

Branes and Relations in Holography

by

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This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Chapter 2 of this thesis consists of materials from

The Giant Graviton Expansion

Davide Gaiotto and Ji Hoon Lee

[\[1\]](#)

Chapter 3 of this thesis consists of materials from

Trace Relations and Open String Vacua

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[\[2\]](#)

Abstract

In this thesis, we propose and study the implications of a holographic relation between the states of certain D-branes in anti-de Sitter space and trace relations in the dual gauge theory.

In the first part, we propose and test a novel conjectural relation satisfied by the superconformal index of maximally supersymmetric $U(N)$ gauge theory in four dimensions. Analogous relations appear to be also valid for the superconformal indices of a large collection of other gauge theories, as well as for a broad class of index-like generating functions. The relation expresses the finite N index as a systematic series of corrections to a large N answer. Individual corrections have a holographic interpretation as the analytic continuation of contributions from giant graviton branes fixed by a specific symmetry generator.

In the second part, we suggest a physical explanation for the empirical observation in the first part and study its implications. We compute the BPS states of D3 giant graviton branes in $\text{AdS}_5 \times S^5$ and provide strong evidence that these states are holographically dual to certain auxiliary ghost states that compensate for the presence of finite N trace relations in $U(N)$ $\mathcal{N} = 4$ super Yang-Mills. The complex formed from spaces of states of bulk D3 giants is observed to furnish a BRST-like resolution of the half-BPS Hilbert space of $U(N)$ $\mathcal{N} = 4$ SYM at finite N . We argue that the identification between the states of certain bulk D-branes and the auxiliary ghosts in the boundary holds rather generally at vanishing 't Hooft coupling $\lambda = 0$. We propose that a complex, which furnishes a BRST-like resolution of the finite N Hilbert space of a boundary $U(N)$ gauge theory at $\lambda = 0$, should be identified as the space of states of the string dual in the $\alpha' \rightarrow \infty$ limit. The Lefschetz trace formula provides the holographic map in this stringy regime, where bulk observables are computed by taking the alternating sum of the expectation values in an ensemble of states built on each open string vacuum. The giant graviton expansion is recovered and generalized in a limit of the resolution.

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Chapter 1

Introduction

1.1 Gauge Fields and Strings

The idea that a large N gauge theory admits a reorganization into a string theory has a long and fruitful history, dating back from 't Hooft [3] and coming to full-bloom in the AdS/CFT correspondence [4–6].

Perhaps the most basic entry in the gauge-string dictionary is that a state labelled by gauge-invariant words in a large N gauge theory is dual to a state consisting of closed strings propagating in a bulk spacetime:

$$\mathrm{Tr}(F^2) \ \mathrm{Tr}(XY\partial^n X) \longleftrightarrow \text{closed strings}$$

This statement is quantified in the AdS/CFT correspondence as a state-counting equivalence

$$Z_{\text{gauge}} = Z_{\text{AdS}}, \tag{1.1.1}$$

where

$$Z_{\text{gauge}}(\beta) = \mathrm{Tr}_{\mathcal{H}_{S^{d-1}}} e^{-\beta H} \tag{1.1.2}$$

is the thermal partition function of a gauge theory on $S^1_\beta \times S^{d-1}$ and Z_{AdS} is the partition function of the string dual on a background with conformal boundary $S^1_\beta \times S^{d-1}$. Currently

available methods only allow the computation of Z_{AdS} in the weak curvature limit $\lambda = L^4/\alpha'^2 \gg 1$ of the asymptotically anti-de Sitter space, and, in situations where Z_{AdS} can be computed, its state-counting interpretation is often unclear.

The gauge theory is, of course, most tractable at weak 't Hooft coupling $\lambda = g_{\text{YM}}^2 N \ll 1$. In conformal (or free) gauge theories, the state-operator correspondence allows one to map states on a spatial S^{d-1} of the theory on $\mathbb{R} \times S^{d-1}$ to operators placed at the origin of the theory on $\mathbb{R}^d$. Therefore, computing a thermal partition function of a four-dimensional $U(N)$ gauge theory at $\lambda = 0$ becomes the combinatorial problem of enumerating gauge-invariant texts built from fundamental fields that appear in a given UV Lagrangian description. In dimensions two, three, and five, one needs also to account for the presence of superselection sectors labelled by vortices, monopoles, and instantons.

The solution [7–9] to this combinatorial problem, for a free four-dimensional gauge theory with fields that transform only in the adjoint representation of the gauge group $U(N)$, is captured in the unitary integral over a holonomy matrix U :

$$Z_N(\beta) = \int_{U(N)} dU \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} f_n(q^n) \text{Tr } U^n \text{Tr } U^{-n} \right), \quad (1.1.3)$$

where $q = e^{-\beta}$. Let us elaborate on the ingredients. The single-letter partition function

$$f_n(q^n) = f_B(q^n) + (-1)^{n+1} f_F(q^n) \quad (1.1.4)$$

is defined in terms of the thermal partition functions $f_B(q), f_F(q)$ of each bosonic and fermionic field and its derivatives, modulo relations from equations of motion and gauge constraints. The $U(N)$ adjoint character

$$\chi_{\text{adj}}(U) = \text{Tr } U \text{Tr } U^{-1} \quad (1.1.5)$$

supplements the single letters with adjoint weights under gauge transformations. The plethystic exponential

$$\text{PE}[h(q)] \equiv \exp \left(\sum_{n=1}^{\infty} \frac{1}{n} h(q^n) \right) \quad (1.1.6)$$

counts multi-letter states built from the single letters. Finally, the integral over $U(N)$ projects $Z_N(\beta)$ onto gauge singlets.

Using the state-counting equivalence $Z_{\text{gauge}} = Z_{\text{AdS}}$, one can deduce much about the large N thermodynamic properties of string theory in the regime $\lambda = L^4/\alpha'^2 \ll 1$ where

the anti-de Sitter background is highly-curved. The large N saddles

$$Z_N(\beta) \sim \sum_{\text{critical points}} e^{-S_0} \times (\text{loop contributions}) \quad (1.1.7)$$

of the matrix integral (1.1.3) correspond to the thermodynamic phases of the string dual. For instance [7, 9, 10], the deconfinement transition of the gauge theory on $S^1_\beta \times S^3$, marked by a turnover from $\mathcal{O}(1)$ to $\mathcal{O}(N^2)$ behavior of the free energy $F = -\log Z_N$, maps in the bulk to a stringy analog of the Hawking-Page transition for AdS black holes [11].

The interpretation of large N saddles (1.1.7) as a sum over bulk geometries can be made precise in supersymmetric sectors. For a superconformal gauge theory with non-anomalous R-symmetry, one may define the superconformal index [12]

$$Z_N(y_i) = \text{Tr}_{\mathcal{H}_N} \left[(-1)^F e^{-\beta\{\mathcal{Q}, \mathcal{Q}^\dagger\}} y_1^{\mathcal{C}_1} y_2^{\mathcal{C}_2} \dots \right], \quad (1.1.8)$$

where y_i are fugacities dual to charges $\mathcal{C}_i$ that commute with the relevant Poincaré/superconformal supercharges $\mathcal{Q}, \mathcal{Q}^\dagger$. The index is computed via the unitary integral (1.1.3) by using the single-letter index $f(y_i) = f_B(y_i) - f_F(y_i)$. The superconformal index counts the number of bosonic states minus the number of fermionic states that satisfy the BPS condition $\{\mathcal{Q}, \mathcal{Q}^\dagger\} = 0$. This difference is a topological invariant under smooth deformations of the theory [13, 14], such as the change in the 't Hooft coupling λ .

Due to the invariance property, the large N superconformal index of $U(N)$ $\mathcal{N} = 4$ super Yang-Mills may be computed at $\lambda = 0$ and found to be in agreement with the index of S^5 Kaluza-Klein modes of IIB supergravity on $\text{AdS}_5 \times S^5$ [12]. In recent years, the superconformal index of $\mathcal{N} = 4$ SYM was shown to capture the entropy of supersymmetric AdS black holes by considering its growth in the space of complex fugacities [15–17].

1.2 Non-Perturbative Aspects

What happens to the statement, “A state labelled by gauge-invariant words in a large N gauge theory is dual to a state consisting of closed strings propagating in a bulk spacetime,” when N takes a finite, integer value?

At finite N , there is an apparent tension regarding the spectrum of states of the bulk and boundary theories. In the boundary gauge theory, states formed by exciting the vacuum with single trace operators are orthogonal at large N but become constrained by trace

relations when N is finite. For example, at $N = 2$ a single 2×2 matrix-valued scalar X satisfies the trace identities

$$\begin{aligned} (\mathrm{Tr} X)^3 - 3(\mathrm{Tr} X)(\mathrm{Tr} X^2) + 2(\mathrm{Tr} X^3) &= 0 \\ (\mathrm{Tr} X)^4 - 6(\mathrm{Tr} X)^2(\mathrm{Tr} X^2) + 3(\mathrm{Tr} X^2)^2 + 8(\mathrm{Tr} X)(\mathrm{Tr} X^3) - 6(\mathrm{Tr} X^4) &= 0 \end{aligned} \quad (1.2.1)$$

and so on, where every $\mathrm{Tr}(X^{>2})$ can be decomposed in terms of sums of products of lower traces. Therefore, the total number of states, at fixed values of the charges and λ , decreases as N is decreased. On the other hand, in the bulk, non-perturbative states such as black holes and D-branes become lighter at finite g_s . Such objects, which are present in addition to the perturbative string states, would naively result in many more states at finite g_s compared to those at small g_s , at fixed charges and $\lambda = L^4/\alpha'^2$.

In order to reconcile these differing expectations, it is typically posited that non-perturbative objects in string theory represent highly redundant descriptions of the same set of quantum states—so much so that the non-perturbative effects actually reduce the total number of states! The expected upper bound on the single-string spectrum is called the *stringy exclusion principle* [18–20]. While this principle is thought to be a generic feature of string theory, there is little quantitative understanding of the mechanism underlying the principle.

In this thesis, we propose and study the implications of a holographic relation between the states of certain D-branes in anti-de Sitter space and trace relations in the dual gauge theory. Our findings conjecturally provide a mechanism by which the stringy exclusion principle is realized in holographic dualities.

In Part I, we propose and test a novel conjectural relation satisfied by the superconformal index of maximally supersymmetric $U(N)$ gauge theory in four dimensions. Analogous relations appear to be also valid for the superconformal indices of a large collection of other gauge theories, as well as for a broad class of index-like generating functions. The relation expresses the finite N index as a systematic series of corrections to a large N answer. Individual corrections have a holographic interpretation as the analytic continuation of contributions from giant graviton branes fixed by a specific symmetry generator.

In Part II, we suggest a physical explanation for the empirical observation in Part I and study its implications. We compute the BPS states of D3 giant graviton branes in $\mathrm{AdS}_5 \times S^5$ and provide strong evidence that these states are holographically dual to certain auxiliary ghost states that compensate for the presence of finite N trace relations in $U(N)$ $\mathcal{N} = 4$ super Yang-Mills. The complex formed from spaces of states of bulk D3 giants is observed to furnish a BRST-like resolution of the half-BPS Hilbert space of $U(N)$ $\mathcal{N} = 4$ SYM at finite N . We argue that the identification between the states of certain bulk D-branes and

the auxiliary ghosts in the boundary holds rather generally at vanishing 't Hooft coupling $\lambda = 0$. We propose that a complex, which furnishes a BRST-like resolution of the finite N Hilbert space of a boundary $U(N)$ gauge theory at $\lambda = 0$, should be identified as the space of states of the string dual in the $\alpha' \rightarrow \infty$ limit. The Lefschetz trace formula provides the holographic map in this stringy regime, where bulk observables are computed by taking the alternating sum of the expectation values in an ensemble of states built on each open string vacuum. The giant graviton expansion is recovered and generalized in a limit of the resolution.

Part I

Observation

Chapter 2

Giant Graviton Expansion

2.1 Introduction

The superconformal index [12, 21, 22] is a powerful observable of supersymmetric conformal field theories (SCFTs). The superconformal index is a generating function for the Witten indices c_γ of the spaces of local operators of fixed charge γ under an appropriate collection of $U(1)$ generators.¹ Schematically, we can write

$$Z(y) = \sum_{\gamma} c_{\gamma} y^{\gamma} \tag{2.1.1}$$

where $y^{\gamma} \equiv \prod_i y_i^{\gamma_i}$. The calculation is set up so that the Witten indices receive contributions only from operators which are annihilated by a certain nilpotent supercharge Q as well as an hermitean conjugate superconformal charge S . Their scaling dimension saturates a BPS bound. The superconformal index is invariant under continuous deformations and can be readily computed from an appropriate UV Lagrangian description.²

The calculation is perturbative and combinatorial in four dimensions: one can turn off continuous couplings and compute the index of local operators which are BPS at tree level. These are polynomials in a certain collection of BPS (derivatives of) elementary fields, aka

¹It is also possible to make the index into an actual function of the fugacities by defining it as a partition function on a twisted $S^3 \times S^1$ geometry, but we will mostly focus on the generating function perspective.

²In an SCFT the index can be computed or defined equivalently in terms of local operators or in terms of states on a sphere. The latter definition can be extended to situations where conformal invariance is lost with the help of supersymmetric sphere compactifications [23]. This helps insure invariance along the RG flow. The former definition may also be extended by considering the cohomology of Q .

“letters”. In other dimensions one has to account for non-trivial topological sectors of the theory: vortices in 2d, monopoles in 3d, instantons in 5d.

The large N behaviour of the superconformal index

$$Z_N(y) = \sum_{\gamma} c_{N;\gamma} y^{\gamma} \quad (2.1.2)$$

of four-dimensional $U(N)$ $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory can be compared with holographic calculations. There are various regimes of interest:

- If we keep γ fixed as N is increased, the corresponding index $c_{N;\gamma}$ stabilizes to some limiting value $c_{\infty;\gamma}$ which agrees the index Z_{∞} of supergravity/string modes in AdS [12, 22].
- If we allow γ to grow linearly in N , we enter a regime where one needs to include the contribution of supersymmetric configurations of D-branes in AdS, usually dubbed “giant gravitons” [20, 24–27]. There have been multiple efforts to find the correct prescription to count the states associated to these configurations [28–30]. For simple configurations a match is possible [31–36] but a general story is missing.
- If we allow γ to grow quadratically in N , the indices grows exponentially in a manner consistent with the entropy of a black hole in the dual geometry [15–17, 37–48].

Our initial motivation was to study a special case of the linear regime. In order to formulate our objective and results, it is useful to refine our notations a bit. We pick a specific letter, a bosonic scalar X transforming in the adjoint representation of $U(N)$. We denote as x the fugacity for a global symmetry $U(1)_x$ under which the field X has charge 1 and denote as y the rest of the fugacities. We are interested in the large N behaviour of $c_{N;\gamma}$ for arbitrarily large $U(1)_x$ charge, keeping all other charges finite. In practice, this means looking at indices counting gauge-invariant operators built from any number of X ’s but only a finite number of other letters.

Running some computational experiments, we find an interesting pattern to the large N corrections to Z_{∞} in this regime. Corrections first occur for $U(1)_x$ charge of order N . The form of the corrections only depends on N through an overall factor of x^N , up to $U(1)_x$ charge of order $2N$. This means we can find an auxiliary generating function $\hat{Z}_1$ such that Z_N agrees with an improved approximation

$$Z_{\infty} + x^N \hat{Z}_1 Z_{\infty} \quad (2.1.3)$$

up to $U(1)_x$ charges of order $2N$. The overall factor of Z_∞ is included for future convenience: it makes the analytic properties of $\hat{Z}_1$ particularly nice.

We then observe that corrections to the improved formula occur for $U(1)_x$ charge of order $2N$ and also depend on N only through an overall factor of x^{2N} , up to $U(1)_x$ charge of order $3N$. This means we can find a second auxiliary generating function $\hat{Z}_2$ such that Z_N agrees with an improved approximation

$$Z_\infty + x^N \hat{Z}_1 Z_\infty + x^{2N} \hat{Z}_2 Z_\infty \quad (2.1.4)$$

up to $U(1)_x$ charges of order $3N$. The pattern continues on and allows us to derive a sequence of improved approximations

$$\left(1 + x^N \hat{Z}_1 + x^{2N} \hat{Z}_2 + \cdots + x^{kN} \hat{Z}_k\right) Z_\infty \quad (2.1.5)$$

The specific $U(1)_x$ charge at which corrections to these formulae first occur decreases progressively as we increase the power of y fugacities we are focussing on. This has two consequences:

- Increasingly large values of N are required to derive a given coefficient of the auxiliary $\hat{Z}_k$.
- For a fixed power of y and sufficiently low N some of the corrections will involve negative powers of x

Our initial expectation was that the sequence of improved approximations would simply break down when the power of the y fugacities becomes of order N .

To our surprise, this does not appear to be the case. As we increase the order k of our approximation, the spurious terms with negative powers of x cancel out and the sequence of approximants eventually stabilizes to the correct value of $c_{N;\gamma}$, even for powers of y which are greater than N and for small N . We thus find an exact relation

$$\frac{Z_N(x; y)}{Z_\infty(x; y)} = 1 + \sum_{k=1}^{\infty} x^{kN} \hat{Z}_k(x; y) \quad (2.1.6)$$

valid for all N , involving a new infinite sequence of auxiliary series $\hat{Z}_k(x; y)$. The sum on the right hand side even makes sense for negative N : the coefficient of every given monomial vanishes as soon as we include enough terms.

It is instructive to look at the simplest specialization of the superconformal index, which counts half-BPS operators. These are gauge-invariant polynomials built from a single letter X . We have finite N indices

$$Z_N(x) = \frac{1}{\prod_{n=1}^N (1 - x^n)} \quad (2.1.7)$$

and a systematic expansion

$$\frac{Z_N(x)}{Z_\infty(x)} = 1 + \sum_{k=1}^{\infty} x^{kN} \hat{Z}_k(x) \quad (2.1.8)$$

with an exactly computable

$$\hat{Z}_k(x) = \frac{(-1)^k x^{\frac{k(k+1)}{2}}}{\prod_{n=1}^N (1 - x^n)} \quad (2.1.9)$$

Both the original and the auxiliary indices are resummed here to rational functions of x . There is a remarkable relation

$$\hat{Z}_k(x) = Z_k(x^{-1}) \quad (2.1.10)$$

We find that this relation is a special example of a more general result valid for the full index. Indeed, we find experimentally that the coefficient of any given power of y in the auxiliary indices $\hat{Z}_k(x; y)$ can be re-summed to a rational function of x with a denominator which depends on the power of y but has zeroes only at n -th roots of unity, with $n \leq k$.

It thus makes sense to define yet another sequence of auxiliary indices $\tilde{Z}_k(\tilde{x}; y)$ obtained from $\hat{Z}_k(\tilde{x}^{-1}; y)$ by re-expanding these rational functions into a power series in $\tilde{x} = x^{-1}$. Strikingly, $\tilde{Z}_k(\tilde{x}; y)$ is related to the original indices $Z_k(\tilde{x}; \tilde{y})$ by a simple reorganization $y \rightarrow \tilde{y}$ of the fugacities.

We can look at holography to gain some insight on this result. The best understood example of half-BPS giant gravitons are D3 branes which wrap the maximal $S^3 \subset S^5$ fixed by $U(1)_x$. They have $U(1)_x$ charge N . These are only special examples of a continuous one-parameter family which wraps smaller S^3 's and carry $U(1)_x$ charge smaller than N . The corresponding radial fluctuation mode in the world-volume theory of the maximal D3 branes carries $U(1)_x$ charge -1 .

If we consider a stack of k maximal giants, the radial fluctuation mode becomes a $k \times k$ matrix of scalars and the world-volume theory has $U(k)$ gauge invariance. The gauge-invariant half-BPS excitations of branes will be counted by an auxiliary $U(k)$ index which is precisely $Z_k(x^{-1})$. Thus $\hat{Z}_k(x)$ is a re-summed form of the index counting infinitesimal fluctuations of a maximal giant graviton. It may be possible to formulate some equivariant localization argument to explain (2.1.8) holographically as a sum over fixed points of

the $U(1)_x$ action consisting of k maximal giant gravitons, weighed by the index for local fluctuations. Re-summations of rational fluctuation determinants are not uncommon in localization calculations.

We will not attempt to formalize such an argument. Instead, we simply observe that the full relation (2.1.6) appears to have a similar holographic interpretation. The infinitesimal fluctuations of k maximal giant gravitons are controlled by a twisted version of four-dimensional $U(k)$ $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory. The identification between the symmetry generators of the $U(k)$ auxiliary theory and of the original $U(N)$ theory is precisely the $y \rightarrow \tilde{y}$ reorganization relating $Z_k(\tilde{x}; \tilde{y})$ to $\tilde{Z}_k(\tilde{x}; y)$. Thus $\tilde{Z}_k(\tilde{x}; y)$ is the index for infinitesimal fluctuations of k giant gravitons and individual terms in (2.1.6) would be reproduced by re-summation of a contribution from k maximal giants.

We can repeat the analysis leading to (2.1.6) for the indices of other four-dimensional supersymmetric gauge theories with a nice large N limit. Indeed, we can apply it to a large class of index-like quantities $Z_N(x; y)$ which have analogous combinatorial properties but are *not* the indices of actual SCFTs. For example, we could take $Z_N(x; y)$ to count $U(N)$ gauge-invariant polynomials built from any reasonable collection of letters including X , counted by some single-letter index $f(x; y)$.

We find that the formula (2.1.6) holds in great generality, together with the re-summation relation $\hat{Z}_k(x; y) \leftrightarrow \tilde{Z}_k(\tilde{x}; y)$ involving a second collection of index-like quantities built from a single-letter index $\tilde{f}(\tilde{x}; y)$ with $x\tilde{x} = 1$ and

$$(1 - f)(1 - \tilde{f}) = (1 - x)(1 - \tilde{x}) \quad (2.1.11)$$

The relation is symmetric: the new $\tilde{Z}_k(\tilde{x}; y)$ indices satisfy an analogue of (2.1.6) involving on the right hand side the analytic continuation of the original $Z_N(x; y)$ indices.

Generalizations of this relation are available if we add (anti)-fundamental letters or consider quiver gauge theories with X being an adjoint or bi-fundamental field. In all cases we checked where the $Z_N(x; y)$ have an holographic interpretation we find that the re-summed auxiliary indices $\tilde{Z}_k(\tilde{x}; y)$ coincide with the indices of appropriate giant graviton worldvolume theories.

The exceedingly broad applicability of these relations is a bit confusing at first. It is rather clear that (2.1.6) and its generalizations should have a purely combinatorial explanation. In all examples we considered where the formula works, the x dependence of the index is essentially controlled by the number of ways an operator in some generic representation of $U(N)$ can be combined in a gauge-invariant way with an arbitrary number of X 's. If that restricted counting problem satisfies an appropriate analogue of (2.1.6), the full indices will do so as well.

Such a combinatorial explanation may appear in tension with the apparently physical relation between the right hand side of (2.1.6) and the indices of world-volume theories for giant gravitons. The tension is ameliorated if we recall that *any* $U(N)$ gauge theory is dual to some kind of string theory, simply because of 't Hooft large N combinatorics. Operators of finite charge are associated to string excitations and operators of size N can be associated to D-branes [49, 50]. From that perspective, it is perhaps natural that the world-volume index for such D-branes could be determined in a purely combinatorial fashion, irrespectively of the existence of a weakly-curved holographic dual.

In order to probe the combinatorial nature of the dual $\tilde{Z}_k(\tilde{x}; y)$ indices, we make contact with the known gauge theory description of giant gravitons. The maximal giant gravitons are dual to operators of the form $\det X$. Powers $(\det X)^k$ represent a stack of k coincident D3 branes. There is a well-defined strategy [51, 52] to construct finite modifications of a $(\det X)^k$ operator which may be matched to small fluctuations of the stack of D3 branes, governed by an appropriate world-volume $U(k)$ gauge theory.

Following this strategy, we will solve an associated combinatorial problem: compute (in a theory-agnostic fashion) the index counting the operators defined at large N as finite modifications of a given product of determinants, in a manner analogous to the way one can count single trace operators built from a finite number of fields. The counting problem is slightly ill-defined at tree level. If we assume a certain non-linear correction to the tree-level supercharge Q we can regularize the problem and find that the finite modifications of $(\det X)^k$ are counted precisely by $\tilde{Z}_k(\tilde{x}; y)$. Without that assumption we can still recover $\tilde{Z}_k(\tilde{x}; y)$ by a formal manipulation whose meaning is somewhat unclear.

Another potential puzzle associated to (2.1.6) is the absence of further corrections of order N^2 which could be associated to non-trivial gravitational saddles such as black holes. In the initial regime of interest this is not a complete surprise: SUSY black holes involve multiple large charges, rather than just $U(1)_x$. Still, there are rich families of half-BPS supergravity solutions [53] which could in principle shown up in a large N expansion for large $U(1)_x$ charges. The simplest potential explanation is that (2.1.6) is a purely stringy result, associated to a regime of small 't Hooft coupling where non-trivial geometries and even black holes may be fully described as large stacks of D-brane and thus do not need to be separately accounted for.

We will also discuss in some detail the 3d version of the story, looking at the superconformal index for the SCFTs associated to stacks of M2 branes. The contribution of monopole sectors make the analysis more intricate, but the payoff is a new insight on the superconformal index of stacks of M5 branes, which appear as giant gravitons in the setup.

While this chapter was concluded another work appeared [54] which has some overlap

with our analysis. In the language of the current chapter, the work computes the indices for the $U(n_1) \times U(n_2) \times U(n_3)$ world-volume theory of three stacks of maximal giant gravitons of different orientations, which roughly speaking would be associated to modifications of a product of determinants $(\det X)^{n_1} \times (\det Y)^{n_2} \times (\det Z)^{n_3}$. The objective is to write a formula (3) which is somewhat analogous to our (2.1.6), but involves a triple summation over analytically-continued indices for all values of the three n_i .³

The reader may well wonder how could our (2.1.6) and the formula (3) from the reference [54] be simultaneously true, as our formula is essentially a sum over a subset of the terms in (3). In lieu of an explanation, we provide a simplified and tractable example of this phenomenon which appears in Section 2.5, involving a certain simplified index $Z_N(x, y)$ of the 3d SCFTs associated to M2 branes. We have a simple expression

$$\sum_{N=0}^{\infty} \zeta^N Z_N(x, y) = \frac{1}{\prod_{a \geq 0, b \geq 0} (1 - \zeta x^a y^b)} \quad (2.1.12)$$

We can evaluate $Z_N(x, y)$ as a contour integral in the ζ plane, giving rise to an exact analogue of (3)

$$Z_N(x, y) = \sum_{k, k' \geq 0} Z_N^{k, k'}(x, y) \quad (2.1.13)$$

valid for finite values of x and y within in the unit circle, with

$$Z_N^{k, k'}(x, y) = x^{kN} y^{k'N} \frac{1}{\prod_{a \geq -k, b \geq -k' | (a, b) \neq (0, 0)} (1 - x^a y^b)} \quad (2.1.14)$$

These individual terms have denominators such as $(x - y)$ which cannot be expanded unambiguously into a power series in x and y . The ambiguous denominators must go away when multiple terms, perhaps with the same value of $k + k'$, are added up.

On the other hand, we can also find experimentally that the analogue of (2.1.6)

$$Z_N(x, y) = \sum_{k \geq 0} Z_N^{k, 0}(x, y) \quad (2.1.15)$$

holds as well. Here the right hand side is expanded first into a power series in y and then into a power series in x . We leave a further investigation of these wall-crossing-like mathematical phenomena to future work.

³A technical difference is that the analytic continuation is done within the integral over $U(n_1) \times U(n_2) \times U(n_3)$ fugacities with the help of a pole-selection rule which appears to require the combination of different terms with the same $n = n_1 + n_2 + n_3$.

2.1.1 Structure of the Chapter

We devote Section 2.2 to a review of the “naive” large N limit $Z_N \rightarrow Z_\infty$. We then proceed in an acausal manner and devote Section 2.3 to the counting problem for small fluctuations of a product of determinant operators. This allows us to derive immediately our conjectural formula for the auxiliary $\tilde{Z}_k(\tilde{x}; y)$ and $\hat{Z}_k(x; y)$ indices. Prior knowledge of the $\hat{Z}_k(x; y)$ greatly simplifies our work in Section 2.4, as an explicit calculation of Z_N for very large N is challenging, but (2.1.6) can be tested at small N as long as candidate $\hat{Z}_k(x; y)$ are available. We devote Section 2.5 to the case of 3d gauge theories.

In Appendix A.1 we count the number of ways an operator in the adjoint representation of $U(N)$ can be combined in a gauge-invariant way with an arbitrary number of X ’s. In Appendix A.2, we write a generating function for the Schur index that greatly facilitates their computation. We present tests of the giant graviton expansion for the Schur index in Appendix A.3. We review a useful formula for $U(N)$ gauge theory indices in terms of S_n characters in Appendix A.4. We present tests of expansion for a specialized $\mathcal{N} = 4$ index and for general $U(N)$ indices in Appendix A.5 and A.6.

2.2 Single Trace Operators in the Large N Limit

We begin by reviewing the large N limit of the Witten indices $c_{N,\gamma}$ at fixed γ .

2.2.1 $U(N)$ Gauge Theory with Adjoint Matter

The superconformal indices of a $U(N)$ gauge theory with adjoint matter has a simple combinatorial interpretation: it counts with signs the gauge-invariant operators built from a collection of adjoint BPS letters L_a with Grassmann parity $|L_a|$. The letters consist of all the elementary fields and their derivatives which are in the tree-level cohomology of the selected supercharge Q .

The letters L_a have some specific charge γ^a under the $U(1)$ global symmetries which organize the index. We denote the single-letter index as

$$f(y) = \sum_a (-1)^{|L_a|} y^{\gamma^a} \quad (2.2.1)$$

In a physical theory the charges γ^a satisfy some unitarity bounds and only a finite collection of monomials in the L_a can contribute to the space of BPS local operators with any given

charge. Whenever we consider more general counting problems of this form which do not arise from an SCFT we should impose a similar constraint by hand. For example, $f(x) = x + x^{-1}$ would not give a good counting problem: any power of the product of the two letters would contribute to the charge 0 index.

The index can be expressed as an integral over the unitary group. In terms of eigenvalues μ_a of an $U(N)$ group element, we can write

$$Z_N(y) = \frac{1}{N!} \int \frac{d\mu_a}{2\pi i \mu_a} \prod_{a \neq b} (1 - \mu_a / \mu_b) \text{PE} \left[\left(\sum_a \mu_a \right) \left(\sum_b \mu_b^{-1} \right) f(y) \right] \quad (2.2.2)$$

where PE denotes the Plethystic Exponential:

$$\text{PE} \left[\sum_a n_a \rho_a \right] = \frac{1}{\prod_a (1 - \rho_a)^{n_a}}. \quad (2.2.3)$$

Here ρ_a are monomials in the fugacities and n_a are integers. The right hand side is expanded into a formal power series in the ρ_a . The plethystic exponential in the integrand counts polynomials in the adjoint letters.

We write

$$Z_N(y) = \sum_{\gamma} c_{N;\gamma} y^{\gamma} \quad (2.2.4)$$

Gauge-invariant operators built from a number of letters which is kept finite as N is increased can be written uniquely as a product of traces

$$\text{Tr } L_{a_1} L_{a_2} \cdots L_{a_n} \quad (2.2.5)$$

of cyclic words built from the letters in the alphabet: any relations between trace involve a product of at least N letters and can thus be disregarded.

This implies that the coefficients $c_{N;\gamma}$ will stabilize to some limit $c_{\infty;\gamma}$ for any given γ as N increases. The stabilization occurs as soon as N is larger than the maximum number of letters which can appear in an operator of charge γ . We will denote it as $Z_{\infty}(y)$ the generating function of the $c_{\infty;\gamma}$, i.e. the index counting polynomials in single trace operators. It can be written explicitly as an infinite product:

$$Z_{\infty}(y) = \prod_{n=1}^{\infty} \frac{1}{1 - f(y^n)}. \quad (2.2.6)$$

This is also the result of a large N saddle-point evaluation of $Z_N(y)$, around a saddle for which the μ_a are distributed uniformly on the unit circle [12].

The single letter partition function for the superconformal index of $\mathcal{N} = 4$ SYM is

$$f(x, y, z, p, q) = 1 - \frac{(1-x)(1-y)(1-z)}{(1-p)(1-q)} \quad (2.2.7)$$

The fugacities are restricted by the relation $pq = xyz$.

This expression for f is a shorthand for

$$f(x, y, z, p, q) = 1 - (1-x)(1-y)(1-z) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p^n q^m \quad (2.2.8)$$

Concretely, the three complex scalars X, Y, Z have fugacities x, y, z , while the holomorphic derivatives $\partial_{\alpha+}$ have fugacities p and q .

The factorization of $1 - f$ leads to a particularly simple large N index:

$$Z_{\infty}(x, y, z, p, q) = \prod_{n=1}^{\infty} \frac{(1-p^n)(1-q^n)}{(1-x^n)(1-y^n)(1-z^n)} \quad (2.2.9)$$

There are a variety of simplifying limits which keep the letter X :

- We can send all fugacities except x to 0 to get the half-BPS index, counting local operators built from X alone: $f = x$. The large N answer

$$Z_{\infty} = \prod_{n=1}^{\infty} \frac{1}{1-x^n} \quad (2.2.10)$$

counts polynomials in the $\text{Tr} X^n$ single trace operators.

- We can send p, q and z to zero to get a $\frac{1}{4}$ -BPS index, with $f = x + y - xy$. The individual letters are now two scalar fields X and Y , as well as a fermion ζ with the same charge as the commutator $[X, Y]$. The supercharge indeed maps $\zeta \rightarrow [X, Y]$, so that the large N Q-cohomology includes symmetrized traces $\text{STr} X^n Y^m$. The large N answer

$$Z_{\infty} = \prod_{n=1}^{\infty} \frac{1}{(1-x^n)(1-y^n)} = \frac{\prod_{n=0}^{\infty} \prod_{m=0}^n (1-x^{n-m+1} y^{m+1})}{\prod_{n=1}^{\infty} \prod_{m=0}^n (1-x^{n-m} y^m)} \quad (2.2.11)$$

can be interpreted as counting polynomials built from symmetrized traces as well as the fermionic operators $\text{STr} \zeta X^n Y^m$.

- We can send p , y and z to zero to get

$$f = \frac{x - q}{1 - q} \quad (2.2.12)$$

- We can set $p = z$ to get the Schur index, with $q = xy$:

$$f = \frac{x + y - 2q}{1 - q} \quad (2.2.13)$$

We could also consider some purely combinatorial problems which do not occur as an index, such as counting operators built from two letters X and Y only, i.e. $f = x + y$. Finally, we could consider problems where X is not present, such the limit where x, y, z and p are sent to 0, leaving a single fermionic letter.

2.2.2 $U(N)$ Gauge Theory with Adjoint and (Anti)Fundamental Matter

There is a simple generalization of the large N formula which is applicable in the situation where (anti)fundamental letters are present. The integral is now

$$\begin{aligned} Z_N(y) = & \frac{1}{N!} \int \frac{d\mu_a}{2\pi i \mu_a} \prod_{a \neq b} (1 - \mu_a / \mu_b) \text{PE} \left[\left(\sum_a \mu_a \right) \left(\sum_b \mu_b^{-1} \right) f(y) \right] \cdot \\ & \cdot \text{PE} \left[\left(\sum_a \mu_a \right) v(y) \right] \text{PE} \left[\left(\sum_b \mu_b^{-1} \right) \bar{v}(y) \right] \end{aligned} \quad (2.2.14)$$

with $v(y)$ and $\bar{v}(y)$ the indices counting the fundamental and anti-fundamental letters.

Gauge-invariant operators built from a number of letters which is kept finite as N is increased are built from both traces and “mesons”, i.e. sequences of adjoint letters sandwiched between a fundamental and an anti-fundamental letters. These operators are counted by

$$Z_\infty(y) = \frac{1}{\prod_{n=1}^{\infty} (1 - f(y^n))} \text{PE} \left[\frac{\bar{v}(y)v(y)}{1 - f(y)} \right] \quad (2.2.15)$$

which is also the saddle-point evaluation of $Z_N(y)$ around a saddle for which the μ_a are distributed uniformly on the unit circle.

As a physical example, consider the index for an half-BPS (aka $(4, 4)$) 2d surface defect in 4d SYM, defined by coupling the theory to a 2d hypermultiplet transforming in the fundamental representation of the gauge group. This defects breaks some supersymmetry, but preserves the one employed in the definition of the superconformal index. The choice of geometric orientation and unbroken R-symmetry breaks the symmetry between p and q and part of the symmetry between x , y and z .

With a specific choice of orientation we find the contribution of the 2d hypermultiplets:

$$v = p^{-1} z \frac{(1-x)(1-y)}{1-q} s^{-1} \quad \bar{v} = \frac{(1-x)(1-y)}{1-q} s \quad (2.2.16)$$

with s being the fugacity for the $U(1)_s$ flavour symmetry rotating the 2d hypermultiplets. Here q is associated to rotations in the plane of the 2d defect.

We obtain

$$Z_\infty(x, y, z, p, q) = \prod_{n=1}^{\infty} \frac{(1-p^n)(1-q^n)}{(1-x^n)(1-y^n)(1-z^n)} \text{PE} \left[z p^{-1} \frac{(1-p)(1-x)(1-y)}{(1-z)(1-q)} \right] \quad (2.2.17)$$

The surface defect is expected to be holographically dual to a D3 brane wrapping $AdS_3 \times S^1$. It would be nice to reproduce the right hand side in that description. As a rational function, the argument of the plethystic exponential on the right hand side is the usual f with p and z replaced respectively by z^{-1} and p^{-1} .⁴

2.2.3 Quiver Gauge Theories

We can employ this formula recursively to study quiver gauge theories, with some $\prod_r U(N_r)$ gauge group and adjoint, bifundamental and (anti)fundamental letters.

For example, consider an $U(N_1) \times U(N_2)$ gauge theory with adjoint letters $f_{11}(y_i)$ and $f_{22}(y_i)$, bifundamental $f_{12}(y_i)$ and $f_{21}(y_i)$. Taking N_2 large first we compute $Z_{N_1, \infty}(y_i)$, which is the product of $\prod_{n=1}^{\infty} \frac{1}{1-f_{22}(y_i^n)}$ and of an integral over $U(N_1)$ fugacities, taking the form of the index of a $U(N_1)$ gauge theory with adjoint index $f(y_i) = f_{11}(y_i) + \frac{f_{12}(y_i)f_{21}(y_i)}{1-f_{22}(y_i)}$. Taking N_1 large as well, we get a neat final expression

$$Z_{\infty, \infty}(y_i) = \prod_{n=1}^{\infty} \frac{1}{1 - f_{11}(y_i^n) - f_{22}(y_i^n) + f_{11}(y_i^n)f_{22}(y_i^n) - f_{12}(y_i^n)f_{21}(y_i^n)} \quad (2.2.18)$$

⁴Another interesting observation is that the plethystic exponential on the right hand side contains a $\prod_n (1-z^n)$ factor which cancels the same factor on the right hand side. This likely means that operators of the form $\text{Tr} Z^n$ are not BPS anymore in the presence of the defect. This is compatible with the fact that the coupling to the 2d hypermultiplets mixes Z with the moment map operators for the hypermultiplets.

which counts operators built from a finite number of adjoint and bifundamental letters, with no trace relations.

We can write this as

$$Z_{\infty,\infty}(y_i) = \prod_{n=1}^{\infty} \frac{1}{\det_{2 \times 2} [1 - \hat{f}(y_i^n)]} \quad (2.2.19)$$

where $\hat{f}$ is the 2×2 matrix with entries f_{rs} . This matches a direct saddle point evaluation of the full integral [55].

If we also had fundamental letters counted by v_1 and v_2 as well as anti-fundamental counted by $\bar{v}_1$ and $\bar{v}_2$, the effective $U(N_1)$ index which appears after the large N_2 limit has $v = v_1(y_i) + \frac{f_{12}(y_i)v_2(y_i)}{1-f_{22}(y_i)}$ and $\bar{v} = \bar{v}_1(y_i) + \frac{\bar{v}_2(y_i)f_{21}(y_i)}{1-f_{22}(y_i)}$. The final answer neatly simplifies to

$$Z_{\infty,\infty}(y_i) = \prod_{n=1}^{\infty} \frac{1}{1 - f_{11}(y_i^n) - f_{22}(y_i^n) + f_{11}(y_i^n)f_{22}(y_i^n) - f_{12}(y_i^n)f_{21}(y_i^n)} \cdot \text{PE} \left[\frac{(1 - f_{22})\bar{v}_1 v_1 + (1 - f_{11})\bar{v}_2 v_2 + f_{12}\bar{v}_1 v_2 + f_{21}\bar{v}_2 v_1}{1 - f_{11} - f_{22} + f_{11}f_{22} - f_{12}f_{21}} \right] \quad (2.2.20)$$

We can write this as

$$Z_{\infty,\infty}(y_i) = \frac{1}{\prod_{n=1}^{\infty} \det_{2 \times 2} [1 - \hat{f}(y_i^n)]} \text{PE} \left[\sum_{r,s} \bar{v}_r [1 - \hat{f}(y_i)]_{rs}^{-1} v_s \right] \quad (2.2.21)$$

The same expressions, with a $n \times n$ matrix $\hat{f}$, can be derived for quivers with n nodes. It matches a direct saddle point evaluation of the full integral [55].

2.3 Modifications of a Determinant

The objective of this section is to introduce the auxiliary indices $\tilde{Z}_k$ which in later section will provide the analytic continuation of the $\hat{Z}_k$. In this section we introduce the $\tilde{Z}_k$ as indices of open string fluctuations of a collection of determinants. The definition and computation seem to be valid under somewhat restrictive sets of assumptions, even though the expression we derive appears to provide the analytic continuation of the $\hat{Z}_k$ in a much broader context.

2.3.1 $U(N)$ Gauge Theory with Adjoint Matter

Next, we pick a specific bosonic letter X with fugacity x . We now use L_a to denote the remaining letters and y_i to denote the remaining fugacities. We will thus refer to the single-letter partition function as $f(x; y_i)$ and to the partition function as $Z_N(x; y_i)$. We denote as $Z_\infty(x; y_i)$ the count of operators of finite size from the previous section.

We want to count “finite” gauge-invariant modification of the operator $\det X$. Loosely speaking, the modifications are obtained from $\det X$ by replacing a finite collection of X symbols with some other finite strings of letters. For example, replacing a single X with $L_1 L_2 L_1$ gives an operator

$$\epsilon^{i_1 i_2 i_3 \dots} \epsilon_{j_1 j_2 j_3 \dots} (L_1 L_2 L_1)_{i_1}^{j_1} X_{i_2}^{j_2} X_{i_3}^{j_3} \dots \quad (2.3.1)$$

etcetera. In the holographic/large N perspective, these modifications represent open strings attached to a giant graviton.

As we will illustrate later on with the help of a fermionization trick, the finite modifications of a determinant in the large N limit are obtained by a finite collection of replacements $X \rightarrow \dots$ as well as multiplication by some overall polynomial of the traces. These contribution of the closed traces will give rise to an overall power of Z_∞ , which we will strip off in this section to focus on $X \rightarrow \dots$ modifications only.

This description involves some overcounting, due to a class of relations which survive in the large N limit. Indeed, if we replace X by XW for some word W , antisymmetry allows one to rewrite the answer as $\text{Tr } W \det X$. The same occurs for a WX modification. These appear to be the only non-trivial relations surviving at large N . We will momentarily devise a strategy to avoid over-counting the possible $X \rightarrow \dots$ modifications.

