Amalgamated free products of tracial von Neumann algebras

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Abstract

We motivate and construct the amalgamated free product of tracial von Neumann algebras, originally introduced by Popa in [Pop93]. In this thesis, we describe various properties of the amalgamated free product, including when it yields a II_1 factor, and use the amalgamated free product to construct both Ueda's HNN extension of σ-finite von Neumann algebras in [Ued05] in the case where all von Neumann algebras are tracial, as well as the II_1 factor in [CIK23].

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Dedication

To my Nana and Grandma.

Table of Contents

Introduction **1**

1 Preliminaries on finite von Neumann algebras **2**

1.1 Tracial von Neumann algebras 2

1.2 Examples 8

1.3 Ultraproducts of von Neumann algebras 11

1.4 1-Bounded Entropy 15

2 Motivation **21**

2.1 Amalgamated free products of groups 21

2.2 Algebras and operator valued random variables 23

3 Construction of amalgamated free products **27**

3.1 Hilbert bimodules and relative tensor products 27

3.2 Construction of amalgamated free products 30

3.3 Free products and Free Group Factors 35

3.4 Intertwining by bimodules 37

4 Applications and Constructions **42**

4.1 HNN extensions of von Neumann algebras 42

4.2 Chifan-Ioana-Kunnawalkam Elayavalli's lifting lemma 50

4.3 The exotic II_1 factor 55

Bibliography **65**

Introduction

Some Constructions of tracial von Neumann algebras, and more generally unital C^*-algebras can be motivated by well-known constructions in group theory. One such construction is given by free products and amalgamated free products. Amalgamated free products of C^*-algebras were first introduced by Voiculescu in [Voi85] in the context of free probability. The amalgamated free product of tracial von Neumann algebras was introduced by Popa in [Pop93] as a way to construct subfactors with arbitrary (within reason) Jones index. Since their introduction, amalgamated free products have been studied in several capacities.

The goal of this thesis is to motivate the construction of the amalgamated free product, and to outline various properties. More specifically, we are interested in when certain amalgamated free products have prescribed properties. Chapter 1 will introduce the necessary preliminaries on tracial von Neumann algebras, as well as elementary equivalence and 1-bounded entropy.

The purpose of Chapter 2 is to motivate and provide intuition for the construction of the amalgamated free product of tracial von Neumann algebras. This chapter introduces the amalgamated free product of groups and unital associative algebras, as well as outlines Voiculescu's construction in [Voi95] via operator-valued free probability.

Chapter 3 introduces the construction of the amalgamated free product and proves a universal characterization. Free products (described by amalgamation over $\mathbb{C}$) and the interpolated free group factors, introduced by Rădulescu and Dykema in [Răd94] and [Dyk94] respectively, are also discussed. Popa's theory of intertwining by bimodules is used to prove that under fairly general conditions, the amalgamated free product is a II_1 factor.

Chapter 4 consists of some auxiliary constructions using the amalgamated free product. This includes the HNN extension of tracial von Neumann algebras, introduced by Ueda in [Ued05] (although rephrased for the tracial setting in place of the σ-finite setting), as well as the constructions in [CIK23] of a non-Γ separable II_1 factor for which every ultrapower has non-positive 1-bounded entropy, and thus is not elementarily equivalent to $L(\mathbb{F}_n)$ for any $n \geq 2$.

Chapter 1

Preliminaries on finite von Neumann algebras

Much of this section benefits heavily from [Pet25], [KR97], and [Tak79].

1.1 Tracial von Neumann algebras

Throughout this section, we will let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$.

Definition 1.1.1. Suppose $(T_i)_{i\in I}$ is a net of operators in $\mathbb{B}(\mathcal{H})$ and $T \in \mathbb{B}(\mathcal{H})$. We define the following kinds of convergence:

- We say $T_i \to T$ in the norm topology when $\|T_i - T\|_{\text{op}} \to 0$
- We say $T_i \to T$ in the strong operator topology (SOT) when $\forall \xi \in \mathcal{H}$, $\|(T_i - T)\xi\| \to 0$
- We say $T_i \to T$ in the weak operator topology (WOT) when $\forall \xi, \eta \in \mathcal{H}$, $\langle T_i\xi, \eta\rangle \to \langle T\xi, \eta\rangle$.

We remark that the SOT and WOT closures of convex subsets of $\mathbb{B}(\mathcal{H})$ agree.

Definition 1.1.2 (von Neumann algebra)**.** Let $M \subseteq \mathbb{B}(\mathcal{H})$. We say that M is a *von Neumann algebra* when M a WOT-closed unital selfadjoint subalgebra (called a $*$-subalgebra). Since algebras are convex, it follows that M is also SOT-closed. Since von Neumann algebras have to be unital subalgebras of $\mathbb{B}(\mathcal{H})$, we use the term *von Neumann subalgebra* to refer to a unital inclusion of von Neumann algebras.

Note that $\mathbb{B}(\mathcal{H})$ is itself a von Neumann algebra. For $A \subset \mathbb{B}(\mathcal{H})$, we let $W^*(A)$ denote the smallest von Neumann subalgebra of $\mathbb{B}(\mathcal{H})$ containing A. If $M_1, M_2 \subset \mathbb{B}(\mathcal{H})$ are von Neumann algebras, we let $M_1 \vee M_2 = W^*(M_1 \cup M_2)$, and more generally for a collection of von Neumann subalgebras $(M_i)_{i\in I}$, we let

$$\bigvee_{i\in I} M_i := W^*\left(\bigcup_{i\in I} M_i\right)$$

If M_1, M_2 are von Neumann subalgebras of a von Neumann algebra M, then we still use $M_1 \vee M_2$ to refer to the von Neumann subalgebra of M generated by both M_1 and M_2.

This notation requires that the two von Neumann algebras be subsets of a larger ambient algebra. If we have $M_1 \subseteq \mathbb{B}(\mathcal{H}_1)$ and $M_2 \subseteq \mathbb{B}(\mathcal{H}_2)$ two von Neumann algebras acting on distinct Hilbert spaces, then the notation $M_1 \vee M_2$ is no longer well-defined. In fact, there are multiple non-isomorphic ways to construct some von Neumann algebra M generated by (isomorphic copies of) M_1 and M_2. This thesis provides a presentation of one such method, and its properties and applications.

Definition 1.1.3 (Commutant)**.** Let $A \subseteq \mathbb{B}(\mathcal{H})$. We define the *commutant* of A, denoted A' by

$$A' := \{T \in \mathbb{B}(\mathcal{H}) : ST = TS, \forall S \in A\}$$

Note that when A is selfadjoint, A' is always a unital $*$-subalgebra of $\mathbb{B}(\mathcal{H})$. The *bicommutant* of A is defined by $A'' = (A')'$. We always have that $A \subseteq A''$.

Lemma 1.1.4. Let $A \subseteq \mathbb{B}(\mathcal{H})$ be a unital $*$-subalgebra. Then, $\forall \xi_1, \ldots, \xi_n \in \mathcal{H}$, $x \in A''$, and $\varepsilon > 0$, $\exists y \in A$ such that $\|(x-y)\xi_i\| < \varepsilon$ for each $i = 1, \ldots, n$.

Proof. We begin by considering the $n = 1$ case.

Let $\xi \in \mathcal{H}$, $x \in A''$, and $\varepsilon > 0$. Consider the cyclic subspace $\mathcal{K} := \overline{A\xi}$. Let $p : \mathcal{H} \to \mathcal{K}$ be the orthogonal projection. We first claim that $p \in A'$. If this is the case, then we have that $p\xi = \xi$ and thus $x\xi = xp\xi = px\xi \in \mathcal{K}$. Thus, $\exists \eta \in A\xi$ such that $\|x\xi - \eta\| < \varepsilon$. By definition, $\eta = y\xi$ for some $y \in A$, and thus $\exists y \in A$ such that $\|(x-y)\xi\| < \varepsilon$.

Now, we show $p \in A'$. Let $\eta \in \mathcal{H}$. Then, $\exists (x_n)_{n\in\mathbb{N}}$ a sequence in A such that $x_n\xi \to p\eta$. Further, for any $y \in A$, we have $yx_n\xi \to yp\eta$, so $yp\eta \in \mathcal{K}$ and thus $pyp\eta = yp\eta$. Since this holds $\forall \eta \in \mathcal{H}$, it follows that $pyp = yp$ for all $y \in A$. Since A is selfadjoint, we may take y^* in place of y and acquire $py^*p = y^*p$ as well. Then, since p is selfadjoint, we have,

$$py = (y^*p)^* = (py^*p)^* = pyp = yp$$

Thus, $p \in A'$ as desired.

To show the general case, we employ the following "matrix trick". We can view $\mathbb{B}(\mathcal{H})$ as a $*$-subalgebra of $\mathbb{B}(\mathcal{H}^{\oplus n}) \cong \mathbb{M}_n(\mathbb{B}(\mathcal{H}))$ by sending any $x \in \mathbb{B}(\mathcal{H})$ to the diagonal matrix $\mathrm{diag}(x, x, \ldots, x)$. Then, A can be viewed as a diagonal $*$-subalgebra of $\mathbb{B}(\mathcal{H}^{\oplus n})$. Let $\xi_1, \ldots, \xi_n \in \mathcal{H}, x \in A'', \varepsilon > 0$. Then, we view $\xi = (\xi_1, \ldots, \xi_n) \in \mathcal{H}^{\oplus n}$. By the above, we can find some $y = \mathrm{diag}(y, y, \ldots, y) \in A$ such that $\|(x-y)\xi\|_{\oplus n} < \varepsilon$. Therefore, for each $i = 1, \ldots, n$, we have $\|(x-y)\xi_i\| < \varepsilon$. □

Theorem 1.1.5 (von Neumann's bicommutant theorem)**.** Let $A \subseteq \mathbb{B}(\mathcal{H})$ be a unital $*$-subalgebra. Then, $A'' = \overline{A}^{\mathrm{WOT}}$.

Proof. Firstly, let $(x_i)_{i\in I}$ be a net in A' with $x_i \to x \in \mathbb{B}(\mathcal{H})$ WOT. Then, for any $a \in A$

and any $\xi, \eta \in \mathcal{H}$, we have

$$\begin{aligned}\langle (xa - ax)\xi, \eta \rangle &= \langle xa\xi, \eta \rangle - \langle x\xi, a^*\eta \rangle \\ &= \lim_i \langle x_i a\xi, \eta \rangle - \langle x_i \xi, a^*\eta \rangle \\ &= \lim_i \langle (x_i a - a x_i)\xi, \eta \rangle \\ &= 0\end{aligned}$$

Hence A' is WOT-closed. Thus, A'' is WOT-closed and we have $\overline{A}^{\text{WOT}} \subseteq A''$. Since the WOT- and SOT-closures of convex subsets of $\mathbb{B}(\mathcal{H})$ agree, they must also agree on $*$-subalgebras. In particular, $\overline{A}^{\text{SOT}} = \overline{A}^{\text{WOT}}$. We continue by showing $A'' \subseteq \overline{A}^{\text{SOT}}$.

Let $I := \{(F, \varepsilon) : F \subseteq \mathcal{H} \text{ finite}, \varepsilon > 0\}$. We can induce an ordering on I given by $(F, \varepsilon) \geq (G, \delta)$ when $F \subseteq G$ and $\varepsilon \leq \delta$. This forms a directed set. Fix some $x \in A''$. For each $i = (F_i, \varepsilon_i) \in I$, the above lemma states that we can find some $x_i \in A$ such that for every $\xi \in F_i$, we have $\|(x - x_i)\xi\| < \varepsilon_i$. We claim that $x_i \to x$ in SOT. To see this, fix $\xi \in \mathcal{H}$ and $\varepsilon > 0$. Then, $i_0 = (\{\xi\}, \varepsilon) \in I$ and thus $\forall i \geq i_0$ we have $\|(x - x_i)\xi\| < \varepsilon$. Thus, $x \in \overline{A}^{\text{SOT}}$ and hence $A'' \subseteq \overline{A}^{\text{SOT}}$. □

Definition 1.1.6 (Projections)**.** Let $M \subseteq \mathbb{B}(\mathcal{H})$ be a von Neumann algebra. We say an element $p \in M$ is a *projection* when $p^2 = p = p^*$. Each projection p is associated to a closed subspace of $\mathcal{H}$, given by $p\mathcal{H}$. For projections $p_1, p_2 \in M$, we define the infimum $p_1 \wedge p_2$ be the orthogonal projection onto $p_1\mathcal{H} \cap p_2\mathcal{H}$. Similarly, we define the supremum $p_1 \vee p_2$ to be the orthogonal projection onto the closed span of $p_1\mathcal{H} \cup p_2\mathcal{H}$. If $p_1\mathcal{H}$ and $p_2\mathcal{H}$ are orthogonal as closed subspaces of $\mathcal{H}$, then we say p_1 and p_2 are *orthogonal*, denoted $p_1 \perp p_2$, and in this case we have $p_1 \vee p_2 = p_1 + p_2$.

We say that p_1 and p_2 are *Murray von Neumann equivalent* (or just equivalent), denoted $p_1 \sim p_2$, when $\exists v \in M$ a partial isometry such that $v^*v = p_1$ and $vv^* = p_2$. We say p_1 is *subequivalent* to p_2, denoted $p_1 \preceq p_2$, when $\exists v \in M$ a partial isometry such that $v^*v = p_1$ and $vv^* \leq p_2$. Subequivalence can be thought of as an ordering coming from the dimensions of closed subspaces. That is to say, $p_1 \preceq p_2$ if and only if $\dim(p_1\mathcal{H}) \leq \dim(p_2\mathcal{H})$. Murray von Neumann equivalence defines an equivalence relation on the space of projections in M, and subequivalence defines a partial ordering. Further, we have $p_1 \preceq p_2$ and $p_2 \preceq p_1$ implies $p_1 \sim p_2$.

Note that we also have another partial ordering on projections, where $p_1 \leq p_2$ if and only if $p_1\mathcal{H} \subseteq p_2\mathcal{H}$. In this case, we say p_1 is a *subprojection* of p_2. In particular, the subequivalence $p_1 \preceq p_2$ means there's some projection q which is equivalent to p_1 and satisfies $q \leq p_2$.

Definition 1.1.7 (Central support of projection)**.** Let $M \subseteq \mathbb{B}(\mathcal{H})$ be a von Neumann algebra and $p \in M$ a projection. We define the *central support* by

$$z_M(p) := \bigwedge \{q : q \in Z(M) \text{ a projection}, qp = pq = p\}$$

If $A \subseteq M$ is a von Neumann subalgebra and $p \in A$, we use $z_A(p)$ to denote the central support inside of A. In general, we will have $z_M(p) \preceq z_A(p)$.

Definition 1.1.8 (Types of projections and von Neumann algebras)**.** Let $M \subset \mathbb{B}(\mathcal{H})$ be a von Neumann algebra and $0 \neq p \in M$ a projection. We introduce the following notation:

- p is *minimal* if $q \leq p$ implies $q = 0$ or $q = p$ for any projection $q \in M$.
- p is *abelian* when pMp is abelian.
- p is *finite* if whenever $q \leq p$ and $q \sim p$, we have $q = p$ for any projection $q \in M$.

For any von Neumann algebra M, we say that M is of type I when every nonzero projection has a nonzero abelian subprojection. We say M is type II_1 if M has no nonzero abelian projections and 1 is a finite projection in M.

Definition 1.1.9 (Tracial von Neumann algebra)**.** Let $M \subseteq \mathbb{B}(\mathcal{H})$ be a von Neumann algebra and $\tau : M \to \mathbb{C}$ a linear functional. We say that τ is a *trace* when it satisfies the following three properties.

(i) $\tau(x^*x) \geq 0$ for all $x \in M$ (we say τ is *positive*).

(ii) $\tau(1) = 1$ (we say τ is *unital*).

(iii) $\tau(xy) = \tau(yx)$ for all $x, y \in M$ (we say τ is *tracial*).

If $\varphi : M \to \mathbb{C}$ is linear and satisfies only the first two, we say that φ is a *state*. In addition, we have the following two other properties.

(iv) $\tau(x^*x) = 0 \implies x = 0$, for all $x \in M$ (we say τ is *faithful*).

(iiv) τ is WOT-continuous on $(M)_1$ the norm-closed unit ball (we say τ is *normal*)

We say a von Neumann algebra M is *tracial* when M has a normal faithful trace τ (from here on out abbreviated just to "trace"), and we denote tracial von Neumann algebras as the tuple (M, τ). Further, we say $(B, \tau_B) \subseteq (M, \tau_M)$ is an inclusion of tracial von Neumann algebras to mean that $B \subseteq M$ is a von Neumann subalgebra and that $\tau_M|_B = \tau_B$.

If M is a *factor*, meaning $M' \cap M = \mathbb{C}1$, then M has a unique trace. If (M, τ) is a tracial von Neumann algebra and M is a factor, then it is either of type I or type II_1. Further, whenever M is a II_1 factor, the subequivalence relation is completely characterized by comparison of traces. That is to say, for any projections $p, q \in M$, we have $p \prec q$ if and only if $\tau(p) \leq \tau(q)$, and $p \sim q$ if and only if $\tau(p) = \tau(q)$. Further, for any $t \in [0, 1]$, $\exists p_t \in M$ a projection with $\tau(p_t) = t$.

Definition 1.1.10 (GNS construction)**.** Let (M, τ) be a tracial von Neumann algebra. We can define a Hilbert space for which the action of M is "canonical" in some sense. Towards this, we can introduce an inner product on M given by $\langle x, y \rangle = \tau(y^*x)$. This is sesquilinear, with conjugate linearity in the second argument. Further, it's positive definite, since τ is positive and faithful. Therefore, this inner product induces a norm $\|\cdot\|_2$ on M, and we define $L^2(M, \tau) := \overline{M}^{\|\cdot\|_2}$. Then, there is a dense copy of M as vectors in $L^2(M, \tau)$, and we will distinguish $x \in M$ being a vector in $L^2(M, \tau)$ by writing $\hat{x}$. We also have a copy of M inside $\mathbb{B}(L^2(M, \tau))$ defined as follows. For any $x \in M$, we can define $x \cdot \hat{1} := \hat{x}$, and more generally $x \cdot \hat{y} = \widehat{xy}$. This is defined on a dense subspace of $L^2(M, \tau)$ and thus extends uniquely to an action on all of $L^2(M, \tau)$. It is then clear that $\hat{1}$ is a *cyclic* and *separating*

vector for M, meaning that $M\hat{1}$ is dense, and $x\hat{1} = 0 \implies x = 0$. Further, the trace is completely recovered from the inner product by $\tau(x) = \langle x\hat{1}, \hat{1}\rangle$. This implies that there is a canonical trace-preserving embeding of M into $\mathbb{B}(L^2(M,\tau))$ as desired.

Now, we have for any $x \in M$ that

$$\|\hat{x}\|_2^2 = \tau(x^*x) = \tau(xx^*) = \|\widehat{x^*}\|_2^2$$

So in particular, the adjoint map $M \to M$ defined by $x \mapsto x^*$ extends to a map $J : L^2(M,\tau) \to L^2(M,\tau)$, and by the above, this map is an antilinear isometry and satisfies $J^2 = \mathrm{id}$. For any $x \in M$, JxJ is a linear operator on $\mathbb{B}(L^2(M,\tau))$ which acts in the following way:

$$JxJ\hat{y} = Jx\widehat{y^*} = J\widehat{xy^*} = \widehat{yx^*} = \hat{y} \cdot x^*$$

In particular, $x \mapsto JxJ$ defines a right action of M which commutes with the left action, so we have $JMJ \subseteq M' \subseteq \mathbb{B}(L^2(M,\tau))$.

Theorem 1.1.11. Let (M,τ) be a tracial von Neumann algebra acting on $L^2(M,\tau)$. Then, $JMJ = M'$.

Proof. Firstly, note that for any $x, y \in M$, we have $\langle J\hat{x}, J\hat{y}\rangle = \tau(yx^*) = \langle \hat{y}, \hat{x}\rangle$, so extending densely yields that $\langle J\xi, J\eta\rangle = \langle \eta, \xi\rangle$ for all $\xi, \eta \in L^2(M,\tau)$.

Now, for any $x \in M$ and $y \in M'$, we have

$$\begin{aligned}
\langle JyJ\hat{1}, \hat{x}\rangle &= \langle yJ\hat{1}, J\hat{x}\rangle \\
&= \langle J\hat{x}, yJ\hat{1}\rangle \\
&= \langle x^*\hat{1}, y\hat{1}\rangle \\
&= \langle y^*x^*\hat{1}, \hat{1}\rangle \\
&= \langle y^*\hat{1}, \hat{x}\rangle
\end{aligned}$$

Thus, $JyJ\hat{1} = y^*\hat{1}$ for any $y \in M'$. For $z \in M'$, we further have

$$\begin{aligned}
\langle z(JyJ)\hat{1}, \hat{x}\rangle &= \langle zy^*\hat{1}, x\hat{1}\rangle \\
&= \langle x^*zy^*\hat{1}, \hat{1}\rangle \\
&= \langle zy^*x^*\hat{1}, \hat{1}\rangle \\
&= \langle y^*x^*\hat{1}, z^*\hat{1}\rangle \\
&= \langle y^*x^*\hat{1}, JzJ\hat{1}\rangle \\
&= \langle x^*\hat{1}, y(JzJ)\hat{1}\rangle \\
&= \langle y(JzJ)\hat{1}, \hat{x}\rangle
\end{aligned}$$

Thus, $zJyJ\hat{1} = JyJz\hat{1}$ for all $y \in M'$. Taking $a, b \in M'$, since $J^2 = \mathrm{id}$, we have that

$$z(JabJ)\hat{1} = (JabJ)z\hat{1} = (JaJ)(JbJ)z\hat{1}$$

and hence

$$z(JaJ)Jb\hat{1} = (JaJ)z(Jb)\hat{1}$$

Now, as $J(JMJ)\hat{1} = \widehat{M}$, it follows that $JM'\hat{1}$ is dense in $L^2(M,\tau)$, hence $\{Jb\hat{1} : b \in M'\}$ is a dense set of vectors where applying $z(JaJ)$ and $(JaJ)z$ agree. Hence, $JM'J \subseteq M'' = M$. □

Theorem 1.1.12 ([Ume54])**.** Let $B \subseteq (M,\tau)$ be an inclusion of tracial von Neumann algebras. Then, there is a unique normal linear map $E_B : M \to B$ such that $E_B|_B = \mathrm{id}$, $\tau \circ E_B = \tau$, and for any $b_1, b_2 \in B$ and $x \in M$, we have $E_B(b_1 x b_2) = b_1 E_B(x) b_2$. This map E_B is called the (trace-preserving) conditional expectation from M to B.

Proof. Consider the GNS constructions $L^2(M,\tau)$ and $L^2(B,\tau)$. We can recover the latter as a closed subspace of the former by $\overline{B\hat{1}} \subseteq L^2(M,\tau)$. As a result, there is an orthogonal projection $e_B : L^2(M,\tau) \to L^2(B,\tau)$, known as the Jones projection. By the above theorem, we have $JBJ = B' \subseteq \mathbb{B}(L^2(B,\tau))$. Since $L^2(B,\tau)$ is invariant under the action of B, we have that e_B commutes with JbJ for all $b \in B$.

We define the conditional expectation E_B by $E_B(x) = e_B x e_B$. To show the range of E_B is B, let $x \in M$ and $b \in B$. Then,

$$\begin{aligned} E_B(x)JbJ &= (e_B x e_B)JbJ \\ &= e_B x JbJ e_B \\ &= e_B JbJ x e_B \\ &= JbJ e_B x e_B \\ &= JbJ E_B(x) \end{aligned}$$

Hence, $E_B(x) \in (JBJ)' = B$. To see this is trace preserving, note

$$\tau(E_B(x)) = \langle e_B x e_B \hat{1}, \hat{1}\rangle = \langle x e_B \hat{1}, e_B \hat{1}\rangle = \langle x\hat{1}, \hat{1}\rangle = \tau(x)$$

To see that E_B is normal, note that $e_B^* = e_B$ so that $\langle e_B x e_B \xi, \eta\rangle = \langle x e_B \xi, e_B \eta\rangle$ for all $x \in M, \xi, \eta \in L^2(M,\tau)$. Thus, if $(x_i)_{i\in I}$ is a net in $(M)_1$ converging WOT to some x, then $\langle x_i e_B \xi, e_B\eta\rangle \to \langle x e_B\xi, e_B\eta\rangle = \langle E_B(x)\xi, \eta\rangle$. Finally, if E were another such trace preserving conditional expectation, then for $x \in M$ and $b \in B$, we have

$$\tau(E(x)y) = \tau(E(xy)) = \tau(xy) = \tau(E_B(xy)) = \tau(E_B(x)y)$$

Thus, take $y = E(x) - E_B(x) \in B$. We have that

$$\begin{aligned} 0 &= \tau(E(x)y^*) - \tau(E_B(x)y^*) \\ &= \tau(E(x)E(x)^* - E(x)E_B(x)^*) - \tau(E_B(x)E(x)^* - E_B(x)E_B(x)^*) \\ &= \tau(yy^*) \end{aligned}$$

By faithfulness of τ, we get $y^* = 0$, and hence $y = 0$. Thus, $E(x) = E_B(x)$. □

1.2 Examples

As stated earlier $\mathbb{B}(\mathcal{H})$ is always a von Neumann algebra for any Hilbert space $\mathcal{H}$. In particular, this gives examples of finite dimensional von Neumann algebras, being $\mathbb{M}_n(\mathbb{C})$ for some n (these will be factors) or more generally multimatrix algebras.

$$\bigoplus_{i=1}^{k} \mathbb{M}_{n_i}(\mathbb{C})$$

These are all tracial von Neumann algebras. When $\mathbb{B}(\mathcal{H})$ is infinite-dimensional, there is no longer a trace.

If $(M, \tau) \subseteq \mathbb{B}(\mathcal{H})$ is a commutative tracial von Neumann algebra, then there is some separable complete metrizable topological space X and a regular Borel probability measure μ (such a probability space (X, μ) is called a *standard* probability space) such that there is a trace-preserving normal $*$-isomorphism $M \to L^\infty(X, \mu)$, where the trace on $L^\infty(X, \mu)$ is given by integration.

Definition 1.2.1 (Group von Neumann algebra)**.** Let G be a discrete group. Form the Hilbert space $\ell^2(G)$ with the orthonormal basis $\{\delta_g\}_{g \in G}$. Let $\lambda : G \to \mathcal{U}(\ell^2(G))$ be the left regular representation, given by $\lambda_g \delta_h = \delta_{gh}$ and $\lambda_g^* = \lambda_{g^{-1}}$. Then, we can extend λ linearly to a map $\lambda : \mathbb{C}G \to \mathbb{B}(\ell^2(G))$, where $\lambda(c_1 g_1 + \cdots + c_n g_n) = c_1 \lambda_{g_1} + \cdots + c_n \lambda_{g_n}$. Then, $\lambda(\mathbb{C}G)$ is a unital $*$-subalgebra of $\mathbb{B}(\ell^2(G))$. The unit is given by λ_1 where $1 \in G$ is the identity. Therefore, $L(G) := \lambda(\mathbb{C}G)'' \subseteq \mathbb{B}(\ell^2(G))$ is a von Neumann algebra, which we call the *group von Neumann algebra* of G. $L(G)$ has a normal faithful trace, given by $\tau(x) = \langle x\delta_e, \delta_e \rangle$.

Proposition 1.2.2. If every nontrivial conjugacy class of G is infinite (we say G is i.c.c), then $L(G)$ is a factor.

Proof. Let $x \in L(G)' \cap L(G)$. Then, for any $g \in G$, we have $\lambda_g x \lambda_g^* = x$. In particular, if $x = \sum_{h \in G} c_h \lambda_h \in \lambda(\mathbb{C}G)$. Then, this implies $c_h = c_{ghg^{-1}}$ for all $g \in G$. Therefore, the coefficient functions on $\lambda(\mathbb{C}G)$ are constant on conjugacy classes. Since all nontrivial conjugacy classes are infinite, this implies $c_g = 0$ for all $g \neq e$. In particular, $x = c\lambda_e$ for some $c \in \mathbb{C}$, and thus $x \in \mathbb{C}1$. If x is not in $\lambda(\mathbb{C}G)$, then we can take WOT-approximations. □

Definition 1.2.3 (The hyperfinite II_1 factor)**.** Let $(M_0, \tau_0) = (\mathbb{M}_2(\mathbb{C}), \mathrm{tr})$ act on the Hilbert space $\mathcal{H}_0 = L^2(M_0, \tau_0)$, where tr is normalized. Then, we can let $M_n = \mathbb{M}_{2^n}(\mathbb{C}) = \mathbb{M}_2(\mathbb{C}) \otimes \mathbb{M}_{2^{n-1}}(\mathbb{C})$. In particular, this gives an embedding of $\mathcal{H}_n$ into $\mathcal{H}_{n+1}$ by viewing $\mathcal{H}_{n+1} = \mathcal{H}_n \otimes \mathcal{H}_0$ and letting $\xi \mapsto \xi \otimes 1$. We can recover the trace on M_n by

$$\tau_n(x) = \langle x(\hat{1} \otimes \cdots \otimes \hat{1}), \hat{1} \otimes \cdots \otimes \hat{1} \rangle$$

Then, we can define the Hilbert space and $*$-algebra

$$\mathcal{H} := \overline{\bigcup_{n \geq 0} \mathcal{H}_n}, \qquad \mathcal{R}_0 := \bigcup_{n \geq 0} M_n \subseteq \mathbb{B}(\mathcal{H})$$

We get a trace on $\mathcal{R}_0$ through the vector state defined by $\tau(x) = \langle x\xi_0, \xi_0\rangle$, where $\xi_0 = \hat{1} \otimes \hat{1} \otimes \cdots$. This $*$-algebra $\mathcal{R}_0$ can be thought of as countable tensors in M_0 for which all but finitely many tensor elements are not 1. We then define $\mathcal{R} := \overline{\mathcal{R}_0}^{WOT}$. This was the first example of a II_1 factor, provided by Murray and von Neumann. They later showed in [MN43] that this tracial von Neumann algebra $(\mathcal{R}, \tau)$ is the unique II_1 factor with an increasing sequence of finite dimensional von Neumann subalgebras with dense union (called *hyperfinite*). We therefore call $\mathcal{R}$ the *hyperfinite* II_1 *factor*.

