using the unsigned alphabet  $[n]$ ,*

$$J_{\mu}(\mathbf{x}; q, t) = \sum_{T \in \text{NAF}_{[n]}(\mu)} \mathbf{x}^T q^{\text{maj}(T)} t^{\text{coinv}(T)} \prod_{\substack{u \in \text{dg}(\mu) \\ T(u)=T(d(u))}} (1 - q^{1+\text{leg}(u)} t^{1+\text{arm}(u)}) \prod_{\substack{u \in \text{dg}(\mu) \\ T(u) \neq T(d(u)) \text{ or } u \text{ in row 1}}} (1 - t). \quad (3.16)$$

*Proof ([HHL05, p. 750]).* Let  $S \in \text{NAF}_{[n]_{\pm}}(\mu)$  be a non-attacking signed filling, and let  $|S| \in \text{NAF}_{[n]}(\mu)$  be the non-attacking unsigned filling obtained from  $S$  by taking the absolute value of each entry.

For any triple  $(u, v, w)$ , since  $u$  attacks  $v$  and  $v$  attacks  $w$ , the non-attacking condition forces  $|S(u)| \neq |S(v)|$  and  $|S(v)| \neq |S(w)|$ . Define the *absolute coinversion number*  $\text{acoinv}(S)$  to be the number of coinversion triples  $(u, v, w)$  in  $S$  for which  $|S(u)| \neq |S(w)|$  (and consequently  $|S(u)|$ ,  $|S(v)|$ , and  $|S(w)|$  are all distinct). Define the *signed coinversion number*  $\text{scoinv}(S)$  to be the number of coinversion triples  $(u, v, w)$  in  $S$  for which  $|S(u)| = |S(w)|$ . Then, we have

$$\text{coinv}(S) = \text{acoinv}(S) + \text{scoinv}(S).$$

Note that, since  $\chi_{aba} = \chi_{ab} + \chi_{ba} - \chi_{aa} = 1$  for all distinct  $a, b \in [n]_{\pm}$ , we have  $\text{scoinv}(|S|) = 0$  and consequently  $\text{acoinv}(S) = \text{acoinv}(|S|) = \text{coinv}(|S|)$ .

Similarly, define the *absolute major index*  $\text{amaj}(S)$  and the *signed major index*  $\text{smaj}(S)$  by

$$\text{amaj}(S) = \sum_{\substack{u \in \text{Des}(S) \\ |S(u)| \neq |S(d(u))|}} (1 + \text{leg}(u)) \quad \text{and} \quad \text{smaj}(S) = \sum_{\substack{u \in \text{Des}(S) \\ |S(u)| = |S(d(u))|}} (1 + \text{leg}(u)).$$

Then, we have

$$\text{maj}(S) = \text{amaj}(S) + \text{smaj}(S).$$

Note that, since  $\chi_{aa} = 0$  for all  $a \in [n]_{\pm}$ , we have  $\text{smaj}(|S|) = 0$  and consequently  $\text{amaj}(S) = \text{amaj}(|S|) = \text{maj}(|S|)$ .

We group the fillings  $S \in \text{NAF}_{[n]_{\pm}}(\mu)$  by their absolute filling  $|S|$  to rewrite (3.13) as

$$J_{\mu}(\mathbf{x}; q, t) = \sum_{T \in \text{NAF}_{[n]}(\mu)} \sum_{\substack{S \in \text{NAF}_{[n]_{\pm}}(\mu) \\ |S|=T}} \mathbf{x}^{|S|} q^{\text{maj}(S)} t^{\text{coinv}(S)} (-t)^{\text{neg}(S)}.$$

Fix a non-attacking unsigned filling  $T \in \text{NAF}_{[n]}(\mu)$ . Using the decompositions of  $\text{maj}(S)$  and  $\text{coinv}(S)$  above, the inner sum corresponding to  $T$  can be rewritten as

$$\sum_{\substack{S \in \text{NAF}_{[n]\pm}(\mu) \\ |S|=T}} \mathbf{x}^{|S|} q^{\text{maj}(S)} t^{\text{coinv}(S)} (-t)^{\text{neg}(S)} = \mathbf{x}^T q^{\text{maj}(T)} t^{\text{coinv}(T)} \sum_{\substack{S \in \text{NAF}_{[n]\pm}(\mu) \\ |S|=T}} q^{\text{smaj}(S)} t^{\text{scoinv}(S)} (-t)^{\text{neg}(S)}.$$

Let  $u \in \text{dg}(\mu)$  such that  $T(u) \neq T(d(u))$  or  $u$  is in row 1. Then, in any signed filling  $S$  with  $|S| = T$ , the box  $u$  does not contribute to  $\text{smaj}(S)$ , and any triple  $(u, v, w)$  does not contribute to  $\text{scoinv}(S)$ . The entry in box  $u$  contributes to  $\text{neg}(S)$  if and only if  $u \in [n]_-$ .

Now, let  $u \in \text{dg}(\mu)$  such that  $T(u) = T(d(u))$ , and let  $S$  be a signed filling with  $|S| = T$ . The box  $u$  contributes to  $\text{smaj}(S)$  if and only if  $\chi_{S(u)S(d(u))} = 1$ , which occurs if and only if  $S(u) \in [n]_-$ . Moreover, any triple  $(u, v, w)$  contributes to  $\text{scoinv}(S)$  if and only if  $\chi_{S(u)S(v)S(w)} = 0$ , which occurs if and only if  $S(u) \in [n]_-$ . Recall that there are  $\text{arm}(u)$  such triples  $(u, v, w)$ . Finally, the entry in box  $u$  contributes to  $\text{neg}(S)$  if and only if  $S(u) \in [n]_-$ .

Note that all of these contributions only depend on the sign of the entry in box  $u$ , and are independent of the signs of the entries in other boxes. Therefore, we can factor the sum over all  $S$  with  $|S| = T$  as a product over all boxes  $u \in \text{dg}(\mu)$ . If  $T(u) \neq T(d(u))$  or  $u$  is in row 1, the contribution from box  $u$  is

$$1 + (-t) = 1 - t,$$

while if  $T(u) = T(d(u))$ , the contribution from box  $u$  is

$$1 + q^{1+\text{leg}(u)} t^{\text{arm}(u)} (-t) = 1 - q^{1+\text{leg}(u)} t^{1+\text{arm}(u)}.$$

Therefore, the inner sum corresponding to a fixed  $T \in \text{NAF}_{[n]}(\mu)$  is

$$\sum_{\substack{S \in \text{NAF}_{[n]\pm}(\mu) \\ |S|=T}} q^{\text{smaj}(S)} t^{\text{scoinv}(S)} (-t)^{\text{neg}(S)} = \prod_{\substack{u \in \text{dg}(\mu) \\ T(u)=T(d(u))}} (1 - q^{1+\text{leg}(u)} t^{1+\text{arm}(u)}) \prod_{\substack{u \in \text{dg}(\mu) \\ T(u) \neq T(d(u)) \text{ or } u \text{ in row 1}}} (1 - t),$$

and adding over all  $T \in \text{NAF}_{[n]}(\mu)$  gives the desired result.  $\square$

### 3.3.3 Symmetric Macdonald polynomials $P_\mu$

Recall that the symmetric Macdonald polynomials  $P_\mu$  are a family of polynomials indexed by partitions  $\mu$ . The integral form Macdonald polynomials  $J_\mu$  is a scalar multiple of  $P_\mu$ , with relationship given by (2.2):

$$J_\mu(\mathbf{x}; q, t) = c_\mu(q, t) P_\mu(\mathbf{x}; q, t),$$

where the normalization factor  $c_\mu(q, t)$  is given in (2.1).

Therefore, from (3.16), we obtain a tableaux formula for  $P_\mu$  as

$$P_\mu(\mathbf{x}; q, t) = \frac{d_\mu(q, t)}{c_\mu(q, t)} \sum_{T \in \text{NAF}_{[n]}(\mu)} \mathbf{x}^T q^{\text{maj}(T)} t^{\text{coinv}(T)} \prod_{\substack{u \in \text{dg}(\mu) \\ T(u) \neq T(d(u))}} \frac{1-t}{1-q^{1+\text{leg}(u)}t^{1+\text{arm}(u)}},$$

where the factor  $d_\mu(q, t)$  is given in (2.4).

The normalization factors make this formula less appealing. See [CHMMW22, Section 4], [Man25a, Section 5] and [Man25b] for more details and other tableaux formulas for  $P_\mu$ .

### 3.3.4 Permuted-basement Macdonald polynomials $E_\alpha^\sigma$

Recall that the permuted-basement Macdonald polynomials  $E_\alpha^\sigma$  are a family of polynomials indexed by a composition  $\alpha$  and a permutation  $\sigma$ . Ferreira [Fer11, Definition 4.4.2] gives a tableau formula for permuted-basement Macdonald polynomials  $E_\alpha^\sigma$  as

$$E_\alpha^\sigma(\mathbf{x}; q, t) = \sum_{T \in \text{NAF}_{[n]}(\alpha, \sigma)} \mathbf{x}^T q^{\text{maj}(T)} t^{\text{coinv}(T)} \prod_{\substack{u \in \text{dg}(\alpha) \\ T(u) \neq T(d(u))}} \frac{1-t}{1-q^{1+\text{leg}(u)}t^{1+\text{arm}(u)}}. \quad (3.17)$$

### 3.3.5 Integral form permuted-basement Macdonald polynomials $\mathcal{E}_\alpha^\sigma$

Recall that the integral form permuted-basement Macdonald polynomials  $\mathcal{E}_\alpha^\sigma$  are defined in (2.5) by

$$\mathcal{E}_\alpha^\sigma(\mathbf{x}; q, t) = \left( \prod_{u \in \text{dg}(\alpha)} (1 - q^{1+\text{leg}(u)}t^{1+\text{arm}(u)}) \right) E_\alpha^\sigma(\mathbf{x}; q, t).$$

Therefore, from (3.17), we obtain a tableaux formula for  $\mathcal{E}_\alpha^\sigma$  as

$$\mathcal{E}_\alpha^\sigma(\mathbf{x}; q, t) = \sum_{T \in \text{NAF}_{[n]}(\alpha, \sigma)} \mathbf{x}^T q^{\text{maj}(T)} t^{\text{coinv}(T)} \prod_{\substack{u \in \text{dg}(\alpha) \\ T(u)=T(d(u))}} (1 - q^{1+\text{leg}(u)}t^{1+\text{arm}(u)}) \prod_{\substack{u \in \text{dg}(\alpha) \\ T(u) \neq T(d(u))}} (1-t). \quad (3.18)$$

Note the similarity of (3.18) for the integral form permuted-basement Macdonald polynomials  $\mathcal{E}_\alpha^\sigma$  with (3.16) for the integral form symmetric Macdonald polynomials  $J_\mu$ , the third of three formulas for  $J_\mu$  given in Section 3.3.2.

We can repeat the arguments used in [Propositions 3.5](#) and [3.4](#) (in reverse order) to obtain formulas for  $\mathcal{E}_\alpha^\sigma$  that are similar to [\(3.13\)](#), the second formula for  $J_\mu$ , and [\(3.12\)](#), the first formula for  $J_\mu$ . The arguments are nearly identical, with some minor adjustments to account for the basement  $\sigma$  and to account for the reverse order of the steps. We nevertheless fully reproduce them below for self-containment.

**Proposition 3.6.** *For a composition  $\alpha = \alpha_1 \dots \alpha_n$  and a permutation  $\sigma$  of  $[n]$ , the integral form permuted-basement Macdonald polynomial  $\mathcal{E}_\alpha^\sigma(\mathbf{x}; q, t)$  can be expressed as a sum over all non-attacking augmented fillings using the signed alphabet  $[n]_\pm$ ,*

$$\mathcal{E}_\alpha^\sigma(\mathbf{x}; q, t) = \sum_{T \in \text{NAF}_{[n]_\pm}(\alpha, \sigma)} \mathbf{x}^{|T|} q^{\text{maj}(T)} t^{\text{coinv}(T)} (-t)^{\text{neg}(T)}. \quad (3.19)$$

*Proof.* Let  $S \in \text{NAF}_{[n]_\pm}(\mu, \sigma)$  be a non-attacking signed augmented filling, and let  $|S| \in \text{NAF}_{[n]}(\mu, \sigma)$  be the non-attacking unsigned augmented filling obtained from  $S$  by taking the absolute value of each entry.

For any triple  $(u, v, w)$ , since  $u$  attacks  $v$  and  $v$  attacks  $w$ , the non-attacking condition forces  $|S(u)| \neq |S(v)|$  and  $|S(v)| \neq |S(w)|$ . Define the *absolute coinversion number*  $\text{acoinv}(S)$  to be the number of coinversion triples  $(u, v, w)$  in  $S$  for which  $|S(u)| \neq |S(w)|$  (and consequently  $|S(u)|$ ,  $|S(v)|$ , and  $|S(w)|$  are all distinct). Define the *signed coinversion number*  $\text{scoinv}(S)$  to be the number of coinversion triples  $(u, v, w)$  in  $S$  for which  $|S(u)| = |S(w)|$ . Then, we have

$$\text{coinv}(S) = \text{acoinv}(S) + \text{scoinv}(S).$$

Note that, since  $\chi_{aba} = \chi_{ab} + \chi_{ba} - \chi_{aa} = 1$  for all distinct  $a, b \in [n]$ , we have  $\text{scoinv}(|S|) = 0$  and consequently  $\text{acoinv}(S) = \text{acoinv}(|S|) = \text{coinv}(|S|)$ .

Similarly, define the *absolute major index*  $\text{amaj}(S)$  and the *signed major index*  $\text{smaj}(S)$  by

$$\text{amaj}(S) = \sum_{\substack{u \in \text{Des}(S) \\ |S(u)| \neq |S(d(u))|}} (1 + \text{leg}(u)) \quad \text{and} \quad \text{smaj}(S) = \sum_{\substack{u \in \text{Des}(S) \\ |S(u)| = |S(d(u))|}} (1 + \text{leg}(u)).$$

Then, we have

$$\text{maj}(S) = \text{amaj}(S) + \text{smaj}(S).$$

Note that, since  $\chi_{aa} = 0$  for all  $a \in [n]_\pm$ , we have  $\text{smaj}(|S|) = 0$  and consequently  $\text{amaj}(S) = \text{amaj}(|S|) = \text{maj}(|S|)$ .

We group the augmented fillings  $S \in \mathbf{NAF}_{[n]\pm}(\mu, \sigma)$  by their absolute augmented filling  $|S|$  to rewrite the right-hand side of (3.19) as

$$\sum_{T \in \mathbf{NAF}_{[n]}(\mu, \sigma)} \sum_{\substack{S \in \mathbf{NAF}_{[n]\pm}(\mu, \sigma) \\ |S|=T}} \mathbf{x}^{|S|} q^{\mathbf{maj}(S)} t^{\mathbf{coinv}(S)} (-t)^{\mathbf{neg}(S)}.$$

Fix a non-attacking unsigned augmented filling  $T \in \mathbf{NAF}_{[n]}(\mu, \sigma)$ . Using the decompositions of  $\mathbf{maj}(S)$  and  $\mathbf{coinv}(S)$  above, the inner sum corresponding to  $T$  can be rewritten as

$$\sum_{\substack{S \in \mathbf{NAF}_{[n]\pm}(\mu, \sigma) \\ |S|=T}} \mathbf{x}^{|S|} q^{\mathbf{maj}(S)} t^{\mathbf{coinv}(S)} (-t)^{\mathbf{neg}(S)} = \mathbf{x}^T q^{\mathbf{maj}(T)} t^{\mathbf{coinv}(T)} \sum_{\substack{S \in \mathbf{NAF}_{[n]\pm}(\mu, \sigma) \\ |S|=T}} q^{\mathbf{smaj}(S)} t^{\mathbf{scoinv}(S)} (-t)^{\mathbf{neg}(S)}.$$

Let  $u \in \widehat{\mathbf{dg}}(\mu, \sigma)$  such that  $u$  is in row 0. Then, in any signed augmented filling  $S$  with  $|S| = T$ , the box  $u$  does not contribute to  $\mathbf{smaj}(S)$  since there are no descents in row 0, neither to  $\mathbf{neg}(S)$  since the basement entries are entries of  $\sigma$  and thus positive. Moreover, there are no triples  $(u, v, w)$  with  $u$  in row 0, so there is no contribution to  $\mathbf{scoinv}(S)$  from such boxes.

Let  $u \in \mathbf{dg}(\mu)$  such that  $T(u) \neq T(d(u))$ . Then, in any signed augmented filling  $S$  with  $|S| = T$ , the box  $u$  does not contribute to  $\mathbf{smaj}(S)$ , and any triple  $(u, v, w)$  does not contribute to  $\mathbf{scoinv}(S)$ . The entry in box  $u$  contributes to  $\mathbf{neg}(S)$  if and only if  $u \in [n]_-$ .

Now, let  $u \in \mathbf{dg}(\mu)$  such that  $T(u) = T(d(u))$ , and let  $S$  be a signed augmented filling with  $|S| = T$ . The box  $u$  contributes to  $\mathbf{smaj}(S)$  if and only if  $\chi_{S(u)S(d(u))} = 1$ , which occurs if and only if  $S(u) \in [n]_-$ . Moreover, any triple  $(u, v, w)$  contributes to  $\mathbf{scoinv}(S)$  if and only if  $\chi_{S(u)S(v)S(w)} = 0$ , which occurs if and only if  $S(u) \in [n]_-$ . Recall that there are  $\mathbf{arm}(u)$  such triples  $(u, v, w)$ . Finally, the entry in box  $u$  contributes to  $\mathbf{neg}(S)$  if and only if  $S(u) \in [n]_-$ .

Note that all contributions to  $\mathbf{smaj}(S)$ ,  $\mathbf{scoinv}(S)$ , and  $\mathbf{neg}(S)$  are counted exactly once in the cases above, and their contributions only depend on the sign of the entry in box  $u$ . Therefore, we can factor the sum over all  $S$  with  $|S| = T$  as a product over all boxes  $u \in \mathbf{dg}(\mu)$  (which excludes row 0, because there is no choice of sign and no contribution from those boxes). If  $T(u) \neq T(d(u))$ , the contribution from box  $u$  is

$$1 + (-t) = 1 - t,$$

while if  $T(u) = T(d(u))$ , the contribution from box  $u$  is

$$1 + q^{1+\mathbf{leg}(u)} t^{\mathbf{arm}(u)} (-t) = 1 - q^{1+\mathbf{leg}(u)} t^{1+\mathbf{arm}(u)}.$$

Therefore, the inner sum corresponding to a fixed  $T \in \text{NAF}_{[n]}(\mu, \sigma)$  is

$$\sum_{\substack{S \in \text{NAF}_{[n]\pm}(\mu, \sigma) \\ |S|=T}} q^{\text{maj}(S)} t^{\text{scoinv}(S)} (-t)^{\text{neg}(S)} = \prod_{\substack{u \in \text{dg}(\mu) \\ T(u)=T(d(u))}} (1 - q^{1+\text{leg}(u)} t^{1+\text{arm}(u)}) \prod_{\substack{u \in \text{dg}(\mu) \\ T(u) \neq T(d(u))}} (1 - t),$$

and adding over all  $T \in \text{NAF}_{[n]}(\mu, \sigma)$  gives the desired result.  $\square$

**Proposition 3.7.** *For a composition  $\alpha = \alpha_1 \dots \alpha_n$  and a permutation  $\sigma$  of  $[n]$ , the integral form permuted-basement Macdonald polynomial  $\mathcal{E}_\alpha^\sigma(\mathbf{x}; q, t)$  can be expressed as a sum over all augmented fillings using the signed alphabet  $[n]_\pm$ ,*

$$\mathcal{E}_\alpha^\sigma(\mathbf{x}; q, t) = \sum_{T \in \text{F}_{[n]\pm}(\alpha, \sigma)} \mathbf{x}^{|T|} q^{\text{maj}(T)} t^{\text{coinv}(T)} (-t)^{\text{neg}(T)}. \quad (3.20)$$

*Proof.* We define a sign-reversing involution on the set of attacking augmented fillings in  $\text{F}_{[n]\pm}(\alpha, \sigma) \setminus \text{NAF}_{[n]\pm}(\alpha, \sigma)$ . Recall that, in  $[n]_\pm$ , we have  $1 < \bar{1} < 2 < \bar{2} < \dots < n < \bar{n}$ , which has the useful properties given in (3.14) and (3.15).

Given an attacking augmented filling  $T \in \text{F}_{[n]\pm}(\alpha, \sigma) \setminus \text{NAF}_{[n]\pm}(\alpha, \sigma)$ , let  $a$  be the smallest label which is the absolute value of two attacking entries in  $T$ , which exists since  $T$  is attacking. Let  $v$  be the earlier box (in reading order) with entry of absolute value  $a$  that is attacked by another box with entry of absolute value  $a$ . Let  $u$  be the earlier box (in reading order) which attacks  $v$  and has absolute value  $a$ . Since the entries in row 0 are the basement entries given by  $\sigma$ , which are all distinct and positive, we have that  $u$  is not in row 0. Define  $T'$  to be the filling obtained from  $T$  by switching the sign of the entry in box  $u$ . Since  $u$  is not in row 0,  $T'$  has basement  $\sigma$ . Then,  $T'$  is an attacking augmented filling in  $\text{F}_{[n]\pm}(\alpha, \sigma) \setminus \text{NAF}_{[n]\pm}(\alpha, \sigma)$ .

Since the definitions of  $a$ ,  $v$  and  $u$  only depend on the absolute values of the entries in  $T$ , we have that applying the same procedure to  $T'$  returns  $T$ . Hence, the map  $T \mapsto T'$  is an involution.

Without loss of generality, assume that  $T(u) = a$  and  $T'(u) = \bar{a}$ . We have  $\mathbf{x}^{|T'|} = \mathbf{x}^{|T|}$  and  $\text{neg}(T') = \text{neg}(T) + 1$ .

I claim that  $\text{Des}(T') = \text{Des}(T)$ , which implies that  $\text{maj}(T') = \text{maj}(T)$ . Let  $w$  be the box directly above  $u$ , if it exists. For any box  $b \notin \{w, u\}$ , the fillings  $T$  and  $T'$  agree on the entries in boxes  $b$  and  $d(b)$ , so  $b$  is a descent in  $T$  if and only if it is a descent in  $T'$ . For  $w$ , if it exists, we can apply (3.14) to conclude that  $w$  is a descent in  $T$  if and only if it is a descent in  $T'$ . For  $u$ , note that  $d(u)$  is attacked by  $v$  and is earlier than  $v$  in reading order. Therefore, by minimality of  $v$ , we have  $|T(d(u))| \neq a$ , so we can apply (3.15) to

conclude that  $u$  is a descent in  $T$  if and only if it is a descent in  $T'$ . Thus, we have shown that  $\text{Des}(T') = \text{Des}(T)$ , which implies that  $\text{maj}(T') = \text{maj}(T)$ .

I claim that  $\text{inv}(T') = \text{inv}(T) + 1$ . Using [Proposition 3.2](#), since  $\text{Des}(T') = \text{Des}(T)$ , it suffices to show that  $|\text{IP}(T')| = |\text{IP}(T)| + 1$ . Any pair of attacking boxes that does not involve box  $u$  is an inversion pair in  $T$  if and only if it is an inversion pair in  $T'$ . Moreover, by [\(3.14\)](#), any box  $b$  that attacks  $u$  forms an inversion pair  $(b, u)$  in  $T$  if and only if it forms an inversion pair  $(b, u)$  in  $T'$ . Consider a box  $b$  that is attacked by  $u$ . If  $b = v$ , then  $(u, v)$  is an inversion pair in  $T'$  but not in  $T$ . If  $b \neq v$ , then by minimality of  $u$ , we have  $|T(b)| \neq a$ , so by [\(3.15\)](#),  $(u, b)$  is an inversion pair in  $T$  if and only if it is an inversion pair in  $T'$ . Therefore,  $\text{IP}(T')$  has exactly one more inversion pair than  $\text{IP}(T)$ , namely  $(u, v)$ , which proves the claim. Consequently,  $\text{coinv}(T') = \text{coinv}(T) - 1$ .

