Reflected and nonsymmetric crystal graphs

by

Harper Niergarth

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

This thesis is comprised of two projects. The first studies a certain composition of crystal operators on semistandard Young tableaux, which we term raised reflection operators, and are related to a sign-reversing involution used to prove the Littlewood–Richardson rule. In particular, we investigate the graph defined by these crystal operators. Our main result is that this graph is balanced bipartite, giving another proof of the Littlewood–Richardson rule. We do so by giving a set of local rules that this graph satisfies and showing that any graph satisfying these rules is balanced bipartite.

The second studies crystal operators on multiline queues. Certain multiline queues, called non-wrapping multiline queues, are in bijection with semistandard Young tableaux but are better equipped to study nonsymmetric polynomials called Demazure atoms. Indeed, each multiline queue comes equipped with a weak composition, called the type, and summing over all multiline queues of a fixed type yields a Demazure atom. Crystal operators on multiline queues do not preserve type. Our main result characterizes how crystal operators interact with the type of a multiline queue. In particular, we show that these operators may only change the type by a simple transposition and that the type changes if and only if the multiline queue is in a specific configuration.

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Chapter 1

Introduction

1.1 Introduction

Crystal graphs were originally introduced by Kashiwara [14] to study the representation theory of quantum groups and other related algebras. These graphs carry rich combinatorial structure that allow us to study these representations combinatorially. Since their introduction, they have also become powerful tools in Algebraic Combinatorics, separate from their origins in representation theory. In this thesis, we will approach our study of crystal graphs purely from the viewpoint of Algebraic Combinatorics, leaving out the representation theory. For a more complete picture of crystal graphs in representation theory, we refer the reader to [3] and [14].

Symmetric functions play a key role in Algebraic Combinatorics and have many connections to other fields such as Enumerative Combinatorics, Representation Theory, Algebraic Geometry, and Group Theory. A central theme in the theory of symmetric functions is to give manifestly positive combinatorial formulas for the coefficients that appear when expanding a symmetric function in various bases of the ring of symmetric functions. Perhaps the most famous basis is the basis of Schur functions s_λ, which are generating functions for semistandard Young tableaux. Proving that a symmetric function expands positively in the Schur basis has many implications for representation theory and algebraic geometry (see [9, 24]) and in general is a "canary in the coal mine" for beautiful combinatorics.

1.2 Methods for Schur-positivity

One of the reasons that crystal graphs have had success in Algebraic Combinatorics is that they give a novel strategy to prove Schur-positivity. From now on, we use "crystal graph" to refer to a type A Stembridge crystal (see [28]). Informally, every connected component of a crystal graph corresponds to an individual Schur function, so we may prove Schur-positivity of a symmetric generating function F of some combinatorial set C by putting a graph structure on the set C and proving it is indeed a crystal graph. Then every connected component of this graph then corresponds to a Schur function and the whole graph corresponds to F, proving the Schur-positivity of F. The difficulty comes from guessing the right graph structure on C. Once we have the correct guess, there are tools to prove that it is actually a crystal graph, as an example, see [28].

We review two different ways of doing this. The first is to define a map φ from C to a set that already has a crystal graph structure on it and prove that φ is an isomorphism of crystals in some formal sense. To see an example of this method in use, see Section 2.3.2, where we prove the Schur-positivity of the skew Schur function $s_{\lambda/\mu}$ by mapping skew semistandard Young tableaux to semistandard Young tableaux. The other method is due to Stembridge. In [28], Stembridge gave a set of local rules that characterize crystal graphs. That is, if a graph satisfies the local rules, then it is a crystal graph. Thus, if we check this set of local rules on the graph we defined on C, then it must be a crystal graph, proving Schur-positivity. For an example of this, see [23].

Another strategy to prove Schur positivity is to write the coefficients in the Schur expansion of F as a signed sum, and then construct a sign-reversing involution to cancel the negative terms. We can in fact write these coefficients as a signed sum for any symmetric generating function, even ones that are not Schur positive. We do this by expanding F in the monomial basis, expanding the Schur function via the Jacobi–Trudi formula, and using a property of the Hall inner product. For an example of this being done, see Proposition 3.1.1, where we do it for the skew Schur function $s_{\lambda/\mu}$. The argument is exactly the same for a general symmetric generating function. This method has been used to give short proofs of the Littlewood-Richardson rule in [11, 27, 25]. This method has also been used to prove the Schur-positivity of the chromatic symmetric functions of incompatibility graphs of $(3+1)$-free posets in [7, 10]. While this method is not inherently tied to crystal graphs, we point out that [7] and [25] both use a composition of crystal operators to define their sign-reversing involution. To see how one uses crystals to construct a sign-reversing involution, see Section 3.1, where we present the proof of the Littlewood-Richardson rule given in [25].

1.3 Nonsymmetric generalizations of Schur functions

There are many generalizations of Schur functions (see [20, 12, 24]). One such generalization is to drop all assumptions of symmetry and instead work with nonsymmetric polynomials. While we lose much of the structure, there is still much to be said. There are many famous nonsymmetric polynomials, such as Schubert polynomials, fundamental slide polynomials, key polynomials, and Demazure atoms, all of which have rich combinatorics underpinning them. For an overview of these families, see [24]. Our focus will be on the basis of Demazure atoms $\mathcal{A}_\alpha$, which are generating functions for semistandard augmented fillings and indexed by weak compositions. Algebraically, these are a nonsymmetric refinement of Schur functions in the sense that

$$s_\lambda = \sum_{\mathtt{sort}(\alpha)=\lambda} \mathcal{A}_\alpha.$$

Just as the Demazure atoms decompose the Schur functions, we may decompose the crystal graph into nonsymmetric subgraphs, each of which correspond to a Demazure atom. Understanding these subgraphs is more challenging than understanding the full crystal, in part because these subgraphs may be disconnected. Indeed, the crystal operators on semistandard augmented fillings do not always preserve the combinatorial structure of the filling, as was the case for semistandard Young tableaux (we make this formal in Section 2.4).

Thus, it is important to have a better understanding of how these crystal operators act on semistandard augmented fillings. Instead of working with these fillings, we may pass to related combinatorial objects called non-wrapping multiline queues, which are in bijection with semistandard augmented fillings. Multiline queues were originally introduced in [8] to compute stationary distributions of a certain particle process from Statistical Physics and have been connected to symmetric Macdonald polynomials in [5] and [6]. For more on multiline queues and their combinatorics, see [19]. Demazure atoms are a specialization of a nonsymmetric refinement of the Macdonald polynomials and so this connection yields a multiline queue formula for Demazure atoms. It is this model that we study in Chapter 4.

The rest of the thesis is organized as follows. Chapter 2 reviews the relevant background on crystal operators, symmetric functions, and nonsymmetric polynomials. In particular, we give proofs based on crystal operators/graphs of two symmetric function identities; the Littlewood–Richardson rule and the Jacobi–Trudi formula. We also review other combinatorial algorithms on semistandard Young tableaux and semistandard augmented fillings such as jeu de taquin and skyline insertion.

In Chapter 3, we present the proof of the Littlewood–Richardson rule given by Remmel and Shimozono in [25]. Then we move on to studying the graph defined by a certain

composition of crystal operators $\widehat{r}_i$ that were used to define the sign-reversing involution in [25]. The body of the chapter is dedicated to providing a set of local rules that this graph satisfies and using them to prove that the graph is balanced bipartite. This gives another proof of the Littlewood–Richardson rule that relies more on the graph structure while still proving it via a sign-reversing involution. These local rules do not characterize these graphs as Stembridge's local rules do, but they are enough to recover some interesting properties.

In Chapter 4, we review the connection between semistandard augmented fillings and non-wrapping multiline queues. We then study crystal operators on multiline queues and, in particular, characterize how they interact with the type of a multiline queue (this corresponds to the shape of a semistandard augmented filling). This characterization allows us to state precisely when crystal operators change the shape of a semistandard augmented filling. We conclude with a discussion of how one may use this to prove Demzaure atom positivity.

Chapter 2

Background

In this chapter, we begin by reviewing some basic combinatorial definitions, then we move on to the basics of crystal operators, and finally we give a more detailed combinatorial background for symmetric functions and nonsymmetric polynomials. For a detailed exposition on crystals, see [3], for symmetric functions, see [17], and for nonsymmetric polynomials, see [24].

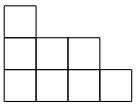
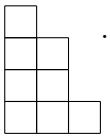
2.1 Notation and conventions

2.1.1 Compositions and partitions

A **weak composition** $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a finite sequence of nonnegative integers. Each integer α_i is called a **part** of α. The number of parts of α is called the **length** of α and is denoted $\ell(\alpha)$. The sum of the parts $\alpha_1 + \cdots + \alpha_n$ of α is called the **size** of α. A **partition** is a weak composition with parts sorted in decreasing order. Let $\mathcal{P}$ denote the set of all partitions. For any weak composition α, we let $\texttt{sort}(\alpha)$ denote the partition obtained by sorting α into weakly decreasing order and removing any zeros.

Example 2.1.1. *The weak composition* $\alpha = (3, 0, 5, 2, 2, 0, 4)$ *has length* 7 *and size* 16 *with* $\texttt{sort}(\alpha) = (5, 4, 3, 2, 2)$.

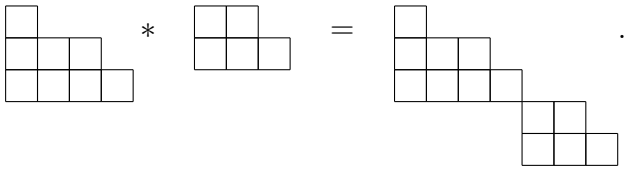
Each partition λ corresponds to a **diagram**, which consists of λ_i left justified boxes on the ith row. We use the convention that the rows are labelled from the bottom to top, consistent with the French notation. The **conjugate** λ' of a partition λ is the partition obtained by reflecting the Young diagram of λ along its main diagonal. For example,

$\lambda = (4,3,1) =$  and $\lambda' = (3,2,2,1) =$  .

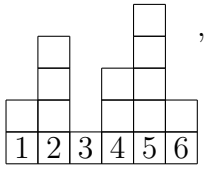
There are two partial orders on partitions that we will find useful. The first is **containment order**. Given two partitions λ and μ we say λ **contains** μ and write $\lambda \supseteq \mu$ if $\lambda_i \geq \mu_i$ for all i. The second is **dominance order**. Given two partitions λ and μ, we say λ **dominates** μ and write $\lambda \geq \mu$ if

$$\lambda_1 + \cdots + \lambda_k \geq \mu_1 + \cdots + \mu_k,$$

for all k. A skew partition is a pair of partitions λ, μ such that $\lambda \supseteq \mu$ and is denoted by λ/μ. Each skew partition corresponds to a **skew diagram**, which consists of all boxes in the diagram of λ that are not in the diagram of μ. Given any two partitions λ, μ, define their product $\lambda * \mu$ as the skew partition obtained by identifying the bottom right corner of λ with the top left corner of μ. For example,



Each weak composition $\alpha = (\alpha_1, \ldots, \alpha_n)$ corresponds to a **skyline diagram**, which consists of α_i bottom justified boxes in the ith column. We use the convention that the columns are labelled from left to right and the rows are labelled bottom to top. We also define the **augmented skyline diagram** of α as the diagram of α fitted with an extra bottom row of n boxes filled with integers $1, \ldots, n$ from left to right. We call this bottom row the **basement** and consider it as the zeroth row of the diagram. For example, if $\alpha = (1,3,0,2,4,1)$, its augmented skyline diagram is



the basement is the zeroth row and so the top row is the fourth row.

2.1.2 Permutations

Let S_n denote the symmetric group of size $n!$, the group of all permutations of $[n] = \{1, 2, \ldots, n\}$. We represent π in **one-line notation** as $[\pi(1), \pi(2), \ldots, \pi(n)]$. For any two permutations $\pi, \sigma \in S_n$, we define their product as $(\sigma\pi)(i) = \sigma(\pi(i))$. The symmetric group is generated by the set of **simple transpositions** $s_1, s_2, \ldots, s_{n-1}$ where s_i is the permutation that swaps i and $i+1$. The **length** $\ell(\pi)$ of a permutation π is the smallest number of simple transpositions needed to express π as a product of simple transpositions. The **sign** of a permutation π is defined as $\mathtt{sgn}(\pi) = (-1)^{\ell(\pi)}$.

For any permutation $\pi \in S_n$ and weak composition $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, the **left action** of π on α is defined as $\pi \cdot \alpha = (\alpha_{\pi^{-1}(1)}, \alpha_{\pi^{-1}(2)}, \ldots, \alpha_{\pi^{-1}(n)})$. This definition ensures that $\sigma \cdot (\pi \cdot \alpha) = (\sigma\pi) \cdot \alpha$. The reader should think of π acting on α by sending ith entry of α to position $\pi(i)$.

Example 2.1.2. *Let* $\sigma = [1, 3, 2, 4]$ *and* $\pi = [3, 4, 2, 1]$. *Then their product is*

$$\sigma\pi = [2, 4, 3, 1].$$

If $\alpha = (3, 1, 4, 4)$ *then the left action of* π *on* α *is*

$$\pi \cdot \alpha = (4, 4, 3, 1).$$

2.1.3 Graphs

A **graph** is a pair (V, E) where V is the (possibly infinite) set of **vertices** and E is the set of **edges**. An edge $e = \{u, v\}$ is a unordered pair of distinct vertices. We say that u and v are **adjacent** to each other and that u and v are **incident** to e. The **degree** $\deg(v)$ of a vertex v is the number of edges incident to v. A **bipartite graph** $G = (V, E)$ is a graph whose vertex set may be partitioned into two sets X, Y such that no two vertices within the same set are adjacent. A bipartite graph with bipartition $V = X \cup Y$ is **balanced** if $\#X = \#Y$. A **directed graph** $G = (V, E)$ is a graph whose edges are ordered pairs of vertices. For an edge $e = (u, v)$, we say that e goes from u to v. We call u the **tail** of e and denote it $\mathtt{tail}(e)$. We call v the **head** of e and denote it $\mathtt{head}(e)$. An **edge-coloring** of a graph G is an assignment $c : E \to \mathbb{Z}_{>0}$ of positive integers to the edges of G. An edge-coloring is **proper** if for every vertex v, the colors of the edges incident to v are all distinct.

When we want to refer the vertices of a graph G, we will write $V(G)$. A subgraph of G induced by $S \subseteq V(G)$ is the graph with vertex set S and edges between $u, v \in S$ if there is an edge between u and v in G. All graphs in this thesis will be directed and edge-colored, but the edge-coloring will not always be proper.

2.2 Crystal operators on words

In this section, we will introduce crystal operators on words (which we call **classical crystal operators**). Fix a positive integer n and let $\mathcal{W} = [n]^*$ be the set of all words with letters the alphabet $[n]$.

Definition 2.2.1 (Bracketing rule)**.** *Let $w \in \mathcal{W}$. For $1 \leq i \leq n-1$, let $\pi_i(w)$ be the word in the alphabet $\{(,)\}$ formed by scanning across w from left to right, and replacing each i with "$)$" and each $i+1$ with "$($". Now match up pairs of adjacent brackets "$()$", ignore the previously matched brackets, and repeat until no more matching can be done. This pairing rule for matching i's with $i+1$'s is called the **bracketing** or the **signature** rule.*

Definition 2.2.2 (Classical crystal operators)**.**

- *We define the **raising operator** $E_i : \mathcal{W} \to \mathcal{W} \sqcup \{\varnothing\}$ as follows: if $\pi_i(w)$ has no unmatched $i+1$'s then $E_i(w) = \varnothing$, if it does, then $E_i(w)$ is the word obtained by changing the leftmost unmatched $i+1$ in w to i.*
- *We define the **lowering operator** $F_i : \mathcal{W} \to \mathcal{W} \sqcup \{\varnothing\}$ as follows: if $\pi_i(w)$ has no unmatched i's then $F_i(w) = \varnothing$, if it does, then $F_i(w)$ is the word obtained by changing the rightmost unmatched i in w to $i+1$.*
- *We define the **reflection operator** $R_i : \mathcal{W} \to \mathcal{W}$ as follows: let a and b be the number of unmatched i's and $i+1$'s in $\pi_i(w)$ respectively. Then $R_i(w)$ is the word obtained by replacing the subword $i^a(i+1)^b$ in w with $i^b(i+1)^a$.*

Example 2.2.3. *Let $w = 221133121223132$ and $i = 2$. We use "$_$" to denote the entries which are not i or $i+1$, and highlight the unmatched brackets in red. Therefore,*

$$\pi_2(w) =) \;) \; _ \; _ \; (\; (\; _ \;) \; _ \;) \;) \; (\; _ \; (\;)$$

Underlining the entries corresponding to unmatched brackets: $w = \underline{2}\;\underline{2}\;1\;1\;3\;3\;1\;2\;1\;2\;\underline{2}\;\underline{3}\;1\;3\;2$. Therefore,

$$\begin{aligned} E_2(w) &= \underline{2}\;\underline{2}\;1\;1\;3\;3\;1\;2\;1\;2\;\underline{2}\;\underline{2}\;1\;3\;2, \\ F_2(w) &= \underline{2}\;\underline{2}\;1\;1\;3\;3\;1\;2\;1\;2\;\underline{3}\;\underline{3}\;1\;3\;2, \\ R_2(w) &= \underline{2}\;\underline{3}\;1\;1\;3\;3\;1\;2\;1\;2\;\underline{3}\;\underline{3}\;1\;3\;2. \end{aligned}$$

When $E_i(T) = \varnothing$ or $F_i(T) = \varnothing$, we will often say $E_i(T)$ or $F_i(T)$ is **not defined**. We may also realize R_i as a composition of E_i's or F_i's. For a word $w \in \mathcal{W}$, define $\varepsilon_i(w)$ as the largest number k such that $E_i^k(w) \neq \varnothing$ and define $\phi_i(w)$ as the largest number k such that $F_i^k(w) \neq \varnothing$. Then the following is immediate from the definition of R_i.

Proposition 2.2.4. *For any $w \in \mathcal{W}$, let $k = \varepsilon_i(w) - \phi_i(w)$. Then,*

$$R_i(w) = \begin{cases} E_i^k(w) & \text{if } k \geq 0, \\ F_i^{-k}(w) & \text{if } k \leq 0, \end{cases}$$

with the convention that $E_i^0(w) = F_i^0(w) = w$.

For a word w, we define the **weight** of w, $\mathtt{wt}(w) = (\mathtt{wt}_1(w), \mathtt{wt}_2(w), ...)$ where $\mathtt{wt}_i(w)$ is the number of i's in w. It is an amazing fact that the R_i's give an S_n-action on $\mathcal{W}$ [14] with the R_i acting on the weight of the word. That is, the R_i's commute when $|i-j| > 1$ ($R_iR_j = R_jR_i$), and satisfy the braid relations when $|i-j| = 1$ ($R_iR_jR_i = R_jR_iR_j$).

2.3 Symmetric functions

Symmetric functions are formal power series of bounded degree in an infinite set of variables $X = \{x_1, x_2, \ldots\}$ that are invariant under permutation of variables. More formally, a function f in the variables X is symmetric if for all $n \geq 1$,

$$f(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)}, x_{n+1} \ldots) = f(x_1, x_2, \ldots, x_n, x_{n+1}, \ldots)$$

holds for all $\pi \in S_n$. Let Λ denote the **ring of symmetric functions** in the variables X with coefficients in $\mathbb{Q}$.

We begin by reviewing two commonly used bases of Λ, which are indexed by partitions. Throughout, we let λ, μ, and ν denote partitions. For any weak composition $\alpha = (\alpha_1, \ldots, \alpha_n)$, we define the monomial $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$. Then the **monomial symmetric functions** are defined as

$$m_\lambda = \sum_{\substack{\alpha \text{ any weak composition} \\ \mathtt{sort}(\alpha) = \lambda}} x^\alpha.$$

The **complete homogeneous symmetric functions** are defined as

$$h_k = \sum_{0 \leq i_1 \leq i_2 \leq \cdots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k}, \quad h_\lambda = \prod_{i=1}^{\ell(\lambda)} h_{\lambda_i}.$$

Then $\{m_\lambda\}$ and $\{h_\lambda\}$ are both bases of Λ indexed by partitions $\lambda \in \mathcal{P}$. For a proof of these facts, we refer the reader to [17].

