Waring rank, Border rank and support concentration of partials

by

Abhiroop Sanyal

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis, we study the classical problem of decomposition of a homogeneous polynomial into sums of powers of linear forms with minimal summands (also known as the *Waring Rank* of the polynomial) and related problems. The case of quadratic polynomials and binary forms was studied by Sylvester [45] in his seminal paper in 1851 and the case for generic polynomials was resolved more than a century later by Alexander and Hirschowitz [3] in 1995. The problem is NP-hard computationally and finding the Waring rank for several interesting classes of polynomials, for example, the general $n \times n$ symbolic determinant/permanent, remains an open problem.

An important parameter in the study of this problem is the dimension D of the vector space of partial derivatives of the given polynomial. It is known that if the Waring rank of a polynomial in n variables of degree d is s, then D is at most $s(d+1)$. A longstanding conjecture states that given D, the Waring rank is upper bounded by a polynomial in n, d and D.

To study this conjecture, we restrict to a very special class of polynomials with no redundant variables (also called concise), which we call 1-support concentrated polynomials, that are defined by the following property: Given such a polynomial f, all its partial derivatives can be obtained as linear combinations of derivatives with respect to powers of a fixed set of n linearly independent linear forms. A crucial property of such f is that the dimension of partial derivatives of f at any degree is at most n. We show that the converse is true: any concise polynomial for which dimension of partial derivatives at any degree is $\leq n$, is also 1-support concentrated. We also generalize an example given by Stanley to give an explicit class of concise polynomials $ST_{n,d}$ in $\binom{n+d-2}{d-1} + n$ variables and degree d that is 1-support concentrated.

A polynomial is a *direct sum* if it can be written as a sum of two polynomials in distinct sets of variables, up to a linear change of variables. A polynomial f is a *limit of direct sums* if there is a sequence of polynomials, each a direct sum, converging to f. Necessary and sufficient conditions for a polynomial to be a direct sum or a limit of direct sums was extensively studied by Buczyńska et al. [16] and Kleppe [33]. We show that any concise 1-support concentrated polynomial with degree $d \geq 2n+1$ is a limit of direct sums. We also show that $ST_{n,d}$ (which does not satisfy the previous degree hypothesis) is a limit of direct sums.

The border rank of a homogeneous polynomial f is the minimal r such that there is a sequence of polynomials, each with Waring rank at most r, converging to f. The *debordering* question is as follows: given f with border Waring rank r, what is the best upper bound for Waring rank of f in terms of n, r and d? The best known bound is due to [20]. In context of this problem, it is interesting to find examples f for which Waring rank of f is strictly greater than its border Waring rank. We show that $ST_{3,4}$ and $ST_{2,d}$, for any $d \geq 3$, have this property.

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Finally, I would like to thank my parents for their unwavering support over my entire life, especially during the past few years. I look forward to seeing you soon!

Dedication

This thesis is dedicated to my grandparents.

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Chapter 1

Introduction

The Waring problem for polynomials involves decomposing a homogeneous polynomial into a sum of powers of linear forms with minimal summands. The problem is also interesting as it is the symmetric version of determining the minimum additive decomposition of tensors (Appendix A) and has many applications in signal processing, algebraic statistics and other problems. Recently, [38] showed a surprising connection of this problem to the areas of parameterized and exact algorithms, producing faster algorithms for approximately counting the number of bounded subgraphs of a given treewidth and other problems.

This chapter is organized as follows: In Section 1.1, we define the notions of Waring rank and Border Waring rank of homogeneous polynomials and mention some classes of polynomials for which either bounds or exact ranks are known. In Section 1.2, we define notions of cactus and smoothable ranks of a homogeneous polynomial and survey the known relationships among the various notions of rank. In Section 1.3, we describe the problem of determining the direct sum decomposability of a given homogeneous polynomial, which is the second major problem we are interested in and mention some known results. In Section 1.4, we state the original results of this thesis.

1.1 Waring Rank and Border rank of forms

We denote by R, the polynomial ring $\mathbb{C}[x_1, \cdots, x_n]$.

Definition 1.1.1. The Waring rank of a polynomial $f \in \mathbb{C}[x_1, \cdots, x_n]_d := R_d$ is defined as the smallest integer r such that f can be written as:

$$f = \sum_{i=1}^{r} \ell_i^d$$

for some $\ell_i \in R_1 \; \forall \; i$. We denote it as $\mathrm{WR}(f)$. We'll call a decomposition such as above witnessing $\mathrm{WR}(f)$ to be a *Waring decomposition* of f.

For any topological space X, we say $f : X \longrightarrow \mathbb{R}$ is *lower semicontinuous* if $f^{-1}([a, \infty])$ is open $\forall\ a \in \mathbb{R}$. It follows from the continuity of the determinant that the rank of a matrix is lower semicontinuous. However, Waring rank is not lower semicontinuous. This is observed by considering the polynomial $x_1^2x_2 \in \mathbb{C}[x_1, x_2]_3$. A Waring decompostion for f is as follows:

$$x_1^2x_2 = \left(\frac{x_1 + x_2}{\sqrt[3]{6}}\right)^3 + \left(\frac{x_2 - x_1}{\sqrt[3]{6}}\right)^3 + \left(\frac{\sqrt[3]{-2}x_2}{\sqrt[3]{6}}\right)^3$$

On the other hand consider the identity:

$$x_1^2x_2 = \lim_{\epsilon \to 0} \frac{1}{3}\left(\left(\frac{x_1}{\epsilon} + \epsilon^2 x_2\right)^3 - \left(\frac{x_1}{\epsilon}\right)^3\right)$$

The above is an example of a sequence of polynomials $(f_\epsilon)_{\epsilon>0}$ converging to $x_1^2x_2$ such that $\mathrm{WR}(f_\epsilon) = 2\ \forall\ \epsilon > 0$, thus clearly violating lower semicontinuity. Thus, one can define the *border Waring rank* as the semicontinuous closure of the Waring rank of a polynomial.

Definition 1.1.2. The border Waring rank of $f \in {}_d$ is defined to be the smallest r such that f can be written as:

$$f = \lim_{\epsilon \to 0} \sum_{i=1}^{r} \ell_i^d$$

where $\ell_i \in \mathbb{C}(\epsilon)[x_1, \cdots, x_n]_1$. We denote it as $\underline{\mathrm{WR}}(f)$. We'll call a decomposition such as above witnessing $\underline{\mathrm{WR}}(f)$ to be a *border Waring decomposition* of f.

It follows from the definition that $\underline{\mathrm{WR}}(f) \leq \mathrm{WR}(f)$. The example discussed above can be generalized to show that $2 = \underline{\mathrm{WR}}\big(x_1^{d-1}x_2\big) < \mathrm{WR}\big(x_1^{d-1}x_2\big) = d$. Computing the Waring rank of a polynomial is NP-hard, as shown by [42, Theorem 6.]. However, we do know the Waring rank for a generic form and for some explicit classes of polynomials . Some of these we discuss below.

1. (Folklore) Given any $f \in R_2$, we can find a symmetric matrix A such that $f = \overline{X}A\overline{X}^T$, where $\overline{X} = [x_1, \cdots, x_n]$. Using the fact that any complex symmetric matrix can be decomposed as UDU^T (Takagi decomposition) where U is unitary and D is diagonal, we can conclude that $\mathrm{WR}(f)$ is exactly the rank of the corresponding symmetric matrix.

2. [45] Any generic form $f \in \mathbb{C}[x_1, x_2]_d$ can be written as a sum of $\lceil \frac{d+1}{2} \rceil$ powers of linear forms.

3. [3, Theorem 2.] Any generic form $f \in \mathbb{C}[x_1, \cdots, x_n]_d$ can be written as a sum of $\left\lceil \frac{\binom{n+d-1}{d}}{n} \right\rceil$ powers of linear forms except the following cases:

   - $f \in \mathbb{C}[x_1, \cdots, x_n]_2^{gen}$: $\mathrm{WR}(f) = n$.
   - $f \in \mathbb{C}[x_1, x_2, x_3]_4^{gen}$: $\mathrm{WR}(f) = 6$.
   - $f \in \mathbb{C}[x_1, \cdots, x_4]_4^{gen}$: $\mathrm{WR}(f) = 10$.

- $f \in \mathbb{C}[x_1, \cdots, x_5]_3^{gen}$: $\mathrm{WR}(f) = 8$.
- $f \in \mathbb{C}[x_1, \cdots, x_5]_4^{gen}$: $\mathrm{WR}(f) = 17$.

4. [43, Theorem 1.1.] For the 3×3 symbolic permanent per_3: $\mathrm{WR}(\mathrm{per}_3) = 16$.

5. [18, Proposition 3.1.] For $f = \prod_{i=1}^{n} y_i^{d_i}$ where $d_i \leq d_{i+1}$ and $y_i \in \{x_1, \cdots, x_n\}$ $\forall$ i: $\mathrm{WR}(f) = \prod_{i=2}^{n} (d_i + 1)$.

6. [18, Theorem 3.2.] For monomials m_i, $1 \leq i \leq k$, in pairwise disjoint sets of variables and $f = \sum_{i=1}^{k} m_i$: $\mathrm{WR}(f) = \sum_{i=1}^{k} \mathrm{WR}(m_i)$.

The last example is a resolution of Strassen's Conjecture on Waring rank for sum of monomials. The general conjecture is stated below.

Conjecture 1.1.3 (Strassen's Conjecture)**.** *For a form f, if $f = \sum_{i=1}^{k} f_i$ where f_i are forms in pairwise disjoint sets of variables, then we have:*

$$\mathrm{WR}(f) = \sum_{i=1}^{k} \mathrm{WR}(f_i)$$

Given the results on generic rank of a form $f \in R_d$ (denoted as $\mathrm{WR}_{\mathrm{gen}}(f)$), one may wonder what the worst possible upper bound of a form in R_d can be. Let us denote by $\mathrm{WR}_{max}(n, d)$, the maximum Waring Rank of a form in $f \in R_d$. By taking a basis of a power of linear forms, we can conclude that

$$\mathrm{WR}_{max}(n, d) \leq \binom{n+d-1}{n}$$

[30, Theorem 6] shows:

$$\mathrm{WR}_{\mathrm{max}}(n, d) \leq \binom{n+d-2}{d-1} - \binom{n+d-3}{d-6}$$

which [7, Remark 4.18] improved to:

$$\mathrm{WR}_{\mathrm{max}}(n, d) \leq \binom{n+d-2}{d-1} - \binom{n+d-6}{d-3} - \binom{n+d-7}{d-4}$$

This bound implies that $\mathrm{WR}_{\mathrm{max}}(f)$ is asymptotically $\left(\frac{nd}{n+d-1}\right)$ times $\mathrm{WR}_{\mathrm{gen}}(f)$. However, [12, Theorem 1] shows the following:

$$\mathrm{WR}_{\mathrm{gen}}(f) \leq \mathrm{WR}_{\mathrm{max}}(f) \leq 2 \cdot \mathrm{WR}_{\mathrm{gen}}(f)$$

Combining the above with Item 3, we get:

$$\mathrm{WR}_{\max}(n,d) \leq 2 \cdot \left\lceil \frac{\binom{n+d-1}{d}}{n} \right\rceil$$

Except for small cases, like $n = 2$ or $d \leq 2$, the above bound is much better than the bound given by [7, Remark 4.18].

For the border Waring rank the situation is more complicated. While some results are known for small monomials and other polynomials of a small constant number of variables or degree [34], the exact border Waring of a large class of well-known polynomials in any number of variables/degree is not known except the following cases:

- For the general case of $f = \prod_{i=1}^{n} y_i^{d_i}$ where $d_i \leq d_{i+1}$ and $y_i \in \{x_1, \cdots, x_n\}$ $\forall$ i, only an exact result is known in the restricted case of: $d_n \geq \sum_{i=1}^{n-1} d_i$ [34, Theorem 11.3]. In particular: $\underline{\mathrm{WR}}(f) = \prod_{i=1}^{n-1} d_i$. It is known [34, Theorem 11.2] that $\prod_{i=1}^{n-1} d_i$ is an upper bound for any monomial ($d_i \geq d_{i-1}$). It is conjectured [34] that this result is true for all monomials.
- [22] showed that for $f = \left(x_1^2 + x_2^2 + x_3^2\right)^d$, we have $\underline{\mathrm{WR}}(f) = \binom{d+2}{2}$.

1.2 Some more notions of rank and known Debordering Results

The exposition in this section borrows heavily from parts of [20, Appendix A].

The Hilbert function of a homogeneous ideal $I \subset S$ is a function $\mathrm{HF} : \mathbb{N} \longrightarrow \mathbb{N}$ defined as $\mathrm{HF}(S/I, k) := \dim_{\mathbb{C}}(S/I)_k = \dim_C(S_k) - \dim_{\mathbb{C}}(I_k)$.

We denote by $f^{\perp}$, the ideal defined by all polynomials g such that $g\left(\partial_{x_1}, \cdots, \partial_{x_n}\right) f = 0$. An ideal I is said to be *apolar* to f is $I \subseteq f^{\perp}$. A *scheme* [21, Section 1.2] is said to be apolar to f if its defining ideal is apolar to f. A polynomial is said to be *concise* if $\left(f^{\perp}\right)_1 = 0$.

We discuss the apolarity lemma for Waring rank in Chapter 2 Lemma 2.1.4. The following result from [15] gives an apolarity result for border rank.

Lemma 1.2.1 ([15]). *Consider a homogeneous polynomial $f \in R_d$. Then $\underline{\mathrm{WR}}(f) \leq r$ if and only if f is apolar to an ideal I that is a limit of ideals of r points.*

The notion of limits in the above lemma corresponds to taking limits in the *multigraded hilbert schemes* ([15, Section 3] and [20, Appendix A.4 and A.5]).

In order to define the notion of "rank" more generally, we need the following definition of *Krull dimension*.

Definition 1.2.2 (Krull dimension)**.** Let $\mathcal{R}$ be any commutative ring with unity. The *Krull dimension* of $\mathcal{R}$ is the supremum of all non-negative integers n such that there is an increasing chain:

$$P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$$

of prime ideals of $\mathcal{R}$.

In general, one can consider the following template definition for rank [20, Definition 23].

Definition 1.2.3 (Rank-template)**.** Let $\mathcal{I}$ be a class of ideals with Krull dimension 1. For $f \in R_d$, define the $\mathcal{I}$-rank of f as the minimal r such that $\exists\, I \subset \mathcal{I}$ apolar to f with length r.

We get Waring rank and border Waring rank by considering ideals of points and limits of ideals of points, respectively, as the class $\mathcal{I}$ in the above definition.

The saturation I^{sat} of a homogeneous ideal $I \subset S$ is defined as follows: $I^{\text{sat}} := \left\{ g \in S \mid x_i^k g \in I \text{ for some } k \in \mathbb{N} \text{ and } \forall\, i \right\}$. An ideal I is said to be saturated if $I^{\text{sat}} = I$ (for more on this, consult [24, Section 5.5]).

We define two more notions of rank below:

- Cactus Rank (denoted by $\text{CR}(f)$): Obtained by considering the class of saturated ideals with constant Hilbert function in Definition 1.2.3 (by [24, Corollary 5.5.11] these are the same as ideals of zero-dimensional schemes).
- Smoothable Rank (denoted by $\text{SR}(f)$): Obtained by considering saturated limits of ideals of points in Definition 1.2.3.
- Border Cactus Rank (denoted by $\underline{\text{CR}}(f)$): Obtained by considering limits of saturated ideals similar to Lemma 1.2.1.

The following table from [20] gives the various definitions of rank obtained by taking different classes of ideals I.

Class of ideals	Rank	Notation
Ideals of points (radical saturated ideals)	Waring rank	$\text{WR}(f)$
Limits of ideals of points	Border Waring rank	$\underline{\text{WR}}(f)$
Smoothable ideals (saturated limits of ideals of points)	Smoothable rank	$\text{SR}(f)$
Saturated ideals	Cactus rank	$\text{CR}(f)$
Saturable ideals (limits of saturated ideals)	Border cactus rank	$\underline{\text{CR}}(f)$

Table 1.1: Various notions of rank.

[8] shows the following relationship between the various notions of rank.

Theorem 1.2.4. *Given $f \in R_d$ for some $d \in \mathbb{N}$, we have:*

$$\underline{\mathrm{CR}}(f) \leq \mathrm{CR}(f) \leq \mathrm{SR}(f) \leq \mathrm{WR}(f).$$

and

$$\underline{\mathrm{CR}}(f) \leq \underline{\mathrm{WR}}(f) \leq \mathrm{SR}(f) \leq \mathrm{WR}(f)$$

From the example $f_1 = x_1^{d-1}x_2$, it is clear that $\mathrm{WR}(f_1) > \underline{\mathrm{WR}}(f_1)$ is possible. [8, Example 2.8] shows that for the polynomial $f_2 = x_1^2x_3 + 6x_2^2x_4 - 3\left(x_1 + x_2\right)^2 x_5$ has $\underline{\mathrm{WR}}(f_2) = 5$ while $\mathrm{SR}(f_2) \geq 6$. This example also shows $\mathrm{CR}(f_2) > \underline{\mathrm{WR}}(f_2)$. [10, Example 1] gives an example of a cubic polynomial f such that $\mathrm{SR}(f) > \mathrm{CR}(f)$. For a generic cubic form in n variables, we have $\mathrm{WR}(f) = \underline{\mathrm{WR}}(f) = \left\lceil \frac{\binom{n+2}{3}}{n+1} \right\rceil$ [3]. [10, Theorem 4] proves that the cactus rank of a generic cubic form in n variables is at most $2n$. Thus, for $n = 8$, we get that $\mathrm{CR}(f) < \underline{\mathrm{WR}}(f)$ for a generic cubic form f in n variables.