Putting this together, we have

$$\begin{aligned} \mathbf{x}^{|T'|} q^{\text{maj}(T')} t^{\text{coinv}(T')} (-t)^{\text{neg}(T')} &= \mathbf{x}^{|T|} q^{\text{maj}(T)} t^{\text{coinv}(T)-1} (-t)^{\text{neg}(T)+1} \\ &= -\mathbf{x}^{|T|} q^{\text{maj}(T)} t^{\text{coinv}(T)} (-t)^{\text{neg}(T)}, \end{aligned}$$

which implies that

$$\sum_{T \in \mathcal{F}_{[n]_{\pm}}(\alpha, \sigma) \setminus \mathcal{NAF}_{[n]_{\pm}}(\alpha, \sigma)} \mathbf{x}^{|T|} q^{\text{maj}(T)} t^{\text{coinv}(T)} (-t)^{\text{neg}(T)} = 0.$$

and consequently

$$\sum_{T \in \mathcal{F}_{[n]_{\pm}}(\alpha, \sigma)} \mathbf{x}^{|T|} q^{\text{maj}(T)} t^{\text{coinv}(T)} (-t)^{\text{neg}(T)} = \sum_{T \in \mathcal{NAF}_{[n]_{\pm}}(\alpha, \sigma)} \mathbf{x}^{|T|} q^{\text{maj}(T)} t^{\text{coinv}(T)} (-t)^{\text{neg}(T)}.$$

The right-hand side equals  $\mathcal{E}_{\alpha}^{\sigma}(\mathbf{x}; q, t)$  by [Proposition 3.6](#), so the result follows.  $\square$



# Chapter 4

## Bijjective combinatorics for Macdonald polynomials

This chapter is split into two sections. [Section 4.1](#) provides bijections on non-augmented fillings to obtain formulas for the modified Macdonald polynomials  $\tilde{H}_\mu(\mathbf{x}; q, t)$ . [Section 4.2](#) provides bijections on augmented fillings to obtain formulas for the permuted-basement Macdonald polynomials  $E_\alpha^\sigma(\mathbf{x}; q, t)$ .

In [Section 4.1](#), we study a rearrangement property of the modified Macdonald polynomials  $\tilde{H}_\mu(\mathbf{x}; q, t)$ , [Theorem 4.14](#), which states that, for any partition  $\mu$  and any rearrangement  $\alpha$  of  $\mu$ ,

$$\sum_{T \in \mathbb{F}_{\mathbb{Z}_{>0}}(\mu)} q^{\text{maj}(T)} t^{\text{inv}(T)} \mathbf{x}^T = \sum_{T \in \mathbb{F}_{\mathbb{Z}_{>0}}(\alpha)} q^{\text{maj}(T)} t^{\text{inv}(T)} \mathbf{x}^T. \quad (4.1)$$

The left-hand sum over  $\mathbb{F}_{\mathbb{Z}_{>0}}(\mu)$  is the celebrated combinatorial expression for  $\tilde{H}_\mu(\mathbf{x}; q, t)$  from [\[HHL05\]](#) (see [Section 3.3.1](#)), where they show that the sum satisfies the conditions of [Proposition 2.4](#). In [\[HHL08\]](#), the rearrangement property was established by proving that the right-hand sum over  $\mathbb{F}_{\mathbb{Z}_{>0}}(\alpha)$  is also equal to  $\tilde{H}_\mu(\mathbf{x}; q, t)$ , by showing that it also satisfies the conditions of [Proposition 2.4](#).

We present a bijective proof of [\(4.1\)](#), without going through the algebraic characterization of  $\tilde{H}_\mu(\mathbf{x}; q, t)$  from [Proposition 2.4](#). The key step is [Theorem 4.12](#), which guarantees the existence of a bijection  $\phi: \mathbb{F}(\alpha) \rightarrow \mathbb{F}(s_i \cdot \alpha)$  that preserves the content, the inversion number, and the major index. An auxiliary bijection  $\psi: \mathbb{F}_\mathcal{A}(\alpha) \rightarrow \mathbb{F}_\mathcal{A}(s_i \cdot \alpha)$  is also defined, that preserves content but changes inv and maj in a controlled way, as specified in [Theorem 4.13](#).

In [Section 4.2](#), we prove [Theorem 4.15](#), which gives an identity for permuted-basement

Macdonald polynomials of the form

$$E_{\alpha}^{\sigma}(\mathbf{x}; q, t) = E_{s_i \cdot \alpha}^{\sigma s_i}(\mathbf{x}; q, t) + c_{\alpha, \sigma, i}(q, t) E_{s_i \cdot \alpha}^{\sigma}(\mathbf{x}; q, t), \quad (4.2)$$

whenever  $\alpha_i > \alpha_{i+1}$ , where  $c_{\alpha, \sigma, i}(q, t)$  is an explicit rational function in  $q$  and  $t$ .

We present a bijective proof of (4.2). We use the bijection  $\phi$  on non-augmented fillings from Section 4.1 and an auxiliary bijection  $\psi$  as a building blocks for a bijection on augmented fillings that implies (4.2).

## 4.1 Bijections on non-augmented fillings and the rearrangement property of $\tilde{H}_{\mu}$

In this section, we define the bijection  $\phi: F(\alpha) \rightarrow F(s_i \cdot \alpha)$  that preserves the statistics  $\text{inv}$  and  $\text{maj}$ . We assume  $\alpha_i \neq \alpha_{i+1}$  throughout this section, since the case  $\alpha_i = \alpha_{i+1}$  has  $\alpha = s_i \cdot \alpha$  and the identity map serves as a suitable choice for  $\phi$ .

There are two equivalent constructions of such bijections  $\phi$ . The first construction partitions the filling into blocks and decides how to transform each block based on local conditions. The second construction is a reinterpretation of an iterative row-by-row algorithm defined by Kim, Lee, and Oh [KLO22]. Each construction has its advantages, and we include both for completeness. We prove that both algorithms coincide in Section 4.1.3, and then prove that  $\phi$  satisfies the desired properties in Section 4.1.4 using the first construction.

As an auxiliary construction, we also define another bijection  $\psi: F(\alpha) \rightarrow F(s_i \cdot \alpha)$  that changes the statistics  $\text{inv}$  and  $\text{maj}$  in a controlled way, as specified in Theorem 4.13. This secondary map  $\psi$  is a crucial contribution of [KLO22], since  $\psi$  will play a central role in Section 4.2.

Both of  $\phi$  and  $\psi$  follow a very rigid structure. For a non-augmented filling  $T \in F(\alpha)$  mapped to a filling  $U \in F(s_i \cdot \alpha)$  by one of these maps, then: all columns except  $i$  and  $i+1$  are identical in  $T$  and  $U$ , that is,

$$T(j, r) = U(j, r) \quad \text{for all } j \in [n] \setminus \{i, i+1\} \text{ and all } r \in [\alpha_j]; \quad (4.3)$$

and the entries in each row  $r \in [\max(\alpha_i, \alpha_{i+1})]$  of columns  $i$  and  $i+1$  in  $T$  and  $U$  are the same, that is,

$$\{T(i, r), T(i+1, r)\} = \{U(i, r), U(i+1, r)\} \quad \text{for all } r \in [\max(\alpha_i, \alpha_{i+1})]. \quad (4.4)$$

We refer to these as the *local restrictions*.

When  $r > \min(\alpha_i, \alpha_{i+1})$  in (4.4), exactly one of the two entries involved is undefined in each filling, so the condition should be interpreted as stating that the defined entry in  $T$  is copied to the corresponding defined position in  $U$ .

By requiring that  $\phi$  and  $\psi$  satisfy these local restrictions, the only remaining freedom in determining  $\phi$  and  $\psi$  is to decide, for each  $r \in [\min(\alpha_i, \alpha_{i+1})]$ , whether to set

$$\begin{aligned} (U(i, r), U(i+1, r)) &= (T(i, r), T(i+1, r)) \text{ or} \\ (U(i, r), U(i+1, r)) &= (T(i+1, r), T(i, r)). \end{aligned}$$

In the first case, we say the entries of  $T$  and  $U$  in row  $r$  of columns  $i$  and  $i+1$  are *not swapped*; in the second case, we say they are *swapped*. Below, we present two different algorithms to decide, for each  $r \in [\min(\alpha_i, \alpha_{i+1})]$ , whether the entries in row  $r$  of columns  $i$  and  $i+1$  are swapped or not swapped. See Figure 4.1 for an illustration of the local restrictions.

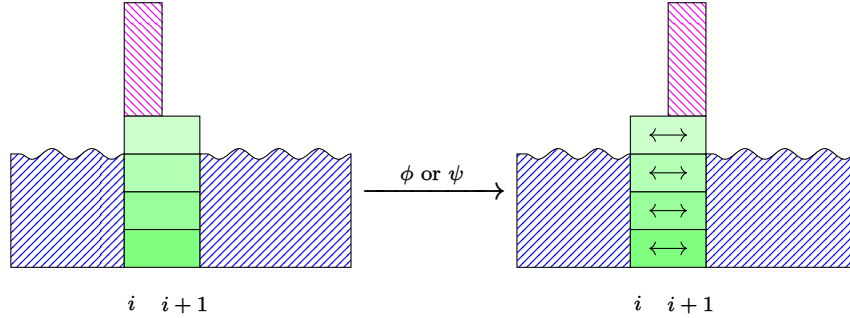


Figure 4.1: Illustration of the local restrictions for  $\phi$  and  $\psi$ . The blue region ( $\square$ ) indicates the fixed columns  $j \notin \{i, i+1\}$  from (4.3). The magenta region ( $\square$ ) shows the rows  $r > \min(\alpha_i, \alpha_{i+1})$  where the unique defined entry is copied. The green region ( $\square$ ) marks the rows  $r \leq \min(\alpha_i, \alpha_{i+1})$  where the entries may be kept or swapped according to (4.4).

#### 4.1.1 Block structure algorithm

Let  $h = \min(\alpha_i, \alpha_{i+1})$ . For the block structure algorithm, we split the rows  $[h]$  into intervals, called *blocks*, and then decide, for each block, whether to swap all its rows or not swap all its rows.

To define the blocks, we define a criterion for when two consecutive rows are grouped together. Since our goal is to construct bijections that preserve the inversion statistic  $\text{inv}$  and inversion triples occur between two consecutive rows, the condition we use is based on the behavior of  $\chi$  on four elements. Before proceeding, we prove some properties of  $\chi$  when applied to four elements  $a, b, c, d \in \mathcal{A}$ .

**Proposition 4.1.** *Let  $a, b, c, d \in \mathcal{A}$  be four elements of an alphabet  $\mathcal{A}$ . Then,*

$$\chi_{ad} + \chi_{bc} - \chi_{ac} - \chi_{bd} \in \{-1, 0, 1\}.$$

*Proof.* Since  $\chi_{xy} \in \{0, 1\}$  for all  $x, y \in \mathcal{A}$ , it's immediate that the expression only take values in  $\{-2, -1, 0, 1, 2\}$ . For its absolute value to be 2, we need  $\chi_{ad} = \chi_{bc} = 1 - \chi_{ac} = 1 - \chi_{bd}$ . This implies that  $a \geq d \geq b \geq c \geq a$  or  $a \leq d \leq b \leq c \leq a$ ; either way, we get  $a = b = c = d$ , for which the expression evaluates to 0, a contradiction. Thus, the expression only take values in  $\{-1, 0, 1\}$ .  $\square$

For distinct elements  $a, b, c, d \in \mathcal{A}$ , the value of  $\chi_{ad} + \chi_{bc} - \chi_{ac} - \chi_{bd}$  equals 0 if and only if, when placing the four elements on a line in increasing order and drawing arcs between  $a$  and  $b$ , and between  $c$  and  $d$ , the arcs do not cross. See Figure 4.2 for two examples. We have Propositions 4.2 and 4.3 that give some other equivalent characterizations of whether the expression is 0 or  $\pm 1$ , which work even when the elements are not distinct.

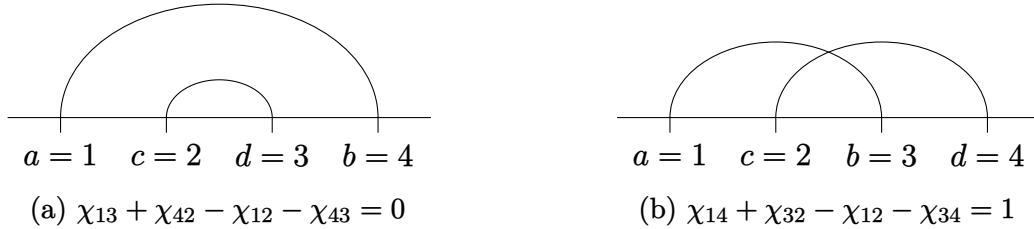


Figure 4.2: Interpretation of  $\chi_{ad} + \chi_{bc} - \chi_{ac} - \chi_{bd}$  for distinct  $a, b, c, d$  as crossing or non-crossing arcs.

**Proposition 4.2.** *Let  $a, b, c, d \in \mathcal{A}$ . The following statements are equivalent:*

- (I)  $\chi_{ad} + \chi_{bc} - \chi_{ac} - \chi_{bd} = 0$ ;
- (II)  $\chi_{abc} = \chi_{abd}$ ;
- (III)  $\chi_{acd} = \chi_{bcd}$ .

*Proof.* Expanding the definition of  $\chi_{xyz}$ , Item (II) is equivalent to  $\chi_{ab} + \chi_{bc} - \chi_{ac} = \chi_{ab} + \chi_{bd} - \chi_{ad}$ , which rearranges to Item (I). The equivalence between Item (III) and Item (I) is analogous.  $\square$

**Proposition 4.3.** *Let  $a, b, c, d \in \mathcal{A}$ . The following statements are equivalent to the negation of those in [Proposition 4.2](#):*

- (IV)  $\chi_{ad} + \chi_{bc} - \chi_{ac} - \chi_{bd} = \pm 1$ ;
- (V)  $\chi_{ad} + \chi_{bc} + \chi_{ac} + \chi_{bd}$  is odd;
- (VI)  $\chi_{abc} = \chi_{bcd} = \chi_{adc}$  and  $a \neq b$  and  $c \neq d$ ;

*Proof.* The equivalence between [Item \(IV\)](#) and the negation of [Item \(I\)](#) is immediate from [Proposition 4.1](#), as well as the equivalence between [Item \(IV\)](#) and [Item \(V\)](#) using modulo 2. Computational verification shows that [Item \(IV\)](#) and [Item \(VI\)](#) are satisfied for exactly the same relative configurations of  $a, b, c, d$ , namely one of the following 40 configurations:

$$\begin{aligned}
& a \overset{\pm}{=} c > b \overset{\pm}{=} d, \quad a \overset{\pm}{=} c > b \overset{\pm}{=} d, \quad b \overset{\pm}{=} d > a \overset{\pm}{=} c, \quad b \overset{\pm}{=} d > a \overset{\pm}{=} c, \\
& a \overset{\pm}{=} c > b > d, \quad a \overset{\pm}{=} c > d > b, \quad b > a \overset{\pm}{=} c > d, \quad b > d > a \overset{\pm}{=} c, \\
& d > a \overset{\pm}{=} c > b, \quad d > b > a \overset{\pm}{=} c, \quad a \overset{\pm}{=} d > b \overset{\pm}{=} c, \quad a \overset{\pm}{=} d > b \overset{\pm}{=} c, \\
& b \overset{\pm}{=} c > a \overset{\pm}{=} d, \quad b \overset{\pm}{=} c > a \overset{\pm}{=} d, \quad a > b \overset{\pm}{=} c > d, \quad a > d > b \overset{\pm}{=} c, \\
& b \overset{\pm}{=} c > a > d, \quad b \overset{\pm}{=} c > d > a, \quad d > a > b \overset{\pm}{=} c, \quad d > b \overset{\pm}{=} c > a, \\
& a \overset{\pm}{=} d > b > c, \quad a \overset{\pm}{=} d > c > b, \quad b > a \overset{\pm}{=} d > c, \quad b > c > a \overset{\pm}{=} d, \\
& c > a \overset{\pm}{=} d > b, \quad c > b > a \overset{\pm}{=} d, \quad a > b \overset{\pm}{=} d > c, \quad a > c > b \overset{\pm}{=} d, \\
& b \overset{\pm}{=} d > a > c, \quad b \overset{\pm}{=} d > c > a, \quad c > a > b \overset{\pm}{=} d, \quad c > b \overset{\pm}{=} d > a, \\
& a > c > b > d, \quad a > d > b > c, \quad b > c > a > d, \quad b > d > a > c, \\
& c > a > d > b, \quad c > b > d > a, \quad d > a > c > b, \quad d > b > c > a,
\end{aligned}$$

where  $x \overset{\pm}{=} y$  means  $x = y \in \mathcal{A}_+$  and  $x \overset{\pm}{=} y$  means  $x = y \in \mathcal{A}_-$ . □

Let  $r \in [h - 1]$ . Let  $a = T(i, r + 1)$ ,  $b = T(i + 1, r + 1)$ ,  $c = T(i, r)$ , and  $d = T(i + 1, r)$ . We say that rows  $r$  and  $r + 1$  of  $T$  are *not intertwined* if  $a, b, c, d$  satisfy the equivalent conditions of [Proposition 4.2](#), and *intertwined* if they satisfy the equivalent conditions of [Proposition 4.3](#). Note that the condition is invariant under swapping  $a$  with  $b$ , as well as invariant under swapping  $c$  with  $d$ .

The interval of rows  $[h]$  is uniquely partitioned into intervals, called *blocks*, such that two consecutive rows in the same block are intertwined, and two consecutive rows in different

|   |     |       |  |
|---|-----|-------|--|
| 7 | 3   |       |  |
| 6 | 2   | 5     |  |
| 5 | 4   | 1     |  |
| 4 | 5   | 8     |  |
| 3 | 1   | 6     |  |
| 2 | 3   | 7     |  |
| 1 | 4   | 6     |  |
|   | $i$ | $i+1$ |  |

} third block  
 } second block  
 } first block

Figure 4.3: Columns  $i$  and  $i + 1$  of a filling  $T$  with three blocks.

blocks are not intertwined. The partition of  $[h]$  into blocks is called the *block structure (of columns  $i$  and  $i + 1$ ) of the filling  $T$* . Figure 4.3 shows a filling and its blocks.

Note that, if the two entries of row  $r$  of columns  $i$  and  $i + 1$  in  $T$  are equal, then rows  $r - 1$  and  $r$  are not intertwined, and rows  $r$  and  $r + 1$  are not intertwined (if these rows exist). Therefore, such row  $r$  forms a block of size 1 by itself, and swapping or not swapping it has the same effect.

Let's interpret the intertwining condition via **Item (VI)** of **Proposition 4.3**. The number  $\chi_{abc}$  is the indicator of whether the triple  $(i, r + 1), (i + 1, r + 1), (i, r)$  is a Type A inversion triple in  $T$ . The number  $\chi_{bcd}$  is the indicator of whether the triple  $(i + 1, r + 1), (i, r), (i + 1, r)$  is a Type B inversion triple in  $U$  if rows  $r$  and  $r + 1$  are not swapped. The number  $\chi_{adc}$  is the indicator of whether the triple  $(i, r + 1), (i + 1, r), (i, r)$  is a Type B inversion triple in  $U$  if rows  $r$  and  $r + 1$  are swapped.

Therefore, if we require that rows in the same block are either all swapped or all not swapped in  $U$ , then we guarantee that the number of inversion triples within each block is the same in  $T$  and  $U$ , regardless of whether the block is swapped or not swapped.

The remaining freedom is to decide, for each block, whether to swap it or not swap it. We use this freedom to guarantee that the number of inversion triples between different blocks is also the same in  $T$  and  $U$ .

Consider a block  $[r_1, r_2]$  of  $T$ . Let

$$\begin{aligned}
 e_1 &= T(i, r_2 + 1), & e_2 &= T(i + 1, r_2 + 1), \\
 a &= T(i, r_2), & b &= T(i + 1, r_2), \\
 c &= T(i, r_1), & d &= T(i + 1, r_1), \\
 f_1 &= T(i, r_1 - 1), \text{ and } & f_2 &= T(i + 1, r_1 - 1).
 \end{aligned} \tag{4.5}$$

Edge cases are when the block is the bottommost one, i.e.  $r_1 = 1$ , in which case  $f_1 = f_2 = \infty$  by convention; and when the block is the topmost one, i.e.  $r_2 = h$ , in which case one of  $e_1, e_2$  is undefined. For any well-defined  $e \in \{e_1, e_2\}$  and  $f \in \{f_1, f_2\}$ , we say that  $(a, b, c, d, e, f)$  are *signature entries* of the block. We remark that a block may have multiple choices of signature entries. See [Figure 4.4](#) for an illustration of the entries  $a, b, c, d, e_1, e_2, f_1, f_2$ .

|          |       |
|----------|-------|
| $e_1$    | $e_2$ |
| $a$      | $b$   |
| $\vdots$ |       |
| $c$      | $d$   |
| $f_1$    | $f_2$ |

Figure 4.4: A block of a filling  $T$ , with the entries  $a, b, c, d, e_1, e_2, f_1, f_2$  indicated. Any of the four tuples  $(a, b, c, d, e_1, f_1)$ ,  $(a, b, c, d, e_1, f_2)$ ,  $(a, b, c, d, e_2, f_1)$ , and  $(a, b, c, d, e_2, f_2)$  are signature entries of the block, as long as the entries are well-defined.

If the block is not the bottommost one, then rows  $r_1 - 1$  and  $r_1$  are not intertwined. Therefore,  $c, d, f_1, f_2$  satisfy the equivalent conditions of [Proposition 4.2](#), in particular,  $\chi_{cdf_1} = \chi_{cdf_2}$ . If the block is the bottommost one, then we have  $\chi_{cdf_1} = \chi_{cdf_2} = \chi_{cd}$  by convention (as expected from  $f_1 = f_2 = \infty$ ). Therefore, in all cases, the value  $\chi_{cdf}$  doesn't depend on the choice of  $f \in \{f_1, f_2\}$ .

If the block is not the topmost one, then rows  $r_2$  and  $r_2 + 1$  are not intertwined. Therefore,  $a, b, e_1, e_2$  satisfy the equivalent conditions of [Proposition 4.2](#), in particular,  $\chi_{e_1ab} = \chi_{e_2ab}$ . If the block is the topmost one, then exactly one of  $e_1$  and  $e_2$  is defined, so there is no ambiguity in choosing  $e$  and in computing  $\chi_{eab}$ . Therefore, in all cases, the value  $\chi_{eab}$  doesn't depend on the choice of  $e \in \{e_1, e_2\}$  (as long as  $e$  is defined).

In all cases, the values  $\chi_{cdf}$  and  $\chi_{eab}$  are uniquely defined for a block, independently of the choice of signature entries  $(a, b, c, d, e, f)$ .

**Definition 4.4** (The bijection  $\phi$  via blocks). Let  $T \in F(\alpha)$ . We define  $U = \phi(T) \in F(s_i \cdot \alpha)$  by requiring the local restrictions (i.e., [\(4.3\)](#) and [\(4.4\)](#)) and requiring that, for each block of  $T$  with signature entries  $(a, b, c, d, e, f)$ , the rows of the block are swapped in  $U$  if and only if

$$\chi_{cdf} \neq \chi_{eab}.$$

**Definition 4.5** (The bijection  $\psi$  via blocks). Let  $T \in F(\alpha)$ . We define  $U = \psi(T) \in F(s_i \cdot \alpha)$  by requiring the local restrictions (i.e., [\(4.3\)](#) and [\(4.4\)](#)) and requiring that, for each block

of  $T$  with signature entries  $(a, b, c, d, e, f)$ , the rows of the block are swapped in  $U$  if and only if

$$\chi_{cdf} \neq \chi_{eab} \quad \text{xor} \quad (\text{the block is the bottommost block}),$$

where xor denotes the exclusive or operation, whose result is true if and only if an odd number of its operands are true.

To make things explicitly, both  $\phi$  and  $\psi$  swap a non-bottommost block if and only if  $\chi_{cdf} \neq \chi_{eab}$ . The difference is in the bottommost block, which  $\phi$  swaps if and only if  $\chi_{cd} \neq \chi_{eab}$ , while  $\psi$  swaps it if and only if  $\chi_{cd} = \chi_{eab}$ .

Figure 4.5 shows the image of the filling in Figure 4.3 under the map  $\phi$ , following this definition.

|   |     |       |                                                         |
|---|-----|-------|---------------------------------------------------------|
| 7 |     | 3     |                                                         |
| 6 | 2   | 5     | } not swapped, since $\chi_{415} = 1 = \chi_{325}$      |
| 5 | 4   | 1     |                                                         |
| 4 | 8   | 5     | } swapped, since $\chi_{258} = 0 \neq 1 = \chi_{374}$   |
| 3 | 6   | 1     |                                                         |
| 2 | 7   | 3     | } not swapped, since $\chi_{46\infty} = 0 = \chi_{746}$ |
| 1 | 4   | 6     |                                                         |
|   | $i$ | $i+1$ |                                                         |

Figure 4.5: Columns  $i$  and  $i + 1$  of the filling  $U$  derived from the filling  $T$  in Figure 4.3.

In Propositions 4.8 and 4.9 and Theorem 4.12, we prove that  $\phi$  is a bijection that preserves content, inv, and maj. Let's highlight the parts that are straightforward from the definition via blocks.

We verify that both  $\phi$  and  $\psi$  are bijections by noticing that  $\phi(\phi(T)) = T$  and  $\psi(\psi(T)) = T$ . The local restrictions guarantee that  $\text{content}(T) = \text{content}(U)$ .

