A counterexample to a conjecture about triangle-free induced subgraphs of graphs with large chromatic number

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Abstract

We prove that for every n, there is a graph G with $\chi(G) \geq n$ and $\omega(G) \leq 3$ such that every induced subgraph H of G with $\omega(H) \leq 2$ satisfies $\chi(H) \leq 4$.

This disproves a well-known conjecture. Our construction is a digraph with bounded clique number, large dichromatic number, and no induced directed cycles of odd length at least 5.

1 Introduction and preliminaries

In this paper, we disprove the following conjecture (its origin appears somewhat unclear;* it is attributed to Louis Esperet in [10], while the authors of [11] state that "we could not find a reference"):

Conjecture 1.1 ([10, 11]) *For all $k, r \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that for every graph G with $\chi(G) \geq n$ and $\omega(G) \leq k$, there is an induced subgraph H of G with $\chi(H) \geq r$ and $\omega(H) = 2$.*

Here, $\chi(G)$ denotes the *chromatic number* of a graph G and $\omega(G)$ denotes the *clique number*. This conjecture is the induced-subgraph analogue of the following theorem:

Theorem 1.2 ([8]) *For every $r \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that for every graph G with $\chi(G) \geq n$, there is a (not necessarily induced) subgraph H of G with $\chi(H) \geq r$ and $\omega(H) = 2$.*

We will show:

Theorem 1.3 *For every $n \in \mathbb{N}$, there is a graph G with $\chi(G) \geq n$ and $\omega(G) \leq 3$ such that every induced subgraph H of G with $\omega(H) \leq 2$ satisfies $\chi(H) \leq 4$.*

This is answers Conjecture 1.1 in the negative for all $r \geq 5$. The following shows that the case $r = 4$ is the only case of Conjecture 1.1 which remains open:

Theorem 1.4 ([9]) *There is a function $f : \mathbb{N} \to \mathbb{N}$ such that for every graph G with no induced cycle of odd length at least 5, we have $\chi(G) \leq f(\omega(G))$.*

Letting f as in Theorem 1.4, it follows that every graph G with $\chi(G) > f(\omega(G))$ contains an induced cycle of odd length at least 5, and therefore contains an induced subgraph H with $\omega(H) = 2$ and $\chi(H) = 3$.

Our construction is based on a construction of [5] and produces a digraph with large dichromatic number, which we define below. Throughout this paper, we only consider simple digraphs D, that is, for every two distinct vertices u and v, the digraph D contains either an edge from u to v, or an edge from v to u, or neither; but not both. For a digraph, we write uv for an edge from u to v. Given a digraph D, we define its *underlying undirected graph* G to be that graph with $V(G) = V(D)$ and in which $u, v \in V(G)$ are adjacent if D contains an edge from u to v or from v to u. The *clique number* $\omega(D)$ of a digraph D is defined as the clique number of the underlying undirected graph of D.

An analogue of chromatic number for directed graphs was introduced in [3, 7]. A digraph is *acyclic* if it contains no directed cycle. For $k \in \mathbb{N}$, a *k-dicoloring* of a digraph D is a function $f : V(D) \to \{1, \dots, k\}$ such that for every $i \in \{1, \dots, k\}$, the induced subdigraph of D with vertex set $\{v \in V(D) : f(v) = i\}$ is acyclic. The *dichromatic number* $\vec{\chi}(D)$ is the smallest integer k such that D has a k-dicoloring.

We show that the digraph analogue of Theorem 1.4 does not hold:

Theorem 1.5 *For every n, there is a digraph D with $\vec{\chi}(D) \geq n$, $\omega(D) \leq 3$ and with no induced directed cycle of odd length at least 5.*

*Recently, Vojtěch Rödl pointed out to us that the problem had first appeared in [6] attributed to Fred Galvin and Vojtěch Rödl.