What we mean by "*debordering results*" would be results of the following form: Suppose X and Y are notions of rank such that for all homogeneous polynomials f, X-rank of $f \leq$ Y-rank of f. Then, if X-rank of f (in n variables and degree d) is r then Y-rank of f is at most $N(r, n, d)$ where N is some function from $\mathbb{N}^3 \longrightarrow \mathbb{N}$.

In the context of debordering, the following conjecture was proposed by [6, Question 1].

Conjecture 1.2.5 ([6, Question 1])**.** *Suppose $f \in R_d$ for some $d \in \mathbb{N}$. Then, we have:*

$$\mathrm{WR}(f) \leq d\left(\underline{\mathrm{WR}}(f) - 1\right)$$

Note that the above conjecture is clearly true for monomials using [34]. Using the $\mathrm{WR}_{\max}$ upper bound from [12] stated earlier in the previous section, we get $\mathrm{WR}(f) \leq \frac{d(d+1)(d+2)}{9}$ for $f = \left(x_1^2 + x_2^2 + x_3^2\right)^d$ and $d \geq 5$. Since $\underline{\mathrm{WR}}(f) = \binom{d+2}{2}$ by [22] and

$$\frac{d(d+1)(d+2)}{9} < d \cdot \binom{d+2}{2} - d$$

Conjecture 1.2.5 is true for f. More evidence for this conjecture can be found in (for $\underline{\mathrm{WR}}(f) = 3$) [9, Theorem 32 and Theorem 37],(for $\underline{\mathrm{WR}}(f) = 4$) [5, Theorem 1] and (for $\underline{\mathrm{WR}}(f) = 5$ and $\deg(f) \geq 9$) [4, Theorem 1].

For $f \in R_d$, if $\underline{\mathrm{WR}}(f) = r$, then f can be transformed into a polynomial with r variables by a linear change of coordinates, by Lemma 2.2.9 and Lemma 2.2.7. Since the maximum possible Waring rank of a form in r variables of degree d is $O\left(\max\left\{r^d, d^r\right\}\right)$, this gives an upper bound for $\mathrm{WR}(f)$. The best general result known for Waring rank vs. Border Waring rank of a form is due to [20, Theorem 1.]

Theorem 1.2.6 ([20, Theorem 1])**.** *Let $f \in R_d$ with $\underline{\mathrm{WR}}(f) = r$. Then,* $\mathrm{WR}(f) \leq 4^r \cdot d$.

Note that we cannot get an upper bound only in terms of r as evidenced by the polynomial $x_1^{d-1}x_2$, where $\mathrm{WR}(f) = d$ while $\underline{\mathrm{WR}}(f) = 2$.

A conjecture for smoothable rank was also proposed in [6, Question 3].

Conjecture 1.2.7 ([6, Question 3])**.** *Let $f \in R_d$ for some $d \in \mathbb{N}$. Then, we have:*

$$\mathrm{WR}(f) \leq (d-1)\mathrm{SR}(f) - d + 2$$

It is also known ([14, Proposition 2.5]) that for high enough degree, border Waring rank is the same as smoothable rank and border cactus rank is the same as cactus rank.

Theorem 1.2.8 ([14, Proposition 2.5])**.** *Suppose $f \in R_d$ for some $d \in \mathbb{N}$. Then,*

$$\deg(f) \geq \underline{\mathrm{WR}}(f) - 1 \implies \underline{\mathrm{WR}}(f) = \mathrm{SR}(f)$$

$$\deg(f) \geq \underline{\mathrm{CR}}(f) - 1 \implies \underline{\mathrm{CR}}(f) = \mathrm{CR}(f)$$

[20, Theorem 32 and Theorem 33] also provides an exact characterization of polynomials with border Waring rank 2 or 3.

Theorem 1.2.9 ([20, Theorem 32])**.** *Suppose $f \in R_d$ for some $d \in \mathbb{N}$ and $\underline{\mathrm{WR}}(f) = 2$. Then, f must be of the form $l_1^d + l_2^d$ or $l_1^{d-1}l_2$, where $l_1, l_2 \in R_1$.*

Theorem 1.2.10 ([20, Theorem 33])**.** *Suppose $f \in R_d$ for some $d \in \mathbb{N}$ and $\underline{\mathrm{WR}}(f) = 3$. Then, f must be one of three forms:*

- $\ell_1^d + \ell_2^d + \ell_3^d$
- $\ell_1^d + \ell_2^{d-1}\ell_3$
- $\ell_1^{d-1}\ell_2 + \ell_1^{d-2}\ell_3^2$

where $\ell_1, \ell_2, \ell_3 \in R_1$.

It is also known that for $f \in R_d$, the quantity $\sum_{k=0}^{d} \mathrm{HF}(S/I, k)$ computes the dimension of the vector space spanned by all the partial derivatives of f (Lemma 2.2.3). It is easy to see that for any polynomial of $\mathrm{WR}(f) \leq s$, we have $\sum_{k=0}^{d} \mathrm{HF}(S/I, k) \leq s(d+1)$. However, a converse to this is not known.

Conjecture 1.2.11. *There exists a constant $k \in \mathbb{N}$, such that for any $f \in R_d$, with $\sum_{j=0}^{d} \mathrm{HF}(S/I, j) = D$, we have:*

$$\mathrm{WR}(f) \leq (ndD)^k$$

1.3 Direct Sum decomposition of polynomials

A polynomial $f \in R_d$ is said to be a direct sum (denoted by $f \in \mathrm{DirSum}$) if $\exists\ f_1 \in \mathbb{C}[x_1, \cdots, x_t]_d$ and $f_2 \in \mathbb{C}[x_{t+1}, \cdots, x_n]$ and independent linear forms $\{\ell_1, \cdots, \ell_n\}$ such that:

$$f = f_1(\ell_1, \cdots, \ell_t) + f_2(\ell_{t+1}, \cdots, \ell_n)$$

If f is a limit of direct sums, then we say $f \in \overline{\mathrm{DirSum}}$. It can be shown that the polynomial $f = x_1^2 x_2$ is not a direct sum but a limit of direct sums.

[16, Theorem 1] proves the following result.

Theorem 1.3.1 ([16, Theorem 1])**.** *Let $n \geq 2$ and $d \geq 3$. Given $f \in R_d$ if $f^\perp$ has a minimum generator of degree d, then $f \in \overline{\mathrm{DirSum}}$. Conversely, for every concise polynomial $\in \overline{\mathrm{DirSum}}$, $f^\perp$ has a minimal generator of degree d.*

We discuss this result and more general results proved in [33] in Section 2.4.

1.4 Our Results

We focus on a very special case of Conjecture 1.2.11. In particular, we restrict our attention to concise polynomials $f \in R_d$ such that there are n independent linear forms $\{\ell_1, \cdots, \ell_n\}$ for which the following holds true:

$$\mathrm{HF}(S/I, k) = \dim\left(\mathcal{V}\left\{\partial_{\ell_i^k} f\right\}_{i=1}^{n}\right) \quad \forall 1 \leq k \leq d$$

where $\mathcal{V}(T)$ denotes the vector space spanned by the set T of vectors. Such polynomials are called *1*-support concentrated with respect to $\{\ell_1, \cdots, \ell_n\}$. It is clear from the definition that $\mathrm{HF}(S/I, k) \leq n$, $\forall\ 1 \leq k \leq d$ for any 1-support concentrated polynomial. We show that the converse is true.

Theorem 1.4.1. *Suppose $f \in R_d$ concise and satisfies $\mathrm{HF}(S/I, k) \leq n$ for all $k \in \mathbb{N}$, where $I = f^\perp$. Then f is 1-*support concentrated *w.r.t some linearly independent forms $\{\ell_1, \cdots, \ell_n\}$.*

We originally conjectured that for any concise and 1-support concentrated $f \in R_d$, $\underline{\mathrm{WR}}(f) = n$. However, this turns out not to be true. Assume we are in the regime $d \geq 2n + 1$. Then, the following holds:

Proposition 1.4.2. *Suppose $f \in R_d$ where $d \geq 2n + 1$. Also, assume f is concise and is 1-*support concentrated *w.r.t n independent linear forms $\ell_1, ..., \ell_n$. Then we have $HF(S/I, k) = n$ for $1 \leq k \leq d - 1$ where $I = f^\perp$.*

The proofs of the above results are presented in Section 3.1.

[16, Proposition 5.9] proved the following result (they use the notion of cactus rank but this equivalent proposition holds).

Theorem 1.4.3 ([16, Proposition 5.9]). *Suppose $n \geq 14$ and $d \geq 2n-1$. Then, $\exists$ concise $f \in R_d$ such that $\mathrm{HF}(S/I,k) = n$ for $1 \leq k \leq d-1$ (where $I = f^{\perp}$) but $\underline{\mathrm{WR}}(f) > n$.*

Combining this with Theorem 1.4.1, we get:

Corollary 1.4.4. *Suppose $n \geq 14$ and $d \geq 2n-1$. Then $\exists$ a concise and 1-*support concentrated *$f \in R_d$ such that $\underline{\mathrm{WR}}(f) > n$.*

To demonstrate a specific class of polynomials for which the hypothesis is true, we generalize an example of [44, Example 4.3] to get a class of polynomial $ST_{n,d}$ of degree d in $\binom{n+d-2}{d-1} + n$ variables (for definition, check Example 2.3.8 and the paragraph following it).

Lemma 1.4.5. *For every $d \geq 4$, $ST_{n,d}$ is concise and $\mathrm{HF}(S/I,k) \leq \binom{n+d-2}{d-1} + n \ \forall\ 1 \leq k \leq d$, where $I = (ST_{n,d})^{\perp}$.*

Concerning the problem of direct sum decomposition, we show the following result (proof in Section 3.1).

Theorem 1.4.6. *Let $d \geq 2n+1$ and suppose $f \in R_d$ is concise and 1-*support concentrated. *Then, $f \in \overline{\mathrm{DirSum}}$.*

Note that for $ST_{n,d}$, since $\binom{n+d-2}{d-1} + n > d$, the above no longer applies. However, we show that $ST_{n,d}$ is still expressible as a limit of direct sums.

Theorem 1.4.7. *$\left(ST_{n,d}^{\perp}\right)$ has exactly $\binom{n+d-1}{d}$ minimal generators of degree d. In particular, $ST_{n,d} \in \overline{\mathrm{DirSum}}$.*

In light of the question of debordering of Waring Rank, it is interesting to find explicit examples of polynomials f where there is a separation of $\mathrm{WR}(f)$ and $\underline{\mathrm{WR}}(f)$. Some examples include the polynomials $x_1^{d-1}x_2$ and [34, Proposition 7.1, 7.4 and 11.10]. We show another class of examples where a separation is observed.

Lemma 1.4.8. *For any $f \in ST_{n,d}$ and $d \geq 3$, we have $\mathrm{WR}(f) \geq 2\binom{n+d-2}{d-1} + n$.*

We use the above result and explicit constructions for $\underline{\mathrm{WR}}(f)$ to separate the Waring and border Waring rank for the following examples.

Theorem 1.4.9. *The following polynomials have $\mathrm{WR}(f) > \underline{\mathrm{WR}}(f)$:*

- *For $f = ST_{3,4}$, we have: $\mathrm{WR}(f) \geq 23$ while $\underline{\mathrm{WR}}(f) = 13$.*
- *For $f = ST_{2,d}$, for any $d \geq 3$ we have: $\mathrm{WR}(f) \geq 2d+2$ while $\underline{\mathrm{WR}}(f) = d+2$.*

Since in both cases above, the border Waring rank was equal to the number of variables, we make the following bold conjecture.

Conjecture 1.4.10. *$\underline{\mathrm{WR}}\left(ST_{n,d}\right) = \binom{n+d-2}{d-1} + n$.*

The proofs of all results concerning $ST_{n,d}$ are presented in Section 3.2.

Chapter 2

Preliminaries

This chapter is organized as follows: Section 2.1 explains the apolarity lemma and properties of Gorenstein Artin Algebras. Section 2.2 describes properties of Hilbert functions of apolar ideals, the notion of essential variables of a polynomial, and growth estimates for Hilbert functions of ideals. Section 2.3 deals with the main conjecture of this thesis: a polynomial upper bound on Waring rank in terms of partial derivatives. In the process, we define the 1-support concentrated class of polynomials and state original results about the Hilbert functions of apolar ideals of such polynomials. We also generalize an example of Stanley to give an explicit class of 1-support concentrated polynomials. Section 2.4 introduces the problem of direct sum decomposition of polynomials and states known results. We state our result that 1-support concentrated polynomials of high enough degree are direct sums as well as the result that Stanley polynomials are direct sums. We present the more general version of the problem due to [33] and sketch the proofs of his results. We also present a randomized algorithm due to [32] for testing if a polynomial is a direct sum. In the final section i.e Section 2.5 we state some results that are used to prove the results in this thesis in Chapter 3.

2.1 Apolarity and Artinian Gorenstein Algebras

One of the main ingredients used in proving upper bounds of Waring and border Waring ranks of forms is the Apolarity Lemma. Before stating the lemma, we define some important algebraic objects essential to the study of Waring/border Waring rank.

Definition 2.1.1 (Apolarity action)**.** Consider the polynomial ring $S := \mathbb{C}[y_1, \cdots, y_n]$ and given $g \in S$ and $f \in R$ define the *apolarity action* $\circ$ of S on R as follows:

$$g \circ f := g\left(\partial_{x_1}, \cdots, \partial_{x_n}\right) f$$

For example, if $f = x_1^5 + x_1 x_2^4$ and $g = y_1 y_2$ then $g \circ f = 4x_2^3$.

Given a vector space V over $\mathbb{C}$, a bilinear form $B : V \times V \longrightarrow \mathbb{C}$ is called *non-degenerate* if for any $0 \neq u \in V\ \exists\ 0 \neq w \in V$ such that $B(u, w) \neq 0$. It is a known linear-algebraic fact that if V is a finite-dimensional vector space then $\exists$ an orthogonal basis of V with respect to any non-degenerate bilinear product. Since $S_i \cong R_i$ as finite-dimensional vector spaces, one can show that the apolarity action $\circ$ induces a non-degenerate bilinear form: $\circ : S_i \times R_i \longrightarrow \mathbb{C}$. Hence, given a homogeneous ideal $I \subset R$, one can define the orthogonal space $I_i^{\perp} \subset S_i$ as follows:

$$I_i^{\perp} = \{g \in S_i \mid g \circ f = 0 \ \forall f \in I_i\}$$

For a single polynomial homogeneous polynomial, we define the *annihilator ideal/apolar ideal* as follows.

Definition 2.1.2 (Apolar Ideal). Let $f \in R_d$. The apolar ideal of f, denoted by $f^{\perp}$, is defined as:

$$f^{\perp} := \{g \in S \mid g \circ f = 0 \ \}$$

We are now ready to state the apolarity Lemma. This form of the apolarity lemma and it's proof has been taken from [33, Lemma 1.4.].

Lemma 2.1.3 ([33, Lemma 1.4.]). *Let $f_1, \cdots, f_s, g_1, \cdots, g_t \in R_d$ and $\mathcal{V}(f_1, \cdots, f_s)$ denote the vector space spanned by the polynomials f_i, $1 \leq i \leq s$ (analogously for the g_i's), Then, the following are equivalent:*

1. $\mathcal{V}(f_1, \cdots, f_s) \subseteq \mathcal{V}(g_1, \cdots, g_t)$.
2. $\bigcap_{i=1}^{s} \left(f_i^{\perp}\right) \supseteq \bigcap_{i=1}^{t} \left(g_i^{\perp}\right)$.
3. $\bigcap_{i=1}^{s} \left(f_i^{\perp}\right)_d \supseteq \bigcap_{i=1}^{t} \left(g_i^{\perp}\right)_d$.

Proof. If $f_k = \sum_{j=1}^{t} \alpha_{kj} g_j$ and for some $h \in S_d$, we have $h \circ g_j = 0 \ \forall \ 1 \leq j \leq t$, then $h \circ f = 0$ by linearity of the apolar action in the second argument. This proves $1 \implies 2$. $2 \implies 3$ by simply passing to degree d in the inequality (the graded parts are pairwise disjoint by definition). For a vector space W spanned by forms, the orthogonal space coincides with the intersection of the apolar ideals of the individual forms. Using the fact that for any finite-dimensional vector space W with a non-degenerate bilinear product, we have $\left(W^{\perp}\right)^{\perp} = W$, we have $3 \implies 1$. □

The more well-known form of the apolarity lemma is as follows.