Proposition 1.2.4. Let $H \leq G$ be an inclusion of groups. Then, there is a trace-preserving conditional expectation $E_{L(H)} : L(G) \to L(H)$. For any $g \in G$, $E_{L(H)}(\lambda_g) = 0$ happens if and only if $g \notin H$. Further, if $x \in L(G)$ satisfies $E_{L(H)}(x) = 0$, then there is a net in $\lambda(\mathbb{C}G)$ WOT-converging to x for which every λ_g satisfies $g \in G \setminus H$.

Proof. Since $L(H) = \overline{\lambda(\mathbb{C}H)}^{\mathrm{WOT}} \subseteq \overline{\lambda(\mathbb{C}G)}^{\mathrm{WOT}}$, we can realize $L(H)$ as a tracial von Neumann subalgebra of $L(G)$. Thus, we get a trace-preserving conditional expectation $E_{L(H)} : L(G) \to L(H)$. Firstly, let $g \in G$. Then, we can look at $E_{L(H)}(\lambda_g)$. By definition, we have

$$E_{L(H)}(\lambda_g)\delta_e = e_{L(H)}\lambda_g e_{L(H)}\delta_e = e_{L(H)}\delta_g = \begin{cases} \delta_g & g \in H \\ 0 & g \notin H \end{cases}$$

Thus, $E_{L(H)}(\lambda_g) = \lambda_g$ whenever $g \in H$ and $E_{L(H)}(\lambda_g) = 0$ otherwise. For elements of $\lambda(\mathbb{C}G)$ this extends as follows:

$$E_{L(H)}\left(\sum_{g\in F} c_g\lambda_g\right) = \sum_{g\in F\cap H} c_g\lambda_g$$

for any $F \subseteq G$ finite. If $F \cap H = \varnothing$, then we take this right-hand side to be 0.

Let $x \in L(G)$. For each $g \in G$, we can define $c_g := \tau(\lambda_{g^{-1}}x)$. For $F \subseteq G$ finite, define

$$x_F := \sum_{g\in F} c_g\lambda_g$$

Let $\mathcal{F} := \{F \subseteq G : F \text{ finite}\}$ be a directed set ordered by inclusion. Then, $\{x_F\}_{F\in\mathcal{F}}$ is a uniformly bounded net converging WOT to x. Uniformly bounded follows from $\|x_F\|_{\mathrm{op}} \leq \|x\|_{\mathrm{op}}$ and WOT-convergence comes from noting that

$$x\delta_e = \sum_{g\in G} \langle x\delta_e, \delta_g\rangle\, \delta_g = \sum_{g\in G} a_g\delta_g$$

Since $E_{L(H)}$ is normal and the above net is uniformly bounded, we have that

$$E_{L(H)}(x) = \lim_{F\in\mathcal{F}} E_{L(H)}(x_F) = \lim_{F\in\mathcal{F}} x_{F\cap H} = \sum_{h\in H} c_h\lambda_h$$

Note this is trace-preserving since $\tau(E_{L(H)}(x)) = c_1 = \tau(x)$, where 1 is the identity in G.

Finally, if $E_{L(H)}(x) = 0$, then $c_h = 0$ for all $h \in H$, and thus

$$x = \sum_{g \in G \setminus H} c_g \lambda_g$$

Alternatively, we can define $\mathcal{F}' := \{F \subseteq G : F \text{ finite}, F \cap H = \varnothing\}$ and then x will be the WOT-limit of $\{x_F\}_{F \in \mathcal{F}'}$ where $\mathcal{F}'$ is ordered by inclusion. □

Example 1.2.5 (Amplifications and Compressions)**.** Let (M, τ) be a tracial von Neumann algebra and let $n \in \mathbb{N}$. We can construct the set of all $n \times n$ matrices with entries in M, denoted $\mathbb{M}_n(M)$. This is a von Neumann algebra with a trace given by $\tau \circ \mathrm{tr}$, where tr is the normalized trace on $\mathbb{M}_n(\mathbb{C})$. M then embeds into $\mathbb{M}_n(M)$ by sending each element $x \in M$ to the diagonal matrix with x in each entry. If M is a II_1 factor, then so is $\mathbb{M}_n(M)$. We call $\mathbb{M}_n(M)$ the *matrix amplification* of M by n.

If $p \in M$ is an element satisfies $p^2 = p = p^*$, we say that p is a *projection* in M. The set $pMp = \{pxp : x \in M\}$ is a von Neumann algebra with trace $\tau_p = p^{-1}\tau$. If p is not 1, then pMp is not a von Neumann subalgebra of M, since the unit of pMp is p and not 1. We call pMp the *compression* of M by p. When $M \subseteq \mathbb{B}(\mathcal{H})$, $pMp \subseteq \mathbb{B}(p\mathcal{H})$.

Suppose M is a II_1 factor. For any $t > 0$, we can define the *amplification* of M as follows: If $0 < t \leq 1$, pick $p \in M$ a projection with $\tau(p) = t$, and let $M^t := pMp$. If $t > 1$, then let $n \in \mathbb{N}$ be such that $n > t$. Then, pick a projection $p \in \mathbb{M}_n(M)$ with $(\tau \circ \mathrm{tr})(p) = t/n$. Then, we let $M^t := p\mathbb{M}_n(M)p$. Since both compressions and matrix amplifications of II_1 factors are II_1 factors, the same extends to any amplification of a II_1 factor.

Example 1.2.6 (Tensor products)**.** Let (M_1, τ_1) and (M_2, τ_2) be two tracial von Neumann algebras acting by their standard representations on $L^2(M_1, \tau_1)$ and $L^2(M_2, \tau_2)$. Let $\{e_i^{(1)}\}_{i \in I}$ and $\{e_j^{(2)}\}_{j \in J}$ be orthonormal bases for $L^2(M_1, \tau_1)$ and $L^2(M_2, \tau_2)$ respectively. Then, we can define the set $\{e_i^{(1)} \otimes e_j^{(2)}\}_{(i,j) \in I \times J}$ and declare this an orthonormal basis for the Hilbert space $\mathcal{H} := L^2(M_1, \tau_1) \overline{\otimes} L^2(M_2, \tau_2)$. For $T \in \mathbb{B}(L^2(M_1, \tau_1))$ and $S \in \mathbb{B}(L^2(M_2, \tau_2))$, we can define the operator $T \otimes S \in \mathbb{B}(\mathcal{H})$ by

$$(T \otimes S)(\xi \otimes \eta) := T\xi \otimes S\eta$$

Then, we define $M_1 \overline{\otimes} M_2 := \overline{\mathrm{span}}^{\mathrm{WOT}}\{x \otimes y : x \in M_1, y \in M_2\} \subseteq \mathbb{B}(\mathcal{H})$.

This forms a von Neumann algebra with a trace defined by

$$\tau(x) = \langle x(\hat{1} \otimes \hat{1}), \hat{1} \otimes \hat{1} \rangle$$

Further, on the elementary tensors $x \otimes y$, the trace acts by $\tau(x \otimes y) = \tau_1(x)\tau_2(y)$.

Note that M_1 and M_2 commute in $M_1 \overline{\otimes} M_2$, since

$$\begin{aligned}(x \otimes 1)(1 \otimes y)(\xi \otimes \eta) &= (x \otimes 1)(\xi \otimes y\eta) \\ &= (x\xi \otimes y\eta) \\ &= (1 \otimes y)(x\xi \otimes \eta) \\ &= (1 \otimes y)(x \otimes 1)(\xi \otimes \eta)\end{aligned}$$

Since $M_1 \overline{\otimes} M_2$ is generated as a von Neumann algebra by M_1 and M_2, we can take this to be the unique von Neumann algebra generated by M_1 and M_2 for which their actions commute. If N is another such von Neumann algebra with trace-preserving embeddings $\lambda_{1,2} : M_{1,2} \hookrightarrow N$, and such that $[\lambda_1(M_1), \lambda_2(M_2)] = 0$, then the map $\psi : x \otimes y \mapsto \lambda_1(x)\lambda_2(y)$ extends to a $*$-homomorphism $\psi : \text{span}\{x \otimes y : x \in M_1, y \in M_2\} \to N$. This is injective and trace-preserving, since λ_1 and λ_2 are injective and trace-preserving. Since $N = \lambda_1(M_1) \vee \lambda_2(M_2)$, ψ extends to a surjection on the WOT-closure, and hence a $*$-isomorphism $\psi : M_1 \overline{\otimes} M_2 \to N$.

Remark 1.2.7. This partially motivates the notion of a free product of von Neumann algebras, which is a canonical way to acquire a new von Neumann algebra generated by (M_1, τ_1) and (M_2, τ_2) and for which the actions of M_1 and M_2 don't commute. In order to make this unique, we require one additional condition which will be detailed later.

1.3 Ultraproducts of von Neumann algebras

Many constructions and properties in tracial von Neumann algebras can be described using ultraproducts and ultrapowers. This section introduces the basics of such constructions and some techniques in approximation.

Definition 1.3.1 (Ultrafilter)**.** Let $\mathcal{U} \subseteq 2^{\mathbb{N}}$ be a collection of nonempty subsets of $\mathbb{N}$. We say $\mathcal{U}$ is a *filter* when it satisfies the following properties:

(i) For $A, B \in \mathcal{U}$, $A \cap B \in \mathcal{U}$.

(ii) For $A \in \mathcal{U}$, any set $B \subseteq \mathbb{N}$ containing A satisfies $B \in \mathcal{U}$.

We say that a filter $\mathcal{U}$ is an *ultrafilter* when for every $A \subseteq \mathbb{N}$, either $A \in \mathcal{U}$ or $\mathbb{N} \setminus A \in \mathcal{U}$.

We say an ultrafilter is *principal* when $\exists n \in \mathbb{N}$ such that $\mathcal{U} = \{A \subseteq \mathbb{N} : n \in A\}$. Any ultrafilter $\mathcal{U}$ not of this form is called *free*. Free ultrafilters do not contain any finite subsets of $\mathbb{N}$. The existence of free ultrafilters depends on the axiom of choice.

Definition 1.3.2 (Ultralimit)**.** Let (X, d) be a metric space and $\mathcal{U}$ an ultrafilter on $\mathbb{N}$. A point $x \in X$ is called a $\mathcal{U}$-limit point of a sequence $(x_n)_{n\in\mathbb{N}}$ when for every $\varepsilon > 0$, we have $\{n \in \mathbb{N} : d(x_n, x) < \varepsilon\} \in \mathcal{U}$. If such a $\mathcal{U}$-limit point exists, then it is unique,and is denoted by

$$\lim_{n\to\mathcal{U}} x_n$$

Further, if $(x_n)_n$ converges to a point $x \in X$ in the metric, then this limit will be the $\mathcal{U}$-limit point of any ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

Definition 1.3.3 (Tracial ultraproducts)**.** Suppose for each $n \in \mathbb{N}$, we have a tracial von Neumann algebra (M_n, τ_n). Let $\ell^\infty(M_n) := \{(x_n)_n : x_n \in M_n, \sup_n \|x_n\| < \infty\}$. Then, $\ell^\infty(M_n)$ is a C^*-algebra with pointwise multiplication. For a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we can define a (non-faithful) trace τ on $\ell^\infty(M_n)$ given by

$$\tau((x_n)_n) = \lim_{n\to\mathcal{U}} \tau_n(x_n)$$

We can then define the ideal

$$I_{\mathcal{U}} := \{x \in \ell^\infty(M_n) : \lim_{n\to\mathcal{U}} \tau_n(x_n^* x_n) = 0\}$$

This is a left ideal, and since τ is tracial, also a right ideal, and hence closed. We then define the ultraproduct

$$\prod_{n\to\mathcal{U}} (M_n, \tau_n) := \ell^\infty(M_n)/I_{\mathcal{U}}$$

The equivalence class of an element $(x_n)_n \in \ell^\infty(M_n)$ is denoted $(x_n)_{\mathcal{U}}$ inside the ultraproduct. This ultraproduct is a tracial von Neumann algebra. If each M_n is the same tracial von Neumann algebra M, then we denote the ultraproduct by $M^{\mathcal{U}}$ and call it an ultrapower.

Note that in the ultrapower case, we get a diagonal embedding $M \hookrightarrow M^{\mathcal{U}}$ given by sending an element x to the element with constant sequence representative $x = x_n$ for all $n \in \mathbb{N}$.

Proposition 1.3.4 (Tracial ultraproduct is a von Neumann algebra)**.** The tracial ultraproduct

$$(M, \tau) := \prod_{n\to\mathcal{U}} (M_n, \tau_n)$$

defined above is a tracial von Neumann algebra.

Proof. The trace τ on M given by the ultralimit of the sequence τ_n is a normal traical state, and modding out by $I_{\mathcal{U}}$ makes this a faithful trace. Thus, we only need to show that (M, τ) is a von Neumann algebra. Towards this, we the unit ball of M is 2-norm complete. If this is the case, then by τ extends to a trace on the SOT-closure of M, and by Kaplansky's density theorem, the topologies on the unit ball of M and the unit ball on its SOT-closure will agree, making M SOT-closed.

Let $(x^{(k)})_{k\in\mathbb{N}}$ be a Cauchy sequence in the unit ball of M, where each $x^{(k)}$ has a representative $(x_n^{(k)})_{n\in\mathbb{N}}$ with $\|x_n^{(k)}\|_{\text{op}} \leq 1$. By passing to a subsequence if necessary, we may assume that $\|x^{(k)} - x^{(k+1)}\|_2 < 2^{-k}$. We have that

$$\lim_{n\to\mathcal{U}} \|x_n^{(k)} - x_n^{(k+1)}\|_2 = \|x^{(k)} - x^{(k+1)}\|_2 < 2^{-k}$$

Thus, we have

$$\{n \in \mathbb{N} : \|x_n^{(k)} - x_n^{(k+1)}\|_2 \geq 2^{-k}\} \notin \mathcal{U}$$

Hence, on this set we may change the representative $x_n^{(k)}$ to be $x_n^{(k-1)}$, since it's not witnessed by the ultrafilter. Through induction, we can find a representative such that for every $n \in NN$ we have

$$\|x_n^{(k)} - x_n^{(k+1)}\|_2 < 2^{-k}$$

Then, for a fixed n, the sequence $(x_n^{(k)})_k \in M_n$ is a Cauchy sequence in the unit ball of M_n, and thus has a 2-norm limit since the unit ball is 2-norm complete. Denote this limit by y_n. We will also have that

$$\|x_n^{(k)} - y_n\|_2 < 2^{-k+2}$$

Then, $y = (y_n)_{\mathcal{U}}$ is in the unit ball of M, and we have

$$\|x^{(k)} - y\|_2 = \lim_{n \to \mathcal{U}} \|x_n^{(k)} - y_n\|_2 < 2^{-k+2}$$

Hence, $x^{(k)} \to y$ in 2-norm, and so the unit ball of M is 2-norm complete. □

What we want now is for certain classes of elements (i.e. projections, normal elements, and unitaries) of the ultraproduct M to have representative sequences made of those classes. Towards this, we use the following lemma.

Lemma 1.3.5. Let $x \in (M, \tau)$ be normal, meaning that x commutes with its adjoint x^*. Pick a representative $x = (x_n)_{\mathcal{U}}$ with $\|x_n\|_{\text{op}} \leq \|x\|_{\text{op}}$. Let $f : \mathbb{C} \to \mathbb{C}$ be a Borel function which is continuous in a neighbourhood of the spectrum $\sigma(x)$, then $f((x_n)_{\mathcal{U}}) = (f(x_n))_{\mathcal{U}}$.

Proof. We may assume f is defined only on the closed ball B of radius $\|x\|_{\text{op}} + 1$ centered at 0.

If f is a polynomial, then this just follows from the operations on M being pointwise with respect to representative sequences. If f is continuous, it can be approximated uniformly by polynomials.

Now, pick G an open neighbourhood of the spectrum $\sigma(x)$ for which f is continuous on $\overline{G}$.By Tietze's extension theorem, we can find a continuous function g which agrees with f on $\overline{G}$. By Urysohn's lemma, we can find $0 \leq h \leq 1$ such that $h(\sigma(x)) = 0$ and $h(B \setminus \overline{G}) = 1$. We have that $|g - f| \leq \|g - f\|_\infty h$. Thus,

$$\|(g - f)(x_n)\|_2 \leq \|g - f\|_\infty \|h(x)\|_2$$

But $h(\sigma(x)) = \{0\}$ so $\lim_{n \to \mathcal{U}} \|h(x_n)\|_2 = 0$, and we have

$$(f(x_n))_{\mathcal{U}} = (g(x_n))_{\mathcal{U}} = g(x) = f(x)$$

since g and f agree on $\sigma(x)$. □

Proposition 1.3.6. Let $p \in M$ be a projection, then p has a sequence rperesentative $(p_n)_{\mathcal{U}}$ for which $p_n \in M_n$ is a projection.

Proof. Suppose $\|x_n\|_{\text{op}} \leq 1$. Since adjoints are WOT-continuous, we can consider $y_n = \frac{1}{2}(x_n + x_n^*)$. Then, y_n is selfadjoint. We can then consider the characteristic function $\mathbb{1}_{[1/2,1]}$. By Borel functional calculus, we have $p_n = \mathbb{1}_{[1/2,1]}(y_n)$ and by the above lemma, p_n is a representative of p. □

Corollary 1.3.7. If each M_n is a factor, then so is M.

Proof. Let $p, q \in M$ be projections with representatives $(p_n)_{\mathcal{U}}$ and $(q_n)_{\mathcal{U}}$, where $p_n, q_n \in M_n$ are projections. Since M_n is a factor, the projections are comparable. Suppose $\tau(p_n) \leq \tau(q_n)$. Then, we can find a partial isometry $v_n \in M_n$ such that $v_n v_n^* = p_n$ and $v_n^* v_n \leq q_n$. Otherwise, if $\tau(p_n) > \tau(q_n)$, then we can choose v_n so that $v_n^* v_n = q_n$ and $v_n v_n^* \leq p_n$. We can then define sets

$$\mathcal{A} := \{n \in \mathbb{N} : \tau(p_n) \leq \tau(q_n)\}, \qquad \mathcal{B} := \{n \in \mathbb{N} : \tau(p_n) > \tau(q_n)\}$$

Since this forms a partition of $\mathbb{N}$, one such set must appear in the ultrafilter $\mathcal{U}$. Thus, one such condition holds everywhere witnessed by the ultrafilter, and we can define $v := (v_n)_{\mathcal{U}}$. If $\mathcal{A} \in \mathcal{U}$ then we get $vv^* = p$ and $v^*v \leq q$, and instead if $\mathcal{B} \in \mathcal{U}$, then we get $vv^* \leq p$ and $v^*v = q$. Either way, any two projection in M are comparable, so M is a factor. □

Corollary 1.3.8. If $x \in M$ is normal, then it has a representative sequence $(x_n)_{\mathcal{U}}$ where each x_n is normal.

Proof. Using the same method of proof in (1.3.4), we get that $\{x \in (M)_1 : x^*x = xx^*\}$ is 2-norm complete, where $(M)_1$ is the unit ball. We then only need to check this is true on a dense subset, which will be given by linear combinations of pairwise orthogonal projections. Towards this, let $p^{(1)}, \ldots, p^{(k)} \in M$ be pairwise orthogonal projections and $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$. Then, let

$$x = \sum_{i=1}^{k} \alpha_i p^{(i)}$$

Pick representative $p^{(i)} = (p_n^{(i)})_{\mathcal{U}}$ for each i, where each $p_n^{(i)} \in M_n$ is a projection. Further, we can pick the projections such that for a fixed n, $p_n^{(1)}, \ldots, p_n^{(k)}$ are pairwise orthogonal. Then, let

$$x_n = \sum_{i=1}^{k} \alpha_i p_n^{(i)}$$

This x_n is normal and $(x_n)_{\mathcal{U}} = x$. □

Remark 1.3.9. All of these constructions work for arbitrary ultrafilters on any directed set I. If an ultrafilter $\mathcal{U}$ is principal, say with principal element $n \in \mathbb{N}$, then $\prod_{n \to \mathcal{U}} (M_n, \tau_n) \cong M_n$, since $\{n\}$ is witnessed by $\mathcal{U}$. In the case where I is not $\mathbb{N}$, we can look at *countably cofinal* ultrafilters, where if $\mathcal{U}$ is an ultrafilter on I, we say that $\mathcal{U}$ is countably cofinal when there is a sequence $(A_n)_{n \in \mathbb{N}}$ with each $A_n \in \mathcal{U}$ and with $\bigcap_n A_n = \varnothing$. Every free ultrafilter on $\mathbb{N}$ is countably cofinal.

Definition 1.3.10 (Property Γ)**.** We say that a II_1 factor (M, τ) has property Γ if for any free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, we have that $M' \cap M^{\mathcal{U}} \neq \mathbb{C}$.

Being a factor means that the only elements inside M which commute with M are the scalars, while not having property Γ means that the only elements of M which asymptotically commute with M are the scalars. While the definition makes sense for an arbitrary tracial von Neumann algebra (M, τ), if M is not a factor, we can find some $0 \neq z \in Z(M)$ with $z \notin \mathbb{C}1$. Then, the constant sequence $(z)_{\mathcal{U}}$ inside $M^{\mathcal{U}}$ commutes with the entire diagonal embedding of M, and thus $M' \cap M^{\mathcal{U}} \neq \mathbb{C}$.

Definition 1.3.11 (McDuff's property)**.** We say that a tracial von Neumann algebra (M, τ) has the *McDuff property* when $M \overline{\otimes} \mathcal{R} \cong M$. This is equivalent to requiring the von Neumann algebra $M' \cap M^{\mathcal{U}}$ to be non-abelian.

We now work towards the notion of elementary equivalence for tracial von Neumann algebras, and discuss varying elementary equivalence classes.

Definition 1.3.12 (Elementary equivalence)**.** Let M and N be two tracial von Neumann algebras. We say that M and N are *elementarily equivalent*, denoted $M \equiv N$, when they share the same first-order sentences. By a variant of the Keisler-Shelah theorem (see [Kei61] and [She71]), this is equivalent to when there are free ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on some arbitrary index sets for which $M^{\mathcal{U}} \cong N^{\mathcal{V}}$. It is this latter statement we will accept as definition.

The one theorem we will borrow from the model-theoretic definition is the downward Löwenheim-Skølem theorem, which states that any tracial von Neumann algebra M is elementarily equivalent to a separable tracial von Neumann algebra, so it suffices to only consider separable von Neumann algebras when discussing elementary equivalence.

We say some property P of (separable) tracial von Neumann algebras is an *elementary property* when if $M \equiv N$, then M has P if and only if N has P. Through this ultrapower definition, such a property P is an elementary property whenever M having P is equivalent to $M^{\mathcal{U}}$ having P for some free ultrafilter $\mathcal{U}$.

Example 1.3.13. In [FGL06] and [FHS14], it is shown that property Γ and the McDuff property are both elementary properties. In particular, $\mathcal{R}$ and $L(\mathbb{F}_2)$ are not elementarily equivalent. [FHS14] shows that any II_1 factor is elementarily equivalent to uncountably many nonisomorphic II_1 factors.

1.4 1-Bounded Entropy

In this section, we introduce an invariant for diffuse tracial von Neumann algebras called the *1-bounded entropy*. The notion of a non-commutative entropy was introduced and studied by Voiculescu in a series of six papers (see [Voi93]) in tandem with random matrix theory. Functionally, the various kinds of free entropy are each ways to measure how well the moments of a tuple of selfadjoint operators can be approximated by moments of tuples of selfadjoint matrices. As a result, these can each be thought of as a measurement of how well a tracial von Neumann algebra M generated by d selfadjoint elements $(x_1, \ldots, x_d)$ can be embedded into a matricial ultraproduct.

The most modern version of such a quantity is called 1-bounded entropy, and was introduced explicitly in [Hay18], although the notion appeared implicitly in [Jun07]. The goal of this section is to introduce 1-bounded entropy.

Remark 1.4.1 (Laws of random variables)**.** Let (X, μ) be a standard probability space. Then, for $d \in \mathbb{N}$ and a tuple $F := (f_1, \ldots, f_d) \in L^\infty(X, \mu)^d$, we can define a *law* or *probability distribution* of F as follows. We can treat F as a measurable map $F : X \to \mathbb{C}^d$, and thus define the pushforward measure ν on $\mathbb{C}^d$ by $\nu(E) := \mu(F^{-1}(E))$.

Fix a single $f \in L^\infty(X, \mu)$. For each $n \in \mathbb{N}$, we can define the *n-th moment* m_n of f by

$$m_n := \int_X f(x)^n \mathrm{d}\mu(x)$$

To generalize this to a tuple F as above, we let each component take on its own power,

and these are referred to as *joint moments* or *mixed moments*, and are given by

$$m(n_1, \ldots, n_d) = \int_X f_1(x)^{n_1} \cdots f_d(x)^{n_d} \mathrm{d}\mu(x)$$

For each $n_1, \ldots, n_d \in \mathbb{N}$. Linearity allows for this to be extended to a linear functional on $\mathbb{C}[t_1, \ldots, t_d]$ (the set of all $\mathbb{C}$-polynomials in the variables $t_1, \ldots, t_d$ and their conjugates). We define this map $\varphi : \mathbb{C}[t_1, \ldots, t_d] \to \mathbb{C}$ by

$$\varphi(p) := \int_X p(f_1(x), \ldots, f_d) \mathrm{d}\mu(x)$$

Using the pushforward measure ν, we get that

$$\varphi(p) = \int_{\mathbb{C}^d} p(z) \mathrm{d}\nu(z)$$

In this sense, φ completely encapsulates the mixed moments of F. We refer to φ as the *law* of F, and note that the pushforward measure ν and the law φ of F encode the same data.

We can treat tracial von Neumann algebras as a non-commutative verson of $L^\infty(X, \mu)$, especially given that all commutative tracial von Neumann algebras are of this form. In doing so, if (M, τ) is a tracial von Neumann algebra, we can look at a tuple $x = (x_1, \ldots, x_d) \in M^d$. Unlike in the commutative case, we cannot always recover a measure in the same way as with the pushforward, but the map φ can always be defined.

Definition 1.4.2 (Law of a tuple)**.** Let (M, τ) be a tracial von Neumann algebra and let $x = (x_1, \ldots, x_d) \in M^d$ for some $d \in \mathbb{N}$. We define the *law* of x to be the linear functional $\ell_x : \mathbb{C}\langle t_1, \ldots, t_d \rangle \to \mathbb{C}$ given by $\ell_x(p) := \tau(p(x))$, where $\mathbb{C}\langle t_1, \ldots, t_d \rangle$ is taken to be $\mathbb{C}$-polynomials in the variables $t_1, \ldots, t_d$ and their adjoints, where we no longer assume that $t_1, \ldots, t_d$ commute with each other. That is, we take $\mathbb{C}\langle t_1, \ldots, t_d \rangle$ to be the free $*$-algebra on d variables.

Note that any $a \in M$ can be written as a sum of selfadjoint elements, so without any loss we may always assume to be working with a selfadjoint tuple $x = (x_1, \ldots, x_d) \in M_{\mathrm{sa}}^d$, where M_{sa} is the set of all selfadjoint elements in M. In this way, we may assume the free $*$-algebra $\mathbb{C}\langle t_1, \ldots, t_d \rangle$ is taken on selfadjoint generators.

For a fixed $R > 0$, we denote by $\Sigma_{R;d}$ the space of all laws of selfadjoint d-tuple x in some tracial von Neumann algebra with $\|x\|_\infty < R < \infty$.

Example 1.4.3. Suppose (M, τ) is a tracial von Neumann algebra. We say a unitary element $u \in M$ is a *Haar unitary* to mean that $\tau(u^n) = 0$ for all $n \in \mathbb{Z} \setminus \{0\}$. Then, the laws ℓ_u and ℓ_{u+u^*} both induce compactly supported probability measures on $\mathbb{C}$ and $\mathbb{R}$ respectively (see [NS06] Lecture 1). In this setting, the non-commutative laws can recover genuine measures, much like in the commutative case.

Proposition 1.4.4. Let $d \in \mathbb{N}$ and $R > 0$. Then, $\Sigma_{R;d}$ is a compact weak*-metrizable subset of $\mathbb{C}\langle t_1, \ldots, t_d \rangle^*$

Proof. For compactness, we define

$$C := \mathbb{C}\langle t_1, \ldots, t_d \rangle \subseteq C^*(t_1, \ldots, t_d : t_i = t_i^*, \|t\|_\infty \leq R) =: A$$

Then, let T denote the set of tracial states on A. Note that T is weak*-closed and contained in the unit ball (since traces have norm 1, and the tracial state conditions are closed conditions), so T is weak*-compact. Then, $\Sigma_{R;d}$ is just the restriction of T to C. Further, since A is separable, T is weak*-metrizable, and hence $\Sigma_{R;d}$ is also metrizable. □

Definition 1.4.5. Let $n, d \in \mathbb{N}$, $\varepsilon > 0$, and $\Omega, \Xi \subset \mathbb{M}_n(\mathbb{C})^d$ be nonempty. We say that Ξ is an $(\varepsilon, \|\cdot\|_p)$-cover of Ω if for every $A \in \Omega$, there is a $B \in \Xi$ such that $\|A - B\|_p < \varepsilon$. We denote the covering number of Ω by $K_\varepsilon(\Omega, \|\cdot\|_p)$, defined by the minimal cardinality of a set which $(\varepsilon, \|\cdot\|_p)$-covers Ω.

We say that Ξ is an orbital ε-cover of Ω when $\forall A \in \Omega$, $\exists B \in \Xi, U \in \mathcal{U}(n)$ such that $\|A - UBU^*\|_2 < \varepsilon$, and denote by $K_\varepsilon^{orb}(\Omega)$ the minimal cardinality of a set which orbitally ε-covers Ω.