Before we move on, we need to discuss a potential obstruction to our counting objective. A determinant modification which replaces X with “1” has effectively the opposite charge to X and changes the overall fugacity of the operator by x^{-1} . On the other hand, a determinant modification containing X fields in the middle of the word, such as $X \rightarrow L_1 X^n L_2$, will change the fugacity of the operator schematically by $y_1 y_2 x^n$. This creates a problem: for every integer n we can combine the $X \rightarrow L_1 X^n L_2$ modification with n other $X \rightarrow 1$ modifications to get a distinct modified determinant operator with total fugacity $x^N y_1 y_2$. This gives an infinite class of operators which are distinct in the large N limit and have the same fugacity, ruining the counting problem.

Concretely, we can define a “single modification” index which counts individual modifications of the determinant, but the computation will break down when we try to count

multiple modifications by a Plethystic Exponential. The problem is somewhat similar to what would have happened in the usual index calculation if we had included by mistake letters which were not BPS at tree level by dropping some terms in the tree-level supercharge: the Witten index would not, strictly speaking, change, but we may have added infinitely many pairs of states which would have cancelled out in the correct calculation. Our interpretation of the situation here is that there must be some non-linear part of the supercharge Q which enforces the cancellation of these infinitely many determinant modifications to recover the count of open strings attached to a giant graviton.

As this phenomenon occurs in a situation where we have N copies of X in the background, it is natural to look at some non-linear piece of Q which can give an effective linear contribution in a background of X 's. Suppose that all the letters come in pairs L_a, L'_a with opposite Grassmann parity, such that L'_a has the same charge as XL_a . Suppose also that the non-linear part of the supercharge Q includes terms which map $L'_a \rightarrow [X, L_a]$. This is indeed the case in important physical examples. This term in the supercharge would allow one to commute X along a chain of symbols, up to Q -exact operators. Then we could bring X to the very left of a modification and eliminate it by the relation discussed above. Modifications of the form $L_1 X^n L_2$ would then drop out of the cohomology.

In this situation, we would have

$$f = x + h(x, y) - xh(x, y) \quad (2.3.2)$$

where h counts the L_a letters. Equivalently, we would be able to factorize

$$1 - f = (1 - x)(1 - h) \quad (2.3.3)$$

We will indeed see that this factorization leads to important simplifications below and makes it possible to exponentiate the single-modification index.

We are ready to introduce a powerful fermionization trick. Write

$$\det X = \int d\psi d\bar{\psi} e^{\bar{\psi} X \psi} \quad (2.3.4)$$

for some (anti)fundamental fermions $\bar{\psi}$ and ψ . Modifications of the determinant can be written as expressions of the form

$$\int d\psi d\bar{\psi} e^{\bar{\psi} X \psi} (\bar{\psi} W_1 \psi) (\bar{\psi} W_2 \psi) \cdots \quad (2.3.5)$$

This gives a quick explanation for the large- N combinatorics of determinants: inserting a determinant in a correlation function is the same as introducing auxiliary (anti)fundamental

variables $\bar{\psi}$ and ψ , which add boundaries to the 't Hooft ribbon graphs and thus add a D-brane to the dual string theory. Fermion bilinears of the form $\bar{\psi}L_1\cdots L_s\psi$ represent open strings attached to the D-brane, in the same way as $\text{Tr}L_1\cdots L_s$ represents closed strings.

The relations we mentioned above are just the Ward identities for the fermions:

$$\int d\psi d\bar{\psi} e^{\bar{\psi}X\psi} (\bar{\psi}XW_1\psi)(\bar{\psi}W_2\psi)\cdots = \int d\psi d\bar{\psi} e^{\bar{\psi}X\psi} \left(\frac{d}{d\psi}W_1\psi\right)(\bar{\psi}W_2\psi)\cdots \quad (2.3.6)$$

where the derivative acts on the whole expression to the right of it.

For counting purposes, we can attempt to impose these Ward identities by adding some bosonic antifields $\bar{u}$ and u , with a BRST differential δ mapping the antifields to the Ward identities:

$$\begin{aligned} \delta X &= \delta\psi = \delta\bar{\psi} = 0 \\ \delta u &= X\psi \\ \delta\bar{u} &= \bar{\psi}X. \end{aligned} \quad (2.3.7)$$

The BRST cohomology in ghost number 0 will reproduce the operators counting we are interested in: ghost number 0 operators do not contain the antifields and are all BRST closed. The BRST exact operators are precisely these which will vanish in the fermion integral.

The main problem with this approach is that there is cohomology in non-zero ghost number. This is closely related to the fact that the expression $\bar{\psi}X\psi$ is set to zero by two distinct Ward identities: it appears in the variation of $\bar{\psi}u$ as well as $\bar{u}\psi$. The combination $\bar{\psi}u + \bar{u}\psi$ is thus BRST closed, but it is not BRST exact. It gives a fermionic zeromode with ghost number -1 and trivial fugacity. Other operators which are cancelled by two Ward identities, such as $\bar{\psi}XX\psi$, do not give this problem: this appears in the BRST image of both $\bar{\psi}Xu$ and $\bar{u}X\psi$, but $\bar{\psi}Xu + \bar{u}X\psi$ is itself in the image of $\bar{u}u$ and cancels out in cohomology.

We find that the large N operator counting in the enlarged theory vanishes: every BRST-closed operator in ghost number 0 we are interested in can be multiplied by the zeromode to give an operator in ghost number -1 with the same charges. This appears to exhaust the large N BRST cohomology. We thus conjecture that this problem can be addressed in a simple manner: we remove the offending fermionic zeromode by hand.

We thus arrive at the following conjectural statement: in the large N limit, finite modifications of $\det X$ can be counted as polynomials in single trace operators as well as

mesons built from fermions and anti-fields, with the $\bar{\psi}u + \bar{u}\psi$ meson subtracted off. These auxiliary fundamental letters are counted by $v = (x-1)\lambda$ and the anti-fundamental by $\bar{v} = (1-x^{-1})\lambda^{-1}$. Here λ denotes a fugacity for an extra symmetry $U(1)$ which only acts on these auxiliary variables. It will actually drop out of calculations, but will be useful later on.

The index counting individual mesons is thus

$$\tilde{f} = 1 + \frac{\bar{v}v}{1-f} = 1 - \frac{(1-x)(1-x^{-1})}{1-f} \quad (2.3.8)$$

We eliminated the contribution of the spurious zeromode by adding 1.

The expression simplifies and becomes more transparent if we have the factorization $(1-f) = (1-x)(1-h)$ discussed above:

$$1 - \tilde{f} = \frac{1-x^{-1}}{1-h} = 1 - \frac{x^{-1}}{1-h} + \frac{h}{1-h} \quad (2.3.9)$$

We tentatively recognize $\frac{x^{-1}}{1-h}$ as the contribution of $X \rightarrow L_1 \cdots L_s$ modifications which do not include any X 's. The extra term must be a bit more mysterious.

In any case, we can introduce the $\tilde{x} = x^{-1}$ fugacity for the $X \rightarrow 1$ modifications and write the relation as

$$\frac{1-\tilde{f}}{1-\tilde{x}} = \frac{1-x}{1-f} \quad (2.3.10)$$

Curiously, this is an involution.

We can exponentiate $\tilde{f}$ to get the counting function for any number of modifications:

$$\tilde{Z}_1(\tilde{x}, y) = \text{PE}[\tilde{f}] \quad (2.3.11)$$

Consider the example of $\mathcal{N} = 4$ SYM. We have

$$\tilde{f}(\tilde{x}, y, z, p, q) = \frac{(1-\tilde{x})(1-p)(1-q)}{(1-y)(1-z)} = f(\tilde{x}, p, q, y, z) \quad (2.3.12)$$

This is just the single-letter index of an $U(1)$ $\mathcal{N} = 4$ SYM with fugacities transformed as $x \rightarrow x^{-1}$, $y \leftrightarrow p$, $z \leftrightarrow q$.

As we mentioned in the introduction, the $\det X$ operator is holographically dual to a giant graviton wrapping an $\mathbb{R} \times S^3$ submanifold in $\text{AdS}_5 \times S^5$. The low energy worldvolume

theory on the giant graviton is thus precisely $U(1) \mathcal{N} = 4$ SYM compactified on S^3 , just as one would employ to compute the index by a state-operator map. The fugacities y and z of the original theory, though, now control the angular momenta on S^3 , while the fugacities p and q are associated to scalar fields describing fluctuations in AdS_5 . This motivates the redefinition of the fugacities. Fluctuations in the size of the S^3 inside S^5 actually lower the x charge and thus have fugacity $\tilde{x} = x^{-1}$.

We should stress that even after making sense of the index for determinant modifications with the single trace contribution stripped out, we would not be able to actually combine it with $Z_\infty(x; y_i)$ to produce an index counting all modifications of the determinant: $Z_\infty(x; y_i)$ is a formal power series in x but $\tilde{Z}_1(\tilde{x}, y)$ is a power series in x^{-1} , so the product is ill-defined.

2.3.2 $U(N)$ Gauge Theory with Adjoint and (Anti)Fundamental Matter

Next, we can add (anti)fundamental letters in the original gauge theory. These will also contribute determinant modifications. Using the (anti)fundamental letters we could write gauge-invariant expressions which have different numbers of ψ and $\bar{\psi}$ letters. These do not actually correspond to determinant modifications. Indeed, they vanish when inserted into the auxiliary $\int d\psi d\bar{\psi} e^{\bar{\psi} X \psi}$ integral. We can take this constraint into account by projecting onto operators invariant under the extra $U(1)_\lambda$ symmetry acting on ψ and $\bar{\psi}$ with opposite charge.

Including both types of (anti)fundamental letters in the large N formula 2.2.15 by replacing $v \rightarrow v + (x - 1)\lambda$ and $\bar{v} \rightarrow \bar{v} + (1 - x^{-1})\lambda^{-1}$ there, we arrive at our conjectural formula for the large N index counting finite modifications of $\det X$:

$$\tilde{Z}_1(x; y_i) = \text{PE}[\tilde{f}] \oint \frac{d\lambda}{2\pi i \lambda} \text{PE} \left[\frac{x-1}{1-f} \bar{v} \lambda \right] \text{PE} \left[\frac{1-x^{-1}}{1-f} v \lambda^{-1} \right] \quad (2.3.13)$$

The corrections thus take the form of the index for an auxiliary $U(1)$ gauge theory.

The formula can be written in a more compact form by a shift $\lambda \rightarrow x^{-\frac{1}{2}} \lambda$:

$$\tilde{Z}_1(x; y_i) = \text{PE}[\tilde{f}] \oint \frac{d\lambda}{2\pi i \lambda} \text{PE} \left[\frac{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}{1-f} (v \lambda^{-1} + \bar{v} \lambda) \right] \quad (2.3.14)$$

This takes the form of the index for an auxiliary $U(1)$ gauge theory, with (anti)fundamental letters counted by indices

$$\tilde{v} = \frac{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}{1 - f} v \quad \quad \tilde{\bar{v}} = \frac{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}{1 - f} \bar{v} \quad (2.3.15)$$

Note the simplification when we factorize $1 - f = (1 - x)(1 - h)$:

$$\tilde{v} = -\frac{x^{-\frac{1}{2}}}{1 - h} v \quad \quad \tilde{\bar{v}} = -\frac{x^{-\frac{1}{2}}}{1 - h} \bar{v} \quad (2.3.16)$$

As an example, consider the index for an half-BPS (aka $(4, 4)$) 2d surface defect in 4d SYM, defined by coupling the theory to a 2d hypermultiplet transforming in the fundamental representation of the gauge group:

$$v = p^{-1} z \frac{(1 - x)(1 - y)}{1 - q} s \quad \quad \bar{v} = \frac{(1 - x)(1 - y)}{1 - q} s^{-1} \quad (2.3.17)$$

so that $\text{Tr} X^n$ and presumably $\det X$ survive as protected operators on the defect and it makes sense to consider their fluctuations. We get a correction factor

$$\text{PE} \left[x p^{-1} \frac{(1 - x^{-1})(1 - p)}{1 - y} s^{-1} \lambda \right] \text{PE} \left[y \frac{(1 - x^{-1})(1 - p)}{1 - y} s \lambda^{-1} \right] \quad (2.3.18)$$

A shift of s gives

$$\text{PE} \left[q \frac{(1 - x^{-1})(1 - p)}{1 - y} s^{-1} \lambda \right] \text{PE} \left[z^{-1} \frac{(1 - x^{-1})(1 - p)}{1 - y} s \lambda^{-1} \right] \quad (2.3.19)$$

On the holographic dual side, the giant graviton brane and the $AdS_3 \times S^1$ brane intersect along a 2d submanifold $\mathbb{R} \times S^1$. Hence we find the usual 2d bifundamental hypermultiplets coupled to the two brane worldvolume theories, precisely as our index calculation shows.

2.3.3 Bifundamental Determinants

It is straightforward to generalize these formulae to modifications of the determinant of an adjoint field in quiver gauge theories.

Another simple generalization is to consider modifications of the determinant of a bifundamental field. These can be accommodated simply by taking ψ and $\bar{\psi}$ to be charged under distinct gauge groups. The rest of the discussion proceeds essentially as before.

2.3.4 Powers of a Determinant

Next, consider the modifications of $(\det X)^k$. For simplicity, we consider first a theory with adjoints only.

We can remove some X 's from any determinants and add extra words attached to any pair of indices in the ϵ tensors. We fermionize the determinants by adding k sets of auxiliary fermions, $\bar{\psi}^\alpha$ and ψ_α with $\alpha = 1, \dots, k$, allowing us to write operators of the form

$$\int d\psi d\bar{\psi} e^{\bar{\psi}^\alpha X \psi_\alpha} (\bar{\psi}^{\beta_1} W_1 \psi_{\gamma_1}) (\bar{\psi}^{\beta_2} W_2 \psi_{\gamma_2}) \dots \quad (2.3.20)$$

representing such general modifications.

Although we have a lot of dangling indices, because of the explicit $U(k)$ global symmetry in the exponent the integral will produce some linear combination of $U(k)$ -invariant tensors. Any linear combination of operators which is not $U(k)$ invariant will vanish. At the end of our calculation, we will thus need to impose $U(k)$ invariance by hand.

We add again antifields u_α and $\bar{u}^\alpha$ to impose the Ward identities for the fermions. There is a bit of redundancy, as $\bar{\psi}^\beta X \psi_\gamma$ is eliminated twice. As a result, we have k^2 spurious fermionic zero-modes, which we will remove by hand. Now $v = (x-1) \sum_a \lambda_a$ counts the fundamental letters and $\bar{v} = (1-x^{-1}) \sum_a \lambda_a^{-1}$ counts the anti-fundamental letters and the partition function is

$$\tilde{Z}_k(x; y_i) = \frac{1}{k!} \int \frac{d\lambda_a}{2\pi i \lambda_a} \prod_{a \neq b} (1 - \lambda_a / \lambda_b) PE \left[\tilde{f} \left(\sum_a \lambda_a \right) \left(\sum_b \lambda_b^{-1} \right) \right] \quad (2.3.21)$$

The integral over the λ_a fugacities takes precisely the form of the index for an auxiliary $U(k)$ gauge theory built from $U(k)$ adjoint letters counted by $\tilde{f}$.

If we apply this to $U(N)$ $\mathcal{N} = 4$ SYM, we find that, as expected, $\tilde{Z}_k(x; y_i)$ counts the worldvolume fluctuations of k coincident giant gravitons, giving rise to an auxiliary $U(k)$ $\mathcal{N} = 4$ SYM theory on $\mathbb{R} \times S^3$.

If we include (anti)fundamentals, we get

$$\begin{aligned} Z_\infty^{(\det X)^k}(x; y_i) &= \frac{1}{k!} \int \frac{d\lambda_a}{2\pi i \lambda_a} \prod_{a \neq b} (1 - \lambda_a / \lambda_b) PE \left[\tilde{f} \left(\sum_a \lambda_a \right) \left(\sum_b \lambda_b^{-1} \right) \right] \\ &\quad \cdot PE \left[\frac{x-1}{1-f} \bar{v} \sum_a \lambda_a \right] PE \left[\frac{1-x^{-1}}{1-f} v \sum_a \lambda_a^{-1} \right] \end{aligned} \quad (2.3.22)$$

i.e. we have an auxiliary $U(k)$ theory with (anti)fundamental letter indices

$$\tilde{v} = \frac{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}{1-f}v \quad \bar{\tilde{v}} = \frac{x^{\frac{1}{2}} - x^{-\frac{1}{2}}}{1-f}\bar{v} \quad (2.3.23)$$

2.3.5 Counting Modifications of Products of Distinct Determinants

Next, we pick two bosonic letters X_1 and X_2 with fugacities x_1 and x_2 . This situation will not be needed in the rest of the chapter, but we include it for completeness. We can study modifications of $\det X_1 \det X_2$. We start from the case with no (anti)fundamental letters.

Using the fermionic trick we have auxiliary $v = (x_1 - 1)\lambda_1 + (x_2 - 1)\lambda_2$ and $\bar{v} = (1 - x_1^{-1})\lambda_1^{-1} + (1 - x_2^{-1})\lambda_2^{-1}$. We get an index

$$\begin{aligned} & \text{PE} \left[1 + x_1^{-1} \frac{(1 - x_1)^2}{1 - f} \right] \text{PE} \left[1 + x_2^{-1} \frac{(1 - x_2)^2}{1 - f} \right] \\ & \cdot \oint \frac{d\lambda_1}{2\pi i \lambda_1} \frac{d\lambda_2}{2\pi i \lambda_2} \text{PE} \left[\frac{(x_1 - 1)(1 - x_2^{-1})}{1 - f} \lambda_2^{-1} \lambda_1 \right] \text{PE} \left[\frac{(x_2 - 1)(1 - x_1^{-1})}{1 - f} \lambda_1^{-1} \lambda_2 \right] \end{aligned} \quad (2.3.24)$$

The auxiliary theory is an $U(1) \times U(1)$ quiver gauge theory.

It is easy to generalize this to include (anti)fundamental fields. It is also straightforward to extend this to modifications of $(\det X_1)^{k_1} (\det X_2)^{k_2}$ described by the index of an auxiliary $U(k_1) \times U(k_2)$ gauge theory, as well as to products of several types of determinants.

Similarly, one can extend this to the case of quiver gauge theories, including both determinants of adjoint fields and determinants of bifundamental fields. All of these can be treated with the fermionization trick.

As a concrete example, consider $(\det X)(\det Z)$ in $\mathcal{N} = 4$ SYM:

$$\begin{aligned} & \text{PE} \left[1 - \frac{(1 - x^{-1})(1 - p)(1 - q)}{(1 - y)(1 - z)} \right] \text{PE} \left[1 - \frac{(1 - z^{-1})(1 - p)(1 - q)}{(1 - x)(1 - y)} \right] \\ & \cdot \oint \frac{d\lambda_1}{2\pi i \lambda_1} \frac{d\lambda_2}{2\pi i \lambda_2} \text{PE} \left[z^{-1} \frac{(1 - p)(1 - q)}{1 - y} \lambda_2^{-1} \lambda_1 \right] \text{PE} \left[x^{-1} \frac{(1 - p)(1 - q)}{1 - y} \lambda_1^{-1} \lambda_2 \right] \end{aligned} \quad (2.3.25)$$

In the holographic dual geometry, the two giant gravitons wrap distinct maximal S^3 's in S^5 , which intersect along an S^1 . At the intersection the open strings stretched between

the giant graviton D3 branes give 2d hypermultiplets in a bifundamental representation of the gauge groups.

In the index we see precisely the contributions of these hypermultiplets:

$$\text{PE} \left[z^{-1} \frac{(1-p)(1-q)}{1-y} \lambda \right] \text{PE} \left[x^{-1} \frac{(1-p)(1-q)}{1-y} \lambda^{-1} \right] \quad (2.3.26)$$

2.4 Systematic Corrections to Z_∞

As we discussed in Section 2.2, the coefficients of Z_N approach limiting values from Z_∞ as N becomes sufficiently large. More precisely, the coefficient for some charge γ stabilizes whenever the maximum number of letters contributing to it is smaller than N .

In this section we pick a specific bosonic field X with fugacity x and focus on operators built with an arbitrary number of X letters but a finite fixed number of other letters. For convenience, we will now denote the fugacities as (x, y_i) and the charges as a pair (n, γ) , writing the index as

$$Z_N(x; y) = \sum_{n, \gamma} c_{N; n, \gamma} x^n y^\gamma = \sum_{\gamma} Z_{N; \gamma}(x) y^\gamma \quad (2.4.1)$$

Concretely, we will study the large N behaviour of the $Z_{N; \gamma}(x)$ power series at fixed γ and the relation to the naive large N limit

$$Z_\infty(x; y) = \sum_{n, \gamma} c_{\infty; n, \gamma} x^n y^\gamma = \sum_{\gamma} Z_{\infty; \gamma}(x) y^\gamma \quad (2.4.2)$$

We will now describe some structure which appears to hold experimentally for the $\mathcal{N} = 4$ SYM index and its simplifications, but also for index-like quantities built from a variety of unphysical f 's.

The discrepancies between $Z_{N; \gamma}(x)$ and $Z_{\infty; \gamma}(x)$ will begin at some order $x^{N+n_\gamma^{(1)}}$ for some integer $n_\gamma^{(1)}$. We find that the difference in the coefficients of x^{N+n} for finite n is independent of N at large N . This statement only fails when $n - N$ hits a certain finite value.

2.4.1 A Systematic Sequence of Corrections

For future convenience, we will work with the ratio $\frac{Z_N(x; y)}{Z_\infty(x; y)}$ and write

$$\frac{Z_N(x; y)}{Z_\infty(x; y)} \sim_x 1 + \sum_{\gamma, n \geq n_\gamma^{(1)}} d_{N; n, \gamma}^{(1)} x^{N+n} y^\gamma \quad (2.4.3)$$

The $\sim_x$ denotes the fact that the structure on the right hand side is only expected to hold at finite γ and for sufficiently large N .

Our observation means that $d_{N; n, \gamma}^{(1)}$ is N -independent until we hit some $n = n_\gamma^{(2)}$ and thus we can define some auxiliary power series $\hat{Z}_1(x; y)$ such that

$$\frac{Z_N(x; y)}{Z_\infty(x; y)} \sim_x 1 + x^N \hat{Z}_1(x; y) + \sum_{\gamma, n \geq n_\gamma^{(2)}} d_{N; n, \gamma}^{(2)} x^{2N+n} y^\gamma \quad (2.4.4)$$

We then find that $d_{N; n, \gamma}^{(2)}$ is N -independent until we hit some $n = n_\gamma^{(3)}$ and thus we can define some auxiliary power series $\hat{Z}_2(x; y)$ such that

$$\frac{Z_N(x; y)}{Z_\infty(x; y)} \sim_x 1 + x^N \hat{Z}_1(x; y) + x^{2N} \hat{Z}_2(x; y) + \sum_{\gamma, n \geq n_\gamma^{(2)}} d_{N; n, \gamma}^{(2)} x^{2N+n} y^\gamma \quad (2.4.5)$$

Assuming that the pattern continues, given sufficient computational power one could identify a systematic sequence of auxiliary functions $\hat{Z}_k(x; y)$ and write

$$\frac{Z_N(x; y)}{Z_\infty(x; y)} \sim_x 1 + \sum_{k=1}^{\infty} x^{kN} \hat{Z}_k(x; y) + \dots \quad (2.4.6)$$

The functions $\hat{Z}_k(x; y)$ are determined from the large N behaviour of the indices for powers of x which scale linearly in N . At this point, we do not try to account for powers of x which scale faster than N . We indicate this omission by the final ellipsis.

After we have computed the $\hat{Z}_k(x; y)$, we could try to compute both sides of this relation at finite N . This only makes sense for N such that the overall powers $n_\gamma^{(k)} + kN$ grow with k so that the sum is sensible for each γ . Experimentally, $n_\gamma^{(k)}$ grows quadratically in k , so this is the case for all N . We thus find the surprising result described in the introduction: an exact equality

$$\frac{Z_N(x; y)}{Z_\infty(x; y)} = 1 + \sum_{k=1}^{\infty} x^{kN} \hat{Z}_k(x; y) \quad (2.4.7)$$

valid for all N . This is our “giant graviton expansion” for the index. The right hand side is well-defined even for negative N , and we find that large cancellations between $\hat{Z}_k(x; y)$ force the right hand side to vanish at negative N .

We will now demonstrate this expansion for various specializations of the $\mathcal{N} = 4$ index as well as for some index-like examples. We refer to Appendix A.6 for extensive checks of the expansion formula for the full $\mathcal{N} = 4$ index.

2.4.2 The Single-Matrix Index

As a basic example, consider the case with a single letter X , which coincides with the half-BPS sector. The index is just

$$Z_N(x) = \frac{1}{\prod_{n=1}^N (1 - x^n)} \quad (2.4.8)$$

and correspondingly

$$Z_\infty(x) = \frac{1}{\prod_{n=1}^\infty (1 - x^n)}. \quad (2.4.9)$$

We can write

$$Z_N(x) = Z_\infty(x) \prod_{n=1}^N (1 - x^{N+n}) \quad (2.4.10)$$

Expanding the product at the leading order in x^N , we have

$$Z_N(x) = Z_\infty(x) - \frac{x^{N+1}}{1 - x} Z_\infty(x) + O(x^{2N}) \quad (2.4.11)$$

We see that the differences begin at order x^{N+1} , where the first trace relations occur, and have a structure which is actually independent of N up to order x^{2N} .

In this simple example we can readily do a systematic expansion:

$$\frac{Z_N(x)}{Z_\infty(x)} = \prod_{n=1}^\infty (1 - x^{N+n}) = \sum_{k=0}^\infty (-1)^k x^{kN} \frac{x^{\frac{k(k+1)}{2}}}{\prod_{n=1}^k (1 - x^n)} \quad (2.4.12)$$

Now we make a crucial observation: as rational functions of x , we have

$$(-1)^k \frac{x^{\frac{k(k+1)}{2}}}{\prod_{n=1}^k (1 - x^n)} = \frac{1}{\prod_{n=1}^k (1 - x^{-n})} = Z_k(x^{-1}) = \tilde{Z}_k(\tilde{x}) \quad (2.4.13)$$

In other words, although the “giant graviton” index $\tilde{Z}_k(\tilde{x})$ solves the problem of counting determinant modifications when expanded in powers of $\tilde{x} = x^{-1}$, it appears to also provide the systematic corrections $\hat{Z}_k(x)$ to the large N expansion when analytically continued as a meromorphic function to a power series in x .

This pattern will persist in other examples.

2.4.3 The $\frac{1}{4}$ -BPS Index

Next, consider

$$f = x + y - xy \quad (2.4.14)$$

so that $1 - f = (1 - x)(1 - y)$ and

$$Z_\infty(x; y) = \frac{1}{\prod_{n>0} (1 - x^n)(1 - y^n)} \quad (2.4.15)$$

We can look first at the coefficient of y in $Z_1(x; y), \dots, Z_6(x; y)$:

$$\begin{aligned} & 1 + O(x^{18}) \\ & 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + O(x^{12}) \\ & 1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + 5x^8 + 5x^9 + 6x^{10} + 6x^{11} + O(x^{12}) \\ & 1 + x + 2x^2 + 3x^3 + 4x^4 + 5x^5 + 7x^6 + 8x^7 + 10x^8 + 12x^9 + 14x^{10} + 16x^{11} + O(x^{12}) \\ & 1 + x + 2x^2 + 3x^3 + 5x^4 + 6x^5 + 9x^6 + 11x^7 + 15x^8 + 18x^9 + 23x^{10} + 27x^{11} + O(x^{12}) \\ & 1 + x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 10x^6 + 13x^7 + 18x^8 + 23x^9 + 30x^{10} + 37x^{11} + O(x^{12}) \end{aligned} \quad (2.4.16)$$

We see the stabilization clearly, with first corrections at order x^N .

Subtracting off Z_∞ we get the first sequence of corrections

$$\begin{aligned} & -x - 2x^2 - 3x^3 - 5x^4 - 7x^5 - 11x^6 - 15x^7 - 22x^8 - 30x^9 - 42x^{10} - 56x^{11} + O(x^{12}) \\ & -x^2 - 2x^3 - 4x^4 - 6x^5 - 10x^6 - 14x^7 - 21x^8 - 29x^9 - 41x^{10} - 55x^{11} - 76x^{12} + O(x^{13}) \\ & -x^3 - 2x^4 - 4x^5 - 7x^6 - 11x^7 - 17x^8 - 25x^9 - 36x^{10} - 50x^{11} - 70x^{12} - 94x^{13} + O(x^{14}) \\ & -x^4 - 2x^5 - 4x^6 - 7x^7 - 12x^8 - 18x^9 - 28x^{10} - 40x^{11} - 58x^{12} - 80x^{13} - 111x^{14} + O(x^{15}) \\ & -x^5 - 2x^6 - 4x^7 - 7x^8 - 12x^9 - 19x^{10} - 29x^{11} - 43x^{12} - 62x^{13} - 88x^{14} - 122x^{15} + O(x^{16}) \\ & -x^6 - 2x^7 - 4x^8 - 7x^9 - 12x^{10} - 19x^{11} - 30x^{12} - 44x^{13} - 65x^{14} - 92x^{15} - 130x^{16} + O(x^{17}) \end{aligned} \quad (2.4.17)$$

We see clearly the stabilization to a power series

$$x^N (-1 - 2x - 4x^2 - 7x^3 - 12x^4 - 19x^5 - 30x^6 + O(x^7)) \quad (2.4.18)$$

up to corrections which start at order x^{2N+1} :

$$\begin{aligned} & x^3 + 2x^4 + 5x^5 + 8x^6 + 15x^7 + 23x^8 + 37x^9 + 55x^{10} + 83x^{11} + 118x^{12} + 171x^{13} + 238x^{14} + O(x^{15}) \\ & x^5 + 2x^6 + 5x^7 + 9x^8 + 16x^9 + 26x^{10} + 42x^{11} + 63x^{12} + 95x^{13} + 138x^{14} + 198x^{15} + O(x^{16}) \\ & x^7 + 2x^8 + 5x^9 + 9x^{10} + 17x^{11} + 27x^{12} + 45x^{13} + 68x^{14} + 104x^{15} + 151x^{16} + O(x^{17}) \\ & x^9 + 2x^{10} + 5x^{11} + 9x^{12} + 17x^{13} + 28x^{14} + 46x^{15} + 71x^{16} + 109x^{17} + O(x^{18}) \\ & x^{11} + 2x^{12} + 5x^{13} + 9x^{14} + 17x^{15} + 28x^{16} + 47x^{17} + O(x^{18}) \\ & x^{13} + 2x^{14} + 5x^{15} + 9x^{16} + 17x^{17} + O(x^{18}) \end{aligned} \quad (2.4.19)$$

We still see a stabilization to a power series

$$x^{2N} (x + 2x^2 + 5x^3 + 9x^4 + 17x^5 + 28x^6 + 47x^7 + O(x^8)) \quad (2.4.20)$$

up to corrections which start at order x^{3N+3} .

We can repeat the exercise for the coefficients of higher powers y^s of y . At each stage, the corrections occur earlier as s increases, at order

$$kN + \frac{k(k+1)}{2} - ks \quad (2.4.21)$$

so that we need to go to higher N to see the coefficients stabilize. But if we blindly subtract off these corrections from the low N indices, even at the cost to introduce negative powers of x , the initial order at which the order k corrections kick in becomes increasingly positive as k increase, so that the expansion

$$\frac{Z_N(x; y)}{Z_\infty(x; y)} = 1 + \sum_{k=1}^{\infty} x^{kN} \hat{Z}_k(x; y) \quad (2.4.22)$$

holds exactly for any value of N .

Next, we can compare the correction coefficients $\hat{Z}_k(x; y)$ with the dual indices $\tilde{Z}_k(\tilde{x}, y)$ computed from

$$1 - \tilde{f} = \frac{1 - \tilde{x}}{1 - y} \quad (2.4.23)$$

i.e.

$$\tilde{f} = \tilde{x} - y + \tilde{x}y + \cdots \quad (2.4.24)$$

The analytic continuation $\tilde{x} \rightarrow x^{-1}$ is made possible by the following observation: the coefficient $\tilde{Z}_{k;s}(\tilde{x})$ of y^s in $\tilde{Z}_k(\tilde{x}, y)$ is a rational function with denominator which divides $\prod_{i=1}^k (1 - \tilde{x}^i)$. Thus

$$\tilde{P}_{k;s}(\tilde{x}) \equiv \tilde{Z}_{k;s}(\tilde{x}) \prod_{i=1}^k (1 - \tilde{x}^i) \quad (2.4.25)$$

is a Laurent polynomial in $\tilde{x}$ and we can define the analytic continuation of $\tilde{Z}_{k;s}(\tilde{x})$ as the power series expansion in x of

$$(-1)^k \frac{x^{k(k+1)/2}}{\prod_{i=1}^k (1 - x^i)} \tilde{P}_{k;s}(x^{-1}) \quad (2.4.26)$$

We find that this agrees with the coefficient $\hat{Z}_{k;s}(x)$ of y^s in $\hat{Z}_k(x; y)$. Indeed, having an explicitly computable candidate expression for $\hat{Z}_{k;s}(x)$ allows us to check (2.4.22) to rather high powers of x and y even for small N . For negative N , the right hand side magically vanishes order by order in x and y .

2.4.4 A Fictitious Two-Letter Index

As the next example, we consider a problem which does not occur as a limit of 4d SYM: the index counting gauge-invariant operators built from two bosonic matrices X and Y , with no extra relations: $f = x + y$.

Now the naive large N index is

$$Z_\infty(x; y) = \frac{1}{\prod_{n>0} (1 - x^n - y^n)} \quad (2.4.27)$$

We can look first at the coefficient of y in $Z_1(x; y), \dots, Z_6(x; y)$:

$$\begin{aligned}
& 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + O(x^{12}) \\
& 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + 9x^8 + 10x^9 + 11x^{10} + 12x^{11} + O(x^{12}) \\
& 1 + 2x + 4x^2 + 6x^3 + 9x^4 + 12x^5 + 16x^6 + 20x^7 + 25x^8 + 30x^9 + 36x^{10} + 42x^{11} + O(x^{12}) \\
& 1 + 2x + 4x^2 + 7x^3 + 11x^4 + 16x^5 + 23x^6 + 31x^7 + 41x^8 + 53x^9 + 67x^{10} + 83x^{11} + O(x^{12}) \\
& 1 + 2x + 4x^2 + 7x^3 + 12x^4 + 18x^5 + 27x^6 + 38x^7 + 53x^8 + 71x^9 + 94x^{10} + 121x^{11} + O(x^{12}) \\
& 1 + 2x + 4x^2 + 7x^3 + 12x^4 + 19x^5 + 29x^6 + 42x^7 + 60x^8 + 83x^9 + 113x^{10} + 150x^{11} + O(x^{12})
\end{aligned} \tag{2.4.28}$$

We see the stabilization to the infinite N value:

$$1 + 2x + 4x^2 + 7x^3 + 12x^4 + 19x^5 + 30x^6 + O(x^7) \tag{2.4.29}$$

up to corrections appearing at order x^N :

$$\begin{aligned}
& -x - 3x^2 - 6x^3 - 11x^4 - 18x^5 - 29x^6 - 44x^7 - 66x^8 - 96x^9 - 138x^{10} - 194x^{11} + O(x^{12}) \\
& -x^2 - 3x^3 - 7x^4 - 13x^5 - 23x^6 - 37x^7 - 58x^8 - 87x^9 - 128x^{10} - 183x^{11} - 259x^{12} + O(x^{13}) \\
& -x^3 - 3x^4 - 7x^5 - 14x^6 - 25x^7 - 42x^8 - 67x^9 - 103x^{10} - 153x^{11} - 223x^{12} - 317x^{13} + O(x^{14}) \\
& -x^4 - 3x^5 - 7x^6 - 14x^7 - 26x^8 - 44x^9 - 72x^{10} - 112x^{11} - 170x^{12} - 250x^{13} - 361x^{14} + O(x^{15}) \\
& -x^5 - 3x^6 - 7x^7 - 14x^8 - 26x^9 - 45x^{10} - 74x^{11} - 117x^{12} - 179x^{13} - 267x^{14} - 389x^{15} + O(x^{16}) \\
& -x^6 - 3x^7 - 7x^8 - 14x^9 - 26x^{10} - 45x^{11} - 75x^{12} - 119x^{13} - 184x^{14} - 276x^{15} - 406x^{16} + O(x^{17})
\end{aligned} \tag{2.4.30}$$

This is stabilizing to

$$x^N (-1 - 3x - 7x^2 - 14x^3 - 26x^4 - 45x^5 - 75x^6 + O(x^7)) \tag{2.4.31}$$

up to corrections which appear at order x^{2N+1} :

$$\begin{aligned}
& x^3 + 3x^4 + 8x^5 + 16x^6 + 31x^7 + 54x^8 + 91x^9 + 146x^{10} + 229x^{11} + 347x^{12} + 518x^{13} + O(x^{14}) \\
& x^5 + 3x^6 + 8x^7 + 17x^8 + 33x^9 + 59x^{10} + 101x^{11} + 164x^{12} + 259x^{13} + 397x^{14} + 595x^{15} + O(x^{16}) \\
& x^7 + 3x^8 + 8x^9 + 17x^{10} + 34x^{11} + 61x^{12} + 106x^{13} + 174x^{14} + 278x^{15} + 429x^{16} + 649x^{17} + O(x^{18}) \\
& x^9 + 3x^{10} + 8x^{11} + 17x^{12} + 34x^{13} + 62x^{14} + 108x^{15} + 179x^{16} + 288x^{17} + O(x^{18}) \\
& x^{11} + 3x^{12} + 8x^{13} + 17x^{14} + 34x^{15} + 62x^{16} + 109x^{17} + O(x^{18}) \\
& x^{13} + 3x^{14} + 8x^{15} + 17x^{16} + 34x^{17} + O(x^{18})
\end{aligned} \tag{2.4.32}$$

This is stabilizing up to corrections at order x^{3N+3} , etc.

We can analyze in a similar manner higher powers of y and find the usual general pattern, with systematic corrections controlled by $\hat{Z}_{k;s}(x)$ starting at order

$$kN + \frac{k(k+1)}{2} - ks \quad (2.4.33)$$

The analytic continuation procedure also seems to work, with the $\hat{Z}_{k;s}(x)$ taking the form of nice rational functions. The main difference with the previous example is that the denominator is slightly more complicated: the $(1-x)$ factor is replaced by $(1-x)^s$ for $s > 0$. The resulting $\tilde{Z}_k(\tilde{x}, y)$ take the form of fictitious $U(k)$ indices with

$$\tilde{f} = \tilde{x} \frac{1 - \tilde{x} + y}{1 - \tilde{x} + \tilde{x}y} \quad (2.4.34)$$

to be expanded out as a power series in $\tilde{x}$ and y in the plethystic exponential.

This function does satisfy the expected relation

$$(1-f)(1-\tilde{f}) = (1-x)(1-\tilde{x}) \quad (2.4.35)$$

as a rational function, but we should stress that $\tilde{f}$ is *not* expanded in a manner compatible with describing determinant modifications, which would involve a series expansion around $x = 0$.

Hence the $\tilde{Z}_k(\tilde{x}, y)$ here have a more mysterious meaning and cannot be directly related to determinant modifications. Indices related by analytic continuation of the argument of a plethystic exponent do occur in some circumstances, such as wall-crossing of holomorphic blocks in [56]. Perhaps something similar is happening here.

2.4.5 Schur Index

The Schur index is a specialization of the full $\mathcal{N} = 4$ index, setting $p = z$. It is also the torus partition function of the $U(N)$ -gauged $\beta\gamma$ system, the 2d chiral algebra associated to $\mathcal{N} = 4$ super Yang-Mills. We consider Schur index in the Ramond sector for simplicity, to avoid powers of $q^{1/2}$ in our formulae. The index in Neveu-Schwarz sector can be obtained from the Ramond index by a shift of the $SU(2)_R$ fugacity $x \rightarrow \sqrt{q}x$.

The single letter index in this example is

$$f = \frac{1}{1-q} \left(x + \frac{q}{x} - 2q \right) = 1 - \frac{(1-x)(1-q/x)}{(1-q)}, \quad (2.4.36)$$

with the naive large N index

$$Z_\infty(x; q) = \prod_{n>0} \frac{(1 - q^n)}{(1 - x^n)(1 - q^n/x^n)}. \quad (2.4.37)$$

The Schur index reduces to the $\frac{1}{2}$ -BPS single matrix index for $q \rightarrow 0$ and has a $\mathbb{Z}_2$ Weyl symmetry exchanging $x \leftrightarrow q/x$. We can efficiently compute Schur indices with the help of a generating function derived in Appendix A.3.

Repeating the analysis in previous examples, we can look at the coefficients of q in $Z_1(x; q), \dots, Z_6(x; q)$:

$$\begin{aligned} & \frac{1}{x} - 1 + O(x^{14}) \\ & \frac{1}{x} + O(x^{14}) \\ & \frac{1}{x} + x + x^3 + x^5 + x^7 + x^9 + x^{11} + x^{13} + O(x^{14}) \\ & \frac{1}{x} + x + x^2 + x^3 + x^4 + 2x^5 + x^6 + 2x^7 + 2x^8 + 2x^9 + 2x^{10} + 3x^{11} + 2x^{12} + 3x^{13} + O(x^{14}) \\ & \frac{1}{x} + x + x^2 + 2x^3 + x^4 + 3x^5 + 2x^6 + 4x^7 + 3x^8 + 5x^9 + 4x^{10} + 7x^{11} + 5x^{12} + 8x^{13} + O(x^{14}) \\ & \frac{1}{x} + x + x^2 + 2x^3 + 2x^4 + 3x^5 + 3x^6 + 5x^7 + 5x^8 + 7x^9 + 7x^{10} + 10x^{11} + 10x^{12} + 13x^{13} + O(x^{14}) \end{aligned} \quad (2.4.38)$$

The coefficients stabilize to the infinite N value

$$\frac{1}{x} + x + x^2 + 2x^3 + 2x^4 + 4x^5 + 4x^6 + 7x^7 + 8x^8 + 12x^9 + 14x^{10} + 21x^{11} + 24x^{12} + 34x^{13} + O(x^{14}) \quad (2.4.39)$$

up to corrections appearing at order x^{N-1} :

$$\begin{aligned} & -1 - x - x^2 - 2x^3 - 2x^4 - 4x^5 - 4x^6 - 7x^7 - 8x^8 - 12x^9 - 14x^{10} - 21x^{11} - 24x^{12} - 34x^{13} + O(x^{14}) \\ & -x - x^2 - 2x^3 - 2x^4 - 4x^5 - 4x^6 - 7x^7 - 8x^8 - 12x^9 - 14x^{10} - 21x^{11} - 24x^{12} - 34x^{13} + O(x^{14}) \\ & -x^2 - x^3 - 2x^4 - 3x^5 - 4x^6 - 6x^7 - 8x^8 - 11x^9 - 14x^{10} - 20x^{11} - 24x^{12} - 33x^{13} + O(x^{14}) \\ & -x^3 - x^4 - 2x^5 - 3x^6 - 5x^7 - 6x^8 - 10x^9 - 12x^{10} - 18x^{11} - 22x^{12} - 31x^{13} + O(x^{14}) \\ & -x^4 - x^5 - 2x^6 - 3x^7 - 5x^8 - 7x^9 - 10x^{10} - 14x^{11} - 19x^{12} - 26x^{13} + O(x^{14}) \\ & -x^5 - x^6 - 2x^7 - 3x^8 - 5x^9 - 7x^{10} - 11x^{11} - 14x^{12} - 21x^{13} + O(x^{14}). \end{aligned} \quad (2.4.40)$$

We see that the corrections are stabilizing to

$$x^N \left(-\frac{1}{x} - 1 - 2x - 3x^2 - 5x^3 - 7x^4 - 11x^5 - 15x^6 - 22x^7 + O(x^8) \right), \quad (2.4.41)$$

up to further corrections appearing at order x^{2N} :

$$\begin{aligned} & x^2 + x^3 + 3x^4 + 3x^5 + 7x^6 + 8x^7 + 14x^8 + 18x^9 + 28x^{10} + 35x^{11} + 53x^{12} + 67x^{13} + O(x^{14}) \\ & x^4 + x^5 + 3x^6 + 4x^7 + 7x^8 + 10x^9 + 16x^{10} + 21x^{11} + 32x^{12} + 43x^{13} + O(x^{14}) \\ & x^6 + x^7 + 3x^8 + 4x^9 + 8x^{10} + 10x^{11} + 18x^{12} + 23x^{13} + O(x^{14}) \\ & x^8 + x^9 + 3x^{10} + 4x^{11} + 8x^{12} + 11x^{13} + O(x^{14}) \\ & x^{10} + x^{11} + 3x^{12} + 4x^{13} + O(x^{14}) \\ & x^{12} + x^{13} + O(x^{14}). \end{aligned} \quad (2.4.42)$$

This in turn stabilizes up to corrections at order x^{3N+2} , and so on.

