Algorithms for Analytic Combinatorics: Positivity Bounds and D-finite Operators

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

This work is based on the following papers that I co-authored during my time at the University of Waterloo.

- SagePeriods: A SageMath Package for Fast Diagonals of Rational Functions [94]
- Multivariate Asymptotics of Rational Functions with Computable Error Terms, joint work with Stephen Melczer (in progress)

Abstract

Analytic combinatorics is concerned with describing limiting behavior of families of combinatorial structures. While this is well-studied in the univariate case, the last two decades have seen the development of *analytic combinatorics in several variables* (ACSV) treating the same problem in the multivariate case. One advantage of the way ACSV is formulated is that, at least in the simplest cases, its methods are amenable to explicit computation. This thesis contributes to an ongoing effort to automate the results of ACSV by providing developments in two related areas: computing D-finite operators for diagonals of rational functions, and computing explicit error bounds for ACSV in the so-called smooth rational case.

First, we provide a SageMath implementation of an algorithm of Lairez for computing periods of rational integrals. Since diagonals of rational functions are rational periods, computing operators of periods is of great importance to practitioners of algebraic and analytic combinatorics. While Lairez gave a MAGMA implementation of his algorithm, our implementation provides full-fledged documentation, robustness, and feature enhancements aimed at combinatorialists – such as computing diagonal operators for arbitrary directions.

Second, we discuss how to find explicit error bounds for asymptotics of rational diagonals, as opposed to the Big-O asymptotics typically provided by ACSV. One motivation for this is the *coefficient positivity problem*; having explicit bounds allows one to reduce positivity of coefficient sequences to checking asymptotic positivity and finitely many initial sequence terms. We provide fully constructive versions of ACSV arguments in the simplest case, then use these to derive an index N so that positivity of our asymptotic implies positivity of our diagonal for all $n \geq N$, under some small additional assumptions about the form of our asymptotic. We then explore the consequences and caveats of this reduction, exhibiting some classes of functions where asymptotic positivity can be known a priori.

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Dedication

To Mom, Dad, Peter, and Hannah. Thank you for all your love and encouragement.

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Chapter 1

Introduction

Combinatorics, or *the art of counting*, is concerned with finding ways to determine the size of a given finite set of objects, where this size is dependent on some discrete parameter n. Oftentimes these objects are of mathematical or scientific interest – they could be tilings of the plane, maps, or partitions, or they could be phylogenetic trees, DNA/RNA sequences, or configurations of particles in a lattice. Each type of object has underlying rules determining how to form valid instances of said object, often from simpler instances of the same type through a recursive process. While these rules sometimes yield simple formulas describing the growth of the number of objects as a function of n, this is usually not the case; most interesting objects (i.e., ones that mathematicians or scientists study in practice) don't allow for nice counting formulas.

Thus a reasonable question to ask is, can one determine formulas that hold *asymptotically*, in other words which approximately describe the number of the objects in the limit as their size $n \to \infty$, with the approximation becoming more accurate as n grows larger. This is the defining question of *analytic combinatorics* (AC), the field concerned with using complex analysis to derive asymptotic formulas for combinatorial objects.

Perhaps the best way to quickly illustrate its power is by an example. We use basic results and terminology from complex analysis in one variable, which can be found in standard references such as [2, 37, 29].

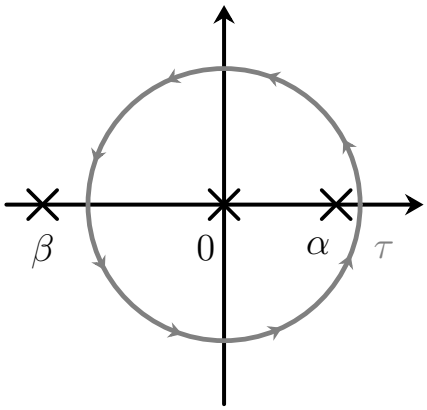


Figure 1.1: Contour of radius $\alpha < \tau < |\beta|$

Example 1 (Binary words with no two consecutive zeros[1])**.** *Let a_n count binary words of length $n \geq 0$ with no occurrence of* 00*. Such a word is a sequence of blocks*

$$1, \quad 01$$

possibly followed by a final 0*, and using the Symbolic Method [40] its generating function can be computed as*

$$A(z) = \sum_{n \geq 0} a_n z^n = \frac{1+z}{1-z-z^2}.$$

The denominator of A has roots

$$\alpha = \frac{\sqrt{5}-1}{2}, \quad \beta = -\frac{\sqrt{5}+1}{2}.$$

Note that the positive root $\alpha \approx 0.618$ is the unique singularity of smallest modulus. Now, by the Cauchy integral and residue theorems from complex variables, for any positive $\varepsilon < \alpha$ we have

$$a_n = \frac{1}{2\pi i} \int_{|z|=\varepsilon} \frac{A(z)}{z^{n+1}} \mathrm{d}z = \mathrm{Res}_{z=0} \left(\frac{A(z)}{z^{n+1}} \right). \tag{1.1}$$

[1]Those familiar with the subject might recognize this class's counting sequence as satisfying a very famous recurrence for which an almost equally famous closed-form expression is immediately available. Despite having such a formula (and hence no need for asymptotics), this example still illustrates the approach towards analytic derivations in a simple case that will generalize far, including to several variables.

We can deform the contour of our integral and as long as this deformation doesn't cross any singularities the value of the integral will remain unchanged. But crossing the first singularity causes something interesting to happen. Let $\alpha < \tau < |\beta|$ *and define the integral*

$$I_n = \frac{1}{2\pi i} \int_{|z|=\tau} \frac{A(z)}{z^{n+1}} \mathrm{d}z.$$

As can be seen from Figure 1.1, our contour now contains two singularities, those at 0 *and* α*. Since* A *is analytic on* $|z| = \tau$*, bounding the integral by the length of the contour* $\{|z| = \tau\}$ *times the maximum modulus of the integrand implies*

$$I_n = O\left(\tau^{-n}\right), \tag{1.2}$$

so the univariate Residue Theorem gives

$$I_n = \mathrm{Res}_{z=0}\left(\frac{A(z)}{z^{n+1}}\right) + \mathrm{Res}_{z=\alpha}\left(\frac{A(z)}{z^{n+1}}\right).$$

Substituting in (1.1) *and* (1.2)*, and rearranging,*

$$a_n = -\mathrm{Res}_{z=\alpha}\left(\frac{A(z)}{z^{n+1}}\right) + O\left(\tau^{-n}\right).$$

As $z = \alpha$ *is a simple pole,*

$$\mathrm{Res}_{z=\alpha}\left(\frac{A(z)}{z^{n+1}}\right) = \mathrm{Res}_{z=\alpha}\left(\frac{1+z}{(1-z-z^2)\, z^{n+1}}\right) = \frac{1+\alpha}{-(1+2\alpha)\alpha^{n+1}},$$

and so

$$a_n = \frac{1+\alpha}{(1+2\alpha)\alpha^{n+1}} + O(\tau^{-n})$$

or, in numeric terms,

$$a_n = (0.72360679775...) \cdot (1.6180339887...)^{n+1} + O(\tau^{-n}).$$

This is an asymptotic expression for a_n*.*

The procedure used in Example 1 is illustrative of the general procedure used for asymptotic derivations, as well as some general principles for AC. The idea is to take a sequence f_n and encode it as the coefficients of the *generating function* (GF)

$$F(z) = \sum_{n \geq 0} f_n z^n.$$

In favorable cases, this formal power series will *also* represent an analytic function, and in these cases there is an observed correspondence between the analytic properties of F and the asymptotics of its power series coefficients (f_n). We quote the following two principles from [40], oft regarded as the analytic combinatorialist's bible:

> **First Principle of Coefficient Asymptotics:** The *location* of a [generating] function's singularities dictates the exponential growth of its power series coefficients.

and

> **Second Principle of Coefficient Asymptotics:** The *nature* of a function's singularities determines the associated sub-exponential growth [of its asymptotic].

The Vivanti-Pringsheim Theorem [90, Proposition 2.4] says that power series whose coefficients are nonnegative have a singularity at $z = R > 0$, where R is their radius of convergence (assuming R is finite). This means that with an expression for our GF $F(z)$ in-hand, we can usually follow a general "4-Step Process" to determine asymptotics:

1. Find the smallest $r > 0$ such that $z = r$ is a singularity of $F(z)$. Such a singularity is called *dominant*, and dictates exponential growth by Principle 1;
2. Find all other dominant singularities, i.e. those on the circle $|z| = r$;
3. If F admits a finite number of dominant singularities, try to determine the asymptotic contribution of each by examining the local behavior of F near these singularities;
4. Add the results to get dominant asymptotics of f_n.

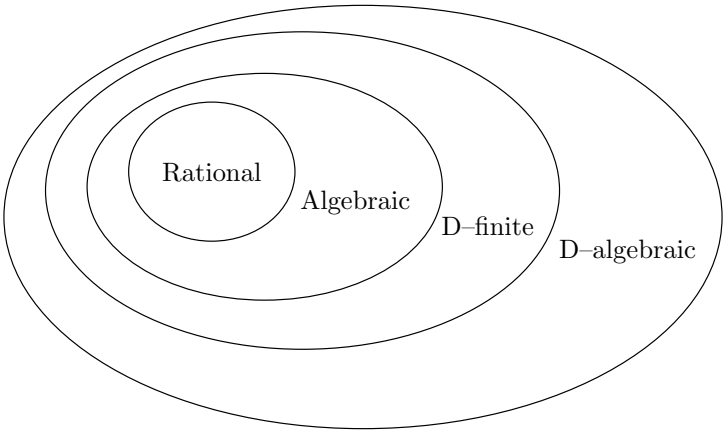


Figure 1.2: GF class hierarchy

Much of the development of analytic combinatorics has gone into determining the conditions under which the above process applies, as well as describing the asymptotic contributions resulting from various types of singularities. Most types of singularities appearing in practice have associated *transfer theorems* [39, 101] characterizing what types of asymptotic contribution they produce. Thus following the 4-Step Process reduces to performing Steps 1,2 and 4 (which are mostly effective through methods of computer algebra) and Step 3, which is essentially a table lookup. Recently, Pivoteau and Salvy [111] have automated the process of deriving asymptotics in the univariate case, starting from a combinatorial description of the family derived using the Symbolic Method.

Generally, it is known that the type of singularity giving rise to an asymptotic term is well-aligned with the *representation* for the GF it is associated with. Not all generating functions are rational; in fact, an overwhelming majority of GFs, especially those from applications, come from combinatorial classes whose underlying structure is simply too complicated to be representable by a rational function. Such functions can be algebraic, D-finite, D-algebraic, or something else entirely.[2] Figure 1.2 shows a hierarchy of containment for some important GF classes.

Finding where a specific function lies in this hierarchy is of great importance, because

[2] Algebraic functions F are those for which there exists a d and polynomials $p_0(z)$, $p_1(z), \ldots, p_d(z) \in \mathbb{C}[z]$ with $p_d(z) \neq 0$ so that

$$p_0(z) + p_1(z)F(z) + p_2(z)F(z)^2 + \cdots + p_d(z)F(z)^d = 0,$$

and we discuss D-finite and D-algebraic functions in the next section.

it can tell us *universal* structural properties that the function and its coefficient sequence must have. For instance, rational GFs can have only a finite number of singularities, and all of them are *poles* yielding asymptotic contributions of the form

$$f_n \sim C\rho^{-n}n^{k-1} \tag{1.3}$$

where ρ is a dominant singularity and k is the order of the pole of F at ρ. Algebraic GFs can have other singularities like *branch points* which are locations where a choice must be made to make functions like square-roots proper $\mathbb{C}$-valued functions. Such algebraic branch points yield asymptotic contributions of

$$f_n \sim C\rho^{-n}n^{-\alpha-1}, \tag{1.4}$$

where now ρ is a branch point, and α is a rational number not equal to a nonnegative integer.

Conversely, sometimes properties we derive from a function can tell us things about its place in the hierarchy. For example, it was shown by André [119, Theorem 1.1] that if a_n counts the number of odd-length alternating permutations of $\{1, ..., n\}$ then its *exponential generating function* is

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n = \tan(z).$$

It is known that D-finite functions admit only finitely many singularities. Thus, since $\tan(z)$ has singularities at all odd-multiples of $\pi/2$, we can rule out it being D-finite.

It is also true that the class in the hierarchy to which a function belongs has bearing on whether we can hope to derive asymptotics for it *automatically*. It has been shown (see [108, Section 3.1] and [90, Section 3.5], for instance) that while the problem of determining asymptotics from a rational or algebraic GF is decidable, that same problem is undecidable for the class of D-algebraic functions [32, Theorem 4.2]. For rational and algebraic GFs, work has been done in making the process of deriving asymptotics effective and providing implementations [38, 63, 116].

1.1 D-finite Functions & Connection Coefficient Problem

Besides rational functions, the most important univariate GF class in this thesis is the class of D-finite functions. As can be seen from Figure 1.2, D-finite functions are the next step up in the hierarchy of GFs from algebraic functions.

Definition 1 (D-finite function)**.** *A function is called D-finite if it satisfies a linear (ordinary) differential equation with polynomial coefficients. In other words, there exist a positive integer d and polynomials $p_0(z), p_1(z), \ldots, p_d(z) \in \mathbb{C}[z]$ with $p_d(z) \neq 0$ such that*

$$p_d(z)F^{(d)}(z) + \cdots + p_2(z)F''(z) + p_1(z)F'(z) + p_0(z)F(z) = 0.$$

Alternatively, D-finite functions are those whose derivatives span a vector space of finite dimension over $\mathbb{C}(z)$. Note that by repeated differentiation, this definition is equivalent to one allowing a polynomial on the right-hand side.

Many important functions are D-finite. Some examples include exponentials, logarithms, $\sin(z)$ and $\cos(z)$, and special functions such as the Airy, Bessel, and Gaussian error functions. The containment of the class of algebraic functions in the class of D-finite functions, like all containments appearing in Figure 1.2, is proper. For example, $F(z) = e^z$ is D-finite but not algebraic. We can see this by observing that it satisfies $F'(z) - F(z) = 0$, but its asymptotic behavior $f_n \sim \frac{1}{n!}$ is not of the form admissible by an algebraic function, comparing with Equation (1.4).

D-finite functions are of interest because they are on the edge of what is known regarding deciding asymptotics; asymptotic decidability for D-finite functions is still open. Nevertheless, they have been shown to have many aspects amenable to computation. These include closure properties, a rigorous zero test, recurrence guessing from a finite list of initial terms, and fast numerical algorithms to evaluate and analytically continue D-finite functions [65, 115]. They are also an ideal setting for many of the algorithms of *creative telescoping*, first pioneered by Zeilberger in [131] and now a flourishing area of computer algebra.[3]

A D-finite function F can be represented by a finite set of data consisting of an annihilating operator $\mathcal{L}$ and $\text{ord}(\mathcal{L})$ initial values. It is sometimes possible to get asymptotics for F's coefficients using its D-finite representation. This can be done by using the methods of Fabry [36] or Birkhoff-Trjitzinsky [17, 16] to form a basis of solutions $\Lambda_1(z), ..., \Lambda_\ell(z)$ for the differential equation $\mathcal{L}F = 0$, so that for some unknown set of constants $\lambda_1, ..., \lambda_\ell \in \mathbb{C}$ we have

$$F(z) = \lambda_1\Lambda_1(z) + \cdots + \lambda_\ell\Lambda_\ell(z),$$

and one can compute asymptotic expansions for the coefficient sequences of the Λ_i, thereby giving an asymptotic expansion of F's coefficients. The λ_i are called the *connection coefficients* associated with the basis (Λ_i), and solving for them is called the *Connection*

[3] Part of our work is an implementation of an algorithm for creative telescoping for rational periods, which are D-finite; see below, as well as Chapter 4.

Coefficient Problem (CCP) for D-finite functions. Unfortunately, there is currently no zero test for the connection coefficients λ_i; while we can compute tens of thousands of digits using software packages like [96], there is no known way to conclude that a coefficient is exactly zero. Thus we can't, in general, solve a CCP exactly. As a consequence, this means decidability of even the dominant growth rate for D-finite asymptotics is unknown.

1.1.1 Multivariate GFs & Rational Diagonals

One way D-finite functions relate to combinatorics is via the notion of *rational diagonals*, which are of central importance in the study of *analytic combinatorics in several variables* (ACSV). This field seeks to generalize the question posed on the first page (as well as its answer) to the multivariate setting. While we discuss ACSV more in Chapter 2, we now introduce multivariate generating functions and their diagonals.

Let $d \geq 2$ be an integer and $\boldsymbol{z} = (z_1, ..., z_d)$ be complex variables. The *multivariate generating function* for the *multivariate sequence* $(f_{\boldsymbol{j}})_{\boldsymbol{j} \in \mathbb{N}^d}$ is the formal power series:

$$F(\boldsymbol{z}) = \sum_{\boldsymbol{j} \in \mathbb{N}^d} f_{\boldsymbol{j}} \boldsymbol{z}^{\boldsymbol{j}}.$$

Here $\boldsymbol{z}^{\boldsymbol{j}}$ stands for $z_1^{j_1} \cdots z_d^{j_d}$. We can now define the class of generating functions which are diagonals of multivariate rational functions.

Definition 2 ($\boldsymbol{r}$-Diagonal of a Multivariate Rational Function)**.** *Fix a direction vector $\boldsymbol{r} \in \mathbb{N}^d$. The $\boldsymbol{r}$-diagonal of F is the univariate series*

$$\Delta_{\boldsymbol{r}} F(t) = \sum_{n=0}^{\infty} f_{n\boldsymbol{r}} t^n,$$

and when $\boldsymbol{r} = \mathbf{1}$ we call $\Delta_{\boldsymbol{r}} F(t)$ the main diagonal of F and denote it by $\Delta F(t)$.

In other words, the $\boldsymbol{r}$-diagonal picks out the univariate subsequence of $f_{\boldsymbol{j}}$ with indices multiples of $\boldsymbol{r}$. It turns out that the class of multivariate rational diagonals lies between algebraic functions, whose coefficient asymptotics are decidable, and D-finite functions, where the problem is still open [26] [81, Theorem 1] [31, Theorem 6.2]. Thus they make an excellent subject for the study of effective asymptotics.

Rational diagonals have many algorithmic closure properties, combinatorial interpretations (e.g. in lattice path theory [91, 92], irrational tilings [44] and Kronecker coefficients

[98]) as well as applications in physics [19, 127], probability [107, Sections 4.10, 4.12], and graph theory [26, Sections 5.1, 6.2, 6.3].

Moreover, rational diagonals can be cast as rational *period integrals* depending on a parameter t, i.e. expressions of the form

$$P(t) = \oint_{\gamma} R(\boldsymbol{x}, t)\mathrm{d}\boldsymbol{x},$$

for $R \in \mathbb{C}(\boldsymbol{x}, t)$, and γ a deformation of a small product of circles about the origin.

It is a classical fact that rational period integrals are D-finite. ACSV is focused on deriving asymptotics for the univariate sequence $(f_{n\boldsymbol{r}}) = (f_{nr_1,nr_2,\ldots,n_rd})_n$ whose generating function is $\Delta_{\boldsymbol{r}}F(t)$ – in other words, we derive an asymptotic for $(f_{\boldsymbol{j}})_{\boldsymbol{j}\in\mathbb{N}^d}$ along the *direction* $\boldsymbol{r}$. While this has recently been made effective in many multivariate *rational* cases [49, 48], both the current theory and implementations of ACSV yield asymptotic statements with Big-O bounds. Chapter 3 revolves around replacing these Big-O statements with *computable error bounds*. Among other things, having such bounds allows one to certify *positivity* (see Sections 1.2 and 3.3) of the sequence $(f_{n\boldsymbol{r}})$ past a certain point.

1.1.2 Other GF classes

Beyond D-finite functions in the hierarchy of Figure 1.2 are the D-algebraic functions, which satisfy an equation of the form

$$P\left(z, F(z), F'(z), F''(z), ..., F^{(d)}(z)\right) = 0$$

for some positive integer d and polynomial $P \in \mathbb{C}[z, y_0, ..., y_d]$. A simple example of a function which is D-algebraic and not D-finite is $\tan(z)$, which satisfies the D-algebraic equation $F'(z) = 1 + F(z)^2$.

While there is a modest amount of work regarding computing with D-algebraic functions [124], and certain things can be said about their combinatorial characterizations [12] and singularities and asymptotics [87, 112], much less is known compared to D-finite functions. And almost nothing can be said about *hypertranscendental* functions, which are all the GFs that aren't D-algebraic.

There are also other classes that lie in the hierarchy between rational and D-algebraic. For instance, Jiménez-Pastor, Pillwein, and Singer [3, 60, 4] have recently initiated a program investigating D^2*-finite functions* – D-finite functions whose coefficients are themselves

D-finite. This can be iterated to form D^n or even D^∞-finite functions. The class of D^∞-finite functions lies between D-finite and D-algebraic; it is unclear whether this class is strictly smaller than D-algebraic, but most of the computational properties of D-finite functions are preserved by D^n-finite functions. Similar constructions have been made on the sequence side by the previous authors, along with Nuspl and Kauers [59, 62, 100]. This class of *C^n-finite sequences* is computationally tractable and admits closure properties, order bounds, and asymptotic and generating function characterizations.

1.2 Positivity Problems and the Skolem Problem

Very important to our work is the problem of deciding if a power series has positive coefficients. We first define the *total* positivity problem for multivariate power series, and its variants.

Problem 1 (Coefficient Positivity Problems (CPP))**.** *Let $F(\boldsymbol{z})$ be a power series with expansion*

$$F(\boldsymbol{z}) = \sum_{\boldsymbol{i} \in \mathbb{N}^d} f_{\boldsymbol{i}} \boldsymbol{z}^{\boldsymbol{i}}.$$

We ask the following questions.

- ***Total Positivity Problem (TPP):*** *Are all $f_{\boldsymbol{i}} > 0$?*
- ***Directional Positivity Problem (DPP):*** *Given $\boldsymbol{r} \in \mathbb{N}^d$, are all $f_{n\boldsymbol{r}} > 0$?*
- ***Ultimate Positivity Problem (UPP):*** *In either of the above settings, are our coefficients eventually positive as parameters grow large?*

The total positivity problem was first motivated by H. Lewy and K. Friedrichs in the 1930s, who were studying finite difference approximations of the wave equation. In the course of their work, they needed to prove that the function

$$\frac{1}{(1-x)(1-y)+(1-x)(1-z)+(1-y)(1-z)} = \sum_{k,m,n \geq 0} a(k,m,n) x^k y^m z^n$$

was totally positive, i.e. that $a(k,m,n) > 0$. Not knowing how to proceed, they sent their question to G. Szegő who, using advanced tools from special function theory, was able to solve it [123, Satz I, p. 674] while remarking (paraphrased from German) [121],

> "The used tools, however, are disproportionate to the simplicity of the statement."

Shortly thereafter, T. Kaluza gave an elementary proof of positivity of Lewy and Friedrichs' function [61]. After that, the problem became interesting in its own right, with authors sporadically finding new classes of functions for which positivity could be proven, sometimes with applications. For more history on positivity theory as well as a survey of its results, see [122].

The techniques used in "classical" positivity proofs vary widely: special functions [6, 5], integral methods and transforms [7, 58], determinantal polynomials [117], combinatorial identities [130], positivity-preserving operators [69, 121], and computer algebra [64, 109] have all been employed to one degree or another in such proofs.

The varied nature of these proofs (as well as a diverse author list, low frequency of publications on the subject and high number of applications) suggest that the CPP variants are hard, in general. Some more evidence of this difficulty is CPP's relation to the *Skolem Problem* for rational functions and C-finite sequences, which has been open for almost a century.

1.2.1 The Skolem Problem

The Skolem Problem is defined for univariate rational functions.

Problem 2 (Skolem Problem)**.** *Let (f_n) be a C-finite sequence, i.e. of the form*

$$f_{n+k} = c_{k-1}f_{n+k-1} + \cdots + c_1 f_{n+1} + c_0 f_n$$

with k some fixed integer and rational constants $c_0, c_1, ..., c_{k-1}$. Can one decide if (f_n) has some index n for which $f_n = 0$?

Skolem showed ([118]; see [15, Theorem 1] for a modern statement) that if a sequence (f_n) is C-finite (equivalent to its GF being rational) then the set $\{n : f_n = 0\}$ consists of a union of a finite set and finitely many arithmetic progressions. While we know how to detect when an arithmetic progression is present [13, Proposition 1, Theorem 2], bounds on the index of the finitely many sporadic zeros do not appear in any of the proofs of Skolem's theorem. While for small cases like sequences of orders 3 and 4 these bounds are known [97, 126], even the order 5 problem is still open [71], and the general problem has been open for almost 100 years.

Nevertheless, there has been recent progress. For instance, an algorithm has been given for deciding the Skolem problem for C-finite sequences whose characteristic roots are simple, contingent on Schanuel's conjecture and the Skolem conjecture [15, Theorem 3]. The notion of a Universal Skolem Set – one for which decidability is automatic – has been defined, and nontrivial examples given [86, Theorem 3] [84, Proposition 4 and Theorem 7] [83, Theorems 9 and 13]. Sparsity criteria for finding large sporadic zeros (not belonging to any arithmetic progression) have been found [85] and, most recently, Dong and Shafrir have solved the Skolem problem in rings of positive characteristic [34, Theorem 1.1].

The Skolem Problem can be reduced to a CPP since the square (f_n^2) of a C-finite sequence (f_n) is also C-finite. Even decidability of the *univariate* UPP is open, in general[4], with Ouaknine and Worrell proving decidability for univariate rational functions with squarefree denominators [103, Theorem 4.3] and arbitrary rational functions with denominator degree at most 5 [102, Theorem 4.2]. In this work they also showed [102, Theorem 5.1] that being able to decide UPP for functions with denominator degree at most 6 would allow for the computation of the *Lagrange constant* for a large collection of transcendental numbers, an important breakthrough in analytic number theory.

1.2.2 Related Work: Positivity

While classical methods of proving positivity were ad-hoc, the advent of ACSV has opened up a new avenue for proving *eventual directional positivity* more systematically. This is done by reducing the problem to checking positivity of a directional asymptotic which we derive using ACSV. The paper [9] is the first to explicitly pursue this idea. In it, the authors prove eventual positivity of rational diagonals for some instances of a certain subclass of rational functions.

Our work was inspired by this paper and attempts to push their agenda one step further by extending from *eventual* directional positivity proofs to *total* directional positivity proofs by introducing explicit error bounds. We also attempt to expand the class of functions for which we can show that their asymptotic is positive.

Other approaches to positivity include reducing the positivity problem for P-recursive sequences to minimality, and/or a zero test for periods [70], computing explicit error bounds for P-recursive sequences in the univariate case [33, Theorem 3.1], and inductive procedures involving verifying certain sequences of inequalities using Cylindrical Algebraic Decompo-

[4] In fact, decidability of the eventual *nonnegativity* of a C-finite sequence is open.

sition[5] [68]. More recently, there has been a promising algorithmic method for proving positivity of a large class of P-recursive sequences developed by Ibrahim and Salvy in [55, 56]. It works through an extension of the Perron-Frobenius theory of matrices and will construct an index guaranteeing positivity of the sequence on a cone contained in the positive orthant.

1.3 Multivariate Analytic Combinatorics

While we give a high-level overview of the theory and practice of ACSV in Chapter 2, we first say a bit about the motivation for the multivariate asymptotics problem and early history of the theory. As we already alluded to, rational diagonals are useful because they form a relatively general class of univariate generating series which encode many interesting combinatorial structures. A prime example of this is [92], where asymptotics were provided for a family of lattice walks in the positive orthant with step sets symmetric on every axis by encoding their counting functions as the main diagonal of a multivariate rational function using a generalization of the kernel method [91]. Then the authors use the techniques of smooth ACSV to analyze the singular set of the rational function to produce asymptotics; see Section 2.3 for an example of using their results to derive lattice path asymptotics. Later, Melczer and Wilson [95] extended the analysis to the non-smooth case to derive asymptotics for walks with step sets symmetric in all but one axis.

