

On the Distributions of Prime Divisor Counting Functions

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

Let k and n be natural numbers. Let $\omega(n)$ denote the number of distinct prime factors of n , $\Omega(n)$ denote the total number of prime factors of n , and $\omega_k(n)$ denote the number of distinct prime factors of n with multiplicity k .

Let $h \geq 2$ be a natural number. We say n is h -free if every prime factor of n has multiplicity less than h , and is h -full if all prime factors have multiplicity at least h .

In [23], Hardy and Ramanujan proved that $\omega(n)$ and $\Omega(n)$ have normal orders $\log \log n$ over naturals. In this thesis, using a new counting argument, we establish the first and second moments of all these arithmetic functions over h -free and h -full numbers. We show that the normal order of $\omega(n)$ is $\log \log n$ for both h -free and h -full numbers. For $\Omega(n)$, the normal order is $\log \log n$ over h -free numbers and $h \log \log n$ over h -full numbers. We also show that $\omega_1(n)$ has normal order $\log \log n$ over h -free numbers and $\omega_h(n)$ has normal order $\log \log n$ over h -full numbers. Moreover, we prove that the functions $\omega_k(n)$ with $1 < k < h$ do not have normal order over h -free numbers and $\omega_k(n)$ with $k > h$ do not have normal order over h -full numbers.

In their seminal work [15], Erdős and Kac showed that $\omega(n)$ is normally distributed over the natural numbers. Later, Liu [35] extended this result by proving a subset generalization of the Erdős-Kac theorem. In this thesis, we leverage Liu's framework to establish the Erdős-Kac theorem for both h -free and h -full numbers. Additionally, we show that $\omega_1(n)$ satisfies the Erdős-Kac theorem over h -free numbers, while $\omega_h(n)$ satisfies it over h -full numbers.

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Dedication

To my family and friends, for their unwavering love, support, and belief in me throughout this journey.

Table of Contents

Examining Committee Membership	ii
Author's Declaration	iii
Abstract	iv
Acknowledgements	v
Dedication	vi
List of Tables	x
List of Symbols	xi
1 Introduction	1
1.1 History and Motivation	1
1.2 History of distributions of prime divisor counting functions	4
1.3 Normal order of prime divisor counting functions	6
1.4 Generalizations of Erdős-Kac theorem	11
2 Normal order of $\omega(n)$ over h-free and h-full numbers	13
2.1 Introduction	13
2.2 Lemmata	16

2.3	The first and the second moment of $\omega(n)$ over h -free numbers	18
2.4	The first and the second moment of $\omega(n)$ over h -full numbers	24
2.5	Normal order of $\omega(n)$ over h -free and h -full numbers	35
3	Normal order of $\Omega(n)$ over h-free and h-full numbers	40
3.1	Introduction	40
3.2	The moments of $\Omega(n)$ over h -free numbers	42
3.3	The moments of $\Omega(n)$ over h -full numbers	49
3.4	Normal order of $\Omega(n)$ over h -free and h -full numbers	63
4	Moments of $\omega_k(n)$ over h-free and h-full numbers	67
4.1	Introduction	67
4.2	Lemmata	72
4.3	Distribution of $\omega_k(n)$ over h -free numbers	72
4.3.1	For $\omega_1(n)$	73
4.3.2	For $\omega_k(n)$ with $2 \leq k \leq (h - 1)$	76
4.4	Distribution of $\omega_k(n)$ over h -full numbers	78
4.4.1	The first moment of $\omega_k(n)$ over h -full numbers	79
4.4.2	The second moment of $\omega_k(n)$ over h -full numbers	80
4.5	No normal orders	83
5	Generalizations of Erdős-Kac Theorems	86
5.1	Introduction	86
5.2	The Erdős-Kac theorem over h -free and h -full numbers	89
5.3	The Erdős-Kac theorems for $\omega_k(n)$	93
6	Generalizations and Future Research	98
6.1	Normal order over abelian monoids	99
6.1.1	Application to set of ideals in number fields	101

6.1.2	Application to set of effective divisors in global function fields . . .	102
6.1.3	Application to the set of effective 0-cycles in geometrically irreducible projective varieties of dimension r	104
6.2	Study of refinements over function fields	104
6.3	Follow-up research	105
References		107

List of Tables

1.1	For moments of prime divisor counting functions	7
1.2	References for constants involved in the moments of prime divisor counting functions	8
1.3	For normal orders of prime divisor counting functions	10
1.4	For analogs of Erdős-Kac theorem for prime divisor counting functions . .	11

List of Symbols

We introduce the following notations and definitions that we will use throughout the thesis.

- $\mathbb{N}, \mathbb{Z}, \mathbb{R}$, and $\mathbb{C}$ denote the sets of natural numbers, integers, real numbers, and complex numbers, respectively.
- $\mathbb{R}_{\geq 0}$ denotes the sets of non-negative real numbers.
- The letters n, m, h, k , and l are used to denote natural numbers, the letters p and q are used to denote prime numbers, α, τ, r, x , and y are used to denote real numbers, and s is used to denote complex numbers.
- The sums $\sum_{n \leq x}$ and $\sum_{n \geq x}$ denote the sums over all natural numbers less than or equal to the real number x , and greater than or equal to the real number x , respectively.
- The sum $\sum_{p \leq x}$ and $\sum_{p \geq x}$ denote the sum over all prime numbers less than or equal to the real number x , and greater than or equal to the real number x , respectively.
- The sum $\sum_{\substack{p, q \\ pq \leq x}}$ denotes the sum over all prime pairs (p, q) such that $pq \leq x$, and counts the pair (p, q) as distinct from (q, p) when $p \neq q$.
- $p|n$ denotes p divides n , and $p^k || n$ denotes p^k divides n but p^{k+1} doesn't divide n .
- Letters B, C, D, γ , and $\mathcal{C}$ with appropriate subscripts are used to denote real constants that depend on the parameter h .
- Letters $\mathcal{S}$ and $\mathcal{N}$ with appropriate subscripts are used to denote subsets of natural numbers. In particular, $\mathcal{S}_h$ denotes the set of h -free numbers and $\mathcal{N}_h$ denotes the set of h -full numbers.
- For a subset $\mathcal{S} \subseteq \mathbb{N}$, $|\mathcal{S}|$ denotes the cardinality of $\mathcal{S}$, and $\mathcal{S}(x)$ denotes the set of elements of $\mathcal{S}$ that are less than or equal to the real number x .
- $\zeta(s)$ denotes the Riemann ζ -function.
- Suppose $f, g : \mathbb{R} \rightarrow \mathbb{R}$ be two functions with $g \geq 0$. We say

1. $f(x) = O(g(x))$ if there exist $x_0 \in \mathbb{R}$ such that $|f(x)| \leq Cg(x)$ for some $C > 0$ and for all $x \geq x_0$.

Moreover, $f(x) = O_X(g(x))$ if the implied constants x_0 and C in the big-O notation depend on the variable set X .

2. $f(x) = o(g(x))$ meaning $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} \rightarrow 0$.
3. $f(x) \sim g(x)$ meaning $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} \rightarrow 1$.

- We introduce the following prime counting function:

$$\pi(x) := \sum_{p \leq x} 1.$$

- We introduce the following prime divisor counting functions: (for $k, n \in \mathbb{N}$)

$$\omega(n) = \sum_{\substack{p \\ p|n}} 1, \quad \Omega(n) = \sum_{\substack{p \\ p^k || n}} k, \quad \text{and} \quad \omega_k(n) = \sum_{\substack{p \\ p^k || n}} 1.$$

Chapter 1

Introduction

1.1 History and Motivation

Let $n \in \mathbb{N}$ and let $\omega(n)$ denote the number of distinct prime divisors of n . For example, $\omega(72) = 2$. This function exhibits irregular behavior; for instance, it attains the value N when n is a product of N distinct prime factors, yet it can drop to 1 for the very next integer if that integer is a prime or a prime power.

However, when averaged over all integers up to x (where $x \in \mathbb{R}$), $\omega(n)$ follows a well-behaved asymptotic trend (see [24, Theorem 430]):

$$\frac{1}{x} \sum_{n \leq x} \omega(n) \sim \log \log x.$$

In [23], Hardy and Ramanujan proved a stronger result that $\omega(n)$ has normal order $\log \log n$. In particular, they showed, that for any $\epsilon > 0$,

$$|\{n \leq x : |\omega(n) - \log \log n| > \epsilon \log \log n\}| = o(x).$$

In other words, the number of integers $n \leq x$ for which $\omega(n)$ significantly deviates from $\log \log n$ is asymptotically negligible.

The proof by Hardy and Ramanujan is intricate, but their result closely parallels the probabilistic law of large numbers, which states that the sample mean of a sequence of independent random variables converges to the expected value. This connection naturally

raises the question of whether the normal order of $\omega(n)$ can be established using probabilistic techniques. Surprisingly, such a proof was later discovered by Turán, despite his limited background in probability theory at the time.

Turán in [49] provided a simplified proof of the normal order result by using the second moment of $\omega(n)$:

$$\frac{1}{x} \sum_{n \leq x} (\omega(n) - \log \log n)^2 = O(\log \log x).$$

Later, Diaconis, Mosteller, and Onishi in [10, Theorem 2] (also see Saidak [44]) made the result more explicit:

$$\frac{1}{x} \sum_{n \leq x} (\omega(n) - \log \log n)^2 = \log \log x + C + O\left(\frac{\log \log x}{\log x}\right),$$

where C is an explicitly determined absolute constant. Such explicit results are crucial for obtaining precise numerical bounds on arithmetic functions and for bridging the gap between theoretical insights and practical applications, including those in cryptography.

After Turán's paper appeared, Kac began contemplating the limiting distribution of $\omega(n)$. During a lecture at Princeton in 1939, Kac proposed that the normalized quantity

$$\frac{\omega(n) - \log \log n}{\sqrt{\log \log n}}$$

might follow a Gaussian or normal distribution. Remarkably, Erdős, who was in the audience, completed the proof of this conjecture before the lecture ended. Kac later recalled:

"If I remember it correctly I first stated (as a conjecture) the theorem on the normal distribution of the number of prime divisors during a lecture in Princeton in March 1939. Fortunately for me and possibly for Mathematics, Erdős was in the audience, and he immediately perked up. Before the lecture was over he had completed the proof, which I could not have done not having been versed in the number-theoretic methods, especially those related to the sieve."

Some number theorists credit this to the magical moment when Probabilistic Number Theory was born.

Erdős's swift response to Kac's question was no coincidence. At the time, he and Wintner [16] were deeply engaged in studying the limiting distributions of families of additive arithmetic functions, aiming to establish the necessary and sufficient conditions for

their existence. Nevertheless, the Princeton encounter remains one of the most celebrated and romantic episodes in number theory.

Erdős and Kac in [15] announced their celebrated result which is famously known as the Erdős-Kac theorem: for any real number a ,

$$\lim_{x \rightarrow \infty} \frac{1}{x} \left| \left\{ n \leq x : \frac{\omega(n) - \log \log n}{\sqrt{\log \log n}} \leq a \right\} \right| = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^a e^{-t^2/2} dt. \quad (1.1)$$

This proof relied on advanced techniques such as Brun's sieve and the central limit theorem. Alternative proofs were later provided by Selberg [45], Halberstam [20], Shapiro [46], Billingsley [2], Liu [33, 35], and Granville and Soundararajan [19], each employing different methods. Notably, Halberstam's approach offered a natural probabilistic interpretation of Erdős and Kac's proof by analyzing all moments of $\omega(n)$. Billingsley refined this by introducing probabilistic techniques involving sums of independent random variables, thereby simplifying some of the more intricate calculations in Halberstam's proof. Liu further generalized Billingsley's argument, presenting a proof that required only the first moment of $\omega(n)$. By effectively controlling the error in the estimation of $\omega(n)$ and the corresponding sum of random variables, her method circumvented the need for second-moment and higher-order calculations. In this thesis, we adopt a subset generalization of her approach to investigate the Erdős-Kac theorem within restricted subsets of natural numbers that satisfy additional constraints.

We introduce new subsets of natural numbers, which are the main focus of this thesis, in particular, subsets that generalize the square-free and the square-full numbers.

For a natural number n , let the prime factorization of n be given as

$$n = p_1^{s_1} \cdots p_r^{s_r}, \quad (1.2)$$

where p_i 's are its distinct prime factors and s_i 's are their respective multiplicities. Let $h \geq 2$ be an integer. We say n is *h -free* if $s_i \leq h - 1$ for all $i \in \{1, \dots, r\}$, and we say n is *h -full* if $s_i \geq h$ for all $i \in \{1, \dots, r\}$. Let $\mathcal{S}_h$ denote the set of h -free numbers and $\mathcal{N}_h$ denote the set of h -full numbers. The set of h -free and the set of h -full numbers are well-studied in the literature (see [25], [26], and [27]).

Let $\zeta(s)$ denote the classical Riemann ζ -function given by

$$\zeta(s) := \sum_n \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s} \right)^{-1} \quad (\Re(s) > 1), \quad (1.3)$$

where the sum runs over all natural numbers and the product runs over all primes.

Let $x > 2$. The set of h -free numbers has the distribution (see [26, (4)])

$$\sum_{\substack{n \in \mathcal{S}_h \\ n \leq x}} 1 = \frac{x}{\zeta(h)} + O_h(x^{\frac{1}{h}}), \quad (1.4)$$

where O_X denotes that the implied big-O constant depends on the variable set X . This implies that the set of h -free numbers has a positive density of $1/\zeta(h)$.

For the distribution of h -full integers, Ivić and Shiu in [25, Equation (1.15) & Lemma 1] proved that set of h -full numbers has the distribution

$$\sum_{\substack{n \in \mathcal{N}_h \\ n \leq x}} 1 = \gamma_{0,h} x^{\frac{1}{h}} + O_h\left(x^{\frac{1}{h+1}}\right), \quad (1.5)$$

where $\gamma_{0,h}$ is the constant defined as

$$\gamma_{0,h} := \prod_p \left(1 + \frac{p - p^{1/h}}{p^2(p^{1/h} - 1)}\right). \quad (1.6)$$

Thus, the set of h -full integers is a zero-density subset of the natural numbers.

Every natural number that is not a h -free or a h -full number can be written uniquely as a product of two co-prime numbers, one being h -free, and the other being h -full. Thus, understanding the behavior of an arithmetic function over the set of h -free and the set of h -full numbers provides greater insights into its behavior over all natural numbers.

This thesis studies the behavior of the prime divisor counting functions and their refinements over the restricted sets of h -free and h -full numbers. We establish analogs of Turán's theorem, Hardy-Ramanujan's normal order result, and Erdős-Kac theorem over these subsets.

1.2 History of distributions of prime divisor counting functions

Let n be a natural number. Let $\omega(n)$ denote the total number of distinct prime factors in the factorization of n , let $\Omega(n)$ denote the total number of prime factors in the factorization of n , and for any integer $k \geq 1$, let $\omega_k(n)$ denote the total number of distinct prime factors

in the factorization of n , with multiplicity k . Note that $\omega_k(n)$ are refinements of $\omega(n)$ in the sense that

$$\omega(n) = \sum_{k \geq 1} \omega_k(n).$$

Moreover,

$$\Omega(n) = \sum_{k \geq 1} k \cdot \omega_k(n).$$

Let B_1 be the Mertens constant given by

$$B_1 = \gamma + \sum_p \left(\log \left(1 - \frac{1}{p} \right) + \frac{1}{p} \right) \approx 0.2615, \quad (1.7)$$

with $\gamma \approx 0.57722$, the Euler-Mascheroni constant, and where the sum runs over all primes p . Let

$$B_2 = B_1 + \sum_p \frac{1}{p(p-1)} \approx 1.0347. \quad (1.8)$$

The following first-moment estimations for $\omega(n)$ and $\Omega(n)$ over natural numbers are well-known (see [24, Theorem 430] and [17, Section 1.4.4]):

$$\sum_{n \leq x} \omega(n) = x \log \log x + B_1 x + O \left(\frac{x}{\log x} \right), \quad (1.9)$$

and

$$\sum_{n \leq x} \Omega(n) = x \log \log x + B_2 x + O \left(\frac{x}{\log x} \right). \quad (1.10)$$

The first-moment estimates for $\omega_k(n)$ over natural numbers were proved by Elma and Liu [12, Theorem 1.1] as

$$\sum_{n \leq x} \omega_1(n) = x \log \log x + (B_1 - P(2)) x + O \left(\frac{x}{\log x} \right), \quad (1.11)$$

and for $k \geq 2$,

$$\sum_{n \leq x} \omega_k(n) = (P(k) - P(k+1)) x + O \left(x^{\frac{k+1}{3k-1}} (\log x)^2 \right), \quad (1.12)$$

where $P(k)$ denote the convergent sum given by

$$P(k) := \sum_p \frac{1}{p^k}. \quad (1.13)$$

Moreover, the first-moment estimates for $\Omega(n)$ over h -free and over h -full numbers have been studied. Let D_3 be the constant defined as

$$D_3 := h(B_2 - \log h) + \sum_p \frac{(h+1)p^{1+1/h} - hp - 2hp^{2/h} + (2h-1)p^{1/h}}{(p-1)(p^{1/h}-1)(p^{1+1/h} + p^{1/h} - p)}. \quad (1.14)$$

In [26, Theorem 1 and Theorem 2], Jakimczuk and Lalín established that

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \Omega(n) = \frac{1}{\zeta(h)} x \log \log x + O_h(x), \quad (1.15)$$

and

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \Omega(n) = h\gamma_{0,h}x^{1/h} \log \log x + D_3\gamma_{0,h}x^{1/h} + O_h\left(\frac{x^{1/h}}{\sqrt{\log x}}\right), \quad (1.16)$$

where $\gamma_{0,h}$ is defined in (1.6).

The results in (1.9), (1.10), (1.11) and (1.12) suggest that $\omega(n)$, $\Omega(n)$ and $\omega_1(n)$ share a similar asymptotic distribution with only a difference in the coefficient of the second main term, whereas the average distribution of $\omega_k(n)$ with $k \geq 2$ is smaller than that of $\omega(n)$. In this thesis, we establish similar behavior over h -free and h -full numbers.

1.3 Normal order of prime divisor counting functions

In this thesis, we establish the first and the second moments of $\omega(n)$, $\Omega(n)$ and $\omega_k(n)$ over the h -free and the h -full numbers. In particular, we prove:

Theorem 1.3.1. *Let $x > 2$ be any real number and let $h \geq 2$ and $k \geq 1$ be any integers. Let $\mathcal{S}$ denote the sets $\mathcal{S}_h$ or $\mathcal{N}_h$. Let $\mathcal{S}(x)$ denote the set of elements of $\mathcal{S}$ less than or equal*

to x . Let f denote the functions $\omega(n)$, $\Omega(n)$ or $\omega_k(n)$. Let $\alpha, \beta, \gamma, \kappa, \nu, \tau \in \mathbb{R}_{\geq 0}$. Then, for each tuple $(f, \mathcal{S}, \alpha, \beta, \gamma, \kappa, \nu, \tau, X)$ given in Table 1.1, we have

$$\sum_{n \in \mathcal{S}(x)} f(n) = \mathcal{C}_{1,f,\mathcal{S}} x^\alpha (h^\beta \log \log x)^\kappa + \mathcal{C}_{2,f,\mathcal{S}} x^\alpha + O_X \left(\frac{x^\gamma (\log \log x)^\nu}{\log x} \right),$$

and

$$\begin{aligned} \sum_{n \in \mathcal{S}(x)} f^2(n) &= \mathcal{C}_{1,f,\mathcal{S}} x^\alpha (h^\beta \log \log x)^{2\kappa} + \mathcal{C}_{3,f,\mathcal{S}} x^\alpha (h^\beta \log \log x)^\kappa + \mathcal{C}_{4,f,\mathcal{S}} x^\alpha \\ &\quad + O_X \left(\frac{x^\gamma (\log \log x)^\tau}{\log x} \right), \end{aligned}$$

where for each fixed f and $\mathcal{S}$, $\mathcal{C}_{1,f,\mathcal{S}}, \mathcal{C}_{2,f,\mathcal{S}}, \mathcal{C}_{3,f,\mathcal{S}}$, and $\mathcal{C}_{4,f,\mathcal{S}}$ are constants given in Table 1.2.

$f(n)$	$\mathcal{S}$	α	β	γ	κ	ν	τ	X
$\omega(n)$ or $\Omega(n)$ or $\omega_1(n)$	$\mathcal{S}_h$	1	0	1	1	0	1	$\{h\}$
$\omega_k(n)$, $2 \leq k \leq h-1$	$\mathcal{S}_h$	1	0	$1/k$	0	0	1	$\{h, k\}$
$\omega(n)$ or $\omega_h(n)$	$\mathcal{N}_h$	$1/h$	0	$1/h$	1	0	1	$\{h\}$
$\Omega(n)$	$\mathcal{N}_h$	$1/h$	1	$1/h$	1	0	1	$\{h\}$
$\omega_{h+1}(n)$	$\mathcal{N}_h$	$1/h$	0	$1/(h+1)$	0	1	2	$\{h\}$
$\omega_k(n)$, $k \geq h+2$	$\mathcal{N}_h$	$1/h$	0	$1/(h+1)$	0	0	0	$\{h, k\}$

Table 1.1: For moments of prime divisor counting functions

The proof of Theorem 1.3.1 follows from accumulating results from Chapters 2, 3, and 4. Theorem 2.1.1 and Theorem 2.1.2 establish the results for $\omega(n)$. Theorem 3.1.1 and Theorem 3.1.2 establish the results for $\Omega(n)$. Finally, Theorem 4.1.1, Theorem 4.1.2, Theorem 4.1.3, and Theorem 4.1.4 establish the results for $\omega_k(n)$. Together, these theorems establish Theorem 1.3.1.

Although the distribution results for $\omega(n)$, $\Omega(n)$, and $\omega_k(n)$ can be expressed in the form presented in Theorem 1.3.1, their proofs require a careful, case-by-case analysis of these arithmetic functions. In what follows, we outline the key challenges in establishing results of the form given in Theorem 1.3.1, along with the methods we developed to address these difficulties and obtain the best known bounds in this direction:

$f(n)$	$\mathcal{S}$	$\mathcal{C}_{1,f,\mathcal{S}}$	$\mathcal{C}_{2,f,\mathcal{S}}$	$\mathcal{C}_{3,f,\mathcal{S}}$	$\mathcal{C}_{4,f,\mathcal{S}}$
$\omega(n)$	$\mathcal{S}_h$	(2.3)	(2.3)	(2.3)	(2.3)
$\Omega(n)$	$\mathcal{S}_h$	(3.1)	(3.1)	(3.1)	(3.1)
$\omega_1(n)$	$\mathcal{S}_h$	(4.2)	(4.2)	(4.2)	(4.2)
$\omega_k(n), 2 \leq k \leq h-1$	$\mathcal{S}_h$	0	(4.3)	0	(4.4)
$\omega(n)$	$\mathcal{N}_h$	(2.7)	(2.7)	(2.7)	(2.7)
$\Omega(n)$	$\mathcal{N}_h$	(3.3)	(3.3)	(3.3)	(3.3)
$\omega_h(n)$	$\mathcal{N}_h$	(4.7)	(4.7)	(4.10)	(4.10)
$\omega_{h+1}(n)$	$\mathcal{N}_h$	0	(4.8)	0	(4.11)
$\omega_k(n), k > h+1$	$\mathcal{N}_h$	0	(4.9)	0	(4.12)

Table 1.2: References for constants involved in the moments of prime divisor counting functions

- To obtain (1.15) for $\Omega(n)$, the authors used the generating series for h -free numbers. For (1.16), they used the completely additive property of $\Omega(n)$ to split the sum into sums over divisors of n distinguished by their multiplicity, and then used generating series arguments to estimate these sums. This method for the h -full case cannot be extended to the study of $\omega(n)$ or $\omega_k(n)$, since they are not completely additive. This limitation of their approach is presented in [29, Page 33, Line 15] where the authors established the function field analog of [26].

To study the moments over h -free and h -full numbers, we use a novel counting argument. At its core, our method relies on counting natural numbers divisible by a given set of prime powers in a bounded range. This new counting argument not only establishes the first and the second moments for $\omega(n)$ and $\omega_k(n)$ over h -free and h -full numbers but also improves results from [26]. In particular, it makes the coefficient of x in (1.15) explicit and improves the error term in (1.16) to $x^{1/h}/\log x$.

- The h -free numbers form a dense set in $\mathbb{N}$ and have the following indicator function associated with them:

$$\sum_{\substack{d \in \mathbb{N} \\ d^h | n}} \mu(d) = \begin{cases} 1 & \text{if } n \text{ is } h\text{-free,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\mu(n)$ denote the Möbius μ -function that takes values ± 1 on squarefree numbers and 0 otherwise. One can produce results similar to Theorem 1.3.1 over h -free numbers by carefully studying the action of $\mu(n)$ on elements of this set. However, no comparable identity is available for h -full numbers. To address this, we introduce

a novel counting argument that allows us to decompose the sum over natural numbers into a sum over primes within a bounded range. For each prime, the associated summand is itself a sum over natural numbers coprime to that prime.

- To produce explicit coefficients for each term in the distribution, in particular, for the second and the third main terms, we require explicit results on the distribution of h -free and h -full numbers that are coprime to a given set of prime numbers (see Lemma 2.2.6 for h -free numbers and Lemma 2.4.1 for h -full numbers). While the distribution of h -free numbers has been studied in the literature [27], our results on the distribution of h -full numbers are new.
- While the counting technique used to study the distribution of $\omega(n)$ and $\Omega(n)$ remains essentially the same, additional care is required to account for the multiplicity of prime powers dividing a natural number n . For example, we employ the expressions (3.22) and (3.31) to isolate the main terms and control the error terms in the distributions of $\Omega(n)$ over h -full numbers. Moreover, we establish a new identity for sums over primes, as in Lemma 3.3.1, which plays a crucial role in bounding the error terms that arise in these distribution results.

Next, we introduce the definition of the normal order of an arithmetic function over a subset of natural numbers. This definition is inspired by the work of Elma and Liu [12] over naturals where they extended the definition of normal order using an increasing function (see [23]) to a definition involving a non-decreasing function. Let $\mathcal{S} \subseteq \mathbb{N}$ and $\mathcal{S}(x)$ denote the set of natural numbers belonging to $\mathcal{S}$ and less than or equal to x . Let $|\mathcal{S}(x)|$ denote the cardinality of $\mathcal{S}(x)$. Let $f, F : \mathcal{S} \rightarrow \mathbb{R}_{\geq 0}$ be two functions such that F is non-decreasing. Then, $f(n)$ is said to have normal order $F(n)$ over $\mathcal{S}$ if for any $\epsilon > 0$, the number of $n \in \mathcal{S}(x)$ that do not satisfy the inequality

$$(1 - \epsilon)F(n) \leq f(n) \leq (1 + \epsilon)F(n)$$

is $o(|\mathcal{S}(x)|)$ as $x \rightarrow \infty$.

The distributions in (1.9) and (1.10) establish that, on average, $\omega(n)$ and $\Omega(n)$ behave like $\log \log n$ over the natural numbers. Hardy and Ramanujan strengthened this result in [23] by proving that both functions have $\log \log n$ as their normal order. Another proof of this result, based on the variance of $\omega(n)$ (see [24, Theorem 431]), can be found in [24, Section 22.11]. This approach, inspired by Turán's work, is the foundation for our use of variance methods to study normal order in this thesis.

Using Theorem 1.3.1, we establish that $\omega(n)$ has normal order $\log \log n$ within the subsets of h -free and h -full numbers. We also show that $\Omega(n)$ has normal order $\log \log n$

within the subset of h -free and normal order $h \log \log n$ within the subset of h -full numbers. For the refinements, we show that $\omega_1(n)$ has normal order $\log \log n$ over h -free numbers and $\omega_h(n)$ has normal order $\log \log n$ over h -full numbers. In particular, we prove:

Theorem 1.3.2. *Let $x > 2$ be any real number and let $h \geq 2$ and $k \geq 1$ be any integers. Let $\mathcal{S}$ denote the sets $\mathcal{S}_h$ or $\mathcal{N}_h$. Let $\mathcal{S}(x)$ denote the set of elements of $\mathcal{S}$ less than or equal to x . Let f denote the functions $\omega(n)$, $\Omega(n)$ or $\omega_k(n)$. Let $\alpha \in \{0, 1\}$. Then, for each tuple $(f, \mathcal{S}, \alpha)$ given in Table 1.3 and for any $\epsilon > 0$, the number of $n \in \mathcal{S}(x)$ that do not satisfy the inequality*

$$(1 - \epsilon)h^\alpha \log \log n \leq f(n) \leq (1 + \epsilon)h^\alpha \log \log n$$

is $o(|\mathcal{S}(x)|)$ as $x \rightarrow \infty$.

$f(n)$	$\mathcal{S}$	α
$\omega(n)$ or $\Omega(n)$ or $\omega_1(n)$	$\mathcal{S}_h$	0
$\omega(n)$ or $\omega_h(n)$	$\mathcal{N}_h$	0
$\Omega(n)$	$\mathcal{N}_h$	1

Table 1.3: For normal orders of prime divisor counting functions

The proof of Theorem 1.3.2 for $\omega(n)$ follows from the proofs of Theorem 2.1.3 and Theorem 2.1.4, and the proofs for $\Omega(n)$ follows from that of Theorem 3.1.3 and Theorem 3.1.4. The proofs of normal orders of $\omega_1(n)$ over h -free numbers and $\omega_h(n)$ over h -full numbers are similar to that of $\omega(n)$ and we discuss this, without proof, at the end of Section 4.1.

We also establish that $\omega_k(n)$ with $1 < k < h$ does not have a normal order over h -free numbers, and $\omega_k(n)$ with $k > h$ does not have a normal order over h -full numbers. Combining Theorem 4.1.5 and Theorem 4.1.6 proves this result.

While the outcomes over h -free numbers were anticipated due to their positive density, the findings for h -full numbers are more surprising. This contrasts with the behavior observed over the set of all prime powers, which has zero density and where $\omega(n)$ has normal order 1. However, it aligns well with Erdős's result [14] on the zero-density set $\{p - 1 : p \text{ prime}\}$, where the normal order of $\omega(n)$ is $\log \log n$. Furthermore, Theorem 1.3.1 and Theorem 1.3.2 reveal that $\omega(n)$ behaves similarly to $\omega_1(n)$ over h -free numbers and to $\omega_h(n)$ over h -full numbers, highlighting the crucial role of the minimal prime power present in a subset in shaping the behavior of $\omega(n)$.

1.4 Generalizations of Erdős-Kac theorem

In this thesis, we show that $\omega(n)$ satisfies the Erdős-Kac theorem over h -free and h -full numbers. Moreover, we showed that the $\omega_1(n)$ satisfies the Erdős-Kac theorem over h -free numbers and $\omega_h(n)$ satisfies the Erdős-Kac theorem over h -full numbers. More precisely, we prove:

Theorem 1.4.1. *Let $x > 2$ be any real number and let $h \geq 2$ and $k \geq 1$ be any integers. Let $\mathcal{S}$ denote the sets $\mathcal{S}_h$ or $\mathcal{N}_h$. Let $\mathcal{S}(x)$ denote the set of elements of $\mathcal{S}$ less than or equal to x . Let f denote the functions $\omega(n)$ or $\omega_k(n)$. Then, for each tuple $(f, \mathcal{S})$ given in Table 1.4, and for any $\gamma \in \mathbb{R}$, we have*

$$\lim_{x \rightarrow \infty} \frac{1}{|\mathcal{S}(x)|} \left| \left\{ n \in \mathcal{S}(x) : n \geq 3, \frac{f(n) - \log \log n}{\sqrt{\log \log n}} \leq \gamma \right\} \right| = \Phi(\gamma),$$

where Φ is the cumulative distribution function of a standard normal distribution given by

$$\Phi(\gamma) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\gamma} e^{-u^2/2} du. \quad (1.17)$$

$f(n)$	$\mathcal{S}$
$\omega(n)$ or $\omega_1(n)$	$\mathcal{S}_h$
$\omega(n)$ or $\omega_h(n)$	$\mathcal{N}_h$

Table 1.4: For analogs of Erdős-Kac theorem for prime divisor counting functions

The above theorem is jointly proved in Chapter 6 combining Theorem 5.1.2, Theorem 5.1.3, Theorem 5.1.4, and Theorem 5.1.5. In fact, in Theorem 5.1.1, we establish the Erdős-Kac theorem over subsets of natural numbers satisfying additional conditions.

To establish Theorem 1.4.1 for $\omega(n)$, we build on results from Liu's work [35]. We begin by outlining the framework for subsets of natural numbers introduced in [35], state her main result for such subsets as in Theorem 5.1.1, and then apply these results to prove Erdős-Kac-type theorems for h -free and h -full numbers. Finally, by adapting techniques from [12, Proof of Theorem 1.3], we extend these results to obtain Erdős-Kac analogs for $\omega_1(n)$ over h -free numbers and $\omega_h(n)$ over h -full numbers.

Finally, we provide an overview of the thesis. Chapter 2 examines the first and second moments of $\omega(n)$ over h -free and h -full numbers, establishing the normal order of $\omega(n)$ within these subsets. The results of this chapter have been published in the International Journal of Number Theory [8].

Chapter 3 investigates the first and second moments of $\Omega(n)$ over h -free and h -full numbers, and determines its normal order in these cases. The author and collaborators are currently preparing a manuscript that extends these results.

Chapter 4 focuses on refinements $\omega_k(n)$, studying their first and second moments over h -free and h -full numbers and establishing corresponding normal order results. The findings presented in this chapter have been published in the Journal of Number Theory [9].

Chapter 5 develops analogs of the Erdős-Kac theorem for $\omega(n)$ and $\omega_1(n)$ over h -free numbers, as well as for $\omega(n)$ and $\omega_h(n)$ over h -full numbers. The results from this chapter have been submitted for publication [7].