The ring of symmetric functions Λ comes equipped with an inner product $\langle \cdot, \cdot \rangle$ called the **Hall inner product**. We define it by asserting $\langle h_\lambda, m_\mu \rangle = \delta_{\lambda,\mu}$ and extending linearly. That is, for symmetric functions $f, g \in \Lambda$ with expansions

$$f = \sum_\lambda f_\lambda h_\lambda, \quad g = \sum_\lambda g_\lambda m_\lambda$$

in the $\{h_\lambda\}$ and $\{m_\lambda\}$ bases, their inner product is defined as $\langle f, g \rangle = \sum_\lambda f_\lambda g_\lambda$. In general, if we have two bases of Λ, $\{u_\lambda\}$ and $\{v_\lambda\}$, we say that they are **dual bases** if $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda,\mu}$ holds for all partitions λ, μ. Dual bases have much utility due to the following proposition.

Proposition 2.3.1. *Let $\{u_\lambda\}$ and $\{v_\lambda\}$ be dual bases of Λ. Then for any symmetric function $f \in \Lambda$, the coefficient of u_μ in f is $\langle f, v_\mu \rangle$.*

Proof. Let $f = \sum_\lambda f_\lambda u_\lambda$ be the expansion of f in $\{u_\lambda\}$ in the basis. Then

$$\langle f, v_\mu \rangle = \left\langle \sum_\lambda f_\lambda u_\lambda, v_\mu \right\rangle = \sum_\lambda f_\lambda \langle u_\lambda, v_\mu \rangle = \sum_\lambda f_\lambda \delta_{\lambda,\mu} = f_\mu.$$

□

We now move onto the main basis of study in this thesis, the basis of Schur functions.

2.3.1 Schur functions and semistandard Young tableaux

The most important family of symmetric functions for this thesis is the basis of **Schur functions**. They may be defined combinatorially as the generating function of semistandard Young tableaux, which we introduce now.

A **Young tableau** T of shape λ/μ is a filling of the skew diagram of λ/μ with positive integers. A Young tableau is **semistandard** (SSYT) if the entries weakly increase left to right along rows and strictly increase bottom to top along columns. If the shape of an SSYT T is a partition (i.e. μ is the empty partition), we say T is **straight**. Otherwise we say T is **skew**. See below for some examples of SSYT. Here T is straight while T' is skew.

$$T = \begin{array}{|c|c|c|c|c|}\hline 4 & 5 & & & \\ \hline 2 & 2 & 4 & 5 & \\ \hline 1 & 1 & 3 & 4 & 4 \\ \hline \end{array} \qquad T' = \begin{array}{|c|c|c|c|c|}\hline 3 & 5 & 6 & & \\ \hline 2 & 4 & 4 & 5 & 6 \\ \hline \multicolumn{1}{c}{} & 1 & 2 & 4 & 4 \\ \cline{2-5} \multicolumn{3}{c}{} & 2 & 3 & 3 \\ \cline{4-6} \end{array}$$

The **row word** $\mathrm{rw}(T)$ of an SSYT T is the word obtained by recording the entries of T from left to right and top to bottom. The **weight** of an SSYT T is the weak composition $\mathtt{wt}(T) = (\mathtt{wt}(T)_1, \mathtt{wt}(T)_2, \ldots)$ where $\mathtt{wt}(T)_i$ counts the occurrences of i in T. We represent the weight of an SSYT T as the monomial $x^T = x^{\mathtt{wt}(T)}$. Let $\mathrm{SSYT}(\lambda/\mu)$ denote the set of all semistandard Young tableaux of shape λ/μ, let $\mathrm{SSYT}(\lambda/\mu, \alpha)$ denote the subset with weight α, and let $\mathrm{SSYT}_n(\lambda/\mu)$ denote the subset with entries at most n.

We may also define **crystal operators on SSYT**. For an SSYT T define $e_i(T)$, $f_i(T)$, and $r_i(T)$ as the Young tableau obtained from applying the classical crystal operators E_i, F_i, or R_i to the row word $\mathrm{rw}(T)$ of T and changing the entries in place. If E_i or F_i are not defined when applied to the row word, we say $e_i(T)$ or $f_i(T)$ are not defined.

Proposition 2.3.2. *When they are defined, $e_i(T)$, $f_i(T)$, and $r_i(T)$ are semistandard.*

Proof. If i and $i+1$ are in the same column, they will be matched in $\pi_i(\mathrm{rw}(T))$, preserving the column condition. Also, the unmatched i's are the rightmost i's on their row and the unmatched $i+1$'s are the leftmost $i+1$'s of their row. Thus, when these are changed by the crystal operator, the row condition is preserved. □

The **Schur functions** are defined as

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^T,$$

and form a basis for Λ since they satisfy the following upper triangularity condition with $\{m_\lambda\}$ under dominance order [17]

$$s_\lambda = m_\lambda + \sum_{\mu < \lambda} K_{\lambda,\mu} m_\mu.$$

where $K_{\lambda,\mu}$ is the cardinality of $\#\mathrm{SSYT}(\lambda, \mu)$. We also define **skew Schur functions** as

$$s_{\lambda/\mu} = \sum_{T \in \mathrm{SSYT}(\lambda/\mu)} x^T.$$

While these do not form a basis, they are nonetheless important to the theory, as we will see in Section 2.3.2.

From this definition, it is far from clear that $s_{\lambda/\mu}$ is symmetric. But this is simply a corollary of Proposition 2.3.2

Corollary 2.3.3. *The skew Schur function $s_{\lambda/\mu}$ is a symmetric function.*

Proof. We have that $\#\mathrm{SSYT}(\lambda/\mu, \alpha) = \#\mathrm{SSYT}(\lambda/\mu, s_i \cdot \alpha)$ since the crystal reflection operator r_i is a bijection between these sets by Proposition 2.3.2. The result follows by induction on the length of the permutation. □

This of course implies the Schur function s_λ is symmetric as well. The usual proof of Proposition 2.3.3 is done by defining a different bijection called the **Bender-Knuth involutions** [26]. These are similar to the crystal reflection operators but, importantly, do *not* give us an S_n-action on $\mathrm{SSYT}(\lambda/\mu)$ as they do not satisfy the braid relations. This gives some credence to the claim that the involutions r_i are "special" but we do not make this statement formal.

The basis $\{s_\lambda\}$ also enjoys the particularly nice property of being orthonormal with respect to the Hall inner product. That is $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$ for all λ, μ. This is equivalent to saying the basis $\{s_\lambda\}$ is **self-dual**. This amazing fact has some beautiful combinatorics behind it (see the Robinson-Schensted-Knuth correspondence) but we omit its proof as it is outside the scope of this thesis. We refer the reader to [4].

Let $K_{\lambda/\mu,\nu}$ denote the number of SSYT's of shape λ/μ and weight ν. These are called the **skew Kostka numbers** are are the coefficients appearing in the monomial expansion of $s_{\lambda/\mu}$. Indeed,

$$s_{\lambda/\mu} = \sum_\nu K_{\lambda/\mu,\nu} m_\nu.$$

Proposition 2.3.4. *The complete homogeneous symmetric function h_λ expands in the $\{s_\lambda\}$ basis as*

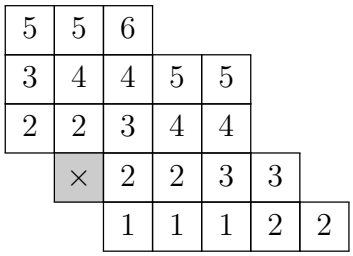
$$h_\lambda = \sum_\mu K_{\mu,\lambda} s_\mu.$$

Proof. The fact that the basis $\{s_\lambda\}$ is self-dual along with Proposition 2.3.1 imply that the coefficient of s_μ in h_λ is $\langle h_\lambda, s_\mu \rangle$. If we instead apply Proposition 2.3.1 with $\{m_\lambda\}$ and $\{h_\lambda\}$ as the dual bases, we get that the coefficient of m_λ in s_μ is $\langle s_\mu, h_\lambda \rangle = \langle h_\lambda, s_\mu \rangle$. But of course, from the first claim, this is precisely $K_{\mu,\lambda}$. □

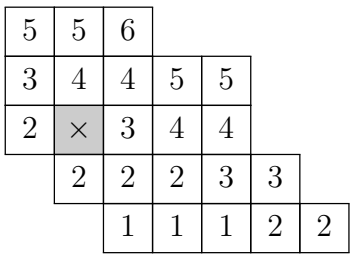
We now discuss **sliding** or **jeu de taquin** (JDT), a procedure that takes in a skew SSYT and returns one that is "a little less skew".

A **corner** of a partition μ is a box in the diagram of μ that has no other box of μ above it or to its right. The JDT procedure goes as follows. Given a skew SSYT of shape λ/μ pick a corner a of μ. Then the corner a swaps repeatedly with the box above it or the box to its right until neither exist. If both exist, it swaps with the box that contains the smaller entry, with ties being broken by the one above it. See Example 2.3.5.

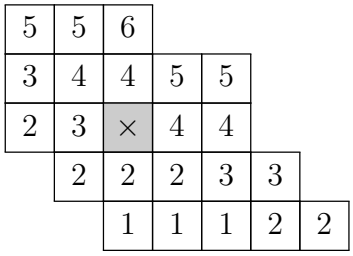
Example 2.3.5. *Let T be the SSYT seen below with the chosen corner marked with "$\times$".*



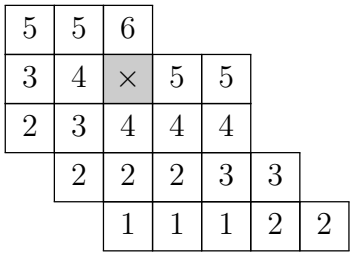
There is a 2 above $\times$ and a 2 to the right of $\times$. In the case of a tie, $\times$ swaps places with the box above it.



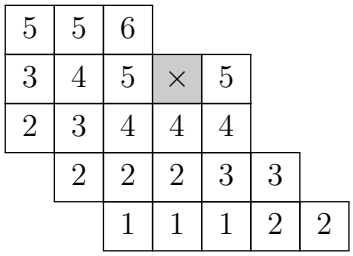
There is a 4 above $\times$ and a 3 to the right of $\times$. The 3 is smaller so $\times$ swaps places with the box to its right.



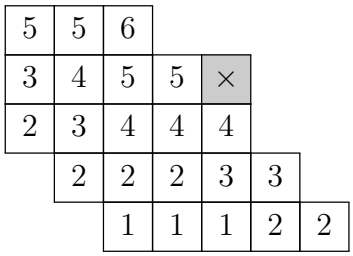
There is a 4 above $\times$ and a 4 to the right of $\times$. Again, in the case of a tie, $\times$ swaps places with the box above it.



There is a 6 *above* $\times$ *and a* 5 *to the right of* $\times$*. The* 5 *is smaller so* $\times$ *swaps places with the box to its right.*



At this point, there is no box above $\times$ *so it swaps places with the box to its right.*

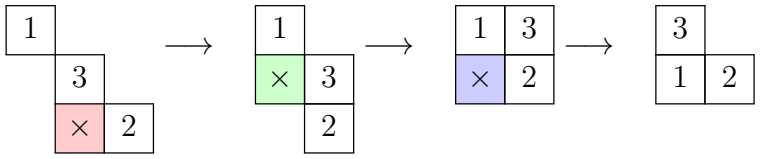


Finally, there is no box above or to the right of $\times$ *so we have completed the slide.*

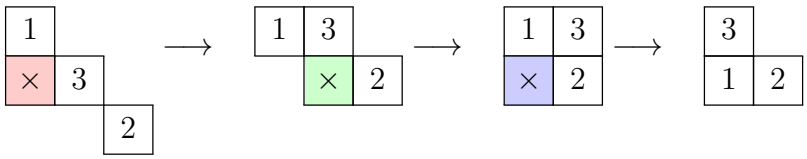
If we keep track of where the corner ends up, it is not difficult to see that this process is actually reversible. Indeed, the reverse process is as follows: the ending corner a swaps repeatedly with with the box below it or the box to its left until neither exists. If both exist, it swaps with the box containing the larger entry, with ties being broken by the one below it.

We may take any skew SSYT and perform slides until we end up with a straight SSYT. See Example 2.3.6.

Example 2.3.6. *The following is a choice of slides that send a skew tableau* T *to a straight SSYT. At each step, we mark the chosen corner with* $\times$ *and a different color.*



We could have instead chosen a different sequence of slides to send T *to a straight SSYT.*



But either choice of slides sends T to the same straight SSYT.

The resulting straight SSYT is called the **rectification** of T and is denoted by $\texttt{rect}(T)$. It is an amazing theorem that $\texttt{rect}(T)$ is well-defined, that is, the resulting straight tableau is independent of the sequence of slides. Again, the proof of this is beyond the scope of this thesis. We refer the reader to [26].

We will mention one more amazing theorem related to slides. The reader may note that while slides change the shape of an SSYT but preserve the content, crystal operators preserve the shape of an SSYT but change the content.

Theorem 2.3.7. *Slides commute with crystal operators.*

The proof of this is also beyond the scope of this thesis, see [16]. The last thing we will discuss in this section is the notion of a crystal graph.

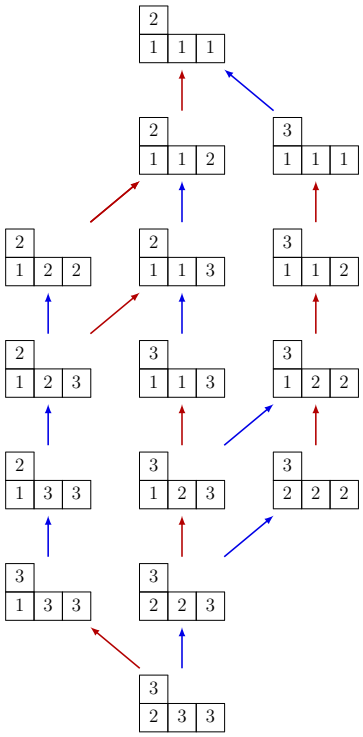


Figure 2.1: The crystal graph $\Gamma((3,1))$ truncated to $\mathrm{SSYT}_3((3,1))$. The red edges correspond to e_1 and the blue edges correspond to e_2.

Definition 2.3.8. *The **crystal graph** $\Gamma(\lambda/\mu)$ is the directed, edged-colored graph with vertex set* $\mathrm{SSYT}(\lambda/\mu)$*, and colored edges* $T \xrightarrow{i} e_i(T)$*.*

Under this definition, $\Gamma(\lambda/\mu)$ has an infinite number of vertices. We may truncate $\Gamma(\lambda/\mu)$ by taking the subgraph induced by the vertex set $\mathrm{SSYT}_n(\lambda/\mu)$ for some n. We do so when we wish to visualize this graph but otherwise, we keep it infinite as the proofs do not change. See Figure 2.1 for an example of such a truncated crystal graph.

An SSYT T is a **Littlewood–Richardson tableau** (or LR tableau) if $e_i(T) = \varnothing$ for all i. In otherwords, the LR tableaux are precisly the sinks of $\Gamma(\lambda/\mu)$. Let $\mathrm{LRT}(\lambda/\mu, \nu)$ denote the set of all LR tableaux of shape λ/μ and weight ν. Lemma 2.3.9 classifies the sinks of the crystal graph $\Gamma(\nu)$.

Lemma 2.3.9. *If ν is a partition, then $\Gamma(\nu)$ has a unique sink, which is the unique SSYT of shape ν and content ν.*

Proof. Suppose that T is a sink of $\Gamma(\nu)$. It suffices to prove that every entry of T on row i is an i. We proceed by induction on the number of rows. For the base case, assume the last entry in the first row is $i > 1$. Then this entry will not be matched in $\pi_{i-1}(\mathrm{rw}(T))$ implying that $e_{i-1}(T) = U$ is defined. Therefore, the crystal $\Gamma(\nu)$ has an edge $T \xrightarrow{i-1} U$ so T is not a sink. Now assume the first $k-1$ rows of T have entries matching their row index. Assume that the last entry in the kth row is $i > k$. Then this entry will not be matched in $\pi_{i-1}(\mathrm{rw}(T))$ since all entries after it in the row word of T are strictly less than $k - 1 < i - 1$ by the induction hypothesis. This implies that $e_{i-1}(T) = U$ is defined and again proves that T is not a sink of the crystal $\Gamma(\nu)$. Therefore, T must have shape and content ν, as desired. □

This gives us some immediate corollaries that we will use to prove the Littlewood–Richardson rule, which we introduce in Section 2.3.2.

Corollary 2.3.10.

(a) The crystal $\Gamma(\nu)$ is connected.

(b) If $\mathcal{C}$ is a connected component of $\Gamma(\lambda/\mu)$, then $\mathcal{C} \cong \Gamma(\nu_{\mathcal{C}})$ for some partition $\nu_{\mathcal{C}}$ with the isomorphism being $T \to \mathtt{rect}(T)$.

(c) We have a bijection between components of $\Gamma(\lambda/\mu)$ such that $\nu_{\mathcal{C}} = \nu$ and $\mathrm{LRT}(\lambda/\mu, \nu)$.

Proof. (a) follows from Lemma 2.3.9 since for any two vertices, $T, U \in \Gamma(\nu)$, there is a path from T to the unique sink of $\Gamma(\nu)$ and from U to the same sink. (b) follows from Theorem 2.3.7 and the fact that slides are reversible. (c) follows from Lemma 2.3.9 and (b). □

2.3.2 The Littlewood–Richardson rule

In this section, we provide a crystal proof of the **Littlewood–Richardson rule**, a positive combinatorial rule for the coefficient of s_ν in the skew Schur function $s_{\lambda/\nu}$.

Theorem 2.3.11 (The Littlewood–Richardson rule)**.** *The coefficient $c^\lambda_{\mu,\nu}$ in the expansion*

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu,\nu} s_\nu.$$

is equal to $\#\mathrm{LRT}(\lambda/\mu, \nu)$. In particular, $c^\lambda_{\mu,\nu} \geq 0$.

Proof of Theorem 2.3.11. By definition, we have

$$s_{\lambda/\mu} = \sum_{T \in \mathrm{SSYT}(\lambda/\mu)} x^T = \sum_{T \in V(\Gamma(\lambda/\mu))} x^T \tag{2.1}$$

Splitting this sum over connected components $\mathcal{C}$ of $\Gamma(\lambda/\mu)$ and applying the content preserving isomorphism $T \to \mathtt{rect}(T)$ from Corollary 2.3.10 (b) to each connected component, (2.1)

$$\sum_{\mathcal{C}} \sum_{T \in V(\mathcal{C})} x^T = \sum_{\mathcal{C}} \sum_{T \in \mathrm{SSYT}(\nu_\mathcal{C})} x^T \tag{2.2}$$

The inner sum on the RHS of (2.2) is by definition equal to the Schur function $s_{\nu_\mathcal{C}}$ and so by applying the bijection from Corollary 2.3.10 (c), (2.2) is equal to

$$\sum_{\mathcal{C}} s_{\nu_\mathcal{C}} = \sum_\nu \sum_{T \in \mathrm{LRT}(\lambda/\mu, \nu)} s_\nu = \sum_\nu \#\mathrm{LRT}(\lambda/\mu, \nu) s_\nu$$

completing the proof. □

2.3.3 The Jacobi–Trudi formula

In this section, we provide a crystal proof of the Jacobi–Trudi formula, a determinantal formula that expresses a Schur function s_λ in the $\{h_\lambda\}$ basis. This proof is based on [25], where they used this technique to prove the LR rule. We present their proof of the LR rule in the next section but build up their ideas in the context of the Jacobi–Trudi formula.

Theorem 2.3.12 (The Jacobi–Trudi formula)**.** *For a partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$

$$s_\lambda = \det(h_{\lambda_i+j-i})_{i,j=1}^{\ell}$$

where we set $h_0 = 1$ *and* $h_{-1} = h_{-2} = \cdots = 0$.