Other instances of objects encoded as multivariate rational diagonals are:

- **Binomial sums:** In [22, Theorem 3.5], it is shown that the class of *binomial sums* (the smallest $\mathbb{Q}$-algebra containing the binomial coefficients and closed under some basic operations, including indefinite summation) is precisely the main diagonals of multivariate rational functions.
- **Irrational Tilings:** In [44], a combinatorial classification of the class of $\mathbb{N}$-rational functions (a subset of rational functions which seem [23] to contain the generating functions of all combinatorial objects having rational generating function) is given in terms of irrational tilings.
- **Many more:** In [107], the authors give 20 examples of combinatorial sequences arising from the study of trees and networks, biology, quantum computation, orthogonal polynomials, Diophantine equations, and many other things.

[5] The algorithms presented in this paper aren't guaranteed to terminate, but the authors do provide subclasses of P-recursive sequences for which termination is guaranteed.

There are also many situations in which one may want *genuinely* multivariate asymptotics for a sequence – for instance, [113] and [82]. ACSV can sometimes help here using *uniformity over directions*, which says that there exists a neighborhood $\mathcal{R}$ of $\boldsymbol{r}$ in $\mathbb{R}^d_{\geq 0}$ in which we can perturb our direction $\boldsymbol{s}$ such that asymptotics for F along $\boldsymbol{s}$ will vary smoothly from those of $\boldsymbol{r}$ as a function of the perturbation. Using this, it has been shown that *central limit theorems* exist for multivariate combinatorial families – see [90, Section 5.3.3] as well as [46] for a discussion of central limit theorems in ACSV.

Early work on multivariate asymptotics came from a probabilistic analysis of their GF coefficients, begun by Bender and Richmond in the 1970s and 80s [10, 11] and continued by others through the late 1990s [43, 35, 53, 54]. A majority of this work focused on limit theorems for the coefficients, but at around the same time people began looking at computing coefficient asymptotics in specific directions using complex residues for bivariate rational functions [80] and rational functions whose denominators are products of linear factors [79, 14]. While other authors used ideas that would later make their way into ACSV theory (e.g. [73], who used residue computations to reduce the derivation of asymptotics to evaluation of saddle-point integrals), it was Pemantle and Wilson [105, 106] who first attempted to bring together these results to give a systematic treatment of directional asymptotics for multivariate rational functions. A few years later, they were joined by Baryshnikov, Melczer and others who transferred ACSV from its earlier "surgery"-style approach (making simple and explicit deformations of Cauchy integral contours in order to perform the analysis) to its full generality using the results of stratified Morse theory and homology computations. They also focused on making the results of ACSV more explicit, to the point where the simpler cases could be fully automated, resulting in software packages like [49], which allow one to input a function and a direction and receive in return an asymptotic expression for $\Delta_{\boldsymbol{r}} F$.

As of now the theory is still in development, with the newest edition of the definitive reference work on the subject having been released in 2024 [108] and new developments for the theory (and its applications) being advertised at

https://acsvproject.com

The goal now is to extend the methods to more complicated types of singularities – including those that result from multivariate algebraic and D-finite GFs, as well as to find applications in fields needing directional multivariate asymptotics.

1.4 Our Contributions

Here we outline the remainder of the thesis and state our contributions.

Chapter 2 contains the background needed to understand smooth ACSV, with a focus on motivation from the univariate case, examples, and stating the results in as explicit a form as possible. Our presentation and notation will mirror Chapter 5 of [90], which has a similar focus.

This explicit formulation of the results will be useful for Chapter 3, where we state our results associated with explicit error bounds for smooth ACSV and an application to the sequence positivity problem. These include constructive versions of arguments which allow us to approximate our diagonal coefficient sequence $(f_{n\boldsymbol{r}})$ by a saddle-point integral $\chi(n)$ (Propositions 2 and 3), and then approximate this saddle-point integral (Propositions 4 and 5). We then provide bounds for the error $|f_{n\boldsymbol{r}} - \lambda(n)|$ between $f_{n\boldsymbol{r}}$ and our asymptotic expression λ (Theorem 2 and 3), and use these to derive an index N at which positivity of λ guarantees positivity of $f_{n\boldsymbol{r}}$ for $n \geq N$, with some light restrictions on the form of our asymptotic (Corollaries 1 and 2). This reduces positivity to checking positivity of λ and finitely many initial terms $f_{\boldsymbol{0}}, f_{\boldsymbol{r}}, f_{2\boldsymbol{r}}, \ldots, f_{(N-1)\boldsymbol{r}}$; we then provide a couple of example classes – Classes 1 and 2 – of functions which a priori guarantee λ will be positive, as well simple criteria for checking various hypotheses of ACSV in the symmetric multi-affine setting.

Chapter 4 contains the history of the creative telescoping problem for rational periods, as well as the mathematical background needed to understand our SageMath implementation, `sage_periods`. This includes motivation from the univariate and smooth multivariate cases, together with a high-level understanding of the machinery needed to extend the reduction-based algorithm to the general (nonsmooth) multivariate case. This algorithm was discovered by Pierre Lairez in [77], and much of our presentation is paraphrased from that paper. We also discuss some of the features and basic design of our implementation, and demonstrate it on examples of practical interest (Examples 6, 7).

Chapter 2

ACSV Background

Before we jump into learning ACSV, we motivate it with some examples from the univariate case. In the univariate case, to perform an asymptotic estimate we deform our Cauchy contour close to singularities and evaluate their contribution using the Residue Theorem plus our pole formula. While we saw how to do this in Example 1, we still need to discuss the notion of *saddle-point*, and how this can be used to estimate asymptotics.

2.1 Univariate motivation: Saddle-Point Method

As we saw in our motivating example, when a generating function F for (f_n) is available to us, and this GF is analytic at the origin, we first make sure that there are finitely many dominant singularities. When there are, these will be the ones around which a singularity analysis *may* yield asymptotics. The caveat implied by this "may" is, roughly speaking, the following:

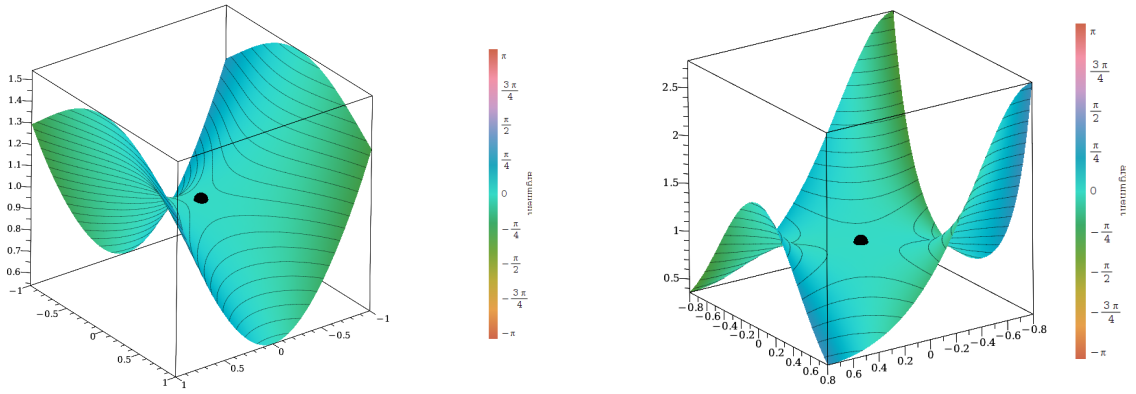
> "In order for singularity analysis of F near a singular point w to yield an asymptotic contribution, the growth of $|F|$ near w must be moderate."

Singularity analysis will not work for singularities around which we have rapid growth or decay; here *moderate* growth means polynomial, and *rapid* growth means exponential or greater. Moreover, singularity analysis will not work when F is entire, for then there are no singularities at finite distance from the origin to analyze!

The *Saddle-Point Method* (SPM) steps in to fill this gap; it is able to provide asymptotics for entire functions, as well as those exhibiting rapid growth or decay near singularities.[1] Furthermore, bounds derived from the SPM are used in deriving limit laws for combinatorial structures depending on several parameters. This is the first genuinely multivariate setting we see, and the usefulness of the SPM here should motivate us to try and exploit it in other multivariate scenarios. As it turns out, this is the right approach for ACSV, and the procedure used to estimate the integral we get by localizing to "dominant" singularities turns out to be a multivariate generalization of the SPM.

But before we get there, we should first say what saddle-points are, and explain how they are used to estimate contour integrals.

Definition 3 (Saddle-Point)**.** *Let Ω be an open connected subset of $\mathbb{C}$ and let $f(z)$ be analytic on Ω. A point $z_0 \in \Omega$ is called a saddle-point of f if $f(z_0) \neq 0$ but $f'(z_0) = 0$.*



$f(z) = \cosh(z)$ $\qquad$ $f(z) = \exp(z^3)$

Figure 2.1: Some examples of magnitude plots, with saddle-points at the origin. The left figure is an example of a 'simple' saddle-point, the right is a 'monkey saddle.'

From looking at a plot of $|f(z)|$ near a saddle-point z_0, it is clear where the name comes from: such points visually resemble the inner part of a saddle or the geographical pass between two mountains. See Figure 2.1 for examples.

Surfaces formed from the modulus of an analytic function $|f(z)|$ have only 3 types of points:

- *Ordinary points*, those where $f(z_0), f'(z_0) \neq 0$

[1] For instance, *essential* singularities, like $z = 1$ in $e^{1/(1-z)}$.

- *Zeros*, where $f(z_0) = 0$, and
- *Saddle points*, those where $f(z_0) \neq 0$ and $f'(z_0) = 0$.

Locally, saddle-points are the common apex of two or more "valleys," where the modulus of the function decreases moving away from the saddle-point. Using this intuition (formalized in Sections VIII.1 and 2 of [40]), one can derive quite general bounds for integrals of the form

$$I = \int_A^B F(z)dz,$$

where F analytic in some open connected set containing the endpoints A and B, by noting that if A and B are in opposite valleys of a saddle-point σ, then the path $\mathcal{C}$ between them minimizing the path functional

$$\Phi(\mathcal{C}) = \sup_{z \in \mathcal{C}} |F(z)|$$

will:

- Pass through σ; since if it went to either side it would pass through a point of magnitude greater than $|F(\sigma)|$.
- Take the *path of steepest ascent* to σ – i.e., the path which passes perpendicular to all level sets of $|F|$ as $\mathcal{C}$ approaches σ; since if it took another path, it is possible that it passes through a point of higher magnitude.

These bounds can be applied to Cauchy integral expressions of the form

$$a_n = \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{F(z)}{z^{n+1}} \mathrm{d}z$$

with $F(z) = e^{f(z)}$ analytic about 0 with *phase* f possibly depending[2] on n, and $\mathcal{C}$ a simple closed curve oriented counterclockwise containing no other singularities besides the one at 0. Saddle points correspond locally to maxima of our integrand modulus along $\mathcal{C}$. Thus, if $\mathcal{C}$ passes through a saddle-point σ we can expect a majority of the "contribution" to the value of a_n to occur in a small neighborhood of σ. For this reason, we split $\mathcal{C}$ into two parts, $\mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1$, with $\mathcal{C}_0$ passing near σ and $\mathcal{C}_1 = \mathcal{C} \setminus \mathcal{C}_0$. The key is to choose the size of $\mathcal{C}_0$ to be dependent on the integrand, large enough so that it contains a majority

[2]The saddle-point method can also accommodate $F = ae^f$, with $a = O(1)$ as $n \to \infty$. The function a is called the *amplitude* of F.

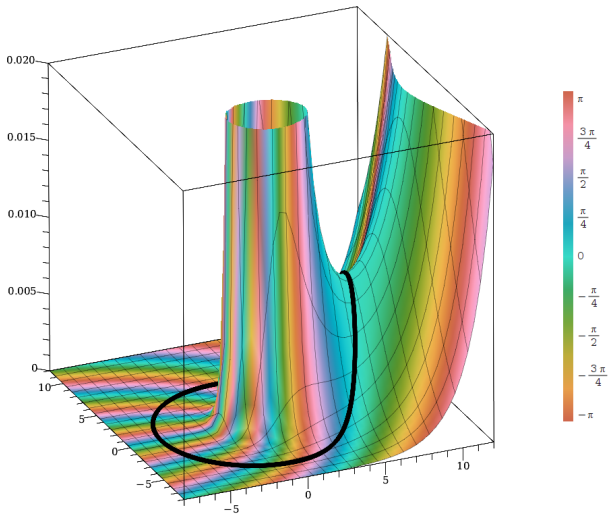


Figure 2.2: Plot of the Cauchy integrand with $n = 6$, together with saddle-point contour.

of a_n's contribution, but small enough so that ϕ can be well-approximated by a quadratic function. Parameterizing, this causes our integral to be asymptotically equivalent to a Gaussian integral over a *finite* symmetric interval about over $\mathcal{C}_1$ has an exponentially small error, then we can add back this error and instead estimate a *standard* Gaussian integral over the entire real line, which we know how to evaluate in closed form.

We illustrate with an example, taken from Section VIII.3 of [40]. The reader will notice the similarities[3] between this example and the resulting (explicit) saddle-point derivation in Chapter 3. For more on the univariate SPM, and general saddle-point bounds, see Chapter VIII of [40].

Example 2 (Univariate saddle-point asymptotics)**.** *We estimate* $\frac{1}{n!} = [z^n]\, e^z$*, starting from*

$$I_n = \frac{1}{2\pi i} \int_{|z|=\epsilon} e^z \frac{dz}{z^{n+1}},$$

the integration being performed along some circle centered about the origin of radius ϵ*. Note by the Cauchy integral formula that for all* n*,* $I_n = [z^n]e^z$*. We resize our circle so that it passes through a neighborhood of the saddle-point of* $e^z z^{-n-1}$ *at* $\sigma := n + 1$*, taking our radius* ϵ *to be equal to* n*. See Figure 2.2 for a plot of the magnitude of our integrand, along with our contour.*

[3] Albeit, with some refactorizations of the derivation to account for finding an explicit error.

Switching to polar coordinates (so that our contour radius becomes $r = n$) and letting $z = ne^{i\theta}$, we obtain

$$I_n = \frac{e^n}{2\pi n^n} \int_{-\pi}^{\pi} e^{n\left(e^{i\theta}-1-i\theta\right)} d\theta \qquad (*)$$

Set $\phi(\theta) = e^{i\theta} - 1 - i\theta$. As outlined by the SPM, we should isolate a small portion of the contour near σ. We thus introduce $B_n := n^{-\alpha}$, for some $\alpha \in \mathbb{Q} \cap (1/3, 1/2)$, and let

$$I_n^{(0)} = \int_{-B_n}^{+B_n} e^{n\phi(\theta)} d\theta, \quad I_n^{(1)} = \int_{B_n}^{2\pi - B_n} e^{n\phi(\theta)} d\theta.$$

The choice of B_n is motivated by our need to have ϕ well-approximated by its order 2 Taylor series expansion. Practically, as we will see, this will translate to the concrete requirement of needing $nB_n^2 \to \infty$ and $nB_n^3 \to 0$ as $n \to \infty$. ***For the sake of concreteness, we take*** *$\alpha = 2/5$* ***for the remainder of this example.***

First, let's approximate the tail integral $I_n^{(1)}$. To do this, note that $|e^{n\phi(z)}| = e^{n\Re(\phi)(z)}$, and since $\Re(\phi) = \Re(e^{i\theta}) - 1 = \cos(\theta) - 1$, unimodality properties of cos *and the univariate maximum modulus integral bound imply*

$$|I_n^{(1)}| = O\left(e^{n(\cos(B_n)-1)}\right) = O\left(e^{-Cn^{1/5}+O(1)}\right).$$

As for the central integral $I_n^{(0)}$, taking a Taylor expansion near $\theta = 0$, one has $\phi(\theta) \equiv e^{i\theta} - 1 - i\theta = -\frac{1}{2}\theta^2 + O(\theta^3)$, so that, for $|\theta| \leq B_n$,

$$e^{n\phi(\theta)} = e^{-n\theta^2/2 + O\left(n\theta^3\right)} = e^{-n\theta^2/2}\left(1 + O\left(nB_n^3\right)\right).$$

Thus, since $B_n = n^{-2/5}$, we have

$$I_n^{(0)} = \int_{-n^{-2/5}}^{n^{-2/5}} e^{-n\theta^2/2} d\theta \left(1 + O\left(n^{-1/5}\right)\right)$$

which, by the change of variables $t = \theta\sqrt{n}$, becomes

$$I_n^{(0)} = \frac{1}{\sqrt{n}} \int_{-n^{1/10}}^{n^{1/10}} e^{-t^2/2} dt \left(1 + O\left(n^{-1/5}\right)\right)$$

Therefore $I_n^{(0)}$ is well-approximated by an incomplete Gaussian integral. For any $\eta > 0$

$$\int_{\eta}^{+\infty} e^{-t^2/2} dt = O\left(e^{-\eta^2/2}\right),$$

so by symmetry

$$I_n^{(0)} = \frac{1}{\sqrt{n}}\left(\underbrace{\int_{-\infty}^{\infty} e^{-t^2/2}\mathrm{d}t}_{=\sqrt{2\pi}} + O\left(e^{-\frac{1}{2}n^{1/5}}\right)\right)\left(1 + O\left(n^{-1/5}\right)\right),$$

and $I^{(1)}$ is exponentially negligible. Combining $I_n^{(0)} + I_n^{(1)}$ and collecting the error terms, we end up with

$$I_n^{(0)} + I_n^{(1)} \sim \sqrt{\frac{2\pi}{n}},$$

hence

$$I_n = \frac{1}{2\pi}\frac{e^n}{n^n}\left(I_n^{(0)} + I_n^{(1)}\right) \sim \frac{e^n}{n^n\sqrt{2\pi n}}.$$

□

Remark 1. *Note that nothing here was special about reducing the integral $I_n^{(0)}$ to a standard Gaussian; higher powers work too. In general, one can use Laplace's method (See Section 6 of Appendix B of [40]) to get, for any positive integer M, an asymptotic expansion of the form*

$$I_n^{(0)} \sim \frac{1}{\sqrt{n}}\left(\sum_{k=0}^{M-1}\frac{d_k}{n^{k/2}} + O\left(n^{-M/2}\right)\right),$$

for some computable constants d_k, with $d_0 = \sqrt{2\pi}$.

We are now ready to begin our discussion of ACSV, where a generalized multivariate SPM will make an appearance.

2.2 ACSV

Here we describe some of the basic notions and terminology of ACSV, as well as a high-level overview of the procedure that we follow to derive asymptotics. Fix a positive integer d and let F be a rational function in complex variables $\boldsymbol{z} = (z_1, ..., z_d)$. We write F as

$$F(\boldsymbol{z}) = \frac{G(\boldsymbol{z})}{H(\boldsymbol{z})},$$

with the polynomial $H(\boldsymbol{z})$ nonzero at the origin. Completely analogous to the theory for a single complex variable, we have a power series representation for F which converges

absolutely on some *polydisk* $D(\boldsymbol{a}) = \{\boldsymbol{z} \in \mathbb{C}^d : |z_i| \leq |a_i|, 1 \leq i \leq d\}$ centered at $\mathbf{0}$, of *polyradius* $\boldsymbol{a}$. In other words, F is *analytic* at the origin with series expansion

$$F(\boldsymbol{z}) = \sum_{\boldsymbol{j} \in \mathbb{N}^d} f_{\boldsymbol{j}} \boldsymbol{z}^{\boldsymbol{j}},$$

the series coefficients $f_{\boldsymbol{j}}$ being described by a suitably generalized form of the Cauchy integral formula:

$$f_{\boldsymbol{j}} = \frac{1}{(2\pi i)^d} \int_{T(\boldsymbol{b})} \frac{F(\boldsymbol{z})}{\boldsymbol{z}^{\boldsymbol{j}+\mathbf{1}}} d\boldsymbol{z}.$$

Here $\boldsymbol{b}$ is any point in $D(\boldsymbol{a})$, and $T(\boldsymbol{b}) = \{\boldsymbol{z} \in \mathbb{C}^d : |z_i| = |b_i|, 1 \leq i \leq d\}$ is the *polytorus* centered at $\mathbf{0}$ of polyradius $\boldsymbol{b}$. In addition to the shorthand $\boldsymbol{z}^{\boldsymbol{j}} = z_1^{j_1} \cdots z_d^{j_d}$ we shall henceforth take $|\boldsymbol{z}|$ to mean the vector $(|z_1|, ..., |z_d|)$ in $\mathbb{R}^d_{\geq 0}$.

Now fix a *direction vector*[4] $\boldsymbol{r} = (r_1, ..., r_d)$ in $\mathbb{N}^d$. We wish to derive an asymptotic expansion for the univariate sequence $(f_{n\boldsymbol{r}})_{n \in \mathbb{N}}$. As is the case for univariate analytic combinatorics, we have a "4-Step Plan" which will often yield asymptotics for $f_{n\boldsymbol{r}}$. But, its steps are different:

1. Bound $\rho = \limsup_{n \to \infty} |f_{n\boldsymbol{r}}|^{1/n}$, the exponential growth of the coefficients. This is the dominant growth rate of our sequence.

2. Determine the singularities of F that contribute to our asymptotic behavior.

3. Write down the Cauchy integral for our sequence; "localize" and compute residues to isolate the portion of the integral immediately surrounding contributing singularities; other singularities are then captured by an error growing smaller than ρ.

4. Estimate the localized integral using the Saddle-Point method; the result is our asymptotic.

In order for us to carry out this process, F must meet some additional requirements. The first of these is that the numerator and denominator polynomials G and H are coprime, so the *singular variety* $\mathcal{V}$ of F, comprising the non-analytic points of F, is the zero set of H. Also, we may assume that F itself is not a polynomial. Because the singular set of $F(\boldsymbol{z})$ depends only on the distinct irreducible factors of H, *not* their multiplicity, it

[4]We can actually take $\boldsymbol{r} \in \mathbb{Q}^d$ and all of the following will still work, but it suffices to consider $\boldsymbol{r} \in \mathbb{N}^d$ by clearing the denominators of the coordinates of $\boldsymbol{r}$.

suffices to consider the *squarefree part* of H, denoted $H^{\mathfrak{s}}$, which is simply the product of H's irreducible polynomial factors with the factorization being over the complex numbers.

The methods work for functions whose singular varieties are complex manifolds near the singularities determining asymptotics – so-called *smooth* singular varieties. To be precise, a *smooth point* of $\mathcal{V}$ is a point where some partial derivative of $H^{\mathfrak{s}}$ does not vanish. $\mathcal{V}$ is called a *smooth variety* if each of its points is smooth.

Now, our goal is to estimate our coefficient sequence $f_{n\boldsymbol{r}}$ via its Cauchy Integral representation

$$f_{n\boldsymbol{r}} = \frac{1}{(2\pi i)^d} \int_{T(\boldsymbol{w})} \frac{F(\boldsymbol{z})}{\boldsymbol{z}^{n\boldsymbol{r}+\mathbf{1}}} d\boldsymbol{z},$$

where $\boldsymbol{w}$ is some point in the domain of convergence $\mathcal{D}$ of the series F. Applying the maximum modulus integral bound to this equality gives us an upper bound on $|f_{n\boldsymbol{r}}|$ for each point $\boldsymbol{w}$ of $\overline{\mathcal{D}}$. We hope to minimize this upper bound on $\overline{\mathcal{D}}$ to determine our exponential growth rate,

$$\limsup_{n\to\infty} |f_{n\boldsymbol{r}}|^{1/n},$$

and search for the singularities in $\partial\mathcal{D}$ which give the minimum (if it exists).

Such points on the boundary are the multivariate analogue of the *dominant singularities* of univariate AC, which contribute to asymptotics.

Definition 4. *A singularity $\boldsymbol{w} \in \mathcal{V}$ is called a* minimal point *if it lies in $\partial\mathcal{D} \cap \mathcal{V}$. Equivalently, $\boldsymbol{w}$ is minimal if no other singularity $\boldsymbol{z}$ of F satisfies $|z_j| < |w_j|$ for all $1 \le j \le d$.*

Minimal singularities are those "closest to the origin" in this multivariate sense; just as in the univariate case, we can deform our Cauchy contour close to them without crossing singularities of F. Note that if $\boldsymbol{w} \in \mathcal{V}$ is minimal, so is any point in the set $T(\boldsymbol{w}) \cap \mathcal{V}$. Our asymptotic analysis becomes easier when we require that there are a finite number of such points.

Definition 5. *A minimal point* $\boldsymbol{w}$ *is called*

- Finitely minimal *if* $T(\boldsymbol{w}) \cap \mathcal{V}$ *is finite. In other words,* $\boldsymbol{w}$ *is finitely minimal if there are only finitely many other singularities of* F *sharing the same coordinate-wise modulus.*
- Strictly minimal *if* $T(\boldsymbol{w}) \cap \mathcal{V} = \{\boldsymbol{w}\}$*. In other words,* $\boldsymbol{w}$ *is strictly minimal if it is minimal and no other singularity of* F *has the same coordinate-wise modulus.*

However, just because a point is minimal – or even finitely or strictly minimal – does not mean it contributes to asymptotics. Our singular variety, in general, will have infinitely many points on $\partial\mathcal{D}$. What we need is another criterion to "cut down" the set of singularities in consideration to a finite size, then perform a singularity analysis on the points in this set to determine asymptotics, as in the univariate case.

The needed criterion turns out to be a geometric: namely, the gradient of H is parallel to the gradient of the monomial function $\Phi_{\boldsymbol{r}} = \boldsymbol{z}^{\boldsymbol{r}}$. This is characterized algebraically as follows.

Definition 6 (Critical Points, Smooth Critical Points)**.** *The* critical point equations *(CPEs) for the pair* $(F, \boldsymbol{r})$ *are:*

$$H^{\mathfrak{s}}(\boldsymbol{w}) = 0$$
$$r_j w_1 H^{\mathfrak{s}}_{z_1}(\boldsymbol{w}) - r_1 w_j H^{\mathfrak{s}}_{z_j}(\boldsymbol{w}) = 0 \quad (2 \le j \le d).$$

Note that any point of $\mathcal{V}$ *where all partials of* $H^{\mathfrak{s}}$ *vanish trivially satisfies the CPEs. But we only discuss* smooth critical points*, CPE solutions with* $H^{\mathfrak{s}}_{z_j}(\boldsymbol{w}) \neq 0$ *for some* $1 \le j \le d$*. We denote the set of smooth critical points of* H *in the direction* $\boldsymbol{r}$ *by* $\mathsf{crit}(\boldsymbol{r})$*.*

In our setup, smooth critical points are exactly the points we need to locally approximate our Cauchy integral using saddle-point integrals. They also are desirable in that we can introduce asymptotically negligible integrals around said points, needed in our residue computations.

Thus, we care about finding minimal smooth critical points, as these determine our asymptotics. Suppose we've found a strictly minimal smooth critical point $\boldsymbol{w}$. In all that follows, we will assume without loss of generality that $\boldsymbol{w}$ has $H_{z_d}(\boldsymbol{w}) \neq 0$. Let $\rho = |w_d|$, adopt the shorthand $\widehat{\boldsymbol{w}}$ for the $(d-1)$-sub-vector of $\boldsymbol{w}$ not containing the last entry $\widehat{\boldsymbol{w}} = (w_1, ..., w_{d-1})$, and define the $(d-1)$-torus $\mathcal{T} = T(\widehat{\boldsymbol{w}})$.