Definition 1.4.6 (Microstates space)**.** Let (M, τ) be a tracial von Neumann algebra and let $x \in M_{\text{sa}}^d$ with $\|x\|_\infty < R$. Let $\ell_x \in \Sigma_{R;d}$ be the law of x and let $\ell_x \in \mathcal{O} \subset \Sigma_{R;d}$ be an open neighbourhood of ℓ_x. We define a microstate space of x to be

$$\Gamma_{R,n}(\mathcal{O}) := \{X \in \mathbb{M}_n(\mathbb{C})_{sa}^d : \ell_X \in \mathcal{O}\}$$

Remark 1.4.7. Note that $\Gamma_{R,n}(\mathcal{O})$ is invariant under unitary conjugation. If $U \in \mathcal{U}(n)$ and $Y = (Y_1, \ldots, Y_d) \in \Gamma_{R,n}(\mathcal{O})$ then, for any polynomial p we have

$$p(UY_1U^*, \ldots, UY_dU^*) = Up(Y_1, \ldots, Y_d)U^*$$

so

$$\ell_{UYU^*}(p) = \frac{1}{n}\operatorname{tr}(Up(Y)U^*) = \frac{1}{n}\operatorname{tr}(p(Y)) = \ell_Y(p)$$

Due to this, the orbital covering number provides a more robust notion of "size" for $\Gamma_{R,n}(\mathcal{O})$ in comparison to the usual covering number.

Definition 1.4.8 (Microstates sequence)**.** Given (M, τ) a tracial von Neumann algebra and $x \in M_{\text{sa}}^d$. A *microstates sequence* for x is a sequence $(X^{(n)})_{n\in\mathbb{N}}$ with $X^{(n)} \in \mathbb{M}_{k_n}(\mathbb{C})_{\text{sa}}^d$ such that $\sup_n \|X^{(n)}\|_\infty < \infty$ and such that $\ell_{x^{(n)}} \to \ell_x$.

These can be constructed using the microstates spaces above. Given $(\mathcal{O}_n)_{n\in\mathbb{N}}$ a collection of open neighbourhoods of ℓ_x with diameter converging to 0, if $\Gamma_{R,n}(\mathcal{O}_n) \neq \varnothing$, for infinitely many n, we can pick some $X^{(n)} \in \Gamma_{R,n}(\mathcal{O}_n)$, and let $X^{(n)} = 0$ whenenver $\Gamma_{R,n}(\mathcal{O}_n) = \varnothing$.

Remark 1.4.9. Using the notation as above, let $A := \{n \in \mathbb{N} : X^{(n)} \text{ exists}\}$. If A is infinite, we can pick a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$ containing A. We can look at the tracial ultraproduct

$$(\mathbb{M}, \operatorname{tr}) := \prod_{n\to\mathcal{U}} (\mathbb{M}_n(\mathbb{C}), \operatorname{tr}_n)$$

where tr_n is the normalized trace. The sequence $(X^{(n)})$ then becomes a selfadjoint d-tuple $X := (X^{(n)})_\omega \in \mathbb{M}_{\text{sa}}^d$, and since $\ell_{X^{(n)}} \to \ell_x$, we have $\ell_X = \ell_x$. In particular, the unital

$*$-subalgebras $\mathbb{C}\langle x_1, \ldots, x_d\rangle$ and $\mathbb{C}\langle X_1, \ldots, X_d\rangle$ are isomorphic, where x_i and X_i are the components of the d-tuples x and X respectively. Taking WOT-closures on both, we get an isomorphism $W^*(x) \cong W^*(X)$. In particular, if $W^*(x) = M$, then this provides an embedding $M \hookrightarrow \mathbb{M}$.

In this respect, we can use the size of $\Gamma_{R,n}(\mathcal{O})$ in order to construct a "measure of embeddability" of a tracial von Neumann algebra into a matricial ultraproduct. This is formalized in the following definition.

Definition 1.4.10 ([Hay18] 1-bounded entropy)**.** Let (M, τ) be a II_1 factor with $x \in M_{\mathrm{sa}}^d$ satisfying $\|x\|_\infty \leq R < \infty$ and $W^*(x) = M$. We define the 1-bounded entropy in the following manner. Firstly, let $\mathcal{O}$ be an open neighbourhood of ℓ_x, and let $\varepsilon > 0$.

$$\begin{aligned} h_{R,\varepsilon}(\mathcal{O}) &:= \limsup_{n\to\infty} \frac{1}{n^2} \log K_\varepsilon^{orb}(\Gamma_{R,n}(\mathcal{O})) \\ h_{R,\varepsilon}(x) &:= \inf\{h_{R,\varepsilon}(\mathcal{O}) : \mathcal{O} \subset \Sigma_{R;d} \text{ an open nbhd of } \ell_x\} \\ h_R(x) &:= \sup_{\varepsilon>0} h_{R,\varepsilon}(x) \end{aligned}$$

Remark 1.4.11. In [Hay18], it is proven that the quantity $h_R(x)$ is independent of R so long as $R \geq \|x\|_\infty$, as well as that $h_R(x)$ is only dependent on the von Neumann algebra generated by the tuple x. so we can drop the subscript and write $h(W^*(x), \tau)$. When M is a finitely generated factor, we can write $h(M)$.

Definition 1.4.12 (Relative 1-bounded entropy)**.** Given an inclusion of tracial von Neumann algebras $N \subseteq (M, \tau)$, we can define an entropy to the inclusion, denoted $h(N : M)$. We do this as follows.

Firstly, let $s \in \mathbb{N}$ and $y \in M_{\mathrm{sa}}^s$. Let $(Y^{(n)})_{n\in\mathbb{N}}$ be a microstates sequence for y. Let $d \in \mathbb{N}$ and $x \in M_{\mathrm{sa}}^d$ be another tuple, and let $R > \max(\|x\|_\infty, \sup_n \|Y^{(n)}\|_\infty)$. We consider the d-tuples of selfadjoint matrices which extend $Y^{(n)}$ to some $(X, Y^{(n)}$ for which $\ell_{(X,Y^{(n)})}$ is in an open neighbourhood of the law $\ell_{(x,y)}$. Explicitly, let $\mathcal{O}$ be a weak*-open neighbourhood of $\ell_{(x,y)}$, where $(x, y) \in M_{\mathrm{sa}}^{s+d}$. We define

$$\Gamma_{R,n}(\mathcal{O}|Y^{(n)}) := \{X \in \mathbb{M}_n(\mathbb{C})_{\mathrm{sa}}^d : (X, Y^{(n)}) \in \Gamma_{R,n}(\mathcal{O})\}$$

We repeat this process with an additional selfadjoint tuple $a \in M_{\mathrm{sa}}^t$ with microstates sequence $(A^{(n)})$ for a. We further assume that $W^*(a)$ is diffuse and hyperfinite, and that $W^*(a) \subseteq W^*(x)$. Let $R > \max(\|x\|_\infty, \|y\|_\infty, \sup_n \|A^{(n)}\|_\infty)$. We will consider the microstates space of the sequence (x, y, a). Let $\pi : \mathbb{M}_n(\mathbb{C})_{\mathrm{sa}}^{d+s} \to \mathbb{M}_n(\mathbb{C})_{\mathrm{sa}}^d$ be the projection onto the first d coordinates. Let $\mathcal{O}$ be a weak*-open neighbourhood of $\ell_{(x,y,a)}$. Then, we define

$$\Gamma_{R,n}(x|A^{(n)} : \mathcal{O}) := \pi(\Gamma_{R,n}(\mathcal{O}|A^{(n)}))$$

And call this a microstates space for x relative to a in the presence of y. We now define

the following

$$h_{R,\varepsilon}(x|(A^{(n)})_n : \mathcal{O}) := \limsup_{n\to\infty} \frac{1}{n^2} \log K_\varepsilon^{orb}(\Gamma_{R,n}(x|A^{(n)} : \mathcal{O}), \|\cdot\|_2)$$
$$h_{R,\varepsilon}(x|(A^{(n)})_n : y) := \inf\{h_{R,\varepsilon}(x|A^{(n)} : \mathcal{O}) : \mathcal{O} \text{ an open neighbourhood of } \ell_{(x,y,a)}\}$$
$$h(x : y) := \sup_{\varepsilon>0} h_{R,\varepsilon}(x|(A^{(n)})_n : y)$$

When $W^*(a)$ is diffuse and hyperfinite, this quantity does not depend on a, and by the same methods as the usual 1-bounded entropy, it can be shown that this does not depend on the choice of R, nor the choice of selfadjoint generators of $W^*(x)$ and $W^*(y)$ (see [Hay18]). Given this, when $W^*(x, y) = M$ and $W^*(x) = N$, we define $h(N : M) := h(x : y)$ and call this the 1-bounded entropy of N relative to M.

Remark 1.4.13. We now list some properties of the 1-bounded entropy and the relative 1-bounded entropy.

- $h(M) = h(M : M)$ for every diffuse tracial von Neumann algebra M.
- $h(N : M) \geq 0$ if $N \subseteq M$ is diffuse and M embeds into a matricial ultraproduct. If M does not embed into a matricial ultraproduct, then $h(N : M) = -\infty$.
- $h(N : M) \leq 0$ whenever $N \subseteq M$ is diffuse and hyperfinite
- $h(L(\mathbb{F}_n)) = \infty$ for all $n \geq 2$ (see [Jun07, corollary 3.5]).
- If $(N_i)_{i\in I}$ is an increasing chain of diffuse subalgebras of M, with $N := \bigvee_{i\in I} N_i$, then $h(N : M) = \sup_i h(N_i : M)$ (see [Hay18, lemma A.10]).
- Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Then, if N is diffuse, $h(N : M) = H(N : M^{\mathcal{U}})$ (see [Hay18, proposition 4.5]).
- Let M be a II_1 factor and $t > 0$, then $h(M^t) = t^{-2}h(M)$ (see [HJK25, proposition 1.3]).
- Let $N_1, N_2 \subseteq M$ with $N_1 \cap N_2$ diffuse, then $h(N_1 \vee N_2 : M) \leq h(N_1 : M) + h(N_2 : M)$. In particular, $h(N_1 \vee N_2) \leq h(N_1) + h(N_2)$ (see[Hay18, Lemma A.10]).
- Suppose $N_1 \subseteq N_2 \subseteq M_2 \subseteq M_1$, with N_1 diffuse. Then, $h(N_1 : M_1) \leq h(N_2 : M_2)$ (see[HJK25, p. 2.3.3]).

Further, the following properties of von Neumann algebras algebras imply nonpositive 1-bounded entropy (see [HJNS21]):

- Being hyperfinite
- Being a factor with property Γ
- Having diffuse centre
- Having diffuse regular hyperfinite subalgebras (e.g. Cartan subalgebras)

Additionally, the main result of [HJK25] shows that II_1 factors with property (T) have finite 1-bounded entropy.

Definition 1.4.14 (Strongly 1-bounded)**.** We say that a tracial von Neumann algebra (M, τ) is *strongly* 1*-bounded* when $h(M, \tau) < \infty$. The original notion of strongly 1-bounded was introduced in as an invariant of a tracial von Neumann algebra [Jun07], and this was proven to be equivalent in [Hay18].

Chapter 2

Motivation

Amalgamated free products are a construction primarily found in groups, but which have been generalized to many purely algebraic structures. Such structures are typically easier to work with, as the amalgamated free product can be uniquely determined by a pushout. When considering tracial von Neumann algebras or more generally, C^*-algebras, topological and analytical constraints tend to restrict how much freedom we have with such constructions.

This section will review amalgamated free products in the algebraic setting and provide some intuition for the correct generalization of amalgamated free products to tracial von Neumann algebras.

2.1 Amalgamated free products of groups

Let G_1, G_2, and H be groups. Suppose we have injective homomorphisms $\theta_i : H \hookrightarrow G_i$ for $i = 1, 2$. We can then view H as a common subgroup within G_1 and G_2 by identifying it with its image inside each group. The amalgamated free product, denoted $G := G_1 *_H G_2$, is a group generated by G_1 and G_2 such that the common subalgebra H is shared between them, and not just isomorphic. We call H the *amalgam* or *amalgamated subgroup*. There are two main ways to construct the amalgamated free product of groups, both of which we want analogues for in the tracial von Neumann algebra setting.

Definition 2.1.1 (Categorical construction)**.** Let G_1, G_2, and H be as above. Suppose we have the following commuting square,

$$\begin{array}{ccc} H & \xhookrightarrow{\theta_1} & G_1 \\ {\scriptstyle\theta_2}\big\downarrow & & \big\uparrow{\scriptstyle\gamma_1} \\ G_2 & \xrightarrow[\gamma_2]{} & G \end{array}$$

Then, the amalgamated free product is the pushout G, which is unique up to isomorphism. G then satisfies the following universal property: If G_0 is another group with homomorphisms $\varphi_i : G_i \to G_0$ which agree on H, then $\exists! \psi : G \to G_0$ such that $\psi \circ \gamma_1 = \varphi_1$ and $\psi \circ \gamma_2 = \varphi_2$.

In particular, if $\mathcal{C}$ is the category whose objects are groups containing a subgroup isomorphic to H, and whose morphisms are H-preserving group homomorphisms, then the coproduct in this category is the amalgamated free product, amalgamated over H.

In contrast, the free product $G_1 * G_2$ is the coproduct in the category of groups and group homomorphisms. The free product can be recovered from the amalgamated free product by amalgamating over the trivial subgroup $\{1\}$.

We will show that the amalgamated free products for tracial von Neumann algebras also satisfy this universal property. The other construction of the amalgamated free product is done via generators and relations.

Definition 2.1.2 (Relations construction)**.** Let H, G_1, and G_2 be as above. Suppose we have presentations $G_1 = \langle S_1 : R_1 \rangle$ and $G_2 = \langle S_2 : R_2 \rangle$, where S_1 and S_2 are generating sets, and R_1 and R_2 are relations. Then, the amalgamated free product G is defined by

$$G = \langle S_1 \cup S_2 : R_1 \cup R_2,\ \theta_1(h) = \theta_2(h),\ \forall h \in H \rangle$$

We treat the unions $S_1 \cup S_2$ and $R_1 \cup R_2$ as disjoint. This is a direct representation of what is meant by G_1 and G_2 sharing the copy of H inside G. This group can be seen as a quotient of the free product $G_1 * G_2 = \langle S_1 \cup S_2 : R_1 \cup R_2 \rangle$ by the normal subgroup generated by $\{\theta_1(h)\theta_2(h)^{-1} : h \in H\}$. In this manner, we can provide a characterization of the elements of $G_1 *_H G_2$.

Definition 2.1.3 (Normal form)**.** Consider a tuple $(c_1, \ldots, c_n)$ for $n \geq 0$ in the free product $G_1 * G_2$. We say that $(c_1, \ldots, c_n)$ is a *reduced sequence* when

(i) Each c_j is in one of G_1 or G_2

(ii) For $c_j \in G_{i_j}$, we have $i_j \neq i_{j+1}$, so that consecutive elements are not from the same group.

(iii) If $n > 1$ then $c_i \notin H$ for all i.

(iv) If $n = 1$, then $c_1 \neq 1$

If $(c_1, \ldots, c_n)$ is a reduced sequence, then the product $c_1 \cdots c_n$ is an element of $G_1 *_H G_2$. Further, every element of $G_1 *_H G_2$ can be represented as the product of a reduced sequence, although not necessarily uniquely.

To ensure the product is unique, we need to fix transversals (coset representatives) of H inside G_1 and G_2. Let T_1, T_2 be such transversals, and let 1 be the coset representative of H in T_1 and T_2. We define a *normal form* to be a sequence $(h, c_1, \ldots, c_n)$ such that

(i) $h \in H$

(ii) $c_j \in T_{i_j} \setminus \{1\}$ for all j

(iii) $(c_1, \ldots, c_n)$ is a reduced sequence.

After fixing transversals, every element in $G_1 *_H G_2$ can be represented *uniquely* as the product of a normal form (see [Kur03]). Further, we acquire a normal form theorem.

Theorem 2.1.4 ([LS01], Normal Form theorem)**.** If $(c_1, \ldots, c_n)$ is a normal form with $n \geq 1$, then $c_1 \cdots c_n \neq 1$, so that $G_1, G_2 \hookrightarrow G_1 *_H G_2$ by the canonical maps. Note we are letting $h = 1$ in the normal form representation $(h, c_1, \ldots, c_n)$.

From this theorem, it's easy to see that the construction from generators and relations does indeed satisfy the universal property. For any element $g \in G_1 *_H G_2$, we may write g as a unique normal form $g = hc_1 \cdots c_n$. Then, if K is any other group with H-preserving homomorphisms $\varphi_i : G_i \to K$, we can define a map $\psi : G_1 *_H G_2 \to K$ by $\psi(g) = \varphi_1(h)\varphi_{i_1}(c_1) \cdots \varphi_{i_n}(c_n)$. This map is unique, and by construction of the normal form, will satisfy $\psi(g) = \varphi_i(g)$ whenever $g \in G_i$.

The normal form theorem provides a much more straightforward analogy for constructing the amalgamated free product of tracial (and more generally, σ-finite) von Neumann algebras. The elements of $(G_1 *_H G_2) \setminus H$ become the analogue of the elements which will be orthogonal to the amalgam.

Remark 2.1.5 (Multiplying normal forms)**.** Let $(c_1, \ldots, c_n)$ and $(s_1, \ldots, s_m)$ be normal forms for $n, m > 1$. We have the following algorithm for reducing the concatenated products $c_1 \cdots c_n s_1 \cdots s_m$. We consider the sequence $(c_1, \ldots, c_n, s_1, \ldots, s_m)$. If c_n and s_1 are not in the same group, then this is already a normal form, and so the product $(c_1 \cdots c_n)(s_1 \cdots s_m) = c_1 \cdots c_n s_1 \cdots s_m$. Otherwise, suppose $c_n, s_m \in G_i$ and let $x \in T_i$ be the representative of $c_n s_m$. We can then replace the sequence with $(c_1, \ldots, c_{n-1}, x, s_2, \ldots, s_m)$. If $x = 1$ then the product collapses to $c_1 \cdots c_{n-1} s_2 \cdots s_m$. Otherwise, the sequence becomes $(c_1, \ldots, c_{n-1}, x, s_2, \ldots, s_m)$. If this sequence is in normal form, then the resulting product is just $c_1 \cdots c_{n-1} x s_2 \cdots s_m$. If this is not the case, then we note that the length of this sequence $(c_1, \ldots, c_{n-1}, x, s_2, \ldots, s_m)$ is one shorter than the concatenated sequence we started with. Therefore, this process terminates after finitely many repetitions, so only finitely many reductions are necessary to acquire a normal form for the product $(c_1 \cdots c_n)(s_1 \cdots s_m)$.

2.2 Algebras and operator valued random variables

Throughout this section, all algebras are assumed to be unital associative $\mathbb{C}$-algebras. Any inclusion of algebras is always assumed to be a unital inclusion. The algebra setting does not differ too much from the group setting for amalgamated free products. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are algebras with a common subalgebra $\mathcal{B}$, then we can form the amalgamated free product $\mathcal{A}_1 *_{\mathcal{B}} \mathcal{A}_2$ as the pushout of $\mathcal{A}_1$ and $\mathcal{A}_2$ with respect to the inclusions of $\mathcal{B}$.

The important distinction here is in the construction of normal forms. Since the category of algebras has a free object, we can also construct $\mathcal{A}_1 *_{\mathcal{B}} \mathcal{A}_2$ with the same generators and relations method for groups as well.

Definition 2.2.1 (Reduced word space)**.** Suppose $M_1, \ldots, M_n$ are nonempty sets. The notation $\mathrm{Word}(M_1, \ldots, M_n)$ or more generally $\mathrm{Word}(M_i : i = 1, \ldots, n)$ will be used to denote the vector space generated by alternating words in M_i, so that

$$\mathrm{Word}(M_1, \ldots, M_n) = \mathrm{span}_{\mathbb{C}}\{x_1 \cdots x_n : n \in \mathbb{N}, x_j \in M_{i_j}, i_j \neq i_{j+1}\}$$

When the M_i have a vector space structure, this has a decomposition as the following direct sum:

$$\mathrm{Word}(M_1, \ldots, M_n) = \bigoplus_{n \geq 1} \bigoplus_{i_1 \neq \cdots \neq i_n} M_{i_1} \otimes \cdots \otimes M_{i_n}$$

where the notation $i_1 \neq \cdots \neq i_n$ means that only adjacent indices are nonequal, but non-adjacent indices can be the same.

Definition 2.2.2. Let $\mathcal{A}_1, \mathcal{A}_2$, and $\mathcal{B}$ be as above. Let $\mathcal{A}_i^\circ$ be the complemented subspaces of $\mathcal{B}$, so that $\mathcal{A}_i = \mathcal{B} \oplus \mathcal{A}_i^\circ$ as vector spaces. The amalgamated free product is then defined by

$$\begin{aligned} \mathcal{A}_1 *_{\mathcal{B}} \mathcal{A}_2 &:= \mathcal{B} \oplus \mathrm{Word}(\mathcal{A}_1^\circ, \mathcal{A}_2^\circ) \\ &= \mathcal{B} \oplus \bigoplus_{n \geq 1} \bigoplus_{i_1 \neq \cdots \neq i_n} \mathcal{A}_{i_1}^\circ \otimes \cdots \otimes \mathcal{A}_{i_n}^\circ \end{aligned}$$

This latter representation as a direct sum will precisely mirror the decomposition in the amalgamated free product of tracial von Neumann algebras. Products are then handled via concatenation. If we represent a tensor product $a_1 \otimes \cdots a_n$ as $a_1 \cdots a_n$, this becomes a bit more clear.

This direct sum decomposition is the analogue to the normal form for the case of groups. Picking transversals is analogous to picking the vector space complements $\mathcal{A}_i^\circ$, and this direct sum decomposition then states that every element in $\mathcal{A}_1 *_{\mathcal{B}} \mathcal{A}_2$ is a linear combination of elements of $\mathcal{B}$ and reduced alternating words in the complements of $\mathcal{B}$ within $\mathcal{A}_1$ and $\mathcal{A}_2$. The proof of this construction satisfying the universal property is the same as with groups.

Remark 2.2.3. We can generalize this construction to arbitrary algebras containing $\mathcal{B}$. If $\mathcal{B} \subseteq \mathcal{A}_i$ is a collection of algebras for $i \in I$ some index set, we can again pick complements $\mathcal{A}_i^\circ$ such that $\mathcal{A}_i^\circ \oplus \mathcal{B} = \mathcal{A}_i$ as a vector space. Then, the amalgamated free product is given by

$$\underset{\mathcal{B}; i \in I}{*} \mathcal{A}_i := \mathcal{B} \oplus \mathrm{Word}(\mathcal{A}_i^\circ : i \in I)$$

This will have the exact same direct sum decomposition as above.

The amalgamated free products of algebras become more interesting when we introduce the notion of a conditional expectation.

Definition 2.2.4 ([Voi95], Algebraic Conditional Expectation)**.** Suppose $\mathcal{B} \subseteq \mathcal{A}$ is an inclusion of algebras. A linear map $\varphi : \mathcal{A} \to \mathcal{B}$ is called an (algebraic) conditional expectation when

(i) $\varphi|_{\mathcal{B}} = \mathrm{id}$

(ii) $\varphi(b_1 a b_2) = b_1 \varphi(a) b_2$ for all $b_1, b_2 \in \mathcal{B}$ and $a \in \mathcal{A}$. In this case, we say φ is $\mathcal{B}$-bimodular.

We call a the tuple $(\mathcal{A}, \varphi)$ an operator-valued (or $\mathcal{B}$-valued) non-commutative probability space, and the elements of $\mathcal{A}$ are called $\mathcal{B}$-valued random variables.

The specification of a conditional expectation means that we have a canonical choice for the complements $\mathcal{A}_i^\circ$ in the above construction. If $(\mathcal{A}_i, \varphi_i)$ are $\mathcal{B}$-valued probability spaces, then the choice for $\mathcal{A}_i^\circ$ can be taken to be $\ker \varphi_i$.

In [Voi95], Voiculescu introduces these operator-valued probability spaces as a method for calculating the spectra of certain convolution operators. The notion of freeness in operator-valued non-commutative probability spaces will provide a more concrete analogy to the notion of a normal form, and an analogue to the normal form theorem for groups.

Definition 2.2.5 (Free with amalgamation)**.** Suppose $\mathcal{B}$ is an algebra, $(\mathcal{A}, \varphi)$ is a $\mathcal{B}$-valued probability space, and let $\mathcal{B} \subseteq \mathcal{A}_i \subseteq \mathcal{A}$ be a collection of subalgebras, for $i \in I$ some index set. Each $\mathcal{A}_i$ turns into a $\mathcal{B}$-valued probability space with the conditional expectation $\varphi_i := \varphi|_{\mathcal{A}_i}$. We say such a collection $(\mathcal{A}_i, \varphi_i)_{i \in I}$ is *free* in $\mathcal{A}$ when $\mathrm{Word}(\ker \varphi_i : i \in I) \subseteq \ker \varphi$. This is to say, for any alternating word $x_1 \cdots x_n$ with $x_j \in \mathcal{A}_{i_j}$, $i_j \neq i_{j+1}$, and with $\varphi_j(x_j) = 0$, we have $\varphi(x_1 \cdots x_n) = 0$.

This condition on the kernel is what will complete our notion of a normal form in the purely algebraic setting. If $\mathcal{A}$ is the (amalgamated) free product of $\mathcal{A}_i$ over $\mathcal{B}$ with the complements $\mathcal{A}_i^\circ$ taken to be $\ker \varphi_i$, then every element in $\mathcal{A}$ has a unique decomposition into linear combinations of elements of $\mathcal{B}$ and words in the kernels $(\ker \varphi_i)_{i \in I}$. Further, any word in the kernels $x_1 \cdots x_n$ with $n \geq 1$ is "orthogonal" to $\mathcal{B}$ in the sense that $\varphi(x_1 \cdots x_n) = 0$. This latter orthogonality condition is the analogue to the condition $c_1 \cdots c_n \neq 1$ for a normal form $(c_1, \ldots, c_n)$ in the group setting.

Proposition 2.2.6 ([Voi95])**.** Let $\mathcal{B}$ be an algebra and $(\mathcal{A}, \varphi)$ is a $\mathcal{B}$-valued probability space. If $(\mathcal{A}_i)_{i \in I}$ is a family of subalgebras which are free in $\mathcal{A}$ and such that $\mathcal{A}$ is generated by $\bigcup_{i \in i} \mathcal{A}_i$, then the conditional expectation φ is uniquely determined by the restrictions $\varphi_i := \varphi|_{\mathcal{A}_i}$.

Proof. By linearity, we only need to consider words $a_1 \cdots a_n$ with $a_j \in A_{i_j}$. We may assume that $i_j \neq i_{j+1}$ for all j, since otherwise we may collapse the terms $a_j a_{j+1}$ to a single element. Now, we proceed by induction on n. If $n = 1$, then $\varphi(a_1) = \varphi_{i_1}(a_1)$, so we can evaluate all words of length 1. Suppose we know how to evaluate all words of length $n-1$ and consider a word $a_1 \cdots a_n$. Let k be the smallest such integer with $\varphi_{i_k}(a_k) \neq 0$. For each $j \geq k$, write $a_j = b_j + e_j$ with $b_j = \varphi_{i_j}(a_j)$ and $e_j = a_j - b_j$. We can then expand the word to get

$$a_1 \cdots a_n = a_1 \cdots a_{k-1}(b_k + e_k)(b_{k+1} + e_{k+1}) \cdots (b_n + e_n)$$

When expanding, any term with a factor of b_j will have length at most $n-1$, since any adjacent $\mathcal{B}$-terms $b_j b_{j+1}$ collapse into a single element of $\mathcal{B}$ and any term $b_j e_{j+1}$ collapses to a single element of $\mathcal{A}_{i_{j+1}}$. Thus, the only term of length n remaining is $a_1 \cdots e_k e_{k+1} \cdots e_n$. By assumption, $\varphi_{i_j}(a_j) = 0$ for all $1 \leq j < k$ and now $\varphi_{i_j}(e_j) = 0$ for all $j \geq k$. By construction we get $\varphi(a_1 \cdots a_{k-1} e_k \cdots e_n) = 0$, so any word of length n can be completely evaluated.

Hence by induction and linearity, φ is completely determined on any element of $\mathcal{A}$ by the collection $\{\varphi_i\}_{i\in I}$. $\square$

Remark 2.2.7. Using group von Neumann algebras as examples of operator-valued probability spaces makes the jump in analogy from groups to von Neumann algebras a bit more tangible. Suppose we have an inclusion of groups $H \leq G$, and consider $M := L(G)$ the group von Neumann algebra, with $E_H : M \to L(H)$ the trace-preserving conditional expectation. (M, E_H) forms an operator-valued probability space. For an element $g \in G$, we have that either $g \in H$ or $g \notin H$. With respect to the left regular representation, we either have $E_H(\lambda_g) = \lambda_g$ or $E_H(\lambda_g) = 0$, which perfectly matches the group setting. Even though for a general $x \in M$, this is no longer the case (i.e. we can have $x \notin L(H)$ but $E_H(x) \neq 0$), the analogy is still carried over. The notion of "disjointness from H" in the group setting (i.e. in reduced sequences) becomes a notion of orthogonality in the operator-valued probability space setting.

Once again, the choice of transversals for subgroups of a group G becomes replaced with the choice of conditional expectations. In particular, if T is such a transversal, then $T \setminus \{1\}$ is the equivalent to the kernel $\ker \varphi$, for a conditional expectation φ.

Chapter 3

Construction of amalgamated free products

3.1 Hilbert bimodules and relative tensor products

Much of the information here follows from [Pet25] and Appendix B of Chapter 5 of [Con94]. The presentation is motivated by that of Hilbert C^*-modules in [BO08] sections 4.3.

The goal of this section is to address following problem: Suppose (B, τ) is a common von Neumann subalgebra of tracial von Neumann algebras (M_1, τ_1) and (M_2, τ_2). Then, the tensor product $L^2(M_1, \tau_1)\overline{\otimes}L^2(M_2, \tau_2)$ has two distinct copies of $L^2(B, \tau)$. Further, each component of the tensor product has a left and right action of B, so in total this space has four distinct B-actions. In order to mirror the construction in (2.2.2), we want to cut down this space so that the actions of B on each component of the tensor product are the same, or equivalently so that the two copies of $L^2(B, \tau)$ become the same subspace. This will be achieved by introducing the notion of a "relative" tensor product.

