SIEVE METHODS IN RANDOM GRAPH THEORY

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Abstract. In this paper, we apply the Turán sieve and the simple sieve developed by R. Murty and the first author to study problems in random graph theory. In particular, we obtain upper and lower bounds on the probability of a graph on n vertices having diameter 2 (or diameter 3 in the case of bipartite graphs) with edge probability p where the edges are chosen independently . An interesting feature revealed in these results is that the Turán sieve and the simple sieve "almost completely" complement each other. As a corollary to our result, we note that the probability of a random graph having diameter 2 approaches 1 as $n \to \infty$ for constant edge probability $p = 1/2$.

1. Introduction

For the purpose of analyzing the random graphs in this paper, we first introduce two sieves known as the simple sieve and the Turán sieve, which were introduced in [7]. These sieves can be described in terms of a bipartite graph. Let X be a bipartite graph with finite partite sets A and B. For $a \in A$ and $b \in B$, we denote by $a \sim b$ if there is an edge that joins a and b. Define

$$\deg b = \#\{a \in A : a \sim b\} \quad \text{and} \quad \omega(a) = \#\{b \in B : a \sim b\}.$$

For $b_1, b_2 \in B$, we define

$$n(b_1, b_2) = \#\{a \in A : a \sim b_1, a \sim b_2\}.$$

In [7], R. Murty and the first author derived an elementary sieve method, called *the simple sieve*, which states that

$$\#\{a \in A : \omega(a) = 0\} \geq |A| - \sum_{b \in B} \deg b.$$

In the same paper, they also adopted Turán's proof [9] on Hardy and Ramanujan's result on the normal order of distinct prime factors of a natural

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number [3] to prove that

$$\#\{a \in A : \omega(a) = 0\} \leq |A|^2 \cdot \frac{\sum_{b_1,b_2 \in B} n(b_1, b_2)}{(\sum_{b \in B} \deg b)^2} - |A|.$$

The above result is called *the Turán sieve*.

The combinatorial extension of sieve methods has been investigated before the work of Murty and the first author. For example, Wilson [10] and Chow [2] formulated the combinatorial Selberg sieve. However, since the M$\acute{}$obius function of a lattice is difficult to compute in an abstract setting, the applications of the combinatorial Selberg sieve have been limited. The obstruction is eliminated by the Turán sieve and the simple sieve as they do not use the M$\acute{}$obius function. Hence one can apply them to various combinatorial problems, including graph colouring, Latin squares, tournaments, etc (see [7], [5], [6]). In this paper, we apply both the simple sieve and the Turán sieve to study problems about random graph theory. First, we need the following definition.

Definition 1.1. The *diameter* of a graph G is defined as the maximum number of edges in G that are needed to traverse from one vertex to another in G where we exclude paths that backtrack, detour, and loop.

Let $G(n, p)$ denote the set of all simple graphs on n vertices where each edge is chosen independently with probability p. In 1981, Bollobás [1] obtained sharp asymptotic results for the probability of a random graph from $G(n, p)$ vertices having diameter d for any fixed $d \geq 2$ with $n \to \infty$. Here we extend his results and obtain concrete upper and lower bounds on the probability of a random graph from $G(n, p)$ having diameter d where n, p, and d are fixed. The results of Bollobás's follow if we let $n \to \infty$. We also study analogous questions for random k-partite graphs having diameter d with $k \geq 2$. Although our approaches work for general diameter d, to better illustrate the methods, most of this paper will be dedicated to stating and proving our results for diameter 2 or diameter 3 in the case of random bipartite graphs. Here is one of the main theorems of the paper.

Theorem 1.2. *Let $G(n, p)$ denote the set of all simple graphs on n vertices where each edge is chosen independently with probability p. Also, let $P(diamG(n, p) = 2)$ be the probability of a graph from $G(n, p)$ having diameter 2. Then*

$$P(diamG(n, p) = 2) \geq 1 - \frac{n^2(1 - p^2)^{n-2}(1 - p)}{2}$$

and

$$P(diamG(n,p)=2) \leq \frac{2}{(n-1)^2(1-p^2)^n(1-p)} + \frac{8}{n\,(1-p^2)^2}\left(1+\frac{p^3}{(1-p)^2}\right)^n.$$

Corollary 1.3. *Let $P(diamG(n,p)=2)$ be defined as in Theorem 1.2. If $p=\frac{1}{2}$, then we have*

$$P(diamG(n,1/2)=2) \geq 1 - \frac{4n^2(3/4)^n}{9}.$$

In the case $p=\frac{1}{2}$, Gilbert [4] showed that 'almost all' graphs are connected. Since a graph with diameter 2 is connected, the above result provides an explicit bound for Gilbert's result.

In the situation where the edge probability $p \to 0$ as $n \to \infty$, we will show the following corollary.

Corollary 1.4. *Let $P(diamG(n,p)=2)$ be defined as in Theorem 1.2. Let $\lim_{n\to\infty} p = 0$. We have*

$$P(diamG(n,p)=2) \geq 1 - (1+o(1))\frac{n^2}{2}e^{-np^2} \tag{1.1}$$

and

$$P(diamG(n,p)=2) \leq (1+o(1))\left(\frac{2}{n^2}e^{np^2}\right)\left(1+4ne^{np^2\left(p^2-1\right)}\right). \tag{1.2}$$

Suppose further that

$$\lim_{n\to\infty}(2\log n - np^2 - \log 2) = c$$

for some $c \in \mathbb{R}\backslash\{0\}$.
1) If $c>0$, we have

$$P(diamG(n,p)=2) \leq (1+o(1))e^{-c}.$$

2) If $c<0$, we have

$$P(diamG(n,p)=2) \geq (1+o(1))(1-e^{c}).$$

We will also study analogous problems for a random directed graph, and a random k-partite graph having diameter 2 with $k \geq 3$, or diameter 3 in the case of bipartite graphs. We can also similarly obtain analogous results for directed graphs and Turán graphs and generalise these results for any given diameter d. We give the details in a long arxiv note where we prove such results in full generality [8].

As we noted in Corollary 1.4, the upper bound we obtained through the Turán sieve works effectively for $c>0$, while the lower bound we obtained

through the simple sieve gives a non-trivial result for $c < 0$. It is interesting to see that the Turán sieve and the simple sieve "almost completely" complement each other in this way.

2. Graphs with Diameter 2 with the Sieves

In this section, we use the Turán sieve and the simple sieve to prove Theorem 1.2.

Proof of Theorem 1.2. For a fixed $n \in \mathbb{N}$, let $G(n,p)$ denote the set of all graphs on n vertices with edge probability p, and let $P(diamG(n,p) = 2)$ be the probability of a graph from $G(n,p)$ having diameter 2. Consider the function $g_n : [0,1] \to [0,1]$ defined as $g_n(x) := P(diamG(n,x) = 2)$. There are $2^{\frac{n(n-1)}{2}}$ graphs in total in $G(n,p)$. Let us say M of these have diameter 2 and label these as G_1, G_2,...,G_M. For $1 \leq i \leq M$, let k_i denote the number of edges in G_i. Then the probability of selecting the graph G_i from $G(n,p)$ according to the edge probability x is $x^{k_i}(1-x)^{\frac{n(n-1)}{2}-k_i}$. Therefore,

$$g_n(x) = x^{k_1}(1-x)^{\frac{n(n-1)}{2}-k_1} + x^{k_2}(1-x)^{\frac{n(n-1)}{2}-k_2} + \cdots + x^{k_M}(1-x)^{\frac{n(n-1)}{2}-k_M}.$$

Thus, for each $n \in \mathbb{N}$ the function g_n is continuous. Therefore, we may assume that $p \in \mathbb{Q} \cap (0,1)$ since $\mathbb{Q} \cap (0,1)$ is dense in $[0,1]$.