The implicit function theorem and strict minimality of $\boldsymbol{w}$ imply existence of a neighborhood $\mathcal{N}$ of $\widehat{\boldsymbol{w}}$ in $\mathcal{T}$, as well as an analytic parameterization $g : \mathcal{N} \to \mathbb{C}$ of z_d and a positive number δ so that for all $\widehat{\boldsymbol{z}}$ in $\mathcal{N}$,

$$H(\boldsymbol{z}) = 0 \iff z_d = g(\widehat{\boldsymbol{z}}) \tag{C1}$$

$$\rho - \delta < |g(\widehat{\boldsymbol{z}})| < \rho + \delta. \tag{C2}$$

Furthermore, for all $\widehat{\boldsymbol{z}} \in \mathcal{T}$,

$$(\ \rho - \delta \le |z_d| \le \rho + \delta\ ) \ \wedge\ (\ H(\widehat{\boldsymbol{z}}, z_d) = 0\ ) \implies (\ \widehat{\boldsymbol{z}} \in \mathcal{N} \wedge z_d = g(\widehat{\boldsymbol{z}})\ ). \tag{C3}$$

If δ is small enough, the neighborhood $\mathcal{N}$ can be taken to be of the form

$$\mathcal{N} = \left\{ \widehat{\boldsymbol{w}} e^{i\boldsymbol{\theta}} : \boldsymbol{\theta} \in [-\delta, \delta]^{d-1} \right\},$$

where we use the notation $\widehat{\boldsymbol{w}} e^{i\boldsymbol{\theta}} = (w_1 e^{i\theta_1}, ..., w_{d-1} e^{i\theta_{d-1}}) \in \mathbb{C}^{d-1}$. We can define the following integrals, whose values are finite:

$$I = \frac{1}{(2\pi \mathrm{i})^d} \int_{\mathcal{T}} \left(\int_{|z_d| = \rho - \delta} F(\boldsymbol{z}) \frac{\mathrm{d}z_d}{z_d^{nr_d+1}} \right) \frac{\mathrm{d}\widehat{\boldsymbol{z}}}{\widehat{\boldsymbol{z}}^{n\widehat{\boldsymbol{r}}+\mathbf{1}}} \tag{2.2}$$

$$I^{\mathrm{loc}} = \frac{1}{(2\pi \mathrm{i})^d} \int_{\mathcal{N}} \left(\int_{|z_d| = \rho - \delta} F(\boldsymbol{z}) \frac{\mathrm{d}z_d}{z_d^{nr_d+1}} \right) \frac{\mathrm{d}\widehat{\boldsymbol{z}}}{\widehat{\boldsymbol{z}}^{n\widehat{\boldsymbol{r}}+\mathbf{1}}} \tag{2.3}$$

$$I^{\mathrm{out}} = \frac{1}{(2\pi \mathrm{i})^d} \int_{\mathcal{N}} \left(\int_{|z_d| = \rho + \delta} F(\boldsymbol{z}) \frac{\mathrm{d}z_d}{z_d^{nr_d+1}} \right) \frac{\mathrm{d}\widehat{\boldsymbol{z}}}{\widehat{\boldsymbol{z}}^{n\widehat{\boldsymbol{r}}+\mathbf{1}}} \tag{2.4}$$

and

$$\chi = I^{\mathrm{loc}} - I^{\mathrm{out}} \tag{2.5}$$

$$= \frac{-1}{(2\pi \mathrm{i})^d} \int_{\mathcal{N}} \left(\int_{|z_d| = \rho + \delta} F(\boldsymbol{z}) \frac{\mathrm{d}z_d}{z_d^{nr_d+1}} - \int_{|z_d| = \rho - \delta} F(\boldsymbol{z}) \frac{dz_d}{z_d^{nr_d+1}} \right) \frac{\mathrm{d}\widehat{\boldsymbol{z}}}{\widehat{\boldsymbol{z}}^{n\widehat{\boldsymbol{r}}+\mathbf{1}}}. \tag{2.6}$$

Note that the integral I *is* our coefficient sequence $f_{n\boldsymbol{r}}$, since the polytorus $\mathcal{T} \times \{|z_d| = \rho - \delta\}$ lies in $\mathcal{D}$. This is just the Cauchy integral formula.

In this situation, we know [90, Lemma 5.1] that $\chi = \chi(n)$ *well-approximates* our coefficient sequence $f_{n\boldsymbol{r}}$, meaning

$$\forall n, \quad |f_{n\boldsymbol{r}} - \chi| = O(\tau^n) \quad \text{for some } \tau < |\boldsymbol{w}|^{-\boldsymbol{r}}. \tag{2.7}$$

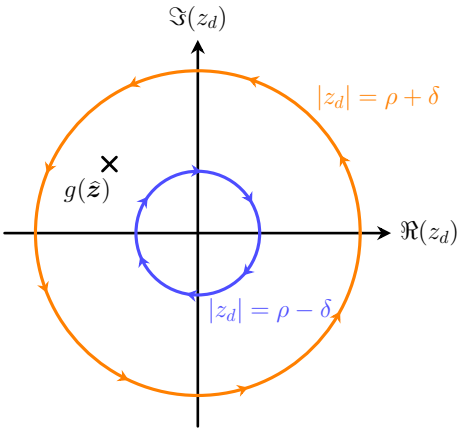


Figure 2.3: Inner integral difference in χ's contours. The difference in handedness comes from the sign on the second integral.

It turns out that under our hypotheses[5] on F and $\boldsymbol{w}$ the integral χ *is* something we know how to approximate, *if* $\boldsymbol{w}$ satisfies one additional criterion.

Definition 7. *For* $1 \leq i, j \leq d$, *define*

$$U_{i,j} = \frac{w_i w_j H^{\mathfrak{s}}_{z_i z_j}(\boldsymbol{w})}{w_d H^{\mathfrak{s}}_{z_d}(\boldsymbol{w})} \quad \text{and} \quad V_i = \frac{w_i H^{\mathfrak{s}}_{z_i}(\boldsymbol{w})}{w_d H^{\mathfrak{s}}_{z_d}(\boldsymbol{w})} = \frac{r_i}{r_d}.$$

The point $\boldsymbol{w}$ *is called* nondegenerate *if the* $(d-1) \times (d-1)$ *matrix* $\mathcal{H}$ *with entries*

$$\mathcal{H}_{i,j} = \begin{cases} V_i V_j + U_{i,j} - V_j U_{i,d} - V_i U_{j,d} + V_i V_j U_{d,d} & : i \neq j \\ V_i + V_i^2 + U_{i,i} - 2V_i U_{i,d} + V_i^2 U_{d,d} & : i = j \end{cases}$$

is non-singular.

If $\boldsymbol{w}$ is a *nondegenerate* strictly minimal smooth contributing point, then $\chi = \chi(n)$ is what is known as a *Fourier-Laplace* integral or *saddle-point integral*, i.e. something of the form

$$I_n = \int_{\mathcal{C}} f_n(\boldsymbol{x}) e^{g_n(\boldsymbol{x})} d\boldsymbol{x}$$

[5] Of course, the similar condition applies in the case when $\mathsf{contrib}(\boldsymbol{r}) \cap \partial\mathcal{V}$ is finitely minimal.

for functions $f, g : \mathbb{C}^d \to \mathbb{C}$ analytic and $\mathcal{C}$ a real d-chain. Here n is assumed to be a large parameter, and $f = f_n$ and $g = g_n$ are called the *magnitude* and *phase* of I, respectively. Saddle point integrals are so-named because their univariate counterparts are estimated using the Saddle-Point Method of the previous section; see Theorem VIII.3 and Figure VIII.4 of [40] for an algorithmic description of the SPM. The analysis generalizes to the multivariate case, and is well-studied – see Section 3.2 for references.

To see χ written as a saddle-point integral, we should first examine the inner integral difference

$$\int_{|z_d|=\rho+\delta} F(\boldsymbol{z}) \frac{\mathrm{d}z_d}{z_d^{nr_d+1}} - \int_{|z_d|=\rho-\delta} F(\boldsymbol{z}) \frac{dz_d}{z_d^{nr_d+1}}.$$

A depiction of the contours for these integrals can be found in Figure 2.3.

Since $\widehat{\boldsymbol{z}} \in \mathcal{N}$ is fixed and $|z_d| \in [\rho - \delta, \rho + \delta]$, we know (by (C2) and (C3)) that there is a unique pole $z_d = g(\widehat{\boldsymbol{z}})$ between these two circles. Thus, we have the perfect situation in which to apply the univariate Residue Theorem. Doing so, after first collecting one of the powers of $\frac{1}{2\pi i}$ from the outer integrals, we see

$$\frac{1}{2\pi i}\left(\int_{|z_d|=\rho+\delta} F(\boldsymbol{z}) \frac{\mathrm{d}z_d}{z_d^{nr_d+1}} - \int_{|z_d|=\rho-\delta} F(\boldsymbol{z}) \frac{dz_d}{z_d^{nr_d+1}}\right) = \mathrm{Res}_{z_d=g(\widehat{\boldsymbol{z}})}\left(\frac{F(\boldsymbol{z})}{z_d^{nr_d+1}}\right).$$

Similar to Example 1, if we know the order p of the pole $z_d = g(\widehat{\boldsymbol{z}})$ we can evaluate this residue. To get p, one can simply note that p is the smallest positive integer so that $\partial^p H/\partial z_d^p(\boldsymbol{w}) \neq 0$. In general, the formula is:

$$\mathrm{Res}_{z_d=g(\widehat{\boldsymbol{z}})}\left(\frac{F(\boldsymbol{z})}{z_d^{nr_d+1}}\right) = \sum_{j=0}^{p-1} \frac{(-1)^j \left(nr_d + p - 1 - j\right)_{(p-j-1)}}{j!(p-j-1)!} g(\hat{\boldsymbol{z}})^{j-n} \lim_{z_d \to g(\hat{\boldsymbol{z}})} \partial_d^j \left(\left(z_d - g(\hat{\boldsymbol{z}})\right)^p F(\boldsymbol{z})\right),$$

which for a simple pole ($p = 1$) simplifies greatly to

$$\mathrm{Res}_{\mathrm{z_d=g(\widehat{z})}}\left(\frac{\mathrm{F(z)}}{\mathrm{z_d^{nr_d+1}}}\right) = \frac{G(\widehat{\boldsymbol{z}}, g(\widehat{\boldsymbol{z}}))}{g(\widehat{\boldsymbol{z}})^{nr_d+1} H_{z_d}(\widehat{\boldsymbol{z}}, g(\widehat{\boldsymbol{z}}))}.$$

Thus, we've reduced the d-dimensional χ to the $(d-1)$-dimensional integral,

$$\frac{-1}{(2\pi i)^{d-1}} \int_{\mathcal{N}} \frac{\mathrm{Res}_{z_d=g(\widehat{\boldsymbol{z}})}\left(F(\boldsymbol{z})/z_d^{nr_d+1}\right)}{\widehat{\boldsymbol{z}}^{n\widehat{\boldsymbol{r}}+\mathbf{1}}} \mathrm{d}\widehat{\boldsymbol{z}}.$$

Now, make the change of variables $z_j \mapsto w_j e^{i\theta_j}$ for $1 \leq j \leq d-1$ to get

$$\chi = \frac{\boldsymbol{w}^{-n\boldsymbol{r}}}{(2\pi)^{d-1}} \int_{\widetilde{\mathcal{N}}} A_n(\boldsymbol{\theta}) e^{-nr_d \phi(\boldsymbol{\theta})} d\boldsymbol{\theta},$$

where $\boldsymbol{\theta} \in \widetilde{\mathcal{N}} := [-\delta, \delta]^{d-1}$,

$$A_n(\boldsymbol{\theta}) = \frac{(-1)^p}{g\left(\widehat{\boldsymbol{w}} e^{\boldsymbol{i\theta}}\right)^p} \Gamma_n\left(\widehat{\boldsymbol{w}} e^{\mathrm{i}\boldsymbol{\theta}}\right) \tag{2.8}$$

$$\phi(\boldsymbol{\theta}) = \log\left(\frac{g\left(\widehat{\boldsymbol{w}} e^{\mathrm{i}\boldsymbol{\theta}}\right)}{g(\widehat{\boldsymbol{w}})}\right) + \mathrm{i}(\hat{\boldsymbol{r}} \cdot \boldsymbol{\theta})/r_d \tag{2.9}$$

and

$$\Gamma_n(\boldsymbol{z}) = \mathrm{Res}_{z_d = g(\hat{\boldsymbol{z}})}\left(\frac{F(\boldsymbol{z})}{z_d^{nr_d+1}}\right) \tag{2.10}$$

$$= \sum_{j=0}^{p-1} \frac{(-1)^j \left(nr_d + p - 1 - j\right)_{(p-j-1)}}{j!(p-j-1)!} g(\hat{\boldsymbol{z}})^{j-n} \lim_{z_d \to g(\hat{\boldsymbol{z}})} \partial_d^j \left(\left(z_d - g(\hat{\boldsymbol{z}})\right)^p F(\boldsymbol{z})\right). \tag{2.11}$$

This is, evidently, a saddle-point integral. Through a procedure following that in [104, Theorem 4.1], for any nonnegative integer M we can find an asymptotic in n for χ of the form

$$\chi = \frac{\boldsymbol{w}^{-n\boldsymbol{r}}}{(2\pi)^{d-1}} \left(\frac{2\pi}{n}\right)^{(d-1)/2} \det(r_d \mathcal{H})^{-1/2} \left(\sum_{k=0}^{M} K_j (r_d n)^{-k} + O(n^{-M-1})\right),$$

where the constants K_k are given by some complicated but explicit formula with $K_0 = A(\mathbf{0})$ and $\mathcal{H}$ is the *Hessian matrix* of ϕ with entries described by Definition 7 above.

Thus, invoking Equation (2.7) and doing a bit of algebraic manipulation (mostly to deal with normalizing the function $A_n(\boldsymbol{\theta})$, which is a polynomial in n of degree at most $p-1$), we end up with the following.

Theorem 1 (Smooth Asymptotic Formula, Theorem 5.3 of [90])**.** *Suppose that $F(\boldsymbol{z}) = G(\boldsymbol{z})/H(\boldsymbol{z})$ admits a nondegenerate strictly minimal smooth contributing point $\boldsymbol{w} \in \mathbb{C}_*^d$ in the direction $\boldsymbol{r} \in \mathbb{R}_*^d$, such that $H_{z_d}^{\mathfrak{s}}(\boldsymbol{w}) \neq 0$, and let p be the smallest integer such that $\left(\partial_d^p H\right)(\boldsymbol{w}) \neq 0$. Then for any nonnegative integer M there exist computable constants $C_0, \ldots, C_M$ such that*

$$f_{n\boldsymbol{r}} = \boldsymbol{w}^{-n\boldsymbol{r}} n^{p-1+(1-d)/2} \frac{(2\pi)^{(1-d)/2}}{\sqrt{\det\left(r_d \mathcal{H}\right)}} \left(\sum_{j=0}^{M} C_j \left(r_d n\right)^{-j} + O\left(n^{-M-1}\right)\right). \tag{2.12}$$

The constants C_j are determined by expanding $A_n(\boldsymbol{\theta})$ in as a polynomial in n and applying our saddle-point estimate to the resulting sum of integrals. The leading constant C_0 in this series has the value

$$C_0 = \frac{(-1)^p G(\boldsymbol{w}) p}{w_d^p \left(\partial_d^p H\right)(\boldsymbol{w})},$$

which is nonzero whenever $G(\boldsymbol{w}) \neq 0$. The asymptotic expansion (2.12) holds uniformly in neighborhoods $\mathcal{R} \subset \mathbb{R}_*^d$ of $\boldsymbol{r}$ where there is a smoothly varying contributing point satisfying the same conditions as $\boldsymbol{w}$.

Thus, to derive asymptotics for a given $(F, \boldsymbol{r})$, all we have to do is find a nonzero strictly minimal smooth contributing point $\boldsymbol{w}$, and compute the quantities used in the formula. It is worth doing a couple of examples to illustrate the power of this result.

Remark 2. *All the arguments here work in the finitely minimal case, as well. Suppose we have a finite set E of nondegenerate, minimal smooth contributing points. Then, for each $\boldsymbol{w} \in E$ we obtain a set of integrals localized about $\boldsymbol{w}$ similar to (2.4) and (2.6); call them $I_{\boldsymbol{w}}, I_{\boldsymbol{w}}^{\text{loc}}, I_{\boldsymbol{w}}^{\text{out}}$ and $\chi_{\boldsymbol{w}}$, respectively. Moreover, there exist constants $\tau_{\boldsymbol{w}} < |\boldsymbol{w}|^{-\boldsymbol{r}}$ making (2.7) hold for $I_{\boldsymbol{w}}$, and $\chi := \sum_{\boldsymbol{w}} \chi_{\boldsymbol{w}}$ is a sum of saddle-point integrals. To get an asymptotic we apply the results of (2.12) to each $\boldsymbol{w}$, then sum these asymptotic contributions to get the total asymptotic for $f_{n\boldsymbol{r}}$.*

2.3 Examples of computing directional asymptotics

The following example comes from [92].

Example 3. *(Lattice Path asymptotics) Consider lattice walks in d dimensions with steps in the set $\mathcal{S} = \{0, \pm 1\}^d \setminus \{\mathbf{0}\}$ restricted to the positive orthant. With*

$$S(\boldsymbol{z}) = \sum_{\boldsymbol{i} \in \mathcal{S}} \boldsymbol{z}^{\boldsymbol{i}},$$

it is shown that the generating function for the number of walks can be represented as the main diagonal

$$\Delta\left(\frac{G(\boldsymbol{z}, t)}{H(\boldsymbol{z}, t)}\right) = \Delta\left(\frac{(1+z_1)(1+z_2)\cdots(1+z_d)}{1 - t(z_1 \cdots z_d) S(\boldsymbol{z})}\right).$$

The variety of the denominator $H(\boldsymbol{z}, t) = 1 - t(z_1 \cdots z_d) S(\boldsymbol{z})$ can then be analyzed to find finitely minimal smooth contributing points, and get asymptotics with Theorem 1.

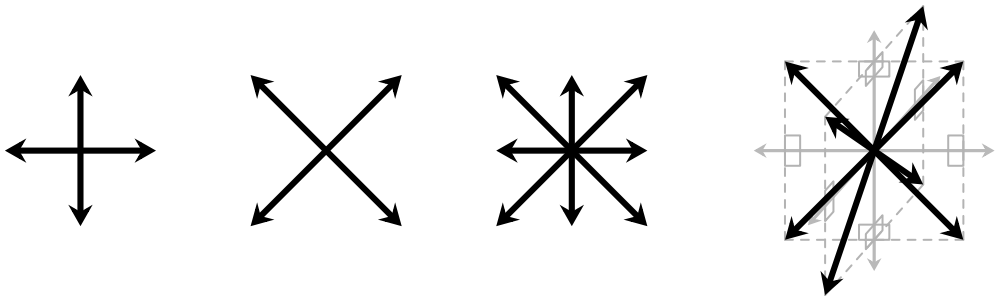


Figure 2.4: Example step sets defining lattice walks in 2D and 3D. The methods of [92] provide asymptotics for these and other step sets which are symmetric along all axes.

For instance, consider the 2D walk defined by the step set

$$\mathcal{S} = \{(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1), (\pm 1, \mp 1)\}$$

pictured third from the left in Figure 2.4. Letting x and y represent steps right and up, respectively, and $\bar{x} := 1/x$ and $\bar{y} := 1/y$ represent steps left and down, we have

$$S(x, y) = x + \bar{x} + y + \bar{y} + xy + \bar{x}y + x\bar{y} + \bar{x}\bar{y}$$

and the generating function $F(t)$ counting the number of walks with steps in $\mathcal{S}$ of length n is then

$$\begin{aligned} F(t) &= \frac{G(x, y, t)}{H(x, y, t)} \\ &= \Delta\left(\frac{(1+x)(1+y)}{1 - txyS(x, y)}\right) \\ &= \Delta\left(\frac{(1+x)(1+y)}{1 - t(1 + x + y + x^2 + y^2 + x^2y + y^2x + x^2y^2)}\right). \end{aligned}$$

Computing a Gröbner basis for the ideal generated by H and its partial derivatives gives (1), the entire ring. Thus the variety cut out by this ideal is empty, so $\mathcal{V}$ is smooth.

Next, we compute a Gröbner basis for the CPEs, then solve this simplified set of polynomial equations to see that we have a set of 5 smooth contributing points. While these points aren't all comparable in terms of the coordinate-wise order, we can check which of them are minimal via Proposition 5.4 of [90]. All we need to do is check, for each $\boldsymbol{w}$ in our set of 7 points, if there is a root $s \in (0, 1)$ to the univariate polynomial equation $H(s|\boldsymbol{w}|) = 0$.

Doing this, we see that we have a unique minimal critical point

$$\boldsymbol{w} = (1/8, 1, 1).$$

We can verify it's strictly minimal using Proposition 5.5 of [90], and to check that $\boldsymbol{w}$ *is nondegenerate we may plug directly into the matrix formulas for* $\mathcal{H}$ *given in Lemma 5.5 of [90]. These are implemented as part of the* `sage_acsv` *package. Doing so, we see the Hessian determinant is positive, thus* $\boldsymbol{w}$ *is a nondegenerate strictly minimal smooth contributing point. Noting that* H *is squarefree and* $p = 1$*, we can apply Theorem 5.3 to conclude that*

$$f_n = \frac{8}{3\pi} \cdot \frac{8^n}{n} + O\left(\frac{8^n}{n^2}\right).$$

□

It is worth noting that all of the above can be done automatically in `sage_acsv`, using a single command:

$$\texttt{diagonal_asymptotics_combinatorial}\left(\frac{(1+x)(1+y)}{1-t(1+x+y+x^2+y^2+x^2y+y^2x+x^2y^2)}\right).$$

Now, we include an example family first studied by Gillis, Reznick, and Zeilberger in 1983 [45]. This function family is important in the study of positivity – in the original paper, the authors claim (though they omit the proof, due to length) that for $d \geq 4$, the positivity of the function

$$F_d(\boldsymbol{z}) = \frac{1}{1 - \sum_{i=1}^{d} z_i + d! \prod_{i=1}^{d} z_i}$$

is implied by the positivity of the diagonal $\Delta F_d(t)$. They also showed that the diagonal coefficients $f_{n,n,\ldots,n} = f_{n\boldsymbol{r}}$ with $\boldsymbol{r} = \mathbf{1}$ could be reduced to the sum

$$f_{n\boldsymbol{r}} = \sum_{j=0}^{n} (-1)^j \frac{(nd - (d-1)j)!(d!)^j}{(n-j)!^d j!},$$

and checked positivity for terms up to 220.

Around three and a half decades later, Baryshnikov et al. [9] showed that this function was eventually positive for $d \geq 4$. The following year, positivity for $d \geq 8$ was proved

by Yaming Yu [130, Theorem 1] using combinatorial identities and a careful analysis of roots of certain associated polynomials. Simultaneously, Pillwein [109] used Cylindrical Algebraic Decomposition to check positivity of ΔF_d for d up to 17, thereby proving total positivity for all d at least 4.[6]

Here we reproduce the derivation of the asymptotic for $f_{n\boldsymbol{r}}$, hence that $\Delta F_d(t)$ is eventually positive, in much the same way as in Section 4 of [9].

Example 4. *(GRZ Class) We only need to check all the ACSV hypotheses; assuming that F_d satisfies them, we can then use our asymptotic statement from Theorem 1. Throughout, we will take advantage of the fact that $F_d(\boldsymbol{z})$ is a symmetric function in its variables $\boldsymbol{z}$, and that we're looking along the main diagonal $\boldsymbol{r} = \mathbf{1}$.*

First, observe the following general facts:

Lemma 1. *For all $d \geq 4$, we have*

$$d! < (d-1)^{d-1}.$$

Lemma 1 is provable by induction, and has an immediate consequence.

Lemma 2. *For all $d \geq 4$ and $\rho \in \left(\frac{1}{d}, \frac{1}{d-1}\right)$*

$$d!\rho^{d-1} < 1.$$

Next, introduce the diagonal polynomial

$$\delta^H(t) := H(t, t, ..., t) = 1 - dt + d!t^d.$$

Via Rouché's theorem, one can prove the following.

Lemma 3 (Proposition 20 of [9])**.** *For any $d \geq 4$, the polynomial δ^H has a unique root ρ of modulus less than $\frac{1}{d-1}$, and this root is real with $\rho \in [\frac{1}{d}, \frac{1}{d-1}]$.*

Next, they invoke the Grace-Walsh-Szegő theorem [25, Proposition 1], which implies that we are guaranteed to have a diagonal minimal contributing point, i.e. one whose coordinates are all equal.

[6] For $d \leq 3$, there must necessarily be negative coefficients.

Proposition 1 (Lemma 15 of [9]). *Let $F = 1/H$ with H a* symmetric *polynomial in its variables that is* multi-affine *(of degree at most 1 in each variable).*[7] *Let ρ be a zero of δ^H of minimal modulus. Then $\boldsymbol{\rho} := (\rho, \rho, ..., \rho)$ is a minimal point of F in $\mathsf{crit}(\boldsymbol{r})$.*

In fact, via [9, Lemma 18] we know that for this function all critical points are diagonal. Once we have this, we can show directly that:

- *H is squarefree. (In fact, it's is irreducible.)*
- *We can solve for g explicitly as a rational function in $\widehat{\boldsymbol{z}} = (z_1, ..., z_d)$:*

$$g(\widehat{\boldsymbol{z}}) = \frac{1 - \sum_{i=1}^{d-1} z_i}{1 - d! \prod_{i=1}^{d-1} z_i}.$$

- *The pole order is $p = 1$.*

We then show that for $d \geq 4$ the point $\boldsymbol{\rho}$ is nondegenerate by noting that, due to symmetry, the Hessian matrix formula from Lemma 5.5 of [90] simplifies tremendously into the compact form

$$\mathcal{H} = (1 - U)(I + \mathbf{1}\mathbf{1}^T)$$

with $U = \frac{d!\rho^{d-1}}{-1+d!\rho^{d-1}} < 0$ by Lemma 2. Thus

$$\det \mathcal{H} = d \cdot (1 - U)^{d-1} > 0,$$

hence $\boldsymbol{\rho}$ is nondegenerate.

*Finally, we must show that $\boldsymbol{\rho}$ is **strictly** minimal.*[8] *To do this, let $\boldsymbol{z}$ be any point in $\mathcal{V} \cap T(\boldsymbol{\rho})$. Write $z_j = \rho u_j$ with $|u_j| = 1$ for all j, and put $\tilde{U} = \prod_j u_j$. Since $0 < d!\rho^{d-1} < 1$ by Lemmas 2 and 3, and $\delta^H(\rho) = 0$,*

$$\rho^{-1} + d!\rho^{d-1} = d. \qquad (*)$$

Since $H(z) = 0$ we have, after dividing by ρ,

$$\rho^{-1} - \sum_j u_j + d!\rho^{d-1}\tilde{U} = 0,$$

[7] Note: These two conditions together are equivalent to H being of the form $H = c_0 + \sum_{k=1}^{d} c_i e_i$, the e_i being the elementary symmetric polynomials in variables $\boldsymbol{z} = (z_1, ..., z_d)$.

[8] This is not shown in [9]; their theorem statement for asymptotics requires a slightly different set of hypotheses, only requiring a finite set of minimal critical points on the torus, and their Lemma 18 plus Proposition 20 give exactly that.

and subtracting this latter equation from $(*)$ *yields*

$$\sum_j (1-u_j) = d!\rho^{d-1}(1-\tilde{U}). \tag{$\dagger$}$$

Now, apply the following inequality, which is a simple generalization of a problem appearing in the 2020 Putnam exam [76, Problem B5]:

$$\left|1-\prod_j u_j\right| \leq \left|\sum_j (1-u_j)\right|, \quad |u_j| \leq 1. \tag{2.13}$$

Applying this to $(\dagger)$ *gives* $|1-\tilde{U}| \leq d!\rho^{d-1}|1-\tilde{U}|$*, and since* $d!\rho^{d-1} < 1$ *this means* $\tilde{U} = 1$*. Therefore* $\sum_j (1-u_j) = 0$*, so* $\sum_j u_j = d$*. Taking real parts forces every* $\Re(u_j) = 1$*, hence* $u_j = 1$ *for all* j*. Thus* $\boldsymbol{z} = \boldsymbol{\rho}$*, giving us strict minimality of* $\boldsymbol{\rho}$*.*

We have now verified that the hypotheses of Theorem 1 apply; we obtain the asymptotic

$$f_{n\boldsymbol{r}} = \rho^{-dn} n^{(1-d)/2} \frac{(2\pi)^{(1-d)/2}}{\sqrt{d(1-U)^{d-1}}} \cdot \frac{1}{\rho - d!\rho^d} \left(1 + O(1/n)\right),$$

which is manifestly positive by Lemma 2 and the fact that ρ *lies between* $\frac{1}{d}$ *and* $\frac{1}{d-1}$*.* □

Besides considering the GRZ class, the authors of [9] derived diagonal asymptotics for a *generic* symmetric multilinear function in 3 variables [9, Theorem 7]. Depending on the situation, these asymptotics could be analyzed to prove eventual positivity of these diagonals. While these arguments show asymptotic positivity, in the next chapter we will show how to extend positivity checks to enough initial terms to guarantee *total* positivity of the diagonal.