The final chapter outlines several potential directions for future research, including generalizations of the results in this thesis and open questions that can be explored using its methods.

Chapter 2

Normal order of $\omega(n)$ over h -free and h -full numbers

2.1 Introduction

In this chapter, we study the moments of $\omega(n)$ over the subsets of h -free and h -full numbers and establish the normal order of $\omega(n)$ over these subsets.

We define the constants (depending on h)

$$C_1 := B_1 - \sum_p \frac{p-1}{p(p^h-1)}, \quad (2.1)$$

and

$$C_2 := C_1^2 + C_1 - \zeta(2) - \sum_p \left(\frac{p^{h-1}-1}{p^h-1} \right)^2, \quad (2.2)$$

where B_1 is defined in (1.7), $\zeta(s)$ is the Riemann ζ -function, and where the sums run over all primes.

We prove the first and the second moments of $\omega(n)$ over h -free numbers as the following:

Theorem 2.1.1. *Let $x > 2$ be a real number. Let $h \geq 2$ be an integer. Let $\mathcal{S}_h(x)$ be the set of h -free numbers less than or equal to x . Then, we have*

$$\sum_{n \in \mathcal{S}_h(x)} \omega(n) = \mathcal{C}_{1,\omega,\mathcal{S}_h} x \log \log x + \mathcal{C}_{2,\omega,\mathcal{S}_h} x + O_h \left(\frac{x}{\log x} \right),$$

and

$$\sum_{n \in \mathcal{S}_h(x)} \omega^2(n) = \mathcal{C}_{1,\omega,\mathcal{S}_h} x (\log \log x)^2 + \mathcal{C}_{3,\omega,\mathcal{S}_h} x \log \log x + \mathcal{C}_{4,\omega,\mathcal{S}_h} x + O_h \left(\frac{x \log \log x}{\log x} \right),$$

where

$$\mathcal{C}_{1,\omega,\mathcal{S}_h} = \frac{1}{\zeta(h)}, \quad \mathcal{C}_{2,\omega,\mathcal{S}_h} = \frac{C_1}{\zeta(h)}, \quad \mathcal{C}_{3,\omega,\mathcal{S}_h} = 2\mathcal{C}_{2,\omega,\mathcal{S}_h} + \mathcal{C}_{1,\omega,\mathcal{S}_h}, \text{ and } \mathcal{C}_{4,\omega,\mathcal{S}_h} = \frac{C_2}{\zeta(h)}. \quad (2.3)$$

Let $\mathcal{L}_h(r)$ be the convergent sum defined for $r > h \geq 2$ as

$$\mathcal{L}_h(r) := \sum_p \frac{1}{p^{(r/h)-1} (p - p^{1-1/h} + 1)}. \quad (2.4)$$

Using this, we define two new constants (depending on h)

$$D_1 := B_1 - \log h + \mathcal{L}_h(h+1) - \mathcal{L}_h(2h), \quad (2.5)$$

and

$$D_2 := D_1^2 + D_1 - \zeta(2) - \sum_p \left(\frac{1}{p - p^{1-1/h} + 1} \right)^2. \quad (2.6)$$

For the distributions of $\omega(n)$ over h -full numbers, we prove:

Theorem 2.1.2. *Let $x > 2$ be a real number. Let $h \geq 2$ be an integer. Let $\gamma_{0,h}$ be defined in (1.6). Let $\mathcal{N}_h(x)$ be the set of h -full numbers less than or equal to x . We have*

$$\sum_{n \in \mathcal{N}_h(x)} \omega(n) = \mathcal{C}_{1,\omega,\mathcal{N}_h} x^{1/h} \log \log x + \mathcal{C}_{2,\omega,\mathcal{N}_h} x^{1/h} + O_h \left(\frac{x^{1/h}}{\log x} \right),$$

and

$$\begin{aligned} \sum_{n \in \mathcal{N}_h(x)} \omega^2(n) &= \mathcal{C}_{1,\omega,\mathcal{N}_h} x^{1/h} (\log \log x)^2 + \mathcal{C}_{3,\omega,\mathcal{N}_h} x^{1/h} \log \log x + \mathcal{C}_{4,\omega,\mathcal{N}_h} x^{1/h} \\ &\quad + O_h \left(\frac{x^{1/h} \log \log x}{\log x} \right), \end{aligned}$$

where

$$\mathcal{C}_{1,\omega,\mathcal{N}_h} = \gamma_{0,h}, \quad \mathcal{C}_{2,\omega,\mathcal{N}_h} = D_1 \gamma_{0,h}, \quad \mathcal{C}_{3,\omega,\mathcal{N}_h} = 2\mathcal{C}_{2,\omega,\mathcal{N}_h} + \mathcal{C}_{1,\omega,\mathcal{N}_h}, \quad \text{and} \quad \mathcal{C}_{4,\omega,\mathcal{N}_h} = D_2 \gamma_{0,h}. \quad (2.7)$$

Note that $\omega(n)$ may exhibit different normal orders over different subsets of $\mathbb{N}$. For instance, $\omega(p) = 1$ for any prime p , and thus $\omega(n)$ has normal order 1 over the set of all primes. However, in this work, we establish that $\omega(n)$ still has the normal order $\log \log n$ over h -free and over h -full numbers. Since the set of h -free numbers has a positive density in $\mathbb{N}$, the proof of normal order of $\omega(n)$ over h -free numbers follows from its classical case over $\mathbb{N}$. This chapter proves this result using the variance of $\omega(n)$ over h -free numbers. We prove:

Theorem 2.1.3. *Let $h \geq 2$ be an integer. Let C_2 be defined in (2.2). Let $\mathcal{S}_h(x)$ be the set of h -free numbers less than or equal to x . Then, the variance of $\omega(n)$ over h -free numbers is given by*

$$\sum_{n \in \mathcal{S}_h(x)} (\omega(n) - \log \log n)^2 = \frac{1}{\zeta(h)} x \log \log x + \frac{C_2}{\zeta(h)} x + O_h \left(\frac{x \log \log x}{\log x} \right).$$

Let $g(x)$ be an increasing function such that $g(x) \rightarrow \infty$ as $x \rightarrow \infty$. Then the set of natural numbers $n \in \mathcal{S}_h(x)$ such that

$$\frac{|\omega(n) - \log \log n|}{\sqrt{\log \log n}} \geq g(x),$$

is $o(|\mathcal{S}_h(x)|)$. As a consequence, $\omega(n)$ has normal order $\log \log n$ over h -free numbers.

On the other hand, the set of h -full numbers has density zero in $\mathbb{N}$ and thus does not directly follow from the classical result. However, writing an h -full number n as $n = r_0^h r_1$ where r_0 is the product of all distinct prime factors of n with multiplicity h , one can use the classical result on $\omega(r_0)$ to establish the normal order of $\omega(n)$ over h -full numbers. Here, we provide another proof of this result using the variance of $\omega(n)$ over h -full numbers. We prove:

Theorem 2.1.4. *Let $h \geq 2$ be an integer. Let $\gamma_{0,h}$ be defined by (1.6) and D_2 be defined by (2.6). Let $\mathcal{N}_h(x)$ be the set of h -full numbers less than or equal to x . Then, the variance of $\omega(n)$ over h -full numbers is given by*

$$\sum_{n \in \mathcal{N}_h(x)} (\omega(n) - \log \log n)^2 = \gamma_{0,h} x^{1/h} \log \log x + D_2 \gamma_{0,h} x^{1/h} + O_h \left(\frac{x^{1/h} \log \log x}{\log x} \right).$$

Let $g(x)$ be an increasing function such that $g(x) \rightarrow \infty$ as $x \rightarrow \infty$. Then the set of natural numbers $n \in \mathcal{N}_h(x)$ such that

$$\frac{|\omega(n) - \log \log n|}{\sqrt{\log \log n}} \geq g(x),$$

is $o(|\mathcal{N}_h(x)|)$. As a consequence, $\omega(n)$ has normal order $\log \log n$ over h -full numbers.

2.2 Lemmata

In this section, we list several results regarding sums over primes required to study the first and the second moment of $\omega(n)$ over h -free and over h -full numbers.

Lemma 2.2.1. [1, Lemma 1.2] *If $\tau > 1$ be any real number. Then*

$$\sum_{p \geq x} \frac{1}{p^\tau} = \frac{1}{(\tau - 1)x^{\tau-1}(\log x)} + O\left(\frac{1}{x^{\tau-1}(\log x)^\tau}\right).$$

Lemma 2.2.2. *Let α be any real number satisfying $0 < \alpha < 1$. Then*

$$\sum_{p \leq x} \frac{1}{p^\alpha} = O_\alpha\left(\frac{x^{1-\alpha}}{\log x}\right).$$

Proof. Recall that the elementary Chebyshev's estimate (see [3]) yields

$$\pi(x) := \sum_{p \leq x} 1 = O\left(\frac{x}{\log x}\right). \quad (2.8)$$

Thus, using partial summation and integration by parts, we obtain

$$\sum_{p \leq x} \frac{1}{p^\alpha} = \frac{\pi(x)}{x^\alpha} + \alpha \int_{2^-}^x \frac{\pi(t)}{t^{\alpha+1}} dt = O_\alpha\left(\frac{x^{1-\alpha}}{\log x}\right),$$

which completes the proof. □

Lemma 2.2.3. [37, Exercise 9.4.4] For $x > 2$, we have

$$\sum_{p \leq x/2} \frac{1}{p \log(x/p)} = O\left(\frac{\log \log x}{\log x}\right).$$

Lemma 2.2.4. [10, Lemma 1] Let p and q denote prime numbers. For $x > 2$, we have

$$\sum_{\substack{p, q \\ pq \leq x}} \frac{1}{pq} = (\log \log x)^2 + 2B_1 \log \log x + (B_1^2 - \zeta(2)) + O\left(\frac{\log \log x}{\log x}\right),$$

where the sum is over prime pairs (p, q) and counts the pair (p, q) as distinct from (q, p) when $p \neq q$.

Lemma 2.2.5. Let $h \geq 2$ be a fixed integer. Let $y > 2$ and $r > h$ be fixed real numbers. Then

$$\sum_{p \leq y} \frac{1}{p^{r/h} (1 - p^{-1/h} + p^{-1})} = \mathcal{L}_h(r) + O_{h,r}\left(\frac{1}{y^{\frac{r}{h}-1}(\log y)}\right),$$

where $\mathcal{L}_h(r)$ is the convergent sum defined in (2.4) as

$$\mathcal{L}_h(r) := \sum_p \frac{1}{p^{(r/h)-1} (p - p^{1-1/h} + 1)},$$

and $O_{h,r}$ means that the implied constant depends on both h and r .

Proof. Note that

$$\sum_{p \leq y} \frac{1}{p^{r/h} (1 - p^{-1/h} + p^{-1})} = \mathcal{L}_h(r) + O_h\left(\sum_{p > y} \frac{1}{p^{r/h}}\right).$$

Applying Lemma 2.2.1 with $k = r/h$ completes the proof. $\square$

Next, we recall the following results regarding the density of certain sequences of h -free numbers:

Lemma 2.2.6. [27, Lemma 3] Let $x > 2$ be a real number. Let $h \geq 2$ be an integer. Let $\mathcal{S}_h$ be the set of h -free numbers. Let $q_1, \dots, q_r$ be distinct prime numbers. Then, we have

$$\sum_{\substack{n \leq x, n \in \mathcal{S}_h \\ (n, q_1) = \dots = (n, q_r) = 1}} 1 = \prod_{i=1}^r \left(\frac{q_i^h - q_i^{h-1}}{q_i^h - 1} \right) \frac{x}{\zeta(h)} + O_h(2^r x^{1/h}).$$

2.3 The first and the second moment of $\omega(n)$ over h -free numbers

In this section, we establish the asymptotic distribution of $\omega(n)$ over h -free numbers.

Proof of Theorem 2.1.1. Let $p^k || n$ denote the property that p^k divides n but p^{k+1} does not divide n . Then writing $n = p^k m$ with $(m, p) = 1$ for such n and using $r_p = \min \left\{ h - 1, \left\lfloor \frac{\log x}{\log p} \right\rfloor \right\}$, we obtain

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \omega(n) = \sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \sum_{\substack{p \\ p | n}} 1 = \sum_{p \leq x} \sum_{k=1}^{r_p} \sum_{\substack{n \leq x \\ n \in \mathcal{S}_h, p^k || n}} 1 = \sum_{p \leq x} \sum_{k=1}^{r_p} \sum_{\substack{m \leq x/p^k \\ m \in \mathcal{S}_h, (m, p)=1}} 1.$$

Using Lemma 2.2.6 for a single prime p to the above and using Lemma 2.2.2 to bound the error term, we obtain

$$\begin{aligned} \sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \omega(n) &= \sum_{p \leq x} \sum_{k=1}^{r_p} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \frac{x}{\zeta(h)} + O_h \left(x^{1/h} \sum_{p \leq x} \frac{1}{p^{1/h}} \right) \\ &= \sum_{p \leq x} \sum_{k=1}^{r_p} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \frac{x}{\zeta(h)} + O_h \left(\frac{x}{\log x} \right). \end{aligned} \quad (2.9)$$

To complete the proof, we only need to estimate the main term above. Consider the set

$$R := \left\{ p \leq x : \left\lfloor \frac{\log x}{\log p} \right\rfloor < h - 1 \right\}.$$

Using the bound $\lfloor x \rfloor \geq x - 1$, we can write

$$\begin{aligned} \sum_{p \leq x} \sum_{k=1}^{r_p} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) &= \sum_{p \leq x} \sum_{k=1}^{h-1} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) - \sum_{\substack{p \leq x \\ p \in R}} \sum_{k=\left\lfloor \frac{\log x}{\log p} \right\rfloor + 1}^{h-1} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \\ &= \sum_{p \leq x} \sum_{k=1}^{h-1} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) + O \left(\sum_{p \leq x} \frac{1}{p^{\frac{\log x}{\log p}}} \right). \end{aligned}$$

Using Chebyshev's estimate given in (2.8) on the above, we obtain

$$\begin{aligned} \sum_{p \leq x} \sum_{k=1}^{r_p} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) &= \sum_{p \leq x} \sum_{k=1}^{h-1} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) + O \left(\frac{1}{x} \sum_{p \leq x} 1 \right) \\ &= \sum_{p \leq x} \sum_{k=1}^{h-1} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) + O \left(\frac{1}{\log x} \right). \end{aligned} \quad (2.10)$$

Now, using Lemma 2.2.1 with $k = h$ and the well-known Mertens' second theorem given as

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + B_1 + O \left(\frac{1}{\log x} \right), \quad (2.11)$$

where B_1 is defined in (1.7), we obtain

$$\begin{aligned} \sum_{p \leq x} \sum_{k=1}^{h-1} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) &= \sum_{p \leq x} \left(\frac{1}{p} - \frac{p-1}{p(p^h - 1)} \right) \\ &= \log \log x + B_1 + O \left(\frac{1}{\log x} \right) - \sum_p \frac{p-1}{p(p^h - 1)} + O_h \left(\frac{1}{x^{h-1} \log x} \right) \\ &= \log \log x + B_1 - \sum_p \frac{p-1}{p(p^h - 1)} + O_h \left(\frac{1}{\log x} \right). \end{aligned}$$

Combining the above with (2.9) and (2.10) completes the first part of the proof.

Next, we prove the second-moment estimate. Notice that,

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \omega^2(n) = \sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \left(\sum_{\substack{p \\ p|n}} 1 \right)^2 = \sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \omega(n) + \sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \sum_{\substack{p, q \\ p|n, q|n, p \neq q}} 1, \quad (2.12)$$

where p and q above denote primes. The first sum on the right-hand side is the first moment estimated above. For the second sum, we consider $r_p = \min \left\{ h-1, \left\lfloor \frac{\log x}{\log p} \right\rfloor \right\}$ and $r_q = \min \left\{ h-1, \left\lfloor \frac{\log x}{\log q} \right\rfloor \right\}$. We rewrite the sum and use Lemma 2.2.6 for two primes p and

q to obtain

$$\begin{aligned}
& \sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \sum_{\substack{p, q \\ p|n, q|n, p \neq q}} 1 \\
&= \sum_{\substack{p, q \\ p \neq q, pq \leq x}} \sum_{k=1}^{r_p} \sum_{l=1}^{r_q} \sum_{\substack{n \leq \mathcal{S}_h(x/(p^k q^l)) \\ (p, n) = (q, n) = 1}} 1 \\
&= \sum_{\substack{p, q \\ p \neq q, pq \leq x}} \sum_{k=1}^{r_p} \sum_{l=1}^{r_q} \left(\left(\frac{p^h - p^{h-1}}{p^k (p^h - 1)} \right) \left(\frac{q^h - q^{h-1}}{q^l (q^h - 1)} \right) \frac{x}{\zeta(h)} + O_h \left(\frac{x^{1/h}}{p^{k/h} q^{l/h}} \right) \right). \tag{2.13}
\end{aligned}$$

We employ (2.11) for $h = 2$ and Lemma 2.2.2 with $\alpha = 2/h$ otherwise below to obtain

$$\begin{aligned}
x^{1/h} \sum_{\substack{p, q \\ p \neq q, pq \leq x}} \frac{1}{(pq)^{1/h}} &= x^{1/h} \sum_{\substack{p, q \\ pq \leq x}} \frac{1}{(pq)^{1/h}} - x^{1/h} \sum_{p \leq x^{1/2}} \frac{1}{p^{2/h}} \\
&= x^{1/h} \sum_{\substack{p, q \\ pq \leq x}} \frac{1}{(pq)^{1/h}} + O_h(x^{1/2} \log \log x).
\end{aligned}$$

We use Lemma 2.2.2 with $\alpha = 1/h$ and Lemma 2.2.3 to obtain

$$\begin{aligned}
x^{1/h} \sum_{\substack{p, q \\ pq \leq x}} \frac{1}{(pq)^{1/h}} &= x^{1/h} \sum_{p \leq x/2} \frac{1}{p^{1/h}} \sum_{q \leq x/p} \frac{1}{q^{1/h}} \\
&\ll_h x \sum_{p \leq x/2} \left(\frac{1}{p \log(x/p)} \right) \\
&\ll_h \frac{x \log \log x}{\log x}. \tag{2.14}
\end{aligned}$$

Thus, the error term in (2.13) is estimated by

$$\sum_{\substack{p, q \\ p \neq q, pq \leq x}} \sum_{k=1}^{r_p} \sum_{l=1}^{r_q} \frac{x^{1/h}}{p^{k/h} q^{l/h}} \ll x^{1/h} \sum_{\substack{p, q \\ pq \leq x}} \frac{1}{(pq)^{1/h}} \ll_h \frac{x \log \log x}{\log x}. \tag{2.15}$$

Thus, it remains to estimate the main term of (2.13), in particular, the term

$$\sum_{\substack{p, q \\ p \neq q, pq \leq x}} \sum_{k=1}^{r_p} \sum_{l=1}^{r_q} \left(\frac{p^h - p^{h-1}}{p^k (p^h - 1)} \right) \left(\frac{q^h - q^{h-1}}{q^l (q^h - 1)} \right).$$

Consider again the set

$$R := \left\{ p \leq x : \left\lfloor \frac{\log x}{\log p} \right\rfloor < h-1 \right\}.$$

Using the definition of r_p and r_q , we can rewrite

$$\begin{aligned} & \sum_{\substack{p,q \\ p \neq q, pq \leq x}} \sum_{k=1}^{r_p} \sum_{l=1}^{r_q} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \left(\frac{q^h - q^{h-1}}{q^l(q^h - 1)} \right) \\ &= \sum_{\substack{p,q \\ p \neq q, pq \leq x}} \sum_{k=1}^{h-1} \sum_{l=1}^{h-1} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \left(\frac{q^h - q^{h-1}}{q^l(q^h - 1)} \right) - I_1 - I_2 + I_3, \end{aligned} \quad (2.16)$$

where

$$\begin{aligned} I_1 &= \sum_{\substack{p,q \\ p \neq q, pq \leq x \\ p \in R}} \sum_{k=\lfloor \frac{\log x}{\log p} \rfloor + 1}^{h-1} \sum_{l=1}^{h-1} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \left(\frac{q^h - q^{h-1}}{q^l(q^h - 1)} \right), \\ I_2 &= \sum_{\substack{p,q \\ p \neq q, pq \leq x \\ q \in R}} \sum_{k=1}^{h-1} \sum_{l=\lfloor \frac{\log x}{\log q} \rfloor + 1}^{h-1} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \left(\frac{q^h - q^{h-1}}{q^l(q^h - 1)} \right), \\ I_3 &= \sum_{\substack{p,q \\ p \neq q, pq \leq x \\ p, q \in R}} \sum_{k=\lfloor \frac{\log x}{\log p} \rfloor + 1}^{h-1} \sum_{l=\lfloor \frac{\log x}{\log q} \rfloor + 1}^{h-1} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \left(\frac{q^h - q^{h-1}}{q^l(q^h - 1)} \right). \end{aligned}$$

The sum I_1 , I_2 , and I_3 contribute to the error term. In fact, using $\lfloor x \rfloor \geq x - 1$, (2.8), and Lemma 2.2.3, we estimate

$$I_1 \ll \sum_{\substack{p,q \\ p \neq q, pq \leq x \\ p \in R}} \frac{1}{p^{\frac{\log x}{\log p}}} \frac{1}{q} \ll \sum_{q \leq x/2} \frac{1}{q} \left(\frac{1}{x} \sum_{p \leq x/q} 1 \right) \ll \sum_{q \leq x/2} \frac{1}{q^2 \log(x/q)} \ll \frac{\log \log x}{\log x}.$$

Similarly,

$$I_2 \ll \frac{\log \log x}{\log x}.$$

Finally, for I_3 , using $\lfloor x \rfloor \geq x - 1$, (2.8), and Lemma 2.2.3 again, we have

$$I_3 \ll \sum_{\substack{p,q \\ p \neq q, pq \leq x \\ p,q \in R}} \frac{1}{p^{\frac{\log x}{\log p}}} \frac{1}{q^{\frac{\log x}{\log q}}} \ll \frac{1}{x} \sum_{q \leq x/2} \left(\frac{1}{x} \sum_{p \leq x/q} 1 \right) \ll \frac{1}{x} \sum_{q \leq x/2} \frac{1}{q \log(x/q)} \ll \frac{\log \log x}{x \log x}.$$

We next estimate the main term in (2.16). First, we rewrite the sum as

$$\begin{aligned} & \sum_{\substack{p,q \\ p \neq q, pq \leq x}} \sum_{k=1}^{h-1} \sum_{l=1}^{h-1} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \left(\frac{q^h - q^{h-1}}{q^l(q^h - 1)} \right) \\ &= \sum_{\substack{p,q \\ p \neq q, pq \leq x}} \left(\frac{p^{h-1} - 1}{p^h - 1} \right) \left(\frac{q^{h-1} - 1}{q^h - 1} \right) \\ &= \sum_{\substack{p,q \\ pq \leq x}} \left(\frac{p^{h-1} - 1}{p^h - 1} \right) \left(\frac{q^{h-1} - 1}{q^h - 1} \right) - \sum_{\substack{p \\ p \leq x^{1/2}}} \left(\frac{p^{h-1} - 1}{p^h - 1} \right)^2. \end{aligned} \quad (2.17)$$

The second sum above is estimated using Lemma 2.2.1 as

$$\begin{aligned} \sum_{\substack{p \\ p \leq x^{1/2}}} \left(\frac{p^{h-1} - 1}{p^h - 1} \right)^2 &= \sum_p \left(\frac{p^{h-1} - 1}{p^h - 1} \right)^2 + \sum_{\substack{p \\ p > x^{1/2}}} \left(\frac{p^{h-1} - 1}{p^h - 1} \right)^2 \\ &= \sum_p \left(\frac{p^{h-1} - 1}{p^h - 1} \right)^2 + O\left(\frac{1}{x^{1/2} \log x} \right). \end{aligned} \quad (2.18)$$

For the first sum on the right-hand side in (2.17), using $\frac{p^{h-1}-1}{p^h-1} = \frac{1}{p} - \frac{p-1}{p(p^h-1)}$, and the symmetry in p and q , we have

$$\begin{aligned} & \sum_{\substack{p,q \\ pq \leq x}} \left(\frac{p^{h-1} - 1}{p^h - 1} \right) \left(\frac{q^{h-1} - 1}{q^h - 1} \right) \\ &= \sum_{\substack{p,q \\ pq \leq x}} \frac{1}{pq} - 2 \sum_{\substack{p,q \\ pq \leq x}} \frac{1}{p} \left(\frac{q-1}{q(q^h-1)} \right) + \sum_{\substack{p,q \\ pq \leq x}} \left(\frac{p-1}{p(p^h-1)} \right) \left(\frac{q-1}{q(q^h-1)} \right). \end{aligned} \quad (2.19)$$

We estimate the sums on the right-hand side above separately. For the first sum, we use

Lemma 2.2.4. For the third sum, we use Lemma 2.2.1 and Lemma 2.2.3 to obtain

$$\begin{aligned}
& \sum_{\substack{p,q \\ pq \leq x}} \left(\frac{p-1}{p(p^h-1)} \right) \left(\frac{q-1}{q(q^h-1)} \right) \\
&= \sum_{\substack{p \\ p \leq x/2}} \left(\frac{p-1}{p(p^h-1)} \right) \sum_{\substack{q \\ q \leq x/p}} \left(\frac{q-1}{q(q^h-1)} \right) \\
&= \sum_{\substack{p \\ p \leq x/2}} \left(\frac{p-1}{p(p^h-1)} \right) \left(\sum_p \left(\frac{p-1}{p(p^h-1)} \right) + O\left(\frac{1}{(x/p) \log(x/p)} \right) \right) \\
&= \left(\sum_p \frac{p-1}{p(p^h-1)} \right)^2 + O\left(\frac{\log \log x}{x \log x} \right). \tag{2.20}
\end{aligned}$$

Now, for the second sum in (2.19), we use Lemma 2.2.1, and (2.11) to obtain

$$\begin{aligned}
& \sum_{\substack{p,q \\ pq \leq x}} \frac{1}{p} \left(\frac{q-1}{q(q^h-1)} \right) \\
&= \sum_{\substack{p \\ p \leq x/2}} \frac{1}{p} \sum_{\substack{q \\ q \leq x/p}} \left(\frac{q-1}{q(q^h-1)} \right) \\
&= \sum_{\substack{p \\ p \leq x/2}} \frac{1}{p} \left(\sum_p \frac{p-1}{p(p^h-1)} + O\left(\frac{1}{(x/p) \log(x/p)} \right) \right) \\
&= \left(\sum_p \frac{p-1}{p(p^h-1)} \right) \left(\log \log(x/2) + B_1 + O\left(\frac{1}{\log x} \right) \right) + O\left(\frac{1}{x} \sum_{\substack{p \\ p \leq x/2}} \frac{1}{\log(x/p)} \right).
\end{aligned}$$

Using $\log \log(x/2) = \log \log x + O(1/\log x)$ and (2.8) above, we obtain

$$\sum_{\substack{p,q \\ pq \leq x}} \frac{1}{p} \left(\frac{q-1}{q(q^h-1)} \right) = \left(\sum_p \frac{p-1}{p(p^h-1)} \right) (\log \log x + B_1) + O\left(\frac{1}{\log x} \right). \tag{2.21}$$

Combining (2.19), (2.20), (2.21) and Lemma 2.2.4, we obtain

$$\begin{aligned}
& \sum_{\substack{p,q \\ pq \leq x}} \left(\frac{p^{h-1} - 1}{p^h - 1} \right) \left(\frac{q^{h-1} - 1}{q^h - 1} \right) \\
&= (\log \log x)^2 + 2B_1 \log \log x + B_1^2 - \zeta(2) - 2 \left(\sum_p \frac{p-1}{p(p^h-1)} \right) (\log \log x + B_1) \\
&\quad + \left(\sum_p \frac{p-1}{p(p^h-1)} \right)^2 + O\left(\frac{\log \log x}{\log x} \right).
\end{aligned}$$

Combining (2.12), (2.13), (2.15), (2.16), (2.17), and (2.18) with the above equation and the first moment of $\omega(n)$, we obtain the required second moment for $\omega(n)$ over h -free numbers. This completes the proof of Theorem 2.1.1. $\square$

2.4 The first and the second moment of $\omega(n)$ over h -full numbers

Let $h \geq 2$ be an integer. Recall that $\mathcal{N}_h$ is the set of h -full numbers and $\mathcal{N}_h(x)$ is the set of h -full numbers less than or equal to x . Note that, if $n \in \mathcal{N}_h(x)$, then for any prime $p|n$, we must have $p^h|n$. Moreover, the maximal power of p that divides any such $n \leq x$ is $\lfloor \log x / \log p \rfloor$. Using this and $n = p^k m$ with $(m, p) = 1$ when $p^k || n$, we can write

$$\sum_{n \in \mathcal{N}_h(x)} \omega(n) = \sum_{n \in \mathcal{N}_h(x)} \sum_{\substack{p \\ p|n}} 1 = \sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \sum_{\substack{m \in \mathcal{N}_h(x/p^k) \\ (m,p)=1}} 1. \quad (2.22)$$

Let q be a fixed prime. We intend to estimate the sum

$$A_{q,h}(y) = \sum_{\substack{n \in \mathcal{N}_h(y) \\ (n,q)=1}} 1. \quad (2.23)$$

To prove the estimate for the above sum, we will use techniques from the work of Ivić and Shiu [25]. Let's recall some of the results about h -full numbers from their work:

Notice that the generating series for the h -full numbers is defined on $\Re(s) > 1/h$ as:

$$N_h(s) = \sum_{n \in \mathcal{N}_h} \frac{1}{n^s} = \prod_p \left(1 + \frac{1}{p^{hs}} + \cdots + \frac{1}{p^{ks}} + \cdots \right) = \prod_p \left(1 + \frac{p^{-hs}}{1 - p^{-s}} \right).$$

Let $A_h(y)$ denote the number of h -full integers not exceeding y . In [25, Section 1], Ivić and Shiu proved that

$$N_h(s) = \zeta(hs)\zeta((h+1)s) \cdots \zeta((2h-1)s)\zeta^{-1}((2h+2)s)\phi_h(s), \quad (2.24)$$

where $\zeta(s)$ is the Riemann ζ -function, where $\phi_h(s)$ satisfies

$$\prod_p \left(1 - p^{-(2h+2)s} + \sum_{r=2h+3}^{(3h^2+h-2)/2} a_{r,h} p^{-rs} \right) = \zeta^{-1}((2h+2)s)\phi_h(s), \quad (2.25)$$

and where $a_{r,h}$ satisfy the identity

$$\left(1 + \frac{v^h}{1-v} \right) (1-v^h) \cdots (1-v^{2h-1}) = 1 - v^{2h+2} + \sum_{r=2h+3}^{(3h^2+h-2)/2} a_{r,h} v^r. \quad (2.26)$$

Note that $|a_{r,h}| \leq h2^h$, $\phi_2(s) = 1$, and $\phi_h(s)$ has a generating series, also called a Dirichlet series, with abscissa of absolute convergence equal to $1/(2h+3)$ if $h > 2$. Moreover, it is also established that

$$N_h(s) = G_h(s)K_h(s),$$

where

$$K_h(s) = \sum_{n=1}^{\infty} k_h(n)n^{-s} = \zeta(hs)\zeta((h+1)s) \cdots \zeta((2h-1)s), \quad (2.27)$$

and

$$G_h(s) = \sum_{n=1}^{\infty} g_h(n)n^{-s} = \frac{\phi_h(s)}{\zeta((2h+2)s)},$$

is a Dirichlet series converging absolutely in $\Re(s) > 1/(2h+2)$. Thus, one can write

$$A_h(y) = \sum_{mn \leq y} g_h(m)k_h(n).$$

Additionally, let $I_h(y)$ denote the sum

$$I_h(y) := \sum_{n \leq y} k_h(n). \quad (2.28)$$

In [25, Theorem 1], Ivić and Shiu proved that

$$I_h(y) = \sum_{r=h}^{2h-1} C_{r,h} y^{1/r} + \Delta_h^*(y), \quad (2.29)$$

where

$$C_{r,h} = \prod_{j=h, j \neq r}^{2h-1} \zeta(j/r), \quad (2.30)$$

and where

$$\Delta_h^*(y) \ll y^{\eta_h}, \quad (2.31)$$

with a constant η_h satisfying $1/(2h+2) < \eta_h < 1/(2h-1)$. Using the above pre-requisite results, we establish an asymptotic result for $A_{q,h}(y)$ as defined in (2.23). We prove:

Lemma 2.4.1. *Let $x > 2$. Let $h \geq 2$ be any integer. Let $q_1, q_2, \dots, q_r$ be distinct primes. Let*

$$A_{q_1, \dots, q_r, h}(x) := \sum_{\substack{n \in \mathcal{N}_h(x) \\ (n, q_1) = \dots = (n, q_r) = 1}} 1. \quad (2.32)$$

Then, we have

$$A_{q_1, \dots, q_r, h}(x) = \gamma_{q_1, \dots, q_r, 0, h} x^{\frac{1}{h}} + \gamma_{q_1, \dots, q_r, 1, h} x^{\frac{1}{h+1}} + \dots + \gamma_{q_1, \dots, q_r, h-1, h} x^{\frac{1}{2h-1}} + O_h(x^{\eta_h}),$$

where η_h is a constant satisfying $1/(2h+2) < \eta_h < 1/(2h-1)$, and for $i \in \{0, 1, \dots, h-1\}$,

$$\gamma_{q_1, \dots, q_r, i, h} = C_{h+i, h} \frac{\phi_h(1/(h+i))}{\zeta((2h+2)/(h+i)) \left(\prod_{j=1}^r \left(1 + \frac{q_j^{-h/(h+i)}}{1 - q_j^{-1/(h+i)}} \right) \right)}, \quad (2.33)$$

with $C_{h+i, h}$ defined in (2.30) and $\phi_h(s)$ satisfying (2.25).