2 The construction

We construct a sequence of digraphs $\{D_n\}$ as follows. Let D_1 be the digraph with a single vertex. For $n \geq 2$, we take $n-1$ disjoint copies of the digraph D_{n-1} and call them $D_{n-1}^1, \dots, D_{n-1}^{n-1}$. Let $\mathcal{T}$ be the set of all sequences $T = (x_1, \dots, x_{n-1})$ with $x_i \in V(D_{n-1}^i)$ for all $i \in \{1, \dots, n-1\}$. Now, for every $T = (x_1, \dots, x_{n-1}) \in \mathcal{T}$ we create a vertex v_T and for every $i \in \{1, \dots, n-1\}$, we add an edge from x_i to v_T. The resulting digraph with vertex set

$$V(D_{n-1}^1) \cup \dots \cup V(D_{n-1}^{n-1}) \cup \{v_T : T \in \mathcal{T}\}$$

and edge set

$$E(D_{n-1}^1) \cup \dots \cup E(D_{n-1}^{n-1}) \cup \{x_i v_T : i \in \{1, \dots, n-1\}, T = (x_1, \dots, x_{n-1}) \in \mathcal{T}\}$$

is called D_n.

We note that the graph D_n is the graph of red edges in the proof of Theorem 3 of [5], where the following was proved:

Lemma 2.1 ([5]) *For all $n \in \mathbb{N}$, we have:*

- *D_n is acyclic;*
- *for every two vertices $u, v \in V(D_n)$ there is at most one directed path from u to v in D_n.*

Proof. We include a proof for completeness. For $n \geq 1$, let us define a partition of $V(D_n)$ into sets $T_1^n, \dots, T_n^n$ as follows: For $n = 1$, let $T_1^1 = V(D_1)$. For $n > 1$ and $i \in \{1, \dots, n-1\}$, let T_i^n be the union of the sets T_i^{n-1} in $D_{n-1}^1, \dots, D_{n-1}^{n-1}$, and let T_n^n be the set of remaining vertices (and thus T_n^n is the set of vertices v_T added when constructing D_n).

By construction we have that for all $i \in \{1, \dots, n\}$, the set T_i^n is a stable set and the only edges between T_i^n and $T_1^n \cup \dots \cup T_{i-1}^n$ are edges from $T_1^n \cup \dots \cup T_{i-1}^n$ to T_i^n. It follows that D_n is acyclic, as desired.

For the second bullet, note that every edge is from T_i^n to T_j^n for some $i < j$. Now, suppose we have vertices u, v such that there exists a directed path P from u to v. Then it follows that $u \in T_i^n$ and $v \in T_j^n$ for $i < j$, and the vertex set of P is contained in $T_i^n \cup \dots \cup T_j^n$. Let H be the copy of D_{j-1} that u is contained in from the construction of D_j. By construction, every edge of D_n with one end in H and one end x in $V(D_n) \setminus V(H)$ satisfies $x \in T_k^n$ for some $k \geq j$. Since v is the only vertex of P in $T_j^n \cup T_{j+1}^n \cup \dots \cup T_n^n$, it follows that all vertices of $P \setminus v$ are contained in H. Note that v has exactly one in-neighbor in H; let that in-neighbor be w. It follows that any directed path from u to v must go through w. By induction on n (since $P \setminus v$ is contained in a copy of D_{j-1} with $j \leq n$), we have that there is at most one directed path from u to w, so it follows that there is at most one directed path from u to v, as desired. This completes the proof. ∎

We define the *length* of a (directed) path as its number of edges. Now, we construct a sequence of digraphs $\{D'_n\}$ as follows. We take a copy of D_n, and create a new graph D'_n with $V(D'_n) = V(D_n)$, and the following edges. For every two vertices u, v where there exists a directed path in D_n from u to v,

- we add an edge from u to v if that path has length equal to 1 modulo 3; and
- we add an edge from v to u if that path has length equal to 2 modulo 3.

From Lemma 2.1, it follows that D'_n is well-defined and a simple digraph. In our analysis, it will be useful to consider a partition of the edges of D'_n into two sets, positive and negative, which we call the *sign* of an edge. Let us call an edge *positive* if it was added as a result of the first bullet above, and *negative* if it was added as a result of the second bullet. Clearly, this is a partition of the edges of D'_n. Note that in particular, if $uv \in E(D_n)$, then the edge uv is added to D'_n according to the first bullet, and hence D_n is a (non-induced) subdigraph of the positive edges of D'_n.