Chapter 3

ACSV Computable Error Bounds

In this chapter we strengthen the usual ACSV analysis by giving explicit error bounds. Recall from the previous chapter that there are 3 major steps involved in computing diagonal asymptotics:

1. *Localization* of the integral I to (finitely) minimal, contributing, nondegenerate singularities.
2. *Evaluation of residues*, to obtain a saddle-point integral formulation of χ.
3. *Estimation* of χ via the *Saddle-Point Method*.

The second step is straightforward, and formulas for the evaluated residues (hence χ) can be found in Melczer [90, Lemma 5.2], so our results are concerned with constructivizing Steps 1 and 3. When done, we will have *explicit* error bounds for our asymptotics instead of Big-O statements.

3.1 Bounding Localization

We first constructivize the result stating that $f_{n\boldsymbol{r}}$ is well-approximated by χ (Equation (2.7)), meaning $|f_{n\boldsymbol{r}} - \chi|$ grows at an exponentially slower rate than $f_{n\boldsymbol{r}}$. Note that the non-explicit version of this result is stated as Lemma 5.1 in [90].

In order to do this, it is helpful to first prove lemmas constructing our analytic parameterization g, and showing that it satisfies properties (C1)-(C3).

Before going on, we generalize our notation for polydisks and polytori to accommodate center-points different than the origin. For $\boldsymbol{c} = (c_1, ..., c_d) \in \mathbb{C}^d$ and $\boldsymbol{r} = (r_1, ..., r_d) \in \mathbb{R}^d_{>0}$, let

$$D_{\boldsymbol{c}}(\boldsymbol{r}) := \{\boldsymbol{z} \in \mathbb{C}^d : |z_j - c_j| \leq r_j, 1 \leq j \leq d\}$$
$$\text{and} \quad T_{\boldsymbol{c}}(\boldsymbol{r}) := \{\boldsymbol{z} \in \mathbb{C}^d : |z_j - c_j| = r_j, 1 \leq j \leq d\}.$$

When a radius is of the form $r\mathbf{1} = (r, r, ..., r)$ for some $r > 0$, we shall abuse notation and often write $T_{\boldsymbol{c}}(r)$ instead of $T_{\boldsymbol{c}}(r\mathbf{1})$ and $D_{\boldsymbol{c}}(r)$ instead of $D_{\boldsymbol{c}}(r\mathbf{1})$.

Lemma 4 (Constructing an analytic parameterization)**.** *Let $F(\boldsymbol{z}) = G(\boldsymbol{z})/H(\boldsymbol{z})$ be rational with $H(\mathbf{0}) \neq 0$ and $\gcd(G, H) = 1$. Further suppose that in the direction $\boldsymbol{r}$, F admits a strictly minimal smooth contributing point $\boldsymbol{w}_0$ with $H_{z_d}(\boldsymbol{w}_0) \neq 0$. Then, we can construct $\delta' > 0$ and a function $g : \overline{D}_{\widehat{\boldsymbol{w}}_0}(\delta') \to \mathbb{C}$ which is the unique analytic parameterization for z_d on $\overline{D}_{\widehat{\boldsymbol{w}}_0}(\delta')$ with $g(\widehat{\boldsymbol{w}}_0) = \boldsymbol{w}_{0,d}$, hence on the closure of*

$$\mathcal{N} = \left\{\widehat{\boldsymbol{w}}_0 e^{i\boldsymbol{\theta}} : \boldsymbol{\theta} \in (-\delta', \delta')^{d-1}\right\}.$$

Proof. First, make the change of variables $\boldsymbol{u} = \widehat{\boldsymbol{z}} - \widehat{\boldsymbol{w}}_0$, $v = z_d - \boldsymbol{w}_{0,d}$ so that the polynomial $H(\boldsymbol{u}, v)$ is zero at $(\boldsymbol{u}, v) = (\mathbf{0}, 0)$. Write

$$H(\boldsymbol{u}, v) = av + \sum_{(\boldsymbol{\alpha},k)\neq(\mathbf{0},1)} c_{\boldsymbol{\alpha},k}\boldsymbol{u}^{\boldsymbol{\alpha}}v^k,$$

noting that $a \neq 0$ since $H_{z_d}(\boldsymbol{w}_0) \neq 0$. Next, for $r, s > 0$ let

$$M(r, s) = \sum_{(\boldsymbol{\alpha},k)\neq(\mathbf{0},1)} |c_{\boldsymbol{\alpha},k}|\ r^{|\boldsymbol{\alpha}|}s^k.$$

Observe that if

$$M(r, s) < |a|s, \tag{$*$}$$

Rouché's theorem [2, Chapter 4, Section 5.2] implies a unique parameterization for v as a function of $\boldsymbol{u}$ on $|v| \leq s$ for all $|\boldsymbol{u}| \leq r$. Moreover, if r is small enough condition $(*)$ will hold. Thus it remains to find an $r, s > 0$ witnessing $(*)$. To do this, first split $M(r, s)$ into $M(r, s) = Y(s) + X_s(r)$, with

$$Y(s) = \sum_{k\geq 2} |c_{\boldsymbol{\alpha},k}|s^k \quad \text{and} \quad X_s(r) = \sum_{|\boldsymbol{\alpha}|\geq 1, k\geq 0} |c_{\boldsymbol{\alpha},k}|r^{|\boldsymbol{\alpha}|}s^k.$$

Next, fix $\epsilon \in (0, 1)$ and pick[1] an $s > 0$ such that $Y(s) < (1 - \epsilon)/2 \cdot |a|s$, then choose an $r > 0$ such that $X_s(r) < \frac{(1-\epsilon)}{2}|a|s$. This gives us a unique parameterization.

[1] You can compute such an s, for instance, by finding the smallest root of $(1-\epsilon)/2 \cdot |a|s - Y(s)$ (necessarily positive real, by Vivanti-Pringsheim), and then taking something slightly smaller.

Note that this zero is simple, hence the implicit function theorem – (see, for instance, [90, Proposition 3.1]) – gives local analytic branches. Uniqueness causes these to agree on overlaps, giving us a unique analytic g on our polydisk. □

Remark 3. *It is worth emphasizing that nothing about this proof explicitly required us to restrict to points in $\mathcal{N}$ itself. We actually have a complex neighborhood of $\widehat{\boldsymbol{w}}_0$ on which the statement holds. This becomes important, for instance, when using Taylor expansions to compute uniform error bounds for $g_{\boldsymbol{w}_0}$ on $\mathcal{N}$.*

Lemma 5 (Constructing δ certifying (C3))**.** *Let δ', $\mathcal{N}$, g be the constants constructed in Lemma 4. Then we can restrict δ' to some δ for which the following three properties hold:*

$$\forall \widehat{\boldsymbol{z}} \in \mathcal{N}, \quad (H(\boldsymbol{z}) = 0 \iff z_d = g(\widehat{\boldsymbol{z}})) \tag{C1}$$

$$\forall \widehat{\boldsymbol{z}} \in \mathcal{N}, \quad (\rho - \delta < |g(\widehat{\boldsymbol{z}})| < \rho + \delta) \tag{C2}$$

$$\forall \widehat{\boldsymbol{z}} \in \mathcal{T}, \tag{3.1}$$

$$[(\ \rho - \delta \leq |z_d| \leq \rho + \delta\ ) \ \wedge\ (\ H(\widehat{\boldsymbol{z}}, z_d) = 0\ ) \implies (\ \widehat{\boldsymbol{z}} \in \mathcal{N} \wedge z_d = g(\widehat{\boldsymbol{z}})\ )]\,. \tag{C3}$$

Proof. The standard statement of the implicit function theorem – (see, for example [90, Proposition 3.1]) – plus our proof of Lemma 4 gives us (C1). As for (C2), one can take the angular radius δ' gotten by Lemma 4 and restrict it, if necessary, to bound[2] $|g(\widehat{\boldsymbol{z}})|$ on $\mathcal{N}$ using continuity of g. Furthermore, when $\widehat{\boldsymbol{z}} \in \mathcal{N}$ we know that (C3) holds with our newfound δ. Thus it suffices to prove (C3) for the case when $\widehat{\boldsymbol{z}} \in \mathcal{N}' = \mathcal{T} \setminus \mathcal{N}$.

By minimality of $\boldsymbol{w}_0$ and $\mathcal{N}'$ being a subset of a torus in the first $d-1$ coordinates, we see that $H(\widehat{\boldsymbol{z}}, z_d) \neq 0$ when $\widehat{\boldsymbol{z}} \in \mathcal{N}'$ and $|z_d| \in [\rho - \delta', \rho)$. It remains to construct $\delta < \delta'$ such that there are no zeros on $|z_d| \in [\rho, \rho + \delta)$.

But by strict minimality of the point $\boldsymbol{w}_0$, we know that for $\widehat{\boldsymbol{z}} \in \mathcal{N}'$ we have $H(\widehat{\boldsymbol{z}}, z_d) \neq 0$ when $|z_d| = \rho$. Thus, since H is continuous and the circle $|z_d| = \rho$ compact,

$$m^{\text{out}} = \min_{\widehat{\boldsymbol{z}} \in \mathcal{N}', |z_d| = \rho} |H(\widehat{\boldsymbol{z}}, z_d)|$$

exists and is positive. Similarly, we note that $H_{z_d}(\widehat{\boldsymbol{z}}, z_d)$ is continuous on the annulus $|z_d| \in [\rho, \rho + \delta']$. Thus, again by smoothness and compactness,

$$M^{\text{out}} = \max_{\widehat{\boldsymbol{z}} \in \mathcal{N}', |z_d| \in [\rho, \rho + \delta']} |H_{z_d}(\widehat{\boldsymbol{z}}, z_d)|$$

[2]Computationally, this can be done with interval arithmetic in Sage: simply subdivide $\mathcal{N}$ into angular intervals, then on each interval evaluate $|g|$ using the `ComplexIntervalField` (`CIF`) and/or `RealIntervalField` (`RIF`) classes. The result will be an interval containing the true upper and lower bounds of the magnitude on that patch of the domain, and maximizing the upper bounds across all interval evaluations will, therefore, bound $|g|$.

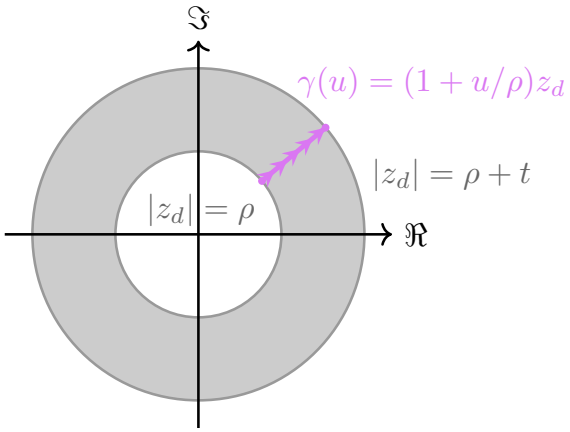


Figure 3.1: Radial parameterization used to isolate pole in $\mathcal{N}'$, property (C3).

is finite. M^{out} is also positive, since if it were zero that would imply the polynomial H_{z_d} is identically zero on an annulus, meaning it's the zero polynomial. Let $\rho = |\boldsymbol{w}_{0,d}|$ and

$$T = \min\left\{\delta, \frac{m^{\text{out}}}{2M^{\text{out}}}\right\},$$

and fix $\widehat{\boldsymbol{z}} \in \mathcal{N}'$ and set $f_t(z_d) = H\left(\widehat{\boldsymbol{z}}, \left(1 + \frac{t}{\rho}\right) z_d\right)$ for fixed $t \in [0, T]$ and $|z_d| = \rho$.

Then the segment γ from z_d to $(1 + t/\rho)z_d$ (see Figure 3.1) is contained in the annulus defined by the interval $[\rho, \rho + t] \subset [\rho, \rho + \delta]$, so applying the fundamental theorem of line integrals to γ

$$f_t(z_d) - f_0(z_d) = \int_\gamma \frac{\partial}{\partial u} f_u(z_d) du$$

and bounding this integral gives

$$|f_t(z_d) - f_0(z_d)| \leq M^{\text{out}} t \leq \frac{m^{\text{out}}}{2} < |f_0(z_d)| \quad \text{on } |z_d| = \rho.$$

Applying Rouché's theorem to this inequality tells us that f_t and f_0 have the same number of zeros in this disk $|z_d| < \rho$. This in turn implies that $H(\widehat{\boldsymbol{z}}, z_d)$ has the same number of zeros in $|z_d| \leq \rho$ and $|z_d| < \rho + t$, and since $H(\widehat{\boldsymbol{z}}, z_d)$ has no zeros with $|z_d| \leq \rho$ (by strict minimality of $\boldsymbol{w}_0$ and the fact that $\widehat{\boldsymbol{z}} \notin \mathcal{N}$), letting $t \to T$ we find that restricting δ so that $\delta < (1 - \epsilon)\frac{m^{\text{out}}}{2M^{\text{out}}}$ for some $\epsilon \in (0, 1)$ gives us (C3) over all of $\mathcal{N}'$.

□

With these two lemmas in hand, we are ready to show that χ is a good approximation of $f_{n\boldsymbol{r}}$.

Proposition 2 (Localized integrals well-approximate our coefficients, strictly minimal version)**.** *Let F, $\boldsymbol{r}$ be as above, and suppose $\boldsymbol{w}_0$ is strictly minimal smooth contributing. Further, let χ be the integral defined in Chapter 2. Then, for all n, $|f_{n\boldsymbol{r}} - \chi| < c\tau^n$ for some $c > 0$ and $0 < \tau < |\boldsymbol{w}_0|^{-\boldsymbol{r}}$ which we construct.*

Proof. Our constructive proof of the existence of τ and c follows in five steps:

1. First, use Lemmas 4 and 5 to construct $\delta', \delta > 0$ and $g : \mathcal{N} \to \mathbb{C}$ such that properties (C1)-(C3) hold. Then, the following integrals converge:

$$\begin{aligned} f_{n\boldsymbol{r}} = I &= \frac{1}{(2\pi\mathrm{i})^d} \int_{\mathcal{T}} \left(\int_{|z_d|=\rho-\delta} F(\boldsymbol{z}) \frac{dz_d}{z_d^{nr_d+1}} \right) \frac{d\widehat{\boldsymbol{z}}}{\widehat{\boldsymbol{z}}^{n\widehat{\boldsymbol{r}}+\widehat{\mathbf{1}}}} \\ I^{\mathrm{loc}} &= \frac{1}{(2\pi\mathrm{i})^d} \int_{\mathcal{N}} \left(\int_{|z_d|=\rho-\delta} F(\boldsymbol{z}) \frac{dz_d}{z_d^{nr_d+1}} \right) \frac{d\widehat{\boldsymbol{z}}}{\widehat{\boldsymbol{z}}^{n\widehat{\boldsymbol{r}}+\widehat{\mathbf{1}}}} \\ I^{\mathrm{out}} &= \frac{1}{(2\pi\mathrm{i})^d} \int_{\mathcal{N}} \left(\int_{|z_d|=\rho+\delta} F(\boldsymbol{z}) \frac{dz_d}{z_d^{nr_d+1}} \right) \frac{d\widehat{\boldsymbol{z}}}{\widehat{\boldsymbol{z}}^{n\widehat{\boldsymbol{r}}+\widehat{\mathbf{1}}}}. \end{aligned}$$

2. Moreover, by the proof of Lemma 5, we know that for any $\widehat{\boldsymbol{z}} \in \mathcal{N}'$ and $|z_d| \in [\rho - \delta, \rho + \delta]$, we have $H(\widehat{\boldsymbol{z}}, z_d) \neq 0$. Equivalently, for all $\widehat{\boldsymbol{z}}$ in $\mathcal{N}' := \mathcal{T} \setminus \mathcal{N}$, the univariate series in z_d for $F(\widehat{\boldsymbol{z}}, z_d)$ is analytic when $|z_d| \in [\rho - \delta, \rho + \delta]$.

3. Thus, for any $\widehat{\boldsymbol{z}} \in \mathcal{N}'$ we can deform the inner integral of $I - I^{\mathrm{loc}}$ to instead lie on the outer circle of this annulus. In other words, we have

$$\int_{|z_d|=\rho-\delta} F(\boldsymbol{z}) \frac{dz_d}{z_d^{nr_d+1}} = \int_{|z_d|=\rho+\delta} F(\boldsymbol{z}) \frac{dz_d}{z_d^{nr_d+1}}.$$

   Applying the maximum modulus integral bound to the above, we have

$$\left| \int_{|z_d|=\rho-\delta} F(\boldsymbol{z}) \frac{dz_d}{z_d^{nr_d+1}} \right| \leq \max_{|z_d|=\rho+\delta} |F(\widehat{\boldsymbol{z}}, z_d)| \frac{2\pi}{(\rho+\delta)^{nr_d}},$$

and applying the same estimate to the difference $|I - I^{\mathrm{loc}}|$ gives:

$$\begin{aligned}
|I - I^{\mathrm{loc}}| &= \frac{1}{(2\pi)^d} \left| \int_{\mathcal{N}'} \left(\int_{|z_d|=\rho-\delta} F(\boldsymbol{z}) \frac{dz_d}{z_d^{r_d n+1}} \right) \frac{d\widehat{\boldsymbol{z}}}{\widehat{\boldsymbol{z}}_j^{n\widehat{\boldsymbol{r}}+\widehat{\mathbf{1}}}} \right| \\
&\leq \frac{1}{(2\pi)^d} \max_{\widehat{\boldsymbol{z}} \in \mathcal{N}'} \left| \int_{|z_d|=\rho-\delta} F(\boldsymbol{z}) \frac{dz_d}{z_d^{r_d n+1}} \right| \frac{\mathrm{area}(\mathcal{N}')}{|\widehat{\boldsymbol{w}}_0|^{n\widehat{\boldsymbol{r}}+\widehat{\mathbf{1}}}} \\
&\leq (1 - (\delta/\pi)^{d-1}) \max_{\widehat{\boldsymbol{z}} \in \mathcal{N}'} \max_{|z_d|=\rho+\delta} |F(\widehat{\boldsymbol{z}}, z_d)| (|\widehat{\boldsymbol{w}}_0|^{\widehat{\boldsymbol{r}}} (\rho+\delta)^{r_d})^{-n}.
\end{aligned}$$

4. Similarly, the max modulus bound shows that

$$|I^{\mathrm{out}}| \leq (\delta/\pi)^{d-1} \max_{\widehat{\boldsymbol{z}} \in \mathcal{N}} \max_{|z_d|=\rho+\delta} |F(\widehat{\boldsymbol{z}}, z_d)| (|\widehat{\boldsymbol{w}_0}|^{\widehat{\boldsymbol{r}}} (\rho+\delta)^{r_d})^{-n}.$$

5. Combining Steps 3 and 4 with the triangle inequality, we have for all n that:

$$\begin{aligned}
|f_{n\boldsymbol{r}} - \chi| &= |I - (I^{\mathrm{loc}} - I^{\mathrm{out}})| \\
&\leq |I - I^{\mathrm{loc}}| + |I^{\mathrm{out}}| \\
&\leq \max_{\widehat{\boldsymbol{z}} \in \mathcal{T}} \max_{|z_d|=\rho+\delta} |F(\widehat{\boldsymbol{z}}, z_d)| \cdot \frac{1}{(|\widehat{\boldsymbol{w}_0}|^{\widehat{\boldsymbol{r}}} (\rho+\delta)^{r_d})^n}.
\end{aligned}$$

Taking

$$c = \max_{\widehat{\boldsymbol{z}} \in \mathcal{T}} \max_{|z_d|=\rho+\delta} |F(\widehat{\boldsymbol{z}}, z_d)|, \quad \tau = \frac{1}{|\widehat{\boldsymbol{w}_0}|^{\widehat{\boldsymbol{r}}} (\rho+\delta)^{r_d}},$$

we see that $c, \tau > 0$ satisfy the conclusions of the theorem. In a similar manner as to the above, a rigorous bound for c can be derived via interval arithmetic involving Sage's `RIF` and `CIF` classes.

□

Note that the argument above extends to the *finitely minimal* case as well (when E is an arbitrary finite set); one need only localize around each point $\boldsymbol{w} \in E$ and construct $(\delta'_{\boldsymbol{w}}, \delta_{\boldsymbol{w}})$, then take

$$\mathcal{N} = \bigcup \mathcal{N}_{\boldsymbol{w}}$$

and, ensuring our $\delta'_{\boldsymbol{w}}$ are small enough to make all $\mathcal{N}_{\boldsymbol{w}}$ disjoint, take

$$\delta' < \min_E \delta'_{\boldsymbol{w}} \quad \text{and} \quad \delta < \min_E \delta_{\boldsymbol{w}}.$$

With a bit more bookkeeping, one can even consider the case where each point $\boldsymbol{w}$ of E has associated to it a (possibly different) index $j = j(\boldsymbol{w})$ so that $H_{z_j}(\boldsymbol{w}) \neq 0$.

Proposition 3 (Localized integrals well-approximate our coefficients, finitely minimal version)**.** *Let F, $\boldsymbol{r}$, E, and χ be as above, this time with E finite, with each $\boldsymbol{w}$ of E having an index $j(\boldsymbol{w})$ so that $H_{z_j}(\boldsymbol{w}) \neq 0$. Then for all n, $|f_{n\boldsymbol{r}} - \chi| < c\tau^n$ for some $c > 0$ and $0 < \tau < |\boldsymbol{w}|^{-\boldsymbol{r}}$ which one can construct.*

Once we've computed the set E of minimal contributing points and verified it is finite, and have in hand our δ, c and τ from the results of Propositions 2 or 3, we can take the integral[3] χ and evaluate the residue in the inner integral, reducing dimension. Making the change of variables $\widehat{\boldsymbol{z}} \mapsto \widehat{\boldsymbol{w}}_0 e^{i\boldsymbol{\theta}}$ puts χ in saddle-point form:

$$\chi = \frac{\boldsymbol{w_0}^{-n\boldsymbol{r}}}{(2\pi)^{d-1}} \int_{\widetilde{\mathcal{N}}} A(\boldsymbol{\theta}) e^{-nr_d \phi(\boldsymbol{\theta})} d\boldsymbol{\theta},$$

with A_n, ϕ, and $\widetilde{\mathcal{N}}$ defined as in Equations (2.8) and (2.10). This is a saddle-point integral; such integrals are well-studied (E.g. [104, 120, 128]). Our next set of results give a constructive formulation of the estimate for the saddle-point integral $\chi = \chi(n)$ as it is presented in ACSV. This can be viewed as a multivariate analogue of the process we carried out in Example 2, with all Big-O terms replaced by explicit error bounds.

3.2 Estimating Saddle-Point Integrals

Before stating our results, it is worth emphasizing just how much scrutiny multivariate saddle-point estimates have been given over the years; hence it is quite surprising (and unfortunate for us) that no suitable results in the literature were found which could be used to estimate χ directly.

Many papers, like [89, 72, 42, 57, 88, 132, 74, 129] focus on the real case; several of them even give explicit bounds for the error terms. A lesser number [52, 30, 104] focus on the complex case. These latter papers are all fully general, describing their asymptotics in terms of complex algebraic geometry. But none of them readily yield explicit bounds amenable to our situation. The best match [30] does allege a formula for error terms for the case of polynomial magnitude and phase, but this formula (Section 5, equations (68)–(72)) depends on the analytic continuation of the Borel transform and intersection numbers of Lefschetz thimbles, making explicit computation difficult. Moreover, the *global* nature of the analysis – using singularities away from the minimal ones to compute our remainder error – seems to clash with the *local* spirit of the saddle-point Method.

[3] Or sum of integrals, in the case where $|E| > 1$.

For these reasons, we have decided to re-derive the bounds from scratch. In an upcoming work, we hope to generalize and preserve these bounds for posterity, so that combinatorialists or numerical analysts looking for explicit error bounds for saddle-point integrals with complex amplitude and phase can find a reference, stated in plain language.

Proposition 4 (Constructive saddle-point integral estimate, strictly minimal case)**.** *Let F and r $\boldsymbol{w}_0, g, \mathcal{N}, \delta'$ be as above, with $\boldsymbol{w}_0$ also* nondegenerate*. Also assume (without loss of generality) that $\boldsymbol{r}$ has been normalized so that $r_d = 1$. Suppose we have the saddle-point integral*

$$I(n) = \int_{\widetilde{\mathcal{N}}} A(\boldsymbol{\theta}) e^{-n\phi(\boldsymbol{\theta})} d\boldsymbol{\theta}$$

where

$$A(\boldsymbol{\theta}) = A_n(\boldsymbol{\theta}) = \frac{(-1)^p}{g\left(\widehat{\boldsymbol{w}} e^{i\boldsymbol{\theta}}\right)^p} \Gamma_n\left(\widehat{\boldsymbol{w}} e^{\mathrm{i}\boldsymbol{\theta}}\right),$$

$\Gamma_n(\boldsymbol{z})$ the residue at $g(\widehat{\boldsymbol{z}})$ of our Cauchy integrand,[4] *and*

$$\phi(\boldsymbol{\theta}) = \log \frac{g\left(\widehat{\boldsymbol{w}}_0 e^{(\boldsymbol{\theta})}\right)}{g(\widehat{\boldsymbol{w}}_0)} + i(\widehat{\boldsymbol{r}} \cdot \boldsymbol{\theta})$$

as seen in Chapter 2. Further, letting $\mathcal{H}$ is the matrix in from Definition 7 – the Hessian of ϕ at $\mathbf{0}$ – suppose that $\lambda_{\min}(\Re(\mathcal{H}))$, the eigenvalue of $\Re(\mathcal{H})$ of minimal magnitude, is positive real. Then, for any integer $M \geq 0$ the integral I has an explicit asymptotic expansion

$$S_M(n) = n^{p-1-(d-1)/2} \frac{(2\pi)^{(d-1)/2}}{\det(\mathcal{H})^{1/2}} \sum_{\ell=0}^{M-1} C_\ell n^{-\ell},$$

where p is the order of the pole of F/z_d^{n+1} at $g(\widehat{\boldsymbol{z}})$ and the C_ℓ are explicit constants depending on a certain polynomial and the entries of $\mathcal{H}$. Furthermore, there is an explicit error $E(n) = E_M(n)$ so that

$$|I(n) - S_M(n)| \leq E(n)$$

with $E(n) = O\left(n^{p-1-(d-1)/2-M}\right)$.

Again, a similar result holds in the finitely minimal case. We won't prove it here.

Proposition 5 (Constructive saddle-point integral estimate, finitely minimal case)**.** *If instead $\chi = \sum_{\boldsymbol{w} \in E} \chi_{\boldsymbol{w}}$ is a sum of saddle-point integrals over our set of (finitely) minimal smooth contributing points, and all points in E are nondegenerate with corresponding matrices $\mathcal{H}_{\boldsymbol{w}}$ having positive real minimal eigenvalues, then one can obtain a result similar to the one in Proposition 4 for any $M \in \mathbb{N}$ by:*

[4] $\Gamma_n(\boldsymbol{z})$ is also a polynomial of degree $p-1$ in n.

- *Forming, for each* $\boldsymbol{w} \in E$*, the analogous integral* $I_{\boldsymbol{w}}(n)$ *to the integral* $I(n)$ *appearing in the statement of Proposition 4.*
- *Computing, for all* $\boldsymbol{w}$ *the expansion* $S_M^{\boldsymbol{w}}(n)$ *about the point* $\boldsymbol{w}$ *and the error* $E_{\boldsymbol{w}}(n)$ *by following along in the proof of Proposition 4.*
- *Letting*

$$I(n) = \sum_{\boldsymbol{w} \in E} I_{\boldsymbol{w}}(n),$$
$$S_M(n) = \sum_{\boldsymbol{w} \in E} S_M^{\boldsymbol{w}}(n),$$
$$E(n) = \sum_{\boldsymbol{w} \in E} E_{\boldsymbol{w}}(n),$$

and noting that

$$|I(n) - S_M(n)| \leq E(n)$$

with $E(n) = O(n^{p-1-(d-1)/2-M})$.