Continuing the procedure for higher powers of q^s , we observe that the corrections controlled by $\hat{Z}_{k;s}(x)$ occur at order

$$N(k+1) + \frac{k(k+3)}{2} - (k+2)s + 1 \quad (2.4.43)$$

of x . In other words, at $O(q^s)$, the truncated formula

$$\frac{Z_N(x; q)}{Z_\infty(x; q)} = 1 + x^N \hat{Z}_1(x; q) + x^{2N} \hat{Z}_2(x; q) + \cdots + x^{kN} \hat{Z}_k(x; q) \quad (2.4.44)$$

is valid as power series in x up to order (2.4.43) minus 1. At large s and small N , one may need many correction terms on the right hand side of (2.4.44) to see the convergence of coefficients. Our claim, however, is that the right hand side of (2.4.44) converges to the left to arbitrarily high orders of x given sufficient number k of corrections.

Upon analytic continuation $\tilde{x} \rightarrow x^{-1}$, the auxiliary indices $\hat{Z}_k(x; q)$ agree with the dual indices $\tilde{Z}_k(\tilde{x}; q)$ defined from

$$1 - \tilde{f} = \frac{(1-q)(1-\tilde{x})}{(1-q\tilde{x})}. \quad (2.4.45)$$

The above definition of $\tilde{f}$ coincides with taking $x \rightarrow \tilde{x}$, $q \rightarrow q\tilde{x}$ in f . This observation allows us to perform extensive checks of the formula

$$\frac{Z_N(x; q)}{Z_\infty(x; q)} = 1 + \sum_{k=1}^{\infty} x^{kN} \hat{Z}_k(x; q) \quad (2.4.46)$$

at finite N for the Schur index in Appendix A.3. In particular, we find that (2.4.46) holds even for negative integer N and for orders of x well beyond $O(x^{N^2})$, at any order of q . These properties suggest that the above expansion is indeed exact rather than asymptotic in N .

Before moving on to $\mathcal{N} = 4$ SYM, we note another striking feature of (2.4.46). The Schur index has a $\mathbb{Z}_2$ Weyl symmetry exchanging $x \leftrightarrow y = q/x$. It would be natural to seek an alternative formula which treats x and y symmetrically and involves a sum of correction terms controlled by some $x^{kN}y^{k'N}$. This was indeed done in [35]. We strongly suspect that both formulae may be simultaneously true, as discussed at the end of the introduction.

2.4.6 A Final Specialization of the $\mathcal{N} = 4$ SYM Index

For a final demonstration, we specialize the $\mathcal{N} = 4$ index as

$$p = q = r, \quad y = z = \frac{r}{\sqrt{x}}. \quad (2.4.47)$$

This examples includes operators which carry both non-zero spin and R-charges. We refer the reader to Appendix A.5 for more detailed checks for this specialization.

Let us look at the coefficients of r in $Z_1(x; r), \dots, Z_5(x; r)$:

$$\begin{aligned} & \frac{2}{\sqrt{x}} - 2 + O(x^{11/2}) \\ & \frac{2}{\sqrt{x}} - 2 + 2\sqrt{x} - 2x + 2x^{3/2} - 2x^2 + 2x^{5/2} - 2x^3 + 2x^{7/2} - 2x^4 + 2x^{9/2} - 2x^5 + O(x^{11/2}) \\ & \frac{2}{\sqrt{x}} - 2 + 2\sqrt{x} - 2x + 4x^{3/2} - 4x^2 + 4x^{5/2} - 4x^3 + 6x^{7/2} - 6x^4 + 6x^{9/2} - 6x^5 + O(x^{11/2}) \\ & \frac{2}{\sqrt{x}} - 2 + 2\sqrt{x} - 2x + 4x^{3/2} - 4x^2 + 6x^{5/2} - 6x^3 + 8x^{7/2} - 8x^4 + 10x^{9/2} - 10x^5 + O(x^{11/2}) \\ & \frac{2}{\sqrt{x}} - 2 + 2\sqrt{x} - 2x + 4x^{3/2} - 4x^2 + 6x^{5/2} - 6x^3 + 10x^{7/2} - 10x^4 + 12x^{9/2} - 12x^5 + O(x^{11/2}) \end{aligned} \quad (2.4.48)$$

The coefficients stabilize to the infinite N value

$$\frac{2}{\sqrt{x}} - 2 + 2\sqrt{x} - 2x + 4x^{3/2} - 4x^2 + 6x^{5/2} - 6x^3 + 10x^{7/2} - 10x^4 + 14x^{9/2} - 14x^5 + O(x^{11/2}) \quad (2.4.49)$$

up to corrections that appear at order $x^{N-\frac{1}{2}}$:

$$\begin{aligned}
& -2\sqrt{x} + 2x - 4x^{3/2} + 4x^2 - 6x^{5/2} + 6x^3 - 10x^{7/2} + 10x^4 - 14x^{9/2} + 14x^5 + O(x^{11/2}) \\
& -2x^{3/2} + 2x^2 - 4x^{5/2} + 4x^3 - 8x^{7/2} + 8x^4 - 12x^{9/2} + 12x^5 - 20x^{11/2} + 20x^6 + O(x^{13/2}) \\
& -2x^{5/2} + 2x^3 - 4x^{7/2} + 4x^4 - 8x^{9/2} + 8x^5 - 14x^{11/2} + 14x^6 - 22x^{13/2} + 22x^7 + O(x^{15/2}) \\
& -2x^{7/2} + 2x^4 - 4x^{9/2} + 4x^5 - 8x^{11/2} + 8x^6 - 14x^{13/2} + 14x^7 - 24x^{15/2} + 24x^8 + O(x^{17/2}) \\
& -2x^{9/2} + 2x^5 - 4x^{11/2} + 4x^6 - 8x^{13/2} + 8x^7 - 14x^{15/2} + 14x^8 - 24x^{17/2} + 24x^9 + O(x^{19/2}).
\end{aligned} \tag{2.4.50}$$

We see that the corrections are stabilizing to

$$x^N \left(-\frac{2}{\sqrt{x}} + 2 - 4\sqrt{x} + 4x - 8x^{3/2} + 8x^2 - 14x^{5/2} + 14x^3 - 24x^{7/2} + 24x^4 + O(x^{9/2}) \right) \tag{2.4.51}$$

up to further corrections appearing at order $x^{2N+\frac{1}{2}}$

$$\begin{aligned}
& 2x^{5/2} - 2x^3 + 4x^{7/2} - 4x^4 + 10x^{9/2} - 10x^5 + 16x^{11/2} - 16x^6 + 30x^{13/2} - 30x^7 + O(x^{15/2}) \\
& 2x^{9/2} - 2x^5 + 4x^{11/2} - 4x^6 + 10x^{13/2} - 10x^7 + 18x^{15/2} - 18x^8 + 32x^{17/2} - 32x^9 + O(x^{19/2}) \\
& 2x^{13/2} - 2x^7 + 4x^{15/2} - 4x^8 + 10x^{17/2} - 10x^9 + 18x^{19/2} - 18x^{10} + 34x^{21/2} - 34x^{11} + O(x^{23/2}) \\
& 2x^{17/2} - 2x^9 + 4x^{19/2} - 4x^{10} + 10x^{21/2} - 10x^{11} + 18x^{23/2} - 18x^{12} + 34x^{25/2} - 34x^{13} + O(x^{27/2}) \\
& 2x^{21/2} - 2x^{11} + 4x^{23/2} - 4x^{12} + 10x^{25/2} - 10x^{13} + 18x^{27/2} - 18x^{14} + 34x^{29/2} - 34x^{15} + O(x^{31/2}).
\end{aligned} \tag{2.4.52}$$

This in turn stabilizes up to corrections at order $x^{3N+\frac{5}{2}}$, and so on.

For general powers r^s , the corrections controlled by $\hat{Z}_{k;s}(x)$ occur at order

$$N(k+1) + \frac{1}{2}k(k+3) - \frac{1}{2}(2k+3)s + 1 \tag{2.4.53}$$

of x . That is, at $O(r^s)$, the truncated formula

$$\frac{Z_N(x; r)}{Z_\infty(x; r)} = 1 + x^N \hat{Z}_1(x; r) + x^{2N} \hat{Z}_2(x; r) + \cdots + x^{kN} \hat{Z}_k(x; r) \tag{2.4.54}$$

is valid as power series in x up to order (2.4.53) minus 1. We find that the right hand side of (2.4.54) converges to the left to arbitrarily high orders of x given sufficient number k of corrections, at any order of r .

2.4.7 Testing the Limits of the Giant Graviton Expansion

We conclude with a couple of examples of fictitious indices for which the giant graviton expansion does *not* work. A natural question to ask at this point is whether our prescription for $\tilde{Z}_k$ and the associated giant graviton expansion are compatible with any index-like quantity. In other words, do $\tilde{Z}_k(\tilde{x})$ defined through

$$\frac{1 - \tilde{f}}{1 - \tilde{x}} = \frac{1 - x}{1 - f} \quad (2.4.55)$$

always yield the right corrections in the expansion formula? We expect the answer to be negative: the whole structure should apply only to situations where we have a single bosonic letters of fugacity x and we can suppress other letters by positive powers of other fugacities.

To demonstrate how our prescription breaks down, we study two more indices $f = -x$ and $f = 2x$. The former is the index of a theory with a single fermionic matrix, while the latter is a specialization of the two-letter index studied earlier.

The integral over gauge fugacities for the fermion matrix index $f = -x$ gives the exact expression

$$Z_N(x) = \prod_{n=1}^N (1 - x^{2n-1}). \quad (2.4.56)$$

It is simple to repeat the exercise of finding corrections to $Z_\infty(x)$ based on stabilizing coefficients. Experimentally, we find that the corrections appear only at even powers of x^N :

$$\frac{Z_N(x)}{Z_\infty(x)} = 1 + x^{2N} \frac{x}{(1 - x^2)} + x^{4N} \frac{x^2}{(1 - x^2)(1 - x^4)} + x^{6N} \frac{x^3}{(1 - x^2)(1 - x^4)(1 - x^6)} + \cdots, \quad (2.4.57)$$

This is readily understood. A fermionic adjoint operator ξ has vanishing determinant, but $\det \xi^2$ is a non-trivial gauge-invariant operator. We can tentatively attribute the sequence of corrections to modifications of $(\det \xi^2)^2$. It would be nice to develop this idea further.

It is also instructive to study the case $f = 2x$. The indices $Z_1(x), \dots, Z_5(x)$ are

$$\begin{aligned}
& 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + 9x^8 + 10x^9 + 11x^{10} + O(x^{11}) \\
& 1 + 2x + 6x^2 + 10x^3 + 20x^4 + 30x^5 + 50x^6 + 70x^7 + 105x^8 + 140x^9 + 196x^{10} + O(x^{11}) \\
& 1 + 2x + 6x^2 + 14x^3 + 29x^4 + 56x^5 + 107x^6 + 186x^7 + 320x^8 + 530x^9 + 851x^{10} + O(x^{11}) \\
& 1 + 2x + 6x^2 + 14x^3 + 34x^4 + 68x^5 + 144x^6 + 276x^7 + 534x^8 + 974x^9 + 1774x^{10} + O(x^{11}) \\
& 1 + 2x + 6x^2 + 14x^3 + 34x^4 + 74x^5 + 159x^6 + 324x^7 + 657x^8 + 1286x^9 + 2488x^{10} + O(x^{11}).
\end{aligned} \tag{2.4.58}$$

They stabilize to the infinite N value

$$1 + 2x + 6x^2 + 14x^3 + 34x^4 + 74x^5 + 166x^6 + 350x^7 + 746x^8 + 1546x^9 + 3206x^{10} + O(x^{11}), \tag{2.4.59}$$

up to corrections that appear at order x^{N+1} :

$$\begin{aligned}
& -3x^2 - 10x^3 - 29x^4 - 68x^5 - 159x^6 - 342x^7 - 737x^8 - 1536x^9 - 3195x^{10} + O(x^{11}) \\
& -4x^3 - 14x^4 - 44x^5 - 116x^6 - 280x^7 - 641x^8 - 1406x^9 - 3010x^{10} + O(x^{11}) \\
& -5x^4 - 18x^5 - 59x^6 - 164x^7 - 426x^8 - 1016x^9 - 2355x^{10} + O(x^{11}) \\
& -6x^5 - 22x^6 - 74x^7 - 212x^8 - 572x^9 - 1432x^{10} + O(x^{11}) \\
& -7x^6 - 26x^7 - 89x^8 - 260x^9 - 718x^{10} + O(x^{11}).
\end{aligned} \tag{2.4.60}$$

The coefficients here *do not* stabilize to definite values, but they do stabilize to N -dependent values:

$$\begin{aligned}
& -x^N \left((N+2)x + (4N+6)x^2 + (15N+14)x^3 + (48N+20)x^4 \right. \\
& \quad \left. + (146N-12)x^5 + (416N-232)x^6 + O(x^7) \dots \right).
\end{aligned} \tag{2.4.61}$$

It may be possible to develop a systematic expansion in powers of x^N and N . We will not attempt to do so.

2.5 The M2 Brane Index

In this section we study the superconformal index of 3d SCFTs with holographic duals. As a representative example we look at the SCFTs which describe the low energy world-volume theory of N coincident $M2$ branes.

The world-volume theory of $M2$ branes is a strongly coupled 3d SCFT. It can be given a variety of distinct UV descriptions:

- The 3d $U(N)$ N=8 SYM
- The 3d ADHM N=4 SYM with $N_f = 1$
- The ABJM theory with level $k = 1$

The three descriptions preserve different symmetries of the IR theory. Only the latter two are suitable for index calculations, and only the second one has a known brane interpretation: a stack of D2 branes in the presence of a D6 brane, which gives a dual description of M2 branes at the center of a Taub-NUT geometry. We will focus on that description.

The main challenge is that 3d gauge theories admit protected monopole operators. The index is thus written as an infinite sum of modified $U(N)$ fugacity integrals, one for each monopole charge [57]. The individual integrals can be treated as we did in the rest of the chapter, but each monopole sector has to be treated separately.

The UV gauge theory description breaks the IR $\text{Spin}(8)_R$ symmetry to $\text{Spin}(4) \times \text{Spin}(4)$, where each subgroup contains an $SU(2)$ R-symmetry factor and an $SU(2)$ global symmetry. The permutation of the two $\text{Spin}(4)$ is a self-mirror symmetry in 3d $\mathcal{N} = 4$ language. It exchanges determinant operators and Abelian monopole operators. This gives us two independent ways to study operators dual to giant gravitons.

In this setting, giant gravitons are M5 branes wrapping $S^5 \subset S^7$. Hence our analysis can shed light on the superconformal index of the 6d SCFT living on the worldvolume of M5 branes. Past efforts to compute that M5 index focussed on a KK circle reduction of the S^5 space geometry to a $\mathbb{CP}^2$ [58]. This involves a $\mathbb{CP}^2$ compactification of $U(k)$ 5d SYM. The main computational challenge is to properly account for the contribution of instanton configurations on $\mathbb{CP}^2$.

In our analysis below we will find conjectural expressions for the M5 SCFT index which appear to represent the index of an auxiliary $U(k)$ gauge theory with instanton sectors, but does not appear to be 5d SYM on $\mathbb{CP}^2$. Given the Taub-NUT geometry behind the ADHM quiver description, we suspect that the formulae we find involve a different, singular KK compactification of the space S^5 down to an S^4 , with a circle of Taub-NUT singularities. Equivalently, the M5 index should be identified with the index of 5d SYM in the presence of a codimension 3 defect involving coupling to 2d chiral fermions. We leave a detailed analysis to future work. It would also be interesting to determine which formulation of the M5 index may emerge from a large N analysis of the index in the ABJM formulation.

2.5.1 The ADHM Theory

The ADHM theory is a 3d $U(N)$ $\mathcal{N} = 4$ gauge theory with a single adjoint hypermultiplet as well as a single (anti)fundamental hypermultiplet. We refer to [59] and references therein for several examples of index calculations for 3d $\mathcal{N} = 4$ gauge theories and of mirror symmetry relations.

The hypermultiplet single letter index is usually written as

$$f_H = \frac{q^{\frac{1}{4}}ts + q^{\frac{1}{4}}ts^{-1} - q^{\frac{3}{4}}t^{-1}s - q^{\frac{3}{4}}t^{-1}s^{-1}}{1 - q} \quad (2.5.1)$$

with s being a flavour fugacity and the vectormultiplet as

$$f_V = \frac{q^{\frac{1}{2}}t^{-2} - q^{\frac{1}{2}}t^2}{1 - q} \quad (2.5.2)$$

The vectormultiplet together with the adjoint hypermultiplet combine into

$$f = \frac{q^{\frac{1}{4}}ts + q^{\frac{1}{4}}ts^{-1} + q^{\frac{1}{2}}t^{-2} - q^{\frac{1}{2}}t^2 - q^{\frac{3}{4}}t^{-1}s - q^{\frac{3}{4}}t^{-1}s^{-1}}{1 - q} \quad (2.5.3)$$

i.e. in a factorized form

$$1 - f = \frac{(1 - q^{\frac{1}{4}}ts)(1 - q^{\frac{1}{4}}ts^{-1})(1 - q^{\frac{1}{2}}t^{-2})}{1 - q} \quad (2.5.4)$$

The extra (anti)fundamental fields give

$$v = \frac{q^{\frac{1}{4}}t - q^{\frac{3}{4}}t^{-1}}{1 - q} \quad \bar{v} = \frac{q^{\frac{1}{4}}t - q^{\frac{3}{4}}t^{-1}}{1 - q} \quad (2.5.5)$$

Notice that the single-letter index for the adjoint fields is symmetric under permutation of the three complex scalars, with fugacities $q^{\frac{1}{4}}ts^{\pm 1}$ and $q^{\frac{1}{2}}t^{-2}$. The symmetry is broken by the (anti)fundamentals.

We will also introduce a fugacity r associated to the monopole charge. Monopoles are weighed by powers of $q^{\frac{1}{4}}t^{-1}r^{\pm 1}$. Self-mirror symmetry exchanges r and s and inverts t .

We can simplify the notation by defining

$$\begin{aligned}
x &= q^{\frac{1}{4}}ts \\
y &= q^{\frac{1}{4}}ts^{-1} \\
z &= q^{\frac{1}{4}}t^{-1}r \\
w &= q^{\frac{1}{4}}t^{-1}r^{-1}
\end{aligned} \tag{2.5.6}$$

and keep q as well, with the constraint $q = xyzw$.

Then we can write:

$$\begin{aligned}
1 - f &= \frac{(1-x)(1-y)(1-zw)}{1-q} \\
v &= \sqrt{xy} \frac{1-zw}{1-q} \\
\bar{v} &= \sqrt{xy} \frac{1-zw}{1-q}
\end{aligned} \tag{2.5.7}$$

The full index will be invariant under the exchange of the pairs of fugacities (x, y) and (z, w) , because of the self-mirror property of the ADHM theory. The low energy SUSY enhancement to $\mathcal{N} = 8$ actually implies that the index should be invariant under any permutation of the x, y, z and w fugacities. We will verify the symmetry enhancement below in some simplified examples.

Our basic conjecture is that the full index $Z_N(x, y, z, w)$ would admit a giant graviton expansion in terms of an improved collection of dual indices $\tilde{Z}_k(\tilde{x}, y, z, w)$, which coincide with the superconformal index for the 6d theory of k coincident M5 branes, with y, z and w being roughly fugacities for rotational symmetries and $\tilde{x}, q$ for R-symmetry generators.

If we could ignore the contribution from monopole sectors, the index would reduce to an $U(N)$ fugacity integral which could be analyzed as we did in 4d. The resulting dual index for an expansion based on the x fugacity would be built from

$$\begin{aligned}
1 - \tilde{f} &= \frac{(1-\tilde{x})(1-q)}{(1-y)(1-zw)} \\
\tilde{v} &= -\sqrt{\tilde{x}y} \frac{1}{1-y} \\
\tilde{\bar{v}} &= -\sqrt{\tilde{x}y} \frac{1}{1-y}
\end{aligned} \tag{2.5.8}$$

with $yzw = \tilde{x}q$. The adjoint contributions have a denominator with two factors, which is naturally associated to the presence of two planes of rotation. The (anti)fundamentals, though, only have a single denominator factor and look like the contributions of 2d chiral fermions.

Two-dimensional chiral fermions can appear when M5 branes are compactified on a Taub-NUT geometry. A reduction along the circle fiber of Taub-NUT reduces the 6d SCFT to 5d $U(k)$ SYM away from the origin but leaves a co-dimension three defect where 2d chiral fermions are coupled to the 5d gauge fields. BPS operators in the 6d SCFT will descend to BPS operators in the 5d theory, but operators which are charged under rotations of the fiber will descend to operators which carry instanton charge. Concretely, that means that operators which are perturbative in 5d couple only to the combination zw of the z and w fugacities associated to rotations within Taub-NUT.

We thus conjecture that the above $\tilde{f}$ and $\tilde{v}$, $\tilde{\bar{v}}$ arise from a formulation of the 6d index in terms of 5d SYM with a co-dimension 3 defect associated to a Taub-NUT compactification. The corresponding $U(k)$ index should be reproduced by a perturbative 5d calculation while the monopole corrections in 3d should give rise to instanton corrections in 5d. We leave a detailed match to future work.

Very Simplified Indices

There are several simplified versions of the index where one sends to zero certain fugacities. An extreme choice is to send y , z and w to zero. This eliminates monopoles and anti-fundamentals. Gauge invariance also eliminates fundamentals. We are left with the familiar count of a single matrix X with fugacity x .

It is interesting to observe the mirror limit sending x , y , w to zero. Then all matter contributions drop out and we are left with pure monopole operators. Monopole sectors are labelled by a collections of N integers m_i . We only need to consider non-negative m_i here, weighed by a power $z^{\sum_i m_i}$: negative m_i would contribute powers of $w = 0$.

Non-zero monopole charges modify the Vandermonde determinant. In this particular limit they simply eliminate $(1 - \mu_i/\mu_j)$ factors when $m_i \neq m_j$. If the magnetic charges consist of n_m copies of m , we effectively get an $\prod_m U(n_m)$ Vandermonde integration measure.

Including an overall $\binom{N}{n_1, \dots}$ combinatorial factor we find that the $\prod_m U(n_m)$ integral gives simply 1 for every choice of a partition $N = \sum_m n_m$. We are left with the overall

power of z . We can readily compute the generating function for all Z_N indices:

$$\sum_{N=0}^{\infty} \zeta^N Z_N(z) = \sum_{n_m \geq 0} \zeta^{\sum_m n_m} z^{\sum_m m n_m} = \frac{1}{\prod_{m \geq 0} (1 - \zeta z^m)} \quad (2.5.9)$$

which indeed is a generating function for the index

$$Z_N(z) = \frac{1}{\prod_{m=1}^N (1 - z^m)} \quad (2.5.10)$$

expected from mirror symmetry.

We are left with this important insight: the combinatorics of a single matrix X can be reproduced by the combinatorics of pure monopole operators of positive charge. It is actually possible to match individual mirror-symmetric operators. In particular, the operator $\det X$ is mirror to a purely Abelian monopole, with all $m_i = 1$.

The monopole version of the counting of determinant modifications is rather intuitive. In the large N limit, operators of finite charge must have a large value of n_0 . The sum over the remaining, unconstrained, n_m gives the usual Z_∞ . If we were to study fluctuations of a purely Abelian monopole, with all $m_i = k$, we would consider a situation where n_k is very large and all other n_m are finite. Summing over these unconstrained n_m gives formally the expected $Z_\infty \hat{Z}_k$.

Higgs Branch Index

The limit sending z and w to 0 fully suppresses monopole contributions. The reduced index counts ‘‘Higgs’’ branch operators. We have a variant of a familiar problem

$$\begin{aligned} 1 - f &= (1 - x)(1 - y) \\ v &= \sqrt{xy} \\ \bar{v} &= \sqrt{xy} \end{aligned} \quad (2.5.11)$$

and thus

$$Z_\infty(x; y) = \frac{1}{\prod_{n=1}^{\infty} (1 - x^n)(1 - y^n)} \text{PE} \left[\frac{xy}{(1 - x)(1 - y)} \right] \quad (2.5.12)$$

i.e.

$$Z_\infty(x; y) = \frac{1}{\prod_{n, m \geq 0 | (n, m) \neq (0, 0)} (1 - x^n y^m)} \quad (2.5.13)$$

The giant graviton expansion proceeds as discussed in previous sections, with

$$\begin{aligned} 1 - \tilde{f} &= \frac{1 - \tilde{x}}{1 - y} \\ v &= -\frac{\sqrt{y}}{1 - y} \\ \bar{v} &= -\frac{\sqrt{y}}{1 - y} \end{aligned} \tag{2.5.14}$$

and large k limit

$$\tilde{Z}_\infty(\tilde{x}; y) = \prod_{n=1}^{\infty} \frac{1 - y^n}{1 - \tilde{x}^n} \text{PE} \left[\frac{y}{(1 - \tilde{x})(1 - y)} \right] \tag{2.5.15}$$

i.e.

$$\tilde{Z}_\infty(\tilde{x}; y) = \frac{1}{\prod_{n>0, m \geq 0} (1 - \tilde{x}^n y^m)} \tag{2.5.16}$$

This is indeed the large N limit of an index counting certain BPS operators of the form $\partial_z^m W_n$ in the M5 SCFT, which give rise to a 2d W-algebra upon Ω deformation [60].

At this point we could readily conjecture explicit expressions for $Z_N(x; y)$ and $\tilde{Z}_k(\tilde{x}; y)$. Instead, we will derive these expressions from the mirror description of the problem.

Coulomb Branch Index

In the limit x, y to zero the matter contributions to the index are still mostly suppressed. We have an adjoint scalar Φ of fugacity zw and monopole contributions with m_i which can be any integer. As $q = 0$, the effect of non-zero m_i on the integrand is still to reduce it to the integrand to that of $\prod_m U(n_m)$, with the same f . We can thus write explicitly a generating function for all indices:

$$\sum_{N=0}^{\infty} \zeta^N Z_N(z; w) = \sum_{n_m \geq 0} \zeta^{\sum_m n_m} z^{\sum_{m>0} m n_m} w^{\sum_{m<0} (-m) n_m} \frac{1}{\prod_m \prod_{k_m=1}^{n_m} (1 - (zw)^{k_m})} \tag{2.5.17}$$

which factorizes into three parts

$$\begin{aligned} & \sum_{n_0 \geq 0} \zeta^{n_0} \frac{1}{\prod_{k_0=1}^{n_0} (1 - (zw)^{k_0})} \\ & \prod_{m > 0} \left[\sum_{n_m \geq 0} (z^m \zeta)^{n_m} \frac{1}{\prod_{k_m=1}^{n_m} (1 - (zw)^{k_m})} \right] \\ & \prod_{m < 0} \left[\sum_{n_m \geq 0} (w^{-m} \zeta)^{n_m} \frac{1}{\prod_{k_m=1}^{n_m} (1 - (zw)^{k_m})} \right] \end{aligned} \quad (2.5.18)$$

giving

$$\sum_{N=0}^{\infty} \zeta^N Z_N(z; w) = \frac{1}{\prod_{k_0 \geq 0} (1 - \zeta(zw)^{k_0})} \prod_{m > 0} \frac{1}{\prod_{k_m \geq 0} (1 - z^m \zeta(zw)^{k_m})} \prod_{m < 0} \frac{1}{\prod_{k_m \geq 0} (1 - w^{-m} \zeta(zw)^{k_m})} \quad (2.5.19)$$

which can be rearranged to a very simple form

$$\sum_{N=0}^{\infty} \zeta^N Z_N(z; w) = \frac{1}{\prod_{a \geq 0, b \geq 0} (1 - \zeta z^a w^b)} \quad (2.5.20)$$

These $Z_N(z; w)$ indices actually coincide with the Higgs branch indices from the previous section.

We will now take a small detour which gives some insight on different possible expansions for the index. Suppose for a moment that z and w have finite values in the unit disk. We can now compute Z_N by a contour integral on the unit circle in the ζ plane, which can be closed at infinity leading to a sum over poles at $z^{-k} w^{-k'}$. The contribution from the (k, k') pole is

$$Z_N^{k, k'}(z; w) = z^{kN} w^{k'N} \frac{1}{\prod_{a \geq 0, b \geq 0, (a, b) \neq (k, k')} (1 - z^a w^b)} \quad (2.5.21)$$

The natural holographic interpretation of this sum would involve contributions from giant gravitons wrapping two different sub-manifolds in S^7 . In a mirror picture, that would represent finite modifications of a product $(\det X)^k (\det Y)^{k'}$ of determinants of different fields. Notice that $Z_N^{k, k'}$ contains denominator factors of the form $(z - w)$ which are somewhat ambiguous when we attempt a power series expansion in z and w . Presumably, the ambiguity disappears when multiple terms are added together.

The terms $(k, 0)$ have the form we expect from a giant graviton expansion:

$$Z_N^{k,0}(z; w) = z^{kN} \frac{1}{\prod_{a=1}^k \prod_{b \geq 0} (1 - z^{-a} w^b)} \frac{1}{\prod_{a \geq 0, b \geq 0 | (a,b) \neq (0,0)} (1 - z^a w^b)} \quad (2.5.22)$$

with the expected $Z_\infty(z; w)$ and

$$\tilde{Z}_k(\tilde{z}; w) = \frac{1}{\prod_{a=1}^k \prod_{b \geq 0} (1 - \tilde{z}^a w^b)} \quad (2.5.23)$$

Experimentally, we find that the giant graviton expansion works at finite N with the above $\tilde{Z}_k(\tilde{z}; w)$: the sum over $Z_N^{k,0}(z; w)$ only, when expanded first in powers of w and then of z , reproduces Z_N .

These $\tilde{Z}_k(\tilde{z}; w)$ match the expectation from the M5 brane SCFT: we see the contribution of chiral algebra operators $\partial_z^b W_a$ for $a = 1, \dots, k$.

We can also run the giant graviton expansion in reverse. We can write

$$\frac{\tilde{Z}_N(\tilde{z}; w)}{\tilde{Z}_\infty(\tilde{z}; w)} = \prod_{a=1}^\infty \prod_{b \geq 0} (1 - \tilde{z}^{N+a} w^b) \quad (2.5.24)$$

and expand out

$$\frac{\tilde{Z}_N(\tilde{z}; w)}{\tilde{Z}_\infty(\tilde{z}; w)} = \sum_{k_b=0}^\infty \tilde{z}^{N \sum_b k_b} w^{\sum_b b k_b} \prod_{b \geq 0} \frac{1}{\prod_{n_b=1}^{k_b} (1 - \tilde{z}^{-1})^{n_b}} \quad (2.5.25)$$

so that

$$\hat{\tilde{Z}}_k(\tilde{z}; w) = \sum_{k_b=0 | \sum_b k_b=k}^\infty w^{\sum_b b k_b} \prod_{b \geq 0} \frac{1}{\prod_{n_b=1}^{k_b} (1 - \tilde{z}^{-1})^{n_b}} \quad (2.5.26)$$

In the next section, we will recognize this as being the same as $Z_k(z; w)$, as expected.

A Mixed Index

Finally, we can send y and w to zero for a peculiar mixed limit which keeps both X and positive monopole charges. Proceeding in the same manner as before, we get

$$\sum_{N=0}^\infty \zeta^N Z_N(x, z) = \sum_{n_m \geq 0} \zeta^{\sum_m n_m} z^{\sum_m m n_m} \frac{1}{\prod_m \prod_{k_m=1}^{n_m} (1 - x^{k_m})} \quad (2.5.27)$$

and thus

$$\sum_{N=0}^{\infty} \zeta^N Z_N(x, z) = \prod_{m \geq 0} \frac{1}{\prod_{k_m \geq 0} (1 - z^m \zeta x^{k_m})} \quad (2.5.28)$$

which is just another version of the usual

$$\sum_{N=0}^{\infty} \zeta^N Z_N(x, z) = \frac{1}{\prod_{a \geq 0, b \geq 0} (1 - \zeta x^a z^b)} \quad (2.5.29)$$

Indeed, this limit is related to the previous ones by the symmetries of the IR theory.

A Minimally Reduced Index

We could also consider setting w to 0 only. We now have a rich matter contribution

$$\begin{aligned} 1 - f &= (1 - x)(1 - y) \\ v &= \sqrt{xy} \\ \bar{v} &= \sqrt{xy} \end{aligned} \quad (2.5.30)$$

but also include the effect of positive monopole operators. Proceeding in the same manner as before, we get

$$\sum_{N=0}^{\infty} \zeta^N Z_N(x, y, z) = \sum_{n_m \geq 0} \zeta^{\sum_m n_m} z^{\sum_m m n_m} Z_{n_m}^{\text{Higgs}}(x, y) \quad (2.5.31)$$

i.e.

$$\sum_{N=0}^{\infty} \zeta^N Z_N(x, y, z) = \prod_{m \geq 0} \frac{1}{\prod_{k_m \geq 0, k'_m \geq 0} (1 - z^m \zeta x^{k_m} y^{k'_m})} \quad (2.5.32)$$

which gives a nicely symmetric expression

$$\sum_{N=0}^{\infty} \zeta^N Z_N(x, y, z) = \frac{1}{\prod_{a \geq 0, b \geq 0, c \geq 0} (1 - \zeta x^a y^b z^c)} \quad (2.5.33)$$

The same contour integral analysis as before gives a tentative

$$\tilde{Z}_k(\tilde{x}; y, z) = \frac{1}{\prod_{a=1}^k \prod_{b \geq 0, c \geq 0} (1 - \tilde{x}^a y^b z^c)} \quad (2.5.34)$$

It would be nice to verify that an appropriate reduced sector of the M5 theory has corresponding generators $\partial_{z_1}^b \partial_{z_2}^c W_a$.

2.5.2 The Full Index

In the full index, the monopole charges modify the integrand by a rescaling $\mu_a/\mu_b \rightarrow \mu_a/\mu_b q^{\frac{|m_a-m_b|}{2}}$ of the Vandermonde and adjoint contributions, $\mu_a \rightarrow \mu_a q^{\frac{|m_a|}{2}}$ of the fundamentals, $\mu_a^{-1} \rightarrow \mu_a^{-1} q^{\frac{|m_a|}{2}}$ of the anti-fundamentals. The powers of z and w are as in the simplified models, the sum of positive m_i and $-m_i$ respectively.

The index in each monopole sector thus takes the form of a complicated $\prod_m U(n_m)$ quiver gauge theory, with

$$\begin{aligned} f_{m,m'} &= q^{\frac{|m-m'|}{2}}(f-1) + \delta_{m,m'} \\ v_m &= q^{\frac{|m|}{2}}v \\ \bar{v}_m &= q^{\frac{|m|}{2}}\bar{v} \end{aligned} \tag{2.5.35}$$

The following analysis will be blind to the precise form of f and v .

If we consider a large N limit with operators of finite charge, only a finite number of m_i can be non-zero. Thus only n_0 will be large. We can deal with this as we learned in previous sections. The $U(n_0)$ theory has (anti)fundamental indices

$$v + \sum_m q^{\frac{|m|}{2}}(f-1)\bar{\chi}_m \qquad \bar{v} + \sum_m q^{\frac{|m|}{2}}(f-1)\chi_m \tag{2.5.36}$$

where χ_m and $\bar{\chi}_m$ are the appropriate characters of gauge fugacities for $U(n_m)$. The large n_0 limit leaves behind new effective adjoint and (anti)fundamental indices for the remaining $U(n_m)$ nodes at $m \neq 0$:

$$\begin{aligned} f_{m,m'} &= (q^{\frac{|m-m'|}{2}} - q^{\frac{|m|+|m'|}{2}})(f-1) + \delta_{m,m'} \\ v_m &= 0 \\ \bar{v}_m &= 0 \end{aligned} \tag{2.5.37}$$

In other words, nodes with different signs for m and m' are completely disconnected. The (anti)fundamental degrees of freedom are effectively gone. The adjoint contributions and the bifundamentals between nodes with the same sign have a simplification which eliminates the $1-q$ denominator: a finite collection of letters appear in the effective quivers.

Unfortunately, the simplification is not sufficient to make Z_∞ fully explicit. This makes a further analysis trickier. We leave it to future work and conclude with a few observations.

Determinants and Giant Gravitons

The natural way to insert a determinant in the large N limit is to add (anti)fundamental fermions and antifields coupled to the $U(n_0)$ node of the effective quiver. Indeed, in the presence of the monopole, only the $n_0 \times n_0$ components of a scalar field X are available to build an operator.

The heuristic analysis of determinant fluctuations in the large n_0 limit then proceeds as before, leading to an effective $U(k)$ gauge theory which is coupled to the remaining $U(n_m)$ nodes at $m \neq 0$. The $U(n_0)$ theory has (anti)fundamental indices

$$v + (x^{\frac{1}{2}} - x^{-\frac{1}{2}})\bar{\chi}_\lambda + \sum_m q^{\frac{|m|}{2}}(f-1)\bar{\chi}_m \quad \bar{v} + (x^{\frac{1}{2}} - x^{-\frac{1}{2}})\chi_\lambda + \sum_m q^{\frac{|m|}{2}}(f-1)\chi_m \quad (2.5.38)$$

so the new effective bifundamental fields have $f_{m,*} = -(x^{\frac{1}{2}} - x^{-\frac{1}{2}})q^{\frac{|m|}{2}}$.

Abelian Monopoles and Giant Gravitons

The Coulomb branch perspective on giant gravitons is also interesting. We can study finite modification of an Abelian monopole operator of charge k simply by taking n_k to be large and all other n_m finite. The $U(n_k)$ theory has (anti)fundamental indices

$$q^{\frac{|k|}{2}}v + \sum_m q^{\frac{|m-k|}{2}}(f-1)\bar{\chi}_m \quad q^{\frac{|k|}{2}}\bar{v} + \sum_m q^{\frac{|m-k|}{2}}(f-1)\chi_m \quad (2.5.39)$$

where χ_m and $\bar{\chi}_m$ are the appropriate characters of gauge fugacities for $U(n_m)$. The large n_k limit leaves behind new effective adjoint and (anti)fundamental indices for the remaining $U(n_m)$ nodes at $m \neq k$:

$$\begin{aligned} f_{m,m'} &= (q^{\frac{|m-m'|}{2}} - q^{\frac{|m-k|+|m'-k|}{2}})(f-1) + \delta_{m,m'} \\ v_m &= (q^{\frac{|m|}{2}} - q^{\frac{|k|}{2}}q^{\frac{|m-k|}{2}})v \\ \bar{v}_m &= (q^{\frac{|m|}{2}} - q^{\frac{|k|}{2}}q^{\frac{|m-k|}{2}})\bar{v} \end{aligned} \quad (2.5.40)$$

2.6 Future Directions

We conclude with a brief list of open questions which are left open by our work:

1. It should be possible to test (2.1.6) and the physical identification of $\tilde{Z}_k$ in a systematic manner in all 4d SCFTs theories with holographic duals, possibly in the presence of defects of various co-dimension.
2. The contribution of monopole sectors in 3d SCFTs is well understood but offers some extra computational challenges which prevent us from deriving an explicit prediction for the corresponding $\tilde{Z}_k$. Overcoming this obstruction would give novel explicit expressions for the indices of the corresponding giant graviton theories, which are often strongly-coupled 6d SCFTs.
3. We have not explored the analogues of (2.1.6) in 2d SCFTs or in dimension greater than four.
4. The expansion (2.1.6) can be interpreted as a non-perturbative string theory result. It would be interesting to explore this perspective further.
5. A localization argument for (2.1.6) could help explain the coexistence of many different expansion formulae which take into account distinct collections of giant graviton configurations.
6. It may be possible to motivate (2.1.6) by a saddle expansion of the superconformal index in a regime where x is finite and other y 's are infinitesimal.

Part II

Explanation

Chapter 3

On Branes and Trace Relations

3.1 Introduction

A central theme in our study of holography and gauge-string duality is the idea that the Feynman diagrams of a large N gauge theory reorganize into the genus expansion of *some* string theory [3–6]. A particularly striking aspect of 't Hooft's proposal is its generality: any theory of gauged large N matrices exhibits such a topological expansion at weak coupling.

To what extent, and in what form, does the notion of gauge-string duality persist at finite N ?

At finite N , one is confronted with conceptual issues regarding the spectrum of the bulk and boundary theories related via holographic duality. In the boundary gauge theory, the total number of states—at fixed values of the conserved charges and the 't Hooft coupling λ —decreases as the rank N of the gauge group is decreased. This occurs because states formed by exciting the vacuum with single trace operators are orthogonal at large N but become constrained by trace relations when N is finite. On the other hand, in the bulk string theory, non-perturbative states such as black holes and D-branes become lighter at finite string coupling g_s . Such objects, which are present in addition to the perturbative string states, would naively result in many more states at finite g_s compared to those at small g_s , at fixed charges and L^4/α'^2 .

How does one reconcile these differing expectations? The correct resolution of the apparent conflict is to posit that non-perturbative objects in string theory represent highly redundant descriptions of the same set of quantum states: a state composed of N fundamental

strings may admit a description as a brane, and a state composed of N D-branes admits a description in terms of the backreacted geometry. The intuition that non-perturbative effects in string theory place an upper bound on the single-string spectrum is known as the stringy exclusion principle [18–20]. While this principle is thought to be a generic feature of string theory, there is little quantitative understanding of the mechanism underlying the principle.

The giant graviton expansion, a recently-proposed formula that relates the BPS sectors of gauge theories and their string duals, suggests how the stringy exclusion principle is realized in the spectrum of the bulk string field theory [1, 32, 54]:¹

$$Z_N(q) = Z_\infty(q) \sum_{k=0}^{\infty} q^{kN} \hat{Z}_k(q). \quad (3.1.1)$$

The formula expresses $Z_N(q)$, the superconformal index [12] of a $U(N)$ gauge theory, as a sum over BPS indices $q^{kN} Z_\infty(q) \hat{Z}_k(q)$ of the bulk string theory on backgrounds labelled by k giant graviton branes. $Z_\infty(q)$ is the index of Kaluza-Klein modes and $q^{kN} \hat{Z}_k(q)$ is the index of k giant graviton branes and their open-string excitations. Giant gravitons are BPS branes on $\mathbb{R} \times S^{m-2} \subset \text{AdS}_n \times S^m$ that are supported via a large angular momentum induced from the background Ramond-Ramond potential [20, 24, 25]. See, for example, [35, 36, 61–71] for related works.