Definition 3.1.1 (Hilbert bimodules)**.** Let M and N be von Neumann algebras. We say a Hilbert space $\mathcal{H}$ is a *Hilbert M-N bimodule* when there are normal commuting $*$-homomorphisms $\pi_L : M \to \mathbb{B}(\mathcal{H})$ and $\pi_R : N^{\mathrm{op}} \to \mathbb{B}(\mathcal{H})$, where N^{op} is the von Neumann algebra with the same elements as N and where multiplication is reversed, so $x^{\mathrm{op}}y^{\mathrm{op}} = (yx)^{\mathrm{op}}$. While in general, bimodules need not have normal representations, we will always have this assumption. When these maps π_L and π_R are normal, the Hilbert space $\mathcal{H}$ is called a *correspondence*.

We denote that $\mathcal{H}$ is an M-N bimodule by writing ${}_M\mathcal{H}_N$ to explicitly display the left and right actions. Since π_L and π_R commute, for any $\xi \in \mathcal{H}$, $x \in M$, and $y \in N$, we have $\pi_L(x)\pi_R(y)\xi = \pi_R(y)\pi_L(x)\xi$, and thus we may write $x\xi y$ to mean either such equality. As such, we notationally omit the maps π_L and π_R in what follows.

Example 3.1.2. Suppose (M, τ) is a tracial von Neumann algebra. Then, $L^2(M, \tau)$ is an M-M bimodule. If $B \subseteq M$ is a von Neumann subalgebra, then $L^2(M, \tau)$ is also a B-B bimodule, and breaks down into a direct sum $L^2(B, \tau) \oplus (L^2(M, \tau) \ominus L^2(B, \tau))$. Since $L^2(B, \tau)$ is invariant under both left and right actions of B, it follows that $L^2(M, \tau) \ominus L^2(B, \tau)$ is also a B-B bimodule.

If (N, φ) is another tracial von Neumann algebra, then we can look at the Hilbert space tensor product of $L^2(M,\tau)$ and $L^2(N,\varphi)$ $\mathcal{H} := L^2(M,\tau)\bar{\otimes}L^2(N,\varphi)$. This is forms an M-N bimodule, where the multiplication is given by $x(\xi \otimes \eta)y = (x\xi) \otimes (\eta y)$ for $x \in M, \xi \in L^2(M,\tau), y \in N, \eta \in L^2(N,\tau)$.

Definition 3.1.3 (Right-bounded operators)**.** Let M and N be tracial von Neumann algebras and let ${}_M\mathcal{H}_N$ be an M-N bimodule. For $\xi \in \mathcal{H}$, we say that ξ is a *Right-bounded operator* for N if $\exists K > 0$ such that for all $y \in N$, we have $\|\xi y\| \leq K\|y\|_2$. This is to say, a right-bounded operator ξ is a vector in $\mathcal{H}$ for which the action $N \ni y \mapsto \xi y \in \mathcal{H}$ extends to a bounded operator $L_\xi : L^2(N) \to \mathcal{H}$. We denote the subspace of all right-bounded operators by $\mathcal{H}^R$. Note that $\mathcal{H}^R$ is invariant under left multiplication by M. Defining $\mathcal{H}^L$, the left-bounded operators for M analogously, we may set ${}^L\mathcal{H}^R$ to be the subspace of all vectors which are both left- and right-bounded. This is invariant under both left and right actions of M and N respectively, and hence forms a subbimodule of $\mathcal{H}$.

Consider the case $M = N$ and $\mathcal{H} = L^2(M)$. The vectors $\hat{x}$ are always right-bounded for any $x \in M$, since $\|\widehat{xy}\|_2 \leq \|x\|_{\text{op}}\|y\|_2$. This also follows from the fact that $y \ni M \mapsto \widehat{xy} \in L^2(M)$ is implemented by the action of M on $L^2(M)$. In particular, on $L^2(M)$, the right-bounded operators form a dense subspace of $L^2(M)$, since $\hat{1}$ is cyclic. Popa proved the more general case, that for any Hilbert M-N bimodule ${}_M\mathcal{H}_N$, the subspace $\mathcal{H}^R$ is dense in $\mathcal{H}$.

Remark 3.1.4. Let M and N be tracial von Neumann algebras and let ${}_M\mathcal{H}_N$ be an M-N bimodule with actions π_L and π_R of M and N^{op} respectively. Suppose $\xi \in \mathcal{H}^R$ is a right-bounded operator. Then,

$$L_{a\xi b} = \pi_L(a)L_\xi b \tag{1}$$

and

$$L_\xi Jb^*J = \pi_R(b^{\text{op}})L_\xi \tag{2}$$

for any $a \in M$ and $b \in N$. Further, if $\eta \in \mathcal{H}^R$ is another right-bounded operator, we have $L_\eta^*\pi_L(a)L_\xi \in N$. From (1), we have $\pi_L(a)L_\xi = L_{a\xi}$. Thus, $L_\eta^* L_{a\xi} \in \mathbb{B}(L^2N)$ is bounded. To see that this commutes with with $JN'J$, we have

$$\begin{aligned} L_\eta^*\pi_L(a)L_\xi Jb^*J &= L_\eta^*\pi_L(a)\pi_R(b^{\text{op}})L_\xi \\ &= L_\eta^*\pi_R(b^{\text{op}})\pi_L(a)L_\xi \\ &= Jb^*JL_\eta^*\pi_L(a)L_\xi \end{aligned}$$

Hence, $L_\eta^* L_{a\xi}$ is a bounded operator on L^2N commuting with $JN'J$, and thus must be an element of N.

Definition 3.1.5 (Operator-valued inner product)**.** Let M and N be tracial von Neumann algebras and ${}_M\mathcal{H}_n$ an M-N bimodule. We can define an inner product on $\mathcal{H}^R$ which takes values in N as follows. For $\xi, \eta \in \mathcal{H}^R$, we define $\langle \xi, \eta \rangle_N := L_\xi^* L_\eta$. This inner product is linear in the second argument and antilinear in the first. For $x \in M$ and $y_1, y_2 \in N$, we have

$$\langle x\xi, \eta \rangle_N = \langle \xi, x^*\eta \rangle, \quad \langle \xi y_1, \eta y_2 \rangle_N = y_1^* \langle \xi, \eta \rangle_N y_2$$

We also have that

$$(\langle \xi, \eta \rangle_N)^* = \langle \eta, \xi \rangle_N$$

Note that $\langle \cdot, \cdot \rangle_N$ being "positive semidefinite" means that $\langle \xi, \xi \rangle_N = L_\xi^* L_\xi$ is a positive operator, and further if $L_\xi^* L_\xi = 0$, then for any vector $\zeta \in L^2 N$, we have

$$0 = \left\langle L_\xi^* L_\xi \zeta, \zeta \right\rangle = \langle L_\xi \zeta, L_\xi \zeta \rangle = \|L_\xi \zeta\|^2$$

Hence for any vector, we have $L_\xi \zeta = 0$, and thus $L_\xi = 0$, so the inner product is positive definite.

Example 3.1.6. Consider an inclusion of tracial von Neumann algebras $B \subseteq (M, \tau)$. Then, $L^2(M, \tau)$ is a B-B bimodule, so we can talk about the right-bounded vectors of $L^2(M, \tau)$ for B. We know that to each $x \in M$, $\hat{x}$ is a right-bounded vector, since $x\hat{b} = \widehat{xb}$ and $\|\widehat{xb}\|_2 \leq \|x\|_{\text{op}} \|b\|_2$ for any $b \in B$. Now, we can consider the operator $L_{\hat{x}} : L^2(B, \tau) \to L^2(M, \tau)$. Let $b \in B$ and $y \in M$. Firstly, note that $\hat{y} = y\hat{1}$ and $\hat{1} = e_B \hat{1}$, where e_B is the Jones projection. Hence, $\hat{y} = y e_B \hat{1}$, and so $e_B \hat{y} = E_B(y)$. Then,

$$\begin{aligned}
\langle \hat{b}, e_B x^* \hat{y} \rangle &= \langle \hat{b}, E_B(x^* y) \hat{1} \rangle \\
&= \tau(E_B(y^* x) b) \\
&= \tau(E_B(y^* x b)) \\
&= \tau(y^* x b) \\
&= \langle \widehat{xb}, \hat{y} \rangle \\
&= \langle L_x \hat{b}, \hat{y} \rangle
\end{aligned}$$

Hence, $L_x^* = e_B x^*$, and thus $L_x^* \hat{y} = E_B(x^* y)\hat{1}$, so the adjoint of any such operator induced by a right-bounded vector $\hat{x} \in \hat{M}$ is itself the operator associated to the right-bounded operator after an orthogonal projection, $e_B \hat{x^*}$. Further, this means that $L_{\hat{x}}^* L_{\hat{y}} = E_B(x^* y)$ for any $x, y \in M$.

Definition 3.1.7 ([Sau83] Relative tensor products)**.** Let M, N, and P be tracial von Neumann algebras. Let ${}_M\mathcal{H}_N$ and ${}_N\mathcal{K}_P$ be M-N and N-P bimodules respectively. We can construct a semi-inner product on the algebraic tensor product $\mathcal{H}^R \odot \mathcal{K}$ given by

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \xi_1, \langle \eta_1, \eta_2 \rangle_N \xi_2 \rangle$$

The algebraic tensor product then inherits commuting actions of M and P^{op} by $x(\xi \otimes \eta)y := (x\xi) \otimes (\eta y)$, making this an M-P bimodule. The semi-inner product is then linear in the first argument and positive semi-definite. The separation-completion of $\mathcal{H}^R \odot \mathcal{K}$ is then denoted $\mathcal{H} \otimes_N \mathcal{K}$, and is called the *relative tensor product* of $\mathcal{H}$ and $\mathcal{K}$, noting that $\mathcal{H}^R$ is dense in $\mathcal{H}$. According to [She03], this was also referred to in an unpublished manuscript as "Connes fusion".

This construction is associative. If R is another tracial von Neumann algebra and ${}_P\mathcal{L}_R$ is a P-R bimodule, the map

$$(\xi \otimes \eta) \otimes \zeta \mapsto \xi \otimes (\eta \otimes \zeta)$$

Extends to an isomorphism $(\mathcal{H} \otimes_N \mathcal{K}) \otimes_P \mathcal{L} \to \mathcal{H} \otimes_N (\mathcal{K} \otimes_P \mathcal{L})$.

Example 3.1.8. Let $B \subseteq (M, \tau)$ be an inclusion of tracial von Neumann algebras. Consider the B-B bimodule $\mathcal{H} := L^2(M, \tau) \otimes_B L^2(M, \tau)$. Let $x_1, y_1, x_2, y_2 \in M$, we have

that

$$\begin{aligned}\langle \hat{x}_1 \otimes \hat{y}_1, \hat{x}_2 \otimes \hat{y}_2 \rangle &= \langle \hat{x}_1, \langle \hat{y}_1, \hat{y}_2 \rangle_B \hat{x}_2 \rangle \\ &= \langle \hat{x}_1 E_B(y_1^* y_2) \hat{x}_2 \rangle \\ &= \tau(x_2^* E_B(y_2^* y_1) x_1)\end{aligned}$$

In particular, this means that $\|\hat{x} \otimes \hat{y}\|^2 = \tau(x^* E_B(y^* y) x)$. Further, we have that for any $b \in B$, $\hat{b} \otimes \hat{1} = \hat{1} \otimes \hat{b}$. To see this, we have

$$\|\hat{b} \otimes \hat{1} - \hat{1} \otimes \hat{b}\|_2^2 = \langle \hat{b} \otimes \hat{1}, \hat{b} \otimes \hat{1} \rangle - \langle \hat{b} \otimes \hat{1}, \hat{1} \otimes \hat{b} \rangle - \langle \hat{1} \otimes \hat{b}, \hat{b} \otimes \hat{1} \rangle + \langle \hat{1} \otimes \hat{b}, \hat{1} \otimes \hat{b} \rangle$$

Evaluating each such inner product gives $\tau(b^* b)$, so that $\|\hat{b} \otimes \hat{1} - \hat{1} \otimes \hat{b}\|_2^2 = 0$, and thus we have equality. In particular, this means that $L^2(B, \tau) \otimes \hat{1} = \hat{1} \otimes L^2(B, \tau)$.

3.2 Construction of amalgamated free products

The purpose of the amalgamated free product is to construct a von Neumann algebra M with copies of M_1 and M_2 such that M_1 and M_2 are free with amalgamation over B. In general, we assume M_1 and M_2 are tracial von Neumann algebras with a copy of B such that the traces agree when restricted to B. In the construction, we wish to build the amalgamated free product M so that the copy of B inside M_1 is the same copy inside M_2.

Construction 3.2.1 (The amalgamated free product)**.** The following construction is a modification of the construction from [BD00] and the construction of the reduced amalgamated free product for unital C^*-algebras in section 4.7 of [BO08].

Let $(M_i, \tau_i)_{i \in I}$ be a collection of tracial von Neumann algebras with common von Neumann subalgebra (B, τ) and trace-preserving conditional expectations $E_i : M_i \to B$. Let $(L^2(M_i, \tau_i), \pi_i, \xi_i)$ be the GNS construction, where $\pi_i : M_i \hookrightarrow \mathbb{B}(L^2(M_i, \tau_i))$ is the left action, and where $\xi_i = \hat{1}$ is the unit cyclic separating vector for $L^2(M_i, \tau_i)$. We decompose $L^2(M_i, tau_i) \cong L^2(B, \tau_i) \oplus \mathcal{H}_i^\circ$ with $\mathcal{H}_i^\circ := L^2(M_i, \tau_i) \ominus L^2(B, \tau_i)$. Note each of the spaces $L^2(B, \tau_i)$ are isomorphic to $L^2(B, \tau)$, since $\tau_i|_B = \tau$ for any $i \in I$. We will therefore denote this space by $L^2(B, \tau)$, with ξ denoting the unit cyclic separating vector $\xi = \hat{1}$ in $L^2(B, \tau)$. The space $L^2(B, \tau_i)$ will specifically be used to refer to the subspace of $L^2(M_i, \tau_i)$. We then define the following Hilbert B-modules:

$$\mathcal{H} := L^2(B, \tau) \oplus \bigoplus_{n \geq 1} \bigoplus_{i_1 \neq \cdots \neq i_n} \mathcal{H}_{i_1}^\circ \otimes_B \cdots \otimes_B \mathcal{H}_{i_n}^\circ$$

and

$$\mathcal{H}(i) := L^2(B, \tau) \oplus \bigoplus_{n \geq 1} \bigoplus_{i \neq i_1 \neq \cdots \neq i_n} \mathcal{H}_{i_1}^\circ \otimes_B \cdots \otimes_B \mathcal{H}_{i_n}^\circ$$

We can then define unitaries $u_i : L^2(M_i, \tau_i) \otimes_B \mathcal{H}(i) \to \mathcal{H}$ by the following rules:

$$u_i : \begin{cases} L^2(B,\tau_i) \otimes_B L^2(B,\tau) & \longmapsto & L^2(B,\tau) \\ \mathcal{H}_i^\circ \otimes_B L^2(B,\mathrm{id}) & \longmapsto & \mathcal{H}_i^\circ \\ L^2(B,\tau_i) \otimes_B (\mathcal{H}_{i_1}^\circ \otimes_B \cdots \otimes_B \mathcal{H}_{i_n}^\circ) & \longmapsto & \mathcal{H}_{i_1}^\circ \otimes_B \cdots \otimes_B \mathcal{H}_{i_n}^\circ \\ \mathcal{H}_i^\circ \otimes_B (\mathcal{H}_{i_1}^\circ \otimes_B \cdots \otimes_B \mathcal{H}_{i_n}^\circ) & \longmapsto & \mathcal{H}_i^\circ \otimes_B \mathcal{H}_{i_1}^\circ \otimes_B \cdots \otimes_B \mathcal{H}_{i_n}^\circ \end{cases}$$

We then have $*$-representations $\lambda_i : M_i \to \mathbb{B}(\mathcal{H})$ by $\lambda_i = u_i(\pi_i \otimes \mathrm{id})u_i^*$. Then, $\lambda_i|_B$ is the canonical left B-action on $\mathcal{H}$, so that $\lambda_i|_B = \lambda_j|_B$ for $i \neq j$. We will therefore omit the representations when considering the B-actions on $\mathcal{H}$. We then define the (reduced) amalgamated free product $M := *_{B,i}(M_i, \tau_i)$ by

$$M := \bigvee_{i \in I} \lambda_i(M_i) \subseteq \mathbb{B}(\mathcal{H})$$

Note further that since for each $i \in I$, $\lambda_i|_B$ implements the canonical left action of B, that M has one (canonical) copy of B as a von Neumann subalgebra, which is a common von Neumann subalgebra of all $\lambda_i(M_i)$. This is to say, within M, we have $\lambda_i(M_i) \cap \lambda_j(M_j) = B$ for all $i \neq j \in I$.

We will now work towards equipping M with a normal conditional expectation onto B, and a normal faithful trace.

Proposition 3.2.2. The vector ξ is cyclic and separating for M.

Proof. We start with showing that ξ is separating for M. It suffices to show that ξ is separating for $\lambda_i(M_i)$ for all i. Thus, let $x \in M_i$ and decompose it as $x = E_i(x) + x^\perp$ where $E_i(x^\perp) = 0$. We have that

$$\begin{aligned} \lambda_i(x)\xi &= \lambda_i(E_i(x))\xi + \lambda_i(x^\perp)\xi \\ &= u_i(\pi_i(E_i(x)) \otimes \mathrm{id})u_i^*\xi + u_i(\pi_i(x^\perp) \otimes \mathrm{id})\xi \\ &= u_i(\pi_i(E_i(x)) \otimes \mathrm{id})(\xi_i \otimes \xi) + u_i(\pi_i(x^\perp) \otimes \mathrm{id})(\xi_i \otimes \xi) \\ &= u_i(\pi_i(E_i(x))\xi_i \otimes \xi) + u_i(\pi_i(x^\perp)\xi_i \otimes \xi) \\ &= E_i(x)\xi + \pi_i(x^\perp)\xi_i \\ &\in L^2(B,\mathrm{id}) \oplus \mathcal{H}_i^\circ \end{aligned}$$

If $\lambda_i(x)\xi = 0$, then $E_i(x)\xi = -\pi_i(x^\perp)\xi_i$, but $E_i(x)\xi \in L^2(B,\tau)$ and $-\pi_i(x^\perp)\xi_i \in \mathcal{H}_i^\circ$, so this forces both to be 0. Since ξ is separating for $L^2(B,\tau)$, $E_i(x)\xi = 0 \implies E_i(x) = 0$. Similarly, since ξ_i is separating for $\lambda_i(M_i)$, it follows that $\pi_i(x^\perp)\xi_i = 0 \implies \pi_i(x^\perp) = 0$. Hence, $\lambda_i(x) = 0$, and ξ is separating for all of M.

It suffices to show that

$$\bigcup_{n \geq 1} \bigcup_{i_1 \neq \cdots \neq i_n} \lambda_{i_1}(M_{i_1}) \cdots \lambda_{i_n}(M_{i_n})\xi$$

is dense in $\mathcal{H}$.

To see this, we first consider the case $\lambda_j(M_j)\lambda_i(M_i)\xi$ for $i \neq j$. Let $y \in M_j$ and $x \in M_i$. Decompose these into $y = E_j(y) + y^\perp$ and $x = E_i(x) + x^\perp$ respectively. We can

consider the action of $E_j(y)$ and $y^\perp$ on the components $L^2(B,\tau)$ and $\mathcal{H}_i^\circ$ separately. Since $E_i(x)\xi \in L^2(B,\tau)$, we will have that $\lambda_j(E_j(y))E_i(x)\xi = E_j(y)E_i(x)\xi$. Similarly, since $\pi_j(y^\perp)\xi_j \otimes \pi_i(x^\perp) \otimes \xi_i \in \mathcal{H}_j^\circ \otimes_B \mathcal{H}_i^\circ$, we will have $\lambda_j(y^\perp)\lambda_i(x^\perp)\xi = \pi_j(y)\xi_j \otimes \pi_i(x)\xi_i$.

Since $\mathcal{H}_i^\circ$ is a B-B bimodule, and $\lambda_j(E_j(y))$ implements left multiplication by $E_j(y)$, we will have that $\lambda_j(E_j(y))\pi_i(x^\perp)\xi_i \in \mathcal{H}_i^\circ$. Finally, for $\lambda_j(y^\perp)E_i(x)\xi$, we can again expand this to get

$$\begin{aligned}
\lambda_j(y^\perp)E_i(x)\xi &= u_j(\pi_j(y^\perp)\otimes \mathrm{id})u_j^* E_i(x)\xi \\
&= u_j(\pi_j(y^\perp)\otimes \mathrm{id})(\xi_j \otimes E_i(x)\xi) \\
&= u_j(\pi_j(y^\perp)\xi_j \otimes E_i(x)\xi) \\
&= \pi_j(y^\perp)E_i(x)\xi_j \\
&\in \mathcal{H}_j^\circ
\end{aligned}$$

Therefore, we have that

$$\lambda_j(y)\lambda_i(x) \in L^2(B,\tau) \oplus \mathcal{H}_i^\circ \oplus \mathcal{H}_j^\circ \oplus \mathcal{H}_j^\circ \otimes_B \mathcal{H}_i^\circ$$

Since each ξ_k is cyclic for M_k, we have that

$$\overline{\lambda_j(M_j)\lambda_i(M_i)\xi} = \overline{\lambda_j(M_j)(L^2(B,\tau)\oplus \mathcal{H}_i^\circ)} = L^2(B,\tau) \oplus \mathcal{H}_i^\circ \oplus \mathcal{H}_j^\circ \oplus \mathcal{H}_j^\circ \otimes_B \mathcal{H}_i^\circ$$

Thus, we have

$$\mathcal{H}_{i_1}^\circ \otimes_B \cdots \otimes_B \mathcal{H}_{i_n}^\circ \subseteq \overline{\lambda_{i_1}(M_{i_1})\cdots\lambda_{i_n}(M_{i_n})\xi}$$

and hence the result follows. □

Proposition 3.2.3. *M has a normal faithful trace given by $\tau_M(x) := \langle x\xi, \xi\rangle$.*

Proof. Note that $x \mapsto \langle x\xi,\xi\rangle$ defines a normal faithful state, since ξ separating, so it suffices to check traciality. We do this on the dense subset

$$\mathcal{A} := \bigcup_{n\geq 1} \bigcup_{i_1\neq\cdots\neq i_n} \lambda_{i_1}(M_{i_1})\cdots\lambda_{i_n}(M_{i_n})$$

Pick two words $x_1\cdots x_n$ and $x_{n+1}\cdots x_m$ from the above set. For each x_j, we have $x_j = E_{i_j}(x_j) + x_j^\circ$, where $E_{i_j}(x_j^\circ) = 0$. Write $b_j := E_{i_j}(x_j)$. We then expand

$$x_1\cdots x_m = (b_1 + x_1^\circ)\cdots(b_m + x_m^\circ)$$

Let w be a term in the above expansion not equal to $b_1\cdots b_m$. Then, there is some index j for which the jth term in w is x_j°. Thus, $w\xi$ lies in a direct summand with a tensor product

component of $\mathcal{H}_{i_j}^\circ$, and hence $w\xi$ is orthogonal to $L^2(B,\tau)$. In particular, this means

$$\begin{aligned}\langle x_1\cdots x_m\xi,\xi\rangle &= \langle b_1\cdots b_m\xi,\xi\rangle\\ &= \tau(b_1\cdots b_m)\\ &= \tau((b_{n+1}\cdots b_m)(b_1\cdots b_n))\\ &= \langle b_{n+1}\cdots b_m b_1\cdots b_n\xi,\xi\rangle\\ &= \langle x_{n+1}\cdots x_m x_1\cdots x_n\xi,\xi\rangle\end{aligned}$$

By linearity, this passes to the $*$-algebra generated by $\mathcal{A}$, and since that $*$-algebra is WOT-dense in M, by Kaplansky's density theorem, the traciality property extends to M. □

Corollary 3.2.4. Since M has a normal faithful trace τ_M defined by the vector state implemented by ξ, it follows that the Hilbert space $\mathcal{H}$ is isomorphic in a canonical sense to $L^2(M,\tau_M)$. Further, we get a unique trace-preserving conditional expectation $E : M \to B$.

Remark 3.2.5. In proposition (2.2.6), it's shown that in the algebraic setting, the conditional expectation E of the algebraic amalgamated free product is uniquely determined by the conditional expectations from each component. This is to say, treating the inclusion $(B,\tau) \subseteq (M_i,\tau_i)_{i\in I}$ as one of B-valued $*$-probability spaces, we can construct the amalgamated free product $\mathcal{M}$, which will come equipped with a conditional expectation $\mathcal{E} : \mathcal{M} \to B$ uniquely determined by the conditional expectations $E_B^{M_i} : M_i \to B$.

In particular, since $\mathcal{M}$ will be WOT-dense in the tracial amalgamated free product M, one only needs to check a few properties (e.g. normality, faithfulness, positivity) in order to verify that $\mathcal{E}$ extends to the genuine conditional expectation $E_B : M \to B$ in the tracial setting. One can then recover the trace τ_M on M by $\tau \circ E_B$.

Remark 3.2.6. Given that $L^2(M,\tau_M)$ has a countably infinite direct sum decomposition, it follows that $L^2(M,\tau_M)$ is infinite dimensional, and thus M is infinite dimensional, regardless of whether the collection $(M_i)_{i\in I}$ were infinite dimensional or not. Thus, no finite dimensional tracial von Neumann algebra can be represented as a nontrivial amalgamated free product.

Further, the von Neumann subalgebras M_i can never commute in M. Suppose $x \in \ker E_i$ and $y \in \ker E_j$ for $i \neq j$. Then, $\lambda_j(y)\lambda_i(x)\xi = y\xi_j \otimes x\xi_i$. If the actions commuted, then we would have

$$y\xi_j \otimes x\xi_i = x\xi_i \otimes y\xi_j \in \mathcal{H}_i^\circ \otimes_B \mathcal{H}_j^\circ \cap \mathcal{H}_j^\circ \otimes_B \mathcal{H}_i^\circ = \{0\}$$

Since these are distinct direct summands, it follows that $y\xi_j \otimes x\xi_i = 0$, which implies one of x or y must be 0. Therefore, no abelian von Neumann algebras can be expressed as a nontrivial amalgamated free product.

"Nontrivial" in both cases means that the amalgamated free product is not of the form $M *_N N$ or $M *_M N$ for an inclusion of tracial von Neumann algebras $N \subseteq M$ or $M \subseteq N$. In these cases, the amalgamated free product will be isomorphic to the larger of the two amalgamation components, and thus may be abelian or finite-dimensional if the component is.

Proposition 3.2.7. The trace τ_M extends the traces τ_i on M_i. This is to say $\tau_M|_{M_i} = \tau_i$.

Proof. Let $x \in M_i$. Then, we have

$$\tau_M(\lambda_i(x)) = \langle E_i(x)\xi, \xi\rangle = \tau(E_i(x))$$

Since E_i is trace-preserving, we have $\tau(E_i(x)) = \tau_i(x)$, and thus $\tau_M(\lambda_i(x)) = \tau_i(x)$ as desired. □

Corollary 3.2.8. The conditional expectation $E : M \to B$ extends the conditional expectations E_i, so that $E|_{M_i} = E_i$. This follows from $E|_{M_i}$ being a τ_i-preserving conditional expectation from M_i onto B, which by uniqueness means it must equal M_i.

Proposition 3.2.9. The von Neumann subalgebras $(M_i)_{i\in I}$ inside M are free with amalgamation over B.

Proof. Let $a_1 \cdots a_n$ be an alternating word with $a_j \in \ker E_{i_j}$ and $i_j \neq i_{j+1}$ for all j. Then, we have

$$a_1 \cdots a_n \xi = a_1\xi_{i_1} \otimes \cdots \otimes a_n\xi_{i_n} \perp L^2(B, \tau)$$

Let $e_B : \mathcal{H} \to L^2(B, \tau)$ be the orthogonal projection (this is, up to the above-mentioned isomorphism, the same as the Jones projection). We have that

$$E(a_1 \cdots a_n)\xi = e_B a_1 \cdots a_n \xi = e_B(a_1\xi_{i_1} \otimes \cdots \otimes a_n\xi_{i_n}) = 0$$

Hence, $E(a_1 \cdots a_n) = 0$. By linearity, this means that $\mathrm{Word}(\ker E_i : i \in I) \subseteq \ker E$. □

Theorem 3.2.10. Let $(B, \tau) \subseteq (M_i, \tau_i) \subseteq (N, \tau_0)$ be an inclusion of tracial von Neumann algebras. Let $E_i : M_i \to B$ denote the trace-preserving conditional expectations. The following conditions are enough to uniquely characterize the amalgamated free product up to isomorphism.

(1) B is a von Neumann subalgebra of N and has a trace-preserving conditional expectation $E_B^N : N \to B$.

(2) Each M_i is a von Neumann subalgebra of N containing the same copy of B and $N = \bigvee_{i\in I} M_i$

(3) $E|_{M_i} = E_i$.

(4) The family $(M_i)_{i\in I}$ are free with amalgamation over B inside N.

Proof. Let $\mathcal{M}$ denote the algebraic amalgamated free product from (2.2.3). Let $M := *_{B;i\in I} M_i$ be the tracial amalgamated free product and let $E : M \to B$ denote the canonical trace-preserving conditional expectation. Since $\mathcal{M}$ is a coproduct, the inclusions $M_i \hookrightarrow M$ and $M_i \hookrightarrow N$ extend to inclusions $\pi_1 : \mathcal{M} \hookrightarrow M$ and $\pi_2 : \mathcal{M} \hookrightarrow N$. Since M and N are generated by their copies of the M_i, it follows that the images of $\mathcal{M}$ inside M and N are WOT-dense.

Let $\Phi_1 := E \circ \pi_1$ and $\Phi_2 := E_B^N \circ \pi_2$. These are two algebraic conditional expectations on $\mathcal{M}$. We claim that $\Phi_1 = \Phi_2$. If this is the case, $\exists u : L^2(M, E) \to L^2(N, E_B^N)$ defined by extending $u : \pi_1(\mathcal{M})\hat{1} \to \pi_2(\mathcal{M})\hat{1}$ to a map $M\hat{1} \to N\hat{1}$. Since Φ_1 and Φ_2 are both trace-preserving, and the GNS construction is unique, we can then conclude $M \cong N$.

