Degenerate Parabolic Diffusion Equations: Theory and Applications in Climatology

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

Diffusion models describe the spread of particles, energy, or other entities within a medium. Perturbations of mechanical systems, random walks(discrete case), and Brownian motion(continuous-time stochastic process) are some classical methods used to model diffusion. Among these, those generated by stochastic processes have been extensively studied by employing the Fokker-Planck equation—a one-dimensional parabolic partial differential equation—to examine these systems by analyzing the probability density function. Given the incomplete theory surrounding degenerate diffusion equations, our objective is to generalize and expand existing results for degenerate diffusion processes by examining cases where weak degeneracy occurs at the boundaries, utilizing a Fokker-Planck-like equation. More precisely, we first address the well-posedness results, which ensure the existence and uniqueness of a solution and are critical for investigating other qualitative properties such as controllability, observability, stabilization, and optimal control. Additionally, we explore the interval of the existence or absence of stationary states, which is fundamental in the analysis of mechanical or physical systems. To this end, we examine sufficient conditions for both the non-existence and existence of stationary points. Furthermore, to verify and illustrate our analytical results, we delve into the Budyko-Sellers model, a climate model, providing results on its well-posedness and addressing the inverse problem of determining the insolation function. Throughout this study, we primarily employ semigroup theory, operator and functional analysis, and weighted Sobolev spaces to manage the non-ellipticity of the diffusion and well-posedness of the parabolic equation, while using the theory of Lyapunov functions to ensure the existence of stationary states.

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Dedication

This is dedicated to my family and partner for their consistent support in my work.

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6.2 The figures show the energy balance of the Earth. Specifically, they display the annual distribution of global temperatures across different latitudinal belts, illustrating the system's energy balance and heat transport across various latitudes. 69

List of Abbreviations

EBM Energy Balance Model 63–65

FP Fokker-Planck 13–17, 20–25, 54–58, 60, 61, 77

GCM Global Climate Models 64

ODE Ordinary Differential Equation 1, 24, 55

PDE Partial Differential Equation 1, 10, 12–15, 25, 27, 29, 40, 43, 63, 74, 77, 78

PDF Probability Density Function 14–16, 18, 19, 21, 22, 24, 57, 77

Pr Probability 12

RAD Reaction-Advection-Diffusion 1, 25

SDE Stochastic Differential Equation 2, 3, 12, 13, 15, 25, 55, 57, 60, 61, 77

SDeg Strong Degeneracy 26–28, 61

SP Stochastic Process 14, 60, 77

WDeg Weak Degeneracy 26–29

List of Symbols

C Continuous functions 43

C^1 Once continuously differentiable functions 14

$C^{1,\beta}_{\mathbf{loc}}$ Functions that are locally in $C^{1,\beta}$, i.e., continuously differentiable with locally β-Hölder continuous derivatives 57

C^2 Twice continuously differentiable functions 56

$C^{2,\beta}_{\mathbf{loc}}$ Functions that are locally in $C^{2,\beta}$, i.e., twice continuously differentiable with locally β-Hölder continuous derivatives 57

C^∞_0 Infinitely differentiable functions with compact support 56

C^k k-times continuously differentiable functions 5

$C^{k,\alpha}$ Functions that are k-times continuously differentiable with α-Hölder continuous derivatives 5

$\mathbb{E}$ The mean value 17

ess sup Essential supremum 6

H^1 Functions with square-integrable first derivatives 7

H^2 Functions with square-integrable second derivatives 7

L^1 Integrable functions 31

L^2 Square-integrable functions 28

$\mathcal{L}(\mathbf{X})$ Bounded linear operators from X to X 11

L^∞ Essentially bounded functions 6

L^p p-integrable functions 6

$W^{1,p}$ Functions with first-order weak derivatives in L^p 6

$W^{1,q}_{\mathbf{loc}}$ Functions that have locally integrable first-order weak derivatives in L^q 57

$W^{k,p}$ Functions with k-th order weak derivatives in L^p 7

Chapter 1

Introduction

Our world is full of exciting events, some easily detectable while others require careful observation. The speed, magnitude, and frequency are vital characteristics of these phenomena. To understand and interpret these events deeply, we must focus not only on the results but also on the study of the building blocks of the evolutionary process. This is crucial for the development of scientific knowledge.

One way to achieve this in-depth perception is by utilizing simple or advanced mathematical tools. The nature and complexity of a system dictate the systematic method one must choose to work with. Dynamical systems, which evolve according to specific laws, are particularly suitable for tackling the challenges inherent in these processes. We can address a broad spectrum of dynamical systems, including Ordinary Differential Equation (ODE), neural networks, Partial Differential Equation (PDE), or even coupled systems consisting of ODEs and PDEs. Each problem has its source of complexity, demanding a different treatment approach, which is why classification among these problems is essential. A fundamental way of classification is the distinction between linear and non-linear equations. In the first case, there is a very good understanding of these systems, and they are usually referred to as the classical case in the literature. Non-linear equations, on the other hand, represent the most complex case and often pose significant challenges.

Moreover, PDE systems are a significant research topic in contemporary science. They model a wide range of processes across various scientific fields such as physics, engineering, biology and climatology. Each of these sciences benefits from the use of PDEs in ways unique to their specific challenges and goals. It is well-known that only a few PDEs can be solved analytically. Even when an analytic solution is possible, it often does not significantly contribute to our analysis or research objectives. Therefore, instead of solving the PDE system directly, it is crucial to study the qualitative characteristics of the equations, such as well-posedness, stability, controllability, and more. To investigate these qualitative aspects, we employ advanced mathematical tools like functional or operator analysis. These tools help address the complex questions that arise and enable us to expand our understanding of a more general class of PDE systems. Through this approach, we can gain insights that are not only theoretically profound but also practically applicable across various scientific domains.

A major motivation of this thesis is the study of a special class of second-order partial differential equations known as Reaction-Advection-Diffusion (RAD) equations. These equations are fundamental in applied sciences due to their multifactorial nature, encom-

passing numerous physical processes. They are particularly useful in modeling phenomena across various disciplines.

In flow dynamics, reaction-advection-diffusion equations are employed to model collective movements, such as the flow of fluids or the migration patterns of organisms. In evolutionary biology, these equations describe the evolution of populations, capturing the effects of natural selection, genetic drift, and spatial distribution. In climatology, they are used to model the evolution of global temperature, taking into account both external influences, like solar radiation, and internal factors, such as ocean currents and atmospheric composition.

These equations often involve different combinations of reaction, advection (also known as the drift term), and diffusion processes. For example, some models may focus solely on reaction-diffusion equations, which describe processes where chemical reactions are coupled with diffusion, such as the spread of pollutants in air or water. Others may use advection-diffusion equations, which describe the transport of substances under the influence of a velocity field, like the dispersion of contaminants in a river.

In this thesis, we are particularly interested in the diffusion terms of these equations. Diffusion processes are a subset of these equations where the primary focus is on the spreading of particles, heat, or other quantities due to random motion. The mathematical representation of diffusion processes involves terms that describe how substances spread over time, driven by concentration gradients.

A critical aspect of diffusion processes is the incorporation of random motion, often referred to as noise perturbations. These perturbations add an extra layer of complexity to the system, as they account for the unpredictable and stochastic nature of the particles' movement. One of the most common models for this randomness is white noise, which represents random fluctuations that can occur in any direction at any time. Over the years, significant developments have occurred in tackling different sources of noise, varying in magnitude and regularity. The stochastic term added to the system can result in a deterministic system or a fully stochastic system. Many advanced tools have been developed to study these systems, which include Stochastic Differential Equation (SDE) that model how random fluctuations influence diffusion and other processes.

By exploring these diffusion terms and incorporating noise perturbations, we aim to gain deeper insights into the underlying mechanisms of various physical and biological systems. Understanding how diffusion interacts with other processes, such as advection and reaction, and how noise influences these interactions can lead to more accurate models and predictions, which are crucial for advancements in applied sciences.

The major difficulty with the noise term is its non-differentiability (e.g., Brownian motion, Markov processes, Weierstrass functions), which prevents the use of classical tools. However, this has not hindered the development of stochastic mathematics. Many scientists have developed advanced mathematical tools, redefining integrals adapted to stochastic processes and ultimately establishing Itô calculus. Another significant contribution to this process is that a large class of stochastic processes exhibits Hölder continuity, which has proven pioneering in this field. Hölder continuity is a condition that provides a weaker form of continuity compared to differentiability and is significant in the study of stochastic processes.

Additionally, another way to study diffusion systems containing stochastic terms is by examining the behavior of certain statistical quantities, such as the probability density

function of the evolutionary state variable. This approach allows us to gain insights into the overall dynamics of the system under the influence of randomness.

A well-known and extensively studied tool for this purpose is the Fokker-Planck equation. The Fokker-Planck equation provides a deterministic framework for describing the time evolution of the probability density function associated with the SDE governing the system. By connecting the SDE with the probability density function, the Fokker-Planck equation enables us to understand how random perturbations influence the distribution of states over time.

The Fokker-Planck equation is particularly powerful because it transforms the problem into a fully deterministic one, even though the underlying system is stochastic. This allows for the application of various analytical and numerical methods to study the diffusion process more systematically and rigorously.

We will delve deeper into the details of the Fokker-Planck equation and its applications in the following chapters (Chapter 3,4,5).

Chapter 2

Basic Definitions

2.1 Fundamental Definitions and Sobolev Spaces

We will begin by introducing fundamental concepts and terminologies that form the foundation of our analysis. These definitions will provide the essential background needed to understand and follow the subsequent discussions and results presented in this thesis.

Initially, we will use two notations for the partial derivative: either $(\cdot)_x$ or $\frac{\partial}{\partial x}$, depending on the context and specific needs. Specifically, in Chapters 4, 5, and 6, we will use the former notation, while in Chapter 3, we will use both notations.

Let X be a general functional space. The definition of positive or semi-positive definite is given by the following two definitions.

Definition 1. *A function $a : X \to \mathbb{R}$ is called **positive definite** if for all $x \in X$ and any non-zero vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$:*

$$\xi^\top a(x)\xi = \sum_{i,j=1}^{n} \xi_i \xi_j a(x) > 0.$$

Definition 2. *A function $a : X \to \mathbb{R}$ is called **positive semi-definite** if for all $x \in X$ and any vector $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$, $n \in \mathbb{N}$:*

$$\xi^\top a(x)\xi = \sum_{i,j=1}^{n} \xi_i \xi_j a(x) \geq 0.$$

Moreover, for the special case $X = \mathbb{R}$, the above definitions take the form:

Definition 3. *A function $a : \mathbb{R} \to \mathbb{R}$ is called **positive definite** if for all $x \in \mathbb{R}$,*

$$a(x) > 0.$$

Definition 4. *A function $a : \mathbb{R} \to \mathbb{R}$ is called **positive semi-definite** if for all $x \in \mathbb{R}$,*

$$a(x) \geq 0.$$

Let us now define what it means for a matrix to be **positive definite** in $\mathbb{R}^n$, where$n \in \mathbb{N}$.

Definition 5. *A matrix $C = (c_{i,j})_{i,j=1,2,\ldots,n}$ is called **positive definite** if for all non-zero vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)^\top \in \mathbb{R}^n$, the following condition holds:*

$$\mathbf{x}^\top C\mathbf{x} > 0$$

That is,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} c_{i,j}x_i x_j > 0$$

for all non-zero $\mathbf{x} \in \mathbb{R}^n$.

Remark 1. *In the case of a **positive semi-definite** matrix, the condition:*

$$\mathbf{x}^\top C\mathbf{x} \geq 0$$

must be satisfied for all vectors $\mathbf{x} \in \mathbb{R}^n$.

Continuous Function Spaces

Continuous function spaces deal with continuous functions on a subset $\Omega \subset \mathbb{R}^n, n \in \mathbb{N}$.

Definition 6. *The space $C^k(\Omega)$ for an integer $k \geq 0$ is defined as:*

$$C^k(\Omega) := \{u : \Omega \to \mathbb{R} \mid \partial^\alpha u \text{ exists and is continuous for all multi-indices } |\alpha| \leq k\},$$

where $\alpha = (\alpha_1, ..., \alpha_n)$ and $|a| = \alpha_1 + ... + \alpha_n$, for $n \in \mathbb{N}$.

The norm in $C^k(\Omega)$ is given by:

$$\|u\|_{C^k(\Omega)} := \sum_{|\alpha|\leq k} \sup_{x\in\Omega} |\partial^\alpha u(x)|.$$

Definition 7. *The space $C^{k,\alpha}(\Omega)$ for an integer $k \geq 0$ and $0 < \alpha \leq 1$ is defined as:*

$$C^{k,\alpha}(\Omega) := \left\{u \in C^k(\Omega) \mid \partial^\beta u \text{ is Hölder continuous of order } \alpha \text{ for all } |\beta| = k\right\}.$$

The norm in $C^{k,\alpha}(\Omega)$ is given by:

$$\|u\|_{C^{k,\alpha}(\Omega)} := \|u\|_{C^k(\Omega)} + \sum_{|\beta|=k} \sup_{x\neq y} \frac{|\partial^\beta u(x) - \partial^\beta u(y)|}{|x-y|^\alpha}.$$

Using Definitions 6 and 7, we have the following two special cases.

Definition 8. *The space $C^{1,0}(\overline{\Omega})$ is the space of functions with continuous first derivatives and is defined as:*

$$C^{1,0}(\overline{\Omega}) := \left\{u \in C^1(\overline{\Omega}) \mid \partial_i u \in C(\overline{\Omega}) \text{ for all } i = 1, \ldots, n\right\}.$$

Definition 9. *The space $C^{1,2}(\overline{\Omega})$ is the space of functions with continuous first and second partial derivatives, defined as:*

$$C^{1,2}(\overline{\Omega}) := \left\{u \in C(\overline{\Omega}) \mid \partial_i u, \partial_{ij} u \in C(\overline{\Omega}) \text{ for all } i, j = 1, \ldots, n\right\}.$$

These function spaces form the foundation for studying various problems in partial differential equations, functional analysis, and other areas of mathematics.

Hilbert Spaces

Definition 10. *A Hilbert space is a linear space H equipped with an inner product (f, g), $f, g \in H$, which is complete with respect to the norm $\|f\|_H = (f, f)^{\frac{1}{2}}$.*

We know that in Hilbert spaces, the Cauchy-Schwarz inequality holds:

$$|\langle x, y\rangle| \leq \|x\|\|y\|, \quad \forall x, y \in H.$$

Additionally, another fundamental inequality is the triangle inequality:

$$\|x + y\| \leq \|x\| + \|y\|, \quad \forall x, y \in H.$$

Lebesgue Spaces

Lebesgue spaces are a class of function spaces that generalize the notion of integrable functions.

Definition 11. *The Lebesgue space $L^p(\Omega)$ for $1 \leq p < \infty$ is defined as:*

$$L^p(\Omega) := \left\{ u : \Omega \to \mathbb{R} \mid \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p} < \infty \right\}.$$

The norm in $L^p(\Omega)$ is given by:

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p \, dx \right)^{1/p}.$$

Definition 12. *The space $L^\infty(\Omega)$ is defined as:*

$$L^\infty(\Omega) := \left\{ u : \Omega \to \mathbb{R} \mid \operatorname{ess\,sup}_{x \in \Omega} |u(x)| < \infty \right\},$$

where the essential supremum ess sup is the smallest number M such that $u(x) \leq M$ almost everywhere. The norm in $L^\infty(\Omega)$ is given by:

$$\|u\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x \in \Omega} |u(x)|.$$

Sobolev Spaces

Sobolev spaces are a fundamental class of function spaces used to study partial differential equations and functional analysis. These spaces generalize the concept of differentiability and integrability.

Definition 13. *The Sobolev space $W^{1,p}(\Omega)$ for $1 \leq p \leq \infty$ is defined as:*

$$W^{1,p}(\Omega) := \{ u \in L^p(\Omega) \mid \partial_i u \in L^p(\Omega) \text{ for } i = 1, \ldots, n \},$$

where $\partial_i u$ denotes the weak partial derivative of u. The norm in $W^{1,p}(\Omega)$ is given by:

$$\|u\|_{W^{1,p}(\Omega)} := \left(\|u\|^p_{L^p(\Omega)} + \sum_{i=1}^{n} \|\partial_i u\|^p_{L^p(\Omega)} \right)^{1/p}.$$

Definition 14. *The Sobolev space $W^{k,p}(\Omega)$ for an integer $k \geq 1$ and $1 \leq p \leq \infty$ is defined as:*

$$W^{k,p}(\Omega) := \{u \in L^p(\Omega) \mid \partial^\alpha u \in L^p(\Omega) \text{ for all multi-indices } |\alpha| \leq k\},$$

where $\partial^\alpha u$ denotes the weak partial derivative of order $|\alpha|$. The norm in $W^{k,p}(\Omega)$ is given by:

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \leq k} \|\partial^\alpha u\|^p_{L^p(\Omega)} \right)^{1/p}.$$

Spaces $W^{k,p}(\Omega)$ with $p = 2$ are especially significant in control theory and our analysis.

Definition 15. *The Sobolev space $H^1(\Omega)$ is defined as:*

$$H^1(\Omega) := W^{1,2}(\Omega) = \left\{u \in L^2(\Omega) \mid \partial_i u \in L^2(\Omega) \text{ for } i = 1, \ldots, n\right\},$$

where $\partial_i u$ denotes the weak partial derivative of u. The norm in $H^1(\Omega)$ is given by:

$$\|u\|_{H^1(\Omega)} := \left(\|u\|^2_{L^2(\Omega)} + \sum_{i=1}^{n} \|\partial_i u\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Definition 16. *The Sobolev space $H^2(\Omega)$ is defined as:*

$$H^2(\Omega) := W^{2,2}(\Omega) = \left\{u \in L^2(\Omega) \mid \partial^\alpha u \in L^2(\Omega) \text{ for all multi-indices } |\alpha| \leq 2\right\},$$

where $\partial^\alpha u$ denotes the weak partial derivative of order $|\alpha|$. The norm in $H^2(\Omega)$ is given by:

$$\|u\|_{H^2(\Omega)} := \left(\sum_{|\alpha| \leq 2} \|\partial^\alpha u\|^2_{L^2(\Omega)} \right)^{1/2}.$$

Let's now introduce the foundational definitions of several key classes of functions that are pivotal to this work. First, we will define Lipschitz functions in the context of the Euclidean space $\mathbb{R}^n$, for $n \in \mathbb{N}$. Following that, we will provide definitions for two other important classes of functions, but these will be framed within the broader and more abstract setting of a general space X.

Definition 17. *Let $V : D \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$, D is a closed set. We assume there exists $k \geq 0$ such that for every $(t, x_1), (t, x_2) \in D$, it holds that $|V(t, x_1) - V(t, x_2)| \leq k|x_1 - x_2|$. Then we say v satisfies the **Lipschitz condition** concerning x on D, and k is a Lipschitz constant for the function $V(t, x)$ over the set D.*

Definition 18. *[49, Def 2.2] Assume $V \in C(D)$, is a non-negative function, where D can be bounded, unbounded, or the entire space X, and the essential upper bound $M = \sup_{x \in D} V(x)$, then the function V is said to be **compact** if*

1. $V(x) < M, \quad x \in D$, *and*
2. $\lim_{x \to \partial D} V(x) = M$.

Definition 19. *[51, Def 2.3(3)] Let $C = (c_{i,j})_{i,j=1,2,\dots,n}$ be a positive semi-definite $n \times n$ matrix-valued function on the space X. A compact function $V \in C^1(X)$ is said to be* ***of class*** $\mathbf{B}^*(\mathbf{A})$ *if $\exists m \in (0, M)$, $M > 0$ and $\exists G \geq 0$, a locally bounded function on $[m, M]$ such that:*

1. $\nabla V(x) \neq 0$, $x \in V^{-1}(q)$ *a.e.* $q \in [m, M)$.
2. $c_{i,j}\partial_i V(x)\partial_j V(x) \leq G(q)$, $x \in V^{-1}(q)$ *a.e.* $q \in [m, M)$.
3. $\int_{q_0}^{M} \frac{1}{G(s)}\, ds = \infty$, $\forall q_0 \in (m, M)$.

Operator Analysis

Definition 20. *An operator A on a vector space X over $\mathbb{R}$ is* ***linear*** *if it satisfies:*

$$A(\alpha u + \beta v) = \alpha A(u) + \beta A(v),$$

for all $u, v \in X$ and $\alpha, \beta \in \mathbb{R}$.

Linear operators form the foundation of mathematical analysis, extensively studied in both theoretical and applied contexts.

Definition 21. *An operator A on a normed vector space X is* ***bounded*** *if there exists a constant $C \geq 0$ such that $\|A(u)\| \leq C\|u\|$ for all $u \in X$, where $\|\cdot\|$ denotes the norm on X.*

Bounded operators ensure stable transformations within normed spaces, foundational in functional analysis.

Remark 2. *The study of bounded operators has led to significant results such as the Hille-Yosida and Stone's theorems, which play pivotal roles in later analyses.*

Definition 22. *An operator A is* ***unbounded*** *if there is no constant $C \geq 0$ such that $\|A(u)\| \leq C\|u\|$ holds for all $u \in X$.*

Unbounded operators present challenges due to their potential to produce arbitrarily large outputs for finite inputs in normed spaces.

Definition 23. *An operator A on a Banach space X is* ***closed*** *if its graph $\{(u, A(u)) \mid u \in \mathcal{D}(A)\}$ is closed in the product topology of $X \times X$, where $\mathcal{D}(A)$ denotes the domain of A.*

Closed operators ensure continuity and completeness in Banach spaces, crucial for rigorous mathematical analysis.

Definition 24. *An operator A on a Hilbert space X is* ***self-adjoint*** *if $A = A^*$, where A^* denotes the adjoint (or Hermitian adjoint) of A.*

Definition 25. *An operator A on a Hilbert space X is* ***symmetric****, if $\langle Au, v\rangle = \langle u, Av\rangle$ for all $u, v \in X$.*

Definition 26. *An operator A on a Hilbert space X is* ***negative*** *if $\langle Au, u\rangle \leq 0$ for all $u \in \mathcal{D}(A)$, where $\mathcal{D}(A)$ is the domain of A.*

Negative operators model dissipative phenomena and stability, crucial in the analysis of dynamical systems within Hilbert spaces.

Definition 27. *A subset D of a topological space X is* ***dense*** *in X if every point in X either belongs to D or is a limit point of D.*

Dense subsets ensure that spaces are rich enough for approximation and convergence in topological contexts.

Definition 28. *The domain $\mathcal{D}(A)$ of an operator A on a Hilbert space X is* ***dense*** *if $\overline{\mathcal{D}(A)} = X$, where $\overline{\mathcal{D}(A)}$ denotes the closure of $\mathcal{D}(A)$ in X.*

A dense domain ensures that operators can be continuously extended and fully defined within Hilbert spaces, essential for completeness and functional analysis.

2.2 Semigroup Theory

A notable feature of infinite-dimensional systems is that the corresponding operator A is not always bounded. Therefore, we focus on operators whose domains do not cover the entire space. Generators of semigroups serve as prime examples of such unbounded operators. This section introduces various concepts of semigroup theory and presents key existence and uniqueness results.

Considering an abstract Sobolev space X, we start with some essential definitions.

Definition 29 ([55], Def 2.2.1)**.** *A* ***strongly continuous semigroup*** *(or C_0-**semigroup**) $S(t)$ on a Banach space X is a family of operators $\{S(t)\}_{t\geq 0}$ such that:*

- *$S(0) = I$ (the identity operator),*
- *$S(t+s) = S(t)S(s)$ for all $t, s \geq 0$,*
- *$S(t)x$ is continuous in t for each fixed $x \in X$.*

Definition 30 ([55], Def 2.2.1)**.** *The operator A is called the* ***infinitesimal generator*** *of the semigroup $S(t)$ if*

$$Ax = \lim_{t\to 0^+} \frac{S(t)x - x}{t},$$

for all x in the domain $D(A)$, where the limit is taken in the norm of X.

In the study of differential equations, various types of solutions are considered, each offering different levels of generality and applicability. Classical solutions are the most restrictive, requiring the solution to be continuously differentiable and to satisfy the differential equation in the traditional sense at every point. Mild solutions, on the other hand, relax these requirements by employing semigroup theory and integral equations, thus allowing solutions that may not be differentiable but still capture the system's dynamics over time. Weak solutions further generalize the concept by using integral forms

and test functions, making it possible to work with functions that are only integrable and not necessarily continuous.

While these solutions may seem distinct, under certain conditions they can coincide; for instance, a classical solution is also a mild and weak solution, but the converse is not always true. In the following lines, we will present the definitions of these solutions and explore their differences in detail.

Definition 31 ([55], Def 2.2.1)**.** *A continuously differentiable function $u : [0, \infty) \to X$ is termed a* ***classical solution*** *of the Cauchy homogeneous problem:*

$$u_t(t) = Au(t), \quad u(0) = u_0, \tag{2.2.1}$$

if $u(t) \in D(A)$ for all $t \geq 0$ and the equation (2.2.1) holds.

Lemma 1 ([55], Lemma 2.2.2)**.** *Consider A as the infinitesimal generator of the strongly continuous semigroup $S(t)$. For any $u_0 \in D(A)$, the function $u(t) = S(t)u_0$ uniquely solves the problem (2.2.1).*

Next, we introduce a more general definition of the solution to the system (2.2.1), where the condition $u_0 \in D(A)$ is not required.

Definition 32 ([55], Def 2.2.4)**.** *The continuous function $u : [0, \infty) \to X$ defined by $u(t) := S(t)u_0$ is called the* ***mild solution****.*

Let's now extend the concept of solutions to the abstract inhomogeneous Cauchy problem

$$u_t(t) = Au(t) + g(t), \quad u(0) = u_0, \tag{2.2.2}$$

where $g : C([0, T); X)$, $T > 0$.

Definition 33 ([55], Def 5.5.1)**.** *A continuously differentiable function $u : [0, \infty) \to X$ is termed a* ***classical solution*** *of the Cauchy inhomogeneous problem (2.2.2), if $u(t) \in D(A)$ for all $t \in (0, T)$ and the equation (2.2.2) holds for all $t \in (0, T]$.*

The mild solution in this case is defined in the following lines:

Definition 34 ([55], Def 5.1.4)**.** *Let $g \in L^p([0, T]; X)$,for a $p \geq 1$. The continuous function $u : [0, \infty) \to X$ defined by $u(t) := S(t)u_0 + \int_0^t S(t-\tau)g(\tau)\, d\tau$ is called the* ***mild solution****.*

In fact, our concept of a mild solution corresponds to what is commonly referred to as a weak solution in the realm of PDEs.