Example 2.3.13. *By the Jacobi–Trudi formula,*

$$s_{(2,1)} = \begin{pmatrix} h_2 & h_3 \\ h_0 & h_1 \end{pmatrix} = \begin{pmatrix} h_2 & h_3 \\ 1 & h_1 \end{pmatrix} = h_{21} - h_3.$$

We will prove this in the following three steps: first, we rewrite the determinant as a signed sum of h_μ's. Then we expand each h_μ in the $\{s_\lambda\}$ basis. Lastly, we construct a sign reversing involution to cancel every Schur function except for s_λ.

Let $\rho^{(n)} = (n-1, n-2, \ldots, 1, 0)$ denote the staircase partition. When n is clear from context, we drop the superscript, simply writing ρ. For $\pi \in S_n$, we define $\pi \bullet \alpha = \pi \cdot (\alpha + \rho) - \rho$ for any weak composition α of length n. This is referred to as the dot action in the literature (see [1, 22]). For us, it gives a compact way of rewriting the determinant in the Jacobi–Trudi formula.

Lemma 2.3.14. *For a partition* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$,

$$\det(h_{\lambda_i+j-i})_{i,j=1}^{n} = \sum_{\pi \in S_n} \mathtt{sgn}(\pi) h_{\pi \bullet \lambda}.$$

Proof. Expanding the determinant,

$$\det(h_{\lambda_i+j-i})_{i,j=1}^{n} = \sum_{\pi \in S_n} \mathtt{sgn}(\pi) \prod_{i}^{n} h_{\lambda_{\pi^{-1}(i)}+i-\pi^{-1}(i)},$$

whereas $(\pi \bullet \lambda)_i = \lambda_{\pi^{-1}(i)} + (n - \pi^{-1}(i)) - (n-i) = \lambda_{\pi^{-1}(i)} + i - \pi^{-1}(i)$. □

Next, we use Proposition 2.3.4 to expand these $h_{\pi \bullet \lambda}$'s in the $\{s_\lambda\}$ basis.

Lemma 2.3.15. *For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$,*

$$\det(h_{\lambda_i+j-i})_{i,j=1}^n = \sum_{\mu} \sum_{\substack{(\pi,T) \\ \pi \in S_n \\ T \in \mathrm{SSYT}(\mu, \pi \bullet \lambda)}} \mathtt{sgn}(\pi) s_\mu.$$

Proof. Combining Lemma 2.3.14 and Proposition 2.3.4, and expanding the skew Kostka number, we obtain

$$\det(h_{\lambda_i+j-i})_{i,j=1}^n = \sum_{\pi \in S_n} \mathtt{sgn}(\pi) \sum_{\mu} K_{\mu, \pi\bullet\lambda} s_\mu = \sum_{\mu} \sum_{\substack{(\pi,T) \\ \pi \in S_n \\ T \in \mathrm{SSYT}(\mu, \pi \bullet \lambda)}} \mathtt{sgn}(\pi) s_\mu.$$

□

What is left is for us to construct a sign reversing involution on the set

$$V(\lambda) = \bigcup_{\mu} \{(\pi, T) : \pi \in S_n, T \in \mathrm{SSYT}(\mu, \pi \bullet \lambda)\},$$

whose single fixed point is of the form (id, T) where T is the unique straight LR tableau of shape μ. The basic building block of this involution will be the composition of crystal operators $r_i e_i$ which we call the **raised reflection operators**. As we will see in Lemma 2.3.16, it is an involution when defined and acts on the weight of an SSYT via the dot action.

Lemma 2.3.16. *For an SSYT T such that $e_i(T)$ is defined,*

(a) $r_i e_i(T) = f_i r_i(T)$,

(b) $(r_i e_i)^2(T) = T$,

(c) $\mathtt{wt}(r_i e_i(T)) = s_i \bullet \mathtt{wt}(T)$.

Proof. For (a), let $k = \varepsilon_i(T) - \phi_i(T)$ and $k' = \varepsilon_i(e_i(T)) - \phi_i(e_i(T))$ and note $k' = k - 2$. Proposition 2.2.4 implies

$$r_i e_i(T) = \begin{cases} e_i^{k'}(e_i(T)) & \text{if } k' \geq 0, \\ f_i^{-k'}(e_i(T)) & \text{if } k' \leq 0. \end{cases} = \begin{cases} e_i^{k-1}(T) & \text{if } k \geq 2, \\ f_i^{-k+1}(T) & \text{if } k \leq 2. \end{cases} = \begin{cases} e_i^{k-1}(T) & \text{if } k \geq 1, \\ f_i^{-k+1}(T) & \text{if } k \leq 1. \end{cases}$$

On the other hand, Proposition 2.2.4 also implies

$$f_i r_i(T) = \begin{cases} f_i(e_i^k(T)) & \text{if } k \geq 0, \\ f_i(f_i^{-k}(T)) & \text{if } k \leq 0. \end{cases} = \begin{cases} e_i^{k-1}(T) & \text{if } k \geq 0, \\ f_i^{-k+1}(T) & \text{if } k \leq 0. \end{cases} = \begin{cases} e_i^{k-1}(T) & \text{if } k \geq 1, \\ f_i^{-k+1}(T) & \text{if } k \leq 1. \end{cases}$$

Now (b) follows directly from (a) as e_i and f_i are inverses when defined, and r_i is its own inverse. Lastly, we prove (c). Let $\mathbf{e}_i$ denote the ith standard basis vector. Then,

$$\begin{aligned} \mathtt{wt}(r_i e_i(T)) &= s_i \cdot (\mathtt{wt}(T) + \mathbf{e}_i - \mathbf{e}_{i+1}) \\ &= s_i \cdot (\mathtt{wt}(T) + \mathbf{e}_i - \mathbf{e}_{i+1}) + \rho - \rho \\ &= s_i \cdot (\mathtt{wt}(T) + \mathbf{e}_i - \mathbf{e}_{i+1} + s_i \cdot \rho) - \rho \\ &= s_i \cdot (\mathtt{wt}(T) + \rho) - \rho \\ &= s_i \bullet \mathtt{wt}(T). \end{aligned}$$

The only step of interest is jumping from line three to four. This follows from the fact that $s_i \cdot \rho = \rho + \mathbf{e}_{i+1} - \mathbf{e}_i$ so we may replace the $\mathbf{e}_i - \mathbf{e}_{i+1} + s_i \cdot \rho$ with ρ. □

For the rest of this thesis, we use the notation $\widehat{r}_i = r_i e_i$ to refer to the raised reflection operators.

Before defining the involution, we give a slightly different characterization of LR tableaux. We say a word w is **reverse lattice** if every suffix of w contains at least as many i's as $i+1$'s for all i. It is a fact that a (possibly skew) SSYT T is LR if and only if $\mathrm{rw}(T)$ is reverse lattice.

Definition 2.3.17. *For a word w that is not reverse lattice, we let $\mathtt{fail}(w)$ denote the rightmost entry of w such that the shortest suffix containing $\mathtt{fail}(w)$ is not reverse lattice. For any SSYT T that is not LR, define $\mathtt{fail}(T) = \mathtt{fail}(\mathrm{rw}(T))$.*

We may now define the involution. We do so in the most general setting (skew shapes, any content) and specialize it to $V(\lambda)$ after.

Definition 2.3.18 (The Remmel-Shimozono involution)**.** *Define a map $\mathcal{RS} : \mathrm{SSYT}(\lambda/\mu) \to \mathrm{SSYT}(\lambda/\mu)$ as follows: if T is LR, then $\mathcal{RS}(T) = T$, if T is not LR, then $\mathcal{RS}(T) = \widehat{r}_i(T)$ where $i = \mathtt{fail}(T) - 1$.*

Now we must prove that $\mathcal{RS}$ is indeed an involution.

Lemma 2.3.19. *$\mathcal{RS}$ is an involution.*

Proof. Let $T \in \mathrm{SSYT}(\lambda/\mu)$ be an SSYT that is not LR and let $i = \mathtt{fail}(T)-1$. By Lemma 2.3.16 (a), it is enough to show that $e_i(T)$ is defined and that $\mathtt{fail}(\mathcal{RS}(T)) = \mathtt{fail}(T)$. For the first point, note that by definition $\mathtt{fail}(T)$ has no unmatched i's to its right in $\mathrm{rw}(T)$ and is therefore unmatched in $\pi_i(\mathrm{rw}(T))$.

For the second point, note that $\mathtt{fail}(T)$ is the rightmost unmatched $i+1$ in $\mathrm{rw}(T)$ and is therefore not changed when applying $\widehat{r}_i$. Further, everything to the right of $\mathtt{fail}(T)$ in $\mathrm{rw}(T)$ is matched in $\pi_i(\mathrm{rw}(T))$ and therefore also does not change when applying $r_i e_i$. □

Proof of Theorem 2.3.12. Define the map $\mathcal{RS}' : V(\lambda) \to V(\lambda)$ by $(\pi, T) \to (s_i\pi, \mathcal{RS}(T))$. This is clearly sign reversing, it is well-defined by Lemma 2.3.16 (c), and it is an involution by Lemma 2.3.19. Moreover, the fixed points are the LR tableaux of shape μ but by Lemma 2.3.9, there is only one such tableau and it has content μ. Thus, $\mathcal{RS}'$ has one fixed point, (id, T) where T has content μ. Applying $\mathcal{RS}'$ to the terms in the sum in Lemma 2.3.15 completes the proof. □

2.4 Nonsymmetric polynomials

In this section, we restrict ourselves to a finite set of variables $\{x_1, \ldots, x_n\}$ and discuss symmetric polynomials rather than symmetric functions in X. Let Λ_n denote the set of symmetric polynomials in the variables $\{x_1, \ldots, x_n\}$, bases of which are indexed by partitions of length at most n. A **nonsymmetric polynomial** is simply any polynomial in $\mathbb{Q}[x_1, \ldots, x_n]$. Bases of this space are indexed by weak compositions of length n and may be related to bases of the space of symmetric polynomials as follows. Let $\{A_\alpha\}$ and $\{B_\lambda\}$ be bases of $\mathbb{Q}[x_1, \ldots, x_n]$ and Λ_n respectively. Then we say $\{A_\alpha\}$ is a **nonsymmetric refinement** of $\{B_\lambda\}$ if the following is true for all partitions λ of length at most n,

$$B_\lambda = \sum_{\mathtt{sort}(\alpha)=\lambda} A_\alpha.$$

For example, the set of all monomials forms a basis for $\mathbb{Q}[x_1, \ldots, x_n]$ and is a nonsymmetric refinement of $\{m_\lambda\}$ in the variables $\{x_1, \ldots, x_n\}$. The focus of this section will be on certain nonsymmetric functions $\mathcal{A}_\alpha$ called **Demazure atoms**. They form a basis of $\mathbb{Q}[x_1, \ldots, x_n]$ and, as we will see, are a nonsymmetric refinement of $\{s_\lambda\}$.

2.4.1 Demazure atoms and semistandard augmented fillings

Demazure atoms were originally introduced by Lascoux and Schützenberger [15], where they were defined via certain divided difference operators. Later, Mason [21] proved that

they have a combinatorial definition as the generating functions of semistandard augmented fillings, which we recall now.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a weak composition with largest part m. An **augmented filling** F of shape α is a filling of the augmented skyline diagram of α with positive integers. Let $F(r, i)$ denote the entry of F in row r and column i. If there is no such box in F, let $F(r, i) = 0$. Importantly, we use the convention that the basement is the zeroth row. Then an augmented filling F is **semistandard** if the following three conditions hold:

- The entries in the columns of F weakly increase when read top to bottom,
- The entries in the first row of F match their column index.
- *(Triple Rule)* For $2 \leq r \leq m$ and $1 \leq i < j \leq n$, if $F(r, j) \neq 0$ and $F(r, i) \leq F(r, j)$, then this implies $F(r-1, i) < F(r, j)$.

We will use SSAF to abbreviate "semistandard augmented filling". Pictorially, we may check the triple rule by choosing a box of F containing an entry a and scanning that row from left to right, checking all triples of the form

$$\begin{array}{|c|}\hline b \\ \hline c \\ \hline \end{array} \qquad \begin{array}{|c|}\hline a \\ \hline \end{array}$$

to confirm that if $b \leq a$ then $c < a$. Note that, importantly, b and c may be zero or in other words, not part of the augmented skyline diagram. Also note that we need not check the triple rule for any boxes that are in the basement (this is inherent in the third bullet point).

Remark 2.4.1. *This is not the original definition of SSAFs that appears in Mason's paper [20]. In her paper, the definition of SSAFs requires checking two kinds of triples instead of one. Instead, the definition we have given comes from [12] in which they define related objects called* ***composition tableaux*** *which are more suitable to study certain quasisymmeric functions. They show that these objects are simply SSAFs with their basement removed and their columns compressed. Thus, we may translate their definition of composition tableaux to give an alternative definition of SSAFs. The only real changes we need to make is to add the basement and change the second bullet point. In [12], their second bullet point states that the entries of the the first row strictly increase when read left to right.*

The **row word** $\mathrm{rw}(F)$ of an SSAF F is the word obtained by recording the entries of F from left to right and top to bottom. The **weight** of an SSAF F is the weak composition $\mathtt{wt}(F) = (\mathtt{wt}(F)_1, \mathtt{wt}(F)_2, \ldots)$ where $\mathtt{wt}(F)_i$ counts the occurrences of i in F *not including*

basement entries. Let $\mathrm{SSAF}(\alpha)$ denote the set of all semistandard augmented fillings of shape α.

Then the **Demazure atoms** are defined as

$$\mathcal{A}_\alpha = \sum_{F \in \mathrm{SSAF}(\alpha)} x^{\mathtt{wt}(F)}$$

and form a basis for $\mathbb{Q}[x_1, \ldots, x_n]$ as they satisfy an upper triangularity condition with the basis of monomials similar to the one that Schur functions satisfy:

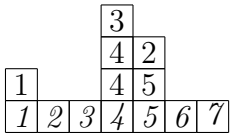
$$\mathcal{A}_\alpha = x^\alpha + \sum_{\beta < \alpha} c_{\alpha,\beta} x^\beta.$$

2.4.2 Skyline insertion

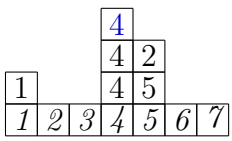
To show that $\{\mathcal{A}_\alpha\}$ is a nonsymmetric refinement of $\{s_\lambda\}$ we need to define an insertion procedure on SSAFs, called **skyline insertion** introduced in [20]. Given an SSAF F and a positive integer a, define $a \to F$ as the augmented filling obtained by the following procedure. Scan F in reading order (left to right, top to bottom) to find the first entry b that is greater than or equal to a. If there is no entry on top of b, place a on top of b to complete the insertion. If the entry directly above b is weakly greater than a, continue to the next entry that is greater than or equal to a and repeat. If the entry c directly above b is strictly less than a, replace it with a and continue to the next entry that is greater than or equal to c and repeat (this time with the new entry c).

Example 2.4.2. *We show the sequence of steps of inserting a* 4 *into the SSAF* F *seen below.*

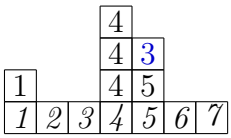
We include the basement entries in italics. We highlight the entry we are actively inserting in red, and the entry we just changed in blue.



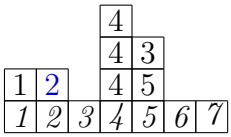
To insert 4*, we scan* F *left to right, top to bottom. The first entry we come across that is greater than or equal to* 4 *is the* 4 *on the second row. Looking above it, we see* 3 *which is weakly greater than* 4*, so we replace this* 3 *with our* 4*.*



We continue to scan, this time inserting 3 *starting at the* 2 *in the second row. The first entry we come across that is greater than or equal to* 3 *is the* 4 *on the first row. Looking above it, we see* 4 *which is not weakly greater than* 3 *and so we continue. The next entry we come across is the* 5 *in the first row. Looking above it, we see a* 2 *which is weakly greater than* 3*, so we replace this* 2 *with our* 3*.*



We continue to scan, this time inserting 2 *starting at the* 1 *in the basement. The first entry we come across that is greater than or equal to* 2 *is the* 2 *in the basement. Looking above it, we see nothing, so we place the* 2*.*



This completes the insertion yielding $4 \to F$*.*

In [20], Mason proves that the resulting augmented filling $F \leftarrow a$ is indeed semistandard and defines a bijection

$$\Psi : \mathrm{SSYT}(\lambda) \to \bigcup_{\mathtt{sort}(\alpha)=\lambda} \mathrm{SSAF}(\alpha)$$

which we describe now. Given an SSYT T, consider its row word $\mathrm{rw}(T)$ and an empty SSAF ε with basement containing all positive integers. Insert the rightmost letter a_1 in $\mathrm{rw}(T)$ to the the empty SSAF to obtain $a_1 \to \varepsilon$. Then let a_2 be the next rightmost letter in $\mathrm{rw}(T)$ and insert this into $a_1 \to \varepsilon$ to obtain $a_1 \to (a_1 \to \varepsilon)$. Continue this process until every letter in $\mathrm{rw}(T)$ is inserted. The outcome is an SSAF whose shape is a rearrangement of λ.

Corollary 2.4.3. *[20, 21] The Demazure atoms* $\{\mathcal{A}_\alpha\}$ *are a nonsymmetric refinement of the Schur polynomials* $\{s_\lambda\}$*. That is,*

$$s_\lambda = \sum_{\mathtt{sort}(\alpha)=\lambda} \mathcal{A}_\alpha.$$

Mason also used skyline insertion to define **crystal operators on SSAFs**. In the following proposition, we let SSAF also denote the set of all SSAFs.

Proposition 2.4.4. *[21] There exists a map $\tilde{e}_i : \mathrm{SSAF} \to \mathrm{SSAF}$ such that the following diagram commutes for all SSYT T,*

$$\begin{array}{ccc} T & \xrightarrow{e_i} & T' \\ \Psi \downarrow & & \downarrow \Psi \\ F & \xrightarrow[\tilde{e}_i]{} & F' \end{array}$$

Further, if $F \in \mathrm{SSAF}(\alpha)$, then either $\tilde{e}_i(F) \in \mathrm{SSAF}(\alpha)$ or $\tilde{e}_i(F) \in \mathrm{SSAF}(s_i \cdot \alpha)$

In particular, this gives us a way to partition the crystal graph $\Gamma(\lambda)$ into nonsymmetric subgraphs, each of which corresponding to a Demazure atom. More formally, the **nonsymmetric subgraph** of $\Gamma(\lambda)$ corresponding to α is the subgraph induced by $\{T : \Psi(T) \in \mathrm{SSAF}(\alpha)\}$. We denote this subgraph by $\Gamma(\lambda; \alpha)$ and give some examples in Figure 2.2. The $\Gamma(\lambda; \alpha)$'s correspond to a Demazure atom in the sense that

$$\sum_{F \in V(\Gamma(\lambda;\alpha))} x^F = \mathcal{A}_\alpha.$$

This subgraph is more difficult to study because the crystal operators on SSAFs do not preserve the shape of an SSAF. It is because of this that we are motivated to study when crystal operators change the shape of an SSAF. In Chapter 4, we will give a characterization of exactly this, going through related combinatorial objects called multiline queues.