Lemma 2.2 *Let $u, v, w \in V(D'_n)$. If uv and vw are edges of D'_n of the same sign, then wu is an edge of D'_n of the opposite sign.*

Proof. Suppose first that uv and vw are positive edges. Then by definition there exists a path P_1 from u to v in D_n with length equal to 1 modulo 3, and a path P_2 from v to w in D_n with length equal to 1 modulo 3. Then clearly $P_3 = P_2 \cup P_1$ is a directed walk from u to w, and since D_n is acyclic by Lemma 2.1, it follows that P_3 is the unique directed path from u to w. Then P_3 has length equal to 2 modulo 3, so it follows that wu is a negative edge, as desired.

Suppose instead that uv and vw are negative edges. Then there exists a path P_1 from v to u and a path P_2 from w to v such that P_1 and P_2 both have length equal to 2 modulo 3. Then clearly $P_3 = P_2 \cup P_1$ is a directed walk from w to u, and since D_n is acyclic by Lemma 2.1, it follows that P_3 is the unique path from w to u. Then P_3 has length equal to 1 modulo 3 and it follows that wu is a positive edge, as desired. This completes the proof. ∎

Lemma 2.3 *Let $u, v, w \in V(D'_n)$. Then not all of uv, vw, uw are edges of D'_n.*

Proof. We only consider the case when uw is positive; the case when uw is negative is analogous. It follows that there is a directed path P_1 from u to w of length congruent to 1 modulo 3. By Lemma 2.2, we may assume that uv and vw do not have the same sign. We consider two cases.

If uv is negative, then vw is positive. It follows that there is a directed path P_2 from v to u of length congruent to 2 modulo 3. Now $P_3 = P_2 \cup P_1$ is a directed walk and since D_n is acyclic by Lemma 2.1, a directed path, from v to w. But P_3 has length congruent to 0 modulo 3, and so from the construction of D'_n, it follows that v and w are not adjacent in either direction, a contradiction.

Now uv is positive, and vw is negative. It follows that there is a directed path P_2 from w to v of length congruent to 2 modulo 3. Now $P_3 = P_1 \cup P_2$ is a directed walk and since D_n is acyclic by Lemma 2.1, a directed path from u to v. But P_3 has length congruent to 0 modulo 3, and so from the construction of D'_n, it follows that v and u are not adjacent in either direction, a contradiction. ∎

Now, we are ready to prove our main theorem, which we restate.

Theorem 1.3 *For every $n \in \mathbb{N}$, there is a graph G with $\chi(G) \geq n$ and $\omega(G) \leq 3$ such that every induced subgraph H of G with $\omega(H) \leq 2$ satisfies $\chi(H) \leq 4$.*

Proof. Let $\{G_n\}$ be the sequence of graphs such that G_n is the underlying undirected graph of D'_n. Then we claim that taking $G = G_n$ will show the desired result.

Indeed, we first show that $\chi(G_n) \geq n$. Since D_n is a subgraph of D'_n, it suffices to show, by induction, that the underlying undirected graph H_n of D_n has chromatic number at least n (which was also shown in [5], and follows from the fact that the n-th Zykov graph [12] is a subgraph of H_n; here we give the short proof for completeness). The base case is trivial. By induction, we know that the underlying undirected graphs H_{n-1} of the $n-1$ copies of D_{n-1} that were used to build D_n all have chromatic number at least $n-1$. So, if we take a coloring of H_n with colors $\{1, \ldots, n-1\}$, it follows that for every $i \in \{1, \ldots, n-1\}$, there exists a vertex $x_i \in V(D^i_{n-1})$ which receives color i. Then, letting $T = (x_1, \ldots, x_{n-1})$, the corresponding vertex v_T must receive a color not in $\{1, \ldots, n-1\}$, and it follows that the coloring uses at least n colors. Thus, $\chi(H_n) \geq n$ for all $n \geq 1$, as claimed.