Proof. Let us consider the sum defined on $\Re(s) > 1/h$ as

$$N_{q_1, \dots, q_r, h}(s) = \sum_{\substack{n \in \mathcal{N}_h \\ (n, q_1) = \dots = (n, q_r) = 1}} \frac{1}{n^s} = \prod_p \left(1 + \frac{p^{-hs}}{1 - p^{-s}} \right).$$

By (2.24), we have

$$N_{q_1, \dots, q_r, h}(s) = \zeta(hs) \zeta((h+1)s) \cdots \zeta((2h-1)s) \frac{\zeta^{-1}((2h+2)s) \phi_h(s)}{\prod_{j=1}^r \left(1 + \frac{q_j^{-hs}}{1-q_j^{-s}}\right)}.$$

Thus, we can write

$$N_{q_1, \dots, q_r, h}(s) = G_{q_1, \dots, q_r, h}(s) K_h(s),$$

where $K_h(s)$ is defined in (2.27) and

$$G_{q, h}(s) = \sum_{n=1}^{\infty} g_{q_1, \dots, q_r, h}(n) n^{-s} = \frac{\phi_h(s)}{\zeta((2h+2)s) \left(\prod_{j=1}^r \left(1 + \frac{q_j^{-hs}}{1-q_j^{-s}}\right) \right)}.$$

By (2.25) and (2.26), we have

$$\begin{aligned} & \frac{\phi_h(s)}{\zeta((2h+2)s) \left(\prod_{j=1}^r \left(1 + \frac{q_j^{-hs}}{1-q_j^{-s}}\right) \right)} \\ &= \prod_{\substack{p \\ p \neq q_1, \dots, q_r}} \left(1 - p^{-(2h+2)s} + \sum_{r=2h+3}^{(3h^2+h-2)/2} a_{r,h} p^{-rs} \right) \left(\prod_{j=1}^r \prod_{r=h}^{2h-1} (1 - q_j^{-rs}) \right). \end{aligned} \quad (2.34)$$

Note that, for $\Re(s) > 1/(2h+2)$ and for all sufficiently large p , using $|a_{r,h}| \leq h2^h$, we have

$$\left| p^{-(2h+2)s} + \sum_{r=2h+3}^{(3h^2+h-2)/2} a_{r,h} p^{-rs} \right| < 1.$$

Moreover, $|q_j^{-rs}| < 1$ for all $j = 1, \dots, r$ and for all $r \in \{h, h+1, \dots, 2h-1\}$. Thus, taking the logarithm of the right-hand side of (2.34) in the region $\Re(s) > 1/(2h+2)$ and using

$|a_{r,h}| \leq h2^h$ again, we obtain

$$\begin{aligned}
& \log \left(\prod_{\substack{p \\ p \neq q_1, \dots, q_r}} \left(1 - p^{-(2h+2)s} + \sum_{r=2h+3}^{(3h^2+h-2)/2} a_{r,h} p^{-rs} \right) \left(\prod_{j=1}^r \prod_{r=h}^{2h-1} (1 - q_j^{-rs}) \right) \right) \\
&= \sum_{\substack{p \\ p \neq q_1, \dots, q_r}} \log \left(1 - p^{-(2h+2)s} + \sum_{r=2h+3}^{(3h^2+h-2)/2} a_{r,h} p^{-rs} \right) + \sum_{j=1}^r \sum_{r=h}^{2h-1} \log (1 - q_j^{-rs}) \\
&\ll_{h,r} \sum_p \frac{1}{p^{(2h+2)\Re(s)}} + \frac{1}{2^{h\Re(s)}}.
\end{aligned}$$

The last term of the previous equation is bounded with $\ll_h \zeta((2h+2)\Re(s)) + 1/2^{h\Re(s)}$ and thus is finite for $\Re(s) > 1/(2h+2)$. Hence, by the theory of convergence of infinite products, $G_{q,h}(s)$ converges absolutely in $\Re(s) > 1/(2h+2)$.

Now, we can write

$$A_{q_1, \dots, q_r, h}(x) = \sum_{mn \leq x} g_{q_1, \dots, q_r, h}(m) k_h(n) = \sum_{m \leq x} g_{q_1, \dots, q_r, h}(m) I_h(x/m),$$

where I_h is defined in (2.28). Further, using (2.29), we obtain

$$A_{q_1, \dots, q_r, h}(x) = \sum_{r=h}^{2h-1} C_{r,h} x^{1/r} \left(\sum_{m \leq x} \frac{g_{q_1, \dots, q_r, h}(m)}{m^{1/r}} \right) + \sum_{m \leq x} g_{q_1, \dots, q_r, h}(m) \Delta_h^*(x/m).$$

We need to estimate the sums above. Note that

$$\sum_{n \leq x} g_{q_1, \dots, q_r, h}(n) n^{-\eta_h} = O_h(1).$$

Thus, for each $r = h, h+1, \dots, 2h-1$, using partial summation, we obtain

$$\sum_{m \leq x} \frac{g_{q_1, \dots, q_r, h}(m)}{m^{1/r}} = G_{q_1, \dots, q_r, h}(1/r) - \sum_{m > x} \frac{g_{q_1, \dots, q_r, h}(m)}{m^{1/r}} = G_{q_1, \dots, q_r, h}(1/r) + O_h \left(x^{\eta_h - \frac{1}{r}} \right),$$

and by (2.31),

$$\sum_{m \leq x} g_{q_1, \dots, q_r, h}(m) \Delta_h^*(x/m) \ll x^{\eta_h} \sum_{m \leq x} |g_{q_1, \dots, q_r, h}(m)| m^{-\eta_h} \ll_h x^{\eta_h}.$$

Combining the above results, we obtain

$$\begin{aligned} A_{q_1, \dots, q_r, h}(x) &= \sum_{r=h}^{2h-1} C_{r,h} G_{q_1, \dots, q_r, h}(1/r) x^{1/r} + O_h(x^{\eta_h}) \\ &= \gamma_{q_1, \dots, q_r, 0, h} x^{1/h} + \dots + \gamma_{q_1, \dots, q_r, h-1, h} x^{1/(2h-1)} + O_h(x^{\eta_h}), \end{aligned}$$

which completes the proof. $\square$

Remark 2.4.1. *Note that*

$$\gamma_{q_1, \dots, q_r, 0, h} = \frac{\gamma_{0, h}}{\prod_{j=1}^r \left(1 + \frac{q_j^{-1}}{1 - q_j^{-1/h}} \right)}, \quad (2.35)$$

where

$$\gamma_{0, h} = C_{h, h} \frac{\phi_h(1/h)}{\zeta((2h+2)/h)} = \prod_p \left(1 + \frac{p - p^{1/h}}{p^2(p^{1/h} - 1)} \right), \quad (2.36)$$

with the product formula described in [26, Page 3].

We use Lemma 2.4.1 with one prime to prove the first and the second moment estimates for $\omega(n)$ over h -full numbers. We prove:

Proof of Theorem 2.1.2. Inserting the formula for $A_{q, h}(y)$ given in Lemma 2.4.1 with $y = x/p^k$ in (2.22), we obtain

$$\begin{aligned} &\sum_{n \in \mathcal{N}_h(x)} \omega(n) \\ &= \sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} A_{p, h}(x/p^k) \\ &= \sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \left(\gamma_{p, 0, h}(x/p^k)^{1/h} + \dots + \gamma_{p, h-1, h}(x/p^k)^{1/(2h-1)} + O_h((x/p^k)^{\eta}) \right). \end{aligned} \quad (2.37)$$

Let us study the main term above given as

$$\sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \gamma_{p,0,h}(x/p^k)^{1/h} = \gamma_{0,h} x^{1/h} \sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \left(\frac{1}{p^{k/h} \left(1 + \frac{p^{-1}}{1-p^{-1/h}} \right)} \right).$$

We obtain

$$\begin{aligned} & \gamma_{0,h} x^{1/h} \sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \left(\frac{1}{p^{k/h} \left(1 + \frac{p^{-1}}{1-p^{-1/h}} \right)} \right) \\ &= \gamma_{0,h} x^{1/h} \sum_{p \leq x^{1/h}} \frac{1}{p(1-p^{-1/h}+p^{-1})} - \gamma_{0,h} x^{1/h} \sum_{p \leq x^{1/h}} \frac{(p^{-1/h})^{\lfloor \frac{\log x}{\log p} \rfloor - h + 1}}{p(1-p^{-1/h}+p^{-1})}. \end{aligned} \quad (2.38)$$

Using $\lfloor x \rfloor \geq x - 1$, and (2.8) with $y = x^{1/h}$, we bound the second term above with

$$\gamma_{0,h} x^{1/h} \sum_{p \leq x^{1/h}} \frac{(p^{-1/h})^{\lfloor \frac{\log x}{\log p} \rfloor - h + 1}}{p(1-p^{-1/h}+p^{-1})} \ll_h \frac{x^{1/h}}{\log x}. \quad (2.39)$$

Thus, it remains to study the first term in (2.38). A little manipulation yields

$$\sum_{p \leq x^{1/h}} \frac{1}{p(1-p^{-1/h}+p^{-1})} = \sum_{p \leq x^{1/h}} \frac{1}{p} + \sum_{p \leq x^{1/h}} \frac{p^{-1/h} - p^{-1}}{p(1-p^{-1/h}+p^{-1})}. \quad (2.40)$$

For the first sum on the right-hand side above, we use (2.11) to obtain

$$\sum_{p \leq x^{1/h}} \frac{1}{p} = \log \log x - \log h + B_1 + O_h \left(\frac{1}{\log x} \right). \quad (2.41)$$

For the second sum on the right-hand side of (2.40), we use the convergence of the corresponding infinite series and Lemma 2.2.5 to obtain

$$\sum_{p \leq x^{1/h}} \frac{p^{-1/h} - p^{-1}}{p(1-p^{-1/h}+p^{-1})} = \mathcal{L}_h(h+1) - \mathcal{L}_h(2h) + O_h \left(\frac{1}{x^{1/h^2}(\log x)} \right),$$

where $\mathcal{L}_h(\cdot)$ is defined in (2.4).

Combining the last three results, we obtain

$$\sum_{p \leq x^{1/h}} \frac{1}{p(1 - p^{-1/h} + p^{-1})} = \log \log x + D_1 + O_h \left(\frac{1}{\log x} \right), \quad (2.42)$$

where D_1 is defined in (2.5). Combining the above result with (2.37), (2.38), and (2.39), we obtain

$$\begin{aligned} \sum_{n \in \mathcal{N}_h(x)} \omega(n) &= \gamma_{0,h} x^{1/h} \log \log x + D_1 \gamma_{0,h} x^{1/h} + O_h \left(\frac{x^{1/h}}{\log x} \right) \\ &\quad + O_h \left(\sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} (x/p^k)^{1/(h+1)} \right). \end{aligned} \quad (2.43)$$

For the error term above, using Lemma 2.2.2 with $\alpha = h/(h+1)$, we obtain

$$\sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} (x/p^k)^{1/(h+1)} \ll x^{1/(h+1)} \sum_{p \leq x^{1/h}} \frac{1}{p^{h/(h+1)}} \ll_h \frac{x^{1/h}}{\log x}.$$

Inserting the above back into (2.43) completes the first part of the proof.

Now, note that

$$\sum_{n \in \mathcal{N}_h(x)} \omega^2(n) = \sum_{n \in \mathcal{N}_h(x)} \left(\sum_{\substack{p \\ p|n}} 1 \right)^2 = \sum_{n \in \mathcal{N}_h(x)} \omega(n) + \sum_{n \in \mathcal{N}_h(x)} \sum_{\substack{p,q \\ p|n, q|n, p \neq q}} 1. \quad (2.44)$$

The first sum on the right side above is the first moment studied above. For the second sum, we first rewrite the sum as

$$\begin{aligned} \sum_{n \in \mathcal{N}_h(x)} \sum_{\substack{p,q \\ p|n, q|n, p \neq q}} 1 &= \sum_{\substack{p,q \\ p \neq q, pq \leq x^{1/h}}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \sum_{l=h}^{\lfloor \frac{\log x}{\log q} \rfloor} \sum_{\substack{n \in \mathcal{N}_h(x) \\ p^k | n, q^l | n}} 1 \\ &= \sum_{\substack{p,q \\ p \neq q, pq \leq x^{1/h}}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \sum_{l=h}^{\lfloor \frac{\log x}{\log q} \rfloor} A_{p,q,h}(x/(p^k q^l)). \end{aligned}$$

Then, we use Lemma 2.4.1 with two distinct primes p and q , and use (2.35) to obtain

$$\begin{aligned}
& \sum_{n \in \mathcal{N}_h(x)} \sum_{\substack{p, q \\ p|n, q|n, p \neq q}} 1 \\
&= \gamma_{0,h} x^{1/h} \sum_{\substack{p, q \\ p \neq q, pq \leq x^{1/h}}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \sum_{l=h}^{\lfloor \frac{\log x}{\log q} \rfloor} \frac{1}{p^{k/h} \left(1 + \frac{p^{-1}}{1-p^{-1/h}}\right)} \frac{1}{q^{l/h} \left(1 + \frac{q^{-1}}{1-q^{-1/h}}\right)} \\
&+ O_h \left(x^{\frac{1}{h+1}} \sum_{\substack{p, q \\ p \neq q, pq \leq x^{1/h}}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \sum_{l=h}^{\lfloor \frac{\log x}{\log q} \rfloor} \frac{1}{p^{k/(h+1)} q^{l/(h+1)}} \right). \tag{2.45}
\end{aligned}$$

The error term above is bounded by using Lemma 2.2.2 and Lemma 2.2.3 as the following

$$\begin{aligned}
x^{\frac{1}{h+1}} \sum_{\substack{p, q \\ p \neq q, pq \leq x^{1/h}}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \sum_{l=h}^{\lfloor \frac{\log x}{\log q} \rfloor} \frac{1}{p^{k/(h+1)} q^{l/(h+1)}} &\ll_h x^{\frac{1}{h+1}} \sum_{p \leq x^{1/h}/2} \frac{1}{p^{h/(h+1)}} \sum_{q \leq x^{1/h}/p} \frac{1}{q^{h/(h+1)}} \\
&\ll_h x^{\frac{1}{h}} \sum_{p \leq x^{1/h}/2} \frac{1}{p \log(x^{1/h}/p)} \\
&\ll_h \frac{x^{\frac{1}{h}} \log \log x}{\log x}. \tag{2.46}
\end{aligned}$$

Next, we estimate the main term in (2.45). First note that

$$\sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \frac{1}{p^{k/h} \left(1 + \frac{p^{-1}}{1-p^{-1/h}}\right)} = \frac{1}{p(1 - p^{-1/h} + p^{-1})} - \frac{p^{-\frac{1}{h}(\lfloor \frac{\log x}{\log p} \rfloor - h + 1)}}{p(1 - p^{-1/h} + p^{-1})}.$$

Thus, using a similar result for a prime q as above and the symmetry of two primes p and q , we deduce

$$\begin{aligned}
& \gamma_{0,h} x^{1/h} \sum_{\substack{p, q \\ p \neq q, pq \leq x^{1/h}}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \sum_{l=h}^{\lfloor \frac{\log x}{\log q} \rfloor} \frac{1}{p^{k/h} \left(1 + \frac{p^{-1}}{1-p^{-1/h}}\right)} \frac{1}{q^{l/h} \left(1 + \frac{q^{-1}}{1-q^{-1/h}}\right)} \\
&= \gamma_{0,h} x^{1/h} \sum_{\substack{p, q \\ p \neq q, pq \leq x^{1/h}}} \frac{1}{p(1 - p^{-1/h} + p^{-1})} \frac{1}{q(1 - q^{-1/h} + q^{-1})} - 2J_1 + J_2,
\end{aligned}$$

where

$$J_1 = \gamma_{0,h} x^{1/h} \sum_{\substack{p,q \\ p \neq q, pq \leq x^{1/h}}} \frac{1}{p(1 - p^{-1/h} + p^{-1})} \frac{q^{-\frac{1}{h}(\lfloor \frac{\log x}{\log q} \rfloor - h + 1)}}{q(1 - q^{-1/h} + q^{-1})},$$

and

$$J_2 = \gamma_{0,h} x^{1/h} \sum_{\substack{p,q \\ p \neq q, pq \leq x^{1/h}}} \frac{p^{-\frac{1}{h}(\lfloor \frac{\log x}{\log p} \rfloor - h + 1)}}{p(1 - p^{-1/h} + p^{-1})} \frac{q^{-\frac{1}{h}(\lfloor \frac{\log x}{\log q} \rfloor - h + 1)}}{q(1 - q^{-1/h} + q^{-1})}.$$

Using $\lfloor x \rfloor \geq x - 1$, (2.8) and Lemma 2.2.3, we obtain

$$J_1 \ll_h \sum_{\substack{p \\ p \leq x^{1/h}/2}} \frac{1}{p(1 - p^{-1/h} + p^{-1})} \left(\sum_{\substack{q \\ q \leq x^{1/h}/p}} 1 \right) \ll_h \frac{x^{1/h} \log \log x}{\log x}.$$

Similarly, we obtain

$$J_2 \ll_h \sum_{\substack{p \\ p \leq x^{1/h}/2}} \frac{p^{-\frac{1}{h}(\lfloor \frac{\log x}{\log p} \rfloor - h + 1)}}{p(1 - p^{-1/h} + p^{-1})} \left(\sum_{\substack{q \\ q \leq x^{1/h}/p}} 1 \right) \ll_h \frac{\log \log x}{\log x}.$$

Combining the last three results, we obtain

$$\begin{aligned} & \gamma_{0,h} x^{1/h} \sum_{\substack{p,q \\ p \neq q, pq \leq x^{1/h}}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \sum_{l=h}^{\lfloor \frac{\log x}{\log q} \rfloor} \frac{1}{p^{k/h} \left(1 + \frac{p^{-1}}{1 - p^{-1/h}}\right)} \frac{1}{q^{l/h} \left(1 + \frac{q^{-1}}{1 - q^{-1/h}}\right)} \\ &= \gamma_{0,h} x^{1/h} \sum_{\substack{p,q \\ p \neq q, pq \leq x^{1/h}}} \frac{1}{p(1 - p^{-1/h} + p^{-1})} \frac{1}{q(1 - q^{-1/h} + q^{-1})} + O_h \left(\frac{x^{1/h} \log \log x}{\log x} \right). \end{aligned} \tag{2.47}$$

Thus to complete the proof, we only require to estimate the main term in (2.47). Using

Lemma 2.2.1, we have

$$\begin{aligned}
& \sum_{\substack{p,q \\ p \neq q, pq \leq x^{1/h}}} \frac{1}{p(1 - p^{-1/h} + p^{-1})} \frac{1}{q(1 - q^{-1/h} + q^{-1})} \\
&= \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{1}{p(1 - p^{-1/h} + p^{-1})} \frac{1}{q(1 - q^{-1/h} + q^{-1})} - \sum_p \left(\frac{1}{p - p^{1-1/h} + 1} \right)^2 \\
&+ O_h \left(\frac{1}{x^{1/(2h)} \log x} \right). \tag{2.48}
\end{aligned}$$

Now, using

$$\frac{1}{p(1 - p^{-1/h} + p^{-1})} = \frac{1}{p} + \frac{p^{-1/h} - p^{-1}}{p(1 - p^{-1/h} + p^{-1})},$$

a similar result for another prime q , and the symmetry of primes p and q , we write the first sum on the right-hand side of (2.48) as

$$\begin{aligned}
& \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{1}{p(1 - p^{-1/h} + p^{-1})} \frac{1}{q(1 - q^{-1/h} + q^{-1})} \\
&= \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{1}{pq} + 2 \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{q^{-1/h} - q^{-1}}{pq(1 - q^{-1/h} + q^{-1})} \\
&+ \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{p^{-1/h} - p^{-1}}{p(1 - p^{-1/h} + p^{-1})} \frac{q^{-1/h} - q^{-1}}{q(1 - q^{-1/h} + q^{-1})}.
\end{aligned}$$

The first sum on the right-hand side above is estimated using Lemma 2.2.4. For the second sum, we use Lemma 2.2.3, Lemma 2.2.5, and (2.11) to obtain

$$\begin{aligned}
& \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{q^{-1/h} - q^{-1}}{pq(1 - q^{-1/h} + q^{-1})} \\
&= \sum_p \frac{1}{p} \sum_{\substack{q \\ q \leq x^{1/h}/p}} \frac{q^{-1/h} - q^{-1}}{q(1 - q^{-1/h} + q^{-1})} \\
&= (\mathcal{L}_h(h+1) - \mathcal{L}_h(2h)) (\log \log x + B_1 - \log h) + O_h \left(\frac{\log \log x}{\log x} \right),
\end{aligned}$$

and similarly, for the third sum, we obtain

$$\begin{aligned} & \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{p^{-1/h} - p^{-1}}{p(1 - p^{-1/h} + p^{-1})} \frac{q^{-1/h} - q^{-1}}{q(1 - q^{-1/h} + q^{-1})} \\ &= (\mathcal{L}_h(h+1) - \mathcal{L}_h(2h))^2 + O_h\left(\frac{\log \log x}{x^{1/h^2} \log x}\right). \end{aligned}$$

Combining the last three results with Lemma 2.2.4, we obtain

$$\begin{aligned} & \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{1}{p(1 - p^{-1/h} + p^{-1})} \frac{1}{q(1 - q^{-1/h} + q^{-1})} \\ &= (\log \log x)^2 + 2D_1 \log \log x + D_1^2 - \zeta(2) + O_h\left(\frac{\log \log x}{\log x}\right). \end{aligned}$$

Combining the above with (2.44), (2.45), (2.46), (2.47), and (2.48), and using the first moment proved above, we obtain the the second moment of $\omega(n)$ over h -full numbers. This completes the proof of Theorem 2.1.2. $\square$

2.5 Normal order of $\omega(n)$ over h -free and h -full numbers

In this section, using the variance of $\omega(n)$ over the set of h -free and the set of h -full numbers, we prove that $\omega(n)$ has normal order $\log \log n$ when restricted to these sets.

Proof of Theorem 2.1.3. Notice that

$$\sum_{n \in \mathcal{S}_h(x)} (\omega(n) - \log \log n)^2 = \sum_{n \in \mathcal{S}_h(x)} \omega^2(n) - 2 \sum_{n \in \mathcal{S}_h(x)} \omega(n) \log \log n + \sum_{n \in \mathcal{S}_h(x)} (\log \log n)^2. \quad (2.49)$$

By (1.4), we have

$$|\mathcal{S}_h(x)| = \frac{x}{\zeta(h)} + O_h(x^{1/h}). \quad (2.50)$$

Using the above with the first moment of $\omega(n)$ over h -free numbers given in Theorem 2.1.1 and partial summation, we obtain

$$\sum_{n \in \mathcal{S}_h(x)} \omega(n) \log \log n = \frac{1}{\zeta(h)} x (\log \log x)^2 + \frac{C_1}{\zeta(h)} x \log \log x + O_h\left(\frac{x \log \log x}{\log x}\right),$$

and

$$\sum_{n \in \mathcal{S}_h(x)} (\log \log n)^2 = \frac{1}{\zeta(h)} x (\log \log x)^2 + O_h \left(\frac{x \log \log x}{\log x} \right).$$

Using the above two results and the second moment of $\omega(n)$ over h -free numbers given in Theorem 2.1.1 in (2.49), we obtain

$$\sum_{n \in \mathcal{S}_h(x)} (\omega(n) - \log \log n)^2 = \frac{1}{\zeta(h)} x \log \log x + \frac{C_2}{\zeta(h)} x + O_h \left(\frac{x \log \log x}{\log x} \right).$$

For the second part, let $g(x)$ be an increasing function such that $g(x) \rightarrow \infty$ as $x \rightarrow \infty$, and let E_{free}^h be the set of natural numbers $n \in \mathcal{S}_h$ with $x/\log x < n \leq x$ such that

$$\frac{|\omega(n) - \log \log n|}{\sqrt{\log \log n}} \geq g(x).$$

Let $|E_{\text{free}}^h|$ be the cardinality of E_{free}^h . Notice that

$$\begin{aligned} \sum_{n \in \mathcal{S}_h(x)} (\omega(n) - \log \log n)^2 &\geq \sum_{n \in E_{\text{free}}^h} (\omega(n) - \log \log n)^2 \\ &> g^2(x/\log x) \sum_{n \in E_{\text{free}}^h} \log \log n \\ &> g^2(x/\log x) |E_{\text{free}}^h| \log \log(x/\log x). \end{aligned}$$

Using the asymptotic result for the left-hand side above, we deduce

$$\frac{|E_{\text{free}}^h|}{x} \ll_h \frac{\log \log x}{g^2(x/\log x) \log \log(x/\log x)}.$$

Since $g(x) \rightarrow \infty$ as $x \rightarrow \infty$, the right-hand side above goes to 0 as $x \rightarrow \infty$. Hence $|E_{\text{free}}^h| = o(x)$. Moreover, by (2.50), $|\mathcal{S}_h(x)| \sim x/\zeta(h)$. Thus, the set of natural numbers $n \in \mathcal{S}_h$ with $x/\log x < n \leq x$ such that

$$\frac{|\omega(n) - \log \log n|}{\sqrt{\log \log n}} \geq g(x),$$

is $o(|\mathcal{S}_h(x)|)$. Next, we choose ϵ' such that $0 < \epsilon' < 1/2$ and choose the function $g(x) = (\log \log x)^{\epsilon'}$. Let $\epsilon > 0$ be arbitrary. Since $\epsilon' < 1/2$, there exist $x_0 \in \mathbb{R}$ such that

$$\frac{(\log \log x)^{\epsilon'}}{(\log \log n)^{1/2}} < \frac{(\log \log x)^{\epsilon'}}{(\log \log(x/\log x))^{1/2}} \leq \epsilon$$

for all $x \geq x_0$ and for all n with $x/\log x < n \leq x$. Hence as $x \rightarrow \infty$, the numbers $n \in \mathcal{S}_h$ with $x/\log x < n \leq x$ that satisfy the inequality

$$\frac{|\omega(n) - \log \log n|}{\log \log n} \geq \epsilon$$

is $o(|\mathcal{S}_h(x)|)$. Again by (2.50), the remaining set of natural numbers $n \in \mathcal{S}_h(x/\log x)$ is $o(|\mathcal{S}_h(x)|)$. Together, this implies that $\omega(n)$ has normal order $\log \log n$ over h -free numbers, which completes the proof. $\square$

Remark 2.5.1. *An alternative proof of the normal order result can be obtained as follows: Note that*

$$\sum_{n \in \mathcal{S}_h(x)} \left(\frac{\omega(n) - \log \log n}{\sqrt{\log \log n}} \right)^2 = O(|\mathcal{S}_h(x)|).$$

Let $g(x)$ be an increasing function of x such that $g(x) \rightarrow \infty$ as $x \rightarrow \infty$. Let

$$S_g(x) := \left| \left\{ n \in \mathcal{S}_h(x) : \left| \frac{\omega(n) - \log \log n}{\sqrt{\log \log n}} \right| \geq g(x) \right\} \right|.$$

Thus, by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} g(x)^2 S_g(x)^2 &\leq \left(\sum_{\substack{n \in \mathcal{S}_h(x) \\ \left| \frac{\omega(n) - \log \log n}{\sqrt{\log \log n}} \right| \geq g(x)}} \left| \frac{\omega(n) - \log \log n}{\sqrt{\log \log n}} \right| \right)^2 \\ &\leq S_g(x) \sum_{n \in \mathcal{S}_h(x)} \left(\frac{\omega(n) - \log \log n}{\sqrt{\log \log n}} \right)^2 \\ &\ll S_g(x) |\mathcal{S}_h(x)|. \end{aligned}$$

Thus,

$$\frac{S_g(x)}{|\mathcal{S}_h(x)|} \ll \frac{1}{g(x)^2},$$

where the right side approaches 0 as $x \rightarrow \infty$. This approach avoids the need to restrict to the subset $(x/\log x, x]$, as was done in the proof above. Furthermore, a similar argument yields an alternative proof of Theorem 2.1.4. We are grateful to the external examiner for suggesting this alternate argument.

For the normal order of $\omega(n)$ over h -full numbers, we prove the following:

Proof of Theorem 2.1.4. Notice that

$$\sum_{n \in \mathcal{N}_h(x)} (\omega(n) - \log \log n)^2 = \sum_{n \in \mathcal{N}_h(x)} \omega^2(n) - 2 \sum_{n \in \mathcal{N}_h(x)} \omega(n) \log \log n + \sum_{n \in \mathcal{N}_h(x)} (\log \log n)^2. \quad (2.51)$$

By (1.5), we have

$$|\mathcal{N}_h(x)| = \gamma_{0,h} x^{1/h} + O_h(x^{1/(h+1)}), \quad (2.52)$$

where $\gamma_{0,h}$ is defined in (1.6). Using the above with the first moment of $\omega(n)$ over h -full numbers given in Theorem 2.1.2 and the partial summation formula, we obtain

$$\sum_{n \in \mathcal{N}_h(x)} \omega(n) \log \log n = \gamma_{0,h} x^{1/h} (\log \log x)^2 + \gamma_{0,h} D_1 x^{1/h} \log \log x + O_h \left(\frac{x^{1/h} \log \log x}{\log x} \right),$$

and

$$\sum_{n \in \mathcal{N}_h(x)} (\log \log n)^2 = \gamma_{0,h} x^{1/h} (\log \log x)^2 + O_h \left(\frac{x^{1/h} \log \log x}{\log x} \right).$$

Using the above two results and the second moment of $\omega(n)$ over h -full numbers given in Theorem 2.1.2 in (2.51), we obtain

$$\sum_{n \in \mathcal{N}_h(x)} (\omega(n) - \log \log n)^2 = \gamma_{0,h} x^{1/h} \log \log x + D_2 \gamma_{0,h} x^{1/h} + O_h \left(\frac{x^{1/h} \log \log x}{\log x} \right).$$

For the second part, let $g(x)$ be an increasing function such that $g(x) \rightarrow \infty$ as $x \rightarrow \infty$, and let E_{full}^h be the set of natural numbers $n \in \mathcal{N}_h$ with $x/\log x < n \leq x$ such that

$$\frac{|\omega(n) - \log \log n|}{\sqrt{\log \log n}} \geq g(x).$$

Let $|E_{\text{full}}^h|$ be the cardinality of E_{full}^h . Then

$$\sum_{n \in \mathcal{N}_h(x)} (\omega(n) - \log \log n)^2 > g^2(x/\log x) |E_{\text{full}}^h| \log \log(x/\log x).$$

Using the asymptotic result for the left-hand side above, we deduce

$$\frac{|E_{\text{full}}^h|}{x^{1/h}} \ll_h \frac{\log \log x}{g^2(x/\log x) \log \log(x/\log x)}.$$

Since $g(x) \rightarrow \infty$ as $x \rightarrow \infty$, the right-hand side above goes to 0 as $x \rightarrow \infty$. Thus, $|E_{\text{full}}^h| = o(x^{1/h})$. Note that, by (2.52), $|\mathcal{N}_h(x)| \sim \gamma_{0,h} x^{1/h}$. Thus the set of natural numbers $n \in \mathcal{N}_h$ with $x/\log x < n \leq x$ such that

$$\frac{|\omega(n) - \log \log n|}{\sqrt{\log \log n}} \geq g(x),$$

is $o(|\mathcal{N}_h(x)|)$. Again, by (2.52), the remaining set of natural numbers $n \in \mathcal{N}_h(x/\log x)$ is $o(|\mathcal{N}_h(x)|)$. Together, this proves the first part of the theorem. Next, we choose ϵ' such that $0 < \epsilon' < 1/2$ and choose the function $g(x) = (\log \log x)^{\epsilon'}$. Let $\epsilon > 0$ be arbitrary. Since $\epsilon' < 1/2$, there exists $x_0 \in \mathbb{R}$ such that for all $x \geq x_0$ and for all n with $x/\log x < n \leq x$,

$$\frac{(\log \log x)^{\epsilon'}}{(\log \log n)^{1/2}} < \frac{(\log \log x)^{\epsilon'}}{(\log \log(x/\log x))^{1/2}} \leq \epsilon.$$

Hence as $x \rightarrow \infty$, the numbers $n \in \mathcal{N}_h$ with $x/\log x \leq n \leq x$ that satisfy the inequality

$$\frac{|\omega(n) - \log \log n|}{\log \log n} \geq \epsilon$$

is $o(|\mathcal{N}_h(x)|)$. This together with the fact that the remaining set of natural numbers $n \in \mathcal{N}_h(x/\log x)$ is $o(|\mathcal{N}_h(x)|)$ implies that $\omega(n)$ has normal order $\log \log n$ over h -full numbers. $\square$

Chapter 3

Normal order of $\Omega(n)$ over h -free and h -full numbers

3.1 Introduction

The counting argument used for $\omega(n)$ in the last chapter can be employed in the study of the distribution of $\Omega(n)$ and produce improvement to (1.15) and (1.16) as mentioned in Chapter 1. In this chapter, we explore these improvements.