Let $p = \frac{r}{s}$ where $r = r(n), s = s(n) \in \mathbb{N}$. We let A be the set of all graphs in $G(n,p)$, allowing for a number of duplicates of each possible graph. The reason for this construction of A is to be able to have a uniformly chosen graph in A correctly represent the random graph $G(n,p)$. We accomplish this by letting there be $r^{\binom{n}{2}}$ copies of the complete graph, $r^{\binom{n}{2}}\left(\frac{s}{r}-1\right)$ copies of each graph with $\binom{n}{2}-1$ edges, $r^{\binom{n}{2}}\left(\frac{s}{r}-1\right)^2$ copies of each graph with $\binom{n}{2}-2$ edges, and so on. By the binomial theorem we have

$$|A| = \sum_{k=0}^{\binom{n}{2}} \binom{\binom{n}{2}}{k} r^k (s-r)^{\binom{n}{2}-k} = s^{\binom{n}{2}}.$$

We let B be all pairs of vertices so $|B| = \binom{n}{2}$. For a graph $a \in A$ and a pair of vertices $b \in B$, we say $a \sim b$ if the pair of vertices b in a do not share a common neighbouring vertex and are not neighbours themselves. Thus, we will have $\omega(a) = 0$ if and only if a is connected with diameter at most 2.

Pick a pair of vertices $b \in B$ and call them v_1 and v_2. To calculate $\deg b$, we need to calculate the number of graphs in A such that the pair of vertices do not have a common neighbouring vertex and are not neighbours themselves. Consider first how we construct such a graph. First, for any pair of vertices that do not include either v_1 and v_2 we do not care if there is an

edge between them or not. We do not, however, want an edge between v_1 and v_2 since otherwise v_1 and v_2 will be neighbours. For each of the $n-2$ vertices that are not either v_1 or v_2, we have a pair of potential edges to take into consideration as well. We know for each of these pairs of edges either one or neither will be in the graph. By the binomial theorem, we have

(2.1)

$$\begin{aligned} D(r,s,n) := \deg b &= \sum_{k_1=0}^{\binom{n-2}{2}} \sum_{k_2=0}^{n-2} \binom{\binom{n-2}{2}}{k_1} \binom{n-2}{k_2} 2^{k_2} r^{k_1+k_2} (s-r)^{\binom{n}{2}-k_1-k_2} \\ &= (s-r)^{\binom{n}{2}} \sum_{k_1=0}^{\binom{n-2}{2}} \binom{\binom{n-2}{2}}{k_1} \left(\frac{r}{s-r}\right)^{k_1} \sum_{k_2=0}^{n-2} \binom{n-2}{k_2} \left(\frac{2r}{s-r}\right)^{k_2} \\ &= (s-r)^{\binom{n}{2}} \left(\frac{s}{s-r}\right)^{\binom{n-2}{2}} \left(\frac{s+r}{s-r}\right)^{n-2} \\ &= (s-r)^{\binom{n}{2}} \left(\frac{s}{s-r}\right)^{\binom{n-2}{2}} \left(\frac{s^2-r^2}{(s-r)^2}\right)^{n-2} \\ &= (s-r) s^{\binom{n-2}{2}} \left(s^2-r^2\right)^{n-2} \end{aligned} \tag{2.2}$$

It follows that

$$\sum_{b \in B} \deg b = \frac{s^{\binom{n}{2}} n(n-1)(1-p^2)^{n-2}(1-p)}{2}.$$

By the simple sieve, we obtain

$$\begin{aligned} P(diamG(n,p)=2) &\geq 1 - \frac{n(n-1)(1-p^2)^{n-2}(1-p)}{2} \\ &> 1 - \frac{n^2(1-p^2)^{n-2}(1-p)}{2} \end{aligned} \tag{2.3}$$

since the random graph $G(n,p)$ is uniform over the set A.

We now try to get an upper bound for $P(diamG(n,p)=2)$, in which we need to estimate $\sum_{b_1,b_2 \in B} n(b_1,b_2)$. In the following, we calculate $n(b_1,b_2)$, depending on how many vertices b_1 and b_2 have in common.

Case 1. Suppose that b_1 and b_2 are two pairs of vertices that have no vertices in common, i.e., b_1 and b_2 consist of 4 distinct vertices. For each of b_1 and b_2, the probability that the pair of vertices in question are not connected by an edge nor have any common neighbouring vertices is

$$\frac{D(r,s,n)}{s^{\binom{n}{2}}}.$$

Let v_1 and v_2 be the pair of vertices in b_1 and v_3 and v_4 be the pair of vertices in b_2. To calculate $n(b_1,b_2)$, we need to calculate the number of graphs in

A such that the each pair of vertices do not have a common neighbouring vertex and are not neighbours themselves.

Consider first how we construct such a graph. First, for any pair of vertices that do not include any of the four vertices v_1, v_2, v_3, v_4 we do not care if there is an edge between them or not. We do not, however, want an edge between v_1 and v_2 and not an edge between v_3 and v_4. Before we consider potential edges involving any other vertices, there are the four potential edges between v_1 and v_3, between v_1 and v_4, between v_2 and v_3, and between v_2 and v_4. Our random graph could have none of these edges in it, exactly one of these edges in it, or exactly two of these edges in it. Notice that it cannot have more than two of these edges in it because then v_1 and v_2 would have a common neighbour vertex between either v_3 and v_4. Also, if we have exactly two of these edges in it, then we have two choices of these two edges: either we have the edges between v_1 and v_3 and between v_2 and v_4 or we have the edges between v_1 and v_4 and between v_2 and v_3.

For each of the $n-4$ vertices that are not any of v_1, v_2, v_3, v_4, we have a group of four potential edges to take into consideration as well. The four edges in any specific group can be split into two pairs such that for each pair either one or neither of the edges will be in the graph. Putting this all together, by the binomial theorem, we have

(2.4)

$$n(b_1,b_2) = \left((s-r)^4 + 4r(s-r)^3 + 2r^2(s-r)^2\right)$$

$$\cdot \sum_{k_1=0}^{\binom{n-4}{2}} \sum_{k_2=0}^{2n-8} \binom{\binom{n-4}{2}}{k_1} \binom{2n-8}{k_2} 2^{k_2} r^{k_1+k_2} (s-r)^{\binom{n}{2}-k_1-k_2-4} \tag{2.5}$$

$$= (s-r)^{\binom{n}{2}} \left(1 + \frac{4r}{s-r} + \frac{2r^2}{(s-r)^2}\right) \tag{2.6}$$

$$\cdot \sum_{k_1=0}^{\binom{n-4}{2}} \binom{\binom{n-4}{2}}{k_1} \left(\frac{r}{s-r}\right)^{k_1} \sum_{k_2=0}^{2n-8} \binom{2n-8}{k_2} \left(\frac{2r}{s-r}\right)^{k_2}$$

$$= (s-r)^{\binom{n}{2}} \left(1 + \frac{4r}{s-r} + \frac{2r^2}{(s-r)^2}\right) \left(\frac{s}{s-r}\right)^{\binom{n-4}{2}} \left(\frac{s+r}{s-r}\right)^{2n-8}$$

$$= (s-r)^{\binom{n}{2}} \left(1 + \frac{4r}{s-r} + \frac{2r^2}{(s-r)^2}\right) \left(\frac{s}{s-r}\right)^{\binom{n-4}{2}} \left(\frac{s^2-r^2}{(s-r)^2}\right)^{2n-8}$$