While the proof of Proposition 4 is based on a multivariate generalization of the saddle-point method, as seen in Example 2 (univariate) or [90, Propositions 5.3] (multivariate), the proof has undergone significant changes in order to produce *good* bounds on the relative error between $I(n)$ and $S_M(n)$. The usual proof of this general estimate (see, e.g. [108, Theorem 5.2]) uses the so-called Complex Morse Lemma [108, Lemma 5.9] to derive Big-O bounds. This doesn't work for finding explicit error bounds, as during the course of the proof $\widetilde{\mathcal{N}}$ is deformed to a complex chain and we would need to explicitly bound the error caused by deforming it back into $\mathbb{R}^{d-1}$, which is hard.

Instead, we introduce a certain decomposition of the integral I and then carefully bound the errors introduced by Taylor approximations. For the expansion S_M, we reduce evaluations of each integral appearing within to the Isserlis-Wick Theorem [99, Theorem 1] for monomial moments of Gaussians. For the error, we make use of the following lemma, which expresses certain Gaussian-type integrals in terms of incomplete Gamma functions [1, 8.35].

Lemma 6. *Let* $a, b > 0$ *and* q, m *be positive integers. Then, letting* $\boldsymbol{x}$ *be a vector of* q *real variables, we have*

$$\int_{\mathbb{R}^q} \|\boldsymbol{x}\|^m e^{-a\|\boldsymbol{x}\|^2} d\boldsymbol{x} = \frac{\pi^{q/2}}{\Gamma\left(\frac{q}{2}\right)} a^{-\left(\frac{q+m}{2}\right)} \Gamma\left(\frac{q+m}{2}\right)$$

Moreover, we can evaluate truncated or tail versions of this integral, in terms of the lower *and* upper incomplete Gamma functions $\gamma(\cdot,\cdot)$ *and* $\Gamma(\cdot,\cdot)$*, respectively:*

$$\int_{\|\boldsymbol{x}\|\leq b} \|\boldsymbol{x}\|^m e^{-a\|\boldsymbol{x}\|^2} d\boldsymbol{x} = \frac{\pi^{q/2}}{\Gamma\left(\frac{q}{2}\right)} a^{-\left(\frac{q+m}{2}\right)} \gamma\left(\frac{q+m}{2}, ab^2\right)$$
$$\int_{\|\boldsymbol{x}\|\geq b} \|\boldsymbol{x}\|^m e^{-a\|\boldsymbol{x}\|^2} d\boldsymbol{x} = \frac{\pi^{q/2}}{\Gamma\left(\frac{q}{2}\right)} a^{-\left(\frac{q+m}{2}\right)} \Gamma\left(\frac{q+m}{2}, ab^2\right).$$

Proof. It is clear that all 3 integrals exist. Let's evaluate the tail variant,

$$\int_{\|\boldsymbol{x}\|\geq b} \|\boldsymbol{x}\|^m e^{-a\|\boldsymbol{x}\|^2} dx.$$

The idea is to transform to (generalized) spherical coordinates [18], where our volume element $d\boldsymbol{x}$ factors into a radial part and angular part

$$d\boldsymbol{x} = r^{q-1} dr d\Omega_n,$$

with $r = \|\boldsymbol{x}\|$ the 2-norm and $d\Omega_n$ the surface area element of the unit $(q-1)$-sphere. Under this change of variables, the integral also factors into an angular and radial component:

$$\int_{\{\|\boldsymbol{x}\|\geq b\}} \|\boldsymbol{x}\|^m e^{-a\|\boldsymbol{x}\|^2} d\boldsymbol{x} = \left(\int_{S^{q-1}} d\Omega_n\right) \int_b^\infty r^m e^{-ar^2} r^{q-1} dr.$$

The angular component represents the surface area of the unit $(q-1)$-sphere, and evaluates to

$$\int_{S^{q-1}} d\Omega_n = \frac{2\pi^{q/2}}{\Gamma\left(\frac{q}{2}\right)}.$$

For the radial component we can make the substitution $u = ar^2$ to reduce the exponent to a standard linear form. Doing so,

$$\begin{aligned}\int_b^\infty r^{q+m-1} e^{-ar^2} dr &= \int_{ab^2}^\infty \left(\frac{u}{a}\right)^{\frac{q+m-1}{2}} e^{-u} \frac{1}{2a} \left(\frac{a}{u}\right)^{1/2} du \\ &= \frac{1}{2} a^{-\left(\frac{q+m}{2}\right)} \int_{ab^2}^\infty u^{\frac{q+m}{2}-1} e^{-u} du,\end{aligned}$$

and this last integral can be recognized as a value of the upper incomplete Gamma function [1, 8.35], namely $\Gamma\left(\frac{q+m}{2}, ab^2\right)$, giving the result. The other two integrals are similar, except that after letting $u = ar^2$ in the radial part and transforming, we end up with a lower incomplete Gamma value for the truncated case and a regular Gamma function for the $\mathbb{R}^q$ case. □

We'll also need some facts about ϕ and its Hessian at $\mathbf{0}$, denoted $\mathcal{H}$, in order to make sure we satisfy the needed hypotheses to evaluate our saddle-point integral.

Lemma 7. *With ϕ and $\mathcal{H}$ as in Proposition 4 we have*

- $\phi(\mathbf{0}) = 0$,
- $\nabla\phi(\mathbf{0}) = \mathbf{0}$,
- $\Re(\phi) \geq 0$ *on* $\widetilde{\mathcal{N}}$ *with* $\Re(\phi)(\boldsymbol{\theta}) = 0$ *only at* $\boldsymbol{\theta} = \mathbf{0}$, *and*
- *The matrix* $\Re(\mathcal{H}) = \frac{\mathcal{H}+\mathcal{H}^H}{2}$ *is symmetric positive definite.*

Proof. The first two claims follow from a direct calculation involving ϕ. The third follows from the fact that, by strict minimality of $\boldsymbol{w}_0$ (or finite, assuming the neighborhood $\mathcal{N}_{\boldsymbol{w}_0}$ is small enough to be disjoint with all other such neighborhoods), the analytic function g parameterizing z_d on $\mathcal{V} \cap \mathcal{N}$ has $|g|$ achieve a strict minimum on $\mathcal{N}$ at $\mathbf{0}$. As for the fourth, we know that $\mathcal{H}$ is symmetric; by direct calculation one can see that $\Re(\mathcal{H}) = \frac{\mathcal{H}+\mathcal{H}^H}{2}$ is real and symmetric. $\Re(\phi)$ having a strict minimum at $\mathbf{0}$ implies positive *semi*definiteness of $\Re(\mathcal{H})$, and the "manual check" hypothesis that $\lambda_{\min}(\Re(\mathcal{H})) > 0$ strengthens this to positive definiteness. □

With these results in hand, we can prove Proposition 4. The proof proceeds in 4 main steps. Throughout it, we will often suppress the dependence of $A_n(\boldsymbol{\theta})$ on n and simply write $A(\boldsymbol{\theta})$. We will also use the normalized amplitude $\tilde{A}_n(\boldsymbol{\theta}) = n^{-(p-1)}A_n(\boldsymbol{\theta})$, in order to account for the polynomial dependence of A on n, noting that $\tilde{A}$ is $O(1)$ in n.

Step 1: Compute $S_M(n)$

First, let $q = d - 1$ and write $\tilde{A}(\boldsymbol{\theta}) = \sum_{j\geq 0} A_j(\boldsymbol{\theta})$ and (using Lemma 7)

$$\phi(\boldsymbol{\theta}) = \frac{1}{2}\boldsymbol{\theta}^T\mathcal{H}\boldsymbol{\theta} + \underbrace{\sum_{j\geq 3}\phi_j(\boldsymbol{\theta})}_{\psi(\boldsymbol{\theta})},$$

with A_j and ϕ_j homogeneous of degree j in $\boldsymbol{\theta}$. Set $\boldsymbol{\theta} = \boldsymbol{x}/\sqrt{n}$, so that

$$I(n) = n^{p-1-q/2}\int_{\sqrt{n}\widetilde{\mathcal{N}}} \tilde{A}(\boldsymbol{x}/\sqrt{n})e^{-\frac{1}{2}\boldsymbol{x}^T\mathcal{H}\boldsymbol{x}} \cdot e^{-n\psi(\boldsymbol{x}/\sqrt{n})}d\boldsymbol{x}.$$

Set $\varepsilon = n^{-1/2}$ so that the integrand is $e^{-\frac{1}{2}\boldsymbol{x}^T\mathcal{H}\boldsymbol{x}}$ times

$$\begin{aligned}\mathcal{F}(\varepsilon, \boldsymbol{x}) &:= \tilde{A}(\varepsilon\boldsymbol{x})e^{-\varepsilon^{-2}\psi(\varepsilon\boldsymbol{x})} \\ &= \sum_{v\geq 0} \varepsilon^v P_v(\boldsymbol{x}),\end{aligned}$$

with P_v a polynomial that can be computed as

$$[\varepsilon^v]\left(\sum_{j\geq 0}\varepsilon^j A_j(\boldsymbol{x})\right)\exp\left(-\sum_{j\geq 3}\varepsilon^{j-2}\phi_j(\boldsymbol{x})\right)$$

using the homogeneity of A_j and ϕ_j.

We then define

$$S_M(n) = n^{p-1-q/2}\sum_{v=0}^{2M-1} n^{-v/2}\int_{\mathbb{R}^q} P_v(\boldsymbol{x})e^{-\frac{1}{2}\boldsymbol{x}^T\mathcal{H}\boldsymbol{x}}d\boldsymbol{x},$$

noting that the integral in this expression will converge since $\Re(\mathcal{H})$ is positive definite (Lemma 7), and will vanish if v is odd, since it can be shown that $P_v(-\boldsymbol{x}) = (-1)^v P_v(\boldsymbol{x})$ for general v and $\boldsymbol{x}$.

The Isserlis-Wick Theorem [99, Theorem 1], together with the basic properties of expectation, implies that if $v = 2\ell$ is even and

$$P_v(\boldsymbol{x}) = \sum_{\boldsymbol{\alpha}} c_{v,\boldsymbol{\alpha}} x^{\boldsymbol{\alpha}},$$

then

$$\int_{\mathbb{R}^q} P_{2\ell}(x)e^{-\frac{1}{2}\boldsymbol{x}^T\mathcal{H}\boldsymbol{x}}dx = (2\pi)^{q/2}\det(\mathcal{H})^{-1/2}\sum_{|\boldsymbol{\alpha}|\text{ even}} c_{2\ell,\boldsymbol{\alpha}}\sum_{\pi\in\text{Pair}([|\boldsymbol{\alpha}|])}\prod_{\{a,b\}\in\pi}\left(\mathcal{H}^{-1}\right)_{a,b}$$

where $[k] = \{1, ..., k\}$ for any number k and for any finite set I of natural numbers $\text{Pair}(I)$ denotes the perfect pairings of the elements of I – i.e. all partitions of I into pairs.

Thus we have an explicit expansion $S_M(n)$. Now, we decompose the integral difference $I(n) - S_M(n)$ into a sum of integrals and take the magnitude, applying the triangle

inequality to obtain:

$$
\begin{aligned}
|I(n) - S_M(n)| \;\leq\; & \underbrace{n^{p-1}\left|\int_{\widetilde{\mathcal{N}}} \tilde{A}(\boldsymbol{\theta})e^{-n\phi(\boldsymbol{\theta})}d\boldsymbol{\theta} - \int_{D_{\delta''}} \tilde{A}(\boldsymbol{\theta})e^{-n\phi(\boldsymbol{\theta})}d\boldsymbol{\theta}\right|}_{|I_{\text{out}}(n)|} \\
& + \underbrace{n^{p-1-q/2}\left|\int_{\sqrt{n}D_{\delta''}} (\mathcal{F}(\varepsilon,\boldsymbol{x}) - \mathcal{F}_{\leq 2M-1}(\varepsilon,\boldsymbol{x}))e^{-\frac{1}{2}\boldsymbol{x}^T\mathcal{H}\boldsymbol{x}}d\boldsymbol{x}\right|}_{|I_{\text{rem}}(n)|} \\
& + \underbrace{n^{p-1-q/2}\left|\int_{\mathbb{R}^q\setminus\sqrt{n}D_{\delta''}} \mathcal{F}_{\leq 2M-1}(\varepsilon,\boldsymbol{x})e^{-\frac{1}{2}\boldsymbol{x}^T\mathcal{H}\boldsymbol{x}}d\boldsymbol{x}\right|}_{|I_{\text{tail}}(n)|}.
\end{aligned}
$$

Here, $0 < \delta'' < \delta'$ is some constant which we construct below, $D_{\delta''} = [-\delta'', \delta'']^q$, and $\mathcal{F}_{\leq 2M-1}$ is the truncation of $\mathcal{F}(\varepsilon, \boldsymbol{x})$ to its first $2M-1$ terms in $\boldsymbol{x}$. The remaining 3 steps in the proof consist of computing bounds $E_{\text{out}}(n)$, $E_{\text{rem}}(n)$, and $E_{\text{tail}}(n)$ for $|I_{\text{out}}|$, $|I_{\text{rem}}|$, and $|I_{\text{tail}}|$, respectively.

Step 2: Bounding $I_{\text{out}} = n^{p-1}\int_{\widetilde{\mathcal{N}}\setminus D_{\delta''}} \tilde{A}e^{-n\phi}$

Setting

$$u(\boldsymbol{\theta}) = \log\left|\frac{g\left(\widehat{\boldsymbol{w}}_0 e^{i\boldsymbol{\theta}}\right)}{g(\widehat{\boldsymbol{w}}_0)}\right|$$

we can (using the fact that $\Re(\phi) > 0$ on $\widetilde{\mathcal{N}} \setminus \{\mathbf{0}\}$ and $\tilde{A}$ analytic on $\widetilde{\mathcal{N}}$) compute[5] an $\eta_{\delta''} > 0$ and a $m^{\text{out}} > 0$ such that

$$
\begin{aligned}
\eta_{\delta''} &\leq \inf_{\boldsymbol{\theta}\in\widetilde{\mathcal{N}}\setminus D_{\delta''}} u(\boldsymbol{\theta}) \\
m^{\text{out}} &\geq \sup_{\boldsymbol{\theta}\in\widetilde{\mathcal{N}}\setminus D_{\delta''}} |\tilde{A}(\boldsymbol{\theta})|.
\end{aligned}
$$

Then, by the maximum modulus integral bound, $|I_{\text{out}}| \leq E_{\text{out}}(n)$ with

$$E_{\text{out}} := n^{p-1}m^{\text{out}}[(2\delta')^q - (2\delta'')^q]e^{-n\eta_{\delta''}}.$$

Step 3: Bounding $I_{\text{rem}} = n^{p-1-q/2}\int_{\sqrt{n}D_{\delta''}} (\mathcal{F}(\varepsilon,\boldsymbol{x}) - \mathcal{F}_{\leq 2M-1}(\varepsilon,\boldsymbol{x}))e^{-\frac{1}{2}\boldsymbol{x}^T\mathcal{H}\boldsymbol{x}}d\boldsymbol{x}$:

[5]Note that even though $\tilde{A}$ depends on n, it depends polynomially on $1/n$, thus there is a uniform bound over all n, hence it makes sense to suppress the n-dependence of the sup in m^{out}.

Recall that $\Re(\mathcal{H})$ is positive definite and let $\lambda > 0$ denote its minimal eigenvalue. Pick $\rho > \sqrt{q}\delta''$ such that $\tilde{A}$ and ϕ are analytic on the closed ball $B_\rho = \{\boldsymbol{\theta} : \|\boldsymbol{\theta}\| \leq \rho\}$. Also compute uniform bounds[6]

$$\begin{aligned} m_{\tilde{A}}(\rho) &\geq |\tilde{A}(\boldsymbol{\theta})|, && \text{and} \\ m_\psi(\rho) \text{ with } |\psi(\boldsymbol{\theta})| &\leq m_\psi(\boldsymbol{\theta})\|\boldsymbol{\theta}\|^3 && \text{for } \boldsymbol{\theta} \in B_\rho. \end{aligned}$$

Define $\mu = \lambda/2 - m_\psi(\rho)\rho$. We need $\mu > 0$. If this is not the case, we can make it so by reducing ρ (and possibly δ''), because $m_\psi(\rho)\rho \to 0$ as $\rho \to 0$.

Note that $\|\boldsymbol{x}/\sqrt{n}\| \leq \sqrt{q}\delta'' < \rho$ for any[7] $\boldsymbol{x} \in \sqrt{n}D_{\delta''}$ and hence, equivalently, $|\varepsilon| < \rho/\|\boldsymbol{x}\|$. Because of this constraint on $|\varepsilon|$,

$$|\tilde{A}(\varepsilon\boldsymbol{x})| \leq m_{\tilde{A}}(\rho)$$

and so

$$\begin{aligned} & |\psi(\varepsilon\boldsymbol{x})| \leq m_\psi(\rho)\|\varepsilon\boldsymbol{x}\|^3 \leq m_\psi(\rho)\rho^3 \\ \implies & \left|\varepsilon^{-2}\psi(\varepsilon\boldsymbol{x})\right| \leqslant m_\psi(\rho)\rho\|\boldsymbol{x}\|^2 \\ \implies & \left|e^{-\varepsilon^{-2}\psi(\varepsilon\boldsymbol{x})}\right| \leqslant e^{m_\psi(\rho)\rho\|\boldsymbol{x}\|^2}. \end{aligned}$$

Thus for such ε and $\boldsymbol{x}$,

$$|\mathcal{F}(\varepsilon, \boldsymbol{x})| \leq \underbrace{m_{\tilde{A}}(\rho)e^{m_\psi(\rho)\rho\|\boldsymbol{x}\|^2}}_{m_x},$$

so applying the Cauchy bound to $\mathcal{F}(\varepsilon, \boldsymbol{x})$ in ε yields

$$|P_v(\boldsymbol{x})| \leq m_{\boldsymbol{x}} \left(\frac{\rho}{\|\boldsymbol{x}\|}\right)^{-v}.$$

From this, we bound the tail of $\mathcal{F}(\varepsilon, \boldsymbol{x})$ as

$$\begin{aligned} |\mathcal{F}(\varepsilon, \boldsymbol{x}) - \mathcal{F}_{\leq 2m-1}(\varepsilon, \boldsymbol{x})| &\leq \sum_{v=2M}^{\infty} |P_V(\boldsymbol{x})|\,\varepsilon^v \\ &\leq m_{\boldsymbol{x}} \sum_{v=2M}^{\infty} \left(\frac{\varepsilon}{\rho/\|\boldsymbol{x}\|\ |}\right)^v \\ &= m_{\boldsymbol{x}} \frac{(\|\boldsymbol{x}\|\varepsilon/\rho)^{2M}}{1 - \|\boldsymbol{x}\|\varepsilon/\rho}. \end{aligned}$$

[6] Taylor's Theorem with remainder implies that a valid choice for $m_\psi(\rho)$ is $\sum_{|\boldsymbol{\alpha}|=3} \frac{1}{\boldsymbol{\alpha}!} \sup_{\boldsymbol{\theta} \in B_\rho} |\partial^{\boldsymbol{\alpha}}\phi(\boldsymbol{\theta})|$, with $\boldsymbol{\alpha}! = \alpha_1! \cdots \alpha_q!$. And again, $\tilde{A}$'s polynomial dependence in $1/n$ means we can collect in powers of $1/n$, bound each coefficient, then sum the results to get an n-uniform bound for $\tilde{A}$.

[7] In what follows, we may assume $\boldsymbol{x} \neq 0$ as $\mathcal{F}(\varepsilon, 0) = \mathcal{F}_{\leq j}(\varepsilon, 0)$ for all j.

Let $\sigma = \frac{\sqrt{q}\delta''}{\rho} < 1$. Since $\boldsymbol{x} \in \sqrt{n}D_{\delta''}$ we have $\|\boldsymbol{x}\| \leq \sqrt{nq}\delta''$ so that

$$\frac{\|\boldsymbol{x}\|\varepsilon}{\rho} = \frac{\|\boldsymbol{x}\|}{\rho\sqrt{n}} \leq \frac{\sqrt{q}\delta''}{\rho} = \sigma < 1,$$

and thus $\frac{1}{1-\frac{\|\boldsymbol{x}\|\varepsilon}{\rho}} \leq \frac{1}{1-\sigma}$. Therefore,

$$\left|\mathcal{F}\left(\varepsilon, \boldsymbol{x}\right) - \mathcal{F}_{\leq 2M-1}\left(\varepsilon, \boldsymbol{x}\right)\right| \leq \frac{m_{\tilde{A}}(\rho)}{1-\sigma} \cdot \frac{\|\boldsymbol{x}\|^{2M}}{\rho^{2M} \cdot n^{M}} \cdot e^{m_{\psi}(\rho)\cdot\rho\cdot\|\boldsymbol{x}\|^2}$$

for such n. We can then sub this estimate into the expression for $|I_{\text{rem}}|$ to get

$$|I_{\text{rem}}(n)| \leq n^{p-1-q/2}\frac{m_{\tilde{A}}(\rho)}{(1-\sigma)\rho^{2M}n^{M}} \int_{\sqrt{n}D_{\delta''}} \|\boldsymbol{x}\|^{2M} \left|e^{-\frac{1}{2}\boldsymbol{x}^T\mathcal{H}\boldsymbol{x}}\right| e^{m_{\psi}(\rho)\rho\|\boldsymbol{x}\|^2} d\boldsymbol{x}.$$

Now,

$$\begin{aligned}\left|e^{-\frac{1}{2}\boldsymbol{x}^T\mathcal{H}\boldsymbol{x}}\right| &= e^{-\frac{1}{2}\Re\left(\boldsymbol{x}^T\mathcal{H}x\right)} \\ &= e^{-\frac{1}{2}\boldsymbol{x}^T\Re(\mathcal{H})\boldsymbol{x}} \\ &\leq e^{-\lambda\|\boldsymbol{x}\|^2/2},\end{aligned}$$

the last inequality coming from the Rayleigh-Ritz theorem[51, Theorem 4.2.2] $\boldsymbol{x}^T\mathcal{H}\boldsymbol{x} \geq \lambda\|\boldsymbol{x}\|^2$. So

$$\left|e^{-\frac{1}{2}\boldsymbol{x}^T\mathcal{H}\boldsymbol{x}}\right| e^{m_{\psi}(\rho)\rho\|\boldsymbol{x}\|^2} \leq e^{-\left(\lambda/2 - m_{\psi}(\rho)\rho\right)\|\boldsymbol{x}\|^2} = e^{-\mu\|\boldsymbol{x}\|^2},$$

which explains why we need $\mu > 0$. Therefore

$$|I_{\text{rem}}(n)| \leq n^{p-1-q/2-M}\frac{m_{\tilde{A}}(\rho)}{(1-\sigma)\rho^{2M}} \int_{\sqrt{n}D_{\delta''}} \|\boldsymbol{x}\|^{2M} e^{-\mu\|\boldsymbol{x}\|^2} d\boldsymbol{x}$$

and it remains to bound this last integral. But

$$\int_{\sqrt{n}D_{\delta''}} \|\boldsymbol{x}\|^{2M} e^{-\mu\|\boldsymbol{x}\|^2} d\boldsymbol{x} \leq \int_{\|\boldsymbol{x}\| \leq \sqrt{qn}\delta''} \|\boldsymbol{x}\|^{2M} e^{-\mu\|\boldsymbol{x}\|^2} d\boldsymbol{x},$$

and using Lemma 6 this evaluates to

$$\frac{\pi^{q/2}}{\Gamma(q/2)}\mu^{-(q+2M)/2} \cdot \gamma\left(\frac{q+2m}{2}, \mu q\delta^2 n\right).$$

Thus,

$$E_{\text{rem}}(n) := n^{p-1-q/2-M} \cdot \frac{m_{\tilde{A}}(\rho)}{(1-\sigma)\rho^{2M}} \cdot \frac{\pi^{q/2}}{\Gamma(q/2)} \cdot \mu^{-(q+2M)/2}\gamma\left(\frac{q+2m}{2}, \mu q\delta^2 n\right)$$

works as our error, when $n > 1/\delta''$.

Step 4: Bounding $I_{\text{tail}} = n^{p-1-q/2}\int_{\mathbb{R}^q\setminus\sqrt{n}D_{\delta''}} \mathcal{F}_{\leq 2M-1}(\varepsilon, \boldsymbol{x})e^{-\frac{1}{2}\boldsymbol{x}^T\mathcal{H}\boldsymbol{x}}d\boldsymbol{x}$

We have

$$|I_{\text{tail}}| \leq n^{p-1-q/2}\sum_{v=0}^{2M-1} n^{-v/2}\left|\int_{\mathbb{R}^q\setminus\sqrt{n}D_{\delta''}} P_v(\boldsymbol{x})e^{-\frac{1}{2}\boldsymbol{x}^T\mathcal{H}\boldsymbol{x}}d\boldsymbol{x}\right|.$$

Again, the Rayleigh-Ritz inequality implies that $\left|e^{-\frac{1}{2}\boldsymbol{x}^T\mathcal{H}x}\right| \leq e^{-\frac{\lambda}{2}\|\boldsymbol{x}\|^2}$ for $\boldsymbol{x} \in \mathbb{R}^q$, and now $|P_v(\boldsymbol{x})| \leq \sum_{\boldsymbol{\alpha}} |c_{v,\boldsymbol{\alpha}}| \, \|\boldsymbol{x}\|^{|\boldsymbol{\alpha}|}$. Since

$$\begin{aligned}
\left|\int_{\mathbb{R}^q\setminus\sqrt{n}D_{\delta''}} P_v(\boldsymbol{x})e^{-\frac{1}{2}\boldsymbol{x}^T\mathcal{H}\boldsymbol{x}}d\boldsymbol{x}\right| &\leq \sum_{\boldsymbol{\alpha}} |c_{v,\boldsymbol{\alpha}}| \int_{\mathbb{R}^q\setminus\sqrt{n}D_{\delta''}} \|\boldsymbol{x}\|^{|\boldsymbol{\alpha}|}\left|e^{-\frac{1}{2}\boldsymbol{x}^T\mathcal{H}\boldsymbol{x}}\right| d\boldsymbol{x} \\
&\leq \sum_{\boldsymbol{\alpha}} |c_{v,\boldsymbol{\alpha}}| \int_{\|\boldsymbol{x}\|\geq\delta''\sqrt{n}} \|\boldsymbol{x}\|^{|\boldsymbol{\alpha}|}e^{-\frac{\lambda}{2}\|\boldsymbol{x}\|^2}d\boldsymbol{x} \\
&= \sum_{\boldsymbol{\alpha}} |c_{v,\boldsymbol{\alpha}}| \frac{\pi^{q/2}}{\Gamma(q/2)}\left(\frac{2}{\lambda}\right)^{\frac{q+|\boldsymbol{\alpha}|}{2}}\Gamma\left(\frac{q+|\boldsymbol{\alpha}|}{2}, \frac{\lambda(\delta'')^2 n}{2}\right),
\end{aligned}$$

the last equality from Lemma 6, we get

$$E_{\text{tail}}(n) := n^{p-1-q/2}\sum_{v=0}^{2M-1} n^{-v/2}\sum_{\boldsymbol{\alpha}} |c_{v,\boldsymbol{\alpha}}|\left(\frac{\pi^{q/2}}{\Gamma(q/2)}\right)\left(\frac{2}{\lambda}\right)^{\frac{q+|\boldsymbol{\alpha}|}{2}}\Gamma\left(\frac{q+|\boldsymbol{\alpha}|}{2}, \frac{\lambda(\delta'')^2 n}{2}\right).$$

Final Bound: Combining Steps 1-4, we have

$$|I(n) - S_M(n)| \leq E_{\text{rem}}(n) + E_{\text{tail}}(n) + E_{\text{out}}(n)$$

with

$$\begin{aligned}
E_{\text{rem}}(n) &= n^{p-1-q/2-M} \cdot \frac{m_{\tilde{A}}(\rho)}{(1-\sigma)\rho^{2M}} \cdot \frac{\pi^{q/2}}{\Gamma(q/2)} \cdot \mu^{-(q+2M)/2}\gamma\left(\frac{q+2m}{2}, \mu q(\delta'')^2 n\right), \\
E_{\text{tail}}(n) &= n^{p-1-q/2}\sum_{v=0}^{2M-1} n^{-v/2}\sum_{\boldsymbol{\alpha}} |c_{v,\boldsymbol{\alpha}}|\left(\frac{\pi^{q/2}}{\Gamma(q/2)}\right)\left(\frac{2}{\lambda}\right)^{\frac{q+|\boldsymbol{\alpha}|}{2}}\Gamma\left(\frac{q+|\boldsymbol{\alpha}|}{2}, \frac{\lambda(\delta'')^2 n}{2}\right), \\
E_{\text{out}}(n) &= n^{p-1}m^{\text{out}}[(2\delta')^q - (2\delta'')^q]e^{-n\eta_{\delta''}},
\end{aligned}$$

valid when $n > 1/\delta''$ – (needed so that $\varepsilon < \sqrt{n}\delta''$) – and with free parameters δ'', ρ, σ chosen so that the following constraints are satisfied

$$\begin{aligned} 0 < \delta'' < \delta' \\ \rho > \sqrt{q}\delta'' \\ \sigma = \sqrt{q}\delta''/\rho \end{aligned}$$

along with the following computed quantities

$$\begin{aligned} \lambda &= \lambda_{\min}(\Re(\mathcal{H})) \\ \mu &= \lambda/2 - m_\psi(\rho)\rho \\ m_{\tilde{A}}(\rho) &\geq \sup_{\|\boldsymbol{\theta}\| \leq \rho} |\tilde{A}(\boldsymbol{\theta})| \\ m_\psi(\rho) &\geq \sup_{\|\boldsymbol{\theta}\| \leq \rho} |\psi(\boldsymbol{\theta})|/\|\boldsymbol{\theta}\|^3 \\ \eta_{\delta''} &\leq \inf_{\tilde{\mathcal{N}} \setminus D_{\delta''}} u(\boldsymbol{\theta}) \\ m^{\text{out}} &\geq \sup_{\tilde{\mathcal{N}} \setminus D_{\delta''}} |\tilde{A}(\boldsymbol{\theta})| \end{aligned}$$

which satisfy the further requirements

$$\begin{aligned} \lambda > 0 \quad &\text{(always true)}, \\ \mu > 0, \\ \eta_{\delta''} > 0. \end{aligned}$$

While theoretically these latter inequalities are guaranteed to be met, insufficiently small choices of ρ or not enough subdivisions of our domain in the numerical bounding of $\eta_{\delta''}$ can lead to them not being met.