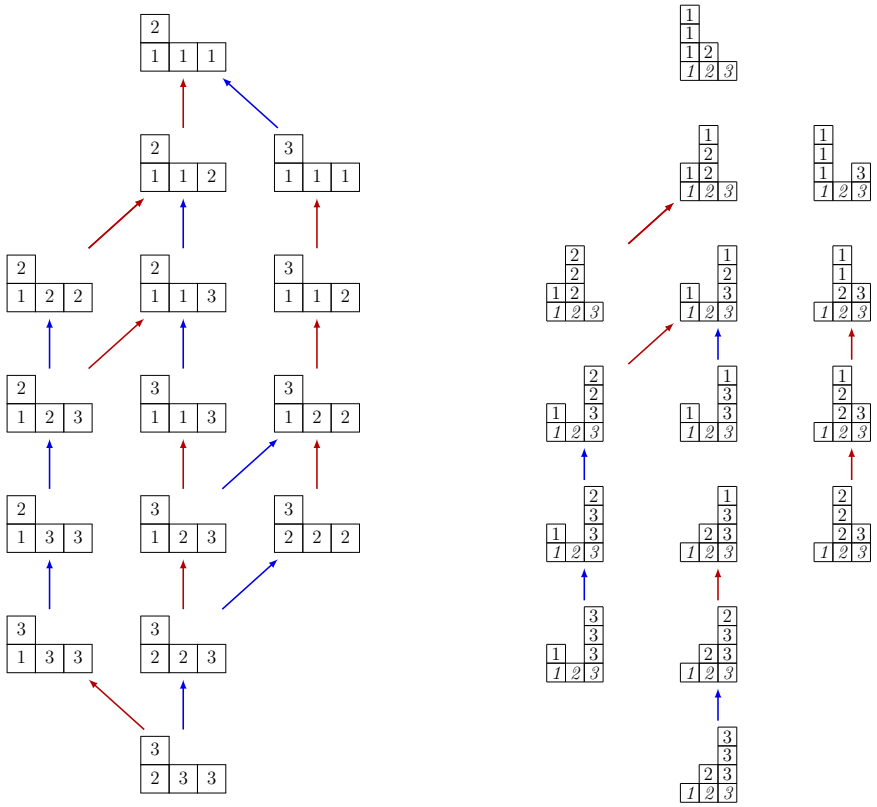


Figure 2.2: The crystal graph appearing in Figure 2.1 along with the six nonsymmetric subgraphs obtained by removing any shape-changing edges.

Chapter 3

Reflected crystal graphs

In [25], Remmel and Shimozono prove the Littlewood–Richardson Rule using the sign-reversing involution $\mathcal{RS}$. We begin this chapter by presenting their proof. The remainder of the chapter is dedicated to studying the graph defined by the $\widehat{r}_i$'s used to define $\mathcal{RS}$. As our main result, we prove that this graph is balanced bipartite, giving another proof of the LR rule that avoids having to construct an explicit involution.

3.1 Remmel and Shimozono's proof

In this section we present Remmel and Shimozono's proof of the LR rule. Most of the setup has been done in Section 2.3.3. The biggest step is to interpret $c^{\lambda}_{\mu\nu}$ as a signed sum. To do this, we apply the Jacobi–Trudi formula and a property of the Hall inner product.

Proposition 3.1.1. *For partitions λ, μ, ν, we have that*

$$c^{\lambda}_{\mu,\nu} = \sum_{\substack{(\pi,T)\\ \pi\in S_n\\ T\in \mathrm{SSYT}(\lambda/\mu,\pi\bullet\nu)}} \mathtt{sgn}(\pi).$$

Proof. By Proposition 2.3.1, $c^{\lambda}_{\mu\nu} = \langle s_{\lambda/\mu}, s_\nu\rangle$. Expanding $s_{\lambda/\mu}$ in the $\{m_\lambda\}$ basis and expanding s_ν in the $\{h_\lambda\}$ basis by the Jacobi–Trudi formula, we have

$$\langle s_{\lambda/\mu}, s_\nu\rangle = \left\langle \sum_\theta K_{\lambda/\mu,\theta} m_\theta, \sum_{\pi\in S_n} \mathtt{sgn}(\pi) h_{\pi\bullet\nu}\right\rangle = \sum_\theta \sum_{\pi\in S_n} \mathtt{sgn}(\pi) K_{\lambda/\mu,\theta}\langle m_\theta, h_{\pi\bullet\nu}\rangle.$$

Since $\{m_\lambda\}$ and $\{h_\lambda\}$ are dual bases, $\langle m_\theta, h_{\pi\bullet\nu}\rangle = \delta_{\theta,\pi\bullet\nu}$ yielding

$$\langle s_{\lambda/\mu}, s_\nu\rangle = \sum_{\pi\in S_n} \mathtt{sgn}(\pi) K_{\lambda/\mu,\pi\bullet\nu} = \sum_{\substack{(\pi,T)\\ \pi\in S_n\\ T\in \mathrm{SSYT}(\lambda/\mu,\pi\bullet\nu)}} \mathtt{sgn}(\pi).$$

□

Proof of Theorem 2.3.11. What we must do is define a sign-reversing involution on the set

$$U^\lambda_{\mu,\nu} = \{(\pi, T) : \pi \in S_n, T \in \mathrm{SSYT}(\lambda/\mu, \pi \bullet \nu)\}.$$

as this is the set the sum in Proposition 3.1.1 is over. Define the map $\mathcal{RS}'' : U^\lambda_{\mu,\nu} \to U^\lambda_{\mu,\nu}$ by $(\pi, T) \to (s_i\pi, \mathcal{RS}(T))$. This is a well-defined, sign-reversing involution on $U^\lambda_{\mu,\nu}$ for the same reasons $\mathcal{RS}'$ is. Further, its fixed points are (id, T) where T is an LR tableaux of shape λ/μ and content ν. Thus, $c^\lambda_{\mu,\nu} = \#\mathrm{LRT}(\lambda/\mu, \nu)$. □

3.2 Reflected crystal graphs

In this section we study the graph whose vertices are a certain subset of $\mathrm{SSYT}(\lambda/\mu)$ and whose edges are defined by the $\widehat{r}_i$'s used to define $\mathcal{RS}$, which we term the **reflected crystal graph**. We offer an alternative proof of the Littlewood–Richardson rule that is more graph theoretic in nature and circumvents the need to define an explicit sign-reversing involution.

Recall the definition of $U^\lambda_{\mu,\nu}$. The elements of this set are pairs (π, T) but this next proposition shows that we may determine π from ν and T.

Proposition 3.2.1. *For a partition λ and permutations $\pi, \sigma \in S_n$, $\pi \bullet \lambda = \sigma \bullet \lambda$ if and only if $\pi = \sigma$.*

Proof. If $\pi = \sigma$, it is clear that $\pi \bullet \lambda = \sigma \bullet \lambda$. Thus, assume that $\pi \neq \sigma$. Then, without loss of generality, there exists an i such that $\pi^{-1}(i) < \sigma^{-1}(i)$. As λ is a partition, this means $\lambda_{\pi^{-1}(i)} \geq \lambda_{\sigma^{-1}(i)}$ and so

$$(\pi \bullet \lambda)_i = \lambda_{\pi^{-1}(i)} + i - \pi^{-1}(i) > \lambda_{\sigma^{-1}(i)} + i - \sigma^{-1}(i) = (\sigma \bullet \lambda)_i.$$

□

Define the set $V^{\lambda}_{\mu,\nu} = \bigcup_{\pi \in S_n} \mathrm{SSYT}(\lambda/\mu, \pi \bullet \nu)$ and for $T \in V^{\lambda}_{\mu,\nu}$ let π_T be the permutation such that $\mathtt{wt}(T) = \pi_T \bullet \nu$ which was proved to be unique by Proposition 3.2.1. Then for any $(\pi, T) \in U^{\lambda}_{\mu,\nu}$, $\pi = \pi_T$. This is really just another way of keeping track of the same information but it will be more connivent for our proofs. We may now define the main object of study for this chapter.

Definition 3.2.2. *For partitions λ, μ, ν, the **reflected crystal graph** $\Gamma^{\lambda}_{\mu,\nu}$ is the directed, edge-colored graph with vertex set $V^{\lambda}_{\mu,\nu}$ and colored edges $T \xrightarrow{i} \widehat{r}_i(T)$ whenever $\mathtt{wt}(T)_i \leq \mathtt{wt}(T)_{i+1}$.*

Note that since $\widehat{r}_i$ is an involution when defined, the edge coloring of $\Gamma^{\lambda}_{\mu,\nu}$ is proper; no vertex is incident to two edges of the same color. See Figure 3.1 for an example of this graph.

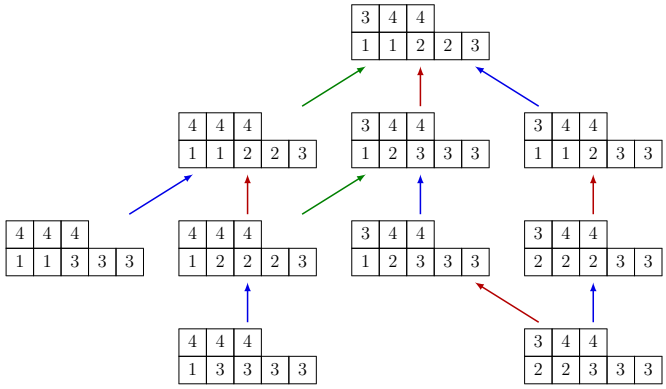


Figure 3.1: A connected component of the refelcted crystal graph $\Gamma^{(5,3)}_{\varnothing,(2,2,2,2)}$. The red edges correspond to $\widehat{r}_1$, the blue edges correspond to $\widehat{r}_2$, and the green edges correspond to $\widehat{r}_3$.

It is here where we depart from Remmel and Shimozono's proof of the LR rule. Instead of constructing an explicit sign reversing involution on the set $V^{\lambda}_{\mu,\nu}$, we prove simply that such an involution exists in the following theorem.

Theorem 3.2.3. *Every connected component of $\Gamma^{\lambda}_{\mu,\nu}$ of size at least two is balanced bipartite. The connected components of size one are precisely the Littlewood–Richardson tableaux.*

The Littlewood–Richardson rule follows as a corollary.

Proof of Theorem 2.3.11. By Proposition 3.1.1,

$$c^{\lambda}_{\mu,\nu} = \sum_{(\pi,T)\in V^{\lambda}_{\mu,\nu}} \mathtt{sgn}(\pi).$$

Splitting this sum over connected components $\mathcal{C}$ of $\Gamma^{\lambda}_{\mu,\nu}$,

$$\sum_{T\in V^{\lambda}_{\mu,\nu}} \mathtt{sgn}(\pi_T) = \sum_{\mathcal{C}} \sum_{T\in V(\mathcal{C})} \mathtt{sgn}(\pi_T).$$

By Lemma 2.3.16, $\widehat{r_i}$ flips the sign of π_T so Theorem 3.2.3 implies that for any connected component $\mathcal{C}$ of size at least two,

$$\sum_{T\in V(\mathcal{C})} \mathtt{sgn}(\pi_T) = 0.$$

Theorem 3.2.3 says further that the connected components that remain are precisely the LR tableaux, which have positive sign since they have partition weight. Thus,

$$c^{\lambda}_{\mu,\nu} = \sum_{T\in \mathrm{LRT}(\lambda/\mu,\nu)} 1 = \#\mathrm{LRT}(\lambda/\mu,\nu).$$

□

The next few sections are dedicated to proving Theorem 3.2.3. We do so by giving a set of local rules satisfied by the connected components of $\Gamma^{\lambda}_{\mu,\nu}$ and proving that any graph satisfying these rules must be balanced bipartite.

3.2.1 Connected components of $\Gamma^{\lambda}_{\mu,\nu}$

To start, we review the necessary background on graphs and prove a few preliminary results about the connected components of $\Gamma^{\lambda}_{\mu,\nu}$. The graph $\Gamma^{\lambda}_{\mu,\nu}$ is almost always disconnected.[1]. We will find it more convenient to study the connected components of $\Gamma^{\lambda}_{\mu,\nu}$. We take a slight detour to define some necessary tools.

The **weak Bruhat order** is a partial order on S_n with cover relations defined by $\pi \lessdot s_i\pi$ whenever $\ell(s_i\pi) = \ell(\pi) + 1$. The **Hasse graph** $\mathcal{H}_n$ of the weak Bruhat order on S_n is

[1]The graph $\Gamma^{\lambda}_{\mu,\nu}$ is connected iff $|\mathrm{SSYT}(\lambda/\mu,\nu)| = 1$ It would be interesting to characterize the ν for which this is true. This was done for straight shapes in [2] but to our knowledge, this has not been extended to skew shapes.

the directed, properly edge-colored, acyclic graph with vertex set S_n and edges $\pi \xrightarrow{i} s_i\pi$ whenever $\pi \lessdot s_i\pi$. For a subset $I \subseteq [n]$, the **truncated Hasse graph** $\mathcal{H}_n^I$ is the subgraph of $\mathcal{H}_n$ obtained in the following way: delete any edges whose color does not appear in I to obtain a possibly disconnected graph, and choose the connected component that contains the original sink of $\mathcal{H}_n$.

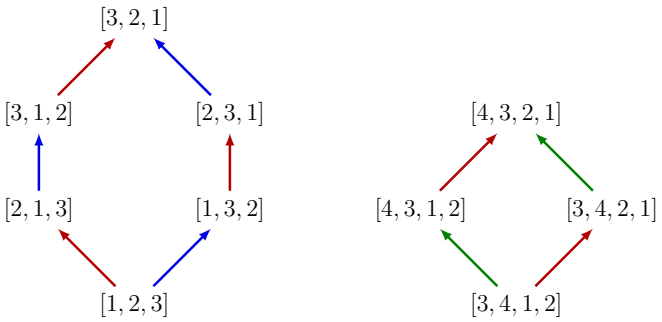


Figure 3.2: The Hasse graph $\mathcal{H}_3$ (left) and the truncated Hasse graph $\mathcal{H}_4^{\{1,3\}}$ (right). The red edges correspond to s_1, the blue edges correspond to s_2 and the green edges correspond to s_3.

The following lemma shows that when $\widehat{r}_i$ is defined, it "acts like" r_i.

Lemma 3.2.4. *For $T \in \mathrm{SSYT}(\lambda/\mu)$ such that $e_i(T)$ is defined, let m be the maximum entry in T, and let P be the unique Littlewood–Richardson tableau of shape $\rho = \rho^{(m)}$. Then*

$$r_i(P * T) = P * \widehat{r}_i(T).$$

Proof. We first show $r_i(P * T) = P * U$ for some tableau U. Observe that $\mathrm{rw}(P) = m^1 \cdots 2^{m-1}1^m$ so letting $w = \mathrm{rw}(T)$ and $w' = \mathrm{rw}(P * T)$,

$$\pi_i(w') = (i+1)^{m-i} i^{m-i+1} \pi_i(w). \tag{3.1}$$

In the prefix appearing in (3.1), every $i+1$ is matched with an i, leaving only one i unmatched. Further, since this i is strictly to the left of $\pi_i(w)$, and therefore any $i+1$, it is unmatched in $\pi_i(w')$. It follows that this i is the leftmost unmatched i. Hence for it to change to an $i+1$ in $r_i(P * T)$, there would need to be no unhidden $i+1$'s in $\pi_i(w')$, or equivalently, in $\pi_i(w)$. But, since we assumed $e_i(T) \neq \varnothing$, this cannot be the case. As this is the only unhidden entry in the left factor of $P * T$, the claim is proven.

It remains to show that $U = \widehat{r}_i(T)$. To do so, we turn our attention to the tableaux T and $P * T$. Let B and B' be the set of boxes in the diagrams of λ and $\rho * \lambda$ that contain

the *unmatched* entries of $\pi_i(w)$ and $\pi_i(w')$ respectively. There is exactly one box $\alpha \in B'$ that lies in the left factor of $\rho * \lambda$, the box corresponding to the single unmatched i in the prefix of (3.1). As the prefix has no effect on the matchings in the suffix $\pi_i(w)$ of (3.1) the boxes in $B' \setminus \{\alpha\}$ are in the same location of λ as the boxes in B (here we are identifying the right factor of $\rho * \lambda$ with λ).

Now, record the entries of the boxes in B and B' in reading order. We obtain $i^a(i+1)^b$ and $i^{a+1}(i+1)^b$ respectively for nonnegative integers a and b. In fact, $b > 0$ as we assumed $e_i(T) \neq \varnothing$. Applying $\widehat{r}_i$ to T and r_i to $P * T$, these subwords become $i^{b-1}(i+1)^{a+1}$ and $i^b(i+1)^{a+1}$. But recall that the leftmost i in the latter subword is contained in the box $\alpha \in B'$, the single box in the left factor of $\rho * \lambda$. Hence, U has the same entries appearing in the same boxes as $\widehat{r}_i(T)$ which is enough to conclude that they are the same tableau. □

The reader should note that one may recover the results in Lemma 2.3.16 directly from Lemma 3.2.4.

An **upwards closed set** of a directed, edge-colored, acyclic graph $G = (V, E)$ is a subset $S \subseteq V$ of vertices such that for every $v \in S$, if there exists an edge e such that $\texttt{tail}(e) = v$, then $\texttt{head}(e) \in S$. Then the following is a consequence of Lemma 3.2.4 and the fact that $\widehat{r}_i$ can always be applied to an SSYT with more $i+1$'s than i's.

Proposition 3.2.5. *Let $\mathcal{C}$ be a connected component of $\Gamma^{\lambda}_{\mu,\nu}$ and let I be the set of colors appearing in $\mathcal{C}$. Then $\mathcal{C}$ is isomorphic to an upwards closed set of $\mathcal{H}^I_n$.*

Remark 3.2.6. *It may be possible to strengthen Proposition 3.2.5 in the following way. The strong Bruhat order is defined in the same way as the weak Bruhat order, but we allow for any transposition rather than only simple transpositions. One may then define the analogous version of $\mathcal{H}^I_n$ by letting I denote the set of colors appearing in $\mathcal{C}$ and all pairs of those colors. Then we may ask if the connected components of $\Gamma^{\lambda}_{\mu,\nu}$ are also isomorphic to upwards closed sets of this graph. And indeed, we have checked this computationally and it is true for all λ, μ, ν such that $|\lambda| - |\mu| \leq 8$ and $|\nu| \leq 8$.*

Lemma 3.2.7. *Let $\mathcal{C}$ be a connected component of $\Gamma^{\lambda}_{\mu,\nu}$. Then $\mathcal{C}$ has a unique sink S, which may be identified as the only tableau in $\mathcal{C}$ with weight ν.*

Proof. By definition, if T does not have partition content, there is an edge $T \xrightarrow{i} \widehat{r}_i(T)$ in $\Gamma^{\lambda}_{\mu,\nu}$. Thus, all sinks have partition content. Further, there is only one sink in $\mathcal{H}^I_n$ and so by Proposition 3.2.5, there is only one sink in $\mathcal{C}$. □

This allows us to characterize the sinks of each connected component $\mathcal{C}$ of $\Gamma^{\lambda}_{\mu,\nu}$. Given $S \in \mathrm{SSYT}(\lambda/\mu, \nu)$, we let $\mathcal{C}_S$ denote the connected component of $\Gamma^{\lambda}_{\mu,\nu}$ whose unique sink is S.

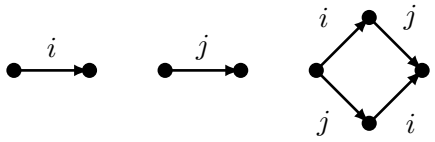
3.2.2 Local rules

Given a directed, properly edge-colored graph $G = (V, E)$, we wish to give a list of local rules such that if G satisfies these, then G is balanced bipartite. Before stating our rules, we need one piece of notation. For an edge colored graph $G = (V, E)$, let E_i be the set of edges of color i. For a subset $E' \subseteq E$, let (V, E') be the subgraph of G induced by E'.

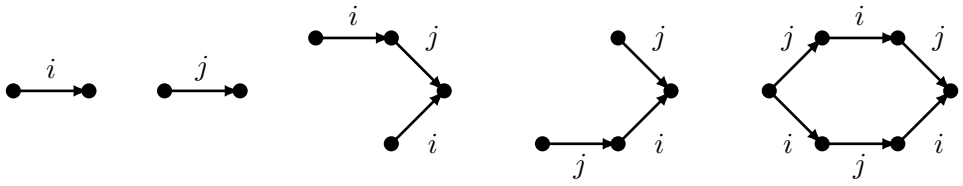
Definition 3.2.8. *For a directed, properly edge-colored graph G of size at least two, we say G is **locally balanced** if the following hold:*

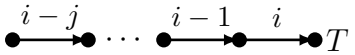
(L0) G is bipartite.

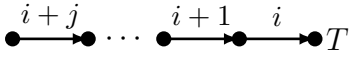
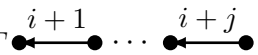
(L1) If $|i-j| > 1$, every connected component of $(V, E_i \cup E_j)$ appears in the following list:



(L2) If $|i-j| = 1$, every connected component of $(V, E_i \cup E_j)$ appears in the following list:

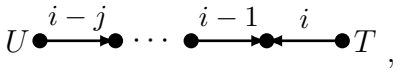


(L3) If G contains a subgraph isomorphic to  *where T is a sink in $(V, E_{i-j} \cup \cdots \cup E_i)$, then G also contains a subgraph isomorphic to* 

(L4) If G contains a subgraph isomorphic to  *where T is a sink in $(V, E_{i+j} \cup \cdots \cup E_i)$, then G also contains a subgraph isomorphic to* .