Now, by construction we have that $\mathcal{M} := B \oplus \text{Word}(M_i^\circ : i \in I)$. For every $x \in \mathcal{M}$, we have that $x - \Phi_1(x) \in \text{Word}(M_i^\circ : i \in I)$. Since E and E_B^N extend the conditional expectations E_i, we have that

$$\Phi_2(x) = \Phi_2(\Phi_1(x)) = \Phi_1(x)$$

and hence $\Phi_2 = \Phi_1$ for all $x \in \mathcal{M}$. □

Remark 3.2.11. The above theorem is the analogue to the normal form theorem for the group-theoretic setting. The notion of a reduced sequence for groups becomes an alternating word in the kernels $\ker E_i$.

3.3 Free products and Free Group Factors

Proposition 3.3.1. Let $H \leq (G_i)_{i \in I}$ be an inclusion of discrete groups, and let $G := *_{H;i} G_i$ be their amalgamated free product. Then, $L(G) \cong *_{L(H);i} L(G_i)$.

Proof. We utilize the prior theorem. Since $H \leq G_i$ is an inclusion of groups, there is a corresponding inclusion of group von Neumann algebras $L(H) \subseteq L(G_i)$, which admits a trace-preserving conditional expectation $E_i : L(G_i) \to L(H)$. Since G is generated as a group by the collection G_i, it follows that $\lambda(G)$ is generated by $\lambda(G_i)$ and thus $L(G)$ is generated by $L(G_i)$.

Since $H \leq G$, we also have a trace-preserving conditional expectation $E_{L(H)} : L(G) \to L(H)$. Furthermore, if it were not the case that E_H extends the E_i, then $E_{L(H)}|_{L(G_i)}$ would define a distinct trace-preserving conditional expectation onto $L(H)$ and thus contradict uniqueness.

Next, note that if $g \in G \setminus H$, then $E_H(g) = 0$. If $(h, c_1, \ldots, c_n)$ is a normal form with $n \geq 1$, then

$$E_{L(H)}(\lambda_{hc_1 \cdots c_n}) = E_{L(H)}(\lambda_h \lambda_{c_1} \cdots \lambda_{c_n}) = \lambda_h E_{L(H)}(\lambda_{c_1 \cdots c_n}) = 0$$

Thus, the normal forms of length at least 2 are in $\text{Word}(\ker E_i)$. To finish the proof, we show that any alternating word in $(\ker E_i)$ is the WOT-limit of linear combinations of such normal forms.

Next, let $F_1 \subseteq G_j \setminus H$, $F_2 \subseteq G_k \setminus H$ be two finite sets and consider the operators

$$x = \sum_{g \in F_1} c_g \lambda_g, \qquad y = \sum_{g \in F_2} a_g \lambda_g$$

For $c_g, a_g \in \mathbb{C}$. We can look at the element $xy \in L(G)$, which will be given by

$$xy = \sum_{g \in F_1} \sum_{k \in F_2} c_g a_k \lambda_{gk}$$

Since $g, k \notin H$ for each $g \in F_1, k \in F_2$, it follows that $(1, g, k)$ is a normal form. In particular, this means that for $gk \in G$, $gk \notin H$. Thus, $E_{L(H)}(xy) = 0$.

Finally, for the general case, pick $x_1 \cdots x_n \in L(G)$ with $x_j \in \ker E_{i_j}$ and $i_j \neq i_{j+1}$. Then, for each x_j, pick a uniformly bounded net $\{x_{j_r}\}_{r \in R_j}$ with $x_{j_r} \in \lambda(\mathbb{C}G_{i_j})$ and $E_{L(H)}(x_{j_r}) = 0$. Then, by the above, we have that $x_{1_r} \cdots x_{n_r} \to x_1 \cdots x_n$ WOT (from uniform boundedness) and $E_{L(H)}(x_{1_r} \cdots x_{n_r}) = 0$. Thus, $E_{L(H)}(x_1 \cdots x_n) = 0$. □

Corollary 3.3.2. Let $\mathbb{F}_n$ denote the free group on n generators. Let $1 \leq d < n, m$. Then, the above shows $L(\mathbb{F}_n) *_{L(\mathbb{F}_d)} L(\mathbb{F}_m) \cong L(\mathbb{F}_{n+m-d})$.

Remark 3.3.3. Let M_1 and M_2 be tracial von Neumann algebras. Then, $\mathbb{C}1$ is a von Neumann subalgebra of both M_1 and M_2, and so we can consider $M_1 *_{\mathbb{C}1} M_2$. This is the free product $M_1 * M_2$, and corresponds in the group setting to amalgamating over the trivial subgroup $\{1\}$. Free with amalgamation over $\mathbb{C}$ is referred to as freeness or free independence, and provides the analogue to classical independence for non-commutative probability. The free product $M := M_1 * M_2$ then constructs a larger tracial von Neumann algebra for which M_1 and M_2 are freely independent inside of M. Clearly, we have $L(G_1) * L(G_2) = L(G_1 * G_2)$, just by noting that $L(\{1\}) = \mathbb{C}$. In particular, $L(\mathbb{F}_n) * L(\mathbb{F}_m) = L(\mathbb{F}_{n+m})$ for any $n, m \in \mathbb{N}$.

Proposition 3.3.4. Let $(B, \tau) \subseteq (M_i, \tau_i)$ for $i = 1, 2$ be an inclusion of tracial von Neumann algebras. Suppose $A_i \subseteq M_i$ is a von Neumann subalgebra such that A_i is free from B for $i = 1, 2$. Then, the subalgebra $A = A_1 \vee A_2 \subseteq (M_1 *_B M_2)$ is isomorphic to the free product $A_1 * A_2$.

Proof. The condition for freeness with amalgamation over $\mathbb{C}$ becomes a condition on the trace given by the following: Let $a_1 \cdots a_n \in A$ be such that $a_j \in A_{i_j}$, with $i_j \neq i_{j+1}$ for all j, and such that $\tau(a_j) = 0$. Then, $\tau(a_1 \cdots a_n) = 0$.

Since $\tau(a_j) = 0$, we have that $E_B(a_j) = 0$, and thus $E_B(a_1 \cdots a_n)$. Since E_B is trace-preserving, this implies $\tau(a_1 \cdots a_n) = 0$, so $A \cong A_1 * A_2 \subseteq M_1 *_B M_2$. □

Definition 3.3.5 (Interpolated free group factors)**.** In [Răd94], Rădulescu defines an extension of the free group factors $L(\mathbb{F}_n)$ to allow indices $1 < r \leq \infty$. These interpolated free group factors satisfy the following two conditions:

$$L(\mathbb{F}_r) * L(\mathbb{F}_s) \cong L(\mathbb{F}_{r+s}), \quad (1 < r, s \leq \infty)$$

and

$$L(\mathbb{F}_r)^\gamma = L(\mathbb{F}_{1+(r-1)/\gamma^2}), \quad (1 < r \leq \infty, 0 < \gamma < \infty)$$

Using the second formula, we get that

$$L(\mathbb{F}_r) = L(\mathbb{F}_2)^{1/\sqrt{r-1}}$$

Therefore, these interpolated free group factors are II_1 factors. Further, we recover the free group factors $L(\mathbb{F}_n)$ whenever $r = n \in \mathbb{N}$. The notation $L(\mathbb{F}_r)$ does not generally make sense as the group von Neumann algebra of a free group, since for $r \notin \mathbb{N}$, $\mathbb{F}_r$ does not define a group. $L(\mathbb{F}_r)$ is simply carried over from the free group case to this more general family.

In [Dyk94], Dykema presents an alternative characterization of these interpolated free group factors given as follows. Let (M, τ) be a II_1 factor. Fix some $1 < r < \infty$. Let $\{s_i\}_{i \in I}$ be a family of free semicircular elements free from some copy of $\mathcal{R}$ inside M (such a family

can be found by considering $M * L(\mathbb{F}_I)$ instead. This will contain a copy of M, and thus $\mathcal{R}$, and $L(\mathbb{F}_I)$ can be seen to be generated by such a free semicircular family). For each index $i \in I$, we can let p_i be a projection in $\mathcal{R}$ such that

$$1 + \sum_{i \in I} \tau(p_i)^2 = r$$

Then, we have

$$L(\mathbb{F}_r) \cong (\mathcal{R} \cup \{p_i s_i p_i : i \in I\})''$$

This construction is independent of the choice of II_1 factor.

In [Dyk93], Dykema proved that $\mathcal{R} * L(\mathbb{Z}) \cong L(\mathbb{F}_2)$, which was boosted to $\mathcal{R} * \mathcal{R} \cong L(\mathbb{Z}) * L(\mathbb{Z}) = L(\mathbb{F}_2)$ by using results from [Dyk94].

While the interpolated free group factors are not group von Neumann algebras of free group factors themselves (for $r \notin \mathbb{N}$), they can be represented as group von Neumann algebras in general.

Theorem 3.3.6 ([PS20])**.** Consider the the symmetric groups S_n (with the convention that $S_0 = \{1\}$) and their direct limit S_∞. For any $1 < r \leq \infty$, we can find some sequence $(k_n)_{n \geq 0}$ of nonnegative integers such that

$$1 + \sum_{n \geq 0} \frac{k_n}{n!} = r$$

Let $H_n := S_n \times \mathbb{F}_{k_n}$. Define $G_0 = H_0 = \mathbb{F}_{k_0}$ and for each $n > 0$, $G_{n+1} := G_n *_{S_n} H_{n+1}$. Since each G_n embeds into G_{n+1}, this sequence has a direct limit, denoted G. Then, $L(G) \cong L(\mathbb{F}_r)$.

More generally, this works for any increasing sequence of finite groups $(Q_n)_{n \geq 0}$ with $|Q_n| \to \infty$, $Q = \bigcup_n Q_n$ ICC, and with k_n chosen so that

$$r = 1 + \sum_{n \geq 0} \frac{k_n}{|Q_n|}$$

The groups H_n and G_n can then be defined analogously.

The proof relies on the result of [MN43] which shows that the group von Neumann algebra of any locally finite ICC group G is isomorphic to $\mathcal{R}$. This result was improved in [Con76], which implies that the group von Neumann algebra of any amenable ICC group is isomorphic to $\mathcal{R}$.

3.4 Intertwining by bimodules

In this section, we introduce the notion of *intertwining by bimodules*, first defined by Popa in [Pop06a] and [Pop06b], and discuss some applications to amalgamated free products, including generalizing a result of Dykema's on when certain amalgamated free products are II_1 factors.

Definition 3.4.1 (Jones basic construction)**.** Suppose $B \subseteq (M, \tau)$ is an inclusion of tracial von Neumann algebras. Let $e_B : L^2(M, \tau) \to L^2(B, \tau)$ be the Jones projection. This projection e_B is a bounded operator in $\mathbb{B}(L^2(M, \tau))$, and we can consider the von Neumann algebra $\langle M, e_B \rangle := W^*(M \cup \{e_B\}) \subseteq \mathbb{B}(L^2(M, \tau))$. That is to say, $\langle M, e_B \rangle$ is the von Neumann subalgebra of $\mathbb{B}(L^2(M, \tau))$ generated by M and the orthogonal projection e_B.

Proposition 3.4.2. Take B and M as above. Then, the set $\text{span}\{xe_By : x, y \in M\}$ is a WOT-dense $*$-subalgebra of the Jones basic construction $\langle M, e_B \rangle$.

Proof. Let $E_B : M \to B$ be the trace-preserving conditional expectation. For $x \in M$, this is given by $E_B(x) = e_B x e_B$. As e_B is a projection, we have $e_B = e_B^2$, so $e_B x e_B = E_B(x) e_B$ for any $x \in M$. For any $w, x, y, z \in M$, we have $(we_Bx)(ye_Bz) = wE_B(xy)e_Bz$. Since $E_B(xy) \in M$, it follows that $wE_B(xy) \in M$, and hence the set $\{xe_By : x, y \in M\}$ is closed under multiplication. As a result, $\mathcal{A} := \text{span}\{xe_By : x, y \in M\}$ forms a $*$-subalgebra of $\langle M, e_B \rangle$.

Now, consider the closed subspace $X := \overline{\mathcal{A}L^2(M)} = \overline{\text{span}}\{xe_By\xi : x, y \in M, \xi \in L^2(M)\}$. Let $p : L^2(M) \to X$ be the corresponding orthogonal projection. Note that $\hat{1} \in L^2(B) \subseteq L^2(M)$, so $e_B\hat{1} = \hat{1}$, and thus for any $x \in M$, we have $x\hat{1} = xe_B\hat{1} \in X$. Since $M\hat{1}$ is dense in $L^2(M)$, it follows that $X = L^2(M)$, and so $p = 1$.

Let $e \in \overline{\mathcal{A}}^{\text{WOT}}$ be the identity element, so that $X \subseteq eL^2(M)$. By the above, this forces $e = 1$, so $\overline{\mathcal{A}}^{\text{WOT}}$ is a unital $*$-subalgebra of $\mathbb{B}(L^2(M))$. By the bicommutant theorem, we get that

$$\mathcal{A}' = (M \cup \{e_B\})' = \langle M, e_B \rangle'$$

Thus, $\mathcal{A}'' = \langle M, e_B \rangle'' = \langle M, e_B \rangle$, so $\mathcal{A}$ is WOT-dense in $\langle M, e_B \rangle$. □

Remark 3.4.3. We can define a normal semifinite trace tr on the Jones basic construction $\langle M, e_B \rangle$ by defining $\text{tr}(xe_By) = \tau(xy)$ for any $x, y \in M$ and extending densely to the positive elements of $\langle M, e_B \rangle$. The details of showing tr is well-defined and tracial can be found in chapter 4.3 of [SS08].

Definition 3.4.4 ([Pop06a])**.** Let (M, τ) be a tracial von Neumann algebra and let $P, Q \subset M$ be von Neumann subalgebras of M. We say that *A corner of P embeds into Q inside M*, denoted $P \prec_M Q$ when $\exists p \in P, q \in Q$ nonzero projections, a unital $*$-homomorphism $\theta : pPp \to qQq$, and a partial isometry $v \in M$ such that

- $vv^* \in (pPp)' \cap pMp$,
- $v^*v \in \Psi(pPp)' \cap qMq$, and
- $xv = v\theta(x)$ for all $x \in pPp$.

Theorem 3.4.5 ([Pop06b])**.** Let P, Q, M be as above. Then, the following are equivalent:

(i) $P \prec_M Q$

(ii) $\exists r \in P' \cap \langle M, e_Q \rangle$ a nonzero projection such that $\text{tr}(r) < \infty$

(iii) $\exists \mathcal{H} \subseteq L^2(M)$ a closed subspace forming a P-Q bimodule, and such that $\mathcal{H}$ has finite dimension as a right Q-module.

Further, the condition $P \not\prec_M Q$ is equivalent to the following: $\exists (u_i)_{i\in I}$ a net of unitaries in P such that for any $x, y \in M$, $\|E_Q(xu_iy)\|_2 \to 0$.

Example 3.4.6. Let (M, τ) be a II_1 factor and $B \subseteq M$ a finite dimensional von Neumann subalgebra. Then, $M \not\prec_M B$.

Suppose we did have $M \prec_M B$. Then, let p, q, θ, v be as in the definition. The kernel $\ker\theta = \{x \in pMp : xv = 0\}$ is then a 2-sided ideal in pMp. Since pMp is a II_1 factor, the only such ideals in pMp are $\{0\}$ and pMp. It can't be the case that $\ker\theta = \{0\}$ since that would imply θ is injective, but pMp is not finite dimensional so this can't happen. Therefore, we would have $\ker\theta = pMp$, so $\theta = 0$, which contradicts the requirement $xv = v\theta(x)$ on any nonzero $x \in pMp$.

In [Dyk11], Dykema describes an algorithm to recover the amalgamated free product $M = M_1 *_B M_2$ of two tracial von Neumann algebras $(M_1, \tau_1), (M_2, \tau_2)$ with common finite dimensional von Neumann subalgebra (B, τ). This is done through the following description:

Theorem 3.4.7. Let (M_1, τ_1) and (M_2, τ_2) be tracial von Neumann algebras with common finite dimensional von Neumann subalgebra (B, τ). Suppose M_2 is a II_1 factor and let $M := M_1 *_B M_2$. Then, M is a II_1 factor, and can be described as follows.

(a) Let $q \in B$ be an abelian projection in B. Then,

$$qMq \cong qM_1q *_{qBq} qM_2q$$

So we may assume without any loss that B is also commutative.

(b) Let $p \in B$ be a minimal projection in B. Define,

$$\mathcal{N}_1 := W^*((1-p)M_1(1-p) \cup (1-p)M_2(1-p)) \subseteq M$$

Then, $\mathcal{N}_1$ is a II_1 factor and

$$\mathcal{N}_1 \cong (1-p)M_1(1-p) *_{(1-p)B(1-p)} (1-p)M_2(1-p)$$

(c) Let $q \in B$ be another minimal projection with $\tau(p) \leq \tau(q)$. Let $r := z_{M_1}(1-p)p$, where $z_{M_1}(1-p)$ is the central support of $1-p$ viewed as a projection in M_1. Define

$$\mathcal{A} := (1-p)M_1(1-p) \oplus \mathbb{C}r \oplus (p-r)M_1(p-r)$$

and let $\mathcal{N}_2 := W *^{(} \mathcal{A} \cup M_2)$. Since M_2 is a II_1 factor and $p, q \in B \subseteq M_2$, we can find a partial isometry $v \in M_2$ such that $v^*v = p$ and $vv^* \leq q$. Let $f := v^*v$ Then,

$$f\mathcal{N}_2 f \cong f\mathcal{N}_1 f * p\mathcal{A}p$$

and $\mathcal{N}_2$ is a II_1 factor.

(d) Suppose $r \neq 0$ above, then

$$rMr \cong r\mathcal{N}_2 r * L(\mathbb{Z})$$

We will prove that M is a II_1 factor using the weaker condition $M_2 \not\prec_{M_2} B$. Towards this, we need the following lemma:

Lemma 3.4.8. Suppose $M_2 \not\prec_{M_2} B$. Then, $\exists (u_i)_{i\in I}$ a net of unitaries in M_2 such that for any $s,t \in M_2$ we have $\|E_B(su_it)\|_2 \to 0$. Let $z \in M$ be such that $E_B(z) = 0$. Then, $u_izu_i^*$ converges WOT to 0.

Proof. We check this on the dense subalgebra $M\hat{1}$. Pick $x, y \in M$, then we have

$$\begin{aligned}|\langle u_izu_i^*\hat{x}, \hat{y}\rangle| &= |\tau(y^*u_izu_i^*x)| \\ &= |\tau(u_izu_i^*xy^*)| \\ &= |\tau(E_B(u_izu_i^*xy^*))|\end{aligned}$$

Now, write $xy^* = E_B(xy^*) + w$ with $E_B(w) = 0$. Then, we have

$$|\tau(E_B(u_izu_i^*xy^*))| \leq |\tau(E_B(u_izu_i^*E_B(xy^*)))| + |\tau(E_B(u_izu_i^*w))|$$

Consider the first term in this sum. By Cauchy-Schwarz, we have

$$|\tau(E_B(u_izu_i^*E_B(xy^*)))| \leq \|E_B(u_izu_i^*E_B(xy^*))\|_2$$

Now, $E_B(xy^*)$ is an element of B, so this norm takes the form $\|E_B(su_it)\|$, where $s = 1 \in M_2$ and $t = zu_i^*E_B(xy^*) \in M_2$, so by the assumption, this converges to 0.

To deal with the second term, note that $w \in \ker E_B$. Hence, we can approximate w WOT by a net in $\mathrm{Word}(\ker E_1, \ker E_2)$, where $E_i : M_i \to B$ are the conditional expectations. By linearity and the triangle inequality, it suffices to consider the case where w is an individual word, with $w = x_1 \cdots x_n$. We start by considering the cases where $x_1 \in M_2$ or $x_n \in M_2$.

Towards this, write $u_izu_i^* = E_B(u_izu_i^*) + w_0$. For notational purposes, write $b_0 := E_B(u_izu_i^*)$. We can then expand the second term as

$$|\tau(E_B(u_izu_i^*x_1 \cdots x_n))| \leq |\tau(E_B(b_0x_1 \cdots x_n))| + |\tau(E_B(w_0x_1 \cdots x_n))|$$

By bimodularity, we get $|\tau(E_B(b_0x_1 \cdots x_n))| = |\tau(b_0E_B(x_1 \cdots x_n))|$, and since $E_B(x_1 \cdots x_n) = 0$ this term is 0. Now, we note that $w_0 \in M_2$. Suppose that $x_1 \in M_1$. Then, $E_B(w_0x_1 \cdots x_n) = 0$ so the second term is 0. If instead we have $x_1 \in M_2$ but also $x_n \in M_1$, then $E_B(x_1 \cdots x_nw_0) = 0$, and from traciality and E_B being trace-preserving we get

$$|\tau(E_B(w_0x_1 \cdots x_n))| = |\tau(E_B(x_1 \cdots x_nw_0))| = 0$$

Thus, in the cases where $x_1 \in M_2$ or $x_n \in M_2$, we have

$$|\tau(E_B(u_izu_i^*x_1 \cdots x_n))| \to 0$$

Now, we consider the case where $x_1 \in M_2$. Expand $u_i^*x_1 \in M_2$ as $u_i^*x_1 = E_B(u_i^*x_1) + w_1$ with $E_B(w_1) = 0$. Write $b_1 = E_B(u_i^*x_1)$. Then, write $u_izb_1 = E_B(u_izb_1) + w_2$, with

$E_B(w_2) = 0$, and take $b_2 = E_B(u_i z b_1)$. Again by linearity and the triangle inequality, we have

$$\begin{aligned}|\tau(E_B(u_i z u_i^* x_1 \cdots x_n))| &\leq |\tau(E_B(b_2 x_2 \cdots x_n))| + |\tau(E_B(w_2 x_2 \cdots x_n))| \\ &= |\tau(b_2 E(x_2 \cdots x_n))| + |\tau(E_B(w_2 x_2 \cdots x_n))|\end{aligned}$$

Note that $E_B(x_2 \cdots x_n) = 0$ so the first term in this sum is 0, and note that $w_2 \in M_2$ and since $x_1 \in M_2$ we have $x_2 \in M_1$ by necessity, so $E_B(w_2 x_2 \cdots x_n) = 0$, and hence the second term is 0. Therefore, by uniformly approximating w by linear combinations of such words, we get

$$|\tau(E_B(u_i z u_i^* w))| \to 0$$

And hence, we get that $|\langle u_i z u_i^* \hat{x}, \hat{y}\rangle| \to 0$ for all $x, y \in M$. Since $\hat{1}$ is cyclic for $L^2 M$, this implies that $u_i z u_i^*$ converges WOT to 0. □

Corollary 3.4.9. Using the notation above, we have the 2-norm closed convex hull

$$\mathcal{C} := \overline{\text{conv}}^{\|\cdot\|_2}\{u_i z u_i^* : i \in I\}$$

contains 0.

Proof. Suppose this weren't the case. Then, by Hahn-Banach separation and Riesz-Representation, $\exists \xi \in L^2(M, \tau)$, $\gamma > 0$, such that the linear functional $\varphi_\xi(\eta) := \langle \eta, \xi \rangle$ satisfies that for any $x \in \mathcal{C}$, we have $\text{Re}\, \varphi_\xi(x) > \gamma$. By construction though, φ_ξ is normal, and hence $\varphi_\xi(u_i z u_i^*) \to 0$. This contradicts $\text{Re}\, \varphi_\xi(x) > \gamma$ for all $x \in \mathcal{C}$. □

Theorem 3.4.10. Let $(B, \tau) \subseteq (M_1, \tau_1), (M_2, \tau_2)$ be an inclusion of tracial von Neumann algebras. Let $M := M_1 *_B M_2$. Suppose that M_2 is a II_1 factor and $M_2 \not\prec_{M_2} B$. Then, M is a II_1 factor.

Proof. Let $a \in Z(M)$ and set $z := a - E_{M_2}(a)$. For any unitary $u \in M_2$, we have $uau^* = auu^* = a$, and by bimodularity of the conditional expectation, we get $uE_{M_2}(a)u^* = E_{M_2}(uau^*) = E_{M_2}(a)$, so $uzu^* = z$ for any such u. By the above, we have that the closed convex hull

$$\mathcal{C} := \overline{\text{conv}}^{\|\cdot\|_2}\{uzu^* : u \in M_2 \text{ a unitary}\}$$

contains 0. However, $uzu^* = z$ for all such u, so $\mathcal{C} = \{z\}$, which implies $z = 0$. Thus, $a = E_{M_2}(a)$, and since $a \in Z(M)$, we get $a \in Z(M_2) = \mathbb{C}$. Hence, M is a II_1 factor. □

Chapter 4

Applications and Constructions

While amalgamated free products are interesting in their own right, they can be used to construct new kinds of von Neumann algebras with their own interesting properties. In this section, we will go over two such constructions. The first being HNN extensions, and the second being of a specific II_1 factor satisfying certain properties.

4.1 HNN extensions of von Neumann algebras

The HNN extension of von Neumann algebras was first introduced by Ueda in [Ued05] for σ-finite von Neumann algebras. Much of the proofs within their paper is focused on the existence of normal conditional expectations which preserve a given state. In the tracial setting, these proofs becomes immediate, and the construction is much more straightforward. We will begin by introducing HNN extensions of groups in order to motivate some of the construction following chapter 4 of [LS01].

Definition 4.1.1 (Higman-Neumann-Neumann extensions of groups)**.** Let $H \leq G$ be an inclusion of groups and suppose we have a group injections $\theta : H \hookrightarrow G$ which is distinct from the inclusion. The purpose of the amalgamated free product was to align the isomorphic subgroups $\theta(H)$ and H, so that they would be the same subset. This required the consideration of two distinct groups which both contained H. For the Higman-Neumann-Neumann extension (the two Neumanns were married!), the goal is to make the isomorphism between H and $\theta(H)$ implemented by conjugation. Towards this, if $G = \langle S : R \rangle$ is the group presentation of G, we define a new group

$$\check{G} := \left\langle S, t : R,\ t\theta(h)t^{-1} = h,\ \forall h \in H \right\rangle$$

Where t is a new generator disjoint from S and referred to as a *stable letter*. This new group $\check{G}$ is called the Higman-Neumann-Neumann (from here on, abbreviated to HNN) extension of G, and denote it $\check{G} := \mathrm{HNN}(G, \theta, H)$. Much like the amalgamated free product case, we get the following notion of a reduced sequence and normal form.

Note that this construction means that the HNN extension has a copy of $\mathbb{Z}$ as a subgroup. If we instead had n isomorphic copies of H, then the HNN extension would contain a copy of the free group $\mathbb{F}_n$ as a subgroup.

Definition 4.1.2 (Reduced sequence for HNN extensions)**.** Using the notation as above, consider a sequence $(g_0, t^{\varepsilon_1}, g_1, \ldots, t^{\varepsilon_n}, g_n)$ with $n \geq 0$, $g_i \in G$, and $\varepsilon_i \in \{1, -1\}$ for all $i = 1, \ldots, n$. We say that such a sequence is *reduced* when neither (t^{-1}, g_i, t) with $g \in H$, nor (t, g_i, t^{-1}) with $g \in \theta(H)$ appear as a consecutive subsequence.

Lemma 4.1.3 (Britton's Lemma)**.** Consider a reduced sequence $(g_0, t^{\varepsilon_1}, g_1, \ldots, t^{\varepsilon_n}, g_n)$ with $n \geq 1$. Then, $g_0 t^{\varepsilon_1} g_1 \cdots t^{\varepsilon_n} g_n \neq 1 \in \check{G}$.

Definition 4.1.4 (Normal form for HNN extensions)**.** Let T_1, T_θ be transversals for H and $\theta(H)$ respectively, with both containing the identity. We say a sequence $(g_0, t^{\varepsilon_1}, g_1, \ldots, t^{\varepsilon_n}, g_n)$ for $n \geq 0$ is a *normal form* when

(i) $g_0 \in G$.

(ii) If $\varepsilon_i = -1$, then $g_i \in T_1$

(iii) If $\varepsilon_i = 1$, then $g_i \in T_\theta$

(iv) There is no consecutive subsequence $(t^{\varepsilon}, 1, t^{-\varepsilon})$.

Much like the group case, we get a normal form theorem which characterizes the elements of $\check{G}$.

Theorem 4.1.5 ([HNN49] Normal form theorem for HNN extensions)**.**

(I) The map $G \to \check{G}$ given by $g \mapsto g$ is an embedding.

(II) If $g_0 t^{\varepsilon_1} \cdots t^{\varepsilon_n} g_n = 1$ for $n \geq 1$, then the sequence $(g_0, t^{\varepsilon_1}, g_1, \ldots, t^{\varepsilon_n}, g_n)$ is not reduced.

(III) Every element of $\check{G}$ has a unique representation as the product of a normal form.

For von Neumann algebras, we wish to recreate the same kind of normal form theorem. Much like in the case with amalgamated free products, normal forms of nontrivial length will correspond to elements which are orthogonal to some natural component in the construction.

Using this uniqueness, we can prove a universal property of HNN extensions similar to the pushout for amalgamated free products.

Theorem 4.1.6 ([YCo] Universal property of HNN extensions)**.** Suppose K is a group with fixed element $k \in K$, and with $\varphi : G \to K$ such that $\varphi(h) = k\varphi(\theta(h))k^{-1}$ for all $h \in H$. Then, there is a unique homomorphism $\psi : \check{G} \to K$ such that $\psi(t) = k$ and $\psi|_G = \varphi$.

Proof. The two requirements $\psi|_G = \varphi$ and $\psi(t) = k$ determine uniquely the image of any normal form, and thus any element of $\check{G}$. In particular, for a normal form $(g_0, t^{\varepsilon_0}, g_1, \ldots, t^{\varepsilon_n}, g_n)$, we have that

$$\psi(g_0 t^{\varepsilon_0} g_1 \cdots t^{\varepsilon_n} g_n) = \varphi(g_0) k^{\varepsilon_0} \varphi(g_1) \cdots k^{\varepsilon_n} \varphi(g_n)$$

Hence, ψ is a well defined group homomorphism and is unique. □

There is another way to help motivate Ueda's construction of the HNN extension for von Neumann algebras: Suppose (B, τ) is a tracial von Neumann algebra, (N, τ) is a II_1 factor, and $\theta_1, \theta_2 : B \hookrightarrow N$ are two trace-preserving embeddings. When does there exist a unitary $u \in N$ such that $u\theta_1(b)u^* = \theta_2(b)$ for every $b \in B$? This is to say, when are two isomorphic von Neumann subalgebras unitarily conjugate? While this answer to this question is unfortunately "not always", we can force such an inner automorphism to exist within a von Neumann algebra slightly larger than N, that being a von Neumann algebra generated by N and such a unitary u.