Moreover, it is straightforward to verify that the number of inversion triples within columns  $i$  and  $i + 1$  is the same in  $T$  and  $\phi(T)$ , since the number of inversion triples within each block is preserved, and the choice of swapping or not swapping each block is made precisely to ensure that the number of inversion triples between different blocks is also preserved. It is not hard to see that this implies that  $\text{inv}(T) = \text{inv}(\phi(T))$ .

The fact that  $\text{maj}(T) = \text{maj}(\phi(T))$  is less straightforward, since we haven't made any direct attempt to control the major index in our definition of  $\phi$ . We prove this in Theorem 4.12.

### 4.1.2 Row-by-row algorithm

It turns out that the bijections  $\phi$  and  $\psi$  are the same as the bijections defined in [KLO22]. In this subsection, we present a reinterpretation of the algorithm from [KLO22]. Instead of the twelve cases in [KLO22], we compactify the algorithm into four cases that are characterized by the relationships among the indicators  $\chi_{ab}, \chi_{ac}, \chi_{ad}, \chi_{bc}, \chi_{bd}, \chi_{cd}$ .

Let  $h = \min(\alpha_i, \alpha_{i+1})$ . Let  $T \in F(\alpha)$ . We describe the algorithm to compute  $U = \phi(T)$  or  $U = \psi(T)$ . The algorithm uses a *coin* with two sides,  $\phi$  and  $\psi$ , as a bookkeeping device. Each row from 1 to  $h$  is, at some point of the algorithm, assigned a *coin value* in  $\{\phi, \psi\}$ . Initially, assign the coin value of row 1 to be  $\phi$  if computing  $U = \phi(T)$ , or  $\psi$  if computing  $U = \psi(T)$ . We denote by  $\text{coin}_r$  the coin value assigned to row  $r$ .

In addition to the coin value, each row from 1 to  $h$  is labeled from  $\{N, S, C\}$ . A label of  $N$  means the entries in row  $r$  of columns  $i$  and  $i + 1$  in  $T$  and  $U$  are *not* swapped; a label of  $S$  means they are *swapped*; and a label of  $C$  indicates that the behavior for row  $r$  should *copy* the behavior for the row above it. We denote by  $\text{label}_r$  the label assigned to row  $r$ .

We process the rows iteratively from row 1 up to row  $h$ . For each  $r \in \{1, 2, \dots, h - 1\}$ , we assume that the coin value  $\text{coin}_r$  has already been assigned. Let  $a = T(i, r + 1)$ ,  $b = T(i + 1, r + 1)$ ,  $c = T(i, r)$ , and  $d = T(i + 1, r)$ . Note that  $\chi_{ad} = \chi_{bc} \neq \chi_{ac} = \chi_{bd}$  is impossible by transitivity, thus  $\chi_{ad} + \chi_{bc} - \chi_{ac} - \chi_{bd} \in \{1, 0, -1\}$ .

- Case I.** If  $\chi_{ad} + \chi_{bc} - \chi_{ac} - \chi_{bd} = \pm 1$ , (cases 3, 4, 9, 10 in [KLO22]), then set  $\text{label}_r = C$ . Moreover, set  $\text{coin}_{r+1} = \text{coin}_r$  if  $\chi_{ab} = \chi_{cd}$  (cases 3, 9 in [KLO22]), and set  $\text{coin}_{r+1} \neq \text{coin}_r$  if  $\chi_{ab} \neq \chi_{cd}$  (cases 4, 10 in [KLO22]).
- Case II.** If  $\chi_{ac} = \chi_{ad} = \chi_{bc} = \chi_{bd}$  (cases 1, 2, 11, 12 in [KLO22]), then set  $\text{label}_r = N$  if  $\text{coin}_r = \phi$ , and set  $\text{label}_r = S$  if  $\text{coin}_r = \psi$ . Moreover, set  $\text{coin}_{r+1} = \phi$ .
- Case III.** If  $\chi_{ac} = \chi_{bc} \neq \chi_{ad} = \chi_{bd}$  (cases 7, 8 in [KLO22]), then set  $\text{label}_r = S$  if  $\text{coin}_r = \phi$ , and set  $\text{label}_r = N$  if  $\text{coin}_r = \psi$ . Moreover, set  $\text{coin}_{r+1} = \phi$ .
- Case IV.** If  $\chi_{ac} = \chi_{ad} \neq \chi_{bc} = \chi_{bd}$  (cases 5, 6 in [KLO22]), then set  $\text{label}_r = N$  if  $\text{coin}_r = \phi$ , and set  $\text{label}_r = S$  if  $\text{coin}_r = \psi$ . Moreover, set  $\text{coin}_{r+1} = \psi$ .

Finally, we process row  $r = h$ , assuming that the coin value  $\text{coin}_h$  has already been assigned. Let  $a = T(i, h + 1)$  if  $h = \alpha_{i+1} < \alpha_i$  or  $a = T(i + 1, h + 1)$  if  $h = \alpha_i < \alpha_{i+1}$ , i.e.,  $a$  is the only entry in row  $h + 1$  and in one of the two columns  $i$  or  $i + 1$ . Let  $c = T(i, h)$ , and let  $d = T(i + 1, h)$ .

- Base I.** If  $\chi_{ac} \neq \chi_{ad}$ , then set  $\text{label}_r = S$  if  $\text{coin}_r = \phi$ , and set  $\text{label}_r = N$  if  $\text{coin}_r = \psi$ .
- Base II.** If  $\chi_{ac} = \chi_{ad}$ , then set  $\text{label}_r = N$  if  $\text{coin}_r = \phi$ , and set  $\text{label}_r = S$  if  $\text{coin}_r = \psi$ .



$2, \dots, h$ . Once all labels are determined, construct  $U$  in a final pass by applying the rules for  $\mathbf{N}, \mathbf{S}, \mathbf{C}$  as before.

It is straightforward to verify that both the iterative and recursive descriptions of the algorithm are equivalent. This recursive interpretation is particularly useful for proving properties about the bijections  $\phi$  and  $\psi$ , as it allows us to use induction on the number of rows in the filling.

### 4.1.3 Equivalence of the two definitions

In this subsection, we show that the block structure algorithm and the row-by-row algorithm define the same bijections  $\phi$  and  $\psi$ . We achieve this goal by applying the row-by-row algorithm and analyzing its behavior on each block.

**Proposition 4.7.** *The definitions of  $\phi$  and  $\psi$  via blocks (Definitions 4.4 and 4.5) are equivalent to the definitions of  $\phi$  and  $\psi$  via the row-by-row algorithm (Definition 4.6), respectively.*

*Proof.* Let  $\phi$  be defined as in Definition 4.6. We show that  $\phi$  satisfies the conditions in Definition 4.4. Let  $T \in \mathbf{F}(\alpha)$ . Consider a block  $[r_1, r_2]$  of  $T$  with signature entries  $(a, b, c, d, e, f)$ .

First, if  $r_1 > 1$ , consider the row-by-row algorithm at row  $r_1 - 1$ , and let's compute the coin value for row  $r_1$ . Since  $r_1 - 1$  and  $r_1$  are not intertwined, we are in Case II, III, or IV. Therefore, we have

$$\text{coin}_{r_1} = \textcircled{\phi} \quad \text{if and only if} \quad \chi_{cf} - \chi_{df} = 0. \quad (4.6)$$

If  $r_1 = 1$ , then by definition,  $\text{coin}_{r_1} = \textcircled{\phi}$ , and by convention,  $\chi_{cf} - \chi_{df} = 0$  since  $f = \infty$ . Thus, (4.6) holds independently of whether  $r_1 = 1$  or  $r_1 > 1$ .

Second, for each  $r \in \{r_1, r_1 + 1, \dots, r_2 - 1\}$ , since rows  $r$  and  $r + 1$  are intertwined, the row-by-row algorithm falls into Case I,  $\text{label}_r = \mathbf{C}$ . Thus, the rows in the block all copy the behavior of row  $r_2$  and the label of row  $r_2$  determines whether the block is swapped or not swapped, that is, the entire block is swapped if and only if  $\text{label}_{r_2} = \mathbf{S}$ .

Moreover,  $\text{coin}_{r+1} = \text{coin}_r$  if and only if  $\chi_{T(i,r+1)T(i+1,r+1)} = \chi_{T(i,r)T(i+1,r)}$ . Stringing these together, we have

$$\text{coin}_{r_2} = \text{coin}_{r_1} \quad \text{if and only if} \quad \chi_{ab} - \chi_{cd} = 0. \quad (4.7)$$

Third, consider the row-by-row algorithm at row  $r_2$ , and let's compute the label for row  $r_2$ . If  $r_2 = h$ , then we are in Base I or II, otherwise, rows  $r_2$  and  $r_2 + 1$  are not intertwined, and thus we are in Case II, III, or IV. In all of these cases, we have

$$\text{label}_{r_2} = \mathbf{S} \quad \text{if and only if} \quad (\chi_{ea} - \chi_{eb} = 0) \text{ xor } (\text{coin}_{r_2} = \psi), \quad (4.8)$$

where xor denotes the exclusive or operation, whose result is true if and only if an odd number of its operands are true.

Finally, using that  $\text{coin}_{r_2} = \psi$  if and only if  $(\text{coin}_{r_2} = \text{coin}_{r_1}) \text{ xor } (\text{coin}_{r_1} = \phi)$ , we have

$$\begin{aligned} \text{label}_{r_2} = \mathbf{S} &\iff (\chi_{ea} - \chi_{eb} = 0) \text{ xor } (\text{coin}_{r_2} = \text{coin}_{r_1}) \text{ xor } (\text{coin}_{r_1} = \phi) \\ &\iff (\chi_{ea} - \chi_{eb} = 0) \text{ xor } (\chi_{ab} - \chi_{cd} = 0) \text{ xor } (\chi_{cf} - \chi_{df} = 0) \\ &\iff \chi_{ea} - \chi_{eb} + \chi_{ab} - \chi_{cd} + \chi_{cf} - \chi_{df} \equiv 1 \pmod{2} \\ &\iff \chi_{eab} \neq \chi_{cdf}. \end{aligned}$$

which is the same swapping condition as in [Definition 4.4](#). Therefore, the two definitions of  $\phi$  are equivalent.

The analysis for  $\psi$  is analogous. For  $\psi$ , (4.6) becomes

$$\text{coin}_{r_1} = \phi \quad \text{if and only if} \quad (\chi_{cf_1} - \chi_{df_1} = 0) \text{ xor } (r_1 = 1),$$

while (4.7) and (4.8) remain unchanged. Therefore, the conclusion becomes

$$\text{label}_{r_2} = \mathbf{S} \iff (\chi_{e_1ab} \neq \chi_{cdf_1}) \text{ xor } (r_1 = 1),$$

which is the same swapping condition as in [Definition 4.5](#). Therefore, the two definitions of  $\psi$  are equivalent.  $\square$

#### 4.1.4 Properties of $\phi$ and $\psi$ via the block structure algorithm

As an abuse of notation, let  $\phi$  denote both the function  $\phi: F(\alpha) \rightarrow F(s_i \cdot \alpha)$  and the function  $\phi: F(s_i \cdot \alpha) \rightarrow F(\alpha)$ . The intended use of  $\phi$  is always clear from the shape of the input filling.

**Proposition 4.8.** *The maps  $\phi: F(\alpha) \rightarrow F(s_i \cdot \alpha)$  and  $\phi: F(s_i \cdot \alpha) \rightarrow F(\alpha)$  are inverses of each other. Moreover,  $\phi$  is a bijection.*

*Proof.* Let  $T \in F(\alpha)$ . Let  $U = \phi(T)$  and  $T' = \phi(U)$ . Since the condition of two rows being intertwined only depends on the set of entries in two rows, not the order of the entries, the block structure of  $T$ ,  $U$ , and  $T'$  are the same. Let  $[r_1, r_2]$  be a block of  $T \in F(\alpha)$  with signature entries  $(a, b, c, d, e, f)$ .

If  $a = b$  or  $c = d$ , then the block consists of a single row, since a row of equal entries cannot be intertwined with any other row. Therefore, the columns  $i$  and  $i + 1$  of this row are identical in  $T$ ,  $U$ , and  $T'$ . Thus, we may assume that  $a \neq b$  and  $c \neq d$ .

By [Definition 4.4](#), the rows in  $[r_1, r_2]$  are swapped between  $T$  and  $U$  if and only if  $\chi_{cdf} \neq \chi_{eab}$ .

If this condition is false, then the rows in  $[r_1, r_2]$  are not swapped between  $T$  and  $U$ . Therefore,  $[r_1, r_2]$  is a block of  $U$  with signature entries  $(a, b, c, d, e, f)$ , regardless of whether the blocks immediately above or below are swapped or not. Now, again by [Definition 4.4](#), the rows in  $[r_1, r_2]$  are swapped between  $U$  and  $T'$  if and only if  $\chi_{cdf} \neq \chi_{eab}$ , which is false. Therefore, the rows in  $[r_1, r_2]$  are not swapped between  $U$  and  $T'$ , and consequently,  $T$  and  $T'$  are identical in this block.

If this condition is true, then the rows in  $[r_1, r_2]$  are swapped between  $T$  and  $U$ . Therefore,  $[r_1, r_2]$  is a block of  $U$  with signature entries  $(b, a, d, c, e, f)$ , regardless of whether the blocks immediately above or below are swapped or not. Now, by [Definition 4.4](#), the rows in  $[r_1, r_2]$  are swapped between  $U$  and  $T'$  if and only if  $\chi_{dcf} \neq \chi_{eba}$ . Because  $a \neq b$  and  $c \neq d$ , we have  $\chi_{dcf} = 1 - \chi_{cdf}$  and  $\chi_{eba} = 1 - \chi_{eab}$ . Therefore,  $\chi_{dcf} \neq \chi_{eba}$  is equivalent to  $\chi_{cdf} \neq \chi_{eab}$ , which is true. Thus, the rows in  $[r_1, r_2]$  are swapped between  $U$  and  $T'$ . Since the rows in  $[r_1, r_2]$  are also swapped between  $T$  and  $U$ , then  $T$  and  $T'$  are identical in this block.

Therefore, for each block of  $T$ , the corresponding block of  $T'$  is identical. All other entries of  $T$  and  $T'$  outside of blocks are equal by the local restrictions. Thus,  $T = T'$ , and consequently  $\phi: F(\alpha) \rightarrow F(s_i \cdot \alpha)$  is the inverse of  $\phi: F(s_i \cdot \alpha) \rightarrow F(\alpha)$ . Since  $\phi$  has an inverse, it is a bijection.  $\square$

**Proposition 4.9.** *The maps  $\psi: F(\alpha) \rightarrow F(s_i \cdot \alpha)$  and  $\psi: F(s_i \cdot \alpha) \rightarrow F(\alpha)$  are inverses of each other. Moreover,  $\psi$  is a bijection.*

The proof of [Proposition 4.9](#) is analogous to the proof of [Proposition 4.8](#), with a minor modification to account for the different swapping condition of the bottommost block.

In [Theorems 4.12](#) and [4.13](#), we describe how the bijections  $\phi$  and  $\psi$  interact with the statistics  $\text{inv}$  and  $\text{maj}$ . Before then, we need [Lemmas 4.10](#) and [4.11](#) that describe certain cancelation properties of the leg and arm statistics.

Given a box  $u = (j, r)$ , let  $s_i \cdot u = (s_i(j), r)$  be the box obtained by applying the transposition  $s_i$  to the column index of  $u$ .

**Lemma 4.10.** *Let  $\alpha$  be a composition. Let  $u, v \in \mathbf{dg}(\alpha)$  be boxes in columns  $i$  and  $i + 1$ , respectively, such that  $u, v \in \mathbf{dg}(s_i \cdot \alpha)$ . Then,*

$$\mathbf{leg}_\alpha(u) + \mathbf{leg}_\alpha(v) = \mathbf{leg}_{s_i \cdot \alpha}(u) + \mathbf{leg}_{s_i \cdot \alpha}(v).$$

*Proof.* Let  $u = (i, r)$  and  $v = (i + 1, r')$ . Recall that  $\mathbf{leg}_\alpha(i, r) = \alpha_i - r$ . Therefore, both quantities are equal to  $\alpha_i + \alpha_{i+1} - r - r'$ .  $\square$

**Lemma 4.11.** *Let  $\alpha$  be a composition. Let  $u, v \in \mathbf{dg}(\alpha)$  be boxes in columns  $i$  and  $i + 1$ , respectively, such that  $u, v \in \mathbf{dg}(s_i \cdot \alpha)$ . Then,*

$$\mathbf{arm}_\alpha(u) + \mathbf{arm}_\alpha(v) = \mathbf{arm}_{s_i \cdot \alpha}(u) + \mathbf{arm}_{s_i \cdot \alpha}(v).$$

*Proof.* Let  $u = (i, r)$  and  $v = (i + 1, r')$ . Recall that

$$\mathbf{arm}_\alpha(i, r) = |\{j < i : \alpha_j \in [r - 1, \alpha_i - 1]\}| + |\{j > i : \alpha_j \in [r, \alpha_i]\}|.$$

Let  $\chi_S(x)$  be the indicator function that is 1 if  $x \in S$  and 0 otherwise. Therefore, for all both quantities, the contribution from  $j < i$  is

$$\chi_{[r-1, \alpha_i-1]}(\alpha_j) + \chi_{[r', \alpha_{i+1}-1]}(\alpha_j) = \chi_{[r-1, \alpha_{i+1}-1]}(\alpha_j) + \chi_{[r', \alpha_i-1]}(\alpha_j),$$

the contribution from  $j > i + 1$  is

$$\chi_{[r, \alpha_i]}(\alpha_j) + \chi_{[r'-1, \alpha_{i+1}]}(\alpha_j) = \chi_{[r', \alpha_i]}(\alpha_j) + \chi_{[r-1, \alpha_{i+1}]}(\alpha_j),$$

and the combined contribution from  $j = i$  and  $j = i + 1$  is one. Hence, all three quantities are equal.  $\square$

**Theorem 4.12** ([KLO22, Proposition 4.3]). *The bijection  $\phi$  preserves the statistics  $\mathbf{maj}$  and  $\mathbf{inv}$ ; that is, for all  $T \in \mathbf{F}(\alpha)$ , if  $U = \phi(T)$ , then*

$$\mathbf{maj}(T) = \mathbf{maj}(U) \tag{4.9}$$

and

$$\mathbf{inv}(T) = \mathbf{inv}(U). \tag{4.10}$$

Kim, Lee, and Oh [KLO22] prove Theorem 4.12 using the row-by-row algorithm. We provide an alternative proof using the block structure algorithm, which is split into two parts: (4.9) and (4.10).

*Proof of (4.9).* Without loss of generality, assume that  $\alpha_i > \alpha_{i+1}$ . Let  $T \in F(\alpha)$  and  $U = \phi(T)$ . We first study how  $\text{Des}(T)$  and  $\text{Des}(U)$  relate to each other.

Clearly, for all boxes  $u$  not in columns  $i$  or  $i + 1$ , we have  $u \in \text{Des}(T)$  if and only if  $u \in \text{Des}(U)$ , by (4.3). Since  $\text{leg}_\alpha(u) = \text{leg}_{s_i \cdot \alpha}(u)$  for such boxes  $u$ , these descents contribute equally to  $\text{maj}(T)$  and  $\text{maj}(U)$ . Moreover, for all boxes  $u = (i, r)$  with  $r \geq \alpha_{i+1} + 2$ , we have  $u \in \text{Des}(T)$  if and only if  $s_i \cdot u \in \text{Des}(U)$ , by (4.4). Note that  $\text{leg}_\alpha(u) = \alpha_i - r = \text{leg}_{s_i \cdot \alpha}(s_i \cdot u)$  for such boxes  $u$ , so these descents contribute equally to  $\text{maj}(T)$  and  $\text{maj}(U)$ . Therefore, it suffices to show that the contribution to  $\text{maj}$  from the boxes in columns  $i$  and  $i + 1$  at rows 2 through  $\alpha_{i+1} + 1$  are the same in both  $T$  and  $U$ .

We say that a row  $r \in [2, \alpha_{i+1}]$  is *balanced* in  $T$  if both or neither of the boxes  $(i, r)$  and  $(i + 1, r)$  are descents of  $T$ ; is *left-unbalanced* in  $T$  if  $(i, r)$  is a descent of  $T$  but  $(i + 1, r)$  is not; is *right-unbalanced* in  $T$  if  $(i + 1, r)$  is a descent of  $T$  but  $(i, r)$  is not.

Let  $r \in [2, \alpha_{i+1}]$  be a balanced row in  $T$ . Let  $a = T(i, r)$ ,  $b = T(i + 1, r)$ ,  $c = T(i, r - 1)$ , and  $d = T(i + 1, r - 1)$ . The number of descents at row  $r$  in  $T$  is  $\chi_{ac} + \chi_{bd} \in \{0, 2\}$ . On the one hand, if rows  $r$  and  $r - 1$  have the same swapping behavior in  $T$  and  $U$ , then number of descents at row  $r$  in  $U$  is also  $\chi_{ac} + \chi_{bd}$ , the same as in  $T$ . On the other hand, if rows  $r$  and  $r - 1$  have different swapping behavior between  $T$  and  $U$ , then  $r$  and  $r - 1$  are not intertwined in  $T$ , so  $\chi_{ac} + \chi_{bd} = \chi_{ad} + \chi_{bc}$ . In this case, the number of descents at row  $r$  in  $U$  is given by  $\chi_{ad} + \chi_{bc}$ , which is equal to  $\chi_{ac} + \chi_{bd}$ , the same as in  $T$ . Therefore, for any balanced row  $r$ , the number of descents at row  $r$  is the same in  $T$  and  $U$  (either 0 or 2). Moreover, by Lemma 4.10, these descents contribute equally to  $\text{maj}(T)$  and  $\text{maj}(U)$ .

Therefore, it suffices to show that the contribution to  $\text{maj}$  from unbalanced rows in  $T$  is the same as the contribution from unbalanced rows in  $U$ . Each of the unbalanced rows in  $[2, \alpha_{i+1} + 1]$  will be *assigned* to a unique block of  $T$ , which is not necessarily the block that contains that row. We describe how the assignment is done.

As an edge case, row  $\alpha_{i+1} + 1$  is assigned to the topmost block. Let  $r \in [2, \alpha_{i+1}]$ . Let  $a = T(i, r)$ ,  $b = T(i + 1, r)$ ,  $c = T(i, r - 1)$ , and  $d = T(i + 1, r - 1)$ . See Figure 4.7.

|       |     |     |       |
|-------|-----|-----|-------|
|       | $r$ | $a$ | $b$   |
|       |     | $c$ | $d$   |
| $r-1$ |     |     |       |
|       |     | $i$ | $i+1$ |

Figure 4.7: Entries involved in determining the assignment of row  $r$ .

First, assume rows  $r - 1$  and  $r$  are not intertwined in  $T$ , that is  $\chi_{ac} + \chi_{bd} = \chi_{ad} + \chi_{bc}$ . Note that row  $r$  is balanced in  $T$  if and only if  $\chi_{ac} = \chi_{ad} = \chi_{bd} = \chi_{bc}$ . Assume row  $r$  is unbalanced in  $T$ , so that  $\chi_{ac} + \chi_{bd} = 1$ . Hence, we also have  $\chi_{ad} + \chi_{bc} = 1$ , so independently

of whether each of the blocks is swapped or not, exactly one of  $(i, r)$  and  $(i + 1, r)$  is a descent of  $U$  as well. If  $\chi_{ac} = \chi_{ad} \neq \chi_{bd} = \chi_{bc}$ , then the unique descent in  $T$  and the unique descent in  $U$  are in the column that contains the largest of  $a$  and  $b$ . In other words,  $r$  is unbalanced on the same side in both  $T$  and  $U$  if  $r$  is not swapped between  $T$  and  $U$ , and  $r$  is unbalanced on opposite sides in  $T$  and  $U$  if  $r$  is swapped between  $T$  and  $U$ . In this case, we *assign* row  $r$  to the block  $B_1$ , which is the block that determines the swapping behavior of row  $r$ . Otherwise, if  $\chi_{ac} = \chi_{bc} \neq \chi_{ad} = \chi_{bd}$ , then the unique descent in  $T$  and the unique descent in  $U$  are in the column that contains the smallest of  $c$  and  $d$ . In other words,  $r$  is unbalanced on the same side in both  $T$  and  $U$  if  $r - 1$  is not swapped between  $T$  and  $U$ , and  $r$  is unbalanced on opposite sides in  $T$  and  $U$  if  $r - 1$  is swapped between  $T$  and  $U$ . In this case, we *assign* row  $r$  to the block  $B_2$ , which is the block that determines the swapping behavior of row  $r - 1$ .