Now we must show a locally balanced graph is balanced bipartite. We will prove this by induction on the number of colors but need some preliminary lemmas first. While long, the proofs of these two lemmas boil down to using the local rules repeatedly to imply the existence of specific edges in G.

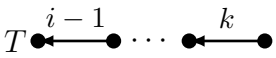
Lemma 3.2.9. *If G is locally balanced, contains a subgraph isomorphic to*



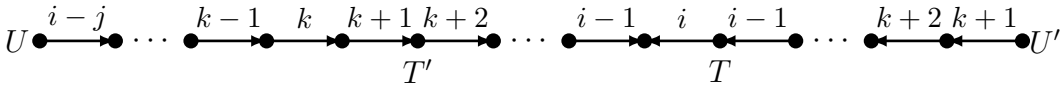
and $\deg(U) = 1$*, then G also contains a subgraph isomorphic to*



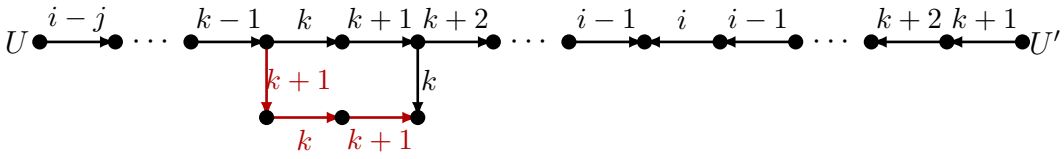
Proof. Proceed by contradiction. Let $k \in [i-1, i-j]$ be the largest color such that G does not contain a subgraph isomorphic to



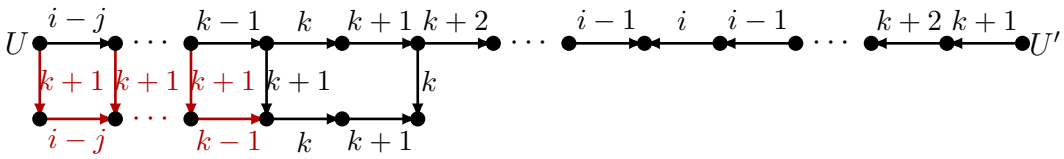
Then, G contains a subgraph isomorphic to the following



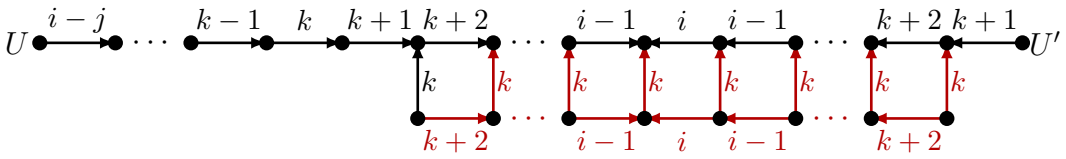
Consider the vertex T'. (L2) implies there is a k-colored edge incident to T'. First assume T' is the tail of this edge. Then (L2) further implies that the following red edges must be present:



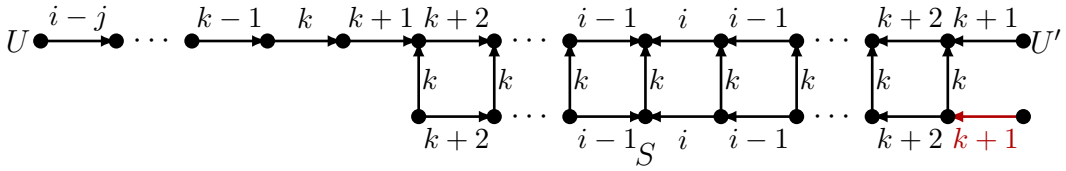
Repeated application of (L1) implies that the following red edges must be present:



But we assumed $\deg(U) = 1$, a contradiction. Now assume that T' is the head of this edge. Repeated applications of (L1) implies that the following red edges must be present:

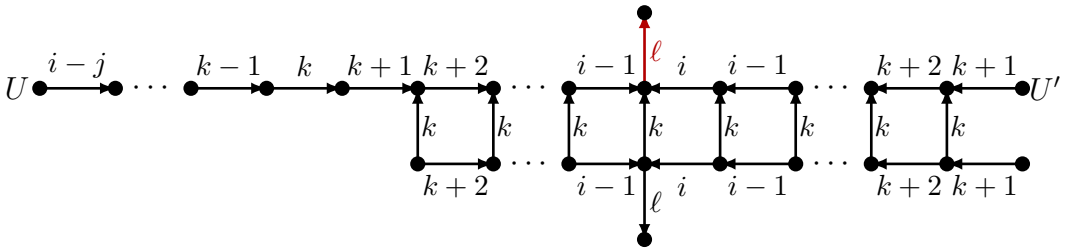


By assumption, there is no k-colored edge whose head is U'. Thus, (L2) implies that the following red edge must be present:

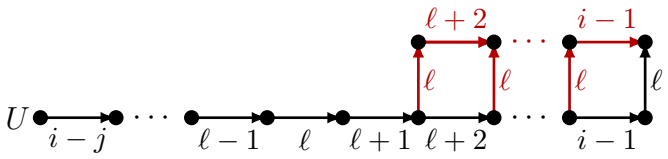


(3.2)

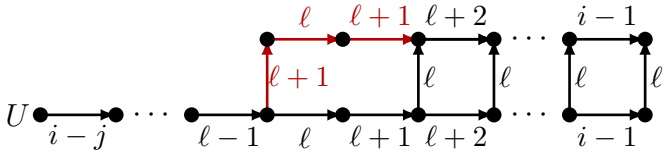
Now, we want to apply (L3) to the bottom row. In order to do that, we will prove that for any $\ell \in [i, k+1]$, there is no ℓ-colored edge e with $\text{tail}(e) = S$. Assume for the sake of contradiction that there is such an edge. Clearly ℓ is not i or $i-1$. Now, an application of (L1) if $\ell \neq k+1$ or (L2) if $\ell = k+1$ implies that the following red edge must be present:



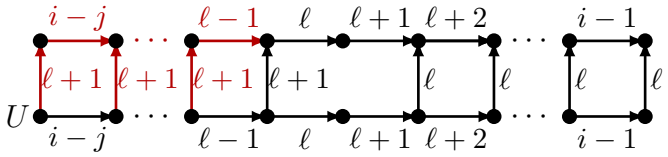
Focus on the top row left of the red ℓ-colored edge. Repeated application of (L1) implies that the following red edges must be present:



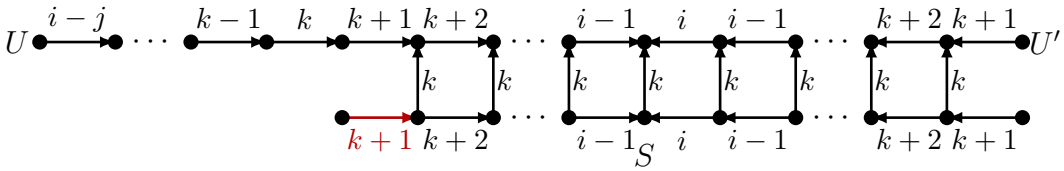
(L2) implies that the following red edges must be present:



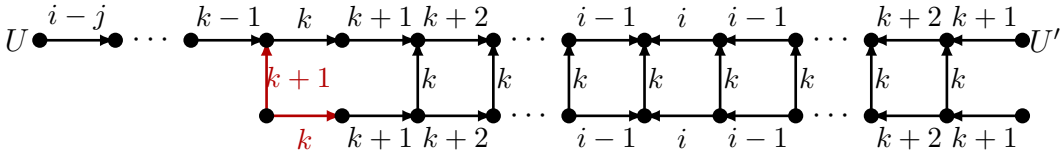
Repeated application of (L1) then implies that the following red edges must be present:



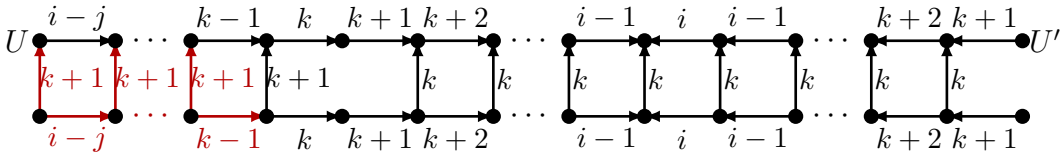
But this contradicts the assumption that $\deg(U) = 1$. Thus, we may apply (L3) to the bottom row of (3.2) which implies that the following red edge must be present:



With this new $k+1$-colored edge, (L2) implies that the following red edges must be present:



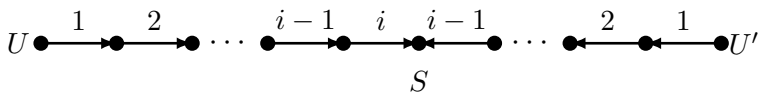
Lastly, repeated applications of (L1) implies that the following red edges must be present:



Again, we arrive at a contradiction since we assumed $\deg(U) = 1$. This completes the proof. □

This next lemma will give us a natural way to pair certain degree one vertices.

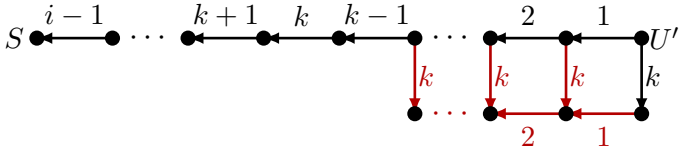
Lemma 3.2.10. *If G is locally balanced, contains a subgraph isomorphic to*



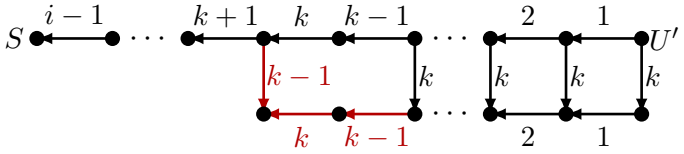
and there does not exist an $(i+1)$-colored edge incident to S, then $\deg(U) = 1$ if and only if $\deg(U') = 1$.

Proof. We assume $\deg(U') \neq 1$ and will show that this implies $\deg(U) \neq 1$. By assumption, there exists some k-colored edge e with $k \neq 1$ incident to U'. If $k > i+1$, repeated applications of (L1) implies there is a k-colored edge incident to U. If $k = i+1$, then repeated applications of (L1) imply there is an $(i+1)$-colored edge incident to S, a contradiction.

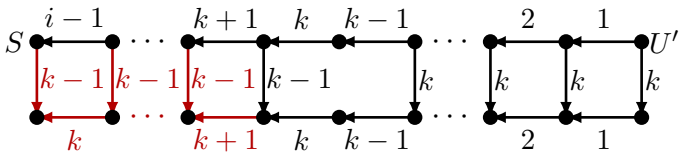
Thus, we may assume $k \in [2, i]$. First consider the case where $\texttt{tail}(e) = U'$ and focus on the path to the right of S. Repeated applications of (L1) imply the following red edges must be present:



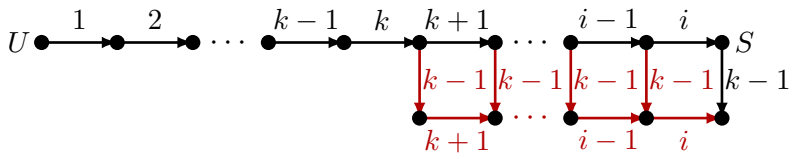
Then (L2) implies that the following red edges must be present:



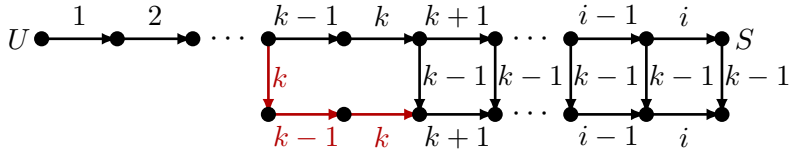
Again, repeated applications of (L1) imply the following red edges must be present:



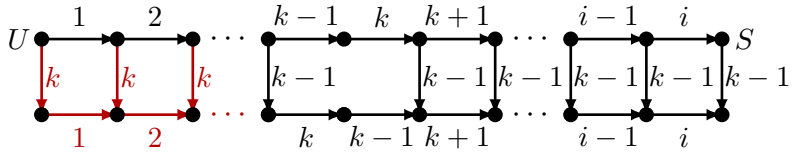
With the knowledge that there is a $(k-1)$-colored edge whose tail is S, now focus on the path left of S. Repeated applications of (L1) imply the following red edges must be present:



Then (L2) implies that the following red edges must be present:

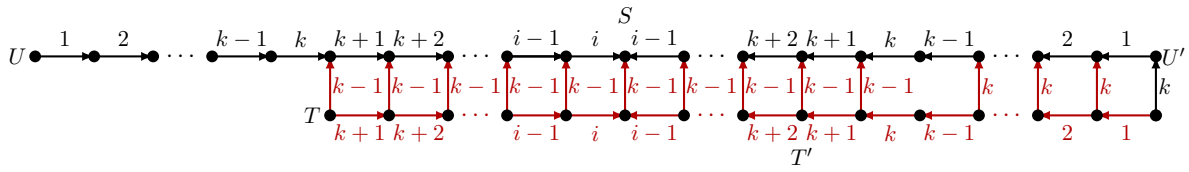


And again, repeated applications of (L1) imply the following red edges must be present:



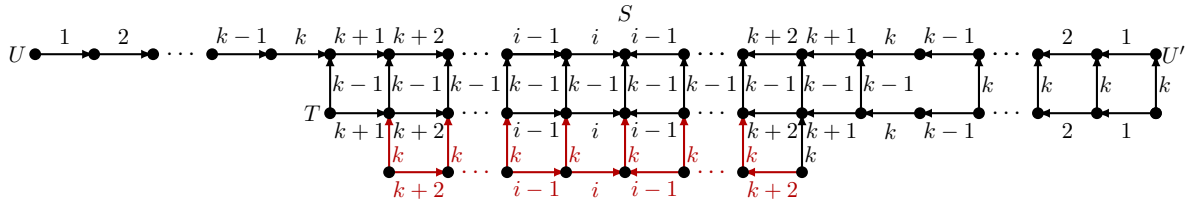
(3.3)

Thus, we have a k-colored edge incident to U. Now consider the other case where $\texttt{head}(e) = U'$. An almost identical argument as the one above implies that the following red edges must be present:

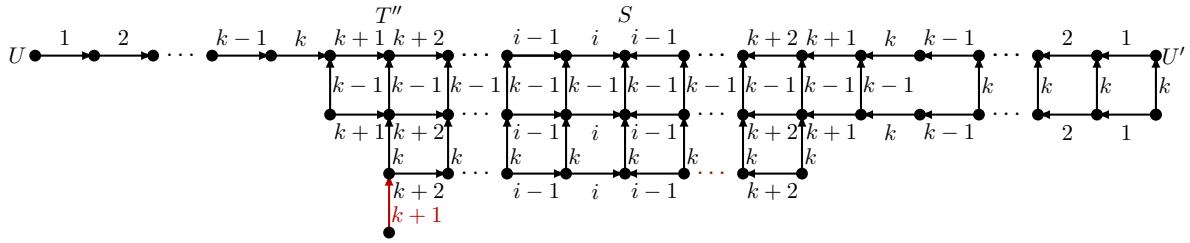


Observe that if there is a k-colored edge incident to T, then an application of (L2) followed by repeated applications of (L1) imply that there is a k-colored edge incident to U. Thus, assume there is no k-colored edge incident to T. Now turn attention to the vertex T'. (L2) implies there is a k-colored edge e' incident to T'. First consider the case where $\texttt{head}(e') = T'$. Then repeated applications of (L1) imply the following red edges must be

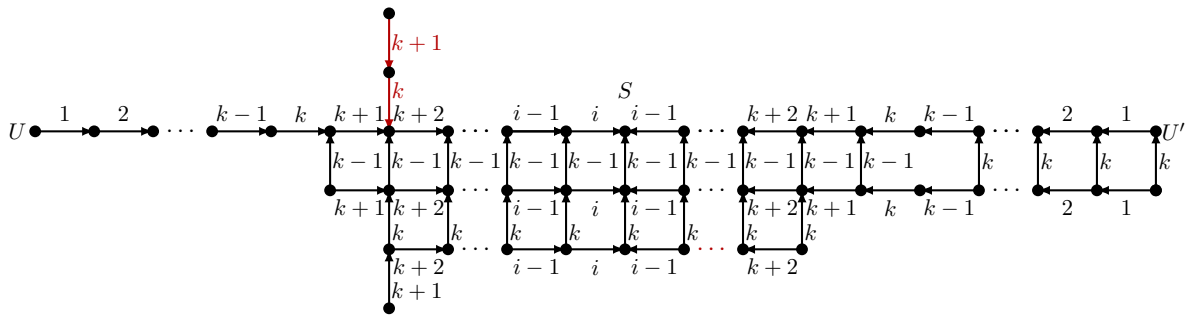
present:



Now, since that there is no k-colored edge incident to T, (L2) implies that the following red edge must be present:



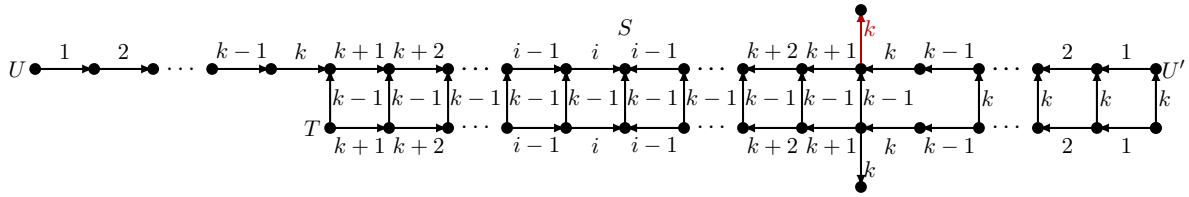
Now switch focus again to the vertex T''. (L2) implies there is a k-colored edge incident to T''. If the tail of the edge is T'', then an application of (L2) followed by repeated applications of (L1) imply that there is a k-colored edge incident to U. Thus, assume the head of the edge is T''. Then T'' is a source in the graph $(V, E_{k-1} \cup E_k \cup E_{k+1})$ and so an application of (L4) implies that the following red edges must be present:



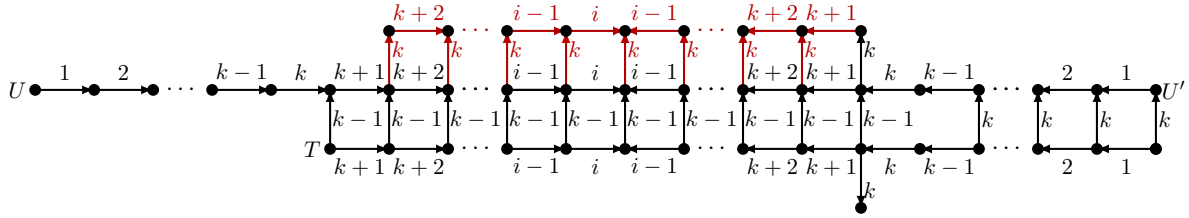
Finally, an application of (L2) followed by repeated applications of (L1) imply that there is a $(k+1)$-colored edge incident to U.