Construction 4.1.7 (HNN extensions of tracial von Neumann algebras)**.** Let (N, τ) be a tracial von Neumann algebra with $B \subseteq N$ a von Neumann subalgebra. Let $\theta_1, \ldots, \theta_n : B \hookrightarrow N$ be trace-preserving embeddings, with θ_1 being the inclusion map $B \subseteq N$. For each θ_i, let $E_i^N : N \to \theta_i(B)$ be the trace-preserving conditional expectation. We begin by considering the matrix amplification $\mathbb{M}_n(N)$, which contains two distinct but isomorphic von Neumann subalgebras:

$$B \oplus B \oplus \cdots \oplus B, \qquad B \oplus \theta_2(B) \oplus \cdots \oplus \theta_n(B)$$

Where each direct sum is n-fold. These two subalgebras will then induce two more conditional expectations $E_1 : \mathbb{M}_n(N) \to B \oplus \cdots \oplus B$, and $E_\theta : \mathbb{M}_n(N) \to B \oplus \cdots \oplus \theta_n(B)$. We then construct the amalgamated free product

$$(\mathcal{N}, \mathcal{E}) := (\mathbb{M}_n(N), E_\theta) *_{D_n(B)} (\mathbb{M}_n(N), E_1)$$

where $D_n(B)$ denotes the two isomorphic diagonal subalgebras, and where $\mathcal{E} : \mathcal{N} \to D_n(B)$ is the trace-preserving conditional expectation. The HNN extension will be constructed as a von Neumann subalgebra of a corner of this amalgamated free product. We denote the two embeddings of $\mathbb{M}_n(N)$ corresponding to each component by λ_θ and λ_1 respectively. By construction, this means $\lambda_\theta|_{D_n}(B) = \lambda_1|_{D_n(B)} =: \lambda$. For each $i = 2, \ldots, n$, we define a partial isometry $u_i := \lambda_1(E_{1i})\lambda_\theta(E_{i1})$, where E_{ij} are the canonical matrix units. We then have:

$$\begin{aligned} u_i^* u_i &= \lambda_\theta(E_{1i})\lambda_1(E_{i1})\lambda_1(E_{1i})\lambda_\theta(E_{i1}) \\ &= \lambda_\theta(E_{1i})\lambda(E_{ii})\lambda_\theta(E_{i1}) \\ &= \lambda_\theta(E_{11}) \\ &= \lambda(E_{11}) \end{aligned}$$

We similarly acquire $\lambda(E_{11}) = u_i u_i^*$. Thus, letting $p := \lambda(E_{11})$, the partial isometries u_i become unitaries in the corner $p\mathcal{N}p$. Further, for each $b \in B$, we have $\lambda_\theta(\theta_i(b) \otimes E_{ii}) = \lambda(b \otimes E_{ii}) = \lambda_1(b \otimes E_{ii})$, where the tensor product notation is used under the identification $\mathbb{M}_n(N) \cong N \bar{\otimes} \mathbb{M}_n(\mathbb{C})$. We therefore have that

$$u_i \lambda_\theta(\theta_i(b) \otimes E_{11}) u_i^* = \lambda_\theta(b \otimes E_{11}), \quad \forall b \in B \tag{1}$$

Now, define an embedding $\pi : N \to p\mathcal{N}p$ given by $\pi(x) = \lambda_\theta(x \otimes E_{11})$. It follows that

$$u_i \pi(\theta_i(b)) u_i^* = \pi(b), \quad \forall b \in B \tag{2}$$

We can then construct the von Neumann algebra $M := \pi(N) \vee \{u_i : i = 2, \dots, n\}'' \subseteq p\mathcal{N}p$. The conditional expectation $\mathcal{E}_\theta : \mathcal{N} \to \lambda_\theta(\mathbb{M}_n(N))$ restricts to a conditional expectation $E^M_{\pi(N)} : M \to \pi(N)$ given by $E^M_{\pi(N)} := \mathcal{E}_\theta|_M$. Note that this follows from $M \subseteq p\mathcal{N}p$. Given any $x \in M$, we can write $x = pxp$, and using bimodularity of the conditional expectation, we get

$$\mathcal{E}_\theta(x) = p\mathcal{E}_\theta(x)p \in p\lambda_\theta(\mathbb{M}_n(N))p = \pi(N)$$

Using this, we can define a trace on M given by $\tau_M := \tau \circ E^M_{\pi(N)}$. We call the triple $(M, E^M_{\pi(N)}, \tau_M)$ the HNN extension of N by $\theta_2, \dots, \theta_n$, and denote this

$$(M, E^M_{\pi(N)}, \tau_M) := \mathrm{HNN}(N, \theta_i, B)$$

Now that we have our HNN extension, we can start working towards normal forms.

Definition 4.1.8 (Normal form)**.** Let $w \in M$ be of the following form:

$$w = \pi(x_0)u_{i_1}^{\varepsilon_1}\pi(x_1)u_{i_1}^{\varepsilon_1} \cdots u_{i_n}^{\varepsilon_n}\pi(x_n)$$

with $x_0, \dots, x_n \in N$, $\varepsilon_i \in \{1, *\}$, and $n \geq 0$. We say that w is in *normal form* if whenever $i_{j-1} = i_j$ and $\varepsilon_{j-1} \neq \varepsilon_j$, then one of the following occurs:

- If $\varepsilon_{j-1} = 1$, then $x_j \in \ker E_j$.
- If $\varepsilon_{j-1} = *$, then $x_j \in \ker E_1$.

Concretely, if we have $u_j\pi(x)u_j^*$ occurring in the above product, then we want x to be orthogonal to the copy of $\theta_j(B)$ inside N, which means we want $x \in \ker E_j$. If instead we have $u_j^*\pi(x)u_j$, then we want x to be orthogonal to B, which requires $x \in \ker E_1$. As mentioned earlier, these orthogonality conditions correspond to the "disjointness from the subgroup H or $\theta(H)$" conditions for groups.

Remark 4.1.9. Consider a reduced word of the form $w = u_{i_0}^{\varepsilon_0} \cdots u_{i_n}^{\varepsilon_n}$ with $\varepsilon_i \in \{1, -1\}$, so for each $\pi(x_i)$ in the above formula, we take $x_i = 1$. Then, w is reduced if and only if the corresponding word $a_{i_0}^{\varepsilon_0} \cdots a_{i_n}^{\varepsilon_n}$ is reduced in the free group $\mathbb{F}_{n-1} = \langle a_2, \dots, a_n \rangle$. The indices here start at 2 because the unitary u_1 is the identity element of M, and more generally of $p\mathcal{N}p$.

If w is reduced, then any j satisfying $i_j = i_{j+1}$ forces $\varepsilon_j = \varepsilon_{j+1}$, since $u_{i_j}^{\varepsilon_j}u_{i_{j+1}}^{\varepsilon_{j+1}} = u_{i_j}^{\varepsilon_j}u_{i_j}^{-\varepsilon_j} = 1$. On the other hand, if $a_{i_0}^{\varepsilon_0} \cdots a_{i_n}^{\varepsilon_n}$ is a reduced word in $\mathbb{F}_{n-1}$, then any j satisfying $i_j = i_{j+1}$ also forces $\varepsilon_j = \varepsilon_{j+1}$ for the same reason.

Proposition 4.1.10 (All reduced words are orthogonal to $\pi(N)$)**.** Let $w = \pi(x_0)u_{i_1}^{\varepsilon_1} \cdots u_{i_n}^{\varepsilon_n}\pi(x_n)$ be a reduced word in M. Then, $E^M_{\pi(N)}(w) = 0$.

Proof. Let $\mathcal{N}_\theta := \lambda_\theta(\mathbb{M}_n(N))$ and $\mathcal{N}_1 := \lambda_1(\mathbb{M}_n(N))$. Let $\mathcal{N}_\theta^\circ := \mathcal{N}_\theta \cap \ker \mathcal{E} = \lambda_\theta(\ker E_\theta)$. Similarly, let $\mathcal{N}_1^\circ := \lambda_1(\ker E_1)$.

Consider a word of the form $u_i^*\pi(x)u_j$. Expanding definitions, this becomes

$$u_i^*\pi(x)u_j = \lambda_\theta(E_{1i})\lambda_1(E_{i1})\lambda_\theta(x \otimes E_{11})\lambda_1(E_{1j})\lambda_\theta(E_{j1})$$

If $x_i \in \ker E_1$, then this word lies in $\mathcal{N}_\theta^\circ \mathcal{N}_1^\circ \mathcal{N}_\theta^\circ \mathcal{N}_1^\circ \mathcal{N}_\theta^\circ$, since the matrix units E_{pq} are orthogonal to the diagonal subalgebra when $p \neq q$, and since x is orthogonal to B.

Otherwise, we can decompose $x = E_1(x) + x^\circ$, with $x^\circ \in \ker E_1$, and acquire the following.

$$\begin{aligned} u_i^* \pi(x) u_j = \; & \lambda_\theta(E_{1i}) \lambda_1(E_{i1}) \lambda_\theta(E_1(x) \otimes E_{11}) \lambda_1(E_{1j}) \lambda_\theta(E_{j1}) \\ & + \lambda_\theta(E_{1i}) \lambda_1(E_{i1}) \lambda_\theta(x^\circ \otimes E_{11}) \lambda_1(E_{1j}) \lambda_\theta(E_{j1}) \end{aligned}$$

Since $E_1(x) \otimes E_{11} \in D_n(B)$ the diagonal subalgebra, we have the equality $\lambda_\theta(E_1(x) \otimes E_{11}) = \lambda_1(E_B^N(x) \otimes E_{11})$, and hence $\lambda_1(E_{i1}) \lambda_1(E_1(x) \otimes E_{11}) \lambda_1(E_{ij}) = \lambda_1(E_1(x) \otimes E_{ij})$, which if $i \neq j$ will also be in $\mathcal{N}_1^\circ$. Assuming $i \neq j$, we have $u_i^* \pi(x) u_j \in \mathcal{N}_\theta^\circ \mathcal{N}_1^\circ \mathcal{N}_\theta^\circ + \mathcal{N}_\theta^\circ \mathcal{N}_1^\circ \mathcal{N}_\theta^\circ \mathcal{N}_1^\circ \mathcal{N}_\theta^\circ$. We have $E_{\pi(N)}^M(u_i^* \pi(x) u_j) = 0$.

We have a similar process for words of the form $u_i \pi(x) u_j^*$. This expands to

$$u_i \pi(x) u_j^* = \lambda_1(E_{1i}) \lambda_\theta(x \otimes E_{ij}) \lambda_1(E_{j1})$$

which will be in $\mathcal{N}_1^\circ \mathcal{N}_\theta^\circ \mathcal{N}_1^\circ$ whenever $x \in \ker E_1$ or whenever $i \neq j$, so we again have $E_{\pi(N)}^M(u_i \pi(x) u_i^*) = 0$. From this, it's clear that if w is a reduced word, then it belongs to the span of all words of length ≥ 2 alternating in $\mathcal{N}_1^\circ$ and $\mathcal{N}_\theta^\circ$, and hence in the kernel of $E_{\pi(N)}^M$. □

Theorem 4.1.11 (Characterization of HNN extensions)**.** Suppose $(\mathcal{M}, \tau_\mathcal{M})$ is a tracial von Neumann algebra with embedding $\psi : N \hookrightarrow \mathcal{M}$ and unitaries $v_2, \ldots, v_n$ satisfying the following:

(i) $v_i \psi(\theta_i(b)) v_i^* = \psi(b)$, for all $b \in B$ and $i = 2, \ldots, n$.

(ii) Every reduced word is orthogonal to $\psi(N)$.

(iii) $\mathcal{M}$ is generated by $\psi(N)$ and the unitaries $v_2, \ldots, v_n$

Then, $\mathcal{M}$ is isomorphic to M.

Proof. Using a similar process to the corresponding theorem with the amalgamated free product, we will show that M and $\mathcal{M}$ have isomorphic dense $*$-subalgebras given by the $*$-algebra generated by $\pi(N)$ and the unitaries $u_2, \ldots, u_n$.

Consider a word $w = \pi(x_0) u_{i_1}^{\delta_1} \cdots u_{i_n}^{\delta_n} \pi(x_n)$ for $\delta_j \in \mathbb{Z} \setminus \{0\}$ and $x_j \in N$. It suffices to show that $E_{\pi(N)}^M(w)$ is uniquely determined by these two conditions. By condition (M), any reduced word w satisfies $E_{\pi(N)}^M(w) = 0$, so it suffices to only consider non-reduced words.

Towards this, first consider some word $w = u_i \pi(x) u_i^*$. If w is not reduced, then we require $x \notin \ker E_i$, so that we have a decomposition $x = E_i(x) + x^\circ$. Expanding the word w, we get

$$\begin{aligned} E_{\pi(N)}^M(w) &= E_{\pi(N)}^M(u_i E_i(x) u_i^*) + E_{\pi(N)}^M(u_i x^\circ u_i^*) \\ &= E_{\pi(N)}^M(\pi(E_i(x))) + 0 \\ &= \pi(E_i(x)) \end{aligned}$$

Similarly, if we started with $w = u_i^* \pi(x) u_i$ with $x \notin \ker E_1$, then we get $E^M_{\pi(N)}(w) = \pi(E_1(x))$.

Since all words of length less than 3 are automatically reduced, if $\text{length}(w) < 3$ we have $E^M_{\pi(N)}(w) = 0$. We prove that for a word w, either $E^M_{\pi(N)}(w) = 0$ or $E^M_{\pi(N)}(w)$ is a word in $\pi(\theta_i(B))$ for $i = 1, \ldots, n$.

Suppose for all words of length $< n$ the above holds, and let w be a word of length n. If w is reduced, we're done. Otherwise, $\exists j$ such that $w = \cdots u_j^{\varepsilon_1} \pi(x) u_j^{\varepsilon_2} \cdots$, where $\varepsilon_1 \neq \varepsilon_2$, and x is in the appropriate kernel. Assume that j is the minimal such index. If $\varepsilon_1 = *$, then we can decompose $x = E_1(x) + x^\circ$, and otherwise as $x = E_j(x) + [x]_j^\circ$. WLOG assume $\varepsilon_1 = *$. Then, we can expand the word w to get

$$w = \underbrace{\cdots \pi(\theta_j(x)) \cdots}_{=w_0} + \underbrace{\cdots u_j^* \pi(x^\circ) u_j \cdots}_{=w_1}$$

Now, $\text{length}(w_0) < \text{length}(w)$, so either $E^M_{\pi(N)}(w_0) = 0$ or is a word in $\pi(\theta_i(B))$. If j was unique, then w_1 is reduced, and hence $E^M_{\pi(N)}(w_1) = 0$. Otherwise, $\exists j_1 > j$ for which $w_1 = \cdots u_{j_1}^{\varepsilon_1} \pi(x_1) u_{j_1}^{\varepsilon_2} \cdots$ can be reduced, and we can repeat the process. If w_1 is not reduced, it splits into $w_{11} + w_{12}$ with $\text{length}(w_{11}) < \text{length}(w_1) = \text{length}(w)$, and with w_{12} either reduced, or having some index $j_2 > j_1 > j$ for which a reduction can occur. Since this can only happen with finitely many indices $j < j_1 < \cdots < j_k$, this process must terminate with some word w' such that $\text{length}(w') = \text{length}(w)$ and w' reduced, or with $\text{length}(w') < \text{length}(w)$ and hence either reduced or $E^M_{\pi(N)}(w')$ being a word in $\pi(\theta_i(B))$. We can then write

$$w = w_0 + w_{11} + \cdots + w'$$

and have that $E^M_{\pi(N)}(w)$ is a sum of words in $\pi(\theta_i(B))$. □

Corollary 4.1.12. There is a natural group embedding of $\mathbb{F}_{n-1} = \langle a_2, \ldots, a_n \rangle$ into $\mathcal{U}(M)$ given by sending $a_i \mapsto u_i$. We denote the image of some $g \in \mathbb{F}_n$ by u_g. If $\ell(\cdot)$ denotes the length function on $\mathbb{F}_{n-1}$ with respect to the generating set, then we have that for any word $w = \pi(x_0) u_{i_1}^{\varepsilon_1} \cdots u_{i_n}^{\varepsilon_n} \pi(x_n)$, then if $g = a_{i_1}^{\varepsilon_1} \cdots a_{i_n}^{\varepsilon_n}$ satisfies $\ell(g) \neq 0$, we have $E^M_{\pi(N)}(w) = 0$ from (4.1.9) and (4.1.10).

Therefore, the family u_i form a free family of Haar unitaries (since the trace φ on M is also given by $\tau \circ E^M_{\pi(N)}$, and these unitaries u_i are orthogonal to $\pi(N)$), and so we have an embedding of $L\mathbb{F}_n$ into M.

Theorem 4.1.13. Let G be a group with isomorphic subgroups H and $\theta(H)$, where $\theta : H \hookrightarrow G$ is an embedding that isn't the inclusion. Form the group HNN extension $\check{G} := HNN(G, \theta, H)$, with $t \in \check{G}$ being the stable letter, satisfying $t\theta(h)t^{-1} = h$ for every $h \in H$. Then, the group von Neumann algebra $L(\check{G})$ is isomorphic to $\text{HNN}(L(G), \theta, L(H))$, where θ is an appropriate extension of the group homomorphism.

Proof. The unitary $u := \lambda_t$ satisfies that for any $h \in H$, that $\text{Ad}(u)\lambda_{\theta(h)} = \lambda_h$, hence $\text{Ad}(u)$ extends to an inner automorphism on $L(\overline{G})$. If we extend θ to the group von Neumann algebras $\theta : L(\theta(H)) \hookrightarrow L(G)$, then $\text{Ad}(u)$ satisfies $\text{Ad}(u)(\theta(x)) = x \in L(H) \subseteq L(G)$. Thus, it suffices to check that all reduced words are orthogonal to $L(G)$.

Pick some word $w \in \overline{G}$ written in normal form as $w = g_0 t^{\varepsilon_0} g_1 \cdots t^{\varepsilon_{n-1}} g_n t^{\varepsilon_n}$. We first want that λ_w is reduced. By the normal form theorem for group HNN extensions, if $n > 0$ here then $w \notin G$ and hence $E_{L(G)}(\lambda_w) = 0$. λ_w is then reduced when viewed as

$$\lambda_w = \lambda_{g_0} u^{\varepsilon_0} \lambda_{g_1} \cdots u^{\varepsilon_n}$$

Now, consider a subword of a reduced word $u^* x u$, being reduced, this forces $x \in \ker E_{L(H)}$. By (1.2.4), we can write x as a WOT-limit $x = \lim_i \sum_k c_{i,k} \lambda_{g_k}$ with $g_k \notin H$. Thus, $t^{-1} g_k t$ is of normal form for every $k \geq k_0$. In doing this, we can write any reduced word as an WOT-limit of normal forms with $n > 0$. By the above, this yields that all reduced words are orthogonal to $L(G)$. □

We now want to determine when an HNN extension is a factor.

Remark 4.1.14. Let $B \subseteq (N, \tau)$ be a von Neumann subalgebra of a II_1 factor N. Let $n \in \mathbb{N}$ and let $D_n(B) := B \oplus \cdots \oplus B \subseteq \mathbb{M}_n(N)$ be the diagonal subalgebra inside the matrix amplification. Then, $N \not\prec_N B$ implies $\mathbb{M}_n(N) \not\prec_{\mathbb{M}_n(N)} D_n(B)$.

To see this, we let $(u_i)_{i \in I}$ be the net of unitaries acquired from the condition $N \not\prec_N B$. Making the identification $\mathbb{M}_n(N) \cong N \overline{\otimes} \mathbb{M}_n(\mathbb{C})$, we can view $D_n(B) \cong B \overline{\otimes} \ell_n^\infty$, where ℓ_n^∞ is the diagonal subalgebra of $\mathbb{M}_n(\mathbb{C})$. Then, we have $E_{D_n(B)} = E_B \otimes E_{\ell_n^\infty}$, where $E_B : N \to B$ and $E_{\ell_n^\infty} : \mathbb{M}_n(\mathbb{C}) \to \ell_n^\infty$ are the trace-preserving conditional expectations. We can then lift the unitaries u_i to unitaries in $\mathbb{M}_n(N)$ by considering $u_i \otimes I$. For $m, n \in \mathbb{M}_n(N)$, we write $m = x \otimes A$ and $n = y \otimes B$ with $x, y \in N$ and $A, B \in \mathbb{M}_n(\mathbb{C})$. Then, we have

$$\|E_{D_n(B)}(a(u_i \otimes I)b)\|_2 = \|E_B(x u_i y)\|_2 \|E_{\ell_n^\infty}(AB)\|_2 \to 0$$

Therefore, we have $\mathbb{M}_n(N) \not\prec \mathbb{M}_n(N) D_n(B)$ as desired.

Corollary 4.1.15. If (N, τ) is a II_1 factor with $B \subseteq N$ and $\theta_i : B \hookrightarrow N$ trace-preserving embbeddings for $i = 2, \ldots, n$. Then, the HNN extension $\mathrm{HNN}(N, \theta, B)$ is also a II_1 factor.

Towards this, we use the following theorem:

Theorem 4.1.16. Suppose $v, v_\theta \in N$ are Haar unitaries such that

$$E_1(v^n) = E_2(v^n) = E_2(v_\theta^n) = 0, \quad \forall n \in \mathbb{Z} \setminus \{0\}$$

Let $V := v \oplus v_\theta \in \lambda_\theta(\mathbb{M}_2(N))$. Let $\mathcal{U} \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter on $\mathbb{N}$. Then, for any $x \in W^*(V)' \cap \mathcal{N}^{\mathcal{U}}$, we have

$$\|u_2(x - \mathcal{E}_\theta^{\mathcal{U}}(x))\|_2 \leq \|[u_2, x]\|_2$$

Where the 2-norm is taken on $L^2(\mathcal{N}^{\mathcal{U}})$, $\mathcal{E}_\theta^{\mathcal{U}}$ is the extension of $\mathcal{E}_\theta$ to the ultrapower, and where u_2 is viewed as its image through the diagonal embedding $u_2 = (u_2)_{\mathcal{U}}$.

Now, we can prove that the HNN extension M is a factor whenenver N is a factor.

Theorem 4.1.17. HNN extensions of II_1 factors are again II_1 factors.

Proof. The proof will use the following three steps: Firstly, we start by showing that

$$W^*(\pi(v), u_2)' \cap M^{\mathcal{U}} \subseteq \pi(N)^{\mathcal{U}} \cong N^{\mathcal{U}} \tag{1}$$

Then, we can upgrade this result to get rid of the ultrapowers:

$$W^*(\pi(v), u_2)' \cap M \subseteq \pi(N) \tag{2}$$

Finally, we can apply this to the center to show that

$$Z(M) = W^*(u_2)' \cap \pi(Z(N)) \cong \{x \in B \cap \theta(B) \cap N' : \theta(x) = x\} \tag{3}$$

Then, N being a factor implies that $Z(M) \subseteq \pi(Z(N)) = \mathbb{C}$.

For the first part, we use (4.1.14). Since $\pi(v) = pV = Vp = pVp$, we have that any $x \in W^*(\pi(v), u_2)' \cap M^{\mathcal{U}}$ satisfies that

$$\|u_2(x - \mathcal{E}_\theta^{\mathcal{U}}(x))\|_2 \leq \|[u_2, x]\|_2 = 0$$

Thus, $\|u_2(x - \mathcal{E}_\theta^{\mathcal{U}}(x))\|_2 = 0$, and since u_2 is a unitary, we get that $\|x - \mathcal{E}_\theta^{\mathcal{U}}(x)\|_2 = 0$. Hence, $x \in \lambda_\theta(\mathbb{M}_2(N))^{\mathcal{U}}$. Now, since $x \in M^{\mathcal{U}}$, we can express each $x = (x_n)_{\mathcal{U}}$, with $x_n \in M$. Then, the x_n will satisfy $x_n = px_np$, and hence $x = (px_np)_{\mathcal{U}} \in \pi(N)^{\mathcal{U}}$.

Now, for any $(x_n)_{\mathcal{U}} \in M \cap \pi(N)^{\mathcal{U}} \subseteq M^{\mathcal{U}}$, we have $x_n \in \pi(N)$, and hence

$$\left(E^M_{\pi(N)}\right)^{\mathcal{U}}((x_n)_{\mathcal{U}}) = \left(E^M_{\pi(N)}(x_n)\right)_{\mathcal{U}}$$

Since this holds for any representative sequence, and since $(x_n)_{\mathcal{U}}$ is taken inside the diagonal embedding of M, we can pick a constant sequence representative $x_n = x \in M$. The above then shows $x = E^M_{\pi(N)}(x)$, and hence $x \in \pi(N)$. Thus, $M \cap \pi(N)^{\mathcal{U}} \in \pi(N) \cap \pi(N)^{\mathcal{U}}$. Thus, (2) holds when viewing $W^*(\pi(v), u_2)' \cap M$ inside $M^{\mathcal{U}}$.

To show (3), note that any $x \in M$ commuting with all of M has to in particular commute with $\pi(N)$ and u_2, and hence with $W^*(u_2)$. Thus, we have $Z(M) = \pi(Z(N)) \cap W^*(u_2)'$. Now, any $x \in \pi(Z(N)) \cap W^*(u_2)'$ satisfies $u_2\pi(x) = \pi(x)u_2$. Thus,

$$0 = \|u_2\pi(x) - \pi(x)u_2\|_2 = \|u_2\pi(x)u_2^* - x\|_2$$

Hence, $\pi(x) = u_2\pi(x)u_2^*$. Similarly, we also get $\pi(x) = u_2^*\pi(x)u_2$. More specifically, we have $\theta(x) = x$ if and only if $\pi(x)$ commutes with u_2. Now, write $x = E_2(x) + y$, where $E_2(y) = 0$. We can calculate $\pi(x)$ using (4.1.11) as follows. We have that

$$\begin{aligned}
\pi(x) &= E^M_{\pi(N)}(x)(u_2\pi(x)u_2^*) \\
&= E^M_{\pi(N)}(u_2\pi(E_2(x))u_2^*) + E^M_{\pi(N)}(u_2\pi(y)u_2^*) \\
&= E^M_{\pi(N)}(\pi(\theta^{-1}(E_2(x)))) && = \pi(\theta^{-1}(E_2(x))) \in \pi(B)
\end{aligned}$$

We can repeat this with the equality $\pi(x) = u_2^*\pi(x)u_2$ to get that $\pi(x) \in \pi(\theta(B))$. Thus, $x \in B \cap \theta(B) \cap N'$. This shows both inclusions, hence (3) holds. □

Corollary 4.1.18. If N does not have property Γ, then neither does the HNN extension M.

Proof. We can repeat the same process as above by showing

$$M' \cap M^{\mathcal{U}} = M' \cap \pi(N)^{\mathcal{U}} \cong \{x \in B^{\mathcal{U}} \cap \theta^{\mathcal{U}}(B^{\mathcal{U}}) \cap N' : \theta^{\mathcal{U}}(x) = x\}$$

Note the first equality comes (1) in (4.1.15). The isomorphism can then be shown through the same method as the calculation in done at the end of the proof of (4.1.15). Namely, for $x \in \pi(Z(N))^{\mathcal{U}} \cap M'$, we have $x(u_2)_{\mathcal{U}} = (u_2)_{\mathcal{U}}x$ if and only if $(u_2)_{\mathcal{U}}x(u_2)^*_{\mathcal{U}} = x$, and so the same calculation works to show that $x \in \pi(B)^{\mathcal{U}} \cap \pi(\theta(B))^{\mathcal{U}} \cap \pi(N)'$.

Thus, if N does not have property Γ, then $M' \cap M^{\mathcal{U}} \subseteq \pi(N)' \cap \pi(N)^{\mathcal{U}} = \mathbb{C}$, so M does not have property Γ either. □

Corollary 4.1.19. Suppose $H \leq G$ is an inclusion of groups and $\theta : H \hookrightarrow G$ is an embedding distinct from the inclusion. If G is ICC, then $\check{G} := \mathrm{HNN}(G, \theta, H)$ is ICC.

This follows from (4.1.13), (4.1.15), and the fact that a group G is ICC if and only if $L(G)$ is a factor.

4.2 Chifan-Ioana-Kunnawalkam Elayavalli's lifting lemma

The construction of Chifan, Ioana, and Kunnawalkam Elayavalli's exotic II_1 factor requires the following lifting lemma, which will be stated as a theorem, being the main focus of this section. We first introduce the following definition.

Definition 4.2.1. Let $N_1, N_2 \subseteq (M, \tau)$ be an inclusion of tracial von Neumann algebras. For an $n \in \mathbb{N}$ with $n \geq 2$, we say that P and Q are n-free or n-freely independent when for any $m \leq n$ we have $\tau(x_1 \cdots x_m) = 0$ for every $x_j \in N_{i_j}$ with $i_j \neq i_{j+1}$ and $\tau(x_j) = 0$ for all j. In particular, this implies that for $y_j \in N_{i_j}$ with $i_j \neq i_{j+1}$ that $\tau(y_1 \cdots y_m) = \tau(y_1) \cdots \tau(y_m)$.

While free independence gives control over moments of all orders, n-free independence only controls evaluation of moments up to a certain order. Thus, n-freeness is not really a notion of "independence" in the traditional sense.

One condition that will be common among both this section and the following section will be 2-freeness.