Next, we claim that $\omega(G_n) \leq 3$. Suppose not; then G_n contains a clique K of size 4. Let $u \in K$ with at least two outneighbors in the digraph induced by K in D'_n (which is possible, since the average outdegree in this four-vertex digraph is 1.5), and let v, w be two outneighbours of u in K. By symmetry, we may assume that vw is an edge of D'_n. But now uv, vw, uw are all edges of D'_n, contrary to Lemma 2.3.

Now, suppose that we have an induced subgraph H of G_n with $\omega(H) \leq 2$. If we look at the corresponding induced subdigraph H' of D'_n, it follows by Lemma 2.2 that H' does not contain a directed 2-edge path with both edges of the same sign as a subdigraph. Thus we can partition the vertices of H' (and H) into two sets A, B such that every vertex in A is not the head of a positive edge and every vertex in B is not the tail of a positive edge. Then note that there can be no positive edges between any two vertices in A, and also there are no positive edges between any two vertices in B. Likewise, we can find a similar partition $V(H') = A' \cup B'$ for the negative edges. Now $(A \cap A', A \cap B', B \cap A', B \cap B')$ is a partition of the vertices of H into four stable sets, and thus $\chi(H) \leq 4$, as claimed. This completes the proof. ∎

The collection of digraphs $\{D'_n\}$ also gives the following result on $\vec{\chi}$-boundedness, which we restate.

Theorem 1.5 *For every n, there is a digraph D with $\vec{\chi}(D) \geq n$, $\omega(D) \leq 3$ and with no induced directed cycle of odd length at least 5.*

Proof. We claim that taking $D = D'_{4n}$ gives the desired result. Indeed, we know from the previous proof that $\omega(D) \leq 3$. Furthermore, suppose that D contains an induced odd directed cycle of length at least 5. Then it follows that there exist two consecutive edges in that cycle of the same sign; but now Lemma 2.2 gives a third edge which contradicts the fact that the cycle is induced.

It remains to show that $\vec{\chi}(D) = \vec{\chi}(D'_{4n}) \geq n$. Indeed, note that any acyclic induced subdigraph H' of D satisfies $\omega(H') \leq 2$ by Lemma 2.3. Now, let H be the underlying undirected graph of H'. Then the argument from the previous proof shows that $\chi(H) \leq 4$.

Since $\chi(G_{4n}) \geq 4n$ it follows that if $V(D)$ is partitioned into t sets which induce acyclic subdigraphs, then $\chi(G_{4n}) \leq 4t$; therefore, we must have $t \geq n$. Thus, $\vec{\chi}(D) \geq n$, as claimed. This completes the proof. ∎

3 Further work

Since $r = 4$ is now the only open case of Conjecture 1.1, it is natural to ask the following, which was first asked by James Davies (private communication):

Question 3.1 *Are there graphs with clique number 3 and arbitrarily large chromatic number whose all triangle-free induced subgraphs have chromatic number at most 3?*

Our construction was originally motivated by questions about the dichromatic number of graphs with bounded clique number. In view of Theorem 1.5, it is natural to ask:

Question 3.2 *Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that for every digraph D with no induced directed cycle of odd length, we have $\vec{\chi}(D) \leq f(\omega(D))$?*

Question 3.3 *Is there a function $f : \mathbb{N} \to \mathbb{N}$ such that for every digraph D with no induced directed cycle of length at least 4, we have $\vec{\chi}(D) \leq f(\omega(D))$?*

Question 3.3 asks about a directed analogue of chordal graphs and their dichromatic number. Since the first version of this paper, Question 3.3 has been answered in the negative by Aboulker, Bousquet, and de Verclos [1]. We also ask the following more general question:

Question 3.4 *For which l is there a function $f : \mathbb{N} \to \mathbb{N}$ such that for every digraph D with no induced directed cycle of length not equal to l, we have $\vec{\chi}(D) \leq f(\omega(D))$?*

Building on ideas of [1], we also answered Questions 3.2 and 3.4 in the negative [2].

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