Let B_1 and B_2 be defined in (1.7) and (1.8) respectively. Let $h \geq 2$ be any integer and let C_3 , C'_3 , and C_4 be three new constants (dependent on h) defined as

$$C_3 := B_2 - \sum_p \frac{h}{p^h - 1},$$

$$C'_3 := B_1 + \sum_p \frac{p^h(3p-1) + (p-1)^2 - p(h^2p^2 + (-2h^2 + 2h + 1)p + (h-1)^2)}{p(p-1)^2(p^h-1)},$$

and

$$C_4 := C_3^2 + C'_3 - \zeta(2) - \sum_p \left(\frac{p^h - hp + h - 1}{(p-1)(p^h-1)} \right)^2.$$

We improve (1.15) as the following:

Theorem 3.1.1. *Let $x > 2$ be a real number. Let $h \geq 2$ be an integer. Let $\mathcal{S}_h(x)$ be the set of h -free numbers less than or equal to x . Then, we have*

$$\sum_{n \in \mathcal{S}_h(x)} \Omega(n) = \mathcal{C}_{1,\Omega,\mathcal{S}_h} x \log \log x + \mathcal{C}_{2,\Omega,\mathcal{S}_h} x + O_h \left(\frac{x}{\log x} \right),$$

and

$$\sum_{n \in \mathcal{S}_h(x)} \Omega^2(n) = \mathcal{C}_{1,\Omega,\mathcal{S}_h} x (\log \log x)^2 + \mathcal{C}_{3,\Omega,\mathcal{S}_h} x \log \log x + \mathcal{C}_{4,\Omega,\mathcal{S}_h} x + O_h \left(\frac{x \log \log x}{\log x} \right),$$

where

$$\mathcal{C}_{1,\Omega,\mathcal{S}_h} = \frac{1}{\zeta(h)}, \quad \mathcal{C}_{2,\Omega,\mathcal{S}_h} = \frac{C_3}{\zeta(h)}, \quad \mathcal{C}_{3,\Omega,\mathcal{S}_h} = 2\mathcal{C}_{2,\Omega,\mathcal{S}_h} + \mathcal{C}_{1,\Omega,\mathcal{S}_h}, \quad \text{and} \quad \mathcal{C}_{4,\Omega,\mathcal{S}_h} = \frac{C_4}{\zeta(h)}. \quad (3.1)$$

Recall D_3 from (1.14). Let D'_3 and D_4 be new constants (dependent on h) defined as

$$D'_3 = h^2(B_1 - \log h) + \sum_p \left(\frac{(2h^2 + 2h - 1)p^{\frac{1+h}{h}} - (1+h)^2 p^{\frac{2+h}{h}} - h^2(p - p^{\frac{1}{h}} + 2p^{\frac{2}{h}} - p^{\frac{3}{h}})}{p(-p^{\frac{1+h}{h}} - p^{\frac{1}{h}} + p)(-1 + p^{\frac{1}{h}})^2} \right), \quad (3.2)$$

$$D_4 := D_3^2 + D'_3 - h^2 \zeta(2) - \sum_p \left(\frac{h(p^{\frac{1}{h}} - 1) + 1}{(p^{\frac{1}{h}} - 1)(p - p^{1-\frac{1}{h}} + 1)} \right)^2.$$

Adopting this definition, we improve (1.16) as the following:

Theorem 3.1.2. *Let $x > 2$ be a real number. Let $h \geq 2$ be an integer. Let $\mathcal{N}_h(x)$ be the set of h -full numbers less than or equal to x . Then, we have*

$$\sum_{n \in \mathcal{N}_h(x)} \Omega(n) = h\mathcal{C}_{1,\Omega,\mathcal{N}_h} x^{1/h} \log \log x + \mathcal{C}_{2,\Omega,\mathcal{N}_h} x^{1/h} + O_h \left(\frac{x^{1/h}}{\log x} \right),$$

and

$$\begin{aligned} \sum_{n \in \mathcal{N}_h(x)} \Omega^2(n) &= h^2 \mathcal{C}_{1,\Omega,\mathcal{N}_h} x^{1/h} (\log \log x)^2 + h \mathcal{C}_{3,\Omega,\mathcal{N}_h} x^{1/h} \log \log x + \mathcal{C}_{4,\Omega,\mathcal{N}_h} x^{1/h} \\ &\quad + O_h \left(\frac{x^{1/h} \log \log x}{\log x} \right), \end{aligned}$$

where

$$\mathcal{C}_{1,\Omega,\mathcal{N}_h} = \gamma_{0,h}, \quad \mathcal{C}_{2,\Omega,\mathcal{N}_h} = D_3\gamma_{0,h}, \quad \mathcal{C}_{3,\Omega,\mathcal{N}_h} = 2\mathcal{C}_{2,\Omega,\mathcal{N}_h} + h\mathcal{C}_{1,\Omega,\mathcal{N}_h}, \quad \text{and} \quad \mathcal{C}_{4,\Omega,\mathcal{N}_h} = D_4\gamma_{0,h}. \quad (3.3)$$

Using the last two theorems and the method of variance demonstrated for $\omega(n)$ in the last chapter, we establish the normal order of $\Omega(n)$ over h -free and h -full numbers as the following:

Theorem 3.1.3. $\Omega(n)$ has normal order $\log \log n$ over h -free numbers.

Theorem 3.1.4. $\Omega(n)$ has normal order $h \log \log n$ over h -full numbers.

3.2 The moments of $\Omega(n)$ over h -free numbers

Using our counting argument, we can make the result in (1.15) explicit by proving the following:

Proof of Theorem 3.1.1. Let $p^k || n$ denote the property that p^k divides n but p^{k+1} does not divide n . Then writing $n = p^k m$ with $(m, p) = 1$ for such n and using $r_p = \min \{h-1, \lfloor \log x / \log p \rfloor\}$, we obtain

$$\sum_{n \in \mathcal{S}_h(x)} \Omega(n) = \sum_{n \in \mathcal{S}_h(x)} \sum_{\substack{p \\ p^k || n}} k = \sum_{p \leq x} \sum_{k=1}^{r_p} \sum_{\substack{n \in \mathcal{S}_h(x) \\ p^k || n}} k = \sum_{p \leq x} \sum_{k=1}^{r_p} k \sum_{\substack{m \in \mathcal{S}_h(x/p^k) \\ (m,p)=1}} 1.$$

Now, using Lemma 2.2.6 for a single prime p to the above and Lemma 2.2.2, we obtain

$$\begin{aligned} \sum_{n \in \mathcal{S}_h(x)} \Omega(n) &= \sum_{p \leq x} \sum_{k=1}^{r_p} k \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \frac{x}{\zeta(h)} + O_h \left(x^{1/h} \sum_{p \leq x} \frac{1}{p^{1/h}} \right) \\ &= \sum_{p \leq x} \sum_{k=1}^{r_p} k \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \frac{x}{\zeta(h)} + O_h \left(\frac{x}{\log x} \right). \end{aligned} \quad (3.4)$$

To complete the proof, we only need to estimate the main term above. Recall the set

$$R := \left\{ p \leq x : \left\lfloor \frac{\log x}{\log p} \right\rfloor < h-1 \right\}.$$

Using the bound $\lfloor x \rfloor \geq x-1$ and (2.8), we can write

$$\begin{aligned} \sum_{p \leq x} \sum_{k=1}^{r_p} k \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) &= \sum_{p \leq x} \sum_{k=1}^{h-1} k \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) - \sum_{\substack{p \leq x \\ p \in R}} \sum_{k=\lfloor \frac{\log x}{\log p} \rfloor + 1}^{h-1} k \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \\ &= \sum_{p \leq x} \sum_{k=1}^{h-1} k \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) + O_h \left(\frac{1}{\log x} \right). \end{aligned} \quad (3.5)$$

Now, using

$$\frac{p^h - p^{h-1}}{p^h - 1} \sum_{k=1}^{h-1} \frac{k}{p^k} = \frac{p^h - hp + h - 1}{(p-1)(p^h - 1)} = \frac{1}{p} + \frac{p^h - hp^2 + hp - 1}{p(p-1)(p^h - 1)},$$

the formula for the reciprocal of primes given in (2.11), the formula

$$C_3 = B_1 + \sum_p \frac{p^h - hp^2 + hp - 1}{p(p-1)(p^h - 1)} = B_2 - \sum_p \frac{h}{p^h - 1}, \quad (3.6)$$

and using Lemma 2.2.1 with $\tau = h$, we obtain

$$\sum_{p \leq x} \sum_{k=1}^{h-1} k \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) = \log \log x + C_3 + O_h \left(\frac{1}{\log x} \right).$$

Combining the above with (3.4) and (3.5) completes the first part of the proof.

Next, we prove the second-moment estimate for $\Omega(n)$ over h -free numbers. Notice that

$$\sum_{n \in \mathcal{S}_h(x)} \Omega^2(n) = \sum_{n \in \mathcal{S}_h(x)} \left(\sum_{\substack{p \\ p^k \parallel n}} k \right)^2 = \sum_{n \in \mathcal{S}_h(x)} \sum_{\substack{p \\ p^k \parallel n}} k^2 + \sum_{n \in \mathcal{S}_h(x)} \sum_{\substack{p, q \\ p^k \parallel n, q^l \parallel n, p \neq q}} (kl), \quad (3.7)$$

where p and q above denote primes. Let $r_p = \min \left\{ h-1, \left\lfloor \frac{\log x}{\log p} \right\rfloor \right\}$. To estimate the first sum on the right-hand side above, we rewrite the sum as

$$\sum_{n \in \mathcal{S}_h(x)} \sum_{\substack{p \\ p^k \parallel n}} k^2 = \sum_{p \leq x} \sum_{k=1}^{r_p} \sum_{\substack{n \in \mathcal{S}_h(x) \\ p^k \parallel n}} k^2 = \sum_{p \leq x} \sum_{k=1}^{r_p} k^2 \sum_{\substack{m \in \mathcal{S}_h(x/p^k) \\ (m, p)=1}} 1.$$

Again, similar to (3.4), using Lemma 2.2.6 for a single prime p and Lemma 2.2.2, we obtain

$$\sum_{p \leq x} \sum_{k=1}^{r_p} k^2 \sum_{\substack{m \in S_h(x/p^k) \\ (m,p)=1}} 1 = \sum_{p \leq x} \sum_{k=1}^{r_p} k^2 \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \frac{x}{\zeta(h)} + O_h \left(\frac{x}{\log x} \right),$$

and, similarly to (3.5), we obtain

$$\sum_{p \leq x} \sum_{k=1}^{r_p} k^2 \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) = \sum_{p \leq x} \sum_{k=1}^{h-1} k^2 \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) + O_h \left(\frac{1}{\log x} \right).$$

Note that

$$\begin{aligned} & \sum_{k=1}^{h-1} k^2 \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \\ &= \frac{1}{p} + \frac{p^h(3p-1) + (p-1)^2 - p(h^2p^2 + (-2h^2 + 2h + 1)p + (h-1)^2)}{p(p-1)^2(p^h - 1)}. \end{aligned}$$

Thus, using (2.11) and Lemma 2.2.1 with $\tau = 2$, we obtain

$$\sum_{k=1}^{h-1} k^2 \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) = \log \log x + C'_3 + O_h \left(\frac{1}{\log x} \right),$$

where

$$C'_3 = B_1 + \sum_p \frac{p^h(3p-1) + (p-1)^2 - p(h^2p^2 + (-2h^2 + 2h + 1)p + (h-1)^2)}{p(p-1)^2(p^h - 1)}. \quad (3.8)$$

Combining the last five results, we obtain

$$\sum_{n \in S_h(x)} \sum_{\substack{p \\ p^k || n}} k^2 = \frac{1}{\zeta(h)} x \log \log x + \frac{C'_3}{\zeta(h)} x + O_h \left(\frac{x}{\log x} \right). \quad (3.9)$$

Next, for the second sum in (3.7), we rewrite the sum and use Lemma 2.2.6 for two

primes p and q to obtain

$$\begin{aligned}
& \sum_{n \in \mathcal{S}_h(x)} \sum_{\substack{p, q \\ p^k || n, q^l || n, p \neq q}} (kl) \\
&= \sum_{\substack{p, q \\ p \neq q, pq \leq x}} \sum_{k=1}^{r_p} \sum_{l=1}^{r_q} (kl) \sum_{\substack{n \leq x/(p^k q^l) \\ n \in \mathcal{S}_h \\ (p, n) = (q, n) = 1}} 1 \\
&= \sum_{\substack{p, q \\ p \neq q, pq \leq x}} \sum_{k=1}^{r_p} \sum_{l=1}^{r_q} \left(\left(\frac{k(p^h - p^{h-1})}{p^k(p^h - 1)} \right) \left(\frac{l(q^h - q^{h-1})}{q^l(q^h - 1)} \right) \frac{x}{\zeta(h)} + O_h \left(\frac{x^{1/h}}{p^{k/h} q^{l/h}} \right) \right). \quad (3.10)
\end{aligned}$$

The error term above is estimated by (2.15) as

$$\sum_{\substack{p, q \\ p \neq q, pq \leq x}} \sum_{k=1}^{r_p} \sum_{l=1}^{r_q} \frac{x^{1/h}}{p^{k/h} q^{l/h}} \ll_h \frac{x \log \log x}{\log x}. \quad (3.11)$$

Thus, it remains to estimate the main term, in particular, the term

$$\sum_{\substack{p, q \\ p \neq q, pq \leq x}} \sum_{k=1}^{r_p} \sum_{l=1}^{r_q} \left(\frac{k(p^h - p^{h-1})}{p^k(p^h - 1)} \right) \left(\frac{l(q^h - q^{h-1})}{q^l(q^h - 1)} \right).$$

Consider again the set

$$R := \left\{ p \leq x : \left\lfloor \frac{\log x}{\log p} \right\rfloor < h - 1 \right\}.$$

Using the definition of r_p and r_q , we can rewrite

$$\begin{aligned}
& \sum_{\substack{p, q \\ p \neq q, pq \leq x}} \sum_{k=1}^{r_p} \sum_{l=1}^{r_q} \left(\frac{k(p^h - p^{h-1})}{p^k(p^h - 1)} \right) \left(\frac{l(q^h - q^{h-1})}{q^l(q^h - 1)} \right) \\
&= \sum_{\substack{p, q \\ p \neq q, pq \leq x}} \sum_{k=1}^{h-1} \sum_{l=1}^{h-1} \left(\frac{k(p^h - p^{h-1})}{p^k(p^h - 1)} \right) \left(\frac{l(q^h - q^{h-1})}{q^l(q^h - 1)} \right) - I_1 - I_2 + I_3, \quad (3.12)
\end{aligned}$$

where

$$I_1 = \sum_{\substack{p, q \\ p \neq q, pq \leq x \\ p \in R}} \sum_{k=\left\lfloor \frac{\log x}{\log p} \right\rfloor + 1}^{h-1} \sum_{l=1}^{h-1} \left(\frac{k(p^h - p^{h-1})}{p^k(p^h - 1)} \right) \left(\frac{l(q^h - q^{h-1})}{q^l(q^h - 1)} \right),$$

$$I_2 = \sum_{\substack{p,q \\ p \neq q, pq \leq x \\ q \in R}} \sum_{k=1}^{h-1} \sum_{l=\lfloor \frac{\log x}{\log q} \rfloor + 1}^{h-1} \left(\frac{k(p^h - p^{h-1})}{p^k(p^h - 1)} \right) \left(\frac{l(q^h - q^{h-1})}{q^l(q^h - 1)} \right),$$

$$I_3 = \sum_{\substack{p,q \\ p \neq q, pq \leq x \\ p,q \in R}} \sum_{k=\lfloor \frac{\log x}{\log p} \rfloor + 1}^{h-1} \sum_{l=\lfloor \frac{\log x}{\log q} \rfloor + 1}^{h-1} \left(\frac{k(p^h - p^{h-1})}{p^k(p^h - 1)} \right) \left(\frac{l(q^h - q^{h-1})}{q^l(q^h - 1)} \right).$$

The sum I_1 , I_2 , and I_3 contribute to the error term. In fact, using $\lfloor x \rfloor \geq x - 1$, (2.8), and Lemma 2.2.3, we estimate

$$I_1 \ll_h \sum_{\substack{p,q \\ p \neq q, pq \leq x \\ p \in R}} \frac{1}{p^{\frac{\log x}{\log p}}} \frac{1}{q} \ll_h \sum_{q \leq x/2} \frac{1}{q} \left(\frac{1}{x} \sum_{p \leq x/q} 1 \right) \ll_h \sum_{q \leq x/2} \frac{1}{q^2 \log(x/q)} \ll_h \frac{\log \log x}{\log x}.$$

Similarly,

$$I_2 \ll_h \frac{\log \log x}{\log x}.$$

Finally, for I_3 , using $\lfloor x \rfloor \geq x - 1$, (2.8), and Lemma 2.2.3 again, we have

$$I_3 \ll_h \sum_{\substack{p,q \\ p \neq q, pq \leq x \\ p,q \in R}} \frac{1}{p^{\frac{\log x}{\log p}}} \frac{1}{q^{\frac{\log x}{\log q}}} \ll_h \frac{1}{x} \sum_{q \leq x/2} \left(\frac{1}{x} \sum_{p \leq x/q} 1 \right) \ll_h \frac{1}{x} \sum_{q \leq x/2} \frac{1}{q \log(x/q)} \ll_h \frac{\log \log x}{x \log x}.$$

We next estimate the main term in (3.12). First, using

$$\frac{p^h - p^{h-1}}{p^h - 1} \sum_{k=1}^{h-1} \frac{k}{p^k} = \frac{p^h - hp + h - 1}{(p - 1)(p^h - 1)},$$

and a similar result for q , we rewrite the sum as

$$\begin{aligned} & \sum_{\substack{p,q \\ p \neq q, pq \leq x}} \sum_{k=1}^{h-1} \sum_{l=1}^{h-1} \left(\frac{k(p^h - p^{h-1})}{p^k(p^h - 1)} \right) \left(\frac{l(q^h - q^{h-1})}{q^l(q^h - 1)} \right) \\ &= \sum_{\substack{p,q \\ p \neq q, pq \leq x}} \left(\frac{p^h - hp + h - 1}{(p - 1)(p^h - 1)} \right) \left(\frac{q^h - hq + h - 1}{(q - 1)(q^h - 1)} \right) \\ &= \sum_{\substack{p,q \\ pq \leq x}} \left(\frac{p^h - hp + h - 1}{(p - 1)(p^h - 1)} \right) \left(\frac{q^h - hq + h - 1}{(q - 1)(q^h - 1)} \right) - \sum_{\substack{p \\ p \leq x^{1/2}}} \left(\frac{p^h - hp + h - 1}{(p - 1)(p^h - 1)} \right)^2. \end{aligned} \quad (3.13)$$

The second sum above is estimated using Lemma 2.2.1 as

$$\begin{aligned} \sum_{\substack{p \\ p \leq x^{1/2}}} \left(\frac{p^h - hp + h - 1}{(p-1)(p^h-1)} \right)^2 &= \sum_p \left(\frac{p^h - hp + h - 1}{(p-1)(p^h-1)} \right)^2 + \sum_{\substack{p \\ p > x^{1/2}}} \left(\frac{p^h - hp + h - 1}{(p-1)(p^h-1)} \right)^2 \\ &= \sum_p \left(\frac{p^h - hp + h - 1}{(p-1)(p^h-1)} \right)^2 + O\left(\frac{1}{x^{1/2} \log x} \right). \end{aligned} \quad (3.14)$$

For the first sum on the right-hand side in (3.13), using

$$\frac{p^h - hp + h - 1}{(p-1)(p^h-1)} = \frac{1}{p} + \frac{p^h - hp^2 + hp - 1}{p(p-1)(p^h-1)},$$

and the symmetry in p and q , we have

$$\begin{aligned} &\sum_{\substack{p,q \\ pq \leq x}} \left(\frac{p^h - hp + h - 1}{(p-1)(p^h-1)} \right) \left(\frac{q^h - hq + h - 1}{(q-1)(q^h-1)} \right) \\ &= \sum_{\substack{p,q \\ pq \leq x}} \frac{1}{pq} + 2 \sum_{\substack{p,q \\ pq \leq x}} \frac{1}{p} \frac{q^h - hq^2 + hq - 1}{q(q-1)(q^h-1)} + \sum_{\substack{p,q \\ pq \leq x}} \left(\frac{p^h - hp^2 + hp - 1}{p(p-1)(p^h-1)} \right) \left(\frac{q^h - hq^2 + hq - 1}{q(q-1)(q^h-1)} \right). \end{aligned} \quad (3.15)$$

We estimate the sums on the right-hand side above separately. For the first sum, we use Lemma 2.2.4. For the third sum, we use Lemma 2.2.1 to obtain

$$\begin{aligned} &\sum_{\substack{p,q \\ pq \leq x}} \left(\frac{p^h - hp^2 + hp - 1}{p(p-1)(p^h-1)} \right) \left(\frac{q^h - hq^2 + hq - 1}{q(q-1)(q^h-1)} \right) \\ &= \sum_{\substack{p \\ p \leq x/2}} \left(\frac{p^h - hp^2 + hp - 1}{p(p-1)(p^h-1)} \right) \sum_{\substack{q \\ q \leq x/p}} \left(\frac{q^h - hq^2 + hq - 1}{q(q-1)(q^h-1)} \right) \\ &= \sum_{\substack{p \\ p \leq x/2}} \left(\frac{p^h - hp^2 + hp - 1}{p(p-1)(p^h-1)} \right) \left(\sum_p \left(\frac{p^h - hp^2 + hp - 1}{p(p-1)(p^h-1)} \right) + O\left(\frac{1}{(x/p) \log(x/p)} \right) \right) \\ &= \left(\sum_p \frac{p^h - hp^2 + hp - 1}{p(p-1)(p^h-1)} \right)^2 + O\left(\sum_{p \leq x/2} \frac{1}{p^2} \right) + O\left(\frac{1}{x} \sum_{p \leq x/2} \frac{1}{p \log(x/p)} \right). \end{aligned}$$

Using Lemma 2.2.1 with $\tau = 2$ again and Lemma 2.2.3 to the above error terms respectively, we obtain

$$\begin{aligned} & \sum_{\substack{p,q \\ pq \leq x}} \left(\frac{p^h - hp^2 + hp - 1}{p(p-1)(p^h-1)} \right) \left(\frac{q^h - hq^2 + hq - 1}{q(q-1)(q^h-1)} \right) \\ &= \left(\sum_p \frac{p^h - hp^2 + hp - 1}{p(p-1)(p^h-1)} \right)^2 + O\left(\frac{\log \log x}{x \log x} \right). \end{aligned} \quad (3.16)$$

Now, for the second sum in (3.15), we use Lemma 2.2.1 to obtain

$$\begin{aligned} & \sum_{\substack{p,q \\ pq \leq x}} \frac{1}{p} \left(\frac{q^h - hq^2 + hq - 1}{q(q-1)(q^h-1)} \right) \\ &= \sum_{\substack{p \\ p \leq x/2}} \frac{1}{p} \sum_{\substack{q \\ q \leq x/p}} \left(\frac{q^h - hq^2 + hq - 1}{q(q-1)(q^h-1)} \right) \\ &= \sum_{\substack{p \\ p \leq x/2}} \frac{1}{p} \left(\sum_p \left(\frac{p^h - hp^2 + hp - 1}{p(p-1)(p^h-1)} \right) + O\left(\frac{1}{(x/p) \log(x/p)} \right) \right). \end{aligned}$$

Using (2.11) to the above error term, we obtain

$$\begin{aligned} & \sum_{\substack{p,q \\ pq \leq x}} \frac{1}{p} \left(\frac{q^h - hq^2 + hq - 1}{q(q-1)(q^h-1)} \right) \\ &= \left(\sum_p \frac{p^h - hp^2 + hp - 1}{p(p-1)(p^h-1)} \right) \left(\log \log(x/2) + B_1 + O\left(\frac{1}{\log x} \right) \right) + O\left(\frac{1}{x} \sum_{\substack{p \\ p \leq x/2}} \frac{1}{\log(x/p)} \right). \end{aligned}$$

Using $\log \log(x/2) = \log \log x + O(1/\log x)$ and (2.8) above, we obtain

$$\sum_{\substack{p,q \\ pq \leq x}} \frac{1}{p} \left(\frac{q^h - hq^2 + hq - 1}{q(q-1)(q^h-1)} \right) = \left(\sum_p \frac{p^h - hp^2 + hp - 1}{p(p-1)(p^h-1)} \right) (\log \log x + B_1) + O\left(\frac{1}{\log x} \right). \quad (3.17)$$

Combining (3.15), (3.16), and (3.17) with Lemma 2.2.4 and (3.6), we obtain

$$\begin{aligned}
& \sum_{\substack{p,q \\ pq \leq x}} \left(\frac{p^h - hp + h - 1}{(p-1)(p^h-1)} \right) \left(\frac{q^h - hq + h - 1}{(q-1)(q^h-1)} \right) \\
&= (\log \log x)^2 + 2B_1 \log \log x + B_1^2 - \zeta(2) + 2 \left(\sum_p \frac{p^h - hp^2 + hp - 1}{p(p-1)(p^h-1)} \right) (\log \log x + B_1) \\
&\quad + \left(\sum_p \frac{p^h - hp^2 + hp - 1}{p(p-1)(p^h-1)} \right)^2 + O\left(\frac{\log \log x}{\log x} \right) \\
&= (\log \log x)^2 + 2C_3 \log \log x + C_3^2 - \zeta(2) + O\left(\frac{\log \log x}{\log x} \right),
\end{aligned}$$

where

$$C_3 = B_1 + \sum_p \frac{p^h - hp^2 + hp - 1}{p(p-1)(p^h-1)} = B_2 - \sum_p \frac{h}{p^h - 1}.$$

Combining (3.7), (3.9), (3.10), (3.11), (3.12), (3.13), and (3.14) with the above equation, we obtain the required second moment for $\Omega(n)$ over h -free numbers as

$$\sum_{n \in \mathcal{S}_h(x)} \Omega^2(n) = \frac{1}{\zeta(h)} x (\log \log x)^2 + \frac{2C_3 + 1}{\zeta(h)} x \log \log x + \frac{C_4}{\zeta(h)} x + O_h \left(\frac{x \log \log x}{\log x} \right),$$

where

$$C_4 = C_3^2 + C_3' - \zeta(2) - \sum_p \left(\frac{p^h - hp + h - 1}{(p-1)(p^h-1)} \right)^2,$$

with C_3' defined in (3.8). This completes the proof of Theorem 3.1.1. $\square$

3.3 The moments of $\Omega(n)$ over h -full numbers

To prove the moments of $\Omega(n)$ over h -full numbers, we recall the constants

$$\begin{aligned}
D_3 &= h(B_2 - \log h) + \sum_p \frac{(h+1)p^{1+1/h} - hp - 2hp^{2/h} + (2h-1)p^{1/h}}{(p-1)(p^{1/h}-1)(p^{1+1/h} + p^{1/h} - p)}, \\
D_3' &= h^2(B_1 - \log h) + \sum_p \left(\frac{(2h^2 + 2h - 1)p^{\frac{1+h}{h}} - (1+h)^2 p^{\frac{2+h}{h}} - h^2(p - p^{\frac{1}{h}} + 2p^{\frac{2}{h}} - p^{\frac{3}{h}})}{p(-p^{\frac{1+h}{h}} - p^{\frac{1}{h}} + p)(-1 + p^{\frac{1}{h}})^2} \right),
\end{aligned}$$

and

$$D_4 = D_3^2 + D_3' - h^2 \zeta(2) - \sum_p \left(\frac{h(p^{1/h} - 1) + 1}{(p^{1/h} - 1)(p - p^{1-1/h} + 1)} \right)^2.$$

Moreover, we need the following additional lemma:

Lemma 3.3.1. *For $x > 2$, we have*

$$\sum_{p \leq x/2} \frac{1}{p \log^2(x/p)} = O\left(\frac{1}{\log x}\right).$$

Proof. We partition the interval $[1, x/2]$ into subintervals of the form $I_j = [e^{j-1}, e^j]$. Over this interval, we have

$$\sum_{p \in I_j} \frac{1}{p \log^2(x/p)} \ll \frac{1}{\log^2(x/e^j)} \sum_{p \in I_j} \frac{1}{p}.$$

Moreover, by (2.8), we have

$$\sum_{p \in I_j} \frac{1}{p} \ll \frac{1}{e^j} \frac{e^j}{j} \ll \frac{1}{j}.$$

Putting the above together with the fact that

$$\sum_{\substack{p \\ e^{\lfloor \log(x/2) \rfloor} < p \leq x/2}} \frac{1}{p \log^2(x/p)} \ll \frac{1}{\log^2 2} \sum_{\substack{p \\ e^{\lfloor \log(x/2) \rfloor} < p \leq x/2}} \frac{1}{p} \ll \frac{1}{x} \sum_{\substack{p \\ p \leq x/2}} 1 \ll \frac{1}{\log x},$$

where in the second last inequality we use (2.8), we obtain

$$\begin{aligned} \sum_{p \leq x/2} \frac{1}{p \log^2(x/p)} &= \sum_{j=1}^{\lfloor \log(x/2) \rfloor} \sum_{p \in I_j} \frac{1}{p \log^2(x/p)} + O\left(\frac{1}{\log x}\right) \\ &= \sum_{j=1}^{\lfloor \log(x/2) \rfloor} \frac{1}{j \log^2(x/e^j)} + O\left(\frac{1}{\log x}\right) \\ &= \sum_{j=1}^{\lfloor \log(x/2) \rfloor} \frac{1}{j(\log x - j)^2} + O\left(\frac{1}{\log x}\right). \end{aligned} \tag{3.18}$$

Note that

$$\frac{1}{j(\log x - j)^2} = \frac{1}{(\log x)^2} \left(\frac{1}{j} + \frac{1}{\log x - j} \right) + \frac{1}{(\log x)(\log x - j)^2}.$$

Moreover,

$$\begin{aligned} \sum_{j=1}^{\lfloor \log(x/2) \rfloor} \frac{1}{j} &\ll \int_{t=1}^{\log(x/2)} \frac{dt}{t} \ll \log \log x, \\ \sum_{j=1}^{\lfloor \log(x/2) \rfloor} \frac{1}{\log x - j} &\ll \int_{t=1}^{\log(x/2)} \frac{dt}{\log x - t} \ll \log \log x, \end{aligned}$$

and

$$\sum_{j=1}^{\lfloor \log(x/2) \rfloor} \frac{1}{(\log x - j)^2} \ll \int_{t=1}^{\log(x/2)} \frac{dt}{(\log x - t)^2} \ll 1.$$

As a result,

$$\sum_{j=1}^{\lfloor \log(x/2) \rfloor} \frac{1}{j(\log x - j)^2} \ll \frac{\log \log x}{(\log x)^2} + \frac{1}{\log x} \ll \frac{1}{\log x}.$$

Putting the above back into (3.18) completes the proof. $\square$

Using this result, we prove:

Proof of Theorem 3.1.2. Using $n = p^k m$ with $(m, p) = 1$ when $p^k \parallel n$, we can write

$$\sum_{n \in \mathcal{N}_h(x)} \Omega(n) = \sum_{n \in \mathcal{N}_h(x)} \sum_{\substack{p \\ p^k \parallel n}} k = \sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \sum_{\substack{n \in \mathcal{N}_h(x) \\ p^k \parallel n}} k = \sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} k \sum_{\substack{m \in \mathcal{N}_h(x/p^k) \\ (m, p)=1}} 1. \quad (3.19)$$

Note that the last sum above is the same as $A_{p,h}(x/p^k)$ as defined in (2.32). Inserting the formula for $A_{p,h}(x/p^k)$ derived in Lemma 2.4.1, we obtain

$$\begin{aligned} &\sum_{n \in \mathcal{N}_h(x)} \Omega(n) \\ &= \sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} k \left(\gamma_{p,0,h}(x/p^k)^{1/h} + \cdots + \gamma_{p,h-1,h}(x/p^k)^{1/(2h-1)} + O_h((x/p^k)^{\eta_h}) \right), \quad (3.20) \end{aligned}$$

where $\frac{1}{2h+2} < \eta_h < \frac{1}{2h-1}$. First, we study the main term above given by

$$\sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} k \gamma_{p,0,h}(x/p^k)^{1/h} = \gamma_{0,h} x^{1/h} \sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \left(\frac{k}{p^{k/h} \left(1 + \frac{p^{-1}}{1-p^{-1/h}}\right)} \right), \quad (3.21)$$

where we used (2.35) to rewrite $\gamma_{p,0,h}$. Note that

$$\sum_{k=h}^r k a^k = \frac{a^{r+1}(ar - r - 1) - a^h((h-1)a - h)}{(1-a)^2} = \frac{ha^h}{(1-a)} + \frac{a^{h+1}}{(1-a)^2} + \frac{a^{r+1}(ar - r - 1)}{(1-a)^2}. \quad (3.22)$$

Using the above formula with $r = \lfloor \frac{\log x}{\log p} \rfloor$ and $a = p^{-1/h}$ in (3.21), we obtain

$$\begin{aligned} & \sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} k \gamma_{p,0,h}(x/p^k)^{1/h} \\ &= \gamma_{0,h} x^{1/h} \sum_{p \leq x^{1/h}} \left(\frac{h}{p(1 - p^{-1/h} + p^{-1})} + \frac{p^{-1/h}}{p(1 - p^{-1/h})(1 - p^{-1/h} + p^{-1})} \right) \\ & \quad + O_h \left(x^{1/h} \sum_{p \leq x^{1/h}} \frac{\left(\lfloor \frac{\log x}{\log p} \rfloor \right) p^{-(\lfloor \frac{\log x}{\log p} \rfloor + 1)/h}}{(1 - p^{-1/h})(1 - p^{-1/h} + p^{-1})} \right). \end{aligned} \quad (3.23)$$

By partial summation, it follows that for any natural number k ,

$$\sum_{p \leq y} \frac{1}{(\log p)^k} \ll \frac{y}{(\log y)^{k+1}}. \quad (3.24)$$

Using (3.24) with $k = 1$ and $y = x^{1/h}$, and $x - 1 < \lfloor x \rfloor \leq x$, we can estimate the error term in (3.23) as

$$x^{1/h} \sum_{p \leq x^{1/h}} \frac{\left(\lfloor \frac{\log x}{\log p} \rfloor \right) p^{-(\lfloor \frac{\log x}{\log p} \rfloor + 1)/h}}{(1 - p^{-1/h})(1 - p^{-1/h} + p^{-1})} \ll_h (\log x) \sum_{p \leq x^{1/h}} \frac{1}{\log p} \ll_h \frac{x^{1/h}}{\log x}.$$