It follows that

$$n(b_1, b_2) = s^{\binom{n-4}{2}}(s-r)^2\left((s-r)^4 + 4r(s-r)^3 + 2r^2(s-r)^2\right)\left(s^2-r^2\right)^{2n-8}$$

so that

$$n(b_1, b_2) = \frac{D(r,s,n)^2}{s^{\binom{n}{2}}} \cdot \frac{s^4((s-r)^4 + 4r(s-r)^3 + 2r^2(s-r)^2)}{(s^2-r^2)^4},$$

and thus

$$\sum_{b_1, b_2 \in B,\ 4 \text{ vertices}} n(b_1, b_2) \tag{2.7}$$

$$< \binom{n}{2}^2 \frac{D(r,s,n)^2}{s^{\binom{n}{2}}} \tag{2.8}$$

$$\cdot \frac{p^{-4}((p^{-1}-1)^4 + 4(p^{-1}-1)^3 + 2(p^{-1}-1)^2)}{(p^{-2}-1)^4} \tag{2.9}$$

$$< \binom{n}{2}^2 \frac{D(r,s,n)^2}{s^{\binom{n}{2}}} \cdot \left(1 + \frac{4p^3}{(1-p)^2}\right). \tag{2.10}$$

Case 2. Take two pairs of vertices b_1 and b_2 that have exactly one vertex in common, i.e., b_1 and b_2 consist of 3 distinct vertices. Let the vertices in b_1 be v_1 and v_2 and the vertices in b_2 be v_2 and v_3. We can do a similar kind of analysis of edge selection as in Case 1 as follows. First, for any pailr of vertices that do not include any of the three vertices v_1, v_2, v_3 we do not care if there is an edge between them or not. We do not, however, want an edge between v_1 and v_2 and not an edge between v_2 and v_3. Given this, however, we do not care if there is an edge between v_1 and v_3.

For each of the $n-3$ vertices that are not any of v_1, v_2, v_3, we have a group of three potential edges to take into consideration as well. In each group, either none of the edges appear in the graph, exactly one of the edges appears in the graph, or the two edges having v_1 and v_3 as endpoints are in the graph, but the edge with v_2 as an endpoint is not in the graph. Putting this all together, by the binomial theorem, we have

$$
\begin{aligned}
n(b_1,b_2) &= (s-r)^2 \sum_{k_1=0}^{\binom{n-3}{2}+1} \binom{\binom{n-3}{2}+1}{k_1} \sum_{k_2=0}^{n-3} \binom{n-3}{k_2} \\
&\quad \cdot \sum_{k_3=0}^{n-3-k_2} \binom{n-3-k_2}{k_3} 3^{k_2} r^{k_1+k_2+2k_3} (s-r)^{\binom{n-3}{2}+3n-8-k_1-k_2-2k_3} \qquad (2.11)\\
&= (s-r)^2 \sum_{k_1=0}^{\binom{n-3}{2}+1} \binom{\binom{n-3}{2}+1}{k_1} \sum_{k_2=0}^{n-3} \binom{n-3}{k_2} \\
&\quad \cdot 3^{k_2} r^{k_1+k_2} (s-r)^{\binom{n-3}{2}+3n-8-k_1-k_2} \left(1+\frac{r^2}{(s-r)^2}\right)^{n-3-k_2} \\
&= (s-r)^2 \left(1+\frac{r^2}{(s-r)^2}+\frac{3r}{s-r}\right)^{n-3} \sum_{k_1=0}^{\binom{n-3}{2}+1} \binom{\binom{n-3}{2}+1}{k_1} r^{k_1} (s-r)^{\binom{n-3}{2}+3n-8-k_1} \\
&= (s-r)^2 \left(1+\frac{r^2}{(s-r)^2}+\frac{3r}{s-r}\right)^{n-3} (s-r)^{\binom{n-3}{2}+3n-8} \left(\frac{s}{s-r}\right)^{\binom{n-3}{2}+1} \\
&= (s-r)^2 s^{\binom{n-3}{2}+1} \left((s-r)^3+r^2(s-r)+3r(s-r)^2\right)^{n-3} \\
&= \frac{D(r,s,n)^2}{s^{\binom{n}{2}}} \cdot \frac{s^{n+1}\left((s-r)^3+r^2(s-r)+3r(s-r)^2\right)^{n-3}}{\left(s^2-r^2\right)^{2n-4}}
\end{aligned}
$$

We deduce

$$
\begin{aligned}
&\sum_{b_1,b_2\in B,\ 3 \text{ vertices}} n(b_1,b_2) \\
&= \frac{D(r,s,n)^2 n(n-1)(n-2)}{s^{\binom{n}{2}}\left(1-p^2\right)^2} \left(1+\frac{1}{p^{-3}+p^{-2}-p^{-1}-1}\right)^{n-3} \\
&\le \frac{D(r,s,n)^2 n(n-1)(n-2)}{s^{\binom{n}{2}}\left(1-p^2\right)^2} \left(1+\frac{p^3}{(1-p)}\right)^{n-3}. \qquad (2.12)
\end{aligned}
$$

Case 3. Suppose b_1 and b_2 have two vertices in common. Then the two pairs are identical, and we have

$$n(b_1,b_2) = \deg b.$$

It follows that

$$\sum_{b_1,b_2\in B,\ 2 \text{ vertices}} n(b_1,b_2) = \sum_{b\in B} \deg b = \frac{s^{\binom{n}{2}} n(n-1)(1-p^2)^{n-2}(1-p)}{2}.$$

Combining Cases $1-3$, we get

$$\sum_{b_1,b_2\in B} n(b_1,b_2) < \binom{n}{2}^2 \frac{D(r,s,n)^2}{s^{\binom{n}{2}}} \cdot \left(1+\frac{4p^3}{(1-p)^2}\right) + \frac{D(r,s,n)^2 n(n-1)(n-2)}{s^{\binom{n}{2}}\left(1-p^2\right)^2}\left(1+\frac{p^3}{(1-p)}\right)^{n-3} + \frac{s^{\binom{n}{2}} n(n-1)(1-p^2)^{n-2}(1-p)}{2}.$$

By the Turán sieve, we deduce

$$\begin{aligned} &P(diamG(n,p)=2) \\ &\leq \frac{2}{n(n-1)(1-p^2)^{n-2}(1-p)} + \frac{4}{n\left(1-p^2\right)^2}\left(1+\frac{p^3}{(1-p)}\right)^{n-3} \\ &\quad + \frac{4p^3}{(1-p)^2}. \end{aligned}$$

Notice that

$$\frac{p^3}{(1-p)^2} < \frac{1}{n}\left(1+n\frac{p^3}{(1-p)^2}\right) < \frac{1}{n}\left(1+\frac{p^3}{(1-p)^2}\right)^n.$$

It follows that

(2.13)

$$P(diamG(n,p)=2) \leq \frac{2}{(n-1)^2(1-p^2)^n(1-p)} + \frac{8}{n\left(1-p^2\right)^2}\left(1+\frac{p^3}{(1-p)^2}\right)^n.$$

By (2.3) and (2.13) Theorem 1.2 follows. □

We now prove Corollary 1.4.

Proof. Using the inequality $1-x \leq e^{-x}$, valid for $0 \leq x \leq 1$, $x = p^2$ in the lower bound in Theorem 1.2 gives (1.1). We can see that the upper bound in Theorem 1.2 is non-trivial only if $np^2 < 2\log n - \log 2$. Then, using the inequality $e^{-x} \leq 1 - x + \frac{x^2}{2}$, valid for $0 \leq x \leq 1$, for $x = p^2$, and the inequality $1 + x \leq e^x$, valid for $x \geq 0$, for $x = \frac{p^3}{(1-p)^2}$, and working out the asymptotics gives in the upper bound in Theorem 1.2 gives (1.2). □

Remark 2.1. Assume that $n \geq 200$ and $p \leq 1/2$. The $o(1)$ in (1.1) can be made explicit as $4p^2$ and the $o(1)$ in (1.2) can be made explicit as $\frac{4(\log n)^2+2}{n} + p + \frac{3e^8(2\log n)^{3/2}}{n^{1/2}}$.

3. Analysis of k-partite graphs

Here we apply our analysis to k-partite graph sets for $k \geq 3$. First, we present a definition.

Definition 3.1. Let $k \geq 2$. A *simple k-partite graph* is an undirected graph whose vertices can be divided into k sets, such that there are no edges between two vertices in the same set.