Looking back to (3.3), consider the other case where $\texttt{tail}(e) = T$. Then (L2) implies

that the following red edge (and three more that we omit) must be present:



Again, repeated application of (L1) implies that the following red edges must be present:

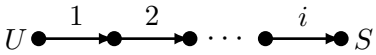


From here, an application of (L2) followed by repeated applications of (L1) implies there is a $(k+1)$-colored edge incident to U'. We may finish the proof by noting that an identical proof suffices to show that $\deg(U) \neq 1$ implies $\deg(U') = 1$. Indeed, for every $k \neq i+1$ as we may switch the direction of the one $(i+1)$-colored edge in all of the arguments. If $k = i+1$, then as before repeated applications of (L1) followed by an application of (L2) implies there is an $(i+1)$-colored edge incident to S, the same contradiction as before. □

Theorem 3.2.11. *If G is locally balanced, then it is balanced bipartite.*

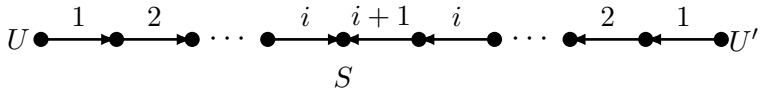
Proof. By (L0), $G = (V, E)$ is bipartite. Let $V = A \cup B$ be the bipartition and induct on the number of colors. Let G' be the graph obtained by deleting all 1-colored edges and let A_0 and B_0 be the sets of isolated vertices of G' in A and B respectively. Every connected component of G' of size at least two is balanced bipartite by the induction hypothesis so it suffices to show that $\#A_0 = \#B_0$. The sets A_0 and B_0 are equivalently described as the degree one vertices in G incident to a 1-colored edge and this is how the reader should consider them.

Let $U \in A_0$ be a degree one vertex incident to a 1-colored edge e. If $\texttt{head}(e) = U$ then G is isomorphic to $U \bullet\!\!\leftarrow\!\!1\!\!-\!\!\bullet$ so we are done. Indeed, if we consider $(V, E_1 \cup E_i)$ for any other color i, then this is the only subgraph allowed by (L1) and (L2). Thus we may assume $\texttt{tail}(e) = U$. Take a maximal increasing directed path of the form

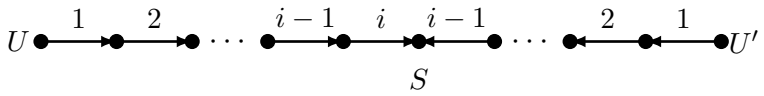


and let S be the vertex at the end of this path. If this path is of length one and there is no 2-colored edge whose head is S, then G is isomorphic to $U \bullet\!\!-\!\!1\!\!\rightarrow\!\!\bullet\, S$. Indeed, if not, there is an i-colored edge incident to S with $i \neq 1, 2$ so (L1) implies that U is not degree one. By maximality of the path, there is no $(i+1)$-colored edge whose tail is S and so we have two cases: either there is an $(i+1)$-colored edge whose head is S, or there is no $(i+1)$-colored edge incident to S at all.

If such an edge exists, then we may apply Lemma 3.2.9 to see that G contains a subgraph isomorphic to



If no such edge exists, then we may apply (L3) to see that G contains a subgraph isomorphic to



Now, define a bijection $\varphi : A_0 \to B_0$ that takes U to U' as constructed above. Since this path is of odd length, $U' \in B$ and by Lemma 3.2.10, U' is degree one in either case so $U' \in B_0$. In particular, this implies φ is well-defined. It is also not difficult to see that φ is its own inverse so φ is in fact a bijection. □

3.2.3 $\Gamma^{\lambda}_{\mu,\nu}$ is locally balanced

In this section, we will show that $\Gamma^{\lambda}_{\mu,\nu}$ is locally balanced. For notation, let $V = V^{\lambda}_{\mu,\nu}$, let E_i be the set of edges of color i in $\Gamma^{\lambda}_{\mu,\nu}$, and let $\widehat{\mathtt{wt}}(T) = \mathtt{wt}(T) + \rho$. Note that for any $T \in V$, $\widehat{\mathtt{wt}}(T)$ has no repeated entries.

We begin by giving two useful lemmas that will simplify our proof.

Lemma 3.2.12. *Let $T \in \mathrm{SSYT}(\lambda/\mu)$. If there exists a $U \in \mathrm{SSYT}(\lambda/\mu)$ such that $\widehat{r}_i(U)$ is defined, $T^{(i)} \subseteq U^{(i)}$, and $T^{(i+1)} \supseteq U^{(i+1)}$, then $\widehat{r}_i(T)$ is defined.*

Proof. Let $i^a(i+1)^b$ be the subword of $\pi_i(U)$ containing the unmatched i's and $i+1$'s. Similarly, let $i^c(i+1)^d$ be the subword of $\pi_i(T)$ containing the unmatched i's and $i+1$'s. Then there is at least one unmatched $i+1$ in $\pi_i(\mathrm{rw}(U))$, or in other words $b > 0$, since

$\widehat{r}_i(U)$ is defined. In the case when both $T^{(i)} = U^{(i)}$ and $T^{(i+1)} = U^{(i+1)}$, then $\pi_i(\mathrm{rw}(U)) = \pi_i(\mathrm{rw}(T))$ and so $\widehat{r}_i(T)$ must be defined. Thus, assume as least one of these equalities does not hold.

First assume $T^{(i)} \subset U^{(i)}$. By removing i's from $\pi_i(U)$, we either unmatch a previously matched $i+1$, in which case $b < d$ and $a = c$, or we match a previously unmatched i, in which case $a > c$ and $b = d$. In particular, $0d \geq b > 0$ and so $\widehat{r}_i(T)$ is defined. Instead assume $T^{(i+1)} \supset U^{(i+1)}$. By adding $i+1$'s to $\pi_i(U)$, this new $i+1$ either does not get matched, in which case $b < d$ and $a = c$, or it matches with a previously unmatched i, in which case, $a > c$ and $b = d$. We reach the same conclusion as in the first case. □

Lemma 3.2.13. *Let $\mathcal{C}$ be a connected component of $(V, E_i \cup E_{i+1})$. Then there exists a straight, two row SSYT S with entries in $\{i, i+1, i+2\}$ such that $\mathcal{C} \cong \mathcal{C}_S$.*

Proof. By Lemma 2.3.9, $\mathcal{C}$ has a unique source U. Let U' be the skew tableau obtained by deleting every box that does not contain an i, $i+1$, or $i+2$. Clearly, $\mathcal{C} \cong \mathcal{C}_{U'}$. Since crystal operators commute with slides, $\mathcal{C}_{U'} \cong \mathcal{C}_{U''}$ where $U'' = \mathtt{rect}(U')$. Lastly, if U has three rows, let ℓ be the length of its third row. Define S to be the SSYT obtained by deleting the first ℓ boxes in every row of U''. Clearly, S has two rows. And $\mathcal{C}_{U''} \cong \mathcal{C}_S$ since the first ℓ entries in the second and third row will always be matched together in $\pi_2(T)$ and similarly, the first ℓ entries in the first and second row will always be matched together in $\pi_1(T)$. □

We will now prove that $\Gamma^{\lambda}_{\mu,\nu}$ satisfies the local rules.

Theorem 3.2.14. *$\Gamma^{\lambda}_{\mu,\nu}$ is locally balanced.*

Proof. (L0) follows directly from the fact that $\widehat{r}_i$ flips the sign of π_T. Now we go slightly out of order and attend to (L3). It suffices to prove the following claim:

Claim 3.2.15. *If $\widehat{r}_{i-j} \cdots \widehat{r}_{i-1}\widehat{r}_i(T)$ is defined and $\widehat{\mathtt{wt}}(T)_{i-j} > \cdots > \widehat{\mathtt{wt}}(T)_i > \widehat{\mathtt{wt}}(T)_{i+1}$, then $\widehat{r}_{i-j} \cdots \widehat{r}_{i-1}(T)$ is defined.*

Induct on j. For the base case, let $U = \widehat{r}_i(T)$. Then $T^{(i-1)} = U^{(i-1)}$ and since $\widehat{\mathtt{wt}}(T)_i > \widehat{\mathtt{wt}}(T)_{i+1}$, $\widehat{r}_i$ changes some number of i's to $i+1$'s so $T^{(i)} \supseteq U^{(i)}$. Thus, since $\widehat{r}_{i-1}(U)$ is defined we may apply Lemma 3.2.12 to see that $\widehat{r}_{i-1}(T)$ is defined. Now for $k > 1$, let $U = \widehat{r}_{i-k} \cdots \widehat{r}_{i-1}\widehat{r}_i(T)$ and assume that $S = \widehat{r}_{i-k} \cdots \widehat{r}_{i-1}(T)$ is defined. We must show that $\widehat{r}_{i-k-1}(S)$ is defined. Clearly, $S^{(i-k-1)} = U^{(i-k-1)}$ so if we show that $S^{(i-k)} \supseteq U^{(i-k)}$, we may apply Lemma 3.2.12, proving the claim. To this end, let $U' = \widehat{r}_{i-k+1} \cdots \widehat{r}_{i-1}\widehat{r}_i(T)$ and $S' = \widehat{r}_{i-k+1} \cdots \widehat{r}_{i-1}(T)$. Consider the following three facts: $S'^{(i-k)} = T^{(i-k)} = U'^{(i-k)}$,

$\widehat{\mathtt{wt}}(U')_{i-k+1} = \widehat{\mathtt{wt}}(T)_{i+1}$, and $\widehat{\mathtt{wt}}(S')_{i-k+1} = \widehat{\mathtt{wt}}(T)_i$. The first follows from the definitions of S' and U' and the second and third follow from Lemma 2.3.16. Together, these facts imply

$$\widehat{\mathtt{wt}}(U')_{i-k} = \widehat{\mathtt{wt}}(T)_{i-k} > \widehat{\mathtt{wt}}(T)_{i+1} = \widehat{\mathtt{wt}}(U')_{i-k+1}, \tag{3.4}$$

and

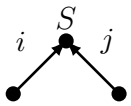
$$\widehat{\mathtt{wt}}(S')_{i-k} = \widehat{\mathtt{wt}}(T)_{i-k} > \widehat{\mathtt{wt}}(T)_{i} = \widehat{\mathtt{wt}}(S')_{i-k+1}, \tag{3.5}$$

and

$$\widehat{\mathtt{wt}}(S')_{i-k+1} = \widehat{\mathtt{wt}}(T)_{i} > \widehat{\mathtt{wt}}(T)_{i+1} = \widehat{\mathtt{wt}}(U')_{i-k+1}. \tag{3.6}$$

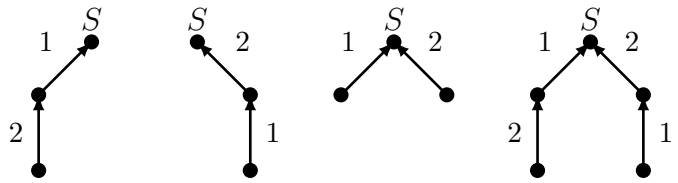
Now, (3.4) and (3.5) imply that when applying $\widehat{r}_{i-k}$ to U' and S' to obtain U and S respectively, $\widehat{r}_{i-k}$ will change some number of $i-k$'s to $i-k+1$'s. Further, (3.6) implies that S' will have had fewer $i-k$'s changed to $i-k+1$'s than U', or in other words, S has more $i-k$'s than U does. Hence, since $S'^{(i-k)} = U'^{(i-k)}$, we conclude that $S^{(i-k)} \supseteq U^{(i-k)}$.

The proof of (L4) is almost identical to the proof of (L3) so we move on to (L1). Let i, j be such that $|i-j| > 1$ and let $\mathcal{C}$ be a connected component of $(V, E_i \cup E_j)$. By Proposition 3.2.5, $\mathcal{C}$ is an upwards closed set of $\mathcal{H}_n^{\{i,j\}}$ (see Figure 3.2). Enumerating all upwards closed sets of this graph, we see that is suffices to prove that $\mathcal{C}$ is not isomorphic to the following graph:



This follows from the fact that $|i-j| > 1$. Indeed, since $\widehat{r}_i(S)$ is defined, $\pi_i(\mathrm{rw}(S))$ must have at least one unmatched $i+1$ and since $|i-j| > 1$, $\pi_i(\mathrm{rw}(\widehat{r}_j(S)))$ will also at least one unmatched $i+1$. Thus $\widehat{r}_j\widehat{r}_i(S)$ is defined so $\mathcal{C}$ cannot be isomorphic to the above graph.

Lastly, we attend to (L2). Let $\mathcal{C}$ be a connected component of $(V, E_i \cup E_{i+1})$. By Lemma 3.2.13, there exists a straight, two row SSYT S with entries in $\{i, i+1, i+2\}$ such that $\mathcal{C} \cong \mathcal{C}_S$. We may assume without loss of generality that $i = 1$, so S has entries in $\{1, 2, 3\}$. By Proposition 3.2.5, $\mathcal{C}_S$ is isomorphic to an upwards closed set of $\mathcal{H}_3$ (see Figure 3.2). Enumerating all upwards closed sets of this, we see that it suffices to prove that $\mathcal{C}_S$ is not isomorphic to the following graphs:



Above, we proved that $\mathcal{C}_S$ must satisfy (L4), ruling out the first graph, and (L3), ruling out the second graph. Assume that $\mathcal{C}_S$ is isomorphic to the third graph. The key observation is that for any straight, two row SSYT T with entries in $\{1, 2, 3\}$, $\widehat{r}_1(T)$ is defined if and only if T has 2's on the first row.

Since $\widehat{r}_1(S)$ is defined, there are 2's on the first row of S. Since $\widehat{r}_1\widehat{r}_2(S)$ is not defined, there are no 2's on the first row of $\widehat{r}_2(S)$. Thus, our application of $\widehat{r}_2$ to S must have changed every 2 on the top row to a 3. But this implies there can be no 3's on the second row of S. Combining this with the fact that $\widehat{r}_2(S)$ is defined, there must be at least one 3 on the first row of S. In turn, this implies there must be at least one 3 on the first row of $\widehat{r}_1(S)$ and so $\widehat{r}_2\widehat{r}_1(S)$ must be defined, contradicting that $\mathcal{C}_S$ is isomorphic to the third graph.

Lastly, assume $\mathcal{C}_S$ is isomorphic to the fourth graph. Given the direction of the 1-colored edge incident to S, $\widehat{r}_1(S)$ has had at least one 1 of S changed to 2. In particular, $\widehat{r}_1(S)$ has at least one 2 on its first row. Since $\widehat{r}_1\widehat{r}_2\widehat{r}_2(S)$ is not defined, $\widehat{r}_2\widehat{r}_1(S)$ has no 2's on its first row. Thus, our application of $\widehat{r}_2$ to $\widehat{r}_1(S)$ must have changed every 2 on the top row to a 3. But this could only happen if there are no 3's on the bottom row of $\widehat{r}_1(S)$ and there is exactly one 3 on the top row of $\widehat{r}_1(S)$. As $\widehat{r}_1$ does not affect 3's, this must also be true of S. But then this implies that in the application of $\widehat{r}_2$ to S, every 2 on the top row of S is changed to a 3, implying that $\widehat{r}_1\widehat{r}_2(S)$ is no defined, contradicting that $\mathcal{C}_S$ is isomorphic to the fourth graph. □

Remark 3.2.16. *In their PhD thesis [13], Iveson uses the same strategy Remmel and Shimozono used in [25] to prove a special case of the k-Littlewood–Richardson rule for a certain definition of k-Schur functions (a generalizations of Schur functions and the LR rule; see [13]). Iveson restricted to a special case since the analogue of $\mathcal{RS}$ fails to be an involution in the general case. The author notes that, as our proof of the LR rule does not require one to explicitly construct a sign-reversing involution, it may be possible to instead use the approach appearing in this thesis to extend the results of [13], although no attempt has been made at this from the author.*

Chapter 4

Nonsymmetric sub-crystals

In this chapter we investigate the crystal operators on multiline queues, objects related to SSAFs. A certain subset of multiline queues is in bijection with SSAFs. We first introduce multiline queues, then relate them to SSAFs, and finally we characterize how crystal operators interact with the type of a multiline queue (this corresponds to the shape of an SSAF).

4.1 Multiline queues

Let λ be a partition, $L = \lambda_1$ and $n > \ell(\lambda)$ be an integer. A **multiline queue** (MLQ) of shape (λ, n) is an arrangement of balls in an $L \times n$ grid with rows indexed from bottom to top such that row j contains λ'_j balls. The columns are indexed from left to right. The site (r, i) of M refers to the cell in column i of row r of M. The row word $\mathrm{rw}(M)$ of an MLQ M is the word obtained by recording the column index of the balls left to right, bottom to top. The column word $\mathrm{cw}(M)$ of an MLQ M is the word obtained by recording the row index of the balls top to bottom, left to right. We denote the set of multiline queues of shape (λ, n) by $\mathrm{MLQ}(\lambda, n)$.

Example 4.1.1. *Below is a multiline queue* $M \in \mathrm{MLQ}((3, 3, 2, 1), 5)$.

$$M = \begin{array}{|c|c|c|c|c|}\hline & & \bullet & & \bullet \\ \hline & \bullet & \bullet & \bullet & \\ \hline \bullet & & \bullet & \bullet & \bullet \\ \hline \end{array}$$

Its row word $\mathrm{rw}(M)$ *and column word* $\mathrm{cw}(M)$ *are then,*

$$\mathrm{rw}(M) = 1\ 3\ 4\ 5 \mid 2\ 3\ 4 \mid 3\ 5 \quad \mathrm{cw}(M) = 1 \mid 2 \mid 3\ 2\ 1 \mid 2\ 1 \mid 3\ 1\ .$$

Next, we enhance each multiline queue by giving labels to each ball via the **Ferrari—Martin** (FM) algorithm [8]. The FM algorithm is a sequential labelling procedure in which balls on row i are paired with balls on row $i-1$, one at a time with respect to a certain priority order. This produces a **labelling** L_M of the MLQ as well as pairings between rows that we call **strands**. The reader should note that while the set of pairings depends on the pairing order and may not be unique (see Remark 4.1.3), the labelling L_M is uniquely determined.

Definition 4.1.2 (FM algorithm)**.** *Let* $M \in \mathrm{MLQ}(\lambda, n)$ *with* $L = \lambda_1$*. For each row* $r = L, \ldots, 2$*:*

- *Every unlabeled ball is labeled "*r*".*
- *Once all balls in the* r*-th row are labeled, each of them is sequentially paired to the first unlabeled ball weakly to its right in the* $(r-1)$*st row, wrapping around from column* n *to the first column if necessary. The order in which balls are paired from the* r*-th row to the* $(r-1)$*st row is from the largest label "L" to the smallest label "*r*", and (by convention) from left to right among balls with the same label.*

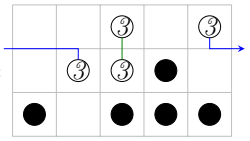
*To complete the process, all unpaired balls in the first row are labeled "*1*". See Example 4.1.4 for an example of this algorithm.*

The **type** of a multiline queue is the composition $\mathtt{type}(M)$ obtained by reading the labels on the first row of L_M and recording 0 whenever we encounter an empty site.

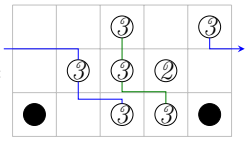
Remark 4.1.3. *While different pairing orders among balls of the same label may change the pairings, they do not change which balls are paired to in the row below. Thus, the labels assigned remain the same (see [19, Remark 3.14]). To make the pairing procedure deterministic, we use a left–to–right pairing convention.*

Example 4.1.4. *Here we show the sequence of steps of the FM algorithm for the multiline queue in Example 4.1.1.*

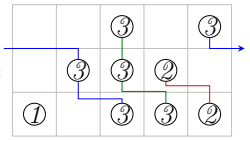
For $r = 3$*, we label all balls in the third row with a "3". For* $\ell = 3$*, we pair balls with label* ℓ *in the third row, from left to right, to a ball that has not already been paired to in the second row, weakly to the right:*

$M =$ 

Next, for $r = 2$, label all unlabelled balls in row 2 with "2". Pair balls with label $\ell = 3$ in row 2 to unpaired balls in row 1:

$M =$ 

To complete the pairings for $r = 2$, we pair balls with label $\ell = 2$ in the second row to unpaired balls weakly to the right in the first row. Finally, for $r = 1$, we label all unlabeled balls in the first row with a "1", which completes the pairing procedure:

$M =$ 

Reading the labels off the bottom row, we get $\texttt{type}(M) = (1, 0, 3, 3, 2)$.