Definition 35 ([55], Def 5.1.9)**.** *For $g \in L^p([0, T]; X)$ with $p \geq 1$, we define u as a* ***weak solution*** *of (2.2.2) on the interval $[0, T)$ if it satisfies the following conditions for every $u^1 \in D(A^*)$:*

- *$u(t)$ is continuous on $[0, T)$ with the initial condition $u(0) = u_0$,*
- *the integral $\int_0^t u(\tau)u^1\, d\tau$ is absolutely continuous on $[0, T)$,*

- *for almost every $t \in [0, \tau)$, the following equation holds:*

$$\int_0^t u_t(\tau)u^1\, d\tau = \int_0^t u(\tau)A^*u^1\, d\tau + \int_0^t g(\tau)u^1\, d\tau,$$

where A^* denotes the adjoint operator of A.

The existence of a strongly continuous semigroup implies the existence of unique classical solutions. Conversely, if A has a non-empty resolvent set, the existence of unique classical solutions implies that A generates a strongly continuous semigroup.

Theorem 1 ([55], Theorem 2.2.3)**.** *Let $A : D(A) \to X$ be a linear operator such that $(\lambda I - A)$ is an invertible bounded operator for some $\lambda \in \mathbb{C}$. If, for all $u_0 \in D(A)$, the differential equation (2.2.1) has a unique classical solution, then A generates a strongly continuous semigroup $S(t)$.*

The concept of semigroups and their generators is pivotal in the study of evolution equations. This theoretical framework offers powerful tools for analyzing the existence, uniqueness, and regularity of solutions to differential equations within an abstract setting. One of the cornerstones of this theory is the Hille-Yosida theorem, which provides a comprehensive understanding of the inhomogeneous case of the problem.

The Hille-Yosida theorem is particularly significant because it characterizes the generators of strongly continuous semigroups, offering essential insights into the solution properties of evolution equations. Assume the following evolution system:

$$\begin{cases} u_t(t) + Au(t) = g(t), \quad T > 0, \\ u(0) = u_0. \end{cases} \tag{2.2.3}$$

where A is a linear operator defined on a Banach space X, $g(t)$ is a given function and $u_0 \in X$ is the initial condition. Let's now state the theorem.

Theorem 2. *[40, Theorem 7.10] Assume that A has a dense domain ($\overline{D(A)} = X$) and for all $\lambda > 0$, the operator $I + \lambda A$ is bijective from $D(A)$ onto X, with $\|(I + \lambda A)^{-1}\|_{\mathcal{L}(x)} \leq 1$, where $\mathcal{L}(\mathbf{X})$ is the space of linear bounded operators from X to X. Then for every $u_0 \in D(A)$ and every $g \in C^1([0, T]; X)$, there exists a unique solution of the problem (2.2.3) with,*

$$u \in C^1([0, T]; X) \cap C([0, T]; D(A)).$$

Moreover, u is given by the formula:

$$u(t) := S(t)u_0 + \int_0^t S(t - \tau)g(\tau)\, d\tau.$$

Chapter 3

The Fokker-Planck Equation

3.1 Fokker-Planck Equation and the Connection with Stochastic Differential Equations

Every system encounters intricate external or internal influences known as noise or fluctuations, stemming from countless small disturbances that cause unpredictable yet slight variations in the system's variables. Let's take for example a dynamical system, given by the equation:

$$x_t = b(x), \quad x \in \mathcal{U} \subset \mathbb{R}^n, \tag{3.1.1}$$

where x is the studying process.

Here, the set $\mathcal{U}$ may be: 1) bounded, 2) unbounded, or 3) the entire space $\mathbb{R}^n$, where $n \in \mathbb{N}$. The function $b = b^i$ represents the vector field or drift field. By incorporating a noise perturbation term $\sigma(x)dW_t$, where $\sigma = (\sigma^{i,j})_{n\times m}$, W_t is the Wiener process, defined as $\xi(t) = \dfrac{dW_t}{dt}$, where $\xi(t)$ is white noise and $m \in \mathbb{N}$ is a matrix-valued function defined on $\mathcal{U}$, we derive the following Itô SDE system:

$$dx = b(x)\,dt + \sigma(x)\,dW_t, \quad x \in \mathcal{U} \subset \mathbb{R}^n. \tag{3.1.2}$$

Provided that both b and σ are locally Lipschitz-continuous in $\mathcal{U}$, the SDE generates a local (in time) diffusion process within $\mathcal{U}$. Recalling the basic theory of SDEs([52],[26],[28],[33],[31]), and employing the Distribution-Based Approach, equation (3.1.2) is connected with a second order linear PDE. More precisely, for any bounded, non-negative, measurable function $u_0 \in \mathcal{U}$ satisfying $\displaystyle\int_{\mathcal{U}} u_0(x)\,dx = 1$ and supposing that the transition probability:

$$p(t,\zeta,x) := Pr(x(t+h) = \zeta | x(t) = x), \quad \text{as} \quad h \to 0,$$

exists,Probability (Pr) , the function

$$u(x,t) := \int_{\mathcal{U}} p(t,\zeta,x)u_0(\zeta)\,d\zeta, \tag{3.1.3}$$

satisfies a second-order PDE, known as the Fokker-Planck (FP) equation, under the following conditions:

$$\begin{cases} u_t(t,x) = \dfrac{1}{2}\displaystyle\sum_{i,j=1}^{m}\left((\sigma^{i,j}(x))^2 u(t,x)\right)_{x_i x_j} - \sum_{i=1}^{m}\left(b^i(x)u(t,x)\right)_{x_i}, & x \in \mathcal{U},\ t>0, \\ u(0,x) = u_0(x), & x \in \mathcal{U}, \\ u(t,x) \geq 0, & x \in \mathcal{U},\ t>0, \\ \displaystyle\int_{\mathcal{U}} u(t,x)\,dx = 1, \end{cases} \tag{3.1.4}$$

where $x = (x_1, ..., x_n) \in \mathbb{R}^n$. Equation (3.1.4) is a differential equation governing the distribution function $u(x,t)$ of n-macroscopic variables x. This approach effectively captures the characteristics of the Itô SDE (3.1.2).

In summary, the transition probability density function $p(t,\zeta,x)$, if it exists, $u(t,x)$ serves as a fundamental solution to the FP equation corresponding to (3.1.2). We define the term as $a(x) = \dfrac{1}{2}\sigma^*(x)\sigma(x)$ and we will focus on the case where $n = 1$. Then, equation (3.1.4) becomes:

$$u_t(t,x) = (a(x)u(t,x))_{xx} - (b(x)u(t,x))_x\,. \tag{3.1.5}$$

Remark 3. *Notice that equation (3.1.2) can be written in the following form:*

$$dx = b(x)\,dt + \sqrt{2a(x)}\,dW_t, \qquad x \in \mathcal{U} \subset \mathbb{R}. \tag{3.1.6}$$

Within the FP equation, $b(x)$ is referred to as the **drift coefficient**, and $a(x)$ is known as the **diffusion coefficient**. Essentially, the FP equation can be described as a diffusion equation augmented by an additional term consisting of a first-order derivative concerning x. Let's now define some useful operators, to help us with the analysis in the following chapters.o The **Fokker-Plank operator** is defined as:

$$Au(t,x) := (a(x)u(t,x))_{xx} - (b(x)u(t,x))_x\,. \tag{3.1.7}$$

In the following Chapters, the adjoint problem will be under consideration. Therefore we define the **adjoint Fokker-Plank operator** as:

$$A^*u(t,x) := a(x)u_{xx}(t,x) + b(x)u_x(t,x). \tag{3.1.8}$$

Finally, the family of adjoint FP operators is crucial for our analysis:

Definition 36. *[51] We call the operator*

$$A^*_{b,a} := a\partial^2 + b\partial. \tag{3.1.9}$$

*the **family of adjoint FP operators**, where $b \in \mathcal{B}$, $a \in \mathcal{A}$, and:*

- *$\mathcal{B}$: is the class of continuous functions $\mathcal{U} \to \mathbb{R}^n$,*
- *$\mathcal{A}$: is the set of $n \times n$ valued matrices,*

- $\mathcal{U}$: *is the class of everywhere positive definite functions of* $W^{1,q}_{loc}$, $q > 1$ *under the topology* $W^{1,q}$*-convergence of any compact subset of* $\mathcal{U}$.

The FP equation addresses fluctuations by providing a framework to understand how such disturbances affect the evolution of system variables over time. Initially introduced by Fokker [2] and Planck [3] to model the phenomenon of Brownian motion in particles, the FP equation, also known as the Kolmogorov forward equation, is a parabolic PDE. It describes the time evolution of the Probability Density Function (PDF) of a Stochastic Process (SP). Exploring the origins of the FP equation provides valuable insights into the fundamental nature of these equations and their relevance to the dynamics of complex systems. The complete solution of a macroscopic system would involve solving all the microscopic equations governing the system. However, since this is often impractical, we resort to a stochastic description. In this approach, we characterize the system using macroscopic variables that exhibit stochastic fluctuations.

FP equation is among the simplest equations governing continuous macroscopic variables. It commonly arises in systems with small but macroscopic subsystems, such as the position and velocity of a small particle undergoing Brownian motion, the current in an electrical circuit, or the electric field in a laser[52]. When the subsystem is larger, fluctuations can often be ignored, leading to deterministic equations. However, in cases where deterministic equations become unstable, a stochastic description becomes necessary, even for large systems.

3.2 The Weak Form of the Fokker-Planck Equation

Let's recast the FP equation in the flux formulation, also known as divergence form. By introducing the quantity $b^*(x) = a_x(x) - b(x)$ into the equation (3.1.5), assuming that $a \in C^1(\mathcal{U})$, we get:

$$u_t = \left(a(x)u_x(t,x)\right)_x + \left(b^*(x)u(t,x)\right)_x, \quad x \in \mathcal{U},\ t > 0. \tag{3.2.1}$$

In this thesis, we will conduct an in-depth analysis of these parabolic-type equations, focusing on a specific class known as degenerate parabolic PDEs. These equations require less regularity of the coefficients, necessitating the consideration of the weak formulation of the FP equation. Specifically, the function $b^*(x)$ implies that $a(x)$ must be differentiable over the entire domain. However, in the class of degenerate equations that will be presented in the next chapter, this condition is not satisfied. Consequently, we need to develop a weak formulation of the FP equation.

To connect the FP equation (3.2.1) with the weak formulation, let's consider the domain $Q = [0,T] \times \partial\mathcal{U}$, $T > 0$ and an additional source term $f : Q \to \mathbb{R}$. By integrating (3.1.5) over time and space and multiplying by a test function $v \in W^{1,2}(\mathcal{U})$ with boundary conditions $v(x) = 0, x \in \partial\mathcal{U}$, we obtain:

$$\iint_Q \left(u_t - \left(a(x)u_x(t,x) + b^*(x)u(t,x)\right)_x\right) v\, dx\, dt = \iint_Q f(t,x)v\, dx\, dt. \tag{3.2.2}$$

Then, every u that satisfies the above equation, in combination with Definition 34, is a weak solution of the system (3.2.1).

3.3 Historic Development

Control and dynamic systems involving nonlinearities and subject to stochastic disturbances present high complexity in terms of mathematical analysis. The special case of Gaussian white noise disturbances has been extensively studied by many authors using various approaches and methodologies. For such systems, constructing the associated FP equation, which describes the evolution of the distribution in the state space, is meaningful. Recent results have clarified how to derive the PDE governing the evolution of the PDF for systems perturbed by white noise.

Many scientists in the late 1800s and early 1900s began to treat random processes in physics using what is now known as the random walk. The primary goal of this work was to derive the FP equation. Notable contributions include Rayleigh's work [45] on the theory of gases and Bachelier's [1] studies on the stock exchange and gambling.

Einstein integrated probability theory and Brownian motion into a more rigorous mathematical framework. This connection led to numerous generalizations for more complex systems. Key scientists in this field include Fokker, Planck, Burger, Smoluchowski, Ornstein, and Langevin.

Kolmogorov took a more advanced and abstract approach in 1931, studying continuous Markov processes and deriving the FP equation, also known as the Fokker-Planck-Kolmogorov equation. This work spurred a wave of applications in nonlinear control systems and electronic engineering, with contributions from Arnold, Stratonovich, Chang, Kazda, Barrett, and Lyon[5]. For a modern perspective, please refer to the following references[26],[52],[16],[28],[53].

In this work, we assume the derivation of the FP equation from the SDE is already known. We focus on solving the steady-state problem, showing that the results follow a specific pattern, with the PDF of the steady-state conforming to the Maxwell-Boltzmann distribution.

3.4 Steady-State of the Fokker-Planck Equation

Another focus of this thesis is the analysis of the steady-state of these complex systems. In many cases, the high complexity of the dynamics or the irregularity of components makes it difficult or even impossible to derive an exact solution for the steady-state. However, it is possible to find the exact solution for systems with nonlinearities in either their deterministic or stochastic components.

By using the FP equation, we will concentrate on these specific cases and aim to find explicit solutions for the steady-state. More precisely, studying the steady-state of the equation means finding solutions where:

$$u_t = 0, \tag{3.4.1}$$

or, using equation (3.1.5),

$$(a(x)u)_{xx} - (b(x)u)_x = 0. \tag{3.4.2}$$

In such cases, we are interested in finding the stationary PDF that u approaches as t becomes large ($t \to \infty$). It will be shown that the steady-state solutions are closely related to the Maxwell-Boltzmann distribution, which is a fundamental concept in classical statistical mechanics.

In the following subsection, we will provide examples of systems for which an explicit form can be derived, referencing [5], [21], and [24]. As we progress, the complexity of the systems presented will gradually increase. For each case, we will present the SDE and the associated FP equation, and investigate the stationary solutions. The initial examples focus on Hamiltonian systems perturbed by white noise forces. The latter examples will explore feedback systems where nonlinearity occurs in the drift term.

Notation for Partial Derivatives

In this section, we will be dealing with multifactorial derivatives involving many variables. To maintain clarity and efficiency in our expressions, we will use the operator notation $\dfrac{\partial u}{\partial x}$ instead of the subscript notation u_x.

3.4.1 Hamiltonian Systems

Consider a Hamiltonian system with n-degrees of freedom. The system is described using generalized coordinates q_i and conjugate momenta, where $i = 1, 2, ..., n$ and the state of the system at any time is given by the $2n$-dimensional vector. The equations of motion for the Hamiltonian system are given by Hamilton's equations:

$$\begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i}, \\ \dot{p}_i = -\dfrac{\partial H}{\partial q_i}. \end{cases} \tag{3.4.3}$$

Definition 37. *[24] If there are n integral of motion $H_1, H_2, ..., H_n$, $H_1 = H$ satisfying the equation:*

$$[H_i, H_j] = 0 \implies \frac{\partial H_i}{\partial p_k}\frac{\partial H_j}{\partial q_k} - \frac{\partial H_i}{\partial q_k}\frac{\partial H_j}{\partial p_k} = 0,$$

then the Hamiltonian System (3.4.3) is called ***integrable****.*

The Poisson bracket, denoted by $[H_i, H_j]$, is defined as:

$$[H_i, H_j] := \sum_{k=1}^{n} \left(\frac{\partial H_i}{\partial p_k}\frac{\partial H_j}{\partial q_k} - \frac{\partial H_i}{\partial q_k}\frac{\partial H_j}{\partial p_k} \right)$$

The concept of integrability for a Hamiltonian system can also be defined in the following remark.

Remark 4. *If we can find a set of Canonical Transformations:*

$$\begin{cases} L_i := L_i(p, q), \\ \psi_i := \psi_i(p, q) \end{cases},$$

such that they satisfy the following system:

$$\begin{cases} \dot{L}_i = -\dfrac{\partial}{\partial \psi_i} H'(L) = 0, \\ \dot{\psi}_i = \dfrac{\partial}{\partial L_i} H'(L) = \theta_i(L), \end{cases}$$

where H' is a transformation of the Hamiltonian independent of ψ then the Hamiltonian System (3.4.3) is said to be ***integrable****.*

Second Order Systems

By using the FP equation, we will try to find the steady-state of mechanical nonlinear, non-conservative, and stochastic systems that are given in the following form:

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + \frac{\partial G(x)}{\partial x} = \xi(t), \tag{3.4.4}$$

where x is a vector in $\mathbb{R}^n$, r represents the damping term, G is the potential energy, and $\xi(t)$ is the vector white noise. We can substitute $z_i = m\dfrac{dx_i}{dt}, \forall i = 1, ..., n$, then the above system becomes:

$$\begin{cases} \dfrac{dx_i}{dt} = \dfrac{z_i}{m}, \\ \dfrac{dz_i}{dt} = -\dfrac{\partial G(x)}{dx_i} - \dfrac{r}{m} z_i + \xi_i. \end{cases} \tag{3.4.5}$$

Taking the Hamiltonian-Total energy as $H(x, z) = G(x) + \dfrac{z^2}{2m}$, equation (3.4.5) becomes:

$$\begin{cases} \dfrac{dx_i}{dt} = \dfrac{\partial H}{\partial z_i}, \\ \dfrac{dz_i}{dt} = -\dfrac{\partial H}{\partial x_i} - r_i \dfrac{\partial H}{\partial z_i} + \xi_i, \end{cases} \tag{3.4.6}$$

for $i = 1, 2, ..., n$. Suppose that the damping term r_i is constant and we assume that the term $\alpha_{i,j}$ that indicates the second moment rate of ξ_i and is defined by:

$$\mathbb{E}(\xi_i(t)\xi_j(t+\tau)) = 2\alpha_{i,j}\delta(\tau), \qquad i, j = 1, 2, ...n,$$

where $\mathbb{E}$ denotes the mean value and yields:

$$\alpha_{i,j} = 0, \quad \forall i \neq j,$$

or

$$\alpha_{i,j} = \alpha_i = C, \quad \forall i = j,$$

where $\mathbb{E}$ represents the mean value of a quantity. The FP equation for the steady-states becomes:

$$\sum_{i=1}^{n}\left[-\frac{\partial}{\partial x_i}(r_i\frac{\partial H}{\partial z_i}u)+\frac{\partial}{\partial z_i}(\frac{\partial H}{\partial x_i}u)\right]+\sum_{i=1}^{n}\left[\frac{\partial}{\partial z_i}(r_i\frac{\partial H}{\partial z_i}u)+\frac{\alpha_i}{2}\frac{\partial^2 u}{\partial z_i^2}\right]=0. \tag{3.4.7}$$

We look for a solution of the form:

$$u=u(H). \tag{3.4.8}$$

When we apply equation (3.4.8) on (3.4.7) the first summation (which is the conservation system without noise and damping) cancels out, as we will show in the following lines:

$$\begin{aligned}-\frac{\partial}{\partial x_i}(\frac{\partial H}{\partial z_i}u(H))+\frac{\partial}{\partial z_i}(\frac{\partial H}{\partial x_i}u(H)) &= -\frac{\partial^2 H}{\partial x_i\partial z_i}u(H)\\ -\frac{\partial H}{\partial z_i}\frac{\partial u(H)}{\partial x_i}+\frac{\partial^2 H}{\partial x_i\partial z_i}u(H)+\frac{\partial H}{\partial x_i}\frac{\partial u(H)}{\partial z_i} &\\ =\frac{\partial H}{\partial z_i}\frac{\partial u(H)}{\partial x_i}+\frac{\partial H}{\partial x_i}\frac{\partial u(H)}{\partial z_i}.&\end{aligned}$$

It is well known that for conservative systems the PDF would be stationary if $u=u(H)$, therefore, applying the chain rule to the last result, we have:

$$\frac{\partial H}{\partial z_i}\frac{\partial u(H)}{\partial x_i}+\frac{\partial H}{\partial x_i}\frac{\partial u(H)}{\partial z_i}=\frac{\partial H}{\partial z_i}\frac{\partial u(H)}{\partial H}\frac{\partial H}{\partial z_i}+\frac{\partial H}{\partial x_i}\frac{\partial u(H)}{\partial H}\frac{\partial H}{\partial z_i}=0.$$

From the last result and Definition 37, we can deduce that the free-damping ($r_i=0$),free-noise ($a_i=0$) system has infinite stationary states. We go back to equation (3.4.7). The remaining part that we have to deal with is:

$$\sum_{i=0}^{n}\left(\frac{\partial}{\partial z_i}(r_i\frac{\partial H}{\partial z_i}u)+\frac{\alpha_i}{2}\frac{\partial^2 u}{\partial z_i^2}\right)=0.$$

Therefore, we require:

$$\frac{\partial}{\partial z_i}(r_i\frac{\partial H}{\partial z_i}u)+\frac{\alpha_i}{2}\frac{\partial^2 u}{\partial z_i^2}=0.$$

Substituting $u=u(H)$ we have:

$$r_i\frac{\partial H}{\partial z_i}u(H)+\frac{\alpha_i}{2}\frac{\partial u(H)}{\partial H}\frac{\partial H}{\partial z_i}=M(x_i). \tag{3.4.9}$$

We assume that $u\to 0$ and $\dfrac{\partial u}{\partial z_i}\to 0$ as $z\to 0$, then $M(x_i)=0$ for $i=1,...,n$.

$$r_i\frac{\partial H}{\partial z_i}u(H)+\frac{\alpha_i}{2}\frac{\partial u}{\partial H}\frac{\partial H}{\partial z_i}=0.$$

Assuming that $\dfrac{\partial H}{\partial z_i}\neq 0$ we get:

$$r_i u(H)+\frac{\alpha_i}{2}\frac{\partial u}{\partial H}=0.$$

This is a differential equation that can be solved easily by multiplying the factor: $\exp\{\dfrac{2r_i}{\alpha_i}H\}$ thus, we get the solution:

$$u(H) = D_i \exp\{\frac{2r_i}{\alpha_i} H\}, \quad D_i \in \mathbb{R}. \tag{3.4.10}$$

If additionally, assume that (3.4.10) is self-consistence we should choose:

$$\frac{2r_1}{\alpha_1} = ... = \frac{2r_n}{\alpha_n} = \lambda,$$

we conclude that the steady-state is given by the form:

$$u(x, z) = D \exp\{\lambda H(x, z)\}, \quad D \in \mathbb{R}. \tag{3.4.11}$$

The last expression of u is the density of Maxwell-Boltzmann distribution, which occurs frequently in classical mechanics.

Extension to Generalized Damping systems

To generalize the result of (3.4.11), we consider the following system, which, although more abstract, is closely related to the previous one:

$$\begin{cases} \dfrac{dx}{dt} = \dfrac{\partial H}{\partial z}, \\ \dfrac{dz}{dt} = -\dfrac{\partial H}{\partial x} - R(x, z) + \xi, \end{cases} \tag{3.4.12}$$

where the form of the damping term $R(x, z)$ has to be determined. We drop off the assumption that $\alpha_{i,j}$ is constant, so we have:

$$\alpha := \alpha(x, z). \tag{3.4.13}$$

We require solutions of the form $\rho = \rho(H)$ and we will try to obtain the conditions that R has to satisfy, to find an explicit form of the PDF. The Fokker-Plank equation is given by:

$$\left[-\frac{\partial}{\partial x}(\frac{\partial H}{\partial z}\rho) + \frac{\partial}{\partial z}(\frac{\partial H}{\partial x}\rho)\right] + \left[\frac{\partial}{\partial z}(R\rho) + \frac{1}{2}\frac{\partial^2(\alpha\rho)}{\partial z^2}\right] = 0. \tag{3.4.14}$$

We have already shown that the first two terms are canceled, so the remaining terms are:

$$\frac{\partial}{\partial z}(R\rho) + \frac{1}{2}\frac{\partial^2(\alpha\rho)}{\partial z^2} = 0, \tag{3.4.15}$$

or

$$\begin{aligned} &\frac{\partial}{\partial z}(Ru) + \frac{1}{2}\frac{\partial^2\alpha}{\partial z^2}u + \frac{1}{2}\frac{\partial^2 u}{\partial z^2}\alpha = 0 \\ \implies &Ru + \frac{1}{2}\frac{\partial\alpha}{\partial z}u + \frac{1}{2}\frac{\partial u}{\partial z}\alpha = 0 \\ \implies &\frac{R}{\alpha} + \frac{1}{2}\frac{\partial\alpha}{\partial z}\frac{1}{\alpha} = -\frac{1}{2}\frac{1}{u}\frac{\partial u}{\partial z} \end{aligned}$$

$$\implies \frac{R}{\alpha} + \frac{1}{2}\frac{\partial}{\partial z}(\ln\alpha) = -\frac{1}{2}\frac{\partial}{\partial z}(\ln u). \tag{3.4.16}$$

The left-hand side of the above equation must be the derivative of a function $Y = Y(H)$:

$$\frac{R}{\alpha} + \frac{1}{2}\frac{\partial}{\partial z}(\ln \alpha) = \frac{\partial}{\partial z}Y(H) = \frac{\partial H}{\partial H}\frac{\partial H}{\partial z}. \tag{3.4.17}$$

Solving for $R(x, z)$ we get:

$$R(x, z) = -\frac{1}{2}\frac{\partial \alpha}{\partial z} + \alpha\left(Y(H) - Y(0)\right)\frac{\partial H}{\partial z}. \tag{3.4.18}$$

This is the form that R has to satisfy. Applying (3.4.18) into (3.4.16) we have:

$$-\frac{1}{2}\frac{\partial}{\partial z}\ln \rho = \frac{\partial}{\partial z}Y(H),$$

$$\rho = C\exp\{-2Y(H)\}, C \in \mathbb{R}. \tag{3.4.19}$$

If we set: $Y(H) - Y(0) := \int_0^H D(\tau)d\tau$, Y: is integrable, equation (3.4.19) becomes:

$$\rho = A\exp\{-\int_0^H D(\tau)d\tau\}, A \in \mathbb{R}. \tag{3.4.20}$$

Ultimately, we establish that the density function (3.4.20) constitutes the steady-state of the system:

$$\begin{cases} \frac{dx}{dt} = \frac{\partial H}{\partial z}, \\ \frac{dz}{dt} = -\frac{\partial H}{\partial x} + \frac{1}{2}\frac{\partial \alpha}{\partial z} - \alpha D(H)\frac{\partial H}{\partial z} + \xi, \end{cases} \tag{3.4.21}$$

maintaining a similar form to the Maxwell-Boltzmann distribution observed in the previous system.

3.4.2 Integrable Hamiltonian Systems

Consider the system:

$$\begin{cases} dq_i = K(q)\frac{\partial H}{\partial p_i}dt, \\ dp_i = -[A(q)\frac{\partial H}{\partial q_i} + c_{i,j}\frac{\partial H]}{\partial p_j}]dt + G_{i,j}dW_j(t), \end{cases} \tag{3.4.22}$$

where $H, G_{i,j} \in C^2$, $K(q)$ is a suitable, arbitrary function,$c_{i,j} \in C^1$. This general stochastic system illustrates an extensive class of problems and it does not been restricted to integral or non integral Hamiltonian systems.