We exclude the bipartite case ($k = 2$) because the only bipartite graph that has diameter 2 is the bipartite graph; we analyze that case by itself in the next section.

Convention 3.2. For each k-partite graph, we label the k partite sets of the graph in a non-decreasing order in terms of the number of vertices each set contains. Thus, the ith set is a set containing n_i vertices.

Theorem 3.3. *Fix $k \geq 3$ and for each $n \in \mathbb{N}$, $n \geq k+2$, pick $n_1, n_2, ..., n_k \in \mathbb{N}$ such that $n_1 \leq n_2 \leq \ldots \leq n_k$, $n_{k-1} \geq 2$, and $n_1 + n_2 + \cdots + n_k = n$. Let $\mathbf{n^{(k)}} = (n_1, n_2, \ldots, n_k)$ and let $G(\mathbf{n^{(k)}}, p)$ denote the set of all k-partite graphs with the partite sets having $n_1, n_2, \ldots, n_k$ vertices respectively where each edge is chosen independently with probability p. Also, let $P(diamG(\mathbf{n^{(k)}}, p) = 2)$ be the probability of a graph from $G(\mathbf{n^{(k)}}, p)$ having diameter 2. Then*

$$\begin{aligned} &P(diamG(\mathbf{n^{(k)}}, p) = 2) \\ &\geq 1 - \frac{n_k^2(1-p^2)^{n-n_k}}{2} \\ &\quad \cdot \left(1 + \frac{2n_{k-1}(1-p^2)^{-n_{k-1}}}{n_k} + \frac{7k^2 n_{k-1}^2 (1-p^2)^{n_k - n_{k-1} - n_{k-2}}}{3n_k^2}\right) \end{aligned}$$

and

$$\begin{aligned} &P(diamG(\mathbf{n^{(k)}}, p) = 2) \\ &\leq \frac{2}{n_k(n_k-1)(1-p^2)^{n-n_k}} \left(1 + \frac{2n_{k-1}(1-p^2)^{-n_{k-1}}(1-p)}{(n_k-1)}\right)^{-1} \\ &\quad + \frac{3k^3 \left(1 + \frac{p^3}{(1-p)}\right)^{n-n_k} (1-p^2)^{-2}}{(n_{k-1}-1)}. \end{aligned}$$

Proof of Theorem 3.3. As in the proof of Theorem 1.2, we may assume that $p \in \mathbb{Q} \cap (0,1)$ for all $n \in \mathbb{N}$.

Let $p = \frac{r}{s}$ where $r, s \in \mathbb{N}$. As in the proof of Theorem 1.2, we let A be the set of all graphs in $G(\mathbf{n^{(k)}}, p)$, allowing for a number of duplicates of each possible graph. As in the proof of Theorem 1.2 this construction of A allows us to have a uniformly chosen graph in A correctly represent the random graph $G(\mathbf{n^{(k)}}, p)$. Since the complete k-partite graph has $t := \sum_{1 \leq i < j \leq k} n_i n_j$ edges, we have r^t copies of the complete k-partite graph and $|A| = s^t$.

We let B be all pairs of vertices. Thus, $|B| = \frac{n(n-1)}{2}$. For a graph $a \in A$ and a pair of vertices $b \in B$, we say $a \sim b$ if the pair of vertices b in a do not share a common neighbouring vertex and are not connected by a single edge. Thus, we will have $\omega(a) = 0$ if and only if a is connected with diameter at most 2. Suppose that $b \in B$ is a pair of vertices that are in the ith partite set for some $1 \leq i \leq k$. Then this pair of vertices has $n - n_i$ potential common neighbouring vertices. Similarly to how we derived (2.2) from (2.1) we again use the binomial theorem on summing all the different possible graphs with their weights to obtain

$$\begin{aligned} D(r,s,n,n_i) &:= \deg b \\ &= (s^2 - r^2)^{n-n_i} s^{t-2n+2n_i} \\ &= (1-p^2)^{n-n_i} s^t, \end{aligned} \tag{3.1}$$

where we replaced $\binom{n-2}{2}$ with $t - 2n + 2n_i$ (the number of potential edges not touching either of the vertices in b) and $n-2$ with $n - n_i$ (the number of potential common neighbouring vertices in both cases).

Suppose that $b \in B$ is a pair of vertices with one vertex being in the ith partite set and the other in the jth partite set, where $i < j$. Then this pair of vertices has $n - n_i - n_j$ potential common neighbouring vertices. Again, similarly to how we derived (2.2) from (2.1) we again use the binomial theorem on summing all the different possible graphs with their weights to obtain

$$\begin{aligned} D(r,s,n,n_i,n_j) &:= \deg b \\ &= \left(s^2 - r^2\right)^{n-n_i-n_j} s^{t-2n+2n_i+2n_j}(1-p) \\ &= (1-p^2)^{n-n_i-n_j}(1-p)s^t \end{aligned} \tag{3.2}$$

since in this case the number of potential edges no touching either of the vertices in b is $t - 2n + 2n_i + 2n_j$ and the number of potential common neighbouring vertices is $n - n_i - n_j$.

It follows that

$$\begin{aligned} &\sum_{b \in B} \deg b \\ &= s^t \sum_{i=1}^{k} \binom{n_i}{2} (1-p^2)^{n-n_i} + s^t \sum_{1 \leq i < j \leq k} n_i n_j (1-p^2)^{n-n_i-n_j}(1-p). \end{aligned}$$

By the simple sieve, we obtain

$$\begin{aligned} &P(diamG(\mathbf{n^{(k)}}, p) = 2) \\ &\geq 1 - \frac{n_k^2(1-p^2)^{n-n_k}}{2} \\ &\quad \cdot \left(1 + \frac{2n_{k-1}(1-p^2)^{-n_{k-1}}}{n_k} + \frac{7k^2 n_{k-1}^2 (1-p^2)^{n_k - n_{k-1} - n_{k-2}}}{3n_k^2}\right) \end{aligned}$$

since the random graph $P(diamG(\mathbf{n^{(k)}}, p) = 2)$ is uniform over the set A.

To get an upper bound for $P(diamG(\mathbf{n^{(k)}}, p) = 2)$, we need to estimate $\sum_{b_1, b_2 \in B} n(b_1, b_2)$. Similar to the proof of Theorem 1.2, we calculate $n(b_1, b_2)$ based on the number of vertices b_1 and b_2 have in common. Suppose first that b_1 and b_2 have no vertices in common. Then, assuming the respective complete k-partite graph has four edges between the vertices in b_1 and the vertices in b_2, we have

$$\begin{aligned} n(b_1, b_2) = &\left((s-r)^4 + 4r(s-r)^3 + 2r^2(s-r)^2\right) \\ &\cdot \sum_{k_1=0}^{t_1} \sum_{k_2=0}^{t_2} \sum_{k_3=0}^{t_3} \binom{t_1}{k_1} \binom{t_2}{k_2} \binom{t_3}{k_3} 2^{k_2+k_3} r^{k_1+k_2+k_3} (s-r)^{t-k_1-k_2-k_3-4}, \end{aligned}$$