Lastly, one will notice that the incomplete Γ and γ in E_{tail} and E_{rem} depend on n in the second entry. But we can use the trivial bound for $\gamma(\cdot, \cdot)$ given by the complete Gamma function:

$$\gamma\left(\frac{q+2M}{2}, \mu q(\delta'')^2 n\right) \leq \Gamma\left(\frac{q+2M}{2}\right).$$

As for $\Gamma(\cdot, \cdot)$, we may use Theorem 2.2 of [41]. Let $a = \frac{q+|\boldsymbol{\alpha}|}{2}$ and $x = \frac{\lambda(\delta'')^2 n}{2}$; denoting $m = \lfloor a \rfloor$ and $\theta = a - m$, and

$$H(m, x) = \sum_{j=0}^{m-1} \frac{(m-1)!}{(m-1-j)!} x^{-j},$$

we have

$$\Gamma(a,x) \leq x^{a-1}e^{-x}H(m,x)^{1-\theta}H(m+1,x)^{\theta}.$$

Notice that the e^{-x} is what gives us the $O(n^{-M})$ factor in the final bound[8] for E_{tail}. This completes the proof of Proposition 4. □

Remark 4. *It should be noted that the proof of Proposition 4 was the culmination of several different proof attempts, plus help from our advisor. But these prior approaches, including the naive one presented in Sections 5.1 and 5.2 of [90], yielded astronomically large constants after recombining error terms, even on small examples.*

For said approaches, we also implemented the intermediate bounds (analogous to $m_{\tilde{A}}, m_{\psi}$, $\eta_{\delta''}$, and m^{out}) in Sage using interval arithmetic. Due to lack of time, we could not include such implementations for the current bound here. Implementing the results here and testing them on real examples is a primary objective, going forward.

Now we need to combine the results of Propositions 2 and 4 in order to derive an explicit error bound for $f_{n\boldsymbol{r}}$, relative to an M-order asymptotic expansion $\lambda(n) = \lambda_M(n)$ for $f_{n\boldsymbol{r}}$. To glue together the asymptotic contributions of Propositions 2 and 4, we need a small calculation involving the *Lambert-W function*, also known as the *product log*.

Lemma 8. *Let β and C be positive real numbers. In the setting of Proposition 2, we can construct an $\hat{N}$ so that for $n \geq \hat{N}$,*

$$c\tau^n \leq C|\boldsymbol{w}_0|^{-n\boldsymbol{r}}n^{-\beta}.$$

Proof. We already know from earlier that $0 < \tau < |\boldsymbol{w}_0|^{-\boldsymbol{r}}$. We define

$$q := \frac{\tau}{|\boldsymbol{w}_0|^{-\boldsymbol{r}}} \in (0,1), \quad a := \log\frac{|\boldsymbol{w}_0|^{-\boldsymbol{r}}}{\tau} > 0, \quad K := \frac{c}{C} > 0.$$

The desired inequality $c\tau^n \leq C|\boldsymbol{w}_0|^{-n\boldsymbol{r}}n^{-\beta}$ is equivalent to $Kq^n \leq n^{-\beta}$, or $Kn^{\beta}e^{-an} \leq 1$. Since exponentials decay faster than powers, it's clear that this will be true for large enough n, but we must find a bound on the largest real solution of the "threshold equation"

$$Kn^{\beta}e^{-an} = 1.$$

[8] Also, one can see that the rational functions (in n) $H(m,x)^{1-\theta}$ and $H(m+1,x)^{\theta}$ will be bounded by a constant as long as $n > \frac{q+|\boldsymbol{\alpha}|-2}{\lambda(\delta'')^2}$, but this makes the computed constant N^* in Theorem 2 significantly worse.

Taking β-th roots and dividing across gives

$$ne^{-(a/\beta)n} = K^{-1/\beta},$$

and multiplying by $-a/\beta$ yields

$$-\frac{a}{\beta}ne^{-(a/\beta)n} = -\frac{a}{\beta}K^{-1/\beta}.$$

Now, such equations of the form $we^w = z$ have solutions $w = W_k(z)$, where k is an integer and W_k is the kth branch of the Lambert W-function. But since all quantities involved here are real, it suffices to consider $k \in \{0, -1\}$.

Therefore

$$-\frac{a}{\beta}n = W_k\left(-\frac{a}{\beta}K^{-1/\beta}\right), \quad k \in \{0, -1\}$$

are our relevant product-log forms, and since the argument to W_k is negative we know that the $k = -1$ branch will have larger magnitude than the $k = 0$ branch.[9] Thus our larger-magnitude root will be

$$n = -\frac{\beta}{a}W_{-1}\left(-\frac{a}{\beta}K^{-1/\beta}\right),$$

and subbing back in our definitions of a and K yields a final bound of

$$\hat{N} = \max\left(2, \left\lceil \frac{-\beta}{\log(|\boldsymbol{w}_0^{-\boldsymbol{r}}|/\tau)} W_{-1}\left(\frac{\log(|\boldsymbol{w}_0^{-\boldsymbol{r}}|/\tau)}{-\beta}\left(\frac{C}{c}\right)^{1/\beta}\right)\right\rceil\right).$$

□

At last, we're ready to state our main theorem:

Theorem 2 (Computable error bounds for Smooth ACSV, strictly minimal case)**.** *Let $F = G/H$ be a rational function analytic at the origin with* $\gcd(G, H) = 1$*, and let $\boldsymbol{r} = (r_1, ..., r_d)$ be a direction vector with positive rational entries normalized so that $r_d = 1$. Suppose that F admits a nondegenerate, strictly minimal critical point $\boldsymbol{w}_0$ (smooth in the*

[9] There is one small edge case: if $K\left(\frac{\beta}{ae}\right)^{\beta} \leq 1$, then $Kn^{\beta}e^{-an} \leq 1$ for all $n > 0$, so the simpler choice $\hat{N} = 2$ already works.

dth coordinate) in the direction $\boldsymbol{r}$*, and let* p *be the smallest positive integer such that* $\partial_{z_d}^p H(\boldsymbol{w}_0) \neq 0$*. Then, for each integer* $M \geq 0$ *there is an asymptotic expansion*

$$\lambda_M(n) := \frac{\boldsymbol{w}_0^{-n\boldsymbol{r}}}{(2\pi)^{(d-1)/2}} n^{p-1-(d-1)/2} (\det \mathcal{H})^{-1/2} \sum_{\ell=0}^{M-1} C_\ell n^{-\ell}$$

where $\mathcal{H}$ *is the matrix obtained from Definition 7 and the* C_ℓ *are explicit constants depending on a certain polynomial and the entries of* $\mathcal{H}$*. Moreover, there exists some explicit error* $E_M(n) = O\left(\left|\boldsymbol{w}_0^{-\boldsymbol{r}}\right|^n n^{p-1-(d-1)/2-M}\right)$ *so that*

$$|f_{n\boldsymbol{r}} - \lambda_M(n)| \leq E_M(n)$$

whenever $n \geq N^*$*, for some* N^* *which we can compute.*

Remark 5. *Again, the particularity of the index* d *is not important in the statement of the theorem above. Indeed, if instead* $\boldsymbol{w}_0$ *were such that* $\partial_{z_j}^p H(\boldsymbol{w}_0) \neq 0$ *for some* $j \in [d]$ *and* $p \geq 1$*, then you could just replace* z_d *with* z_j *and* r_d *with* r_j *in the Theorem statement (and proof), and all the machinery would still work.*

Proof. Using Proposition 2, find a c and τ such that $|f_{n\boldsymbol{r}} - \chi| \leq c\tau^n$ for all n, where

$$\chi = \frac{\boldsymbol{w}_0^{-n\boldsymbol{r}}}{(2\pi)^{d-1}} \int_{\widetilde{\mathcal{N}}} A(\boldsymbol{\theta}) e^{-n\phi(\boldsymbol{\theta})} d\boldsymbol{\theta},$$

A and ϕ having the formulas from Chapter 2. Next, apply Proposition 4 to the integral

$$I(n) = \int_{\widetilde{\mathcal{N}}} A(\boldsymbol{\theta}) e^{-n\phi(\boldsymbol{\theta})} d\boldsymbol{\theta}$$

appearing in χ to get an expansion $S_M(n)$ and an error $\tilde{E}_M(n) = O(n^{p-1-(d-1)/2-M})$ so that for any n greater than $\lceil 1/\delta'' \rceil$, $|I(n) - S_M(n)| \leq \tilde{E}_M(n)$. Multiplying both sides of this inequality by $|\boldsymbol{w}_0|^{-n\boldsymbol{r}}/(2\pi^{d-1})$ and putting $\lambda := \frac{\boldsymbol{w}_0^{-n\boldsymbol{r}}}{(2\pi)^{d-1}} S$ and $\hat{E} := \frac{|\boldsymbol{w}_0|^{-n\boldsymbol{r}}}{(2\pi)^{d-1}} \tilde{E}$ gives us

$$|\chi - \lambda| \leq \hat{E}$$

with $\hat{E}_M(n) = O\big(|\boldsymbol{w}_0|^{-n\boldsymbol{r}} n^{p-1-(d-1)/2-M}\big)$. Decomposing $\hat{E}$ into $\hat{E}_{\text{out}} + \hat{E}_{\text{rem}} + \hat{E}_{\text{tail}}$ according to Proposition 4 and applying the bounds for $\Gamma(\cdot,\cdot)$ and $\gamma(\cdot,\cdot)$, we can look at the remainder component and do the following. First, note that $\hat{E}_{\text{rem}}$ has the form

$$\hat{E}_{\text{rem}}(n) = KR^n n^\alpha \gamma(a, bn)$$

for some $K, R, a, b > 0$ and $\alpha \in \mathbb{Q}$. For n greater than the N_0 from Proposition 4 we have

$$\gamma(a, bn) \geq \gamma\left(a, bN_0\right) =: g_0 > 0,$$

hence

$$\hat{E}_{\text{rem}}\left(n\right) \geq K g_0 R^n n^\alpha.$$

Applying Lemma 8 produces an N_1 such that

$$c\tau^n \leq K g_0 R^n n^\alpha \leq \hat{E}_{\text{rem}}\left(n\right) \leq \hat{E}_M(n)$$

for $n \geq N_1$. Consequently, for $n \geq \max\{N_0, N_1\}$,

$$|f_{n\mathrm{r}} - \lambda_M(n)| \leq 2\hat{E}_M(n),$$

so we can take $E_M(n) = 2\hat{E}_M(n)$ and we're done.

□

Again, a similar results holds in the finitely minimal case.

Theorem 3 (Explicit error bounds for Smooth ACSV, finitely minimal case)**.** *Let $F = G/H$ be a rational function analytic at the origin with $\gcd(G, H) = 1$, and let $\boldsymbol{r} = (r_1, ..., r_d)$ be a direction vector with positive rational entries. Suppose that F admits a finite set E of nondegenerate, finitely minimal smooth critical points in the direction $\boldsymbol{r}$ such that for each $\boldsymbol{w} \in E$, there is a $j = j(\boldsymbol{w})$ and a smallest $p = p(\boldsymbol{w})$ so that $\partial_{z_j}^p H(\boldsymbol{w}) \neq 0$. Then, for each integer $M \geq 0$ we can find an order-M asymptotic expansion λ for $f_{n\boldsymbol{r}}$ by letting $\lambda_{\boldsymbol{w}}$ be the result of the formula for λ in Theorem 2, then setting $\lambda := \sum_{\boldsymbol{w} \in E} \lambda_{\boldsymbol{w}}$.*

The asymptotic has associated with it the error bound

$$|f_{n\boldsymbol{r}} - \lambda(n)| \leq E(n),$$

where $E(n)$ is obtained by summing over the contributions $E_{\boldsymbol{w}}$ from the bound of Theorem 2 applied to each point of E.

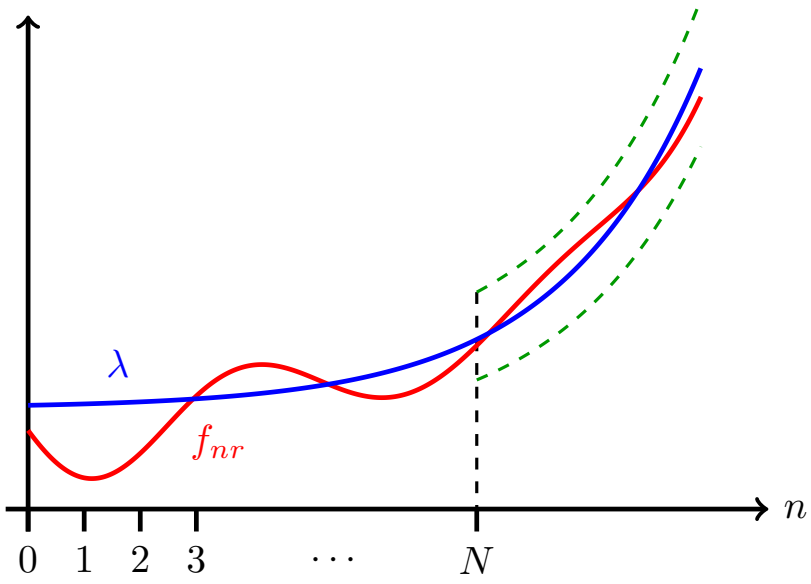


Figure 3.2: To compute N certifying positivity of $f_{n\boldsymbol{r}}$ for all n, first show that $\lambda > 0$ for all n then bound $|\lambda - f_{n\boldsymbol{r}}|$ explicitly.

3.3 Application: Coefficient Positivity

Using Theorems 2 and 3 we can find conditions for determining when $f_{n\boldsymbol{r}}$ is positive, conditional on the positivity of λ. For example, we have the following.

Corollary 1 (Diagonal Positivity Criterion, $M = 1$ case)**.** *Let F be a function and $\boldsymbol{r}$ a direction to which Theorem 2 applies. Then there exists an $N \in \mathbb{N}$ large enough so that if the leading-order asymptotic $\lambda_1(n)$ of $f_{n\boldsymbol{r}}$ is positive for all n, then to prove $f_{n\boldsymbol{r}}$ positive for all n it suffices to check $f_{\boldsymbol{0}}, f_{\boldsymbol{r}}, f_{2\boldsymbol{r}}, ..., f_{(N-1)\boldsymbol{r}}$ for positivity.*

Proof. Use the result of the theorem to find N^* so that for n greater than or equal to N^*,

$$|f_{n\boldsymbol{r}} - \lambda| \leq E(n) \leq C|\boldsymbol{w}_0|^{-n\boldsymbol{r}} n^{p-1-(d-1)/2-M}$$

for some $C > 0$. Write

$$\lambda = \tilde{C}\boldsymbol{w}_0^{-n\boldsymbol{r}} n^{p-1-(d-1)/2} C_0,$$

lumping the (positive) constant prefactors of λ into $\tilde{C}$. Since $\lambda = |\lambda| > 0$, we know that for all

$$n > \frac{C}{|\tilde{C}C_0|}$$

we will have

$$\begin{aligned} Cn^{-1} &< |\tilde{C}| \cdot |C_0| \\ \implies Cn^{-M} &< |\tilde{C}| \cdot |C_0| n^{-(M-1)} \\ \implies C|\boldsymbol{w}_0|^{-n\boldsymbol{r}} n^{p-1-(d-1)/2-M} &< |\lambda|, \end{aligned}$$

so $|f_{n\boldsymbol{r}} - \lambda| < |\lambda|$, and the definition of absolute value gives the result. □

Corollary 2 (Diagonal Positivity Criterion, C_0-dominant case)**.** *Let F be a function and $\boldsymbol{r}$ a direction to which Theorem 2 applies. Suppose also that the constant C_0 associated with the leading-order expansion λ_1 is positive. Then, for any $M \in \mathbb{Z}_+$ there exists an $N \in \mathbb{N}$ large enough so that if λ_M is positive for every n, it suffices to check $f_{\boldsymbol{0}}, f_{\boldsymbol{r}}, f_{2\boldsymbol{r}}, ..., f_{(N-1)\boldsymbol{r}}$ for positivity to prove positivity of $f_{n\boldsymbol{r}}$ for all n.*

Proof. For each $M \in \mathbb{Z}_+$ we have, as in the previous proof, an error

$$|f_{n\boldsymbol{r}} - \lambda| \leq E(n) \leq C|\boldsymbol{w}_0|^{-n\boldsymbol{r}} n^{p-1-(d-1)/2-M}$$

valid for $n \geq N^*$, and we may write

$$\lambda = \tilde{C}\boldsymbol{w}_0^{-n\boldsymbol{r}} n^{p-1-(d-1)/2} \left(C_0 + C_1 n^{-1} + \cdots + C_{M-1} n^{-M+1}\right).$$

Since $C_0 > 0$, we can find N_1 such that for larger n,

$$\sum_{\ell=1}^{M-1} |C_\ell| n^{-\ell} \leq \frac{C_0}{2},$$

so then $\lambda_M \geq \frac{C_0}{2}\tilde{C}\boldsymbol{w}_0^{-n\boldsymbol{r}} n^{p-1-(d-1)/2}$. Next, compute N_2 such that the total error $|f_{n\boldsymbol{r}} - \lambda| \leq (C_0/4)\tilde{C}\boldsymbol{w}_0^{-n\boldsymbol{r}} n^{p-1-(d-1)/2}$. For $n \geq \max\{N^*, N_1, N_2\}$,

$$f_{n\boldsymbol{r}} \geq (C_0/4)\tilde{C}\boldsymbol{w}_0^{-n\boldsymbol{r}} n^{p-1-(d-1)/2} > 0,$$

as desired. □

Note that these corollaries are somewhat limited in scope; we can either only consider the leading asymptotic, or add an additional hypothesis in order to guarantee that the

higher-order correction terms can be ignored in the lower bounding of $|\lambda|$ by our error. Besides these limitations, there are two caveats implicit in the statement of *any* such corollary. The first is in the condition, "*If* λ is positive...". When $\boldsymbol{w}_0$ is strictly minimal and we're considering the leading asymptotic λ_1, this is an easy condition to check for a given F and $\boldsymbol{r}$ *a posteriori*: we only need to compute $\boldsymbol{w}_0^{-\boldsymbol{r}}$ and C_0 and compare their signs. What is more interesting (and sometimes complicated) is finding large classes of functions which *a priori* have such a $\boldsymbol{w}_0$ with these conditions met, for different asymptotic orders. When $\boldsymbol{r} = (1, 1, ..., 1)$ and F has *symmetric*, *multi-affine* denominator, i.e. is of the form

$$F(\boldsymbol{z}) = \frac{G(\boldsymbol{z})}{H(\boldsymbol{z})}$$

with $G(\boldsymbol{z}) = 1$ and $H(\boldsymbol{z}) = 1 - \sum_{j=1}^{d} a_j e_j(\boldsymbol{z})$, each a_j real and e_j the jth elementary symmetric polynomial in $\boldsymbol{z}$,[10] we can take advantage of symmetry to find instances of such classes for the leading-order asymptotic λ_1. This was the approach used in [9]. In the next subsection we shall describe some classes different from the ones found in that paper.

When $\boldsymbol{w}_0$ is *finitely* minimal, the situation is even more complicated. Different points $\boldsymbol{w} \in E$ will contribute dominant growth rates with the same magnitude but different phase. In this case it is difficult to tell how these various contributions interfere with one another to produce an (eventually) positive dominant asymptotic term. This relates to Skolem's problem, discussed earlier.

The second caveat is more practical: even when we have an asymptotic that is positive, is the N obtained from the theorem small enough so that positivity of $f_{\boldsymbol{0}}, ..., f_{(N-1)\boldsymbol{r}}$ can be verified in a reasonable amount of time? In order to answer this question, we need to have implemented our bounds. While we had an implementation for previous versions of our results,[11] the results turned out to be quite unsatisfactory, yielding N on the order of $\sim 10^{18}$ for a simple bivariate example, and $N \sim 10^{21}$ for the GRZ class with $d = 4$. Once our new bounds are implemented, we will investigate the feasibility of using these results to prove sequence positivity for practical problems of interest.

[10] This is sufficient, since allowing arbitrary polynomials G and $H = c - \sum_{k=1}^{d} a_k e_k$ for some real $c \neq 1$ affect asymptotics in a trivial way.

[11] Specifically, a version involving an earlier rendition of Proposition 2 for leading-order asymptotics having only fractional power decay $n^{-\beta}$ for $1/3 < \beta < 1/2$, and a version of Lemma 4 involving the Banach fixed point theorem.

3.3.1 Positivity along main diagonal for symmetric multi-affine denominators

As mentioned above, our results apply when $\boldsymbol{r} = (1, ..., 1)$ to those functions F with with $G(\boldsymbol{z}) = 1$ and $H(\boldsymbol{z}) = 1 - \sum_{j=1}^{d} a_j e_j(\boldsymbol{z})$, the $a_j \in \mathbb{R}$. In this setting, there are some simple computational checks to determine if we have a strictly minimal smooth contributing point $\boldsymbol{w}_0$, hence the results of ACSV (and Theorems 2 or 3) apply.

The first thing to check is whether or not H is squarefree, i.e. $H = H^{\mathfrak{s}}$. We use the following result:

Lemma 9. *Let c_k be real numbers, not all zero, for k ranging in $\{1, ..., d\}$. Further let*

$$H(\boldsymbol{z}) = 1 + \sum_{k=1}^{d} c_k e_k(\boldsymbol{z}).$$

Then H is irreducible in $\mathbb{C}[z_1, \ldots, z_d]$ unless $c_k = a^k$ for some $a \neq 0$.

Proof. Suppose H factors into irreducibles as $H = cf_1 \cdots f_r$. Because H is of degree 1 in each variable, that variable can appear in exactly one of the f_i. This gives us a partition $E_1, ..., E_r$ of the variables, with cell E_i containing those variables in factor f_i. But by symmetry of H and uniqueness of our irreducible decomposition, if we permute the variables, in order to maintain the same partition we must either have $r = 1$ or $r = d$. In the first case, H is irreducible. In the second case, each f_i is a linear form containing only z_i. The fact that all z_i appear in H and the constant term of H is 1 mean our factorization must be of the form

$$H = (1 + az_1) \cdots (1 + az_d),$$

as needed.

□

With $c_0 = 1$, $c_k = -a_k$ for $k \geq 1$, we know that either H admits a trivial factorization, or $H = H^{\mathfrak{s}}$ and by Proposition 3.2 of [93] that $\mathcal{V} = \{\boldsymbol{z} \in \mathbb{C}^d : H(\boldsymbol{z}) = 0\}$.

Next, we hope to find a *diagonal* nondegenerate minimal smooth critical point. But we already have this, by Proposition 1 of Chapter 2 – namely, the point $\boldsymbol{w}_0 := \rho(1, 1, ..., 1)$, where ρ is the unique real minimal modulus root of the diagonal polynomial

$$\delta^H(t) = 1 - \sum_{k=1}^{d} a_k \binom{d}{k} t^k.$$

Noting also that *non*-smoothness of the point $\boldsymbol{w}_0$ reduces to it being a solution of the following system of 2 scalar equations,

$$\delta^H(t) = 0$$
$$\gamma^H(t) := -\sum_{k=1}^{d} a_k \binom{d-1}{k-1} t^{k-1} = 0,$$

we get the following sufficient condition for smoothness of $\boldsymbol{w}_0$ immediately from the definition of the resultant.

Lemma 10. *If* $\operatorname{Res}_t\left(\gamma^H(t), \delta^H(t)\right) \neq 0$ *then every root of* δ^H *is simple, thus every diagonal point* $\boldsymbol{w}_0 \in \mathcal{V}$ *is smooth.*

To check that $\boldsymbol{w}_o$ is *nondegenerate*, we use Definition 7 and symmetry, which provides us with a generalization of the U we saw in 4.

Lemma 11 (Explicit form for Hessian Determinant)**.** *In the above setting, let*

$$U := \frac{\rho\left(a_2 + \sum_{k=3}^{d} a_k \binom{d-2}{k-2} \rho^{k-2}\right)}{a_1 + \sum_{k=2}^{d} a_k \binom{d-1}{k-1} \rho^{k-1}}.$$

Then $\mathcal{H} = (1-U)(I + \mathbf{1}\mathbf{1}^T)$*, and hence* $\det \mathcal{H} = d \cdot (1-U)^{d-1}$.

Thus $\boldsymbol{w}_0$ is nondegenerate iff $U \neq 1$, and it can be used as point around which to center our asymptotic calculations. But we still need to consider those singularities sharing the same polytorus as $\boldsymbol{w}_0$, i.e. are also minimal. Are they critical, and are there finitely many of them? While we don't give a theoretical answer to this question, one can simply *solve* the smooth critical point equations for the full set of critical points $\mathsf{crit}(\boldsymbol{r})$, using Gröbner bases. Generically, there will be finitely many solutions, and one can just check which of their moduli equal $|\boldsymbol{w}_0|$. Thus we assume for the remainder of this subsection that $E = \{\boldsymbol{w}_0\}$.

Now, to guarantee positivity of the resulting asymptotic we must show that the root ρ of δ^H is positive real. In order to enforce strict minimality of $\boldsymbol{w}_0$, we will also show that this root is unique among minimal modulus roots. Moreover, we seek to isolate ρ on some interval within $(0, \infty)$ – this we do to make bounds involving ρ explicit.

Our primary means of establishing such results is Rouché's theorem, akin to what the authors of [9] did in their Proposition 20. In both of the following restricted classes of functions, we let $b_k := a_k \binom{d}{k}$ for $1 \leq k \leq d$. Our first class corresponds to polynomials δ^H whose linear term grows faster than the sum of all other terms.