To construct the steady-state probability density, we have to evaluate the FP equation for the system (3.4.22).

$$-\frac{\partial}{\partial q_i}\left(K(q)\frac{\partial H}{\partial p_i}u^*\right) + \frac{\partial}{\partial p_i}\left(K(q)\frac{\partial H}{\partial q_i}u^* + c_{i,j}\frac{\partial H}{\partial p_j}u^*\right) + \frac{1}{2}\frac{\partial^2(\alpha_{i,j}u^*)}{\partial p_i \partial p_j} = 0. \tag{3.4.23}$$

Applying the derivation and taking into account the property of Poisson brackets:

$$u^*[K(q), H] + A[u^*, H] = [K(q)u^*, H].$$

We finally get the expression of the FP equation:

$$[K(q)u^*, H] + \frac{\partial}{\partial p_i}\left(c_{i,j}\frac{\partial H}{\partial p_j}u^*\right) + \frac{1}{2}\frac{\partial^2(\alpha_{i,j}u^*)}{\partial p_i \partial p_j} = 0. \tag{3.4.24}$$

In addition, we choose the boundary conditions $|p|, |q| \to \infty$ as:

$$\begin{cases} K(q)\dfrac{\partial H}{\partial p_i}u^* = 0 \\ \left(K(q)\dfrac{\partial H}{\partial q_i} + c_{i,j}\dfrac{\partial H}{\partial p_j}\right)u^* + \dfrac{1}{2}\dfrac{\partial(\alpha_{i,j}u^*)}{\partial p_j} = 0. \end{cases} \tag{3.4.25}$$

In order to solve (3.4.24) with the boundary conditions (3.4.25) we assume a solution:

$$u(p,q) = \frac{C}{K(q)}\exp\{-w(H_1, .., H_n)\}|_{H_k = H_k(p,q)}, \tag{3.4.26}$$

where w is a functional to be determined. For simplicity we assume $K(q) = 1$ so we get the solution:

$$u(p,q) = C\exp\{-w(H_1, .., H_n)\}|_{H_k = H_k(p,q)}. \tag{3.4.27}$$

Using equations (3.4.24),(3.4.25)) and (3.4.27) we get the following relation:

$$c_{i,j}\frac{\partial H}{\partial p_j} + \frac{\partial \alpha_{i,j}}{\partial p_j} - \alpha_{i,j}\frac{\partial H}{\partial p_i}\frac{\partial w}{\partial H_l} = 0, \forall i, j, l = 1, ..., n.$$

Setting

$$\begin{cases} \beta_{i,j} := b_{i,j}\dfrac{\partial H}{\partial p_i}, \\ \gamma_{i,j} := c_{i,j}\frac{\partial H}{\partial p_i} + \dfrac{\partial}{\partial p_j}b_{i,j}. \end{cases} \tag{3.4.28}$$

Consequently, we have to solve the following linear system:

$$\beta_{i,j}\frac{\partial w}{\partial H_l} = \gamma_{i,j}. \tag{3.4.29}$$

To obtain the solution we also assume that the function w satisfies the compatibility condition:

$$\frac{\partial^2 w}{\partial H_l \partial H_k} = \frac{\partial^2 w}{\partial H_k \partial H_l}. \tag{3.4.30}$$

Thus, function w has to be in the form:

$$w = w_0 + \int_0^{(H_1,...,H_n)} \frac{\partial w}{\partial H_l} dH_l, \tag{3.4.31}$$

and the steady-state PDFin this case(K(q)=1) is given by:

$$u(p,q) = C\exp\left\{-\int_0^{(H_1,...,H_n)} \frac{\partial w}{\partial H_l}\, dH_l\right\}. \tag{3.4.32}$$

Remark 5. *In the case of non-integrable systems, we take the PDF as:*

$$u(p,q) = C \exp\{-w(H)\}|_{H=H(p,q)}.$$

Finally, dropping of the assumption of simplicity, $K(q)$, we get the steady-state PDF, as:

$$u(p,q) = \frac{C}{K(q)} \exp\{-\lambda(L_1(p,q), ..., L_n(p,q))\}, \tag{3.4.33}$$

where λ has been defined for the special case $K(q) = 1$ and L_i are the canonical variables of the Hamiltonian system.

3.4.3 Nonlinear Feedback Systems Perturbed by Noise

In the last subsection we will deal with Itô stochastic systems of the matrix form:

$$dx = (Dx + bf(c^T x))dt + BdW, \tag{3.4.34}$$

where $x, b, c \in \mathbb{R}^n$, f assumed to be a differentiable (piecewise) function, W is m-dimensional Wiener process, and D, B are matrices with dimensions adjusted to suit the context or requirements of the system under consideration.

System (3.4.34) corresponds to feedback systems of Lur'e type perturbed by white noise. These systems belong to a class of nonlinear dynamical systems characterized by interconnected linear and nonlinear components. The presence of both linear and nonlinear elements introduces challenges in analysis and control design. However, they also provide opportunities for modeling and understanding complex physical systems.

System (3.4.34) has no restriction to be in a gradient type, so we can not take advantage of the previous results that refer to Hamiltonian systems. Equation (3.4.34) is a nonlinear one, due to the factor $bf(c^T x)$, so we split the linear and the nonlinear parts of (3.4.34), and we get:

$$dx = \underbrace{Dxdt + BdW}_{\text{Linear Part}} + \underbrace{bf(c^T x)dt}_{\text{Non-Linear Part}}.$$

In addition, we make the following assumptions:

- All eigenvalues of D have negative real parts,
- (D, B) is a controllable pair.

Furthermore, the FP operator is defined by the form:

$$\begin{aligned} Au = -(trD + \sum_{i=1}^{n} b_i c_i f'(c^T x))u - \sum_{i,j}^{n} D_{i,j} x_i u_{x_i} + \\ \sum_{i=0}^{n} (b_i f(c^T x) u_{x_i}) + \frac{1}{2} \sum_{j,k}^{n} (BB^T)_{j,k} u_{x_i} u_{x_k}. \end{aligned} \tag{3.4.35}$$

Removing the nonlinear part by substituting $b = 0$ into (3.4.34) we get a linear system:

$$dx = Dxdt + BdW,$$

a classical case that has been studied extensively. More precisely, under the above assumptions for the matrices D, G there is a unique steady-state probability density, given by:

$$u(x) = \frac{1}{\sqrt{(2\pi)^n det(Q)}} \exp\{(-\frac{1}{2})x^T Q^{-1} x\},$$

where Q is called **Steady-State Variance** with $Q > 0$ and symmetric, and satisfies the Lyapunov function:

$$DQ + QD^T = -BB^T. \tag{3.4.36}$$

At this point, we introduce the definition of the quantizer q, a piecewise constant function, that will contribute to the work of generalizing our results for the required density function.

Definition 38. *[21] Given $K > 0$ and $\Delta \geq 0$, we define the **quantizer**, $q \in \mathbb{R} \to \mathbb{Z}$ as:*

$$q(x) := \begin{cases} 1, & x \geq (K + \frac{1}{2})\Delta, \\ -1, & x < -(K + \frac{1}{2})\Delta, \\ \lfloor \frac{x}{\Delta} + \frac{1}{2} \rfloor, & otherwise, \end{cases} \tag{3.4.37}$$

where $\lfloor \cdot \rfloor$ is the integer part and Δ, K are the sensitivity and the saturation value, respectively.

In addition, we introduce one compatibility condition that has to be satisfied to get a steady-state.

Definition 39. *[21] The system: $dx = Dxdt + BdW + bf(c^T x)dt$ is **compatible** if the following condition is satisfied:*

$$b = \lambda DQc, \lambda \in \mathbb{R}, \tag{3.4.38}$$

where Q refers to the matrix we defined earlier and satisfies equation (3.4.36).

Making the substitution $f(x) = q(x)$ into (3.4.34) outcomes the quantize feedback system:

$$dx = Dxdt + BdW + bq(c^T x)dt. \tag{3.4.39}$$

Having the main two definitions, quantizers and the compatibility condition, we can now derive the following Theorem:

Theorem 3. *[21, Theorem 2] If the compatibility condition is satisfied, the above process admits a steady-state probability distribution with a piecewise continuous Gaussian density, given by:*

$$u(x) = M \exp\{-\frac{1}{2}x^T Q^{-1} x - \lambda \int_0^{c^T x} q(v)dv\}.$$

Proof. Taking the expression of the FP operator, which is presented at (3.4.35) we have:

$$Au = -(trD + \sum_{i=1}^{n} b_i c_i q'(c^T x))u - \sum_{i,j}^{n} D_{i,j} x_i u_{x_i} +$$
$$\sum_{i=1}^{n} b_i q(c^T x u_{x_i}) + \frac{1}{2} \sum_{j,k}^{n} (BB^T)_{j,k} u_{x_i} u_{x_k},$$

and applying the Lyapunov function (3.4.36) and the compatibility condition (3.4.38) we get:

$$Au = -(trD + -\sum_{i,j}^{n} D_{i,j}x_i u_{x_i} + \sum_{i=1}^{n}(\lambda DQc_i q(c^T x)u_{x_i}) + \frac{1}{2}\sum_{j,k}^{n}(DQ + QD^T)_{j,k}u_{x_i}u_{x_k}.$$

Thus, if we choose the PDF:

$$u(x) = M\exp\{-\frac{1}{2}x^T Q^{-1}x - \lambda\int_0^{c^T x} q(v)dv\},$$

we get $Au = 0$ thus, $u(x)$ is a steady-state density function.

□

The research presented here focuses on systems where the terms in the stationary FP equation mutually neutralize, leading to the simplification of the equation into an ODE with readily available solutions. However, the systems under study mimic conservative systems with the inclusion of damping and noise terms, which limits their applicability. Nonetheless, certain nonlinear stochastic control and vibration systems can be explicitly analyzed within this framework. Nevertheless, the extension of these findings to more complex and moderately general systems presents a formidable challenge in deriving closed-form solutions.

Chapter 4

Degenerate Parabolic Equations in Divergence Form

4.1 Introduction to Degeneracy

Equations of the following form:

$$u_t - (a(x)u_x(x,t))_x - (b(x)u(x,t))_x + c(x)u(x,t) = f(x,t), \tag{4.1.1}$$

have been extensively studied under various domains and suitable boundary conditions. They are well-known as RAD equations. Notice that, if one chooses $c(x) = 0$, then equation (4.2.1) has the same form as the FP equation (3.2.1) and can be associated with the SDE (3.1.2).

By establishing different assumptions on the coefficients or the functional setup, a wide range of results on the existence and uniqueness of various solutions, stability, and controllability are known for these problems.

We focus on PDEs where the coefficient $a(x)$ is assumed to be a positive definite function within the domain but becomes zero at the boundaries. This characteristic transforms the system into a degenerate parabolic problem that standard methods cannot address. Classical theory typically assumes the coefficient of the highest order to be positive definite throughout the entire domain, satisfying the ellipticity condition $a(x) > 0$ for all x. Numerous results exist for such equations, as illustrated by Evans (2010), in [41]. In contrast, degenerate problems require additional regularity assumptions on the coefficients to establish the equation's existence and/or uniqueness.

Many authors have explored numerous approaches to tackle such problems, resulting in rich literature that provides insights into the behavior and solutions of these degenerate parabolic equations.

Many authors have dealt with degenerate problems using a semigroup approach, On [14], the regularity of the solutions to an abstract Cauchy problem associated with the operator $Au = au_{xx}$ was established. Furthermore, [18] discusses the evolution equation $u_t = (a(x)u_x)_x - b(x)u$ on the unit interval, utilizing Neumann boundary conditions. Additional insights are provided in [13], [34], and [38]. Recent work on boundary controllability can be found in [57], while [56] explores applications to climate science. Lastly, [27] offers a detailed analysis of null controllability in degenerate heat equations.

One can also deal with the degenerate case utilizing a probabilistic approach. For instance, Chen and Weih-Wadman [54] discussed the fundamental solution to a 1D degenerate diffusion equation. Cherny and Engelbert [32] provided an extensive study on singular stochastic differential equations. Ethier [9] explored a class of degenerate diffusion processes in population genetics. Krylov [8] presented estimates on the distribution of stochastic integrals. Kumar [39] investigated degenerate stochastic differential equations with non-Lipschitz coefficients. Stroock and Varadhan [7] analyzed diffusion processes with continuous coefficients.

In the context of our study, we delve into the specifics of these equations, applying advanced techniques to establish new results and further understanding in this area of mathematical analysis.

4.2 Types of Degeneracy Under Different Boundary Conditions

An important inspiration for this work stems from the contributions of Cannarsa et al. [38] and Alabau et al. [34]. In [34], the authors studied the following system in divergence form:

$$u_t - (a(x)u_x(t,x))_x + c(x)u(t,x) = f(t,x), \tag{4.2.1}$$

where they explored two cases of degeneracy: the Weak Degeneracy (WDeg) and the Strong Degeneracy (SDeg), localized specifically at the left boundary $x = 0$ of the interval $[0,1]$. For the weakly degenerate case, the conditions imposed on the function $a(x)$ are:

$$\begin{aligned} &a \in C([0,1]) \cap C^1((0,1]), \quad a > 0 \quad \text{in} \quad (0,1] \quad \text{and} \quad a(0) = 0, \\ &\exists M \in [0,1): \quad xa_x(x) \leq Ma(x) \quad \forall x \in (0,1]. \end{aligned}$$

A model example of such a degeneracy is the function x^α, with the choice: $\alpha \in (0,1)$. From these conditions, it follows that $\dfrac{1}{a(x)} \in L^1(0,1)$, which significantly supports the analysis in this scenario.

Moreover, the SDeg case has been studied utilizing the following assumptions:

$$\begin{aligned} &a \in C([0,1]), \quad a > 0 \quad \text{in} \quad (0,1] \quad \text{and} \quad a(0) = 0, \\ &\exists M \in [1,2): \quad xa_x(x) \leq Ma(x) \quad \forall x \in (0,1]. \end{aligned}$$

We note that additional assumptions have been made for the SDeg case, which we omit here due to their technical details being irrelevant to this study. In the SDeg case, a model example would be, x^α, with the choice: $\alpha \in (1,2)$. In this scenario, it is evident that the property $\dfrac{1}{a(x)} \in L^1(0,1)$ does not apply. However, due to the given inequality, it is straightforward to observe that $\dfrac{1}{\sqrt{a(x)}} \in L^1(0,1)$, a crucial property influencing the choice of functional space.

Another critical consideration for these problems is the choice of boundary conditions, as the degeneracy of $a(x)$ can lead to issues of ill-posedness. In [34], the authors addressed this by employing Dirichlet boundary conditions for the WDeg case, resulting in the following problem formulation:

$$\begin{cases} u_t + (a(x)u_x(t,x))_x - c(x)u(t,x) = f(t,x), & (t,x) \in (0,T) \times (0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,T). \end{cases} \tag{4.2.2}$$

Mixed-type boundary conditions are employed on the SDeg case: Dirichlet boundary conditions are applied in the non-degenerate regions, while Neumann boundary conditions are chosen specifically at the degeneracy point $x = 0$. Then, the system in the SDeg case becomes:

$$\begin{cases} u_t + (a(x)u_x(t,x))_x - c(x)u(t,x) = f(t,x), & (t,x) \in (0,T) \times (0,1), \\ \begin{cases} u(t,1) = 0, \\ au_x(t,0) = 0, \end{cases} & t \in (0,T). \end{cases} \tag{4.2.3}$$

This distinction is essential for the analysis of the SDeg case, contrasting with the implications seen in the WDeg case discussed earlier.

The main observation for the SDeg case is that the domain of the operator automatically implies the Neumann conditions at the degeneracy point.[34, Proposition 2.4.]

So far, we have discussed the two types of degeneracy and their distinctions. Now, it is essential to address the nature of the solutions—specifically, the regularity, existence, and uniqueness of the solutions. To study these types of PDEs, we will employ the semigroup approach. This approach necessitates defining suitable operators and their domains. To get more insights into this method, let's present some advancements that have been made by the two papers [38] and [34].

In the latter work by [34], they introduce the operator A_1 defined as:

$$\begin{cases} A_1 u := (a(x)u_x)_x, \\ D(A_1) := H_a^2. \end{cases}$$

Here, the domain $D(A_1)$ varies depending on whether the situation corresponds to WDeg or SDeg, where H_a^2 is defined, using the auxiliary functional spaces H_a^1, as follows:

$$H_a^1 := \big\{ u \in L^2(0,1) \mid u \text{ is locally absolutely continuous in } [0,1], \\ \sqrt{a}u_x \in L^2(0,1), \quad \text{and} \quad u(0) = u(1) = 0 \big\}, \quad \text{if WDeg},$$

$$H_a^1 := \big\{ u \in L^2(0,1) \mid u \text{ is locally absolutely continuous in } (0,1], \\ \sqrt{a}u_x \in L^2(0,1), \quad \text{and} \quad u(1) = 0 \big\}, \quad \text{if SDeg},$$

and

$$H_a^2 := \{u \in H_a^1 \mid au_x \in H^1(0,1)\}.$$

Notice that in the WDeg space, the domain for absolute continuity is closed, in contrast to the SDeg case, where it is open at the degenerate point. An additional crucial point is that both spaces consider only Dirichlet boundary conditions, not Neumann boundary conditions. Choosing different boundary conditions and defining the domain of absolute continuity differently would lead to different behavior in the problem. As we will see in the following sections, if one chooses the open set $(0,1)$ and imposes additional conditions, the functional space will lead the system to follow Neumann boundary conditions.

The main conclusion of this process is that it has been proven that the operator A_1 is an infinitesimal generator. Consequently, using the theory of semigroups, A_1 generates a strongly continuous semigroup on $L^2(0,1)$. Furthermore, given that A acts as a generator and the operator $B(t)$, defined as $B(t)u = -c(t,\cdot)u$, represents a bounded perturbation of A, within the specified functional spaces, equations (4.2.2) and (4.2.3) are guaranteed to be well-posed according to semigroup theory.

This result is significant as it implies the well-posedness of the corresponding evolution equation.

Similar work has been done in [38], where they deal with the problem of non-divergence form:

$$\begin{cases} u_t - a(x)u_{xx}(t,x)) - b(x)u_x(t,x) = f(t,x), & (t,x) \in (0,T)\times(0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1), \end{cases} \tag{4.2.4}$$

where $a, b \in C^0([0,1])$ and $a(0) = a(1) = 0$. As one can see, degeneracy occurs on both boundary conditions, and a suitable example would be, $a(x) = x(1-x)$, for $x \in (0,1)$.

Unlike the approach in [34], we cannot treat the advection term as a bounded perturbation. So, an alternative strategy would need to be made. Let's introduce the two functions:

$$\eta(x) := \exp\{\int_{\frac{1}{2}}^{1} \frac{b(y)}{a(y)}\, dy\},$$
$$k(x) := a(x)\eta^{-1}(x),$$

and the operator

$$A_2 u := a(x)u_{xx} + b(x)u_x,$$

we have the form of the problem:

$$\begin{cases} u_t - A_2 u(t,x) = f(t,x), & (t,x) \in (0,T)\times(0,1), \\ u(t,0) = u(t,1) = 0, & t \in (0,T), \\ u(0,x) = u_0(x), & x \in (0,1). \end{cases} \tag{4.2.5}$$

Pointing out that the operator A_2 can be written in a simpler form, using the function $k(x)$, they get a problem in divergence form with only the diffusion term appearing on the equation. More precisely, one can prove:

$$A_2 u = k(x)(\eta^{-1}(x)u_x)_x.$$

Therefore, the problem becomes very similar to the case [34], as they achieve a very similar formation of the evolution equation. Due to the component $k(x)$ occurring, Cannarsa et. al. [38] introduce some new weighted functional spaces, that we present on the following lines;

$$\begin{aligned} L_{1/k} &:= \left\{ u \in L^2(0,1) \mid \int_0^1 u^2 \frac{1}{k(s)}\, ds < \infty \right\}, \\ H^1_{1/k} &:= L^2_{1/k} \cap H^1_0, \\ H^2_{1/k} &:= \left\{ u \in H^1_{1/k}(0,1) \mid A_2 u \in L^2_{1/k} \right\}, \end{aligned}$$

and the domain of the operator takes the form:

$$D(A_2) := \left\{ u \in H^1_{1/k}(0,1) \mid A_2 u \in L^2_{1/k} \right\}.$$

Utilizing the proposed functional spaces alongside the properties of the operator A_2, we can ultimately establish a well-posedness result.

Our objective is to apply the methodologies previously discussed to handle more complex cases by considering coefficients with lower regularity. Specifically, we will endeavor to enhance our understanding of functional spaces, drawing inspiration from the methods outlined in the earlier papers and their associated references.

4.3 The Dirichlet Boundary Conditions

In this section, we will try to study a degenerate parabolic PDE that is closely related to the FP equation in a weak sense, as it is been presented in Chapter 3, utilizing Dirichlet boundary conditions.

4.3.1 Functional Framework and Well-Posedness

Consider the problem in the divergence form:

$$u_t - (a(x)u_x(t,x))_x - (b(x)u(t,x))_x = f(t,x)\mathbb{1}_\omega(x), \quad (t,x) \in (0,T) \times (-1,1), \tag{4.3.1}$$

where $\omega = (\alpha, \beta) \subset\subset (-1,1)$ and $a(x) \geq 0$, and the source term $f : [0,T] \times (-1,1) \to \mathbb{R}$, $T > 0$. In this section, we will focus specifically on the case where:

- $a(x) := (1-x^2)^p, \quad p \in \mathbb{N},$
- $b(x) \in C^1(-1,1).$

The classical theory of uniformly parabolic equations does not apply in this situation because $a(-1) = a(1) = 0$.

In the following sections, we will deal with the WDeg problem, that is $\dfrac{1}{a(x)} \in L^1(-1,1)$ and implies that: $p \in (0,1)$.

The results we obtain can be extended to more general diffusion coefficients $a(x)$ exhibiting the same type of degeneracy. Hence, in the subsequent discussion, we will prefer to use the notation $a(x)$ in place of $(1-x^2)^p$. To start the analysis, let's set the function:

$$\begin{aligned} \eta(x) &:= a(x)\exp\{\int_0^x \frac{b(y)}{a(y)}dy\}, \\ k(x) &:= \exp\{-\int_0^x \frac{b(y)}{a(y)}dy\}, \end{aligned} \qquad x \in (-1,1).$$

It is natural to choose the following Dirichlet boundary conditions, so we have the system:

$$\begin{cases} u_t(t,x) - (a(x)u_x(t,x))_x + (b(x)u(t,x))_x = f(t,x), & t>0, x\in I=(-1,1), \\ u(t,x) = 0, & t>0, x\in \partial I, \\ u(x,t_0) = u_0(x), & x\in I. \end{cases} \tag{4.3.2}$$

The problem achieves well-posedness by using weighted spaces instead of the traditional Sobolev spaces.

$$\begin{aligned} L^2_{1/k}(-1,1) &:= \left\{ u\in L^2(-1,1) \mid \int_{-1}^1 \frac{1}{k(x)}u^2\,dx < \infty \right\}, \\ H^1_{0,a} &:= \big\{ u\in L^2_{1/k} \mid u \text{ is absolutely continuous in } [-1,1] \text{ and} \\ &\quad \sqrt{a(x)}u_x \in L^2(-1,1), \quad u(-1)=u(1)=0 \big\}, \end{aligned}$$

associated with the following norms:

$$\begin{aligned} \|u\|^2_{1/k} &:= \int_{-1}^1 \frac{1}{k}u^2 dx, \\ \|u\|^2_1 &:= \|u\|^2_{1/k} + \int_{-1}^1 \frac{a(x)}{k(x)}\Big(u_x + \frac{b}{a}u\Big)^2 dx, \end{aligned}$$

and inner product:

$$\begin{aligned} (u,v)_{1/k} &:= \int_{-1}^1 \frac{1}{k}uv\,dx, \\ (u,v)_1 &:= (u,v)_{1/k} + \int_{-1}^1 \frac{a(x)}{k(x)}\Big(u_x+\frac{b}{a}u\Big)\Big(v_x+\frac{b}{a}v\Big)\,dx. \end{aligned}$$

Take the operator:

$$Au := \Big(a(x)u_x + b(x)u\Big)_x$$

and let $D(A)$ refers to the the domain of the operator A, then:

$$D(A) := \{u\in H^1_{0,a} \mid a(x)u_x + b(x)u \in H^1(-1,1)\},$$

with the associated norm:

$$\|u\|_{D(A)} := \|u\|^2_1 + \|Au\|^2_{1/k},$$

and the inner product:

$$(u,v)_{D(A)} := (u,v)_1 + (Au,Av)_{1/k}.$$

4.3.2 Embedding Theorem and Existence of Limits

In this section, we will explore various properties that characterize the nature of the spaces defined in the previous section. These properties will help us understand the structure and behavior of these spaces in greater detail.

Proposition 1. *The following inclusion is satisfied:*

$$L^2_{1/k} \hookrightarrow L^2.$$

Moreover, assume $u \in L^2_{1/k}$*, then:*

$$\|u\|^2_{1/k} \leq M\|u\|^2_{L^2}, \qquad M > 0.$$

Proof. Let $u \in L^2_{1/k}$. Notice that

$$\frac{1}{k(x)} = \exp \int_0^x \frac{b(s)}{a(s)} ds,$$

where $\dfrac{b(x)}{a(x)} \in L^1(-1,1)$ which implies that there is a constant $M > 0$ satisfying:

$$0 < \exp \int_0^x \frac{b(s)}{a(s)} ds < M,$$

and then

$$\int_{-1}^{1} \frac{1}{k(s)} u^2 \, ds \leq M \int_{-1}^{1} u^2 \, ds \implies \|u\|^2_{1/k} \leq M\|u\|^2_{L^2}.$$

□

Remark 6. *This embendding theorem ensures that the operator* $Au \in L^2_{1/k}(-1,1)$*, if* $u \in D(A)$*.*

Proposition 2. *If* $u \in D(A)$*, then* $\forall x \in (-1,1)$ *the quantity* $x(1-x^2)^q u$ *for* $q \in (-1,0)$ *is* $L^1(-1,1)$ *function.*

Proof. From the definition of the set $D(A)$ one has that $(1-x^2)^p u_x$, for $p \in (0,1)$, is absolutely continuous on $[-1,1]$. Thus the following integral has the L^1 property:

$$\left| \int_{-1}^{1} (1-x^2)^p u_x \, dx \right| < \infty.$$

Applying one-time integration by parts and the boundary conditions, one gets:

$$\left| \int_{-1}^{1} 2px(1-x^2)^{p-1} u \, dx \right| < \infty.$$

Due to the fact, $p \in (0,1)$ the exponential on the previous expression is between $(-1,0)$. Therefore, one can rewrite it using the following:

$$\left| \int_{-1}^{1} 2px(1-x^2)^q u \, dx \right| < \infty, \quad q \in (-1,0).$$

□

Understanding the behavior of solutions at boundary points is pivotal in problem analysis. Proving the existence or non-existence of certain quantities at boundary points is crucial for establishing the validity of solutions or theorems. It can help ensure the mathematical model's robustness and accuracy and the choice of a suitable functional setup. The subsequent proposition is of fundamental importance for proof of the main result.