Theorem 4.2.2 ([CIK23] lemma 3.1)**.** Let $(M_n)_{n\in\mathbb{N}}$ be a family of II_1 factors and let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Let $M := \prod_{n\to\mathcal{U}} M_i$. Suppose we have projections $p, q^{(1)}, q^{(2)}, q^{(3)} \in M$. Let $P := W^*(p)$ and $Q := W^*(q^{(1)}, q^{(2)}, q^{(3)})$. Suppose that $q^{(1)} + q^{(2)} + q^{(3)} = 1$ and that P and Q are 2-freely independent. Then, there are representations $p = (p_n)_{\mathcal{U}}$ and $q^{(j)} = (q_n^{(j)})_{\mathcal{U}}$ for which $q_n^{(1)} + q_n^{(2)} + q_n^{(3)} = 1$ and $W^*(p_i)$ is 2-freely independent from $W^*(q_n^{(1)}, q_n^{(2)}, q_n^{(3)})$ for all $i \in I$.

To remark on why we consider only 2-freeness instead of genuine free independence, consider the above theorem in the case where each M_n is isomorphic to the hyperfinite II_1 factor $\mathcal{R}$. One can define unitaries $u = 2p - 1$ and $v = q^{(1)} + \zeta q^{(2)} + \zeta^2 q^{(3)}$ where $\zeta = e^{2\pi i/3}$. In doing so, we have $u^2 = v^3 = 1$, and this translation between 2-free unitaries of finite order and a projection 2-free from a partition of unity. If we replace 2-freeness

with genuine freeness, then u and v generate freely independent von Neumann subalgebras, and hence $W^*(u,v) \cong L(\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z})$. Since this is the group von Neumann algebra of a non-amenable group, $W^*(u,v)$ is itself non-amenable. This group von Neumann algebra does embed into an ultraproduct of $\mathcal{R}$, but replacing 2-freeness with freeness in the above theorem would give an embedding of $W^*(u,v)$ into $\mathcal{R}$ itself, which would imply that $\mathcal{R}$ has a non-amenable subalgebra.

We will approach the proof of this theorem with a series of lemmas.

Lemma 4.2.3. Let (M,τ) be a diffuse tracial von Neumann algebra. Let $x \in M_{\text{sa}}$ such that $|\tau(x)| < \tau(|x|) =: \|x\|_1$. Then, for $c > \varepsilon > 0$ with $|\tau(x)| \leq \varepsilon$ and $\|x\|_1 > c$, $\exists p \in M$ a projection with $\tau(xp) = 0$ and $\tau(p) > (c+\varepsilon)^{-1}(c-\varepsilon)$.

Proof. Since x is selfadjoint, we can write $x = x_+ - x_-$ for $x_+, x_- \in M_+$ positive elements. Let $q \in M$ be the support of x_+, so that $x_+ \in qMq$ and $x_- \in q^\perp M q^\perp$.

Assume that $\tau(x) > 0$ (otherwise, consider $-x$ instead). Then, we have $\tau(x_+) - \tau(x_-) \leq \varepsilon$ and $\tau(x_+) + \tau(x_-) > c$. Thus, $2\tau(x_+) < c + \varepsilon$ and $2\tau(x_-) > c - \varepsilon$. In particular, letting $s = (c+\varepsilon)^{-1}(c-\varepsilon)$, we get $\tau(x_+)s < \tau(x_-)$.

Taking a sequence of simple functions f_n on $\sigma(x_+)$ converging to the identity on $\sigma(x_+)$, have that $\|x_+ - f_n(x_+)\|_1 \to 0$. As a result, each $f_n(x_+)$ has finite spectrum, since each f_n is a linear combination of indicator functions. Thus, $\exists n \in \mathbb{N}$ such that

$$\|x_+ - f_n(x_+)\|_1 < \tau(x_-) - \tau(x_+)s \tag{1}$$

We take $y := f_n(x_+)$. Now, since f_n is a simple function, we can write y as a linear combination of a partition of unity. I.e.

$$y = \sum_{k=1}^{m} \alpha_k p_k$$

Where $\{p_k\}_{k=1}^m$ are pairwise orthogonal and sum to 1 (this latter condition comes from the fact that for sufficiently large n f_n can only be nonzero on 0 and a nullset of $\sigma(x_+)$, so it particular, the indicators have to partition $\sigma(x_+)$ up to a nullset). Then, each corner $p_k qMq p_k$ is diffuse, and so for any $t \in [0,1]$, there's some projection $e_t^{(k)} \in p_k qMq p_k$ such that $\tau_q(e_t^{(k)}) = t\tau_q(p_k)$, where τ_q is the trace on the corner qMq. We can then define

$$e_t := \sum_{k=1}^{m} e_t^{(k)}$$

We have that

$$\tau_q(e_t) = \sum_{k=1}^{m} \tau_q(e_t^{(k)})\tau_q(p_k)$$

Thus, $\tau_q(e_t) = t$ for all $t \in [0,1]$, so in particular $e_1 = q$ (the unit in qMq), and $e_0 = 0$. Further, since the p_k are pairwise orthogonal, and each $e_t^{(k)}$ is a subprojection of p_k, we have

$$ye_t = \sum_{k=1}^{m} \alpha_k e_t^{(k)}$$

Thus,

$$\tau_q(ye_t) = \sum_{k=1}^{m} \alpha_k \tau_q(e_t^{(k)}) = \sum_{k=1}^{m} \alpha_k t \tau_q(p_k) = t \sum_{k=1}^{m} \alpha_k \tau_q(p_k) = t\tau_q(y)$$

Now, this lifts to the trace τ on M, and so for any $t \in [0,1]$, we have

$$|\tau(x_+e_t) - t\tau(x_+)| \leq |\tau(x_+e_t) - \tau(ye_t)| + |\tau(x_+) - \tau(y)| \leq 2\|y - x_+\|_1$$

Therefore, we have $\tau(x_+e_t) \leq t\tau(x_+) + 2\|y - x_+\|_1$.

Taking $t = s$, we can combine this with (1) to get $\tau(x_+e_s) < \tau(x_-)$. Since $\tau(x_+e_1) = \tau(x_+q) = \tau(x_+) > \tau(x_-)$, we have that $t \mapsto \tau(x_+e_t)$ is continuous, and so $\exists t \in (s,1)$ for which $\tau(x_+e_t) = \tau(x_-)$. With this t, we take $p = e_t + q^\perp$. Then, $\tau(xp) = \tau(x_+p) - \tau(x_-p) = \tau(x_+e_t) - \tau(x_-) = 0$. Further, $\tau(p) = t\tau(q) + \tau(q^\perp) \geq t > s$, as desired. □

Lemma 4.2.4. Let (M,τ) be a diffuse tracial von Neumann algebra. Let $\varepsilon, c \geq 0$ with $\varepsilon < c^2$. Let $p, r_1, r_2, r_3 \in M$ be projections with $r_1 + r_2 + r_3 = 1$ and such that

$$|\tau(pr_i) - \tau(p)\tau(r_i)| \leq \varepsilon, \quad \|r_i(p - \tau(p))r_i\|_1 > c, \quad \text{for } i = 1,2$$

Then, $\exists q_1, q_2, q_3 \in M$ projections with $q_1 + q_2 + q_3 = 1$ and with

$$\tau(pq_i) = \tau(p)\tau(q_i), \quad \|q_i - r_i\|_1 < \frac{4\varepsilon}{c^2}, \quad \text{for } i = 1,2,3$$

Proof. Let $x_i := r_i(p - \tau(p))r_i \in (r_iMr_i)_{\text{sa}}$ for $i = 1,2$. The hypotheses then become $|\tau(x_i)| \leq \varepsilon$ and $\|x_i\|_1 > c$. Further, we have $\|x_i\|_1 \leq \tau(r_i)\|p - \tau(p)\|_{\text{op}} \leq \tau(r_i)$, since

$$\|x_i\|_1 \leq \|r_i\|_1 \|(p - \tau(p))\|_{\text{op}} \|r_i\|_{\text{op}} = \tau(r_i)\|p - \tau(p)\|_{\text{op}}$$

and from $\|p - \tau(p)\|_{\text{op}} \leq 1$. From this, we can see that $1 \geq \tau(r_i) > c$, and hence

$$\left|\frac{\tau(x_i)}{\tau(r_i)}\right| = \frac{|\tau(x_i)|}{\tau(r_i)} \leq \frac{\varepsilon}{c}, \qquad \frac{\|x_i\|_1}{\tau(r_i)} \geq \|x_i\|_1 > c$$

Where this second inequality comes from noting that $\tau(r_i)^{-1} \geq 1$. We can apply the above lemma (4.2.3) to $x_i \in (r_iMr_i)_{\text{sa}}$ with the following bounds.

$$|\tau_{r_i}(x_i)| \leq c^{-1}\varepsilon, \qquad \|x_i\|_{1,r_i} > c$$

to acquire projections $q_i \in r_iMr_i$ such that

$$\tau_{r_i}(x_iq_i) = 0 = \tau(x_iq_i), \qquad \tau_{r_i}(q_i) > \frac{c - \frac{\varepsilon}{c}}{c + \frac{\varepsilon}{c}}$$

Where τ_{r_i} is the trace on the corner r_iMr_i. The latter inequality becomes

$$\frac{\tau(q_i)}{\tau(r_i)} > \frac{1 - \frac{\varepsilon}{c^2}}{1 + \frac{\varepsilon}{c^2}} > 1 - \frac{2\varepsilon}{c^2} \tag{1}$$

From $\tau(x_iq_i) = 0$ and since $q_i \leq r_i$, we get $\tau((p - \tau(p))q_i) = 0$, and hence $\tau(p)\tau(q_i) = 0$.

Further, $q_i \leq r_i$ implies $\|r_i - q_i\|_1 = \tau(r_i) - \tau(q_i)$. From (1), we have

$$\tau(r_i) - \tau(q_i) \leq \tau(r_i)\left(\frac{2\varepsilon}{c^2}\right) \leq \frac{2\varepsilon}{c^2}$$

Since $q_1 \perp q_2$, have $q_1 + q_2 \in M$ is a projection, so we can let $q_3 := (q_1 + q_2)^\perp = 1 - q_1 - q_2$. Then, $q_1 + q_2 + q_3 = 1$, and we have

$$\begin{aligned}\tau(pq_3) &= \tau(p) - \tau(pq_1) - \tau(pq_2) \\ &= \tau(p)\left(1 - \tau(q_1) - \tau(q_2)\right) \\ &= \tau(p)\tau(q_3)\end{aligned}$$

Since $r_1 + r_2 + r_3 = 1$, we have $r_3 = 1 - r_1 - r_2$, and hence

$$\begin{aligned}\|q_3 - r_3\|_1 &= \|(q_1 + q_2) - (r_1 + r_2)\|_1 \\ &\leq \|q_1 - r_1\|_1 + \|q_2 - r_2\|_1 \\ &< \frac{4\varepsilon}{c^2}\end{aligned}$$

This completes the proof. □

Lemma 4.2.5. For all $\varepsilon > 0$, $\exists \delta > 0$ such that whenever (M, τ) is a II_1 factor with projections $e, r^{(1)}, r^{(2)}, r^{(3)} \in M$ satisfying $r^{(1)} + r^{(2)} + r^{(3)} = 1$ and $|\tau(er^{(j)}) - \tau(e)\tau(r^{(j)})| < \delta$ for all j, then $\exists p, q^{(1)}, q^{(2)}, q^{(3)} \in M$ projections satisfying $q^{(1)} + q^{(2)} + q^{(3)} = 1$ and $\tau(pq^{(j)}) = \tau(p)\tau(q^{(j)})$ for all j, and such that

$$\|p - e\|_2 < \varepsilon, \quad \|q^{(j)} - r^{(j)}\|_2 < \varepsilon \tag{1}$$

Proof. Firstly, for projections $p, q \in M$, we have

$$\|p - q\|_2 = \|p(p - q) + (p - q)q\|_2 \leq 2\|p - q\|_1$$

In particular, replacing ε with $\frac{1}{2}\varepsilon$, it suffices to prove the above but with (1) replaced by the following:

$$\|p - e\|_1 < \varepsilon, \quad \|q^{(j)} - r^{(j)}\|_1 < \varepsilon$$

Finally, by replacing ε again with $\varepsilon/4$, we may change the assumption to

$$\|p - e\|_1 + \sum_{j=1}^{3} \|q^{(j)} - r^{(j)}\|_1 < \varepsilon \tag{1'}$$

We now go by contradiction. Let $\varepsilon > 0$ be the value such that for any $n \in \mathbb{N}$, there is a II_1 factor (M_n, τ_n) with projections $e_n, r_n^{(1)}, r_n^{(2)}, r_n^{(3)} \in M_n$ such that $r_n^{(1)} + r_n^{(2)} + r_n^{(3)} = 1$ and

$$|\tau(e_n, r_n^{(j)}) - \tau(e_n)\tau(r_n^{(j)})| < \frac{1}{n}$$

but for all projections $p_n, q_n^{(1)}, q_n^{(2)}, q_n^{(3)} \in M_n$ with $\tau(p_n q_n^{(j)}) = \tau(p_n)\tau(q_n^{(j)})$ for all j, we have

$$\|p_n - e_n\|_1 + \sum_{j=1}^{3} \|q_n^{(j)} - r_n^{(j)}\|_1 \geq \varepsilon$$

Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and let $(M, \tau) := \prod_{n\to\mathcal{U}}(M_n, \tau_n)$. Let $p := (e_n)_\mathcal{U}, q^{(j)} := (r_n^{(j)})_\mathcal{U}$. Then, $q^{(1)} + q^{(2)} + q^{(3)} = 1$ and $\tau(pq^{(j)}) = \tau(p)\tau(q^{(j)})$ for all j. We define the set

$$\mathcal{A} := \{1 \leq j \leq 3 : q^{(j)}(p - \tau(p))q^{(j)} = 0\}$$

and break into two cases.

Firstly, suppose that $|\mathcal{A}| \leq 1$. By reordering, we may assume that $q^{(j)}(p-\tau(p))q^{(j)} \neq 0$ for $j = 1, 2$. Let $\delta_j := \|q^{(j)}(p - \tau(p))q^{(j)}\|_1$, and $\delta_{j,n} := \|r_n^{(j)}(e_n - \tau_n(e_n))r_n^{(j)}\|_1$ for $j = 1, 2$. Finally, define $\kappa_{j,n} := |\tau_n(e_n r_n^{(j)}) - \tau_n(e_n)\tau_n(r_n^{(j)})|$. By the assumptions, we have that $0 \leq \kappa_{j,n} < \frac{1}{n}$, $\delta_j > 0$, and $\lim_{n\to\mathcal{U}} \delta_{j,n} = \delta_j$. Let $\delta := \min(\delta_1, \delta_2)$. Now, we have

$$J := \{n \in \mathbb{N} : \delta_{j,n} > \delta/2,\ \kappa_{i,n} < \delta_{j,n}^2,\ j = 1, 2\} \in \mathcal{U}$$

By lemma (4.2.4), for any $n \in J$, we can find projections $q_n^{(i)} \in M_n$ for $i = 1, 2, 3$ such that $q_n^{(1)} + q_n^{(2)} + q_n^{(3)} = 1$, $\tau_n(pq_n^{(i)}) = \tau_n(p)\tau_n(q_n^{(i)})$, and

$$\|q_n^{(i)} - r_n^{(i)}\|_1 < \frac{4\kappa_{i,n}}{\delta_{i,n}^2} < \frac{16}{\delta^2 n}$$

for $i = 1, 2$. Since J is infinite, we can find $n \in J$ such that $\frac{16}{\delta^2 n} < \frac{\varepsilon}{3}$. Then, we can pick $p_n = e_n$ and we have

$$\|p_n - e_n\|_1 + \sum_{i=1}^{3} \|q_n^{(i)} - r_n^{(i)}\|_1 < \varepsilon$$

which is a contradiction.

Now, consider the case where $|\mathcal{A}| \geq 2$ and by rearranging if necessary, assume $q^{(3)}(p - \tau(p))q^{(3)} \neq 0$. Let $Q := W^*(p, q^{(1)}, q^{(2)}, q^{(3)})$. For $i \leq 2$, we have $q^{(i)}pq^{(i)} = \tau(p)q^{(i)}$. Then, define $v_i := \tau(p)^{-1/2}q^{(i)}p$. This is a partial isometry with $p_i := v_i^* v_i$. Consider $pQp \subseteq \mathbb{B}(L^2(pMp))$. We have $pQp = W^*(pq^{(1)}p, pq^{(2)}p, pq^{(3)}p) = W^*(p_1, p_2)$. Note that this first equality comes from p being the unit in pMp, and the second comes from noting that $pq^{(1)}p + pq^{(2)}p + pq^{(3)}p = p$, so $pq^{(3)}p = p - pq^{(1)}p - pq^{(2)}p$. Therefore, pQp is generated by two projections and hence of type I. Further, since $q^{(i)}(p^\perp - \tau(p^\perp))q^{(i)} = q^{(i)}(p - \tau(p))q^{(i)} = 0$, we have that $p^\perp Q p^\perp$ is also of type I. Hence, Q must also be of type I.

Since Q is of type I, it is hyperfinite, and inherits the trace on M. Let $\pi_n : Q \to M_n$ be trace-preserving $*$-homomorphisms, which are guaranteed since each M_n is a II_1 factor. Let $\pi : Q \to M$ be the lifting of π_n to the ultraproduct. Since Q is hyperfinite, any two trace-preserving $*$-homomorphisms are unitarily conjugate. Thus, $\exists u_n \in M_n$ unitaries such that for any $x \in Q$, we have $x = (u_n \pi_n(x) u_n^*)_\mathcal{U}$. Thus, we can pick representatives $p = (u_n \pi_n(p) u_n^*)_\mathcal{U}$ and $q^{(i)} = (u_n \pi_n(q_i) u_n^*)_\mathcal{U}$. Then, letting $p_n = u_n \pi_n(p) u_n^*$ and $q_n^{(i)} :=$

$u_n\pi_n(q_i)u_n^*$, we have $q_n^{(1)} + q_n^{(2)} + q_n^{(3)} = 1$. Further,

$$\lim_{n\to\mathcal{U}} \left(\|p_n - e_n\|_1 + \sum_{i=1}^{3} \|q_n^{(i)} - r_n^{(i)}\|_1 \right) = 0$$

And since the π_n are trace-preserving, we have for all i that $\tau_n(p_n q_n^{(i)}) = \tau_n(p_n)\tau_n(q_n^{(i)})$, which yields contradiction. □

Now, we can prove theorem (4.2.2) using the above lemma.

Proof of Theorem 4.2.2. Let $p = (e_n)_\mathcal{U}$ and $q^{(i)} = (r_n^{(i)})_\mathcal{U}$ be representative sequences of projections, with $r_n^{(1)} + r_n^{(1)} + r_n^{(3)} = 1$ for each n (for $q^{(1)}$ and $q^{(2)}$, we can find mutually orthogonal representative sequences, and then set $q_n^{(3)} := 1 - q_n^{(1)} - q_n^{(2)}$ in order to achieve this). Since $W^*(p)$ and $W^*(q^{(1)}, q^{(2)}, q^{(3)})$ are 2-freely independent, it follows that $\tau(pq^{(i)}) = \tau(p)\tau(q^{(i)})$ for $i = 1, 2, 3$. Thus, we have

$$\lim_{n\to\mathcal{U}} |\tau(e_n r_n^{(i)}) - \tau(e_n)\tau(r_n^{(i)})| = 0$$

To each $n \in \mathbb{N}$, let $\delta > 0$ come from lemma (4.2.5) with $\varepsilon = \frac{1}{n}$. Define the set

$$\mathcal{A}_n := \{k \in \mathbb{N} : |\tau(e_k r_k^{(i)}) - \tau(e_k)\tau(r_k^{(i)})| < \delta\}$$

Then, $\mathcal{A}_n \in \mathcal{U}$ for each $n \in \mathbb{N}$. We can pick $p_n, q_n^{(i)} \in M_n$ projections such that $q_n^{(1)} + q_n^{(2)} + q_n^{(3)} = 1$ and $\tau(p_n q_n^{(i)}) = \tau(p_n)\tau(q_n^{(i)})$ for all i, and such that $\|p_n - e_n\|_2, \|q_n^{(i)} - r_n^{(i)}\|_2 < \frac{1}{n}$. Thus, on $\mathcal{A}_n$ we may replace e_n with p_n and $r_n^{(i)}$ with $q_n^{(i)}$. Outside of $\mathcal{A}_n$, we can simply set $p_n = q_n^{(1)} = q_n^{(2)} = 0$ and $q_n^{(3)} = 1$, and we still have $\tau(p_n q_n^{(i)}) = \tau(p_n)\tau(q_n^{(i)})$, as well as $q_n^{(1)} + q_n^{(2)} + q_n^{(3)} = 1$. Since this is not witnessed by the ultrafilter, we still have $p = (p_n)_\mathcal{U}$ and $q^{(i)} = (q_n^{(i)})_\mathcal{U}$.

Now, for each n, we have $\tau(p_n q_n^{(i)}) = \tau(p_n)\tau(q_n^{(i)})$. Since the $q_n^{(i)}$ are mutually orthogonal, we have that for any polynomial $f \in \mathbb{C}\langle q_n^{(1)}, q_n^{(2)}, q_n^{(3)}\rangle$ that $\tau(pf) = \tau(p)\tau(f)$, and by passing to WOT-closures, we have that $W^*(p_n)$ is 2-freely independent from $W^*(q_n^{(1)}, q_n^{(2)}, q_n^{(3)})$ for each n. □

4.3 The exotic II_1 factor

We now detail the construction of a non-Γ separable II_1 which is not elementarily equivalent to $L(\mathbb{F}_n)$ for any $n \geq 2$. While [CIK23] proves this using ultrafilters on arbitrary index sets, by reducing to countably cofinal ultrafilters, much of the detail remains the same when replacing countably cofinal ultrafilters on arbitrary index sets with free ultrafilters on $\mathbb{N}$, outside of a couple technical results which are dealt with in other papers. As a result, we will focus on showing such a II_1 factor M exists for which if N is a tracial von Neumann algebra of positive 1-bounded entropy, then $M^\mathcal{U} \ncong N^\mathcal{V}$ for any two ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$.

We start with the following lemma:

Lemma 4.3.1. Let (M, τ) be a diffuse tracial von Neumann algebra, let $x \in M_{\text{sa}}$ with $\tau(x) = 0$, and let $F := \{p \in M \text{ a projection} : \tau(xp) = 0\}$. Then, $F'' = M$.

Proof. Firstly, assume M is abelian. Let $x = x_+ - x_-$ be the decomposition into positive and negative parts. Then, $\tau(x_+) = \tau(x_-)$. Let $p, q \in M$ be the support projections of x_+ and x_- respectively.

Let $r_1 \in pMp$ be a projection. Since M is diffuse and $\tau(x_+r_1) \leq \tau(x_+) = \tau(x_-) = \tau(x_-q)$, we can find $r_2 \in qMq$ such that $\tau(x_+r_1) = \tau(x_-r_2)$. To see this, note that $M \cong L^\infty(X, \mu)$ for some standard diffuse probability space. Since X is diffuse, we can find some measurable set $E \subseteq X$ for which

$$\int_E x_- \mathrm{d}\mu = \tau(x_+r_1)$$

Then, we can take r_2 to be the the indicator function $\mathbb{1}_E$. Now, since p and q are orthogonal to each other, we have $\tau(x(r_1 - r_2)) = \tau((x_+ - x_-)(r_1 - r_2)) = \tau(x_+r_1) - \tau(x_-r_2) = 0$, so $r_1 - r_2 \in F$. Further, we have $(r_1 - r_2)^2 = r_1 + r_2$ since r_1 and r_2 are orthogonal. Then, $r_1 + r_2 \in F''$. Since $r_1 - r_2 \in F \subseteq F''$, we get $r_1 \in F''$ for any such projection $r_1 \in pMp$. Hence, $pMp \subseteq F''$. Similarly, we get that $qMq \subseteq F''$.

Now, let $f := 1 - p - q$. We have that $xf = 0$, and hence $f \in F$. For any other projection $e \in fMf$, we have $\tau(xe) = 0$, so $e \in F$, and hence $fMf \subseteq F''$. Now, since $M \cong pMp \oplus qMq \oplus fMf$, we have $M \subseteq F''$, and hence $F'' = M$.

Suppose now that M is no longer abelian, and let $N \subseteq M$ be a diffuse abelian von Neumann subalgebra. Then, $\tau(E_N(x)) = 0$, and if $p \in N$ is a projection with $\tau(E_N(x)p) = 0$, we have $p \in F$, since $\tau(E_N(x)p) = \tau(E_N(xp)) = \tau(xp)$. We can then apply the above with N to get that $N \subseteq F''$. Since this holds for every diffuse abelian subalgebra of M, we conclude $F'' = M$. □

Definition 4.3.2 ([PV10])**.** Let $B \subseteq M_1, M_2$ be an inclusion of tracial von Neumann algebras and let $(M, \tau) := M_1 *_B M_2$. For each $\rho \in (0, 1)$, we can define a map $\mathrm{m}_\rho : M \to M$ such that $\mathrm{m}_\rho|_B = \mathrm{id}\,|_B$ and $\mathrm{m}_\rho(\mathrm{x}_1 \cdots \mathrm{x}_\mathrm{n}) = \rho^\mathrm{n}\mathrm{x}_1 \cdots \mathrm{x}_\mathrm{n}$ for $x_j \in M_{i_j} \ominus B$ and $i_j \neq i_{j+1}$ for all j. This map is normal and completely positive. Further, since $(1 - \rho)^2 \leq 1 - \rho \leq 1 - \rho^2$, we have the same inequality with the operator m_ρ. Namely,

$$(1 - \mathrm{m}_\rho)^2 \leq 1 - \mathrm{m}_\rho \leq 1 - \mathrm{m}_\rho^2$$

We now state a variant of one of the theorems of [IPP08] for later use, for which the statement can be found in [PV10, theorem 5.4] and [CIK23, theorem 4.5].

Theorem 4.3.3. Let $B \subseteq M_1, M_2$ be an inclusion of tracial von Neumann algebras and let $(M, \tau) := M_1 *_B M_2$. Let $p \in M$ be a nonzero projection and $Q \subseteq pMp$ a von Neumann subalgebra of the corner. Suppose $\exists \rho \in (0, 1)$ and $\delta > 0$ such that $\|\mathrm{m}_\rho(u)\| \geq \delta$ for all unitaries $u \in Q$. Then, either $Q \prec_M M_1$ or $Q \prec_M M_2$

Again for later use, the following is theorem 1.1 of [IPP08]

Theorem 4.3.4. Let $B \subseteq M_1, M_2$ be an inclusion of tracial von Neumann algebras and let $(M, \tau) := M_1 *_B M_2$. Suppose $q \in M_1$ is a nonzero projection and $Q \subset qM_1q$ is a von

Neumann subalgebra. Suppose $Q \not\prec_{M_1} B$. If $Q \prec_M M_k$ for some $k \in \{1,2\}$, then $k = 1$ and moreover $Q \prec_{M_k} M_k$. Furthermore, $Q' \cap qMq \subset M_1$.

Lemma 4.3.5. Let $B \subseteq (M_i, \tau_i)_{i=1,2}$ be an inclusion of tracial von Neumann algebras, and let $(M, \tau) := M_1 *_B M_2$. Suppose for each i, there are von Neumann subalgebras $A_i \subseteq M_i$ which are 2-freely independent from B, and let $A := A_1 \vee A_2 \subseteq M$. Let $Q \subset M_1$ be a von Neumann subalgebra such that $Q \prec_M A$ and $Q \not\prec_{M_1} B$. Then, $Q \prec_{M_1} A_1$.

Proof. We begin first by showing that A_1 and A_2 are freely independent from each other. Towards this, we consider $\tau(a_1 \cdots a_n)$ where $a_j \in A_{i_j}$ with $\tau_{i_j}(a_j) = 0$ and $i_j \neq i_{j+1}$ for all j. Since A_i are 2-free from B, we have $E_B(a_j) = 0$ for all j, and hence $E_B(a_1 \cdots a_n) = 0$. Since the conditional expectation is trace-preserving, we have $0 = \tau(E_B(a_1 \cdots a_n)) = \tau(a_1 \cdots a_n)$. Thus, $A \cong A_1 * A_2$.

Since $Q \prec_M A$, we can find projections $q \in Q, p \in A$, a nonzero partial isometry $v \in M$, and a map $\theta : qQq \to pAp$ for which $xv = v\theta(x)$ for all $x \in qQq$. Next, by a corollary in [Pop06a], we have that if $u \in M$ is a unitary, then

$$\lim_{\rho \to 1} \sup_{x \in (M)_1} \|\mathrm{m}_\rho(xu) - \mathrm{m}_\rho(x)u\|_2 = 0 \tag{1}$$

and that

$$\lim_{\rho \to 1} \sup_{x \in (M)_1} \|\mathrm{m}_\rho(ux) - u\mathrm{m}_\rho(x)\|_2 = 0 \tag{2}$$

Since the unitary group $\mathcal{U}(M)$ spans M, we have that (1) and (2) still hold when replacing u with any element of M. For $x \in M_1$, we have $\mathrm{m}_\rho(x) = E_B(x) + \rho(x - E_B(x))$. Hence, we have

$$\|\mathrm{m}_\rho(x) - x\|_2 = (1-\rho)\|x - E_B(x)\|_2 \leq (1-\rho)\|x\|_2$$

Taking $x \in (qQq)_1$, we have

$$\lim_{\rho \to 1} \sup_{x \in (qQq)_1} \|x - \mathrm{m}_\rho(x)\|_2 = 0 \tag{3}$$

Now, for $x \in (qQq)1$, we have that

$$\|v\mathrm{m}_\rho(\theta(x)) - xv\|_2 \leq \|v\mathrm{m}_\rho(\theta(x)) - \mathrm{m}_\rho(v\theta(x))\|_2 + \|\mathrm{m}_\rho(v\theta(x)) - xv\|_2$$

Similarly, we have

$$\begin{aligned} \|\mathrm{m}_\rho(v\theta(x)) - xv\|_2 &= \|\mathrm{m}_\rho(xv) - xv\|_2 \\ &\leq \|\mathrm{m}_\rho(xv) - \mathrm{m}_\rho(x)v\|_2 + \|\mathrm{m}_\rho(x)v - xv\|_2 \\ &\leq \|\mathrm{m}_\rho(xv) - \mathrm{m}_\rho(x)v\|_2 + \|\mathrm{m}_\rho(x) - x\|_2\|v\|_{\mathrm{op}} \end{aligned}$$

Using these, combined with (1), (2), and (3), we have that

$$\lim_{\rho \to 1} \sup_{x \in (qQq)_1} \|v\mathrm{m}_\rho(\theta(x)) - xv\|_2 = 0 (4)$$

From (4), we can find $\rho \in (0,1)$ such that $\|v\mathrm{m}_\rho(\theta(x)) - xv\|_2 < \frac{1}{2}\|v\|_2$ for all $x \in (qQq)_1$.