Lemma 2.1.4 (Apolarity Lemma). *Let $f \in R_d$. Then the following are equivalent:*

1. $f = \sum_{i=1}^{r} \ell_i^d$ *with* $\ell_i = \sum_{j=1}^{n} \alpha_{ij} x_j$.
2. *Consider the points $\overline{\alpha}_i = (\alpha_{i1}, \cdots, \alpha_{in})$, $1 \leq i \leq r$. Let I be the vanishing ideal of $S = \{\overline{\alpha}_1, \cdots, \overline{\alpha}_r\}$. Then, $I \subset f^{\perp}$.*

The original polarity lemma follows from the extended version by considering $s = 1$ and $g_i = \ell_i^d$ with $\ell_i = \sum_{j=1}^{n} \alpha_{ij} x_j$. The apolarity lemma thus allows us to establish upper bounds on Waring rank of forms by exhibiting zero-dimensional radical ideals inside the apolar ideal.

Example 2.1.5. *Consider the monomial $f = \prod_{i=1}^{k} x_n$. It is easy to check that $f^\perp = \langle x_1^2, \cdots, x_n^2 \rangle$. The ideal $\langle x_1^2 - x_2^2, x_1^2 - x_3^2, \cdots x_1^2 - x_n^2 \rangle \subset f^\perp$ is the vanishing ideal of the points $Z = \{(1, z_1, z_2, \cdots, z_{n-1}) \mid z_i \in \{1, -1\}\}$. This given an upper bound of 2^{n-1} on the Waring rank and the following formula using Lemma 2.1.4:*

$$\prod_{i=1}^{n} x_i = \frac{1}{2^{n-1} \cdot n!} \sum_{z_1, \cdots, z_{n-1} \in \{1,-1\}} \left(x_1 + \sum_{i=2}^{n} z_{i-1} x_i \right)^n$$

For $I = f^\perp$, the algebra S/I is of a particular type known as an *Artinian Gorenstein algebra*. We first define these terms more generally.

Definition 2.1.6. For any $I \subset S$, an algebra S/I is said to be Artinian if $\dim_C(S/I) < \infty$.

In other words, S/I is Artinian, if $\exists\, d \in \mathbb{N}$ such that for any $D \geq d$, we have $I_D = S_D$.

Lemma 2.1.7. *For $f \in R_d$, and $I = f^\perp$, S/I is Artinian.*

Proof. Since $f \in R_d$, for any $D \geq d+1$ and $g \in S_D$, we have $g \circ f = 0$. Therefore, $I_D = S_D$ $\forall\, D \geq (d+1)$. □

Definition 2.1.8. Consider the maximal ideal $\mathbf{m} = \langle y_1, \cdots, y_n \rangle$. Given an Artin algebra S/I, it's *socle*, denoted by $\mathrm{soc}(S/I)$, is defined as:

$$\mathrm{soc}(S/I) := \{h \in S/I \mid h\mathbf{m} = 0\} := (0 : \mathbf{m})$$

S/I is said to have *socle* degree d if $S/I = \bigoplus_{k=0}^{d} (S/I)_k$.

Definition 2.1.9. An Artin Algebra S/I is said to be *Gorenstein* if $\dim_C(\mathrm{soc}(S/I)) = 1$.

Note that we have $(S/I)_d \subset \mathrm{soc}(S/I)$, since degree of socle is d. Thus, S/I is Gorenstein iff $\mathrm{soc}(A) = (S/I)_d$ and $\dim_C(S/I)_d = 1$.

Lemma 2.1.10 (Euler's Formula)**.** *Let $d > 0$ and $f \in R_d$. Then:*

$$d \cdot f = \sum_{i=1}^{n} x_i \cdot \partial_{x_i} f$$

We want to show that $S/\left(f^\perp\right)$ is Gorenstein for any form f.

Lemma 2.1.11. *For $f \in R_d$ and $I = f^\perp$, S/I is a Gorenstein algebra.*

Proof. Consider the vector space $\mathcal{V}(\{f\}) \subset R_d$. Since the apolarity action $\circ$ gives a degenerate bilinear pairing, we have that $\dim_C(I_d) = dim_C(S_d) - 1$. Therefore, $\dim_C\left((S/I)_d\right) = 1$.

Now, we need to show $\mathrm{soc}(S/I) = (S/I)_d$. Observe that $\circ$ obeys: $y_i \circ (g \circ F) = (g \cdot y_i) \circ F$ $\forall\, 1 \le i \le n$, any $F \in \mathbb{C}[x_1, \cdots, x_n]$ and any $g \in \mathbb{C}[y_1, \cdots, y_n]$. Suppose $\mathrm{soc}(S/I) \neq (S/I)_d$. Then, $\exists\, h \in S_k$, $k < d$, such that $h \notin f^\perp$, but $h\mathbf{m} \subset f^\perp$. Thus, $\forall\, 1 \le i \le n$, $hy_i \in f^\perp$. So, $(hy_i) \circ f = 0$. Therefore, $y_i \circ (h \circ f) = 0\ \forall i$. But $h \circ f \in R_t$ for $t \ge 1$. Euler's formula for forms implies that $\exists\, i$ such that $y_i \circ (h \circ f) \neq 0$, which is a contradiction! □

Combining Lemma 2.1.7 and Lemma 2.1.11, we get $S/\left(f^\perp\right)$ is an Artinian Gorenstein Algebra.

Macaulay gave a converse to the above result.

Theorem 2.1.12 (Macaulay's Theorem)**.** *An Artinian algebra $\mathcal{A}$ of socle degree d is Gorenstein $\Leftrightarrow \exists\, f \in R_d$ such that $\mathcal{A} \cong S/\left(f^\perp\right)$.*

2.2 Hilbert function, partial derivatives and growth estimates

2.2.1 Properties of Hilbert functions

Another crucial tool used in the study of $\mathrm{WR}(f)$ is the Hilbert function of the quotient algebra $S/\left(f^\perp\right)$. We define this object below.

Definition 2.2.1. Given a homogeneous ideal $I \subset S$, the *Hilbert function* of S/I, denoted by $\mathrm{HF}(S/I, \cdot)$ is defined as:

$$\mathrm{HF}(S/I, \cdot) : \mathbb{N} \longrightarrow \mathbb{N}, \qquad \mathrm{HF}(S/I, k) := \dim_{\mathbb{C}}(S/I)_k$$

Corresponding to the Hilbert Function, one can define the Hilbert series of S/I as the power series:

$$\mathrm{HS}(S/I) := \sum_{k \ge 0} \mathrm{HF}(S/I, k) \cdot t^k$$

For any $f \in R_d$, we also want to consider the space of partial derivatives of f.

Definition 2.2.2. For $f \in R_d$ and any $0 \le i \le d$, the k-th partial derivative space, denoted by $S_k(f)$, is the vector space $\mathcal{V}(T)$ spanned by the following set T of degree $(d-k)$ forms:

$$T := \left\{ \partial_{\overline{x}^{\overline{\alpha}}} f \;\middle|\; \overline{\alpha} = (\alpha_1, \cdots, \alpha_n),\ \overline{x}^{\overline{\alpha}} := \prod_{i=1}^{n} x_i^{\alpha_i},\ \sum_{i=1}^{n} \alpha_i = k \right\}$$

The Hilbert function and partial derivatives are related as follows.

Lemma 2.2.3. *Let $f \in R_d$, and $I := f^{\perp} \subset S$. Then, we have:*

$$\mathrm{HF}(S/I, k) = \dim_{\mathbb{C}} S_k(f)$$

Proof. Consider the following linear map (known as the k-th *catalecticant map*) defined on the monomial basis of S_k:

$$\phi : S_k \longrightarrow R_{d-k}, \qquad \phi\left(\overline{y}^{\overline{\alpha}}\right) := \partial_{\overline{x}^{\overline{\alpha}}}(f)$$

We have $Ker(\phi) = \left(f^{\perp}\right)_k$ and $Im(\phi) = S_k(f)$. By rank-nullity, we have

$$\dim_{\mathbb{C}} \left(f^{\perp}\right)_k + \dim_{\mathbb{C}}(S_k(f)) = \dim_{\mathbb{C}}(S_k)$$

From the property of quotient vector spaces, $\dim_{\mathbb{C}}(S_k) - \dim_{\mathbb{C}} \left(f^{\perp}\right)_k = \dim_{\mathbb{C}} \left(S/\left(f^{\perp}\right)\right)_k$. Thus, we have:

$$\dim_{\mathbb{C}}(S_k(f)) = \dim_{\mathbb{C}} \left(S/\left(f^{\perp}\right)\right)_k = \mathrm{HF}(S/I, k)$$

□

The following is a useful fact.

Lemma 2.2.4. *The Hilbert function of an Artinian Gorenstein algebra S/I of socle degree d is symmetric i.e*

$$\mathrm{HF}(S/I, k) = HF(S/I, d-k) \quad \forall\ 0 \leq k \leq d$$

In light of the above theorem, we shall use the notation $\mathrm{HF}(S/I, k)$ to denote $\dim_{\mathbb{C}}(S_k(f))$ to minimize the number of different notations used.

2.2.2 Essential variables and algorithms

A notion we shall require in future discussions is the notion of *essential* variables of a polynomial. The following definition is from [32, Definition 5.].

Definition 2.2.5. Given $f \in R_d$, f is said to not depend on the variable x_i if $\partial_{x_i} f = 0$ i.e no monomial of f contains the variables x_i. f is said to have t essential variables if $\exists$ an invertible linear transformation A such that $f(A\mathbf{x})$ depends only on t variables $\in \{x_1, \cdots, x_n\}$ and no such transformation exists such that f is mapped to fewer than t variables. If the number of essential variables of $f = n$, then f is said to be *concise*. The variables that f does not depend on after a linear transformation are called *redundant* variables.

Example 2.2.6. *Consider the polynomial $f = x_1^3 + 3x_1x_2^2 + 3x_2^2x_1 + x_2^3 + x_3^3$. Rewriting, we get $f = (x_1 + x_2)^3 + x_3^3$. Consider linear map A as follows:*

$$A : \quad x_1 \longrightarrow (x_1 - x_2), \quad x_2 \longrightarrow x_2, \quad x_3 \longrightarrow x_3$$

We get $f(A\mathbf{x}) = x_1^3 + x_3^3$, thus f has at most two essential variables.

[18] gives a simple characterization for the exact number of essential variables. The proof here follows [32, Lemma B.1.]

Lemma 2.2.7 ([32, Lemma B.1.]). *For $f \in R_d$, consider the vector $v_\partial(f) := (\partial_{x_1} f, \cdots, \partial_{x_n} f)$. Define the space $v_\partial(f)^\perp$ as:*

$$v_\partial(f)^\perp := \{\mathbf{a} \in \mathbb{C}^n \mid \mathbf{a} \cdot v(f) = 0\}$$

Then, the number of essential variables of $f = \dim_{\mathbb{C}} \left(v_\partial(f)^\perp\right)$.

Proof. Suppose $\exists$ linear transformation $A = (a_{ij})_{i,j=1}^n$ such that $g := f(A\mathbf{x})$ does not depend on variable x_i. Then, $\partial_{x_i} g = \sum_{j=1}^n a_{ji}(\partial_{x_j} f)(A\mathbf{x}) = 0$. Thus we have $\mathbf{a}_i = (a_{1i}, \cdots, a_{ni}) \in v_\partial(f)^\perp$.

In the other direction, assume $\{\mathbf{a}_1, \cdots, \mathbf{a}_K\}$ form a basis for $v_\partial(f)^\perp$. Extend this independent set to form a basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_{n-K}, \mathbf{a}_1, \cdots, \mathbf{a}_K\}$ of $\mathbb{C}^n$. Construct the invertible matrix A with these columns in order and consider the polynomial $g := f(A\mathbf{x})$. For all the variables x_i, $n - K + 1 \leq i \leq n$, we get:

$$\begin{aligned}\partial_{x_i} g &= \sum_{j=1}^n a_{j(K-n+i)}(\partial_{x_j} f)(A\mathbf{x}) \\ &= (\mathbf{a}_{K-n+i}) \cdot v(f)(A\mathbf{x}) \\ &= 0\end{aligned}$$

This proves the claim. □

Note that if f is concise, then $\left(f^\perp\right)_1 = 0$. This is because if $0 \neq \sum_{i=1}^n a_i x_i = \ell \in \left(f^\perp\right)_1$, then $\partial_\ell f = 0 \implies 0 \neq a = (a_1, \cdots, a_n) \in v_\partial(f)^\perp$, which is a contradiction!

Example 2.2.8. *The polynomial $f = \prod_{i=1}^n x_i$ is concise. To see this note that, for any $l = \sum_{i=1}^n a_i x_i$, we have $\partial_l(f) = a_1 x_2 \cdots x_n + x_1 \cdot g$ for a non-zero polynomial $g \in \mathbb{C}[x_2, \cdots, x_n]$. Thus, if $\partial_l f = 0$, then x_1 divides $x_2 \cdots x_n$, which is impossible.*

We also have the following result. Note that Lemma 3.2.9 refines this for $\mathrm{WR}(f)$.

Lemma 2.2.9. *Let $\mathrm{ess}(f)$ denote the number of essential variables of f. We have the following simple lower bound:*

$$\mathrm{WR}(f) \geq \underline{\mathrm{WR}}(f) \geq \mathrm{ess}(f)$$

2.2.3 Macaulay and Gotzmann growth estimates

Macaulay and Gotzmann's growth estimates are an important tool used to study how $\mathrm{HF}\,(S/I, k)$ grows with k. The material presented here is taken from [25, Chapter 3.].

Definition 2.2.10 (Macaulay representation)**.** Let $m, d \in \mathbb{Z}_{>0}$ be integers. The d^{th} Macaulay representation of m is the unique way of writing

$$m = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_e}{e}$$

where $k_d > k_{d-1} > \cdots > k_e \geq e > 0$ (where uniqueness refers to the uniqueness of the integers k_i). For instance, the 2^{nd} and 3^{rd} Macaulay representations of 14 are as follows:

$$14 = \binom{5}{2} + \binom{4}{1}$$
$$14 = \binom{5}{3} + \binom{3}{2} + \binom{1}{1}$$

Given m, d, we denote by

$$m^{\langle d \rangle} := \binom{k_d + 1}{d+1} + \binom{k_{d-1} + 1}{d} + \cdots + \binom{k_e + 1}{e+1}.$$

With the above definition we are ready to state Macaulay's estimate on the growth of ideals, as given in [25, Chapter 3.].

Theorem 2.2.11 (Macaulay's growth estimate)**.** *Let $I \subset S$ be a homogeneous ideal with the Hilbert function of S/I denoted by $h(k) := \mathrm{HF}(S/I, k)$. Then*

$$h(k+1) \leq h(k)^{\langle k \rangle}.$$

A simple consequence of the above theorem is the following corollary, which is taken from [44, Remark (c) on p.61].

Corollary 2.2.12 ([44, Remark (c) on p.61])**.** *Let $I \subset S$ be a homogeneous ideal with the Hilbert function of S/I denoted by $h(k) := \mathrm{HF}(S/I, k)$. If for some $\ell \in \mathbb{N}$ we have $h(\ell) \leq \ell$, then $h(i+1) \leq h(i)$ for all $i \geq \ell$.*

Proof. Suppose, we have $h(\ell) \leq \ell$ for some $\ell \in \mathbb{N}$. Then, $h(\ell) = \binom{\ell}{\ell} + \binom{\ell-1}{\ell-1} + \cdots + \binom{\ell-h(\ell)+1}{\ell-h(\ell)+1}$ is the unique ℓ^{th} Macaulay representation of $h(\ell)$. Therefore, we also have

$$h(\ell)^{\langle \ell \rangle} = \binom{\ell+1}{\ell+1} + \binom{\ell}{\ell} + \cdots + \binom{\ell - h(\ell) + 2}{\ell - h(\ell) + 2} = h(\ell).$$

By Theorem 2.2.11, $h(\ell+1) \leq h(\ell)^{\langle \ell \rangle} = h(\ell) \leq \ell$. Extending this, we get $h(i+1) \leq h(i)$ $\forall\, i \geq \ell$. □

We can now state Gotzmann's persistence theorem, as given in [25, Chapter 3.].

Theorem 2.2.13 (Gotzmann Persistence Theorem). *Let $I \subset S$ be a homogeneous ideal generated in degrees $\leq d+1$, and $h(k) := \mathrm{HF}(S/I, k)$. If*

$$h(d+1) = h(d)^{\langle d \rangle}$$

then I is d-regular and

$$h(k+1) = h(k)^{\langle k \rangle}, \quad \text{for all } k \geq d.$$

2.3 Partial derivatives vs. Waring Rank and Support concentration of forms

Definition 2.3.1 (Support -Size). The *support-size* of a monomial $m \in \mathbb{C}[x_1, \cdots, x_n]$ is given by $|\mathrm{supp}(m)|$ where:

$$\mathrm{supp(m)} := \{i \in [n] \mid \partial_{x_i} m \neq 0\}$$

The support-size of a polynomial $f \in \mathbb{C}[x_1, \cdots, x_n]$ is given by $|\mathrm{supp}(f)|$ where:

$$\mathrm{supp}(f) := \bigcup_{\text{monomial } m \in f} \mathrm{supp}(m)$$

The notion of *support-concentration* was first introduced in [23, Definition 3.1 in full version] (in their work, this notion was known as *rank-concentration*). Given a vector of polynomials $\overline{\mathbf{f}} = (f_1, \cdots, f_t) \in \mathbb{C}[\mathbf{x}]^t$ and a monomial $\overline{\mathbf{x}}^{\overline{\mathbf{a}}} \in \mathbb{C}[\mathbf{x}]$ we denote:

$$\partial_{\overline{\mathbf{x}}^{\overline{\mathbf{a}}}} \overline{\mathbf{f}} := (\partial_{\overline{\mathbf{x}}^{\overline{\mathbf{a}}}} f_1, \cdots, \partial_{\overline{\mathbf{x}}^{\overline{\mathbf{a}}}} f_t)$$

Definition 2.3.2 (Rank-Concentration.). For a vector of polynomials $\overline{\mathbf{f}} := (f_1, \cdots, f_t) \in \mathbb{C}[\mathbf{x}]_d^t$ and a set of monomials $S \subset \mathbb{C}[\mathbf{x}]$, we say that that $\overline{\mathbf{f}}$ is *rank-concentrated* on S at $\overline{\alpha} \in \mathbb{C}^n$ if the following holds:

$$\left\{\partial_{\overline{\mathbf{x}}^{\overline{\mathbf{a}}}} \overline{\mathbf{f}}\,(\overline{\alpha})\right\}_{\overline{\mathbf{a}} \in S} = \left\{\partial_{\overline{\mathbf{x}}^{\overline{\mathbf{a}}}} \overline{\mathbf{f}}\,(\overline{\alpha})\right\}_{\overline{\mathbf{a}} \in \{0,1,\cdots,d\}^n}$$

If $\left|\bigcup_{\overline{\mathbf{a}} \in S} \mathrm{supp}(\overline{\mathbf{a}})\right| = k$, we say f is support-k rank-concentrated. If $\left|\bigcup_{\overline{\mathbf{a}} \in S} \mathrm{supp}(\overline{\mathbf{a}})\right| \leq k$, we say f is support-$\leq k$ rank-concentrated.