Second, assume rows  $r - 1$  and  $r$  are intertwined in  $T$ , meaning that both rows belong to the same block  $B$ , and that  $\chi_{ac} + \chi_{bd} + \chi_{ad} + \chi_{bc}$  is odd. we have  $\chi_{ac} = \chi_{bd}$ . Using the fact that  $\chi_{ac} + \chi_{bd} + \chi_{ad} + \chi_{bc}$  is odd, we can conclude that  $\chi_{ab} = \chi_{cd}$  in this case, that is,  $\chi_r = \chi_{r-1}$ . If row  $r$  is left-unbalanced with respect to  $T$ , meaning that only  $(i, r)$  is a descent of  $T$  (equivalently  $\chi_{ac} = 1$  and  $\chi_{bd} = 0$ ), then we have  $\chi_r = \chi_{ab} = 1$  and  $\chi_{r-1} = \chi_{cd} = 0$ . If row  $r$  is right-unbalanced with respect to  $T$ , meaning that only  $(i + 1, r)$  is a descent of  $T$  (equivalently  $\chi_{ac} = 0$  and  $\chi_{bd} = 1$ ), then we have  $\chi_r = \chi_{ab} = 0$  and  $\chi_{r-1} = \chi_{cd} = 1$ . Note that we can determine whether such row  $r$  is balanced, left-unbalanced, or right-unbalanced by looking only at  $\chi_{r-1}$  and  $\chi_r$ : if  $\chi_{r-1} = \chi_r$ , then row  $r$  is balanced; if  $\chi_{r-1} = 0$  and  $\chi_r = 1$ , then row  $r$  is left-unbalanced; and if  $\chi_{r-1} = 1$  and  $\chi_r = 0$ , then row  $r$  is right-unbalanced.

Let  $\text{Des}(B; T)$  be the set of descents of  $T$  in rows assigned to the block  $B$ . We define the *major index contributions* of a block  $B$  with respect to the fillings  $T$  and  $U$  as

$$\begin{aligned} \text{maj}(B; T) &= \sum_{u \in \text{Des}(B; T)} (\text{leg}_\alpha(u) + 1), \\ \text{maj}(B; U) &= \sum_{u \in \text{Des}(B; U)} (\text{leg}_{s_i \cdot \alpha}(u) + 1). \end{aligned}$$

We claim that for every block  $B$ , we have  $\text{maj}(B; T) = \text{maj}(B; U)$ . Let  $B = [r_1, r_2]$  be a block of  $T$ .

Assume that  $B$  is swapped between  $T$  and  $U$ . Then, if  $u$  is a descent assigned to  $B$  in  $T$ , then  $s_i \cdot u$  is a descent assigned to  $B$  in  $U$ , and vice versa, regardless of whether the descent is in row  $r_1$ , row  $r_2 + 1$ , or any row in between. Since  $\text{leg}_\alpha(u) = \text{leg}_{s_i \cdot \alpha}(s_i \cdot u)$  for all boxes  $u$  in columns  $i$  or  $i + 1$ , we have  $\text{maj}(B; T) = \text{maj}(B; U)$  for all blocks  $B$  that are swapped between  $T$  and  $U$ .

Now, assume that  $B = [r_1, r_2]$  is not swapped between  $T$  and  $U$ , and that it is not the topmost block. Let  $(a, b, c, d, e, f)$  be signature entries of  $B$ . This means that  $\chi_{cdf} = \chi_{eab}$ , that is,

$$\chi_{cd} + \chi_{df} - \chi_{cf} = \chi_{ea} + \chi_{ab} - \chi_{eb}.$$

We know that unbalanced rows  $r_1 + 1$  through  $r_2$  are assigned to  $B$ . Let's apply the assigning rules to rows  $r_1$  and  $r_2 + 1$ . The only possibility for row  $r_1$  to be unbalanced and assigned to  $B$  is when  $\chi_{cf} \neq \chi_{df}$ , in which case row  $r_1$  is unbalanced to the side that contains the largest of  $c$  and  $d$ . Similarly, the only possibility for row  $r_2 + 1$  to be unbalanced and assigned to  $B$  is when  $\chi_{ea} \neq \chi_{eb}$ , in which case row  $r_2 + 1$  is unbalanced to the side that contains the smallest of  $a$  and  $b$ .

First, assume  $\chi_{ab} = \chi_{cd}$ , that is,  $\chi_{r_2} = \chi_{r_1}$ . In the sequence  $(\chi_{r_1}, \chi_{r_1+1}, \dots, \chi_{r_2})$ , each time the indicator does not change, the corresponding row is balanced, and each time the indicator changes from 0 to 1 (resp. from 1 to 0), the corresponding row is left-unbalanced (resp. right-unbalanced). Since  $\chi_{r_2} = \chi_{r_1}$ , the number of left-unbalanced rows is equal to the number of right-unbalanced rows among rows  $r_1 + 1$  through  $r_2$ . Moreover, we have  $\chi_{df} - \chi_{cf} = \chi_{ea} - \chi_{eb}$ . This means that  $\chi_{df} = \chi_{cf}$  if, and only if,  $\chi_{ea} = \chi_{eb}$ . If both are true, then rows  $r_1$  and  $r_2 + 1$  are not assigned to  $B$ , and consequently, the numbers of left-unbalanced and right-unbalanced rows assigned to  $B$  are equal. If both are false, then rows  $r_1$  and  $r_2 + 1$  are unbalanced and assigned to  $B$ , with row  $r_1$  unbalanced to the side of the largest of  $c$  and  $d$ , and row  $r_2 + 1$  unbalanced to the side of the smallest of  $a$  and  $b$ . Since  $\chi_{ab} = \chi_{cd}$ , the largest of  $c$  and  $d$  is in the opposite column as the smallest of  $a$  and  $b$ , and consequently, row  $r_1$  is left-unbalanced while row  $r_2 + 1$  is right-unbalanced, or vice versa. Therefore, in this case, the number of left-unbalanced rows assigned to  $B$  is equal to the number of right-unbalanced rows assigned to  $B$  as well.

Second, assume  $\chi_{ab} \neq \chi_{cd}$ , that is,  $\chi_{r_2} \neq \chi_{r_1}$ . Moreover, assume  $\chi_{ab} = 1$  and  $\chi_{cd} = 0$ . In the sequence  $(\chi_{r_1}, \chi_{r_1+1}, \dots, \chi_{r_2})$ , each time the indicator does not change, the corresponding row is balanced, and each time the indicator changes from 0 to 1 (resp. from 1 to 0), the corresponding row is left-unbalanced (resp. right-unbalanced). Since this sequence starts with 0 and ends with 1, there is exactly one more left-unbalanced row than right-unbalanced rows among rows  $r_1 + 1$  through  $r_2$ . Moreover, we have  $\chi_{df} - \chi_{cf} \neq \chi_{ea} - \chi_{eb}$ . This means that exactly one of  $\chi_{df} \neq \chi_{cf}$  and  $\chi_{ea} \neq \chi_{eb}$  is true. If  $\chi_{df} \neq \chi_{cf}$  and  $\chi_{ea} = \chi_{eb}$ , then row  $r_1$  is unbalanced to the side of the largest of  $c$  and  $d$ , which is  $d$  since  $\chi_{cd} = 0$ , that is, row  $r_1$  is right-unbalanced; while row  $r_2 + 1$  is not assigned to  $B$ . Therefore, with this extra right-unbalanced row assigned to  $B$ , the numbers of left-unbalanced and right-unbalanced rows assigned to  $B$  are equal. If  $\chi_{df} = \chi_{cf}$  and  $\chi_{ea} \neq \chi_{eb}$ , then row  $r_2 + 1$  is unbalanced to the side of the smallest of  $a$  and  $b$ , which is  $a$  since  $\chi_{ab} = 1$ , that is, row  $r_2 + 1$  is left-unbalanced; while row  $r_1$  is not assigned to  $B$ . Therefore, with this extra

left-unbalanced row assigned to  $B$ , the numbers of left-unbalanced and right-unbalanced rows assigned to  $B$  are equal as well. The argument for the case  $\chi_{ab} = 0$  and  $\chi_{cd} = 1$  is analogous, by swapping the roles of left and right throughout.

Therefore, in all cases, the number of left-unbalanced rows assigned to  $B$  is equal to the number of right-unbalanced rows assigned to  $B$ . This means that we can pair each left-unbalanced row assigned to  $B$  with a unique right-unbalanced row assigned to  $B$ . By [Lemma 4.10](#), the contributions to  $\text{maj}(B; T)$  and  $\text{maj}(B; U)$  from each such pair are equal, and consequently,  $\text{maj}(B; T) = \text{maj}(B; U)$  for all non-topmost blocks  $B$  that are not swapped between  $T$  and  $U$ .

Finally, assume that  $B = [r_1, \alpha_{i+1}]$  is the topmost block of  $T$ , and that it is not swapped between  $T$  and  $U$ . Let  $(a, b, c, d, e, f)$  be signature entries of  $B$ , as in (4.5). This means that  $\chi_{cd} + \chi_{df} - \chi_{cf} = \chi_{ea} + \chi_{ab} - \chi_{eb}$ . We know that unbalanced rows  $r_1 + 1$  through  $r_2$  are assigned to  $B$ . We also know that row  $\alpha_{i+1} + 1$  is assigned to  $B$ , as an edge case. The box  $(i, \alpha_{i+1} + 1)$  is a descent of  $T$  if and only if  $\chi_{ea} = 1$ , in which case its contribution to  $\text{maj}(T)$  is  $\alpha_i - \alpha_{i+1}$ . Similarly, the box  $(i + 1, \alpha_{i+1} + 1)$  is a descent of  $U$  if and only if  $\chi_{eb} = 1$ , in which case its contribution to  $\text{maj}(U)$  is  $\alpha_i - \alpha_{i+1}$ . Finally, just as in the non-topmost case, the only possibility for row  $r_1$  to be unbalanced and assigned to  $B$  is when  $\chi_{cf} \neq \chi_{df}$ , in which case row  $r_1$  is unbalanced to the side that contains the largest of  $c$  and  $d$ .

First, assume  $\chi_{ab} = \chi_{cd}$ , that is,  $\chi_{r_2} = \chi_{r_1}$ . As in the non-topmost case, the number of left-unbalanced rows assigned to  $B$  is equal to the number of right-unbalanced rows assigned to  $B$  among rows  $r_1 + 1$  through  $\alpha_{i+1}$ . As before, these can be paired up so that the contribution to  $\text{maj}(B; T)$  and  $\text{maj}(B; U)$  from each such pair are equal. The remaining contributions to  $\text{maj}(B; T)$  and  $\text{maj}(B; U)$  come from rows  $r_1$  and  $\alpha_{i+1} + 1$ . We have  $\chi_{df} - \chi_{cf} = \chi_{ea} - \chi_{eb}$ , which means that  $\chi_{df} = \chi_{cf}$  if, and only if,  $\chi_{ea} = \chi_{eb}$ . If both are true, then  $r_1$  is not assigned to  $B$ , and the box  $(i, \alpha_{i+1} + 1)$  is a descent of  $T$  if and only if the box  $(i + 1, \alpha_{i+1} + 1)$  is a descent of  $U$ , so their contributions to  $\text{maj}(B; T)$  and  $\text{maj}(B; U)$  are equal (both zero or both  $\alpha_i - \alpha_{i+1}$ ). Assume both are false. If  $\chi_{ab} = \chi_{cd} = 0$ , then we have  $\chi_{ea} = 1$  and  $\chi_{eb} = 0$ , so the box  $(i, \alpha_{i+1} + 1)$  is a descent of  $T$  while the box  $(i + 1, \alpha_{i+1} + 1)$  is not a descent of  $U$ , contributing  $\alpha_i - \alpha_{i+1}$  to  $\text{maj}(B; T)$  and 0 to  $\text{maj}(B; U)$ . Moreover, since  $\chi_{cf} \neq \chi_{df}$ , row  $r_1$  is assigned to  $B$  and is unbalanced to the side of the largest of  $c$  and  $d$ , which is  $d$  since  $\chi_{cd} = 0$ , so row  $r_1$  is right-unbalanced. Thus, row  $r_1$  contributes  $\text{leg}_\alpha(i + 1, r_1) + 1 = \alpha_{i+1} - r_1 + 1$  to  $\text{maj}(B; T)$  and  $\text{leg}_{s_i \cdot \alpha}(i + 1, r_1) + 1 = \alpha_i - r_1 + 1$  to  $\text{maj}(B; U)$ . Therefore, since

$$(\alpha_i - \alpha_{i+1}) + (\alpha_{i+1} - r_1 + 1) = (\alpha_i - r_1 + 1),$$

the contributions to  $\text{maj}(B; T)$  and  $\text{maj}(B; U)$  from rows  $r_1$  and  $\alpha_{i+1} + 1$  are equal as well. If  $\chi_{ab} = \chi_{cd} = 1$ , then we have  $\chi_{ea} = 0$  and  $\chi_{eb} = 1$ , so the box  $(i, \alpha_{i+1} + 1)$  is not a

descent of  $T$  while the box  $(i+1, \alpha_{i+1}+1)$  is a descent of  $U$ , contributing 0 to  $\text{maj}(B; T)$  and  $\alpha_i - \alpha_{i+1}$  to  $\text{maj}(B; U)$ . Moreover, since  $\chi_{cf} \neq \chi_{df}$ , row  $r_1$  is assigned to  $B$  and is unbalanced to the side of the largest of  $c$  and  $d$ , which is  $c$  since  $\chi_{cd} = 1$ , so row  $r_1$  is left-unbalanced. Thus, row  $r_1$  contributes  $\text{leg}_\alpha(i, r_1) + 1 = \alpha_i - r_1 + 1$  to  $\text{maj}(B; T)$  and  $\text{leg}_{s_i, \alpha}(i, r_1) + 1 = \alpha_{i+1} - r_1 + 1$  to  $\text{maj}(B; U)$ . Therefore, since

$$(\alpha_i - r_1 + 1) = (\alpha_{i+1} - \alpha_i) + (\alpha_{i+1} - r_1 + 1),$$

the contributions to  $\text{maj}(B; T)$  and  $\text{maj}(B; U)$  from rows  $r_1$  and  $\alpha_{i+1} + 1$  are equal as well. Therefore, we have  $\text{maj}(B; T) = \text{maj}(B; U)$  in this case.

Second, assume  $\chi_{ab} = 1$  and  $\chi_{cd} = 0$ , that is,  $\chi_{r_1} = 0$  and  $\chi_{r_2} = 1$ . There is exactly one more left-unbalanced row than right-unbalanced rows among rows  $r_1 + 1$  through  $\alpha_{i+1}$ . We have that exactly one of  $\chi_{df} \neq \chi_{cf}$  and  $\chi_{ea} \neq \chi_{eb}$  is true. If  $\chi_{df} \neq \chi_{cf}$  and  $\chi_{ea} = \chi_{eb}$ , then row  $r_1$  is unbalanced to the side of the largest of  $c$  and  $d$ , which is  $d$  since  $\chi_{cd} = 0$ , so row  $r_1$  is right-unbalanced; while the box  $(i, \alpha_{i+1} + 1)$  is a descent of  $T$  if and only if the box  $(i+1, \alpha_{i+1} + 1)$  is a descent of  $U$ , so their contributions to  $\text{maj}(B; T)$  and  $\text{maj}(B; U)$  are equal (both zero or both  $\alpha_i - \alpha_{i+1}$ ). Therefore, with the extra right-unbalanced row  $r_1$  assigned to  $B$ , the numbers of left-unbalanced and right-unbalanced rows assigned to  $B$  are equal, and consequently  $\text{maj}(B; T) = \text{maj}(B; U)$ . If  $\chi_{df} = \chi_{cf}$  and  $\chi_{ea} \neq \chi_{eb}$ , then row  $r_1$  is not assigned to  $B$ ; since  $\chi_{ab} = 1$ , we have  $\chi_{ea} = 0$  and  $\chi_{eb} = 1$ , so the box  $(i, \alpha_{i+1} + 1)$  is not a descent of  $T$  while the box  $(i+1, \alpha_{i+1} + 1)$  is a descent of  $U$ , contributing 0 to  $\text{maj}(B; T)$  and  $\alpha_i - \alpha_{i+1}$  to  $\text{maj}(B; U)$ . We pair each right-unbalanced row assigned to  $B$  with a unique left-unbalanced row assigned to  $B$ , except for one extra left-unbalanced row. This extra left-unbalanced row  $r$  contributes  $\text{leg}_\alpha(i, r) + 1 = \alpha_i - r + 1$  to  $\text{maj}(B; T)$  and  $\text{leg}_{s_i, \alpha}(i, r) + 1 = \alpha_{i+1} - r + 1$  to  $\text{maj}(B; U)$ . Therefore, since

$$(\alpha_i - r + 1) = (\alpha_{i+1} - \alpha_i) + (\alpha_{i+1} - r + 1),$$

the contributions to  $\text{maj}(B; T)$  and  $\text{maj}(B; U)$  from the pair of the extra left-unbalanced row and row  $\alpha_{i+1} + 1$  are equal as well. Therefore, we have  $\text{maj}(B; T) = \text{maj}(B; U)$  in this case as well.

Third, assume  $\chi_{ab} = 0$  and  $\chi_{cd} = 1$ , that is,  $\chi_{r_1} = 1$  and  $\chi_{r_2} = 0$ . There is exactly one more right-unbalanced row than left-unbalanced rows among rows  $r_1 + 1$  through  $\alpha_{i+1}$ . We have that exactly one of  $\chi_{df} \neq \chi_{cf}$  and  $\chi_{ea} \neq \chi_{eb}$  is true. If  $\chi_{df} \neq \chi_{cf}$  and  $\chi_{ea} = \chi_{eb}$ , then row  $r_1$  is unbalanced to the side of the largest of  $c$  and  $d$ , which is  $c$  since  $\chi_{cd} = 1$ , so row  $r_1$  is left-unbalanced; while the box  $(i, \alpha_{i+1} + 1)$  is a descent of  $T$  if and only if the box  $(i+1, \alpha_{i+1} + 1)$  is a descent of  $U$ , so their contributions to  $\text{maj}(B; T)$  and  $\text{maj}(B; U)$  are equal (both zero or both  $\alpha_i - \alpha_{i+1}$ ). Therefore, with the extra left-unbalanced row  $r_1$

assigned to  $B$ , the numbers of left-unbalanced and right-unbalanced rows assigned to  $B$  are equal, and consequently  $\text{maj}(B; T) = \text{maj}(B; U)$ . If  $\chi_{df} = \chi_{cf}$  and  $\chi_{ea} \neq \chi_{eb}$ , then row  $r_1$  is not assigned to  $B$ ; since  $\chi_{ab} = 0$ , we have  $\chi_{ea} = 1$  and  $\chi_{eb} = 0$ , so the box  $(i, \alpha_{i+1} + 1)$  is a descent of  $T$  while the box  $(i + 1, \alpha_{i+1} + 1)$  is not a descent of  $U$ , contributing  $\alpha_i - \alpha_{i+1}$  to  $\text{maj}(B; T)$  and 0 to  $\text{maj}(B; U)$ . We pair each left-unbalanced row assigned to  $B$  with a unique right-unbalanced row assigned to  $B$ , except for one extra right-unbalanced row. This extra right-unbalanced row  $r$  contributes  $\text{leg}_\alpha(i + 1, r) + 1 = \alpha_{i+1} - r + 1$  to  $\text{maj}(B; T)$  and  $\text{leg}_{s_i \cdot \alpha}(i + 1, r) + 1 = \alpha_i - r + 1$  to  $\text{maj}(B; U)$ . Therefore, since

$$(\alpha_{i+1} - r + 1) + (\alpha_i - \alpha_{i+1}) = (\alpha_i - r + 1),$$

the contributions to  $\text{maj}(B; T)$  and  $\text{maj}(B; U)$  from the pair of the extra right-unbalanced row and row  $\alpha_{i+1} + 1$  are equal as well. Therefore, we have  $\text{maj}(B; T) = \text{maj}(B; U)$  in this case as well.

Finally, we have shown that for every block  $B$ , we have  $\text{maj}(B; T) = \text{maj}(B; U)$ . Since each unbalanced row is assigned to exactly one block, we conclude that  $\text{maj}(T) = \text{maj}(U)$ , completing the proof.  $\square$

*Proof of (4.10).* Without loss of generality, assume that  $\alpha_i > \alpha_{i+1}$ . Let  $T \in \mathbf{F}(\alpha)$  and  $U = \phi(T)$ . Let  $\beta = 0 \dots 0\alpha_{i+1}\alpha_{i+1}0 \dots 0$ . Let  $T_1 \in \mathbf{F}(\beta)$  and  $U_1 \in \mathbf{F}(s_i \cdot \beta)$  be the fillings obtained from  $T$  and  $U$  by restricting to columns  $i$  and  $i + 1$ .

Using [Proposition 3.1](#), we decompose the inversion statistic as

$$\text{inv}(T) - \text{inv}(U) = (|\text{IP}(T)| - |\text{IP}(U)|) - \left( \sum_{u \in \text{Des}(T)} \text{arm}_\alpha(u) - \sum_{u \in \text{Des}(U)} \text{arm}_{s_i \cdot \alpha}(u) \right).$$

Note that a pair of boxes both not in columns  $i$  or  $i + 1$  is an inversion pair in  $T$  if and only if it is an inversion pair in  $U$ . Note that a box  $(i', r')$  with  $i' \notin \{i, i + 1\}$  attacks (resp. is attacked by) neither or both of  $(i, r)$  and  $(i + 1, r)$ . Moreover, by [\(4.4\)](#), for each row  $r$ , we have

$$\{T(i, r), T(i + 1, r)\} = \{U(i, r), U(i + 1, r)\}.$$

Therefore, the number of inversion pairs in  $T$  involving a box not in columns  $i$  or  $i + 1$  and a box in column  $i$  or  $i + 1$  is equal to the number of inversion pairs in  $U$  involving a box not in columns  $i$  or  $i + 1$  and a box in column  $i$  or  $i + 1$ . Consequently, the difference  $|\text{IP}(T)| - |\text{IP}(U)| = |\text{IP}(T_1)| - |\text{IP}(U_1)|$ .

Next, we analyze the difference

$$\sum_{u \in \text{Des}(T)} \text{arm}_\alpha(u) - \sum_{u \in \text{Des}(U)} \text{arm}_{s_i \cdot \alpha}(u).$$

Note that a box  $u$  not in columns  $i$  or  $i + 1$  has the same arm length in  $\alpha$  and in  $s_i \cdot \alpha$ . Moreover, by (4.3), such a box  $u$  is a descent of  $T$  if and only if it is a descent of  $U$ . Therefore, the contributions to the above difference from boxes not in columns  $i$  or  $i + 1$  cancel out. Consequently, changing the summations to only consider descents in  $\text{dg}(\beta)$  and  $\text{dg}(s_i \cdot \beta)$ , respectively, does not change the value of the difference. Define

$$\text{arm}'_\alpha(u) = \text{arm}_\alpha(u) - \text{arm}_\beta(u), \quad \text{and} \quad \text{arm}'_{s_i \cdot \alpha}(u) = \text{arm}_{s_i \cdot \alpha}(u) - \text{arm}_{s_i \cdot \beta}(u),$$

which are the contributions to the arm lengths from boxes not in columns  $i$  or  $i + 1$ . This implies that  $\text{arm}'_\alpha(u) = \text{arm}'_{s_i \cdot \alpha}(s_i \cdot u)$  for all boxes  $u \in \text{dg}(\beta)$ . We prove that

$$\sum_{u \in \text{Des}(T) \cap \text{dg}(\beta)} \text{arm}'_\alpha(u) - \sum_{u \in \text{Des}(U) \cap \text{dg}(s_i \cdot \beta)} \text{arm}'_{s_i \cdot \alpha}(u) = 0. \quad (4.11)$$

Consider the definitions of balanced rows, left-unbalanced rows, right-unbalanced, assignment of rows to blocks in the proof of (4.9). Consider a block  $B = [r_1, r_2]$  in  $T$ , and let's analyse the contributions to each side of (4.11) from descents assigned to  $B$ .

If  $B$  is swapped between  $T$  and  $U$ , then for each descent  $u$  in  $T$  assigned to  $B$ , the box  $s_i \cdot u$  is a descent in  $U$  assigned to  $B$ , and vice versa. Since  $\text{arm}'_\alpha(u) = \text{arm}'_{s_i \cdot \alpha}(s_i \cdot u)$ , the contributions to each side of (4.11) from descents assigned to  $B$  are equal.

If  $B$  is not swapped between  $T$  and  $U$  and it is not the topmost block, then in the proof of (4.9), we showed that the number of left-unbalanced rows assigned to  $B$  is equal to the number of right-unbalanced rows assigned to  $B$ . Therefore, we can pair each left-unbalanced row assigned to  $B$  with a unique right-unbalanced row assigned to  $B$ . By Lemma 4.11, the contributions to each side of (4.11) from each such pair are equal.

Finally, if  $B$  is the topmost block and it is not swapped between  $T$  and  $U$ , then in the proof of (4.9), we showed that one of the following three cases holds:

- The number of left-unbalanced rows assigned to  $B$  equals the number of right-unbalanced rows assigned to  $B$ . In this case, we can then pair each left-unbalanced row with a unique right-unbalanced row, and by Lemma 4.11, the contributions to each side of (4.11) from each such pair are equal.