In particular, for a unitary $u \in qQq$, we have that

$$\begin{aligned}\frac{1}{2}\|v\|_2 &> \|v\mathrm{m}_\rho(\theta(u)) - uv\|_2 \\ &\geq |\|v\mathrm{m}_\rho(\theta(u))\|_2 - \|vu\|_2| \\ &= |\|v\mathrm{m}_\rho(\theta(u))\|_2 - \|v\|_2|\end{aligned}$$

In particular, we either have $\|v\mathrm{m}_\rho(\theta(u))\|_2 \geq \frac{3}{2}\|v\|_2$, or that $\|v\mathrm{m}_\rho(\theta(u))\|_2 \geq \frac{1}{2}\|v\|_2$, depending on sign. In either case, we have $\|\mathrm{m}_\rho(\theta(u))\|_2 = \|v\|_{\mathrm{op}}\|\mathrm{m}_\rho(\theta(u))\|_2 \geq \|v\mathrm{m}_\rho(\theta(u))\|_2$. This implies that for any unitary $y \in \theta(qQq)$, we have $\|\mathrm{m}_\rho(y)\|_2 > \|v\|_2$. Hence, by theorem (4.3.3) applied with $\theta(qQq)$, we have either $\theta(qQq) \prec_A A_1$ or $\theta(qQq) \prec_A A_2$.

From $\theta(qQq) \prec_A A_i$ for some i, we can find projections $r \in \theta(qQq) \subseteq pAp$ (hence, $r \leq p$), $s \in A_i$, a $*$-homomorphism $\psi : r\theta(qQq)r \to sA_is$, and a partial isometry $w \in A$ for which $xw = w\psi(x)$ for all $x \in rQr$. Then, note that if $vw = 0$, we would have $pw = E_A(v^*v)w = E_A(v^*vw) = 0$. However, for any $x \in r\theta(qQq)r$, we have $x \in pAp$, and hence $x = pxp$. Since $xw = w\psi(x)$, we have $xw = pxpw = 0$ for all $x \in r\theta(qQq)r$, a contradiction. Hence, $vw \neq 0$. Now, since $r \in \theta(qQq)$ and θ is a $*$-homomorphism, there's some $r_0 \in qQq$ a projection for which $\theta(r_0) = r$. In particular, this means we can restrict θ to r_0Qr_0 (noting $r_0 \leq q$), and then for $\theta_0 := \theta|_{r_0Qr_0}$, we have that $\psi \circ \theta_0 : r_0Qr_0 \to sA_is$ and satisfies that for any $x \in r_0Qr_0$ we have $x(vw) = vw(\psi \circ \theta_0)(x)$. Hence, $Q \prec_M A_i$.

By theorem (4.3.4), the above, combined with the conditions $qQq \subseteq M_1$, $A_i \subseteq M_i$, and $Q \not\prec_{M_1} B$ imply that $Q \prec_{M_1} A_1$ specifically (i.e. that $i = 1$). □

Definition 4.3.6. Let (M, τ) be a tracial von Neumann algebra and $A_1, A_2 \subseteq M$ von Neumann subalgebras. Let $B_1 = B_2 = L(\mathbb{Z})$. Then, we define recursively the following extension of M by A_1 and A_2.

$$\Phi(M, A_1) := M *_{A_1} (A_1 \overline{\otimes} B_1)$$

and

$$\Phi(M, A_1, A_2) := \Phi(M, A_1) *_{A_2 \vee B_1} ((A_2 \vee B_1)\overline{\otimes}B_2)$$

More generally, if $A_1, \ldots, A_n \subseteq M$ are von Neumann subalgebras, we can take $B_1, \ldots, B_n = L(\mathbb{Z})$ and define

$$\Phi(M, A_1, \ldots, A_n) := \Phi(M, A_1, \ldots, A_{n-1}) *_{(A_n \vee B_1 \vee \cdots \vee B_{n-1})} ((A_n \vee B_1 \vee \cdots \vee B_{n-1})\overline{\otimes}B_n)$$

We will only be considering the extension by two subalgebras. Further, if $u_1, u_2 \in M$ are unitaries, we may write $\Phi(M, u_1, u_2)$ in place of $\Phi(M, W^*(u_1), W^*(u_2))$.

Theorem 4.3.7. Let (M, τ) be a tracial von Neumann algebra and $A_1, A_2 \subseteq M$ 2-freely independent von Neumann subalgebras such that for $i = 1, 2$ we have $M \not\prec_M A_i$. Let $P := \Phi(M, A_1, A_2)$. Then,

(i) P contains Haar unitaries v_1, v_2 for which $[v_1, v_2] = 0$ and $v_i \in A_i' \cap P$ for $i = 1, 2$.

(ii) for $Q \subseteq M$ a von Neumann subalgebra with $Q \not\prec_M A_i$ for $i = 1, 2$, we have $Q' \cap P \subset M$

(iii) P is a II_1 factor.

Proof.

(i) Let $v_i \in B_i$ be the Haar unitaries generating B_i. Let $N := \Phi(M, A_1)$ so that $P = N *_{A_2 \vee B_1} ((A_2 \vee B_1)\overline{\otimes}B_2)$. Then, Since $\Phi(M, A_1)$ has a copy of $A_1\overline{\otimes}B_1$, we have $[A_1, B_1] = 0$. Similarly, from the construction of P we have $[A_2, B_2] = \{0\} = [B_1, B_2]$. In particular, this means $[v_1, v_2] = 0$ and $v_i \in A_i' \cap P$.

(ii) Let $Q \subseteq M$ be a von Neumann subalgebra with $Q \not\prec_M A_i$ for $i = 1, 2$. We can consider the contrapositive of lemma (4.3.5) with respect to N. This then states that $Q \not\prec_M A_2$ implies either $Q \prec_M A_1$ or $Q \not\prec_N (A_2 \vee B_1)$. Given that $Q \not\prec_M A_i$ for $i = 1, 2$, we can conclude $Q \not\prec_N (A_2 \vee B_1)$. Then, by theorem (4.3.4) we have that $Q \not\prec_N (A_2 \vee B_1)$ implies $Q' \cap P \subseteq N$. We can apply the theorem again with respect to $Q \not\prec_M A_1$ to get that $Q' \cap N \subseteq M$, and hence $Q' \cap P \subseteq M$.

(iii) Suppose first that M is a II_1 factor. Then, we can take M in place of Q, and since $M \not\prec_M A_i$ for $i = 1, 2$. From (ii) we get that $M' \cap P \subseteq M$. For any $x \in Z(P)$, we have that $x \in M' \cap P \subseteq M$. Similarly, any $x \in P' \cap M$ is already in $Z(P)$. Hence, $Z(P) = P' \cap M \subseteq Z(M)$. Thus, P has trivial center.

It now remains to deal with the case where M isn't necessarily a factor. Firstly, we can apply theorem (4.3.4) with $B_1 \subseteq A_1\overline{\otimes}B_1$. Note that $B_1 \not\prec_{A_1\overline{\otimes}B_1} A_1$, since we can consider the unitaries $\lambda_n \in B_1$ (or $1 \otimes \lambda_n \in B \subseteq A_1\overline{\otimes}B_1$) where λ_n is the left regular representation of $n \in \mathbb{Z}$. Then, on elementary tensors $x = a_1 \otimes b_1$ and $y = a_2 \otimes b_2$, we have

$$\|E_A(xu_ny)\|_2 = \|E_A(a_1a_2 \otimes b_1\lambda_nb_2)\|_2 = \|a_1a_2\tau(b_1\lambda_nb_2)\|_2 = |\tau(b_2b_1\lambda_n)|\|a_1a_2\|_2$$

Write $b := b_2b_1$, and let (b_F) be a uniformly bounded net converging WOT to b, where F is indexed over all finite subsets of $\mathbb{Z}$, and with $b_F := \sum_{m\in F} c_m\lambda_m$. Fix $F \subseteq \mathbb{Z}$ finite. We have that

$$b_F\lambda_n = \sum_{m\in F} c_m\lambda_{m+n}$$

Then, for all sufficiently large n, we have $m + n \neq 0$, and hence $\tau(\lambda_{m+n}) = 0 = \tau(b_F\lambda_n)$. Since for any finite subset F, we can find such an n_0 for which all $n \geq n_0$ satisfy $\tau(b_F\lambda_n) = 0$, and hence $|\tau(b_1b_2\lambda_n)| \to 0$ as $n \to \infty$. By taking WOT-limits of linear combinations of elementary tensors, we get that for any $x, y \in A_1\overline{\otimes}B_1$ that $\|E_A(xu_ny)\|_2 \to 0$. Hence, $B_1 \not\prec_{A_1\overline{\otimes}B_1} A_1$. By theorem (4.3.4) with $q = 1$, we get that $B_1' \cap N \subseteq A_1\overline{\otimes}B_1$. Hence, $B_1' \cap M = B_1' \cap (N \cap M) \subseteq A_1\overline{\otimes}B_1 \cap M = A_1$. Since $[A_1, B_1] = \{0\}$, the reverse inclusion holds, and we get $B_1' \cap M = A_1$. By a similar argument, we get $B_2 \not\prec_{(A_2\vee B_1)\overline{\otimes}B_2} A_2 \vee B_1$, and hence again by theorem (4.3.4) that $B_2' \cap P \subseteq (A_2 \vee B_1)\overline{\otimes}B_2$. In particular, this means $B_2' \cap N = A_2 \vee B_1$, and thus $B_2' \cap M = (A_2 \vee B_1) \cap M$. Using this and the above, we have $Z(P) = P' \cap M \subseteq (B_1' \cap M) \cap (B_2' \cap M) = A_1 \cap (A_2 \vee B_1)$. It therefore suffices to show that A_1 is 2-freely independent from $A_2 \vee B_1$, since that will imply $A_1 \cap (A_2 \vee B_1) = \mathbb{C}1$.

Let $x \in A_1$ and $y_1 \cdots y_n \in A_2 \vee B_1$ be a reducd word with either $y_j \in A_1 \ominus \mathbb{C}1$ or $y_j \in B_1 \ominus \mathbb{C}1$ with y_j and y_{j+1} from different algebras for all $j = 1, \ldots, n$. Since A_1 and A_2 and B_1 are both 2-freely independent from A_1, we have $E_{A_1}(y_j) = 0$ for all j. Hence, for each j, y_j is in either $M \ominus A_1$ or $(A_1\overline{\otimes}B_1) \ominus A_1$. Hence, $E_{A_1}(xy_1) = xE_{A_1}(y_1) = 0$. In particular, $\tau(xy_1) = 0$. Since B_1 and A_1 are 2-freely independent, we have

$$\tau(xy_1 \cdots y_n) = \tau((xy_1)y_2 \cdots y_n) = 0$$

Since the span of all such reduced words $y_1 \cdots y_n$ are dense in $(A_2 \vee B_1) \ominus \mathbb{C}1$, we have that A_1 is 2-freely independent from $A_2 \vee B_1$ as desired. Thus, $A_1 \cap (A_2 \vee B_1) = \mathbb{C}1$ and hence $Z(P) \subseteq \mathbb{C}1$ so P is a II_1 factor. $\square$

Definition 4.3.8. Let M be a separable II_1 factor. Then, the unitary group $\mathcal{U}(M)$ is separable in the 2-norm. Let $\mathcal{V}(M) \subseteq \mathcal{U}(M) \times \mathcal{U}(M)$ be the set of all tuples (u_1, u_2) such that $u_1^2 = u_2^3 = 1$ and such that $W^*(u_1)$ and $W^*(u_2)$ are 2-freely independent. We equip $\mathcal{V}(M)$ with the product topology of $\mathcal{U}(M)$ with the 2-norm. Then, $\mathcal{V}(M)$ is also separable.

We now construct a II_1 factor as an inductive limit of II_1 factors.

Construction 4.3.9. We construct by induction a sequence of separable II_1 factors $(M_n)_{n\in\mathbb{N}}$. Firstly, let $(\sigma_1, \sigma_2) : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ be a bijection with $\sigma_1(n) \leq n$ for all $n \in \mathbb{N}$. Firstly, pick M_1 an arbitrary separable II_1 factor. Assume we've defined $M_1 \subseteq M_2 \subseteq \ldots \subseteq M_n$ for some $n \in \mathbb{N}$. Note that $M_i \subseteq M_{i+1}$ implies $\mathcal{V}(M_i) \subseteq \mathcal{V}(M_{i+1})$. Then, since M_n is separable, $\exists S := \{(u_1^{(n,k)}, u_2^{(n,k)}) : k \in \mathbb{N}\}$ a countable dense subset of $\mathcal{V}(M_n)$. Since $\sigma_1(n) \leq n$, we have that $(u_1^{(\sigma_1(n),\sigma_2(n))}, u_2^{(\sigma_1(n),\sigma_2(n))}) \in S \subseteq \mathcal{V}(M_n)$, and hence we can define

$$M_{n+1} := \Phi(M_n, u_1^{(\sigma_1(n),\sigma_2(n))}, u_2^{(\sigma_1(n),\sigma_2(n))})$$

Then, $M_n \subseteq M_{n+1}$. Since M_n is a II_1 factor, it has no type I direct summand. For any unitary $u \in M$, $W^*(u)$ is abelian and hence of type I, it follows that $M_n \not\prec_{M_n} W^*(u)$, and hence by theorem (4.3.7) we have that M_{n+1} is a II_1 factor. Since M_n and $L(\mathbb{Z})$ are both separable, it follows that M_{n+1} is also separable. We can then define

$$M := \left(\bigcup_{n\geq 1} M_n\right)''$$

and M will be a II_1 factor. Since $M_n \subseteq M_{n+1}$ is trace-preserving, we have $L^2(M_n) \subseteq L^2(M_{n+1})$, and so this bicommutant can be taken in the inductive limit of the Hilbert spaces $\{L^2(M_n)\}_{n\in\mathbb{N}}$. Since M is dependent on the choice of M_1 and (σ_1, σ_2), we denote this construction of M by $\Xi(M_1, \sigma_1, \sigma_2)$, or simply by $\Xi(M_1)$ if the bijection is clear.

Proposition 4.3.10. Take $M = \Xi(M_1)$ as above, for some separable II_1 factor M_1. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$, and let $u^{(1)}, u^{(2)} \in M^{\mathcal{U}}$ with $(u^{(1)})^2 = (u^{(2)})^3 = 1$ and with $W^*(u^{(1)})$ 2-free from $W^*(u^{(2)})$. Assume further that $u^{(1)} \neq \pm 1$. Then, $\exists v^{(1)}, v^{(2)} \in M^{\mathcal{U}}$ Haar unitaries for which $[u^{(i)}, v^{(i)}] = [v^{(1)}, v^{(2)}] = 0$ for $i = 1, 2$.

Proof. Let $p, q^{(1)}, q^{(2)}, q^{(3)} \in M^{\mathcal{U}}$ be projections with $u^{(1)} = 2p - 1$ and $u_2 = q^{(1)} + \zeta q^{(2)} + \zeta^2 q^{(3)}$, where $\zeta = e^{2\pi i/3}$. The condition $u^{(1)} \neq \pm 1$ then implies $p \neq 0, 1$. By [BCI17, lemma 2.2], we have for some subsequence $(k_n)_{n\in\mathbb{N}}$ that $p, q^{(1)}, q^{(2)}, q^{(3)} \in \prod_{n\to\mathcal{U}} M_{n_k}$, where the M_n are as constructed above. Then, since $W^*(u^{(1)})$ and $W^*(u^{(2)})$ are 2-freely independent and equal to $W^*(p)$ and $W^*(q^{(1)}, q^{(2)}, q^{(3)})$ respectively, we can use theorem (4.2.2) (the lifting lemma) in order to acquire sequence representatives $p = (p_n)_{\mathcal{U}}$ and $q^{(i)} = (q_n^{(i)})_{\mathcal{U}}$ for $i = 1, 2, 3$, for which $p_n, q_n^{(i)} \in M_{k_n}$, $q_n^{(1)} + q_n^{(2)} + q_n^{(3)} = 1$, and for which $W^*(p_n)$ is 2-freely independent from $W^*(q_n^{(1)}, q_n^{(2)}, q_n^{(3)})$ for each $n \in \mathbb{N}$. We can then let $u_n^{(1)} := 2p_n - 1$ and $u_n^{(2)} = q_n^{(1)} + \zeta q_n^{(2)} + \zeta^2 q_n^{(3)}$. Then, $u^{(i)} = (u_n^{(i)})$ are sequence representatives for $i = 1, 2$, and

$W^*(u_n^{(1)})$ is 2-freely independent from $W^*(u_n^{(2)})$ for each n. Hence, $(u_n^{(1)}, u_n^{(2)}) \in \mathcal{V}(M_{k_n})$ for each n. Let S_{k_n} be the countable dense subset of $\mathcal{V}(M_{k_n})$ from the construction (4.3.9). Then, $\exists j_n \in \mathbb{N}$ for which

$$\|u_n^{(1)} - u_1^{(k_n,j_n)}\|_2 + \|u_n^{(2)} - u_2^{(k_n,j_n)}\|_2 \leq \frac{1}{n} \tag{1}$$

for every $n \in \mathbb{N}$. Since $(\sigma_1, \sigma_2) : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a bijection, we can find some $s_n \in \mathbb{N}$ for which $k_n = \sigma_1(s_n)$ and $j_n = \sigma_2(s_n)$. By construction, $M_{s_n+1} = \Phi(M_{s_n}, u_1^{(k_n,j_n)}, u_2^{(k_n,j_n)})$. Thus, again by theorem (4.3.7) we can find Haar unitaries $v_n^{(1)}, v_n^{(2)} \in M_{s_n+1} \subseteq M$ for which

$$[u_1^{(k_n,j_n)}, v_n^{(1)}] = [u_2^{(k_n,j_n)}, v_n^{(2)}] = [v_n^{(1)}, v_n^{(2)}] = 0, \quad \forall n \in \mathbb{N} \tag{2}$$

Let $v^{(1)} := (v_n^{(1)})_{\mathcal{U}}$ and $v^{(2)} := (v_n^{(2)})_{\mathcal{U}}$. By (1) and (2), we have that $[u^{(i)}, v^{(i)}] = [v^{(1)}, v^{(2)}] = 0$ for $i = 1, 2$. □

Definition 4.3.11 ([Pop12] Weak spectral gap)**.** Let M be a II_1 factor and $Q \subseteq M$ a von Neumann subalgebra. We say that Q has *weak spectral gap* in M when $\forall \varepsilon > 0, \exists u_1, \ldots, u_n \in Q$ unitaries, and $\exists \delta > 0$ for which whenever $x \in (M)_1$ satisfies $\|[u_i, x]\|_2 \leq \delta$ for $i = 1, \ldots, n$, we have $\|x - E_{Q' \cap M}(x)\|_2 \leq \varepsilon$.

This can be rephrased in terms of ultrapowers. Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$. Then $Q' \cap M^{\mathcal{U}} = (Q' \cap M)^{\mathcal{U}}$ is equivalent to Q having weak spectral gap in M. To see why this definition implies the ultrapower condition, note that $(Q' \cap M)^{\mathcal{U}} \subseteq Q' \cap M^{\mathcal{U}}$, so it suffices to show the reverse direction. Let $x = (x_n)_{\mathcal{U}} \in Q' \cap M^{\mathcal{U}}$ with $\|x\| \leq 1$, and $\varepsilon > 0$. For any unitaries $u_1, \ldots, u_m \in Q$, we have $\|[x_n, u_i]\|_2 = 0$ for any n and $i = 1, \ldots, m$, and hence by the weak spectral gap condition, we have $\|x_n - E_{Q' \cap M}(x_n)\|_2 \leq \varepsilon$. In particular, for any fixed $n \in \mathbb{N}$, we can take $\varepsilon \to 0$ and get that $\|x_n - E_{Q' \cap M}(x_n)\|_2 = 0$, so $x = (x_n)_{\mathcal{U}} \in (Q' \cap M)^{\mathcal{U}}$.

As an example of when weak spectral gap can occur, if Q is a property (T) II_1 factor, then for any inclusions of Q as a subfactor of some II_1 factor M, Q will have weak spectral gap in M.

Proposition 4.3.12. Suppose M_1 has weak spectral gap in $M := \Xi(M_1)$. Then, M does not have property Γ.

Proof. By definition, we have $M_{n+1} := \Phi(M_n, u_1^{(\sigma_1(n),\sigma_2(n))}, u_2^{(\sigma_1(n),\sigma_2(n))})$. Since M_1 is a II_1 factor and $W^*(u)$ is of type I for any unitary u, we have that $M_1 \not\prec_{M_n} W^*(u)$ for the unitaries. By theorem (4.3.7) (ii), we get that $M_1' \cap M_{n+1} = M_1' \cap M_n$, and hence for any $n \in \mathbb{N}$, we have $M_1' \cap M_n = Z(M_1) = \mathbb{C}1$. Since this holds for all n, we get $M_1' \cap M = \mathbb{C}1$, and since M_1 has weak spectral gap in M, we have $M_1' \cap M^{\mathcal{U}} = (M_1' \cap M)^{\mathcal{U}} = \mathbb{C}1$. Hence, $M' \cap M^{\mathcal{U}} = \mathbb{C}1$ and thus, M does not have property Γ. □

Theorem 4.3.13. There exists a separable II_1 factor M without property Γ such that for any free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and $u_1, u_2 \in M^{\mathcal{U}}$ unitaries with $u_1^2 = u_2^3 = 1$ and $W^*(u_1)$ 2-freely independent from $W^*(u_2)$, $\exists v_1, v_2 \in M^{\mathcal{U}}$ Haar unitaries such that $[u_1, v_1] = [u_2, v_2] = [v_1, v_2] = 0$.

Proof. Let M_1 be a separable II_1 factor with M_1 having weak spectral gap in $M := \Xi(M_1)$ (for example, take $M_1 = L(\mathrm{SL}_3(\mathbb{Z}))$, which is a separable property (T) II_1 factor). Let $\mathcal{U}$ be a free ultrafilter on $\mathbb{N}$ and $u_1, u_2 \in M^{\mathcal{U}}$ the described unitaries. Then, proposition (4.3.10) gives the Haar unitaries v_1, v_2 satisfying $[v_1, u_1] = [v_2, u_2] = [v_1, v_2] = 0$, and proposition (4.3.12) implies that M does not have property Γ. □

The remaining goal of this section is to show that for such an M as constructed in theorem (4.3.13), we have that $h(M^{\mathcal{U}}) \leq 0$, where $\mathcal{U}$ is a free ultrafilter on $\mathbb{N}$. Towards this, we start by looking at the relative 1-bounded entropy of a von Neumann subalgebra of $M^{\mathcal{U}}$.

Corollary 4.3.14. Let M_1 and M be from theorem (4.3.13), $\mathcal{U}$ a free ultrafilter on $\mathbb{N}$, and $p, q_1, q_2, q_3 \in M^{\mathcal{U}}$ projections with $q_1 + q_2 + q_3 = 1$ and $W^*(p)$ 2-freely independent from $W^*(q_1, q_2, q_3)$. Let $N := W^*(p, q_1, q_2, q_3) \subseteq M^{\mathcal{U}}$. Then, $h(N : M^{\mathcal{U}}) \leq 0$.

Proof. Let $\zeta = e^{2\pi i/3}$, $u_1 := 2p - 1$ and $u_2 := q_1 + \zeta q_2 + \zeta^2 q_3$. Then, $u_1^2 = u_2^3 = 1$ and $W^*(u_1)$ is 2-freely independent from $W^*(u_2)$. By proposition (4.3.10), we have Haar unitaries $v_1, v_2 \in M^{\mathcal{U}}$ such that $[u_1, v_1] = [u_2, v_2] = [v_1, v_2] = 0$. Let $C_1 := W^*(u_1, v_1)$, $C_2 := W^*(u_2, v_2)$, and $C_3 := W^*(v_1, v_2)$. Then, C_1, C_2, and C_3 are all abelian, and hence $h(C_1), h(C_2), h(C_3) \leq 0$, since each is diffuse and hyperfinite. Since 1-bounded entropy is subadditive on algebras with diffuse intersection, we have

$$h(W^*(u_1, u_2, v_1, v_2)) = h(C_1 \vee C_2 \vee C_3) \leq 0$$

Let $C := W^*(u_1, u_2)$. Then, using the last point in remark (1.4.13) we have

$$h(C : M^{\mathcal{U}}) \leq h(C : W^*(u_1, u_2, v_1, v_2)) \leq h(W^*(u_1, u_2, v_1, v_2)) \leq 0$$

Finally, we have

$$h(N : M^{\mathcal{U}}) = h(W^*(u_1, u_2) : M^{\mathcal{U}}) \leq 0$$

as desired. □

Theorem 4.3.15. Let M_1 and $M = \Xi(M_1)$ be as in theorem (4.3.13). Then, $h(M^{\mathcal{U}}) \leq 0$ for any ultrafilter $\mathcal{U}$ on $\mathbb{N}$.

Proof. If $\mathcal{U}$ is not free, then $M^{\mathcal{U}} \cong M$, and if $\mathcal{V}$ is a free ultrafilter on $\mathbb{N}$, then

$$h(M^{\mathcal{U}}) = h(M) = h(M : M^{\mathcal{V}}) \leq h(M^{\mathcal{V}})$$

Thus, we may assume $\mathcal{U}$ is free.

Let $P := M^{\mathcal{U}}$ and let $G := \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. Note that G is amenable, since $\mathbb{Z}$ is a finite-index cyclic subgroup of G. Hence, the group von Neumann algebra $S := L(G)$ is hyperfinite. Since P is a II_1 factor, we can find a trace-preserving embedding of S into P. Let $p, q \in S$ be two projections of trace $\frac{1}{2}$ generating the two copies of $L(\mathbb{Z}/2\mathbb{Z})$ inside S.

Write $q^\circ := q - \frac{1}{2}$. Then, $\tau(q^\circ) = 0$, and we have

$$\begin{aligned}\tau(q(2p-1)q(2p-1)) &= \tau\left(\left[q^\circ(2p-1) + \frac{1}{2}(2p-1)\right]\left[q^\circ(2p-1) + \frac{1}{2}(2p-1)\right]\right) \\ &= \begin{cases} \tau(q^\circ(2p-1)q^\circ(2p-1)) + 2\left(\frac{1}{2}\tau((2p-1)q^\circ(2p-1))\right) \\ \qquad\qquad + \frac{1}{4}\tau((2p-1)(2p-1)) \end{cases}\end{aligned}$$

Where the second $\tau((2p-1)q^\circ(2p-1))$ term comes from using traciality with the term $\tau(q^\circ(2p-1)(2p-1))$. Since the two copies of $L(\mathbb{Z}/2\mathbb{Z})$ are freely independent, the only nonzero term of this sum is the final one, for which $\tau((2p-1)^2) = 1$. In particular, this means $\|q(2p-1)q\|_2 = \frac{1}{2}$. Similarly, $\|q^\perp(2p-1)q^\perp\|_2 = \frac{1}{2}$. Let $x = q^\perp(2p-1)q^\perp \in q^\perp P q^\perp$. Then, x is nonzero, selfadjoint, and traceless. If F is the set of all projections $r \in q^\perp P q^\perp$ for which $\tau(xr) = 0$.

For any such $r \in F$, we can define $S_r := W^*(p, q, r, 1-q-r)$. We then have

$$\begin{cases} \tau((2p-1)q) = 0 \\ \tau((2p-1)r) = 0 \\ \tau((2p-1)(1-q-r)) = 0 \end{cases}$$

Hence, $W^*(p)$ and $W^*(q, r, 1-q-r)$ are 2-freely independent. By the above corollary, we have $h(S_r : P) \leq 0$ for every $r \in F$. Since S is generated by p and q, it follows that for any $r \in F$, we have $S \subseteq S_r$. Further, S is diffuse, and hence we have

$$h\left(\bigvee_{r \in F} S_r : P\right) \leq 0$$

By lemma (4.3.1) we have that $F'' = q^\perp P q^\perp$ and hence

$$\bigvee_{r \in F} S_r = S \vee q^\perp P q^\perp$$

Hence, $h(S \vee q^\perp P q^\perp : P) \leq 0$, and similarly that $h(S \vee qPq : P) \leq 0$. Since S is diffuse, we get $h(S \vee qPq \vee q^\perp P q^\perp : P) \leq 0$, and since P is a II_1 factor with $q, q^\perp$ Murray von Neumann equivalent projections, we have $S \vee qPq \vee q^\perp P q^\perp = P$, so that $h(P) = h(P : P) \leq 0$. □

We can now prove the main result.

Theorem 4.3.16. There exists a separable II_1 factor without property Γ, which is not elementarily equivalent to any tracial von Neumann algebra of positive 1-bounded entropy.

Proof. Let M_1 be a separable II_1 factor with weak spectral gap in $M := \Xi(M_1)$. By proposition (4.3.12), M does not have property Γ. By theorem (4.3.15), we have $h(M^{\mathcal{U}}) \leq 0$ for any ultrafilter on $\mathbb{N}$.

Now, suppose N is a tracial von Neumann algebra with positive 1-bounded entropy, and let $\mathcal{V}$ be another ultrafilter on $\mathbb{N}$. We have that $h(N : N) = h(N : N^{\mathcal{V}}) \leq h(N^{\mathcal{V}} :$

$N^{\mathcal{V}}) = h(N^{\mathcal{V}})$ by combining two points in remark (1.4.13). If M and N had isomorphic ultrapowers over $\mathbb{N}$, then we would have $0 \geq h(M^{\mathcal{U}}) = h(N^{\mathcal{V}}) > 0$, a contradiction. $\square$

Note that much of the work with free ultrafilters on $\mathbb{N}$ can be directly copied when considering countably cofinal ultrafilters on an arbitrary index set. There are only a few instances where this is not the case, and in both such cases this can be resolved through [BCI17].

Finally, we remark that since the free group factors have positive 1-bounded entropy, we have $M \not\equiv L(\mathbb{F}_n)$ for any $n \geq 2$.

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