Thus, we obtain

$$\begin{aligned}
& \sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} k \gamma_{p,0,h}(x/p^k)^{1/h} \\
&= \gamma_{0,h} x^{1/h} \sum_{p \leq x^{1/h}} \left(\frac{h}{p(1 - p^{-1/h} + p^{-1})} + \frac{p^{-1/h}}{p(1 - p^{-1/h})(1 - p^{-1/h} + p^{-1})} \right) + O_h \left(\frac{x^{1/h}}{\log x} \right) \\
&= h \gamma_{0,h} x^{1/h} \sum_{p \leq x^{1/h}} \frac{1}{p(1 - p^{-1/h} + p^{-1})} + \gamma_{0,h} x^{1/h} \sum_{p \leq x^{1/h}} \frac{1}{(p^{1/h} - 1)(p - p^{1-1/h} + 1)} \\
&\quad + O_h \left(\frac{x^{1/h}}{\log x} \right). \tag{3.25}
\end{aligned}$$

Using Lemma 2.2.1 for $\tau = 1 + (1/h)$, we can estimate the second sum above as

$$\sum_{p \leq x^{1/h}} \frac{1}{(p^{1/h} - 1)(p - p^{1-1/h} + 1)} = \sum_p \frac{1}{(p^{1/h} - 1)(p - p^{1-1/h} + 1)} + O_h \left(\frac{1}{x^{1/h^2}(\log x)} \right).$$

Moreover, for the first sum in (3.25), using (1.8), (2.5), and (2.42), we have

$$\begin{aligned}
& \sum_{p \leq x^{1/h}} \frac{1}{p(1 - p^{-1/h} + p^{-1})} \\
&= \log \log x + B_1 - \log h + \sum_p \frac{p^{-1/h} - p^{-1}}{p(1 - p^{-1/h} + p^{-1})} + O_h \left(\frac{1}{\log x} \right) \\
&= \log \log x + B_2 - \log h - \sum_p \frac{1}{p(p-1)} + \sum_p \frac{p^{-1/h} - p^{-1}}{p(1 - p^{-1/h} + p^{-1})} + O_h \left(\frac{1}{\log x} \right).
\end{aligned}$$

Also, a slight simplification yields

$$\begin{aligned}
& -h \sum_p \frac{1}{p(p-1)} + h \sum_p \frac{p^{-1/h} - p^{-1}}{p(1 - p^{-1/h} + p^{-1})} + \sum_p \frac{1}{(p^{1/h} - 1)(p - p^{1-1/h} + 1)} \\
&= \sum_p \frac{(h+1)p^{1+1/h} - hp - 2hp^{2/h} + (2h-1)p^{1/h}}{(p-1)(p^{1/h} - 1)(p^{1+1/h} + p^{1/h} - p)}. \tag{3.26}
\end{aligned}$$

Combining the last four results with (3.20), we obtain

$$\begin{aligned} \sum_{n \in \mathcal{N}_h(x)} \Omega(n) &= h\gamma_{0,h}x^{1/h} \log \log x + D_3\gamma_{0,h}x^{1/h} + O_h\left(\frac{x^{1/h}}{\log x}\right) \\ &\quad + O_h\left(\sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} k(x/p^k)^{1/(h+1)}\right), \end{aligned} \quad (3.27)$$

where

$$D_3 = h(B_2 - \log h) + \sum_p \frac{(h+1)p^{1+1/h} - hp - 2hp^{2/h} + (2h-1)p^{1/h}}{(p-1)(p^{1/h}-1)(p^{1+1/h} + p^{1/h} - p)}.$$

Thus, to complete the proof for the first moment, it only remains to estimate

$$x^{1/(h+1)} \sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} k(p^{-1/(h+1)})^k.$$

Using (3.22) with $r = \lfloor \frac{\log x}{\log p} \rfloor$ and $a = p^{-1/(h+1)}$, using Lemma 2.2.2 with $\alpha = h/(h+1)$, and using (3.24), we obtain

$$\begin{aligned} &x^{1/(h+1)} \sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} k(p^{-1/(h+1)})^k \\ &\ll_h x^{1/(h+1)} \left(\sum_{p \leq x^{1/h}} \frac{1}{p^{h/(h+1)}} + \sum_{p \leq x^{1/h}} \frac{\left(\lfloor \frac{\log x}{\log p} \rfloor\right) p^{-\left(\lfloor \frac{\log x}{\log p} \rfloor + 1\right)/(h+1)}}{(1 - p^{-1/(h+1)})^2} \right) \\ &\ll_h \frac{x^{1/h}}{\log x} + (\log x) \sum_{p \leq x^{1/h}} \frac{1}{\log p} \\ &\ll_h \frac{x^{1/h}}{\log x}. \end{aligned} \quad (3.28)$$

Finally, inserting the above back into (3.27), we obtain

$$\sum_{n \in \mathcal{N}_h(x)} \Omega(n) = h\gamma_{0,h}x^{1/h} \log \log x + D_3\gamma_{0,h}x^{1/h} + O_h\left(\frac{x^{1/h}}{\log x}\right).$$

This completes the first part of the proof.

For the second part, note that

$$\sum_{n \in \mathcal{N}_h(x)} \Omega^2(n) = \sum_{n \in \mathcal{N}_h(x)} \left(\sum_{\substack{p \\ p^k || n}} k \right)^2 = \sum_{n \in \mathcal{N}_h(x)} \sum_{\substack{p \\ p^k || n}} k^2 + \sum_{n \in \mathcal{N}_h(x)} \sum_{\substack{p, q \\ p^k || n, q^l || n, p \neq q}} (kl), \quad (3.29)$$

For the first sum on the right side above, we rewrite the sum as

$$\sum_{n \in \mathcal{N}_h(x)} \sum_{\substack{p \\ p^k || n}} k^2 = \sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \sum_{\substack{n \in \mathcal{N}_h(x) \\ p^k || n}} k^2 = \sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} k^2 \sum_{\substack{m \in \mathcal{N}_h(x/p^k) \\ (m, p)=1}} 1.$$

Working similarly to (3.20) and (3.21), we obtain

$$\begin{aligned} \sum_{n \in \mathcal{N}_h(x)} \sum_{\substack{p \\ p^k || n}} k^2 &= \gamma_{0,h} x^{1/h} \sum_{p \leq x^{1/h}} \frac{1 - p^{-1/h}}{(1 - p^{-1/h} + p^{-1})} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \left(\frac{k^2}{p^{k/h}} \right) \\ &\quad + O_h \left(x^{1/(h+1)} \sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \frac{k^2}{p^{k/(h+1)}} \right). \end{aligned} \quad (3.30)$$

Note that

$$\begin{aligned} &\sum_{k=h}^r k^2 a^k \\ &= \frac{(a^2 r^2 + (-2r^2 - 2r + 1)a + (r+1)^2) a^{r+1} - a^h ((h-1)^2 a^2 + (-2h^2 + 2h + 1)a + h^2)}{(a-1)^3} \\ &= \frac{h^2 a^h}{(1-a)^3} + \frac{a^{h+1}(-2h^2 + 2h + 1)}{(1-a)^3} + \frac{a^{h+2}(h-1)^2}{(1-a)^3} \\ &\quad + \frac{a^{r+1}(a^2 r^2 + (-2r^2 - 2r + 1)a + (r+1)^2)}{(a-1)^3}. \end{aligned} \quad (3.31)$$

Using the above relation with $r = \lfloor \frac{\log x}{\log p} \rfloor$ and $a = p^{-1/h}$, using $x-1 < [x] \leq x$, and using

(3.24) with $k = 2$ and $y = x^{1/h}$, we obtain

$$\begin{aligned}
& \sum_{p \leq x^{1/h}} \frac{1 - p^{-1/h}}{(1 - p^{-1/h} + p^{-1})} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \left(\frac{k^2}{p^{k/h}} \right) \\
&= \sum_{p \leq x^{1/h}} \frac{h^2 p^{-1} + p^{-1-\frac{1}{h}}(-2h^2 + 2h + 1) + p^{-1-\frac{2}{h}}(h-1)^2}{(1 - p^{-\frac{1}{h}})^2(1 - p^{-\frac{1}{h}} + p^{-1})} + O \left(\sum_{p \leq x^{1/h}} \frac{\left(\frac{\log x}{\log p} \right)^2}{p^{\frac{\log x}{h}}} \right) \\
&= \sum_{p \leq x^{1/h}} \frac{h^2 + p^{-\frac{1}{h}}(-2h^2 + 2h + 1) + p^{-\frac{2}{h}}(h-1)^2}{p(1 - p^{-\frac{1}{h}})^2(1 - p^{-\frac{1}{h}} + p^{-1})} + O \left(\frac{(\log x)^2}{x^{1/h}} \sum_{p \leq x^{1/h}} \frac{1}{(\log p)^2} \right) \\
&= \sum_{p \leq x^{1/h}} \frac{h^2}{p} + \sum_{p \leq x^{1/h}} \left(\frac{(2h^2 + 2h - 1)p^{\frac{1+h}{h}} - (1+h)^2 p^{\frac{2+h}{h}} - h^2(p - p^{\frac{1}{h}} + 2p^{\frac{2}{h}} - p^{\frac{3}{h}})}{p(-p^{\frac{1+h}{h}} - p^{\frac{1}{h}} + p)(-1 + p^{\frac{1}{h}})^2} \right) \\
&\quad + O_h \left(\frac{1}{\log x} \right).
\end{aligned}$$

Notice that the second sum on the right side above is $O(1)$. Thus, using (2.41) and Lemma 2.2.1 with $\tau = 1 + (1/h)$, we obtain

$$\begin{aligned}
& \sum_{p \leq x^{1/h}} \frac{1 - p^{-1/h}}{(1 - p^{-1/h} + p^{-1})} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \left(\frac{k^2}{p^{k/h}} \right) \\
&= h^2 \left(\log \log x - \log h + B_1 + O_h \left(\frac{1}{\log x} \right) \right) \\
&\quad + \sum_p \left(\frac{(2h^2 + 2h - 1)p^{\frac{1+h}{h}} - (1+h)^2 p^{\frac{2+h}{h}} - h^2(p - p^{\frac{1}{h}} + 2p^{\frac{2}{h}} - p^{\frac{3}{h}})}{p(-p^{\frac{1+h}{h}} - p^{\frac{1}{h}} + p)(-1 + p^{\frac{1}{h}})^2} \right) \\
&\quad + O_h \left(\sum_{p > x^{1/h}} \frac{1}{p^{1+\frac{1}{h}}} \right) + O_h \left(\frac{1}{\log x} \right) \\
&= h^2 \log \log x + h^2(B_1 - \log h) \\
&\quad + \sum_p \left(\frac{(2h^2 + 2h - 1)p^{\frac{1+h}{h}} - (1+h)^2 p^{\frac{2+h}{h}} - h^2(p - p^{\frac{1}{h}} + 2p^{\frac{2}{h}} - p^{\frac{3}{h}})}{p(-p^{\frac{1+h}{h}} - p^{\frac{1}{h}} + p)(-1 + p^{\frac{1}{h}})^2} \right) + O_h \left(\frac{1}{\log x} \right).
\end{aligned}$$

Similarly, using (3.31) with $r = \lfloor \frac{\log x}{\log p} \rfloor$ and $a = p^{-1/(h+1)}$, we obtain

$$\begin{aligned}
& x^{1/(h+1)} \sum_{p \leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \frac{k^2}{p^{k/(h+1)}} \\
& \ll_h x^{1/(h+1)} \left(\sum_{p \leq x^{1/h}} \frac{1}{p^{h/(h+1)}} + \sum_{p \leq x^{1/h}} \left(\frac{\log x}{\log p} \right)^2 p^{-\frac{\lfloor \frac{\log x}{\log p} \rfloor + 1}{h+1}} \right) \\
& \ll_h \frac{x^{1/h}}{\log x} + (\log x)^2 \sum_{p \leq x^{1/h}} \frac{1}{(\log p)^2} \\
& \ll_h \frac{x^{1/h}}{\log x}.
\end{aligned}$$

Inserting the last two results into (3.30), we obtain

$$\begin{aligned}
& \sum_{n \in \mathcal{N}_h(x)} \sum_{\substack{p \\ p^k \parallel n}} k^2 \\
& = \gamma_{0,h} x^{1/h} \left(h^2 \log \log x + h^2 (B_1 - \log h) \right. \\
& \quad \left. + \sum_p \left(\frac{(2h^2 + 2h - 1)p^{\frac{1+h}{h}} - (1+h)^2 p^{\frac{2+h}{h}} - h^2(p - p^{\frac{1}{h}} + 2p^{\frac{2}{h}} - p^{\frac{3}{h}})}{p(-p^{\frac{1+h}{h}} - p^{\frac{1}{h}} + p)(-1 + p^{\frac{1}{h}})^2} \right) \right) + O_h \left(\frac{x^{1/h}}{\log x} \right) \\
& = h^2 \gamma_{0,h} x^{1/h} \log \log x + D'_3 \gamma_{0,h} x^{1/h} + O_h \left(\frac{x^{1/h}}{\log x} \right), \tag{3.32}
\end{aligned}$$

where D'_3 is defined in (3.2).

For the second sum in (3.29), we rewrite the sum, use Lemma 2.4.1 with two distinct

primes p and q , and use (2.35) to obtain

$$\begin{aligned}
& \sum_{n \in \mathcal{N}_h(x)} \sum_{\substack{p, q \\ p^k || n, q^l || n, p \neq q}} (kl) \\
&= \sum_{\substack{p, q \\ p \neq q, pq \leq x^{1/h}}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \sum_{l=h}^{\lfloor \frac{\log x}{\log q} \rfloor} \sum_{\substack{n \leq \mathcal{N}_h(x/(p^k q^l)) \\ (n, p) = (n, q) = 1}} (kl) \\
&= \gamma_{0,h} x^{1/h} \sum_{\substack{p, q \\ p \neq q, pq \leq x^{1/h}}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \sum_{l=h}^{\lfloor \frac{\log x}{\log q} \rfloor} \frac{k}{p^{k/h} \left(1 + \frac{p^{-1}}{1-p^{-1/h}}\right)} \frac{l}{q^{l/h} \left(1 + \frac{q^{-1}}{1-q^{-1/h}}\right)} \\
&\quad + O_h \left(x^{\frac{1}{h+1}} \sum_{\substack{p, q \\ p \neq q, pq \leq x^{1/h}}} \left(\sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \frac{k}{p^{k/(h+1)}} \right) \left(\sum_{l=h}^{\lfloor \frac{\log x}{\log q} \rfloor} \frac{l}{q^{l/(h+1)}} \right) \right). \tag{3.33}
\end{aligned}$$

Note that taking $r \rightarrow \infty$ and $0 < a < 1$ in (3.22), we have

$$\sum_{k=h}^{\infty} k a^k \ll_h \frac{a^h}{(1-a)^2}.$$

Thus, using $a = p^{-1/(h+1)}$ and $q^{-1/(h+1)}$, the bounds $p^{-1/(h+1)}, q^{-1/(h+1)} \leq 2^{-1/(h+1)}$, Lemma 2.2.2 and Lemma 2.2.3, we bound the error term in (3.33) with

$$\begin{aligned}
&\ll x^{\frac{1}{h+1}} \sum_{\substack{p, q \\ p \neq q, pq \leq x^{1/h}}} \left(\sum_{k=h}^{\infty} \frac{k}{p^{k/(h+1)}} \right) \left(\sum_{l=h}^{\infty} \frac{l}{q^{l/(h+1)}} \right) \\
&\ll_h x^{\frac{1}{h+1}} \sum_{\substack{p \\ p \leq x^{1/h}/2}} \frac{1}{p^{h/(h+1)}} \sum_{\substack{q \\ q \leq x^{1/h}/p}} \frac{1}{q^{h/(h+1)}} \\
&\ll_h x^{\frac{1}{h}} \sum_{\substack{p \\ p \leq x^{1/h}/2}} \frac{1}{p \log(x^{1/h}/p)} \\
&\ll_h \frac{x^{\frac{1}{h}} \log \log x}{\log x}. \tag{3.34}
\end{aligned}$$

Next, we estimate the main term in (3.33). First note that, by (3.22), we have

$$\sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \frac{k}{p^{k/h} \left(1 + \frac{p^{-1}}{1-p^{-1/h}}\right)} = \frac{h(1-p^{-1/h}) + p^{-1/h}}{p(1-p^{-1/h})(1-p^{-1/h}+p^{-1})} + O_h \left(\frac{\left(\lfloor \frac{\log x}{\log p} \rfloor\right) p^{-(\lfloor \frac{\log x}{\log p} \rfloor + 1)/h}}{(1-p^{-1/h})(1-p^{-1/h}+p^{-1})} \right).$$

Thus, using a similar result for a prime q as above and the symmetry of two primes p and q , we deduce

$$\begin{aligned} & \gamma_{0,h} x^{1/h} \sum_{\substack{p,q \\ p \neq q, pq \leq x^{1/h}}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \sum_{l=h}^{\lfloor \frac{\log x}{\log q} \rfloor} \frac{k}{p^{k/h} \left(1 + \frac{p^{-1}}{1-p^{-1/h}}\right)} \frac{l}{q^{l/h} \left(1 + \frac{q^{-1}}{1-q^{-1/h}}\right)} \\ &= \gamma_{0,h} x^{1/h} \sum_{\substack{p,q \\ p \neq q, pq \leq x^{1/h}}} \frac{h(1-p^{-1/h}) + p^{-1/h}}{p(1-p^{-1/h})(1-p^{-1/h}+p^{-1})} \frac{h(1-q^{-1/h}) + q^{-1/h}}{q(1-q^{-1/h})(1-q^{-1/h}+q^{-1})} + 2J_1 + J_2, \end{aligned}$$

where

$$J_1 \ll_h \gamma_{0,h} x^{1/h} \sum_{\substack{p,q \\ p \neq q, pq \leq x^{1/h}}} \left(\frac{\left(\lfloor \frac{\log x}{\log p} \rfloor\right) p^{-(\lfloor \frac{\log x}{\log p} \rfloor + 1)/h}}{(1-p^{-1/h})(1-p^{-1/h}+p^{-1})} \right) \left(\frac{h(1-q^{-1/h}) + q^{-1/h}}{q(1-q^{-1/h})(1-q^{-1/h}+q^{-1})} \right),$$

and

$$J_2 \ll_h \gamma_{0,h} x^{1/h} \sum_{\substack{p,q \\ p \neq q, pq \leq x^{1/h}}} \left(\frac{\left(\lfloor \frac{\log x}{\log p} \rfloor\right) p^{-(\lfloor \frac{\log x}{\log p} \rfloor + 1)/h}}{(1-p^{-1/h})(1-p^{-1/h}+p^{-1})} \right) \left(\frac{\left(\lfloor \frac{\log x}{\log q} \rfloor\right) q^{-(\lfloor \frac{\log x}{\log q} \rfloor + 1)/h}}{(1-q^{-1/h})(1-q^{-1/h}+q^{-1})} \right).$$

Using $\lfloor x \rfloor \geq x - 1$, (2.11), and (3.24), we obtain

$$\begin{aligned}
J_1 &\ll_h \sum_{\substack{p \\ p \leq x^{1/h}/2}} \frac{\log x}{\log p} \left(\sum_{\substack{q \\ q \leq x^{1/h}/p}} \frac{1}{q} \right) \\
&\ll_h (\log x)(\log \log x) \sum_{\substack{p \\ p \leq x^{1/h}/2}} \frac{1}{\log p} \\
&\ll_h \frac{x^{1/h} \log \log x}{\log x}.
\end{aligned}$$

Moreover, using Lemma 3.3.1 and partial summation, we obtain

$$\begin{aligned}
J_2 &\ll_h \sum_{\substack{p \\ p \leq x^{1/h}/2}} \left(\frac{\left(\left\lfloor \frac{\log x}{\log p} \right\rfloor \right) p^{-\left(\left\lfloor \frac{\log x}{\log p} \right\rfloor - h + 1 \right)/h}}{p(1 - p^{-1/h})^2} \right) \left(\sum_{\substack{q \\ q \leq x^{1/h}/p}} \frac{\log x}{\log q} \right) \\
&\ll_h (\log x)^2 \sum_{\substack{p \\ p \leq x^{1/h}/2}} \frac{1}{p(\log^2(x^{1/h}/p))(\log p)} \\
&\ll_h 1.
\end{aligned}$$

Combining the last three results, we obtain

$$\begin{aligned}
&\gamma_{0,h} x^{1/h} \sum_{\substack{p,q \\ p \neq q, pq \leq x^{1/h}}} \sum_{k=h}^{\left\lfloor \frac{\log x}{\log p} \right\rfloor} \sum_{l=h}^{\left\lfloor \frac{\log x}{\log q} \right\rfloor} \frac{1}{p^{k/h} \left(1 + \frac{p^{-1}}{1 - p^{-1/h}} \right)} \frac{1}{q^{l/h} \left(1 + \frac{q^{-1}}{1 - q^{-1/h}} \right)} \\
&= \gamma_{0,h} x^{1/h} \sum_{\substack{p,q \\ p \neq q, pq \leq x^{1/h}}} \frac{h(1 - p^{-1/h}) + p^{-1/h}}{p(1 - p^{-1/h})(1 - p^{-1/h} + p^{-1})} \frac{h(1 - q^{-1/h}) + q^{-1/h}}{q(1 - q^{-1/h})(1 - q^{-1/h} + q^{-1})} \\
&\quad + O_h \left(\frac{x^{1/h} \log \log x}{\log x} \right). \tag{3.35}
\end{aligned}$$

Thus to complete the proof, we only require to estimate the main term in (3.35). Using

Lemma 2.2.1, we have

$$\begin{aligned}
& \sum_{\substack{p,q \\ p \neq q, pq \leq x^{1/h}}} \frac{h(1 - p^{-1/h}) + p^{-1/h}}{p(1 - p^{-1/h})(1 - p^{-1/h} + p^{-1})} \frac{h(1 - q^{-1/h}) + q^{-1/h}}{q(1 - q^{-1/h})(1 - q^{-1/h} + q^{-1})} \\
&= \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{h(1 - p^{-1/h}) + p^{-1/h}}{p(1 - p^{-1/h})(1 - p^{-1/h} + p^{-1})} \frac{h(1 - q^{-1/h}) + q^{-1/h}}{q(1 - q^{-1/h})(1 - q^{-1/h} + q^{-1})} \\
&\quad - \sum_{\substack{p \\ p \leq x^{1/(2h)}}} \left(\frac{h(1 - p^{-1/h}) + p^{-1/h}}{p(1 - p^{-1/h})(1 - p^{-1/h} + p^{-1})} \right)^2 \\
&= \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{h(1 - p^{-1/h}) + p^{-1/h}}{p(1 - p^{-1/h})(1 - p^{-1/h} + p^{-1})} \frac{h(1 - q^{-1/h}) + q^{-1/h}}{q(1 - q^{-1/h})(1 - q^{-1/h} + q^{-1})} \\
&\quad - \sum_p \left(\frac{h(p^{1/h} - 1) + 1}{(p^{1/h} - 1)(p - p^{1-1/h} + 1)} \right)^2 + O_h \left(\frac{1}{x^{1/(2h)} \log x} \right). \tag{3.36}
\end{aligned}$$

Now, using

$$\frac{h(1 - p^{-1/h}) + p^{-1/h}}{p(1 - p^{-1/h})(1 - p^{-1/h} + p^{-1})} = \frac{h}{p} + \frac{h(p - p^{1-(1/h)} - p^{1/h} + 1) + p}{p(p^{1/h} - 1)(p - p^{1-(1/h)} + 1)},$$

a similar result for another prime q , and the symmetry of primes p and q , we write the first sum on the right-hand side of (3.36) as

$$\begin{aligned}
& \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{h(1 - p^{-1/h}) + p^{-1/h}}{p(1 - p^{-1/h})(1 - p^{-1/h} + p^{-1})} \frac{h(1 - q^{-1/h}) + q^{-1/h}}{q(1 - q^{-1/h})(1 - q^{-1/h} + q^{-1})} \\
&= h^2 \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{1}{pq} + 2h \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{1}{p} \left(\frac{h(q - q^{1-(1/h)} - q^{1/h} + 1) + q}{q(q^{1/h} - 1)(q - q^{1-(1/h)} + 1)} \right) \\
&\quad + \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \left(\frac{h(p - p^{1-(1/h)} - p^{1/h} + 1) + p}{p(p^{1/h} - 1)(p - p^{1-(1/h)} + 1)} \right) \left(\frac{h(q - q^{1-(1/h)} - q^{1/h} + 1) + q}{q(q^{1/h} - 1)(q - q^{1-(1/h)} + 1)} \right).
\end{aligned}$$

The first sum on the right-hand side above is estimated using Lemma 2.2.4. For the second

sum, we use Lemma 2.2.1, (2.11), Lemma 2.2.2 and $\log(x^{1/h}/p) \geq \log 2$ together to obtain

$$\begin{aligned}
& \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{1}{p} \left(\frac{h(q - q^{1-(1/h)} - q^{1/h} + 1) + q}{q(q^{1/h} - 1)(q - q^{1-(1/h)} + 1)} \right) \\
&= \sum_{\substack{p \\ p \leq x^{1/h}/2}} \frac{1}{p} \sum_{\substack{q \\ q \leq x^{1/h}/p}} \frac{h(q - q^{1-(1/h)} - q^{1/h} + 1) + q}{q(q^{1/h} - 1)(q - q^{1-(1/h)} + 1)} \\
&= \sum_{\substack{p \\ p \leq x^{1/h}/2}} \frac{1}{p} \left(\sum_p \frac{h(p - p^{1-(1/h)} - p^{1/h} + 1) + p}{p(p^{1/h} - 1)(p - p^{1-(1/h)} + 1)} + O_h \left(\frac{p^{1/h}}{x^{1/h^2} \log(x^{1/h}/p)} \right) \right) \\
&= \left(\sum_p \frac{h(p - p^{1-(1/h)} - p^{1/h} + 1) + p}{p(p^{1/h} - 1)(p - p^{1-(1/h)} + 1)} \right) (\log \log x + B_1 - \log h) + O_h \left(\frac{1}{\log x} \right).
\end{aligned}$$

Similarly, for the third sum, we obtain

$$\begin{aligned}
& \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \left(\frac{h(p - p^{1-(1/h)} - p^{1/h} + 1) + p}{p(p^{1/h} - 1)(p - p^{1-(1/h)} + 1)} \right) \left(\frac{h(q - q^{1-(1/h)} - q^{1/h} + 1) + q}{q(q^{1/h} - 1)(q - q^{1-(1/h)} + 1)} \right) \\
&= \sum_{\substack{p \\ p \leq x^{1/h}/2}} \frac{h(p - p^{1-(1/h)} - p^{1/h} + 1) + p}{p(p^{1/h} - 1)(p - p^{1-(1/h)} + 1)} \sum_{\substack{q \\ q \leq x^{1/h}/p}} \frac{h(q - q^{1-(1/h)} - q^{1/h} + 1) + q}{q(q^{1/h} - 1)(q - q^{1-(1/h)} + 1)} \\
&= \sum_{\substack{p \\ p \leq x^{1/h}/2}} \frac{h(p - p^{1-(1/h)} - p^{1/h} + 1) + p}{p(p^{1/h} - 1)(p - p^{1-(1/h)} + 1)} \left(\sum_p \frac{h(p - p^{1-(1/h)} - p^{1/h} + 1) + p}{p(p^{1/h} - 1)(p - p^{1-(1/h)} + 1)} \right. \\
&\quad \left. + O_h \left(\frac{p^{1/h}}{x^{1/h^2} \log(x^{1/h}/p)} \right) \right) \\
&= \left(\sum_p \frac{h(p - p^{1-(1/h)} - p^{1/h} + 1) + p}{p(p^{1/h} - 1)(p - p^{1-(1/h)} + 1)} \right)^2 + O_h \left(\frac{\log \log x}{x^{1/h^2} \log x} \right).
\end{aligned}$$

Combining the last three results with Lemma 2.2.4 and simplifying further using

$$\begin{aligned}
D_3 &= h(B_2 - \log h) + \sum_p \frac{(h+1)p^{1+1/h} - hp - 2hp^{2/h} + (2h-1)p^{1/h}}{(p-1)(p^{1/h}-1)(p^{1+1/h} + p^{1/h} - p)} \\
&= h(B_1 - \log h) + \sum_p \frac{h(p - p^{1-(1/h)} - p^{1/h} + 1) + p}{p(p^{1/h} - 1)(p - p^{1-(1/h)} + 1)},
\end{aligned}$$

we obtain

$$\begin{aligned}
& \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{h(1 - p^{-1/h}) + p^{-1/h}}{p(1 - p^{-1/h})(1 - p^{-1/h} + p^{-1})} \frac{h(1 - q^{-1/h}) + q^{-1/h}}{q(1 - q^{-1/h})(1 - q^{-1/h} + q^{-1})} \\
&= h^2 ((\log \log x - \log h)^2 + 2B_1(\log \log x - \log h) + B_1^2 - \zeta(2)) \\
&\quad + 2h \left(\sum_p \frac{h(p - p^{1-(1/h)} - p^{1/h} + 1) + p}{p(p^{1/h} - 1)(p - p^{1-(1/h)} + 1)} \right) (\log \log x + B_1 - \log h) \\
&\quad + \left(\sum_p \frac{h(p - p^{1-(1/h)} - p^{1/h} + 1) + p}{p(p^{1/h} - 1)(p - p^{1-(1/h)} + 1)} \right)^2 + O_h \left(\frac{\log \log x}{\log x} \right) \\
&= h^2 (\log \log x)^2 + 2D_3 h \log \log x + D_3^2 - h^2 \zeta(2) + O_h \left(\frac{\log \log x}{\log x} \right).
\end{aligned}$$

Combining the above with (3.29), (3.32), (3.33), (3.34), (3.35), and (3.36), we obtain

$$\begin{aligned}
\sum_{n \in \mathcal{N}_h(x)} \Omega^2(n) &= h^2 \gamma_{0,h} x^{1/h} (\log \log x)^2 + (2D_3 + h) h \gamma_{0,h} x^{1/h} \log \log x + D_4 \gamma_{0,h} x^{1/h} \\
&\quad + O_h \left(\frac{x^{1/h} \log \log x}{\log x} \right).
\end{aligned}$$

This completes the proof of Theorem 3.1.2. $\square$

3.4 Normal order of $\Omega(n)$ over h -free and h -full numbers

In this section, we prove that $\Omega(n)$ has normal order $\log \log n$ when restricted to h -free numbers and has normal order $h \log \log n$ when restricted to h -full numbers.