where t_1 is the number of potential edges not touching any vertex in b_1 or b_2, t_2 is the number of potential common neighbouring vertices shared by the pair in b_1, and t_3 is the number of potential common neighbouring vertices shared by the pair in b_2. If the respective complete k-partite graph has exactly three edges between the vertices in b_1 and the vertices in b_2, then the factor of $((s-r)^4 + 4r(s-r)^3 + 2r^2(s-r)^2)$ is replaced with $((s-r)^3 + 3r(s-r)^2 + r^2(s-r))$ with the expression $(s-r)^{t-k_1-k_2-k_3-4}$ replaced by $(s-r)^{t-k_1-k_2-k_3-3}$. If the respective complete k-partite graph has exactly two edges between between the vertices in b_1 and the vertices in b_2, then the factor of $((s-r)^4 + 4r(s-r)^3 + 2r^2(s-r)^2)$ is replaced with $((s-r)^2 + 2r(s-r) + 2(s-r)^2)$ with the expression $(s-r)^{t-k_1-k_2-k_3-4}$ replaced by $(s-r)^{t-k_1-k_2-k_3-2}$. If the respective complete k-partite graph has exactly two edges between between the vertices in b_1 and the vertices in b_2, then the factor of $((s-r)^4 + 4r(s-r)^3 + 2r^2(s-r)^2)$ is replaced with $(r + s - r)$ with the expression $(s-r)^{t-k_1-k_2-k_3-4}$ replaced by $(s-r)^{t-k_1-k_2-k_3-1}$. If the respective complete k-partite graph has no edges between between the vertices in b_1 and the vertices in b_2, then the factor of $((s-r)^4 + 4r(s-r)^3 + 2r^2(s-r)^2)$ is replaced with 1 with the expression $(s-r)^{t-k_1-k_2-k_3-4}$ replaced by $(s-r)^{t-k_1-k_2-k_3}$. In all cases, similarly to how we derived (2.10) from (2.5), we can derive

$$n(b_1, b_2) \leq \frac{\deg b_1 \deg b_2}{s^t} \left(1 + \frac{4p^3}{(1-p)^2}\right).$$

Now we consider two pairs of vertices b_1 and b_2 that have exactly one vertex in common, i.e., b_1 and b_2 consist of 3 distinct vertices. First suppose all three vertices are in the same ith partite set. Then the equation for $n(b_1, b_2)$ is the same as in (2.11) with the following minor changes. The factor $(s-r)^2$ is removed, $\binom{n-3}{2}+1$ is replaced with $t - 3(n - n_i)$, $n-3$ is replaced with $n - n_i$, $n-3-k_2$ is replaced with $n - n_i - k_2$, and $\binom{n-3}{2} + 3n - 8 - k_1 - k_2 - 2k_3$ is replaced with $t - k_1 - k_2 - k_3$. We can deduce that in this case

$$\begin{aligned} n(b_1, b_2) &\leq \frac{D(r, s, n, n_i)^2}{s^t} \left(1 + \frac{1}{p^{-3} + p^{-2} - p^{-1} - 1}\right)^{n-n_i} \\ &\leq \frac{D(r, s, n, n_k)^2}{s^t} \left(1 + \frac{1}{p^{-3} + p^{-2} - p^{-1} - 1}\right)^{n-n_k} \\ &\leq \frac{D(r, s, n, n_k)^2}{s^t} \left(1 + \frac{p^3}{(1-p)}\right)^{n-n_k} \end{aligned}$$

the same way we derived (2.12) from (2.11). We now consider the case when the shared vertex between b_1 and b_2 is in the jth partite set and the two other vertices in the pairs are in the ith partite set, where $i \neq j$. For this case, we make the following minor changes to (2.11). The factor $(s-r)^2$ is removed, $\binom{n-3}{2}+1$ is replaced with $t - 3n + 3n_i + 3n_j + 2$, $n-3$ is replaced with $n - n_i - n_j$, $n-3-k_2$ is replaced with $n - n_i - n_j - k_2$, and $\binom{n-3}{2} + 3n - 8 - k_1 - k_2 - 2k_3$ is replaced with $t - k_1 - k_2 - k_3$. We can deduce that in this case

$$\begin{aligned} n(b_1, b_2) &\leq \frac{D(r, s, n, n_i, n_j)^2}{s^t} \left(1 + \frac{1}{p^{-3} + p^{-2} - p^{-1} - 1}\right)^{n-n_i-n_j} \\ &\leq \frac{D(r, s, n, n_k, n_{k-1})^2}{s^t} \left(1 + \frac{1}{p^{-3} + p^{-2} - p^{-1} - 1}\right)^{n-n_k-n_{k-1}} \\ &\leq \frac{D(r, s, n, n_k, n_{k-1})^2}{s^t} \left(1 + \frac{p^3}{(1-p)}\right)^{n-n_k-n_{k-1}} \end{aligned}$$

the same way we derived (2.12) from (2.11). We now consider the case when one of the pair of vertices is in the ith partite set and the other vertex in the other pair is in the jth partite set, where $i \neq j$. For this case, we make the following minor changes to (2.11). The factor $(s-r)^2$ is removed, $\binom{n-3}{2}+1$ is replaced with $t-3n+3n_i+n_j+1$, $n-3$ is replaced with $n-n_i-n_j$,

$n-3-k_2$ is replaced with $n-n_i-n_j-k_2$, and $(s-r)^{\binom{n-3}{2}+3n-8-k_1-k_2-2k_3}$ is replaced with yet another summation

$$\sum_{k_4=0}^{n_j-1}\binom{n_j-1}{k_4}2^{k_4}r^{k_4}(s-r)^{t-k_1-k_2-k_3-k_4}.$$

We can deduce that in this case

$$\begin{aligned}n(b_1,b_2) &\leq \frac{D(r,s,n,n_i)D(r,s,n,n_i,n_j)}{s^t\left(1-p^2\right)}\left(1+\frac{1}{p^{-3}+p^{-2}-p^{-1}-1}\right)^{n-n_i-n_j}\\ &\leq \frac{D(r,s,n,n_k)D(r,s,n,n_k,n_{k-1})}{s^t\left(1-p^2\right)}\left(1+\frac{1}{p^{-3}+p^{-2}-p^{-1}-1}\right)^{n-n_k-n_{k-1}}\\ &\leq \frac{D(r,s,n,n_k)D(r,s,n,n_k,n_{k-1})}{s^t\left(1-p^2\right)}\left(1+\frac{p^3}{(1-p)}\right)^{n-n_k-n_{k-1}}\end{aligned}$$

the same way we derived (2.12) from (2.11). We now consider the case when all three vertices in the two pairs are in three different partite sets, say in the ith, jth, and lth partite sets, where the shared vertex between the pairs is in the jth partite set. For this case, we make the following minor changes to (2.11). The factor $(s-r)^2$ is removed, $\binom{n-3}{2}+1$ is replaced with $t-3n+n_i+n_j+n_l+4$, $n-3$ is replaced with $n-n_i-n_j-n_l$, $n-3-k_2$ is replaced with $n-n_i-n_j-n_l-k_2$, and $(s-r)^{\binom{n-3}{2}+3n-8-k_1-k_2-2k_3}$ is replaced with yet another double summation

$$\sum_{k_4=0}^{n_i-1}\sum_{k_5=0}^{n_l-1}\binom{n_i-1}{k_4}\binom{n_l-1}{k_5}2^{k_4+k_5}r^{k_4+k_5}(s-r)^{t-k_1-k_2-k_3-k_4-k_5}.$$