Class 1 (Linearly-Dominant Rouché class)**.** *Let* $b_1 > 0$ *and* $b_2, \ldots, b_d \in \mathbb{R}$*. Fix* $\epsilon \in (0,1)$ *and set*

$$r = \frac{1-\epsilon}{b_1}, \quad R = \frac{1+\epsilon}{1-\epsilon} \cdot \frac{1}{b_1}.$$

If

$$\sum_{k=2}^{d} |b_k| \, R^{k-1} \leq \epsilon b_1,$$

then $\rho \in (r, R)$ *is the unique root of* δ^H *of modulus less than* R*.*

Proof. On $|t| = R$, $1 - b_1 t$ dominates the tail; by Rouché, δ^H has exactly one zero in $|t| < R$. Moreover, through simple inequalities one can show that $\delta^H(r) > 0$ and $\delta^H(R) < 0$, so there is a unique root $\zeta \in (r, R)$. Thus ρ must lie in the desired interval. □

Our other class's proof involves the Vivanti-Pringsheim theorem, as well as an aperiodicity condition on the coefficients of the polynomial δ^H.

Class 2 (Nonnegative Vivanti-Pringsheim + Aperiodic class)**.** *Suppose that all* $b_k \geq 0$ *and that the* $\gcd\{k : b_k \neq 0\} = 1$*. Pick the minimal* j *such that* $b_j \neq 0$*, and let*

$$R := \frac{1}{b_j^{1/j}}, \quad r := \min\left\{1, \frac{1}{2b_j}, \frac{1}{\sqrt{2\sum_{k\neq j} b_k}}\right\}.$$

Then, $\rho \in [r, R]$ *is the unique root*[12] *of* δ^H *of modulus less than* R*.*

Proof. First, we use our nonnegativity condition on the b_k and Vivanti-Pringsheim to conclude that there is a dominant singularity of $1/\delta^H(t)$, i.e. the root of δ^H of minimal modulus, lying on the positive real axis. Observe that $\sum b_k t^k$ is an *aperiodic series* in the sense of [90, Definition 5.9]. Thus applying [90, Proposition 5.5] in the univariate case we conclude that ρ is the *only* root of δ^H with that modulus.

Now, we show that ρ lies in $[r, R]$. Noting that $\delta^H(t)$ is positive at 0 and strictly monotonically decreasing for all $t > 0$, it suffices to show that $\delta^H(r) \geq 0$ and $\delta^H(R) \leq 0$. Showing the latter is immediate from the definition of R and j. For showing the former, set $C = \sum_{k\neq j} b_k = \sum_{k>j} b_k$ and note that by r's definition we will have

$$b_j r \leq \frac{1}{2}, \quad Cr^2 \leq \frac{1}{2},$$

[12]Since the roots of δ^H are isolated, we can perturb r, R by a small enough ϵ so that $\rho \in (r-\epsilon, R+\epsilon)$.

thus $b_j r + Cr^2 \leq 1$. However, $r \leq 1$ implies that for $k \geq 2$ we have $r^k \leq r^2$, thus

$$b_j r + \sum_{k>j} b_k r^k \leq 1,$$

i.e. $\delta^H(r) \geq 0$, as needed. □

While explicit and seemingly somewhat restrictive, these classes do contain practical examples from the literature. For example, with a choice of ϵ dependent on d, the diagonal of the denominator corresponding to the GRZ-class function $F_d(\boldsymbol{z})$, defined:

$$F_d(\boldsymbol{z}) = \frac{1}{1 - \sum_{j=1}^{d} z_j + d! \prod_{j=1}^{d} z_j}$$

belongs to Class 1 for $d > 10$, as do the Szegő rational function [123]

$$S(x, y, z, w) = \frac{1}{1 - e_1 + 1/6 e_2}.$$

Assuming we've verified finite minimality of the critical point $\boldsymbol{w}_0$ and F and ρ fall into Class 1 or Class 2, we know $\rho > 0$ is the unique root of δ^H of minimal modulus, and so all coordinates of $\boldsymbol{w}_0 = \rho \mathbf{1}$ are positive. Moreover, multi-affineness implies that our pole order $p = 1$, and in this case the constant C_0 reduces[13] to the simple form

$$C_0 = \tilde{A}(\mathbf{0}) = \frac{-G(\boldsymbol{w}_0)}{\rho H_{z_d}(\boldsymbol{w}_0)} = \frac{-1}{\rho H_{z_d}(\boldsymbol{w}_0)}.$$

If $\tilde{A}(\mathbf{0}) > 0$, then $\operatorname{sgn}(\tilde{A}(\mathbf{0})) = \operatorname{sgn}(\boldsymbol{w}_0)$ and we have positivity of the asymptotic leading term $\lambda = \lambda_1$. Moreover, due to symmetry it has the following explicit form:

$$\lambda(n) = \rho^{-(dn+1)} n^{(1-d)/2} \cdot \frac{(2\pi(1-U))^{(1-d)/2} d^{-1/2}}{a_1 + \sum_{k=2}^{d} a_k \binom{d-1}{k-1} \rho^{k-1}}.$$

[13] This simplification holds in general whenever $p = 1$, not just in the symmetric multi-affine setting.

3.3.2 A few remarks

Besides creating a full Sage implementation for our results in their most general form, there are a few potential improvements to our bounds that we are aware of, both theoretical and practical:

- Recombining errors more carefully in the proofs of Proposition 2 and Theorem 2; at all steps, we generally take the simplest bound possible. There are known instances where we have discarded exponentially decaying factors for the sake of making the proof simpler, e.g. being able to apply Lemma 8 directly in Theorem 2. Another likely source of blowup is the trivial (constant) bound in terms of Γ for the $\gamma(\cdot,\cdot)$ in E_{rem}. While there aren't (to our knowledge) direct literature sources for bounds for γ like there are for $\Gamma(\cdot,\cdot)$ [110, 41], we can instead just use the identity $\Gamma(a) = \Gamma(a,x) + \gamma(a,x)$ and bounds for $\Gamma(\cdot)$ and $\Gamma(\cdot,\cdot)$ to obtain decaying bounds for $\gamma(\cdot,\cdot)$ depending on n.

- Better 'balancing' heuristics and free-parameter optimization throughout the proof; there are many places where we can improve choices of intermediate quantities involved in our analysis through an iterative process. For instance, in the proof of Lemma 4, instead of trying to make $Y(s)$ and $X_s(r)$ both less than $(1-\epsilon)/2 \cdot |a|s$, for a small ϵ, we should optimize a choice of (c_1, c_2) so that $c_1 + c_2 = 1 - \epsilon$ and $Y(s) \leq c_1|a|s$ and $X_s(r) \leq c_2|a|s$. As another instance, in the proof of Theorem 2 we choose our $\hat{N}_\bullet$ so that $c\tau^n \leq \hat{E}_\bullet/3$ for $\bullet = \mathrm{tail}, \mathrm{out}, \mathrm{rem}$, respectively.

- Improvements / extensions in numerics associated with our implementation; for certain functions, the usual 53 bits of precision associated with `RIF` will not be enough. Higher-resolution versions of these fields should be used, and we would like to implement heuristics to set the precision based on the complexity of the input.

We hope to implement these changes in a future version of this work, as well as apply our (hopefully then practical) methods to positivity problems in the literature.

Chapter 4

Sage Periods

Recall from Section 1.1.1 how a period

$$I(t) = \int_\gamma R(\boldsymbol{x}, t) d\boldsymbol{x}$$

for a rational function $R = G/H$ satisfies a homogeneous linear differential equation in t with polynomial coefficients. Let $\mathcal{L}_{R,\gamma}$ be some differential operator so that $\mathcal{L}_{R,\gamma} I = 0$. It is known [77, Section 1, Equation 2] that there exists an operator $\mathcal{L}$ which is simultaneously a left multiple of all $\mathcal{L}_{R,\gamma}$ over different cycles γ. The fact that allows the computation of $\mathcal{L}$ to be made *algebraic* is that it is the smallest operator for which

$$\mathcal{L}(R) = \sum_{i=1}^{d} \partial_i(B_i) \qquad (*)$$

for some rational functions B_i whose poles are contained in $\mathcal{V}(\mathrm{H})$. This is the connection to *creative telescoping*[28]. The operator $\mathcal{L}$ is called a *telescoper* of R – the name coming from the fact that the derivative operators are continuous analogues of the "telescoping" forward-difference operator Δ, with $(*)$ implying that $\mathcal{L} \int_\gamma R = \mathcal{L} I = 0$, since $\mathcal{L}$ contains only derivatives ∂_t and the poles of the B_i are among those of R.

Thus we reduce the problem of finding an annihilator for periods to finding a telescoping operator $\mathcal{L}$ for R. This problem is in turn solved by introducing an algorithm relying on a reduction of a function modulo derivatives. We shall see the basics of how the algorithm works in Section 4.3, but first let's examine related work and see examples of our software package in action.

4.1 The SagePeriods Package and Related Software

Prior to the mid-2010s, the main software for computing such D-finite equations were the `Mgfun` package [27] in Maple and the `HolonomicFunctions` package [75] in Mathematica. Lairez [77], building on previous work of Bostan et al. [21], gave the first algorithm for rational period integrals, and thus also diagonals, that is singly exponential in the number of variables. As part of this work, Lairez developed the `Periods` package[1] in MAGMA. Although lacking thorough documentation and requiring a paid MAGMA subscription to use, the package is still the most practical way to compute rational periods and diagonals, having been used to compute hundreds of Picard-Fuchs equations for Calabi-Yau operators [125, 78]. In addition, the Sage `ore_algebra` package provides some basic capabilities for creative telescoping [67], the recent `MultivariateCreativeTelescoping.jl` Julia package [24] gives an efficient implementation of general creative telescoping (which handles, but uses less optimized algorithms for, rational functions), and Ramesh and Weiss [114] provide a sage package to compute exact volumes of semi-algebraic convex sets using specialized Picard-Fuchs equations.

In this section we illustrate the main functionality of the `sage_periods` package, bringing Lairez's algorithm to SageMath, adding full documentation, and providing compatibility with the `ore_algebra` [66, 96] and `sage_acsv` [48, 49] packages for D-finite equations and asymptotics of multivariate rational functions. The `sage_periods` package can be installed in any recent SageMath installation (preferably version `10.2` or later) by downloading its source code from

`https://github.com/ACSVMath/sage_periods`

then navigating into the source directory and running

```
sage -pip install .
```

or simply by running the command

```
sage -pip install git+https://github.com/ACSVMath/sage_periods.git
```

from a terminal with access to the internet. The source code repository also contains a Jupyter notebook running through all examples discussed in this paper, and more. Full documentation is available inside SageMath, and on

`https://acsvmath.github.io/sage_periods/`

[1] Available at `https://github.com/lairez/periods`.

4.2 Illustrating the SagePeriods Package

For the combinatorics user interested in rational diagonals, the most important function is `compute_diagonal_annihilator`, which takes a rational function as an element of the `SymbolicRing` or a multivariate polynomial ring's `FractionField` and returns a representation of a D-finite operator annihilating its (by default) main diagonal. The package detects whether the `ore_algebra` package is available on the user's system: if it is found then the annihilating operator is returned (by default) as an element of the `Univariate Ore algebra in Dt over Univariate Polynomial Ring in t over Rational Field`, while if `ore_algebra` is not found then the operator is returned as an element of the `Ore Polynomial Ring in Dt over Univariate Polynomial Ring in t over Rational Field twisted by d/dt`. Optional arguments allow the user to change the direction vector $\boldsymbol{r}$, the order of the variables used to convert the diagonal to a residue, and the name of the output variable and differential operator.

Example 5. *The computation*

```
sage: from sage_periods import compute_diagonal_annihilator
....:      as diagonal
sage: var('w x y z u')
sage: diagonal(1/(1-x-y))
(t - 1/4)*Dt + 1/2
```

implies that the rational diagonal

$$S(t) = \Delta\left(\frac{1}{1-x-y}\right) = \Delta\left(\sum_{i,j\geq 0}\binom{i+j}{i}x^i y^j\right) = \sum_{n\geq 0}\binom{2n}{n}t^n = (1-4t)^{-1/2}$$

satisfies the D-finite equation $(t-1/4)S'(t) + (1/2)S(t) = 0$.

The ability to use the functionality of `ore_algebra` with the computed differential operators opens up powerful techniques for combinatorics applications.

Example 6. *Here we derive a D-finite operator for a lattice walk constrained to the positive orthant, as in Example 2.3. This time, our step set is* $\mathcal{S} = \{(\pm 1, 0), (0, \pm 1)\}$ *and our diagonal, which counts*

$$s_n = \textit{number of walks starting at 0 in the positive orthant of length } n,$$

is the main diagonal of $F(x, y, u) = (1+x)(1+y)/(1-uxy(1/x+x+1/y+y))$.

Carrying on from the code executed above, the computation

```
sage: deq = diagonal((1+x)*(1+y)/(1-u*x*y*(1/x+x+1/y+y)))
sage: deq
(t^4 - 1/16*t^2)*D^3 + (8*t^3 + 1/2*t^2 - 3/8*t)*D^2 + (14*t^2 + 7/4*t -
    3/8)*D + 4*t + 3/4
```

implies, after a constant scaling, that the $S(t)$ satisfies the D-finite equation

$$(16t^4 - t^2)S'''(t) + (128t^3 + 8t^2 - 6t)S''(t) + (224t^2 + 28t - 6)S'(t) \\ +(64t + 12)S(t) = 0.$$

We can convert this D-finite operator into the corresponding P-recursive sequence satisfied by the diagonal:

```
sage: from ore_algebra import OreAlgebra
sage: rec = deq.to_S(OreAlgebra(QQ['n'], 'Sn'))
sage: rec
(-1/16n^3 - 9/16n^2 - 13/8n - 3/2)Sn^2 + (1/2n^2 + 9/4n + 5/2)Sn + n^3 + 5n
    ^2 + 8n + 4
```

This implies, after a constant scaling, that (s_n) satisfies the recurrence

$$(-n^3 - 9n^2 - 26n - 24)s_{n+2} + (8n^2 + 36n + 40)s_{n+1} + (16n^3 + 80n^2 + 128n + 64)s_n = 0.$$

The recurrence can be used to generate terms very *efficiently. For instance, the command* `rec.to_list([1, 2],10001)[-1]` *gives s_{10000}, which has around* six thousand *digits, in under a second. After importing* `bound_coefficients as BC` *from* `ore_algebra`*, the computation*

```
sage: BC(deq, [1, 2], order=1, prec=20)
1.00000*4^n*(([1.2732 +/- 4.39e-5] + [+/- 2.04e-18]*I)*n^(-1) + B([1678.76
    +/- 2.19e-3]*n^(-2)*log(n), n >= 5))
```

uses [33, Theorem 3.1, Algorithm 1] to prove the existence of a (in this case necessarily real) constant $\lambda \in [1.2732 - 4.39 \cdot 10^{-5}, 1.2732 + 4.39 \cdot 10^{-5}]$ such that

$$\left| s_n - \lambda \frac{4^n}{n} \right| \leq \left(1678.76 + 2.19 \cdot 10^{-3}\right) \frac{4^n \log n}{n^2} \tag{4.1}$$

for all $n \geq 5$. Note that we set a low precision for numerical constants and compute only the leading term in this asymptotic expansion for readability reasons – on a modern laptop

one can compute the first 20 terms in the asymptotic expansion of s_n with all constants certified numerically to a thousand digits of accuracy in around a second.

In this case the singular set of $F(x, y, u)$ forms a smooth manifold and all coefficients in the power series expansion of F (not just those on the main diagonal) are non-negative, so after importing `diagonal_asymptotics_combinatorial as diagonal_asm` *from the* `sage_acsv` *package the computation*

```
sage: diagonal_asm((1+x)*(1+y)/(1-u*x*y*(1/x+x+1/y+y)))
4/pi*4^n*n^(-1) + O(4^n*n^(-2))
```

proves the leading constant $\lambda = 4/\pi$ in (4.1).

The constant λ in (4.1) is a *connection coefficient*. Recall from Section 1.1 that a zero test for connection coefficients is an open problem, meaning the decidability of asymptotics for P-recursive sequences is open. On the other hand, `sage_acsv` computes exact asymptotic expansions, but the difficulty of multivariate techniques means it works under assumptions that may not apply on practical problems. Our `sage_periods` package provides a bridge from multivariate diagonals to D-finite equations, allowing both packages to be used simultaneously to help resolve this tension.

Example 7. *Besides the functions we saw in Example 4 of Chapter 2, Gillis et al. [45] studied functions of the form $F_{c,d} = 1/(1 - \sum_{i=1}^{d} x_i + c\prod_{i=1}^{d} x_i)$ to $c > d!$. For instance, consider the function*

$$F(w, x, y, z) = \frac{1}{1 - w - x - y - z + 27wxyz}.$$

F has a cone-point singularity*, which prevents* `sage_acsv` *from directly computing asymptotics of its main diagonal. Baryshnikov et al. give a criterion [8, Proposition 6] to rule out so-called* critical points at infinity*, which in this case can be used as a black-box to prove that an asymptotic expansion derived from the D-finite equation satisfied by the diagonal is an* integer multiple *of an asymptotic expansion derived directly from the multivariate rational diagonal representation [8, Example 6, Theorem 5].*

After importing the functions `critical_points, get_expansion_terms, and compute_asymptotics_at_points as CAP` *from* `sage_acsv`*, the computation*

```
sage: F = 1/(1 - (w + x + y + z) + 27*w*x*y*z)
# Get asymptotic expansion from D-finite equation
sage: asm = BC(diagonal(F), [1, -3], order=1)
sage: C1 = list(asm.series_factor().summands)[0].coefficient
```

```
# Get asymptotic expansion from rational diagonal
sage: rho = critical_points(F)[1]
sage: asm = get_expansion_terms(CAP(F, [rho]))[0]
sage: C2 = asm.coefficient * asm.pi_factor
# Compare coefficients in both
sage: CIF(C1/C2)
3.00000000000000? + 0.?e-14*I
```

implies that diagonal asymptotics are 3 times the asymptotic expansion with exact coefficients returned by `sage_acsv` *(note that there are two dominant asymptotic terms, and an analogous computation shows the integer factor for the second one is also 3) allowing for the exact computation of the connection coefficients. This combination of univariate and multivariate techniques requires a rational diagonal expression for a P-recursive sequence of interest, but is one of the only general approaches currently known for attacking a D-finite CCP.*

Because the package comes with full documentation, we do not spend more time here describing its basic functions. The Jupyter notebook[2] accompanying our article [94] also contains results of running the algorithm on a representative collection of 50 examples from the textbooks [90, 108]. The total time to compute annihilators of all these examples was approximately 170 seconds, however two of the examples took over half of the total time and nine of the examples together took up 150 seconds.

4.3 Background and Summary of Techniques

In order to describe the reduction algorithm used in our package, we first motivate it by considering similar algorithms for the univariate and smooth multivariate case.

4.3.1 Univariate Case: Hermite Reduction

In 1872, Charles Hermite [50] pioneered *Hermite reduction*, a procedure for computing the algebraic part of the antiderivative of a rational function without finding the roots of its denominator. For a rational function $R \in \mathbb{C}(x)$, the reduction is denoted by $[R]$ and is defined as follows. First, write R in the form a/f^q where $a, f \in \mathbb{C}[x]$, f is squarefree, and

[2] Available at https://github.com/ACSVMath/sage_periods/blob/main/Example%20Notebook.ipynb

q is a natural number. Since f is squarefree we have $a = uf + v'f$ for some $u, v \in \mathbb{C}[x]$, and using the quotient rule we can rewrite R as

$$\frac{a}{f^q} = \frac{u + \frac{1}{q-1}v'}{f^{q-1}} + \left(\frac{-v}{(q-1)f^{q-1}}\right)'.$$

This gives an inductive definition:

$$[R] = \left[\frac{a}{f^q}\right] := \frac{\tilde{a}}{f^q} + \left[\frac{u + \frac{1}{q-1}v'}{f^{q-1}}\right].$$

For $q = 1$ we put $[a/f] := r/f$, where r is our remainder in the polynomial division of a by f. Hermite reduction has the following properties:

- With f fixed, $[R]$ is $\mathbb{C}$-linear.
- R is the derivative of some rational function if and only if $[R] = 0$.
- The reduced partial derivatives $[\partial_x^k R]$, for $k \geq 0$, lie in a finite-dimensional vector space depending only on f.

Thus, we can use Hermite reduction to compute an operator which, when applied to R, gives a derivative of some rational function B. To do so, simply compute reductions $[\partial_x^k R]$ for increasing k until you get a linear dependency over $\mathbb{C}(x)$:

$$\sum_{k=0}^{r} a_k(x)[\partial_x^k R] = 0.$$

For then the above properties imply

$$\sum_{k=0}^{r} a_k(x)\partial_x^k$$

is such an operator for R. In [20], the complexity of algorithms for computing the Hermite reduction of R as well is analyzed, and algorithms for computing both a "telescoper" $\mathcal{L}$ and a certificate B so that $\mathcal{L}R = \partial_x(B)$ are given.

4.3.2 Smooth Multivariate Case: Griffiths-Dwork Reduction

Things are more complicated in the multivariate case. Given $R \in \mathbb{Q}(t)(x_1, ..., x_d)$, we are tasked with finding an $\mathcal{L}$ which telescopes R. Letting A denote the polynomial ring $\mathbb{Q}(t)[x_0, ..., x_d]$ in one extra variable x_0, we first *homogenize* R to obtain a new rational function

$$R_{\text{hom}}(t, x_0, x_1, \ldots, x_d) = x_0^{-d-1} R(t, x_1/x_0, \ldots, x_d/x_0)$$

in the variables of A homogeneous of degree $-d-1$. Working projectively helps control the *pole at infinity* during reductions, and the degree is chosen so that a telescoper for R_{hom} is also a telescoper for R.

Write $R_{\text{hom}} = a/f^q$ for positive integer q and (necessarily homogeneous) polynomials a and f such that f is squarefree; note that a and f may share factors. Reducing a using a Gröbner basis of $\text{Jac}(f) = (\partial_0 f, \ldots, \partial_d f)$ gives a relationship $a = \tilde{a} + \sum_{k=0}^{d} p_k(\boldsymbol{x})(\partial_k f)$ for polynomials $\tilde{a}$ and p_k, and it can be verified that

$$R_{\text{hom}} = \frac{\tilde{a}}{f^q} + \frac{\frac{1}{q-1}\sum_{k=0}^{d} \partial_k p_k}{f^{q-1}} + \sum_{k=0}^{d} \partial_k \left(\frac{-p_k}{(q-1) f^{q-1}} \right). \qquad (*)$$

Again, defining inductively,

$$[R_{\text{hom}}]_{GD} = \left[\frac{a}{f^q} \right]_{GD} = \left[\frac{\frac{1}{q-1}\sum_{k=0}^{d} \partial_k p_k}{f^{q-1}} \right]_{GD}$$

for $q > 1$ and $[a/f]_{GD} = \text{rem}(a, \text{Jac}(f)$ for $q = 1$, we have a reduction $[\,]_{GD}$ which expresses R_{hom}, up to a sum of pure derivatives, as a sum of polynomials reduced modulo $\text{Jac}(f)$ over powers of f.

When $\mathcal{V}(f)$ forms a smooth variety in $\mathbb{P}^d$ one has that $A/\text{Jac}(f)$ is finite-dimensional and, like in the Hermite case, $[R]_{GD} = 0$ iff R is a sum of derivatives of rational functions. Moreover, all $[\partial_t^k R]_{GD}$ lie in a finite-dimensional vector space [21, 47]. Continually taking derivatives of R_{hom} with respect to t and applying $[\,]_{GD}$ eventually gives a non-trivial linear relationship

$$\sum_{k=0}^{r} a_k(t) [\partial_t^k R]$$

between the partial derivatives, encoding the telescoper

$$\sum_{k=0}^{r} a_k(t) \partial_t^k,$$

just like before. The algorithm in the smooth case is thus theoretically simple and relatively easy to implement using Gröbner basis computations and the division algorithm.

4.3.3 Multivariate General Case: Lairez's Reductions

In the non-smooth case, $A/\text{Jac}(f)$ is *not* finite-dimensional. One must reduce modulo *higher order relations*, which initially come from differentials of syzygies of $\text{Jac}(f)$ but at higher orders include differentials of higher-order syzygies as well. Lairez [77] introduces advanced tools from differential and algebraic geometry to study and exploit these relations, but we shall not introduce them here. Instead, we motivate the need for "closing" under such relations by introducing *syzygies* of the Jacobian ideal – that is, polynomials $(b_0, ..., b_d)$ such that $\sum_{k=0}^{d} b_k \partial_k f = 0$. If we have such a situation, then putting $a = \sum_{k=0}^{d} \partial_k b_k$ and substituting in a/f^q to our differentiation formula $(*)$ yields:

$$\sum_{k=0}^{n} b_k \partial_k f = 0 \Rightarrow \sum_{k=0}^{d} \partial_k \left(\frac{b_k}{f^q} \right) = \frac{\sum_{k=0}^{d} \partial_k b_k}{f^q}.$$

In other words, letting the notation A_f denote the *localization* of A at f, there exist choices of $R = a/f^q$ which are in the subspace of partial derivatives $\sum_{k=0}^{d} \partial_i A_f$ but do not come from the Jacobian. Thus the above algorithm for Griffiths-Dwork could fail, since $\text{Jac}(f)$ will not reduce pole order on R from q to $q-1$. This is best illustrated by example.

Example 8 (Section 1.2 of [77])**.** *Let $d = 3$ and*

$$R(x, y, z) = \frac{x^3}{f^2}$$

with $f = xy^2 - z^3$. We can compute a reduced Gröbner basis for $Jac(f)$ of (xy, y^2, z^2). Direct calculation shows $\text{rem}(a, Jac(f)) = x^3$ is a itself, therefore the formula $(*)$ will not reduce pole order on R.

Nevertheless, one can check that

$$\frac{x^3}{f^2} = \partial_x \left(\frac{2/7x^4}{f^2} \right) - \partial_y \left(\frac{1/7x^3 y}{f^2} \right),$$

so R is indeed in our subspace.

Thus we must somehow "add" all relations introduced by syzygy differentials to consideration in our reduction of R. To do this, we start by letting M_q denote the set of rational functions of the form a/f^q, and let W_q^1 be the following subset of $M_q \times M_{q-1}$:

$$W_q^1 = \left\{ \left((q-1)\frac{\sum_{k=0}^n b_i \partial_i f}{f^q}, \frac{\sum_{k=0}^n \partial_i b_i}{f^{q-1}} \right) \middle| \; b_i \in A \right\}.$$

A pair $(R, R') \in W_q^1$ relates a rational function R with a pole of order at most q with another rational function R', with pole order at most $q-1$, and they are equivalent mod derivatives – precisely the behavior captured by the Griffiths-Dwork reduction and the essence of what we want to extend.

Now, notice that W_q^1 contains ordered pairs of the form $(0, R')$. In particular, if $b_0, \ldots, b_d$ is a syzygy, then $(0, \sum_k \partial_k b_k / f^{q-1})$ is in W_q^1. While for every pair $(0, R')$, the rational function R' is in $\sum_k \partial_k A_f$, it could be the case that R' is *not* a part of a pair $(R', R'') \in W_{q-1}^1$. This motivates the definition of W_q^2 as

$$W_q^2 \stackrel{\text{def}}{=} W_q^1 + \left\{ (R, 0) \mid (0, R) \in W_{q+1}^1 \right\}.$$

But this is still not enough; for instance with $f = x_0^4 x_1 - x_0^2 x_1 x_2^2 + x_0 x_2^4$, we have

$$\frac{x_1^7}{f^2} = \frac{1062347}{276480} \frac{89x_0^2 + 96x_0x_1 + 712x_2^2}{f} + \sum_{k=0}^{2} \partial_i \left(\frac{b_i}{f^3} \right)$$

for some polynomials b_i, but x_1^7 is not a sum of an element of $\mathrm{Jac}(f)$ and a differential of a syzygy. Thus we must iterate the construction of W_q^2 to get, for general r,

$$W_q^{r+1} \stackrel{\text{def}}{=} W_q^r + \left\{ (R, 0) \mid (0, R) \in W_{q+1}^r \right\}.$$

Then, by somehow "reducing" R with respect to all $(W_q^r)_{r=1}^\infty$ at once, we will have captured all higher-order relations and are guaranteed to reduce pole order. Defining a map (or, more accurately, a family of maps) to perform this reduction and verifying it has the desired properties is where Lairez introduces advanced tools. Ultimately, one obtains a collection of reduction maps $[\,]_r$ for $r \geq 1$ so that

- Each $[\,]_r$ is $\mathbb{Q}(t)$-linear.
- The maps are *compatible*, in that $[\,]_{r+1} \circ [\,]_r = [\,]_{r+1}$.