In algebraic complexity theory, one of the central problems is to design a deterministic polynomial-time algorithm for the Polynomial Identity Testing (PIT) problem. A set $S \subset \mathbb{C}^n$ is called a *hitting set* for a circuit class $\mathcal{C} \in 2^{R_d}$ if for any $0 \neq f \in \mathcal{C}$, $\exists\, \alpha \in S$ such that $f(\alpha) \neq 0$. Thus, an efficient design for a hitting set for a circuit class leads to an efficient PIT algorithm for that circuit class. The notion of rank-concentration was introduced to design efficient hitting sets for particular classes of commutative circuits in [2], [23], [1] and [26].

In our setting, we shall not look at a vector of polynomials but a single polynomial $f \in R_d$. Therefore, f is support-k-rank concentrated if $\exists$ a non-zero monomial in f of support-size k. [2] give an $(nd)^{O(k)}$ size hitting set for the class of support-$\leq k$-rank concentrated polynomials. This is efficient when k is small.

However, our definition of support-concentration is motivated by a different question. Recall Conjecture 1.2.11.

Conjecture 2.3.3 (Waring Rank vs. Partial derivatives)**.** *There exists a constant $k \in \mathbb{N}$, such that for any $f \in R_d$, with $\sum_{j=0}^{d} \mathrm{HF}(S/I, j) = D$, we have:*

$$\mathrm{WR}(f) \leq (ndD)^k$$

This conjecture along with the previous observation asserts that $\mathrm{WR}(f)$ and the dimension of partial derivatives of f are polynomially related to one another.

The validity of this conjecture has major implications in algebraic complexity theory. In particular, [39, Theorem 1.10] show that if Conjecture 2.3.3 is true then the circuit class of diagonal-ROABPs efficiently simulates the circuit class of commutative-ROABPs (which is a superclass of diagonal-ROABPs). In particular, they show:

Theorem 2.3.4 ([39, Theorem 1.10])**.** *Let $\mathrm{WR}(n, D)$ be the largest possible Waring rank of a polynomial in n variables with $\sum_{j=0}^{d} \mathrm{HF}(S/I, j) \leq D$. Then any commutative-ROABP of with n variables, degree d and width w can be simulated by a diagonal ROABP in n variables, degree d and width at most $\mathrm{WR}(w^2, w^2)nw^4$. Thus, if Conjecture 2.3.3 is true, then the width of the diagonal ABP is $\mathrm{poly}(n, d, w)$.*

As a first step towards solving Conjecture 2.3.3, we can look at a special case where the partial derivatives are spanned by support-1 monomials. This leads to the definition of 1-support concentration which is central to this thesis.

Definition 2.3.5 (1-Support Concentration)**.** Given a set of independent linear forms $\mathcal{L} := \{\ell_1, \dots, \ell_n\} \subset R_1$, we say $f \in R_d$ is 1-support concentrated with respect to $\mathcal{L}$ if for every $1 \leq k < d$, we have

$$S_k(f) = \mathcal{V}\left(\left\{\partial_{\ell_i^k} f\right\}\right)$$

We say that f is 1-support concentrated if $\exists$ some set $\mathcal{L}$ of n linearly independent forms such that the above holds with respect to $\mathcal{L}$.

The condition of being 1-support concentrated depends on the choice of $\mathcal{L}$, as the following example shows.

Example 2.3.6. *Consider* $f = (x+y)^4 - (x-y)^4 = 8x^3y + 8xy^3$. *In this case, we have* $\mathcal{V}(\{\partial_{x^2} f, \partial_{y^2} f\}) = \mathcal{V}(\{xy\})$ *while* $\mathcal{V}(\{\partial_{xy} f\}) = \mathcal{V}(\{x^2+y^2\}) \neq \mathcal{V}(\{xy\})$. *Thus, f is not 1-*support concentrated *with respect to the linear forms x, y. However, direct computation shows that f is 1-*support concentrated *with respect to the linear forms $l_1 = x+y$, $l_2 = x-y$.*

When we say f is 1-support concentrated with respect to *standard basis*, we will mean that f is 1-support concentrated with respect to $\mathcal{L} = \{x_1, \cdots, x_n\}$. Our first result is to show that there is a class of forms f such that f is 1-support concentrated with respect to some basis.

Theorem 2.3.7. *Suppose $f \in R_d$ concise and satisfies* $\mathrm{HF}(S/I, k) \leq n$ *for all* $2 \leq k \leq d-1$, *where* $I = f^{\perp}$. *Then f is 1-*support concentrated.

Therefore, we need to find examples of concise forms that obey $\mathrm{HF}(S/I, k) \leq n$ for $2 \leq k \leq d-1$.

Given any $d \in \mathbb{N}$, [44, Example 4.3] constructs examples of concise forms of degree d in $n = \binom{d+1}{2} + 3$ variables for which $\mathrm{HF}(S/I, k) \leq n$. We outline the construction below.

Example 2.3.8 (Stanley's example.). *Given degree d, consider the ring* $T = \frac{\mathbb{C}[x_1,x_2,x_3]}{(x_1,x_2,x_3)^d}$. *Consider the space of $\mathbb{C}$-linear maps from T to $\mathbb{C}$, denoted* $E := \mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C})$. *$E$ is a T-module (given by $(x\phi)(y) = \phi(xy)$) and can be graded as $E = E_1 \oplus E_2 \oplus \cdots E_{d+1}$, where $E_k = \mathrm{Hom}_{\mathbb{C}}(T_{d-k}, \mathbb{C})$. We can consider the cartesian product $\mathcal{A} = T \times E$ and give $\mathcal{A}$ a commutative ring structure via the operations:*

$$(x, \phi) + (y, \psi) = (x+y, \phi+\psi)$$
$$(x, \phi) \cdot (y, \psi) = (xy, x\psi + y\phi)$$

The above operation is known as idealization of E or the trivial extension of T by E (Nagata introduced the general case for a module in [36]). $\mathcal{A}$ is graded as $\mathcal{A}_k = T_k \times E_k$. Using [40, Corollary 6.], one can show $\mathcal{A}$ is a Gorenstein algebra. By Macaulay's theorem $\exists$ a homogeneous polynomial f of degree d such that $\mathcal{A} \cong S/\left(f^{\perp}\right)$. Using Matlis Duality, [11, Lemma 1.], one can explicitly construct the polynomial f as follows:

Consider the set $\mathcal{M}_{d-1}$ of monomials of degree $d-1$ in 3 variables. Consider a new set of variables $V_e := \{e_m \mid m \in \mathcal{M}_{d-1}\}$. Stanley polynomial $ST_{3,d} \in \mathbb{C}[x, y, z, e_{x^{d-1}}, \cdots, e_{z^{d-1}}]$ is defined as:

$$ST_{3,d} = \sum_{m \in M_{d-1}} m \cdot e_m$$

[44, Example 4.3] showed that $ST_{3,4}$ has the Hilbert series (1,13,12,13,1) and also stated $\mathrm{HF}(S/I, k) \geq \mathrm{HF}(S/I, k+1)\ \forall\ 1 \leq k \leq \lfloor d/2 \rfloor$ *for $I = f^{\perp}$ where $f := ST_{3,d}$.*

It is natural to generalize Stanley's example to an arbitrary number of variables. Define $\mathcal{M}_{n,d-1}$ to be the set of monomials in n $\{x_1, \cdots, x_n\}$ variables of degree $d-1$. Consider a new set of variables $V_e := \{e_m \mid m \in \mathcal{M}_{n,d-1}\}$. In order to distinguish between the two sets of variables, we will call the variables $\{x_i\}_{i=1}^n$ the *primary variables* and the set of variables V_e the *secondary variables*. The Stanley polynomial $ST_{n,d}$ in n primary variables of degree d is defined as:

$$ST_{n,d} := \sum_{m \in \mathcal{M}_{n,d-1}} e_m \cdot m$$

Our next lemma, in conjunction with Theorem 2.3.7, shows that $ST_{n,d}$ is a class of 1-support concentrated polynomials.

Lemma 2.3.9. *For every $d \geq 4$, $ST_{n,d}$ is concise and $\mathrm{HF}(S/I,k) \leq \binom{n+d-2}{d-1} + n \ \forall\ 1 \leq k \leq d-1$, where $I = (ST_{n,d})^{\perp}$.*

2.4 Direct sum decomposition of polynomials

2.4.1 Direct sum, Apolarity and Stanley Polynomials

The problem of direct sum decomposition is as follows: Given $f \in R_d$, does there exist independent linear forms $\ell_1, \cdots, \ell_n$ and $f_1 \in \mathbb{C}[x_1, \cdots, x_t]_d$, $f_2 \in \mathbb{C}[x_{t+1}, \cdots, x_n]_d$ such that:

$$f = f_1(\ell_1, \cdots, \ell_t) + f_2(\ell_{t+1}, \cdots, \ell_n)$$

If a polynomial $f \in R_d$ decomposes as above after a linear change of coordinates, we call f a *direct sum*. We denote the class of all polynomials that are direct sums as DirSum.

[16] deals with this question via apolarity theory.

Definition 2.4.1. Let $f \in R_d$ and $I = f^{\perp}$. An minimal generator of I of degree $k \leq d$ is $g \in I_k$ such that $g \notin (I_{\leq k-1})_k$ (this is the degree k piece of I generated by elements of degree $\leq k$). Any minimal generator is I is called an *apolar* generator. A minimal generator of degree $d = \deg(f)$ is called an *equipotent* apolar generator.

[16, Theorem 1.1] connects apolarity theory and direct sum decomposition as follows.

Theorem 2.4.2 ([16, Theorem 1.1])**.** *If $f \in$ DirSum, then f has an equipotent apolar generator.*

Using this, [16] solve some longstanding questions about direct sum decomposition, in particular the direct sum decomposition of the symbolic determinant. [41, Corollary 2.14] showed that quadratics generate the apolar ideal of the symbolic determinant. Therefore, by Theorem 2.4.2, it cannot be a direct sum.

However, the converse to Theorem 2.4.2 does not hold as shown in [16, Example 1.3]

Example 2.4.3. *Consider $f = x_1 x_2^2 \in \mathbb{C}[x_1, x_2]$. We have $f^{\perp} = (x_1^2, x_2^3)$. Note that $f^{\perp}$ has an equipotent apolar generator. If $f = \ell_1^3 + \ell_2^3$, then we can factorize:*

$$f = (\ell_1 + \ell_2)\left(\ell_1 - \left(\frac{1+\sqrt{3}i}{2}\right)\ell_2\right)\left(\ell_1 - \left(\frac{1-\sqrt{3}i}{2}\right)\ell_2\right)$$

Note that since ℓ_1 and ℓ_2 are independent linear forms, f must have three distinct linear factors after a change of coordinates which is a contradiction! Thus, f is not a direct sum.

In fact, [16, Proposition 2.12] generalizes the above example greatly.

Lemma 2.4.4 ([16, Proposition 2.12])**.** *If $f \in R_d$ has a linear factor, then either f is not a direct sum or $f = \ell_1^d + \ell_2^d$.*

Note that while $x_1x_2^2$ is not a direct sum, as we had observed before, we can rewrite it as:

$$x_1x_2^2 = \lim_{\epsilon \to 0} \frac{1}{3}\left(\left(\frac{x_2}{\epsilon} + \epsilon^2 x_1\right)^3 - \left(\frac{x_2}{\epsilon}\right)^3\right)$$

For any $\epsilon \in \mathbb{C}$, $\left(\frac{x_2}{\epsilon} + \epsilon^2 x_1\right)$ and $\left(\frac{x_2}{\epsilon}\right)$ are linearly independent linear forms. Therefore, $x_1x_2^2$ is a limit of direct sums. The class of all polynomials that are limits of direct sums is denoted as $\overline{\text{DirSum}}$.

[16, Theorem 1.7] characterizes direct sum via apolarity as follows.

Theorem 2.4.5 ([16, Theorem 1.7])**.** *Let $n \geq 2$ and $d \geq 3$. If $f \in R_d$ has an equipotent apolar generator, then $f \in \overline{\text{DirSum}}$. Conversely, every concise polynomial polynomial $\in \overline{\text{DirSum}}$ has an equipotent apolar generator.*

We show that many polynomials (we call this the "high-degree" regime) that we are interested in $\in \overline{\text{DirSum}}$.

Theorem 2.4.6. *Let $d \geq 2n+1$ and suppose $f \in R_d$ is concise and 1-*support concentrated. *Then, $f \in \overline{\text{DirSum}}$.*

As hinted before in the previous section, another specific class of polynomials we are interested in is $ST_{n,d}$ for $n \geq 2$. By Lemma 2.3.9, we know that the number of essential variables of $ST_{n,d}$ is $\binom{n+d-2}{d-1} + n$ so we are no longer in the high-degree regime (i.e $d < \binom{n+d-2}{d-1} + n$). We show that $ST_{n,d} \in \overline{\text{DirSum}}$.

Theorem 2.4.7. *$ST_{n,d}$ has exactly $\binom{n+d-1}{d}$ equipotent apolar generators. In particular, $ST_{n,d} \in \overline{\text{DirSum}}$.*

2.4.2 More general direct sum decomposition

We say $f \in R_d$ splits r times if there are n linearly independent forms $\ell_1, \cdots, \ell_n$ such that:

$$f = f_1(\ell_{t_1}, \cdots, \ell_{t_2-1}) + f_2(\ell_{t_2}, \cdots, \ell_{t_3-1}) + \cdots + f_r(\ell_{t_r}, \cdots, \ell_n)$$

for $f_i \in \mathbb{C}\left[x_{t_i}, \cdots, x_{t_{i+1}-1}\right]_d$ with $t_1 = 1$ and $t_{r+1} = n+1$. We denote this by $f \in \text{DirSum}^r$. Similarly, if f is a limit of direct sums with r summands, then we denote $f \in \overline{\text{DirSum}^r}$.

In particular, let $\mathcal{V}_{\Phi_{d-1}}(g) \subset R_1$ denote the image of g under the catalecticant map of degree $d-1$, where $d = \deg(g)$. Then, the following holds:

$$\mathcal{V}_{\Phi_{d-1}}(f_i) \bigcap \left(\sum_{j \neq i} \mathcal{V}_{\Phi_{d-1}}(f_j) \right) = 0 \qquad \forall 1 \leq i \leq r$$

Observation 2.4.8. *Using the non-degeneracy of the apolarity action, one can show that above identity implies* $\mathcal{V}_{\Phi_{d-k}}(f_i) \bigcap \left(\sum_{j \neq i} \mathcal{V}_{\Phi_{d-k}}(f_j) \right) = 0 \quad \forall 1 \leq i \leq r$ *and* $\forall\ k > 0$.

[33, Theorem 3.7, Theorem 4.5] characterizes when $f \in \mathrm{DirSum}^r$ and $\overline{\mathrm{DirSum}^r}$ respectively. We briefly sketch some necessary background and the results below.

The first step is to compare the annihilator ideal of f to the annihilator ideal of f_i [33, Lemma 2.9].

Lemma 2.4.9 ([33, Lemma 2.9])**.** *Supoose* $f \in \mathrm{DirSum}^r$ *and* f *splits as* $f = \sum_{i=1}^{r} f_i$. *Then, we have:*

$$\left(f^{\perp}\right)_i = \bigcap_{j=1}^{r} \left(f_j^{\perp}\right)_i \qquad \forall i < d$$

Since any element of S that annihilates all f_i must also annihilate f, the direction $\subseteq$ follows. The direction $\supseteq$ follows from Observation 2.4.8.

Lemma 2.4.9 implies that for splitting $f \in R_d$, we must find $G \in R_d$ such that $\left(f^{\perp}\right)_k \subseteq \left(G^{\perp}\right)_k \ \forall\ k < d$. Recall that we denote by $v_{\partial}(f)$ the vector $(\partial_{x_1} f, \cdots, \partial_{x_n} f)$. [33, Lemma 2.12] shows the following.