Proof of Theorem 3.1.3. Notice that

$$\sum_{n \in \mathcal{S}_h(x)} (\Omega(n) - \log \log n)^2 = \sum_{n \in \mathcal{S}_h(x)} \Omega^2(n) - 2 \sum_{n \in \mathcal{S}_h(x)} \Omega(n) \log \log n + \sum_{n \in \mathcal{S}_h(x)} (\log \log n)^2. \tag{3.37}$$

Using the above with the first moment of $\Omega(n)$ over h -free numbers given in Theorem 3.1.1, (2.50) and partial summation, we obtain

$$\sum_{n \in \mathcal{S}_h(x)} \Omega(n) \log \log n = \frac{1}{\zeta(h)} x (\log \log x)^2 + \frac{C_3}{\zeta(h)} x \log \log x + O_h \left(\frac{x \log \log x}{\log x} \right),$$

and

$$\sum_{n \in \mathcal{S}_h(x)} (\log \log n)^2 = \frac{1}{\zeta(h)} x (\log \log x)^2 + O_h \left(\frac{x \log \log x}{\log x} \right).$$

Using the above two results and the second moment of $\Omega(n)$ over h -free numbers given in Theorem 3.1.1 in (3.37), we obtain

$$\sum_{n \in \mathcal{S}_h(x)} (\Omega(n) - \log \log n)^2 = \frac{1}{\zeta(h)} x \log \log x + \frac{C_4}{\zeta(h)} x + O_h \left(\frac{x \log \log x}{\log x} \right).$$

For the second part, let $g(x)$ be an increasing function such that $g(x) \rightarrow \infty$ as $x \rightarrow \infty$, and let E_{free}^h be the set of natural numbers $n \in \mathcal{S}_h$ with $x/\log x < n \leq x$ such that

$$\frac{|\Omega(n) - \log \log n|}{\sqrt{\log \log n}} \geq g(x).$$

Let $|E_{\text{free}}^h|$ be the cardinality of E_{free}^h . Notice that

$$\begin{aligned} \sum_{n \in \mathcal{S}_h(x)} (\Omega(n) - \log \log n)^2 &\geq \sum_{n \in E_{\text{free}}^h} (\Omega(n) - \log \log n)^2 \\ &> g^2(x/\log x) \sum_{n \in E_{\text{free}}^h} \log \log n \\ &> g^2(x/\log x) |E_{\text{free}}^h| \log \log(x/\log x). \end{aligned}$$

Using the asymptotic result for the left-hand side above, we deduce

$$\frac{|E_{\text{free}}^h|}{x} \ll_h \frac{\log \log x}{g^2(x/\log x) \log \log(x/\log x)}.$$

Since $g(x) \rightarrow \infty$ as $x \rightarrow \infty$, the right-hand side above goes to 0 as $x \rightarrow \infty$. Hence $|E_{\text{free}}^h| = o(x)$. Moreover, by (2.50), $|\mathcal{S}_h(x)| \sim x/\zeta(h)$. Thus, the set of natural numbers $n \in \mathcal{S}_h$ with $x/\log x < n \leq x$ such that

$$\frac{|\Omega(n) - \log \log n|}{\sqrt{\log \log n}} \geq g(x),$$

is $o(|\mathcal{S}_h(x)|)$. Next, we choose ϵ' such that $0 < \epsilon' < 1/2$ and choose the function $g(x) = (\log \log x)^{\epsilon'}$. Let $\epsilon > 0$ be arbitrary. Since $\epsilon' < 1/2$, there exist $x_0 \in \mathbb{R}$ such that

$$\frac{(\log \log x)^{\epsilon'}}{(\log \log n)^{1/2}} < \frac{(\log \log x)^{\epsilon'}}{(\log \log(x/\log x))^{1/2}} \leq \epsilon$$

for all $x \geq x_0$ and for all n with $x/\log x < n \leq x$. Hence as $x \rightarrow \infty$, the numbers $n \in \mathcal{S}_h$ with $x/\log x < n \leq x$ that satisfy the inequality

$$\frac{|\Omega(n) - \log \log n|}{\log \log n} \geq \epsilon$$

is $o(|\mathcal{S}_h(x)|)$. Again by (2.50), the remaining set of natural numbers $n \in \mathcal{S}_h(x/\log x)$ is $o(|\mathcal{S}_h(x)|)$. Together, this implies that $\Omega(n)$ has normal order $\log \log n$ over h -free numbers, which completes the proof. $\square$

Proof of Theorem 3.1.4. Notice that

$$\begin{aligned} & \sum_{n \in \mathcal{N}_h(x)} (\Omega(n) - h \log \log n)^2 \\ &= \sum_{n \in \mathcal{N}_h(x)} \Omega^2(n) - 2h \sum_{n \in \mathcal{N}_h(x)} \Omega(n) \log \log n + h^2 \sum_{n \in \mathcal{N}_h(x)} (\log \log n)^2. \end{aligned} \quad (3.38)$$

Using (2.52) with the first moment of $\Omega(n)$ over h -full numbers given in Theorem 3.1.2 and partial summation, we obtain

$$\sum_{n \in \mathcal{N}_h(x)} \Omega(n) \log \log n = h\gamma_{0,h}x^{1/h}(\log \log x)^2 + \gamma_{0,h}D_3x^{1/h} \log \log x + O_h\left(\frac{x^{1/h} \log \log x}{\log x}\right),$$

and

$$\sum_{n \in \mathcal{N}_h(x)} (\log \log n)^2 = \gamma_{0,h}x^{1/h}(\log \log x)^2 + O_h\left(\frac{x^{1/h} \log \log x}{\log x}\right).$$

Using the above two results and the second moment of $\Omega(n)$ over h -full numbers given in Theorem 3.1.2 in (3.38), we obtain

$$\sum_{n \in \mathcal{N}_h(x)} (\Omega(n) - h \log \log n)^2 = h^2\gamma_{0,h}x^{1/h} \log \log x + D_4\gamma_{0,h}x^{1/h} + O_h\left(\frac{x^{1/h} \log \log x}{\log x}\right).$$

For the second part, let $g(x)$ be an increasing function such that $g(x) \rightarrow \infty$ as $x \rightarrow \infty$, and let E_{full}^h be the set of natural numbers $n \in \mathcal{N}_h$ with $x/\log x < n \leq x$ such that

$$\frac{|\Omega(n) - h \log \log n|}{\sqrt{h \log \log n}} \geq g(x).$$

Let $|E_{\text{full}}^h|$ be the cardinality of E_{full}^h . Then

$$\sum_{n \in \mathcal{N}_h(x)} (\Omega(n) - h \log \log n)^2 > hg^2(x/\log x) |E_{\text{full}}^h| \log \log(x/\log x).$$

Using the asymptotic result for the left-hand side above, we deduce

$$\frac{|E_{\text{full}}^h|}{x^{1/h}} \ll_h \frac{\log \log x}{g^2(x/\log x) \log \log(x/\log x)}.$$

Since $g(x) \rightarrow \infty$ as $x \rightarrow \infty$, the right-hand side above goes to 0 as $x \rightarrow \infty$. Moreover, by (2.52), $|E_{\text{full}}^h| = o(x^{1/h})$. Note that $|\mathcal{N}_h(x)| \sim \gamma_{0,h} x^{1/h}$. Thus the set of natural numbers $n \in \mathcal{N}_h$ with $x/\log x < n \leq x$ such that

$$\frac{|\Omega(n) - h \log \log n|}{\sqrt{h \log \log n}} \geq g(x),$$

is $o(|\mathcal{N}_h(x)|)$. Similar to the previous proof, we choose an ϵ' satisfying $0 < \epsilon' < 1/2$ and choose the function $g(x) = (h \log \log x)^{\epsilon'}$. As a result, for any $\epsilon > 0$, the numbers $n \in \mathcal{N}_h$ with $x/\log x \leq n \leq x$ that satisfy the inequality

$$\frac{|\Omega(n) - h \log \log n|}{h \log \log n} \geq \epsilon$$

is $o(|\mathcal{N}_h(x)|)$. Again, by (2.52), the remaining set of natural numbers $n \in \mathcal{N}_h(x/\log x)$ is $o(|\mathcal{N}_h(x)|)$. Together, this implies that $\Omega(n)$ has normal order $h \log \log n$ over h -full numbers. This completes the proof. $\square$

Chapter 4

Moments of $\omega_k(n)$ over h -free and h -hull numbers

4.1 Introduction

Recall that, for natural numbers n and k , $\omega_k(n)$ denotes the number of distinct prime divisors of n , with multiplicity k . Let $h \geq 2$ be an integer. In this chapter, we study the distribution of $\omega_k(n)$ over h -free and over h -full numbers.

To begin, we show that when restricted to the set of h -free numbers, $\omega_1(n)$ behaves similarly to $\omega(n)$ and $\omega_k(n)$ with $k \geq 2$ exhibits a smaller asymptotic size than $\omega(n)$. We begin by studying the distributions of $\omega_1(n)$ over h -free numbers.

Let B_1 be the constant defined in (1.7). We define the constants (depending on h)

$$C_5 := B_1 - \sum_p \frac{p^{h-1} - 1}{p(p^h - 1)}, \quad (4.1)$$

and

$$C_6 := C_5^2 + C_5 - \zeta(2) - \sum_p \left(\frac{p^{h-1} - p^{h-2}}{p^h - 1} \right)^2.$$

For the first and the second moment of $\omega_1(n)$ over h -free numbers, we prove:

Theorem 4.1.1. *Let $x > 2$ be a real number. Let $h \geq 2$ be an integer. Let $\mathcal{S}_h$ be the set of h -free numbers. Then, we have*

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \omega_1(n) = \mathcal{C}_{1,\omega_1,\mathcal{S}_h} x \log \log x + \mathcal{C}_{2,\omega_1,\mathcal{S}_h} x + O_h \left(\frac{x}{\log x} \right),$$

and

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \omega_1^2(n) = \mathcal{C}_{1,\omega_1,\mathcal{S}_h} x (\log \log x)^2 + \mathcal{C}_{3,\omega_1,\mathcal{S}_h} x \log \log x + \mathcal{C}_{4,\omega_1,\mathcal{S}_h} x + O_h \left(\frac{x \log \log x}{\log x} \right),$$

where

$$\mathcal{C}_{1,\omega_1,\mathcal{S}_h} = \frac{1}{\zeta(h)}, \quad \mathcal{C}_{2,\omega_1,\mathcal{S}_h} = \frac{C_5}{\zeta(h)}, \quad \mathcal{C}_{3,\omega_1,\mathcal{S}_h} = 2\mathcal{C}_{2,\omega_1,\mathcal{S}_h} + \mathcal{C}_{1,\omega_1,\mathcal{S}_h}, \quad \text{and} \quad \mathcal{C}_{4,\omega_1,\mathcal{S}_h} = \frac{C_6}{\zeta(h)}. \quad (4.2)$$

Next, we establish the moments of $\omega_k(n)$ with $k \geq 2$ over h -free numbers as the following:

Theorem 4.1.2. *Let $k \geq 2$ and $h \geq 2$ be any integers. Let $\mathcal{S}_h$ be the set of h -free numbers. For $k \leq (h-1)$, we have*

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \omega_k(n) = \mathcal{C}_{2,\omega_k,\mathcal{S}_h} x + O_{h,k} \left(\frac{x^{1/k}}{\log x} \right),$$

and

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \omega_k^2(n) = \mathcal{C}_{4,\omega_k,\mathcal{S}_h} x + O_{h,k} \left(\frac{x^{1/k} \log \log x}{\log x} \right),$$

where $O_{h,k}$ means that the implied constant depends on h and k , and where

$$\mathcal{C}_{2,\omega_k,\mathcal{S}_h} = \frac{1}{\zeta(h)} \sum_p \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right), \quad (4.3)$$

and

$$\mathcal{C}_{4,\omega_k,\mathcal{S}_h} = \frac{1}{\zeta(h)} \left(\left(\sum_p \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \right)^2 - \sum_p \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right)^2 + \sum_p \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \right). \quad (4.4)$$

Remark 4.1.1. Note that if $n \in \mathcal{S}_h$, then $\omega_k(n) = 0$ for all $k \geq h$. Thus, the distribution of $\omega_k(n)$ with $k \geq h$ over the h -free numbers is zero.

Next, we prove the distribution of $\omega_k(n)$ over h -full numbers. We notice that $\omega(n)$ and $\omega_h(n)$ have similar asymptotic distributions over h -full numbers. Moreover, $\omega_k(n)$ with $k > h$ has a smaller asymptotic size than $\omega(n)$. We can thus infer that the smallest power of primes defining the subset of h -full numbers contributes to the main term for the asymptotic distribution of $\omega(n)$ over the subset. This inference also satisfies the behavior observed over h -free numbers. The set of h -free numbers includes the first power of primes, and it is observed that $\omega(n)$ and $\omega_1(n)$, which counts prime factors with multiplicity 1, satisfy similar distributions over h -free numbers. To prove the distribution over h -full numbers, we define two new constants (depending on h)

$$D_5 := B_1 - \log h - \mathcal{L}_h(2h), \quad (4.5)$$

and

$$D_6 := D_5^2 + D_5 - \zeta(2) - \sum_p \left(\frac{p^{1/h} - 1}{p^{1+1/h} - p + p^{1/h}} \right)^2, \quad (4.6)$$

where $\mathcal{L}_h(\cdot)$ is defined in (2.4). For the first moments of $\omega_k(n)$ over h -full numbers, we prove:

Theorem 4.1.3. Let $k \geq 2$ and $h \geq 2$ be any integers. Let $\mathcal{N}_h$ be the set of h -full numbers. Then, we have

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_h(n) = \mathcal{C}_{1, \omega_h, \mathcal{N}_h} x^{1/h} \log \log x + \mathcal{C}_{2, \omega_h, \mathcal{N}_h} x^{1/h} + O_h \left(\frac{x^{1/h}}{\log x} \right),$$

where

$$\mathcal{C}_{1, \omega_h, \mathcal{N}_h} = \gamma_{0,h}, \quad \text{and} \quad \mathcal{C}_{2, \omega_h, \mathcal{N}_h} = D_5 \gamma_{0,h}, \quad (4.7)$$

where $\gamma_{0,h}$ is defined in (1.6), and D_5 is defined in (4.5).

Moreover, we have

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_{h+1}(n) = \mathcal{C}_{2, \omega_{h+1}, \mathcal{N}_h} x^{1/h} + O_h \left(x^{1/(h+1)} \log \log x \right),$$

where

$$\mathcal{C}_{2,\omega_{h+1},\mathcal{N}_h} = (\mathcal{L}_h(h+1) - \mathcal{L}_h(h+2)) \gamma_{0,h}, \quad (4.8)$$

with $\mathcal{L}_h(\cdot)$ is defined in (2.4), and for $k > h+1$, we have

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_k(n) = \mathcal{C}_{2,\omega_k,\mathcal{N}_h} x^{1/h} + O_{h,k}(x^{1/(h+1)}),$$

where

$$\mathcal{C}_{2,\omega_k,\mathcal{N}_h} = (\mathcal{L}_h(k) - \mathcal{L}_h(k+1)) \gamma_{0,h}. \quad (4.9)$$

For the second moments, we obtain:

Theorem 4.1.4. *Under the assumptions as in Theorem 4.1.3, we have*

$$\begin{aligned} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_h^2(n) &= \mathcal{C}_{1,\omega_h,\mathcal{N}_h} x^{1/h} (\log \log x)^2 + \mathcal{C}_{3,\omega_h,\mathcal{N}_h} x^{1/h} \log \log x + \mathcal{C}_{4,\omega_h,\mathcal{N}_h} x^{1/h} \\ &\quad + O_h \left(\frac{x^{1/h} \log \log x}{\log x} \right), \end{aligned}$$

where

$$\mathcal{C}_{3,\omega_h,\mathcal{N}_h} = 2\mathcal{C}_{2,\omega_h,\mathcal{N}_h} + \mathcal{C}_{1,\omega_h,\mathcal{N}_h}, \quad \text{and} \quad \mathcal{C}_{4,\omega_h,\mathcal{N}_h} = D_6 \gamma_{0,h}, \quad (4.10)$$

and where $\mathcal{C}_{1,\omega_h,\mathcal{N}_h}$ and $\mathcal{C}_{2,\omega_h,\mathcal{N}_h}$ are defined in (4.7) with $\gamma_{0,h}$ defined in (1.6), and where D_6 is defined in (4.6).

Moreover, for $k = h+1$, we have,

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_{h+1}^2(n) = \mathcal{C}_{4,\omega_{h+1},\mathcal{N}_h} x^{1/h} + O_h(x^{1/(h+1)} (\log \log x)^2),$$

where

$$\begin{aligned} \mathcal{C}_{4,\omega_{h+1},\mathcal{N}_h} &= \left((\mathcal{L}_h(h+1) - \mathcal{L}_h(h+2))^2 + \mathcal{L}_h(h+1) - \mathcal{L}_h(h+2) \right. \\ &\quad \left. - \sum_p \left(\frac{p^{1/h} - 1}{p^{1+(2/h)} - p^{1+(1/h)} + p^{2/h}} \right)^2 \right) \gamma_{0,h}, \end{aligned} \quad (4.11)$$

with $\mathcal{L}_h(\cdot)$ is defined in (2.4), and for $k > h + 1$, we have

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_k^2(n) = \mathcal{C}_{4, \omega_k, \mathcal{N}_h} x^{1/h} + O_{h,k}(x^{1/(h+1)}),$$

where

$$\begin{aligned} \mathcal{C}_{4, \omega_k, \mathcal{N}_h} = & \left((\mathcal{L}_h(k) - \mathcal{L}_h(k+1))^2 + \mathcal{L}_h(k) - \mathcal{L}_h(k+1) \right. \\ & \left. - \sum_p \left(\frac{p^{1/h} - 1}{p^{(k+1)/h} - p^{k/h} + p^{(k+1-h)/h}} \right)^2 \right) \gamma_{0,h}. \end{aligned} \quad (4.12)$$

Remark 4.1.2. Note that if $n \in \mathcal{N}_h$, then $\omega_k(n) = 0$ for all $k \leq (h-1)$. Thus, the distribution of $\omega_k(n)$ over h -full numbers is zero for $k \leq (h-1)$.

In Chapter 2, we showed that $\omega(n)$ has the normal order $\log \log n$ over h -free and over h -full numbers as well. Note that the set of h -free numbers has positive density, and both ω and ω_1 behave asymptotically similar over h -free numbers. Thus, the proof of $\omega_1(n)$ having normal order $\log \log n$ over h -free numbers follows from the classical case. In particular, working similarly to Theorem 2.1.3, one can establish that for any $\epsilon > 0$, the number of $n \in \mathcal{S}_h(x)$ that do not satisfy the inequality

$$(1 - \epsilon) \log \log n \leq \omega_1(n) \leq (1 + \epsilon) \log \log n$$

is $o(|\mathcal{S}_h(x)|)$ as $x \rightarrow \infty$. On the other hand, the set of h -full numbers has density 0, and thus the proof of normal order of ω_h does not follow from the classical result. However, since ω and ω_h behave asymptotically similar over h -full numbers, the proof of $\omega_h(n)$ having normal order $\log \log n$ over h -full numbers can be inferred in a manner analogous to the proof presented in Theorem 2.1.4 for $\omega(n)$. In particular, one can establish that for any $\epsilon > 0$, the number of $n \in \mathcal{N}_h(x)$ that do not satisfy the inequality

$$(1 - \epsilon) \log \log n \leq \omega_h(n) \leq (1 + \epsilon) \log \log n$$

is $o(|\mathcal{N}_h(x)|)$ as $x \rightarrow \infty$.

Unlike $\omega_1(n)$, we observe that $\omega_k(n)$ for $1 < k < h$ does not have a normal order over h -free numbers. This goes in accordance with the findings of Elma and Liu [12] where they proved that $\omega_k(n)$ for $k > 1$ does not have a normal order over natural numbers. In particular, we prove:

Theorem 4.1.5. *For any integer $h \geq 2$ and any integer k satisfying $1 < k < h$, the function $\omega_k(n)$ does not have normal order $F(n)$ for any non-decreasing function $F : \mathcal{S}_h \rightarrow \mathbb{R}_{\geq 0}$.*

Finally, we show that $\omega_k(n)$ for $k > h$ does not have a normal order over h -full numbers. In particular, we prove:

Theorem 4.1.6. *For any integer $h \geq 2$ and any integer $k > h$, the function $\omega_k(n)$ does not have normal order $F(n)$ for any non-decreasing function $F : \mathcal{N}_h \rightarrow \mathbb{R}_{\geq 0}$.*

4.2 Lemmata

We list the following result regarding the density of certain sequences of h -full numbers necessary for our study:

Lemma 4.2.1. [27, Lemma 17] *Let q be a prime and $k > h \geq 2$ be integers. Let $\mathcal{S}_k$ denote the set of k -free numbers and $\mathcal{N}_h$ denote the set of h -full numbers. Then, for any small $\epsilon > 0$, we have*

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h \cap \mathcal{S}_k, (n, q) = 1}} 1 = \frac{1 - q^{-1/h}}{1 - q^{-1/h} + q^{-1} - q^{-k/h}} \eta_{h,k} x^{1/h} + O\left(x^{\frac{2h+1}{2h(h+1)} + \epsilon}\right),$$

where $\eta_{h,k}$ is the convergent product given by

$$\eta_{h,k} = \prod_p \left(1 - \frac{1}{p}\right) \left(\frac{1 - p^{-1/h} + p^{-1} - p^{-k/h}}{1 - p^{-1/h}}\right), \quad (4.13)$$

and where the product runs over all primes.

4.3 Distribution of $\omega_k(n)$ over h -free numbers

In this section, we study the first moment of $\omega_k(n)$ over h -free numbers.

4.3.1 For $\omega_1(n)$

Proof of Theorem 4.1.1. Writing $n = pm$ with $(m, p) = 1$ for n satisfying $p \parallel n$, i.e., $p \mid n$ but $p^2 \nmid n$, we obtain

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \omega_1(n) = \sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \sum_{\substack{p \\ p \parallel n}} 1 = \sum_{p \leq x} \sum_{\substack{n \leq x \\ n \in \mathcal{S}_h, p \parallel n}} 1 = \sum_{p \leq x} \sum_{\substack{m \leq x/p \\ m \in \mathcal{S}_h, (m, p) = 1}} 1.$$

Now, using Lemma 2.2.6 for a single prime p and Lemma 2.2.2 to the above, we obtain

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \omega_1(n) = \sum_{p \leq x} \left(\frac{p^h - p^{h-1}}{p(p^h - 1)} \right) \frac{x}{\zeta(h)} + O_h \left(\frac{x}{\log x} \right).$$

Using $\frac{p^h - p^{h-1}}{p(p^h - 1)} = \frac{1}{p} - \frac{p^{h-1} - 1}{p(p^h - 1)}$, Lemma 2.2.1 with $\tau = 2$, and Mertens' second theorem given in (2.11), we obtain

$$\sum_{p \leq x} \left(\frac{p^h - p^{h-1}}{p(p^h - 1)} \right) = \log \log x + B_1 - \sum_p \frac{p^{h-1} - 1}{p(p^h - 1)} + O_h \left(\frac{1}{\log x} \right). \quad (4.14)$$

Plugging the above back into the previous equation completes the first part of the proof.

Next, we prove the second moment. Note that

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \omega_1^2(n) = \sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \left(\sum_{\substack{p \\ p \parallel n}} 1 \right)^2 = \sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \omega_1(n) + \sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \sum_{\substack{p, q \\ p \parallel n, q \parallel n, p \neq q}} 1, \quad (4.15)$$

where p and q above denote primes. The first sum on the right-hand side above is the first moment estimated above and for the second sum, using Lemma 2.2.6 for two primes p and q , we obtain

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \sum_{\substack{p, q \\ p \parallel n, q \parallel n, p \neq q}} 1 = \sum_{\substack{p, q \\ p \neq q, pq \leq x}} \left(\left(\frac{p^h - p^{h-1}}{p(p^h - 1)} \right) \left(\frac{q^h - q^{h-1}}{q(q^h - 1)} \right) \frac{x}{\zeta(h)} + O_h \left(\frac{x^{1/h}}{(pq)^{1/h}} \right) \right). \quad (4.16)$$

Next, we bound the above error term using Lemma 2.2.2 with $\alpha = 1/h$ and Lemma 2.2.3

as the following

$$\begin{aligned}
x^{1/h} \sum_{\substack{p,q \\ p \neq q, pq \leq x}} \frac{1}{(pq)^{1/h}} &= x^{1/h} \sum_{p \leq x/2} \frac{1}{p^{1/h}} \sum_{q \leq x/p} \frac{1}{q^{1/h}} \\
&\ll_h x \sum_{p \leq x/2} \left(\frac{1}{p \log(x/p)} \right) \\
&\ll_h \frac{x \log \log x}{\log x}.
\end{aligned} \tag{4.17}$$

Now, we estimate the main term in (4.16). First, we can divide the sum as

$$\begin{aligned}
&\sum_{\substack{p,q \\ p \neq q, pq \leq x}} \left(\frac{p^h - p^{h-1}}{p(p^h - 1)} \right) \left(\frac{q^h - q^{h-1}}{q(q^h - 1)} \right) \\
&= \sum_{\substack{p,q \\ pq \leq x}} \left(\frac{p^h - p^{h-1}}{p(p^h - 1)} \right) \left(\frac{q^h - q^{h-1}}{q(q^h - 1)} \right) - \sum_{\substack{p \\ p \leq x^{1/2}}} \left(\frac{p^h - p^{h-1}}{p(p^h - 1)} \right)^2.
\end{aligned} \tag{4.18}$$

The second sum on the right-hand side above is estimated using Lemma 2.2.1 as

$$\sum_{\substack{p \\ p \leq x^{1/2}}} \left(\frac{p^h - p^{h-1}}{p(p^h - 1)} \right)^2 = \sum_p \left(\frac{p^h - p^{h-1}}{p(p^h - 1)} \right)^2 + O\left(\frac{1}{x^{1/2} \log x} \right). \tag{4.19}$$

Using $\frac{p^h - p^{h-1}}{p(p^h - 1)} = \frac{1}{p} - \frac{p^{h-1} - 1}{p(p^h - 1)}$, and the symmetry in p and q , we have

$$\begin{aligned}
&\sum_{\substack{p,q \\ pq \leq x}} \left(\frac{p^h - p^{h-1}}{p(p^h - 1)} \right) \left(\frac{q^h - q^{h-1}}{q(q^h - 1)} \right) \\
&= \sum_{\substack{p,q \\ pq \leq x}} \frac{1}{pq} - 2 \sum_{\substack{p,q \\ pq \leq x}} \frac{1}{p} \left(\frac{q^{h-1} - 1}{q(q^h - 1)} \right) + \sum_{\substack{p,q \\ pq \leq x}} \left(\frac{p^{h-1} - 1}{p(p^h - 1)} \right) \left(\frac{q^{h-1} - 1}{q(q^h - 1)} \right).
\end{aligned} \tag{4.20}$$

We estimate the sums on the right-hand side above separately. For the first sum, we use Lemma 2.2.4. For the second sum, we use Lemma 2.2.1, (2.11) and the Chebyshev's

inequality given in (2.8) to obtain

$$\begin{aligned}
& \sum_{\substack{p,q \\ pq \leq x}} \frac{1}{p} \left(\frac{q^{h-1} - 1}{q(q^h - 1)} \right) \\
&= \sum_{\substack{p \\ p \leq x/2}} \frac{1}{p} \sum_{\substack{q \\ q \leq x/p}} \left(\frac{q^{h-1} - 1}{q(q^h - 1)} \right) \\
&= \sum_{\substack{p \\ p \leq x/2}} \frac{1}{p} \left(\sum_p \left(\frac{p^{h-1} - 1}{p(p^h - 1)} \right) + O \left(\frac{1}{(x/p) \log(x/p)} \right) \right) \\
&= \sum_p \left(\frac{p^{h-1} - 1}{p(p^h - 1)} \right) (\log \log x + B_1) + O \left(\frac{1}{\log x} \right). \tag{4.21}
\end{aligned}$$

For the third sum, we use Lemma 2.2.1 twice and Lemma 2.2.3 to obtain

$$\sum_{\substack{p,q \\ pq \leq x}} \left(\frac{p^{h-1} - 1}{p(p^h - 1)} \right) \left(\frac{q^{h-1} - 1}{q(q^h - 1)} \right) = \left(\sum_p \left(\frac{p^{h-1} - 1}{p(p^h - 1)} \right) \right)^2 + O \left(\frac{\log \log x}{x \log x} \right). \tag{4.22}$$

Combining (4.20), (4.21), (4.22), and Lemma 2.2.4, we obtain

$$\begin{aligned}
& \sum_{\substack{p,q \\ pq \leq x}} \left(\frac{p^h - p^{h-1}}{p(p^h - 1)} \right) \left(\frac{q^h - q^{h-1}}{q(q^h - 1)} \right) \\
&= (\log \log x)^2 + 2B_1 \log \log x + B_1^2 - \zeta(2) - 2 \sum_p \left(\frac{p^{h-1} - 1}{p(p^h - 1)} \right) (\log \log x + B_1) \\
&\quad + \left(\sum_p \frac{p^{h-1} - 1}{p(p^h - 1)} \right)^2 + O \left(\frac{\log \log x}{\log x} \right).
\end{aligned}$$

Combining (4.15), (4.16), (4.17), (4.18) and (4.19) with the above equation and the first moment of $\omega_1(n)$ over h -free numbers, we obtain the required second moment estimate. This completes the proof. $\square$

4.3.2 For $\omega_k(n)$ with $2 \leq k \leq (h-1)$

Proof of Theorem 4.1.2. Using $n = p^k m$ with $(p, m) = 1$ for n satisfying $p \parallel n$, we obtain

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \omega_k(n) = \sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \sum_{\substack{p \mid n \\ p^k \parallel n}} 1 = \sum_{p \leq x^{1/k}} \sum_{\substack{n \leq x \\ n \in \mathcal{S}_h, p^k \parallel n}} 1 = \sum_{p \leq x^{1/k}} \sum_{\substack{m \leq x/p^k \\ m \in \mathcal{S}_h, (m, p) = 1}} 1. \quad (4.23)$$

Using Lemma 2.2.6 for a single prime p , we obtain

$$\begin{aligned} \sum_{p \leq x^{1/k}} \sum_{\substack{m \leq x/p^k \\ m \in \mathcal{S}_h, (m, p) = 1}} 1 &= \sum_{p \leq x^{1/k}} \left(\frac{1}{\zeta(h)} \left(\frac{p^h - p^{h-1}}{p^h - 1} \right) \frac{x}{p^k} + O_h \left(\frac{x^{1/h}}{p^{k/h}} \right) \right) \\ &= \sum_p \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \frac{x}{\zeta(h)} + \sum_{p > x^{1/k}} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \frac{x}{\zeta(h)} \\ &\quad + O_h \left(x^{1/h} \sum_{p \leq x^{1/k}} \frac{1}{p^{k/h}} \right). \end{aligned} \quad (4.24)$$

Using Lemma 2.2.1 with $\tau = k$, we estimate the second sum on the right side of the above equation as

$$\sum_{p > x^{1/k}} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \frac{x}{\zeta(h)} \ll_{h,k} \frac{x^{1/k}}{\log x}. \quad (4.25)$$

Using Lemma 2.2.2 with $\alpha = k/h$, we obtain

$$\sum_{p \leq x^{1/k}} \frac{1}{p^{k/h}} \ll_{h,k} \frac{x^{\frac{1}{k} - \frac{1}{h}}}{\log x}. \quad (4.26)$$

Combining (4.23), (4.24), (4.25), and (4.26) establishes the first moment of $\omega_k(n)$ over h -free numbers and completes the first part of the proof.

For the second moment, note that

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \omega_k^2(n) = \sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \left(\sum_{\substack{p \mid n \\ p^k \parallel n}} 1 \right)^2 = \sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \omega_k(n) + \sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \sum_{\substack{p, q \\ p^k \parallel n, q^k \parallel n, p \neq q}} 1, \quad (4.27)$$

where p and q above denote primes. The first sum on the right-hand side above is the first moment studied above, and for the second sum, using Lemma 2.2.6, we obtain

$$\sum_{\substack{n \leq x \\ n \in \mathcal{S}_h}} \sum_{\substack{p, q \\ p^k \| n, q^k \| n, p \neq q}} 1 = \sum_{\substack{p, q \\ p \neq q, pq \leq x^{1/k}}} \left(\left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \left(\frac{q^h - q^{h-1}}{q^k(q^h - 1)} \right) \frac{x}{\zeta(h)} + O_h \left(\frac{x^{1/h}}{(pq)^{k/h}} \right) \right). \quad (4.28)$$

Next, we bound the above error term. We employ Lemma 2.2.2 with $\alpha = k/h$, and with $\alpha = 2k/h$ when $2k < h$, (2.11) when $2k = h$, $\sum_p 1/p^{2k/h} = O(1)$ when $2k > h$, and Lemma 2.2.3 below to obtain

$$\begin{aligned} x^{1/h} \sum_{\substack{p, q \\ p \neq q, pq \leq x^{1/k}}} \frac{1}{(pq)^{k/h}} &= x^{1/h} \sum_{\substack{p, q \\ pq \leq x^{1/k}}} \frac{1}{(pq)^{k/h}} - x^{1/h} \sum_{p \leq x^{1/2k}} \frac{1}{p^{2k/h}} \\ &\ll_{k, h} \frac{x^{\frac{1}{k}} \log \log x}{\log x}. \end{aligned} \quad (4.29)$$

Now, we estimate the main term in (4.28). First, we can divide the sum as

$$\begin{aligned} &\sum_{\substack{p, q \\ p \neq q, pq \leq x^{1/k}}} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \left(\frac{q^h - q^{h-1}}{q^k(q^h - 1)} \right) \\ &= \sum_{\substack{p, q \\ pq \leq x^{1/k}}} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \left(\frac{q^h - q^{h-1}}{q^k(q^h - 1)} \right) - \sum_{\substack{p \\ p \leq x^{1/2k}}} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right)^2. \end{aligned} \quad (4.30)$$

The second sum on the right-hand side above is estimated using Lemma 2.2.1 as

$$\sum_{\substack{p \\ p \leq x^{1/2k}}} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right)^2 = \sum_p \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right)^2 + O_k \left(\frac{1}{x^{1-\frac{1}{2k}} \log x} \right). \quad (4.31)$$

For the first sum, we employ Lemma 2.2.1 twice and Lemma 2.2.3 to obtain

$$\begin{aligned}
& \sum_{\substack{p,q \\ pq \leq x^{1/k}}} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \left(\frac{q^h - q^{h-1}}{q^k(q^h - 1)} \right) \\
&= \sum_{\substack{p \\ p \leq x^{1/k}/2}} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \sum_{\substack{q \\ q \leq x^{1/k}/p}} \left(\frac{q^h - q^{h-1}}{q^k(q^h - 1)} \right) \\
&= \sum_{\substack{p \\ p \leq x^{1/k}/2}} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \left(\sum_p \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \right) + O_k \left(x^{\frac{1}{k}-1} \sum_{\substack{p \\ p \leq x^{1/k}/2}} \frac{1}{p \log(x^{1/k}/p)} \right) \\
&= \left(\sum_p \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \right)^2 + O_k \left(\frac{x^{\frac{1}{k}-1}}{\log x} \right) + O_k \left(\frac{x^{\frac{1}{k}-1} \log \log x}{\log x} \right).
\end{aligned}$$

Combining the last three results, we obtain

$$\begin{aligned}
& \sum_{\substack{p,q \\ p \neq q, pq \leq x^{1/k}}} \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \left(\frac{q^h - q^{h-1}}{q^k(q^h - 1)} \right) \frac{x}{\zeta(h)} \\
&= \left(\left(\sum_p \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right) \right)^2 - \sum_p \left(\frac{p^h - p^{h-1}}{p^k(p^h - 1)} \right)^2 \right) \frac{x}{\zeta(h)} + O_{h,k} \left(\frac{x^{1/k} \log \log x}{\log x} \right).
\end{aligned}$$

Combining the above with (4.27), (4.28), and the first moment of $\omega_k(n)$ studied in the first part of the proof, we obtain the second moment of $\omega_k(n)$ over h -free numbers. This completes the proof. $\square$

4.4 Distribution of $\omega_k(n)$ over h -full numbers

In this section, we study the distribution of the function $\omega_k(n)$ over h -full numbers. The definition of h -full numbers enforces that the distribution of $\omega_k(n)$ over h -full numbers is zero for $k \leq h - 1$. Thus, we only need to study the case $k \geq h$.