We say an MLQ M is **non-wrapping** if no pairings wrap around M. For example, the MLQ in Example 4.1.4 is wrapping as the pairing between the ball at site $(5, 3)$ and the ball at site $(2, 2)$ wraps. We let $\mathrm{MLQ}_0(\lambda, n)$ denote the set of non-wrapping MLQs of shape (λ, n). This subset is important as $\mathrm{MLQ}_0(\lambda, n)$ is in bijection with $\mathrm{SSYT}_n(\lambda)$ [19]. Even further, the set of non-wrapping multiline queues of shape (λ, n) and type α, which we denote by $\mathrm{MLQ}_0(\alpha)$, is in bijection with $\mathrm{SSAF}(\alpha)$. We discuss this in further detail in Section 4.1.2.

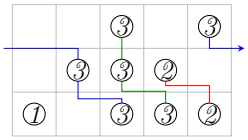
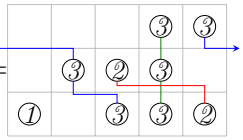
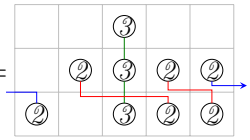
4.1.1 Row and column crystal operators on multiline queues

Unlike the combinatorial objects we have discussed so far, MLQs have two different kinds of crystal operators; ones that move balls between columns, and ones that move balls between rows.

Definition 4.1.5 (Column crystal operators). *The operator $e_i^{\leftarrow}$ acts a multiline queue M by moving the leftmost unmatched entry in $\pi_i(\mathrm{rev}(\mathrm{rw}(M)))$ from column $i+1$ to column i. If no such entry exists, we say $e_i^{\leftarrow}(M)$ is not defined. In the multiline queue M, this corresponds to the topmost unmatched ball in column $i+1$. The operator $f_i^{\rightarrow}$ acts a multiline queue M by moving the rightmost unmatched entry in $\pi_i(\mathrm{rev}(\mathrm{rw}(M)))$ from column $i+1$ to column i. If no such entry exists, we say $f_i^{\rightarrow}(M)$ is not defined. In the multiline queue M, this corresponds to the bottommost unmatched ball in column i.*

Definition 4.1.6 (Row crystal operators). *The operator $e_i^{\downarrow}$ acts a multiline queue M by moving the leftmost unmatched entry in $\pi_i(\mathrm{cw}(M))$ from row $i+1$ to row i. If no such entry exists, we say $e_i^{\downarrow}(M)$ is not defined. In the multiline queue M, this is the leftmost unmatched ball in row $i+1$. The operator $f_i^{\uparrow}$ acts a multiline queue M by moving the rightmost unmatched entry in $\pi_i(\mathrm{cw}(M))$ from row $i+1$ to row i. If no such entry exists, we say $f_i^{\uparrow}(M)$ is not defined. In the multiline queue M, this is the rightmost unmatched ball in row i.*

Example 4.1.7. *We use the multiline queue in Example 4.1.4.*

$M =$  $f_3^{\rightarrow}(M) =$  $e_2^{\downarrow}(M) =$ 

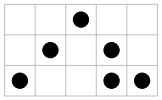
The reader should consider the column crystal operators as the usual ones. These operators are related by the follow fact:

Lemma 4.1.8 (Lemma 5.7 in [19]). *Row and column crystal operators commute.*

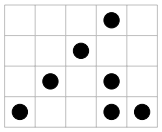
We will use the row crystal operators to define an insertion procedure on non-wrapping MLQs that is called **collapsing** that was introduced in [19].[1] Given a non-wrapping MLQ M and a positive integer a, define $a \to M$ as the non-wrapping MLQ obtained by the following procedure. Append a row to the top of M with a single ball in column a. Say this is row r. Then apply $e_{r-1}^{\downarrow}$ to this (possibly wrapping) MLQ. If $e_{r-1}^{\downarrow}$ is not defined, collapsing is complete. Otherwise, continue by applying $e_{r-2}^{\downarrow}$. Repeat until one of the row crystal operators is not defined or we reach the bottom of the MLQ. The output $a \to M$ is then a non-wrapping MLQ [19]. It is important to note that this collapsing procedure terminates precisely when we obtain a non-wrapping MLQ [19, Proposition 4.8].

[1] The collapsing map that was introduced in [19] is more complicated. See Section 4.3 and [19] for more details.

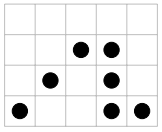
Example 4.1.9. *We show the sequence of steps of inserting a* 4 *into the MLQ* M *seen below.*



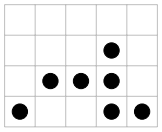
We begin by appending a row on top of M *with a ball in column* 4*.*



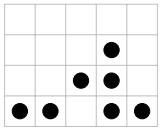
The outcome is indeed still an MLQ, but it is wrapping. Applying $e_3^{\downarrow}$ *to this, we see that the ball at site* $(4,4)$ *is the only unmatched ball in the fourth row and so we move it to the row below.*



Now applying $e_2^{\downarrow}$ *to this, we see that the ball at site* $(3,3)$ *is the only unmatched ball in the third row and so we move it to the row below.*



Applying $e_1^{\downarrow}$ *to this, we see that the ball at site* $(2,2)$ *is the only unmatched ball in the second row and so we move it to the row below.*



At this point, we have reached the bottom row and so the MLQ above is the MLQ $4 \to M$*.*

4.1.2 Relation to SSAFs

One obvious but useful property of MLQs is that they are uniquely determined by their row content. A more surprising fact is that SSAFs are also uniquely determined by their row content [20]. These facts together allow us to state the bijection between non-wrapping MLQs and SSAFs.

Proposition 4.1.10. *Define the map* $\Phi : \mathrm{MLQ}_0(\alpha) \to \mathrm{SSAF}(\alpha)$ *by declaring that* $\Phi(M)$ *is the unique SSAF with the same row content at* M*. Then* Φ *is a bijection.*

Proof. Let $w = \mathrm{rw}(M)$. Each row of M corresponds to a strictly increasing subsequence in w and since M is non-wrapping, every letter in w has at least one letter that is greater than or equal to it before it in the previous strictly increasing subsequence.

Consider collapsing this word into the empty MLQ ϵ, that is $M' = w_n \to (\cdots (w_1 \to \epsilon))$. We claim that each strictly increasing subsequence of w lands in the same row. To see this, observe that $e_i^{\downarrow}$ is not defined on non-wrapping multiline queues for all i. Assume the first $r-1$ strictly increasing subsequences have landed in their respective rows and consider the process of collapsing the rth increasing subsequence of w. When collapsing the first entry w_i of this subsequence, the multiline queue obtained by appending a row with a ball x in column w_i will be non-wrapping and so $e_{r-1}^{\downarrow}$ is not defined. Indeed, since w_i is weakly less than the previous entry we collapsed, there will be a ball on row $r-1$ that is weakly to the right of x. Therefore, x lands on row r after collapsing. When collapsing the kth entry w_{i+k-1} of this subsequence, the multiline queue obtained by appending a row with a ball x in column w_{i+k-1} will be wrapping and $e_{r-1}^{\downarrow}$ will act by moving x to row r. But after this application, the multiline queue will be non-wrapping for the same reason as above, so x lands on row r as well. As M and M' have the same row content, they must be equal.

Now consider inserting this word into the empty SSAF ε with basement containing all positive integers, that is $F' = w_n \to (\cdots (w_1 \to \varepsilon))$. We claim that each strictly increasing subsequence of w lands in the same row as well. Assume it has for the first $r-1$ rows and consider the process of inserting the rth strictly increasing subsequence of w, $w_i < \cdots < w_{i+k}$. Let $1 \le j \le k$. When inserting w_{i+j}, there will always be an entry a on row $r-1$ that has nothing above it since w_{i+j} has at least one letter that is greater than or equal to it before it in the previous strictly increasing subsequence. In the case where w_{i+j} comes across some entry on row $r-1$ with an entry b already on top of it before reaching a, the entry b will be strictly smaller than it so w_{i+j} bumps b. But since b is smaller than w_{i-j}, it will still come across a, and stay on row r. Thus, F' is the unique SSAF that has the same row content as M or in other words, $F' = \Phi(M)$.

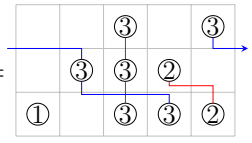
It remains to show that $\texttt{type}(M)$ is equal to the shape of $\Phi(M)$. This claim will follow from the claim that, when collapsing/inserting the rth strictly increasing subsequence of w, the set of entries on row $r-1$ that have balls paired to them/entries placed on top of them are equal. Inducting on the number of rows we have inserted, assume we are now inserting the rth strictly increasing subsequence of w. Assume for the sake of contradiction that there is an entry a on the $r-1$st row of $\Phi(M)$ that had no entries place on top of it but that the ball x at site $(r-1,a)$ in M is paired to a ball on row r. The ball x is paired to a ball on row r if and only if the number of paired balls to its left on row $r-1$ is less than the number of balls weakly to its left on row r. When considering $\Phi(M)$, this implies that the number of entries on row r to the left of a is strictly less than the number of entries on row r that are less than or equal to a. In particular, there is an entry, say b that is less than or equal to a strictly to the right of a in row r. But since there is no entry above a on row r, this violates the triple rule implying that $\Phi(M)$ is not semistandard, a contradiction. □

The reader should note that the SSAF F appearing in Example 2.4.2 is the image of the MLQ M appearing in Example 4.1.9 under this bijection Φ.

4.2 Crystal operators and the type of MLQs

In this section, we describe how column crystal operators interact with the type of a multiline queue. These results originally appeared in [18] and Corollary 4.2.8 is used to prove that a specialization of a quasisymmetric Macdonald polynomial expands positively in a quasisymmetric generalization of the Schur basis. The reader should note that these results apply to all MLQs, not just non-wrapping MLQs.

For notation, we write $L_M(r,i)$ to mean the label of the ball at site (r,i) in M. When there is no ball at site (r,i), we set $L_M(r,i)=0$. We also let L_M refer the array of integers obtained by placing $L_M(r,i)$ at the site (r,i) as illustrated in Figure 4.1.

$M =$ 

$L_M =$

0	0	3	0	3
0	3	3	2	0
1	0	3	3	2

Figure 4.1: An example of a multiline queue M with its corresponding labelled array L_M.

This is, of course, an equivalent way of describing the labelled multiline queue L_M. Our motivation for this is simply to make the statements of the results below more clear. We

begin with some preliminary lemmas. This first lemma states that the labels on any sequence of consecutive balls in a column must be weakly increasing from top to bottom.

Lemma 4.2.1. *If $L_M(s,i) > 0$ for all $r \geq s \geq t$, then $L_M(r,i) \leq L_M(r-1,i) \leq \cdots \leq L_M(t,i)$.*

Proof. We induct on $s-t$. The base case $s=t$ is immediate. Assume the statement is true for $s \geq r \geq k$ for some $k \geq t$. Let x and y be the balls at sites (k,i) and $(k-1,i)$ respectively. Then, during the pairing process, either x pairs to y, in which case $L_M(k,i) = L_M(k-1,i)$, or y was already paired, in which case $L_M(k,i) \leq L_M(k-1,i)$. □

This next lemma states that a ball labelled $\ell > r$ in row r cannot have a smaller labelled ball directly to its left unless there is a ball with the same label directly above.

Lemma 4.2.2. *If $L_M(r,i) > r$ and $L_M(r+1,i) \neq L_M(r,i)$ then either $L_M(r,i-1) = 0$ or $L_M(r,i-1) \geq L_M(r,i)$.*

Proof. If $L_M(r,i-1) = 0$ we are done, so suppose $L_M(r,i-1) > 0$. Since $L_M(r,i) > r$ there is a ball x on row $r+1$ that pairs down to the ball at site (r,i). By assumption, $L_M(r+1,i) \neq L_M(r,i)$ so x is not at site $(r+1,i)$. This implies that x skipped over the ball at site $(r,i-1)$ during the pairing process. Thus, $L_M(r,i-1)$ is at least the label of x and since x pairs down to the ball at site (r,i), $L_M(r,i-1) \geq L_M(r,i)$. □

As we will see, $L_{e_i^{\leftarrow}(M)}$ only differs from L_M in a specific subset of its sites, which we call the i-active region of M.

Definition 4.2.3. *Suppose $e_i^{\leftarrow}(M)$ is defined, let x be the ball at site $(r,i+1)$ that moves to site (r,i) in $e_i^{\leftarrow}(M)$, and let ℓ be the label of x. We define the **i-active region** of M as the collection of sites*

$$\mathtt{act}_i(M) = \{(s,j) : p \leq s \leq r, i \leq j \leq i+1\}$$

*where $p \leq r$ is the maximal row such that $L_M(p-1,i) = 0$ or $L_M(p-1,i) \geq \ell$. If no such site exists, we set $p = 1$ and say M is **i-full**. We also say a row is **i-active** if $\mathtt{act}_i(M)$ contains a site on that row.*

Example 4.2.4. *We give an example of various active regions in the multiline queue M shown in Figure 4.2. The 2-active region is shown in red and, as it extends down to the first row, M is 2-full. The 5-active region is shown in blue and stops at row three since there is no ball at site $(5,2)$. The 7-active region is shown in green and stops at row three since there is a ball with label 3 at site $(7,2)$, which is weakly larger than the label of the ball at site $(8,3)$ (the one moved by $e_7^{\leftarrow}$).*

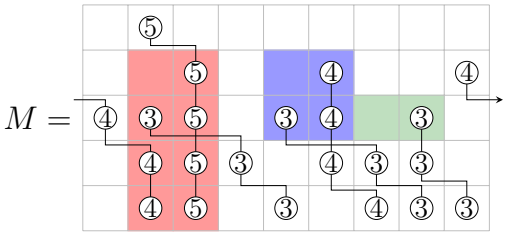


Figure 4.2: A multiline queue M with its 2-, 5-, and 7-active regions highlighted in red, blue, and green respectively. The balls moved by $e_2^{\leftarrow}$, $e_5^{\leftarrow}$, and $e_7^{\leftarrow}$ are in the top right corners of the respective colored regions.

The following lemma states that if x is the ball that moved from M to $e_i^{\leftarrow}(M)$, then M has no vacancies in column i in the active region below x, and the labels of all balls in column $i+1$ are equal to the label of x.

Lemma 4.2.5. *Suppose $e_i^{\leftarrow}(M)$ is defined and let x be the ball at site $(r, i+1)$ that moves to site (r, i) in $e_i^{\leftarrow}(M)$. Then $L_M(s, j) > 0$ for all $(s, j) \in \mathtt{act}_i(M) \setminus \{(r, i)\}$. Moreover, $L_M(r, i+1) = L_M(r-1, i+1) = \cdots = L_M(p, i+1)$ and further, these balls are in the same strand.*

Proof. By construction, $L_M(s, i) > 0$ for all $r > s \geq p$. This and the fact that x exists and is unmatched imply that $L_M(s, i+1) > 0$ for all $r \geq s \geq p$. For the other claim, let $\ell = L_M(r, i+1)$ and apply Lemma 4.2.1 to obtain $L_M(s, i+1) \geq \ell$ for all $r \geq s \geq p$. Assume for sake of contradiction that there exists some $r \geq s \geq p$ such that $L_M(s, i+1) > \ell$. Choose s maximal with respect to this property. Then $L_M(s, i+1) > s$ since by maximality,

$$L_M(s, i+1) > L_M(s+1, i+1) \geq s+1 > s.$$

Thus, we may apply Lemma 4.2.2 and since $L_M(s, i) \neq 0$, this yields $L_M(s, i) \geq L_M(s, i+1)$. But this contradicts the definition of $\mathtt{act}_i(M)$. For the final claim, note that every ball in the ith column of $\mathtt{act}_i(M)$ has label strictly less than ℓ. If there were a ball in column $i+1$ of $\mathtt{act}_i(M)$ that is not paired to the ball above it, then there would have been a ball of label ℓ that skipped over a ball in the ith column that has a label smaller than ℓ. □

We let $L_M^{(r)}$ denote the rth row of L_M. The following lemma will be used as a base case for Lemma 4.2.7

Lemma 4.2.6. *Suppose $e_i^{\leftarrow}(M) \neq M$, and let x be the ball at site $(r, i+1)$ that moves to site (r, i) in $e_i^{\leftarrow}(M)$. Then L_M and $L_{e_i^{\leftarrow}(M)}$ match above row r. Further, $L_{e_i^{\leftarrow}(M)}^{(r)} = s_i \cdot L_M^{(r)}$.*

Proof. The first claim follows directly from the fact that the labelling procedure is carried out from top to bottom and M and $e_i^{\leftarrow}(M)$ match above row r. For the other claim, we must prove the following:

(1) $L_{e_i^{\leftarrow}(M)}(r, i+1) = 0$

(2) $L_{e_i^{\leftarrow}(M)}(r, i) = L_M(r, i+1)$

(3) $L_{e_i^{\leftarrow}(M)}(r, j) = L_M(r, j)$ for $j \neq i, i+1$.

(1) follows directly from the definition of $e_i^{\leftarrow}$. We now show that (2) and (3) are implied by the following statement which we refer to as ($\star$): if there exists a ball at site $(r+1, i+1)$ of M, it does not pair down to x.

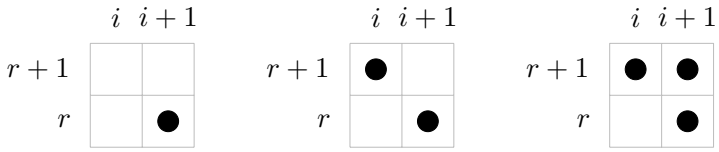


Figure 4.3: The configurations of balls in M under consideration.

First assume there is no ball at site $(r+1, i+1)$, or in other words $L_M(r+1, i+1) = 0$. Then the configuration of balls is either the left or center configuration in Figure 4.3. In the case where no ball on row $r+1$ pairs down to x in M, we are in the left configuration and (2) and (3) are trivial. In the case where there is a ball on row $r+1$ that pairs down to x in M, we could be in either configuration. In either case, letting x' be the ball at site (r, i) in $e_i^{\leftarrow}(M)$, the ball on row that pairs to x in M will also pair to x' in $e_i^{\leftarrow}(M)$. (2) and (3) follow directly from this.

Now assume there is a ball at site $(r+1, i+1)$, or in other words $L_M(r+1, i+1) > 0$. Then the configuration of balls is the right configuration in Figure 4.3. By assumption, the ball y that pairs down to x is not at site $(r+1, i+1)$ of M. Let $(r+1, j)$ be the site containing y in M. Let x' be the ball at site (r, i) of $e_i^{\leftarrow}(M)$. During the pairing process between rows $r+1$ and r in $e_i^{\leftarrow}(M)$, the ball at site $(r+1, j)$ in $e_i^{\leftarrow}(M)$ will still pair to x'. This is because $j \neq i$, guaranteeing that x' will still be weakly to the left of the ball at site $(r+1, j)$ (including possible wrappings), and the fact that $e_i^{\leftarrow}$ does not change the relative order of the balls on row r. Since the pairing involving the ball that moved remained unchanged, the other pairings must have as well, implying (2) and (3).

Now what is left is to prove ($\star$). Let $\ell = L_M(r, i+1)$ be the label of x in M and choose $q \geq r$ to be minimal such that $L_M(q+1, i+1) \neq \ell$. If no such row exists, let q be the

number of rows in M. Observe that $r \leq q \leq \ell$ by our choice of q and since any ball above row ℓ will have been assigned a label strictly greater than ℓ. Therefore, we consider the following four cases:

Case i. *($r = q = \ell$)* Since $r = \ell$, there can not be a ball on row $r+1$ in M that pairs down to x otherwise, x would have a label larger than ℓ.

Case ii. *($r = q < \ell$)* Since $r < \ell$, x is paired above in M and since $r = q$, $L_M(r+1, i+1) \neq \ell$. This implies that the ball that pairs down to x is not at site $(r+1, i+1)$ of M.