We can deduce that in this case

$$\begin{aligned}n(b_1,b_2) &\leq \frac{D(r,s,n,n_i,n_j)D(r,s,n,n_l,n_j)}{s^t\left(1-p^2\right)^2}\left(1+\frac{1}{p^{-3}+p^{-2}-p^{-1}-1}\right)^{n-n_i-n_j-n_l}\\ &\leq \frac{D(r,s,n,n_k,n_{k-1})D(r,s,n,n_k,n_{k-2})}{s^t\left(1-p^2\right)^2}\left(1+\frac{1}{p^{-3}+p^{-2}-p^{-1}-1}\right)^{n-n_k-n_{k-1}-n_{k-2}}\\ &\leq \frac{D(r,s,n,n_k,n_{k-1})D(r,s,n,n_k,n_{k-2})}{s^t\left(1-p^2\right)^2}\left(1+\frac{p^3}{(1-p)}\right)^{n-n_k-n_{k-1}-n_{k-2}}\end{aligned}$$

the same way we derived (2.12) from (2.11). Putting it altogether, we get

$$\sum_{b_1,b_2\in B} n(b_1,b_2)$$
$$\leq \sum_{b\in B}\deg b + \frac{\left(\sum_{b\in B}\deg b\right)^2}{s^t}\left(1+\frac{4p^3}{(1-p)^2}\right)$$
$$+\frac{k^3 n_k n_{k-1}^2 D(r,s,n,n_k,n_{k-1})D(r,s,n,n_k,n_{k-2})}{s^t(1-p^2)^2}\left(1+\frac{p^3}{(1-p)}\right)^{n-n_k-n_{k-1}-n_{k-2}}$$
$$+\frac{k^2 n_k^2 n_{k-1} D(r,s,n,n_k,n_{k-1})^2}{s^t}\left(1+\frac{p^3}{(1-p)}\right)^{n-n_k-n_{k-1}}$$
$$+\frac{k^2 n_k^2 n_{k-1} D(r,s,n,n_k)D(r,s,n,n_k,n_{k-1})}{s^t(1-p^2)}\left(1+\frac{p^3}{(1-p)}\right)^{n-n_k-n_{k-1}}$$
$$+\frac{k n_k^3 D(r,s,n,n_k)^2}{s^t}\left(1+\frac{p^3}{(1-p)}\right)^{n-n_k}.$$

Then, by the Turán sieve, we get

$$P(diamG(\mathbf{n^{(k)}},p)=2) < \frac{2}{n_k(n_k-1)(1-p^2)^{n-n_k}} \cdot\left(1+\frac{2n_{k-1}(1-p^2)^{-n_{k-1}}(1-p)}{(n_k-1)}\right)^{-1} + \frac{3k^3\left(1+\frac{p^3}{(1-p)}\right)^{n-n_k}(1-p^2)^{-2}}{(n_{k-1}-1)}.$$

This completes the proof of Theorem 3.3. □

By substituting $p=\frac{1}{2}$, we deduce from Theorem 3.3 the following.

Corollary 3.4. *Let $P(diamG(\mathbf{n^{(k)}},p)=2)$ be defined as in Theorem 3.3. If $p=\frac{1}{2}$, then we have*

$$P(diamG(\mathbf{n^{(k)}},1/2)=2) \geq 1-\frac{n_k^2(3/4)^{n-n_k}}{2}\left(1+\frac{2n_{k-1}(3/4)^{-n_{k-1}}}{n_k}+\frac{7k^2n_{k-1}^2(3/4)^{n_k-n_{k-1}-n_{k-2}}}{3n_k^2}\right).$$

In the case when $p\to 0$ as $n\to\infty$, we have the following.

Corollary 3.5. *Let $P(diamG(\mathbf{n^{(k)}},p)=2)$ be defined as in Theorem 3.3. Let $\lim_{n\to\infty} p^4(n-n_k)=0$. We have*

$$P(diamG(\mathbf{n^{(k)}},p)=2) \geq 1-\frac{n_k^2 e^{-p^2(n-n_k)}}{2} \cdot\left(1+\frac{2n_{k-1}}{n_k}e^{p^2 n_{k-1}}\left(1+\frac{7k^2 n_{k-1}e^{-p^2(n_k-n_{k-2})}}{6n_k}\right)\right)$$

and

$$
\begin{aligned}
(3.3)\qquad & P(diamG(\mathbf{n^{(k)}},p)=2)\\
&\le (1+o(1))\frac{2e^{p^2(n-n_k)}}{n_k^2}\left(1+\frac{2n_{k-1}}{n_k}e^{p^2n_{k-1}}\right)^{-1}\\
&\quad\cdot\left(1+\frac{3k^3n_k^2e^{\left(p^3-p^2\right)(n-n_k)}}{2(n_{k-1}-1)}+\frac{3k^3n_kn_{k-1}e^{\left(p^3-p^2\right)(n-n_k)+p^2n_{k-1}}}{(n_{k-1}-1)}\right).
\end{aligned}
$$

Suppose further that

$$
\lim_{n\to\infty}\left(\log n_{k-1}-\log n-p^2n_{k-1}\right)=-\infty, \tag{3.4}
$$

$$
\lim_{n\to\infty}\left(2\log n+(p^3-p^2)(n-n_k)-\log n_{k-1}\right)=-\infty, \tag{3.5}
$$

$$
\lim_{n\to\infty}\left(\left(p^3-p^2\right)(n-n_k)+p^2n_{k-1}+\log n\right)=-\infty, \tag{3.6}
$$

and that

$$
\lim_{n\to\infty}\left(2\log n_k-p^2(n-n_k)-\log 2+\log\left(1+\frac{2n_{k-1}}{n_k}e^{p^2n_{k-1}}\right)\right)=c
$$

for some $c\in\mathbb{R}$.
1) If $c<0$, *we have*

$$
P(diamG(\mathbf{n^{(k)}},p)=2)\ge 1-(1+o(1))e^c.
$$

2) If $c>0$, *we have*

$$
P(diamG(\mathbf{n^{(k)}},p)=2)\le (1+o(1))e^{-c}.
$$

Proof. Since $\lim_{n\to\infty}p^4(n-n_k)=0$ and $p^{-2}\ge 1$, for $n\to\infty$, by similar reasoning as in the proof of Theorem 1.2, we have

$$
\begin{aligned}
e^{-p^2(n-n_k)}>\left(1-p^2\right)^{p^{-2}\cdot p^2(n-n_k)}&>e^{-p^2(n-n_k)}\left(1-p^2\right)^{p^2(n-n_k)}\\
&=e^{-p^2(n-n_k)}(1+o(1))
\end{aligned}
$$

and

$$
\begin{aligned}
e^{p^2n_{k-1}}<\left(1-p^2\right)^{-p^{-2}\cdot p^2n_{k-1}}&<e^{p^2n_{k-1}}\left(1-p^2\right)^{-p^2n_{k-1}}\\
&=e^{p^2n_{k-1}}(1+o(1)).
\end{aligned}
$$

Thus the first term in the upper bound of $P(diamG(\mathbf{n^{(k)}}, p) = 2)$ in Theorem 3.3 becomes

$$\frac{2}{n_k(n_k-1)\left(1-p^2\right)^{p^{-2}\cdot p^2(n-n_k)}} \cdot \left(1+\frac{2n_{k-1}\left(1-p^2\right)^{\frac{-n}{np^2}\cdot p^2 n_{k-1}}(1-p)}{(n_k-1)}\right)^{-1} \tag{3.7}$$

$$= (1+o(1))\frac{2e^{p^2(n-n_k)}}{n_k^2}\left(1+\frac{2n_{k-1}}{n_k}e^{p^2 n_{k-1}}\right)^{-1}. \tag{3.8}$$

For the second term, first note that since $\lim_{n\to\infty} p^4(n-n_k) = 0$, we have

$$\lim_{n\to\infty}(n-n_k)p^2\left(\frac{p}{(1-p)}-1\right)-(n-n_k)p^2(p-1) = 0.$$

Thus the second term in the upper bound of $P(diamG(\mathbf{n^{(k)}}, p) = 2)$ in Theorem 3.3 becomes

$$\frac{3k^3 e^{\frac{p^3(n-n_k)}{1-p}}\left(1-p^2\right)^{-2}}{n_{k-1}-1} = \frac{3k^3 e^{p^3(n-n_k)}}{n_{k-1}-1}(1+o(1)). \tag{3.9}$$

Combining Equations (3.8) and (3.9), the upper bound of Corollary 3.5 follows. Also, by Equations (3.4), (3.5), and (3.6), Statements (1) and (2) follow as in the proof of Corollary 1.4. □

Remark 3.6. Similar to Remark 2.1, all $o(1)$ terms in Corollary 3.5 can be made explicit.

4. Bipartite Graphs with Diameter 3

Here we analyze bipartite graphs in a similar way to k-partite graphs, but instead of considering diameter 2, we consider diameter 3 since, except for the complete bipartite graph, all bipartite graphs have diameter at least 3.