A salient feature of the giant graviton expansion is the absence of explicit contributions from gravitational saddles, even in the $\frac{1}{16}$ -BPS sector of IIB string theory and $\mathcal{N} = 4$ super Yang-Mills where supersymmetric black holes are present. That the asymptotic degeneracies of bulk configurations of strings and branes in (3.1.1) reproduce the entropy of large BPS black holes in $\text{AdS}_5 \times S^5$ was demonstrated recently [64, 69].

The following property of the formula realizes the stringy exclusion principle: BPS spectra on string backgrounds labelled by different numbers of giant graviton branes exhibit large cancellations to reproduce the finite N index. This occurs even in sectors of gauge theory that are purely bosonic, such the half-BPS sector of $\mathcal{N} = 4$ SYM:

$$\frac{1}{\prod_{n=1}^N (1 - x^n)} = \frac{1}{\prod_{n=1}^{\infty} (1 - x^n)} \sum_{k=0}^{\infty} (-1)^k x^{kN} \frac{x^{k(k+1)/2}}{\prod_{m=1}^k (1 - x^m)}. \quad (3.1.2)$$

Observe that each summand, supplemented with the half-BPS Kaluza-Klein spectrum, overcounts the half-BPS spectrum of $U(N)$ $\mathcal{N} = 4$ SYM. The overcounting is saved only

¹There exist different but compatible versions of the formula that are related by wall-crossing [1, 61]. We write a simple form of the formula which is a special case of the relation proposed in the current work.

by the overall minus signs $(-1)^k$ that appear with odd numbers of D3 giants. Motivated by this observation, we suggested in [61] that the bulk BPS Hilbert space possesses an extra $\mathbb{Z}_2$ -grading, on top of the usual fermion number grading $(-1)^F$ in the gauge theory, that labels open string vacua associated to different D-brane backgrounds.

Natural questions are then, what is the bulk origin of this grading? How does the $U(N)$ gauge theory reflect the extra bulk grading if the grading is not manifest in its physical Hilbert space? Most importantly, how is the presence of the extra grading consistent with holography?

In Section 3.2, we show in the half-BPS sector that the space of states of k giant graviton branes on $\text{AdS}_5 \times S^5$ is holographically dual to the space $\mathcal{V}_k$ of certain “heavy” ghosts in $\mathcal{N} = 4$ SYM that compensate for null states due to finite N trace relations, with total ghost number k . That is, *open strings on bulk D-branes are not dual to any physical states of a $U(N)$ gauge theory, but rather are objects that systematically cancel out the redundancies among states that arise at finite N .* We find that normalizable states of the worldvolume theory of k D3 giants are gauge-invariant differential k -forms constructed from worldvolume scalars and that such states survive in the large curvature limit of $\text{AdS}_5 \times S^5$ due to a localization mechanism. We identify the form degree k of these states as the extra grading in the bulk that labels open string vacua. The spectrum of normalizable states of half-BPS D3 giants are recovered by that of finite N trace relations, whose generating function is the determinant operator $\det(z - X)$ in the gauge theory. Due to the grading, the normalizable states are identified with auxiliary ghosts that compensate for the presence of finite N null states.²

We use the term *D-brane states* to refer to the states of bulk D-branes obtained by quantizing the open string modes.

Based on the above identification, we show that the spaces of half-BPS D-brane states form a complex that furnishes a BRST-like resolution of the half-BPS Hilbert space $\mathcal{H}_N$ of $U(N)$ $\mathcal{N} = 4$ SYM at finite N . The resolution indicates that open strings on bulk D-branes are encoded in the *relations* between the single-trace generators that arise in the boundary gauge theory at finite N . The giant graviton expansion (3.1.2) in the half-BPS sector is recovered using the fact that the alternating sum of the Hilbert series of modules in a resolution vanishes. Thus, we identify the complex formed from spaces of D-brane states as the half-BPS Hilbert space of IIB string theory in the large curvature limit of $\text{AdS}_5 \times S^5$.

²There exist previous proposals for the holographic dual of open strings on D3 giants in terms of physical operators in the boundary [52, 72, 73]. Note, however, that the spectrum of *normalizable* excitations of half-BPS D3 giants are reproduced by the spectrum of auxiliary ghosts that compensate for trace relations.

While the bulk computations in the half-BPS sector rely on supersymmetry and the presence of a weakly-curved gravity limit, we argue that the identification between D-brane states and auxiliary ghosts is quite general and holds independently of those properties. Trace relations between gauge-invariant operators are present in any finite N gauge theory with adjoint fields, and such a theory may be rewritten as a $N = \infty$ gauge theory supplemented with auxiliary heavy ghosts for trace relations that are present in the original theory, at least when $\lambda = 0$.

Let us state our main proposal here, leaving a review of the mathematical background to Section 3.3: Consider a four-dimensional $U(N)$ gauge theory on $\mathbb{R} \times S^3$ possessing only fields in the adjoint representation of the gauge group, at zero 't Hooft coupling $\lambda = 0$. Let R be the ring of polynomials of traces of fields and their derivatives (before accounting for trace relations). The space of states $\mathcal{H}_N$ of the gauge theory is the quotient of $\mathcal{H}_\infty = R|0\rangle$ by the submodule generated by the set of all trace relations at a given value of N .

Our proposal is that the complex $\mathcal{V}^N$:

$$\cdots \rightarrow \mathcal{V}_3^N \xrightarrow{\hat{Q}} \mathcal{V}_2^N \xrightarrow{\hat{Q}} \mathcal{V}_1^N \xrightarrow{\hat{Q}} \mathcal{H}_\infty \rightarrow 0, \quad (3.1.3)$$

which furnishes a Koszul-Tate resolution $\mathcal{K}$ of $\mathcal{H}_N$ over the ring R , should be identified as the space of states of the string dual in the $\alpha' \rightarrow \infty$ limit. The homology of $\mathcal{V}^N$ is concentrated at ghost number 0 and coincides with $\mathcal{H}_N = \mathcal{H}_\infty / \hat{Q}\mathcal{V}_1^N$. The free R -module $\mathcal{V}_k^N$ is, in a sense that is elaborated, the $\alpha' \rightarrow \infty$ limit of the bulk space of states on a string background with k wrapped D-branes. The observables of the $U(N)$ gauge theory are related to those of its string dual via the Lefschetz trace formula

$$\mathrm{Tr}_{\mathcal{H}_N} \mathcal{O} = \sum_{k=0}^{\infty} (-1)^k \mathrm{Tr}_{\mathcal{V}_k^N} \mathcal{O} \quad (3.1.4)$$

which provides the holographic map in the regime of $\lambda = 0$ and finite N . Bulk observables are computed by taking the alternating sum of the expectation values in an ensemble of states built on each open string vacuum.

In Section 3.3.6, we show how the giant graviton expansion of [1] is recovered and generalized in a certain limit of the complex $\mathcal{V}^N$. In Section 3.3.7, we place a lower bound on the degeneracies of D-brane states using the uniqueness of minimal resolutions. In Section 3.4, we discuss Koszul-Tate resolutions in simple physical examples.

Many questions are left open in our work. For instance, while our proposal suggests how gauge-string duality at weak 't Hooft coupling can be generalized to the regime of finite N , there remains the difficult problem of finding a consistent bulk string theory

with the features suggested by our proposal. We hope that this work contributes to the understanding of some kinematical properties of this theory.

We conclude with a partial list of open questions in the Discussion.

3.2 D-branes as Heavy Ghosts

In this section, we argue that the normalizable states of half-BPS D3 giant gravitons in the bulk are holographically dual to certain auxiliary “ghosts” in $U(N)$ $\mathcal{N} = 4$ super Yang-Mills that compensate for null states due to finite N trace relations. That is, open strings on D3 giants are not dual to any physical states of the $U(N)$ gauge theory, but rather are objects that systematically cancel out the redundancies among states that arise at finite N .

Indeed, giant gravitons were conceived in order to provide a bulk explanation for the truncation of the single trace spectrum at finite N , in terms of the polarization of large charge gravitons into branes. The goal in this section is to understand precisely why the identification between the states of giant graviton branes and ghosts for trace relations should be made.

3.2.1 A Twist on the Worldvolume

Let us show that the background Ramond-Ramond potential induces a twist of the stress tensor on the worldvolume of a D3 giant. The sign of the twist on the D3 worldvolume will later have an important consequence: normalizable states that describe transverse fluctuations of branes in S^5 become valued in gauge-invariant combinations of differential forms constructed from worldvolume scalars.

Let us work in $\text{AdS}_5 \times S^5$ coordinates

$$ds^2 = - \left(1 + \frac{\rho^2}{L^2}\right) dt^2 + \left(1 + \frac{\rho^2}{L^2}\right)^{-1} d\rho^2 + \rho^2 d\hat{\Omega}_{S^3}^2 + \frac{L^2}{L^2 - r^2} dr^2 + r^2 d\theta^2 + (L^2 - r^2) d\Omega_{S^3}^2 \quad (3.2.1)$$

covering half of the five-sphere, with Ramond-Ramond 4-form potential

$$C_4 = \frac{\rho^4}{L} dt \wedge d\hat{\Omega}_{S^3} + (L^2 - r^2)^2 d\theta \wedge d\Omega_{S^3}. \quad (3.2.2)$$

The coordinate $r \in [0, L)$ is the radius on the transverse plane to S^3 inside S^5 . The maximal D3 solution sits at $r = 0$ with $\dot{\theta} = \frac{1}{L}$ and has energy $E = \frac{N}{L}$ and angular momentum $R = N$.³ Despite wrapping a topologically trivial cycle, giant gravitons are stable BPS solutions because they possess a large angular momentum induced from the background Ramond-Ramond potential C_4 .

We are interested in half-BPS transverse fluctuations of a giant on a maximal $S^3 \subset S^5$ fixed under the $U(1)_X$ isometry generated by $R = -i\partial_\theta$. The bosonic DBI+CS action on the worldvolume of a single D3 brane is

$$S_{\text{D3}} = -\frac{N}{2\pi^2 L^4} \int (d^4x \sqrt{-g_{\text{D3}}} - P[C_4]) . \quad (3.2.3)$$

It will be instructive to first work in the stationary frame with respect to $\text{AdS}_5 \times S^5$ to note some properties of the corresponding hamiltonian. The quadratic action S_{D3}^Φ for transverse fluctuations in S^5 in the stationary frame is

$$\begin{aligned} S_{\text{D3}}^\Phi &= \frac{1}{2\pi^2 L^4} \int dt d\Omega_{S^3} \left(|\dot{\Phi}|^2 - |\nabla\Phi|^2 + \frac{3}{L^2} |\Phi|^2 + \frac{2i}{L} (\bar{\Phi}\dot{\Phi} - \dot{\Phi}\Phi) \right) \\ &= \frac{1}{2\pi^2 L^4} \int dt d\Omega_{S^3} \left(|\partial_t(e^{-2it/L}\Phi)|^2 - |\nabla\Phi|^2 - \frac{1}{L^2} |\Phi|^2 \right) \end{aligned} \quad (3.2.4)$$

in local complex coordinates $\Phi = \frac{1}{\sqrt{2}} r e^{i\theta}$ and $\bar{\Phi} = \frac{1}{\sqrt{2}} r e^{-i\theta}$. The potential term $-\frac{1}{L^2} |\Phi|^2$ is present in the second line because of the nonvanishing Ricci scalar on $\mathbb{R} \times S^3$. As was observed in [32], the stationary frame action of a D3 giant is equivalent to the usual action of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$ where its fields possess a time-dependent phase

$$(\text{field}) \rightarrow e^{-\frac{2it}{L}R}(\text{field}) \quad (3.2.5)$$

proportional to the relevant worldvolume R-charge, which we can identify with their charge R under the background $U(1)_X$ R-isometry. The scalars $\Phi, \bar{\Phi}$ have charge ± 1 under $U(1)_X$.

The resulting worldvolume theory is a $U(1)$ $\mathcal{N} = 4$ SYM with a modified potential for fields that carry nonzero R . Let us take gauginos with $SU(4)$ indices $\psi^{A=1,2}$ and $\psi^{B=3,4}$ to have R-charges $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively, and let $\mathcal{A} = 1, \dots, 4$ be the usual $SU(4)$ index. Then under (3.2.5), the fermion lagrangian acquires the mass terms

$$+\frac{1}{L} \bar{\psi}_A \psi^A - \frac{1}{L} \bar{\psi}_B \psi^B. \quad (3.2.6)$$

³The angular velocity at $r = 0$ arises because the maximal D3 solution is a limit of stable solutions as $r \rightarrow 0$ and $\dot{\theta} \rightarrow \frac{1}{L}$.

With these fermion masses, one finds that supersymmetry transformations require conformal Killing spinors ζ_α on $\mathbb{R} \times S^3$ that depend on time [23]. Therefore, the supercharges depend on time in the stationary frame with respect to the $\text{AdS}_5 \times S^5$ background, i.e. the generator of time translations in this system is not supersymmetric.

That the generator H_{D3} of time translations is not supersymmetric has an interesting consequence. In the stationary frame, the hamiltonian is

$$H_{\text{D3}} = \int d\Omega_{S^3} \left[|\Pi|^2 + |\nabla\Phi|^2 + \frac{1}{L^2}|\Phi|^2 + \frac{2i}{L}(\Pi\Phi - \bar{\Phi}\bar{\Pi}) + i\bar{\psi}_A\bar{\sigma}^i\nabla_i\psi^A - \frac{1}{L}\bar{\psi}_A\psi^A + \frac{1}{L}\bar{\psi}_B\psi^B \right] + \dots \quad (3.2.7)$$

where ellipses denote contributions from fields on the worldvolume SYM theory that do not possess $U(1)_X$ R-charges. The conjugate momenta $\Pi = \dot{\Phi} + \frac{2i}{L}\bar{\Phi}$ and $\bar{\Pi} = \dot{\bar{\Phi}} - \frac{2i}{L}\Phi$ and the fermionic counterpart satisfy the equal-time (anti-)commutation relations. Observe that H_{D3} has the shift $+\frac{2}{L}R$ relative to the free $\mathcal{N} = 4$ hamiltonian, where the worldvolume charge R is

$$R = \int d\Omega_{S^3} \left[i(\Pi\Phi - \bar{\Phi}\bar{\Pi}) - \frac{1}{2}(\bar{\psi}_A\psi^A - \bar{\psi}_B\psi^B) \right]. \quad (3.2.8)$$

The hamiltonian H_{D3} indicates that the energies of D3 worldvolume fields must be shifted by two units of their R charges, relative to naive dimensional analysis based on a $\mathcal{N} = 4$ action. That is, the background Ramond-Ramond potential C_4 induced a twist of the stress tensor on the worldvolume of a D3 giant. The definition of R here coincides with its previous definition as the background $U(1)_X$ isometry generator $R = -i\partial_\theta$ upon restricting to bosonic S^3 -zeromodes.

Due to the twist, the only worldvolume field satisfying the half-BPS condition

$$H_{\text{D3}} - \frac{1}{L}R = 0 \quad (3.2.9)$$

is the anti-holomorphic scalar $\bar{\Phi}$ with $H_{\text{D3}}L = R = -1$. The negative mode is an artifact of the hamiltonian H_{D3} not commuting with supercharges in this frame.

3.2.2 Localization at Small Radius

In the following sections, we find the half-BPS states of D3 giant gravitons.

In deriving the worldvolume twist, we considered the D3 action in the stationary frame relative to the $\text{AdS}_5 \times S^5$ background. However, a maximal giant graviton rotates at the

speed of light $\dot{\theta} = \frac{1}{L}$, so expanding about the solution actually entails working in a frame that is rotating with the giant. The worldvolume action about the rotating D3 solution turns out to admit time-independent supercharges for the relevant chiral multiplet, i.e. the time translation generator commutes with these supercharges.

In the worldline quantum mechanics comprising of the lightest S^3 modes, these supercharges will be identified with the Dolbeault differential $\bar{\partial}$ and its adjoint $\bar{\partial}^\dagger$ deformed by an inverted Morse function h . The deformed Dolbeault laplacian $\Delta_h^{\bar{\partial}} = \{\bar{\partial}_h, \bar{\partial}_h^\dagger\}$ can therefore be identified with the hamiltonian of this supersymmetric quantum mechanics. The BPS states are then the ground states of $\Delta_h^{\bar{\partial}}$. We find that these states survive and localize to fixed points of R-isometries in the large curvature limit $L \rightarrow 0$ of $\text{AdS}_5 \times S^5$. This allows us to compare the spectrum of the string states with the spectrum of finite N null states in the boundary gauge theory in Section 3.2.6.

Expanding the action S_{D3}^Φ around the on-shell value $\theta(t) = \frac{1}{L}t + \delta\theta(t)$, we find

$$\begin{aligned} S_{\text{D3}}^\phi &= \frac{1}{2\pi^2 L^4} \int dt d\Omega_{S^3} \left(|\dot{\phi}|^2 - |\nabla\phi|^2 + \frac{i}{L}(\bar{\phi}\dot{\phi} - \dot{\bar{\phi}}\phi) \right) \\ &= \frac{1}{2\pi^2 L^4} \int dt d\Omega_{S^3} \left(|\partial_t(e^{-it/L}\phi)|^2 - |\nabla\phi|^2 - \frac{1}{L^2}|\phi|^2 \right). \end{aligned} \quad (3.2.10)$$

The complex coordinates $\phi = \frac{1}{\sqrt{2}}re^{i\delta\theta}$ and $\bar{\phi} = \frac{1}{\sqrt{2}}re^{-i\delta\theta}$ are now defined in terms of the fluctuation angle $\delta\theta$. The fields now possess the time-dependent phase

$$(\text{field}) \rightarrow e^{-\frac{it}{L}R}(\text{field}) \quad (3.2.11)$$

compared to those in the usual action of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$. This phase differs from that in the stationary frame by a unit of the R-charge.

We again take gauginos with $SU(4)$ indices $\psi^{A=1,2}$ and $\psi^{B=3,4}$ to have R-charges $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively, and let $\mathcal{A} = 1, \dots, 4$ be the usual $SU(4)$ index. Then the fermion lagrangian acquires the mass terms $+\frac{1}{2L}\bar{\psi}_A\psi^A - \frac{1}{2L}\bar{\psi}_B\psi^B$. The total action becomes

$$\begin{aligned} S_{\text{D3}} &= \frac{1}{2\pi^2 L^4} \int dt d\Omega_{S^3} \left[-\partial_\mu \bar{\phi} \partial^\mu \phi + \frac{i}{L}(\bar{\phi}\dot{\phi} - \dot{\bar{\phi}}\phi) \right. \\ &\quad \left. - i\bar{\psi}_A \bar{\sigma}^\mu \nabla_\mu \psi^A + \frac{1}{2L}\bar{\psi}_A\psi^A - \frac{1}{2L}\bar{\psi}_B\psi^B \right] + \dots \end{aligned} \quad (3.2.12)$$

where ellipses denote fields that do not possess $U(1)_X$ R-charges. We will restrict attention

to half of the supersymmetries

$$\begin{aligned}
\delta\phi &= -\sqrt{2}\zeta_A\psi^A \\
\delta\bar{\phi} &= -\sqrt{2}\bar{\zeta}^A\bar{\psi}_A \\
\delta\psi_\alpha^A &= -i\sqrt{2}(\sigma^\mu\bar{\zeta}^A)_\alpha\partial_\mu\phi \\
\delta\bar{\psi}_{A\dot{\alpha}} &= +i\sqrt{2}(\zeta_A\sigma^\mu)_{\dot{\alpha}}\partial_\mu\bar{\phi}
\end{aligned} \tag{3.2.13}$$

that relate the complex scalars $\phi, \bar{\phi}$ for transverse fluctuations of S^3 in S^5 with only half of the gauginos $\psi^A, \bar{\psi}_A$ of positive R-charge. The corresponding conformal Killing spinors $\zeta, \bar{\zeta}$ on $\mathbb{R} \times S^3$ satisfy [23]

$$\begin{aligned}
0 &= \partial_t \zeta_A^\alpha \\
0 &= \partial_t \bar{\zeta}_{\dot{\alpha}}^A \\
0 &= \nabla_i \zeta_A^\alpha + \frac{i}{L}(\zeta_A \sigma_{0i})^\alpha \\
0 &= \nabla_i \bar{\zeta}_{\dot{\alpha}}^A - \frac{i}{L}(\bar{\zeta}^A \bar{\sigma}_{0i})_{\dot{\alpha}}.
\end{aligned} \tag{3.2.14}$$

The spinors $\zeta, \bar{\zeta}$ are time-independent unlike the previous case, so the hamiltonian of this worldvolume theory commutes with the supercharges from (3.2.13).

The new generator H'_{D3} of time translations for the D3 solution

$$\begin{aligned}
H'_{\text{D3}} &= \int d\Omega_{S^3} \left[|\pi|^2 + |\nabla\pi|^2 + \frac{i}{L}(\pi\phi - \bar{\phi}\bar{\pi}) \right. \\
&\quad \left. + i\bar{\psi}_A\bar{\sigma}^i\nabla_i\psi^A - \frac{1}{2L}\bar{\psi}_A\psi^A + \frac{1}{2L}\bar{\psi}_B\psi^B \right] + \dots
\end{aligned} \tag{3.2.15}$$

has the shift $+\frac{1}{L}R$ relative to the free $\mathcal{N}=4$ hamiltonian, where the definition of the R-charge

$$R = \int d\Omega_{S^3} \left[i(\pi\phi - \bar{\phi}\bar{\pi}) - \frac{1}{2}(\bar{\psi}_A\psi^A - \bar{\psi}_B\psi^B) \right]. \tag{3.2.16}$$

remains invariant. The conjugate momenta $\pi = \dot{\bar{\phi}} + \frac{i}{L}\bar{\phi}$ and $\bar{\pi} = \dot{\phi} - \frac{i}{L}\phi$ and the fermionic counterpart satisfy the equal-time (anti-)commutation relations. H'_{D3} has the shift $-\frac{1}{L}R$ relative to the stationary frame hamiltonian H_{D3} upon relating the fields and momenta, so we recognize the new hamiltonian

$$H'_{\text{D3}} = H_{\text{D3}} - \frac{1}{L}R \tag{3.2.17}$$

as the supersymmetric hamiltonian whose ground states are BPS states.

3.2.3 Reduction to Quantum Mechanics

For the half-BPS sector, it actually suffices to consider the supersymmetric quantum mechanics given by consistent truncation to the lightest S^3 -modes related by (3.2.13), because the higher modes do not satisfy the BPS condition. See e.g. [74] for details on the S^3 mode expansion of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times S^3$, though in our case the Killing spinors $\zeta, \bar{\zeta}$ are time-independent. The fluctuation scalars and gauginos have S^3 mode expansions

$$\begin{aligned}\phi(t, \Omega) &= \sum_{j=0}^{\infty} \sum_M \phi_{jM}(t) Y_{jM}(\Omega) \\ \psi_{\alpha}^A(t, \Omega) &= \sum_{j=0}^{\infty} \sum_{\kappa=\pm} \sum_M \psi_{jM\kappa}^A(t) Y_{jM\kappa}^{\kappa}(\Omega).\end{aligned}\tag{3.2.18}$$

j labels the irreps of the S^3 isometry group $SO(4) \simeq SU(2)_L \times SU(2)_R$, which are $(\frac{j}{2}, \frac{j}{2})$ for scalars and $(\frac{j+1}{2}, \frac{j}{2}), (\frac{j}{2}, \frac{j+1}{2})$ for the $\kappa = \pm$ fermions, respectively. M runs over the dimensions of the irreducible representation associated with j . We will be interested in the scalar S^3 zeromode $\phi \equiv \phi_{00}(t)$ and the lightest gaugino mode $\psi \equiv \psi_{0M+}^1(t)$ with $R = +\frac{1}{2}$, $\kappa = +$, and $M = (+\frac{1}{2}, 0)$.⁴ The other choice $M = (-\frac{1}{2}, 0)$ or linear combinations thereof are equally valid. We may also include higher modes to generalize our computation to BPS sectors preserving fewer supersymmetries, but we leave this to future work.

The worldline action for half-BPS excitations of a D3 giant is

$$S_{\text{QM}} = \frac{N}{L} \int dt \left[\dot{\phi} \dot{\phi} + \frac{i}{L} (\bar{\phi} \dot{\phi} - \dot{\phi} \phi) + i \bar{\psi} \dot{\psi} + \frac{1}{L} (\bar{\psi} \psi - \psi \bar{\psi}) \right].\tag{3.2.19}$$

While this action is invariant under simple supersymmetries that descend from (3.2.13), the shift in the definition of conjugate momenta results in an interesting interpretation of the supercharges on the transverse plane to S^3 in S^5 , which is locally $\mathbb{C}$.⁵ In particular, we can identify the supercharges

$$\begin{aligned}Q &= i \bar{\psi} \dot{\phi} = \bar{\psi} (i \bar{\pi} - \frac{1}{L} \phi) \\ \bar{Q} &= -i \psi \dot{\phi} = \psi (-i \pi - \frac{1}{L} \bar{\phi})\end{aligned}\tag{3.2.21}$$

⁴A choice of fermion modes with $R = -\frac{1}{2}$ and $\kappa = -$ would correspond to a different set of supersymmetries.

⁵The S^3 -zeromode conjugate momenta are $\pi = \dot{\phi} + \frac{i}{L} \bar{\phi}$ and $\bar{\pi} = \dot{\phi} - \frac{i}{L} \phi$, and the (anti-)commutation relations

$$[\phi, \pi] = i, \quad [\bar{\phi}, \bar{\pi}] = i, \quad \{\psi, \bar{\psi}\} = 1\tag{3.2.20}$$

hold at the level of the relevant S^3 modes.

as the Dolbeault differential and its adjoint⁶

$$\begin{aligned}\bar{\partial}_h &= e^{-h/L} \bar{\partial} e^{h/L} \\ \bar{\partial}_h^\dagger &= e^{h/L} \bar{\partial}^\dagger e^{-h/L}\end{aligned}\tag{3.2.22}$$

deformed by an *inverted* Morse function $h = -\bar{\phi}\phi$. The inverse radius L^{-1} plays the role of the localization parameter. Then the hamiltonian $H = \{Q, \bar{Q}\}$ of this supersymmetric quantum mechanics

$$H = \bar{\pi}\pi + \frac{1}{L^2}\bar{\phi}\phi + \frac{i}{L}(\phi\pi - \bar{\phi}\bar{\pi}) - \frac{1}{L}[\bar{\psi}, \psi]\tag{3.2.23}$$

is the deformed Dolbeault laplacian

$$\Delta_h^{\bar{\partial}} = \bar{\partial}_h \bar{\partial}_h^\dagger + \bar{\partial}_h^\dagger \bar{\partial}_h.\tag{3.2.24}$$

We also identify the fermions with operations $\bar{\psi} = d\bar{\phi} \wedge$ and $\psi = i_{\partial/\partial\bar{\phi}}$ on differential forms. The BPS states are the eigenstates of $H = \Delta_h^{\bar{\partial}}$ with zero eigenvalue. This system was studied in the context of holomorphic Morse inequalities in [75].

The function h induces a vector field V generating an isometry of $\mathbb{C}$ with fixed points at the critical points of h . Near any fixed point, the Dolbeault laplacian deformed by h takes the form

$$\Delta_h^{\bar{\partial}} = \Delta^{\bar{\partial}} + \frac{1}{L^2}\bar{\phi}\phi - \frac{1}{L}\mathcal{L}_V + \frac{1}{L}(\cdots)\tag{3.2.25}$$

where $\mathcal{L}_V$ is the Lie derivative with respect to the vector field V . The ellipses denote scalar terms that do not contain derivatives. For H , the induced vector field V is

$$V = -\left(\phi\frac{\partial}{\partial\phi} - \bar{\phi}\frac{\partial}{\partial\bar{\phi}}\right),\tag{3.2.26}$$

which coincides with the $U(1)_X$ isometry generator up to a sign. The sign determines the types of states that have support at the fixed point p of V as well as the orientation for the isometry action $\mathcal{L}_V$ that is appropriate for computing the R-charges of these states. We elaborate on this point shortly.

Let us discuss why it suffices to consider only the quadratic fluctuation action of the maximal D3 solution for the purpose of finding BPS states that survive in the small radius limit of $\text{AdS}_5 \times S^5$. Taking this limit is within the regime of validity of the worldline

⁶The adjoint of the Dolbeault differential $\bar{\partial}$ in one complex dimension is defined as $\bar{\partial}^\dagger = -\star\bar{\partial}\star$.

quantum mechanics on $\mathbb{R}$, because we take this limit after having decomposed fields into S^3 harmonics. We also neglect the higher open string modes because they do not satisfy the BPS condition. First we note that since V generates an isometry of the ϕ -plane, we have

$$[\Delta_h^{\bar{\partial}}, \mathcal{L}_V] = 0. \quad (3.2.27)$$

Therefore we can find the relevant BPS states by finding solutions to

$$\Delta_h^{\bar{\partial}} \Psi = 0 \quad (3.2.28)$$

in the small radius limit $L \rightarrow 0$, within each $\mathcal{L}_V$ -eigenspace

$$\mathcal{L}_V \Psi = m \Psi \quad (3.2.29)$$

of fixed charge m . Then the dominant potential $\frac{1}{L^2} \bar{\phi} \phi$ forces all BPS states of charge m to be localized near the fixed point, for any given m in the limit $L \rightarrow 0$. That is, BPS states of D3 giants localize to fixed points of bulk isometries in the small radius limit $L \rightarrow 0$ of $\text{AdS}_5 \times S^5$. The quadratic D3 action was sufficient because $\frac{1}{L^2} \bar{\phi} \phi$ is the dominant potential at any zero of V .

Our localization argument explains why dual D3 giant gravitons expanding in $S^3 \subset \text{AdS}_5$ do not seem to appear in the giant graviton expansion. A dual giant with large R-charge does not sit at a fixed point of the R-isometry, so BPS states of dual giants do not survive the small radius limit.

3.2.4 Normalizable States

To find normalizable BPS states, we allow $\Delta_h^{\bar{\partial}}$ to act on the space of differential forms.

Recall that the Morse index (for real manifolds) of a critical point p is the number of unstable directions at p with respect to the function h . The holomorphic analog n_p of the Morse index [75] is the number of unstable complex directions at the critical point p (with respect to h), or equivalently the number of complex directions for which V acts clockwise in local complex coordinates about the fixed point p . The holomorphic Morse index n_p coincides with the anti-holomorphic form degree of the normalizable state supported near p .

As in [75], the Lie derivative $\mathcal{L}_V$ acting on forms with respect to V computes the R-charge of a BPS state at p , given a choice of the fugacity domain for which the charge spectrum converges. In particular, the operator $\mathcal{L}_V$ computes directly the charge spectrum

of states that is convergent in the fugacity domain by taking into account the orientation at p induced by h . For example, for a fixed point with $n_p = 1$, the normalizable states are anti-holomorphic 1-forms so $\mathcal{L}_V$ computes the R-charge with a reversed orientation for the isometry action.

For the problem at hand, p is the sole fixed point of V at the origin of the ϕ -plane and it has holomorphic Morse index $n_p = 1$. The normalizable tower of BPS states is given by anti-holomorphic 1-forms

$$\Psi_m = \bar{\phi}^m e^{-\bar{\phi}\phi/L} d\bar{\phi} \quad (3.2.30)$$

that satisfy

$$\bar{\partial}_h^\dagger \Psi_m = \left(-\frac{\partial}{\partial \phi} - \frac{1}{L} \bar{\phi} \right) \Psi_m = 0 \quad (3.2.31)$$

or $\Delta_h^\partial \Psi_m = 0$, for $m = 0, 1, 2, \dots$. In the half-BPS sector we are interested in the fugacity domain $|x| < 1$ where each BPS state is counted with weight x^R . We can directly compute the charge spectrum that is convergent for $|x| < 1$:

$$\mathcal{L}_V \Psi_m = (m+1) \Psi_m. \quad (3.2.32)$$

This tower gives the half-BPS spectrum of a single giant

$$\mu x^N \sum_{m=0}^{\infty} x^{m+1} = \mu x^N \frac{x}{1-x}, \quad (3.2.33)$$

where x^N accounts for R-charges carried by the D3 giant and we introduced a fugacity μ for the form degree. This coincides with the $k = 1$ term of the giant graviton expansion in the half-BPS sector with $\mu = -1$.

There is a shift

$$N \rightarrow N + 1 \quad (3.2.34)$$

in the R-charge of the open string vacuum relative to the classical expectation of N . This resulted from the fact that the open string vacuum on a single D3 giant is labelled by an anti-holomorphic 1-form.

Note that a naive count of the charges based on the background R-isometry (and without accounting for form-valued states) would have given the spectrum $x^N \frac{1}{1-x-1}$, whose series expression converges for $|x| > 1$. This series is related to (3.2.33) by analytic continuation.

3.2.5 Generalization to k Coincident Giants

Let us generalize the analysis to k coincident giants. We could diagonalize the hamiltonian

$$\begin{aligned} H_k &= \text{Tr} \left[\bar{\pi}\pi + \frac{1}{L^2}\bar{\phi}\phi + \frac{i}{L}(\phi\pi - \bar{\phi}\bar{\pi}) - \frac{1}{L}[\bar{\psi}, \psi] \right] \\ &= \text{Tr} \left[-\frac{\partial}{\partial\bar{\phi}}\frac{\partial}{\partial\phi} + \frac{1}{L^2}\bar{\phi}\phi + \frac{1}{L}\left(\phi\frac{\partial}{\partial\phi} - \bar{\phi}\frac{\partial}{\partial\bar{\phi}}\right) - \frac{1}{L}(\bar{\psi}\psi - \psi\bar{\psi}) \right] \end{aligned} \quad (3.2.35)$$

in the position representation where the operators are now $k \times k$ matrices, and then look for normalizable $U(k)$ -invariant ground states. We proceed instead via a shortcut, namely by finding form-valued and $U(k)$ -invariant solutions to the supersymmetry conditions

$$\bar{\partial}_h \Psi = \bar{\partial}_h^\dagger \Psi = 0 \quad (3.2.36)$$

under a suitable ansatz for the ground state wavefunction Ψ .

Let us first determine the local form of the Morse potential h in terms of the eigenvalues $\phi_a, \bar{\phi}_b$ of the $k \times k$ matrices $\phi, \bar{\phi}$. Let P and Q be invertible matrices such that $\phi_{ij} = P_{ia}\phi_a P_{aj}^{-1}$ and $\bar{\phi}_{ij} = Q_{ia}\bar{\phi}_a Q_{aj}^{-1}$. We have $Q = P^\dagger$ since ϕ and $\bar{\phi}$ matrices are hermitian conjugates. The relevant part of the hamiltonian H_k containing the induced vector field V is

$$\frac{1}{L}\text{Tr} \left[\phi\frac{\partial}{\partial\phi} - \bar{\phi}\frac{\partial}{\partial\bar{\phi}} \right] = \frac{1}{L} \sum_{i,j=1}^k \left(\phi_{ij}\frac{\partial}{\partial\phi_{ij}} - \bar{\phi}_{ij}\frac{\partial}{\partial\bar{\phi}_{ij}} \right). \quad (3.2.37)$$

We can express the derivatives with respect to matrices $\phi, \bar{\phi}$ in terms of the eigenvalues

$$\frac{\partial}{\partial\phi_{ij}} = \sum_{a=1}^k P_{ai}^{-1} P_{ja} \frac{\partial}{\partial\phi_a}, \quad \frac{\partial}{\partial\bar{\phi}_{ij}} = \sum_{a=1}^k Q_{ai}^{-1} Q_{ja} \frac{\partial}{\partial\bar{\phi}_a}, \quad (3.2.38)$$

so we have

$$\begin{aligned} \frac{1}{L}\text{Tr} \left[\phi\frac{\partial}{\partial\phi} - \bar{\phi}\frac{\partial}{\partial\bar{\phi}} \right] &= \frac{1}{L} \sum_{i,j=1}^k \sum_{a,b=1}^k \left[P_{ia}\phi_a P_{aj}^{-1} P_{bi}^{-1} P_{jb} \frac{\partial}{\partial\phi_b} - Q_{ia}\bar{\phi}_a Q_{aj}^{-1} Q_{bi}^{-1} Q_{jb} \frac{\partial}{\partial\bar{\phi}_b} \right] \\ &= +\frac{1}{L} \sum_{a=1}^k \left(\phi_a \frac{\partial}{\partial\phi_a} - \bar{\phi}_a \frac{\partial}{\partial\bar{\phi}_a} \right). \end{aligned} \quad (3.2.39)$$

The local form of the vector field V about the origin in the space $\mathbb{C}^k$ of eigenvalues, found by comparing the above expression with the Lie derivative $-\frac{1}{L}\mathcal{L}_V$ of $\Delta_h^{\bar{\partial}}$ in (3.2.25), is

$$V = -\sum_{a=1}^k \left(\phi_a \frac{\partial}{\partial \phi_a} - \bar{\phi}_b \frac{\partial}{\partial \bar{\phi}_b} \right). \quad (3.2.40)$$

V is induced from the Morse function $h = -\sum_{a=1}^k \bar{\phi}_a \phi_a$ in local coordinates. The holomorphic Morse index at this point is $n_p = k$, so the state Ψ is an anti-holomorphic k -form. Thus, the normalizable state Ψ must have the expression

$$\Psi = G(\phi_a, \bar{\phi}_b) e^{-\sum_{a=1}^k \bar{\phi}_a \phi_a / L} d\bar{\phi}_1 \wedge \cdots \wedge d\bar{\phi}_k \quad (3.2.41)$$

in local eigenvalue coordinates $(\phi_a, \bar{\phi}_b)$. $G(\phi_a, \bar{\phi}_b)$ is totally antisymmetric under exchanges of the anti-holomorphic eigenvalues $\bar{\phi}_b$.

The supersymmetry condition $\bar{\partial}_h^\dagger \Psi = 0$ imposes further restrictions. Since

$$\bar{\partial}_h^\dagger \Psi \propto \sum_{a=1}^k \frac{\partial G}{\partial \phi_a} d\bar{\phi}_1 \wedge \cdots \wedge \widehat{d\bar{\phi}_a} \wedge \cdots \wedge d\bar{\phi}_k = 0, \quad (3.2.42)$$

where we omit the hatted component, G is independent of the holomorphic eigenvalues ϕ_a . As there are no further constraints, we can now write a $U(k)$ -invariant basis for the supersymmetric ground states of H_k :

$$\Psi_{m_1 \cdots m_k} = \frac{1}{k!} \sum_{a_1, \dots, a_k=1}^k \epsilon_{a_1 \cdots a_k} \bar{\phi}_1^{m_{a_1}} \bar{\phi}_2^{m_{a_2}} \cdots \bar{\phi}_k^{m_{a_k}} e^{-\sum_{a=1}^k \bar{\phi}_a \phi_a / L} d\bar{\phi}_1 \wedge \cdots \wedge d\bar{\phi}_k \quad (3.2.43)$$

where $0 \leq m_1 < m_2 < \cdots < m_k < \infty$. These states have R-charges

$$\mathcal{L}_V \Psi_{m_1 \cdots m_k} = (m_1 + \cdots + m_k + k) \Psi_{m_1 \cdots m_k}, \quad (3.2.44)$$

so the half-BPS spectrum of k giants is

$$\mu^k x^{kN} \sum_{0 \leq m_1 < \cdots < m_k < \infty} x^{m_1 + \cdots + m_k + k} = \mu^k x^{kN} \frac{x^{k(k+1)/2}}{\prod_{n=1}^k (1 - x^n)} \quad (3.2.45)$$

This is the k -th term of the giant graviton expansion in the half-BPS sector with $\mu = -1$.

There is again a shift

$$kN \rightarrow kN + \frac{1}{2}k(k+1) \quad (3.2.46)$$

in the R-charge of the open string vacuum relative to the classical expectation of kN , which originates from the fact that the open string vacuum on k D3 giants is labelled by an anti-holomorphic k -form.

3.2.6 Ghosts for Trace Relations

In our bulk analysis, we have seen that the open string vacuum on k giant graviton branes is labelled by a differential k -form. The form degree of states resulted from the fact that the supersymmetry generators were identified to be Dolbeault differentials deformed by a Morse function h with k unstable complex directions at the fixed point of an isometry. The form degree equips the bulk open string Hilbert space with an extra grading associated to the number of branes in the background.

On the other hand, normalizable BPS states of the boundary $\mathcal{N} = 4$ super Yang-Mills on $\mathbb{R} \times S^3$ are scalar-valued because the corresponding Morse function does not admit any unstable direction. This essentially stems from the fact that the BPS hamiltonian $H_{\text{BPS}} = H_{\mathcal{N}=4} - R$ of the boundary gauge theory possesses an angular momentum term that is opposite in sign compared to that of H'_{D3} in (3.2.15). An analysis similar to that done in Section 3.2.5 shows that the following normalizable wavefunctions

$$\Psi_{m_1 \dots m_N} = \frac{1}{N!} \sum_{\sigma \in S_N} X_1^{m_{\sigma(1)}} X_2^{m_{\sigma(2)}} \dots X_N^{m_{\sigma(N)}} e^{-\sum_{a=1}^N \bar{X}_a X_a}, \quad (3.2.47)$$

with $0 \leq m_1 \leq m_2 \leq \dots \leq m_N < \infty$, form a basis of half-BPS states for the boundary $U(N)$ $\mathcal{N} = 4$ SYM. These states were written in terms of the eigenvalues $(X_a, \bar{X}_b)$ of the $N \times N$ scalars X and $\bar{X}$. They have R-charges

$$\mathcal{L}_V \Psi_{m_1 \dots m_N} = (m_1 + \dots + m_N) \Psi_{m_1 \dots m_N} \quad (3.2.48)$$

with $V = + \sum_{a=1}^N \left(X_a \frac{\partial}{\partial X_a} - \bar{X}_a \frac{\partial}{\partial \bar{X}_a} \right)$, and their counting reproduces the boundary half-BPS spectrum $Z_N = \prod_{n=1}^N \frac{1}{1-x^n}$. This shows that the grading with respect to the form degree of states does not appear in the physical Hilbert space of the boundary gauge theory.

The obvious question is then, how can the presence of the extra grading in the bulk open string Hilbert space be consistent with holography? To answer this question, we need to identify the “states” in the boundary gauge theory that possess an extra grading and also reproduce the spectrum (3.2.45). Identifying such states is made subtle because, as demonstrated above, these states cannot belong to the physical Hilbert space of the gauge theory.

We can obtain clues regarding the nature of these states by examining the “mode” expansion in z of the determinant operator $\det(z - X)$ inserted at a point in $U(N)$ $\mathcal{N} = 4$ SYM. This operator is dual to a family of D3 giant graviton branes wrapped on $S^3 \subset S^5$ with order N units of an R-charge. The auxiliary parameter z has an interpretation as the

bulk insertion point of the brane on the plane transverse to the S^3 in S^5 . More explicitly, according to $\det(z - X)$ the brane must reach its maximal size at $z = 0$ and shrink to zero size at $z \rightarrow \infty$. We can therefore make the identification $z = \frac{\Phi}{1 - \Phi/L^2}$ in coordinates discussed in Section 3.2.1. The spacetime interpretation of z is useful even though we are working in a regime where stringy effects are large.