- There is one extra right-unbalanced row  $r$  assigned to  $B$ , the box  $(i, \alpha_{i+1} + 1)$  is a descent of  $T$ , and the box  $(i + 1, \alpha_{i+1} + 1)$  is not a descent of  $U$ . In this case, we check that

$$\begin{aligned}
\text{arm}'_{\alpha}(i + 1, r) + \text{arm}'_{\alpha}(i, \alpha_{i+1} + 1) &= \\
&= |\{j < i : \alpha_j + 1 \in [r, \alpha_{i+1}]\}| + |\{j > i + 1 : \alpha_j \in [r, \alpha_{i+1}]\}| \\
&\quad + |\{j < i : \alpha_j + 1 \in [\alpha_{i+1} + 1, \alpha_i]\}| + |\{j > i + 1 : \alpha_j \in [\alpha_{i+1} + 1, \alpha_i]\}| \\
&= |\{j < i : \alpha_j + 1 \in [r, \alpha_{i+1}]\}| + |\{j > i + 1 : \alpha_j \in [r, \alpha_{i+1}]\}| \\
&= \text{arm}'_{s_i \cdot \alpha}(i + 1, r),
\end{aligned}$$

so the contributions to each side of (4.11) from the extra left-unbalanced row  $r$  and the row  $\alpha_{i+1} + 1$  are equal.

- There is one extra left-unbalanced row  $r$  assigned to  $B$ , the box  $(i, \alpha_{i+1} + 1)$  is not a descent of  $T$ , and the box  $(i + 1, \alpha_{i+1} + 1)$  is a descent of  $U$ . In this case, we check that

$$\begin{aligned}
\text{arm}'_{\alpha}(i, r) &= |\{j < i : \alpha_j + 1 \in [r, \alpha_i]\}| + |\{j > i + 1 : \alpha_j \in [r, \alpha_i]\}| \\
&= |\{j < i : \alpha_j + 1 \in [r, \alpha_{i+1}]\}| + |\{j > i + 1 : \alpha_j \in [r, \alpha_{i+1}]\}| \\
&\quad + |\{j < i : \alpha_j + 1 \in [\alpha_{i+1} + 1, \alpha_i]\}| + |\{j > i + 1 : \alpha_j \in [\alpha_{i+1} + 1, \alpha_i]\}| \\
&= \text{arm}'_{s_i \cdot \alpha}(i, r) + \text{arm}'_{s_i \cdot \alpha}(i + 1, \alpha_{i+1} + 1),
\end{aligned}$$

so the contributions to each side of (4.11) from the extra left-unbalanced row  $r$  and the row  $\alpha_{i+1} + 1$  are equal.

In all cases, the contributions to each side of (4.11) from descents assigned to  $B$  are equal. Therefore, we have

$$\begin{aligned}
\text{inv}(T) - \text{inv}(U) &= (|\text{IP}(T)| - |\text{IP}(U)|) - \left( \sum_{u \in \text{Des}(T)} \text{arm}_{\alpha}(u) - \sum_{u \in \text{Des}(U)} \text{arm}_{s_i \cdot \alpha}(u) \right) \\
&= (|\text{IP}(T_1)| - |\text{IP}(U_1)|) - \left( \sum_{u \in \text{Des}(T_1)} \text{arm}_{\beta}(u) - \sum_{u \in \text{Des}(U_1)} \text{arm}_{s_i \cdot \beta}(u) \right) \\
&= \text{inv}(T_1) - \text{inv}(U_1).
\end{aligned}$$

Finally, we check that  $\text{inv}(T_1) = \text{inv}(U_1)$ . For each block  $[r_1, r_2]$ , we consider the set of type  $A$  triples

$$\{a_r = ((i, r), (i + 1, r), (i, r - 1)) : r \in [r_1, r_2]\}$$

associated to the block in  $T$ , and the set of type  $B$  triples

$$\{b_r = ((i+1, r+1), (i, r), (i+1, r)) : r \in [r_1, r_2]\}.$$

Note that this associates each triple in  $T_1$  to exactly one block, and similarly for  $U_1$ . For each  $r \in [r_1+1, r_2]$ , since rows  $r$  and  $r-1$  are intertwined, the triple  $a_r$  in  $T$  has the same inversion type as the triple  $b_{r+1}$  in  $U$ . Finally, for the remaining unmatched triples  $a_{r_1}$  in  $T$  and  $b_{r_2}$  in  $U$ , the condition determining whether the block is swapped ensures that the triples  $a_{r_1}$  in  $T$  and  $b_{r_2}$  in  $U$  also have the same inversion type. Therefore, the contributions to  $\text{inv}(T_1)$  and  $\text{inv}(U_1)$  from each block are equal, and consequently,  $\text{inv}(T_1) = \text{inv}(U_1)$ .  $\square$

**Theorem 4.13** ([KLO22, p. 14]). *Assume that  $\alpha_i > \alpha_{i+1}$ . Let  $T \in F(\alpha)$  and  $U = \psi(T)$ . Let  $L = \text{leg}(u) = \alpha_i - \alpha_{i+1} - 1$  and  $A = \text{arm}(u)$  for  $u = (i, \alpha_{i+1} + 1) \in \text{dg}(\alpha)$ . If  $\chi_{T(i,1)T(i+1,1)} = 1$  and  $\chi_{U(i,1)U(i+1,1)} = 1$ , then*

$$\text{maj}(T) = \text{maj}(U) - (L + 1), \quad \text{and} \quad \text{inv}(T) = \text{inv}(U) + (A + 1).$$

*If  $\chi_{T(i,1)T(i+1,1)} = 1$  and  $\chi_{U(i,1)U(i+1,1)} = 0$ , then*

$$\text{maj}(T) = \text{maj}(U), \quad \text{and} \quad \text{inv}(T) = \text{inv}(U) + 1.$$

*If  $\chi_{T(i,1)T(i+1,1)} = 0$  and  $\chi_{U(i,1)U(i+1,1)} = 1$ , then*

$$\text{maj}(T) = \text{maj}(U), \quad \text{and} \quad \text{inv}(T) = \text{inv}(U) - 1.$$

*If  $\chi_{T(i,1)T(i+1,1)} = 0$  and  $\chi_{U(i,1)U(i+1,1)} = 0$ , then*

$$\text{maj}(T) = \text{maj}(U) + (L + 1), \quad \text{and} \quad \text{inv}(T) = \text{inv}(U) - (A + 1).$$

Again, [KLO22] proves **Theorem 4.13** using the row-by-row algorithm. We provide an alternative proof using the block-by-block algorithm.

*Proof of Theorem 4.13.* We assume the notation in the proof of **Theorem 4.12**. Let  $T \in F(\alpha)$  and  $U = \psi(T)$ . Most of the analysis is unchanged, except for the contributions to  $\text{maj}(T)$  and  $\text{inv}(U)$  from the bottommost block  $B = [1, r_2]$ .

For the major index statistic, if the bottommost block  $B$  is swapped between  $T$  and  $U$ , then the contributions to  $\text{maj}(T)$  and  $\text{maj}(U)$  from descents assigned to  $B$  are equal. This establishes the second and third cases of the theorem.

Assume that the bottommost block  $B = [1, r_2]$  is not swapped between  $T$  and  $U$ , and that  $B$  is not the topmost block. Let  $(a, b, c, d, e, f)$  be signature entries of  $B$ . Since  $B$  is

bottommost,  $f = \infty$ , and thus  $\chi_{cdf} = \chi_{cd} = \chi_{T(i,1)T(i+1,1)} = \chi_{U(i,1)U(i+1,1)}$ . Since  $T$  and  $U$  are not swapped under  $\psi$ , we have  $\chi_{eab} \neq \chi_{cd}$ .

First, assume  $\chi_{ab} = \chi_{cd}$ , and consequently  $\chi_{ea} \neq \chi_{eb}$ . As in the proof of [Theorem 4.12](#), the number of left-unbalanced rows assigned to  $B$  is equal to the number of right-unbalanced rows assigned to  $B$  among rows 2 through  $r_2$ , so we can pair each left-unbalanced row assigned to  $B$  with a unique right-unbalanced row assigned to  $B$ . By [Lemma 4.10](#), the contributions to  $\text{maj}(T)$  and  $\text{maj}(U)$  from each such pair are equal. Moreover, we have  $\chi_{ea} \neq \chi_{eb}$ , so row  $r_2 + 1$  is assigned to  $B$  and unbalanced to the side of the smallest of  $a$  and  $b$ . If  $\chi_{cd} = 0$ , then the smallest of  $a$  and  $b$  is  $a$ , so row  $r_2 + 1$  is left-unbalanced, contributing  $\text{leg}_\alpha(i, r_2 + 1) + 1 = \alpha_i - r_2$  to  $\text{maj}(T)$  and  $\text{leg}_{s_i \cdot \alpha}(i, r_2 + 1) + 1 = \alpha_{i+1} - r_2$  to  $\text{maj}(U)$ . Therefore,

$$\text{maj}(T) - \text{maj}(U) = (\alpha_i - r_2) - (\alpha_{i+1} - r_2) = \alpha_i - \alpha_{i+1} = L + 1,$$

which agrees with the fourth case of the theorem. If  $\chi_{cd} = 1$ , then the smallest of  $a$  and  $b$  is  $b$ , so row  $r_2 + 1$  is right-unbalanced, contributing  $\text{leg}_\alpha(i + 1, r_2 + 1) + 1 = \alpha_{i+1} - r_2$  to  $\text{maj}(T)$  and  $\text{leg}_{s_i \cdot \alpha}(i + 1, r_2 + 1) + 1 = \alpha_i - r_2$  to  $\text{maj}(U)$ . Therefore,

$$\text{maj}(T) - \text{maj}(U) = (\alpha_{i+1} - r_2) - (\alpha_i - r_2) = \alpha_{i+1} - \alpha_i = -(L + 1),$$

which agrees with the first case of the theorem.

Second, assume  $\chi_{ab} \neq \chi_{cd}$ , and consequently  $\chi_{ea} = \chi_{eb}$ . As in the proof of [Theorem 4.12](#),  $\chi_{ea} = \chi_{eb}$  implies that row  $r_2 + 1$  is not assigned to  $B$ . If  $\chi_{cd} = 0$  and  $\chi_{ab} = 1$ , as in the proof of [Theorem 4.12](#), there is exactly one more left-unbalanced row than right-unbalanced rows among rows 2 through  $r_2$ , so we can pair each right-unbalanced row assigned to  $B$  with a unique left-unbalanced row assigned to  $B$ . By [Lemma 4.10](#), the contributions to  $\text{maj}(T)$  and  $\text{maj}(U)$  from each such pair are equal. The extra left-unbalanced row  $r$  contributes  $\text{leg}_\alpha(i, r) + 1 = \alpha_i - r + 1$  to  $\text{maj}(T)$  and  $\text{leg}_{s_i \cdot \alpha}(i, r) + 1 = \alpha_{i+1} - r + 1$  to  $\text{maj}(U)$ . Therefore,

$$\text{maj}(T) - \text{maj}(U) = (\alpha_i - r + 1) - (\alpha_{i+1} - r + 1) = \alpha_i - \alpha_{i+1} = L + 1,$$

which agrees with the fourth case of the theorem. The analysis for  $\chi_{cd} = 1$  and  $\chi_{ab} = 0$  is analogous, changing the roles of left-unbalanced and right-unbalanced rows, the computation for  $\text{maj}(T) - \text{maj}(U)$  gives  $-(L + 1)$ , which agrees with the first case of the theorem.

Lastly, it might be that the bottommost block  $B = [1, r_2]$  is also the topmost block. We omit the details, whose change from the details in the proof of [Theorem 4.12](#) are analogous to the changes above: if  $\chi_{cd} = 1$ , then there is one extra left-unbalanced row contributing  $L + 1$  to  $\text{maj}(T) - \text{maj}(U)$ ; and if  $\chi_{cd} = 0$ , then there is one extra right-unbalanced row contributing  $-(L + 1)$  to  $\text{maj}(T) - \text{maj}(U)$ .

Now, we analyze the inversion statistic. As in the proof of [Theorem 4.12](#),  $|\text{IP}(T)| - |\text{IP}(U)| = |\text{IP}(T_1)| - |\text{IP}(U_1)|$ . We decompose the difference  $\text{inv}(T) - \text{inv}(U)$  as

$$\begin{aligned} & (|\text{IP}(T_1)| - |\text{IP}(U_1)|) - \left( \sum_{u \in \text{Des}(T_1)} \text{arm}_\beta(u) - \sum_{u \in \text{Des}(U_1)} \text{arm}_{s_i \cdot \beta}(u) \right) - \left( \sum_{u \in \text{Des}(T_1)} \text{arm}'_\alpha(u) - \sum_{u \in \text{Des}(U_1)} \text{arm}'_{s_i \cdot \alpha}(u) \right) = \\ & = (\text{inv}(T_1) - \text{inv}(U_1)) - \left( \sum_{u \in \text{Des}(T_1)} \text{arm}'_\alpha(u) - \sum_{u \in \text{Des}(U_1)} \text{arm}'_{s_i \cdot \alpha}(u) \right). \end{aligned}$$

On the one hand, as in the proof of [Theorem 4.12](#), the number of inversion pairs in  $T_1$  assigned to each non-bottommost block is equal to the number of inversion pairs in  $U_1$  assigned to the same block. For the bottommost block  $B = [1, r_2]$ , the unmatched triples,  $a_1$  in  $T$  and  $b_{r_2}$  in  $U$ , have different inversion types. If  $\chi_{cd} = 1$ , corresponding to the first and second cases of the theorem, then  $a_1$  is an inversion triple while  $b_{r_2}$  is not, which makes  $|\text{IP}(T_1)| - |\text{IP}(U_1)| = 1$ . If  $\chi_{cd} = 0$ , corresponding to the third and fourth cases of the theorem, then  $a_1$  is not an inversion triple while  $b_{r_2}$  is, which makes  $|\text{IP}(T_1)| - |\text{IP}(U_1)| = -1$ . In summary, we have

$$|\text{IP}(T_1)| - |\text{IP}(U_1)| = \begin{cases} 1, & \text{if } \chi_{T(i,1)T(i+1,1)} = 1, \\ -1, & \text{if } \chi_{T(i,1)T(i+1,1)} = 0. \end{cases} \quad (4.12)$$

On the other hand, for each non-bottommost block, the sum of  $\text{arm}'_\alpha(u)$  over descents  $u$  assigned to the block in  $T$  is equal to the sum of  $\text{arm}'_{s_i \cdot \alpha}(u)$  over descents  $u$  assigned to the block in  $U$ , by the same analysis as in the proof of [Theorem 4.12](#). Therefore, it remains to analyze the contributions to the difference

$$\sum_{u \in \text{Des}(T_1)} \text{arm}'_\alpha(u) - \sum_{u \in \text{Des}(U_1)} \text{arm}'_{s_i \cdot \alpha}(u)$$

from descents assigned to the bottommost block  $B = [1, r_2]$ .

For the bottommost block  $B = [1, r_2]$ , if it is swapped between  $T$  and  $U$ , which corresponds to the second and third cases of the theorem, then the sum of  $\text{arm}'_\alpha(u)$  over descents  $u$  assigned to  $B$  in  $T$  is equal to the sum of  $\text{arm}'_{s_i \cdot \alpha}(u)$  over descents  $u$  assigned to  $B$  in  $U$ .

Assume that the bottommost block  $B = [1, r_2]$  is not swapped between  $T$  and  $U$ . If  $\chi_{cd} = 1$ , which corresponds to the first case of the theorem, then there is one extra left-unbalanced row  $r$  assigned to  $B$  as established above. This extra left-unbalanced

row  $r$  contributes  $\text{arm}'_\alpha(i, r)$  to the sum over descents assigned to  $B$  in  $T$  and contributes  $\text{arm}'_{s_i \cdot \alpha}(i, r)$  to the sum over descents assigned to  $B$  in  $U$ . Therefore, the difference in contributions from the extra left-unbalanced row is

$$\begin{aligned} \text{arm}'_\alpha(i, r) - \text{arm}'_{s_i \cdot \alpha}(i, r) &= \\ &= |\{j < i : \alpha_j + 1 \in [r, \alpha_i]\}| + |\{j > i + 1 : \alpha_j \in [r, \alpha_i]\}| \\ &\quad - |\{j < i : \alpha_j + 1 \in [r, \alpha_{i+1}]\}| - |\{j > i + 1 : \alpha_j \in [r, \alpha_{i+1}]\}| \\ &= |\{j < i : \alpha_j + 1 \in [\alpha_{i+1} + 1, \alpha_i]\}| + |\{j > i + 1 : \alpha_j \in [\alpha_{i+1} + 1, \alpha_i]\}| \\ &= A. \end{aligned}$$

If  $\chi_{cd} = 0$ , which corresponds to the fourth case of the theorem, then there is one extra right-unbalanced row  $r$  assigned to  $B$  as established above. This extra right-unbalanced row  $r$  contributes  $\text{arm}'_\alpha(i + 1, r)$  to the sum over descents assigned to  $B$  in  $T$  and contributes  $\text{arm}'_{s_i \cdot \alpha}(i + 1, r)$  to the sum over descents assigned to  $B$  in  $U$ . Therefore, the difference in contributions from the extra right-unbalanced row is

$$\begin{aligned} \text{arm}'_\alpha(i + 1, r) - \text{arm}'_{s_i \cdot \alpha}(i + 1, r) &= \\ &= |\{j < i : \alpha_j + 1 \in [r, \alpha_{i+1}]\}| + |\{j > i + 1 : \alpha_j \in [r, \alpha_{i+1}]\}| \\ &\quad - |\{j < i : \alpha_j + 1 \in [r, \alpha_i]\}| - |\{j > i + 1 : \alpha_j \in [r, \alpha_i]\}| \\ &= -|\{j < i : \alpha_j + 1 \in [\alpha_{i+1} + 1, \alpha_i]\}| - |\{j > i + 1 : \alpha_j \in [\alpha_{i+1} + 1, \alpha_i]\}| \\ &= -A. \end{aligned}$$

In summary, we have

$$\sum_{u \in \text{Des}(T)} \text{arm}'_\alpha(u) - \sum_{u \in \text{Des}(U)} \text{arm}'_{s_i \cdot \alpha}(u) = \begin{cases} 0, & \text{if } \chi_{T(i,1)T(i+1,1)} \neq \chi_{U(i,1)U(i+1,1)}, \\ -A, & \text{if } \chi_{T(i,1)T(i+1,1)} = \chi_{U(i,1)U(i+1,1)} = 1, \\ A, & \text{if } \chi_{T(i,1)T(i+1,1)} = \chi_{U(i,1)U(i+1,1)} = 0. \end{cases} \quad (4.13)$$

Adding (4.12) and (4.13) gives the desired result.  $\square$

One important corollary of [Theorem 4.12](#) is the identity (4.1), which shows that the combinatorial formula for  $\tilde{H}_\mu(\mathbf{x}; q, t)$  can be taken over  $\mathbb{F}_{\mathbb{Z}_{>0}}(\alpha)$  for any rearrangement  $\alpha$  of the partition  $\mu$ .

**Theorem 4.14** ([[HHL08](#)]). *Let  $\mu$  be a partition and  $\alpha$  be a rearrangement of  $\mu$ . Then*

$$\sum_{T \in \mathbb{F}_{\mathbb{Z}_{>0}}(\mu)} q^{\text{maj}(T)} t^{\text{inv}(T)} \mathbf{x}^T = \sum_{T \in \mathbb{F}_{\mathbb{Z}_{>0}}(\alpha)} q^{\text{maj}(T)} t^{\text{inv}(T)} \mathbf{x}^T, \quad (4.1)$$

where  $\mathbb{F}_{\mathbb{Z}_{>0}}(\mu)$  (resp.  $\mathbb{F}_{\mathbb{Z}_{>0}}(\alpha)$ ) is the set of fillings of shape  $\mu$  (resp.  $\alpha$ ) with labels in the positive integers  $\mathbb{Z}_{>0}$ .

## 4.2 Bijections on augmented fillings and a three-term identity for $E_\alpha^\sigma$

In this section, we present bijections on augmented signed fillings to prove a three-term identity for permuted-basement Macdonald polynomials  $E_\alpha^\sigma(\mathbf{x}; q, t)$ , in [Theorem 4.15](#). We adapt the bijections  $\phi$  and  $\psi$  from [Section 4.1](#) defined on non-augmented fillings to create bijections on augmented fillings to obtain the desired identity.

**Theorem 4.15.** *Let  $\alpha$  be a composition with  $\alpha_i > \alpha_{i+1}$  for some  $i \in [n-1]$ . Then,*

$$E_\alpha^\sigma(\mathbf{x}; q, t) = E_{s_i \cdot \alpha}^{\sigma s_i}(\mathbf{x}; q, t) + \left( (q^{L+1} t^{A+1})^{\chi_{\sigma_{i+1} \sigma_i}} t^{\rho(\alpha, \sigma, i)} \frac{(1-t)}{(1 - q^{L+1} t^{A+1})}, \right) E_{s_i \cdot \alpha}^\sigma(\mathbf{x}; q, t).$$

where  $L = \text{leg}(u) = \alpha_i - \alpha_{i+1} - 1$  and  $A = \text{arm}(u)$  for  $u = (i, \alpha_{i+1} + 1) \in \text{dg}(\alpha)$ , and  $\rho(\alpha, \sigma, i) = \text{twinv}(s_i \cdot \alpha, \sigma) - \text{twinv}(\alpha, \sigma)$ .

Alexandersson [[Ale19](#), Proposition 17] proves [Theorem 4.15](#) for the case where  $\sigma_i = \sigma_{i+1} + 1$ . The specialization for  $q = 0$  is proved by Alexandersson and Sawhney [[AS19](#), Proposition 5.5].

Our proof uses the integral form permuted-basement Macdonald polynomials  $\mathcal{E}_\alpha^\sigma$  defined in [Section 3.3.5](#). Recall that the integral forms have a combinatorial formula in terms of augmented fillings using the signed alphabet  $[n]_\pm$ , as given in [\(3.20\)](#):

$$\mathcal{E}_\alpha^\sigma(\mathbf{x}; q, t) = \sum_{T \in \mathbf{F}_{[n]_\pm}(\alpha, \sigma)} \mathbf{x}^{\text{content}(T)} q^{\text{maj}(T)} t^{\text{coinv}(T)} (-t)^{\text{neg}(T)}.$$

For the purposes of this section, let  $\text{wt}(T) = \mathbf{x}^{|T|} q^{\text{maj}(T)} t^{\text{coinv}(T)} (-t)^{\text{neg}(T)}$ , for an augmented signed filling  $T$ .

Recall from [Section 3.2.3](#) that any augmented filling  $T \in \mathbf{F}(\alpha, \sigma)$  may be identified with a non-augmented filling  $T' \in \mathbf{F}(\alpha + \mathbf{1})$  by moving each box one unit up. This identification preserves  $\text{maj}$ , but changes  $\text{coinv}$ . We have  $\text{symcoinv}(T) = \text{coinv}(T') = \text{coinv}(T) + \text{twinv}(\alpha, \sigma)$ .

We can apply  $\phi$  to  $T'$  to obtain  $U' = \phi(T') \in \mathbf{F}(s_i \cdot \alpha + \mathbf{1})$ , and then identify  $U'$  with an augmented filling  $U$  of shape  $\alpha$ . Note that  $\phi$  might or might not change the entries in positions  $(i, 1)$  and  $(i + 1, 1)$  of  $T'$ , so the basement of  $U$  might be  $\sigma$  or  $\sigma s_i$ .

Define the injection  $\phi: \mathbf{F}(\alpha, \sigma) \rightarrow \mathbf{F}(s_i \cdot \alpha, \sigma) \cup \mathbf{F}(s_i \cdot \alpha, \sigma s_i)$  by applying the bijection  $\phi$  to the non-augmented filling  $T'$  corresponding to  $T$ , and then identifying the result with an augmented filling  $U$ . Similarly, define the injection  $\psi: \mathbf{F}(\alpha, \sigma) \rightarrow \mathbf{F}(s_i \cdot \alpha, \sigma) \cup \mathbf{F}(s_i \cdot \alpha, \sigma s_i)$

by applying the bijection  $\psi$  to the non-augmented filling  $T'$  corresponding to  $T$ , and then identifying the result with an augmented filling  $U$ .

We have immediate coloraries of [Theorem 4.12](#) and [Theorem 4.13](#).