Proposition 3. *If $u \in D(A)$, then the limit $\lim_{x\to\pm1} \frac{u^2}{(1-x^2)^{1-p}}$ exists, and is equal to 0.*

Proof. Using the L'Hopital rule and taking advantage of the Boundary Conditions one can ensure the existence of the limit. More precisely, by taking the limit and noticing that it belongs to the case $\frac{0}{0}$, one can derive the following expression:

$$\begin{aligned}\lim_{x\to\pm1} \frac{u^2}{(1-x^2)^{1-p}} &= \lim_{x\to\pm1} \frac{2u_x u}{-2x(1-p)(1-x^2)^{-p}} \\ &= \lim_{x\to\pm1} \frac{-1}{x(1-p)} \lim_{x\to\pm1} (1-x^2)^p u_x \lim_{x\to\pm1} u.\end{aligned}$$

Since, $u \in D(A)$, means that $a(x)u_x \in H^1(-1,1)$, then $\lim_{x\to\pm1}(1-x^2)^p u_x$ exists. Then applying the boundary conditions to the above limits, one gets:

$$\lim_{x\to\pm1} \frac{u^2}{(1-x^2)^{1-p}} = 0.$$

□

4.3.3 Hardy-Poincaré Type Inequality

This section delves into deriving the Hardy-Poincaré type inequality, a key property that will play a crucial role in proving the fundamental theorem of this chapter. By exploring this inequality, we will lay the groundwork for a deeper understanding of the concepts to follow and demonstrate its essential applications in our analysis.

Theorem 4. *For $p > 0$ the following inequality holds $\forall u \in D(A)$:*

$$\frac{3}{4}(1-p)^2 \int_{-1}^{1} x^2(1-x^2)^{p-2}u^2\,dx \le \int_{-1}^{1}(1-x^2)^p u_x^2\,dx.$$

Proof.

$$0 \le \int_{-1}^{1} \Big((1-x^2)^{p/2}u_x - \frac{p-1}{2}x(1-x^2)^{\frac{p-2}{2}}u\Big)^2 dx =$$

$$\int_{-1}^{1} \Big[(1-x^2)^p u_x^2 + \big(\frac{p-1}{2}\big)^2 x^2(1-x^2)^{p-2}u^2 - \frac{p-1}{2}2x(1-x^2)^{p-1}u_x u\Big]\,dx.$$

Using integration by parts, one can evaluate the third term:

$$\int_{-1}^{1} 2x(1-x^2)^{p-1}u_x u\,dx = (1-x^2)^{p-1}xu^2|_{-1}^{1} +$$

$$\int_{-1}^{1} 2(p-1)x^2(1-x^2)^{p-2}u^2\,dx - \int_{-1}^{1}(1-x^2)^{p-1}u^2\,dx.$$

From Proposition 3 we have seen that the boundary terms of the above equation tend to be 0, therefore we have:

$$\int_{-1}^{1} x(1-x^2)^{p-1}u_x u\,dx = \int_{-1}^{1} 2(p-1)x^2(1-x^2)^{p-2}u^2\,dx - \int_{-1}^{1}(1-x^2)^{p-1}u^2\,dx.$$

Substituting this result to the first inequality, one has:

$$0 \leq \int_{-1}^{1}\left[(1-x^2)^p u_x^2 + \big(\frac{p-1}{2}\big)^2 x^2(1-x^2)^{p-2}u^2\right]dx$$
$$-(p-1)^2\int_{-1}^{1} x^2(1-x^2)^{p-2}u^2\,dx + \frac{p-1}{2}\int_{-1}^{1}(1-x^2)^{p-1}u^2\,dx \implies$$

$$0 \leq \int_{-1}^{1}\left[(1-x^2)^p u_x^2 - \frac{3}{4}\big(p-1\big)^2 x^2(1-x^2)^{p-2}u^2\right]dx + \frac{p-1}{2}\int_{-1}^{1}(1-x^2)^{p-1}u^2\,dx.$$

It is clear now that the following inequality is true since the last term is negative ($p-1<0$):

$$\frac{3}{4}(p-1)^2\int_{-1}^{1} x^2(1-x^2)^{p-2}u^2\,dx \leq \int_{-1}^{1}(1-x^2)^p u_x^2\,dx,$$

and the proof is complete. □

Remark 7. *The setup defined in the earlier sections shows that the Hardy-Poincaré inequality needs to be determined utilizing weighted products. On the previous inequality substitute u with the term: $\dfrac{u}{\sqrt{k(x)}}$ Then:*

$$\frac{3}{4}(1-p)^2\int_{-1}^{1}\frac{1}{k(x)}x^2(1-x^2)^{p-2}u^2\,dx \leq \int_{-1}^{1}(1-x^2)^p\left(u\frac{1}{k(x)}\right)_x^2 dx.$$

By taking the second term, one has:

$$\int_{-1}^{1}(1-x^2)^p\left(u\frac{1}{k(x)}\right)_x^2 dx = \int_{-1}^{1}\frac{a(x)}{k(x)}\left(\frac{b(x)}{2a(x)}u + u_x\right)^2 dx =$$
$$\int_{-1}^{1}\frac{a(x)}{k(x)}\left(\frac{b^2(x)}{4a^2(x)}u^2 + u_x^2 + \frac{b(x)}{a(x)}uu_x\right)dx.$$

By separating the integrals, one can get:

$$\int_{-1}^{1}(1-x^2)^p\left(u\frac{1}{k(x)}\right)_x^2 dx =$$

$$\int_{-1}^{1}\frac{1}{k(x)}\frac{b^2(x)}{4a(x)}u^2\,dx + \int_{-1}^{1}\frac{1}{k(x)}a(x)u_x^2\,dx + \int_{-1}^{1}\frac{1}{k(x)}b(x)uu_x\,dx \leq$$
$$\leq \int_{-1}^{1}\frac{1}{k(x)}\frac{b^2(x)}{2a(x)}u^2\,dx + \int_{-1}^{1}\frac{1}{k(x)}a(x)u_x^2\,dx + \int_{-1}^{1}\frac{1}{k(x)}b(x)\left(\frac{u}{2}\right)_x dx.$$

Therefore, one can apply integration by parts for the third term:

$$\int_{-1}^{1}(1-x^2)^p\left(u\frac{1}{k(x)}\right)_x^2 dx \le \int_{-1}^{1}\frac{1}{k(x)}\frac{b^2(x)}{2a(x)}u^2\,dx+\int_{-1}^{1}\frac{1}{k(x)}a(x)u_x^2\,dx-$$
$$\frac{1}{2}\int_{-1}^{1}\frac{1}{k(x)}\big(b_x(x)+\frac{b^2(x)}{a(x)}u^2\big)\,dx=\int_{-1}^{1}\frac{1}{k(x)}a(x)u_x^2\,dx-\frac{1}{2}\int_{-1}^{1}\frac{1}{k(x)}b_x(x)u^2\,dx.$$

and we conclude to

$$\int_{-1}^{1}(1-x^2)^p\left(u\frac{1}{k(x)}\right)_x^2 dx \le \int_{-1}^{1}\frac{1}{k(x)}a(x)u_x^2\,dx-\frac{1}{2}\int_{-1}^{1}\frac{1}{k(x)}b_x(x)u^2\,dx.$$

Returning to the first inequality:

$$\frac{1}{2}\Big(\int_{-1}^{1}\frac{1}{k(x)}\Big(b_x(x)+\frac{3}{4}(1-p)^2x^2(1-x^2)^{p-2}\Big)u^2\,dx\Big)\le\int_{-1}^{1}\frac{1}{k(x)}a(x)u_x^2\,dx.$$

Choosing $b_x(x)\le\frac{3}{4}(1-p)^2x^2(1-x^2)^{p-2}$ *for* $x\in(-1,1)$*, one could get the weighted Hardy-Poincare inequality:*

$$\int_{-1}^{1}\frac{1}{k(x)}b_x(x)u^2\,dx\le\int_{-1}^{1}\frac{1}{k(x)}a(x)u_x^2\,dx.$$

Remark 8. *Notice that:*

$$\int_{-1}^{1}\frac{1}{k(x)}\Big(a(x)u_x^2-b_xu^2\Big)\,dx=\int_{-1}^{1}\frac{a(x)}{k(x)}\Big(u_x+\frac{b(x)}{a(x)}u\Big)^2\,dx.$$

Therefore, we get directly the inequality:

$$\int_{-1}^{1}\frac{1}{k(x)}b_x(x)u^2\,dx\le\int_{-1}^{1}\frac{1}{k(x)}a(x)u_x^2\,dx.$$

Proof. By applying integration by parts in the following term one has:

$$\int_{-1}^{1}\frac{1}{k(x)}b_x(x)u^2\,dx=\frac{b(x)u^2}{k(x)}\Big|_{x=-1}^{x=1}-\int_{-1}^{1}b(x)\left(\frac{u^2}{k(x)}\right)_x dx=$$
$$-\int_{-1}^{1}\frac{1}{k(x)}b(x)\Big(2uu_x+\frac{b(x)}{a(x)}u^2\Big)\,dx=-\int_{-1}^{1}\frac{1}{k(x)}\Big(2b(x)uu_x+\frac{b^2(x)}{a(x)}u^2\Big)\,dx.$$

Therefore:

$$\int_{-1}^{1}\frac{1}{k(x)}\Big(a(x)u_x^2-b_xu^2\Big)\,dx=\int_{-1}^{1}\frac{1}{k(x)}\Big(a(x)u_x^2+2b(x)uu_x+\frac{b^2(x)}{a(x)}u^2\Big)\,dx=$$
$$\int_{-1}^{1}\frac{a(x)}{k(x)}\Big(u_x^2+2\frac{b(x)}{a(x)}uu_x+\frac{b^2(x)}{a^2(x)}u^2\Big)\,dx$$
$$=\int_{-1}^{1}\frac{a(x)}{k(x)}\Big(u_x+\frac{b(x)}{a(x)}u\Big)^2.$$

The last term is positive, so:

$$\int_{-1}^{1}\frac{1}{k(x)}\Big(a(x)u_x^2-b_xu^2\Big)\,dx\ge0,$$

and it is easy to conclude the required inequality.

□

4.3.4 Basic Properties of the Operator

In this section, we establish the existence of the semigroup by employing a fundamental theorem from semigroup theory. Additionally, we introduce an alternative formulation of the inner product in the space $L^2_{1/k}$, which will be instrumental in our analysis.

Lemma 2. *For all $(u, v) \in D(A) \times D(A)$ we have:*

$$(Au, v)_{1/k} = -\int_{-1}^{1} \frac{1}{k(x)} a(x) u_x v_x \, dx + \int_{-1}^{1} \frac{1}{k(x)} b_x(x) uv \, dx. \tag{4.3.3}$$

Proof. Take the space

$$H^1_s(-1,1) := \left\{ w \in H^1(-1,1) | \quad \sup\{w\} \subset (-1,1) \right\}.$$

One can deduce that $H^1_s(-1,1)$ is dense on $H^1_{0,a}(-1,1)$. By considering the sequence: $(w_n)_{n\geq 8}$ given by: $w_n = \zeta_n w$, for $w \in H^1_{0,a}(-1,1)$, where:

$$\zeta_n := \begin{cases} 0 & ,\text{for } x \in [-1, -1+\frac{2}{n}) \cup (1-\frac{2}{n}, 1], \\ 1 & ,\text{for } x \in [-1+\frac{2}{n}, -1+\frac{4}{n}) \cup (1-\frac{4}{n}, 1-\frac{2}{n}], \\ n|x| - 1 & ,\text{for } x \in [-1+\frac{4}{n}, -\frac{4}{n}) \cup (\frac{4}{n}, 1-\frac{4}{n}], \\ n(1-|x|) - 1 & ,\text{for } x \in [-\frac{4}{n}, \frac{4}{n}], \end{cases}$$

then one can conclude with the convergence result:

$$w_n \to w \in H^1_{0,a}(-1,1).$$

Set:

$$\Psi(w) := \int_{-1}^{1} \frac{1}{k(x)} \Big(Auw - a(x) u_x w_x + b_x(x) uw \Big) \, dx \quad , \quad \forall u \in H^1_{0,a}(-1,1).$$

Then Ψ is a bounded linear function on $H^1_{0,a}(-1,1)$ and $\Psi = 0$ on $H^1_s(-1,1)$. Indeed, one has:

$$\int_{-1}^{1} \frac{1}{k(x)} Auw \, dx = -\int_{-1}^{1} \frac{1}{k(x)} \Big(a(x) u_x w_x - b_x(x) uw \Big) \, dx.$$

Thus, $\Psi = 0$ on $H^1_{0,a}(-1,1)$ and one gets the equation (4.3.3).

□

Theorem 5. *The operator $A : D(A) \to L^2_{1/k}(-1,1)$ is closed, self-adjoint, negative with dense domain.*

From Theorem 5 and the fundamental principles underlying the theory of semigroups, we can deduce that A acts as the infinitesimal generator of a strongly continuous semigroup. This semigroup, denoted by e^{At} resides within the weighted L^2-space, specifically $L^2_{1/k}(-1,1)$. For the proof of Theorem 5, we will employ a similar approach to that used in Cannarsa et al. [27].

Proof. It is obvious that the space $D(A)$ is **dense** on $L^2_{1/k}(-1,1)$
Closed Consider the following sequences: $(u^n)_{n\in\mathbb{N}}$ such that $u^n \in D(A)$ for every $n \in \mathbb{N}$ and $U^n := Au^n$ such that the following convergence is satisfied:

$$u^n \to u \in L^2(-1,1) \quad \text{and} \quad U^n \to U \in L^2(-1,1).$$

We should prove that the limit functions belong to $D(A)$, i.e. $u, U \in D(A)$.

Since $\dfrac{1}{a(x)} \in L^1(-1,1), \forall x \in [-1,1]$, $n, m \in \mathbb{N}$ and using the boundary conditions of the domain $D(A)$: $u^n(-1) = u^n(1) = 0, u^m(-1) = u^m(1) = 0$.

$$u^n(x) - u^m(x) = \int_{-1}^{x} \big(u_x^n(s) - u_x^m(s)\big)\, ds = \int_{-1}^{x} \frac{\sqrt{a(s)k(s)}}{\sqrt{a(s)k(s)}}(u_x^n(s) - u_x^m(s))\, ds.$$

Employing the Holder inequality:

$$|u^n(x) - u^m(x)| \le \Big(\int_{-1}^{x} \frac{1}{a(s)k(s)}\, ds\Big)^{\frac{1}{2}} \Big(\int_{-1}^{x} a(s)k(s)\big(u_x^n(x) - u_x^m(x)\big)^2\, ds\Big)^{\frac{1}{2}}.$$

Setting the term: $M_1 = \Big(\int_{-1}^{x} \dfrac{1}{a(s)k(s)}\, ds\Big)^{\frac{1}{2}}$, we write:

$$|u^n(x) - u^m(x)| \le M_1\Big(\int_{-1}^{1} a(x)k(x)\big((u_x^n)^2 + (u_x^m)^2 - 2u_x^n u_x^m\big)\, dx\Big)^{\frac{1}{2}} =$$

$$M_1\Big(\int_{-1}^{1} a(x)u_x^n k(x)u_x^n\, dx + \int_{-1}^{1} a(x)u_x^m k(x)u_x^m\, dx - 2\int_{-1}^{1} a(x)u_x^n k(x)u_x^m\, dx\Big)^{\frac{1}{2}}$$

Applying integration by parts and using the fact that $a(x)u_x \in H^1(-1,1)$ we have from the last inequality:

$$\begin{aligned}|u^n(x) - u^m(x)| \le M_1\Big(a(x)u_x^n(x)k(x)u^n(x)|_{x=-1}^{1} - \int_{-1}^{1} \big(a(x)u_x^n k(x)\big)_x u^n\, dx + \\ a(x)u_x^m(x)k(x)u^m(x)|_{x=-1}^{1} - \int_{-1}^{1} \big(a(x)u_x^m k(x)\big)_x u^m\, dx - \\ 2\big(a(x)u_x^n(x)k(x)u^m|_{-1}^{1} - \int_{-1}^{1} \big(a(x)u_x^n k(x)\big)_x u^m\, dx\big)\Big)^{\frac{1}{2}}\end{aligned}$$

and the limit terms are vanished:

$$\begin{aligned}|u^n(x) - u^m(x)| \le M_1\Big(\int_{-1}^{1} -k(x)\Big[\big(\big(a(x)u_x^n\big)_x - b(x)u_x^n\big)u^n\, dx + \big(\big(a(x)u_x^m\big)_x - b(x)u_x^m\big)u^m\, dx - \\ 2\big(\big(a(x)u_x^n\big)_x - b(x)u_x^n\big)u^m\Big]\, dx\big)\Big)^{\frac{1}{2}},\end{aligned}$$

or, setting $M_2 = M_1 \max\limits_{x\in[-1,1]}\{k(x)\}$:

$$\begin{aligned}|u^n(x) - u^m(x)| \le M_2\Big(\int_{-1}^{1} \big(\big(a(x)u_x^n\big)_x - b(x)u_x^n\big)u^n\, dx + \big(\big(a(x)u_x^m\big)_x - b(x)u_x^m\big)u^m\, dx - \\ 2\big(\big(a(x)u_x^n\big)_x - b(x)u_x^n\big)u^m\, dx\big)\Big)^{\frac{1}{2}}.\end{aligned}$$

Notice that $\forall n, m \in \mathbb{N}$ and $u^n, u^m \in D(A)$:

$$\int_{-1}^{1} (Au^m)u^n \, dx = \int_{-1}^{1} \big[(a(x)u_x^n)_x - b(x)u_x^n\big]u^m \, dx.$$

Therefore:

$$|u^n(x) - u^m(x)| \leq M_2 \Big[\int_{-1}^{1} Au^n u^n \, dx + \int_{-1}^{1} Au^m u^m \, dx - 2\int_{-1}^{1} (Au^m)u^n \, dx \Big]^{\frac{1}{2}}.$$

Taking the limits we get:

$$|u^n(x) - u^m(x)| \leq M_2 \Big[\int_{-1}^{1} Uu + Uu - 2Uu \, dx \Big] \frac{1}{2} = 0,$$

concluding to the result that u^n converges uniformly to u and the boundary terms become: $u(-1) = u(1) = 0$. Furthermore, it is essential to prove that $a(x)u_x \in H^1(-1,1)$.
Consider a test function $\varphi \in C^\infty([-1,1])$ such that $\varphi(-1) = 0$. Since, $u^n \in D(A)$,then:

$$((au_x^n + bu^n)\varphi)(-1) = 0 \quad \text{and} \quad ((au^n)\varphi_x)(-1) = 0.$$

In addition, taking the next integral it could be proven that u_x^n converges uniformly on $[\eta, 1]$, where $\eta > -1$.

$$\int_{-1}^{x} U^n \phi \, ds = \int_{-1}^{x} \big(a(s)u_x^n + b(s)u^n\big)_x \varphi \, ds =$$
$$a(x)u_x^n(x)\varphi(x) + b(x)u^n(x)\varphi(x) - a(x)\varphi_x(x)u^n(x) - \int_{-1}^{x} \big((a(s)\varphi_x)_x - b(x)\varphi_x\big)u^n \, ds.$$

So, rearranging the terms one has:

$$a(x)u_x^n(x)\varphi(x) = \int_{-1}^{x} U^n\phi \, ds - b(x)u^n(x)\varphi(x) + a(x)\varphi_x(x)u^n(x) + \int_{-1}^{x} \big((a(s)\varphi_x)_x - b(x)\varphi_x\big)u^n \, ds.$$

Noticing that U^n converges in $L^2(-1,1)$ and u^n converges uniformly on $[-1,1]$, it implies that $a(x)\varphi(x)u_x^n(x)$ converges uniformly on $[-1,1]$. Particularly, one can conclude the result: u_x^n converges uniformly on $[\eta, 1]$, $\forall \varepsilon > -1$. Writing:

$$\int_{-1}^{x} U^n \, ds = \int_{-1}^{x} (a(s)u_x^n + b(s)u^n)_x \, ds =$$
$$a(x)u_x^n + b(x)u^n(x) - a(-1)u_x^n(-1) - b(-1)u^n(-1).$$

Since U^n converges on $L^2(-1,1)$ and $u_x^n(x)$ converges uniformly it implies that $a(-1)u_x^n(-1) = \lambda$, that is the limit exists.
Then:

$$a(x)u_x^n(x) = \int_{-1}^{x} U^n \, ds + \lambda - b(x)u^n(x).$$

Concluding to $a(x)u_x \in H^1(-1,1)$.
One now seeks to prove that $\sqrt{a(x)}u_x \in L^2(-1,1)$. By taking the following integral and applying that $a(x)u_x^n \in H^1(-1,1)$, one can see $\forall u^n \in D(A)$:

$$\int_{-1}^{1} Au^n u^n \, dx = \int_{-1}^{1} \Big(- a(x)(u_x^n)^2 + b_x(x)(u^n)^2 \Big) \, dx. \tag{4.3.4}$$

From Remark 8 one has that $\int_{-1}^{1}\Big(-a(x)(u_x^n)^2+b_x(x)(u^n)^2\Big)\,dx\le 0$. Notice also that the equation (4.3.4) implies that:

$$\int_{-1}^{1}\Big(a(x)(u_x^n)^2-b_x(x)(u^n)^2\Big)\to-\int_{-1}^{1}Uu\,dx,$$

almost everywhere in $(-1,1)$. Therefore Fatou's Lemma could be applied as follows:

$$\int_{-1}^{1}\Big(a(x)u_x^2-b_x(x)u^2\Big)\,dx\le\liminf_{n\to\infty}\int_{-1}^{1}\Big(a(x)(u_x^n)^2-b_x(x)(u^n)^2\Big)\,dx<\infty.$$

Thus, $\sqrt{a}u_x\in L^2(-1,1)$. Finally, it still requires validation that the function u is absolutely continuous on the domain $[-1,1]$. It is evident that $u_x\in L^1(-1,1)$, as $\dfrac{1}{a(x)}\in L^1(-1,1)$ and $a(x)u_x\in H^1(-1,1)\subset L^1(-1,1)$. In light of the fact that $u(-1)=0$ and demonstrating that:

$$\forall x\in[-1,1],\quad u(x)=\int_{-1}^{x}u_x(s)ds,$$

it is satisfactory for the verification of absolute continuity.
By considering $x\in(-1,1]$, one has:

$$\forall n\in\mathbb{N}\qquad u^n(x)=\int_{-1}^{x}u_x^n(s)\,ds=\int_{-1}^{\eta}u_x^n(s)\,ds+\int_{\eta}^{x}u_x^n(s)\,ds.$$

By rearranging the terms and considering absolute values can be obtained:

$$|u^n(x)-\int_{-1}^{\eta}u_x^n(s)\,ds|\le|\int_{\eta}^{x}u_x^n(s)\,ds|\le\sqrt{\int_{-1}^{\eta}\frac{1}{a(s)}\,ds\int_{-1}^{\eta}(a(u_x^b)^2)\,ds}\le D\sqrt{\int_{-1}^{\eta}\frac{1}{a(s)}\,ds}.$$

As it has been proven in the previous steps, u^n converges in $C^1([\eta,1])$, $\forall\eta>-1$, one can take the limit in the above relation:

$$|u(x)-\int_{-1}^{\eta}u_x(s)\,ds|\le D\int_{\eta}^{x}\frac{1}{a(s)}\,ds,$$

which directly implies the required relation:

$$\forall x\in[-1,1],\qquad u(x)=\int_{-1}^{x}u_x(s)ds.$$

Self-Adjoint We define the set of adjoin functions:

$$D(A^*)=\{v\in L^2_{1/k}(-1,1)|\exists c>0,\forall u\in D(A),(v,Au)_{1/k}\le c\|u\|_{1/k}\}.$$

For some $u\in D(A)$ and $v\in D(A^*)$:

$$(Au,v)_{1/k}=\int_{-1}^{1}\frac{1}{k(x)}(a(x)u_x+b(x)u)_xvdx=$$

$$\frac{v(a(x)u_x+b(x)u)}{k(x)}|_{-1}^{1}-\int_{-1}^{1}(\frac{1}{k(x)}v)_x(a(x)u_x+b(x)u)dx.$$

By calculating the term: $(\frac{1}{k(x)}v)_x = (v_x + \frac{b(x)}{a(x)}v)\frac{1}{k(x)}$, one has:

$$(Au,v)_{1/k} = -\int_{-1}^{1}\frac{1}{k(x)}\Big(a(x)u_xv_x + b(x)u_xv + b(x)uv_x + \frac{b^2(x)}{a(x)}uv\Big)dx =$$

$$-\int_{-1}^{1}\frac{1}{k(x)}\Big(b(x)uv_x + \frac{b^2(x)}{a(x)}uv\Big) - \int_{-1}^{1}\frac{1}{k(x)}\Big(a(x)v_x + b(x)v\Big)u_x dx.$$

Applying integration by parts for the last term we get:

$$(Au,v)_{1/k} = -\int_{-1}^{1}\frac{1}{k(x)}\Big(b(x)uv_x + \frac{b^2(x)}{a(x)}uv\Big)dx - \frac{(a(x)v_x + b(x)v)u}{k(x)}|_{x=-1}^{1} + \int_{-1}^{1}\Big(\frac{1}{k(x)}a(x)v_x + \frac{1}{k(x)}b(x)v\Big)_x udx.$$

Deleting the boundary terms:

$$(Au,v)_{1/k} = -\int_{-1}^{1}\frac{1}{k(x)}\Big(b(x)uv_x + \frac{b^2(x)}{a(x)}uv\Big)dx + \int_{-1}^{1}\Big(\frac{1}{k(x)}\Big)_x\Big(a(x)v_x + b(x)v\Big)udx + \int_{-1}^{1}\frac{1}{k(x)}\Big(a(x)v_x + b(x)v\Big)_x udx.$$

Notice that $\left(\frac{1}{k(x)}\right)_x = \frac{1}{k(x)}\frac{b(x)}{a(x)}$, so substituting this term into the equation we see that the first two terms are canceled each other, so:

$$(Au,v)_{1/k} = \int_{-1}^{1}\frac{1}{k(x)}(a(x)v_x + b(x)v)_x udx = (u,Av)_{1/k}.$$

We conclude that: $D(A) \subset D(A^*)$.
It remains to be proven: $D(A^*) \subset D(A)$. The inner product of the Hilbert space: $H^1_{0,a}(-1,1)$ is given by:

$$(u,\psi)_1 = \int_{-1}^{1}\frac{1}{k}u\psi\,dx + \int_{-1}^{1}\frac{1}{k}(a(x)u_x\psi_x - b_x(x)u\psi)\,dx.$$

For every $f \in L^2_{1/k}(-1,1)$ there is a unique $u \in H^1_{0,a}(-1,1)$ such that:

$$\forall\psi \in H^1_{0,a}(-1,1) : (u,\psi)_1 = \int_{-1}^{1} f\psi\,dx,$$

which implies that:

$$u - Au = f, \qquad u \in D(A). \tag{4.3.5}$$

It is necessary to show that for $v \in D(A^*)$, $v \in D(A)$ is also true.

$$\exists y \in L^2_{1/k}(-1,1) \ \text{ such that } \ \forall\psi \in D(A) : \int_{-1}^{1}\frac{1}{k}A\psi y\,dx = \int_{-1}^{1}\frac{1}{k}\psi y\,dx.$$

Taking $f = v - y$ from the equation (4.3.5) one has: $y = v - u + Au$. Therefore it is easy to conclude the equation:

$$\int_{-1}^{1} \frac{1}{k}(A\psi - \psi)(v-u)\,dx = 0.$$

For ψ being the solution of (4.3.5) for $f = u - v$ such that: $A\psi - \psi = v - u = 0$ and then $u = v$. That means that $v \in D(A)$ which proves: $D(A^*) \subset D(A)$.