The fluctuation “modes” in z of the determinant operator are

$$\det(z - X) = z^N e^{-\sum_{n=1}^{\infty} \frac{1}{n} z^{-n} \text{Tr } X^n} = z^N \sum_{n=0}^{\infty} \frac{(-1)^n P_n}{z^n}. \quad (3.2.49)$$

where $P_n = P_n(\text{Tr } X^\bullet)$ are defined in terms of the exponential generating function and where the expression is normal-ordered. The first few modes are

$$\begin{aligned} P_1 &= \text{Tr } X \\ P_2 &= \frac{1}{2} ((\text{Tr } X)^2 - (\text{Tr } X^2)) \\ P_3 &= \frac{1}{6} ((\text{Tr } X)^3 - 3(\text{Tr } X)(\text{Tr } X^2) + 2(\text{Tr } X^3)) \\ P_4 &= \frac{1}{24} ((\text{Tr } X)^4 - 6(\text{Tr } X)^2(\text{Tr } X^2) + 3(\text{Tr } X^2)^2 + 8(\text{Tr } X)(\text{Tr } X^3) - 6(\text{Tr } X^4)) \end{aligned} \quad (3.2.50)$$

and so on. At any positive integer value N , the tower of modes

$$\det(z - X) = (-1)^{N+1} \left[\dots + \frac{P_{N+1}}{z} - \frac{P_{N+2}}{z^2} + \frac{P_{N+3}}{z^3} + \dots \right] \quad (3.2.51)$$

starting at P_{N+1} are but the trace relations between gauge-invariant operators that are present in any $U(N)$ gauge theory. In other words, states constructed by acting with modes $P_{N+1}, P_{N+2}, \dots$ on the vacuum $|0\rangle$ are the finite N null states. Since these modes vanish automatically at any integer N , the determinant operator $\det(z - X)$ is a polynomial of degree N in z .

While these null states yield the correct half-BPS spectrum $x^N \frac{x}{1-x}$ of a single D3 giant, we should not identify the set of trace relations directly with the states of bulk D-branes. The reason is that one needs the states to possess an extra grading in order to explain the overall signs $(-1)^k$ that appear in the BPS spectrum on k coincident D-branes, and no such grading distinguishes the finite N null states from the physical states.

Instead, we identify the BPS D-brane states with auxiliary “heavy” ghosts χ , with dimension and R-charge starting at $N + 1$, in a boundary $U(\infty)$ gauge theory that compensate for null states due to trace relations at a finite value of N . The effect of introducing

these anti-commuting ghosts along with a homological charge $\widehat{Q}$ acting as

$$\begin{aligned} [\chi_{-N-a}, \widehat{Q}]_{\pm} &= P_{N+a} \\ [P_a, \widehat{Q}]_{\pm} &= 0, \end{aligned} \tag{3.2.52}$$

where $a = 1, 2, \dots$, is to reduce the $U(\infty)$ theory to a $U(N)$ gauge theory. We assign the “heavy” ghost number 1 to χ_a and -1 to the differential $\widehat{Q} = [\cdot, \widehat{Q}]_{\pm}$. The heavy ghosts should be distinguished from the BRST (anti-)ghosts that one introduces for gauge-fixing and to impose on-shell constraints in a $U(N)$ gauge theory. Rather, the role of the ghost and the differential is to supplement the $U(\infty)$ theory with directions for systematically removing all but the $U(N)$ degrees of freedom.

It is simple to compute the spectrum with ghost number k in the half-BPS sector, because this sector does not have ghosts-for-ghosts. The states with ghost number k are therefore multi-ghost states built out of χ_{-N-a} . We can write a basis of k multi-ghost states as

$$\chi_{-N-a_1} \chi_{-N-a_2} \cdots \chi_{-N-a_k} |0\rangle \tag{3.2.53}$$

where $1 \leq a_1 < a_2 < \cdots < a_k < \infty$, which yields the charge spectrum

$$\mu^k x^{kN} \frac{x^{k(k+1)/2}}{\prod_{n=1}^k (1 - x^n)} \tag{3.2.54}$$

that agrees with our bulk computation (3.2.45). Fugacity μ in the gauge theory now keeps track of the heavy ghost number.

Our findings suggest that the form degree k for the states of k D-branes in the bulk should be identified with the ghost number assigned to auxiliary ghosts that compensate for trace relations in the boundary gauge theory (as opposed to the fermion number F in the gauge theory that is usually defined in terms of other quantum numbers). This is the extra grading in the bulk that is not seen in the physical Hilbert space of a $U(N)$ gauge theory if one works strictly at any positive integer value of N .

3.3 Bulk Space of States as a Resolution of Finite N Modules

Thus far, we found the BPS states of coincident bulk D-branes by quantizing its open string modes. These states possess an additional grading, associated with the brane number k

labelling the open string vacua, whose bulk origin is the form degree of states and whose boundary origin is the ghost number assigned to “heavy” ghosts that compensate for finite N null states.

We have not yet discussed how the space of states built on different D-brane backgrounds are related to each other. That is, we should ask whether there is a natural way to organize the relationships among the perturbative Hilbert spaces built on various D-brane backgrounds.

In this section and the next, we comment on global aspects of the bulk space of states at vanishing 't Hooft coupling λ and finite N , by employing the holographic map between D-brane states and auxiliary ghosts. Our proposal is that a complex formed from the spaces of states of certain bulk D-branes in the $\alpha' \rightarrow \infty$ limit furnishes a BRST-like resolution of the finite N Hilbert space of the boundary gauge theory at $\lambda = 0$. Objects in the complex are labelled by ghost number k . The giant graviton expansion can be derived by computing the Hilbert series of the resolution.

As discussed in Section 3.2.3, taking the small radius limit $L \rightarrow 0$ of D-brane states is well-defined if they are the ground states of a supersymmetric hamiltonian of the worldline quantum mechanics on $\mathbb{R}$. Such states localize to fixed points of bulk isometries at large curvatures because the inverse radius L^{-1} assumes the role of a localization parameter. The consideration of D-branes at finite N is, of course, an extrapolation drawn from our computations at large N . That the extrapolation from large to finite N works in the giant graviton expansion is rather miraculous, and we discuss its implications in Section 3.3.6.

While the bulk computations of Section 3.2 relied on supersymmetry and the presence of a weakly-curved gravity limit, there are reasons to think that the identification between D-brane states and auxiliary heavy ghosts is quite general and holds independently of those properties. Trace relations between gauge-invariant operators are present in any finite N gauge theory with adjoint fields, and such a theory can be rewritten as a $N = \infty$ gauge theory supplemented with an auxiliary ghost for every trace relation in the original theory.⁷

3.3.1 Half-BPS Space of States in the Bulk

Here, we motivate our main proposal regarding the bulk Hilbert space by gathering the lessons of our bulk computations in the half-BPS sector. The half-BPS sector is rather special because the set of trace relations do not satisfy further non-trivial relations. Therefore, one does not need to introduce ghosts-for-ghosts. We generalize these lessons subsequently.

⁷There can certainly be non-trivial relations between trace relations that necessitate ghosts-for-ghosts, and we discuss these cases thoroughly.

In Section 3.2, we found the states of D-branes responsible for the spectrum

$$\mu^k x^{kN} \frac{x^{k(k+1)/2}}{\prod_{m=1}^k (1-x^m)} \quad (3.3.1)$$

on k giant graviton branes. The giant graviton expansion suggests that, to retrieve the half-BPS spectrum of boundary $\mathcal{N} = 4$ SYM at finite N , we should simply sum up the open string spectra $\hat{Z}_k(x)$ of k -brane sectors for all k , each in the presence of half-BPS closed strings $Z_\infty(x)$ of the $\text{AdS}_5 \times S^5$ background:

$$\frac{1}{\prod_{n=1}^N (1-x^n)} = \frac{1}{\prod_{n=1}^\infty (1-x^n)} \sum_{k=0}^\infty (-1)^k x^{kN} \frac{x^{k(k+1)/2}}{\prod_{m=1}^k (1-x^m)}. \quad (3.3.2)$$

We also set the fugacity μ for the form degree grading to $\mu = -1$.

On the other hand, we identified its holographic dual as heavy ghosts χ that compensate for the presence of finite N trace relations. We can study the half-BPS sector at finite N by equipping the $U(\infty)$ gauge theory with ghosts χ and a homological differential $\hat{\mathcal{Q}} = [\cdot, \hat{\mathcal{Q}}]_\pm$ acting on operators via the graded Leibniz rule

$$\begin{aligned} \hat{\mathcal{Q}}(\chi_{-N-a_1} \cdots \chi_{-N-a_k}) &= \sum_{i=1}^k (-1)^{i+1} (\chi_{-N-a_1} \cdots \hat{\mathcal{Q}}(\chi_{-N-a_i}) \cdots \chi_{-N-a_k}) \\ &= \sum_{i=1}^k (-1)^{i+1} P_{N+a_i} (\chi_{-N-a_1} \cdots \chi_{-N-a_{i-1}} \chi_{-N-a_{i+1}} \cdots \chi_{-N-a_k}) \end{aligned} \quad (3.3.3)$$

and $\hat{\mathcal{Q}}^2 = 0$. In particular, the ghosts χ map to trace relations: $\hat{\mathcal{Q}}(\chi_{-N-a}) = P_{N+a}$. Then the space of multi-ghost states, with coefficients in the ring of polynomials of traces

$$R = \mathbb{C}[\text{Tr } X, \text{Tr } X^2, \text{Tr } X^3, \dots], \quad (3.3.4)$$

is given by the complex

$$\cdots \rightarrow \mathcal{V}_3 \xrightarrow{\hat{\mathcal{Q}}} \mathcal{V}_2 \xrightarrow{\hat{\mathcal{Q}}} \mathcal{V}_1 \xrightarrow{\hat{\mathcal{Q}}} \mathcal{H}_\infty \rightarrow 0, \quad (3.3.5)$$

where $\mathcal{V}_k$ is the space of multi-ghost states of heavy ghost number k with coefficients in the ring R . Also, $\mathcal{V}_0 \equiv \mathcal{H}_\infty = R|0\rangle$ is the half-BPS Hilbert space of the gauge theory at infinite N . The ghost degree k of the complex labels the open string vacuum associated to k coincident D3 branes in the bulk.

The complex (3.3.5) has non-vanishing homology only at zero ghost number. Since the image $\widehat{Q}\mathcal{V}_1 \subset \mathcal{H}_\infty$ is the space of trace relations with coefficients in R , the homology at ghost number $k = 0$ is the Hilbert space

$$\mathcal{H}_N = \mathcal{H}_\infty / \widehat{Q}\mathcal{V}_1. \quad (3.3.6)$$

in the half-BPS sector of $U(N)$ $\mathcal{N} = 4$ SYM at finite N .

Let us observe that each $\mathcal{V}_k$, $k = 0, 1, 2, \dots$ in the complex (3.3.5) is freely generated by the multi-ghost generators

$$\chi_{-N-a_1} \chi_{-N-a_2} \cdots \chi_{-N-a_k} |0\rangle \quad (3.3.7)$$

with coefficients in R . In other words, $\mathcal{V}_k$ are free R -modules. This means that, if we augment (3.3.5) with $\mathcal{H}_N$ at the final step, the resulting exact sequence

$$\cdots \rightarrow \mathcal{V}_3 \xrightarrow{\widehat{Q}} \mathcal{V}_2 \xrightarrow{\widehat{Q}} \mathcal{V}_1 \xrightarrow{\widehat{Q}} \mathcal{H}_\infty \rightarrow \mathcal{H}_N \rightarrow 0 \quad (3.3.8)$$

is a free resolution of the half-BPS Hilbert space $\mathcal{H}_N$ at finite N . We will explain these notions in Section 3.3.2. For now, it suffices to realize that the resolution (3.3.8) gives a precise relationship between (1) the half-BPS Hilbert space $\mathcal{H}_N$ of $U(N)$ $\mathcal{N} = 4$ SYM at finite N and (2) the complex (3.3.5) identified as the “bulk” half-BPS Hilbert space of IIB string theory on $\text{AdS}_5 \times S^5$ via the holographic map between D-brane states and auxiliary ghosts. The resolution says that the open strings on bulk D-branes are encoded in the *relations* between the single-trace generators that arise in the boundary gauge theory at finite N .

It is simple to use this observation to derive the giant graviton expansion (3.3.2) in the half-BPS sector. Since an alternating sum of the Hilbert series (graded by the R-charge) of modules in an exact sequence must vanish, we can express the charge spectrum of $\mathcal{H}_N$ in terms of an alternating sum of the charge spectra of $\mathcal{V}_k$ over $k = 0, 1, 2, \dots$:

$$\text{Tr}_{\mathcal{H}_N} x^R = \sum_{k=0}^{\infty} (-1)^k \text{Tr}_{\mathcal{V}_k} x^R. \quad (3.3.9)$$

This yields the half-BPS giant graviton expansion (3.3.2). The closed string spectrum $Z_\infty = \prod_{n=1}^{\infty} \frac{1}{1-x^n}$ will factor out, because $\mathcal{V}_k$ are free R -modules and Z_∞ is the Hilbert series of R . This derivation shows that the resolution (3.3.8) “categorifies” the giant graviton expansion in the half-BPS sector.

What is the physical interpretation of the differential $\widehat{Q}$? It maps a half-BPS state of k coincident D3 giants to linear combinations of states of $k - 1$ coincident D3 giants with

coefficients in the traces (i.e. closed strings). Thus a natural interpretation for $\hat{Q}$ is that it is an instanton that interpolates between a pair of open string vacua with a ghost number difference of 1. An example with $k = 1$ would be a Euclidean D3 instanton on $S^3 \subset S^5$ described by the trajectory $\cos \theta(\tau) = e^{\tau - \tau_0}$ for $-\infty < \tau \leq \tau_0$, where $0 \leq \theta \leq \frac{\pi}{2}$ is a polar angle on S^5 that parametrizes the size of the wrapped S^3 . It would be important to study the radiation due to such processes.

Our analysis here also clarifies why it is appropriate to describe half-BPS giant gravitons in terms of fermionic droplets, as is often done in the literature [53, 76, 77]. We can collect the auxiliary ghosts into a single “field”

$$\chi(z) = \sum_{n \in \mathbb{Z}} \frac{\chi_n}{z^{n+N+1}} \quad (3.3.10)$$

on the z -plane of ghost number 1. This is an anti-commuting field in the local transverse z -plane of S^3 in S^5 whose excitations are the states of D3 giants. It would be interesting to study the effective dynamics of giant gravitons along these lines.

The lessons from the half-BPS sector are the following: (1) we can identify the bulk space of states with the complex of auxiliary ghosts that compensate for finite N trace relations, and (2) this complex furnishes a BRST-like resolution of the finite N Hilbert space $\mathcal{H}_N$ of the boundary gauge theory. These lessons are quite general when taken as statements about string theories dual to weakly-coupled gauge theories. Namely, they do not depend explicitly on supersymmetry or the presence of a weakly-curved gravity regime. That the bulk Hilbert space is a resolution of the gauge theory counterpart only requires the latter to be a theory of gauged $N \times N$ matrices.

3.3.2 Finite N Modules and Koszul-Tate Resolutions

In following sections, we develop the mathematical background required to generalize the lessons from the half-BPS sector. Some useful references are [78–81]. We present the background in the context that is relevant for this work.

Consider a free four-dimensional $U(N)$ gauge theory on $\mathbb{R} \times S^3$ with fields only in the adjoint representation of the gauge group. We choose this manifold to avoid having topological sectors, but similar considerations apply to other manifolds as long as one works within a superselection sector for local operators.

In the limit $N \rightarrow \infty$, the space of states of the gauge theory is the “vacuum” free module $\mathcal{H}_\infty = R|0\rangle$, given by the ring of gauge-invariant polynomials of fields and their

derivatives

$$R = \mathbb{C}[\text{Tr } X, \text{Tr } XY, \text{Tr } \psi F \dots, \text{Tr } \partial \dots \partial \dots X, \dots]. \quad (3.3.11)$$

acting on the vacuum $|0\rangle$. As an R -module, $\mathcal{H}_\infty$ consists only of the identity generator acting on $|0\rangle$ with coefficients in R . $\mathcal{H}_\infty$ is graded with respect to the quantum numbers of the single trace generators of R . Following lessons from the half-BPS sector, we work over the “large N ” ring R generated by all single traces before taking into account the trace relations. In particular, R is independent of N .

To find the space of states of a free gauge theory at finite N , one takes the quotient

$$\mathcal{H}_N = \mathcal{H}_\infty / \mathcal{I}_N \quad (3.3.12)$$

of $\mathcal{H}_\infty$ by the submodule $\mathcal{I}_N = I_N|0\rangle$, where the ideal over R

$$I_N = \langle \text{trace relations at } N \rangle_R \quad (3.3.13)$$

is generated by the set of trace relations that are present at a given value of N . For a single matrix X , these relations are given by the set $P_{N+1}, P_{N+2}, \dots$ as provided explicitly in Section 3.2.6. We refer to the quotient module $\mathcal{H}_N$ over R as the finite N module. What makes the study of $\mathcal{H}_N$ difficult is that $\mathcal{H}_N$ is in general not freely generated, i.e. it is not a Fock space built from single trace generators. This means that in general we can only describe $\mathcal{H}_N$ at best in terms of a set of generators that have relations among themselves.

The half-BPS sector suggests that, in order to find the “bulk” space of states for the string dual on a background labelled by k wrapped D-branes, we should replace the quotient module $\mathcal{H}_N = \mathcal{H}_\infty / \mathcal{I}_N$ by an infinite sequence of free R -modules $\mathcal{V}_k^N$ that resolves $\mathcal{H}_N$. The resulting complex is a free resolution of $\mathcal{H}_N$, also known as the Koszul-Tate resolution.⁸ It is important that the resulting complex is a resolution of $\mathcal{H}_N$, i.e. that the truncated complex as in (3.3.5) has homology only at ghost number 0. Since the extra ghost degree is not present in $\mathcal{H}_N$ of the boundary gauge theory, the presence of a non-vanishing homology at $k \neq 0$ would be at odds with holography.

A *Koszul-Tate resolution* $\mathcal{K}$ of $\mathcal{H}_N$ is an exact sequence of free R -modules $\mathcal{V}_k^N$ equipped with a nilpotent linear differential $\hat{Q}$

$$\dots \rightarrow \mathcal{V}_3^N \xrightarrow{\hat{Q}} \mathcal{V}_2^N \xrightarrow{\hat{Q}} \mathcal{V}_1^N \xrightarrow{\hat{Q}} \mathcal{H}_\infty \rightarrow \mathcal{H}_N \rightarrow 0 \quad (3.3.14)$$

⁸We emphasize that this Koszul-Tate resolution, which implements trace relations as constraints on the large N space of states, is distinct from the complex appearing in the BV-BRST quantization of gauge theories, where the resolution is used to implement the reduction to the on-shell surface. Therefore, the “heavy” ghosts that appear in this context are not the anti-ghosts of the BV-BRST formalism, although the mathematical tools are equivalent.

where $\widehat{Q}\mathcal{V}_1^N = \mathcal{I}_N$, i.e. the image $\widehat{Q}\mathcal{V}_1^N$ is the submodule generated by the trace relations at a given value of N . In addition to the grading by the quantum numbers induced from the generators of the ring R , the complex itself has a “heavy” ghost degree gh under which

$$\text{gh}(\mathcal{V}_k^N) = k, \quad \text{gh}(\widehat{Q}) = -1. \quad (3.3.15)$$

The generators of $\mathcal{V}_k^N$ have quantum numbers that depend on N , because the set of trace relations depends on the value of N . The elements of $\mathcal{V}_k^N$ are termed heavy because the energies/charges of their generators start at order $\sim kN$.

Intuitively, the resolution $\mathcal{K}$ “resolves” the highly-constrained finite N module $\mathcal{H}_N$ by replacing it with an infinite sequence of Fock spaces $\mathcal{V}_k^N$ (starting at $\mathcal{V}_0 \equiv \mathcal{H}_\infty$) that better and better approximate $\mathcal{H}_N$. Since $\mathcal{H}_\infty$ maps to $\mathcal{H}_N$, the zeroth approximation to the finite N module $\mathcal{H}_N$ is the large N module $\mathcal{H}_\infty$. The subsequent free R -modules $\mathcal{V}_k^N$ can be understood as the space of relations among the generators of $\mathcal{V}_{k-1}^N$. For example, $\mathcal{V}_1^N$ is the space of relations among the large N Fock space generators⁹ of $\mathcal{H}_\infty$, i.e. the generators of $\mathcal{V}_1$ map to trace relations under $\widehat{Q}$. Then $\mathcal{V}_2^N$ is the space of relations among trace relations, and so on. Therefore, $\mathcal{V}_k^N$ is the space of k -th order relations among the single-trace generators of the large N ring R .

The resolution $\mathcal{K}$ of $\mathcal{H}_N$ consisting of free R -modules $\mathcal{V}_k^N$ can always be constructed. For a given set of (commuting) quantum numbers $\{\mathcal{C}_i\}$, the resolution $\mathcal{K}$ truncates at finite length because there exists a value of k beyond which all generators of $\mathcal{V}_{k'}^N$ with $k' > k$ have quantum numbers that exceed $\{\mathcal{C}_i\}$. It is important that $\mathcal{K}$ has a notion of uniqueness despite having vanishing homology. We discuss its uniqueness properties in Section 3.3.4.

In physical terms, the Koszul-Tate resolution $\mathcal{K}$ can be understood as the procedure of introducing auxiliary ghosts in a $U(\infty)$ theory that compensate for null states that would have arisen due to trace relations at a given value of N . There may be non-trivial relations between trace relations that require higher ghosts to compensate for them. Doing so in iteration reduces the space of states of a $U(\infty)$ gauge theory to that of a $U(N)$ gauge theory.

Explicitly, let $P_A \in R$ be a trace relation at N , where the label A ranges over the (infinite) set of trace relations. For every P_A , we introduce a heavy ghost χ_A of $\text{gh}(\chi_A) = 1$ with opposite grassmann parity to that of P_A . The nilpotent differential $\widehat{Q} = [\cdot, \widehat{Q}]_\pm$ on operators acts as

$$\widehat{Q}\chi_A = P_A, \quad (3.3.16)$$

⁹“Generators” here refer to the generators of the ring R , rather than the identity generator of $\mathcal{H}_\infty$ as an R -module. But when referring to the generators of $\mathcal{V}_k^N$ for $k \geq 1$, we mean the generators of the R -module.

which makes P_A $\widehat{\mathcal{Q}}$ -exact.

The set of trace relations P_A may not be linearly-independent over the ring R , i.e. P_A may satisfy

$$r_1 P_{A_1} + r_2 P_{A_2} + \cdots + r_s P_{A_s} = 0 \quad (3.3.17)$$

where r_i are non-zero elements of R . (3.3.17) is a relation between trace relations that needs to be compensated via ghosts. Actually, there always exist relations between trace relations of the form

$$P_{A_2} P_{A_1} - P_{A_1} P_{A_2} = 0 \quad (3.3.18)$$

where we set $r_1 = P_{A_2}$, $r_2 = -P_{A_1}$ and assumed P_A are grassmann even. This is a “trivial” relation because it simply states that the trace relations P_{A_1}, P_{A_2} commute. Trivial relations that result from the (anti-)commutation properties of trace relations are automatically accounted for by taking $\widehat{\mathcal{Q}}$ to act on products of ghosts via the graded Leibniz rule:

$$\begin{aligned} \widehat{\mathcal{Q}}(\chi_{A_1} \chi_{A_2}) &= P_{A_1} \chi_{A_2} + (-1)^{\varepsilon_1} P_{A_2} \chi_{A_1} \\ \widehat{\mathcal{Q}}(\chi_{A_1} \chi_{A_2} \chi_{A_3}) &= P_{A_1} \chi_{A_2} \chi_{A_3} + (-1)^{\varepsilon_1} P_{A_2} \chi_{A_1} \chi_{A_3} + (-1)^{\varepsilon_1 + \varepsilon_2} P_{A_3} \chi_{A_1} \chi_{A_2} \end{aligned} \quad (3.3.19)$$

and so on, where $\varepsilon_i = 0$ if χ_{A_i} is grassmann even and $\varepsilon_i = 1$ if χ_{A_i} is grassmann odd. Thus, the (anti-)commutation properties are implemented by the statement that the expression on the RHS of (3.3.19) are $\widehat{\mathcal{Q}}$ -exact. If there are only trivial relations, it is sufficient to consider all multi-ghost generators

$$\chi_{A_1} \chi_{A_2} \cdots \chi_{A_k} |0\rangle \in \mathcal{V}_k^N \quad (3.3.20)$$

with coefficients in R , and the procedure of finding the Koszul-Tate resolution $\mathcal{K}$ terminates.

However, there may exist non-trivial relations (3.3.17) between trace relations P_A that do not just amount to the (anti-)commutation properties of P_A . In this case, we must introduce ghosts-for-ghosts $\chi^{(2)}$ of $\text{hgh}(\chi^{(2)}) = 2$ satisfying

$$\widehat{\mathcal{Q}}(\chi_B^{(2)}) = r_1 \chi_{A_1}^{(1)} + r_2 \chi_{A_2}^{(1)} + \cdots + r_s \chi_{A_s}^{(1)}, \quad (3.3.21)$$

where $\chi_{A_i}^{(1)} \equiv \chi_{A_i}$. This implements non-trivial relations between trace relations, and the subscript B here labels the set of such relations. If there are no further non-trivial relations (again with coefficients in R) between the various relations (3.3.17), the procedure of finding the Koszul-Tate resolution $\mathcal{K}$ terminates. If higher relations do exist, the process iterates until one has accounted for all non-trivial higher relations by making them $\widehat{\mathcal{Q}}$ -exact with

respect to $\chi^{(h)}$ of $\text{hgh}(\chi^{(h)}) = h$.¹⁰ Given a set of quantum numbers $\{\mathcal{C}_i\}$, there are only finitely many generators and relations with charges less than or equal to $\{\mathcal{C}_i\}$.

In general, the differential $\widehat{\mathcal{Q}}$ on operators acts as

$$\widehat{\mathcal{Q}}\left(\chi_{A_1}^{(h_1)}\chi_{A_2}^{(h_2)}\cdots\chi_{A_n}^{(h_n)}\right) = \sum_{i=1}^n (-1)^{\varepsilon_1+\varepsilon_2+\cdots+\varepsilon_{i-1}} \chi_{A_1}^{(h_1)}\chi_{A_2}^{(h_2)}\cdots\widehat{\mathcal{Q}}(\chi_{A_i}^{(h_i)})\cdots\chi_{A_n}^{(h_n)} \quad (3.3.22)$$

where $\varepsilon_i = 0$ if $\chi_{A_i}^{(h_i)}$ is grassmann even and $\varepsilon_i = 1$ if $\chi_{A_i}^{(h_i)}$ is grassmann odd. The Koszul-Tate resolution is a complex built from multi-particling all such ghosts $\chi_{A_i}^{(h_i)}$ with coefficients in the large N ring R . States of total ghost number k

$$\chi_{A_1}^{(h_1)}\chi_{A_2}^{(h_2)}\cdots\chi_{A_n}^{(h_n)}|0\rangle \in \mathcal{V}_k^N \quad (3.3.23)$$

generate the free R -module $\mathcal{V}_k^N$ where $k = \sum_{i=1}^n h_i$.

Once a resolution $\mathcal{K}$ is constructed, we can consider the truncated complex $\mathcal{V}^N$:

$$\cdots \rightarrow \mathcal{V}_3^N \xrightarrow{\widehat{\mathcal{Q}}} \mathcal{V}_2^N \xrightarrow{\widehat{\mathcal{Q}}} \mathcal{V}_1^N \xrightarrow{\widehat{\mathcal{Q}}} \mathcal{H}_\infty \rightarrow 0 \quad (3.3.24)$$

whose homology is concentrated at heavy ghost number 0:

$$\left(\frac{\text{Ker } \widehat{\mathcal{Q}}}{\text{Im } \widehat{\mathcal{Q}}}\right)_{\text{hgh}=0} = \mathcal{H}_N. \quad (3.3.25)$$

So the complex $\mathcal{V}^N$ of free R -modules encodes the data of the finite N module $\mathcal{H}_N$ via its homology. We also refer to the complex $\mathcal{V}^N$ as a resolution of $\mathcal{H}_N$.

We find evidence in Section 3.4 that higher ghosts are required even in rather simple settings, such as the free $\frac{1}{4}$ -BPS sector and the $\frac{1}{4}$ -BPS chiral ring of $\mathcal{N} = 4$ SYM. It would be important to understand the criteria for when they arise and, in situations where they exist, the holographic description of higher ghosts.

3.3.3 Thermal Partition Function and Correlators

Given a Koszul-Tate resolution $\mathcal{K}$, the Lefschetz trace formula [78] tells us how to relate physical quantities computed over $\mathcal{H}_N$ and over its resolution $\mathcal{V}^N$:

$$\text{Tr}_{\mathcal{H}_N} \mathcal{O} = \sum_{k=0}^{\infty} (-1)^k \text{Tr}_{\mathcal{V}_k^N} \mathcal{O} \quad (3.3.26)$$

¹⁰For example, if there exists a relation between relations between trace relations, then the compensating ghost $\chi^{(3)}$ maps to a linear combination of $\chi_B^{(2)}$ and $\chi_{A_1}^{(1)}\chi_{A_2}^{(1)}$ with coefficients in R .

where $\mathcal{O}$ is an operator of heavy ghost number 0 in the $U(N)$ gauge theory that we also take to act on $\mathcal{V}_k^N$. To compute the trace, we need the concept of a dual module $(\mathcal{V}_k^N)^*$, whose elements are related to those of $\mathcal{V}_k^N$ by the hermitian adjoint and carry opposite heavy ghost number hgh and quantum numbers $\{\mathcal{C}_i\}$ to those of $\mathcal{V}_k^N$. We work in an orthonormal basis for $(\mathcal{V}_k^N)^*$ and $\mathcal{V}_k^N$ of definite energy and charge eigenvalues.

The Lefschetz trace formula will provide the holographic map in our proposal of Section 3.3.5, in the regime of $\lambda = 0$ and finite N . Bulk observables are computed by taking the alternating sum of the expectation values in an ensemble of states built on each open string vacuum. For instance, the free thermal partition function of a $U(N)$ gauge theory on $S^1 \times S^3$ is equal to the alternating sum of the free thermal partition functions over $\mathcal{V}_k^N$ of the resolution $\mathcal{V}^N$:

$$\text{Tr}_{\mathcal{H}_N} e^{-\beta H} = \sum_{k=0}^{\infty} (-1)^k \text{Tr}_{\mathcal{V}_k^N} e^{-\beta H}. \quad (3.3.27)$$

The thermal correlation functions are related as

$$\begin{aligned} & \text{Tr}_{\mathcal{H}_N} [e^{-\beta H} \mathcal{O}_1(\tau_1, x_1) \mathcal{O}_2(\tau_2, x_2) \cdots \mathcal{O}_n(\tau_n, x_n)] \\ &= \sum_{k=0}^{\infty} (-1)^k \text{Tr}_{\mathcal{V}_k^N} [e^{-\beta H} \mathcal{O}_1(\tau_1, x_1) \mathcal{O}_2(\tau_2, x_2) \cdots \mathcal{O}_n(\tau_n, x_n)] \end{aligned} \quad (3.3.28)$$

where $0 \leq \tau_n < \cdots < \tau_2 < \tau_1 < \beta$. The hamiltonian on the right-hand side is defined in the $U(\infty)$ theory supplemented with ghosts for trace relations at a value of N .

For superconformal gauge theories with non-anomalous R-symmetry, we can write in addition the relation between the superconformal indices

$$\text{Tr}_{\mathcal{H}_N} [(-1)^F e^{-\beta \Delta} q_1^{\mathcal{C}_1} q_2^{\mathcal{C}_2} \cdots] = \sum_{k=0}^{\infty} (-1)^k \text{Tr}_{\mathcal{V}_k^N} [(-1)^F e^{-\beta \Delta} q_1^{\mathcal{C}_1} q_2^{\mathcal{C}_2} \cdots] \quad (3.3.29)$$

where $\Delta = \frac{1}{2}\{\mathcal{Q}, \mathcal{Q}^\dagger\} = 0$ is the BPS condition and $\mathcal{C}_i$ are conserved charges dual to fugacities q_i that commute with the supercharge $\mathcal{Q}$. In Section 3.3.6, we describe the conditions under which the dependence of the right-hand side on N simplifies and (3.3.29) may be expressed as

$$Z_N(x, y_i) = Z_\infty(x, y_i) \sum_{k=0}^{\infty} x^{kN} \hat{Z}_k(x, y_i), \quad (3.3.30)$$

which is the giant graviton expansion proposed in [1]. Various minus signs from both $(-1)^k$ and $(-1)^F$ are hidden in the definition of $\hat{Z}_k(x, y_i)$. If the bulk has a supergravity regime, the indices on the right of (3.3.30) can also be computed directly in the bulk by counting the fluctuation modes of probe D-branes [32, 54].

3.3.4 (Non-)Uniqueness and Instantons

Here we comment on the uniqueness properties of a Koszul-Tate resolution $\mathcal{K}$ of $\mathcal{H}_N$. First, we discuss why a Koszul-Tate resolution $\mathcal{K}$ of $\mathcal{H}_N$ does not necessarily produce a unique set of free R -modules $\mathcal{V}_k^N$. Then we explain that there nonetheless exists a notion of uniqueness for $\mathcal{K}$, which we use to make physical statements about the bulk spectrum in Section 3.3.7.

It is not hard to understand why there exist infinitely many resolutions of $\mathcal{H}_N$ at any finite value of N . Given a resolution $\mathcal{K}$, one may always introduce an additional pair of spurious ghosts $\xi^{(k)}$ and $\xi^{(k-1)}$ on which the differential $\widehat{\mathcal{Q}}$ act as

$$\begin{aligned}\widehat{\mathcal{Q}}\xi^{(k)} &= \xi^{(k-1)} \\ \widehat{\mathcal{Q}}\xi^{(k-1)} &= 0.\end{aligned}\tag{3.3.31}$$

Contributions from such pairs cancel in the Lefschetz trace formula (3.3.26) due to the relative minus sign $(-1)^k$. So an arbitrary number of them can be introduced without affecting physical quantities computed by (3.3.26). However, doing so still changes the energy/charge spectra associated with individual pairs of free R -modules $\mathcal{V}_k^N$ and $\mathcal{V}_{k-1}^N$.

This situation is analogous to the effect, on the instanton complex in supersymmetric quantum mechanics, of deforming the Morse function h on the target manifold such that the number of critical points of h changes [13]. Under generic deformations, the number of non-degenerate critical points changes by two, and the created/annihilated pair of critical points have Morse indices that differ by 1. While the number of critical points of h with a given Morse index k may change by an arbitrary positive amount, the Euler characteristic remains invariant. The difference between the instanton complex and our resolution $\mathcal{V}^N$, however, is that the instanton complex may have non-trivial cohomology at any form degree while the homology of $\mathcal{V}^N$ is concentrated only at heavy ghost number 0.

Following the analogy with the instanton complex, our interpretation of the situation as follows: At any positive integer N , there is an inherent ambiguity in the spectrum associated to a string background labelled by a collection of k wrapped D-branes, due to the unphysical freedom to introduce spurious pairs of ghosts (3.3.31) between D-brane sectors $\mathcal{V}_k^N$ and $\mathcal{V}_{k-1}^N$ whose ghost numbers differ by 1. Thus, while the number of states with a given ghost degree k and quantum numbers $\{\mathcal{C}_i\}$ may change by an arbitrary positive amount, any physical quantity computed by the Lefschetz trace formula remains invariant.

It is still interesting to ask whether there are physical “invariants” of $\mathcal{K}$ that are non-trivial at each ghost number k . This is because in the bulk we can at best compute only on backgrounds with a fixed set of D-branes. One would like to be able to compare bulk

quantities on a background with k wrapped D-branes with the trace over $\mathcal{V}_k^N$ found via the Koszul-Tate resolution $\mathcal{K}$ of $\mathcal{H}_N$. It turns out that there is indeed a sense in which the modules $\mathcal{V}_k^N$ of $\mathcal{K}$ are uniquely determined:

A *minimal* Koszul-Tate resolution $\widehat{\mathcal{K}}$ (also known as a minimal graded free resolution) is that for which the number of generators of each free R -module $\mathcal{V}_k^N$ in $\mathcal{K}$ is minimal [79], i.e. a resolution $\widehat{\mathcal{K}}$ for which there are no spurious ghosts. Though the modules $\mathcal{V}_k^N$ possess an infinite number of generators, there are only a finite number of generators with a given set of quantum numbers $\{\mathcal{C}_i\}$. Therefore, the minimality of $\widehat{\mathcal{K}}$ is well-defined for any commuting set $\{\mathcal{C}_i\}$. A minimal Koszul-Tate resolution $\widehat{\mathcal{K}}$ of $\mathcal{H}_N$ is unique up to isomorphisms, and any other non-minimal resolution of $\mathcal{H}_N$ may be constructed by taking the direct sum of $\widehat{\mathcal{K}}$ with a trivial complex consisting of spurious ghosts.

3.3.5 Bulk Space of States at $\lambda = 0$

In Section 3.3.1, we provided strong evidence for the following statement in the half-BPS sectors of $U(N)$ $\mathcal{N} = 4$ SYM and IIB string theory on $\text{AdS}_5 \times S^5$:

$$\mathcal{V}_k = \mathcal{H}_\infty^{\text{closed}} \otimes \mathcal{H}_k^{\text{open}} \quad (\text{half-BPS}). \quad (3.3.32)$$

In words, (1) the free R -module $\mathcal{V}_k$ consisting of states of ghost degree k in a resolution (3.3.8) of the half-BPS Hilbert space $\mathcal{H}_N$ at $\lambda = 0$ is holographically dual to (2) the half-BPS open-closed Hilbert space $\mathcal{H}_\infty^{\text{closed}} \otimes \mathcal{H}_k^{\text{open}}$ of IIB strings on $\text{AdS}_5 \times S^5$ with k coincident D3 giant graviton branes, in the $\alpha' \rightarrow \infty$ limit. It is possible to make a statement such as (3.3.32), where there exists a sharp semiclassical bulk description, because of special properties of the half-BPS resolution that we clarify in Section 3.3.6.

In more general circumstances, $\mathcal{V}_k^N$ of the Koszul-Tate resolution $\mathcal{K}$ generalizes the bulk interpretation associated with $\mathcal{H}_\infty^{\text{closed}} \otimes \mathcal{H}_k^{\text{open}}$ in a few important ways. In the limit where N is large and all but one $U(1)$ charge is held fixed, the module $\mathcal{V}_k^N$ asymptotes to (in a sense that will be made precise) the space of string states on k coincident D-branes fixed under the $U(1)$ bulk isometry. This regime includes non-BPS excitations of the half-BPS sector. In other regimes, such as when two or more charges are treated on an equal footing, $\mathcal{V}_k^N$ ceases to have a “good” large N limit: the number of elements of $\mathcal{V}_k^N$ of a given total charge or energy (minus the total background charge kN) diverges as $N \rightarrow \infty$. The module $\mathcal{V}_k^N$ nonetheless remains well-defined at any finite value of N where no semiclassical bulk description may exist, modulo unphysical ambiguities described in Section 3.3.4. It is in this sense that $\mathcal{V}_k^N$ generalizes the notion of the “large curvature limit of the space of string states on a background with k wrapped D-branes”.

The Proposal Consider a four-dimensional $U(N)$ gauge theory on $\mathbb{R} \times S^3$ possessing only fields in the adjoint representation of the gauge group, at zero 't Hooft coupling $\lambda = 0$. Let R be the ring of polynomials of traces of fields and their derivatives (before accounting for trace relations). The space of states $\mathcal{H}_N$ of the gauge theory is the quotient of $\mathcal{H}_\infty = R|0\rangle$ by the submodule generated by the set of all trace relations at a given value of N .

Our proposal is that the complex $\mathcal{V}^N$:

$$\cdots \rightarrow \mathcal{V}_3^N \xrightarrow{\widehat{Q}} \mathcal{V}_2^N \xrightarrow{\widehat{Q}} \mathcal{V}_1^N \xrightarrow{\widehat{Q}} \mathcal{H}_\infty \rightarrow 0, \quad (3.3.33)$$

which furnishes a Koszul-Tate resolution $\mathcal{K}$ of $\mathcal{H}_N$ over the ring R , should be identified as the space of states of the string dual in the $\alpha' \rightarrow \infty$ limit. The homology of $\mathcal{V}^N$ is concentrated at heavy ghost number 0 and coincides with $\mathcal{H}_N = \mathcal{H}_\infty / \widehat{Q}\mathcal{V}_1^N$. The free R -module $\mathcal{V}_k^N$ is, in a sense that is elaborated, the $\alpha' \rightarrow \infty$ limit of the space of string states on a background with k wrapped D-branes. The observables of the $U(N)$ gauge theory are related to those of its string dual via the Lefschetz trace formula

$$\mathrm{Tr}_{\mathcal{H}_N} \mathcal{O} = \sum_{k=0}^{\infty} (-1)^k \mathrm{Tr}_{\mathcal{V}_k^N} \mathcal{O} \quad (3.3.34)$$

which provides the holographic map in the regime of $\lambda = 0$ and finite N . Bulk observables are computed by taking the alternating sum of the expectation values in an ensemble of states built on each open string vacuum.

There exist instanton-like ambiguities for the modules $\mathcal{V}_k^N$ at finite N that arise from the freedom to introduce spurious ghosts in the resolution without affecting physical quantities computed via (3.3.34). We can nonetheless place lower bounds on the spectrum of string states on D-brane backgrounds at any value of N , using the uniqueness of minimal resolutions. In the limit where N is large and all but one $U(1)$ charge is held fixed, we can determine exactly the string spectrum on certain D-branes in the large curvature limit from the counting of trace relations in a $U(N)$ gauge theory.

It is clear that various assumptions for the gauge theory may be relaxed to produce interesting outcomes. We comment on these possibilities in the discussion.

3.3.6 Semiclassics and Large N

Let us discuss the circumstances under which the module $\mathcal{V}_k^N$ acquires a sharp bulk description as the space of states of k coincident giant graviton-like branes. In this limit, the

string spectrum on D-brane backgrounds in the large curvature limit can be determined from the counting of trace relations in the $U(N)$ gauge theory.

We begin by stating some reasonable properties for this regime: For one, N must be large. Moreover, as we saw in Section 3.2, the D-branes that contribute states in the large curvature limit $L \rightarrow 0$ wrap the fixed point of a $U(1)$ bulk isometry. They also possess large charges $\sim kN$ under this $U(1)$ due to the induced potential. Therefore, $\mathcal{V}_k^N$ is well-described as the space of string states on k coincident D-branes at a $U(1)$ fixed-point when N is large and when all but one $U(1)$ charge are held fixed.