Case iii. *($r < q = \ell$)* Since $r < \ell$, x is paired above in M, since $r < q$, $L_M(r+1, i+1) = \ell$ and since $q = \ell$, $L_M(q, i) \geq \ell$. We claim that $L_M(s, i) > 0$ for all $r < s \leq q$. By choice of q, $L_M(s, i+1) > 0$ for all $r \langle s \leq q$. If there were an s such that $L_M(s, i) = 0$, then there is a ball at site $L_M(s, i+1)$ that would be unmatched in M for if not, x would be matched. But that would imply x is not the highest unmatched ball in M, a contraction. Thus, we may apply Lemma 4.2.1 to the portion of column i we are considering which implies that $L_M(r+1, i) \geq \ell$. If $L_M(r+1, i) > \ell$, the ball at site $(r+1, i)$ in M would pair to x, which contradicts x being labelled ℓ. So $L_M(r+1, i) = \ell$ implying that the ball that pairs down to x is not at site $(r+1, i+1)$ of M.

Case iv. *($r < q < \ell$)* Since $r < \ell$, x is paired above in M, since $r < q$, $L_M(r+1, i+1) = \ell$, and since $q < \ell$, the ball at site $(q, i+1)$ is paired above. Further, by definition of q, $L_M(q+1, i+1) \neq \ell$ and so we may apply Lemma 4.2.2 to obtain $L_M(q, i) \geq L_M(q, i+1) = \ell$. Thus, the argument of *Case iii* applies. □

This last lemma states when $e_i^{\leftarrow}$ is defined, the the labels in the active region $\mathtt{act}_i$ swap between columns i and $i+1$.

Lemma 4.2.7. *Suppose $e_i^{\leftarrow}(M) \neq M$ and let p and r be as defined in Definition 4.2.3. Then*

$$L^{(s)}_{e_i^{\leftarrow}(M)} = \begin{cases} s_i \cdot L^{(s)}_M & p \leq s \leq r, \\ L^{(s)}_M & \textit{otherwise.} \end{cases}$$

Proof. We first prove this under the assumption that M is i-full. Induct on the number i-active rows in M. The basecase, when M has just one i-active row, is precisely Lemma 4.2.6. Now assume M has more than one i-active row and let M' be the MLQ obtained by deleting the bottom row of M. As we have assumed M is i-full, the bottom row of M is i-active so M' has one less i-active row than M. Thus, by the induction hypothesis (IH), the statement holds for M' or equivalently, the statement holds for every row of M except row 1. We must prove the following:

(1) $L_{e_i^{\leftarrow}(M)}(1,i) = L_M(1,i+1)$,

(2) $L_{e_i^{\leftarrow}(M)}(1,i+1) = L_M(1,i)$,

(3) $L_{e_i^{\leftarrow}(M)}(1,i') = L_M(1,i')$ for $i' \neq i, i+1$.

Since the first and second rows of M are i-active, there are balls at sites $(1,i)$, $(1,i+1)$, $(2,i)$, and $(2,i+1)$ by Lemma 4.2.5. Also by Lemma 4.2.5 the ball at site $(2,i+1)$ pairs down to the ball at site $(1,i+1)$. Thus, there are only two possible configurations of M that are relevant depending on whether the ball at site $(2,i)$ pairs to the ball directly below it or not. These can be seen in Figure 4.2, keeping in mind that M may contain wrappings (i.e. we allow for $j > i$ and $i > j'$). The reader should note that the proof of (1) will be identical for either configuration, we only distinguish which case we are in for the proofs of (2) and (3).

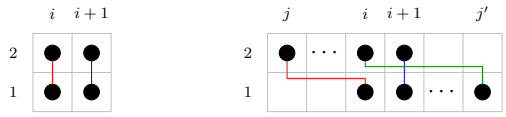


Figure 4.4: The two configurations of M relevant to the induction step.

We also include what we claim the pairings will look like in $e_i^{\leftarrow}(M)$ in Figure 4.5. The blue pairing implies (1), the red pairing implies (2), and the green pairing, along with the first two claims, imply (3). The configuration on the left has no green pairing. This is because the blue and red pairings alone imply (3).

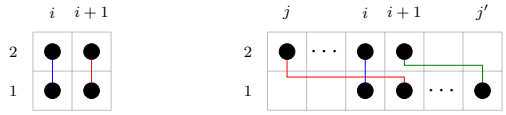


Figure 4.5: The two configurations of $e_i^{\leftarrow}(M)$ and the claimed pairings.

Let $\ell = L_M(r, i+1)$, the label of the ball moved by $e_i^{\leftarrow}$. We first prove (1). By Lemma 4.2.5 and the induction hypothesis (IH), $L_{e_i^{\leftarrow}(M)}(2,i) = \ell$. Assume for sake of contradiction that $L_{e_i^{\leftarrow}(M)}(1,i) \neq \ell$. By Lemma 4.2.1, $L_{e_i^{\leftarrow}(M)}(1,i) > \ell$, implying that there is a ball at some site $(2,j)$ that pairs down to the ball at site $(1,i)$ of $e_i^{\leftarrow}(M)$. Again by the IH, $L_M(2,j) = L_{e_i^{\leftarrow}(M)}(2,j) > \ell$ so there is a ball y at site $(2,j)$ of M with label larger than ℓ that does not pair to any ball before column i. But since the first row of M is i-active, $L_M(1,i) < \ell$ implying that y would pair to the ball at site $(1,i)$ of M so $L_M(1,i) > \ell$, a contradiction.

Next, we prove (2). Let x be the ball at site $(1,i)$ of M, let y be the ball at site $(2,j)$ of M that pairs down to x and let $m = L_M(1,i)$. If $j = i$, then we are in the left configuration in Figure 4.4. In this case, (2) follows by an argument almost identical to that of (1). If $j = i+1$, then $m = \ell$, contradicting the definition of $\mathtt{act}_i(M)$. Therefore, we may assume $j \neq i, i+1$, so we are in the right configuration in Figure 4.4. Let x' be the ball at site $(1,i+1)$ of $e_i^{\leftarrow}(M)$, and let y' be the ball at site $(2,j)$ of $e_i^{\leftarrow}(M)$. By the IH, $L_{e_i^{\leftarrow}(M)}(2,j) = L_M(2,j) = m$. Assume for sake of contradiction that y' does not pair to x'. Then either y' pairs to some new ball that came before x' in the pairing process or some new ball z' pairs to x'. The first scenario can not occur since the only new ball before x' is labelled $\ell > m$ so it will have already been paired. The second scenario also can not occur but requires a more detailed explanation. In order for such a z' to exist, It must come before y' in pairing order and must be in column i. Such a z' does exist, the ball at site $(2,i)$ does have label $\ell > m$. But in the proof of Claim 1, it was shown to pair down to the ball at site $(2,i)$. Thus, y' does pair down to x' and so $L_{e_i^{\leftarrow}(M)}(1,i+1) = m = L_M(1,i)$.

Lastly, we prove (3). If we are in the left configuration in Figure 4.4, this follows from Claim 1 and Claim 2. Thus, assume we are in the right configuration in Figure 4.4. Let u be the ball at site $(2,i)$ of M, let w be the ball at site $(1,j')$ of M that u pairs down to, and let $L_M(2,i) = n$. Since we are in the right configuration, we can not have $j' = i$ and by Claim 1, we can not have $j' = i+1$. Further, with m as defined above, $n \leq m$ by Lemma 4.2.1. Let u' be the ball at site $(2,i+1)$ of $e_i^{\leftarrow}(M)$ and let w' be the ball at site $(1,j')$ of $e_i^{\leftarrow}(M)$. By the IH, $L_{e_i^{\leftarrow}(M)}(2,i+1) = L_M(2,i) = n$. Assume for sake of contradiction that u' does not pair down to w'. Then, again, either u' pairs to some new ball that came before w' in the pairing process or some new ball v' pairs to w'. The first scenario can not occur because the only new ball that could come before w' is the ball at site $(1,i+1)$. But by the proof of (2), the ball at site $(1,i+1)$ of $e_i^{\leftarrow}(M)$ is paired before we determine which ball u' pairs to. The second scenario can not occur since there is not even a candidate for a new ball. Thus, u' does pair down to w' and so $L_{e_i^{\leftarrow}(M)}(1,j') = L_M(1,j')$. Moreover, since the balls pictured on the top row of the right configuration in Figure 4.4 paired to the same set of balls in M and $e_i^{\leftarrow}(M)$, the pairings of the remaining balls will remain the same, proving (3).

We now drop the assumption that M is i-full and prove the statement in full generality. Let p be as in Definition 4.2.3. Then the first $p-1$ rows of M are not i-active. Let M' be the MLQ obtained by deleting the first $p-1$ rows of M. Then M' is i-full so the statement is true for M', or equivalently, for all but the bottom $p-1$ rows of M. Thus, it suffices to show that $L_{e_i^{\leftarrow}(M)}(p-1,i') = L_M(p-1,i')$ for all i'. To this end, restrict attention to rows p and $p-1$ of M. The relevant configuration of balls in M is then as in Figure 4.6 (left). We also include what we claim the pairings will look like in $e_i^{\leftarrow}(M)$ in Figure 4.6 (right).



Figure 4.6: The relevant configuration of M (left), the claimed pairings in $e_i^{\leftarrow}(M)$ (right).

Let x be the ball at site $(p, i+1)$ of M and let y be the ball at site $(p-1, j)$ of M that x pairs down to. Since the statement is true for row p, $L_{e_i^{\leftarrow}(M)}(p, i) = L_M(p, i+1) = \ell$. Let x' and y' be the balls at sites (p, i) and $(p-1, j)$ of $e_i^{\leftarrow}(M)$ respectively. Assume for sake of contradiction that x' does not pair down to y'. Then either x' pairs to some new ball that came before y' in the pairing process or some new ball z' pairs to y'. To show the first scenario can not occur, first note that the only candidate for such a ball would be at site $(p-1, i)$ of $e_i^{\leftarrow}(M)$. If no such ball exists, we are done, so assume $L_{e_i^{\leftarrow}(M)}(p-1, i) > 0$. By definition of $\mathtt{act}_i(M)$, $L_M(p-1, i) > \ell$ and since the labels of balls greater than ℓ on row p of M and $e_i^{\leftarrow}(M)$ are the same, the pairing process will be the same implying that $L_{e_i^{\leftarrow}(M)}(p-1, i) > \ell$. This prohibits the first scenario since x' has label ℓ and therefore will not pair to the ball at site $(p-1, i)$. The second scenario also can not occur since the only candidate for z' would be the ball at site $(p, i+1)$ which has label strictly less than ℓ. Thus, $L_{e_i^{\leftarrow}(M)}(p-1, j) = \ell$.

Let u and v be the balls at sites (p, i) and $(p-1, j')$ of M respectively and let $m = L_M(p, i)$. By definition of $\mathtt{act}_i(M)$, $m < \ell$, and since the statement is true for row p, $L_{e_i^{\leftarrow}(M)}(p, i+1) = m$. Let u' and v' be the balls at sites $(p, i+1)$ and $(p-1, j')$ of $e_i^{\leftarrow}(M)$ respectively. Assume for sake of contradiction that u' does not pair down to v'. Then either u' pairs to some new ball that came before v' in the pairing process or some new ball w' pairs to v'. The first scenario can not occur since u' comes across the same balls as u does. The second scenario can not occur since there is no candidate for w'. Thus, $L_{e_i^{\leftarrow}(M)}(p-1, j') = m$. Lastly, for $k \neq j, j'$, $L_{e^{\leftarrow(i)}}(p-1, k) = L_M(p-1, k)$ follows from the fact that the balls at sites (p, i) and $(p, i+1)$ in $e_i^{\leftarrow}(M)$ paired to balls at the same sites as they did in M and thus the pairings for every ball other than these two will be the same as they were in M, completing the proof. □

Finally, we may characterize how a column crystal operator acts on the type of an MLQ.

Corollary 4.2.8. *For any MLQ M,*

$$\mathtt{type}(e_i^{\leftarrow}(M)) = \begin{cases} s_i \cdot \mathtt{type}(M) & \text{if } M \text{ is } i\text{-full,} \\ \mathtt{type}(M) & \text{otherwise.} \end{cases}$$

We may also translate this to a statement about SSAF via Proposition 4.1.10

Corollary 4.2.9. *For any $F \in \mathrm{SSAF}(\alpha)$, let a be the entry of F corresponding to the leftmost unmatched $i+1$ in $\pi_i(\mathrm{rw}(F))$. Then $\tilde{e}_i(F) \in \mathrm{SSAF}(s_i \cdot \alpha)$ if and only if there is an i in every row below a and each such i is in a column that is strictly shorter than the column containing a.*

In Figure 4.7, we include the crystal graph from Figure 2.1 with SSAF vertices for comparison. We dash the edges corresponding to the $\tilde{e}_i$ that change the shape of the SSAFs. The reader should note that by deleting the dashed edges, we obtain the 6 nonsymmetric subgraphs shown on the right in Figure 2.2.

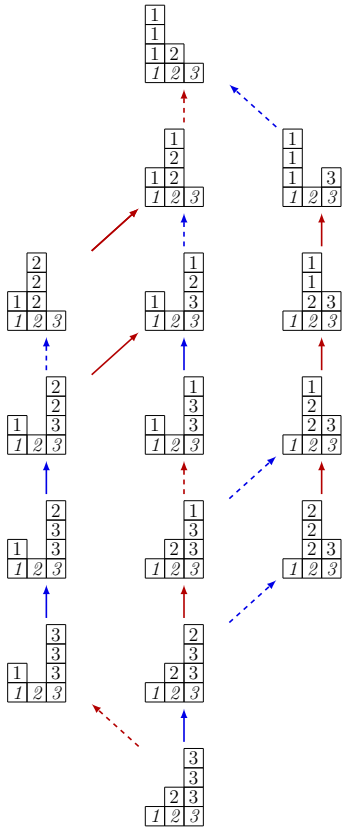


Figure 4.7: The crystal graph appearing in Figure 2.1 under the map Ψ. The red edges correspond to $\tilde{e}_1$ and the blue edges correspond to $\tilde{e}_2$. The edges corresponding to a change of shape are dashed.

4.3 Applications to atom-positivity

In this section, we give an broad overview of how Corollary 4.2.8 can be used to prove Demazure atom-positivity. We refer to this as atom-positivity for brevity. As an explicit example, we discuss atom-positivity of the ASEP polynomial $F_\alpha(X;q,0)$, which was proven in [18]. We give a sketch of their proof and highlight the use of Corollary 4.2.8. We will skip many details in this section but include references throughout. The reader should note that everything in this section has connections to nonsymmetric Macdonald polynomials, although describing these connections is beyond the scope of this thesis. See [6] and [18] for more details.

Fix $n > 0$, let $X = \{x_1, \ldots, x_n\}$ denote a finite set of variables, and let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a weak composition of length n. We define atom-positivity for polynomials $f \in R[X]$ where R is some *polynomial ring* over $\mathbb{Q}$. A polynomial $f \in R[X]$ is **atom-positive** if the coefficients c_α in the expansion

$$f = \sum_\alpha c_\alpha \mathcal{A}_\alpha, \tag{4.1}$$

are *polynomials* with non-negative integer coefficients. In (4.1), the sum is over all weak compositions α of length n. The reader should consider atom-positivity as a nonsymmetric analogue of Schur-positivity.

The **ASEP polynomials** $F_\alpha(X;q,t)$, studied in [6], are a two-parameter deformation of the Demazure atoms. They may be described as the generating function of a generalization of MLQs of type α (see [6]). At the specialization $t = 0$, they reduce to the generating function for the MLQs we defined in Section 4.1:

$$F_\alpha(X;q,0) = \sum_{\substack{M \in \mathrm{MLQ}(\mathtt{sort}(\alpha)) \\ \mathtt{type}(M)=\alpha}} q^{\mathtt{maj}(M)} x^{\mathtt{wt}(M)}, \tag{4.2}$$

where the **major index** $\mathtt{maj}(M)$ of M encodes the pairings of M that wrap [19, Definition 3.9]. In particular, the MLQs that satisfy $\mathtt{maj}(M) = 0$ are precisely the non-wrapping MLQs. Hence, $F_\alpha(X;0,0) = \mathcal{A}_\alpha(X)$.

In [18], Corollary 4.2.8 is used as a key step in proving that $F_\alpha(X;q,0)$ is atom-positive. Recall from the discussion above that this means the coefficients appearing in the expansion

$$F_\alpha(X,q,0) = \sum_\beta K_{\alpha,\beta}(q)\mathcal{A}_\beta(X), \tag{4.3}$$

are *polynomials* in q with non-negative integer coefficients, that is $K_{\alpha,\beta}(q) \in \mathbb{Z}_{\geq 0}[q]$.

To be more specific, the authors in [18] proved that $K_{\alpha,\beta}(q) = \sum_T q^{\mathtt{charge}(T)}$ where the sum is over a certain set of SSYT determined by α and β. See [19] for a discussion of this `charge` statistic. To prove this, the authors use the **collapsing map** ρ, introduced in [19]:

$$\rho : \mathrm{MLQ}(\lambda, n) \longrightarrow \bigcup_{\mu \leq \lambda} \mathrm{MLQ}_0(\mu, n) \times \mathrm{SSYT}(\mu', \lambda'). \tag{4.4}$$

They proved that this map is a bijection such that $\mathtt{maj}(M) = \mathtt{charge}(Q)$ where $(N, Q) =: \rho(M)$ [19, Theorem 4.20]. This map can be extended to all binary matrices [19, Definition 4.7] where it may be considered as an analogue of the RSK correspondence for MLQs.

An important property of ρ is that it is defined as a particular composition of row crystal operators. Thus, ρ commutes with column crystal operators by Lemma 4.1.8. We now give a sketch of the proof of (4.3).

Sketch of proof of (4.3). Fix a $Q \in \mathrm{SSYT}(\mu', \lambda')$. The set $\mathrm{MLQ}_0(\beta, n) \times \{Q\}$ has generating function $q^{\mathtt{charge}(Q)}\mathcal{A}_\beta(X)$. Consider the preimage of this set under ρ:

$$\mathrm{MLQ}_0(\beta, n) \times \{Q\} \xrightarrow{\rho^{-1}} \mathrm{MLQ}(\lambda, n).$$

Our goal is to show that this preimage is wholly contained in $\mathrm{MLQ}(\alpha)$ for some α, as this will prove (4.3).

Let $N, N' \in \mathrm{MLQ}_0(\beta, n)$ be non-wrapping MLQs of type β, let $\lambda = \mathtt{sort}(\beta)$, and let $\alpha = \mathtt{type}(\rho^{-1}(N, Q))$. Our goal may now be restated as showing $\mathtt{type}(\rho^{-1}(N', Q)) = \alpha$. Under the column crystal operators, the crystal graph with vertex set $\mathrm{MLQ}_0(\lambda, n)$ is a connected graph [18]. Thus, there is a sequence of column crystal operators $\mathbf{s} = g_m \cdots g_1$ with $g_j \in \{e_{i_j}^{\leftarrow}, f_{i_j}^{\rightarrow}\}$ such that $\mathbf{s}N = N'$. Since ρ commutes with column crystal operators,

$$\mathtt{type}(\rho^{-1}(N', Q)) = \mathtt{type}(\rho^{-1}(\mathbf{s}N, Q)) = \mathtt{type}(\mathbf{s}\rho^{-1}(N, Q)).$$

Thus, what remains to show is the following implication:

$$\mathtt{type}(\mathbf{s}N) = \mathtt{type}(N) \;\Rightarrow\; \mathtt{type}(\mathbf{s}\rho^{-1}(N, Q)) = \mathtt{type}(\rho^{-1}(N, Q)). \tag{4.5}$$

That is, if $\mathbf{s}$ preserves the type of a non-wrapping MLQ N, then it preserves the type of the MLQ $\rho^{-1}(N, Q)$ in the preimage of ρ. This is stated and proved in full detail as [18, Proposition 2.38]. Corollary 4.2.8 plays an important role in this proof as it allows us to say the following: if there ever is a change in type during the application of $\mathbf{s}$, the MLQ in

the preimage of ρ must be in a very specific configuration, which forces the non-wrapping MLQ N to contain this configuration as well. This yields enough information to prove (4.5), though more careful combinatorial analysis must be done to obtain the result. The reader is encouraged to see [18] for the full details. □

To conclude, having an understanding of when the column crystal operators change the type of an MLQ allow us to use these crystal operators to prove atom-positivity results in the nonsymmetric setting. The author hopes that this understanding will lead to more atom-positivity results in the future.

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