Negative Using the Lemma 2 and the remarks of section: Hardy-Poincare type inequality one can directly derive the negativity of the operator A. □

4.3.5 Energy Estimates

Energy estimates are fundamental tools in the analysis of PDEs and functional spaces. These estimates provide essential insights into the behavior of solutions by bounding their 'energy,' which is typically represented by norms in appropriate function spaces. The necessity of energy estimates arises from their ability to ensure the existence and uniqueness of solutions, demonstrating that solutions do not grow unboundedly and remain stable under small perturbations of initial conditions. Additionally, energy estimates are crucial for establishing the regularity of solutions, indicating the smoothness and differentiability within the functional space.

In numerical analysis, energy estimates play a vital role in controlling the approximation error, making them indispensable for the accuracy and convergence of numerical schemes such as the finite element method. Furthermore, for evolutionary PDEs, these estimates help understand the long-term behavior of solutions, whether they remain bounded, decay, or exhibit periodic behavior over time.

Now, we will establish energy estimates for the weak degenerate case utilizing Dirichlet boundary conditions. This approach will allow us to handle the potential irregularities introduced by the weak degeneracy, ensuring that we can still obtain meaningful bounds on the solution's energy. By integrating these conditions, we will demonstrate the stability and regularity of solutions within the prescribed functional framework.

Theorem 6. *Let $f \in L^2_{1/k}(-1,1)$ and for every $u_0 \in L^2_{1/k}$ the problem has a unique weak solution:*

$$u \in C^0([0,T]; L^2_{1/k}(-1,1)) \cap L^2_{1/k}(0,T; H^1_0),$$

and

$$\sup_{t\in[0,T]} \|u(t)\|^2_{1/k} \leq D_1\Big(\|u_0\|^2_{1/k} + \sup_{t\in[0,T]} \|f(t)\|^2_{1/k}\Big). \tag{4.3.6}$$

Additionally, if one chooses: $u_0 \in D(A)$, the unique solution of the system (4.3.1) satisfies the following relations:

$$u \in C^0([0,T]; H^1_0(-1,1)) \cap L^2_{1/k}(0,T; D(A)) \cap H^1(0,T; L^2_{1/k}(-1,1)), \tag{4.3.7}$$

and

$$\sup_{t\in[0,T]} \|u\|^2_1 + \int_0^T \Big(\|u_t(\tau)\|^2_{1/k} + \|Au(\tau)\|^2_{1/k}\Big)\,d\tau \leq D'_T\Big(\sup_{t\in[0,T]} \|f(t)\|^2_{1/k} + \|u_0\|^2_1\Big), \tag{4.3.8}$$

for $D'_T > 0$.

Proof. For $f \in L^2_{1/k}$, begin with the equation

$$u_t - (a(x)u_x + b(x)u)_x = f(t,x).$$

By multiplying the above equation by $\dfrac{u}{k(x)}$ and integrating both sides over the interval $(-1,1)$, one has, for all $t \in (0,T)$:

$$\int_{-1}^{1} \frac{1}{k(x)} u_t u \, dx - \int_{-1}^{1} \frac{1}{k(x)} \big(a(x)u_x + b(x)u\big)_x u \, dx = \int_{-1}^{1} \frac{1}{k(x)} f u \, dx.$$

Applying one-time integration by parts and taking into account that $a(x)u_x u(-1) = a(x)u_x u(1) = 0$ since $a(x)u_x \in H^1$, one gets:

$$\begin{aligned} &\int_{-1}^{1} \frac{1}{k(x)} u_t u \, dx + \int_{-1}^{1} \frac{1}{k(x)} \big(a(x)u_x + b(x)u\big)\big(\frac{b(x)}{a(x)}u + u_x\big) \, dx = \int_{-1}^{1} \frac{1}{k(x)} f u \, dx \\ &\qquad \Longrightarrow \ \frac{1}{2}\frac{d}{dt}\|u\|^2_{1/k} + \int_{-1}^{1} \frac{a(x)}{k(x)} \big(u_x + \frac{b(x)}{a(x)}u\big)^2 = \int_{-1}^{1} \frac{1}{k(x)} f u \, dx. \end{aligned}$$

Since:

$$\int_{-1}^{1} \frac{1}{k(x)} f u \, dx \le \frac{1}{2}\|f\|^2_{1/k} + \frac{1}{2}\|u\|^2_{1/k},$$

one can derive:

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{1/k} + \int_{-1}^{1} \frac{a(x)}{k(x)} \big(u_x + \frac{b(x)}{a(x)}u\big)^2 \le \frac{1}{2}\|f\|^2_{1/k} + \frac{1}{2}\|u\|^2_{1/k}. \tag{4.3.9}$$

Thus,

$$\frac{d}{dt}\|u\|^2_{1/k} \le \|f\|^2_{1/k} + \|u\|^2_{1/k}.$$

By multiplying both sides with e^{-t} and relocating the term $\|u\|^2_{1/k}$ to the left-hand side, one can refine the expression as follows:

$$\frac{d}{dt}\Big(e^{-t}\|u\|^2_{1/k}\Big) \le \|f(t)\|^2_{1/k}.$$

Integrate between $(0,t)$, $t \le T$ one concludes to the required inequality:

$$\begin{aligned} \sup_{t\in[0,T]} \|u(t)\|^2_{1/k} &\le e^T \Big(\|u_0\|^2_{1/k} + \int_0^T \|f(t)\|^2_{1/k} \, dt\Big) \\ &\le D_1 \Big(\|u_0\|^2_{1/k} + \sup_{t\in[0,T]} \|f(t)\|^2_{1/k}\Big), \end{aligned} \tag{4.3.10}$$

where $D_1 > 0$. From (4.3.9) one has the additional inequality:

$$\int_{-1}^{1} \frac{a(x)}{k(x)} \big(u_x + \frac{b(x)}{a(x)}u\big)^2 \le \frac{1}{2}\|f\|^2_{1/k} + \frac{1}{2}\|u\|^2_{1/k}. \tag{4.3.11}$$

Adding the term $\|u\|^2_{1/k}$:

$$\begin{aligned} &\|u\|^2_{1/k} + \int_{-1}^{1} \frac{a(x)}{k(x)} \big(u_x + \frac{b(x)}{a(x)}u\big)^2 \le \frac{1}{2}\|f\|^2_{1/k} + \frac{3}{2}\|u\|^2_{1/k} \\ &\qquad \Longrightarrow \ \sup_{t\in[0,T]} \|u(t)\|^2_1 \le D_2 \Big(\sup_{t\in[0,T]} \|f(t)\|^2_{1/k} + \|u\|^2_{1/k}\Big), \end{aligned} \tag{4.3.12}$$

where $D_2 > 0$.
From (4.3.13) and (4.3.12) one can conclude to the following inequality:

$$\sup_{t\in[0,T]} \|u(t)\|_1^2 \leq C_1\Big(\sup_{t\in[0,T]} \|f(t)\|_{1/k}^2 + \|u_0\|_{1/k}^2\Big), \tag{4.3.13}$$

where $C_1 > 0$ does not depend on a, f, u_0 or u. It remains to prove the second inequality. Going back to the equation (4.3.1) and multiplying by the term: $-\dfrac{(a(x)u_x + b(x)u)_x}{k(x)}$ one has:

$$\begin{aligned} -\int_{-1}^{1} \frac{(a(x)u_x + b(x)u)_x}{k(x)} u_t\, dx + \int_{-1}^{1} \frac{\Big((a(x)u_x + b(x)u)_x\Big)^2}{k(x)}\, dx \\ = -\int_{-1}^{1} \frac{(a(x)u_x + b(x)u)_x}{k(x)} f(t,x))\, dx \implies \\ -R + \int_{-1}^{1} \frac{1}{k(x)}(a(x)u_x + b(x)u)\big(u_{xt} + \frac{b(x)}{a(x)}u_t\big)\, dx + \|Au\|_{1/k}^2 \\ \leq \frac{1}{2}\|f\|_{1/k}^2 + \frac{1}{2}\|u\|_{1/k}^2, \end{aligned}$$

where $R = (a(x)u_x + b(x)u)\, u_t(x)|_{-1}^{1}$. By rearranging the terms and making some simplifications one gets to the following inequality:

$$\frac{1}{2}\frac{d}{dt}\Big(\|u\|_1^2 - \|u\|_{1/k}^2\Big) + \|Au\|_{1/k}^2 \leq \frac{1}{2}\|f\|_{1/k}^2 + \frac{1}{2}\|u\|_{1/k}^2.$$

Taking the integral over time $(0,T)$, one can conclude to the following inequality:

$$\sup_{t\in[0,T]} \|u\|_1^2 + \int_0^T \|Au(\tau)\|_{1/k}^2\, d\tau \leq D_T\Big(\sup_{t\in[0,T]} \|f(\tau)\|_{1/k}^2 + \|u_0\|_1^2\Big). \tag{4.3.14}$$

Finally, return to the equation (4.3.1) and apply the norms of $L_{1/k}^2$:

$$\|u_t(t)\|_{1/k}^2 = \|Au(t) + f(t)\|_{1/k}^2 \leq 2\|u(t)\|_{1/k}^2 + 2\|f(t)\|_{1/k}^2. \tag{4.3.15}$$

By integrating the above inequality over $(0,T)$:

$$\begin{aligned} \int_0^T \|u_t(\tau)\|_{1/k}^2\, d\tau \leq \int_0^T \Big(2\|u(\tau)\|_{1/k}^2 + 2\|f(\tau)\|_{1/k}^2\Big)\, d\tau \implies \\ \int_0^T \|u_t(\tau)\|_{1/k}^2\, d\tau \leq D_T'\Big(\sup_{t\in[0,T]} \|f(t)\|_{1/k}^2 + \|u_0\|_1^2\Big). \end{aligned} \tag{4.3.16}$$

Combining the equations: (4.3.13), (4.3.14) and (4.3.16) one has the required inequality:

$$\sup_{t\in[0,T]} \|u\|_1^2 + \int_0^T \Big(\|u_t(\tau)\|_{1/k}^2 + \|Au(\tau)\|_{1/k}^2\Big)\, d\tau \leq D_T'\Big(\sup_{t\in[0,T]} \|f(t)\|_{1/k}^2 + \|u_0\|_1^2\Big).$$

□

4.4 The Robin Boundary Condition

In this section, we will investigate the degenerate parabolic PDE, presented in the previous section, with Robin boundary conditions. These conditions are particularly useful in modeling scenarios involving both heat conduction and convective heat transfer at the boundary.

4.4.1 Well-Posedness

Consider the problem in the divergence form:

$$u_t - (a(x)u_x(t,x))_x - (b(x)u(t,x))_x = f(t,x)\mathbb{1}_\omega(x), \quad (t,x) \in (0,T)\times(-1,1), \tag{4.4.1}$$

where $\omega = (\alpha,\beta) \subset\subset (-1,1)$. As we already have seen, equation (4.4.1) is semi-positive definite with degeneracy on the boundaries, meaning that:

$$\begin{cases} a(x) \geq 0, \\ a(-1) = a(1) = 0, \end{cases} \quad \text{on} \quad x \in I = (-1,1).$$

The assumptions for the weak degenerate diffusion problem, $\dfrac{1}{a(x)} \in L^1(-1,1)$, are slightly different from the previous section and are presented below:

- $a(x) := (1-x^2)^p$, $\forall x \in I = (-1,1)$ and $p \in (0,1)$,
- $b(x) \in C(-1,1)$, with $b(-1) \geq 0$ and $b(1) \leq 0$.

We transform the above equation:

$$u(t,x) = w(t,x)e^{-\int_0^x \frac{b(s)}{a(s)}\,ds}, \quad t > 0, x \in I = (-1,1). \tag{4.4.2}$$

Then one gets the 1-dimensional PDE:

$$w_t(t,x) = \Big(a(x)w_x\Big)_x - b(x)w_x,$$

Then, substituting (4.4.2) into (4.4.1), we are interested in the following problem:

$$\begin{cases} u_t(t,x) - \Big(a(x)u_x\Big)_x + b(x)u_x = f(t,x), & t > 0, x \in (-1,1), \\ a(x)u_x(t,x) - b(x)u(t,x) = 0, & t > 0, x \in \partial I, \\ u(x,t_0) = u_0(x), & x \in I. \end{cases} \tag{4.4.3}$$

We introduce the function:

$$k(x) = e^{-\int_0^x \frac{b(s)}{a(s)}\,ds}.$$

Given the presence of a degeneracy in the diffusion coefficient at the boundary, it becomes imperative to introduce a weighted Sobolev space denoted as V to effectively address the well-posedness of the problem.

$$L^2_k(-1,1) := \left\{ u \in L^2(-1,1) \mid \int_{-1}^{1} k(x)u^2\,dx < \infty \right\},$$

$$V := \Big\{ w \in L^2_k \mid w \text{ is absolutely continuous on } [-1,1], \\ \sqrt{a(x)} w_x \in L^2_k \Big\},$$

endowed with the inner product:

$$(w,v)_V := (w,v)_{L^2_k(I)} + (\sqrt{a(x)} w_x, \sqrt{a(x)} v_x)_{L^2_k}, \quad \forall w, v \in V,$$

and the associated norm:

$$\|w\|_V := \|w\|^2_{L^2_k} + \|\sqrt{a(x)} w_x\|^2_{L^2_k}, \qquad \forall w \in V$$

The norm space $(V, \|\cdot\|_V)$ is a Hilbert space and $V \subset H^1(-1,1)$ and $V \subset L^2_k$. We have the following results for the space:

- The space C^∞_0 is dense in V and thus V is dense in L^2_k.(Lemma 2.6: [18])
- $V \hookrightarrow L^2_k$ is compact (Lemma 1 :[17])

Remark 9. *One can observe that equation (4.4.3) can be expressed as:*

$$\begin{cases} u_t(t,x) - \dfrac{1}{k(x)} \Big(k(x) a(x) u_x \Big)_x = f(t,x), & t > 0, x \in (-1,1), \\ a(x) u_x(t,x) - b(x) u(t,x) = 0, & t > 0, x \in \partial I, \\ u(x,t_0) = u_0(x), & x \in I \end{cases} \tag{4.4.4}$$

We now define the operator:

$$\begin{cases} D(A) := \{ u \in V | a(x) u_x \in H^1(-1,1) \text{ and } a(x) u_x(t,x) - b(x) u(t,x) = 0, t > 0, x \in \partial I, \} \\ Au := \dfrac{1}{k(x)} (k(x) a(x) u_x)_x, \forall u \in D(A), \end{cases}$$

with the associated norm:

$$\|u\|_{D(A)} := \|u\|^2_V + \|Au\|^2_k,$$

and the inner product:

$$(u,v)_{D(A)} := (u,v)_V + (Au, Av)_k.$$

Remark 10. *Notice that the operator A in our context bears a resemblance to the operator A_2 discussed in the work by Cannarsa et al. (2007) [38]. However, the main distinction lies in the manner the degenerate diffusion term appears. Specifically, in [38], the operator A_2 is defined as:*

$$A_2 u = k(x) (\eta^{-1}(x) u_x)_x,$$

which simplifies the problem by removing the degeneracy from inside the derivative term ($k(x)$ contains the degeneracy in this case). This structure allows the degeneracy to be handled more easily. In contrast, our operator A retains the degeneracy within the divergence form, making it more challenging to manage.

Remark 11. *Referring to the previous result, Proposition 1, and following a similar method to prove it, we can confirm that the following inclusion holds:*

$$L^2_k \hookrightarrow L^2.$$

The following theorem is the main theorem of this section and is similar to the one we had on the Dirichlet case.

Theorem 7. *The operator $A : D(A) \to L^2_k(-1,1)$ is closed, self-adjoined, and negative with dense domain.*

Hence A is the infinitesimal generator of a strongly continuous semigroup $e^{At} \in L^2_k(-1,1)$.

Remark 12. *The limits concerning the $b(x)u$ exists since b is sufficient smooth function and every $u \in V$ is absolutely continuous function in $[-1,1]$.*

Proof. It is obvious that the space $D(A)$ is **dense** on $L^2_k(-1,1)$.
Closed Consider the following sequences: $(u^n)_{n\in\mathbb{N}}$, such that $u^n \in D(A)$ for every $n \in \mathbb{N}$ and $U^n := Au^n$ such that the following convergence is satisfied:

$$u^n \to u \in L^2_k(-1,1) \quad \text{and} \quad U^n \to U \in L^2_k(-1,1).$$

We should prove that the limit functions belong to $D(A)$, i.e. $u, U \in D(A)$. Since $\dfrac{1}{a(x)} \in L^1(-1,1), \forall x \in [-1,1]$, $n, m \in \mathbb{N}$ and using the boundary conditions of the domain $D(A)$: $a(\pm1)u^n_x(\pm1) - b(\pm1)u^n(\pm1) = 0$, $a(\pm1)u^m_x(\pm1) - b(\pm1)u^m(\pm1) = 0$.

$$u^n(x) - u^m(x) = \int_{-1}^{x} \big(u^n_x(s) - u^m_x(s)\big)\, ds = \int_{-1}^{x} \frac{\sqrt{a(s)k(s)}}{\sqrt{a(s)k(s)}}(u^n_x(s) - u^m_x(s))\, ds.$$

Employing the Holder inequality:

$$|u^n(x) - u^m(x)| \le \Big(\int_{-1}^{x} \frac{1}{a(s)k(s)}\, ds\Big)^{\frac{1}{2}} \Big(\int_{-1}^{x} a(s)k(s)\big(u^n_x(x) - u^m_x(x)\big)^2\, ds\Big)^{\frac{1}{2}}.$$

Setting the term: $M_1 = \Big(\int_{-1}^{x} \dfrac{1}{a(s)k(s)}\, ds\Big)^{\frac{1}{2}}$, we write:

$$|u^n(x) - u^m(x)| \le M_1\Big(\int_{-1}^{1} a(x)k(x)\big((u^n_x)^2 + (u^m_x)^2 - 2u^n_x u^m_x\big)\, dx\Big)^{\frac{1}{2}} =$$

$$M_1\Big(\int_{-1}^{1} a(x)u^n_x k(x)u^n_x\, dx + \int_{-1}^{1} a(x)u^m_x k(x)u^m_x\, dx - 2\int_{-1}^{1} a(x)u^n_x k(x)u^m_x\, dx\Big)^{\frac{1}{2}}.$$

Applying integration by parts and using the fact that $a(x)u_x < \infty$ on ∂I and u is absolutely continuous in $[-1,1]$ we have from the last inequality:

$$\begin{aligned} |u^n(x) - u^m(x)| \le M_1\Big(a(x)u^n_x(x)k(x)u^n(x)|^1_{x=-1} - \int_{-1}^{1} \big(a(x)u^n_x k(x)\big)_x u^n\, dx + \\ a(x)u^m_x(x)k(x)u^m(x)|^1_{x=-1} - \int_{-1}^{1} \big(a(x)u^m_x k(x)\big)_x u^m\, dx - \\ 2\big(a(x)u^n_x(x)k(x)u^m|^1_{x=-1} - \int_{-1}^{1} \big(a(x)u^n_x k(x)\big)_x u^m\, dx\big)\Big)^{\frac{1}{2}}. \end{aligned}$$

Therefore:

$$|u^n(x) - u^m(x)| \le M_1\left(\int_{-1}^{1} k(x)Au^n u^n\, dx + \int_{-1}^{1} k(x)Au^m u^m\, dx - 2\int_{-1}^{1} k(x)(Au^m)u^n\, dx\right)^{\frac{1}{2}}.$$

Taking the limits we get:

$$|u^n(x) - u^m(x)| \leq M_1\Big(\int_{-1}^{1} k(x)\Big(Uu + Uu - 2Uu\Big)\,dx\Big)\frac{1}{2} = 0,$$

concluding to the result that u^n converges uniformly to u and the boundary terms become: $a(x)u_x - b(x)u = 0$, on $x \in I$. Furthermore, it is essential to prove that $a(x)u_x \in H^1(-1,1)$. Consider a test function $\varphi \in C^\infty([-1,1])$ such that $\varphi_x(-1) = 0$. Since, $u^n \in D(A)$,then:

$$((kau_x^n)\varphi)(-1) = 0 \quad \text{and} \quad ((kau^n)\varphi_x)(-1) = 0.$$

In addition, taking the next integral it could be proven that u_x^n converges uniformly on $[\eta, 1]$, where $\eta > -1$.

$$\int_{-1}^{x} k(s)U^n\phi\,ds = \int_{-1}^{x}\Big(\big(k(s)a(s)u_x^n\big)_x\Big)\varphi\,ds =$$
$$k(x)a(x)u_x^n(x)\varphi(x) - k(x)a(x)\varphi_x(x)u^n(x) - \int_{-1}^{x}\big(k(s)a(s)\varphi_x\big)_x u^n\,ds.$$

So, rearranging the terms one has:

$$a(x)u_x^n(x)\varphi(x) = \int_{-1}^{x} k(s)U^n\phi\,ds + k(x)a(x)\varphi_x(x)u^n(x) - \int_{-1}^{x}\big(k(s)a(s)\varphi_x\big)_x\big)u^n\,ds.$$

Noticing that U^n converges in $L_k^2(-1,1)$ and u^n converges uniformly on $[-1,1]$, it implies that $a(x)\varphi(x)u_x^n(x)$ converges uniformly on $[-1,1]$. Particularly, one can conclude the result: u_x^n converges uniformly on $[\eta,1]$, $\forall\varepsilon > -1$. Writing:

$$\int_{-1}^{x} k(s)U^n\,ds = \int_{-1}^{x}(k(s)a(s)u_x^n)_x\,ds = k(x)a(x)u_x^n - k(-1)a(-1)u_x^n(-1). \tag{4.4.5}$$

Then:

$$k(x)a(x)u_x^n(x) = \int_{-1}^{x}\big(U^n\big)\,dx + \lambda, \qquad \lambda \in \mathbb{R}.$$

Concluding to $a(x)u_x \in H^1(-1,1)$. One now seeks to prove that $\sqrt{a(x)}u_x \in L_k^2$. By taking the following integral and applying that $a(x)u_x^n \in H^1(-1,1)$, one can see $\forall u^n \in D(A)$:

$$\int_{-1}^{1} k(x)Au^nu^n\,dx = \int_{-1}^{1}\Big(k(x)a(x)u_x^n\Big)_x u^n\,dx = k(x)a(x)u_x^nu^n|_{x=-1}^{1} \tag{4.4.6}$$
$$-\int_{-1}^{1} k(x)a(x)(u^n)_x^2\,dx = k(x)b(x)(u^n(x))^2|_{x=-1}^{1} - \int_{-1}^{1} k(x)a(x)(u^n)_x^2\,dx. \tag{4.4.7}$$

By choosing $b(-1) \geq 0$ and $b(1) \leq 0$ one has that the above expression (4.4.6) is negative. Notice also that the equation (4.4.6) implies that:

$$\int_{-1}^{1}\Big(k(x)a(x)(u_x^n)^2\Big) \to R - \int_{-1}^{1} k(x)Uu\,dx,$$

where $R = \lim_{n\to\infty} k(x)b(x)(u^n)^2$ almost everywhere in $(-1,1)$. Therefore Fatou's Lemma could be applied as follows:

$$\int_{-1}^{1}\Big(k(x)a(x)u_x^2\Big)_x\,dx \leq \liminf_{n\to\infty}\int_{-1}^{1}\Big(k(x)a(x)(u_x^n)^2\Big)_x\,dx < \infty.$$

Thus, $\sqrt{a}u_x \in L^2_k(-1,1)$. Finally, it still requires validation that the function u is absolutely continuous on the domain $[-1,1]$. It is evident that $u_x \in L^1(-1,1)$, as $\dfrac{1}{a(x)} \in L^1(-1,1)$ and $a(x)u_x \in H^1(-1,1) \subset L^1(-1,1)$. In light of the fact that $u(-1) = \dfrac{au_x(-1)}{b(-1)} = R_1$ and demonstrating that:

$$\forall x \in [-1,1], \qquad u(x) - u(-1) = \int_{-1}^{x} u_x(s)ds,$$

it is satisfactory for the verification of absolute continuity.
By considering $x \in (-1,1]$, one has:

$$\forall n \in \mathbb{N} \qquad u^n(x) - u^n(-1) = \int_{-1}^{x} u_x^n(s)\,ds = \int_{-1}^{\eta} u_x^n(s)\,ds + \int_{\eta}^{x} u_x^n(s)\,ds.$$

By rearranging the terms and considering absolute values can be obtained:

$$|u^n(x) - \int_{-1}^{\eta} u_x^n(s)\,ds - R_1| \leq |\int_{\eta}^{x} u_x^n(s)\,ds|$$
$$\leq \sqrt{\int_{-1}^{\eta} \frac{1}{a(s)}\,ds \int_{-1}^{\eta} (a(s)(u_x^s)^2)\,ds} \leq D\sqrt{\int_{-1}^{\eta} \frac{1}{a(s)}\,ds}.$$

As it has been proven in the previous steps, u^n converges in $C^1([\eta,1])$, $\forall \eta > -1$, one can take the limit in the above relation:

$$|u(x) - \int_{-1}^{\eta} u_x(s)\,ds - R_1| \leq D\int_{\eta}^{x} \frac{1}{a(s)}\,ds,$$

which directly implies the required relation:

$$\forall x \in [-1,1], \qquad u(x) - u(-1) = \int_{-1}^{x} u_x(s)ds.$$

Self-Adjoin We define the set of adjoin functions:

$$D(A^*) = \{v \in L^2_k(-1,1) | \exists c > 0, \forall u \in D(A), (v, Au)_k \leq c\|u\|_k\}.$$

For some $u \in D(A)$ and $v \in D(A^*)$:

$$\int_{-1}^{1} \big(k(x)a(x)u_x\big)_x v\,dx = -\int_{-1}^{1} k(x)a(x)u_x v_x\,dx + k(x)a(x)u_x v|_{x=-1}^{1} =$$
$$\int_{-1}^{1} \big(k(x)a(x)v_x\big)_x u\,dx + k(x)a(x)u_x v|_{-1}^{1} - k(x)a(x)v_x u|_{x=-1}^{1} =$$
$$\int_{-1}^{1} \big(k(x)a(x)v_x\big)_x u\,dx - k(x)b(x)uv|_{x=-1}^{1} + k(x)b(x)vu|_{x=-1}^{1} = \int_{-1}^{1} \big(k(x)a(x)v_x\big)_x u\,dx.$$

Thus, we conclude to:

$$(Au, v) = (u, Av), \qquad \forall u \in D(A), \forall v \in D(A^*), \tag{4.4.8}$$

and $D(A) \subset D(A^*)$. It remains to prove: $D(A) \subset D(A^*)$. The inner product of the Hilbert space: V is given by:

$$(u, \psi)_V = \int_{-1}^{1} k(x) u \psi \, dx + \int_{-1}^{1} k(x) a(x) u_x \psi_x) \, dx.$$

For every $f \in L^2(-1, 1)$ there is a unique $u \in V$ such that:

$$\forall \psi \in V : (u, \psi)_V = \int_{-1}^{1} k(x) f \psi \, dx.$$

By considering $R_2 = k(x)a(x)u_x(x)\psi(x)|_{x=-1}^{1}$, then there is a function $r_2(x) \in L_k^2$ such that: $R_2 = \int_{-1}^{1} k(x) r_2 \psi \, dx$ for any $\psi \in V$. In addition, for any $f \in L_k^2$ there is a unique $u \in V$ such that:

$$\forall \psi \in V, \quad (u, \psi)_V = \int_{-1}^{1} k(x) f \psi \, dx.$$

Applying one-time integration by parts and utilizing the remark from the previous paragraph:

$$\int_{-1}^{1} k(x) u \psi \, dx - \int_{-1}^{1} k(x) A u \psi \, dx + R_2 = \int_{-1}^{1} k(x) u \psi \, dx -$$
$$\int_{-1}^{1} k(x) A u \psi \, dx + \int_{-1}^{1} k(x) r_2 \psi \, dx = \int_{-1}^{1} k(x) f \psi \, dx.$$

Set the value: $\tilde{f} = f - r_2$, where $\tilde{f} \in L_k^2$ one can see that:

$$\forall \psi \in V, \quad \int_{-1}^{1} k(x) u \psi \, dx - \int_{-1}^{1} k(x) A u \psi \, dx = \int_{-1}^{1} k(x) \tilde{f} \psi \, dx,$$

which implies that:

$$u - Au = \tilde{f}, \qquad u \in D(A). \tag{4.4.9}$$

It is necessary to show that for $v \in D(A^*)$, $v \in D(A)$ is also true.

$$\exists y \in L_k^2(-1, 1) \text{ such that } \forall \psi \in D(A) : \int_{-1}^{1} k(x) A \psi y \, dx = \int_{-1}^{1} k(x) \psi y \, dx.$$

Taking $\tilde{f} = v - y$ from the equation (4.4.9) one has: $y = v - u + Au$. Therefore it is easy to conclude the equation:

$$\int_{-1}^{1} (A\psi - \psi)(v - u) \, dx = 0.$$

For ψ being the solution of (4.4.9) for $\tilde{f} = u - v$ such that: $A\psi - \psi = v - u = 0$ and then $u = v$. That means that $v \in D(A)$ which proves: $D(A^*) \subset D(A)$.