This limit corresponds to considering trace relations that contain an arbitrary number of a single adjoint field X and fixed numbers of other adjoint “letters” $W_1, W_2, \dots$. For instance assuming all letters are bosonic, the set of trace relations for $N = 2$ and containing two letters W_1, W_2 starts at

$$\begin{aligned} & \text{Tr } XW_1W_2 + \text{Tr } XW_2W_1 - \text{Tr } X \text{Tr } W_1W_2 \\ & - \text{Tr } W_1 \text{Tr } XW_2 - \text{Tr } W_2 \text{Tr } XW_1 + \text{Tr } X \text{Tr } W_1 \text{Tr } W_2 = 0. \end{aligned} \quad (3.3.35)$$

There is 1 relation with W_1, W_2 and one X . Next, we have

$$\begin{aligned} & 2 \text{Tr } X^2W_1W_2 - 2 \text{Tr } X \text{Tr } XW_1W_2 - \text{Tr } X^2 \text{Tr } W_1W_2 + (\text{Tr } X)^2 \text{Tr } W_1W_2 = 0 \\ & 2 \text{Tr } X^2W_2W_1 + 2 \text{Tr } X \text{Tr } XW_1W_2 - \text{Tr } X^2 \text{Tr } W_1W_2 - (\text{Tr } X)^2 \text{Tr } W_1W_2 \\ & - 2 \text{Tr } X \text{Tr } W_1 \text{Tr } XW_2 - 2 \text{Tr } X \text{Tr } W_2 \text{Tr } XW_1 + 2 (\text{Tr } X)^2 \text{Tr } W_1 \text{Tr } W_2 = 0 \\ & 2 \text{Tr } XW_1XW_2 + \text{Tr } X^2 \text{Tr } W_1W_2 - (\text{Tr } X)^2 \text{Tr } W_1W_2 \\ & - 2 \text{Tr } XW_1 \text{Tr } XW_2 - \text{Tr } X^2 \text{Tr } W_1 \text{Tr } W_2 + (\text{Tr } X)^2 \text{Tr } W_1 \text{Tr } W_2 = 0. \end{aligned} \quad (3.3.36)$$

There are 3 independent relations with W_1, W_2 and two X s, and so on with higher insertions of X s.

What do we mean by the large N limit of the space of trace relations? Let us define

$$Z_{\mathcal{V}_k^N} = \text{Tr}_{\mathcal{V}_k^N} [x^{\mathcal{C}_X} w_1^{\mathcal{C}_1} w_2^{\mathcal{C}_2} \dots] \quad (3.3.37)$$

where $\mathcal{C}_i$ are charges of the letters W_i .¹¹ Since $\mathcal{V}_k^N$ are free R -modules, we can divide $Z_{\mathcal{V}_k^N}$ by the large N partition function

$$Z_\infty = \text{Tr}_{\mathcal{H}_\infty} [x^{\mathcal{C}_X} w_1^{\mathcal{C}_1} w_2^{\mathcal{C}_2} \dots] \quad (3.3.38)$$

¹¹The letters W_i may also carry $\mathcal{C}_X$ charges. We can specialize w_i and $\mathcal{C}_i$ later as long as there are a finite number of W_i with fixed charges.

to find the quantum numbers of the generators of $\mathcal{V}_k^N$. Since the module $\mathcal{V}_1^N$ of a minimal Koszul-Tate resolution is isomorphic to the space of trace relations with coefficients in R , we are interested in the large N behavior of the generators of $\mathcal{V}_1^N$ that map to trace relations containing the letters W_1, W_2 . As N is increased, the number of such generators of the free R -module $\mathcal{V}_1^N$ stabilizes up to an overall shift in $\mathcal{C}_X$ by N units:

$$\left. \frac{Z_{\mathcal{V}_1^{N \rightarrow \infty}}}{Z_\infty} \right|_{w_1 w_2} = x^{N-1} + 3x^N + 5x^{N+1} + 7x^{N+2} + 9x^{N+3} + 11x^{N+4} + 13x^{N+5} + \dots \quad (3.3.39)$$

where $|_{w_1^{c_1} w_2^{c_2} \dots}$ denotes that we take the coefficient of $w_1^{c_1} w_2^{c_2} \dots$. The first two coefficients of the power series in x agree with the counting of independent trace relations at $N = 2$ in the preceding paragraph, but the higher coefficients do not agree:

$$\left. \frac{Z_{\mathcal{V}_1^{N=2}}}{Z_\infty} \right|_{w_1 w_2} = x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 + \dots \quad (3.3.40)$$

Our claim is that these series coefficients converge to those of (3.3.39) as one considers modules $\mathcal{V}_1^N$ in a sequence of minimal Koszul-Tate resolutions of increasing N . As another example, we write the large N limit of the number of independent trace relations that contain letters W_1, W_2, W_3 :

$$\left. \frac{Z_{\mathcal{V}_1^{N \rightarrow \infty}}}{Z_\infty} \right|_{w_1 w_2 w_3} = x^{N-2} + 7x^{N-1} + 19x^N + 37x^{N+1} + 61x^{N+2} + 91x^{N+3} + \dots \quad (3.3.41)$$

The set of trace relations possesses degeneracies in the $U(1)$ charge that stabilizes at large N , with other charges $\mathcal{C}_i$ fixed. As such, we observe that

$$\lim_{N \rightarrow \infty} \mathcal{V}_1^N \Big|_{\mathcal{C}_i} = \mathcal{G}_1 \Big|_{\mathcal{C}_i}, \quad (3.3.42)$$

namely that the space $\mathcal{V}_1^N$ of (ghosts for) trace relations converges in the large N limit to another free R -module $\mathcal{G}_1$ that depends trivially on N . We conjecture that generators of $\mathcal{G}_1$ are holographically dual to states of a single D-brane on the $U(1)$ fixed locus in the $\alpha' \rightarrow \infty$ limit of the string dual.

This observation is actually just the first step at $k = 1$ of the giant graviton expansion [1]

$$Z_N(x, w_i) = Z_\infty(x, w_i) \sum_{k=0}^{\infty} x^{kN} \hat{Z}_k(x, w_i), \quad (3.3.43)$$

for superconformal indices when all but one $U(1)$ charge are held fixed. But now we have an understanding of the $k = 1$ term of (3.3.43) at the level of trace relations and resolution thereof:

$$x^N Z_\infty(x, w_i) \hat{Z}_1(x, w_i) = -\text{Tr}_{\mathcal{G}_1} [(-1)^F x^{\mathcal{C}_X} w_1^{\mathcal{C}_1} w_2^{\mathcal{C}_2} \dots] . \quad (3.3.44)$$

The minus sign on the right side compensates for an overall minus sign that arises from the definition of $\hat{Z}_1$ due to an analytic continuation of the fugacity x . The identification (3.3.44) holds when the letters W_i preserve supersymmetries used to define the index. Note however that our definition (3.3.42) of $\mathcal{G}_1$ based on trace relations does not require that W_i are BPS letters.

An important property of the formula (3.3.43) is that the right hand side depends on N only through the prefactor x^{kN} , i.e. $\hat{Z}_k(x, w_i)$ are independent of N . This suggests that, if we add the number of generators of $\mathcal{G}_1$ to

$$\frac{1}{Z_\infty} \text{Tr}_{\mathcal{H}_N} [x^{\mathcal{C}_X} w_1^{\mathcal{C}_1} w_2^{\mathcal{C}_2} \dots] - 1, \quad (3.3.45)$$

the remaining series starting at $O(x^{2N})$ also stabilizes in the large N limit. We take the remaining series to count (up to overall sign) the number of generators of another free R -module $\mathcal{G}_2$. For example, the generators of $\mathcal{G}_2$ when the trace relations involve arbitrary numbers of X and fixed numbers of letters W_1, W_2 and W_1, W_2, W_3 possess the series

$$\begin{aligned} \left. \frac{Z_{\mathcal{G}_2}}{Z_\infty} \right|_{w_1 w_2} &= x^{2N-1} + 4x^{2N} + 9x^{2N+1} + 16x^{2N+2} + 25x^{2N+3} + 36x^{2N+4} + \dots \\ \left. \frac{Z_{\mathcal{G}_2}}{Z_\infty} \right|_{w_1 w_2 w_3} &= x^{2N-3} + 6x^{2N-2} + 22x^{2N-1} + 56x^{2N} + 114x^{2N+1} + 202x^{2N+2} + \dots \end{aligned} \quad (3.3.46)$$

As the procedure may be continued iteratively, this suggests that there exists a non-minimal Koszul-Tate resolution $\mathcal{G}$ of $\mathcal{H}_N$

$$\dots \rightarrow \mathcal{G}_3 \xrightarrow{\hat{Q}} \mathcal{G}_2 \xrightarrow{\hat{Q}} \mathcal{G}_1 \xrightarrow{\hat{Q}} \mathcal{H}_\infty \rightarrow \mathcal{H}_N \rightarrow 0 \quad (3.3.47)$$

at any value of N (again understood within each $\mathcal{C}_i$ -eigenspace), where the free R -modules $\mathcal{G}_k$ depend on N only via an overall shift in the $\mathcal{C}_X$ charges of its generators.¹²

The non-minimal resolution $\mathcal{G}$ generalizes the expansion (3.3.43) from a statement about superconformal indices

$$\text{Tr}_{\mathcal{H}_N} [(-1)^F x^{\mathcal{C}_X} w_1^{\mathcal{C}_1} w_2^{\mathcal{C}_2} \dots] = \sum_{k=0}^{\infty} (-1)^k \text{Tr}_{\mathcal{G}_k} [(-1)^F x^{\mathcal{C}_X} w_1^{\mathcal{C}_1} w_2^{\mathcal{C}_2} \dots] , \quad (3.3.48)$$

¹²Note that in general $\mathcal{G}_k|_{\mathcal{C}_i} \neq \lim_{N \rightarrow \infty} \mathcal{V}_k^N|_{\mathcal{C}_i}$ for $k > 1$.

to that about the charge spectrum of free gauge theories at finite N

$$\mathrm{Tr}_{\mathcal{H}_N} [x^{c_x} w_1^{c_1} w_2^{c_2} \dots] = \sum_{k=0}^{\infty} (-1)^k \mathrm{Tr}_{\mathcal{G}_k} [x^{c_x} w_1^{c_1} w_2^{c_2} \dots], \quad (3.3.49)$$

because $\mathcal{G}_k$ can be found only based on the analysis of trace relations. We conjecture that the generators of the free R -module $\mathcal{G}_k$ are holographically dual to states of k coincident D-branes on a fixed locus of the $U(1)$ isometry in the $\alpha' \rightarrow \infty$ limit of the dual string theory.

3.3.7 A Lower Bound on D-brane States

At finite N and generic charges, the bulk description of the module $\mathcal{V}_k^N$ as the space of string states on a background with k wrapped D-branes becomes less precise. A reflection of this fact on $\mathcal{V}_k^N$ is that the number of states of $\mathcal{V}_k^N$ of a given *total* charge or energy (minus the total background charge kN) necessarily diverges as $N \rightarrow \infty$, in contrast to the situation in the previous section. So in general the space $\mathcal{V}_k^N$ does not have a “good” large N limit.

Nevertheless, the space $\mathcal{V}_k^N$ in a Koszul-Tate resolution of $\mathcal{H}_N$ remains well-defined at any finite value of N and energies, because $\mathcal{V}_k^N$ is defined through the set of trace relations and their relations in a boundary $U(N)$ gauge theory. It is therefore possible to put constraints on the “quantum” D-brane sectors based on our knowledge of $\mathcal{V}_k^N$.

According to our proposal, the free R -module $\mathcal{V}_k^N$ in $\mathcal{K}$ gives the spectrum of string states on k D-brane background. Due to the freedom to add spurious pairs of ghosts between $\mathcal{V}_k^N$ and $\mathcal{V}_{k-1}^N$, this spectrum is in principle subject only to the global constraint

$$\mathrm{Tr}_{\mathcal{H}_N} e^{-\beta H} = \sum_{k=0}^{\infty} (-1)^k \mathrm{Tr}_{\mathcal{V}_k^N} e^{-\beta H}. \quad (3.3.50)$$

We interpret this freedom as an indication that, at finite N , there may exist different computational schemes in the bulk for computing the spectrum on D-brane backgrounds, where the different schemes are required only to satisfy the global constraint (3.3.50). We discussed an analogy of this situation to that of the instanton complex in Section 3.3.4.

Let us show that it is still possible to make useful statements about the spaces $\mathcal{V}_k^N$ at each degree k . Recall from Section 3.3.4 that a minimal Koszul-Tate resolution $\widehat{\mathcal{K}}$ of $\mathcal{H}_N$ is unique, i.e. any non-minimal resolution $\mathcal{K}$ of $\mathcal{H}_N$ may be constructed by adjoining to $\widehat{\mathcal{K}}$

spurious pairs of ghosts that decouple from $\widehat{\mathcal{K}}$. Given the set of trace relations in a $U(N)$ gauge theory, we use the uniqueness of $\widehat{\mathcal{K}}$ to place lower bounds on the spectrum of states on D-brane backgrounds in the string dual at $\alpha' \rightarrow \infty$ and finite N .

Let $\widehat{\mathcal{K}}$ and $\mathcal{K}$, respectively, be minimal and non-minimal Koszul-Tate resolutions of $\mathcal{H}_N$:

$$\begin{aligned}\widehat{\mathcal{K}} : \dots &\rightarrow \widehat{\mathcal{V}}_3^N \xrightarrow{\widehat{Q}} \widehat{\mathcal{V}}_2^N \xrightarrow{\widehat{Q}} \widehat{\mathcal{V}}_1^N \xrightarrow{\widehat{Q}} \mathcal{H}_\infty \rightarrow \mathcal{H}_N \rightarrow 0 \\ \mathcal{K} : \dots &\rightarrow \mathcal{V}_3^N \xrightarrow{\widehat{Q}} \mathcal{V}_2^N \xrightarrow{\widehat{Q}} \mathcal{V}_1^N \xrightarrow{\widehat{Q}} \mathcal{H}_\infty \rightarrow \mathcal{H}_N \rightarrow 0,\end{aligned}\tag{3.3.51}$$

where $\widehat{Q}$ are defined differently in $\widehat{\mathcal{K}}$ and $\mathcal{K}$. The Lefschetz trace formula says that

$$\mathrm{Tr}_{\mathcal{H}_N} e^{-\beta H} = \sum_{k=0}^{\infty} (-1)^k \mathrm{Tr}_{\widehat{\mathcal{V}}_k^N} e^{-\beta H} = \sum_{k=0}^{\infty} (-1)^k \mathrm{Tr}_{\mathcal{V}_k^N} e^{-\beta H},\tag{3.3.52}$$

which may be further refined by global charges. If we define the Hilbert-Poincaré series

$$\begin{aligned}\widehat{\mathbf{P}}(\mu, q) &= \sum_{k=0}^{\infty} \mu^k \mathrm{Tr}_{\widehat{\mathcal{V}}_k^N} e^{-\beta H} \\ \mathbf{P}(\mu, q) &= \sum_{k=0}^{\infty} \mu^k \mathrm{Tr}_{\mathcal{V}_k^N} e^{-\beta H}\end{aligned}\tag{3.3.53}$$

where $q = e^{-\beta}$, then by the uniqueness of the minimal resolution $\widehat{\mathcal{K}}$, there exists $\mathbf{Q}(\mu, q)$ with non-negative coefficients in the μ - and q -series such that the following holds:

$$\mathbf{P}(\mu, q) - \widehat{\mathbf{P}}(\mu, q) = (1 + \mu) \mathbf{Q}(\mu, q).\tag{3.3.54}$$

This is the statement that $\mathcal{K}$ can be constructed from a minimal resolution $\widehat{\mathcal{K}}$ by introducing spurious *pairs* of heavy ghosts $\xi^{(k)}$ and $\xi^{(k-1)}$ satisfying $\widehat{Q}\xi^{(k)} = \xi^{(k-1)}$ and $\widehat{Q}\xi^{(k-1)} = 0$. The ghosts $\xi^{(k)}$ and $\xi^{(k-1)}$ possess the same energy eigenvalue. (3.3.54) places a lower bound at each energy eigenvalue on the number of generators of $\mathcal{V}_k^N$, i.e. on the degeneracy of open strings on D-branes in the $\alpha' \rightarrow \infty$ limit.

Note that the lower bound (3.3.54) applies when there is a state-counting interpretation for the bulk partition function computed in background with k D-branes. For instance, one situation where the bound does not apply is in the giant graviton-like expansion considered in [62]. It was found in [66] that brane partition functions in this expansion originate from a $U(m|m)$ super-matrix integral, which suggests that the partition function is associated to a more exotic brane system. It would be interesting to understand the expansion of [62] in terms of Koszul-Tate resolutions.

Another situation where the bound does not apply is when analytic properties of the bulk partition functions are used to find the spectrum. For example, the giant graviton expansion for the $\frac{1}{4}$ -BPS chiral ring of $U(N)$ $\mathcal{N} = 4$ SYM can be found by summing over residues of a grand canonical partition function:

$$\begin{aligned} Z_N(x, y) &= Z_\infty \sum_{k, k'=0}^{\infty} \frac{x^{kN} y^{k'N}}{\prod_{n=1}^k \prod_{m=0}^{\infty} (1 - x^{-n} y^m) \prod_{n=1}^{k'} \prod_{m=0}^{\infty} (1 - y^{-n} x^m) \prod_{n=1}^k \prod_{m=1}^{k'} (1 - x^{-n} y^{-m})} \\ &= Z_\infty \sum_{k, k'=0}^{\infty} x^{kN} y^{k'N} \hat{Z}_{(k, k')}. \end{aligned} \quad (3.3.55)$$

The terms $x^{kN} Z_\infty \hat{Z}_{(k, 0)}$ and $y^{k'N} Z_\infty \hat{Z}_{(0, k')}$ have state-counting interpretations and their charge degeneracies are subject to the lower bound (refined with fugacities) at each N . The other terms $\hat{Z}_{(k, k')}$ with $k, k' \neq 0$ do not have state-counting interpretations, because $\hat{Z}_{(k, k')}$ with $k, k' \neq 0$ does not have a power series expression. Computing the right side of (3.3.55), e.g. in the limit $x = y = t$, requires one to use analytic properties of $\hat{Z}_{(k, k')}$ in fugacities x, y to cancel poles that are present in that limit. The spectrum on the RHS at each total value of $k + k'$ therefore does not possess a state-counting interpretation. Wall-crossing phenomena due to the analytic properties of such expansions were discussed in the context of superconformal indices in [61]. We discuss the resolution in this example in Section 3.4.

3.4 Basic Examples

We provide some simple examples of Koszul-Tate resolutions of finite N modules, such as those for a $U(N)$ -gauged free fermion, $\frac{1}{4}$ -BPS sector of free $\mathcal{N} = 4$ SYM, and the $\frac{1}{4}$ -BPS chiral ring of $\mathcal{N} = 4$ SYM. Higher ghosts appear in the free $\frac{1}{4}$ -BPS sector and the $\frac{1}{4}$ -BPS chiral ring.

We hope it is clear to the reader that generalization to more interesting examples, such as $\frac{1}{16}$ -BPS sector of $\mathcal{N} = 4$ SYM or thermal partition function of pure Yang-Mills, is certainly possible in principle. However, this would require a significant improvement in the algorithm and would benefit from a better understanding of their string duals for comparison of the results.

3.4.1 Enumerating Relations

We can enumerate the trace relations in a $U(N)$ gauge theory by defining the function [82]

$$\Delta(M_1, \dots, M_{N+1}) = \sum_{\sigma \in S_{N+1}} \underbrace{\chi_{(1,1,\dots,1)}(\sigma)}_{N+1} (M_1)_{i_1}^{i_{\sigma(1)}} (M_2)_{i_2}^{i_{\sigma(2)}} \dots (M_{N+1})_{i_{N+1}}^{i_{\sigma(N+1)}}. \quad (3.4.1)$$

which consists of $N + 1$ matrices $M_1, M_2, \dots, M_{N+1}$. Here, $\chi_R(\sigma)$ is the character of the symmetric group S_{N+1} in the antisymmetric representation $R = (1, 1, \dots, 1)$. As a consequence of the Cayley-Hamilton theorem, the following relation

$$\Delta(M_1, \dots, M_{N+1}) = 0. \quad (3.4.2)$$

holds identically for any set of $N \times N$ matrices M_i , where each M_i may be grassmann even or odd. Trace relations in a $U(N)$ gauge theory can be found by considering replacements of M_i with (products of) matrix-valued fields in the gauge theory.¹³ Since there are infinitely many trace relations, the set of trace relations must be computed in practice with a cutoff in the charges.

There is a large redundancy in the set of trace relations obtained from making substitutions in $\Delta(M_1, \dots, M_{N+1}) = 0$. We find the minimal set of trace relation generators and the minimal free resolution of $\mathcal{H}_N$, up to a truncation in the charge values, by using a Gröbner basis algorithm in a computer program [84].

3.4.2 Examples

$U(N)$ -Gauged Fermion Consider a free fermion ψ in the adjoint representation of the gauge group $U(N)$. To simplify matters, we consider the Hilbert space built from gauge-invariant operators without derivatives, i.e. traces of an $N \times N$ fermion matrix. Writing down trace relations is straightforward, and the Koszul-Tate resolution of this example can be worked out analytically. Taking the energy of the fermion to be $1/2$, the thermal partition function of the system with anti-periodic boundary conditions is

$$\text{Tr}_{\mathcal{H}_N} e^{-\beta H} = \prod_{n=1}^N (1 + q^{n-\frac{1}{2}}), \quad (3.4.3)$$

¹³See for example [83] for a recent study using the Cayley-Hamilton theorem to determine trace relations.

and with periodic boundary conditions is

$$\mathrm{Tr}_{\mathcal{H}_N}(-1)^F e^{-\beta H} = \prod_{n=1}^N (1 - q^{n-\frac{1}{2}}), \quad (3.4.4)$$

where $q = e^{-\beta}$. The fermion number grading $(-1)^F$ here should be distinguished from the heavy ghost grading $(-1)^k$ in the resolution. It will be interesting to see the effect of having both $(-1)^F$ and $(-1)^k$ for the periodic case.

Let us find the trace relations and the resolution. From the cyclicity of the trace, one finds that traces containing even powers of ψ vanish identically, independently of N :

$$\mathrm{Tr} \psi^{2n} = 0 \quad \text{for } n \geq 1. \quad (3.4.5)$$

So the large N ring R is

$$R = \mathbb{C}[\mathrm{Tr} \psi, \mathrm{Tr} \psi^3, \mathrm{Tr} \psi^5, \dots]. \quad (3.4.6)$$

The trace relations at N are

$$\mathrm{Tr} \psi^{2N+1} = \mathrm{Tr} \psi^{2N+3} = \mathrm{Tr} \psi^{2N+5} = \dots = 0. \quad (3.4.7)$$

We can find the Koszul-Tate resolution of the finite N module

$$\mathcal{H}_N = \mathcal{H}_\infty / \langle \mathrm{Tr} \psi^{2N+1}, \mathrm{Tr} \psi^{2N+3}, \mathrm{Tr} \psi^{2N+5}, \dots \rangle_R \quad (3.4.8)$$

by introducing ghosts φ and a differential $\widehat{\mathcal{Q}} = [\cdot, \widehat{Q}]$ on operators satisfying

$$\widehat{\mathcal{Q}}(\varphi_{-2N-a}) = \mathrm{Tr} \psi^{2N+a} \quad (3.4.9)$$

where $a = 1, 3, 5, \dots$. The anti-commuting differential $\widehat{\mathcal{Q}}$ in a graded free resolution must commute with all quantum numbers other than the ghost degree of the resolution, so the ghosts φ_{-2N-a} are commuting operators with $F = 1 \bmod \mathbb{Z}_2$ and $H = N + \frac{a}{2}$.¹⁴ The differential is nilpotent $\widehat{\mathcal{Q}}^2 = 0$ and acts via the graded Leibniz rule on multi-ghost operators:

$$\begin{aligned} \widehat{\mathcal{Q}}(\varphi_{-2N-a_1} \cdots \varphi_{-2N-a_k}) &= \sum_{i=1}^k (\varphi_{-2N-a_1} \cdots \widehat{\mathcal{Q}}(\varphi_{-2N-a_i}) \cdots \varphi_{-2N-a_k}) \\ &= \sum_{i=1}^k \mathrm{Tr} \psi^{2N+a_i} (\varphi_{-2N-a_1} \cdots \varphi_{-2N-a_{i-1}} \varphi_{-2N-a_{i+1}} \cdots \varphi_{-2N-a_k}). \end{aligned} \quad (3.4.10)$$

¹⁴ φ may both commute and have $F = 1 \bmod \mathbb{Z}_2$ because F is defined through other quantum numbers such as Lorentz spin in higher dimensions.

There are no higher ghosts in this example. This statement can be checked by computing the Lefschetz trace formula using only the multi-ghost states (with coefficients in R)

$$\varphi_{-2N-a_1}\varphi_{-2N-a_2}\cdots\varphi_{-2N-a_k}|0\rangle\in\mathcal{V}_k \quad (3.4.11)$$

where $a = 1, 3, 5, \dots$ with $1 \leq a_1 \leq a_2 \leq \dots \leq a_k < \infty$ and no higher ghosts. Let us compute the ghost partition functions by taking traces with respect to the free R -modules $\mathcal{V}_k$. The k -ghost partition function without $(-1)^F$ is

$$\begin{aligned} \text{Tr}_{\mathcal{V}_k} e^{-\beta H} &= \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}}) \sum_{\substack{1 \leq a_1 \leq \dots \leq a_k < \infty \\ a_i \text{ odd}}} q^{N+\frac{a_1}{2}} q^{N+\frac{a_2}{2}} \cdots q^{N+\frac{a_k}{2}} \\ &= \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}}) \frac{q^{kN} q^{\frac{k}{2}}}{\prod_{m=1}^k (1 - q^m)}, \end{aligned} \quad (3.4.12)$$

and with $(-1)^F$ is

$$\begin{aligned} \text{Tr}_{\mathcal{V}_k} (-1)^F e^{-\beta H} &= \prod_{n=1}^{\infty} (1 - q^{n-\frac{1}{2}}) \sum_{\substack{1 \leq a_1 \leq \dots \leq a_k < \infty \\ a_i \text{ odd}}} (-1)^k q^{N+\frac{a_1}{2}} q^{N+\frac{a_2}{2}} \cdots q^{N+\frac{a_k}{2}} \\ &= \prod_{n=1}^{\infty} (1 - q^{n-\frac{1}{2}}) \frac{(-1)^k q^{kN} q^{\frac{k}{2}}}{\prod_{m=1}^k (1 - q^m)}. \end{aligned} \quad (3.4.13)$$

In the latter, $(-1)^F$ evaluates to $(-1)^k$ because each φ has $F = 1 \bmod \mathbb{Z}_2$ and heavy ghost number 1.

We find that the Lefschetz trace formula is satisfied in both cases. For the fermion ψ with anti-periodic boundary conditions one obtains

$$\prod_{n=1}^N (1 + q^{n-\frac{1}{2}}) = \prod_{n=1}^{\infty} (1 + q^{n-\frac{1}{2}}) \sum_{k=0}^{\infty} (-1)^k q^{kN} \frac{q^{\frac{k}{2}}}{\prod_{m=1}^k (1 - q^m)} \quad (3.4.14)$$

and with periodic boundary conditions one obtains

$$\prod_{n=1}^N (1 - q^{n-\frac{1}{2}}) = \prod_{n=1}^{\infty} (1 - q^{n-\frac{1}{2}}) \sum_{k=0}^{\infty} q^{kN} \frac{q^{\frac{k}{2}}}{\prod_{m=1}^k (1 - q^m)}. \quad (3.4.15)$$

It is interesting that $(-1)^k$ cancels with $(-1)^F$ in the latter. If a string dual of a $U(N)$ -gauged fermion can be formulated, this suggests that D-branes in the anti-periodic case behave with fermionic statistics while D-branes in the periodic case behave as bosons.

Free $\frac{1}{4}$ -BPS Sector Let us consider the $\frac{1}{4}$ -BPS sector of free $U(N)$ $\mathcal{N} = 4$ super Yang-Mills, which consists of gauge-invariants constructed from two $N \times N$ scalars X and Y [85]. Starting with examples involving two matrices, the trace relations become rather involved. We find the trace relations at low values of N up to a truncation in the total dimension of the adjoint fields X and Y . Then we reduce the set of trace relations to a minimal set of generators using [84].

The partition function of this system is

$$Z_N(x, y) = \frac{1}{N!} \oint \prod_{a=1}^N \frac{d\sigma_a}{2\pi i \sigma_a} \frac{\prod_{a>b} (1 - \sigma_a \sigma_b^{-1})(1 - \sigma_b \sigma_a^{-1})}{\prod_{a,b=1}^N (1 - x \sigma_a \sigma_b^{-1})(1 - y \sigma_a \sigma_b^{-1})}. \quad (3.4.16)$$

where x and y are fugacities for charges associated to fields X and Y . The large N ring

$$R = \mathbb{C}[\text{Tr } X, \text{Tr } Y, \text{Tr } X^2, \text{Tr } XY, \dots] \quad (3.4.17)$$

is the ring of polynomials in all single traces constructed from X, Y , where X, Y do not commute. The large N partition function is

$$Z_\infty(x, y) = \prod_{n=1}^{\infty} \frac{1}{1 - x^n - y^n}. \quad (3.4.18)$$

Let us compute Z_N at low values of N to set some expectations regarding the higher relations. We can read off the single-trace generators of the free $\frac{1}{4}$ -BPS sector at low values of N from Z_N . The partition function for $N = 1, 2$ are

$$\begin{aligned} Z_1 &= \frac{1}{(1-x)(1-y)} \\ Z_2 &= \frac{1}{(1-x)(1-y)(1-x^2)(1-xy)(1-y^2)}. \end{aligned} \quad (3.4.19)$$

This suggests that $\mathcal{H}_{N=1}$ is a Fock space built from the generators $\text{Tr } X, \text{Tr } Y$, and that $\mathcal{H}_{N=2}$ is a Fock space built from

$$\text{Tr } X, \text{Tr } Y, \text{Tr } X^2, \text{Tr } XY, \text{Tr } Y^2. \quad (3.4.20)$$

However, the partition function at $N = 3$

$$Z_3 = \frac{(1 - x^6 y^6)}{(\prod_{n=1}^3 (1 - x^n)(1 - y^n)) (1 - xy)(1 - x^2 y)(1 - xy^2)(1 - x^2 y^2)(1 - x^3 y^3)} \quad (3.4.21)$$

says that $\mathcal{H}_{N=3}$ cannot be a Fock space built from single trace generators [86], because there exists a relation of degree (6, 6) in X, Y fields among the generators

$$\text{Tr } X, \text{Tr } Y, \text{Tr } X^2, \text{Tr } XY, \text{Tr } Y^2, \text{Tr } X^3, \text{Tr } X^2Y, \text{Tr } XY^2, \text{Tr } Y^3, \text{Tr } X^2Y^2, \text{Tr } X^3Y^3. \quad (3.4.22)$$

One can think of other traces, e.g. $\text{Tr } XYXY$, $\text{Tr } XYX^2Y^2$, and $\text{Tr } XYXYXY$, at degrees (2, 2) and (3, 3) as having been eliminated by expressing them in terms of $\text{Tr } X^2Y^2$ and $\text{Tr } X^3Y^3$ using the trace relations at $N = 3$. The analysis in this paragraph amounts to using the trace relations that were present at each N to reduce the ring generators to a finite set, but we would like to work over the large N ring R and find whether there exist non-trivial relations between trace relations.

Let $\mathcal{V}_k^N$ be free R -modules consisting a minimal Koszul-Tate resolution $\widehat{\mathcal{K}}$ of $\mathcal{H}_N$. We define the following, which is the Hilbert series of $\mathcal{V}_k^N$:

$$Z_{\mathcal{V}_k^N} = \text{Tr}_{\mathcal{V}_k^N} [x^{c_X} y^{c_Y}]. \quad (3.4.23)$$

By enumerating trace relations, we find that the generators of $\mathcal{V}_1^N$ that map to trace relations possess the following total charges:

$$\begin{aligned} \frac{Z_{\mathcal{V}_1^{N=1}}}{Z_\infty} &= 3t^2 + 4t^3 + 6t^4 + 8t^5 + 14t^6 + 20t^7 + 36t^8 + 60t^9 + 108t^{10} + 188t^{11} + 352t^{12} + \dots \\ \frac{Z_{\mathcal{V}_1^{N=2}}}{Z_\infty} &= 4t^3 + 6t^4 + 8t^5 + 14t^6 + 20t^7 + 36t^8 + 60t^9 + 108t^{10} + 188t^{11} + 352t^{12} + \dots \\ \frac{Z_{\mathcal{V}_1^{N=3}}}{Z_\infty} &= 5t^4 + 8t^5 + 13t^6 + 20t^7 + 36t^8 + 60t^9 + 108t^{10} + 188t^{11} + 352t^{12} + \dots \end{aligned} \quad (3.4.24)$$

where we specialized the fugacities to $x = y = t$.

We would like to find whether there exist higher relations between trace relations. We can do so by checking whether the Lefschetz trace formula is satisfied by multi-ghost states constructed only from single-ghost generators with the above charges. We use Polya theory [7–9] to multi-particle the anti-commuting ghosts of $\mathcal{V}_1^N$:

$$\exp \left[- \sum_{n=1}^{\infty} \frac{\mu^n}{n} \frac{Z_{\mathcal{V}_1^N}}{Z_\infty}(t^n) \right] = \sum_{k=0}^{\infty} (-1)^k \mu^k M_{k,N}(t) \quad (3.4.25)$$

where the series $M_{k,N}(t)$ contains charges of states with total heavy ghost number k . If the $M_{k,N}(t)$ above satisfy

$$\frac{Z_N(t)}{Z_\infty(t)} \stackrel{?}{=} \sum_{k=0}^{\infty} (-1)^k M_{k,N}(t), \quad (3.4.26)$$

it means that there are no ghosts-for-ghosts.

Up to total charge 12, we find that (3.4.26) is satisfied for $N = 1, 2$. There is a discrepancy at $N = 3$ of

$$\frac{Z_3(t)}{Z_\infty(t)} - \sum_{k=0}^{\infty} (-1)^k M_{k,3}(t) = -t^{12} + O(t^{13}) \quad (3.4.27)$$

which indicates that the Koszul-Tate resolution at $N = 3$ involves a higher ghost state at total charge 12. The minus sign, and the fact that the $M_{k=3,3}(t)$ is the final term to contribute to (3.4.27) up to charge 12, suggests the higher ghost $\chi^{(3)}|0\rangle \in \mathcal{V}_{k=3}^{N=3}$ of charge 12 maps under $\widehat{Q}$ to a linear combination of $\chi_{A_1}^{(1)}\chi_{A_2}^{(1)}|0\rangle$ with coefficients in R .

$\frac{1}{4}$ -BPS Chiral Ring A similar computation may be done in the $\frac{1}{4}$ -BPS chiral ring of $U(N)$ $\mathcal{N} = 4$ SYM, which involves adjoint fields X and Y subject to a superpotential constraint $[X, Y] = 0$. The large N ring R consists of traces of commuting fields:

$$R = \mathbb{C}[\text{Tr } X^n Y^m] \quad (3.4.28)$$

where $n, m = 0, 1, 2, \dots$ and $(n, m) \neq (0, 0)$. The grand canonical partition function of the $\frac{1}{4}$ -BPS chiral ring is

$$\mathcal{Z}(p; x, y) = \sum_{N=0}^{\infty} p^N Z_N(x, y) = \prod_{n,m=0}^{\infty} \frac{1}{1 - p x^n y^m} \quad (3.4.29)$$

and the large N partition function is

$$Z_\infty(x, y) = \prod_{\substack{n,m=0 \\ (n,m) \neq (0,0)}}^{\infty} \frac{1}{1 - x^n y^m}. \quad (3.4.30)$$

One finds non-trivial results also starting at $N = 3$, where the generators of $\mathcal{V}_1^{N=3}$ that map to trace relations possess the following total charges:

$$\frac{Z_{\mathcal{V}_1^{N=3}}}{Z_\infty} = 5t^4 + 8t^5 + 10t^6 + 8t^7 + 9t^8 + 10t^9 + 11t^{10} + \dots \quad (3.4.31)$$

There is a discrepancy in (3.4.26) at $N = 3$:

$$\frac{Z_3(t)}{Z_\infty(t)} - \sum_{k=0}^{\infty} (-1)^k M_{k,3}(t) = 3t^8 + O(t^9), \quad (3.4.32)$$

which indicates that three ghost-for-ghost states $\chi^{(2)}|0\rangle \in \mathcal{V}_{k=2}^{N=3}$ exist in the minimal Koszul-Tate resolution at charge 8. It would be nice to understand the higher ghosts in this context by comparing the large N spectrum of $\mathcal{G}_k$ in Section 3.3.6 at fixed $\mathcal{C}_Y$ charges with the spectrum of normalizable states constructed from higher S^3 modes on D3 giants.

3.5 Discussion

We conclude with a partial list of open questions:

1. A path integral realization of the Koszul-Tate resolution implementing trace relations would be conceptually enlightening. This may help understand how structures discussed in this work are realized in the bulk string field theory in the $\alpha' \rightarrow \infty$ limit.
2. To make the bulk grading and the Hilbert space structure manifest, we used canonical quantization to study the states of bulk D-branes. It would be nice to understand how the grading $(-1)^k$ arises in the gravitational path integral in the presence of Euclidean D-branes.
3. Trace relations exist at finite λ . How does the structure of the Koszul-Tate resolution change when one turns on the coupling, and what additional ingredients are required? How do we describe the dynamics of the heavy ghost states, and how are properties of black holes encoded therein? How does the bulk radial coordinate emerge at finite λ ? It would also be nice to understand the recent progress in $\frac{1}{16}$ -BPS states at weak coupling [87–91] using tools presented in this work.
4. More intricate finite N relations appear in gauge theories with (anti-)fundamental and adjoint fields or in gauge theories with surface defects with (anti-)fundamental degrees of freedom. It would be nice to gain an algebraic understanding of various branes in anti-de Sitter space in terms of relations at finite N .
5. It would be important to have a sharper bulk understanding of the higher ghosts (e.g. as strings on multiply-wrapped branes, etc.). Along these lines, it would be helpful to generalize the computation of normalizable states in Section 3.2 to that with higher S^3 modes and fluctuations in AdS directions.
6. It would be nice to understand the relationship between our work and works on the K-theory classification of D-branes [92–97], higher spin holography [98–101], tensionless strings [102–105], and works relating holography and Koszul duality [106–110].

7. The holographic interpretation of the differential $\widehat{Q}$ is an instanton which maps a state on k wrapped D-branes to a linear combination of states on $k - 1$ wrapped D-branes with coefficients in the traces (i.e. closed strings). What are the gauge theory predictions for the tunneling amplitudes between vacua labelled by different D-branes, as well as the effect of turning on λ on this amplitude?

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APPENDICES

Appendix A

Appendices for Chapter 2

A.1 Reflections

Consider the $U(N)$ integrand for $f(x) = x$:

$$Z_N(x) = \frac{(-1)^{\frac{N(N-1)}{2}}}{N!} \int \left[\prod_a \frac{d\mu_a}{2\pi i \mu_a (1-x)} \right] \prod_{a \neq b} \frac{\mu_b - \mu_a}{\mu_b - x\mu_a} \quad (\text{A.1.1})$$

A basic determinant identity for $\det_{a,b} \left[\frac{1}{\mu_b - \nu_a} \right]$ allows one to rewrite that as

$$Z_N(x) = \frac{x^{-\frac{N(N-1)}{2}}}{N!} \int \left[\prod_a \frac{d\mu_a}{2\pi i} \right] \det_{a,b} \frac{1}{\mu_b - x\mu_a} \quad (\text{A.1.2})$$

i.e.

$$Z_N(x) = \frac{x^{-\frac{N(N-1)}{2}}}{N!} \sum_{\sigma \in S_N} (-1)^\sigma \int \left[\prod_a \frac{d\mu_a}{2\pi i} \right] \frac{1}{\mu_{\sigma(a)} - x\mu_a} \quad (\text{A.1.3})$$

For every cycle in the permutation σ we have a sequence of convolutions. Notice that

$$\int \frac{d\mu_2}{2\pi i} \frac{1}{\mu_2 - x^a \mu_1} \frac{1}{\mu_3 - x^b \mu_2} = \frac{1}{\mu_3 - x^{a+b} \mu_1} \quad (\text{A.1.4})$$

so that a cycle of length ℓ in σ ultimately gives a factor of $(-1)^{\ell-1} \frac{1}{1-x^\ell}$ to the final answer:

$$Z_N(x) = x^{-\frac{N(N-1)}{2}} (-1)^N \sum_{n_k | \sum_k k n_k = N} \prod_k (-1)^{n_k} \frac{1}{n_k! k^{n_k}} \frac{1}{(1-x^k)^{n_k}} \quad (\text{A.1.5})$$

We can compute a generating function

$$\sum_N \zeta^N x^{\frac{N(N+1)}{2}} Z_N(x) = \sum_{n_k} \prod_k (-1)^{n_k} (-x\zeta)^{kn_k} \frac{1}{n_k! k^{n_k}} \frac{1}{(1-x^k)^{n_k}} \quad (\text{A.1.6})$$

i.e.

$$\log\left[\sum_k \zeta^k x^{\frac{k(k+1)}{2}} Z_k(x)\right] = -\sum_k (x\zeta)^k \frac{1}{k} \frac{1}{(1-x^k)} = \sum_{n=1}^{\infty} \log(1 + \zeta x^n) \quad (\text{A.1.7})$$

which we used with $\zeta = -x^N$ to derive the giant graviton expansion in the simplest toy model.

Consider now how the answer would change if we inserted in the integrand the character of some representation of $U(N)$. For example, consider an adjoint $\sum_{u,v} \mu_u \mu_v^{-1}$. These extra factors will have interesting effects when inserted in the cycles of σ . For example, the insertion of a single power of μ_a along a cycle gives a vanishing result. For positive n we have

$$\int \frac{d\mu_2}{2\pi i} \frac{\mu_2^n}{\mu_2 - x^a \mu_1} \frac{1}{\mu_3 - x^b \mu_2} = \frac{x^{an} \mu_1^n}{\mu_3 - x^{a+b} \mu_1} \quad (\text{A.1.8})$$

and for negative n we have

$$\int \frac{d\mu_2}{2\pi i} \frac{\mu_2^n}{\mu_2 - x^a \mu_1} \frac{1}{\mu_3 - x^b \mu_2} = \frac{x^{-bn} \mu_3^n}{\mu_3 - x^{a+b} \mu_1} \quad (\text{A.1.9})$$

If both μ_u and μ_v^{-1} are inserted along the same cycle we get a power of x equal to the difference between the location of the insertion of μ_u and μ_v^{-1} .