4.4.1 The first moment of $\omega_k(n)$ over h -full numbers

Proof of Theorem 4.1.3. Note that

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_k(n) = \sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \sum_{\substack{p|n \\ p^k \parallel n}} 1 = \sum_{p \leq x^{1/k}} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_h, p^k \parallel n}} 1 = \sum_{p \leq x^{1/k}} A_{p,h}(x/p^k), \quad (4.32)$$

where $A_{p,h}(y)$ is defined in (2.32). Thus, applying Lemma 2.4.1 with a single prime p , we have

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_k(n) = \gamma_{0,h} x^{1/h} \sum_{p \leq x^{1/k}} \frac{1 - p^{-1/h}}{p^{k/h} (1 - p^{-1/h} + p^{-1})} + O_h \left(x^{1/(h+1)} \sum_{p \leq x^{1/k}} \frac{1}{p^{k/(h+1)}} \right). \quad (4.33)$$

The above formula for $k = h$ yields

$$\begin{aligned} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_h(n) &= \gamma_{0,h} x^{1/h} \left(\sum_{p \leq x^{1/h}} \frac{1}{p (1 - p^{-1/h} + p^{-1})} - \sum_{p \leq x^{1/h}} \frac{1}{p^{1+1/h} (1 - p^{-1/h} + p^{-1})} \right) \\ &\quad + O_h \left(x^{1/(h+1)} \sum_{p \leq x^{1/h}} \frac{1}{p^{h/(h+1)}} \right). \end{aligned}$$

By (2.42), we have

$$\sum_{p \leq x^{1/h}} \frac{1}{p (1 - p^{-1/h} + p^{-1})} = \log \log x + B_1 - \log h + \mathcal{L}_h(h+1) - \mathcal{L}_h(2h) + O_h \left(\frac{1}{\log x} \right).$$

Thus, using Lemma 2.2.2 with $\alpha = h/(h+1)$, and Lemma 2.2.5 with $y = x^{1/h}$ and $r = h+1$, we obtain

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_h(n) = \gamma_{0,h} x^{1/h} \log \log x + \left(B_1 - \log h - \mathcal{L}_h(2h) \right) \gamma_{0,h} x^{1/h} + O_h \left(\frac{x^{1/h}}{\log x} \right).$$

Now, let's consider the case $k > h$. Rewriting (4.33) for $k > h$ and using Lemma 2.2.5, we obtain

$$\begin{aligned} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_k(n) &= \gamma_{0,h} x^{1/h} \left(\mathcal{L}_h(k) - \mathcal{L}_h(k+1) + O_{h,k} \left(\frac{1}{x^{\frac{1}{h} - \frac{1}{k}} (\log x)} \right) \right) \\ &\quad + O_h \left(x^{1/(h+1)} \sum_{p \leq x^{1/k}} \frac{1}{p^{k/(h+1)}} \right). \end{aligned}$$

Note that, for $k = h + 1$,

$$\sum_{p \leq x^{1/k}} \frac{1}{p^{k/(h+1)}} = O(\log \log x)$$

and for $k > h + 1$,

$$\sum_{p \leq x^{1/k}} \frac{1}{p^{k/(h+1)}} = O_k(1).$$

Combining the above results, we obtain

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_{h+1}(n) = (\mathcal{L}_h(h+1) - \mathcal{L}_h(h+2)) \gamma_{0,h} x^{1/h} + O_h(x^{1/(h+1)} \log \log x),$$

and for $k > h + 1$, we obtain

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_k(n) = (\mathcal{L}_h(k) - \mathcal{L}_h(k+1)) \gamma_{0,h} x^{1/h} + O_{h,k}(x^{1/(h+1)}).$$

This completes the proof. □

4.4.2 The second moment of $\omega_k(n)$ over h -full numbers

Proof of Theorem 4.1.4. Note that, for $k \geq h$, we have

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_k^2(n) = \sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \left(\sum_{\substack{p \\ p^k \parallel n}} 1 \right)^2 = \sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_k(n) + \sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \sum_{\substack{p, q \\ p^k \parallel n, q^k \parallel n, p \neq q}} 1, \quad (4.34)$$

where p and q above denote primes. The first sum on the right-hand side above can be estimated using Theorem 4.1.3 and for the second sum, we first rewrite the sum, and use Lemma 2.4.1 with two distinct primes p and q to obtain

$$\begin{aligned}
\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \sum_{\substack{p, q \\ p^k \| n, q^k \| n, p \neq q}} 1 &= \sum_{\substack{p, q \\ p \neq q, pq \leq x^{1/k}}} \sum_{\substack{n \leq x/(pq)^k \\ n \in \mathcal{N}_h \\ (p, n) = (q, n) = 1}} 1 \\
&= \gamma_{0, h} x^{1/h} \sum_{\substack{p, q \\ pq \leq x^{1/k}}} \frac{1}{p^{k/h} \left(1 + \frac{p^{-1}}{1 - p^{-1/h}}\right)} \frac{1}{q^{k/h} \left(1 + \frac{q^{-1}}{1 - q^{-1/h}}\right)} \\
&\quad - \gamma_{0, h} x^{1/h} \sum_p \left(\frac{1}{p^{k/h} \left(1 + \frac{p^{-1}}{1 - p^{-1/h}}\right)} \right)^2 + O_{h, k} \left(\frac{x^{1/2k}}{\log x} \right) \\
&\quad + O_h \left(x^{\frac{1}{h+1}} \sum_{\substack{p, q \\ p \neq q, pq \leq x^{1/h}}} \frac{1}{p^{k/(h+1)} q^{k/(h+1)}} \right). \tag{4.35}
\end{aligned}$$

For bounding the sum in the error term above, we use Lemma 2.2.2 and Lemma 2.2.3 for $k = h$, Lemma 2.2.4 for $k = h + 1$, and $\sum_{p, q} \frac{1}{(pq)^{k/(h+1)}} = O(1)$ for $k > h + 1$. Thus, we obtain

$$x^{\frac{1}{h+1}} \sum_{\substack{p, q \\ p \neq q, pq \leq x^{1/h}}} \frac{1}{p^{k/(h+1)} q^{k/(h+1)}} \ll_{h, k} \begin{cases} \frac{x^{1/h} \log \log x}{\log x} & \text{if } k = h, \\ x^{1/(h+1)} (\log \log x)^2 & \text{if } k = h + 1, \\ x^{1/(h+1)} & \text{if } k > h + 1. \end{cases} \tag{4.36}$$

We study the first sum in the main term in (4.35) by dividing it into cases. We begin with $k = h$ and estimate the sum

$$\sum_{\substack{p, q \\ pq \leq x^{1/h}}} \frac{1 - p^{-1/h}}{p(1 - p^{-1/h} + p^{-1})} \frac{1 - q^{-1/h}}{q(1 - q^{-1/h} + q^{-1})}.$$

Note that

$$\frac{1 - p^{-1/h}}{p(1 - p^{-1/h} + p^{-1})} = \frac{1}{p} - \frac{1}{p^2(1 - p^{-1/h} + p^{-1})}.$$

Using this, a similar result for a prime q and the symmetry in primes p and q , we expand the previous sum as

$$\begin{aligned}
& \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{1 - p^{-1/h}}{p(1 - p^{-1/h} + p^{-1})} \frac{1 - q^{-1/h}}{q(1 - q^{-1/h} + q^{-1})} \\
&= \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{1}{pq} - 2 \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{1}{pq^2(1 - q^{-1/h} + q^{-1})} \\
&\quad + \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{1}{p^2(1 - p^{-1/h} + p^{-1})} \frac{1}{q^2(1 - q^{-1/h} + q^{-1})}.
\end{aligned}$$

We bound the first sum above using Lemma 2.2.3. For the second sum, using Lemma 2.2.5 and (2.8), we have

$$\begin{aligned}
\sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{1}{pq^2(1 - q^{-1/h} + q^{-1})} &= \sum_{p \leq x^{1/h}/2} \frac{1}{p} \sum_{q \leq x^{1/h}/p} \frac{1}{q^2(1 - q^{-1/h} + q^{-1})} \\
&= \sum_{p \leq x^{1/h}/2} \frac{1}{p} \left(\mathcal{L}_h(2h) + O_h \left(\frac{p}{x^{1/h} \log(x^{1/h}/p)} \right) \right) \\
&= \mathcal{L}_h(2h) (\log \log x + B_1 - \log h) + O_h \left(\frac{1}{\log x} \right).
\end{aligned}$$

Similarly, for the third sum, we obtain

$$\sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{1}{p^2(1 - p^{-1/h} + p^{-1})} \frac{1}{q^2(1 - q^{-1/h} + q^{-1})} = (\mathcal{L}_h(2h))^2 + O_h \left(\frac{\log \log x}{x^{1/h} \log x} \right).$$

Combining the last three results with Lemma 2.2.3, we obtain

$$\begin{aligned}
& \sum_{\substack{p,q \\ pq \leq x^{1/h}}} \frac{1 - p^{-1/h}}{p(1 - p^{-1/h} + p^{-1})} \frac{1 - q^{-1/h}}{q(1 - q^{-1/h} + q^{-1})} \\
&= (\log \log x - \log h)^2 + 2B_1(\log \log x - \log h) + B_1^2 - \zeta(2) \\
&\quad - 2(\mathcal{L}_h(2h) (\log \log x + B_1 - \log h)) + (\mathcal{L}_h(2h))^2 + O_h \left(\frac{\log \log x}{\log x} \right).
\end{aligned}$$

Combining the above with (4.34), (4.35), and (4.36) for $k = h$ and using Theorem 4.1.3, we obtain the second moment for $\omega_h(n)$ over h -full numbers.

Now, we focus on the case $k \geq h + 1$. Using the definition of $\mathcal{L}_h(r)$ (2.4), we obtain

$$\begin{aligned} & \sum_{\substack{p, q \\ pq \leq x^{1/k}}} \frac{1 - p^{-1/h}}{p^{k/h} (1 - p^{-1/h} + p^{-1})} \frac{1 - q^{-1/h}}{q^{k/h} (1 - q^{-1/h} + q^{-1})} \\ &= (\mathcal{L}_h(k) - \mathcal{L}_h(k+1))^2 + O_{h,k} \left(\frac{x^{\frac{1}{k} - \frac{1}{h}} \log \log x}{\log x} \right). \end{aligned}$$

Combining the above with (4.34), (4.35), and (4.36) for $k \geq h+1$ and using Theorem 4.1.3, for $k \geq h+1$, we obtain the required second moments for $\omega_k(n)$ for $k \geq h+1$ over h -full numbers. This completes the proof. $\square$

4.5 No normal orders

In this section, we prove Theorem 4.1.5 and Theorem 4.1.6. The former establishes that the functions $\omega_k(n)$ with $1 < k < h$ do not have normal order over h -free numbers, and the latter proves that $\omega_k(n)$ with $k > h$ do not have normal order over h -full numbers.

Proof of Theorem 4.1.5. We first assume that $F(n)$ is not identically 0. Then, there exists $n_0 \in \mathbb{N}$ such that $F(n_0) > 0$ for all $n \geq n_0$. For $x > 2$ and $1 < k < h$, let

$$S_{0,k}^h(x) := \{n \in \mathcal{S}_h(x) : \omega_k(n) = 0\}.$$

Note that

$$S_{0,k}^h(x) \supseteq \mathcal{S}_k(x).$$

Since $|\mathcal{S}_k(x)| \gg x/\zeta(k)$, thus $|S_{0,k}^h(x)| \gg x/\zeta(k)$. In particular, the set of $n \in \mathcal{S}_h(x)$ for which $F(n) > 0$ and $\omega_k(n) = 0$ is not $o(x)$. For all such n , notice that the inequality

$$|\omega_k(n) - F(n)| > \frac{F(n)}{2} \tag{4.37}$$

is satisfied. Thus, we deduce that $\omega_k(n)$ does not have normal order $F(n)$ when $F(n)$ is not identically zero.

Next, we work with the case when $F(n)$ is identically 0. Let

$$S_{1,k}^h(x) := \{n \in \mathcal{S}_h(x) : \omega_k(n) = 1\}.$$

Using Lemma 2.2.6 for the single prime $p = 2$, we deduce

$$\begin{aligned} |S_{1,k}^h(x)| &\geq \sum_{\substack{n \in \mathcal{S}_h(x) \\ 2^k \parallel n \\ p^k \nmid n \text{ for any } p \neq 2}} 1 \\ &= \sum_{\substack{n \in \mathcal{S}_k(x/2^k) \\ (n,2)=1}} 1 \\ &\gg \frac{2^k - 2^{k-1}}{2^k(2^k - 1)} \frac{x}{\zeta(k)}. \end{aligned}$$

Thus, the set of $n \in \mathcal{S}_h(x)$ for which $F(n) > 0$ and $\omega_k(n) = 1$ is not $o(x)$. Also, for all such n , inequality (4.37) is satisfied. Thus, $\omega_k(n)$ does not have normal order $F(n)$ when $F(n)$ is identically zero. This completes the proof. $\square$

Proof of Theorem 4.1.6. We first assume that $F(n)$ is not identically 0. Thus, there exists $n_0 \in \mathbb{N}$ such that $F(n_0) > 0$ for all $n \geq n_0$. For $x > 2$ and $k > h$, let

$$N_{0,k}^h(x) := \{n \in \mathcal{N}_h(x) : \omega_k(n) = 0\}.$$

Note that

$$N_{0,k}^h(x) \supseteq (\mathcal{N}_h \cap \mathcal{S}_k)(x).$$

Moreover, using Lemma 4.2.1 for the prime $p = 2$, we obtain

$$\begin{aligned} |(\mathcal{N}_h \cap \mathcal{S}_k)(x)| &\geq \sum_{\substack{n \leq x \\ n \in \mathcal{N}_h \cap \mathcal{S}_k, (n,2)=1}} 1 \\ &\gg \frac{1 - 2^{-1/h}}{1 - 2^{-1/h} + 2^{-1} - 2^{-k/h}} \eta_{h,k} x^{1/h}, \end{aligned}$$

where $\eta_{h,k}$ is defined in (4.13). Thus, the set of $n \in \mathcal{N}_h(x)$ for which $F(n) > 0$ and $\omega_k(n) = 0$ is not $o(x^{1/h})$. Also, for all such n , inequality (4.37) is satisfied. Thus, $\omega_k(n)$ does not have normal order $F(n)$ when $F(n)$ is not identically 0.

Next, we work with the case when $F(n)$ is identically 0. Let

$$N_{1,k}^h(x) := \{n \in \mathcal{N}_h(x) : \omega_k(n) = 1\}.$$

Using Lemma 2.4.1 for the single prime $p = 2$, we deduce

$$\begin{aligned}
|N_{1,k}^h(x)| &\geq \sum_{\substack{n \in \mathcal{N}_h(x) \\ 2^k \parallel n \\ p^k \nmid n \text{ for any } p \neq 2}} 1 \\
&= \sum_{\substack{n \leq x/2^k \\ n \in \mathcal{N}_h \cap \mathcal{S}_k, (n,2)=1}} 1 \\
&\gg \frac{1 - 2^{-1/h}}{2^{k/h}(1 - 2^{-1/h} + 2^{-1} - 2^{-k/h})} \eta_{h,k} x^{1/h}.
\end{aligned}$$

Therefore, the set of $n \in \mathcal{N}_h(x)$ for which $F(n) > 0$ and $\omega_k(n) = 1$ is not $o(x^{1/h})$. Also, for all such n , inequality (4.37) is satisfied. Thus, $\omega_k(n)$ does not have normal order $F(n)$ when $F(n)$ is identically zero. This completes the proof. $\square$

Chapter 5

Generalizations of Erdős-Kac Theorems

5.1 Introduction

Erdős and Kac [15] used probabilistic means to establish that $\omega(n)$ obeys the Gaussian distribution with mean $\log \log n$ over naturals. In particular, for any $\gamma \in \mathbb{R}$, they proved

$$\lim_{x \rightarrow \infty} \frac{1}{x} \left| \left\{ n \leq x : n \geq 3, \frac{\omega(n) - \log \log n}{\sqrt{\log \log n}} \leq \gamma \right\} \right| = \Phi(\gamma),$$

where Φ is defined in (1.17). Liu in [35] provided a subset generalization of the Erdős-Kac theorem. In this chapter, we apply her work to prove Erdős-Kac theorem analogs over h -free and h -full numbers.

In the following, we mention Liu's setup from [35], and in Theorem 5.1.1 below, we mention her subset generalization result under this setup.

Let $\mathcal{S}$ be a subset of infinitely many elements of $\mathbb{N}$. For $x > 1$, we define

$$\mathcal{S}(x) = \{n \in \mathcal{S} : n \leq x\}.$$

We assume that $\mathcal{S}$ satisfies the following condition:

$$|\mathcal{S}(x^{1/2})| = o(|\mathcal{S}(x)|), \quad \text{for all } x > 1. \tag{5.1}$$

Let $f : \mathcal{S} \rightarrow \mathbb{N}$ be the identity map, where f is defined in [35, Page 159, Line -2]. Thus, we use $\omega(f(n)) = \omega(n)$ for all $n \in \mathcal{S}$ in the setup given in [35, Section 3] to obtain the following:

For each prime p , we write

$$\frac{1}{|\mathcal{S}(x)|} |\{n \in \mathcal{S}(x) : p|n\}| = \lambda_p + e_p(x), \quad (5.2)$$

where λ_p denotes the main term that is independent of x and $e_p(x)$ is the error term. In the following, we will use e_p to abbreviate $e_p(x)$.

For any sequence of distinct primes $p_1, p_2, \dots, p_u$, we write

$$\frac{1}{|\mathcal{S}(x)|} |\{n \in \mathcal{S}(x) : p_i|n \text{ for all } i = 1, \dots, u\}| = \lambda_{p_1} \cdots \lambda_{p_u} + e_{p_1 \dots p_u}(x).$$

In the following, we will use $e_{p_1 \dots p_u}$ to abbreviate $e_{p_1 \dots p_u}(x)$.

Suppose there exists a β with $0 < \beta \leq 1$ and $y = y(x) < x^\beta$ such that the following conditions are satisfied:

- (a) $|\{p \text{ prime} : p > x^\beta, p|n\}| = O_\beta(1)$ for each $n \in \mathcal{S}(x)$.
- (b) $\sum_{y < p \leq x^\beta} \lambda_p = o((\log \log x)^{1/2})$.
- (c) $\sum_{y < p \leq x^\beta} |e_p| = o((\log \log x)^{1/2})$.
- (d) $\sum_{p \leq y} \lambda_p = \log \log x + o((\log \log x)^{1/2})$.
- (e) $\sum_{p \leq y} \lambda_p^2 = o((\log \log x)^{1/2})$.
- (f) For $r \in \mathbb{N}$, let u be any integer picked from $\{1, 2, \dots, r\}$. We have

$$\sum'' |e_{p_1 \dots p_u}| = o((\log \log x)^{-r/2}),$$

where $\sum''$ extends over all u -tuples $(p_1, p_2, \dots, p_u)$ with $p_i \leq y$ for all $i \in \{1, 2, \dots, u\}$ and all p_i 's are distinct.

Under the above conditions, Liu in [35, Theorem 3] provided the following subset generalization of the Erdős-Kac theorem:

Theorem 5.1.1. [35, Theorem 3] *Let $\mathcal{S}$ be a subset of $\mathbb{N}$. For any $x > 1$, let $\mathcal{S}(x)$ be the set of elements in $\mathcal{S}$ less than or equal to x . Assume that $\mathcal{S}$ satisfies the condition (5.1).*

Suppose there exists a β with $0 < \beta \leq 1$ and $y = y(x) < x^\beta$ such that the conditions (a) to (f) above hold. Then for $\gamma \in \mathbb{R}$, we have

$$\lim_{x \rightarrow \infty} \frac{1}{|\mathcal{S}(x)|} \left| \left\{ n \in \mathcal{S}(x) : n \geq 3, \frac{\omega(n) - \log \log n}{\sqrt{\log \log n}} \leq \gamma \right\} \right| = \Phi(\gamma),$$

where $\Phi(\gamma)$ is defined in (1.17).

Remark 5.1.1. Let $\mathcal{S} = \mathbb{N}$. Then by Theorem 5.1.1, we recover the classical Erdős-Kac theorem, as shown in (1.1).

In this chapter, we prove that $\omega(n)$ obeys the Gaussian distribution over h -free and over h -full numbers as well. We achieve this as an application of Theorem 5.1.1. In particular, we prove:

Theorem 5.1.2. Let $x > 2$ be any real number and $h \geq 2$ be any integer. Let $\mathcal{S}_h(x)$ denote the set of h -free numbers less than or equal to x . Then for $a \in \mathbb{R}$, we have

$$\lim_{x \rightarrow \infty} \frac{1}{|\mathcal{S}_h(x)|} \left| \left\{ n \in \mathcal{S}_h(x) : \frac{\omega(n) - \log \log n}{\sqrt{\log \log n}} \leq a \right\} \right| = \Phi(a).$$

Theorem 5.1.3. Let $x > 2$ be any real number and $h \geq 2$ be any integer. Let $\mathcal{N}_h(x)$ denote the set of h -full numbers less than or equal to x . Then for $a \in \mathbb{R}$, we have

$$\lim_{x \rightarrow \infty} \frac{1}{|\mathcal{N}_h(x)|} \left| \left\{ n \in \mathcal{N}_h(x) : \frac{\omega(n) - \log \log n}{\sqrt{\log \log n}} \leq a \right\} \right| = \Phi(a).$$

We extend this result to $\omega_1(n)$ over h -free numbers and $\omega_h(n)$ over h -full numbers. We prove the following two results:

Theorem 5.1.4. Let $x > 2$ be any real number and $h \geq 2$ be any integer. Let $\mathcal{S}_h(x)$ denote the set of h -free numbers less than or equal to x . Then for $a \in \mathbb{R}$, we have

$$\lim_{x \rightarrow \infty} \frac{1}{|\mathcal{S}_h(x)|} \left| \left\{ n \in \mathcal{S}_h(x) : \frac{\omega_1(n) - \log \log n}{\sqrt{\log \log n}} \leq a \right\} \right| = \Phi(a),$$

where $\Phi(a)$ is defined in (1.17).

Theorem 5.1.5. Let $x > 2$ be any real number and $h \geq 2$ be any integer. Let $\mathcal{N}_h(x)$ denote the set of h -full numbers less than or equal to x . Then for $a \in \mathbb{R}$, we have

$$\lim_{x \rightarrow \infty} \frac{1}{|\mathcal{N}_h(x)|} \left| \left\{ n \in \mathcal{N}_h(x) : \frac{\omega_h(n) - \log \log n}{\sqrt{\log \log n}} \leq a \right\} \right| = \Phi(a),$$

where $\Phi(a)$ is defined in (1.17).

5.2 The Erdős-Kac theorem over h -free and h -full numbers

In this section, we prove the Erdős-Kac theorem for $\omega(n)$ over h -free and h -full numbers. We intend to prove the case of h -free numbers given in Theorem 5.1.2 by applying Theorem 5.1.1 with $\mathcal{S} = \mathcal{S}_h$, $\beta = 1$, and $y = x^{1/\log \log x}$. We prove:

Proof of Theorem 5.1.2. Consider the set $\mathcal{S} = \mathcal{S}_h$. From (2.50), we have

$$\mathcal{S}(x) = \frac{x}{\zeta(h)} + O_h(x^{1/h}),$$

where ζ is the classical Riemann ζ -function. Thus,

$$\frac{|\mathcal{S}(x^{1/2})|}{|\mathcal{S}(x)|} \ll \frac{1}{x^{1/2}},$$

and hence the condition (5.1), $|\mathcal{S}(x^{1/2})| = o(|\mathcal{S}(x)|)$ is satisfied. For a fixed prime p , let

$$\mathcal{S}(p, x) := \{n \leq x : n \in \mathcal{S}_h, p|n\}.$$

Using Lemma 2.2.6, we obtain

$$\begin{aligned} |\mathcal{S}(p, x)| &= \sum_{k=1}^{h-1} \sum_{\substack{n \in \mathcal{S}_h(x/p^k) \\ (n, p)=1}} 1 \\ &= \frac{p^{h-1} - 1}{p^h - 1} \frac{x}{\zeta(h)} + O_h\left(\frac{x^{1/h}}{p^{1/h}}\right). \end{aligned} \tag{5.3}$$

Thus,

$$\frac{|\mathcal{S}(p, x)|}{|\mathcal{S}(x)|} = \lambda_p + e_p(x),$$

where $\lambda_p = (p^{h-1} - 1)/(p^h - 1)$ and $e_p(x) = O_h\left(1/(x^{1-\frac{1}{h}} p^{\frac{1}{h}})\right)$.

Next, we choose $y = x^{1/\log \log x}$ and $\beta = 1$, and check that all the conditions in Theorem 5.1.1 hold true. Note that the set in Condition (a) is empty and thus the condition

holds trivially. Note that, a well-known formula for the sum of the reciprocal of primes is given in (2.11) as

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + B_1 + O\left(\frac{1}{\log x}\right),$$

where B_1 is defined in (1.7). Using this, we obtain

$$\sum_{x^{\frac{1}{\log \log x}} < p \leq x} \lambda_p \ll \sum_{x^{\frac{1}{\log \log x}} < p \leq x} \frac{1}{p} \ll \log \log \log x.$$

Since $\log \log \log x = o((\log \log x)^{1/2})$, Condition (b) is satisfied as well. Using Lemma 2.2.2, we have

$$\sum_{x^{\frac{1}{\log \log x}} < p \leq x} |e_p(x)| \ll_h \frac{1}{x^{1-\frac{1}{h}}} \sum_{p \leq x} \frac{1}{p^{\frac{1}{h}}} \ll_h \frac{1}{\log x},$$

which makes Condition (c) true. Using

$$\frac{p^{h-1} - 1}{p^h - 1} = \frac{1}{p} - \frac{p - 1}{p(p^h - 1)}$$

and (2.11) again, we obtain

$$\begin{aligned} \sum_{p \leq x^{\frac{1}{\log \log x}}} \lambda_p &= \sum_{p \leq x^{\frac{1}{\log \log x}}} \frac{1}{p} + O(1) \\ &= \log \log x + O(\log \log \log x), \end{aligned}$$

which makes Condition (d) true. Finally, using $\sum_p 1/p^2 = O(1)$, we have

$$\sum_{p \leq x^{\frac{1}{\log \log x}}} \lambda_p^2 \ll \sum_{p \leq x^{\frac{1}{\log \log x}}} \frac{1}{p^2} \ll O(1).$$

This makes Condition (e) true. Thus, we are only required to verify Condition (f). Using (5.3) and the Chinese Remainder Theorem, we obtain, for distinct primes $p_1, \dots, p_u$,

$$|\{n \in \mathcal{S}(x) : p_i | n \text{ for all } i \in \{1, 2, \dots, u\}\}| = \left(\prod_{i=1}^u \frac{p_i^{h-1} - 1}{p_i^h - 1} \right) \frac{x}{\zeta(h)} + O_h \left(\frac{x^{1/h}}{\prod_{i=1}^u p_i^{1/h}} \right).$$

Thus

$$\frac{|\{n \in \mathcal{S}(x) : p_i | n \text{ for all } i \in \{1, 2, \dots, u\}\}|}{|\mathcal{S}(x)|} = \prod_{i=1}^u \frac{p_i^{h-1} - 1}{p_i^h - 1} + e_{p_1 \dots p_u}(x),$$

where

$$|e_{p_1 \dots p_u}(x)| \ll_h \frac{1}{x^{1-\frac{1}{h}}} \frac{1}{\prod_{i=1}^u p_i^{1/h}}.$$

Let $r \in \mathbb{N}$. By the definition of $\sum''$ in the conditions mentioned before Theorem 5.1.1, and using Lemma 2.2.2 and $x^{1/\log \log x} = o(x^\epsilon)$ for any small $\epsilon > 0$, we have

$$\begin{aligned} \sum'' |e_{p_1 \dots p_u}(x)| &\ll_h \frac{1}{x^{1-\frac{1}{h}}} \left(\sum_{p \leq x^{\frac{1}{\log \log x}}} \frac{1}{p^{\frac{1}{h}}} \right)^u \\ &\ll_h \frac{1}{x^{1-\frac{1}{h}}} \left(\frac{x^{\frac{1}{\log \log x} (1-\frac{1}{h})}}{\log x} \right)^u \\ &\ll_h \frac{1}{x^{1-\frac{1}{h}-\epsilon'}}, \end{aligned}$$

for any small $\epsilon' > 0$. Since, $x^{-(1-(1/h)-\epsilon')} = o((\log \log x)^{-r/2})$, thus Condition (f) holds true as well. Since all the conditions of Theorem 5.1.1 hold true with $\mathcal{S} = \mathcal{S}_h$, $\beta = 1$ and $y = x^{1/\log \log x}$, thus applying Theorem 5.1.1 completes the proof. $\square$

Next, we intend to prove the case of h -full numbers given in Theorem 5.1.3 by using Theorem 5.1.1 with $\mathcal{S} = \mathcal{N}_h$, $\beta = 1/h$, and $y = x^{1/(h \log \log x)}$. We prove:

Proof of Theorem 5.1.3. Consider the set $\mathcal{S} = \mathcal{N}_h$. From (2.52), we have

$$\mathcal{S}(x) = \gamma_{0,h} x^{1/h} + O_h(x^{1/(h+1)}),$$

where $\gamma_{0,h}$ is defined in (1.6) as

$$\gamma_{0,h} = \prod_p \left(1 + \frac{p - p^{1/h}}{p^2(p^{1/h} - 1)} \right).$$

Therefore,

$$\frac{|\mathcal{S}(x^{1/2})|}{|\mathcal{S}(x)|} \ll \frac{1}{x^{1/(2h)}},$$

and hence, $|\mathcal{S}(x^{1/2})| = o(|\mathcal{S}(x)|)$ is satisfied. For a fixed prime p , let

$$\mathcal{S}(p, x) := \{n \leq x : n \in \mathcal{N}_h, p|n\}.$$

Using Lemma 2.4.1, we obtain

$$\begin{aligned} |\mathcal{S}(p, x)| &= \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \sum_{\substack{n \in \mathcal{N}_h(x/p^k) \\ (p, n)=1}} 1 \\ &= \left(\sum_{k=h}^{\infty} \frac{1}{p^{k/h}} \right) \left(\frac{\gamma_{0,h}}{1 + \frac{p^{-1}}{1-p^{-1/h}}} \right) x^{1/h} + O_h \left(x^{1/(h+1)} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \frac{1}{p^{k/(h+1)}} \right) \\ &= \frac{\gamma_{0,h}}{p(1 - p^{-1/h} + p^{-1})} x^{1/h} + O_h \left(\frac{x^{1/(h+1)}}{p^{h/(h+1)}} \right). \end{aligned} \quad (5.4)$$

Thus,

$$\frac{|\mathcal{S}(p, x)|}{|\mathcal{S}(x)|} = \lambda_p + e_p(x),$$

where $\lambda_p = \frac{1}{p(1-p^{-1/h}+p^{-1})}$ and $e_p(x) = O_h \left(\frac{1}{x^{\frac{1}{h(h+1)}} p^{\frac{h}{h+1}}} \right)$.