Negative By choosing $b(-1) \geq 0$ and $b(1) \leq 0$ and substituting u to u^n in the equation (4.4.6), one can directly derive the negativity of the operator A.

□

4.4.2 Energy Estimates for the Robin Boundary Conditions

Theorem 7 implies that the operator A is the infinitesimal generator of a strongly continuous semigroup e^{At} on $L^2_k(-1,1)$ space. Therefore, if the initial condition specifies $u_0 \in L^2_k$, the system (4.4.4) admits a unique solution $u \in C^0([0,T];L^2_k(-1,1))$. If one chooses the initial condition as $u_0 \in D(A)$, then the unique solution would reside on the space: $u \in C^0([0,T];L^2_k(-1,1)) \cap L^2_k(0,T;D(A)) \cap H^1(0,T;L^2_k(-1,1))$.

The following result is necessary for the well-posedness of the problem.

Theorem 8. *Let $f \in L^2_k(-1,1)\times(0,T)$ and for $u_0 \in L^2_k(-1,1)$, problem (4.4.4) admits a unique solution:*

$$u \in C^0([0,T];L^2_k(-1,1)) \cap L^2_k(0,T;V), \tag{4.4.10}$$

and

$$\sup_{t\in[0,T]} \|u(t)\|_{L^2_k(-1,1)} + \int_0^T \|\sqrt{a}u_x(t)\|^2_{L^2_k(-1,1)}dt \leq C_T\Big(\|u_0\|_{L^2_k(-1,1)} + \sup_{t\in[0,T]}\|f\|^2_{L_k}\Big). \tag{4.4.11}$$

Additionally, if one chooses: $u_0 \in D(A)$, the unique solution of the system (4.4.4) satisfies the following relations:

$$u \in C^0([0,T];V) \cap L^2_k(0,T;D(A)) \cap H^1(0,T;L^2_k(-1,1)), \tag{4.4.12}$$

and

$$\sup_{t\in[0,T]} \|u(t)\|^2_k + \int_0^T \Big(\|u_t(\tau)\|^2_k + \|Au(\tau)\|^2_k\Big)\,d\tau \leq D'_T\Big(\int_0^T \|u_0\|^2_V + \sup_{t\in[0,T]}\|f(t)\|^2_k\Big). \tag{4.4.13}$$

Proof. By taking the equation (4.4.4) and multiplying by the factor: $k(x)u$ one get:

$$k(x)u(t,x)u_t(t,x) - k(x)\frac{1}{k(x)}\Big(k(x)a(x)u_x\Big)_x u = f(t,x)u(t,x).$$

By integrating over $(-1,1)$ and employing the notation $R = k(x)a(x)u_x u|^1_{-1}$, one arrives at:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_k + \|\sqrt{a(x)}u_x(t)\|^2_k \leq \frac{1}{2}\|u(t)\|^2_k + \frac{1}{2}\|f(t)\|^2_k + R_1. \tag{4.4.14}$$

Equation (4.4.14) can be reformulated as:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_k + \|\sqrt{a(x)}u_x(t)\|^2_k \leq C_1\Big(\|u(t)\|^2_k + \|f(t)\|^2_k\Big), \quad C_1 > 0. \tag{4.4.15}$$

One can get the following two inequalities:

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt}\|u(t)\|^2_k \leq C_1\Big(\|u(t)\|^2_k + \|f(t)\|^2_k\Big), \\ \|\sqrt{a(x)}u_x(t)\|^2_k \leq C_1\Big(\|u(t)\|^2_k + \|f(t)\|^2_k\Big). \end{cases} \tag{4.4.16}$$

Thus, by solving the first case, one can easily conclude:

$$\sup_{t\in[0,T]} \|u(t)\|^2_k \leq C_T\Big(\|u_0\|^2_k + \int_0^T \|f(s)\|^2_k\,ds\Big). \tag{4.4.17}$$

From (4.4.16) and (4.4.17) it also follows that:

$$\sqrt{a(x)}u_x(t)\|_k^2 \le C_1\Big(\|u_0\|_k^2 + \int_0^T \|f(s)\|_k^2\,ds + \|f(t)\|_k^2\Big) \le C_2\Big(\|u_0\|_k^2 + \|f\|_{k\times(0,T)}^2\Big),$$

and so one concludes to the inequality:

$$\sup_{t\in[0,T]} \|u(t)\|_k^2 + \|\sqrt{a(x)}u_x(t)\|_k^2 \le C\Big(\|u_0\|_k^2 + \|f\|_{k,(0,T)}^2\Big), \quad C>0. \tag{4.4.18}$$

Therefore, (4.5.4) follows from the inequality (4.4.18) if the initial condition satisfies: $u_0 \in L_k^2(-1,1)$. The next step is to prove the second inequality for $u_0 \in D(A)$. Recall that $D(A)$ represents the domain of the operator A, which contains functions of the space V, so one expects to get information about the behavior of the solution, concerning the initial condition. By taking the equation (4.4.4), multiplying by $-(k(x)a(x)u_x)_x$ and integrating over $(-1,1)$ can has:

$$\int_{-1}^1 u_t(k(x)a(x)u_x)_x\,dx + \int_{-1}^1 \frac{1}{k(x)}(k(x)a(x)u_x)_x^2\,dx = \int_{-1}^1 (k(x)a(x)u_x)_x f(t,x)\,dx.$$

By rearranging the terms and applying one-time integration by parts on the first term:

$$\begin{aligned} -R + \int_{-1}^1 k(x)a(x)u_x u_{xt}\,dx + \|Au\|_k^2 \le \frac{1}{2}\|Au\|_k^2 + \frac{1}{2}\|f\|_k^2 \implies \\ \frac{1}{2}\frac{d}{dt}\|\sqrt{a(x)}u_x\|_k^2 + \frac{1}{2}\|Au\|_k^2 \le \frac{1}{2}\|f|\|_k^2 + R \implies \\ \frac{d}{dt}\|\sqrt{a(x)}u_x\|_k^2 + \|Au\|_k^2 \le \|f|\|_k^2 + 2R. \end{aligned}$$

Integrate over $(0,t)$, for $0<t\le T$:

$$\begin{aligned} \|\sqrt{a(x)}u_x(t)\|_k^2 - \|\sqrt{a(x)}u_x(0)\|_k^2 + \int_0^t \|Au(\tau)\|_k^2\,d\tau \le \int_0^T \Big(\|f(\tau)\|_k^2 + 2R\Big)d\tau \implies \\ \sup_{t\in[0,T]} \|\sqrt{a(x)}u_x(t)\|_k^2 + \int_0^t \|Au(\tau)\|_k^2\,d\tau \le \\ D_T\Big(\sup_{t\in[0,T]} \|f(t)\|_k^2 + \|\sqrt{a(x)}u_x(0)\|_k^2 + \|u(0)\|_k^2\Big) = D_T\Big(\sup_{t\in[0,T]} \|f(t)\|_k^2 + \|u_0\|_V^2\Big), \end{aligned}$$

where $D_T>0$. Thus,

$$\begin{cases} \sup\limits_{t\in[0,T]} \|\sqrt{a(x)}u_x(t)\|_k^2 \le D_T\Big(\sup\limits_{t\in[0,T]} \|f(t)\|_k^2 + \|u_0\|_V^2\Big), \\ \int_0^t \|Au(\tau)\|_k^2\,d\tau \le D_T\Big(\sup\limits_{t\in[0,T]} \|f(t)\|_k^2 + \|u_0\|_V^2\Big). \end{cases} \tag{4.4.19}$$

Return to the equation (4.4.4) and take the norm of the space $L_k^2 f$ for both sides:

$$\|u_t\|_k^2 = \|Au(t) + f(t)\|_k^2 \implies \|u_t\|_k^2 \le 2\|Au(t)\|_k^2 + 2\|f\|_k^2.$$

Integrate both sides over $(0,T)$ and using the inequalities (4.4.19), then:

$$\int_0^T \|u_t(\tau)\|_k^2\,d\tau \le D_T'\Big(\int_0^T \|u_0\|_V^2 + \sup_{t\in[0,T]} \|f(t)\|_k^2\Big), \tag{4.4.20}$$

where $D'_T > 0$. By combining the inequalities: (4.4.17),(4.4.19) and (4.4.20) one gets:

$$\sup_{t\in[0,T]} \|u(t)\|_k^2 + \int_0^T \Big(\|u_t(\tau)\|_k^2 + \|Au(\tau)\|_k^2\Big)\, d\tau \leq D'_T\Big(\int_0^T \|u_0\|_V^2 + \sup_{t\in[0,T]} \|f(t)\|_k^2\Big).$$

Utilizing a density argument, given the density of $D(A)$ within $L_k^2(-1,1)$, it follows that if $u_0 \in L_k^2(-1,1)$, then inequality (4.5.7) remains valid, and consequently, $u \in L_k^2(-1,T;V)$. This substantiates the validity of inequality (4.5.6). □

4.5 The Neumann Boundary Condition

Now, it's opportune to employ the Neumann Boundary Condition for the degenerate problem presented in (4.3.1). These conditions also referred to as reflecting boundary conditions, are essential in physical systems where controlling the flux or rate of change of the system's state at the boundaries is crucial, such as in problems involving insulated boundaries in heat conduction or conservation laws.

4.5.1 Well-posedness

Let's reformulate the problem akin to our approach in the section where we applied the Robin boundary condition.

$$\begin{cases} u_t(t,x) - \dfrac{1}{k(x)}\Big(k(x)a(x)u_x\Big)_x = f(t,x), & t>0, x\in I=(-1,1),\\ a(x)u_x(t,x) = 0, & t>0, x\in \partial I,\\ u(x,t_0) = u_0(x), & x\in I. \end{cases} \tag{4.5.1}$$

Continuing with the same framework as in the previous section would prove advantageous for examining the problem's well-posedness. However, transitioning to Neumann boundary conditions introduces a notable difference: the loss of absolute continuity of the solution at the boundaries $\partial I = \{-1,1\}$. Consequently, it becomes imperative to establish certain limits' existence to compensate for this discrepancy. The assumptions for the weak degenerate diffusion problem, $\dfrac{1}{a(x)} \in L^1(-1,1)$, are the same as the previous section and are presented below:

- $a(x) := (1-x^2)^p$, $\forall x \in I = (-1,1)$ and $p \in (0,1)$,
- $b(x) \in C^0(-1,1)$, with $b(-1) \geq 0$ and $b(1) \leq 0$.

At this point, we will introduce the new spaces for the problem (4.5.1).

$$Z := \{w \in L_k^2 \mid w \text{ is absolutely continuous function on } (-1,1),\\ \sqrt{a(x)}w_x \in L_k^2\},$$

endowed with the inner product:

$$(w,v)_Z := (w,v)_{L_k^2(I)} + (\sqrt{a}w_x, \sqrt{a}v_x)_{L_k^2}, \quad \forall w,v \in Z,$$

and the associated norm:

$$\|w\|_Z := \|w\|^2_{L^2_k} + \|\sqrt{a}w_x\|^2_{L^2_k}, \qquad \forall w \in Z.$$

The norm space $(Z, \|\cdot\|_V)$ is a Hilbert space and $Z \subset H^1(-1,1)$ and $Z \subset L^2_k$. We now define the operator:

$$\begin{cases} D(A) := \{u \in Z | a(x)u_x \in H^1(-1,1)\ \}, \\ Au := \dfrac{1}{k(x)}\left(k(x)a(x)u_x\right)_x, \quad \forall u \in D(A), \end{cases}$$

with the associated norm:

$$\|u\|_{D(A)} := \|u\|^2_Z + \|Au\|^2_k,$$

and the inner product:

$$(u,v)_{D(A)} := (u,v)_Z + (Au, Av)_k.$$

As previously noted, the findings regarding well-posedness in this section closely parallel those of Robin's Boundary Conditions. Consequently, our objective will be to formulate supplementary theorems aimed at ensuring well-posedness in this scenario as well.

Lemma 3. *For every function $u, v \in D(A)$ one has that for every $x \in [-1,1]$ the following inequality is satisfied:*

$$|k(x)a(x)u_x(x)| \leq \sqrt{|x|-1}\|Au\|_k. \tag{4.5.2}$$

Proof. This proof is just an application of the fundamental integration theorem and Hölder inequality. By taking the left hand side of (4.5.2) one has:

$$\begin{aligned} |k(x)a(x)u_x(x)| &= \int_1^x \left(k(s)a(s)u_x\right)_x \, ds \\ &\leq \left(\int_1^x 1\, ds\right)^{\frac{1}{2}} \left(\int_1^x \left(k(s)a(s)u_x\right)^2_x \, ds\right)^{\frac{1}{2}} \leq \sqrt{|x|-1}\|Au\|_k, \end{aligned} \tag{4.5.3}$$

where, (4.5.3) agrees with (4.5.2) and the proof is completed.

□

Proposition 4. *Let $u, v \in D(A)$ then the limit:*

$$\lim_{x \to \pm 1} a(x)u_x(x)v(x),$$

exists and it is equal to 0.

Proof. Notice that for every $x_1 \in (0,1)$:

$$\int_{-x_1}^{x_1} (k(x)a(x)u_x)_x v \, dx = k(x)a(x)u_x(x)v(x)|^{x_1}_{x=-x_1} - \int_{-x_1}^{x_1} k(x)a(x)u_x v_x \, dx.$$

Since the functions $u, v \in D(A)$, the quantities satisfy the properties: $Au \in L^1(-1,1)$ and $k(x)a(x)u_x v_x \in L^1(-1,1)$. Then, $k(x)a(x)u_x(x)v(x) \to R \in \mathbb{R}$ as $x \to \pm 1$. Suppose that the limit satisfies: $R \neq 0$. Utilizing Lemma 3 one gets:

$$\frac{D'}{\sqrt{|x|-1}} \leq \frac{|R|}{|2k(x)a(x)u_x|} \leq |v| \implies |v| \to \infty.$$

Therefore, one can conclude to $v \notin L^2_k$. Thus, $R = 0$. □

Applying a similar procedure as outlined in Chapter 4.4.1, one can readily arrive at relevant theorems analogous to Theorem 7 and Theorem 10.

Theorem 9. *The operator $A : D(A) \to L_k^2(-1,1)$ is closed, self-adjoined, and negative with dense domain.*

In addition, the associated energy estimate result is as follows:

Theorem 10. *Let $f \in L_k^2(-1,1) \times (0,T)$ and for $u_0 \in L_k^2(-1,1)$, problem (4.5.1) admits a unique solution:*

$$u \in C^0([0,T]; L_k^2(-1,1)) \cap L_k^2(0,T;Z), \tag{4.5.4}$$

and

$$\sup_{t\in[0,T]} \|u(t)\|_{L_k^2(-1,1)} + \int_0^T \|\sqrt{a}u_x(t)\|^2_{L_k^2(-1,1)} dt \le C_T\Big(\|u_0\|_{L_k^2(-1,1)} + \sup_{t\in[0,T]} \|f\|^2_{L_k}\Big). \tag{4.5.5}$$

Additionally, if one chooses: $u_0 \in D(A)$, the unique solution of the system (4.5.1) satisfies the following relations:

$$u \in C^0([0,T]; Z) \cap L_k^2(0,T; D(A)) \cap H^1(0,T; L_k^2(-1,1)), \tag{4.5.6}$$

and

$$\sup_{t\in[0,T]} \|u(t)\|_k^2 + \int_0^T \Big(\|u_t(\tau)\|_k^2 + \|Au(\tau)\|_k^2\Big)\, d\tau \le D_T'\Big(\int_0^T \|u_0\|_Z^2 + \sup_{t\in[0,T]} \|f(t)\|_k^2\Big). \tag{4.5.7}$$

Chapter 5

Invariant Measures

5.1 Preliminaries

When studying dynamical systems, we are interested in stability results and the steady-states of the system. Much work has been devoted to investigating these states, including their nature, such as asymptotic stability, exponential stability, or instability. A more general approach involves the development of the theory of invariant measures and the existence of these states. These states are deeply connected with the notion of stationary solutions and their domains.

In more technical terms, let's briefly define an invariant set. The set $K \subset L^2$ is said to be invariant if, for every initial condition $x_0 \in L^2$ such that $x_0 \in K$ almost surely, the solution x of the SDE:

$$dx = b(x)\,dt + \sigma(x)\,dW_t,$$

satisfies $x(t) \in K$ for all $t \geq 0$.

Invariant applications are mainly applicable to flows, such as traffic flow. The traffic flow can be formulated as a control problem:

$$\dot{x} = Ax + Bu,$$

where x is the state variable (position and velocity), u is the control function, and A and B are matrices with suitable dimensions.

Suppose we have the problem of maximizing the car capacity of a highway. To achieve this, grouping vehicles into a synchronized stream of vehicles or platoons has been suggested as a possible method. This involves keeping constant distances and velocities between cars that have entered the highway. This is analogous to finding an invariant measure for the state variable x.

Returning to the problem under consideration, we focus on studying the steady-states of the FP equation with degenerate diffusion terms. Our primary goal is to investigate the existence and uniqueness of invariant measures.

The series of research papers of Huang W. et al. "Steady-States of Fokker–Planck Equations",[49],[50], and [51] explores the dynamics of steady-states within the context of FP equations situated in general domains in $\mathbb{R}^n, n \in \mathbf{N}$. The focus is on equations incorporating L^p_{loc} drift terms and $W^{1,p}_{\text{loc}}$ diffusion terms for $p > n$.

The research introduces novel existence results for steady-states, utilizing the level set method and an integral identity from prior literature. These results encompass stationary solutions and measures under Lyapunov-like conditions. Additionally, illustrative examples highlight phenomena such as noise stabilization of unstable equilibria and the establishment of steady-states amidst boundary degeneracy of diffusion within bounded domains.

Further analysis presents non-existence results for stationary measures under conditions involving anti-Lyapunov functions associated with the stationary Fokker-Planck equation. This work establishes necessary and sufficient conditions for the presence of stationary measures, providing insights into noise perturbations in systems of ODEs, particularly concerning stochastic bifurcations.

Finally, the investigation delves into scenarios featuring degenerate diffusion, extending findings to provide existence results for stationary measures under Lyapunov conditions accommodating diffusion degeneracy. This underscores the significance of these conditions in addressing intricate scenarios where diffusion may exhibit degeneracy.

In this thesis, we aim to explore and reproduce key aspects of the work conducted by Huang W. et al., with a particular focus on scenarios involving diffusion degeneracy, particularly in the context of one-dimensional problems. Specifically, we will delve into a detailed examination of the solution to the stationary FP equation and establish connections with the adjoint FP equation, or the Kolmogorov backward equation.

Moreover, we will endeavor to present and elucidate existence and uniqueness results pertinent to these types of equations. By thoroughly analyzing these equations, we aim to shed light on the fundamental properties and characteristics of systems subject to diffusion degeneracy.

Towards the culmination of our analysis, we will provide two illustrative examples of strong and weak degeneracy, respectively. Through these examples, we aim to demonstrate the practical implications and subtle distinctions associated with diffusion degeneracy in real-world systems.

5.2 Invariant Measures

In this chapter, we extend our analysis by delving into an Itô SDE system:

$$dx = b(x)\,dt + \sqrt{2a(x)}\,dW_t, \qquad x \in \mathcal{U} \subset \mathbb{R}^n, \tag{5.2.1}$$

where $\mathcal{U}$ could be: 1) a bounded region, 2) an unbounded region, or 3) the entire space $\mathbb{R}$, with $n \in \mathbb{N}$. As it is already mentioned, the equation:

$$u(t,x) = \int_{\mathcal{U}} p(t,\zeta,x)u_0(\zeta)\,d\zeta,$$

satisfies the FP equation under the following conditions:

$$\begin{cases} u_t(t,x) = Au(t,x), \qquad x \in \mathcal{U},\ t > 0, \\ u(0,x) = u_0(x), \\ u(t,x) \geq 0, \\ \displaystyle\int_{\mathcal{U}} u(t,x)\,dx = 1, \end{cases} \tag{5.2.2}$$

and A denotes the FP operator as defined on (3.1.7). In this chapter, following the approach of [49], [50], and [51], we will extend our assumptions regarding the nature of the drift and diffusion coefficients. We will make the following hypothesis:

$$a \in W^{1,q}_{\text{loc}}(\mathcal{U}), \quad b \in L^p_{\text{loc}}(\mathcal{U}), \text{where } q > n \ . \tag{5.2.3}$$

The associated stationary problem with the system (5.2.2) is:

$$\begin{cases} Au(x) = 0, \\ u \geq 0, \\ \displaystyle\int_{\mathcal{U}} u(t,x)\, dx = 1, \end{cases} \tag{5.2.4}$$

for $x \in \mathcal{U}$. Since we have employed less stringent regularity conditions, or, in other words, weak regular conditions, in Hypothesis (5.2.3), we aim to find weak stationary solutions of the FP equation. This entails identifying continuous functions that fulfill weak solutions of the stationary FP equation.:

$$\begin{cases} \displaystyle\int_{\mathcal{U}} A^* u_0(x) u(x)\, dx = 0, \\ u \geq 0, \\ \displaystyle\int_{\mathcal{U}} u(t,x)\, dx = 1, \end{cases} \tag{5.2.5}$$

for $u_0 \in C^\infty_0(\mathcal{U})$. Let's state some definitions that will be useful for the discussion on the invariant measures.

Definition 40. *[51, Def 2] Let μ be a Borel probability measure defined on the set $\mathcal{U}$. The measure μ is called a **stationary measure** of the FP equation (5.2.2) or a **measure solution** of the FP equation (5.2.5) if it fulfills the following conditions:*

1. *The drift coefficients $b \in L^1_{loc}(\mathcal{U}, \mu)$.*
2. *For any $u_0 \in C^\infty_0(\mathcal{U})$, the integral condition*

$$\int_{\mathcal{U}} A^* u_0(x)\, d\mu(x) = 0, \tag{5.2.6}$$

holds.

To link the weak form of the stationary FP equation (5.2.5) with the definition provided above, we will introduce the concept of a regular stationary measure in the following lines.

Definition 41. *[51] A stationary measure μ of the FP equation is called **regular** if it admits a continuous density function u, such that u is the weak stationary solution of (5.2.2), i.e., $\mu(x) = u(x)\, dx$.*

Let's define the different forms of Lyapunov and anti-Lyapunov functions that will be used as dissipation properties in the following sections.

Definition 42. *[51, Def 2.2] Let $V \in C^2$ be a compact function with essential upper bound M.*

1. *V is called* ***Lyapunov function*** *in the domain $\mathcal{U}$ with respect to (5.2.2) or (5.2.5) if $\exists m \in (0, M)$, such that $\exists l > 0$, Lyapunov constant:*

$$A * V(x) \leq -l, \quad x \in \mathcal{U} \setminus \mathcal{Q}_m, \tag{5.2.7}$$

where $\mathcal{Q}_m := \{x \in \mathcal{U} : V(x) < m\}$.

2. *V is called* ***strong Lyapunov function*** *in $\mathcal{U}$ with respect to (5.2.2) or (5.2.5) if*

$$\lim_{x \to \partial\mathcal{U}} A^* V(x) = -\infty. \tag{5.2.8}$$

3. *V is called* ***anti-Lyapunov function*** *in the domain $\mathcal{U}$ with respect to (5.2.2) or (5.2.5) if $\exists m \in (0, M)$, such that $\exists l > 0$, Lyapunov constant:*

$$A * V(x) \geq -l, \quad x \in \mathcal{U} \setminus \mathcal{Q}_m. \tag{5.2.9}$$

The relationship between stationary measures of the FP equation (5.2.2) and invariant measures of the diffusion process generated by the SDE (5.2.1) can be summarized as follows:

If a measure μ is invariant for the diffusion process X_t generated by the SDE (5.2.1), then the corresponding PDF $p(x)$ satisfies the stationary FP equation (5.2.2). Conversely, if $p(x)$ is a stationary solution to the FP equation, the measure with density $p(x)$ is invariant for the diffusion process. This equivalence highlights the close connection between the long-term behavior of the diffusion process and the stationary solutions of the associated FP equation.