- $R \in \sum_k \partial_k A_f$ iff $[R]_r = 0$ for some r high enough.
- For some r large enough, all $[\partial_t^k R]_r$ lie in a finite-dimensional vector space.

Together, these imply an algorithm for computing telescopers similar in spirit to the Griffiths-Dwork and Hermite cases, but for a general multivariate rational function.

4.3.4 Faster Computation & Modular Methods

While correct, the algorithm alluded to above requires reducing polynomials modulo *very* high-dimensional $\mathbb{Q}(t)$-vector spaces, which is costly. To account for this, Lairez introduces both theoretical and practical advancements, which have been worked into our package.

On the theoretical side, one improvement that can be made is to first discard *trivial syzygies* – those that have a specific form leading them to be already present in W_q^1 – from among the higher-order relations. This reduces the dimensionality of the problem. Another is to reuse the Gröbner-based Griffiths-Dwork reduction, which can be incorporated easily into the general reductions and is faster than plain linear algebra.

Despite being faster, all of these computations are still over $\mathbb{Q}(t)$, where existing implementations for things like polynomial division, Gröbner basis computation, and Gaussian elimination are quite slow. To mitigate this, the algorithm picks random primes p_i and interpolation points $t_{i,k}$ and computes relations in $GF(p_i)$ after evaluating $t = t_{i,k}$. Results can then be interpolated back into $GF(p_i)(t)$, and the differential operators over $\mathbb{Q}(t)$ are reconstructed from those modulo the p_i using the Chinese Remainder Theorem. Unfortunately, because of a lack of tight a priori bounds on the size of the output, the ultimate result is not certified to be correct; a full discussion on this issue and how it can be mitigated is found in Lairez [77, Sections 7.2–7.3].

4.3.5 Future Improvements

Despite being a good preliminary implementation, there is still much to do to improve our package's performance and compatibility with `sage_acsv` and `ore_algebra`. First, although not practical in many cases, we will allow the user to work over $\mathbb{Q}(t)$ to have certified output. Second, we plan to implement the *partial certificates* discussed in [77, Section 7.3] which have more reasonable size than the full regular certificates but still allow for certification of a telescoper. We plan to further automate the approach to computing connection coefficients suggested by Example 7, especially after forthcoming work giving an

algorithmic approach to detecting critical points at infinity is completed (otherwise the user will have to verify the non-existence of such points independently). Third, the telescopers computed by our package may not have minimal order. As seen in Example 6 it is possible to use a telescoper to get a linear recurrence for the coefficient sequence of interest, and then use this recurrence to generate a large number of terms in the sequence; these terms can be used to *guess* a lower order telescoper (if one exists). The `ore_algebra` package can be used to certify a guessed telescoper $\mathcal{M}$ of a function $f(t)$ from a certified telescoper $\mathcal{L}$ by verifying that $\mathcal{M}$ is a right divisor of $\mathcal{L}$ and annihilates a high-enough order truncation of $f(t)$. Fourth, we hope to incorporate results from an upcoming package implementing the algebraic-to-rational-diagonal reduction of Denef and Lipshitz [31, Theorem 6.2] to allow the user to pass in *algebraic* inputs as well as rational. Finally, we plan to explore possible algorithmic improvements, including multiprocessing of certain stages of the computation, an improved algorithm for Cauchy interpolation, and heuristics (some of which are in the package of Lairez) for choosing things like variable order and making monomial changes of variables to simplify computation.

References

[1] 8 - special functions. In Daniel Zwillinger and Victor Moll, editors, *Table of Integrals, Series, and Products (Eighth Edition)*, pages 867–1013. Academic Press, Boston, eighth edition edition, 2015.

[2] Lars Ahlfors. *Complex Analysis: An Introduction to The Theory of Analytic Functions of One Complex Variable*. 1979.

[3] Jiménez-Pastor Antonio and Pillwein Veronika. A Computable Extension for Holonomic Functions: DD-Finite Functions. *Journal of Symbolic Computation*, 94:90–104, September-October 2019.

[4] Jiménez-Pastor Antonio and Pillwein Veronika. Differentially definable functions: a survey. In Gesellschaft für Angewandte Mathematik und Mechanik (GAMM), editor, *Proceedings in Applied Mathematics and Mechanics (PAMM)*, volume 21, pages e202100178–e2021001, 2021.

[5] Richard Askey. Certain rational functions whose power series have positive coefficients. ii. *SIAM Journal on Mathematical Analysis*, 5(1):53–57, February 1974.

[6] Richard Askey and George Gasper. Certain rational functions whose power series have positive coefficients. *The American Mathematical Monthly*, 79(4):327–341, 1972.

[7] Richard Askey and George Gasper. Convolution structures for Laguerre polynomials. *Journal d'Analyse Mathématique*, 31(1):48 – 68, 1977. Cited by: 43.

[8] Yuliy Baryshnikov, Stephen Melczer, and Robin Pemantle. Stationary points at infinity for analytic combinatorics. *Found. Comput. Math.*, 22(5):1631–1664, 2022.

[9] Yuliy Baryshnikov, Stephen Melczer, Robin Pemantle, and Armin Straub. Diagonal asymptotics for symmetric rational functions via acsv. In *29th International Conference on Probabilistic, Combinatorial and Asymptotic Methods for the Analysis of*

Algorithms, volume 110 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 12, 15. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018.

[10] Edward A Bender. Central and local limit theorems applied to asymptotic enumeration. *Journal of Combinatorial Theory, Series A*, 15(1):91–111, 1973.

[11] Edward A Bender and L Bruce Richmond. Central and local limit theorems applied to asymptotic enumeration ii: Multivariate generating functions. *Journal of Combinatorial Theory, Series A*, 34(3):255–265, May 1983.

[12] François Bergeron and Christophe Reutenauer. Combinatorial resolution of systems of differential equations iii: a special class of differentially algebraic series. *European Journal of Combinatorics*, 11(6):501–512, November 1990.

[13] Jean Berstel and Maurice Mignotte. Deux propriétés décidables des suites récurrentes linéaires. *Bulletin de la Société mathématique de France*, 104(2):175–184, 1976.

[14] Andrea Bertozzi and James Mckenna. Multidimensional residues, generating functions, and their application to queueing networks. *SIAM Review*, 35(2):239–268, 1993.

[15] Yuri Bilu, Florian Luca, Joris Nieuwveld, Joël Ouaknine, David Purser, and James Worrell. Skolem Meets Schanuel. In Stefan Szeider, Robert Ganian, and Alexandra Silva, editors, *47th International Symposium on Mathematical Foundations of Computer Science (MFCS 2022)*, volume 241 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 20:1–20:15, Dagstuhl, Germany, 2022. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

[16] George D. Birkhoff. Formal theory of irregular linear difference equations. *Acta Mathematica*, 54(0):205–246, 1930.

[17] George D. Birkhoff and W. J. Trjitzinsky. Analytic theory of singular difference equations. *Acta Mathematica*, 60(0):1–89, 1933.

[18] L. E. Blumenson. A derivation of n-dimensional spherical coordinates. *The American Mathematical Monthly*, 67(1):63–66, 1960.

[19] A Bostan, S Boukraa, G Christol, S Hassani, and J-M Maillard. Ising n-fold integrals as diagonals of rational functions and integrality of series expansions. *Journal of Physics A: Mathematical and Theoretical*, 46(18):185202, April 2013.

[20] Alin Bostan, Shaoshi Chen, Frédéric Chyzak, and Ziming Li. Complexity of creative telescoping for bivariate rational functions. In *Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation*, ISSAC '10, page 203–210, New York, NY, USA, 2010. Association for Computing Machinery.

[21] Alin Bostan, Pierre Lairez, and Bruno Salvy. Creative telescoping for rational functions using the Griffiths-Dwork method. In *ISSAC 2013—Proceedings of the 38th International Symposium on Symbolic and Algebraic Computation*, pages 93–100. ACM, New York, 2013.

[22] Alin Bostan, Pierre Lairez, and Bruno Salvy. Multiple binomial sums. *Journal of Symbolic Computation*, 80:351–386, 2017.

[23] Mireille Bousquet-Mélou. *Rational and algebraic series in combinatorial enumeration*, page 789–826. EMS Press, May 2007.

[24] Hadrien Brochet, Frédéric Chyzak, and Pierre Lairez. Faster multivariate integration in D-modules. *J. Symbolic Comput.*, 137:Paper No. 102569, 2026.

[25] PETTER BRÄNDÉN and DAVID G. WAGNER. A converse to the grace–walsh–szegő theorem. *Mathematical Proceedings of the Cambridge Philosophical Society*, 147(2):447–453, 2009.

[26] Gilles Christol. Diagonales de fractions rationnelles et équations différentielles. *Groupe de travail d'analyse ultramétrique*, 10(2):1–10, 1982-1983.

[27] Frédéric Chyzak. An extension of Zeilberger's fast algorithm to general holonomic functions. *Discrete Math.*, 217(1-3):115–134, 2000.

[28] Frédéric Chyzak. *The ABC of Creative Telescoping — Algorithms, Bounds, Complexity*. Habilitation à diriger des recherches (hdr), University Paris-Sud 11, April 2014.

[29] John B. Conway. *Functions of One Complex Variable I*. Springer Science & Business Media, December 2012.

[30] E. Delabaere and C. J. Howls. Global asymptotics for multiple integrals with boundaries. *Duke Mathematical Journal*, 112(2):199 – 264, 2002.

[31] J Denef and L Lipshitz. Algebraic power series and diagonals. *Journal of Number Theory*, 26(1):46–67, May 1987.

[32] J. Denef and L. Lipshitz. Decision problems for differential equations. *Journal of Symbolic Logic*, 54(3):941–950, 1989.

[33] Ruiwen Dong, Stephen Melczer, and Marc Mezzarobba. Computing error bounds for asymptotic expansions of regular p-recursive sequences. *Accepted to Mathematics of Computation*, 2023.

[34] Ruiwen Dong and Doron Shafrir. The Skolem problem in rings of positive characteristic. In *Proceedings of the 58th Annual ACM Symposium on Theory of Computing*, STOC '26, page 323–329, New York, NY, USA, 2026. Association for Computing Machinery.

[35] Michael Drmota. Asymptotic distributions and a multivariate Darboux method in enumeration problems. *Journal of Combinatorial Theory, Series A*, 67(2):169–184, 1994.

[36] Eugène Fabry. *Sur les intégrales des équations différentielles linéaires à coefficients rationnels*. Thèse, Gauthier-Villars, Paris, 1885.

[37] Stephen D. Fisher. *Complex Variables*. Thomson Brooks/Cole, 1986.

[38] P. Flajolet, B. Salvy, and P. Zimmermann. Lambda-Upsilon-Omega: The 1989 Cookbook. Research Report 1073, Institut National de Recherche en Informatique et en Automatique, August 1989. 116 pages.

[39] Philippe Flajolet and Andrew Odlyzko. Singularity analysis of generating functions. *SIAM Journal on Discrete Mathematics*, 3(2):216–240, May 1990.

[40] Philippe Flajolet and Robert Sedgewick. *Analytic Combinatorics*. Cambridge University Press, January 2009.

[41] Steven G. From and Suthakaran Ratnasingam. New upper and lower bounds for the upper incomplete gamma function. *Results in Applied Mathematics*, 25:100552, 2025.

[42] Ikki Fukuda, Yoshiki Kagaya, and Yuki Ueda. Higher-order asymptotic expansion with error estimate for the multidimensional laplace-type integral under perturbations, 2025.

[43] Zhicheng Gao and L.Bruce Richmond. Central and local limit theorems applied to asymptotic enumeration iv: multivariate generating functions. *Journal of Computational and Applied Mathematics*, 41(1–2):177–186, August 1992.

[44] Scott Garrabrant and Igor Pak. Counting with irrational tiles, 2014.

[45] J. Gillis, B. Reznick, and D. Zeilberger. On elementary methods in positivity theory. *SIAM Journal on Mathematical Analysis*, 14(2):396–398, 1983.

[46] Torin Greenwood, Stephen Melczer, Tiadora Ruza, and Mark C. Wilson. Asymptotics of coefficients of algebraic series via embedding into rational series (extended abstract). *Sém. Lothar. Combin.*, 86B:Art. 30, 12, 2022.

[47] Philip A. Griffiths. On the periods of certain rational integrals: I. *Annals of Mathematics*, 90(3):460–495, 1969.

[48] Benjamin Hackl, Andrew Luo, Stephen Melczer, and Éric Schost. A SageMath package for analytic combinatorics in several variables: beyond the smooth case. *Sém. Lothar. Combin.*, 93B:Art. 159, 12, 2025.

[49] Benjamin Hackl, Andrew Luo, Stephen Melczer, Jesse Selover, and Elaine Wong. Rigorous analytic combinatorics in several variables in sagemath. *Sém. Lothar. Combin.*, 89B:Art. 90, 12, 2023.

[50] Ch. Hermite. Sur l'intégration des fractions rationnelles. *Annales scientifiques de l'École Normale Supérieure*, 2e série, 1:215–218, 1872.

[51] Roger A. Horn and Charles R. Johnson. *Matrix Analysis*. Cambridge University Press, 1985.

[52] C. J. Howls. Hyperasymptotics for multidimensional integrals, exact remainder terms and the global connection problem. *Proceedings: Mathematical, Physical and Engineering Sciences*, 453(1966):2271–2294, 1997.

[53] Hsien-Kuei Hwang. Large deviations for combinatorial distributions. i. central limit theorems. *The Annals of Applied Probability*, 6(1), February 1996.

[54] Hsien-Kuei Hwang. Large deviations of combinatorial distributions. ii. local limit theorems. *The Annals of Applied Probability*, 8(1), February 1998.

[55] Alaa Ibrahim. Positivity proofs for linear recurrences with several dominant eigenvalues, 2025.

[56] Alaa Ibrahim and Bruno Salvy. Positivity proofs for linear recurrences through contracted cones. *Journal of Symbolic Computation*, 132:102463, 2026.

[57] Tadeusz Inglot and Piotr Majerski. Simple upper and lower bounds for the multivariate Laplace approximation. *Journal of Approximation Theory*, 186:1–11, 2014.

[58] Mourad E. H. Ismail and M. V. Tamhankar. A combinatorial approach to some positivity problems. *SIAM Journal on Mathematical Analysis*, 10(3):478–485, 1979.

[59] A. Jimenez-Pastor, P. Nuspl, and V. Pillwein. An extension of holonomic sequences: C^2-finite sequences. *Journal of Symbolic Computation*, 116:400–424, 2023.

[60] A. Jimenez-Pastor, V. Pillwein, and M.F. Singer. Some structural results on D^n-finite functions. *Advances in Applied Mathematics*, 117:0–0, June 2020.

[61] Th Kaluza. Elementarer beweis einer vermutung von K. Friedrichs und H. Lewy. *Mathematische Zeitschrift*, 37(1):689–697, 1933.

[62] M. Kauers, P. Nuspl, and V. Pillwein. Order bounds for C^2-finite sequences. In A.Dickenstein, E. Tsigaridas, and G. Jeronimo, editors, *Proceedings of the 2023 International Symposium on Symbolic and Algebraic Computation*, ISSAC '23, Tromso, Norway, pages 389–397, New York, NY, USA, July 2023. Association for Computing Machinery.

[63] Manuel Kauers. *A Mathematica Package for Computing Asymptotic Expansions of Solutions of P-finite Recurrence Equations*. Linz, Austria, November 2004.

[64] Manuel Kauers. Computer algebra and power series with positive coefficients. In *Proceedings of the FPSAC 2007*, 2006.

[65] Manuel Kauers. *D-Finite Functions*. Springer Nature Switzerland, Cham, 2023b.

[66] Manuel Kauers, Maximilian Jaroschek, and Fredrik Johansson. Ore polynomials in Sage. In *Computer algebra and polynomials*, volume 8942 of *Lecture Notes in Comput. Sci.*, pages 105–125. Springer, Cham, 2015.

[67] Manuel Kauers and Marc Mezzarobba. Multivariate ore polynomials in SageMath. *ACM Commun. Comput. Algebra*, 53(2):57–60, 2019.

[68] Manuel Kauers and Veronika Pillwein. When can we detect that a p-finite sequence is positive? In *Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation*, ISSAC '10, page 195–201, New York, NY, USA, 2010. Association for Computing Machinery.

[69] Manuel Kauers and Doron Zeilberger. Experiments with a positivity preserving operator, 2007.

[70] George Kenison, Oleksiy Klurman, Engel Lefaucheux, Florian Luca, Pieter Moree, Joël Ouaknine, Markus A. Whiteland, and James Worrell. On Positivity and Minimality for Second-Order Holonomic Sequences. In Filippo Bonchi and Simon J. Puglisi, editors, *46th International Symposium on Mathematical Foundations of Computer Science (MFCS 2021)*, volume 202 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 67:1–67:15, Dagstuhl, Germany, 2021. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

[71] George Kenison, Richard Lipton, Joël Ouaknine, and James Worrell. On the skolem problem and prime powers. In *Proceedings of the 45th International Symposium on Symbolic and Algebraic Computation*, page 289–296, New York, NY, USA, 2020. ACM.

[72] William D. Kirwin. Higher asymptotics of Laplace's approximation. *Asymptotic analysis*, 70(3/4):231–248, 2010.

[73] Yaakov Kogan and Andrei Yakovlev. Asymptotic analysis for closed multichain queueing networks with bottlenecks. *Queueing Systems*, 23(1–4):235–258, March 1996.

[74] Vassili N. Kolokoltsov. Rates of convergence in Laplace's integrals and sums and conditional central limit theorems. *Mathematics*, 8(4), 2020.

[75] Christoph Koutschan. A fast approach to creative telescoping. *Math. Comput. Sci.*, 4(2-3):259–266, 2010.

[76] Mark Krusemeyer, Daniel H. Ullman, and Paul Zeitz. The eighty-first william lowell putnam mathematical competition. *The American Mathematical Monthly*, 128(8):675–686, 2021.

[77] Pierre Lairez. Computing periods of rational integrals. *Mathematics of Computation*, 85(300):1719–1752, November 2015.

[78] Pierre Lairez and Pierre Vanhove. Algorithms for minimal Picard-Fuchs operators of Feynman integrals. *Lett. Math. Phys.*, 113(2):Paper No. 37, 45, 2023.

[79] Ben Lichtin. The asymptotics of a lattice point problem associated to a finite number of polynomials i. *Duke Mathematical Journal*, 63(1), 1991.

[80] Ben Lichtin. The asymptotics of a lattice point problem associated to a finite number of polynomials ii. *Duke Mathematical Journal*, 77(3), March 1995.

[81] L Lipshitz. The diagonal of a d-finite power series is d-finite. *Journal of Algebra*, 113(2):373–378, March 1988.

[82] Hexuan Liu, Michael Wallner, and Guan-Ru Yu. A combinatorial framework for the pons-batle identity: Young tableaux, lattice paths, and limit laws, 2026.

[83] Florian Luca, James Maynard, Armand Noubissie, Joël Ouaknine, and James Worrell. Skolem meets Bateman-Horn, 2024.

[84] Florian Luca, Joël Ouaknine, and James Worrell. A Universal Skolem Set of Positive Lower Density. In Stefan Szeider, Robert Ganian, and Alexandra Silva, editors, *47th International Symposium on Mathematical Foundations of Computer Science (MFCS 2022)*, volume 241 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 73:1–73:12, Dagstuhl, Germany, 2022. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

[85] Florian Luca, Joël Ouaknine, and James Worrell. On Large Zeros of Linear Recurrence Sequences. In Paweł Gawrychowski, Filip Mazowiecki, and Michał Skrzypczak, editors, *50th International Symposium on Mathematical Foundations of Computer Science (MFCS 2025)*, volume 345 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 71:1–71:11, Dagstuhl, Germany, 2025. Schloss Dagstuhl – Leibniz-Zentrum für Informatik.

[86] Florian Luca, Joël Ouaknine, and James Worrell. Universal Skolem sets. In *2021 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–6, 2021.

[87] Edmond Maillet. Sur les séries divergentes et les équations différentielles. *Annales scientifiques de l'École normale supérieure*, 20:487–518, 1903.

[88] Piotr Majerski. Simple error bounds for the multivariate Laplace approximation under weak local assumptions, 2016.

[89] J.P. McClure and R. Wong. Error bounds for multidimensional laplace approximation. *Journal of Approximation Theory*, 37(4):372–390, 1983.

[90] Stephen Melczer. *An Invitation to Analytic Combinatorics: From One to Several Variables*. Texts & Monographs in Symbolic Computation. Springer International Publishing, 2021.

[91] Stephen Melczer and Marni Mishna. Singularity analysis via the iterated kernel method. *Discrete Mathematics & Theoretical Computer Science*, DMTCS Proceedings vol. AS,...(Proceedings), January 2013.

[92] Stephen Melczer and Marni Mishna. Asymptotic lattice path enumeration using diagonals. *Algorithmica*, 75(4):782–811, 2015.

[93] Stephen Melczer and Bruno Salvy. Effective coefficient asymptotics of multivariate rational functions via semi-numerical algorithms for polynomial systems. *Journal of Symbolic Computation*, 103:234–279, mar 2021.

[94] Stephen Melczer and John Hunn Smith. SagePeriods: A SageMath package for fast diagonals of rational functions. In *Proceedings of the International Congress on Mathematical Software*. Springer Cham, 2026.

[95] Stephen Melczer and Mark C. Wilson. Higher Dimensional Lattice Walks: Connecting Combinatorial and Analytic Behavior. *SIAM J. Discrete Math.*, 33(4):2140–2174, 2019.

[96] Marc Mezzarobba. Rigorous multiple-precision evaluation of D-finite functions in SageMath, 2016.

[97] Max Mignotte, Tarlok Shorey, and Robert Tijdeman. The distance between terms of an algebraic recurrence sequence. *Journal für die reine und angewandte Mathematik (Crelles Journal)*, 1984(349):63–76, May 1984.

[98] Marni Mishna, Mercedes Rosas, and Sheila Sundaram. Vector partition functions and kronecker coefficients. *Journal of Physics A: Mathematical and Theoretical*, 54(20):205204, May 2021.

[99] Hans Z. Munthe-Kaas, Olivier Verdier, and Gilles Vilmart. A short proof of Isserlis' theorem, 2025.

[100] P. Nuspl and V. Pillwein. Simple C^2-finite Sequences: a Computable Generalization of C-finite Sequences. In Marc Moreno Maza and Lihong Zhi, editors, *ISSAC '22: Proceedings of the 2022 International Symposium on Symbolic and Algebraic Computation*, pages 45–53. Association for Computing Machinery, 2022.

[101] A. M. Odlyzko. *Asymptotic enumeration methods*, page 1063–1229. MIT Press, Cambridge, MA, USA, 1996.

[102] Joël Ouaknine and James Worrell. Positivity problems for low-order linear recurrence sequences. In *Proceedings of the Twenty-Fifth Annual ACM-SIAM Symposium on Discrete Algorithms*, page 366–379, Philadelphia, PA, December 2013. Society for Industrial and Applied Mathematics.

[103] Joël Ouaknine and James Worrell. *Ultimate Positivity is Decidable for Simple Linear Recurrence Sequences*, page 330–341. Springer Berlin Heidelberg, Berlin, Heidelberg, 2014.

[104] Robin Pemantle and Mark Wilson. Asymptotic expansions of oscillatory integrals with complex phase, 2009.

[105] Robin Pemantle and Mark C. Wilson. Asymptotics of multivariate sequences. *Journal of Combinatorial Theory, Series A*, 97(1):129–161, January 2002.

[106] Robin Pemantle and Mark C. Wilson. Asymptotics of multivariate sequences ii: Multiple points of the singular variety. *Combinatorics, Probability and Computing*, 13(4–5):735–761, 2004.

[107] Robin Pemantle and Mark C. Wilson. Twenty combinatorial examples of asymptotics derived from multivariate generating functions. *SIAM Review*, 50(2):199–272, January 2008.

[108] Robin Pemantle, Mark C. Wilson, and Stephen Melczer. *Analytic Combinatorics in Several Variables*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2 edition, 2024.

[109] Veronika Pillwein. On the positivity of the gillis–reznick–zeilberger rational function. *Advances in Applied Mathematics*, 104:75–84, 2019.

[110] Iosif Pinelis. Exact lower and upper bounds on the incomplete gamma function. *Mathematical Inequalities & Applications*, 2(4):1261–1278, 2020.

[111] Carine Pivoteau and Bruno Salvy. Effective asymptotics of combinatorial systems. Technical report, arXiv, 2025.

[112] J. Popken. *Über arithmetische Eigenschaften analytischer Funktionen: proefschrift ...* N.V. Noord-Hollandsche Uitgeversmaatschappij, 1935.

[113] Andrew Elvey Price, Wenjie Fang, Baptiste Louf, and Michael Wallner. Bivariate asymptotics via random walks: application to large genus maps. Preprint, arXiv:2506.06924 [math.CO] (2026), 2026.

[114] Lakshmi Ramesh and Nicolas Weiss. Exact volumes of semi-algebraic convex bodies. *Accepted to Proceedings of the 2026 International Symposium on Symbolic and Algebraic Computation (ISSAC 2026)*, 2026.

[115] Bruno Salvy. Linear differential equations as a data structure. *Foundations of Computational Mathematics*, 19(5):1071–1112, January 2019.

[116] Bruno Salvy and Paul Zimmermann. Gfun. *ACM Transactions on Mathematical Software*, 20(2):163–177, 1994.

[117] Alexander D. Scott and Alan D. Sokal. Complete monotonicity for inverse powers of some combinatorially defined polynomials. *Acta Mathematica*, 213(2):323–392, 2014.

[118] Thoralf Skolem. Ein verfahren zur behandlung gewisser exponentialer gleichungen und diophantischer gleichungen. *Comptes rendus du Huitième Congrès des Mathématiciens Scandinaves*, pages 163–188, 1934.

[119] Richard P Stanley et al. A survey of alternating permutations. *Contemp. Math*, 531:165–196, 2010.

[120] Elias M. Stein and Timothy S. Murphy. *Harmonic analysis: Real-variable methods, orthogonality, and oscillatory integrals*. Princeton University Press, August 1993.

[121] Armin Straub. Positivity of Szegö's rational function. *Advances in Applied Mathematics*, 41(2):255–264, 2008.

[122] Armin Straub and Wadim Zudilin. Positivity of rational functions and their diagonals. *Journal of Approximation Theory*, 195:57–69, jul 2015.

[123] G. Szegö. Über gewisse potenzreihen mit lauter positiven koeffizienten. *Mathematische Zeitschrift*, 37(1):674–688, Dec 1933.

[124] Joris van der Hoeven. Computing with d-algebraic power series. *Applicable Algebra in Engineering, Communication and Computing*, 30(1):17–49, May 2018.

[125] Duco van Straten. Calabi-Yau operators. In *Uniformization, Riemann-Hilbert correspondence, Calabi-Yau manifolds & Picard-Fuchs equations*, volume 42 of *Adv. Lect. Math. (ALM)*, pages 401–451. Int. Press, Somerville, MA, 2018.

[126] N. K. Vereshchagin. Occurrence of zero in a linear recursive sequence. *Mathematical Notes of the Academy of Sciences of the USSR*, 38(2):609–615, August 1985.

[127] Michael S. Waterman. *Some Molecular Biology*, page 5–27. Springer US, Boston, MA, 1995.

[128] R. Wong. *Asymptotic approximations of integrals*. Academic Press, 1989.

[129] R. Wong and J. P. McClure. On a method of asymptotic evaluation of multiple integrals. *Mathematics of Computation*, 37(156):509–521, 1981.

[130] Yaming Yu. Positivity of the rational function of Gillis, Reznick and Zeilberger, 2019.

[131] Doron Zeilberger. The method of creative telescoping. *Journal of Symbolic Computation*, 11(3):195–204, 1991.

[132] Tomasz M. Łapiński. Multivariate laplace's approximation with estimated error and application to limit theorems. *Journal of Approximation Theory*, 248:105305, 2019.