The final answer will thus be a sum of terms where the two eigenvalues fall within a cycle of length k . We get an overall factor of $kn_k \frac{1-x^k}{1-x}$. Thus in the analogue of (A.1.6) we can shift $n_k \rightarrow n_k + 1$ to reabsorb most of this. We are left with an overall factor of $-\frac{(-x\zeta)^k}{1-x}$. Summing over k , this gives an overall factor of

$$\frac{x\zeta}{(1-x)(1+x\zeta)} \quad (\text{A.1.10})$$

so that

$$\sum_k \zeta^k x^{\frac{k(k+1)}{2}} Z_k^{\text{Adj}}(x) = \frac{x\zeta}{1-x} \prod_{n=2}^{\infty} \log(1 + \zeta x^n) \quad (\text{A.1.11})$$

i.e.

$$\sum_k \zeta^k x^{\frac{k(k+1)}{2}} Z_k^{\text{Adj}}(x) = \frac{x\zeta}{1-x} \sum_k x^k \zeta^k x^{\frac{k(k+1)}{2}} Z_k(x) \quad (\text{A.1.12})$$

or

$$\sum_k \zeta^k x^{\frac{k(k+1)}{2}} Z_k^{\text{Adj}}(x) = \frac{1}{1-x} \sum_k \zeta^k x^{\frac{k(k+1)}{2}} Z_{k-1}(x) \quad (\text{A.1.13})$$

so that

$$Z_N^{\text{Adj}}(x) = \frac{1}{1-x} Z_{N-1}(x) \quad (\text{A.1.14})$$

In particular, we have

$$\frac{Z_N^{\text{Adj}}(x)}{Z_\infty(x)} = \sum_{k=0}^{\infty} x^{kN} \frac{x^{-k}}{1-x} Z_k(x^{-1}) \quad (\text{A.1.15})$$

We could dub the functions on the right hand side of this as

$$\tilde{Z}_k^{\text{Adj}}(\tilde{x}) = -\frac{\tilde{x}^{k+1}}{1-\tilde{x}} Z_k(\tilde{x}) \quad (\text{A.1.16})$$

We have a reciprocal relation:

$$\frac{\tilde{Z}_N^{\text{Adj}}(x)}{Z_\infty(x)} = -\frac{x^{N+1}}{1-x} \sum_{k=0}^{\infty} x^{kN} Z_k(x^{-1}) = \sum_{k=1}^{\infty} x^{kN} Z_k^{\text{Adj}}(x^{-1}) \quad (\text{A.1.17})$$

We expect a similar pattern to persist when we replace the adjoint representation by other representations. This would imply that for any index-like quantity $Z_N(x; y)$ we can write a giant-graviton-like expansion for the ratio $Z_N(x; y)/Z_\infty(x)$ and thus for $Z_N(x; y)/Z_\infty(x; y)$.

A.2 Schur Index

The Schur index $Z_N(q, x)$ is a specialization of the $\mathcal{N} = 4$ index. It is also the torus partition function of the $U(N)$ -gauged adjoint $\beta\gamma$ system, which arises as the 2d chiral algebra associated to $\mathcal{N} = 4$ super Yang-Mills. We consider Schur index in the Ramond sector for simplicity, to avoid powers of $q^{1/2}$ in our formulae. The index in Neveu-Schwarz sector can be obtained from the Ramond index by a shift of the $SU(2)_R$ fugacity $x \rightarrow \sqrt{q}x$.

The full expression for the Schur index in Ramond sector is

$$Z_N = \frac{1}{N!} \oint \prod_i \frac{d\sigma_i}{2\pi i \sigma_i} \frac{\prod_{i < j} (1 - \sigma_i \sigma_j^{-1})(1 - \sigma_j \sigma_i^{-1})}{\prod_{i,j} (1 - x \sigma_i \sigma_j^{-1})} \frac{\prod_{i,j} \prod_{n=1}^{\infty} (1 - \sigma_i / \sigma_j q^n)(1 - \sigma_j / \sigma_i q^n)}{\prod_{i,j} \prod_{n=1}^{\infty} (1 - x \sigma_i / \sigma_j q^n)(1 - x^{-1} \sigma_j / \sigma_i q^n)} \quad (\text{A.2.1})$$

This can be understood as the BRST reduction of a module built from a highest weight annihilated by the zero mode of one of the two symplectic bosons, say Y_0 . The zero-energy states are built from the action of $\text{Tr} X_0^n$ on that vector.

Here, we cannot solve the index exactly in N . We can, however, evaluate the indices perturbatively for a given N and see if the statement we made above for the toy index generalizes.

An explicit formula for a family of generating functions of the Schur index is

$$\sum_N \frac{\theta(ux^N)}{\theta(u)} Z_N(q; x) x^{\frac{N^2}{2}} z^N = \prod_{n=-\infty}^{\infty} \left(1 + \frac{zx^n}{1 - uq^n} \right) \quad (\text{A.2.2})$$

to be understood as expanded in the regime $|q| < |x| < 1$. This equality holds for all values of u and can be derived by an application of a determinant identity for theta functions followed by a Fermi Gas analysis, which rewrites the generating function as the determinant of an integral operator.

A.3 Tests for the Schur Index

In this section, we provide evidence in support of the giant graviton expansion of the Schur index. For convenience, we rewrite the proposal here:

$$Z_N(x; q) = Z_\infty(x; q) \left[1 + x^N \hat{Z}_1(x; q) + x^{2N} \hat{Z}_2(x; q) + \cdots \right]. \quad (\text{A.3.1})$$

In practice, let us consider a truncation of the formula at the k -th correction:

$$Z_N(x; q) = Z_\infty(x; q) \left[1 + x^N \hat{Z}_1(x; q) + \cdots + x^{kN} \hat{Z}_k(x; q) \right], \quad (\text{A.3.2})$$

where we also define

$$Z_0 = 1, \quad Z_{-1} = Z_{-2} = \cdots = 0.$$

The following tables enumerate the series coefficients of x of the left and right hand sides of (A.3.2) for a given N , order of q , and truncation level k . Integers n denote the power of x in the series. The top row lists the powers n of x . The middle rows tabulate the coefficients of x^n of the right hand side of (A.3.2) for different values of the truncation k . The final row tabulates the coefficients of x^n of $Z_N(q, x)$.

From the results below, we observe that the order of x to which the formula holds, at a given order s of q , is

$$N + \frac{k}{2}(k + 3 + 2N) - (2 + k)s. \quad (\text{A.3.3})$$

Checks beyond those presented below can be readily conducted by using the generating function [\(A.2.2\)](#) for the Schur index.

At $N = 1$ and $O(q^0)$,

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$k=0$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627
$k=1$	1	1	1	1	1	0	-1	-4	-8	-15	-25	-41	-62	-94	-137	-197	-277	-387	-530	-722	-970
$k=2$	1	1	1	1	1	1	1	1	1	2	3	6	11	20	33	56	88	138	208	311	452
$k=3$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	-1	-4	-9	-19	-34	-60
$k=4$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	2
$k=5$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
LHS	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

At $N = 1$ and $O(q^1)$,

n	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$k=0$	1	0	1	1	2	2	4	4	7	8	12	14	21	24	34	41	55	66	88	105	137	165
$k=1$	1	-1	0	-1	-1	-3	-3	-7	-8	-14	-18	-28	-35	-53	-67	-94	-121	-165	-209	-280	-353	-462
$k=2$	1	-1	0	0	0	0	1	2	4	7	12	20	31	48	72	106	152	218	304	423	578	787
$k=3$	1	-1	0	0	0	0	0	0	0	0	-1	-1	-3	-5	-10	-14	-26	-37	-60	-85	-129	-180
$k=4$	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	3	5	10	16	27
$k=5$	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1
LHS	1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = 1$ and $O(q^2)$,

n	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$k=0$	2	1	2	3	5	6	10	12	18	23	32	40	56	69	92	116	151	187	242	298	379	467
$k=1$	1	-1	-1	-3	-4	-8	-11	-19	-26	-40	-55	-80	-107	-151	-200	-272	-357	-476	-616	-808	-1035	-1337
$k=2$	1	0	0	0	1	2	4	7	12	20	31	48	72	106	152	218	304	423	578	787	1056	1413
$k=3$	1	0	0	-1	0	0	-1	-1	-2	-4	-7	-11	-20	-30	-49	-74	-113	-164	-245	-346	-496	-692
$k=4$	1	0	0	-1	0	0	0	0	0	0	1	1	2	4	7	11	19	30	47	73	111	165
$k=5$	1	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-2	-4	-7	-11	-19
LHS	1	0	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = 1$ and $O(q^3)$,

n	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$k=0$	0	3	1	3	4	7	8	14	16	25	31	44	54	77	93	126	157	206	253	330	403	516
$k=1$	-1	1	-2	-3	-5	-8	-14	-19	-31	-43	-63	-87	-124	-165	-229	-304	-407	-534	-706	-911	-1187	-1519
$k=2$	0	2	1	0	4	4	8	12	21	29	49	67	103	144	210	285	408	547	755	1006	1360	1786
$k=3$	0	1	0	-2	0	-2	-3	-6	-7	-14	-19	-32	-45	-71	-99	-151	-209	-304	-421	-596	-811	-1125
$k=4$	0	1	0	-1	0	0	1	2	3	5	9	13	21	31	48	69	104	146	215	300	429	609
$k=5$	0	1	0	-1	1	0	-1	0	0	0	-1	-1	-2	-3	-6	-8	-15	-20	-34	-47	-73	-102
LHS	0	1	0	-1	1	0	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = 1$ and $O(q^4)$,

n	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$k=0$	0	0	0	5	2	5	7	12	14	24	28	43	54	76	94	133	162	218	273	357	440
$k=1$	0	-1	-2	1	-5	-7	-11	-17	-29	-39	-62	-85	-123	-169	-238	-315	-435	-574	-765	-1000	-1317
$k=2$	1	0	1	4	3	5	11	14	24	35	55	78	120	165	242	335	472	640	889	1186	1609
$k=3$	0	-1	-1	0	-2	-4	-6	-10	-14	-24	-33	-51	-73	-109	-152	-223	-308	-439	-603	-840	-1139
$k=4$	0	0	0	2	1	2	2	4	7	10	15	24	33	52	73	105	149	215	294	417	572
$k=5$	0	0	0	1	0	0	-1	-1	-2	-3	-5	-7	-12	-16	-25	-37	-53	-74	-111	-151	-215
LHS	0	0	0	1	0	0	0	0	0	0	-1	0	0	0	0	0	0	0	0	0	0

At $N = 2$ and $O(q^0)$,

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$k=0$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627
$k=1$	1	1	2	2	3	3	4	3	3	0	-3	-11	-20	-38	-60	-96	-142	-211	-299	-425	-585
$k=2$	1	1	2	2	3	3	4	4	5	5	6	6	8	9	13	18	28	42	66	100	153
$k=3$	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	9	8	6
$k=4$	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11
$k=5$	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11
LHS	1	1	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9	9	10	10	11

At $N = 2$ and $O(q^1)$,

n	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$k=0$	1	0	1	1	2	2	4	4	7	8	12	14	21	24	34	41	55	66	88	105	137
$k=1$	1	0	0	0	0	-1	-1	-3	-4	-7	-10	-16	-21	-32	-43	-60	-80	-110	-143	-192	-248
$k=2$	1	0	0	0	0	0	0	0	0	1	1	3	5	9	13	23	32	50	70	103	141
$k=3$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-3	-5	-10	-15	-26
$k=4$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
$k=5$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
LHS	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = 2$ and $O(q^2)$,

n	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$k=0$	2	1	2	3	5	6	10	12	18	23	32	40	56	69	92	116	151	187	242	298	379
$k=1$	2	0	0	0	-1	-3	-4	-9	-13	-21	-31	-47	-64	-94	-128	-176	-237	-321	-421	-560	-727
$k=2$	2	0	0	1	0	0	1	1	2	5	7	13	22	34	51	81	115	169	240	339	467
$k=3$	2	0	0	1	0	0	0	0	0	0	-1	-1	-2	-4	-8	-11	-21	-32	-52	-78	-120
$k=4$	2	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	12
$k=5$	2	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
LHS	2	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = 2$ and $O(q^3)$,

n	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$k=0$	3	1	2	3	4	7	8	14	16	25	31	44	54	77	93	126	157	206	253	330	403
$k=1$	2	-1	0	-2	-2	-7	-8	-17	-22	-37	-50	-77	-101	-149	-196	-273	-358	-487	-629	-838	-1074
$k=2$	2	0	0	1	1	2	4	5	9	15	22	35	53	78	113	166	231	328	452	623	843
$k=3$	2	0	1	0	0	0	0	-1	-2	-3	-6	-8	-15	-21	-35	-49	-78	-108	-165	-228	-333
$k=4$	2	0	1	0	0	0	1	0	0	0	1	1	2	3	6	9	14	22	34	50	75
$k=5$	2	0	1	0	0	0	1	0	0	0	0	0	0	0	0	-1	-1	-2	-3	-6	-9
LHS	2	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = 3$ and $O(q^0)$,

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$k=0$	1	1	2	3	4	7	8	14	15	22	30	42	56	77	101	135	176	231	297	385	490
$k=1$	1	1	2	3	4	5	7	8	10	11	12	11	10	4	-4	-19	-41	-76	-123	-194	-288
$k=2$	1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	28	32	38	47	59	77
$k=3$	1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37	40	44
$k=4$	1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37	40	44
$k=5$	1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37	40	44
LHS	1	1	2	3	4	5	7	8	10	12	14	16	19	21	24	27	30	33	37	40	44

At $N = 3$ and $O(q^1)$,

n	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$k=0$	1	0	1	1	2	2	4	4	7	8	12	14	21	24	34	41	55	66	88	105	137
$k=1$	1	0	1	0	1	0	1	-1	0	-3	-3	-8	-9	-18	-22	-36	-46	-69	-88	-126	-160
$k=2$	1	0	1	0	1	0	1	0	1	0	1	0	2	1	4	5	10	14	24	34	53
$k=3$	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	0	-1	-2
$k=4$	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
$k=5$	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
LHS	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1

At $N = 3$ and $O(q^2)$,

n	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$k=0$	2	1	2	3	5	6	10	12	18	23	32	40	56	69	92	116	151	187	242	298	379
$k=1$	2	1	1	1	2	0	1	-2	-3	-8	-12	-23	-31	-51	-71	-104	-141	-201	-266	-365	-479
$k=2$	2	1	1	1	2	1	2	1	2	2	3	7	9	15	24	38	56	86	125	182	256
$k=3$	2	1	1	1	2	1	2	1	2	1	2	1	2	1	1	0	0	-3	-6	-11	-19
$k=4$	2	1	1	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	3
$k=5$	2	1	1	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2
LHS	2	1	1	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2

At $N = 3$ and $O(q^3)$,

n	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$k=0$	3	1	2	3	4	7	8	14	16	25	31	44	54	77	93	126	157	206	253	330	403
$k=1$	3	0	1	1	1	-1	-1	-6	-8	-16	-24	-40	-54	-85	-116	-165	-224	-311	-410	-556	-725
$k=2$	3	0	1	2	2	2	2	3	4	6	7	12	18	27	38	62	85	128	179	259	356
$k=3$	3	0	1	2	2	2	1	2	2	2	1	1	0	-1	-5	-6	-14	-20	-36	-50	-80
$k=4$	3	0	1	2	2	2	1	2	2	2	2	2	2	2	2	3	3	4	5	8	12
$k=5$	3	0	1	2	2	2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2
LHS	3	0	1	2	2	2	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2

The pattern of convergence continues, where the expansion in terms of giant graviton branes on the right hand side converges faster in k to Z_N for larger values of N . Some representative examples with $N = 10$ are shown:

At $N = 10$ and $O(q^0)$,

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$k=0$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627
$k=1$	1	1	2	3	5	7	11	15	22	30	42	55	75	97	128	164	212	267	340	423	530
$k=2$	1	1	2	3	5	7	11	15	22	30	42	55	75	97	128	164	212	267	340	423	530
$k=3$	1	1	2	3	5	7	11	15	22	30	42	55	75	97	128	164	212	267	340	423	530
LHS	1	1	2	3	5	7	11	15	22	30	42	55	75	97	128	164	212	267	340	423	530

At $N = 10$ and $O(q^1)$,

n	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$k=0$	1	0	1	1	2	2	4	4	7	8	12	14	21	24	34	41	55	66	88	105	137
$k=1$	1	0	1	1	2	2	4	4	7	8	11	13	19	21	29	34	44	51	66	75	95
$k=2$	1	0	1	1	2	2	4	4	7	8	11	13	19	21	29	34	44	51	66	75	95
$k=3$	1	0	1	1	2	2	4	4	7	8	11	13	19	21	29	34	44	51	66	75	95
LHS	1	0	1	1	2	2	4	4	7	8	11	13	19	21	29	34	44	51	66	75	95

At $N = 10$ and $O(q^2)$,

n	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$k=0$	2	1	2	3	5	6	10	12	18	23	32	40	56	69	92	116	151	187	242	298	379
$k=1$	2	1	2	3	5	6	10	12	18	22	30	37	50	60	78	95	120	143	179	211	259
$k=2$	2	1	2	3	5	6	10	12	18	22	30	37	50	60	78	95	120	143	179	212	260
$k=3$	2	1	2	3	5	6	10	12	18	22	30	37	50	60	78	95	120	143	179	212	260
LHS	2	1	2	3	5	6	10	12	18	22	30	37	50	60	78	95	120	143	179	212	260

At $N = 10$ and $O(q^3)$,

n	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$k=0$	3	1	3	4	7	8	14	16	25	31	44	54	77	93	126	157	206	253	330	403	516
$k=1$	3	1	3	4	7	8	14	16	24	29	41	48	68	78	104	124	159	185	236	272	338
$k=2$	3	1	3	4	7	8	14	16	24	29	41	48	68	78	104	124	159	186	237	275	341
$k=3$	3	1	3	4	7	8	14	16	24	29	41	48	68	78	104	124	159	186	237	275	341
LHS	3	1	3	4	7	8	14	16	24	29	41	48	68	78	104	124	159	186	237	275	341

The expansion also holds at $N \leq 0$:

At $N = 0$ and $O(q^0)$,

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$k=0$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627
$k=1$	1	0	0	-1	-2	-5	-8	-15	-23	-37	-55	-83	-118	-171	-238	-332	-453	-618	-827	-1107	-1460
$k=2$	1	0	0	0	0	0	1	2	5	10	18	31	52	82	127	193	285	415	595	841	1174
$k=3$	1	0	0	0	0	0	0	0	0	0	-1	-2	-5	-10	-20	-34	-60	-97	-157	-242	-371
$k=4$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	2	5	10	20
$k=5$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
LHS	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = 0$ and $O(q^1)$,

n	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$k=0$	1	0	1	1	2	2	4	4	7	8	12	14	21	24	34	41	55	66	88	105	137
$k=1$	0	-1	-1	-2	-3	-5	-7	-11	-15	-22	-30	-42	-56	-77	-101	-135	-176	-231	-297	-385	-490
$k=2$	0	0	0	1	1	3	4	8	11	19	26	41	56	83	112	160	213	295	389	526	686
$k=3$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$k=4$	0	0	0	0	0	0	0	0	0	0	1	1	3	5	10	15	27	40	65	95	145
$k=5$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-3	-5	-10	-16
$k=6$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
LHS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = 0$ and $O(q^2)$,

n	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$k=0$	0	2	1	2	3	5	6	10	12	18	23	32	40	56	69	92	116	151	187	242	298
$k=1$	-1	0	-2	-4	-6	-9	-15	-21	-32	-45	-64	-88	-123	-164	-223	-296	-392	-512	-671	-864	-1116
$k=2$	0	1	1	1	4	6	11	17	28	41	64	91	134	188	267	365	507	682	924	1227	1634
$k=3$	0	0	0	-1	-1	-2	-3	-7	-10	-18	-28	-45	-67	-104	-150	-222	-316	-451	-628	-878	-1199
$k=4$	0	0	0	0	0	0	1	1	2	4	6	11	18	28	44	70	103	156	229	334	477
$k=5$	0	0	0	0	0	0	0	0	0	0	-1	-1	-2	-4	-7	-10	-19	-28	-45	-68	-105
$k=6$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7
$k=7$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
LHS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = 0$ and $O(q^3)$,

n	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$k=0$	0	0	0	3	1	3	4	7	8	14	16	25	31	44	54	77	93	126	157	206	253	330
$k=1$	0	-1	-2	0	-5	-6	-11	-15	-25	-33	-52	-69	-100	-134	-188	-245	-337	-438	-583	-753	-988	-1260
$k=2$	1	0	1	3	4	6	11	16	27	39	60	85	127	175	251	344	478	643	878	1164	1559	2045
$k=3$	0	-1	-1	-1	-2	-5	-7	-12	-16	-29	-39	-63	-88	-134	-185	-273	-373	-533	-724	-1007	-1352	-1849
$k=4$	0	0	0	1	1	2	2	5	8	12	19	29	42	65	93	135	194	278	387	547	755	1043
$k=5$	0	0	0	0	0	0	-1	-1	-2	-3	-5	-8	-14	-19	-31	-45	-68	-95	-144	-198	-288	-397
$k=6$	0	0	0	0	0	0	0	0	0	0	1	1	2	3	6	8	14	21	32	47	70	101
$k=7$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-2	-3	-6	-9	-14
$k=8$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
LHS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = -1$ and $O(q^0)$,

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$k=0$	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490	627
$k=1$	0	-1	-2	-4	-7	-12	-19	-30	-45	-67	-97	-139	-195	-272	-373	-508	-684	-915	-1212	-1597	-2086
$k=2$	0	0	0	1	2	5	9	17	28	47	73	114	170	253	365	525	738	1033	1422	1939	2616
$k=3$	0	0	0	0	0	0	-1	-2	-5	-10	-19	-33	-57	-92	-147	-227	-345	-512	-733	-1043	-1462
$k=4$	0	0	0	0	0	0	0	0	0	0	1	2	5	10	20	35	62	91	141	207	303
$k=5$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-2	-3	-5	-7
$k=6$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
LHS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = -1$ and $O(q^1)$,

n	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$k=0$	0	1	0	1	1	2	2	4	4	7	8	12	14	21	24	34	41	55	66	88	105
$k=1$	-1	0	-2	-2	-4	-5	-9	-11	-18	-23	-34	-44	-63	-80	-111	-142	-190	-242	-319	-402	-522
$k=2$	0	1	1	2	4	6	10	15	23	33	49	68	97	133	184	247	336	444	592	774	1007
$k=3$	0	0	0	-1	-1	-3	-4	-9	-12	-22	-31	-50	-70	-107	-147	-215	-293	-413	-556	-747	-996
$k=4$	0	0	0	0	0	0	1	1	3	5	9	15	25	38	60	90	133	194	269	379	523
$k=5$	0	0	0	0	0	0	0	0	0	0	-1	-1	-3	-5	-10	-15	-28	-40	-62	-89	-131
$k=6$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	5	7
$k=7$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
LHS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = -1$ and $O(q^2)$,

n	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$k=0$	0	0	0	2	1	2	3	5	6	10	12	18	23	32	40	56	69	92	116	151	187
$k=1$	0	-1	-2	-1	-5	-7	-11	-16	-25	-34	-51	-69	-97	-131	-180	-236	-319	-416	-547	-707	-919
$k=2$	1	0	1	4	5	8	15	22	35	52	77	110	160	221	310	425	580	778	1048	1384	1831
$k=3$	0	-1	-1	-1	-3	-6	-9	-16	-24	-40	-59	-91	-132	-196	-277	-398	-553	-774	-1057	-1449	-1949
$k=4$	0	0	0	1	1	2	3	6	10	16	26	41	62	96	142	209	304	438	619	874	1193
$k=5$	0	0	0	0	0	0	-1	-1	-2	-4	-6	-10	-18	-26	-42	-65	-98	-144	-218	-310	-445
$k=6$	0	0	0	0	0	0	0	0	0	0	1	1	2	4	7	10	18	28	43	65	94
$k=7$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	-1	-2	-4	-6	-9
$k=8$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
LHS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = -2$ and $O(q^0)$,

n	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$k=0$	0	1	1	2	3	5	7	11	15	22	30	42	56	77	101	135	176	231	297	385	490
$k=1$	-1	-1	-3	-5	-9	-14	-23	-34	-52	-75	-109	-153	-216	-296	-407	-549	-739	-981	-1300	-1702	-2223
$k=2$	0	1	2	4	8	14	24	39	62	95	144	212	309	442	626	873	1209	1653	2236	3001	3998
$k=3$	0	0	0	-1	-2	-5	-9	-18	-30	-52	-83	-133	-203	-310	-457	-672	-946	-1337	-1850	-2547	-3450
$k=4$	0	0	0	0	0	0	1	2	5	10	19	34	59	97	144	217	317	456	643	898	1232
$k=5$	0	0	0	0	0	0	0	0	0	0	-1	-2	-3	-5	-8	-12	-18	-26	-37	-52	-72
$k=6$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
LHS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = -2$ and $O(q^1)$,

n	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$k=0$	0	0	0	1	0	1	1	2	2	4	4	7	8	12	14	21	24	34	41	55	66
$k=1$	0	-1	-1	-1	-3	-4	-6	-9	-13	-18	-26	-35	-48	-65	-87	-114	-152	-197	-256	-330	-424
$k=2$	1	0	2	3	5	7	13	17	28	38	57	77	112	148	208	275	374	489	655	846	1105
$k=3$	0	-1	-1	-2	-4	-7	-11	-18	-27	-42	-61	-90	-128	-183	-254	-354	-483	-659	-866	-1165	-1528
$k=4$	0	0	0	1	1	3	4	9	13	23	34	55	79	122	172	253	340	483	651	893	1185
$k=5$	0	0	0	0	0	0	-1	-1	-3	-5	-9	-15	-26	-39	-61	-88	-129	-182	-258	-354	-488
$k=6$	0	0	0	0	0	0	0	0	0	0	1	1	3	4	7	10	16	22	33	45	64
$k=7$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
LHS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = -2$ and $O(q^2)$,

n	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$k=0$	0	0	0	0	0	0	2	1	2	3	5	6	10	12	18	23	32	40	56	69	92	116
$k=1$	0	0	0	-1	-2	-3	-4	-8	-12	-18	-26	-38	-53	-75	-102	-140	-188	-252	-332	-439	-571	-742
$k=2$	0	1	1	2	3	7	11	18	26	42	60	90	126	182	250	350	473	647	862	1156	1520	2008
$k=3$	-1	0	-1	-3	-5	-7	-13	-20	-33	-50	-76	-111	-166	-235	-337	-473	-660	-905	-1243	-1677	-2260	-3012
$k=4$	0	1	1	1	3	5	9	14	23	35	56	83	126	184	270	384	552	771	1080	1487	2046	2773
$k=5$	0	0	0	-1	-1	-2	-3	-6	-9	-16	-24	-39	-58	-90	-132	-198	-285	-415	-589	-838	-1169	-1634
$k=6$	0	0	0	0	0	0	1	1	2	4	6	10	17	26	40	63	93	139	205	299	427	609
$k=7$	0	0	0	0	0	0	0	0	0	0	-1	-1	-2	-4	-7	-10	-18	-27	-43	-64	-93	-132
$k=8$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	2	3	5	7	11
$k=9$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
LHS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

A.4 Character Formula for $U(N)$ Indices

We review the derivation of [77, 85], following closely [46], showing that $U(N)$ gauge theory indices can be written in terms of characters of symmetric groups S_n . This will dramatically simplify the computations of $\mathcal{N} = 4$ indices.

We will consider a $U(N)$ gauge theory index of the form

$$Z_N = \int_{U(N)} DU \exp \left(\sum_{j=1}^{\infty} \frac{1}{j} f(x^j) \text{Tr } U^j \text{Tr } U^{-j} \right)$$

where x denotes a collection of fugacities, each of which is exponentiated by j .

Let us rewrite the plethystic exponential as

$$\exp \left(\sum_{j=1}^{\infty} \frac{1}{j} a(x^j) \text{Tr } U^j \text{Tr } U^{-j} \right) = \left(\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \right) \prod_{j=1}^{\infty} \frac{1}{j^{k_j} k_j!} (a(x^j) \text{Tr } U^j \text{Tr } U^{-j})^{k_j}.$$

If we interpret j as an element of a partition P and k_j as the multiplicity of j in P , the collection of sums on the right can be recognized as the sum over *all* partitions P

$$\sum_P = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots,$$

so that the index becomes

$$Z_N = \sum_P \int_{U(N)} DU \prod_{j=1}^{\infty} \frac{1}{j^{k_j} k_j!} (f(x^j) \text{Tr } U^j \text{Tr } U^{-j})^{k_j}.$$

Note that the product over j must be finite for any partition P of a finite number.

Representations of $U(n)$ and of symmetric group S_n share the property that they are both labelled by a partition λ of n . While the characters $\tilde{\chi}_\lambda(U)$ of $U(n)$ are functions of the conjugacy class of U in representation λ , the characters $\chi^\lambda(P)$ of S_n are functions of another partition P of n labelling the conjugacy class (i.e. a cycle shape) of a symmetric group element in S_n . For given U and P , the Frobenius character formula for $U(N)$ relates the characters of $U(n)$ and of S_n as

$$\prod_{j=1}^m (\text{Tr } U^j)^{k_j} = \sum_{\substack{\lambda \vdash |P| \\ l(\lambda) \leq N}} \tilde{\chi}_\lambda(U) \chi^\lambda(P).$$

Here, the weight $|P|$ of P is the integer being partitioned by P , and $l(\lambda)$ is the number of elements (or parts) in a partition λ . The partition P is encoded on the left hand side through the set of positive integers j in P and their multiplicities k_j . Since the product over j must be finite for any partition of a finite number, we denoted the largest integer in P as m .

The Frobenius formula allows us to evaluate the integral over $U(N)$ in the gauge theory index using the orthogonality relation for characters

$$\int DU \tilde{\chi}_\lambda(U) \tilde{\chi}_{\lambda'}(U^\dagger) = \delta_{\lambda\lambda'},$$

so that

$$\int DU \prod_{j=1}^m (\text{Tr } U^j \text{Tr } U^{-j})^{k_j} = \sum_{\substack{\lambda \vdash |P| \\ l(\lambda) \leq N}} \chi^\lambda(P)^2.$$

The expression on the right does not involve a complex conjugation because symmetric group characters are integers.

Combining, we have

$$Z_N(x) = \sum_P f(x)_P \frac{1}{z_P} \sum_{\substack{\lambda \vdash |P| \\ l(\lambda) \leq N}} \chi^\lambda(P)^2 \quad (\text{A.4.1})$$

where we defined

$$z_P := \prod_{j=1}^m j^{k_j} k_j!$$

and

$$f(x)_P := \prod_{j=1}^m f(x^j)^{k_j}.$$

The result of the derivation states that we need only generate character tables of symmetric groups to compute $U(N)$ gauge theory indices perturbatively. As in the Schur case, an appropriate resummation is necessary to obtain $\hat{Z}_N$ from a perturbative expression of Z_N .

A.5 Tests for the Specialized $\mathcal{N} = 4$ Index

In this section, we provide evidence in support of the giant graviton expansion for the specialized $\mathcal{N} = 4$ index discussed in Section 2.4.6. We take $x = w^2$ to avoid square roots

in the expansion:

$$Z_N(w; r) = Z_\infty(w; r) \left[1 + w^{2N} \hat{Z}_1(w; r) + w^{4N} \hat{Z}_2(w; r) + \cdots \right], \quad (\text{A.5.1})$$

where our prescription for the giant graviton corrections $\hat{Z}_k$ amounts to

$$\hat{}: \quad w \rightarrow \frac{1}{w}, \quad r \rightarrow \frac{r}{w}. \quad (\text{A.5.2})$$

To check our proposal, we truncate the expansion at the k -th correction as

$$Z_N(w; r) = Z_\infty(w; r) \left[1 + w^{2N} \hat{Z}_1(w; r) + \cdots + w^{2kN} \hat{Z}_k(w; r) \right], \quad (\text{A.5.3})$$

where we also defined

$$Z_0 = 1, \quad Z_{-1} = Z_{-2} = \cdots = 0.$$

The following tables enumerate the series coefficients of w of the left and right hand sides of (A.5.3) for given N , order of r , and truncation level k . Integers n denote the power of w in the series. The top row lists the powers n of w . The middle rows tabulate the coefficients of w^n of the right hand side of (A.5.3) for different values of the truncation k . The final row tabulates the coefficients of w^n of $Z_N(w; r)$.

From the results below, we observe that the order of w to which the formula holds, at a given order s of r , is

$$1 + k^2 + 2N(1 + k) + k(3 - 2s) - 3s. \quad (\text{A.5.4})$$

Checks beyond those presented below can be conducted by using the relation of $U(N)$ indices to symmetric group characters as reviewed in Appendix A.4.

At $N = 1$ and $O(r^0)$,

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$k=0$	1	0	1	0	2	0	3	0	5	0	7	0	11	0	15	0	22	0	30	0	42	0	56	0	77	0
$k=1$	1	0	1	0	1	0	1	0	1	0	0	0	-1	0	-4	0	-8	0	-15	0	-25	0	-41	0	-62	0
$k=2$	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	2	0	3	0	6	0	11	0
$k=3$	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0
LHS	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0

At $N = 1$ and $O(r^1)$,

n	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$k=0$	2	-2	2	-2	4	-4	6	-6	10	-10	14	-14	22	-22	30	-30	44	-44	60	-60	84	-84	112	-112
$k=1$	2	-2	0	0	0	0	-2	2	-4	4	-10	10	-16	16	-30	30	-46	46	-74	74	-110	110	-166	166
$k=2$	2	-2	0	0	0	0	0	0	0	0	0	0	2	-2	4	-4	10	-10	20	-20	36	-36	62	-62
$k=3$	2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2	2	-4	4
$k=4$	2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
LHS	2	-2	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = 1$ and $O(r^2)$,

n	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$k=0$	5	-4	4	-4	9	-8	13	-12	22	-20	30	-28	48	-44	64	-60	95	-88	128	-120	180
$k=1$	2	0	-6	8	-11	12	-21	24	-36	40	-63	68	-97	104	-155	164	-229	240	-341	356	-490
$k=2$	2	0	-3	4	-3	0	3	-4	8	-12	22	-28	45	-56	85	-100	146	-172	247	-280	388
$k=3$	2	0	-3	4	-3	0	0	0	0	0	0	0	-3	4	-8	12	-22	28	-46	60	-94
$k=4$	2	0	-3	4	-3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3
LHS	2	0	-3	4	-3	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = 2$ and $O(r^0)$,

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$k=0$	1	0	1	0	2	0	3	0	5	0	7	0	11	0	15	0	22	0	30	0	42	0	56	0	77
$k=1$	1	0	1	0	2	0	2	0	3	0	3	0	4	0	3	0	3	0	0	-3	0	-11	0	-20	
$k=2$	1	0	1	0	2	0	2	0	3	0	3	0	4	0	4	0	5	0	5	0	6	0	6	0	8
$k=3$	1	0	1	0	2	0	2	0	3	0	3	0	4	0	4	0	5	0	5	0	6	0	6	0	7
LHS	1	0	1	0	2	0	2	0	3	0	3	0	4	0	4	0	5	0	5	0	6	0	6	0	7

At $N = 2$ and $O(r^1)$,

n	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
$k=0$	2	-2	2	-2	4	-4	6	-6	10	-10	14	-14	22	-22	30	-30	44	-44	60	-60	84	-84	112	-112	154
$k=1$	2	-2	2	-2	2	-2	2	-2	2	-2	0	0	-2	2	-8	8	-16	16	-30	30	-50	50	-82	82	-124
$k=2$	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2
$k=3$	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2
LHS	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2	-2	2

At $N = 2$ and $O(r^2)$,

n	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$k=0$	5	-4	4	-4	9	-8	13	-12	22	-20	30	-28	48	-44	64	-60	95	-88	128	-120	180	-168	238
$k=1$	5	-4	1	0	-1	4	-7	8	-12	16	-28	32	-45	52	-81	88	-124	136	-196	208	-289	308	-432
$k=2$	5	-4	1	0	-1	4	-4	4	-4	4	-4	4	-1	0	4	-8	18	-24	44	-56	86	-104	156
$k=3$	5	-4	1	0	-1	4	-4	4	-4	4	-4	4	-4	4	-4	4	-4	4	-4	4	-7	8	-12
$k=4$	5	-4	1	0	-1	4	-4	4	-4	4	-4	4	-4	4	-4	4	-4	4	-4	4	-4	4	-4
LHS	5	-4	1	0	-1	4	-4	4	-4	4	-4	4	-4	4	-4	4	-4	4	-4	4	-4	4	-4

At $N = 2$ and $O(r^3)$,

n	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$k=0$	10	-10	8	-8	18	-18	26	-26	44	-44	60	-60	96	-96	128	-128	190	-190	256	-256	360
$k=1$	6	-4	-8	18	-26	32	-44	56	-78	96	-136	164	-214	250	-338	394	-508	578	-750	852	-1086
$k=2$	6	-4	-4	12	-14	12	-2	-6	14	-24	46	-64	96	-132	192	-246	334	-424	576	-708	914
$k=3$	6	-4	-4	12	-14	12	-6	0	2	-4	8	-8	4	-2	-4	12	-30	48	-80	116	-184
$k=4$	6	-4	-4	12	-14	12	-6	0	2	-4	8	-8	8	-8	8	-8	8	-8	8	-8	8
LHS	6	-4	-4	12	-14	12	-6	0	2	-4	8	-8	8	-8	8	-8	8	-8	8	-8	8

At $N = 2$ and $O(r^4)$,

n	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$k=0$	0	0	20	-20	15	-16	36	-36	51	-52	87	-88	118	-120	190	-192	252	-256	376	-380
$k=1$	-5	8	-2	20	-62	88	-111	132	-177	224	-298	364	-489	584	-736	868	-1128	1316	-1641	1888
$k=2$	0	0	14	-8	-2	-8	41	-92	171	-236	319	-420	574	-740	979	-1240	1606	-1988	2519	-3088
$k=3$	0	0	9	0	-18	20	-13	-8	29	-20	-14	52	-108	184	-304	436	-623	860	-1198	1576
$k=4$	0	0	9	0	-18	20	-8	-16	45	-48	40	-32	28	-20	19	-28	51	-76	128	-196
LHS	0	0	9	0	-18	20	-8	-16	45	-48	40	-32	23	-12	3	0	-3	8	-8	8

At $N = 3$ and $O(r^0)$,

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$k=0$	1	0	1	0	2	0	3	0	5	0	7	0	11	0	15	0	22	0	30	0	42	0	56	0	77	0
$k=1$	1	0	1	0	2	0	3	0	4	0	5	0	7	0	8	0	10	0	11	0	12	0	11	0	10	0
$k=2$	1	0	1	0	2	0	3	0	4	0	5	0	7	0	8	0	10	0	12	0	14	0	16	0	19	0
$k=3$	1	0	1	0	2	0	3	0	4	0	5	0	7	0	8	0	10	0	12	0	14	0	16	0	19	0
$k=4$	1	0	1	0	2	0	3	0	4	0	5	0	7	0	8	0	10	0	12	0	14	0	16	0	19	0
LHS	1	0	1	0	2	0	3	0	4	0	5	0	7	0	8	0	10	0	12	0	14	0	16	0	19	0

At $N = 3$ and $O(r^1)$,

n	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23
$k=0$	2	-2	2	-2	4	-4	6	-6	10	-10	14	-14	22	-22	30	-30	44	-44	60	-60	84	-84	112	-112	154
$k=1$	2	-2	2	-2	4	-4	4	-4	6	-6	6	-6	8	-8	8	-8	6	-6	6	-6	0	-6	6	-22	22
$k=2$	2	-2	2	-2	4	-4	4	-4	6	-6	6	-6	8	-8	8	-8	10	-10	10	-10	12	-12	12	-12	16
$k=3$	2	-2	2	-2	4	-4	4	-4	6	-6	6	-6	8	-8	8	-8	10	-10	10	-10	12	-12	12	-12	14
$k=4$	2	-2	2	-2	4	-4	4	-4	6	-6	6	-6	8	-8	8	-8	10	-10	10	-10	12	-12	12	-12	14
LHS	2	-2	2	-2	4	-4	4	-4	6	-6	6	-6	8	-8	8	-8	10	-10	10	-10	12	-12	12	-12	14

At $N = 3$ and $O(r^2)$,

n	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
$k=0$	5	-4	4	-4	9	-8	13	-12	22	-20	30	-28	48	-44	64	-60	95	-88	128	-120	180	-168	238	-224
$k=1$	5	-4	4	-4	6	-4	3	0	2	0	-4	8	-10	16	-29	36	-50	60	-91	104	-144	160	-231	252
$k=2$	5	-4	4	-4	6	-4	3	0	2	0	-1	4	-2	4	-5	8	-6	8	-6	8	-2	0	9	-12
$k=3$	5	-4	4	-4	6	-4	3	0	2	0	-1	4	-2	4	-5	8	-6	8	-9	12	-10	12	-13	16
$k=4$	5	-4	4	-4	6	-4	3	0	2	0	-1	4	-2	4	-5	8	-6	8	-9	12	-10	12	-13	16
LHS	5	-4	4	-4	6	-4	3	0	2	0	-1	4	-2	4	-5	8	-6	8	-9	12	-10	12	-13	16

At $N = 3$ and $O(r^3)$,

n	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$k = 0$	10	-10	8	-8	18	-18	26	-26	44	-44	60	-60	96	-96	128	-128	190	-190	256	-256	360	-360
$k = 1$	10	-10	4	-2	2	8	-18	24	-26	38	-62	80	-100	128	-182	218	-276	332	-442	512	-646	748
$k = 2$	10	-10	4	-2	2	8	-14	18	-14	18	-20	18	-8	8	0	-10	34	-50	88	-128	196	-254
$k = 3$	10	-10	4	-2	2	8	-14	18	-14	18	-20	18	-12	14	-12	10	-4	6	-4	2	0	4
$k = 4$	10	-10	4	-2	2	8	-14	18	-14	18	-20	18	-12	14	-12	10	-4	6	-4	2	4	-2
LHS	10	-10	4	-2	2	8	-14	18	-14	18	-20	18	-12	14	-12	10	-4	6	-4	2	4	-2

At $N = 3$ and $O(r^4)$,

n	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$k=0$	20	-20	15	-16	36	-36	51	-52	87	-88	118	-120	190	-192	252	-256	376	-380	505	-512	712
$k=1$	15	-12	-7	24	-41	68	-96	116	-141	188	-267	332	-417	512	-674	804	-1004	1192	-1512	1756	-2169
$k=2$	15	-12	-2	16	-25	40	-36	20	11	-36	81	-128	200	-272	389	-520	711	-916	1222	-1548	1991
$k=3$	15	-12	-2	16	-25	40	-41	28	-5	-8	27	-44	58	-56	56	-48	29	8	-61	128	-238
$k=4$	15	-12	-2	16	-25	40	-41	28	-5	-8	27	-44	63	-64	72	-76	83	-76	75	-76	85
LHS	15	-12	-2	16	-25	40	-41	28	-5	-8	27	-44	63	-64	72	-76	83	-76	75	-76	80

The pattern of convergence persists, where the expansion in terms of giant gravitons on the right hand side converges faster in k to Z_N for larger values of N .