**Corollary 4.16.** *For any  $T \in F(\alpha, \sigma)$  and  $U = \phi(T)$ , we have*

$$\text{symcoinv}(T) = \text{symcoinv}(U) \quad \text{and} \quad \text{maj}(T) = \text{maj}(U).$$

**Corollary 4.17.** *Assume that  $\alpha_i > \alpha_{i+1}$ . Let  $T \in F(\alpha, \sigma)$  and  $U = \psi(T)$ . Let  $L = \text{leg}(u) = \alpha_i - \alpha_{i+1} - 1$  and  $A = \text{arm}(u)$  for  $u = (i, \alpha_{i+1} + 1) \in \text{dg}(\alpha)$ . If  $\sigma_i > \sigma_{i+1}$  and  $\pi_i > \pi_{i+1}$ , then*

$$\text{maj}(T) = \text{maj}(U) - (L + 1), \quad \text{and} \quad \text{symcoinv}(T) = \text{symcoinv}(U) - (A + 1).$$

*If  $\sigma_i > \sigma_{i+1}$  and  $\pi_i < \pi_{i+1}$ , then*

$$\text{maj}(T) = \text{maj}(U), \quad \text{and} \quad \text{symcoinv}(T) = \text{symcoinv}(U) - 1.$$

*If  $\sigma_i < \sigma_{i+1}$  and  $\pi_i > \pi_{i+1}$ , then*

$$\text{maj}(T) = \text{maj}(U), \quad \text{and} \quad \text{symcoinv}(T) = \text{symcoinv}(U) + 1.$$

*If  $\sigma_i < \sigma_{i+1}$  and  $\pi_i < \pi_{i+1}$ , then*

$$\text{maj}(T) = \text{maj}(U) + (L + 1), \quad \text{and} \quad \text{symcoinv}(T) = \text{symcoinv}(U) + (A + 1).$$

**Definition 4.18.** Define  $f: F(\alpha, \sigma) \rightarrow F(s_i \cdot \alpha, \sigma s_i)$  and  $g: F(\alpha, \sigma) \rightarrow F(s_i \cdot \alpha, \sigma)$  by

$$f(T) = \begin{cases} \phi(T), & \text{if } \phi(T) \in F(s_i \cdot \alpha, \sigma s_i), \\ \psi(T), & \text{if } \psi(T) \in F(s_i \cdot \alpha, \sigma s_i), \end{cases} \quad \text{and} \quad g(T) = \begin{cases} \phi(T), & \text{if } \phi(T) \in F(s_i \cdot \alpha, \sigma), \\ \psi(T), & \text{if } \psi(T) \in F(s_i \cdot \alpha, \sigma). \end{cases}$$

**Proposition 4.19.** *The maps  $f$  and  $g$  are well-defined and are bijections.*

*Proof.* Let  $T \in F(\alpha, \sigma)$ . Recall that the swapping conditions for  $\phi$  and  $\psi$  differ only in the bottommost block. Since the first row is always part of the bottommost block, exactly one of  $\phi(T)$  and  $\psi(T)$  has basement  $\sigma$ , and the other has basement  $\sigma s_i$ . Therefore,  $f$  and  $g$  are well-defined. The bijectivity of  $f$  and  $g$  follows from [Propositions 4.8](#) and [4.9](#).  $\square$

We split [Theorem 4.15](#) into two cases, depending on the relative order of  $\sigma_i$  and  $\sigma_{i+1}$ . The case  $\sigma_i > \sigma_{i+1}$  is established in [Theorem 4.22](#), while the case  $\sigma_i < \sigma_{i+1}$  is established in [Theorem 4.25](#).

### 4.2.1 Case $\sigma_i > \sigma_{i+1}$

In this subsection, we prove [Theorem 4.15](#) in the case where  $\sigma_i > \sigma_{i+1}$ . Together with  $\alpha_i > \alpha_{i+1}$ , we compute that in this case

$$\text{twinv}(\alpha, \sigma) = \text{twinv}(s_i \cdot \alpha, \sigma s_i).$$

**Lemma 4.20.** *Assume that  $\alpha_i > \alpha_{i+1}$  and  $\sigma_i > \sigma_{i+1}$ . Let  $L = \text{leg}(u) = \alpha_i - \alpha_{i+1} - 1$  and  $A = \text{arm}(u)$  for  $u = (i, \alpha_{i+1} + 1) \in \text{dg}(\alpha)$ . Let  $T$  be an augmented filling in  $F(\alpha, \sigma)$ . Let  $U = f(T)$  and  $V = g(T)$ . Then*

$$(1 - q^{L+1}t^{A+2}) \text{wt}(T) = (1 - q^{L+1}t^{A+1}) \text{wt}(U) + t^{\rho(\alpha, \sigma, i)}(1 - t) \text{wt}(V). \quad (4.14)$$

*Proof.* First, assume  $U = \phi(T)$  and  $V = \psi(T)$ . On the one hand, by [Corollary 4.16](#),  $\text{symcoinv}(T) = \text{symcoinv}(U)$  and  $\text{maj}(T) = \text{maj}(U)$ . Since  $\text{twinv}(\alpha, \sigma) = \text{twinv}(s_i \cdot \alpha, \sigma s_i)$ , we have

$$\text{coinv}(T) = \text{symcoinv}(T) - \text{twinv}(\alpha, \sigma) = \text{symcoinv}(U) - \text{twinv}(s_i \cdot \alpha, \sigma s_i) = \text{coinv}(U).$$

Thus,  $\text{wt}(U) = \text{wt}(T)$ . On the other hand, by [Corollary 4.17](#), we have  $\text{maj}(T) = \text{maj}(V) - (L + 1)$  and  $\text{symcoinv}(T) = \text{symcoinv}(V) - (A + 1)$ . Thus,

$$\begin{aligned} \text{coinv}(T) &= \text{symcoinv}(T) - \text{twinv}(\alpha, \sigma) \\ &= \text{symcoinv}(V) - (A + 1) - \text{twinv}(\alpha, \sigma) = \text{coinv}(V) - (A + 1) + \rho(\alpha, \sigma, i), \end{aligned}$$

and consequently  $\text{wt}(V) = t^{-\rho(\alpha, \sigma, i)} q^{L+1} t^{A+1} \text{wt}(T)$ . Therefore,

$$\begin{aligned} (1 - q^{L+1}t^{A+1}) \text{wt}(U) + t^{\rho(\alpha, \sigma, i)}(1 - t) \text{wt}(V) &= \\ &= (1 - q^{L+1}t^{A+1}) \text{wt}(T) + t^{\rho(\alpha, \sigma, i)}(1 - t) t^{-\rho(\alpha, \sigma, i)} q^{L+1} t^{A+1} \text{wt}(T) \\ &= (1 - q^{L+1}t^{A+1} + (1 - t) q^{L+1} t^{A+2}) \text{wt}(T) \\ &= (1 - q^{L+1}t^{A+2}) \text{wt}(T). \end{aligned}$$

Second, assume  $U = \psi(T)$  and  $V = \phi(T)$ . On the one hand, by [Corollary 4.17](#),  $\text{maj}(T) = \text{maj}(U)$  and  $\text{symcoinv}(T) = \text{symcoinv}(U) - 1$ . Since  $\text{twinv}(\alpha, \sigma) = \text{twinv}(s_i \cdot \alpha, \sigma s_i)$ , Thus,  $\text{wt}(U) = \text{wt}(T)$ . On the other hand, by [Corollary 4.16](#),  $\text{symcoinv}(T) = \text{symcoinv}(V)$

and  $\text{maj}(T) = \text{maj}(V)$ . Thus,  $\text{coinv}(T) = \text{coinv}(V) + \rho(\alpha, \sigma, i)$ , and consequently  $\text{wt}(V) = t^{-\rho(\alpha, \sigma, i)} \text{wt}(T)$ . Therefore,

$$\begin{aligned} (1 - q^{L+1}t^{A+1}) \text{wt}(U) + t^{\rho(\alpha, \sigma, i)}(1 - t) \text{wt}(V) &= \\ &= (1 - q^{L+1}t^{A+1}) t \text{wt}(T) + t^{\rho(\alpha, \sigma, i)}(1 - t) t^{-\rho(\alpha, \sigma, i)} \text{wt}(T) \\ &= (t - q^{L+1}t^{A+2} + 1 - t) \text{wt}(T) \\ &= (1 - q^{L+1}t^{A+2}) \text{wt}(T). \end{aligned}$$

Therefore, in either case, (4.14) holds.  $\square$

**Corollary 4.21.** *Let  $\alpha$  be a composition such that  $\alpha_i > \alpha_{i+1}$  and  $\sigma \in S_n$  such that  $\sigma_i > \sigma_{i+1}$ . Then,*

$$(1 - q^{L+1}t^{A+2}) \mathcal{E}_\alpha^\sigma = (1 - q^{L+1}t^{A+1}) \mathcal{E}_{s_i \cdot \alpha}^{\sigma s_i} + t^{\rho(\alpha, \sigma, i)}(1 - t) \mathcal{E}_{s_i \cdot \alpha}^\sigma, \quad (4.15)$$

where  $L = \text{leg}(u) = \alpha_i - \alpha_{i+1} - 1$  and  $A = \text{arm}(u)$  for  $u = (i, \alpha_{i+1} + 1) \in \text{dg}(\alpha)$ .

*Proof.* The result follows by summing (4.14) in Lemma 4.20 over all  $T \in F(\alpha, \sigma)$ .  $\square$

**Theorem 4.22.** *Let  $\alpha$  be a composition such that  $\alpha_i > \alpha_{i+1}$  and  $\sigma \in S_n$  such that  $\sigma_i > \sigma_{i+1}$ . Then,*

$$E_\alpha^\sigma(X; q; t) = E_{s_i \cdot \alpha}^{\sigma s_i}(X; q; t) + t^{\rho(\alpha, \sigma, i)} \frac{1 - t}{1 - q^{L+1}t^{A+1}} E_{s_i \cdot \alpha}^\sigma(X; q; t) \quad (4.16)$$

where  $L = \text{leg}(u) = \alpha_i - \alpha_{i+1} - 1$  and  $A = \text{arm}(u)$  for  $u = (i, \alpha_{i+1} + 1) \in \text{dg}(\alpha)$ .

*Proof.* The result follows from Lemma 4.20 by dividing both sides of (4.15) by the appropriate factor to convert the integral forms to non-integral forms.  $\square$

### 4.2.2 Case $\sigma_i < \sigma_{i+1}$

In this subsection, we prove Theorem 4.15 in the case where  $\sigma_i < \sigma_{i+1}$ . Together with  $\alpha_i > \alpha_{i+1}$ , we compute that in this case

$$\text{twinv}(\alpha, \sigma) = \text{twinv}(s_i \cdot \alpha, \sigma s_i) + 1.$$

**Lemma 4.23.** *Assume that  $\alpha_i > \alpha_{i+1}$  and  $\sigma_i < \sigma_{i+1}$ . Let  $L = \text{leg}(u) = \alpha_i - \alpha_{i+1} - 1$  and  $A = \text{arm}(u)$  for  $u = (i, \alpha_{i+1} + 1) \in \text{dg}(\alpha)$ . Let  $T$  be an augmented filling in  $F(\alpha, \sigma)$ . Let  $U = f(T)$  and  $V = g(T)$ . Then*

$$(1 - q^{L+1}t^{A+2}) \text{wt}(T) = (1 - q^{L+1}t^{A+1}) \text{wt}(U) + (q^{L+1}t^{A+1})(1 - t) t^{\rho(\alpha, \sigma, i)} \text{wt}(V). \quad (4.17)$$

*Proof.* First, assume  $U = \phi(T)$  and  $V = \psi(T)$ . On the one hand, by [Corollary 4.16](#),  $\text{maj}(T) = \text{maj}(U)$  and  $\text{symcoinv}(T) = \text{symcoinv}(U)$ . Since  $\text{twinv}(\alpha, \sigma) = \text{twinv}(s_i \cdot \alpha, \sigma s_i) + 1$ , we have

$$\begin{aligned}\text{coinv}(T) &= \text{symcoinv}(T) - \text{twinv}(\alpha, \sigma) \\ &= \text{symcoinv}(U) - \text{twinv}(s_i \cdot \alpha, \sigma s_i) - 1 = \text{coinv}(U) - 1.\end{aligned}$$

Thus,  $\text{wt}(U) = \text{wt}(T)$ . On the other hand, by [Corollary 4.17](#), we have  $\text{maj}(T) = \text{maj}(V) + (L + 1)$  and  $\text{symcoinv}(T) = \text{symcoinv}(V) + (A + 1)$ . Thus,

$$\begin{aligned}\text{coinv}(T) &= \text{symcoinv}(T) - \text{twinv}(\alpha, \sigma) \\ &= \text{symcoinv}(V) + (A + 1) - \text{twinv}(\alpha, \sigma) = \text{coinv}(V) + (A + 1) + \rho(\alpha, \sigma, i),\end{aligned}$$

and consequently  $\text{wt}(V) = t^{-\rho(\alpha, \sigma, i)} q^{-(L+1)} t^{-(A+1)} \text{wt}(T)$ . Therefore,

$$\begin{aligned}(1 - q^{L+1} t^{A+1}) \text{wt}(U) + (q^{L+1} t^{A+1}) (1 - t) t^{\rho(\alpha, \sigma, i)} \text{wt}(V) &= \\ &= (1 - q^{L+1} t^{A+1}) \text{wt}(T) + (q^{L+1} t^{A+1}) (1 - t) t^{\rho(\alpha, \sigma, i)} t^{-\rho(\alpha, \sigma, i)} q^{-(L+1)} t^{-(A+1)} \text{wt}(T) \\ &= (t - q^{L+1} t^{A+2} + q^{L+1} t^{A+1} - q^{L+1} t^{A+2}) \text{wt}(T) \\ &= (1 - q^{L+1} t^{A+2}) \text{wt}(T).\end{aligned}$$

Second, assume  $U = \psi(T)$  and  $V = \phi(T)$ . On the one hand, by [Corollary 4.17](#),  $\text{maj}(T) = \text{maj}(U)$  and  $\text{symcoinv}(T) = \text{symcoinv}(U) + 1$ . Since  $\text{twinv}(\alpha, \sigma) = \text{twinv}(s_i \cdot \alpha, \sigma s_i) + 1$ ,

$$\begin{aligned}\text{coinv}(T) &= \text{symcoinv}(T) - \text{twinv}(\alpha, \sigma) \\ &= \text{symcoinv}(U) + 1 - \text{twinv}(s_i \cdot \alpha, \sigma s_i) - 1 = \text{coinv}(U).\end{aligned}$$

Thus,  $\text{wt}(U) = \text{wt}(T)$ . On the other hand, by [Corollary 4.16](#),  $\text{maj}(T) = \text{maj}(V)$  and  $\text{symcoinv}(T) = \text{symcoinv}(V)$ . Thus,  $\text{coinv}(T) = \text{coinv}(V) + \rho(\alpha, \sigma, i)$ , and consequently  $\text{wt}(V) = t^{-\rho(\alpha, \sigma, i)} \text{wt}(T)$ . Therefore,

$$\begin{aligned}(1 - q^{L+1} t^{A+1}) \text{wt}(U) + (q^{L+1} t^{A+1}) (1 - t) t^{\rho(\alpha, \sigma, i)} \text{wt}(V) &= \\ &= (1 - q^{L+1} t^{A+1}) \text{wt}(T) + (q^{L+1} t^{A+1}) (1 - t) t^{-\rho(\alpha, \sigma, i)} t^{\rho(\alpha, \sigma, i)} \text{wt}(T) \\ &= (1 - q^{L+1} t^{A+1} + (1 - t) q^{L+1} t^{A+1}) \text{wt}(T) \\ &= (1 - q^{L+1} t^{A+2}) \text{wt}(T).\end{aligned}$$

Therefore, in either case, [\(4.17\)](#) holds. □

**Corollary 4.24.** *Let  $i \in [n - 1]$ ,  $\alpha = \alpha_1 \dots \alpha_n$  be a composition with  $\alpha_i > \alpha_{i+1}$ , and  $\sigma = [\sigma_1, \dots, \sigma_n] \in S_n$  be a permutation with  $\sigma_i < \sigma_{i+1}$ . Then, we have*

$$(1 - q^{L+1}t^{A+2}) \mathcal{E}_\alpha^\sigma = (1 - q^{L+1}t^{A+1}) \mathcal{E}_{s_i \cdot \alpha}^{\sigma s_i} + (q^{L+1}t^{A+1})(1 - t)t^{\rho(\alpha, \sigma, i)} \mathcal{E}_{s_i \cdot \alpha}^\sigma, \quad (4.18)$$

where  $L = \text{leg}(u) = \alpha_i - \alpha_{i+1} - 1$  and  $A = \text{arm}(u)$  for  $u = (i, \alpha_{i+1} + 1) \in \text{dg}(\alpha)$ .

*Proof.* The result follows by summing (4.17) in Lemma 4.23 over all  $T \in F(\alpha, \sigma)$ .  $\square$

**Theorem 4.25.** *Let  $\alpha$  be a composition such that  $\alpha_i > \alpha_{i+1}$  and  $\sigma \in S_n$  such that  $\sigma_i < \sigma_{i+1}$ . Then,*

$$E_\alpha^\sigma(X; q; t) = E_{s_i \cdot \alpha}^{\sigma s_i}(X; q; t) + (q^{L+1}t^{A+1})t^{\rho(\alpha, \sigma, i)} \frac{1 - t}{1 - q^{L+1}t^{A+1}} E_{s_i \cdot \alpha}^\sigma(X; q; t) \quad (4.19)$$

where  $L = \text{leg}(u) = \alpha_i - \alpha_{i+1} - 1$  and  $A = \text{arm}(u)$  for  $u = (i, \alpha_{i+1} + 1) \in \text{dg}(\alpha)$ .

*Proof.* The result follows from Lemma 4.23 by dividing both sides of (4.18) by the appropriate factor to convert the integral forms to non-integral forms.  $\square$

# Chapter 5

## Probabilistic bijective combinatorics for Macdonald polynomials

Alexandersson [Ale19, Theorem 22] established a partial symmetry result for permuted-basement Macdonald polynomials under certain conditions. Using [Ale19, Lemma 20 and Proposition 15], we reformulate the result as follows.

**Theorem 5.1** (equivalent to [Ale19, Theorem 22]). *Let  $\alpha = \alpha_1 \dots \alpha_n$  be a composition with  $\alpha_i = \alpha_{i+1}$  for some  $i \in [n - 1]$ . Let  $\sigma \in \mathcal{S}_n$ , such that  $\sigma_{i+1} = \sigma_i \pm 1$ . Then,*

$$E_{\alpha}^{\sigma}(\mathbf{x}; q, t) = E_{\alpha}^{\sigma s_i}(\mathbf{x}; q, t).$$

The main result of this chapter is stated in the following theorem, which extends **Theorem 5.1** by removing the assumption that  $\sigma_i$  and  $\sigma_{i+1}$  are consecutive.

**Theorem 5.2.** *Let  $\alpha = \alpha_1 \dots \alpha_n$  be a composition with  $\alpha_i = \alpha_{i+1}$  for some  $i \in [n - 1]$ , and let  $\sigma \in \mathcal{S}_n$ . Then,*

$$E_{\alpha}^{\sigma}(\mathbf{x}; q, t) = E_{\alpha}^{\sigma s_i}(\mathbf{x}; q, t).$$

The combinatorial interpretation of **Theorem 5.2** is that swapping the basement entries in two adjacent columns of the same height does not change the generating function for non-attacking fillings. Our approach for proving this builds upon a (classical) bijection introduced in [CHMMW22] for unrestricted fillings of partition shape, in which two adjacent entries of the filling in the bottom row of columns of the same height are swapped while preserving the overall weight of the filling. [Man25b] extended this technique to partition-shaped non-attacking fillings by introducing a probabilistic bijection. In this chapter, we

generalize this probabilistic bijection to composition-shaped non-attacking fillings, thereby giving a combinatorial proof of [Theorem 5.2](#).

As implications of [Theorem 5.2](#), certain assumptions can be removed from theorems in [\[Ale19\]](#) and [\[CMW22\]](#), which we summarize in [Section 5.3](#).

## 5.1 Probabilistic bijections

Probabilistic bijections, also referred to as *bijectivization* or *coupling* by [\[BP19\]](#), generalize weight-preserving bijections. We follow the exposition given in [\[FS22\]](#).

**Definition 5.3** (Probability map). Let  $A$  be an algebra. Let  $\mathbf{T}$  and  $\mathbf{U}$  be finite sets. A *probability map* from  $\mathbf{T}$  to  $\mathbf{U}$  is a function  $\text{prob} : \mathbf{T} \times \mathbf{U} \rightarrow A$  such that

$$\sum_{U \in \mathbf{U}} \text{prob}(T, U) = 1$$

for every  $T \in \mathbf{T}$ .

Note that these do not formally correspond to probability maps in the sense of probability theory because the values of  $\text{prob}$  are not required to be in the interval  $[0, 1]$  or even real numbers.

**Definition 5.4** (Probabilistic bijection [\[FS22\]](#)). Let  $A$  be an algebra. Let  $\mathbf{T}$  and  $\mathbf{U}$  be finite sets equipped with weight functions  $\text{wt}_{\mathbf{T}} : \mathbf{T} \rightarrow A$  and  $\text{wt}_{\mathbf{U}} : \mathbf{U} \rightarrow A$ . A *probabilistic bijection* between  $(\mathbf{T}, \text{wt}_{\mathbf{T}})$  and  $(\mathbf{U}, \text{wt}_{\mathbf{U}})$  is a pair of probability maps  $\text{prob}_{\mathbf{T}} : \mathbf{T} \times \mathbf{U} \rightarrow A$  and  $\text{prob}_{\mathbf{U}} : \mathbf{U} \times \mathbf{T} \rightarrow A$  such that

$$\text{wt}_{\mathbf{T}}(T) \text{prob}_{\mathbf{T}}(T, U) = \text{wt}_{\mathbf{U}}(U) \text{prob}_{\mathbf{U}}(U, T)$$

for every  $T \in \mathbf{T}$  and  $U \in \mathbf{U}$ .

A weight-preserving bijection  $f : \mathbf{T} \rightarrow \mathbf{U}$  canonically induces a probabilistic bijection between  $(\mathbf{T}, \text{wt}_{\mathbf{T}})$  and  $(\mathbf{U}, \text{wt}_{\mathbf{U}})$  by setting

$$\text{prob}_{\mathbf{T}}(T, U) = \text{prob}_{\mathbf{U}}(U, T) = \begin{cases} 1 & \text{if } f(T) = U, \\ 0 & \text{otherwise,} \end{cases}$$

explaining the choice of terminology. Moreover, as shown in [Proposition 5.5](#), this generalization retains a key property of weight-preserving bijections: the sum of the weights of  $\mathbf{T}$  and of  $\mathbf{U}$  are equal.

**Proposition 5.5.** *Let  $\text{prob}_{\mathbf{T}}: \mathbf{T} \times \mathbf{U} \rightarrow A$  and  $\text{prob}_{\mathbf{U}}: \mathbf{U} \times \mathbf{T} \rightarrow A$  form a probabilistic bijection between  $(\mathbf{T}, \text{wt}_{\mathbf{T}})$  and  $(\mathbf{U}, \text{wt}_{\mathbf{U}})$ . Then,*

$$\sum_{T \in \mathbf{T}} \text{wt}(T) = \sum_{U \in \mathbf{U}} \text{wt}(U).$$

*Proof.* We compute

$$\begin{aligned} \sum_{T \in \mathbf{T}} \text{wt}_{\mathbf{T}}(T) &= \sum_{T \in \mathbf{T}} \sum_{U \in \mathbf{U}} \text{wt}_{\mathbf{T}}(T) \text{prob}_{\mathbf{T}}(T, U) \\ &= \sum_{U \in \mathbf{U}} \sum_{T \in \mathbf{T}} \text{wt}_{\mathbf{U}}(U) \text{prob}_{\mathbf{U}}(U, T) = \sum_{U \in \mathbf{U}} \text{wt}_{\mathbf{U}}(U). \end{aligned} \quad \square$$

A Markov-theoretical interpretation of a probabilistic bijection is that it defines a bipartite Markov chain on the state space  $\mathbf{T} \cup \mathbf{U}$  with transition probabilities given by the function  $\text{prob}$  under a specific balance condition. In the case of [Definition 5.4](#), this condition is detailed balance, but a probabilistic bijection can also be defined using a more general balance condition.

## 5.2 A probabilistic bijection for non-attacking fillings

In this section, we define a map on non-attacking fillings that allows us to construct a probabilistic bijection between  $\text{NAF}(\alpha, \sigma)$  and  $\text{NAF}(\alpha, \sigma s_i)$  in order to prove [Theorem 5.2](#) in [Section 5.2.1](#).

In [\[CHMMW22\]](#), a weight-preserving bijection was established between fillings in  $F(\lambda)$  with statistics  $\text{maj}$  and  $\text{inv}$ , corresponding to swapping two adjacent entries in the bottom row within columns of the same height. This was done using an operator  $\tau_i$ , originally introduced in [\[LN12\]](#), on fillings in  $F(\lambda)$ . The bijection established the equality of generating functions over weighted fillings for the sets  $\{T \in F(\lambda) : \text{bottom row of } T \text{ is } w\}$  and  $\{T \in F(\lambda) : \text{bottom row of } T \text{ is } w s_i\}$  for an index  $i$  such that  $\lambda_i = \lambda_{i+1}$ . The result was used to obtain a compact formula for the modified Macdonald polynomials.