Theorem 4.1. *For each $n \in \mathbb{N}$, $n \geq 4$, pick $n_1, n_2 \in \mathbb{N}$ such that $2 \leq n_1 \leq n_2$ and $n_1 + n_2 = n$. Let $G(n_1, n_2, p)$ denote the set of all bipartite graphs with the partite sets having n_1 and n_2 vertices respectively where each edge is chosen independently with probability p. Also, let $P(diamG(n_1, n_2, p) = 3)$ be the probability of a graph from $G(n_1, n_2, p)$ having diameter 3. Then*

$$P(diamG(n_1,n_2,p)=3) \geq 1-\frac{n_2^2(1-p^2)^{n_1}}{2}\left(1+\frac{n_1^2(1-p^2)^{n_2-n_1}}{n_2^2}\right)$$

and

$$\begin{aligned} &P(diamG(n_1,n_2,p)=3)\\ &\leq \left(\frac{2}{n_2(n_2-1)(1-p^2)^{n_1}}+\frac{\left(1+\frac{p^3}{(1-p)}\right)^{n_1}}{n_2}\left(8+\frac{8}{(1-p)}\right)\right)\\ &\quad\cdot\left(1+\frac{n_1(n_1-1)(1-p^2)^{n_2-n_1}}{n_2(n_2-1)}\right)^{-1}. \end{aligned}$$

Proof of Theorem 4.1. As in the proof of Theorem 1.2, we may assume that $p\in\mathbb{Q}\cap(0,1)$ for all $n\in\mathbb{N}$.

Let $p=\frac{r}{s}$ where $r,s\in\mathbb{N}$. As in the proofs of Theorems 1.2 3.3, we let A be the set of all graphs in $G(n_1,n_2,p)$, allowing for a number of duplicates of each possible graph. As in the proofs of Theorems 1.2 and 3.3 this construction of A allows us to have a uniformly chosen graph in A correctly represent the random graph $G(n_1,n_2,p)$. Since the complete bipartite graph has n_1n_2 edges, we have $r^{n_1n_2}$ copies of the complete bipartite graph and $|A|=s^{n_1n_2}$.

We let B be the set of all pairs of vertices such that both vertices of a pair occur in the same partite set. Thus, $|B|=\binom{n_1}{2}\binom{n_2}{2}$. For $a\in A$ and $b\in B$, we write $a\sim b$ if the pair of vertices b in the graph a do not share a common neighbouring vertex. Thus, we will have $\omega(a)=0$ if and only if a is connected with diameter at most 3. For each pair of vertices $b\in B$ in the set containing n_1 vertices, there are n_2 potential neighbouring vertices to take into account. Similarly to how we derived (2.2) from (2.1) we again use the binomial theorem on summing all the different possible graphs with their weights to obtain

$$D(r,s,n,n_1):=\deg b=((s-r)^2+2r(s-r))^{n_2}((s-r)+r)^{n_1n_2-2n_2}.$$

For each pair of vertices $b\in B$ in the set containing n_2 vertices, the n_1 and n_2 are switched in the above equality. It follows that

$$\sum_{b\in B}\deg b=\frac{s^{n_1n_2}n_1(n_1-1)(1-p^2)^{n_2}}{2}+\frac{s^{n_1n_2}n_2(n_2-1)(1-p^2)^{n_1}}{2}.$$

By the simple sieve, we obtain

$$\begin{aligned} P(diamG(n_1,n_2,p)=3)&>1-\frac{n_1^2(1-p^2)^{n_2}}{2}-\frac{n_2^2(1-p^2)^{n_1}}{2}\\ &=1-\frac{n_2^2(1-p^2)^{n_1}}{2}\left(1+\frac{n_1^2(1-p^2)^{n_2-n_1}}{n_2^2}\right) \end{aligned}$$

since the random graph $P(diamG(n_1,n_2,p)=3)$ is uniform over the set A.

To get an upper bound for $P(diamG(n_1, n_2, p) = 3)$, we need to estimate $\sum_{b_1,b_2 \in B} n(b_1, b_2)$. Similar to the proofs of Theorems 1.2 and 3.3, we calculate $n(b_1, b_2)$ based on the number of vertices b_1 and b_2 have in common. If b_1 and b_2 occur in different partite sets (and hence have no vertices in common), then we have

$$n(b_1, b_2) = \left((s-r)^4 + 4r(s-r)^3 + 2r^2(s-r)^2\right) \cdot \sum_{k_1=0}^{(n_1-2)(n_2-2)} \sum_{k_2=0}^{2n-8} \binom{(n_1-2)(n_2-2)}{k_1} \binom{n_1+n_2-4}{k_2} 2^{k_2} r^{k_1+k_2} (s-r)^{n_1n_2-k_1-k_2-4}.$$

Then similarly to how we derived (2.10) from (2.5), we can derive

$$n(b_1, b_2) \leq \frac{\deg b_1 \deg b_2}{s^{n_1n_2}} \left(1 + \frac{4p^3}{(1-p)^2}\right).$$

Suppose that b_1 and b_2 both occur in the partite set consisting of n_1 vertices and have no vertices in common. Then

$$n(b_1, b_2) = \sum_{k_1=0}^{(n_1-4)n_2} \sum_{k_2=0}^{2n-8} \binom{(n_1-4)n_2}{k_1} \binom{4n_2}{k_2} 2^{k_2} r^{k_1+k_2} (s-r)^{n_1n_2-k_1-k_2}.$$

Then similarly to how we derived (2.10) from (2.5), we can derive

$$n(b_1, b_2) = \frac{\deg b_1 \deg b_2}{s^{n_1n_2}}$$

which, by symmetry, is what we also have if b_1 and b_2 both occur in the partite set consisting of n_2 vertices and have no vertices in common.

Now suppose that b_1 and b_2 have exactly one vertex in common. Suppose that they occur in the partite set with n_1 vertices. Then the equation for $n(b_1, b_2)$ is the same as in (2.11) with the following minor changes. The factor $(s-r)^2$ is removed, $\binom{n-3}{2} + 1$ is replaced with $(n_1 - 3)\, n_2$, $n-3$ is replaced with n_2, $n-3-k_2$ is replaced with $n_2 - k_2$, and $\binom{n-3}{2} + 3n - 8 - k_1 - k_2 - 2k_3$ is replaced with $n_1n_2 - k_1 - k_2 - k_3$. We can deduce that in this case

$$\begin{aligned} n(b_1, b_2) &\leq \frac{D(r, s, n, n_1)^2}{s^t} \left(1 + \frac{1}{p^{-3} + p^{-2} - p^{-1} - 1}\right)^{n_2} \\ &\leq \frac{D(r, s, n, n_1)^2}{s^t} \left(1 + \frac{p^3}{(1-p)}\right)^{n_2} \end{aligned}$$

the same way we derived (2.12) from (2.11).