The expansion also holds at $N \leq 0$:

At $N = 0$ and $O(r^0)$,

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
$k=0$	1	0	1	0	2	0	3	0	5	0	7	0	11	0	15	0	22	0	30	0	42	0	56	0	77	0
$k=1$	1	0	0	0	0	0	-1	0	-2	0	-5	0	-8	0	-15	0	-23	0	-37	0	-55	0	-83	0	-118	0
$k=2$	1	0	0	0	0	0	0	0	0	0	0	0	1	0	2	0	5	0	10	0	18	0	31	0	52	0
$k=3$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	-2	0	-5	0
$k=4$	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
LHS	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = 0$ and $O(r^1)$,

n	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$k=0$	2	-2	2	-2	4	-4	6	-6	10	-10	14	-14	22	-22	30	-30	44	-44	60	-60	84
$k=1$	0	0	-2	2	-4	4	-8	8	-14	14	-24	24	-38	38	-60	60	-90	90	-134	134	-194
$k=2$	0	0	0	0	0	0	2	-2	4	-4	10	-10	18	-18	34	-34	56	-56	94	-94	146
$k=3$	0	0	0	0	0	0	0	0	0	0	0	0	-2	2	-4	4	-10	10	-20	20	-38
$k=4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2
LHS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = 0$ and $O(r^2)$,

n	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$k=0$	0	0	5	-4	4	-4	9	-8	13	-12	22	-20	30	-28	48	-44	64	-60	95	-88	128
$k=1$	-3	4	-5	8	-16	16	-25	28	-45	48	-71	76	-115	120	-171	180	-260	268	-374	388	-542
$k=2$	0	0	3	-4	8	-12	19	-24	40	-48	71	-84	125	-144	204	-232	328	-368	504	-560	767
$k=3$	0	0	0	0	0	-3	4	-8	12	-22	28	-43	56	-89	108	-154	188	-271	320	-442	
$k=4$	0	0	0	0	0	0	0	0	0	0	0	3	-4	8	-12	22	-28	46	-60	92	
$k=5$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-3	
LHS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	

At $N = 0$ and $O(r^3)$,

n	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11
$k=0$	0	0	0	0	0	0	10	-10	8	-8	18	-18	26	-26	44	-44	60	-60	96	-96	128
$k=1$	0	0	-4	6	-16	26	-34	40	-62	74	-104	122	-170	198	-266	302	-406	462	-602	672	-878
$k=2$	4	-6	8	-14	26	-36	58	-80	120	-154	206	-260	360	-442	576	-700	920	-1098	1398	-1656	2112
$k=3$	0	0	-4	6	-12	20	-34	50	-76	104	-158	212	-296	382	-522	664	-878	1094	-1434	1766	-2258
$k=4$	0	0	0	0	0	0	4	-6	12	-20	38	-56	84	-118	180	-242	342	-454	630	-810	1080
$k=5$	0	0	0	0	0	0	0	0	0	0	0	0	-4	6	-12	20	-38	56	-88	124	-188
$k=6$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	4
LHS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = -1$ and $O(r^0)$,

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
$k=0$	1	0	1	0	2	0	3	0	5	0	7	0	11	0	15	0	22	0	30	0	42	0	56	0	77
$k=1$	0	0	-1	0	-2	0	-4	0	-7	0	-12	0	-19	0	-30	0	-45	0	-67	0	-97	0	-139	0	-195
$k=2$	0	0	0	0	0	0	1	0	2	0	5	0	9	0	17	0	28	0	47	0	73	0	114	0	170
$k=3$	0	0	0	0	0	0	0	0	0	0	0	0	-1	0	-2	0	-5	0	-10	0	-19	0	-33	0	-57
$k=4$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	2	0	5
LHS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = -1$ and $O(r^1)$,

n	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
$k=0$	0	0	2	-2	2	-2	4	-4	6	-6	10	-10	14	-14	22	-22	30	-30	44	-44	60
$k=1$	-2	2	-2	2	-6	6	-10	10	-18	18	-28	28	-46	46	-68	68	-104	104	-150	150	-218
$k=2$	0	0	2	-2	4	-4	8	-8	16	-16	28	-28	48	-48	78	-78	124	-124	190	-190	288
$k=3$	0	0	0	0	0	0	-2	2	-4	4	-10	10	-18	18	-36	36	-60	60	-104	104	-166
$k=4$	0	0	0	0	0	0	0	0	0	0	0	0	2	-2	4	-4	10	-10	20	-20	38
$k=5$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-2
LHS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = -1$ and $O(r^2)$,

n	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12
$k=0$	0	0	0	0	0	0	5	-4	4	-4	9	-8	13	-12	22	-20	30	-28	48	-44	64
$k=1$	0	0	-3	4	-10	12	-15	16	-30	32	-49	52	-80	84	-123	128	-189	196	-276	284	-405
$k=2$	3	-4	5	-8	14	-16	29	-36	55	-64	93	-108	160	-180	252	-284	399	-440	602	-664	904
$k=3$	0	0	-3	4	-8	12	-19	24	-38	48	-75	92	-133	160	-230	272	-376	440	-607	700	-940
$k=4$	0	0	0	0	0	0	3	-4	8	-12	22	-28	43	-56	87	-108	158	-196	279	-336	466
$k=5$	0	0	0	0	0	0	0	0	0	0	0	0	-3	4	-8	12	-22	28	-46	60	-92
$k=6$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	3
LHS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = -2$ and $O(r^0)$,

n	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$k=0$	0	0	1	0	1	0	2	0	3	0	5	0	7	0	11	0	15	0	22	0	30
$k=1$	-1	0	-1	0	-3	0	-5	0	-9	0	-14	0	-23	0	-34	0	-52	0	-75	0	-109
$k=2$	0	0	1	0	2	0	4	0	8	0	14	0	24	0	39	0	62	0	95	0	144
$k=3$	0	0	0	0	0	0	-1	0	-2	0	-5	0	-9	0	-18	0	-30	0	-52	0	-83
$k=4$	0	0	0	0	0	0	0	0	0	0	0	1	0	2	0	5	0	10	0	19	
$k=5$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	-1	
LHS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	

At $N = -2$ and $O(r^1)$,

n	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$k = 0$	0	0	0	0	0	0	2	-2	2	-2	4	-4	6	-6	10	-10	14	-14	22	-22	30
$k = 1$	0	0	-2	2	-4	4	-6	6	-12	12	-20	20	-32	32	-50	50	-76	76	-112	112	-164
$k = 2$	2	-2	2	-2	6	-6	12	-12	22	-22	36	-36	62	-62	96	-96	152	-152	228	-228	342
$k = 3$	0	0	-2	2	-4	4	-8	8	-16	16	-30	30	-52	52	-88	88	-142	142	-226	226	-348
$k = 4$	0	0	0	0	0	0	2	-2	4	-4	10	-10	18	-18	36	-36	62	-62	108	-108	176
$k = 5$	0	0	0	0	0	0	0	0	0	0	0	0	-2	2	-4	4	-10	10	-20	20	-38
$k = 6$	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	2
LHS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

At $N = -2$ and $O(r^2)$,

n	-14	-13	-12	-11	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$k=0$	0	0	0	0	0	0	0	0	0	0	0	0	5	-4	4	-4	9	-8	13	-12	22
$k=1$	0	0	0	0	0	0	-3	4	-10	12	-20	20	-32	32	-54	56	-84	88	-132	136	-197
$k=2$	0	0	3	-4	8	-12	21	-24	34	-40	65	-76	113	-128	186	-208	291	-324	456	-500	681
$k=3$	-3	4	-5	8	-14	16	-27	36	-59	72	-103	124	-180	212	-296	348	-484	556	-753	864	-1163
$k=4$	0	0	3	-4	8	-12	19	-24	38	-48	73	-92	137	-168	238	-288	402	-480	653	-772	1042
$k=5$	0	0	0	0	0	0	-3	4	-8	12	-22	28	-43	56	-87	108	-156	196	-283	344	-474
$k=6$	0	0	0	0	0	0	0	0	0	0	0	0	3	-4	8	-12	22	-28	46	-60	92
LHS	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

A.6 Tests for a General Class of $U(N)$ Indices

Here, we provide evidence in support of the giant graviton expansion for a more general class of $U(N)$ indices

$$Z_N = \int_{U(N)} DU \exp \left(\sum_{j=1}^{\infty} \frac{1}{j} f(x^j) \text{Tr } U^j \text{Tr } U^{-j} \right) \quad (\text{A.6.1})$$

with

$$f(x^j) = x^j + (1 - x^j)h_j. \quad (\text{A.6.2})$$

Under tilde, the quantity h_j transforms as

$$\sim : h_j \rightarrow \tilde{h}_j = \frac{h_j}{h_j - 1}.$$

This class of indices namely includes the $U(N)$ $\mathcal{N} = 4$ SYM index.

We consider the truncated expansion

$$Z_N/Z_{\infty} = 1 + x^N \hat{Z}_1 + x^{2N} \hat{Z}_2 + \cdots + x^{kN} \hat{Z}_k. \quad (\text{A.6.3})$$

with

$$Z_0 = 1, \quad Z_{-1} = Z_{-2} = \cdots = 0.$$

We compare the perturbative series in x that arise as coefficients of monomials of h_j , on each side of (A.6.3).

We compare the series expansion in x on left and right sides of (A.6.3) for given N , truncation level k , and monomials of h_j for which the x -series is a coefficient. For convenience, we write the left and right sides as $L_N[\{h_j\}]$ and $R_N^{(k)}[\{h_j\}]$, where $\{h_j\}$ denotes the monomial in h_j .

Checks beyond those presented below can be conducted by using the relation of $U(N)$ indices to symmetric group characters reviewed in Appendix A.4.

For $N = 1$ and coefficient of 1, we have

$$\begin{aligned}
L_1[1] &= 1 - x^2 - x^3 - x^4 + x^7 + x^8 + x^9 + x^{10} + x^{11} - x^{15} + O(x^{16}) \\
R_1^{(0)}[1] &= 1 + O(x^{16}) \\
R_1^{(1)}[1] &= 1 - x^2 - x^3 - x^4 - x^5 - x^6 - x^7 - x^8 - x^9 - x^{10} - x^{11} - x^{12} - x^{13} - x^{14} - x^{15} + O(x^{16}) \\
R_1^{(2)}[1] &= 1 - x^2 - x^3 - x^4 + x^7 + x^8 + 2x^9 + 2x^{10} + 3x^{11} + 3x^{12} + 4x^{13} + 4x^{14} + 5x^{15} + O(x^{16}) \\
R_1^{(3)}[1] &= 1 - x^2 - x^3 - x^4 + x^7 + x^8 + x^9 + x^{10} + x^{11} - x^{14} - 2x^{15} + O(x^{16}) \\
R_1^{(4)}[1] &= 1 - x^2 - x^3 - x^4 + x^7 + x^8 + x^9 + x^{10} + x^{11} - x^{15} + O(x^{16}) \\
R_1^{(5)}[1] &= 1 - x^2 - x^3 - x^4 + x^7 + x^8 + x^9 + x^{10} + x^{11} - x^{15} + O(x^{16})
\end{aligned}$$

For $N = 1$ and coefficient of h_1 , we have

$$\begin{aligned}
L_1[h_1] &= -x + x^3 + x^4 + x^5 - x^8 - x^9 - x^{10} - x^{11} - x^{12} + O(x^{15}) \\
R_1^{(0)}[h_1] &= O(x^{15}) \\
R_1^{(1)}[h_1] &= -x + O(x^{15}) \\
R_1^{(2)}[h_1] &= -x + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + O(x^{15}) \\
R_1^{(3)}[h_1] &= -x + x^3 + x^4 + x^5 - x^8 - x^9 - 2x^{10} - 2x^{11} - 3x^{12} - 3x^{13} - 4x^{14} + O(x^{15}) \\
R_1^{(4)}[h_1] &= -x + x^3 + x^4 + x^5 - x^8 - x^9 - x^{10} - x^{11} - x^{12} + O(x^{15}) \\
R_1^{(5)}[h_1] &= -x + x^3 + x^4 + x^5 - x^8 - x^9 - x^{10} - x^{11} - x^{12} + O(x^{15})
\end{aligned}$$

For $N = 1$ and coefficient of h_1^2 , we have

$$\begin{aligned}
L_1[h_1^2] &= -\frac{1}{2} + x^2 + \frac{x^3}{2} - \frac{x^5}{2} - \frac{x^6}{2} - \frac{x^7}{2} - \frac{x^8}{2} + \frac{x^{12}}{2} + \frac{x^{13}}{2} + O(x^{14}) \\
R_1^{(0)}[h_1^2] &= O(x^{14}) \\
R_1^{(1)}[h_1^2] &= -\frac{1}{2} - \frac{x}{2} + O(x^{14}) \\
R_1^{(2)}[h_1^2] &= -\frac{1}{2} + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + O(x^{14}) \\
R_1^{(3)}[h_1^2] &= -\frac{1}{2} + x^2 + \frac{x^3}{2} - \frac{x^5}{2} - x^6 - \frac{3x^7}{2} - 2x^8 - \frac{5x^9}{2} - 3x^{10} - \frac{7x^{11}}{2} - 4x^{12} - \frac{9x^{13}}{2} + O(x^{14}) \\
R_1^{(4)}[h_1^2] &= -\frac{1}{2} + x^2 + \frac{x^3}{2} - \frac{x^5}{2} - \frac{x^6}{2} - \frac{x^7}{2} - \frac{x^8}{2} + \frac{x^{10}}{2} + x^{11} + 2x^{12} + 3x^{13} + O(x^{14}) \\
R_1^{(5)}[h_1^2] &= -\frac{1}{2} + x^2 + \frac{x^3}{2} - \frac{x^5}{2} - \frac{x^6}{2} - \frac{x^7}{2} - \frac{x^8}{2} + \frac{x^{12}}{2} + \frac{x^{13}}{2} + O(x^{14})
\end{aligned}$$

For $N = 1$ and coefficient of h_2 , we have

$$\begin{aligned}
L_1[h_2] &= -\frac{1}{2} + \frac{x^3}{2} + x^4 + \frac{x^5}{2} + \frac{x^6}{2} - \frac{x^7}{2} - \frac{x^8}{2} - x^9 - x^{10} - x^{11} - \frac{x^{12}}{2} - \frac{x^{13}}{2} + O(x^{14}) \\
R_1^{(0)}[h_2] &= O(x^{14}) \\
R_1^{(1)}[h_2] &= -\frac{1}{2} - \frac{x}{2} + O(x^{14}) \\
R_1^{(2)}[h_2] &= -\frac{1}{2} + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + O(x^{14}) \\
R_1^{(3)}[h_2] &= -\frac{1}{2} + \frac{x^3}{2} + x^4 + \frac{x^5}{2} - \frac{x^7}{2} - x^8 - \frac{3x^9}{2} - 2x^{10} - \frac{5x^{11}}{2} - 3x^{12} - \frac{7x^{13}}{2} + O(x^{14}) \\
R_1^{(4)}[h_2] &= -\frac{1}{2} + \frac{x^3}{2} + x^4 + \frac{x^5}{2} + \frac{x^6}{2} - \frac{x^7}{2} - \frac{x^8}{2} - x^9 - \frac{x^{10}}{2} - x^{11} + O(x^{14}) \\
R_1^{(5)}[h_2] &= -\frac{1}{2} + \frac{x^3}{2} + x^4 + \frac{x^5}{2} + \frac{x^6}{2} - \frac{x^7}{2} - \frac{x^8}{2} - x^9 - x^{10} - x^{11} - \frac{x^{12}}{2} - \frac{x^{13}}{2} + O(x^{14})
\end{aligned}$$

For $N = 1$ and coefficient of $h_1 h_2$, we have

$$\begin{aligned}
L_1[h_1 h_2] &= \frac{x}{2} - \frac{x^4}{2} - x^5 - \frac{x^6}{2} - \frac{x^7}{2} + \frac{x^8}{2} + \frac{x^9}{2} + x^{10} + x^{11} + x^{12} + O(x^{13}) \\
R_1^{(0)}[h_1 h_2] &= O(x^{13}) \\
R_1^{(1)}[h_1 h_2] &= -\frac{1}{2x} + \frac{x}{2} + O(x^{13}) \\
R_1^{(2)}[h_1 h_2] &= \frac{1}{2} + x + \frac{x^2}{2} + O(x^{13}) \\
R_1^{(3)}[h_1 h_2] &= \frac{x}{2} - \frac{x^2}{2} - \frac{x^3}{2} - \frac{3x^4}{2} - \frac{3x^5}{2} - \frac{3x^6}{2} - \frac{3x^7}{2} - \frac{3x^8}{2} - \frac{3x^9}{2} - \frac{3x^{10}}{2} - \frac{3x^{11}}{2} - \frac{3x^{12}}{2} + O(x^{13}) \\
R_1^{(4)}[h_1 h_2] &= \frac{x}{2} - \frac{x^4}{2} - \frac{x^5}{2} + \frac{x^7}{2} + \frac{3x^8}{2} + 2x^9 + \frac{5x^{10}}{2} + \frac{7x^{11}}{2} + 4x^{12} + O(x^{13}) \\
R_1^{(5)}[h_1 h_2] &= \frac{x}{2} - \frac{x^4}{2} - x^5 - \frac{x^6}{2} - \frac{x^7}{2} + \frac{x^8}{2} + \frac{x^{10}}{2} + O(x^{13})
\end{aligned}$$

For $N = 1$ and coefficient of h_3 , we have

$$\begin{aligned}
L_1[h_3] &= -\frac{2}{3} + \frac{2x^2}{3} + \frac{x^3}{3} + \frac{2x^4}{3} + \frac{x^5}{3} + \frac{x^6}{3} - \frac{x^7}{3} - \frac{2x^8}{3} - \frac{2x^9}{3} - x^{10} - x^{11} - \frac{x^{12}}{3} + O(x^{13}) \\
R_1^{(0)}[h_3] &= O(x^{13}) \\
R_1^{(1)}[h_3] &= -\frac{1}{3x} - \frac{1}{3} - \frac{x}{3} + O(x^{13}) \\
R_1^{(2)}[h_3] &= -\frac{1}{3} + \frac{x^2}{3} + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + O(x^{13}) \\
R_1^{(3)}[h_3] &= -\frac{2}{3} + \frac{x^2}{3} + \frac{x^3}{3} + \frac{2x^4}{3} + \frac{x^5}{3} - \frac{x^6}{3} - \frac{2x^7}{3} - \frac{4x^8}{3} - \frac{5x^9}{3} - \frac{7x^{10}}{3} - \frac{8x^{11}}{3} - \frac{10x^{12}}{3} + O(x^{13}) \\
R_1^{(4)}[h_3] &= -\frac{2}{3} + \frac{2x^2}{3} + \frac{x^3}{3} + \frac{2x^4}{3} + \frac{2x^5}{3} + \frac{x^6}{3} - \frac{x^7}{3} - \frac{x^8}{3} - \frac{2x^9}{3} - \frac{x^{10}}{3} - \frac{x^{11}}{3} + \frac{x^{12}}{3} + O(x^{13}) \\
R_1^{(5)}[h_3] &= -\frac{2}{3} + \frac{2x^2}{3} + \frac{x^3}{3} + \frac{2x^4}{3} + \frac{x^5}{3} + \frac{x^6}{3} + \frac{x^7}{3} - \frac{2x^8}{3} - x^9 - x^{10} - x^{11} - \frac{2x^{12}}{3} + O(x^{13})
\end{aligned}$$

For $N = 2$ and coefficient of 1, we have

$$\begin{aligned}
L_2[1] &= 1 - x^3 - x^4 - x^5 - x^6 + x^9 + x^{10} + 2x^{11} + x^{12} + 2x^{13} + x^{14} + x^{15} + O(x^{16}) \\
R_2^{(0)}[1] &= 1 + O(x^{16}) \\
R_2^{(1)}[1] &= 1 - x^3 - x^4 - x^5 - x^6 - x^7 - x^8 - x^9 - x^{10} - x^{11} - x^{12} - x^{13} - x^{14} - x^{15} + O(x^{16}) \\
R_2^{(2)}[1] &= 1 - x^3 - x^4 - x^5 - x^6 + x^9 + x^{10} + 2x^{11} + 2x^{12} + 3x^{13} + 3x^{14} + 4x^{15} + O(x^{16}) \\
R_2^{(3)}[1] &= 1 - x^3 - x^4 - x^5 - x^6 + x^9 + x^{10} + 2x^{11} + x^{12} + 2x^{13} + x^{14} + x^{15} + O(x^{16}) \\
R_2^{(4)}[1] &= 1 - x^3 - x^4 - x^5 - x^6 + x^9 + x^{10} + 2x^{11} + x^{12} + 2x^{13} + x^{14} + x^{15} + O(x^{16})
\end{aligned}$$

For $N = 2$ and coefficient of h_1 , we have

$$\begin{aligned}
L_2[h_1] &= -x^2 + x^5 + x^6 + x^7 + x^8 - x^{11} - x^{12} - 2x^{13} - x^{14} + O(x^{15}) \\
R_2^{(0)}[h_1] &= O(x^{15}) \\
R_2^{(1)}[h_1] &= -x^2 + O(x^{15}) \\
R_2^{(2)}[h_1] &= -x^2 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + O(x^{15}) \\
R_2^{(3)}[h_1] &= -x^2 + x^5 + x^6 + x^7 + x^8 - x^{11} - x^{12} - 2x^{13} - 2x^{14} + O(x^{15}) \\
R_2^{(4)}[h_1] &= -x^2 + x^5 + x^6 + x^7 + x^8 - x^{11} - x^{12} - 2x^{13} - x^{14} + O(x^{15})
\end{aligned}$$

For $N = 2$ and coefficient of h_1^2 , we have

$$\begin{aligned}
L_2[h_1^2] &= -\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{2} + x^4 + x^5 + \frac{x^6}{2} - \frac{x^8}{2} - x^9 - x^{10} - x^{11} - x^{12} - \frac{x^{13}}{2} + O(x^{14}) \\
R_2^{(0)}[h_1^2] &= O(x^{14}) \\
R_2^{(1)}[h_1^2] &= -\frac{x}{2} - \frac{x^2}{2} + O(x^{14}) \\
R_2^{(2)}[h_1^2] &= -\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{2} + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + O(x^{14}) \\
R_2^{(3)}[h_1^2] &= -\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{2} + x^4 + x^5 + \frac{x^6}{2} - \frac{x^8}{2} - x^9 - \frac{3x^{10}}{2} - 2x^{11} - \frac{5x^{12}}{2} - 3x^{13} + O(x^{14}) \\
R_2^{(4)}[h_1^2] &= -\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{2} + x^4 + x^5 + \frac{x^6}{2} - \frac{x^8}{2} - x^9 - x^{10} - x^{11} - x^{12} - \frac{x^{13}}{2} + O(x^{14})
\end{aligned}$$

For $N = 2$ and coefficient of h_2 , we have

$$\begin{aligned}
L_2[h_2] &= -\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{2} + x^5 + \frac{x^6}{2} + x^7 + \frac{x^8}{2} - x^{11} - x^{12} - \frac{3x^{13}}{2} + O(x^{14}) \\
R_2^{(0)}[h_2] &= O(x^{14}) \\
R_2^{(1)}[h_2] &= -\frac{x}{2} - \frac{x^2}{2} + O(x^{14}) \\
R_2^{(2)}[h_2] &= -\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{2} + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + O(x^{14}) \\
R_2^{(3)}[h_2] &= -\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{2} + x^5 + \frac{x^6}{2} + x^7 + \frac{x^8}{2} - \frac{x^{10}}{2} - x^{11} - \frac{3x^{12}}{2} - 2x^{13} + O(x^{14}) \\
R_2^{(4)}[h_2] &= -\frac{x}{2} - \frac{x^2}{2} + \frac{x^3}{2} + x^5 + \frac{x^6}{2} + x^7 + \frac{x^8}{2} - x^{11} - x^{12} - \frac{3x^{13}}{2} + O(x^{14})
\end{aligned}$$

For $N = 2$ and coefficient of $h_1 h_2$, we have

$$\begin{aligned}
L_2[h_1 h_2] &= -\frac{1}{2} + \frac{x}{2} + x^2 - x^5 - x^7 - \frac{x^8}{2} - \frac{x^9}{2} - \frac{x^{10}}{2} + \frac{x^{12}}{2} + O(x^{13}) \\
R_2^{(0)}[h_1 h_2] &= O(x^{13}) \\
R_2^{(1)}[h_1 h_2] &= -\frac{1}{2} + \frac{x^2}{2} + O(x^{13}) \\
R_2^{(2)}[h_1 h_2] &= -\frac{1}{2} + \frac{x}{2} + x^2 + \frac{x^3}{2} + \frac{x^4}{2} + O(x^{13}) \\
R_2^{(3)}[h_1 h_2] &= -\frac{1}{2} + \frac{x}{2} + x^2 - x^5 - \frac{x^6}{2} - \frac{3x^7}{2} - \frac{3x^8}{2} - \frac{3x^9}{2} - \frac{3x^{10}}{2} - \frac{3x^{11}}{2} - \frac{3x^{12}}{2} + O(x^{13}) \\
R_2^{(4)}[h_1 h_2] &= -\frac{1}{2} + \frac{x}{2} + x^2 - x^5 - x^7 - \frac{x^8}{2} - \frac{x^9}{2} + \frac{x^{11}}{2} + \frac{3x^{12}}{2} + O(x^{13}) \\
R_2^{(5)}[h_1 h_2] &= -\frac{1}{2} + \frac{x}{2} + x^2 - x^5 - x^7 - \frac{x^8}{2} - \frac{x^9}{2} - \frac{x^{10}}{2} + \frac{x^{12}}{2} + O(x^{13})
\end{aligned}$$

For $N = 2$ and coefficient of h_3 , we have

$$\begin{aligned}
L_2[h_3] &= -\frac{1}{3} - \frac{x^2}{3} + \frac{x^4}{3} + x^5 + \frac{2x^6}{3} + \frac{2x^7}{3} + \frac{x^8}{3} - \frac{x^{10}}{3} - x^{11} - \frac{2x^{12}}{3} + O(x^{13}) \\
R_2^{(0)}[h_3] &= O(x^{13}) \\
R_2^{(1)}[h_3] &= -\frac{1}{3} - \frac{x}{3} - \frac{x^2}{3} + O(x^{13}) \\
R_2^{(2)}[h_3] &= -\frac{1}{3} - \frac{x^2}{3} + \frac{x^3}{3} + \frac{x^4}{3} + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + O(x^{13}) \\
R_2^{(3)}[h_3] &= -\frac{1}{3} - \frac{x^2}{3} + \frac{x^4}{3} + x^5 + \frac{x^6}{3} + \frac{2x^7}{3} + \frac{x^8}{3} - \frac{x^9}{3} - \frac{2x^{10}}{3} - \frac{4x^{11}}{3} - \frac{5x^{12}}{3} + O(x^{13}) \\
R_2^{(4)}[h_3] &= -\frac{1}{3} - \frac{x^2}{3} + \frac{x^4}{3} + x^5 + \frac{2x^6}{3} + \frac{2x^7}{3} + \frac{x^8}{3} - x^{11} - \frac{2x^{12}}{3} + O(x^{13}) \\
R_2^{(5)}[h_3] &= -\frac{1}{3} - \frac{x^2}{3} + \frac{x^4}{3} + x^5 + \frac{2x^6}{3} + \frac{2x^7}{3} + \frac{x^8}{3} - \frac{x^{10}}{3} - x^{11} - \frac{2x^{12}}{3} + O(x^{13})
\end{aligned}$$

For $N = 3$ and coefficient of 1, we have

$$\begin{aligned}
L_3[1] &= 1 - x^4 - x^5 - x^6 - x^7 - x^8 + x^{11} + x^{12} + 2x^{13} + 2x^{14} + 2x^{15} + O(x^{16}) \\
R_3^{(0)}[1] &= 1 + O(x^{16}) \\
R_3^{(1)}[1] &= 1 - x^4 - x^5 - x^6 - x^7 - x^8 - x^9 - x^{10} - x^{11} - x^{12} - x^{13} - x^{14} - x^{15} + O(x^{16}) \\
R_3^{(2)}[1] &= 1 - x^4 - x^5 - x^6 - x^7 - x^8 + x^{11} + x^{12} + 2x^{13} + 2x^{14} + 3x^{15} + O(x^{16}) \\
R_3^{(3)}[1] &= 1 - x^4 - x^5 - x^6 - x^7 - x^8 + x^{11} + x^{12} + 2x^{13} + 2x^{14} + 2x^{15} + O(x^{16}) \\
R_3^{(4)}[1] &= 1 - x^4 - x^5 - x^6 - x^7 - x^8 + x^{11} + x^{12} + 2x^{13} + 2x^{14} + 2x^{15} + O(x^{16})
\end{aligned}$$

For $N = 3$ and coefficient of h_1 , we have

$$\begin{aligned}
L_3[h_1] &= -x^3 + x^7 + x^8 + x^9 + x^{10} + x^{11} - x^{14} + O(x^{15}) \\
R_3^{(0)}[h_1] &= O(x^{15}) \\
R_3^{(1)}[h_1] &= -x^3 + O(x^{15}) \\
R_3^{(2)}[h_1] &= -x^3 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + O(x^{15}) \\
R_3^{(3)}[h_1] &= -x^3 + x^7 + x^8 + x^9 + x^{10} + x^{11} - x^{14} + O(x^{15}) \\
R_3^{(4)}[h_1] &= -x^3 + x^7 + x^8 + x^9 + x^{10} + x^{11} - x^{14} + O(x^{15})
\end{aligned}$$

For $N = 3$ and coefficient of h_1^2 , we have

$$\begin{aligned}
L_3[h_1^2] &= -\frac{x^2}{2} - \frac{x^3}{2} + \frac{x^5}{2} + x^6 + x^7 + x^8 + \frac{x^9}{2} - \frac{x^{11}}{2} - x^{12} - \frac{3x^{13}}{2} + O(x^{14}) \\
R_3^{(0)}[h_1^2] &= O(x^{14}) \\
R_3^{(1)}[h_1^2] &= -\frac{x^2}{2} - \frac{x^3}{2} + O(x^{14}) \\
R_3^{(2)}[h_1^2] &= -\frac{x^2}{2} - \frac{x^3}{2} + \frac{x^5}{2} + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + O(x^{14}) \\
R_3^{(3)}[h_1^2] &= -\frac{x^2}{2} - \frac{x^3}{2} + \frac{x^5}{2} + x^6 + x^7 + x^8 + \frac{x^9}{2} - \frac{x^{11}}{2} - x^{12} - \frac{3x^{13}}{2} + O(x^{14}) \\
R_3^{(4)}[h_1^2] &= -\frac{x^2}{2} - \frac{x^3}{2} + \frac{x^5}{2} + x^6 + x^7 + x^8 + \frac{x^9}{2} - \frac{x^{11}}{2} - x^{12} - \frac{3x^{13}}{2} + O(x^{14})
\end{aligned}$$

For $N = 3$ and coefficient of h_2 , we have

$$\begin{aligned}
L_3[h_2] &= -\frac{x^2}{2} - \frac{x^3}{2} + \frac{x^5}{2} + x^7 + x^8 + \frac{x^9}{2} + x^{10} + \frac{x^{11}}{2} - \frac{x^{13}}{2} + O(x^{14}) \\
R_3^{(0)}[h_2] &= O(x^{14}) \\
R_3^{(1)}[h_2] &= -\frac{x^2}{2} - \frac{x^3}{2} + O(x^{14}) \\
R_3^{(2)}[h_2] &= -\frac{x^2}{2} - \frac{x^3}{2} + \frac{x^5}{2} + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + O(x^{14}) \\
R_3^{(3)}[h_2] &= -\frac{x^2}{2} - \frac{x^3}{2} + \frac{x^5}{2} + x^7 + x^8 + \frac{x^9}{2} + x^{10} + \frac{x^{11}}{2} - \frac{x^{13}}{2} + O(x^{14}) \\
R_3^{(4)}[h_2] &= -\frac{x^2}{2} - \frac{x^3}{2} + \frac{x^5}{2} + x^7 + x^8 + \frac{x^9}{2} + x^{10} + \frac{x^{11}}{2} - \frac{x^{13}}{2} + O(x^{14})
\end{aligned}$$

For $N = 3$ and coefficient of $h_1 h_2$, we have

$$\begin{aligned}
L_3[h_1 h_2] &= -\frac{x}{2} + x^3 + \frac{x^4}{2} + \frac{x^5}{2} - \frac{x^7}{2} - x^8 - \frac{x^9}{2} - x^{10} - x^{11} - \frac{x^{12}}{2} + O(x^{13}) \\
R_3^{(0)}[h_1 h_2] &= O(x^{13}) \\
R_3^{(1)}[h_1 h_2] &= -\frac{x}{2} + \frac{x^3}{2} + O(x^{13}) \\
R_3^{(2)}[h_1 h_2] &= -\frac{x}{2} + x^3 + \frac{x^4}{2} + \frac{x^5}{2} + \frac{x^6}{2} + O(x^{13}) \\
R_3^{(3)}[h_1 h_2] &= -\frac{x}{2} + x^3 + \frac{x^4}{2} + \frac{x^5}{2} - \frac{x^7}{2} - x^8 - \frac{x^9}{2} - \frac{3x^{10}}{2} - \frac{3x^{11}}{2} - \frac{3x^{12}}{2} + O(x^{13}) \\
R_3^{(4)}[h_1 h_2] &= -\frac{x}{2} + x^3 + \frac{x^4}{2} + \frac{x^5}{2} - \frac{x^7}{2} - x^8 - \frac{x^9}{2} - x^{10} - x^{11} - \frac{x^{12}}{2} + O(x^{13})
\end{aligned}$$

For $N = 3$ and coefficient of h_3 , we have

$$\begin{aligned}
L_3[h_3] &= -\frac{x}{3} - \frac{x^2}{3} + \frac{x^5}{3} + x^7 + x^8 + \frac{x^9}{3} + x^{10} + \frac{x^{11}}{3} - \frac{x^{12}}{3} + O(x^{13}) \\
R_3^{(0)}[h_3] &= O(x^{13}) \\
R_3^{(1)}[h_3] &= -\frac{x}{3} - \frac{x^2}{3} - \frac{x^3}{3} + O(x^{13}) \\
R_3^{(2)}[h_3] &= -\frac{x}{3} - \frac{x^2}{3} + \frac{x^5}{3} + \frac{x^6}{3} + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + O(x^{13}) \\
R_3^{(3)}[h_3] &= -\frac{x}{3} - \frac{x^2}{3} + \frac{x^5}{3} + x^7 + x^8 + \frac{x^9}{3} + \frac{2x^{10}}{3} + \frac{x^{11}}{3} - \frac{x^{12}}{3} + O(x^{13}) \\
R_3^{(4)}[h_3] &= -\frac{x}{3} - \frac{x^2}{3} + \frac{x^5}{3} + x^7 + x^8 + \frac{x^9}{3} + x^{10} + \frac{x^{11}}{3} - \frac{x^{12}}{3} + O(x^{13})
\end{aligned}$$

For $N = 0$ and coefficient of 1, we have

$$\begin{aligned}
L_0[1] &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + O(x^{16}) \\
R_0^{(0)}[1] &= 1 + O(x^{16}) \\
R_0^{(1)}[1] &= 1 - x - x^2 - x^3 - x^4 - x^5 - x^6 - x^7 - x^8 - x^9 - x^{10} - x^{11} - x^{12} - x^{13} - x^{14} - x^{15} + O(x^{16}) \\
R_0^{(2)}[1] &= 1 - x - x^2 + x^5 + x^6 + 2x^7 + 2x^8 + 3x^9 + 3x^{10} + 4x^{11} + 4x^{12} + 5x^{13} + 5x^{14} + 6x^{15} + O(x^{16}) \\
R_0^{(3)}[1] &= 1 - x - x^2 + x^5 + x^7 - x^{10} - x^{11} - 3x^{12} - 3x^{13} - 5x^{14} - 6x^{15} + O(x^{16}) \\
R_0^{(4)}[1] &= 1 - x - x^2 + x^5 + x^7 - x^{12} + O(x^{16}) \\
R_0^{(5)}[1] &= 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + O(x^{16})
\end{aligned}$$

For $N = 0$ and coefficient of h_1 , we have

$$\begin{aligned}
L_0[h_1] &= -1 + x + x^2 - x^5 - x^7 + x^{12} + O(x^{15}) \\
R_0^{(0)}[h_1] &= O(x^{15}) \\
R_0^{(1)}[h_1] &= -1 + O(x^{15}) \\
R_0^{(2)}[h_1] &= -1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + x^{14} + O(x^{15}) \\
R_0^{(3)}[h_1] &= -1 + x + x^2 - x^5 - x^6 - 2x^7 - 2x^8 - 3x^9 - 3x^{10} - 4x^{11} - 4x^{12} - 5x^{13} - 5x^{14} + O(x^{15}) \\
R_0^{(4)}[h_1] &= -1 + x + x^2 - x^5 - x^7 + x^{10} + x^{11} + 3x^{12} + 3x^{13} + 5x^{14} + O(x^{15}) \\
R_0^{(5)}[h_1] &= -1 + x + x^2 - x^5 - x^7 + x^{12} + O(x^{15})
\end{aligned}$$

For $N = 0$ and coefficient of h_1^2 , we have

$$\begin{aligned}
L_0[h_1^2] &= 0 \\
R_0^{(0)}[h_1^2] &= O(x^{14}) \\
R_0^{(1)}[h_1^2] &= -\frac{1}{2x} - \frac{1}{2} + O(x^{14}) \\
R_0^{(2)}[h_1^2] &= \frac{1}{2} + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + O(x^{14}) \\
R_0^{(3)}[h_1^2] &= -\frac{x^2}{2} - x^3 - \frac{3x^4}{2} - 2x^5 - \frac{5x^6}{2} - 3x^7 - \frac{7x^8}{2} - 4x^9 - \frac{9x^{10}}{2} - 5x^{11} - \frac{11x^{12}}{2} - 6x^{13} + O(x^{14}) \\
R_0^{(4)}[h_1^2] &= \frac{x^5}{2} + x^6 + \frac{3x^7}{2} + \frac{5x^8}{2} + \frac{7x^9}{2} + \frac{9x^{10}}{2} + 6x^{11} + \frac{15x^{12}}{2} + 9x^{13} + O(x^{14}) \\
R_0^{(5)}[h_1^2] &= -\frac{x^9}{2} - x^{10} - \frac{3x^{11}}{2} - \frac{5x^{12}}{2} - 4x^{13} + O(x^{14}) \\
R_0^{(6)}[h_1^2] &= O(x^{14})
\end{aligned}$$

For $N = 0$ and coefficient of h_2 , we have

$$\begin{aligned}
L_0[h_2] &= -1 + x + x^2 - x^5 - x^7 + x^{12} + O(x^{14}) \\
R_0^{(0)}[h_2] &= O(x^{14}) \\
R_0^{(1)}[h_2] &= -\frac{1}{2x} - \frac{1}{2} + O(x^{14}) \\
R_0^{(2)}[h_2] &= -\frac{1}{2} + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + x^{13} + O(x^{14}) \\
R_0^{(3)}[h_2] &= -1 + x + \frac{x^2}{2} - \frac{x^4}{2} - x^5 - \frac{3x^6}{2} - 2x^7 - \frac{5x^8}{2} - 3x^9 - \frac{7x^{10}}{2} - 4x^{11} - \frac{9x^{12}}{2} - 5x^{13} + O(x^{14}) \\
R_0^{(4)}[h_2] &= -1 + x + x^2 - \frac{x^5}{2} - \frac{x^7}{2} + \frac{x^8}{2} + \frac{x^9}{2} + \frac{3x^{10}}{2} + 2x^{11} + \frac{7x^{12}}{2} + 4x^{13} + O(x^{14}) \\
R_0^{(5)}[h_2] &= -1 + x + x^2 - x^5 - x^7 - \frac{x^9}{2} - \frac{x^{11}}{2} + \frac{x^{12}}{2} - x^{13} + O(x^{14}) \\
R_0^{(6)}[h_2] &= -1 + x + x^2 - x^5 - x^7 + x^{12} + O(x^{14})
\end{aligned}$$

For $N = -1$ and coefficient of 1, we have

$$\begin{aligned}
L_{-1}[1] &= 0 \\
R_{-1}^{(0)}[1] &= 1 + O(x^{16}) \\
R_{-1}^{(1)}[1] &= -x - x^2 - x^3 - x^4 - x^5 - x^6 - x^7 - x^8 - x^9 - x^{10} - x^{11} - x^{12} - x^{13} - x^{14} + O(x^{16}) \\
R_{-1}^{(2)}[1] &= x^3 + x^4 + 2x^5 + 2x^6 + 3x^7 + 3x^8 + 4x^9 + 4x^{10} + 5x^{11} + 5x^{12} + 6x^{13} + O(x^{16}) \\
R_{-1}^{(3)}[1] &= -x^6 - x^7 - 2x^8 - 3x^9 - 4x^{10} - 5x^{11} - 7x^{12} + O(x^{16}) \\
R_{-1}^{(4)}[1] &= x^{10} + x^{11} + O(x^{16}) \\
R_{-1}^{(5)}[1] &= O(x^{16})
\end{aligned}$$

For $N = -1$ and coefficient of h_1 , we have

$$\begin{aligned}
L_{-1}[h_1] &= 0 \\
R_{-1}^{(0)}[h_1] &= O(x^{15}) \\
R_{-1}^{(1)}[h_1] &= -\frac{1}{x} + O(x^{15}) \\
R_{-1}^{(2)}[h_1] &= 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12} + O(x^{15}) \\
R_{-1}^{(3)}[h_1] &= -x^2 - x^3 - 2x^4 - 2x^5 - 3x^6 - 3x^7 - 4x^8 - 4x^9 - 5x^{10} - 5x^{11} + O(x^{15}) \\
R_{-1}^{(4)}[h_1] &= x^5 + x^6 + 2x^7 + 3x^8 + 4x^9 + 5x^{10} + O(x^{15}) \\
R_{-1}^{(5)}[h_1] &= -x^9 + O(x^{15}) \\
R_{-1}^{(6)}[h_1] &= O(x^{15})
\end{aligned}$$

For $N = -1$ and coefficient of h_1^2 , we have

$$\begin{aligned}
L_{-1}[h_1^2] &= 0 \\
R_{-1}^{(0)}[h_1^2] &= O(x^{14}) \\
R_{-1}^{(1)}[h_1^2] &= -\frac{1}{2x^2} - \frac{1}{2x} + O(x^{14}) \\
R_{-1}^{(2)}[h_1^2] &= \frac{1}{2x^3} + \frac{1}{2x^2} + \frac{1}{2x} + 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + O(x^{14}) \\
R_{-1}^{(3)}[h_1^2] &= -\frac{1}{2x^2} - \frac{1}{x} - 1 - \frac{3x}{2} - 2x^2 - \frac{5x^3}{2} - 3x^4 - \frac{7x^5}{2} - 4x^6 - \frac{9x^7}{2} - 5x^8 - \frac{11x^9}{2} - 6x^{10} + O(x^{14}) \\
R_{-1}^{(4)}[h_1^2] &= \frac{1}{2} + x + \frac{3x^2}{2} + 2x^3 + 3x^4 + 4x^5 + 5x^6 + \frac{13x^7}{2} + 8x^8 + \frac{19x^9}{2} + O(x^{14}) \\
R_{-1}^{(5)}[h_1^2] &= -\frac{x^3}{2} - x^4 - \frac{3x^5}{2} - \frac{5x^6}{2} - \frac{7x^7}{2} - 5x^8 + O(x^{14}) \\
R_{-1}^{(6)}[h_1^2] &= \frac{x^7}{2} + O(x^{14})
\end{aligned}$$

For $N = -1$ and coefficient of h_2 , we have

$$\begin{aligned}
L_{-1}[h_2] &= 0 \\
R_{-1}^{(0)}[h_2] &= O(x^{14}) \\
R_{-1}^{(1)}[h_2] &= -\frac{1}{2x^2} - \frac{1}{2x} + O(x^{14}) \\
R_{-1}^{(2)}[h_2] &= \frac{1}{2x^3} - \frac{1}{2x^2} + \frac{1}{2x} + 1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + O(x^{14}) \\
R_{-1}^{(3)}[h_2] &= -\frac{1}{2x^2} - \frac{x}{2} - x^2 - \frac{3x^3}{2} - 2x^4 - \frac{5x^5}{2} - 3x^6 - \frac{7x^7}{2} - 4x^8 - \frac{9x^9}{2} - 5x^{10} + O(x^{14}) \\
R_{-1}^{(4)}[h_2] &= \frac{1}{2} + \frac{x^2}{2} + x^4 + x^5 + 2x^6 + \frac{5x^7}{2} + 4x^8 + \frac{9x^9}{2} + O(x^{14}) \\
R_{-1}^{(5)}[h_2] &= -\frac{x^3}{2} - \frac{x^5}{2} - \frac{x^6}{2} - \frac{x^7}{2} - x^8 + O(x^{14}) \\
R_{-1}^{(6)}[h_2] &= \frac{x^7}{2} + O(x^{14})
\end{aligned}$$