Lemma 2.4.10 ([33, Lemma 2.12])**.** *Let* $f, G \in R_d$. *Then, the following are equivalent:*

1. $\left(f^{\perp}\right)_k \subseteq \left(G^{\perp}\right)_k \ \forall\ k < d$.
2. $\exists\ A \in \mathbb{C}^{n \times n}$ *such that* $v_{\partial}(G) = A v_{\partial}(f)$.

Proof. Using non-degeneracy of the apolarity action, one gets $\left(f^{\perp}\right)_{d-1} \subseteq \left(G^{\perp}\right)_{d-1} \Leftrightarrow \left(f^{\perp}\right)_k \subseteq \left(G^{\perp}\right)_k \ \forall\, k < d$. Let $V_1 := \mathcal{V}\left(\partial_{x_1} f, \cdots, \partial_{x_n} f\right) \subseteq R_1$ and $V_2 := \mathcal{V}\left(\partial_{x_1} G, \cdots, \partial_{x_n} G\right) \subseteq R_1$. By the definition of kernel and image of the catalecticant map, we get $\left(f^{\perp}\right)_{d-1} \subseteq \left(G^{\perp}\right)_{d-1} \Leftrightarrow V_2 \subseteq V_1$. Rewriting this inclusion of vector spaces in terms of matrices, we get $V_2 \subseteq V_1 \Leftrightarrow \exists\ A \in \mathbb{C}^{n \times n}$ such that $v_{\partial}(G) = A v_{\partial}(f)$. This completes the proof. □

Given any matrix $M \in R^{n \times n}$, let $I_t(M)$ denote the ideal generated by the $t \times t$ minors of M. We now introduce conditions on $A \in \mathbb{C}^{n \times n}$ and $f \in R_d$ such that we have $G \in R_d$ satisfying $v_{\partial}(G) = A v_{\partial}(f)$ (this gives a converse to Lemma 2.4.10. This result is [33, Lemma 2.13]).

Lemma 2.4.11 ([33, Lemma 2.13])**.** *Let $f \in R_d$ and $A \in \mathbb{C}^{n\times n}$. Let M be the $n\times 2$ matrix with the columns $v_\partial := (\partial_{x_1}, \cdots, \partial_{x_n})^T$ and Av_∂. Then, the following are equivalent:*

1. *$\exists$ unique $G \in R_d$ such that $v_\partial(G) = Av_\partial(f)$.*
2. *$\left(A\partial\partial^T\right)(f)$ is a symmetric matrix, where $\left(\partial\partial^T\right)$ is the operator given $\left(\partial\partial^T\right)_{ij} = \partial_{x_i}\partial_{x_j} = v_\partial v_\partial^T$.*
3. *$I_2(M) \subseteq \left(f^\perp\right)$.*

Proof. $(1 \Leftrightarrow 2)$: We first prove a lifting theorem.

Claim 2.4.12. *Given a set of polynomials $\{g_1, \cdots, g_n\} \in R_{d-1}$, $\exists$ $G \in R_d$ such that:*

$$\partial_{x_i}G = g_i \quad \forall 1 \le i \le n \Leftrightarrow \partial_{x_i}g_j = \partial_{x_j}g_i \quad \forall 1 \le i, j \le n.$$

Proof. ($\Rightarrow$ direction): We have $\partial_{x_i}g_j = \partial_{x_i}\partial_{x_j}G = \partial_{x_j}\partial_{x_i}G = \partial_{x_j}g_i$ $\forall 1 \le i, j \le n$

($\Leftarrow$ direction): Define $G := \frac{1}{(d+1)}\left(\sum_{i=1}^{n} x_i g_i\right)$. Suppose $\partial_{x_i}g_j = \partial_{x_j}g_i$ $\forall 1 \le i, j \le n$. Then, we have:

$$\begin{aligned}
\partial_{x_i}G &= \frac{1}{(d+1)}\left(g_i + \sum_{j=1}^{n} x_j\partial_{x_i}g_j\right) \quad \text{(Product Rule)} \\
&= \frac{1}{(d+1)}\left(g_i + \sum_{j=1}^{n} x_j\partial_{x_j}g_i\right) \quad \text{(by hypothesis)} \\
&= \frac{1}{(d+1)}\left(g_i + d\cdot g_i\right) \quad (\textit{Lemma } 2.1.10) \\
&= g_i
\end{aligned}$$

This completes the proof of claim. ☐

Setting $g_i = (Av_\partial(f))_i$ in the above claim gives us G such that $1 \Leftrightarrow 2$.

$(2 \Leftrightarrow 3)$: 2 holds $\Leftrightarrow$ $\left(A\partial\partial^T - \partial\partial^T A^T\right)(f) = 0$. The (i, j)-th entry of this identity implies that the minor defined by the i-th and j-th row of $M \in f^\perp$, and vice-versa. This completes the proof. ☐

Inspired by this theorem, associated with every $f \in R_d$ we can define the following set of matrices:

$$\mathcal{M}_f := \left\{A \in \mathbb{C}^{n\times n} \mid I_2(M) \subseteq \left(f^\perp\right)\right\}$$

Using Lemma 2.4.10 and Lemma 2.4.11, one can prove the following important properties of $\mathcal{M}_f$ ([33, Proposition 2.21]).

Lemma 2.4.13 ([33, Proposition 2.21]). *Let $d \geq 3$ and $f \in R_d$. Then, $\mathcal{M}_f$ is a $\mathbb{C}$-algebra. If f is concise, then $\mathcal{M}_f$ is a commutative $\mathbb{C}$-algebra.*

Proof. To show $\mathcal{M}_f$ is a $\mathbb{C}$-algebra, it's enough to show that $\mathcal{M}_f$ is closed under multiplication. Let $A, B \in \mathcal{M}_f$. Since $A \in \mathcal{M}_f$, by Lemma 2.4.11, $\exists\ g \in R_d$ such that $v_\partial(g) = Av_\partial(f)$. This implies $\mathcal{V}(\partial_{x_1} g, \cdots, \partial_{x_n} g) \subseteq \mathcal{V}(\partial_{x_1} f, \cdots, \partial_{x_n} f)$ which in turn, by Lemma 2.1.4 implies $\left(g^\perp\right)_2 \supseteq \left(f^\perp\right)_2$. Thus, by part 3 of Lemma 2.4.11, $B \in \mathcal{M}_g$. Thus, by Lemma 2.4.11 again, we have $\exists\ h \in R_d$ such that $v_\partial(h) = Bv_\partial(g) = BAv_\partial(f)$. Using part 3 of Lemma 2.4.11 again, we get $BA \in \mathcal{M}_f$. This proves $\mathcal{M}_f$ is a $\mathbb{C}$-algebra.

Given, $A, B \in \mathcal{M}_f$. By part 2 of Lemma 2.4.11, we know that $AB\partial\partial^T(f)$ is symmetric. Thus, we have:

$$\begin{aligned}
\left(AB\partial\partial^T(f)\right)^T &= \left(\partial\partial^T(f)\right)^T (AB)^T \\
&= \left(\partial\partial^T(f)B^T\right) A^T \quad (\partial\partial^T(f) \text{ is symmetric}) \\
&= \left(B\partial\partial^T(f)\right) A^T \quad (\textit{Lemma } 2.4.11 \text{ and } B \in \mathcal{M}_f\ ) \\
&= B\left(\partial\partial^T(f)A^T\right) \\
&= BA\partial\partial^T(f) \quad (\textit{Lemma } 2.4.11 \text{ and } A \in \mathcal{M}_f\ ) \\
&= AB\partial\partial^T(f) \quad (\textit{Lemma } 2.4.11 \text{ and } AB \in \mathcal{M}_f \text{ by first part of proof})
\end{aligned}$$

Thus, $(AB - BA)\partial\partial^T(f) = 0$. By an application of Lemma 2.1.10, this reduces to $(AB - BA)v_\partial(f) = 0$. Since, f is concise, by Lemma 2.2.7, we get that $AB - BA = 0 \Rightarrow AB = BA$. This completes the proof. □

In [33], the splitting of f is characterized by properties of $\mathcal{M}_f$. The following lemma ([33, Lemma 2.17]) relates the dimension of algebra $\mathcal{M}_f$ to the number of equipotent apolar generators of $\left(f^\perp\right)$.

Lemma 2.4.14 ([33, Lemma 2.17]). *Let r_1 denote the number of generators of $\left(f^\perp\right)_1$ and r_2 be the number of equipotent apolar generators. Then, we have:*

$$\dim_{\mathbb{C}}\left(\mathcal{M}_f\right) = 1 + r_2 + n \cdot r_1$$

Note that if f is concise, then $r_1 = 0$, so $\dim_{\mathbb{C}}\left(\mathcal{M}_f\right) = 1 + r_2$.

Therefore, if f is concise and $\exists$ a non-scalar matrix $A \in \mathcal{M}_f$, then $r_2 > 0$ (since $I \in \mathcal{M}_f$) and by Theorem 2.4.5, $f \in \overline{\text{DirSum}}$.

Example 2.4.15. *As an illustration of the above results, consider the polynomial $ST_{2,3} = x_1^2 e_{x_1^2} + x_2^2 e_{x_2^2} + +x_1x_2e_{x_1x_2}$. By Lemma 2.3.9, we know that $ST_{2,3}$ is a concise polynomial. Thus, we need to exhibit a non-scalar matrix in $\mathcal{M}_f$ to conclude the $ST_{2,3}$ is a direct sum. Consider the following matrix P:*

$$P := \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The matrix M with v_∂^T and Pv_∂^T as columns is as follows:

$$\begin{pmatrix} \partial_{x_1} & \partial_{e_{x_1^2}} \\ \partial_{x_2} & \partial_{e_{x_2^2}} \\ \partial_{e_{x_1^2}} & 0 \\ \partial_{e_{x_2^2}} & 0 \\ \partial_{e_{x_1 x_2}} & 0 \end{pmatrix}$$

Note that $I_2(M) \subsetneq \left(f^\perp\right)$ (verified by direct computation) . Thus, $P \in \mathcal{M}_f$. Clearly, P is not scalar and so $ST_{2,3} \in \overline{\text{DirSum}}$.

Definition 2.4.16 ([33])**.** An element $E \in \mathcal{M}_f$ is called an *idempotent* if $E^2 = E$. A set $\{E_1, \cdots, E_r\} \subseteq \mathcal{M}_f$ is called a set of *orthogonal idempotents* if $E_i^2 = E_i \ \forall\ 1 \leq i \leq r$ and $E_iE_j = 0 \ \forall\ 1 \leq i \neq j \leq r$. Such a set is also called *complete* if $\sum\limits_{i=1}^{n} E_i = 1$. A complete set of orthogonal idempotents is called *coid* for short.

Consider two coids $C_1 = \{E_1, \cdots, E_r\}$ and $C_2 = \{D_1, \cdots, D_s\}$. Then, C_1 refines C_2 if $\exists$ a partition $\{P_1, \cdots, P_s\}$ of $[r]$ such that $D_i = \sum\limits_{j \in P_i} E_j \ \forall\ 1 \leq i \leq s$. We say $\mathcal{M}_f$ has a unique maximal coid if it has a coid of maximal length that refines any other coid.

[33, Lemma 3.5, part 1] proves the following.

Lemma 2.4.17 ([33, Lemma 3.5, part 1])**.** *Suppose $f \in R_d$ is concise. $\mathcal{M}_f$ has a unique maximal coid.*

Proof. Since f is concise, using Lemma 2.4.13, $\mathcal{M}_f$ is a commutative ring with unity. As $\mathcal{M}_f$ has finite dimension as a $\mathbb{C}$-vector space, $\mathcal{M}_f$ is also Noetherian. [33, Lemma 3.2 part b] shows that any commutative ring with unity, which is Artinian has a unique maximal coid. This completes the proof. □

For every $M \in \mathcal{M}_f$, let ϕ_f be the map that sends M to the unique G for which $v_\partial(G) = Mv_\partial(f)$. The following theorem [33, Theorem 3.7] exactly characterizes if $f \in \text{DirSum}^r$.

Theorem 2.4.18 ([33, Theorem 3.7])**.** *Suppose $d \geq 2$ and $f \in R_d$ is concise. Define $\text{Coid}\,(\mathcal{M}_f)$ as the set of all coids $\in \mathcal{M}_f$. Let the set $\text{Spilt}(f)$ be defined as:*

$$\text{Split}(f) = \{\{G_1, \cdots, G_k\} \mid f = \sum_{i=1}^{k} G_i \quad \textit{is a direct sum decomposition of } f\}$$

Then the map $\Phi_f : \text{Coid}(f) \longrightarrow \text{Split}(f)$ given by $\{E_1, \cdots, E_r\} \longrightarrow \{\phi_f(E_1), \cdots, \phi_f(E_r)\}$ is a bijection.

Proof Sketch. Using conciseness of f and non-degeneracy of the apolarity action, one can show $\left\{v_\partial\left(\overline{\partial} f\right) \mid \overline{\partial} \in S_{d-1}\right\} = \mathbb{C}^n$. Since, $v_\partial(G_i) = E_i v_\partial(f)$ we have

$$\left\{v_\partial\left(\overline{\partial} G_i\right) \mid \overline{\partial} \in S_{d-1}\right\} = \operatorname{Im}(E_i)$$

Therefore, $S_{d-1}(G_i) = \left\{\sum_{i=1}^{n} w_i x_i \mid w = (w_1, \cdots, w_n) \in \operatorname{Im}(E_i)\right\}$. Noting that that $\{E_i\}$ is a coid, we get using Euler's theorem, that $f = \sum_{i=1}^{r} G_i$. Disjointness of variables follows from noting that for any set of orthogonal idempotents, we have $\operatorname{Im}(E_i) \bigcap \left(\sum_{j \neq i} E_j\right) = 0$.

Conversely, if f splits as $f = G_1 + \cdots + G_r$, Using Lemma 2.4.11, we know that $\exists\ E_i$ such that $v_\partial(G_i) = E_i v_\partial(f)$. Since,

$$v_\partial(f) = \sum_{i=1}^{r} v_\partial(G_i) = \sum_{i=1}^{r} E_i v_\partial(f)$$

and f is concise, we get $\sum_{i=1}^{r} E_i = I$. Using disjointness of variables, we can infer

$$\operatorname{Im}(E_i) \bigcap \left(\sum_{j \neq i} E_j\right) = 0$$

. Combining these two observations, we can get $E_i E_j = 0$ for $i \neq j$. This completes the proof. ☐

The case of $f \in \overline{\mathrm{DirSum}^{\mathrm{r}}}$ is characterized by the existence of nilpotent matrices in $\mathcal{M}_f$. The index of a nilpotent matrix $M \in \mathbb{C}^{n \times n}$ is defined as the smallest $m \in \mathbb{N}$ such that $A^m = 0$.

[33, Theorem 4.5] proves the following.

Theorem 2.4.19 ([33, Theorem 4.5])**.** *Let $d \geq 3$ and $f \in R_d$. Assume $\exists$ a non-zero nilpotent matrix $M \in \mathcal{M}_f$ and let $r = \operatorname{index}(M) - 1$. Then, $f \in \overline{\mathrm{DirSum}^r}$.*

2.4.3 Algorithm for direct sum decomposition

From an algorithmic perspective, [32, Theorem C.2] provided an efficient randomized polynomial-time algorithm to decide if a polynomial is decomposable, provided the polynomial satisfies certain mild requirements. In the rest of the section, we sketch this algorithm.

- As a first step, suppose a partition of variables (without resorting to linear transformation) exists such that $f = f_1(x_1, \cdots, x_t) + f_2(x_{t+1}, \cdots, x_n)$. We can consider a graph G on n vertices representing the variables $x_1, \cdots, x_n$. Two vertices x_i and x_j are connected if and only if $\exists$ a monomial $\in f$ containing both the variables. The required partition is given by the decomposition of G into its connected components in polynomial-time.

- The Hessian of a polynomial f in n variables is given by the symmetric matrix $H_f(\overline{\mathbf{x}}) := (a_{ij})_{i,j=1}^n$ where:
$$a_{ij} := \partial_{ij} f$$
Note that if $f \in \text{DirSum}$, then $\exists$ an invertible matrix $A \in \mathbb{C}^{n\times n}$ such that $f(A\overline{\mathbf{x}}) = f_1(x_1, \cdots, x_t) + f_2(x_1, \cdots, x_t)$. By rescaling A if necessary, we can assume w.l.o.g that $\det(A) = 1$. Let $g(\overline{\mathbf{x}}) := f(A\overline{\mathbf{x}})$. Then, the Hessian of g has a diagonal block matrix structure with respect to the variable partition in g, giving:
$$\det(H_g)(\overline{\mathbf{x}}) = \det(H_{f_1})(\overline{\mathbf{x}}) \cdot \det(H_{f_2})(\overline{\mathbf{x}})$$

- Noting that $\det(H_{f_1})(\overline{\mathbf{x}})$ is a polynomial in the first t variables and $\det(H_{f_2})(\overline{\mathbf{x}})$ in the remaining $n-t$ variables, we reduce the problem to finding a matrix A such that the above factorization is possible. To do so, one uses Kaltofen's algorithm to compute a factorization of $\det(H_f)(\overline{\mathbf{x}}) = \prod_{i=1}^{k} p_i^{d_i}$ and then compute a basis of the linear spaces $\left((\partial p_i)^\perp\right)^\perp \in \mathbb{C}^n$. Since $\sum_{i=1}^{k} \dim\left(\left((\partial p_i)^\perp\right)^\perp\right) = n$ (assuming $\det(H_f)(\overline{\mathbf{x}})$ is concise, this is the requirement mentioned initially), these bases give an invertible linear transform A $\mathbb{C}^n$.