Putting it altogether, we get

$$\begin{aligned}
&\sum_{b_1,b_2\in B} n(b_1,b_2)\\
&= \binom{n_1}{2}\binom{n_1-2}{2}\frac{D(r,s,n,n_2)^2}{s^{n_1n_2}} + \binom{n_2}{2}\binom{n_2-2}{2}\frac{D(r,s,n,n_1)^2}{s^{n_1n_2}}\\
&\quad + 2\binom{n_1}{2}\binom{n_2}{2}\frac{D(r,s,n,n_1)D(r,s,n,n_2)}{s^{n_1n_2}}\cdot\left(1+\frac{4p^3}{(1-p)^2}\right)\\
&\quad + \frac{D(r,s,n,n_2)^2 n_2(n_2-1)(n_2-2)}{s^{n_1n_2}}\left(1+\frac{p^3}{(1-p)}\right)^{n_1}\\
&\quad + \frac{D(r,s,n,n_1)^2 n_1(n_1-1)(n_1-2)}{s^{n_1n_2}}\left(1+\frac{p^3}{(1-p)}\right)^{n_2}\\
&\quad + \frac{s^{n_1n_2}n_2(n_2-1)(1-p^2)^{n_1}}{2}\left(1+\frac{n_1(n_1-1)(1-p^2)^{n_2-n_1}}{n_2(n_2-1)}\right).
\end{aligned}$$

Then, by the Turán sieve, we can get

$$\begin{aligned}
&P(diamG(n_1,n_2,p)=3)\cdot\left(1+\frac{n_1(n_1-1)(1-p^2)^{n_2-n_1}}{n_2(n_2-1)}\right)^2.\\
&< \frac{2}{n_2(n_2-1)(1-p^2)^{n_1}}\left(1+\frac{n_1(n_1-1)(1-p^2)^{n_2-n_1}}{n_2(n_2-1)}\right)\\
&\quad + \frac{4n_1^3\left(1+\frac{p^3}{(1-p)}\right)^{n_2}(1-p^2)^{2n_2-2n_1}}{n_2^4} + \frac{4\left(1+\frac{p^3}{(1-p)}\right)^{n_1}}{n_2}\\
&\quad + \frac{8n_1^2\left(\frac{p^3}{(1-p)^2}\right)(1-p^2)^{n_2-n_1}}{n_2^2}.
\end{aligned}$$

Notice that

$$\frac{p^3}{(1-p)} < \frac{1}{n_1}\left(1+n_1\frac{p^3}{(1-p)}\right) < \frac{1}{n_1}\left(1+\frac{p^3}{(1-p)}\right)^{n_1}$$

and

$$\left(1+\frac{p^3}{(1-p)}\right)(1-p^2)^2 < 1.$$

It follows that

$$\begin{aligned}
&P(diamG(n_1,n_2,p)=3)\left(1+\frac{n_1(n_1-1)(1-p^2)^{n_2-n_1}}{n_2(n_2-1)}\right)\\
&<\frac{2}{n_2(n_2-1)(1-p^2)^{n_1}}+\frac{4n_1^3\left(1+\frac{p^3}{(1-p)}\right)^{n_2}(1-p^2)^{2n_2-2n_1}}{n_2^4}\\
&\quad+\frac{\left(1+\frac{p^3}{(1-p)}\right)^{n_1}}{n_2}\left(4+\frac{8n_1(1-p^2)^{n_2-n_1}}{n_2(1-p)}\right)\\
&\leq\frac{2}{n_2(n_2-1)(1-p^2)^{n_1}}+\frac{\left(1+\frac{p^3}{(1-p)}\right)^{n_1}}{n_2}\left(8+\frac{8}{(1-p)}\right)
\end{aligned}$$

from which we obtain our upper bound. This completes the proof of Theorem 4.1. □

By substituting in $p=\frac{1}{2}$, we deduce from Theorem 4.1 the following.

Corollary 4.2. *Let $P(diamG(n_1,n_2,p)=3)$ be defined as in Theorem 4.1. If $p=\frac{1}{2}$, then we have*

$$P(diamG(n_1,n_2,p)=3)\geq 1-\frac{n_2^2(3/4)^{n_1}}{2}\left(1+\frac{n_1^2(3/4)^{n_2-n_1}}{n_2^2}\right)$$

and

$$P(diamG(n_1,n_2,p)=3)\leq\left(\frac{2(4/3)^{n_1}}{n_2(n_2-1)}+\frac{24(5/4)^{n_1}}{n_2}\right)\left(1+\frac{n_1(n_1-1)(3/4)^{n_2-n_1}}{n_2(n_2-1)}\right)^{-1}.$$

0

Remark 4.3. The upper bound given for $P(G(n_1,n_2,p),1/2)$ in Corollary 4.2 will in general only be non-trivial when n_2 much larger than n_1. For instance, if $n_1<\min\{\frac{2\log n_2-\log 8}{\log(4/3)},\frac{\log n_2-\log 48}{\log(5/4)}\}$, then the upper bound will be less than 1.

In the situation where the edge probability $p\to 0$ as $n\to\infty$, we will show the following.

Corollary 4.4. *Let $P(diamG(n_1,n_2,p)=3)$ be as in Theorem 4.1. Let $\lim_{n\to\infty}np^4=0$. We have*

$$P(diamG(n_1,n_2,p)=3)\geq 1-\frac{n_2^2e^{-n_1p^2}}{2}\left(1+e^{2\log n_1-2\log n_2-(n_2-n_1)p^2}\right)$$

and

(4.1)

$$\begin{aligned}
P(diamG(n_1,n_2,p)=3)\leq(1+o(1))\left(\frac{2}{n_2^2}e^{n_1p^2}\right)\left(1+e^{2\log n_1-2\log n_2-(n_2-n_1)p^2}\right)^{-1}\\
\cdot\left(1+8n_2e^{n_1p^2(p-1)}\right).
\end{aligned}$$

Suppose further that

$$\lim_{n\to\infty}\left(2\log n_1 - 2\log n_2 - (n_2-n_1)p^2\right) = -\infty,$$

and

$$\lim_{n\to\infty}\left(2\log n_2 - n_1p^2 - \log 2\right) = c$$

for some $c \in \mathbb{R}$.

1) If $c < 0$, *we have*

$$P(diamG(n_1,n_2,p)=3) \geq 1-(1+o(1))e^c.$$

2) If $c > 0$, *we have*

$$P(diamG(n_1,n_2,p)=3) \leq (1+o(1))e^{-c}.$$

Proof. By the upper bound of $P(diamG(n_1,n_2,p)=3)$ in Theorem 4.1, we can get

$$\begin{aligned} &P(diamG(n_1,n_2,p)=3)\left(1+\frac{n_1(n_1-1)\left(1-p^2\right)^{n_2-n_1}}{n_2(n_2-1)}\right) \\ &< \frac{2}{n_2(n_2-1)\left(1-p^2\right)^{n_1}} + \left(8+\frac{8}{1-p}\right)\frac{\left(1+\frac{p^3}{1-p}\right)^{n_1}}{n_2}. \end{aligned}$$

Since $\lim_{n\to\infty} np^4 = 0$, we have $\lim_{n\to\infty} n_1p^4 = 0$ and so

$$\lim_{n\to\infty} n_1p^2\left(\frac{p}{1-p}-1\right) - n_1p^2\left(p-1\right) = 0.$$

Also, since $p^{-2} \geq 1$, we have

$$\left(1-p^2\right)^{n_1} > e^{-n_1p^2}\left(1-p^2\right)^{n_1p^2} = e^{-n_1p^2}(1-o(1))$$

and

$$\frac{\left(1+\frac{p^3}{1-p}\right)^{n_1}}{n_2} < \frac{e^{\frac{n_1p^3}{1-p}}}{n_2} = \frac{e^{n_1p^2}}{n_2^2}\cdot n_2 e^{n_1p^2\left(\frac{p}{1-p}-1\right)}.$$

Also,

$$\begin{aligned} &\frac{n_1^2\left(1-p^2\right)^{n_2-n_1}}{n_2^2\left(1-p^2\right)} \\ &= e^{2\log n_1 - 2\log n_2}\left(1-p^2\right)^{p^{-2}\cdot(n_2-n_1)p^2}\left(1-p^2\right)^{-1} \\ &> e^{2\log n_1 - 2\log n_2 - (n_2-n_1)p^2}\left(1-p^2\right)^{(n_2-n_1)p^2}\left(1-p^2\right)^{-1}. \end{aligned}$$

Since $\lim_{n\to\infty} np^4 = 0$, we have

$$\lim_{n\to\infty}\left(1-p^2\right)^{(n_2-n_1)p^2} = 1.$$

We thus obtain our bounds. Statements (1) and (2) follow as in the proof of Corollary 1.4. □

Remark 4.5. Similar to Remark 2.1, all $o(1)$ terms in Corollary 4.4 can be made explicit.

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