In situations where $\mathcal{U} = \mathbb{R}^n$, and both b and a exhibit local Lipschitz continuity, the SDE (5.2.1) gives rise to a generalized (in time) diffusion process across $\mathbb{R}^n$. It's widely acknowledged that any invariant measure characterizing this diffusion process is a stationary measure for the associated FP equation (5.2.2), and conversely, under specific conditions.

In cases where (5.2.1) fails to generate a diffusion process, possibly due to the inadequate regularity of b and a, a stationary measure originating from (5.2.2) can still act as an invariant measure for a separate generalized diffusion process associated with (5.2.1), provided certain conditions are met.

Below, we outline some classical findings where assuming sufficiently regular conditions on the coefficients leads to the existence of solutions:

- All stationary measures of (5.2.2), under the assumption (5.2.3) and with a being positive definite, demonstrate regularity with a density function $u \in W^{1,q}_{\text{loc}}(\mathcal{U})$.[22]
- If $a \in C^{2,\beta}_{\text{loc}}(\mathcal{U})$ and $b \in C^{1,\beta}_{\text{loc}}(\mathcal{U})$, where $\beta \in (0, 1)$, then u emerges as a classical solution.
- A FP equation on a compact manifold without boundary has unique smooth stationary solutions when (a_{ij}) is a C^2-smooth, everywhere positive definite matrix and the drift field is smooth.[11]

In the above examples, the coefficients of the system not only are sufficiently smooth, but the diffusion term is positive definite. In the case under consideration, degeneracy on diffusion on the boundary, this condition is being violated and cannot be used for our analysis. Therefore, one requires additional conditions to study this peculiar case. In the work [49],[50] and [51] they extend the theory on the degenerate case introducing a dissipation property, as they use the notion of Lyapunov functions, that the coefficients of the system have to follow.

5.3 Existence and Non-Existence of the Stationary Measure

In this section, we will present some foundational results from the works [49, 50, 51]. First, we will establish results for the general case without degeneracy, and then we will provide a theorem focusing on the degenerate case, highlighting the differences between the two scenarios. Finally, we will present a brief overview of the proof for the degenerate case, noting that detailed non-existence results for the degeneracy case are not yet available.

5.3.1 General Case Results

We begin with results that apply to the general, non-degenerate case. These results provide important insights into the existence and uniqueness of stationary measures under various conditions.

Theorem 11. *[49, Theorem A] Assume that $a \in W^{1,2}_{loc}(\mathcal{U})$ and $b \in L^2_{loc}(\mathcal{U})$ hold in $\mathcal{U}$ and the diffusion is positive definite everywhere. If there exists a Lyapunov function concerning the system (5.2.4), then the system (5.2.2), $u \in \mathcal{U}$ for $t > 0$, possesses a stationary measure in $\mathcal{U}$ which is regular and has a positive density in $W^{1,2}_{loc}(\mathcal{U})$. Moreover, if the Lyapunov function is unbounded, the stationary measure is unique on $\mathcal{U}$.*

The following theorem is devoted to the concept of the nonexistence of stationary measures in $\mathcal{U}$.

Theorem 12. *[50, Theorem A] Assume that $a \in W^{1,2}_{loc}(\mathcal{U})$ and $b \in L^2_{loc}(\mathcal{U})$ hold in $\mathcal{U}$ and the diffusion is positive definite everywhere. If there exists an anti-Lyapunov function concerning the system (5.2.4) that is of the class $B^*(a)$, then the system possesses no stationary measure in $\mathcal{U}$.*

5.3.2 Degenerate Case Results

Next, we focus on the case where degeneracy is present. The following theorem addresses the existence of stationary measures under such conditions.

Theorem 13. *[51, Theorem 4.1] Assume that there exists a Lyapunov function V concerning the family of adjoint FPoperator $A^*_{\tilde{b},\tilde{a}}$ where $\tilde{a} \in W^{1,q}_{loc}(\mathcal{U})$ and $\tilde{b} \in C(\mathcal{U}, \mathbb{R}^n)$, and one of the following properties is satisfied:*

- *V is a strongly Lyapunov function with respect to $A^*_{\tilde{b},\tilde{a}}$; or*
- *$V \in B^*(\tilde{a})$.*

Then, the Fokker-Planck equation corresponding to $A = A_{\tilde{b},\tilde{a}}$ admits a stationary measure in $\mathcal{U}$.

In comparing Theorems 11 and 13, we note that the conditions for the existence of stationary measures differ significantly between the general and degenerate cases. The presence of degeneracy requires more refined conditions on the coefficients and the Lyapunov function.

It is also important to note that, currently, we do not have non-existence results for the degenerate case analogous to those presented in Theorem 12 for the general case.

In conclusion, we have established a framework for understanding the existence of stationary measures under different conditions. While the results for the general case are well-understood, the degenerate case presents unique challenges and requires further investigation.

In the following lines, we will delve deeper into the challenges of degeneracy cases, providing a brief overview of the proof techniques and outlining directions for future research Next, we outline the fundamental idea behind the proof of Theorem 13, as presented in [51]. The approach involves perturbing a using a family of positive definite matrices, followed by taking the weak*-limit of the associated stationary measures as these perturbations approach zero.

Proof. Assume V is a Lyapunov function, and the coefficients $\tilde{a} \in W^{1,q}_{\text{loc}}$ and $\tilde{b} \in C(\mathcal{U}, \mathbb{R}^n)$ are positive definite matrices in $\mathcal{U}$. We introduce the function $G^*(l)$ defined by:

$$G^*(l) = \max_{x \in V^{-1}(l)} \tilde{a} \partial V(x) \partial V(x), \quad l \in [0, M].$$

Notice that, if $V \in B^*(a)$, then there exists $G_1 \in [0, M]$ satisfying the following properties:

- For all $l \in [0, M]$,
$$\min_{y \in [0,l]} \{G_1(y) - G^*(y)\} > 0,$$
- Additionally,
$$\int_0^M \frac{1}{G_1(s)}\, ds = \infty.$$

Consider
$$\delta(x) = \frac{\min\{\delta_0, G_1(V(x)) - G^*(V(x))\}}{|\mathcal{D}V(x)| + |\nabla^2 V(x)|^2 + 1}, \quad x \in \mathcal{U},$$
where $\mathcal{D}$ is the Hessian matrix of V, and
$$\delta_0 = \begin{cases} 1, & \text{for} \quad V \notin B^*(a), \\ \dfrac{\theta}{2}, & \text{for} \quad V \in B^*(a). \end{cases}$$

From the earlier discussion, one can deduce that $\delta(x) > 0$ on any compact subset of $\mathcal{U}$. Continuing with the proof, let a, such that $a \in W^{1,q}_{\text{loc}}$,
$$\max_{1 \le i,j \le n} |a(x) - \tilde{a}(x)| \le \delta(x), \quad x \in \mathcal{U},$$

then for each $a \in \mathcal{A}$ we have:

$$\left|\left(A^*_{\tilde{b},a} - A^*_{\tilde{b},\tilde{a}}\right) V(x)\right| \leq \delta_0,$$

and

$$|(a(x) - \tilde{a}(x))\, \partial V(x) \partial V(x)| \leq G_1(V(x)) - G^*(V(x)).$$

It follows that V is a uniform Lyapunov function with respect to the family $\{A^*_{\tilde{b},a} : a \in \mathcal{A}\}$. Additionally, in [51] they prove that $\mathcal{A}$ contains $\tilde{a}$ as a limit point, by constructing a family a_δ with the properties: $a_\delta \in \mathcal{A}$ and $a_\delta \to \tilde{a}$, as $\delta \to 0$. Therefore, by an application of Theorem 11, we have the existence of the FP equation in $\mathcal{U}$ corresponding to A.

□

5.3.3 Examples

In this section, we present two primary examples to illustrate the cases of degeneracy discussed in Chapter 4. We will specifically examine the degenerate diffusion defined by $a(x) = (1 - x^2)^p$ where $p \in [0, \infty)$. By applying the theoretical framework introduced earlier, we aim to establish the existence and uniqueness of invariant measures for the corresponding SPs.

Example 1(The strongly degenerate diffusion)[50]

Consider the following SDE:

$$dx = b(x)dt + \sqrt{2}(1 - x^2)dW_t. \tag{5.3.1}$$

Then, the SDE (5.3.1) is related to the following FP equation:

$$Au = ((1 - x^2)^2 u)_{xx} - (b(x)u)_x,$$

and the adjoint problem is given by:

$$A^*u = (1 - x^2)^2 u_{xx} + b(x)u_x.$$

Let the function $V = -ln(1 - x^2)$ for the space domain $x \in (-1, 1)$. Then, one can easily verify that this function has the following properties:

1. V is C^2 in $(-1, 1)$,
2. $V(x) \to \infty$ as $|x| \to 1$,

and

$$A^*V(x) = 2 + 2x^2 + \frac{2b(x)x}{1 - x^2} \text{ for } x \in (-1, 1).$$

Choosing the function $b(x)$ to satisfy:

- for $x > 0$: $b(x) < 0$,
- for $x \leq 0$: $b(x) > 0$,

one can readily verify that: $A^*V(x) \to -\infty$ indicating that V is indeed an unbounded Lyapunov function concerning the stationary FP equation associated with (5.3.1). Consequently, by applying Theorem 11, we conclude that the FP equation associated with (5.3.1) has a unique stationary measure in the interval $(-1, 1)$. The analytic form of the stationary distribution, choosing $b(x) = -\gamma x$, as has been presented on [42], is given by:

$$u(x) = \frac{1}{1-x^2} e^{-\frac{\gamma}{1-x^2}},$$

where $\gamma > 0$ and the above expression is integrable.

Example 2(The weak degenerate diffusion)

Consider the SDE system:

$$dx = b(x)dt + \sqrt{2}(1-x^2)^{p/2} dW_t, \tag{5.3.2}$$

where we have assumed that $p \in (0, 1)$. Then the SDE is related to the following FP equation:

$$Au = ((1-x^2)^p u)_{xx} - (b(x)u)_x,$$

and the adjoint problem is given by:

$$A^*u = (1-x^2)^p u_{xx} + b(x)u_x.$$

Let the function $V = -ln(1-x^2)^p$ for the space domain $x \in (-1, 1)$. It can be readily shown that this function possesses the following properties:

1. V is C^2 in $(-1, 1)$,
2. $V(x) \to \infty$ as $|x| \to 1$,

and

$$A^*V(x) = \frac{2p(1+x^2)}{(1-x^2)^{2-p}} + \frac{2pb(x)x}{1-x^2} \quad \text{for } x \in (-1, 1).$$

In addition,

1. $A^*V \to \infty$,
2. $V \notin B_*(a)$.

Then, we cannot apply Theorem 13 and we are not able to conclude if the system $u_t = Au$ admits or not a stationary measure.

In our previous discussion, we examined how the existence of an invariant measure for an SDE is influenced by the degree of degeneracy of the diffusion coefficient. We particularly focused on the example where $a(x) = (1-x^2)^2$, which falls under the SDeg category. By applying Theorem 11, we concluded that the system admits a unique stationary measure on the interval $(-1, 1)$.

However, if we slightly modify the exponent in the diffusion coefficient, choosing $a(x) = (1-x^2)^{1.99}$ instead of $(1-x^2)^2$, the situation changes significantly. In this case, the condition

required by Theorem 11 for the existence of a unique stationary measure no longer holds, which prevents us from drawing a definitive conclusion about the invariant measure.

This highlights the sensitivity of the existence and uniqueness of the stationary measure to the specific form of the diffusion coefficient and at the same time the necessity for further development in the theory to handle cases with exponents slightly less than 2. Specifically, additional results or refined techniques are required to determine whether a unique stationary measure exists for diffusion coefficients of the form $a(x) = (1-x^2)^p$ when $1 < p < 2$. This remains an open problem and a potential area for future research in the study of stochastic differential equations and their invariant measures.

Chapter 6

Applications of the Degenerate Diffusion

6.1 Climatology

Dynamical systems described by degenerate PDEs have significant applications in the physical sciences. For instance, the Lighthill-Whitham-Richards (LWR) model for traffic flows incorporates degenerate PDEs to handle discontinuities and maintain stability in vehicle density predictions. Similarly, the inclusion of diffusion terms in these models helps to smooth out abrupt changes, enhancing their applicability to real-world phenomena. Another example is the study of heat conduction in materials with varying properties, where degenerate PDEs are used to model the changes in thermal conductivity. In population dynamics, degenerate PDEs are employed to describe species dispersion in heterogeneous environments, accounting for varying diffusion rates across different regions.

Building on these concepts, we turn our attention to climate modeling. Climate models have distinct characteristics compared to weather prediction models, primarily due to the vastly different time scales they operate on—centuries for climate models versus days or weeks for weather prediction models. Additionally, their main objectives are complementary: weather prediction models are prognostic, focusing on forecasting future weather conditions, while climate models are diagnostic, aiming to understand the mechanisms driving past and future climate changes and their sensitivity to various parameters. Climate models were developed to gain insights into the historical and future climate states and to analyze their dependence on a few critical features, which quantitative analysis reduces to specific parameters.

In this work, we will focus on the mathematical treatment of some horizontal energy balance models, generalizing the models independently introduced by Budyko (1969) and Sellers (1969). These Energy Balance Model (EBM) are essential for understanding the distribution of energy within the Earth's climate system and play a pivotal role in climate studies. Specifically, we will examine the 1D Budyko-Sellers model, where degeneracy occurs due to the transformation to spherical coordinates of the system. This transformation adds complexity to the equations, necessitating a thorough mathematical analysis to fully understand the model's behavior and implications.

6.1.1 Climatic Modeling-Energy Balance Models

EBMs are one-dimensional models predicting surface temperature variation with latitude or longitude. Simplified relationships are used to calculate the terms contributing to the energy balance in each latitude or longitude zone.

At the core of an EBM is the energy equation that balances incoming solar radiation with outgoing terrestrial radiation and other energy losses. The equation typically takes the form:

Internal Energy Flux Variation	=	Absorbed Energy − Reflected Energy +Diffusion

This formula is a straightforward representation of the energy fluxes entering and leaving the Earth system.

Budyko and Sellers published descriptions of models that attempted to simulate the essentials of the climate system simply.([4],[6]). These EBMs drew upon observational data derived from descriptive climatology; for example, the reasons why the major climatic zones are roughly latitudinal. In particular, EBMs are computationally much faster than Global Climate Models (GCM) because instead of calculating the dynamical movement of the atmosphere, using the Navier–Stokes equations, they employ much simpler parameterizations, and typically, much coarser grids. As a consequence of the intrinsically simpler parameterization schemes employed in EBMs, they could be applied to longer time-scale changes than the atmospheric GCM of the time. Although the desire for longer simulation times is still an important driver for the development of simple climate model types, other demands, particularly associated with climate policy needs, are becoming increasingly important.

Diaz [17] meticulously explored and formulated the concept of Earth as a compact two-dimensional manifold without boundary, denoted as $\mathcal{M}$. In this elegant abstraction, let $u(t,x)$ symbolize the annually or seasonally averaged surface temperature. The behavior of this temperature distribution is described through the mathematical framework of a **reaction-diffusion (RD) equation**:

$$c(t,x)\frac{\partial u(t,x)}{\partial t} - \operatorname{div}(k(t,x)\nabla u(t,x)) = R_a(t,x,u(t,x)) - R_e(t,x,u(t,x)), \tag{6.1.1}$$

- c(t,x): is a positive function that is frequently influenced to a significant extent by the presence of oceans. It's noteworthy that approximately 70% of Earth's surface is covered by oceans, making them a major factor in determining the overall heat capacity of the planet. Averaging $c \sim 1.05 \times 10^{23}\,\mathrm{Jm^{-2}K^{-1}}$
- the diffusion operator has a double justification:

  $$\operatorname{div}(k\nabla u) = \operatorname{div}(F_c + F_a),$$

  with:

  - $F_c = k_c \operatorname{div} u$: the conduction heat flux.
  - F_a: the advection heat flux.

- R_a: The solar energy absorbed by the Earth and it of the form:

$$R_a := SQ(t,x)\beta(u),$$

where

  - S: The Solar constant, defined as the annual average amount of radiation energy per unit time passing through a unit area perpendicular to the Sun's rays at the Earth's orbit, remains consistent in our context. Averaging S$\sim 1.370 W/m^2$.
  - $Q(t,x)$: The distribution of solar radiation across the Earth's surface.
  - β(u): The planetary albedo, denoted by β, represents the fraction of solar radiation absorbed by the Earth's surface, taking into account the average temperature. Typically,β is assumed to be a non-decreasing function of$u$, the surface temperature, and it maintains constant values (positive and less than 1) for any given value of$u$. In subsequent sections, we will outline the distinctions in the selection of the β function between the Budyko and Sellers models.

- $R_e(t,x,u)$: The mean emitted energy flux, denoted by R_e, represents the average energy emitted by the Earth's surface. It is empirically determined and influenced by factors such as greenhouse gases, clouds, and water vapour present in the atmosphere. Budyko's model posits a distinct value for R_e compared to Sellers' model, a discrepancy that will be elaborated upon in the ensuing discussion.

The differential operators, divergence (div) and gradient (grad) must be appropriately interpreted in light of the Riemannian metric.

If we assume that $\mathcal{M}$ is the unit sphere of $\mathbb{R}^3$, the Laplace-Beltrami operator (div$(k\nabla)$) becomes:

$$\operatorname{div}(k\nabla u) = \frac{1}{\sin\phi}\left(\frac{\partial}{\partial\phi}(k\sin\phi\frac{\partial u}{\partial\phi}) + \frac{\partial}{\partial\lambda}(k\frac{\partial u}{\partial\lambda})\right), \tag{6.1.2}$$

where ϕ is the colatitude and λ is the longitude. If we take the average of the temperature at $x = \sin\phi$, assuming that $\dfrac{\partial u}{\partial\lambda} = 0$, which seems like a reasonable assumption (Figure 6.2), the level mean zonally averaged temperature $u(t,x)$ on the Earth, satisfies a quasi-linear EBM strongly degenerate problem, in the bounded domain $(-1,1)$, of the following type:

$$\begin{cases} u_t - (a(x)|u_x|^{q-2}u_x)_x = R_a(x,t,u) - R_e(x,t,u), & x \in I, t > 0, \\ a(x)u_x = 0, & x \in \partial I, t > 0, \\ u(x,0) = u_0(x), & x \in I, \end{cases} \tag{6.1.3}$$

where $I = (-1,1)$ and $1 < q < \infty$.

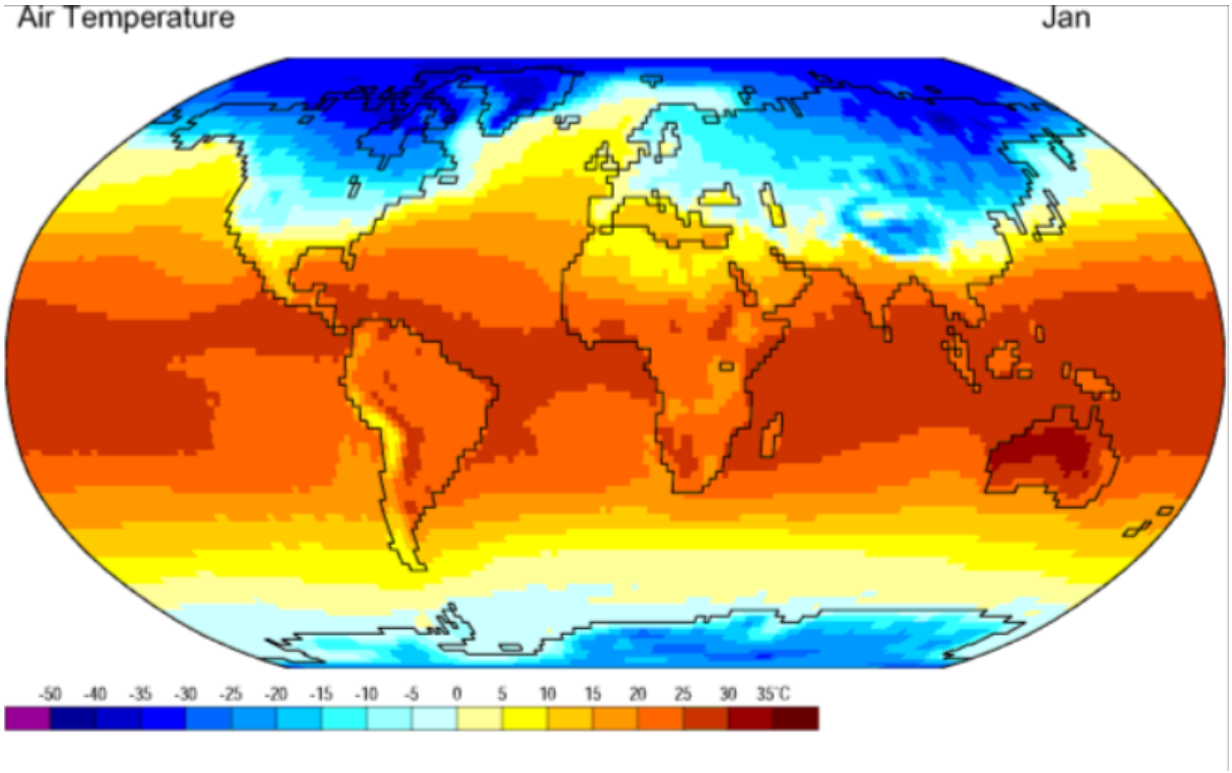

Figure 6.1: The figure shows an example of atmospheric temperature at the Earth's surface in January. Temperatures are averaged over the 1959-1997 period. Red areas correspond to a warm climate and blue areas to a cold climate. (Copyrights:https://polarpedia.eu/en/climate/)

In this work, we will make the following structural assumptions:

- $q := 2$,
- $a(x) := k(1-x^2)$, $k > 0$,
- $R_a := Q(t,x)\beta(u)$,where:
  - $Q \in C([-1,1] \times \mathbb{R}_+)$, and $Q(t,x) > 0$,
  - $\beta(u)$ is a non-decreasing and bounded function function, $|\beta(u)| \leq M, \forall u \in \mathbb{R}$, $M > 0$.

Remark 13. *The occurrence of the degenerate diffusion term* $(1-x^2)$ *comes from the transformation* $x = \sin\phi$ *and the equation (6.1.1)*

According to the above assumption, one has the following RD equation with reflecting Boundary Conditions:

$$\begin{cases} u_t - ((1-x^2)u_x)_x = Q(t,x)\beta(u) - R_e(x,t,u), & x \in I, t > 0, \\ (1-x^2)u_x = 0, & x \in \partial I, t > 0, \\ u(x,0) = u_0(x), & x \in I. \end{cases} \tag{6.1.4}$$

We have not yet made any assumptions about the function R_e, which represents the energy emitted by the Earth. The value of R_e might be influenced by various factors such as the

concentration of greenhouse gases, the presence of clouds and water vapour in the atmosphere, and changes caused by human activities. Next, we will explore the key differences between the Budyko and Seller models. While both models incorporate similar elements like the diffusion term and the Incoming Solar function, there are distinct assumptions in each model that we'll outline. For the **Incoming Solar Flux** we assume $Q(t,x) = q(x)r(t)$ where q is the **insolation function** and:

$$\begin{cases} q \in L^\infty(I), \\ r \in C^1(\mathbb{R}_+) \text{ and } r, r' \in L^\infty(\mathbb{R}_+). \end{cases}$$

The **Budyko type coalbedo** function is given by the form:

$$\beta(u) := \begin{cases} a_i, & u < -10^\circ C, \\ [a_i, a_f], & u = -10^\circ C, \\ a_f, & u > -10^\circ C. \end{cases}$$

In contrast, the **Seller's type coalbedo** is taken as:

$$\beta \in C^2(\mathbb{R}) \quad \text{and} \quad \beta, \beta', \beta'' \in L^\infty(\mathbb{R}).$$

Additionally, both models define the function R_e, which represents the energy emitted by the Earth. In Budyko's model, this radiation function, R_e, is assumed to be a linear function of temperature:

$$R_e(u) := a + bu, \qquad s \in \mathbb{R} \quad \text{and} \quad b > 0. \tag{6.1.5}$$

In contrast, in Seller's work, the radiation function emitted by the Earth follows a Stephan-Boltzmann-type law:

$$R_e(u) := \epsilon(u)|u|^3 u, \tag{6.1.6}$$

meaning that the Earth radiates as a black body. In the formula (6.1.6), $\epsilon(u)$ denotes the **emissivity** and is a positive bounded function taken as:

$$\epsilon(u) := \sigma\big(1 - m \tanh\big(\frac{19u^6}{10^6}\big)\big),$$

where $\sigma > 0$ represents the **emissivity coefficient** and $m > 0$ indicates the **atmospheric opacity**.

Remark 14. *Notice that equation (6.1.5) corresponds to a linear approximation of (6.1.6) near the actual mean temperature of 15°C.*

Remark 15. *A drawback of the model occurs in this step and is related to the direct linkage between albedo and temperature, which completely neglects the long response times of the continental ice sheets.*

We are now able to write the most general form of the two models, in Budyko's and Seller's case.

Budyko's Model

$$\begin{cases} u_t - ((1-x^2)u_x)_x = r(t)q(x)\beta(u) - \alpha + bu, & x \in I, t > 0, \\ (1-x^2)u_x = 0, & x \in \partial I, t > 0, \\ u(x,0) = u_0(x), & x \in I. \end{cases} \tag{6.1.7}$$

Seller's Model

$$\begin{cases} u_t - ((1-x^2)u_x)_x = r(t)q(x)\beta(u) - \varepsilon(u)|u|^3 u, & x \in I, t > 0, \\ (1-x^2)u_x = 0, & x \in \partial I, t > 0, \\ u(x,0) = u_0(x), & x \in I. \end{cases} \tag{6.1.8}$$

6.1.2 Numerical Simulation in Budyko's Climate Model

In this section, we will present numerical results for the Budyko-type model. To ensure that our model is both reasonable and realistic, we will carefully adjust the parameters to reflect current behaviors. Below, we describe the model configuration that we will simulate.

$$cu_t - \frac{D}{\sin\phi}\left(\sin\phi\, u_\phi\right)_\phi = (1-\alpha)Q - (a+bu), \tag{6.1.9}$$

or equivalently:

$$cu_t - D\left((1-x^2)\, u_x\right)_x = (1-\alpha)Q - (a+bu), \tag{6.1.10}$$

where we have chosen:

- $a = 210\ Wm^{-2}$, is the emission at $0°C$,
- $b = 2\ Wm^{-2}\,{}^\circ\mathrm{C}^{-1}$, rise in emissions per degree, associated with the net long-wave climate feedback,
- $D = 0.55\ Wm^{-2}\,{}^\circ\mathrm{C}^{-1}$, constant diffusivity of the system,
- $Q = \overline{Q(\phi)}$, annual mean insolation,
- $\alpha(\phi) = \alpha(\phi, u(\phi)) = \begin{cases} \alpha_0 + \alpha_2 P_2(\sin\phi), & u(\phi) \geq -10°C \ \text{(no ice)} \\ \alpha_i, & u(\phi) < -10°C \ \text{(ice covered)} \end{cases}$, is the albedo function and we made the additional adjustment:
  - $\alpha_0 = 0.3$,
  - $\alpha_2 = 0.078$,
  - $\alpha_i = 0.62$,
  - $P_2(\phi) = \frac{1}{2}(3\phi^2 - 1)$ is the Legander Polynomial.