- Finally, we use the algorithm of the first step to check if $f(A \cdot \overline{\mathbf{x}})$ indeed decomposes.

We know that computing $\text{WR}(f)$ is **NP**-hard. One may ask if a similar hardness result can be proved for testing if $f \in \text{DirSum}$. Let us assume $f \in$ the complexity class **VP** [17, Definition 2.4]. Since polynomial det and the second-order derivatives of $f \in$ **VP** ([35] and [19, Section 2.3.1] respectively), we have $\det(H_f(\overline{\mathbf{x}})) \in$ **VP**. Thus, Kaltofen's algorithm runs in polynomial-time. As a consequence, the above algorithm due to [32] shows that testing $f \in \text{DirSum}$ is in the complexity class **RP** for the polynomial class **VP**. So if the testing problem for **VP** is **NP**-hard, then **NP** = **RP**. This implies that the Polynomial Hierarchy collapses to the second level [31, Theorem 6.1], an implication widely believed to be false.

2.5 More Useful Background

2.5.1 Useful Results

The following are some well-known combinatorial identities and results in algebraic geometry.

Lemma 2.5.1. *Let $V(f)$ denote the set of zeros of a polynomial f. Let $X \subset \mathbb{P}^{n-1}$ be a projective variety and $f \in R$ be a non-constant homogeneous polynomial that does not vanish identically on X. Then* $\dim(X \cap V(f)) = \dim(X) - 1$. *(Here* dim *denotes the Krull dimension of a projective variety).*

Lemma 2.5.2 (Cauchy-Binet formula)**.** *Let $A \in T^{m\times n}$ and $B \in T^{n\times m}$, where T is any commutative ring. Denote by $\binom{[n]}{m}$, the subsets of $\{1,\cdots,n\}$ of size m. For any $S \in \binom{[n]}{m}$, denote by $A_{(m,S)}$ the $m\times m$ submatrix of A using columns from S (consider similarly the $m\times m$ submatrix $B_{(S,m)}$ of B). Then, we have:*

$$\det(AB) = \sum_{S\subset\binom{[n]}{m}} \det\left(A_{(m,S)}\right)\cdot\left(B_{(S,m)}\right)$$

Lemma 2.5.3 (Polynomial-Identity Lemma)**.** *Let $f \in R_d$ and $S \subset \mathbb{C}$ be a finite set. Let $S^n = S\times\cdots\times S$ and assume $|S| > d$. Then, we have:*

$$|V(f)\cap S^n| \le d\cdot|S|^{n-1}$$

Lemma 2.5.4. *Let $N := \binom{n+d-1}{d}$. Then, $\exists$ a set of linear forms $\{l_1,\cdots,l_N\}\subset R_1$ such that the set $\mathcal{B}_{lin} := \left\{l_1^d,\cdots,l_N^d\right\}$ forms a basis for R_d.*

2.5.2 Ring of divided powers

In some cases, as we shall see in the next chapter, it is easier to work over the ring of divided powers $\mathbb{C}[x_1,\cdots,x_n]^{DP}$ (which we denote by R^{DP} for brevity) than the polynomial ring $\mathbb{C}[x_1,\cdots,x_n]$. The exposition in this section borrows from: [33, Section 1.1] and [29, Appendix A].

R_d^{DP} (the degree d homogeneous part of R^{DP}) is defined by defining the monomial basis:

$$R_d^{DP} := \mathcal{V}\left\{\overline{x}^{(\overline{\alpha})} = \prod_{i=1}^{n} x_i^{(\alpha_i)} \;\middle|\; \sum_{i=1}^{n}\alpha_i = d,\ \alpha_i\in\mathbb{N}\ \forall i\right\}$$

The ring structure (i.e multiplication) on R^{DP} is given by:

$$x_i^{(\alpha)}\cdot x_i^{(\beta)} = \binom{\alpha+\beta}{\beta}x_i^{(\alpha+\beta)}$$

where $\alpha,\beta\in\mathbb{N}$. The ring of partials S acts on R^{DP} via the following $\mathbb{C}$-linear action:

$$\overline{\partial}^{(\overline{\alpha})}\left(a\cdot\overline{x}^{(\overline{\beta})}+\overline{x}^{(\overline{\gamma})}\right) = a\cdot\overline{x}^{\left(\overline{\beta}-\overline{\alpha}\right)}+\overline{x}^{(\overline{\gamma}-\overline{\alpha})}$$

where $\overline{\alpha}$, $\overline{\beta}\in\mathbb{N}^n$ and $a\in\mathbb{C}$. Note that under this action, we have:

$$\begin{aligned}
\partial_{x_i}\left(x_i^{(\alpha)}x_i^{(\beta)}\right) &= \binom{\alpha+\beta}{\beta}\partial_{x_i}\left(x_i^{(\alpha+\beta)}\right)\\
&= \binom{\alpha+\beta}{\beta}x_i^{(\alpha+\beta-1)}\\
&= \binom{\alpha+\beta-1}{\beta}x_i^{(\alpha+\beta-1)} + \binom{\alpha+\beta-1}{\beta-1}x_i^{(\alpha+\beta-1)}\\
&= x_i^{(\alpha-1)}\cdot x_i^{(\beta)} + x_i^{(\alpha)}\cdot x_i^{(\beta-1)}\\
&= x_i^{(\beta)}\cdot\partial_{x_i}\left(x_i^{(\alpha)}\right) + x_i^{(\alpha)}\cdot\partial_{x_i}\left(x_i^{(\beta)}\right)
\end{aligned}$$

This shows that ∂_i is indeed a derivation. Combining the above observations, we get the following [29, Proposition A.12. iii)].

Lemma 2.5.5. *Consider the map:*

$$\phi : R^{DP} \longrightarrow R \qquad \phi\left(\overline{x}^{(\overline{\alpha})}\right) = \frac{1}{\alpha_1! \cdots \alpha_n!} \prod_{i=1}^{n} x_i^{\alpha_i}$$

Then, ϕ defines an isomorphism of rings and ϕ commutes with the action of S.

The above holds over all fields K with $\mathrm{char}(K) = 0$, not just $\mathbb{C}$.

Chapter 3

Results

This chapter is organized as follows: Section 3.1 contains the proofs of Theorem 1.4.1, Proposition 1.4.2 and Theorem 1.4.6. Section 3.2 contains the proofs of Lemma 1.4.5, Theorem 1.4.7, Lemma 1.4.8 and Theorem 1.4.9.

3.1 Hilbert Functions and Support Concentration

For any $f \in R_d$ $1 \leq k \leq d$, we let $S_k(f)$ denote the vector space spanned by the k-th order partial derivatives of f.

Theorem 3.1.1. *Suppose $f \in R_d$ is concise and* $\mathrm{HF}(S/I, k) \leq n \; \forall \; 1 \leq k \leq d$*. Then, f is 1*-support concentrated.

Proof. Let $L_i = \sum_{j=1}^{n} y_{ij} x_j$, where y_{ij} $1 \leq i, j \leq n$ are formal variables. We pick n random linear forms by picking random values for y_{ij} from a large enough subset of $\mathbb{C}$.

Suppose $\mathrm{HF}(S/I, k) = r$. Thus, we have, $\forall \; 1 \leq i \leq r$:

$$\partial_{L_i^k} f = \sum_{\overline{e} \in E_k} \alpha_{\overline{e}} \cdot \overline{y}^{\overline{e}} \cdot \partial_{\overline{x}^{\overline{e}}} f$$

for some suitable binomial coefficient $\alpha_{\overline{e}} \in \mathbb{C} \setminus 0 \; \forall \; \overline{e}$. Consider a lexicographic monomial ordering $\succ_{(R,d-k)}$ for R_{d-k} and $\succ_{(R,k)}$ for R_k. Writing this system of equations in matrix form we get:

$$\partial_{\overline{L}^k} = M_k \cdot D(\overline{\alpha}) \cdot N_k$$

where the matrices are defined as follows:

- $\partial_{\overline{L}^k}(i, j) = \mathrm{coeff}_{m_j} \left(\partial_{L_i^k} f \right)$ where m_j is the coefficient of the j-th monomial in the ordering $\succ_{(R,d-k)}$ of R_{d-k}. This is an $r \times |E_{d-k}|$ matrix (where E_i is the set of monomials in n variables of degree i).

- $M_k(i,\overline{e}) = \text{coeff}_{x^{\overline{e}}}\left(L_i^k\right)$ where columns $\overline{x}^{\overline{e}}$ are ordered according to $\succ_{(R,k)}$. This is an $r \times |E_k|$ matrix.

- $D(\overline{\alpha})$ is a $|E_k| \times |E_k|$ diagonal matrix with $D(\overline{\alpha})(\overline{e},\overline{e}) = \alpha_{\overline{e}}$ and columns $\overline{x}^{\overline{e}}$ are ordered according to $\succ_{(R,k)}$.

- $N_k(\overline{e},j) = \text{coeff}_{m_j}\left(\partial_{\overline{x}^{\overline{e}}} f\right)$ where m_j is the j-th monomial in the ordering $\succ_{(R,d-k)}$ of R_{d-k} and columns $\overline{x}^{\overline{e}}$ are ordered according to $\succ_{(R,k)}$. This is an $|E_k| \times |E_{d-k}|$ matrix.

Since $\text{HF}(S/I,k) = r$, we know that the matrix N_k has rank r. Let $N_k(\cdot, j_i)$ for $1 \leq i \leq r$ be r independent columns of N_k and let the corresponding full-rank $|E_k| \times r$ matrix be $\overline{N_k}$. We now pick r independent rows of $\overline{N_k}$ using the following greedy procedure:

- Step 1: Pick the largest $\overline{e}$ such that $N_k(\overline{e},\cdot) \neq 0$ (i.e not a zero row). Call this $\overline{e}_1$.

- Step t (for $2 \leq t \leq r$): Given $\overline{e}_1, \cdots, \overline{e}_{t-1}$, pick the largest $\overline{e}$ such that:

$$N_k(\overline{e},\cdot) \notin \mathcal{V}\left(\{N_k(\overline{e}_1,\cdot), \cdots, N_k(\overline{e}_{t-1},\cdot)\}\right)$$

  Call this $\overline{e}_t$.

Let Z be the set of rows obtained by the above procedure.

Consider the $r \times r$ matrix $U_k = M_k \cdot D(\overline{\alpha}) \cdot \overline{N_k}$. Using Lemma 2.5.2, we get:

$$\begin{aligned}\det(U_k) &= \sum_{A \subset \binom{|E_k|}{r}} \det\left(M_{(k,A)} \cdot D(\overline{\alpha})_{(A,A)} \cdot \overline{N_{(k,A)}}\right) \\ &= \det\left(M_{k,Z} \cdot D(\overline{\alpha})_{Z,Z} \cdot \overline{N_{k,Z}}\right) + \text{other terms}\end{aligned}$$

where $M_{k,A}$ is the $r \times r$ submatrix of M_k obtained by picking the columns corresponding to A and where $\overline{N_{k,A}}$ is the $r \times r$ submatrix of $\overline{N_k}$ obtained by picking the rows corresponding to A. Consider the variable order $y_{11} \succ \cdots y_{1n} \succ y_{21} \succ \cdots \succ y_{nn}$ and let $\mathbb{C}[\overline{y}]$ be lexicographically ordered. Also, let $\overline{y}_i = \{y_{i1}, \cdots, y_{in}\}$. We claim that the leading monomial, up to non-zero constant, of $\det(U_k)$ is $\prod_{i=1}^{r} \overline{y}_i^{\overline{e}_i}$. Note that $\prod_{i=1}^{r} \overline{y}_i^{\overline{e}_i}$ is the leading monomial of the polynomial $\det\left(M_{k,Z} \cdot D(\overline{\alpha})_{Z,Z} \cdot \overline{N_{k,Z}}\right)$ and it appears with non-zero coefficient since both $D(\overline{\alpha})_{Z,Z}$ and $\overline{N_{k,Z}}$ are full rank. Consider a set $Z \neq Z' = \{\overline{e}'_1, \cdots, \overline{e}'_r\}$ of rows of $\overline{N_k}$. Consider the first index t where Z and Z' differ. Because of the greedy procedure, we know $\overline{e}_t \succ \overline{e}'_t$. Therefore, $\prod_{i=1}^{n} \overline{y}_i^{\overline{e}_i} \succ \prod_{i=1}^{n} \overline{y}_i^{\overline{e}'_i}$. This shows that the monomial $\prod_{i=1}^{n} \overline{y}_i^{\overline{e}_i}$ is never produced in the "other terms" part of the sum. This proves the claim. Therefore, $\det(U_k)$ is a non-zero polynomial in the variables y_{ij}.

Considering the polynomial $P := \prod_{k=1}^{d} \det(U_k)$, we get, by Lemma 2.5.3, that with high probability, there is an evaluation $\overline{\beta}$ of the y_{ij} variables such that U_k is full-rank for every $1 \le k \le n$. Therefore, this evaluation renders the product $[M_k \cdot D(\overline{\alpha}) \cdot N_k](\overline{\beta})$ full-rank $\forall$ $1 \le k \le d$. As a consequence, the matrix $\partial_{\overline{L}^k}(\overline{\beta})$ has rank equal to $\mathrm{HF}(S/I, k)$ $\forall$ $1 \le k \le d$. So, f is 1-support concentrated w.r.t $\{L_i(\overline{\beta})\}_{i=1}^{n}$. This completes the proof. $\square$

Observation 3.1.2. *The above theorem also shows that the condition of 1-*support concentrated *is a* generic condition *i.e if a form is 1-*support concentrated *with respect to a basis $\ell_1, \dots, \ell_n$, then there is a Zariski-open set of bases for which it will also be 1-*support concentrated*.*

We show that in the high degree regime $d \ge 2n+1$, the Hilbert function of a concise polynomial is constant.

Proposition 3.1.3. *Suppose $f \in R_d$ where $d \ge 2n+1$. Also, assume f is concise and is 1-*support concentrated *w.r.t n independent linear forms $\ell_1, ..., \ell_n$. Then we have* $\mathrm{HF}(S/I, i) = n$ *for* $1 \le i \le d-1$ *where* $I = \left(f^{\perp}\right)$.

Proof. Since f is concise, we have $\mathrm{HF}(S/I, 1) = n$. By the symmetric nature of the Hilbert function, we have $\mathrm{HF}(S/I, i) = \mathrm{HF}(S/I, d-i)$ $\forall$ $1 \le i \le d-1$. Since f is 1-support concentrated, we know that $\mathrm{HF}(S/I, i) \le n$ $\forall$ $1 \le i \le d-1$. Suppose for some $1 < i \le d/2$, we have $\mathrm{HF}(S/I, i) \le n-1$. By symmetry, $\mathrm{HF}(S/I, d-i) \le n-1 \le d-i$ (since $d \ge 2n$). By Corollary 2.2.12, $\mathrm{HF}(S/I, j) \le n-1$, $\forall$ $j \ge d-i$. In particular, $\mathrm{HF}(S/I, n-1) = \mathrm{HF}(S/I, 1) \le n-1$, which is a contradiction. $\square$

We now prove every concise polynomial in the high-degree regime $\in \overline{\mathrm{DirSum}}$ i.e Theorem 2.4.6.

Theorem 3.1.4. *Let $d \ge 2n+1$ and suppose $f \in R_d$ is concise and 1-*support concentrated*. Then, $f \in \overline{\mathrm{DirSum}}$.*

Proof. Let $I := f^{\perp} \subset S$ and let $h_I(k) := \mathrm{HF}(S/I, k)$. Since f is 1-support concentrated, by Proposition 3.1.3 we have that

$$h(k) = \begin{cases} 1, & \text{if } k \in \{0, d\} \\ n, & \text{if } 1 < k < d \\ 0, & \text{otherwise.} \end{cases}$$

Let $J = (I_{\le n+1})$ be the ideal generated by the forms of degree $\le n+1$ in I. Since

$$h_J(n+1) = h_I(n+1) = n = h_I(n)^{\langle n \rangle} = h_J(n)^{\langle n \rangle},$$

and J is by definition generated in degrees $\le n+1$, by Theorem 2.2.13 we have that $h_J(k+1) = h_J(k)^{\langle k \rangle}$ for all $k \ge n$. By Corollary 2.2.12, we have that $h_J(k) = n$ for all

$k \geq n$. Since $J_k = I_k$ for $k \leq n+1$, we also have $h_J(k) = h_I(k) = n$ for $k \leq n+1$, and thus the Hilbert series of S/J is given by $(1, n, n, n, \dots)$.

As $J \subset I$, we must have that $h_J(k) \geq h_I(k)$ for all $k \geq 0$, with equality iff $I_k = J_k$. Thus, the previous paragraph implies that $I_k = J_k$ for $k < d$, and hence $J = (I_{<d})$. Since $h_I(d) = 1 < n = h_J(d)$, we have that I_d contains $n-1$ forms which are not in J. Thus, these forms are generators of I of degree d.