In our case, proving that  $E_{\alpha}^{\sigma} = E_{\alpha}^{\sigma s_i}$  combinatorially requires showing that the weight generating functions over  $\text{NAF}(\alpha, \sigma)$  and  $\text{NAF}(\alpha, \sigma s_i)$  are equal, which also amounts to finding a bijection on tableaux that swaps basement entries in columns  $i$  and  $i+1$ . However, the operator  $\tau_i$  used in [\[CHMMW22\]](#) does not preserve the non-attacking property of tableaux, so the same strategy could not be applied. This issue was resolved in [\[Man25b\]](#) by introducing a probabilistic operator  $\hat{\tau}_i$  for non-attacking tableaux with partition shape,

in order to obtain a compact formula for  $P_\lambda$ . The latter operator is the map we use to define our probabilistic bijection.

There are two key differences between our setting and the one in [Man25b]. First, our map applies to tableaux with composition shapes, whereas [Man25b] considers only partition shapes. This distinction allows [Man25b] to restrict attention to type A triples, while we consider both types. Second, our fillings include basements, eliminating the need for the so-called *degenerate* triples in [Man25b]. Hence, the probability map in Definition 5.8 and the surrounding lemmas are adapted from the original to fit our setting.

Throughout this section, let  $\alpha = \alpha_1 \dots \alpha_n$  be a composition, let  $\sigma \in \mathcal{S}_n$  be a permutation, and let  $i \in [n-1]$  such that  $\alpha_i = \alpha_{i+1}$ .

**Definition 5.6** ( $\mathfrak{t}_i^{(r)}$  and  $\mathfrak{t}_i^{[0,r]}$ ). Given a filling  $T$  of shape  $\alpha$  and  $r \in [0, \alpha_i]$ , let  $\mathfrak{t}_i^{(r)}(T)$  denote the filling obtained by swapping the entries in boxes  $(i, r)$  and  $(i+1, r)$  in  $T$ . Moreover, let  $\mathfrak{t}_i^{[0,r]} = \mathfrak{t}_i^{(r)} \circ \dots \circ \mathfrak{t}_i^{(1)} \circ \mathfrak{t}_i^{(0)}$ , which represents a sequence of swaps in the first  $r+1$  rows within the columns  $i$  and  $i+1$ .

Note that if  $T \in \mathcal{F}(\alpha, \sigma)$ , then  $\mathfrak{t}_i^{[0,r]}(T) \in \mathcal{F}(\alpha, \sigma s_i)$  for any  $r \geq 0$ . Also, the fillings  $T$  and  $\mathfrak{t}_i^{[0,r]}(T)$  have the same content. However,  $\mathfrak{t}_i^{(r)}$  and  $\mathfrak{t}_i^{[0,r]}$  do not in general preserve the property of a filling being non-attacking.

**Definition 5.7** ( $\rho_i^{(r)}$ ). Let  $T \in \mathcal{NAF}(\alpha, \sigma)$  be a non-attacking filling. Let  $r \in [0, \alpha_i - 1]$ . We define  $\rho_i^{(r)}(T) \in \mathbb{Q}(q, t)$ . Let  $a = T(i, r)$ ,  $b = T(i+1, r)$ ,  $c = T(i, r+1)$ , and  $d = T(i+1, r+1)$ . Let  $A = \text{arm}(i+1, r+1)$  and  $\ell = \text{leg}(i+1, r+1)$ .

If  $|\{a, b, c, d\}| = 4$ , then we split into two cases.

- (I) If  $\chi_{cda} = \chi_{cdb}$ , then  $\rho_i^{(r)}(T) = 0$ .
- (II) If  $\chi_{cda} = \chi_{dcb}$ , then  $\rho_i^{(r)}(T) = 1$ .

If  $|\{a, b, c, d\}| = 3$ , the non-attacking condition implies that  $a \neq b$ ,  $a \neq d$  and  $c \neq d$ , so we have three cases.

- (III) If  $b = c$ , then  $\rho_i^{(r)}(T) = 0$ .
- (IV) If  $b = d$ , then  $\rho_i^{(r)}(T) = 1$ .
- (V) If  $a = c$ , then

$$\rho_i^{(r)}(T) = t^{1-\chi_{dab}} \frac{1 - q^{\ell+1} t^{A+1}}{1 - q^{\ell+1} t^{A+2}}.$$

If  $|\{a, b, c, d\}| = 2$ , then we have one case.

- (VI) If  $a = c$  and  $b = d$ , then  $\rho_i^{(r)}(T) = 1$ .

|       | $T$                                                                                                                 | $\rho_i^{(r)}(T)$                                                  | $1 - \rho_i^{(r)}(T)$                                              |
|-------|---------------------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------|--------------------------------------------------------------------|
| (I)   | $\begin{smallmatrix} r+1 & \boxed{c \mid d} \\ r & \boxed{a \mid b} \end{smallmatrix}$<br>$\chi_{cda} = \chi_{cdb}$ | 0                                                                  | 1                                                                  |
| (II)  | $\begin{smallmatrix} r+1 & \boxed{c \mid d} \\ r & \boxed{a \mid b} \end{smallmatrix}$<br>$\chi_{cda} = \chi_{dcb}$ | 1                                                                  | 0                                                                  |
| (III) | $\begin{smallmatrix} r+1 & \boxed{b \mid d} \\ r & \boxed{a \mid b} \end{smallmatrix}$                              | 0                                                                  | 1                                                                  |
| (IV)  | $\begin{smallmatrix} r+1 & \boxed{c \mid b} \\ r & \boxed{a \mid b} \end{smallmatrix}$                              | 1                                                                  | 0                                                                  |
| (V)   | $\begin{smallmatrix} r+1 & \boxed{a \mid d} \\ r & \boxed{a \mid b} \end{smallmatrix}$                              | $t^{1-\chi_{dab}} \frac{1-q^{\ell+1}t^{A+1}}{1-q^{\ell+1}t^{A+2}}$ | $(q^{\ell+1}t^{A+1})^{\chi_{dab}} \frac{1-t}{1-q^{\ell+1}t^{A+2}}$ |
| (VI)  | $\begin{smallmatrix} r+1 & \boxed{a \mid b} \\ r & \boxed{a \mid b} \end{smallmatrix}$<br>$i \ i+1$                 | 1                                                                  | 0                                                                  |

Table 5.1: The values of  $\rho_i^{(r)}(T)$  and  $1 - \rho_i^{(r)}(T)$  for  $T \in \text{NAF}(\alpha, \sigma)$  and  $r \in [0, \alpha_i - 1]$ , on each case of [Definition 5.7](#).

[Table 5.1](#) summarizes the values of  $\rho_i^{(r)}(T)$  for  $T \in \text{NAF}(\alpha, \sigma)$  and  $r \in [0, \alpha_i - 1]$ .

For ease of computation, we note that, in case (V), we have

$$1 - \rho_i^{(r)}(T) = (q^{\ell+1}t^{A+1})^{\chi_{dab}} \frac{1-t}{1-q^{\ell+1}t^{A+2}},$$

which is straightforward to verify by splitting into cases of  $\chi_{dab}$  being 0 or 1 (compare to [\[Man25b, Lemma 3.3\]](#)).

As a convention, we set  $\rho_i^{(\alpha_i)}(T) = 0$ .

**Definition 5.8** ( $\text{prob}_i$ ). The map

$$\text{prob}_i: \text{NAF}(\alpha, \sigma) \times \text{F}(\alpha, \sigma s_i) \rightarrow \mathbb{Q}(q, t)$$

is defined for  $T \in \text{NAF}(\alpha, \sigma)$  and  $U \in \text{F}(\alpha, \sigma s_i)$  by letting

$$\text{prob}_i(T, U) = \left( \prod_{r=0}^{h-1} \rho_i^{(r)}(T) \right) (1 - \rho_i^{(h)}(T)) \quad (5.1)$$

if there exists  $h \in [0, \alpha_i]$  such that  $U = t_i^{[0, h]}(T)$ , and  $\text{prob}_i(T, U) = 0$  otherwise.

**Proposition 5.9.** *The map  $\text{prob}_i: \text{NAF}(\alpha, \sigma) \times \text{F}(\alpha, \sigma s_i) \rightarrow \mathbb{Q}(q, t)$  is a probability map, that is, for all  $T \in \text{NAF}(\alpha, \sigma)$ ,*

$$\sum_{U \in \mathcal{F}(\alpha, \sigma s_i)} \text{prob}_i(T, U) = 1.$$

*Proof.* Using that  $\rho_i^{(\alpha_i)}(T) = 0$ , it is straightforward to compute

$$\sum_{h=0}^{\alpha_i} \left( \prod_{r=0}^{h-1} \rho_i^{(r)}(T) \right) (1 - \rho_i^{(h)}(T)) = 1,$$

as required.

**Example 5.10.** Let  $T \in \text{NAF}(34400, [5, 1, 3, 4, 2])$  be the non-attacking filling

|   |   |   |   |   |  |
|---|---|---|---|---|--|
|   | 3 | 2 |   |   |  |
| 4 | 1 | 2 |   |   |  |
| 3 | 4 | 1 |   |   |  |
| 3 | 4 | 2 |   |   |  |
| 5 | 1 | 3 | 4 | 2 |  |

The values of  $\rho_2^{(r)}(T)$  and  $1 - \rho_2^{(r)}(T)$  are

$$\begin{array}{ll} \rho_2^{(0)}(T) = 1, & 1 - \rho_2^{(0)}(T) = 0, \\ \rho_2^{(1)}(T) = \frac{1 - q^3 t^1}{1 - q^3 t^2}, & 1 - \rho_2^{(1)}(T) = \frac{q^3 t^1 - q^3 t^2}{1 - q^3 t^2}, \\ \rho_2^{(2)}(T) = 0, & 1 - \rho_2^{(2)}(T) = 1, \\ \rho_2^{(3)}(T) = 1, & 1 - \rho_2^{(3)}(T) = 0, \\ \rho_2^{(4)}(T) = 0, & 1 - \rho_2^{(4)}(T) = 1. \end{array}$$

Hence, the values of  $\text{prob}_2 T, \mathfrak{t}_2^{[0,h]}(T)$  for  $h \in [0, 4]$  are

$$\begin{aligned} \text{prob}_2\left(T, \mathfrak{t}_2^{[0,0]}(T)\right) &= 0, \\ \text{prob}_2\left(T, \mathfrak{t}_2^{[0,1]}(T)\right) &= \frac{q^3 t^1 - q^3 t^2}{1 - q^3 t^2}, \\ \text{prob}_2\left(T, \mathfrak{t}_2^{[0,2]}(T)\right) &= \frac{1 - q^3 t^1}{1 - q^3 t^2}, \\ \text{prob}_2\left(T, \mathfrak{t}_2^{[0,3]}(T)\right) &= 0, \\ \text{prob}_2\left(T, \mathfrak{t}_2^{[0,4]}(T)\right) &= 0. \end{aligned}$$

The two nonzero values of  $\text{prob}_2(T, U)$  are obtained for  $U = t_2^{[0,1]}(T)$  and  $U = t_2^{[0,2]}(T)$ , which are, respectively,

$$\begin{array}{|c|c|c|} \hline & 3 & 2 \\ \hline 4 & 1 & 2 \\ \hline 3 & 4 & 1 \\ \hline 3 & 2 & 4 \\ \hline 5 & 3 & 1 & 4 & 2 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|} \hline & 3 & 2 \\ \hline 4 & 1 & 2 \\ \hline 3 & 1 & 4 \\ \hline 3 & 2 & 4 \\ \hline 5 & 3 & 1 & 4 & 2 \\ \hline \end{array}.$$

By construction, the  $\text{prob}_i$  is nonzero only for pairs of non-attacking fillings.

**Proposition 5.11.** *Let  $T \in \text{NAF}(\alpha, \sigma)$  be a non-attacking filling. Let  $U \in \text{F}(\alpha, \sigma s_i)$  be a filling. If  $\text{prob}_i(T, U) \neq 0$ , then  $U$  is non-attacking.*

*Proof.* Note that attacking pairs not involving columns  $i$  and  $i + 1$  remain unchanged between  $T$  and  $U$ , and attacking pairs involving one box in columns  $i$  or  $i + 1$  in  $T$  correspond to attacking pairs with a box in the same columns in  $U$ , though their positions may be swapped between the two columns. Hence, the only distinction in attacking pairs between  $T$  and  $U$  comes from the attacking boxes in both columns  $i$  and  $i + 1$ .

If  $\text{prob}_i(T, U) \neq 0$ , there exists  $h \in [0, \alpha_i]$  such that  $U = t_i^{[0,h]}(T)$ , and thus by (5.1),  $\rho_i^{(r)}(T) \neq 0$  for all  $r \in [0, h - 1]$  and  $\rho_i^{(h)}(T) \neq 1$ . Therefore, it suffices to show that if  $T$  is non-attacking, then  $U(i, r) \neq U(i + 1, r + 1)$  for all  $r \in [0, h]$ . For  $r \in [0, h - 1]$ , we have  $\rho_i^{(r)}(T) \neq 0$ , so the configuration of **Item (III)** does not occur; thus,  $U(i, r) = T(i + 1, r) \neq T(i, r + 1) = U(i + 1, r + 1)$ . For  $r = h$ , we have  $\rho_i^{(h)}(T) \neq 1$ , so configurations of **Item (IV)** or **Item (VI)** does not occur; thus,  $U(i, h) = T(i + 1, h) \neq T(i + 1, h + 1) = U(i + 1, h + 1)$ . Therefore,  $U$  is indeed non-attacking.  $\square$

**Corollary 5.12.** *The restriction*

$$\text{prob}_i: \text{NAF}(\alpha, \sigma) \times \text{NAF}(\alpha, \sigma s_i) \rightarrow \mathbb{Q}(q, t)$$

*is a probability map.*

As an abuse of notation, we write  $\text{prob}_i$  for both the function on  $\text{NAF}(\alpha, \sigma) \times \text{F}(\alpha, \sigma s_i)$  and the function on  $\text{NAF}(\alpha, \sigma s_i) \times \text{NAF}(\alpha, \sigma)$ . Moreover, we also write  $\text{prob}_i$  for both the function on  $\text{NAF}(\alpha, \sigma) \times \text{NAF}(\alpha, \sigma s_i)$  and the function on  $\text{NAF}(\alpha, \sigma s_i) \times \text{NAF}(\alpha, \sigma)$ .

The main result of this section is **Theorem 5.13**, which generalizes [Man25b, Lemma 4.4] to our setting.

**Theorem 5.13** (adapted from [Man25b, Lemma 4.4]). *The pair of maps defines a probabilistic bijection between  $\text{NAF}(\alpha, \sigma)$  and  $\text{NAF}(\alpha, \sigma s_i)$  with respect to the weight function  $\text{wt}_{q,t}$ . In other words, for all  $T \in \text{NAF}(\alpha, \sigma)$  and  $U \in \text{NAF}(\alpha, \sigma s_i)$ ,*

$$\text{wt}_{q,t}(U) \text{prob}_i(U, T) = \text{wt}_{q,t}(T) \text{prob}_i(T, U).$$

For organization purposes, we split the proof of **Theorem 5.13** into several lemmas.

**Lemma 5.14.** *Let  $a, b, c, d, f \in [n]$ . Then,*

$$\chi_{cfa} + \chi_{dfb} - \chi_{cfb} - \chi_{dfa} = \chi_{cb} + \chi_{da} - \chi_{ca} - \chi_{db}.$$

*Proof.* We directly compute

$$\begin{aligned} \chi_{cfa} + \chi_{dfb} - \chi_{cfb} - \chi_{dfa} &= (\chi_{cf} + \chi_{fa} - \chi_{ca}) + (\chi_{df} + \chi_{fb} - \chi_{db}) \\ &\quad - (\chi_{cf} + \chi_{fb} - \chi_{cb}) - (\chi_{df} + \chi_{fa} - \chi_{da}) \\ &= \chi_{cb} + \chi_{da} - \chi_{ca} - \chi_{db}. \end{aligned} \quad \square$$

The  $q, t$ -weight of a filling can be decomposed as the product of the weights contributed by the descents within each row and the inversions between every pair of rows. Let  $L = \max_i \alpha_i$  be the total number of rows. For  $0 \leq r \leq L - 1$ , define  $\text{dg}^{(r)}(\alpha)$  as the set of boxes in the  $r^{\text{th}}$  row of  $\text{dg}(\alpha)$ . Then, define

$$\text{maj}^{(r+1)}(T) = \sum_{u \in \text{Des}^{(r+1)}(T)} \text{leg}(u) + 1,$$

where  $\text{Des}^{(r+1)}(T) = \text{Des}(T) \cap \text{dg}^{(r+1)}(\alpha)$ . Also, define  $\text{coinv}^{(r,r+1)}(T)$  as the number of inversion triples in  $T$  within the pair of rows  $r$  and  $r + 1$ . Then, the  $r^{\text{th}}$  component of the  $q, t$ -weight of a non-attacking filling  $T \in \text{NAF}(\alpha, \sigma)$  is defined as

$$\text{wt}_{q,t}^{(r)}(T) = q^{\text{maj}^{(r+1)}(T)} t^{\text{coinv}^{(r,r+1)}(T)} \prod_{\substack{u \in \text{dg}^{(r+1)}(\alpha) \\ T(u) \neq T(d(u))}} \frac{1-t}{1-q^{1+\text{leg}(u)}t^{1+\text{arm}(u)}}, \quad (5.2)$$

yielding the decomposition

$$\text{wt}_{q,t}(T) = \prod_{r=0}^{L-1} \text{wt}_{q,t}^{(r)}(T).$$

**Lemma 5.15.** *Let  $T \in \text{NAF}(\alpha, \sigma)$  be a non-attacking filling. Let  $U = \mathfrak{t}_i^{[0, h]}(T)$  for some  $h \in [0, \alpha_i]$ . Then, for all  $r \in [0, h - 1]$ ,*

$$\mathbf{wt}_{q,t}^{(r)}(U) \rho_i^{(r)}(U) = \mathbf{wt}_{q,t}^{(r)}(T) \rho_i^{(r)}(T). \quad (5.3)$$

*Proof.* Let  $r \in [0, h - 1]$ . Let

$$\begin{aligned} c &= T(i, r + 1) = U(i + 1, r + 1), & d &= T(i + 1, r + 1) = U(i, r + 1), \\ a &= T(i, r) = U(i + 1, r), \text{ and} & b &= T(i + 1, r) = U(i, r). \end{aligned}$$

Since  $U$  is non-attacking, we have  $b \neq c$ . We note that  $\mathbf{maj}^{(r+1)}(T) = \mathbf{maj}^{(r+1)}(U)$ . Moreover, triples not involving columns  $i$  and  $i + 1$  are preserved between  $T$  and  $U$ , triples involving column  $i$  in  $T$  correspond to triples involving column  $i + 1$  in  $U$ , and triples involving column  $i + 1$  in  $T$  correspond to triples involving column  $i$  in  $U$ . Hence, the only distinction in the coinversion and inversion statistics between  $T$  and  $U$  comes from the triples involving both columns  $i$  and  $i + 1$ . Therefore,

$$\mathbf{coinv}^{(r, r+1)}(T) - \mathbf{coinv}^{(r, r+1)}(U) = \mathbf{inv}^{(r, r+1)}(U) - \mathbf{inv}^{(r, r+1)}(T) = \chi_{dcb} - \chi_{cda}.$$

We show that (5.3) holds for all cases of  $(a, b, c, d)$  with  $b \neq c$ .

First, suppose  $|\{a, b, c, d\}| = 4$ . Then, the sets  $\{u \in \text{Arm}(i + 1, r + 1) : T(u) \neq T(d(u))\}$  and  $\{u \in \text{Arm}(i + 1, r + 1) : U(u) \neq U(d(u))\}$  are equal.

- On the one hand, if  $\chi_{cda} = \chi_{cdb}$ , then  $\rho_i^{(r)}(T) = 0$  by **Item (I)**. Then, it also holds that  $\chi_{dcb} = \chi_{dca}$ , and hence  $\rho_i^{(r)}(U) = 0$  by **Item (I)**.
- On the other hand, if  $\chi_{cda} = 1 - \chi_{cdb} = \chi_{dcb}$ , then  $\rho_i^{(r)}(T) = 1$  by **Item (II)**. Then, it also holds that  $\chi_{dcb} = \chi_{cda}$ , and hence  $\rho_i^{(r)}(U) = 1$  by **Item (II)**. For the (co)inversion number, we have  $\mathbf{coinv}^{(r, r+1)}(T) - \mathbf{coinv}^{(r, r+1)}(U) = \chi_{dcb} - \chi_{cda} = 0$ .

Thus, in both cases for  $\chi_{cda}$ , (5.3) holds for  $|\{a, b, c, d\}| = 4$ .

Next, suppose  $|\{a, b, c, d\}| = 3$ . Since  $b \neq c$ , we either have  $b = d$  or  $a = c$ . Without loss of generality, assume  $b = d$ , since swapping  $T$  and  $U$  gives the other case. By **Item (IV)**, we have  $\rho_i^{(r)}(T) = 0$ . Moreover,

$$\rho_i^{(r)}(U) = t^{1 - \chi_{cba}} \frac{(1 - q^{\ell+1} t^{A+1})}{1 - q^{\ell+1} t^{A+2}},$$

where  $A = \text{arm}(i+1, r+1)$ , by **Item (V)**.  $\text{coinv}^{(r,r+1)}(T) - \text{coinv}^{(r,r+1)}(U) = \chi_{bcb} - \chi_{cba} = 1 - \chi_{cba}$ . Finally, the ratio

$$\left( \prod_{\substack{u \in \text{dg}^{(r+1)}\alpha \\ T(u) \neq T(d(u))}} \frac{1-t}{1-q^{\text{leg}(u)+1}t^{\text{arm}(u)+1}} \right) / \left( \prod_{\substack{u \in \text{dg}^{(r+1)}\alpha \\ U(u) \neq U(d(u))}} \frac{1-t}{1-q^{\text{leg}(u)+1}t^{\text{arm}(u)+1}} \right)$$

only differs in the boxes  $(i, r+1)$  and  $(i+1, r+1)$ , and thus equals

$$\left( \frac{1-t}{1-q^{\ell+1}t^{A+2}} \right) / \left( \frac{1-t}{1-q^{\ell+1}t^{A+1}} \right) = \frac{1-q^{\ell+1}t^{A+1}}{1-q^{\ell+1}t^{A+2}},$$

where  $A = \text{arm}(i+1, r+1)$ . Therefore, we compute

$$\frac{\text{wt}_{q,t}^{(r)}(T)}{\text{wt}_{q,t}^{(r)}(U)} = t^{1-\chi_{cba}} \frac{1-q^{\ell+1}t^{A+1}}{1-q^{\ell+1}t^{A+2}} = \rho_i^{(r)}(T, U),$$

confirming (5.3).

Finally, if  $|\{a, b, c, d\}| = 2$ , then  $a = c$  and  $b = d$ . Hence,  $\rho_i^{(r)}(T) = \rho_i^{(r)}(U) = 1$ , by **Item (VI)**. For the (co)inversion number, we have  $\text{coinv}^{(r,r+1)}(T) - \text{coinv}^{(r,r+1)}(U) = \chi_{bab} - \chi_{aba} = 1 - 1 = 0$ . Moreover, the sets  $\{u \in \text{Arm}(i+1, r+1) : T(u) \neq T(d(u))\}$  and  $\{u \in \text{Arm}(i+1, r+1) : U(u) \neq U(d(u))\}$  are equal. Thus, (5.3) holds in this final case, as well.  $\square$

**Lemma 5.16.** *Let  $T \in \text{NAF}(\alpha, \sigma)$  be a non-attacking filling. Let  $U = \mathfrak{t}_i^{[0,h]}(T)$  for some  $h \in [0, \alpha_i]$ . Then,*

$$\text{wt}_{q,t}^{(h)}(U) (1 - \rho_i^{(h)}(U)) = \text{wt}_{q,t}^{(h)}(T) (1 - \rho_i^{(h)}(T)). \quad (5.4)$$

*Proof.* If  $h = \alpha_i$ , then  $\text{wt}_{q,t}^{(\alpha_i)}(T) = \text{wt}_{q,t}^{(\alpha_i)}(U) = 1$  and  $1 - \rho_i^{(\alpha_i)}(T) = 1 - \rho_i^{(\alpha_i)}(U) = 1$ , so (5.4) holds trivially. Assume that  $h < \alpha_i$ . Then

$$\begin{aligned} c &= T(i, h+1) = U(i, h+1), & d &= T(i+1, h+1) = U(i+1, h+1), \\ a &= T(i, h) = U(i+1, h), \text{ and} & b &= T(i+1, h) = U(i, h). \end{aligned}$$

Since  $U$  is non-attacking, we have  $b \neq d$ . We note that

$$\text{maj}^{(h+1)}(T) - \text{maj}^{(h+1)}(U) = \chi_{ca} + \chi_{db} - \chi_{cb} - \chi_{da}.$$

Moreover, we have

$$\begin{aligned}\text{coinv}^{(h,h+1)}(T) - \text{coinv}^{(h,h+1)}(U) &= \text{inv}^{(h,h+1)}(U) - \text{inv}^{(h,h+1)}(T) \\ &= \chi_{cdb} - \chi_{cda} + \sum_{\substack{u \in \text{Arm}(i+1, h+1) \\ k=U(u)}} (\chi_{ckb} + \chi_{dka} - \chi_{cka} - \chi_{dkb}),\end{aligned}$$

which simplifies to  $(A+1)(\chi_{ca} + \chi_{db} - \chi_{cb} - \chi_{da})$  using [Lemma 5.14](#), where  $A = \text{arm}(i+1, h+1)$ .