Next, we choose $y = x^{\frac{1}{h \log \log x}}$, and $\beta = 1/h$, and check again that all the conditions in Theorem 5.1.1 hold true. Note that the set in Condition (a) is empty and thus the condition holds trivially. Using (2.11), we obtain

$$\sum_{x^{\frac{1}{h \log \log x}} < p \leq x^{1/h}} \lambda_p \ll \sum_{x^{\frac{1}{h \log \log x}} < p \leq x^{1/h}} \frac{1}{p} \ll \log \log \log x,$$

which makes Condition (b) true. Using Lemma 2.2.2, we have

$$\sum_{x^{\frac{1}{h \log \log x}} < p \leq x^{1/h}} |e_p(x)| \ll_h \frac{1}{x^{\frac{1}{h(h+1)}}} \sum_{p \leq x^{1/h}} \frac{1}{p^{\frac{h}{h+1}}} \ll_h \frac{1}{\log x},$$

which makes Condition (c) true. Using $\frac{1}{p(1-p^{-1/h}+p^{-1})} = \frac{1}{p} + \frac{p^{-1/h}-p^{-1}}{p(1-p^{-1/h}+p^{-1})}$ and (2.11), we obtain

$$\begin{aligned} \sum_{p \leq x^{\frac{1}{h \log \log x}}} \lambda_p &= \sum_{p \leq x^{\frac{1}{h \log \log x}}} \frac{1}{p} + O(1) \\ &= \log \log x + O(\log \log \log x), \end{aligned}$$

which makes Condition (d) true. Finally, again using $\sum_p 1/p^2 = O(1)$, we have

$$\sum_{p \leq x^{\frac{1}{h \log \log x}}} \lambda_p^2 \ll \sum_{p \leq x^{\frac{1}{h \log \log x}}} \frac{1}{p^2} \ll O(1).$$

This makes Condition (e) true. Thus, similar to the case of h -free numbers, we are only required to verify Condition (f). Using (5.4) and the Chinese Remainder Theorem, we obtain, for distinct primes $p_1, \dots, p_u$,

$$\begin{aligned} |\{n \in \mathcal{S}(x) : p_i | n \text{ for all } i \in \{1, 2, \dots, u\}\}| &= \left(\prod_{i=1}^u \frac{1}{p_i(1 - p_i^{-1/h} + p_i^{-1})} \right) \gamma_{0,h} x^{1/h} \\ &\quad + O_h \left(\frac{x^{1/(h+1)}}{\prod_{i=1}^u p_i^{h/(h+1)}} \right). \end{aligned}$$

Thus

$$\frac{|\{n \in \mathcal{S}(x) : p_i | n \text{ for all } i \in \{1, 2, \dots, u\}\}|}{|\mathcal{S}(x)|} = \prod_{i=1}^u \frac{1}{p_i(1 - p_i^{-1/h} + p_i^{-1})} + e_{p_1 \dots p_u}(x),$$

where

$$|e_{p_1 \dots p_u}(x)| \ll_h \frac{1}{x^{\frac{1}{h(h+1)}}} \frac{1}{\prod_{i=1}^u p_i^{\frac{h}{h+1}}}.$$

Let $r \in \mathbb{N}$. By the definition of $\sum''$ in the conditions mentioned before Theorem 5.1.1, and using Lemma 2.2.2 and $x^{1/(h(h+1) \log \log x)} = o(x^\epsilon)$ for any small $\epsilon > 0$, we have

$$\sum'' |e_{p_1 \dots p_u}(x)| \ll_h \frac{1}{x^{\frac{1}{h(h+1)}}} \left(\sum_{p \leq x^{\frac{1}{h \log \log x}}} \frac{1}{p^{\frac{h}{h+1}}} \right)^u \ll_h \frac{1}{x^{\frac{1}{h(h+1)} - \epsilon'}},$$

for any small $\epsilon' > 0$. Since, $x^{-(1/(h(h+1)) - \epsilon')} = o((\log \log x)^{-r/2})$, thus Condition (f) holds true as well. Since all the conditions of Theorem 5.1.1 hold true with $\mathcal{S} = \mathcal{N}_h$, $\beta = 1/h$ and $y = x^{1/(h \log \log x)}$, thus applying Theorem 5.1.1 completes the proof. $\square$

5.3 The Erdős-Kac theorems for $\omega_k(n)$

In this section, we establish the Erdős-Kac theorem for $\omega_1(n)$ over h -free numbers and for $\omega_h(n)$ over h -full numbers. To do this, we employ ideas from [12, Proof of Theorem 1.3]. We prove the following two theorems:

Proof of Theorem 5.1.4. For an arithmetic function f and a natural number $n \geq 3$, let $r_f(n)$ be the ratio

$$r_f(n) := \frac{f(n) - \log \log n}{\sqrt{\log \log n}}. \quad (5.5)$$

In this proof, we will be using f to represent ω and ω_1 when necessary. For $a \in \mathbb{R}$ and a subset $\mathcal{S}$ of natural numbers, let $\mathcal{S}(x)$ denote the set of elements of $\mathcal{S}$ up to x , and

$$D(f, \mathcal{S}, x, a) := \frac{1}{|\mathcal{S}(x)|} |\{n \in \mathcal{S}(x) : r_f(n) \leq a\}| \quad (5.6)$$

be the density function for sufficiently large x . Since $\omega_1(n) \leq \omega(n)$, thus $r_{\omega_1}(n) \leq r_{\omega}(n)$ for all $n \geq 3$. Therefore using $\mathcal{S} = \mathcal{S}_h$,

$$D(\omega, \mathcal{S}_h, x, a) \leq D(\omega_1, \mathcal{S}_h, x, a)$$

for all $x \geq 3$. Thus, by the Erdős-Kac theorem for $\omega(n)$ over h -free numbers (see Theorem 5.1.2), we have

$$\Phi(a) \leq \liminf_{x \rightarrow \infty} D(\omega_1, \mathcal{S}_h, x, a). \quad (5.7)$$

For any $\epsilon > 0$, we define the set

$$A(\mathcal{S}_h, x, \epsilon) := \left\{ n \in \mathcal{S}_h(x) : \frac{\omega(n) - \omega_1(n)}{\sqrt{\log \log n}} \leq \epsilon \right\}.$$

Let $A^c(\mathcal{S}_h, x, \epsilon)$ denote the complement of $A(\mathcal{S}_h, x, \epsilon)$ inside $\mathcal{S}_h(x)$. Note that

$$r_{\omega_1}(n) = r_{\omega}(n) + \frac{\omega_1(n) - \omega(n)}{\sqrt{\log \log n}}.$$

Thus, using the definition of $A(\mathcal{S}_h, x, \epsilon)$, we obtain

$$\begin{aligned} \{n \in \mathcal{S}_h(x) : r_{\omega_1} \leq a\} &= \left\{ n \in \mathcal{S}_h(x) : r_{\omega}(n) + \frac{\omega_1(n) - \omega(n)}{\sqrt{\log \log n}} \leq a \right\} \\ &= \left\{ n \in A(\mathcal{S}_h, x, \epsilon) : r_{\omega}(n) + \frac{\omega_1(n) - \omega(n)}{\sqrt{\log \log n}} \leq a \right\} \\ &\quad \cup \left\{ n \in A^c(\mathcal{S}_h, x, \epsilon) : r_{\omega}(n) + \frac{\omega_1(n) - \omega(n)}{\sqrt{\log \log n}} \leq a \right\} \\ &\subseteq \{n \in \mathcal{S}_h(x) : r_{\omega}(n) \leq a + \epsilon\} \cup A^c(\mathcal{S}_h, x, \epsilon). \end{aligned}$$

Then, by the definition of $D(f, \mathcal{S}_h, x, a)$, we have

$$D(\omega_1, \mathcal{S}_h, x, a) \leq D(\omega, \mathcal{S}_h, x, a + \epsilon) + \frac{|A^c(\mathcal{S}_h, x, \epsilon)|}{|\mathcal{S}_h(x)|}. \quad (5.8)$$

We intend to show that the second summand on the right-hand side above goes to 0 as $x \rightarrow \infty$. By Theorem 2.1.1 and Theorem 4.1.1, we have

$$\sum_{n \in \mathcal{S}_h(x)} (\omega(n) - \omega_1(n)) \ll \frac{x}{\zeta(h)}.$$

Additionally,

$$\begin{aligned} \sum_{n \in \mathcal{S}_h(x)} (\omega(n) - \omega_1(n)) &\geq \sum_{\substack{\frac{x}{\log x} \leq n \leq x \\ n \in A^c(\mathcal{S}_h, x, \epsilon)}} (\omega(n) - \omega_1(n)) \\ &> \epsilon \sum_{\substack{\frac{x}{\log x} \leq n \leq x \\ n \in A^c(\mathcal{S}_h, x, \epsilon)}} \sqrt{\log \log n} \\ &\geq \epsilon \sqrt{\log \log(x/\log x)} |\{n \geq x/\log x : n \in A^c(\mathcal{S}_h, x, \epsilon)\}|. \end{aligned}$$

The above two results imply

$$|\{n \geq x/\log x : n \in A^c(\mathcal{S}_h, x, \epsilon)\}| \ll \frac{1}{\epsilon \cdot \zeta(h)} \frac{x}{\sqrt{\log \log(x/\log x)}},$$

where the right-hand side is $o(x)$. Since $|\mathcal{S}_h(x/\log x)|$ is also $o(x)$ and $|\mathcal{S}_h(x)| \gg x/\zeta(h)$, we obtain

$$\lim_{x \rightarrow \infty} \frac{|A^c(\mathcal{S}_h, x, \epsilon)|}{|\mathcal{S}_h(x)|} = 0.$$

Finally, taking the limits as $x \rightarrow \infty$ on both sides of (5.8), and using the above with the Erdős-Kac theorem for $\omega(n)$ over h -free numbers, we obtain

$$\limsup_{x \rightarrow \infty} D(\omega_1, \mathcal{S}_h, x, a) \leq \Phi(a + \epsilon).$$

Since, $\epsilon > 0$ is arbitrary, combining the above with (5.7) yields

$$\lim_{x \rightarrow \infty} D(\omega_1, \mathcal{S}_h, x, a) = \Phi(a).$$

This completes the proof. □

Proof of Theorem 5.1.5. Recall the definitions for $r_f(n)$ and $D(f, \mathcal{S}, x, a)$ from (5.5) and (5.6) respectively. In this proof, we will be using $\mathcal{S} = \mathcal{N}_h$ and f to represent ω and ω_h when necessary.

Since $\omega_h(n) \leq \omega(n)$, thus $r_{\omega_h}(n) \leq r_\omega(n)$ for all $n \geq 3$. Therefore using $\mathcal{S} = \mathcal{N}_h$,

$$D(\omega, \mathcal{N}_h, x, a) \leq D(\omega_h, \mathcal{N}_h, x, a)$$

for all $x \geq 3$. Thus, by the Erdős-Kac theorem for $\omega(n)$ over h -full numbers (see Theorem 5.1.3), we have

$$\Phi(a) \leq \liminf_{x \rightarrow \infty} D(\omega_h, \mathcal{N}_h, x, a). \quad (5.9)$$

For any $\epsilon > 0$, we define the set

$$B(\mathcal{N}_h, x, \epsilon) := \left\{ n \in \mathcal{N}_h : \frac{\omega(n) - \omega_h(n)}{\sqrt{\log \log n}} \leq \epsilon \right\}.$$

Let $B^c(\mathcal{N}_h, x, \epsilon)$ denote the complement of $B(\mathcal{N}_h, x, \epsilon)$ inside $\mathcal{N}_h(x)$. Note that

$$r_{\omega_h}(n) = r_\omega(n) + \frac{\omega(n) - \omega_h(n)}{\sqrt{\log \log n}}.$$

Thus, we obtain

$$\begin{aligned} \{n \in \mathcal{N}_h(x) : r_{\omega_h} \leq a\} &= \left\{ n \in \mathcal{N}_h(x) : r_\omega(n) + \frac{\omega_h(n) - \omega(n)}{\sqrt{\log \log n}} \leq a \right\} \\ &= \left\{ n \in B(\mathcal{N}_h, x, \epsilon) : r_\omega(n) + \frac{\omega_h(n) - \omega(n)}{\sqrt{\log \log n}} \leq a \right\} \\ &\quad \cup \left\{ n \in B^c(\mathcal{N}_h, x, \epsilon) : r_\omega(n) + \frac{\omega_h(n) - \omega(n)}{\sqrt{\log \log n}} \leq a \right\} \\ &\subseteq \{n \in \mathcal{N}_h(x) : r_\omega(n) \leq a + \epsilon\} \cup B^c(\mathcal{N}_h, x, \epsilon). \end{aligned}$$

Then, by the definition of $D(f, \mathcal{N}_h, x, a)$, we have

$$D(\omega_h, \mathcal{N}_h, x, a) \leq D(\omega, \mathcal{N}_h, x, a + \epsilon) + \frac{|B^c(\mathcal{N}_h, x, \epsilon)|}{|\mathcal{N}_h(x)|}. \quad (5.10)$$

We again intend to show that the second summand on the right-hand side above goes to 0 as $x \rightarrow \infty$. By Theorem 2.1.2 and Theorem 4.1.3, we have

$$\sum_{n \in \mathcal{N}_h(x)} (\omega(n) - \omega_h(n)) \ll_h x^{1/h}.$$

Additionally,

$$\begin{aligned}
\sum_{n \in \mathcal{N}_h(x)} (\omega(n) - \omega_h(n)) &\geq \sum_{\substack{\frac{x}{\log x} \leq n \leq x \\ n \in B^c(\mathcal{N}_h, x, \epsilon)}} (\omega(n) - \omega_h(n)) \\
&> \epsilon \sum_{\substack{\frac{x}{\log x} \leq n \leq x \\ n \in B^c(\mathcal{N}_h, x, \epsilon)}} \sqrt{\log \log n} \\
&\geq \epsilon \sqrt{\log \log(x/\log x)} |\{n \geq x/\log x : n \in B^c(\mathcal{N}_h, x, \epsilon)\}|.
\end{aligned}$$

The above two results imply

$$|\{n \geq x/\log x : n \in B^c(\mathcal{N}_h, x, \epsilon)\}| \ll_h \frac{1}{\epsilon} \frac{x^{1/h}}{\sqrt{\log \log(x/\log x)}},$$

where the right-hand side is $o(x^{1/h})$. Since the size of the set $|\mathcal{N}_h(x/\log x)|$ is also $o(x^{1/h})$ and $|\mathcal{N}_h(x)| \gg \gamma_{0,h} x^{1/h}$, we obtain

$$\lim_{x \rightarrow \infty} \frac{|B^c(\mathcal{N}_h, x, \epsilon)|}{|\mathcal{N}_h(x)|} = 0.$$

Finally, taking the limits as $x \rightarrow \infty$ on both sides of (5.10), and using the above with the Erdős-Kac theorem for $\omega(n)$ over h -full numbers, we obtain

$$\limsup_{x \rightarrow \infty} D(\omega_h, \mathcal{N}_h, x, a) \leq \Phi(a + \epsilon).$$

Since, $\epsilon > 0$ is arbitrary, combining the above with (5.9) yields

$$\lim_{x \rightarrow \infty} D(\omega_h, \mathcal{N}_h, x, a) = \Phi(a).$$

This completes the proof. □

Chapter 6

Generalizations and Future Research

In this thesis, we studied the first and the second moments of $\omega(n)$, its refinements $\omega_k(n)$, and $\Omega(n)$ over h -free and over h -full numbers, establishing their respective normal orders. Such functions can be generalized to any countably generated free abelian monoid. In fact, in [6], the author of this thesis, together with Kuo and Liu, studied the generalized ω -function over such abelian monoids and established analogous results. This new approach allows us to analyze a variety of mathematical objects, such as number fields, global function fields, and geometrically irreducible projective varieties, all within one unified framework.

Let $\mathcal{P}$ be a countable set of elements with a map

$$N : \mathcal{P} \rightarrow \mathbb{N} \setminus \{1\}, \quad \wp \mapsto N(\wp).$$

We call the elements of $\mathcal{P}$ as the *prime elements*, and the map $N(\cdot)$ as the *Norm map*. Let $\mathcal{M}$ be a *free abelian monoid* generated by elements of $\mathcal{P}$. In other words, for each $\mathbf{m} \in \mathcal{M}$, we write

$$\mathbf{m} = \sum_{\wp \in \mathcal{P}} n_{\wp}(\mathbf{m}) \wp,$$

with $n_{\wp}(\mathbf{m}) \in \mathbb{N} \cup \{0\}$ and $n_{\wp}(\mathbf{m}) = 0$ for all but finitely many $\wp$. We call $n_{\wp}(\mathbf{m})$ the *multiplicity* of $\wp$ in $\mathbf{m}$. We extend the norm map N on $\mathcal{M}$ as the following:

$$N : \mathcal{M} \rightarrow \mathbb{N}$$

$$\mathbf{m} = \sum_{\wp \in \mathcal{P}} n_{\wp}(\mathbf{m}) \wp \longmapsto N(\mathbf{m}) := \prod_{\wp \in \mathcal{P}} N(\wp)^{n_{\wp}(\mathbf{m})}.$$

Thus, N can be extended to a monoid homomorphism from $(\mathcal{M}, +)$ to $(\mathbb{N}, \cdot)$. Let X be a countable subset of $\mathbb{Q}$ that contains the image $\text{Im}(N(\mathcal{M}))$ with an extra condition: if $x_1, x_2 \in X$, the fraction x_1/x_2 belongs to X , too. By [33, Theorem 2], without loss of generality, one can assume $X = \mathbb{Q}$ or $X = \{q^z : z \in \mathbb{Z}\}$ where q is a power of a natural number strictly greater than 1.

Given $\mathcal{P}$, $\mathcal{M}$, X , and for sufficiently large $x \in X$, we assume that the following condition hold:

$$\mathcal{M}(x) := \sum_{\substack{\mathfrak{m} \in \mathcal{M} \\ N(\mathfrak{m}) \leq x}} 1 = \kappa x + O(x^\theta), \quad \text{for some } \kappa > 0 \text{ and } 0 \leq \theta < 1. \quad (\star)$$

6.1 Normal order over abelian monoids

For each $\mathfrak{m} \in \mathcal{M}$, we define

$$\omega(\mathfrak{m}) = \sum_{\substack{\wp \in \mathcal{P} \\ n_\wp(\mathfrak{m}) \geq 1}} 1,$$

the number of elements of $\mathcal{P}$ that generates $\mathfrak{m}$, counted without multiplicity.

For a non-zero element $\mathfrak{m} \in \mathcal{M}$, let the prime element factorization of $\mathfrak{m}$ be given as

$$\mathfrak{m} = s_1 \wp_1 + \cdots + s_r \wp_r,$$

where $\wp_i$'s are its distinct prime elements and s_i 's are their respective non-zero multiplicities. Here, $\omega(\mathfrak{m}) = r$ and

$$N(\mathfrak{m}) = N(\wp_1)^{s_1} \cdots N(\wp_r)^{s_r}.$$

Let $h \geq 2$ be an integer. We say $\mathfrak{m}$ is an h -free element if $s_i \leq h-1$ for all $i \in \{1, \dots, r\}$, and we say $\mathfrak{m}$ is an h -full element if $s_i \geq h$ for all $i \in \{1, \dots, r\}$. Let $\mathcal{S}_h$ denote the set of h -free elements and $\mathcal{N}_h$ denote the set of h -full elements.

Next, we introduce the generalized ζ -function which is an analog of the classical Riemann ζ -function as the following:

$$\zeta_{\mathcal{M}}(s) := \sum_{\mathfrak{m}} \frac{1}{(N(\mathfrak{m}))^s} = \prod_{\wp} \left(1 - N(\wp)^{-s}\right)^{-1} \text{ for } \Re(s) > 1,$$

where $\mathfrak{m}$ and $\wp$ respectively range through the non-zero elements in $\mathcal{M}$ and the prime elements in $\mathcal{P}$.

Let $x \in X$. Let $\mathfrak{A}$ and $\mathfrak{B}$ be constants defined as

$$\mathfrak{A} := \lim_{x \rightarrow \infty} \left(\sum_{\substack{\wp \in \mathcal{P} \\ N(\wp) \leq x}} \frac{1}{N(\wp)} - \log \log x \right), \quad (6.1)$$

and

$$\mathfrak{B} := \begin{cases} -\pi^2/6 & \text{if } X = \mathbb{Q} \text{ and} \\ (\log \log q)^2 - \pi^2/6 & \text{if } X = \{q^z : z \in \mathbb{Z}\}. \end{cases} \quad (6.2)$$

The existence of the constant $\mathfrak{A}$ is explained in [34, Lemma 2]. We define the constants

$$\mathfrak{C}_1 := \mathfrak{A} - \sum_{\wp} \frac{N(\wp) - 1}{N(\wp)(N(\wp)^h - 1)}, \quad (6.3)$$

and

$$\mathfrak{C}_2 := \mathfrak{C}_1^2 + \mathfrak{C}_1 + \mathfrak{B} - \sum_{\wp} \left(\frac{N(\wp)^{h-1} - 1}{N(\wp)^h - 1} \right)^2. \quad (6.4)$$

For the distribution of $\omega(\mathfrak{m})$ over h -free elements, we proved:

Theorem 6.1.1. [6, Theorem 1.2] *Let $\mathcal{P}, \mathcal{M}$, and X satisfy the Condition $(\star)$. Let $x \in X$ and $h \geq 2$ be an integer. Let $\mathcal{S}_h(x)$ be the set of h -free elements with norm $N(\cdot)$ less than or equal to x . Then, we have*

$$\sum_{\mathfrak{m} \in \mathcal{S}_h(x)} \omega(\mathfrak{m}) = \frac{\kappa}{\zeta_{\mathcal{M}}(h)} x \log \log x + \frac{\kappa \mathfrak{C}_1}{\zeta_{\mathcal{M}}(h)} x + O_h \left(\frac{x}{\log x} \right),$$

and

$$\sum_{\mathfrak{m} \in \mathcal{S}_h(x)} \omega^2(\mathfrak{m}) = \frac{\kappa}{\zeta_{\mathcal{M}}(h)} x (\log \log x)^2 + \frac{\kappa(2\mathfrak{C}_1 + 1)}{\zeta_{\mathcal{M}}(h)} x \log \log x + \frac{\kappa \mathfrak{C}_2}{\zeta_{\mathcal{M}}(h)} x + O_h \left(\frac{x \log \log x}{\log x} \right).$$

Next, we consider the case of h -full elements. Let $\mathfrak{L}_h(r)$ be the convergent sum defined for $r > h \geq 2$ as

$$\mathfrak{L}_h(r) := \sum_{\wp} \frac{1}{N(\wp)^{(r/h)-1} (N(\wp) - N(\wp)^{1-1/h} + 1)}. \quad (6.5)$$

We also define two new constants

$$\mathfrak{D}_1 := \mathfrak{A} - \log h + \mathfrak{L}_h(h+1) - \mathfrak{L}_h(2h), \quad (6.6)$$

and

$$\mathfrak{D}_2 := \mathfrak{D}_1^2 + \mathfrak{D}_1 + \mathfrak{B} - \sum_{\wp} \left(\frac{1}{N(\wp) - N(\wp)^{1-1/h} + 1} \right)^2. \quad (6.7)$$

Let

$$\gamma_{h,\mathcal{M}} := \prod_{\wp} \left(1 + \frac{N(\wp) - N(\wp)^{1/h}}{N(\wp)^2 (N(\wp)^{1/h} - 1)} \right). \quad (6.8)$$

For the distributions of $\omega(\mathfrak{m})$ over h -full elements, we proved:

Theorem 6.1.2. [6, Theorem 1.3] *Let $\mathcal{P}$, $\mathcal{M}$, and X satisfy the Condition $(\star)$. Let $x \in X$ and $h \geq 2$ be an integer. Let $\mathcal{N}_h(x)$ be the set of h -full elements with norm $N(\cdot)$ less than or equal to x . Let $\gamma_{h,\mathcal{M}}$ be defined in (6.8). Then, we have*

$$\sum_{\mathfrak{m} \in \mathcal{N}_h(x)} \omega(\mathfrak{m}) = \kappa \gamma_{h,\mathcal{M}} x^{1/h} \log \log x + \kappa \gamma_{h,\mathcal{M}} \mathfrak{D}_1 x^{1/h} + O_h \left(\frac{x^{1/h}}{\log x} \right),$$

and

$$\begin{aligned} \sum_{\mathfrak{m} \in \mathcal{N}_h(x)} \omega^2(\mathfrak{m}) &= \kappa \gamma_{h,\mathcal{M}} x^{1/h} (\log \log x)^2 + \kappa \gamma_{h,\mathcal{M}} (2\mathfrak{D}_1 + 1) x^{1/h} \log \log x + \kappa \gamma_{h,\mathcal{M}} \mathfrak{D}_2 x^{1/h} \\ &\quad + O_h \left(\frac{x^{1/h} \log \log x}{\log x} \right). \end{aligned}$$

Using the method of variance demonstrated in this thesis, one can show that $\omega(\mathfrak{m})$ has normal order $\log \log N(\mathfrak{m})$ over h -free and over h -full elements.

6.1.1 Application to set of ideals in number fields

Let $K/\mathbb{Q}$ be a number field of degree $n_K = [K : \mathbb{Q}]$ and $\mathcal{O}_K$ be its ring of integers. Let $\mathcal{P}$ be the set of prime ideals of $\mathcal{O}_K$ and $\mathcal{M}$ be the set of ideals of $\mathcal{O}_K$. Let the norm map be $N : \mathcal{M} \rightarrow \mathbb{N}$ be the standard norm map, i.e., $\mathfrak{m} \mapsto N(\mathfrak{m}) := |\mathcal{O}_K/\mathfrak{m}|$. Let $X = \mathbb{Q}$.

Let κ_K be given by

$$\kappa_K = \frac{2^{r_1}(2\pi)^{r_2}hR}{\nu\sqrt{|d_K|}},$$

with

- r_1 = the number of real embeddings of K ,
- $2r_2$ = the number of complex embeddings of K ,
- h = the class number,
- R = the regulator,
- ν = the number of roots of unity,
- d_K = the discriminant of K .

Landau in [31, Satz 210] proved that

$$\sum_{\substack{\mathfrak{m} \in \mathcal{M} \\ N(\mathfrak{m}) \leq x}} 1 = \kappa_K x + O\left(x^{1-\frac{2}{n_K+1}}\right),$$

which satisfies Condition $(\star)$ with $\kappa = \kappa_K$ and $\theta = 1 - \frac{2}{n_K+1}$. Thus, Theorem 2.1.1 and Theorem 2.1.2 give the distribution of $\omega(\mathfrak{m})$ over h -free and h -full ideals respectively.

Remark 6.1.1. *For the case $K = \mathbb{Q}$, we have $n_K = 1$ and $\kappa_K = 1$. Thus, we can obtain the distribution of $\omega(n)$ over h -free and h -full numbers as shown in Theorem 2.1.1 and Theorem 2.1.2 respectively.*

6.1.2 Application to set of effective divisors in global function fields

Let q be a prime power and $\mathbb{F}_q$ be the finite field with q elements. Let $K/\mathbb{F}_q$ be a global function field. Let G_K be its genus and C_K be its class number. A prime $\wp$ in K is a discrete valuation ring R with maximal ideal P such that $P \subset R$ and the quotient field of R is K . The degree of $\wp$, denoted as $\deg \wp$, is defined as the dimension of R/P over $\mathbb{F}_q$, which is finite. Let $\mathcal{P}$ be the set of all primes in K . Let $\mathcal{M}$ be the free abelian monoid generated by $\mathcal{P}$. For each $\mathfrak{m} \in \mathcal{M}$, we write $\mathfrak{m} = \sum_{\wp \in \mathcal{P}} n_{\wp}(\mathfrak{m})\wp$, with $n_{\wp}(\mathfrak{m}) \in \mathbb{N} \cup \{0\}$ and

$n_\varphi(\mathfrak{m}) = 0$ for all but finitely many φ . We call elements of $\mathcal{M}$ as effective divisors. For an element $\mathfrak{m} \in \mathcal{M}$, we define the degree of $\mathfrak{m}$ as

$$\deg \mathfrak{m} = \sum_{\varphi \in \mathcal{P}} n_\varphi(\mathfrak{m}) \deg \varphi.$$

By [43, Lemma 5.5], for any integer $n \geq 0$, there are finitely many effective divisors of degree n . This proves that $\mathcal{P}$ is a countable set that satisfies the hypothesis of our main theorems. Let the norm map $N : \mathcal{M} \rightarrow \mathbb{N}$ be the q -power map defined as $\mathfrak{m} \mapsto N(\mathfrak{m}) := q^{\deg \mathfrak{m}}$. Let $X = \{q^z : z \in \mathbb{Z}\}$.

By [43, Lemma 5.8 & Corollary 4 to Theorem 5.4], for a non-negative integer n satisfying $n > 2G_K - 2$, the number of effective divisors of degree n is

$$C_K \frac{q^{n-G_K+1} - 1}{q - 1}.$$

Thus, for sufficiently large n , we obtain

$$\sum_{\substack{\mathfrak{m} \in \mathcal{M} \\ N(\mathfrak{m}) \leq q^n}} 1 = \frac{C_K}{q^{G_K}} \left(\frac{q}{q-1} \right)^2 q^n + O(n).$$

This satisfies Condition $(\star)$ with $\kappa = \frac{C_K}{q^{G_K}} \left(\frac{q}{q-1} \right)^2$ and $\theta = \epsilon$ for any $\epsilon \in (0, 1)$. Thus, Theorem 6.1.1 and Theorem 6.1.2 give the distribution of $\omega(\mathfrak{m})$ over h -free and h -full effective divisors in a global function field.

Remark 6.1.2. *For the special case when $K = \mathbb{F}_q(x)$, whose genus and class number are 0 and 1 respectively, we can consider the abelian monoid $\mathcal{M} \subseteq \mathbb{F}_q[x]$ that is the ring of monic polynomials in one variable over $\mathbb{F}_q$. The prime elements of $\mathcal{M}$ are the monic irreducible polynomials in $\mathcal{M}$. The localizations of $\mathcal{M}$ at these prime elements exhaust the set of all primes of K except one, the prime at infinity. Using the fact that there are q^n monic polynomials of degree n , we obtain*

$$\sum_{\substack{N(\mathfrak{m}) \leq q^n \\ \mathfrak{m} \in \mathcal{M}}} 1 = \frac{q}{q-1} q^n + O(1).$$

This satisfies Condition $(\star)$ with $\kappa = q/(q-1)$ and $\theta = 0$. Thus, the distribution of $\omega(\mathfrak{m})$ over h -free and h -full polynomials over finite fields can be deduced from Theorem 2.1.1 and Theorem 2.1.2. Such a result will be equivalent to the ones studied by Lalín and Zhang [29, Theorems 4.1 & 6.1]. Note that, in this special case, $\kappa = q/(q-1)$ instead of $(q/(q-1))^2$ to account for the lack of the prime at infinity analog in its construction.

6.1.3 Application to the set of effective 0-cycles in geometrically irreducible projective varieties of dimension r

In this section, we adopt notation from [34, Example 4 of Section 4].

Let q be a prime power and $\mathbb{F}_q$ be the finite field with q elements. Let r be a positive integer. Let $V/\mathbb{F}_q$ be a geometrically irreducible projective variety of dimension r . Let $\mathcal{P}$ be the set of closed points of $V/\mathbb{F}_q$, which is in bijection with the set of orbits of $V/\mathbb{F}_q$ under the action of $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q)$ (see [36, Proposition 6.9]). For each $\wp \in \mathcal{P}$, we define the degree of $\wp$, $\deg \wp$, to be the length of the corresponding orbit. Let $\mathcal{M}$ be the free abelian monoid generated by $\mathcal{P}$. We call elements of $\mathcal{M}$ as effective 0-cycles. For $\mathbf{m} \in \mathcal{M}$, we have $\mathbf{m} = \sum_{\wp \in \mathcal{P}} n_{\wp}(\mathbf{m}) \wp$ with $n_{\wp}(\mathbf{m}) \in \mathbb{N} \cup \{0\}$ and $n_{\wp}(\mathbf{m}) = 0$ for all but finitely many $\wp$. We define the degree of $\mathbf{m}$ as

$$\deg \mathbf{m} = \sum_{\wp \in \mathcal{P}} n_{\wp}(\mathbf{m}) \deg \wp.$$

By [36, Lemma 3.11], we deduce that $\mathcal{P}$ is countable and satisfies the hypothesis of our main theorems. Let the norm map $N : \mathcal{M} \rightarrow \mathbb{N}$ be the q^r -power map defined as $\mathbf{m} \mapsto N(\mathbf{m}) := q^{r \deg \mathbf{m}}$. Let $X = \{q^{rz} : z \in \mathbb{Z}\}$. In [34, Remark 1 of Section 4], the third author proved that

$$\sum_{\substack{\mathbf{m} \in \mathcal{M} \\ N(\mathbf{m}) \leq q^{rn}}} 1 = \kappa' \left(\frac{q^r}{q^r - 1} \right) q^{rn} + O_r(n \cdot q^{(r-1)n}),$$

where κ' is some positive constant defined explicitly in [34, Lemma 7 of Section 4]. This satisfies Condition $(\star)$ with $\kappa = \kappa' \left(\frac{q^r}{q^r - 1} \right)$ and $\theta = \epsilon$ for any $\epsilon \in (1 - 1/r, 1)$. Thus, Theorem 6.1.1 and Theorem 6.1.2 give the distribution of $\omega(\mathbf{m})$ over h -free and h -full effective 0-cycles in a geometrically irreducible projective variety of dimension r .

6.2 Study of refinements over function fields

In [5], the author of this thesis together with Elma, Kuo, and Liu studied the ω_k -function for any integer $k \geq 1$ in the context of function fields and established normal order results analogous to this thesis. More precisely:

Let $\mathbb{F}_q[x]$ be the ring of polynomials in the variable x with coefficients from the finite field $\mathbb{F}_q$ with q elements. Let $\mathcal{M}$ be the set of monic polynomials in $\mathbb{F}_q[x]$ with degree $\deg(f) \geq 1$ and $\mathcal{P}$ be the set of monic irreducible polynomials in $\mathbb{F}_q[x]$.

For a monic polynomial $f \in \mathcal{M}$, let $\omega_k(f)$ count the number of distinct monic irreducible factors $\ell \in \mathcal{P}$ of f with a given multiplicity $k \geq 1$. In [5], we proved the first and the second moment of $\omega_k(f)$ over $\mathcal{M}$, establishing that $\omega_1(f)$ has normal order $\log \deg(f)$ over $\mathcal{M}$.

6.3 Follow-up research

The research in this thesis raises new questions and research ideas about arithmetic functions, some of which are listed in this section.

- (a) Based on the research in Section 6.2, we plan to construct the generalized ω_k -function for any integer $k \geq 1$ over a countably generated abelian monoid. Our goal is to analyze the distribution of these functions across the monoid and its subsets, ultimately establishing their normal order results. We plan to investigate these ideas further in a follow-up project.
- (b) In this thesis, we proved the Erdős-Kac theorem analogs over h -free and over h -full numbers, and proved an analog for $\omega_1(n)$ over h -free numbers and $\omega_h(n)$ over h -full numbers.

In [7], we proved an analog of the Erdős-Kac theorem over the subsets of number fields, including the sets of h -free and the set of h -full ideals.

In [5], we proved another analog of this theorem for the ω_1 -function over function fields.

This demonstrates the potential of further generalizations to a general abelian monoid. We plan to extend the study of Erdős-Kac theorem for the ω -function and ω_k -functions to general abelian monoids and their subsets, with a particular focus on subsets of h -free and h -full elements.

- (c) In this thesis, we investigate the distribution of three types of additive arithmetic functions: the ω -function, which is strongly additive; the Ω -function, which is completely additive; and the ω_k -functions, which are neither. As a natural extension of this work, we aim to explore the distribution of general additive functions over the natural numbers and its various subsets, with a particular focus on identifying conditions under which an additive function satisfies an Erdős-Kac type theorem or exhibits other limiting distributions. This direction has been pursued in earlier work by Elboim and Gorodetsky [11], as well as by Tenenbaum [47, 48]. In a recent article [4], the author of this thesis studied the limiting distribution of ω_k -functions and other arithmetic functions.

- (d) Building on our analysis of the ω_k -functions and the generalized additive functions studied by Elma and Martin [13], we can extend our investigation to these functions over an abelian monoid and its subsets. In collaboration with Kuo and Liu, the author of this thesis has been examining the limiting behavior of such functions in this broader setting. We plan to develop these ideas further in a future project.

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