By Theorem 2.4.5 and the above, we have that f is the limit of a direct sum. □

3.2 Stanley Polynomials

Lemma 3.2.1. *$ST_{n,d}$ is concise and* $\mathrm{HF}(S/I,k) \leq \binom{n+d-2}{d-1} + n \ \forall\ 1 \leq k \leq d-1$, *where* $I = (ST_{n,d})^{\perp}$ *and* $d \geq 4$.

Proof. We first prove an inequality on binomial coefficients.

Claim 3.2.2. *Let $d \geq 4$. Then, we have:*

$$\binom{n+i-1}{n-1} + \binom{n+d-i-1}{n-1} \leq \binom{n+d-2}{n-1} + n.$$

$\forall\ 1 \leq i \leq d-1$.

Proof. Note that for $i = 1$ and $i = d-1$ we have equality $\forall\ n$ and $d \in \mathbb{N}$. Fix $n \in \mathbb{N}$.We will prove the result by induction on d.

- **Base Case** ($d = 4$): We only need to consider $i = 2$. We have:

$$\binom{n+i-1}{n-1} + \binom{n+d-i-1}{n-1} = 2 \cdot \binom{n+1}{2}$$

  Note that:

$$\binom{n+2}{3} + n - 2 \cdot \binom{n+1}{2} = \frac{n(n-1)(n-2)}{6} \geq 0$$

  This proves the base case.

- **Induction step**: The following combinatorial identity holds:

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$$

  Therefore, we have:

$$\binom{n+d-2}{n-1} + n = \binom{n+d-3}{n-2} + \binom{n+d-3}{n-1} + n$$

By the induction hypothesis we get:

$$\binom{n+d-3}{n-1} + n \geq \binom{n+i-1}{n-1} + \binom{n+d-i-2}{n-1}$$

Using the aforementioned combinatorial identity, we also get:

$$\binom{n+i-1}{n-1} + \binom{n+d-i-1}{n-1} = \binom{n+i-1}{n-1} + \binom{n+d-i-2}{n-1} + \binom{n+d-i-2}{n-2}$$

For the claim to hold, we only need:

$$\binom{n+d-3}{n-2} \geq \binom{n+d-i-2}{n-2}$$

which clearly holds $\forall\ 1 \leq i \leq d-1$. This completes the induction step and hence, the proof. ☐

Denote $F := ST_{n,d}$. Suppose, for the sake of contradiction, $\exists$ linear form ℓ such that $\partial_\ell F = 0$. Divide the set $var(\ell) := S_1 \cup S_2$, where $S_1 \cap V_e = S_2 \cap \{x_1, \cdots, x_n\} = \phi$. Let $F = \ell_1 + \ell_2$, where $var(\ell_1) \subset S_1$ and $var(\ell_2) \subset S_2$. By linearity of derivatives, we have:

$$\partial_\ell F = \partial_{\ell_1+\ell_2} F = \partial_{\ell_1} F + \partial_{\ell_2} F = 0$$

Since F is linear in the variables V_e, we get $var(m) \cap V_e \neq \phi$ for every monomial $m \in \partial_{\ell_1} F$ and $var(m) \cap V_e = \phi$ for every monomial $m \in \partial_{\ell_2} F$. By setting the variables in S_2 to zero, we get $\partial_{\ell_1} F = 0$. Since this substitution does not affect $\partial_{\ell_2} F$ we get $\partial_{\ell_2} F = 0$ by the above equation. By the structure of F, $\partial_{\ell_2} F$ is a sum of distinct monomials in $\{x_1, \cdots, x_n\}$ and hence cannot be zero. This is a contradiction!

Since F is concise, we have $\mathrm{HF}(S/I, 1) = \mathrm{HF}(S/I, d-1) = \binom{n+d-2}{d-1} + n$. Recall the construction of F, by the trivial extension of $T := \frac{\mathbb{C}[x_1,\cdots,x_n]}{(x_1,\cdots,x_n)^d}$ by $E := \mathrm{Hom}_{\mathbb{C}}(T, \mathbb{C})$. Thus, we get that:

$$\begin{aligned}\mathrm{HF}(S/I, k) = \dim(T_k) + \dim(E_k) &= \dim(T_k) + \dim\left(\mathrm{Hom}_{\mathbb{C}}\left(T_{d-k}, \mathbb{C}\right)\right) \\ &= \binom{n+k-1}{n-1} + \binom{n+d-k-1}{n-1}\end{aligned}$$

By Claim 3.2.2, we are done. ☐

Next we prove Theorem 1.4.7 i.e Stanley Polynomials $\in \overline{\mathrm{DirSum}}$. In fact, we give the exact equipotent apolar generators.

Proof of Theorem 1.4.7. We begin by showing that $ST_{n,d}$ has at least one equipotent apolar generator (note that this already establishes inclusion in $\overline{\mathrm{DirSum}}$).

Lemma 3.2.3. *$ST_{n,d}$ has an equipotent apolar generator.*

Proof. We make the following claim.

Claim 3.2.4. *$ST_{n,d}$ has no non-zero degree $d-1$ annihilator $g = \sum_{i=1}^{k} m_i$ (m_i are monomials) with the following property: $\exists\ 1 \leq i \leq k$ such that $supp(m_i) \subseteq \{x_1, ..., x_n\}$.*

Proof. Suppose not. Let $U = \{m_1, .., m_k\}$, $U_1 = \{m \in U | supp(m) \subseteq \{x_1, ..., x_n\}\}$ and $U_2 = U \setminus U_1$. Suppose U_1 is non-empty. Then,

$$0 = \partial_g ST_{n,d} = \underbrace{\sum_{m \in U_1} \partial_m ST_{n,d}}_{f_1} + \underbrace{\sum_{m \in U_2} \partial_m ST_{n,d}}_{f_2}$$

Note that degree in e-variables of f_1 is 1 while degree in e-variables of f_2 is 0. Hence, $f_1 = f_2 = 0$. Let $g_1 = \sum_{m \in U_1} m$. We have:

$$0 = f_1 = \partial_{g_1} ST_{n,d} = \sum_{m \in \mathcal{M}_{n,d-1}} (\partial_{g_1} m) \cdot e_m$$

Comparing coefficients, we get $\partial_{g_1} m = 0\ \forall\ m \in \mathcal{M}_{n,d-1}$. Since the apolar action is a non-degenerate map, we get $g_1 = 0$, which is a contradiction since we assumed U_1 was non-empty. □

Consider any degree d annihilator h in the primary variables $\{x_1, ..., x_n\}$. By Claim 3.2.4, h cannot be generated by degree $(d-1)$ annihilators (Suppose it can. By writing the generating identity and setting the e-variables to zero in the equation, we get a contradiction). □

We assume we are working over the ring of divided powers in the following result. Also, let $\mathbf{Mon}(f)$ denote the set of distinct monomials in the polynomial f.

Corollary 3.2.5. *$ST_{n,d}$ has exactly $\binom{n+d-1}{d}$ equipotent apolar generators.*

Proof. In the proof of Lemma 3.2.3, we showed that every degree d annihilator in the primary variables is minimal. Since all monomials in $\mathcal{M}_{n,d}$ annihilate $ST_{n,d}$, we get that $ST_{n,d}$ has at least $\binom{n+d-1}{d}$ equipotent apolar generators. We now show that these are the only equipotent apolar generators. Let g be an equipotent apolar generator such that $supp(m_i)$ has a secondary variable for some $m_i \in \mathbf{Mon}(g) = \{m_1, ..., m_k\}$. Note that the degree of any such secondary variable has to be 1 because e_m^2 annihilates $ST_{n,d}\ \forall\ m \in \mathcal{M}_{n,d-1}$. Also, since each degree d monomial in only the primary variables is an equipotent apolar generator, we can assume every m_i has a secondary variable with degree 1. Also, consider a monomial $m \cdot e_{m'}$ where $m \neq m'$. This is a degree d annihilator. Consider $x_i \in supp(m) \setminus supp(m')$ (suppose this is non-empty). Then $x_i \cdot e_{m'}$ annihilates $ST_{n,d}$ and generates $m \cdot e_{m'}$ in lower degree. If $supp(m) \setminus supp(m')$ is empty, then there must be a variable x_i in m such that $deg_{x_i}(m) > deg_{x_i}(m')$. Let $a_i = deg_{x_i}(m)$ and suppose $a_i < d-1$. Then, $x_i^{a_i} \cdot e_{m'}$ generates $m \cdot e_{m'}$ in lower degree. Suppose $a_i = d-1$ but $deg_{x_i}(m') < d-2$.

Then, $x_i^{d-2} \cdot e_{m'}$ generates $m \cdot e_{m'}$ in lower degree. This leaves only the following case: $m = x_i^{d-1}$ and $m' = x_i^{d-2}x_j$ for some $i \neq j$. Consider the following expansion:

$$x_i^{d-1} \cdot e_{x_i^{d-2}x_j} = x_i \cdot \left(x_i^{d-2} \cdot e_{x_i^{d-2}x_j} - x_j^{d-2} \cdot e_{x_j^{d-1}}\right) + x_j \cdot \left(x_i x_j^{d-3} \cdot e_{x_j^{d-1}}\right)$$

Each bracketed term is an annihilator of degree $d-1$. Thus, $x_i^{d-1} \cdot e_{x_i^{d-2}x_j}$ is generated in lower degree.

From the above discussion, we can conclude that for any $m_i \in \mathbf{Mon}(g)$, we have $m_i \in \mathbf{Mon}(ST_{n,d})$.

Claim 3.2.6. *For any two monomials $m_1, m_2 \in ST_{n,d}$ the annihilator $m_1 - m_2$ is generated in degree $< d$.*

Proof. Suppose $m_1 = hm_1'$ and $m_2 = hm_2'$ where $h \notin \mathbb{C}$. Then $m_1' - m_2'$ is also an annihilator and so $m_1 - m_2$ is generated in lower degree. Otherwise, m_1 and m_2 have disjoint support. W.l.o.g, suppose $x_1 \in supp(m_1) \setminus supp(m_2)$ and $x_2 \in supp(m_2) \setminus supp(m_1)$. Let $x_1 x_2^{d-2} \cdot e_M \in \mathbf{Mon}(ST_{n,d})$. Then, consider $m_1 - m_2 = \left(m_1 - x_1 x_2^{d-2} \cdot e_M\right) + \left(x_1 x_2^{d-2} \cdot e_M - m_2\right)$. Each term in the bracket is an annihilator of degree d generated in lower degree by the first part of the proof. Hence, $m_1 - m_2$ is generated in degree $< d$. □

Assume $g = \sum_{i=1}^{k} c_i m_i$. Then, $\partial_g ST_{n,d} = \sum_{i=1}^{k} c_i = 0$. Thus, $g = \sum_{i=1}^{k-1} c_i (m_i - m_k)$. By the above claim, g is generated in degree $< d$ which contradicts the assumption that g is an equipotent apolar generator. Hence, the only equipotent apolar generators are $\mathcal{M}_{n,d}$. □

This completes the proof of Theorem 2.4.7 □

Observation 3.2.7. *Note that while we exactly determine what the equipotent apolar generators of $ST_{n,d}$ are, Example 2.4.15 gives us another way of proving $ST_{n,d} \in \overline{\text{DirSum}}$ by looking at the smallest degree component of the apolar ideal. Consider the variable order $x_1 \succ \cdots \succ x_n \succ e_{m_1} \succ \cdots \succ e_{m_{N-n}}$ where $m_1 > \cdots > m_{N-n}$ is a monomial ordering for $\mathcal{M}_{n,d-1}$ $\left(\text{where } N := \binom{n+d-2}{d-1} + n\right)$. Let $v_\partial = \left(\partial_{x_1}, \cdots, \partial_{x_n}, \partial_{e_{m_1}}, \cdots, \partial_{e_{m_{N-n}}}\right)$. Without loss of generality, let $e_1 := x_1^{d-1}$ and $e_k := x_2^{d-1}$ for some $k \in \mathbb{N}$. Let v_1 and v_2 be the $(n+1)$-th and $(n+k)$-th standard basis vectors (written as $1 \times N$ row vectors), respectively, of $\mathbb{C}^N$. Consider the following $N \times N$ matrix with last $(N-2)$ zero rows:*

$$P := \begin{pmatrix} \cdots & \cdots & v_1 & \cdots & \cdots \\ \cdots & \cdots & v_2 & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For this matrix, the matrix M with columns v_∂^T and Pv_∂^T are:

$$\begin{pmatrix} \partial_{x_1} & \partial_{e_{m_1}} \\ \partial_{x_2} & \partial_{e_{m_k}} \\ \vdots & \vdots \\ \partial_{x_n} & 0 \\ \partial_{e_{m_1}} & 0 \\ \vdots & \vdots \\ \partial_{e_{m_{N-n-1}}} & 0 \\ \partial_{e_{m_{N-n}}} & 0 \end{pmatrix}$$

Note that since $m_1 = x_1^{d-1}$ and $m_2 = x_2^{d-2}$, $\partial_{x_1}\partial_{e_{m_k}}(f) = \partial_{x_2}\partial_{e_{m_1}}(f) = 0$. Also, since $ST_{n,d}$ is linear in the variables V_e, $\partial_{e_i}\partial_{e_j}(f) = 0 \; \forall \; 1 \leq i, j \leq N-n$. Therefore, $I_2(M) \subset\neq \left(f^\perp\right)$. Clearly, M is not a scalar matrix. Since $ST_{n,d}$ is concise by Lemma 2.3.9, $ST_{n,d} \in \overline{\mathrm{DirSum}}$ by Lemma 2.4.14 .

The lower bound on $\mathrm{WR}\left(ST_{n,d}\right)$ uses the following well-known result [34, Theorem 1.3.]. Before we state the result, we need the following definition. Thus,

Definition 3.2.8. Given $f \in R_d$, let $V(f) := \{z \in \mathbb{C}^n \mid f(z) = 0\}$. Define $\Sigma_k(f)$ as follows:

$$\Sigma_k(f) := \left\{z \in V(f) \mid \overline{\partial}^{\overline{\alpha}}(f)(z) = 0, \;\; |\overline{\alpha}| = k\right\}$$

Lemma 3.2.9. *Let $f \in R_d$ and suppose f is concise. Let $I = f^\perp$. We use the convention that $\dim(\phi) = -1$, where $\dim$ is the Krull dimension. Then, we have:*

$$\mathrm{WR}(f) \geq \mathrm{HF}(S/I, k) + \dim\left(\Sigma_k(f)\right) + 1$$

We now prove the lower bound on $ST_{n,d}$ i.e Lemma 1.4.8.

Proof of Lemma 1.4.8. Since $ST_{n,d}$ is concise, we know that $N := \mathrm{HF}(S/I, 1) = \binom{n+d-2}{d-1} + n$. Consider an ordering $\succ$ of the variables $x_1 \succ \cdots \succ x_n \succ e_{m_1} \succ \cdots \succ e_{m_{N-n}}$, where $m_1 > \cdots > m_{N-n}$ is a monomial ordering for $\mathcal{M}_{n,d-1}$. Consider the following set T :

$$T := \left\{z \in \mathbb{C}^N \mid z_1 = \cdots = z_n = 0\right\}$$

Since every monomial in $ST_{n,d}$ uses primary variables, $T \subset V\left(ST_{n,d}\right)$. Since $d \geq 3$, the total degree of primary variables in every monomial is ≥ 2. Hence, $T \subset \Sigma_1\left(ST_{n,d}\right)$. By repeated application of Lemma 2.5.1, we see that $\dim\left(\Sigma_1\left(ST_{n,d}\right)\right) \geq \binom{n+d-2}{d-1} - 1$. Using Lemma 3.2.9, we get the required lower bound. □

Next, we prove Theorem 1.4.9.

Proof of Theorem 1.4.9. We prove each result via explicit construction below.