We show that [\(5.4\)](#) holds for all possible cases for  $(a, b, c, d)$  with  $b \neq d$ .

First, suppose  $|\{a, b, c, d\}| = 4$ . Then, the sets  $\{u \in \text{Arm}(i+1, h+1) : T(u) \neq T(d(u))\}$  and  $\{u \in \text{Arm}(i+1, h+1) : U(u) \neq U(d(u))\}$  are equal.

- On the one hand, if  $\chi_{cda} = \chi_{cdb}$ , then  $1 - \rho_i^{(h)}(T) = 1$  by [Item \(I\)](#). Then, it also holds that  $\chi_{dcb} = \chi_{dca}$ , and hence  $1 - \rho_i^{(h)}(U) = 1$  by [Item \(I\)](#). It also holds that

$$\chi_{ca} + \chi_{db} - \chi_{cb} - \chi_{da} = 0,$$

therefore,  $\text{maj}^{(h+1)}(T) = \text{maj}^{(h+1)}(U)$  and  $\text{coinv}^{(h,h+1)}(T) = \text{coinv}^{(h,h+1)}(U)$ .

- On the other hand, if  $\chi_{cda} = 1 - \chi_{cdb} = \chi_{dcb}$ , then  $1 - \rho_i^{(h)}(T) = 0$  by [Item \(II\)](#). Then, it also holds that  $\chi_{dcb} = \chi_{dca}$ , and hence  $1 - \rho_i^{(h)}(U) = 0$  by [Item \(II\)](#).

Thus, in both cases for  $\chi_{cda}$ , [\(5.4\)](#) holds for  $|\{a, b, c, d\}| = 4$ .

Now suppose that  $|\{a, b, c, d\}| = 3$ . Since  $b \neq d$ , without loss of generality we may assume  $b = c$ . By [Item \(III\)](#), we have  $1 - \rho_i^{(h)}(T) = 1$ . Moreover,

$$1 - \rho_i^{(h)}(U) = (q^{L-h}t^{A+1})^{1-\chi_{dba}} \frac{1-t}{1-q^{L-h}t^{A+2}},$$

where  $A = \text{arm}(i+1, h+1)$ , by [Item \(V\)](#). It also holds that

$$\chi_{ba} + \chi_{db} - \chi_{bb} - \chi_{da} = \chi_{dba}.$$

For the major index, we have  $\text{maj}^{(r+1)}(T) - \text{maj}^{(r+1)}(U)$  equals  $(\ell+1)\chi_{dba}$ . For the coinversion number, we have  $\text{coinv}^{(r,r+1)}(T) - \text{coinv}^{(r,r+1)}(U)$  equals  $(A+1)\chi_{dba}$ . Finally, we have

$$\left( \prod_{\substack{u \in \text{dg}^{(r+1)}\alpha \\ T(u) \neq T(d(u))}} \frac{1-t}{1-q^{\text{leg}(u)+1}t^{\text{arm}(u)+1}} \right) / \left( \prod_{\substack{u \in \text{dg}^{(r+1)}\alpha \\ U(u) \neq U(d(u))}} \frac{1-t}{1-q^{\text{leg}(u)+1}t^{\text{arm}(u)+1}} \right) = \frac{(1-t)}{1-q^{\ell+1}t^{A+1}},$$

since  $u = (i, r + 1)$  is the only relevant box. Therefore, we compute

$$\frac{\mathbf{wt}_{q,t}^{(r)}(T)}{\mathbf{wt}_{q,t}^{(r)}(U)} = q^{(\ell+1)\chi_{dba}} t^{(A+1)\chi_{dba}} \frac{(1-t)}{1 - q^{\ell+1} t^{A+1}} = \rho_i^{(r)}(T, U),$$

and thus (5.4) holds for this case as well.

Finally, the case  $|\{a, b, c, d\}| = 2$  does not happen as  $b \neq d$  by the non-attacking property of  $U$ . Therefore, (5.4) holds in all cases.  $\square$

We are now ready to prove the main result of this section, **Theorem 5.13**.

*Proof of Theorem 5.13.* Assume that  $U = \mathfrak{t}_i^{[0,h]}(T)$  for some  $h \in [0, \alpha_i]$ . If such  $h$  doesn't exist, then  $\mathbf{prob}_i(T, U) = \mathbf{prob}_i(U, T) = 0$ , and the lemma holds trivially. For all  $r \in [h + 1, \alpha_i - 1]$ , since the  $r^{\text{th}}$  and  $(r + 1)^{\text{st}}$  rows of  $T$  and  $U$  are equal, we have

$$\mathbf{wt}_{q,t}^{(r)}(T) = \mathbf{wt}_{q,t}^{(r)}(U) \quad (5.5)$$

By **Lemma 5.15**, for all  $r \in [0, h - 1]$ , we have

$$\mathbf{wt}_{q,t}^{(r)}(U) \rho_i^{(r)}(U) = \mathbf{wt}_{q,t}^{(r)}(T) \rho_i^{(r)}(T), \quad (5.6)$$

and by **Lemma 5.16**, we have

$$\mathbf{wt}_{q,t}^{(h)}(U) (1 - \rho_i^{(h)}(U)) = \mathbf{wt}_{q,t}^{(h)}(T) (1 - \rho_i^{(h)}(T)). \quad (5.7)$$

Multiplying (5.5) for each  $r \in [h + 1, \alpha_i - 1]$ , (5.6) for all  $r \in [0, h - 1]$ , and (5.7), we obtain

$$\left( \prod_{r=0}^{\alpha_i-1} \mathbf{wt}_{q,t}^{(r)}(U) \right) \left( \prod_{r=0}^{h-1} \rho_i^{(r)}(U) \right) (1 - \rho_i^{(h)}(U)) = \left( \prod_{r=0}^{\alpha_i-1} \mathbf{wt}_{q,t}^{(r)}(T) \right) \left( \prod_{r=0}^{h-1} \rho_i^{(r)}(T) \right) (1 - \rho_i^{(h)}(T)).$$

By (5.2), and **Definition 5.8**, we obtain

$$\mathbf{wt}_{q,t}(U) \mathbf{prob}_i(U, T) = \mathbf{wt}_{q,t}(T) \mathbf{prob}_i(T, U). \quad \square$$

Recall that  $\mathbf{NAF}(\alpha, \sigma, \beta)$  denotes the set of non-attacking fillings of shape  $\alpha$  with content  $\beta$ . Since  $\mathbf{prob}_i(T, U) = 0$  whenever  $\mathbf{x}^T \neq \mathbf{x}^U$ , we may restrict the map in **Theorem 5.13** to obtain a probabilistic bijection between  $\mathbf{NAF}(\alpha, \sigma, \beta)$  and  $\mathbf{NAF}(\alpha, \sigma s_i, \beta)$ .

**Corollary 5.17.** *The pair of maps defines a probabilistic bijection between  $\mathbf{NAF}(\alpha, \sigma, \beta)$  and  $\mathbf{NAF}(\alpha, \sigma s_i, \beta)$  with respect to the weight function  $\mathbf{wt}_{q,t}$ .*

**Example 5.18.** Let  $n = 4$ ,  $\alpha = 2201$ ,  $\sigma = [3, 1, 2, 4]$ ,  $\beta = 1202$ . Let  $i = 1$  and note that  $\alpha_1 = \alpha_2 = 2$ . Moreover, note that  $\sigma s_i = [1, 3, 2, 4]$ . **Corollary 5.17** implies that the maps  $\text{prob}_1: \text{NAF}(\alpha, \sigma, \beta) \times \text{NAF}(\alpha, \sigma s_1, \beta) \rightarrow \mathbb{Q}(q, t)$  and  $\text{prob}_1: \text{NAF}(\alpha, \sigma s_1, \beta) \times \text{NAF}(\alpha, \sigma, \beta) \rightarrow \mathbb{Q}(q, t)$  form a probabilistic bijection.

The set  $\text{NAF}(\alpha, \sigma, \beta) = \text{NAF}(2201, [3, 1, 2, 4], 1202)$  consists of three fillings,  $T_1$ ,  $T_2$ , and  $T_3$ , and the set  $\text{NAF}(\alpha, \sigma s_i, \beta) = \text{NAF}(2201, [1, 3, 2, 4], 1202)$  consists of two fillings,  $U_1$  and  $U_2$ , shown in the left and right sides of **Figure 5.1**, respectively. **Figure 5.1** shows the values of  $\text{prob}_1(T, U)$  and  $\text{prob}_1(U, T)$  for each pair of fillings.

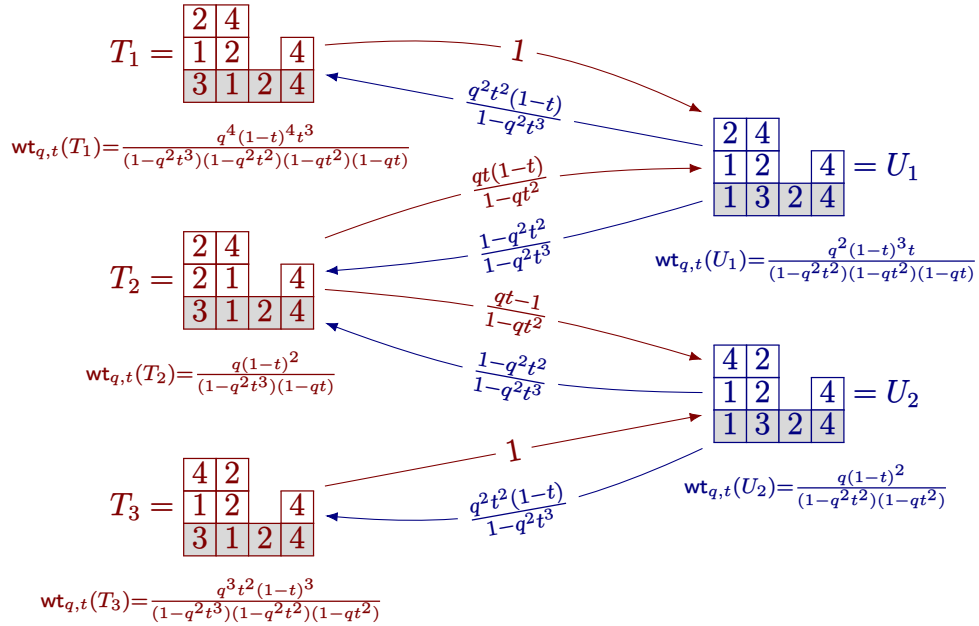


Figure 5.1: For each  $T \in \text{NAF}(2201, [3, 1, 2, 4], 1202)$  (right, in red), and each  $U \in \text{NAF}(2201, [1, 3, 2, 4], 1202)$  (left, in blue), the label of the red arrow from  $T$  to  $U$  is the value of  $\text{prob}_1(T, U)$ , and the label of the blue arrow from  $U$  to  $T$  is the value of  $\text{prob}_1(U, T)$ . The lack of an arrow indicates that the value is zero. Below each filling, we show the value of  $\text{wt}_{q,t}$ .

The fact that the maps form a probabilistic bijection can be checked by computing that, for each  $T \in \{T_1, T_2, T_3\}$  and each  $U \in \{U_1, U_2\}$ , we have

$$\text{prob}_1(T, U) \text{wt}_{q,t}(T) = \text{prob}_1(U, T) \text{wt}_{q,t}(U).$$

Moreover, using the fact that there exists a probabilistic bijection between  $\{T_1, T_2, T_3\}$  and

$\{U_1, U_2\}$ , **Theorem 5.13** implies that their  $q, t$ -weight generating functions are equal, that is,

$$\text{wt}_{q,t}(T_1) + \text{wt}_{q,t}(T_2) + \text{wt}_{q,t}(T_3) = \text{wt}_{q,t}(U_1) + \text{wt}_{q,t}(U_2).$$

This equality can be checked from the known values of  $\text{wt}_{q,t}$  in **Figure 5.1**. Note that the left-hand side is the coefficient  $[\mathbf{x}^{1202}]E_{2201}^{[3,1,2,4]}(\mathbf{x}; q, t)$  and the right-hand side is the coefficient  $[\mathbf{x}^{1202}]E_{2201}^{[1,3,2,4]}(\mathbf{x}; q, t)$ .

### 5.2.1 Bijective proof of **Theorem 5.2**

We now have the tools to give a bijective proof for the main result of this chapter, **Theorem 5.2**, a symmetry theorem for the permuted-basement Macdonald polynomials. Let's recall the statement.

**Theorem 5.2.** *Let  $\alpha = \alpha_1 \dots \alpha_n$  be a composition with  $\alpha_i = \alpha_{i+1}$  for some  $i \in [n-1]$ , and let  $\sigma \in \mathcal{S}_n$ . Then,*

$$E_\alpha^\sigma(\mathbf{x}; q, t) = E_\alpha^{\sigma s_i}(\mathbf{x}; q, t).$$

*Proof.* We write

$$E_\alpha^\sigma(\mathbf{x}; q, t) = \sum_{T \in \text{NAF}(\alpha, \sigma)} \text{wt}(T) = \sum_{\text{compositions } \beta} x^\beta \left( \sum_{T \in \text{NAF}(\alpha, \sigma, \beta)} \text{wt}_{q,t}(T) \right).$$

For each composition  $\beta = \beta_1 \dots \beta_n$ , **Corollary 5.17** implies that there exists a probabilistic bijection between  $\text{NAF}(\alpha, \sigma, \beta)$  and  $\text{NAF}(\alpha, \sigma s_i, \beta)$  with respect to  $\text{wt}_{q,t}$ . Hence, **Proposition 5.5** implies that

$$[x^\beta]E_\alpha^\sigma(\mathbf{x}; q, t) = \sum_{T \in \text{NAF}(\alpha, \sigma, \beta)} \text{wt}_{q,t}(T) = \sum_{U \in \text{NAF}(\alpha, \sigma s_i, \beta)} \text{wt}_{q,t}(U) = [x^\beta]E_\alpha^{\sigma s_i}(\mathbf{x}; q, t),$$

and the result follows.  $\square$

Given a composition  $\alpha = \alpha_1 \dots \alpha_n$ , let  $\mathcal{C}_\alpha$  be the subgroup of  $\mathcal{S}_n(\alpha)$  generated by the transpositions  $\{s_i : \alpha_i = \alpha_{i+1}\}$ . Writing  $\alpha = p_1^{m_1} p_2^{m_2} \dots p_k^{m_k}$ , we have that

$$\mathcal{C}_\alpha = \mathcal{S}_{[1, m_1]} \times \mathcal{S}_{[m_1+1, m_1+m_2]} \times \dots \times \mathcal{S}_{[m_1+m_2+\dots+m_{k-1}+1, n]} \subseteq \mathcal{S}_n(\alpha).$$

For example, if  $\alpha = 112213 = 1^2 2^2 1^1 3^1$ , then  $\mathcal{C}_\alpha$  is the subgroup of  $\mathcal{S}_6(\alpha)$  generated by the transpositions  $s_1$  and  $s_3$ :

$$\mathcal{C}_\alpha = \{[1, 2, 3, 4, 5, 6], [2, 1, 3, 4, 5, 6], [1, 2, 4, 3, 5, 6], [4, 1, 4, 3, 5, 6]\}.$$

The group  $\mathcal{S}_6(\alpha)$  is generated by  $\{s_1, s_3, s_{2,5}\}$ , where  $s_{2,5} = s_2 s_3 s_4 s_3 s_2$ . For instance,  $[5, 2, 3, 4, 1, 6] \in \mathcal{S}_n(\alpha) \setminus \mathcal{C}_\alpha$ .

**Corollary 5.19.** *Let  $\alpha = \alpha_1 \dots \alpha_n$  be a composition. Then, if  $\sigma, \tau \in \mathcal{S}_n$  are such that  $\sigma^{-1}\tau \in \mathcal{C}_\alpha$ , then*

$$E_\alpha^\sigma(\mathbf{x}; q, t) = E_\alpha^\tau(\mathbf{x}; q, t).$$

*Proof.* By induction, it suffices to prove the statement for  $\sigma^{-1}\tau = s_i$  for some  $i$  such that  $\alpha_i = \alpha_{i+1}$ . Then, the result follows from [Theorem 5.2](#).  $\square$

If  $\alpha = p_1^{m_1} p_2^{m_2} \dots p_k^{m_k}$ , then  $\mathcal{C}_\alpha = \mathcal{S}_n(\alpha)$  if and only if  $p_1, p_2, \dots, p_k$  are pairwise distinct. In particular, this is the case when  $\alpha$  is a partition or an antipartition.

**Corollary 5.20.** *Let  $\alpha = \alpha_1 \dots \alpha_n$  be a partition or an antipartition. If  $\sigma, \tau \in \mathcal{S}_n$  are such that  $\sigma \cdot \alpha = \tau \cdot \alpha$ , then*

$$E_\alpha^\sigma(\mathbf{x}; q, t) = E_\alpha^\tau(\mathbf{x}; q, t).$$

*Proof.* Since  $\sigma^{-1}\tau \in \mathcal{S}_n(\alpha)$ , when  $\alpha$  is a partition or an antipartition,  $\sigma^{-1}\tau \in \mathcal{C}_\alpha$ , and [Corollary 5.19](#) applies.  $\square$

## 5.3 Applications

Immediate applications of [Theorem 5.2](#) include the removal of certain assumptions from theorems in [\[Ale19\]](#) and [\[CMW22\]](#).

### 5.3.1 $t$ -atom polynomials and $t$ -key polynomials

Fix a composition  $\alpha = \alpha_1 \dots \alpha_n$  and a permutation  $\sigma \in \mathcal{S}_n$ . The *permuted basement  $t$ -atom polynomial of  $\alpha$  with basement  $\sigma$* , first introduced in [\[HLMW11\]](#) and further studied in [\[AS19\]](#), is defined as

$$\mathcal{A}_\alpha^\sigma(\mathbf{x}; t) = E_\alpha^\sigma(\mathbf{x}; 0, t).$$

Similarly, the  *$t$ -atom polynomial*  $\mathcal{A}_\alpha(\mathbf{x}; t)$  and the  *$t$ -key polynomial*  $\mathcal{K}_\alpha(\mathbf{x}; t)$  are defined as

$$\mathcal{A}_\alpha(\mathbf{x}; t) = \mathcal{A}_\alpha^{\text{id}}(\mathbf{x}; t) \quad \text{and} \quad \mathcal{K}_\alpha(\mathbf{x}; t) = \mathcal{A}_\alpha^{w_0}(\mathbf{x}; t).$$

**Proposition 5.21** (inferred from [Ale19, Proposition 27]). *Let  $\alpha = \alpha_1 \dots \alpha_n$  be a composition. Let  $\sigma \in \mathcal{S}_n$  be the shortest permutation such that  $\sigma \cdot \text{inc}(\alpha) = \alpha$ , and let  $\pi \in \mathcal{S}_n$  be the shortest permutation such that  $\pi \cdot \text{dec}(\alpha) = \alpha$ .*

$$\mathcal{A}_\alpha(\mathbf{x}; t) = \mathcal{A}_{\text{inc}(\alpha)}^\sigma(\mathbf{x}; t) \quad \text{and} \quad \mathcal{K}_\alpha(\mathbf{x}; t) = \mathcal{A}_{\text{dec}(\alpha)}^{w_0\pi}(\mathbf{x}; t).$$

Using Corollary 5.20, we can remove the length restriction on  $\sigma$  and  $\pi$  above.

**Corollary 5.22.** *Let  $\alpha = \alpha_1 \dots \alpha_n$  be a composition. Let  $\tau \in \mathcal{S}_n$  be any permutation such that  $\tau \cdot \text{inc}(\alpha) = \alpha$ , and let  $\pi \in \mathcal{S}_n$  be any permutation such that  $\pi \cdot \text{dec}(\alpha) = \alpha$ . Then,*

$$\mathcal{A}_\alpha(\mathbf{x}; t) = \mathcal{A}_{\text{inc}(\alpha)}^\tau(\mathbf{x}; t) \quad \text{and} \quad \mathcal{K}_\alpha(\mathbf{x}; t) = \mathcal{A}_{\text{dec}(\alpha)}^{w_0\pi}(\mathbf{x}; t).$$

*Proof.* Let  $\sigma \in \mathcal{S}_n$  be the shortest permutation such that  $\sigma \cdot \text{inc}(\alpha) = \alpha$ . By Proposition 5.21, we have  $\mathcal{A}_\alpha(\mathbf{x}; t) = \mathcal{A}_{\text{inc}(\alpha)}^\sigma(\mathbf{x}; t) = E_{\text{inc}(\alpha)}^\sigma(\mathbf{x}; 0, t)$ . Since  $\tau \cdot \text{inc}(\alpha) = \sigma \cdot \text{inc}(\alpha)$ , Corollary 5.20 implies  $E_{\text{inc}(\alpha)}^\sigma(\mathbf{x}; 0, t) = E_{\text{inc}(\alpha)}^\tau(\mathbf{x}; 0, t) = \mathcal{A}_{\text{inc}(\alpha)}^\tau(\mathbf{x}; t)$ .

The proof for  $\mathcal{K}_\alpha$  is analogous, and is omitted.  $\square$

### 5.3.2 ASEP polynomials

In [CMW22], the authors introduce the ASEP polynomial  $F_\alpha(\mathbf{x}; q, t)$ , which are weight-generating functions for multiline queues with bottom row  $\alpha$ . A key result from their work is the following proposition.

**Proposition 5.23** ([CMW22, Proposition 3.1]). *Let  $\alpha = \alpha_1 \dots \alpha_n$  be a composition. Let  $\sigma \in \mathcal{S}_n$  be the longest permutation such that  $\sigma^{-1} \cdot \alpha = \text{inc}(\alpha)$ . Then,*

$$F_\alpha(\mathbf{x}; q, t) = E_{\text{inc}(\alpha)}^\sigma(\mathbf{x}; q, t).$$

By Corollary 5.20, we can remove the length requirement on  $\sigma$ .

**Corollary 5.24.** *Let  $\alpha = \alpha_1 \dots \alpha_n$  be a composition, and let  $\tau \in \mathcal{S}_n$  such that  $\tau^{-1} \cdot \alpha = \text{inc}(\alpha)$ . Then,*

$$F_\alpha(\mathbf{x}; q, t) = E_{\text{inc}(\alpha)}^\tau(\mathbf{x}; q, t).$$

This allows us to realize  $t$ -atom polynomials as specializations of ASEP polynomials.

**Corollary 5.25.** *Let  $\alpha = \alpha_1 \dots \alpha_n$  be a composition. Then,*

$$F_\alpha(\mathbf{x}; 0, t) = \mathcal{A}_\alpha(\mathbf{x}; t).$$

*Proof.* Let  $\tau \in \mathcal{S}_n$  such that  $\tau^{-1} \cdot \alpha = \text{inc}(\alpha)$ . From [Corollary 5.24](#) and [Corollary 5.22](#), we have  $F_\alpha(\mathbf{x}; 0, t) = E_{\text{inc}(\alpha)}^\tau(\mathbf{x}; 0, t) = \mathcal{A}_{\text{inc}(\alpha)}^\tau(\mathbf{x}; t) = \mathcal{A}_\alpha(\mathbf{x}; t)$ .  $\square$

Finally, for a partition  $\lambda = \lambda_1 \dots \lambda_n$ , it was observed in [\[CMW22\]](#) that

$$P_\lambda(\mathbf{x}; q, t) = \sum_{\mu \in S_n \cdot \lambda} E_{\text{inc}(\lambda)}^{\sigma_\mu}(\mathbf{x}; q, t), \quad (5.8)$$

where  $S_n \cdot \lambda$  is the set of rearrangements of  $\lambda$ ,  $\text{inc}(\lambda) = (\lambda_n, \dots, \lambda_1)$ , and  $\sigma_\mu$  is the longest permutation that rearranges  $\text{inc}(\lambda)$  into  $\mu$ .

[Corollary 5.20](#) allows us to rewrite (5.8) without the length restriction on  $\sigma_\mu$ .

**Corollary 5.26.** *Let  $\lambda = \lambda_1 \dots \lambda_n$  be a partition. Then the symmetric Macdonald polynomial satisfies*

$$P_\lambda(\mathbf{x}; q, t) = \sum_{\mu \in S_n \cdot \lambda} E_{\text{inc}(\lambda)}^{\sigma_\mu}(\mathbf{x}; q, t),$$

where  $\sigma_\mu \in S_n$  is any permutation such that  $\sigma_\mu \cdot \text{inc}(\lambda) = \mu$ .



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