Here, we operate under the assumption that in areas where temperatures exceed $-10°C$, ice cover cannot form. Consequently, if the temperature satisfies $u(\phi) > -10°C$ for all $\phi \in [-90, 90]$, it implies a scenario where the Earth is completely devoid of ice-covered regions. Conversely, if $u(\phi) \leq -10°C$ for all $\phi \in [-90, 90]$, it indicates a Snowball Earth scenario, where the planet is entirely encased in ice.

The selection of the diffusion coefficient D carefully considers the pole-to-equator temperature gradient and peak poleward heat transport and is sophisticatedly chosen to study the behavior of temperature under varying solar radiation conditions.

For our analysis, we will use a Python package called "Climlab". I give the credits in the following lines:

Credits The Python package is part of The Climate Laboratory, an open-source textbook developed and maintained by Brian E. J. Rose, University at Albany.

It is licensed for free and open consumption under the Creative Commons Attribution 4.0 International (CC BY 4.0) license.

Running the model for the assumed (default) values we get the following figures

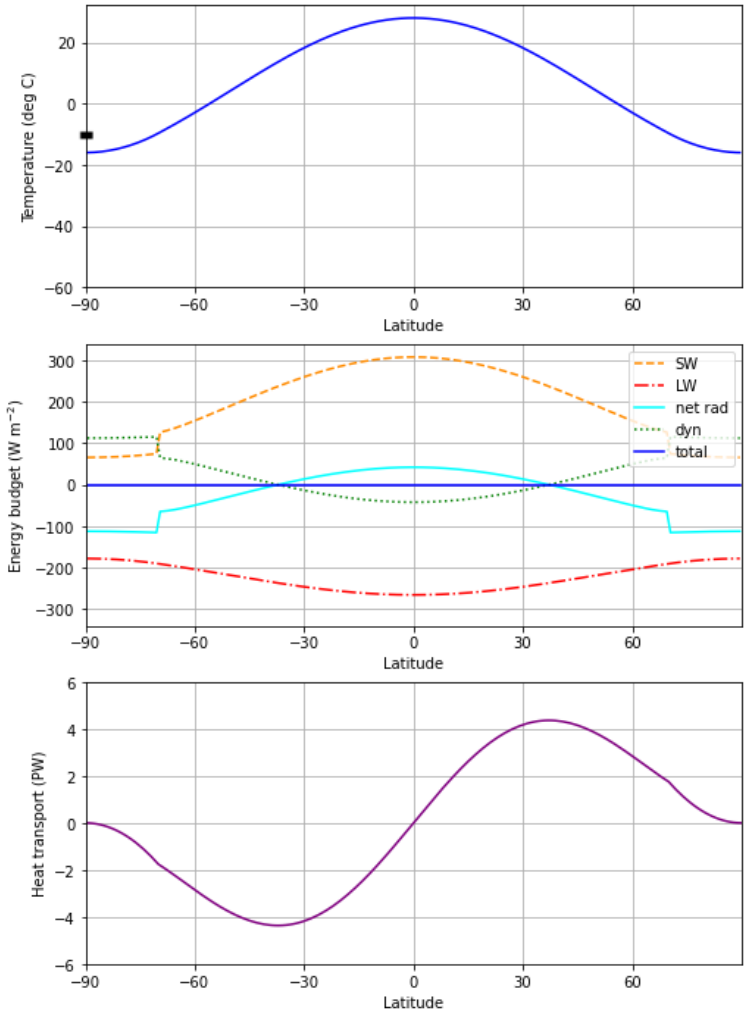

Figure 6.2: The figures show the energy balance of the Earth. Specifically, they display the annual distribution of global temperatures across different latitudinal belts, illustrating the system's energy balance and heat transport across various latitudes.

In the upper figure, we present the annual distribution of global temperatures across different latitudinal belts. This distribution effectively captures a realistic view of the Earth's current climate, showcasing satisfactory model performance.

Additionally, the model accurately identifies the current location of the ice edges within the latitudinal zones of $[-90, -70]$ and $[70, 90]$, providing a robust approximation. It is crucial to note the symmetric behavior of the system, particularly the mild delineation at the Earth's equator, which further enhances our understanding of global climatic patterns.

The other two figures illustrate the behavior of the model concerning energy dynamics. Specifically, the middle figure demonstrates that the multifactorial system maintains a total

Energy Budget of zero, affirming the appropriateness of using the term Energy Balance Model (EBM).

The final figure displays the Heat Transport across different latitudes. Notably, the minimum points occur at the poles and the Equator, indicating a lack of energy transport at these locations. Conversely, the maximum values are observed around ± 40 degrees latitude, suggesting that these regions are pivotal in driving the heating process.

Budyko's model, especially in its one-dimensional form, demonstrates remarkable power and elegance as an EBM. With even a straightforward selection of insolation and emission functions, it performs impressively, aligning closely with the current global temperature scenario. This model's simplicity belies its potential for deeper exploration into Earth's climatic history. By fine-tuning its parameters, one can delve into various historical epochs, predicting phenomena such as the ice edge epoch, the Snowball Earth period, or even envisioning a world with no ice-covered regions. This adaptability highlights Budyko's model as a valuable tool for understanding and forecasting Earth's climatic changes.

6.1.3 Well-Posedness for Seller's Model

In this section, we'll establish the functional framework guiding our analysis. This functional framework, introduced by Tort and Vancostenoble [48], is pivotal due to the degeneracy of the diffusion term. We'll employ weighted Sobolev spaces, following their approach, offering an effective treatment for this particular scenario.

Set:

$$Y := \{y \in L^2(I) : \sqrt{\rho} y_x \in L^2(I)\},$$

where I denotes the interval $(-1, 1)$.

The associated inner product and the corresponding norm are as follows:

$$(u, y)_Y := (u, y)_{L^2(I)} + (\sqrt{a} u_x, \sqrt{a} y_x)_{L^2(I)},$$

and

$$\|u\|_Y := \sqrt{(u, y)_Y}.$$

We aim to recast our system (6.1.4) into the evolution equation:

$$\begin{cases} u_t(t) + Au(t) = P(t, u(t)), & t \in [0, T] \\ u(0) = u_0. \end{cases}$$

To do so, we introduce the following degenerate diffusion operator:

$$\begin{cases} D(A) := \{u \in Y | y \mapsto \displaystyle\int_{-1}^{1} a u_x y_x \, dx \text{ is } L^2(I) \text{ continuous}\}, \\ \forall u \in D(A), \forall y \in Y, \quad (Au, y) = \displaystyle\int_{-1}^{1} a u_x y_x \, dx. \end{cases} \tag{6.1.11}$$

Then one has the following characterization of the operator A:

Theorem 14. *[48, Theorem 3.1] Operator $(A, D(A))$ is the infinitesimal generator of an analytic semigroup $S(t)$.*

Proof. Notice that $\exists \gamma > 0$ and $\theta \in \mathbb{R}$ such that for every u:

$$\int_{-1}^{1} a(u_x)^2 \, dx + \gamma \|u\|^2_{L^2(I)} \geq \theta \|u\|^2_Y.$$

By directly applying Theorem 2.12,p.115 [12], we obtain the required result, thereby completing the proof.

□

Lemma 4. *[48, Lemma 3.1] The couple $(A, D(A))$ satisfies:*

$$\begin{cases} D(A) := \{u \in Y | au_x \in H^1(I)\}, \\ \forall u \in D(A), Au := -(au_x)_x. \end{cases} \tag{6.1.12}$$

Remark 16. *Notice that we haven't included a boundary condition on (6.1.12). That is because of the structure of the space $D(A)$, where the condition $(\rho u_x)_x \in L^2(I)$ implies that u satisfies $(\rho u_x)(x) = 0$ on ∂I. This outcome will be discussed on Lemma's proof.*

Proof. [48] In order to prove Lemma 4 we will introduce the set

$$\mathcal{D} := \{u \in Y | au_x \in H^1(I)\}.$$

We will start by proving that $D(A) \subset \mathcal{D}$:

The $L^2(I)$-valued function P is defined as follows: By applying the operator A onto the element $y \in D(A)$ one gets that $Ay \in L^2(I)$. In addition, let $w \in \mathcal{D}$ and take the following inner product:

$$(Ay, w)_{L^2} = \int_{-1}^{1} (ay_x w_x) \, dx = ay_x w \Big|_{-1}^{1} - \int_{-1}^{1} (ay_x)_x w \, dx \implies$$
$$\implies (Ay, w)_{L^2} = -\int_{-1}^{1} (ay_x)_x w \, dx.$$

Therefore, one has:

$$Ay = -(ay_x)_x, \quad \text{on } \mathcal{D}^*.$$

Thus, from the above result and $Ay \in L^2(I)$ one can conclude to $(ay_x)_x \in L^2(I)$ and therefore $y \in \mathcal{D}$, that proves that if

$$y \in D(A) \implies y \in \mathcal{D},$$

then $D(A) \subset \mathcal{D}$. Let's now prove that $\mathcal{D} \subset D(A)$: Given that ρu_x belongs to $H^1(I)$, it implies that au_x is in $C^0([-1, 1])$. Let's assume that as x approaches -1, $|au_x|$ tends to L, where $L > 0$. Then, there exists a region close to $x = -1$ where $|au_x| \geq L/2$. By rearranging the inequality, we get $|\sqrt{a}u_x| \geq \dfrac{L}{2\sqrt{a}}$. This contradicts the fact that $\sqrt{a}u_x$ is in $L^2(I)$, since $\dfrac{1}{a} \notin L^1(I)$. Hence, we conclude that $L = 0$. By following the same procedure, it can be demonstrated that $au_x(1) = 0$.

Moving forward, we derive the following inequality:

$$|(ay_x)(x)| = |\int_x^1 (ay_x)_x(\sigma)\,d\sigma| \leq \Big(\int_x^1 (ay_x)_x^2(\sigma)\,d\sigma\Big)^{\frac{1}{2}}\Big(\int_x^1 d\sigma\Big)^{\frac{1}{2}} = \\ = \|(ay_x)_x\|_{L^2(I)}\sqrt{1-x}.$$

Then, we get the following useful inequality:

$$|(ay_x)(x)| \leq \|(ay_x)_x\|_{L^2(I)}\sqrt{1-x}. \tag{6.1.13}$$

Consider two points: $x_1, x_2 \in (0,1)$. Since $y \in \mathcal{D}$:

$$\int_{-x_1}^{x_2} ay_x w_x\,dx = ay_x w\Big|_{-x_1}^{x_2} - \int_{-x_1}^{x_2} (ay_x)_x w\,dx.$$

Notice that the following limits:

$$\lim_{x_2\to 1}\int_{-x_1}^{x_2} ay_x w_x\,dx \quad \text{and} \quad \lim_{x_2\to 1}\int_{-x_1}^{x_2} (ay_x)_x w\,dx,$$

are finite.

Therefore $\lim\limits_{x_2\to 1} ay_x w = L$ exists. We want to prove that $L = 0$ using a contradiction statement:

Let$L \neq 0$, then

$$|(ay_x w)(x_2)| > \frac{|L|}{2}, \quad \text{for } x_2 \text{ sufficiently close to 1.}$$

Utilizing the inequality (6.1.13):

$$|(ay_x w)(x_2)| \leq |(ay_x)(x_2)|w(x_2)| \leq \|(ay_x)_x\|_L^2\sqrt{1-x_2}|w(x_2)|.$$

Thus, one can find a constant $C > 0$ such that: $|w(x_2)| > \dfrac{C}{\sqrt{1-x_2}}$ which is a contradiction since $w \in L^2(I)$. Therefore, $L = 0$. One can follow the same process using the limit for $x_1 \to 1$. Then, $\mathcal{D} \subset D(A)$ and one can finally conclude to $\mathcal{D} = D(A)$.

By following the same procedure, it can be demonstrated that $au_x(1) = 0$. □

We define the map:

$$P : \begin{cases} \forall t \in [0,T], u \in Y, \\ (t,x) \mapsto r(t)q\beta(u) - \varepsilon(u)|u|^3 u. \end{cases}$$

Given that the function P is well-defined, we can rewrite the problem (6.1.4) as the following evolution equation:

$$\begin{cases} u_t(t) + Au(t) = P(t,u(t)), \quad t \in [0,T], \\ u(0) = u_0. \end{cases} \tag{6.1.14}$$

and now we can apply the basic theory for the evolutionary equations in $L^2(I)$. Having established the evolutionary equation, we can now derive our initial result concerning the local existence of a solution.

Theorem 15. *[48, Theorem 3.2] For each $u_0 \in D(A)$, there exists a time $T^{\#}(u_0) \in (0, \infty]$ such that for any $0 < T < T^{\#}(u_0)$, the problem (6.1.14) admits a unique solution $u \in C([0,T], D(A)) \cap C^1([0,T], L^2(I))$. Furthermore, if $T^{\#}(u_0) < \infty$, then $\|u(t)\|_Y \to \infty$ as $t \to T^{\#}(u_0)$.*

Having established the existing result, it is now logical to derive a maximum principle to guarantee the boundedness of the solution. Specifically, we aim to obtain a result that ensures the solution remains bounded and does not blow up at the time $T^*(u_0)$.

Theorem 16. *[48, Theorem 3.3] Consider the initial state $u_0 \in D(A) \cap L^\infty(I)$ and $0 < T < T^{\#}(u_0)$. Then*

$$K = \max\{\|u_0\|_{L^\infty(I)}, \Big(\frac{\|q\|_{L^\infty(I)}\|r\|_{L^\infty(\mathbb{R})}\|\beta\|_{L^\infty(\mathbb{R})}}{\epsilon_1}\Big)^{1/4}\},$$

where $\epsilon_1 < \epsilon(u)$

Then the solution of (6.1.14) satisfies:

$$\|u\|_{L^\infty((0,T)\times I)} \leq K.$$

We note that to investigate the existence of regular solutions, the initial state u_0 must be further restricted to the following set:

$$\mathcal{U} := \{u_0 \in D(A) \cap L^\infty(I) : Au_0 \in L^\infty(I)\}.$$

In the work by Tort and Vancostenoble [48], a thorough analysis of this problem is provided, leading to the following conclusion:

Theorem 17. *[48, Theorem 3.5] If $u_0 \in \mathcal{U}$, then the solution to the system (6.1.14) exists for all time $[0, \infty)$, meaning that $T^{\#} = \infty$.*

In the primary reference for this work [48], complete proofs for these theorems are provided. We do not present them analytically here for two reasons: Firstly, the technical aspects of these proofs are not central to the objectives of our study and may detract from the main themes we wish to highlight. Secondly, these detailed proofs are not of primary interest for this work, as our focus lies more on the application and implications of the theorems rather than their derivations.

6.2 Inverse Problem

Inverse problems hold significant importance in climatology. In this section, we will explore these problems and, leveraging the previous well-posedness results, aim to solve a class of inverse problems related to degenerate diffusion. To begin, let's clarify the distinction between direct and inverse problems.

Direct or Forward Problems

Direct or forward problems involve determining the effect given a specific cause. Many practical problems in science and engineering are usually formulated this way. For example, given the temperature, one can determine the conductivity or predict climate behavior.

Inverse Problems

Inverse problems, on the other hand, involve finding the cause given an observed effect. These problems are generally much more challenging than direct problems. Consider a projection problem: the direct problem would involve projecting a three-dimensional object onto a two-dimensional plane. The inverse problem would involve trying to determine the original three-dimensional object from its two-dimensional projection. This problem is not well-defined because multiple objects can produce the same projection.

Therefore, one can understand that it is natural to have multiple causes for a given effect, meaning that we have ill-posed problems. Consequently, it is necessary to determine the appropriate constraints to address these ill-posed issues. The inverse method refers to the process of calculating the causal factors from a set of observations.

There are many references dealing with uniqueness results for parabolic inverse problems, for instance:

- The pioneering work of Bukhgeim and Klibanov [10],
- The books of Isakov [20],
- Klibanov [29],
- Klibanov and Timonov [30], and the references therein.

Let us focus here on a specific result by Imanuvilov and Yamamoto [19] obtained for some standard inverse source problem for the linear heat equation. Their purpose is to retrieve the source term of the equation using some partial measurements of the solution. Their method is based on the global Carleman estimates for the heat equation that Fursikov and Imanuvilov [15] developed and used to solve null controllability issues. The novelty of their work is that they not only solve the uniqueness question but also provide some strong stability results concerning the reconstruction of the source term (more precisely they prove unconditional Lipschitz stability).

In this section, we outline the construction of an inverse problem following the framework provided in [19]. Consider the parabolic PDE on a domain Ω with boundary conditions, where Ω is of class C^2:

$$\begin{cases} u_t(t,x) = -\left(a(x)u_x(t,x)\right)_x + b(x)u_x(t,x) + c(x)u(t,x) + h(t,x), & (t,x) \in [0,T] \times \Omega, \\ \alpha_1(x)u_x + \beta_1(x)u = 0, & (t,x) \in [0,T] \times \partial\Omega, \\ u(\theta,x) = u_0(x), & x \in \Omega, \end{cases}$$

Here, $a(x)$ is a uniformly elliptic operator, and the coefficients of the PDE are assumed to be sufficiently regular over the domain $(t,x) \in [0,T] \times \overline{\Omega}$. The functions $\alpha_1(x)$ and $\beta_1(x)$ satisfy $|\alpha_1(x)| + |\beta_1(x)| > 0$ for all $x \in \partial\Omega$. For convenience, define the differential operator A as follows:

$$Au := -\left(a(x)u_x(t,x)\right)_x + b(x)u_x(t,x) + c(x)u(t,x).$$

We introduce the source term $h(t,x)$ as $h(t,x) := r(t,x)q(x)$, where $r(t,x)$ is a known function that satisfies:

$$|r(\theta, x)| > 0, \qquad \forall x \in \overline{\Omega}, \quad r \in C^{1,0}(\overline{\Omega}),$$

with $\theta \in [0, T]$ fixed. The goal is to determine the unknown function $q(x)$ from overdetermined data collected in a subset of the spatial domain $\omega \subset \Omega$.

There are two primary types of inverse problems to consider:

- Determine $q(x)$ from overdetermined data $u_{[0,T]\times\omega}$ within the equation:

$$\begin{cases} u_t(t,x) = -Au + r(t,x)q(x), & (t,x) \in [0,T] \times \Omega, \\ \alpha_1(x)u_x + \beta_1(x)u = 0, & (t,x) \in [0,T] \times \partial\Omega, \\ u(\theta,x) = u_0(x), & x \in \Omega, \end{cases}$$

- Determine $q(x)$ from overdetermined data $u_{[0,T]\times\Sigma_0}$ and $\nabla u_{[0,T]\times\Sigma_0}$, where $\Sigma_0 \subset \partial\Omega$, in the equation:

$$\begin{cases} u_t(t,x) = -Au + r(t,x)q(x), & (t,x) \in [0,T] \times \Omega, \\ \alpha_1(x)u_x + \beta_1(x)u = 0, & (t,x) \in [0,T] \times \partial\Omega \setminus \Sigma_0, \\ u(\theta,x) = u_0(x), & x \in \Omega. \end{cases}$$

The objective is to identify the function $q(x)$, which is known as the "Inverse source problem for a parabolic equation".

In the referenced work, it is demonstrated that observations can be limited to a small sub-domain ω or a small sub-boundary Σ_0 over any time interval $(0, T)$. These minimal observations are crucial for unique determination and can be interpreted as the limit of observation data when the number of observation points increases.

Here are some notable applications and improvements of the presented theory: Baudouin and Puel [25] on an inverse problem for the Schrödinger equation; Benabdallah, Gaitan, and Le Rousseau [36] on the stability of discontinuous diffusion coefficients and initial conditions; Benabdallah, Dermenjian, and Le Rousseau [37] on Carleman estimates for the one-dimensional heat equation; Cannarsa, Tort, and Yamamoto [42] on determining a source term in a degenerate parabolic equation; Tort [43] on determining source terms from a locally distributed observation; Vancostenoble [46] on improved Hardy–Poincaré inequalities and sharp Carleman estimates; Vancostenoble [44] on sharp Carleman estimates for singular parabolic equations; Vancostenoble [47] on Lipschitz stability in inverse source problems for singular parabolic equations; and Yamamoto and Zou [23] on the simultaneous reconstruction of the initial temperature and heat radiative coefficient.

6.2.1 Lipschitz Stability

The work of Tort and Vancostenoble [48] provides a general method of dealing with a wide class of inverse problems of non-linear parabolic PDEs. In our case, they try to find the unknown coefficient in a non-linear term in the 1D Sellers climate model, by establishing an unconditional global Lipschitz stability. By doing this, we get in addition a unique result, meaning that every solution u of the PDE system corresponds to a unique insolation function q.

To deal with the Lipschitz stability we define the **set of admissible coefficients**:

$$\mathcal{G}_K := \{q \in L^\infty(I) : \|q\|_{L^\infty(I)} \leq K\}.$$

The Inverse Problem refers to the question:

• Is it possible to recover the insolation function **q** from some measurements of the temperature u?

This question can be expressed in mathematical terms as:

• Suppose we have two functions: u and $\tilde{u}$ and the associated insolation functions of its temperature are q and $\tilde{q}$. Then if these quantities satisfy the following systems:

$$\begin{cases} u_t - ((1-x^2)u_x)_x = r(t)q(x)\beta(u) - \varepsilon(u)|u|^3u, & x \in I, t > 0, \\ (1-x^2)u_x = 0, & x \in \partial I, t > 0, \\ u(x,0) = u_0(x), & x \in \partial I. \end{cases}$$

$$\begin{cases} \tilde{u}_t - ((1-x^2)\tilde{u}_x)_x = r(t)\tilde{q}(x)\beta(\tilde{u}) - \varepsilon(\tilde{u})|\tilde{u}|^3\tilde{u}, & x \in I, t > 0, \\ (1-x^2)\tilde{u}_x = 0, & x \in \partial I, t > 0, \\ \tilde{u}(x,0) = \tilde{u}_0(x), & x \in \partial I. \end{cases}$$

can one deduce the following expression:

$$u = \tilde{u} \text{ in a small set } \implies q = \tilde{q} \text{ in a small set?}$$

The following theorem states the Lipschitz stability.

Theorem 18. *[48, Theorem 4.1] Let $t_0 \in (0,T)$ be given and consider the time:*

$$T^* := \frac{T + t_0}{2}.$$

Let $\phi \in (L_1, L_2)$ where $L_1, L_2 \in (-1,1)$ and $L_1 < L_2$. If $u \in \mathcal{U}$, then for all $K > 0$ $\exists D > 0$ such that, for all $q_1, q_2 \in \mathcal{G}_K$, the associated solutions u_1, u_2 of problem (6.1.4) satisfy:

$$\|q_1 - q_2\|_{L^2(I)} \leq D\Big(\|(u_1 - u_2)(T^*, \cdot)\|^2_{D(A)} + \|u_{1,t} - u_{2,t}\|^2_{L^2((t_0,T)\times\phi)}\Big).$$

Theorem 18 not only gives a stability result, but simultaneously, it answers to the inverse problem. Specifically, it states that with some measurements of the temperature u, one can uniquely recover the insolation function. This result has been proven, by utilizing a global Carleman estimate for degenerate PDEs with locally distributed observations. This is a really powerful tool that has been used extensively for inverse problems. One can find references on the derivation of these types of estimates on [35],[34] and the references therein.

Chapter 7

Conclusion

In this work, we explored a broad spectrum of topics, beginning with stochastic systems and the FP equation. We established a connection between SDEs and the ??equation, which describes the time evolution of the PDF of a SP. By introducing the steady-state problem, we examined specific cases of mechanical systems where the FP equation can be solved, leading to solutions closely related to the Maxwell-Boltzmann distribution.

We then transitioned to the study of degenerate PDEs, a class of equations where the diffusion term may vanish at the boundaries. Through this lens, we analyzed a particular degenerate PDE system given by:

$$u_t - (a(x)u_x(t,x))_x - (b(x)u(t,x))_x = f(t,x)\mathbb{1}_\omega(x), \quad (t,x) \in (0,T) \times (-1,1),$$

under various boundary conditions. Here, the diffusion coefficient was chosen as $a(x) = (1-x^2)^p$, with $p \in (0,1)$ and the drift term $b(x)$ was concerned either as a $C^1(-1,1)$ or a $C^0(-1,1)$ function. We addressed the significant challenges of degenerate diffusion equations, including the appropriate selection of boundary conditions. For each scenario, we followed a similar approach in [27], establishing a suitable functional framework to ensure well-posedness and provide energy estimates.

In Chapter 5, the dynamics of steady-states within the context of FP equations in general domains of $\mathbb{R}^n$, $n \in \mathbb{N}$, were examined by presenting the work of Huang W. et al. in [49], [50], and [51]. Theorems from these references were presented, addressing the existence and non-existence of stationary measures in the non-degenerate case and the existence results for the degenerate case. Specifically, for the choice of the diffusion term $a(x) = (1-x^2)^p$, when the diffusion coefficient is strongly degenerate, with $p \in [2,\infty)$, the existence of a stationary measure can be established. However, for the weak degeneracy case, with $p \in (0,1)$, or the less strong degeneracy $p \in [1,2)$, the existence or absence of a stationary measure remains inconclusive.

Finally, to assess the effectiveness of the degenerate PDEs, the Budyko-Sellers model, an application from climate science, was presented. Since the diffusion term of this model is not in a weakly degenerate case, with $\dfrac{1}{a(x)} \notin L^1(-1,1)$, a well-posedness result was established using a slightly different approach than that in Chapter 4. By tuning the parameters of Budyko's model, the results obtained were realistic and consistent with the data. Subsequently, using the results from [48], an invariant problem was investigated, demonstrating that the insolation function can be uniquely recovered using only some temperature measurements u.

In future research, the results of this thesis can serve as a foundation for several advanced studies. One promising direction is the exploration of controllability, observability, and optimal control problems in systems governed by degenerate PDEs. The well-posedness results and functional frameworks established in this work provide a critical starting point for analyzing how control strategies can be effectively applied in such systems, particularly when dealing with weakly degenerate cases.

Additionally, further investigation into more general classes of degenerate PDEs could significantly expand the scope of current findings. By broadening the range of equations studied, new insights into the behavior and solutions of these systems can be uncovered. The methodologies developed here will be instrumental in addressing the challenges posed by more complex degenerate systems.

Finally, this thesis lays the groundwork for developing tools to study inverse problems and invariant measures in weakly degenerate systems. The unique recovery of insolation functions demonstrated in the Budyko-Sellers model suggests that similar approaches can be extended to other applications, providing valuable techniques for identifying and characterizing hidden variables in various scientific and engineering contexts.

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