- To prove the first assertion, we give an explicit construction matching border Waring rank of the polynomial:

$$ST_{3,4} = x_1^3 e_{x_1^3} + x_2^3 e_{x_2^3} + x_3^3 e_{x_3^3} + + x_1 x_2^2 e_{x_1 x_2^2} + x_1^2 x_2 e_{x_1^2 x_2} + x_2^2 x_3 e_{x_2^2 x_3} \\ + x_2 x_3^2 e_{x_2 x_3^2} + x_1^2 x_3 e_{x_1^2 x_3} + x_1 x_3^2 e_{x_1 x_3^2} + x_1 x_2 x_3 e_{x_1 x_2 x_3}$$

  The set of linear forms involved in the construction are as follows:

  1. $\ell_1 := \left(x_2 + x_3 + \frac{\epsilon}{3}\left(e_{x_2^2 x_3} + e_{x_2 x_3^2}\right)\right)$
  2. $\ell_2 := \left(x_2 - x_3 + \frac{\epsilon}{3}\left(e_{x_2^2 x_3} - e_{x_2 x_3^2}\right)\right)$
  3. $\ell_3 := \left(x_1 + x_3 + \frac{\epsilon}{3}\left(e_{x_1^2 x_3} + e_{x_1 x_3^2}\right)\right)$
  4. $\ell_4 := \left(x_1 - x_3 + \frac{\epsilon}{3}\left(e_{x_1^2 x_3} - e_{x_1 x_3^2}\right)\right)$
  5. $\ell_5 := \left(x_1 + x_2 + \frac{\epsilon}{3}\left(e_{x_1^2 x_2} + e_{x_1 x_2^2}\right)\right)$
  6. $\ell_6 := \left(x_1 - x_2 + \frac{\epsilon}{3}\left(e_{x_1^2 x_2} - e_{x_1 x_2^2}\right)\right)$
  7. $\ell_7 := \sqrt[4]{2\omega}\left(x_1 + \frac{\epsilon}{3}\left(e_{x_1^2 x_3} + e_{x_1 x_2^2} - 3e_{x_1^3}\right)\right)$
  8. $\ell_8 := \sqrt[4]{2\omega}\left(x_2 + \frac{\epsilon}{3}\left(e_{x_1^2 x_2} + e_{x_2^2 x_3} - 3e_{x_2^3}\right)\right)$
  9. $\ell_9 := \sqrt[4]{2\omega}\left(x_3 + \frac{\epsilon}{3}\left(e_{x_1 x_3^2} + e_{x_2 x_3^2} - 3e_{x_3^3}\right)\right)$
  10. $\ell_{10} := \sqrt[4]{\frac{\omega}{2}}\left(x_1 + x_2 + x_3 - \frac{\epsilon e_{x_1 x_2 x_3}}{6}\right)$
  11. $\ell_{11} := \sqrt[4]{\frac{\omega}{2}}\left(x_1 - x_2 - x_3 - \frac{\epsilon e_{x_1 x_2 x_3}}{6}\right)$
  12. $\ell_{12} := \sqrt[4]{\frac{\omega}{2}}\left(x_1 + x_2 - x_3 - \frac{\epsilon e_{x_1 x_2 x_3}}{6}\right)$
  13. $\ell_{13} := \sqrt[4]{\frac{\omega}{2}}\left(x_1 - x_2 + x_3 - \frac{\epsilon e_{x_1 x_2 x_3}}{6}\right)$

  where ω is a fourth root of -1. The construction therefore is:

$$ST_{3,4} = \lim_{\epsilon \to 0} \frac{1}{8\epsilon}\left(\ell_1^4 + \ell_2^4 + \ell_3^4 + \ell_4^4 + \ell_5^4 + \ell_6^4 + \ell_7^4 + \ell_8^4 + \ell_9^4 + \ell_{10}^4 + \ell_{11}^4 + \ell_{12}^4 + \ell_{13}^4\right)$$

  Using Lemma 1.4.8, we can conclude $ST_{3,4} \geq 2 \cdot \binom{3+4-2}{4-1} + 3 = 23$.

- We establish a $\underline{\mathrm{WR}}\left(ST_{2,d}\right)$ via explicit construction.

  Lemma 3.2.10. *$ST_{2,d}$ has border Waring rank $d+2$.*

  Proof. We will write $ST_{2,d}$ of the form

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sum_{i=1}^{d+2} L_i^d,$$

where the L_i are linear forms in the $d+2$ variables of $ST_{2,d}$, allowing ε coefficients. Recall for ζ_d the dth root of unity the following identity:

$$\sum_{i=0}^{d-1}(x_1+\zeta_d^i x_2)^d = d(x_1^d + x_2^d).$$

Construct now the polynomial

$$F = \sum_{i=0}^{d-1}(x_1+\zeta_d^i x_2+\varepsilon\ell_i)^d - dx_1^d - dx_2^d,$$

where the ℓ_i are linear forms in the variables e_m for $m \in \mathcal{M}_{2,d-1}$.

By the identity, the constant term of F will vanish. We wish to solve for ℓ_i such that the ε coefficient of F is $ST_{2,d}$, which would then give us precisely the desired limit form above. We will assume that ε does not arise within the ℓ_i. It turns out this simplifying restriction does not affect our ability to solve for the ℓ_i.

Now, the ε coefficient of F is

$$\sum_{i=0}^{d-1} d(x_1+\zeta_d^i x_2)^{d-1}\ell_i,$$

and equating this to $ST_{2,d}$, we get

$$\begin{bmatrix} e_{x_1^{d-1}} \\ e_{x_1^{d-2}x_2} \\ \vdots \\ e_{x_2^{d-1}} \end{bmatrix} = d \begin{bmatrix} \binom{d-1}{0}(\zeta_d^0)^0 & \binom{d-1}{0}(\zeta_d^0)^1 & \cdots & \binom{d-1}{0}(\zeta_d^0)^{d-1} \\ \binom{d-1}{1}(\zeta_d^1)^0 & \binom{d-1}{1}(\zeta_d^1)^1 & \cdots & \binom{d-1}{1}(\zeta_d^1)^{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ \binom{d-1}{d-1}(\zeta_d^{d-1})^0 & \binom{d-1}{d-1}(\zeta_d^{d-1})^1 & \cdots & \binom{d-1}{d-1}(\zeta_d^{d-1})^{d-1} \end{bmatrix} \begin{bmatrix} \ell_0 \\ \ell_1 \\ \vdots \\ \ell_{d-1} \end{bmatrix}$$

$$\begin{bmatrix} \ell_0 \\ \ell_1 \\ \vdots \\ \ell_{d-1} \end{bmatrix} = \frac{1}{d^2} \begin{bmatrix} (\zeta_d^d)^0 & (\zeta_d^d)^1 & \cdots & (\zeta_d^d)^{d-1} \\ (\zeta_d^{d-1})^0 & (\zeta_d^{d-1})^1 & \cdots & (\zeta_d^{d-1})^{d-1} \\ \vdots & \vdots & \ddots & \vdots \\ (\zeta_d^1)^0 & (\zeta_d^1)^1 & \cdots & (\zeta_d^1)^{d-1} \end{bmatrix} \begin{bmatrix} \frac{1}{\binom{d-1}{0}} e_{x_1^{d-1}} \\ \frac{1}{\binom{d-1}{1}} e_{x_1^{d-2}x_2} \\ \vdots \\ \frac{1}{\binom{d-1}{d-1}} e_{x_2^{d-1}} \end{bmatrix},$$

where one can easily verify that the matrix in the second line is the inverse of the one in the first (after the binomial coefficients have been removed), up to a factor of $\frac{1}{d}$. With this, we have the desired decomposition of $ST_{2,d}$.

□

Using Lemma 1.4.8, we know that $\mathrm{WR}\,(ST_{2,d}) \geq 2 \cdot \binom{2+d-2}{d-1} + 2 = 2d+2$. This completes the proof of the second assertion.

□

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APPENDICES

Appendix A

Macaulay2 code for Apolar Ideals and Support Concentration

A.1 Some useful code

A.1.1 Code for computing apolar ideals

We assume for our base field $\mathbb{Z}/p$ where $p = 61$. The following code snippet computes $f^{\perp}$ for $f \in R_d$ and is due to Oneto [37, Algorithm 2.1.16]. This is done by computing the kernel of the catalecticant map for each degree $k \leq d$ and stitching them together.

```
perpId := method();
perpId (RingElement) := F-> (
    P := ring(F);
    D := first degree F;
    L := for j from 0 to D list (
        basis(j,P)*
        mingens kernel transpose
            diff(transpose basis(j,P), diff(basis(D-j,P),F))
        );
    return trim ideal L;
    )
```

A.1.2 Code for computing rank concentration

We can generalize the definition of support concentration to define k-support concentrated polynomials. Given f, the following code finds the minimum k so that f is k-support concentrated with respect to the linear forms $\{x_1, \cdots, x_n\}$.

```
p = 61
R = ZZ/p[x,y,z]
R2 = ZZ/p
-- The function below computes a list of
-- tuples of partial derivatives and the support of the corresponding
-- monomial wrt which the partial derivative has been computed.

 partials = f -> (
      myder := {};
      temp := 0;
      mygens := makebasis(f);
      for m in mygens do (
           if diff(m,f) != 0 then (
           myder = append(myder, (diff(m,f), support(substitute(m,R))));
           );
           );
      myder = delete(first(myder),myder);
      myder = unique myder;
      return myder;
      )

-- The following function computes a list of 2-tuples
-- where the first element in each tuple is a partial derivative and the
 -- second is the support of the monomial
 -- wrt  which the derivative has been computed.

 remdupartials = f -> (
      myuniqpartials:= {};
      temp := ();
      mylistpar := partials(f);
      for m in mylistpar do (
       temp = m;
       for n in mylistpar do (
        if first(temp) == first(n) or
          (isConstant(first temp) and isConstant(first n))  then  (
               if #unique last(temp)>= #unique last(n) then temp = n;
         );
       );
      myuniqpartials = append(myuniqpartials,
           (first temp, unique last temp));
      temp = ();
      );
   return unique myuniqpartials;
   )
```

```
checkspan = (myvector, mymatrix) -> (
     use R2;
     if instance(solve
          (substitute(mymatrix,R2),substitute(myvector,R2)),Nothing)

               then return false else return true ;
   )


makebasis = f -> (
     n := first(degree(f));
     basislist := {};
     for i from 0 to n do basislist =
          join(basislist, flatten entries basis(i,R));
     return basislist;
     )

 makenewlist = f -> (
      partialslist := remdupartials(f);
      basislist := makebasis(f);
      newlist := {};
      for m in partialslist do (
           (M,C) := coefficients(first(m), Monomials=> basislist);
           newlist = append(newlist, (C, #last(m)));
           );
      return newlist;
      )

 supplist = f -> (
     vectorlist := makenewlist(f);
     mysupplist := {};
     for m in vectorlist do mysupplist = append(mysupplist, last(m));
     mysupplist = rsort(mysupplist);
     return mysupplist;
      )

sortedpartials = f -> (
     vectorlist := makenewlist(f);
     mysupplist := supplist(f);
     sortedlist := {};
     for m in mysupplist do (
          for n in vectorlist do (
               if last(n) == m then(
```

```
                    sortedlist = append(sortedlist,n);
                    vectorlist = delete(n, vectorlist);
                    break;
                    );
               );
          );
     return sortedlist;
          )



getvecnlist = vecslist -> (
     if (#vecslist > 1) then (
     myfirst := first(first(vecslist));
     myrest := {};
     for i from 1 to (#vecslist-1) do myrest =
          append(myrest, first(vecslist#i));
     return (myfirst, myrest);
     )
     else return (first(first(vecslist)), {});
     )

makematrix = vectorlist -> (
     temp = 0;
     listvec := {};
     for m in vectorlist do listvec = append(listvec, flatten entries m);
     mymatrix := matrix(listvec);
     return transpose(mymatrix);
     )

---The following code computes rank concentration of the polynomial f.

rankconcentration = f -> (
      myf = substitute(f,R);
      mypartials := sortedpartials(myf);
      for m in mypartials do (
           (v,L) = getvecnlist(mypartials);
           if L == {} then return last(m) else (
           if checkspan(v,makematrix(L)) then
                mypartials = delete(m,mypartials) else return last(m);
           );
      );
      )
```

A.1.3 Code for computing apolar ideals of Boij-Laskov type polynomials

The following code computes the apolar ideals of Boij-Laskov [13] type polynomials of which Stanley polynomials are a subset. This code is due to Tam An Le Quang.

```
    needsPackage "TensorComplexes";

-- Returns the basis of (V : A)_d as a set, where:
--      A = A_0 + A_1 + ... is a graded algebra over the field k = A_0
--      V is a k-subspace of A_c
-- (_ : _) in this setting is defined in
--      Nonunimodality of Graded Gorenstein Artin Algebras
-- by Boij and Laksov

vecQuotientSlice = (A, V, c, d) -> (
    A1 = image basis (c - d, A);
    return flatten entries super basis (d, quotient (V, A1));
);

-- Returns the ideal of A constructed from V, with variables as above
vecIdeal = (A, V, c) -> (
    -- We go up to c + 1 to kill off all higher degree monomials,
    --      as (V : A)_(c + 1) = A_(c + 1)
    gen = for d from 0 to c + 1 list vecQuotientSlice (A, V, c, d);
    return ideal flatten gen;
);

doTheRest = (A, I, c) -> (
    k = coefficientRing A;
    S = A / I;

    -- For each 0 <= i <= c, we want bases_i to be the 1 x dim_k(S_i)
    --      matrix in A of the basis of S_i orthogonal to I_i under the
    --      derivative pairing
    bases = for i from 0 to c list (
        basis (i, A)  *  generators kernel diff (
            -- super gets the actual basis elements and not just the
            --      matrix representation
            transpose super basis (i, I),
            basis (i, A)
        )
    );

    -- Append an additional variable to A for each basis element of S_c
```

```
--      representing the dual elements


R1 = A[apply (flatten entries bases_(-1), m -> e_m)];
dualbas = generators R1;
-- Impose the relations wherein the composition of the dual elements
--      vanishes
R = R1 / ideal (apply (multiSubsets (dualbas, 2), m -> m_0   *
m_1));

-- For each 1 <= i <= c, we want all of the syzygies between the dual
-- elements above, where the coefficients on the dual elements are
-- homogeneous polynomials in A of degree i
-- We are really computing the Nagata idealization of
--S by Hom_k(S, k),



syzygies = flatten for i from 1 to c list (

    flatten entries (

        (basis (i, A)   **   basis (1, R))   *

        generators kernel transpose fold (
            (a, b) -> a || b,
            for e in flatten entries basis (i, A) list (
                sub (sub (e   *
                  bases_(-i - 1), S) // sub (bases_(-1), S), R)
            )
        )
    )
);

-- Pair every dual element with every monomial in A of degree c
topbas = basis (c, A)    **    basis (1, R);
-- Get the single element in the span of topbas orthogonal
-- to every syzygy
-- If there is more than one element up
--to independence then we have a problem
result = (topbas   *
    generators kernel diff (transpose matrix {syzygies}, topbas));
-- return result_0_0;

R3 = k[gens A | gens R1];
```

```
    print result_0_0;
    return ideal ((apply (flatten entries
        gens I, s -> sub (s, R3))) |
            (apply (syzygies, s -> sub (s, R3))) |
            (apply (multiSubsets (dualbas, 2), m -> sub (m_0, R3)
                                                   *   sub (m_1, R3))));
);

stanley = (A, V, c) -> (
    return doTheRest (A, vecIdeal (A, V, c), c);
);

A = QQ[x, y, z];
V = image matrix {{0_A}};
M = stanley (A, V, 3);
print M;
--for i from 0 to 20 do print length flatten entries basis (i, R3 / M);
--minimalBetti M
```

A.2 Tensors and Waring rank

We define the *tensor product* of n vectors $a_1, ..., a_n$, with $a_i \in C^{m_i}$ as $T \in \mathbb{C}^{m_1 \times \cdots \times \mathbb{C}^{m_n}}$ with $T_{j_1,...,j_n} = \prod_{k=1}^{n} (a_k)_{j_k}$. We denote this as $T = a_1 \otimes \cdots \otimes a_n$. T is said to be a rank-1 tensor of order n and size $\prod_{i=1}^{n} m_i$.

Definition A.2.1. A rank of a tensor $T \in \mathbb{C}^{m_1 \times \cdots \times C^{m_n}}$ is defined as the smallest integer r such that such that the following equation holds:

$$T = \sum_{i=1}^{r} a_{1i} \otimes \cdots \otimes a_{ni}$$

where $a_{ki} \in \mathbb{C}^{m_k}\ \forall\ 1 \leq i \leq r$.

Thus, the rank of a tensor T is the smallest r such that T can be written as the sum of r rank-1 tensors. This coincides with the notion of matrix rank (matrices are order-2 tensors). However, while the rank of matrices is easy to compute via Gaussian elimination, the rank of tensors of order ≥ 3 is notoriously difficult to compute. [27] and [28] showed that computing tensor rank is **NP**-hard.

We are interested in a more restricted notion of tensor rank. In particular, we consider the *symmetric rank* of *symmetric tensors*, also known as the *Waring rank* of a homogeneous polynomial.

Definition A.2.2. A tensor $T \in (\mathbb{C}^n)^{\otimes d}$ is said to be *symmetric* if $\forall$ permutations $\sigma \in \mathcal{S}_d$, we have:

$$T_{i_1,\cdots,i_d} = T_{i_{\sigma(1)},\cdots,i_{\sigma(d)}}$$

The set of such tensors T is denoted as $S^{\otimes d}(\mathbb{C}^n)$.

Let us define the polynomial ring $R := \mathbb{C}[x_1, \cdots, x_n]$. From this point onwards, we also will denote the set of all monomials of R of degree d by $\mathcal{M}_d$ and the set of all monomials of degree $\leq d$ by $\mathcal{M}_{\leq d}$. We will also denote the $\mathbb{C}$-vector space spanned by a set of forms S as $\mathcal{V}(S)$.

Lemma A.2.3 (Folklore)**.** $S^{\otimes d}(\mathbb{C}^n) \cong \mathbb{C}[x_1, \cdots, x_n]_d = R_d$.

Thus, to every symmetric tensor $\in S^{\otimes d}(\mathbb{C}^n)$ we can associate a homogeneous polynomial in n-variables of degree d.

Definition A.2.4. The *symmetric* rank of a symmetric tensor $T \in S^{\otimes d}(\mathbb{C}^n)$ is defined as the smallest integer r such that T can be written as follows:

$$T = \sum_{i=1}^{r} (a_i)^{\otimes k}$$

where $a_i \in \mathbb{C}^n \; \forall \; 1 \leq i \leq r$

The above definition and Lemma A.2.3, allow us to define the Waring rank of a homogeneous polynomial as the symmetric rank of the corresponding symmetric tensor.