

Solitons with continuous symmetries

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In this thesis, we develop a framework for classifying symmetric points on moduli spaces using representation theory. We utilize this framework in a few case studies, but it has applications well beyond these cases.

As a demonstration of the power of this framework, we use it to study various symmetric solitons: instantons as well as hyperbolic, singular, and Euclidean monopoles. Examples of these objects are hard to come by due to non-linear constraints. However, by applying this framework, we introduce a linear constraint, whose solution greatly simplifies the non-linear constraints. This simplification not only allows us to easily find a plethora of novel examples of these solitons, it also provides a framework for classifying such symmetric objects. As an example, by applying this method, we prove that the basic instanton is essentially the only instanton with two particular kinds of conformal symmetry.

Additionally, we study the symmetry breaking of monopoles, a part of their topological classification. We prove a straightforward method for determining the symmetry breaking of a monopole and explicitly identify the symmetry breaking for all Lie groups with classical, simple Lie algebras. We also identify methods for doing the same for the exceptional simple Lie groups.

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Dedication

To my wife, who inspires me to reach new heights.

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“But why are we attracted to symmetry?

Why do we human beings delight in seeing perfectly round planets through the lens of a telescope and six-sided snowflakes on a cold winter day?

The answer must be partly psychological.

I would claim that symmetry represents order, and we crave order in this strange universe we find ourselves in.

The search for symmetry, and the emotional pleasure we derive when we find it, must help us make sense of the the seasons and the reliability of friendships.

Symmetry is also economy.

Symmetry is simplicity.

Symmetry is elegance.”

— Alan Lightman, *The Accidental Universe: The World You Thought You Knew*

Chapter 1

Introduction

In this thesis, we examine topological solitons with continuous symmetries. This is done by carefully applying a main theorem, Theorem 1.1.1, which, for any continuous symmetry, provides a linear equation that determines if an object possesses that symmetry. This result allows us to investigate invariant objects, whose equivalence class in the moduli space is fixed under the action of a symmetry group. Not only does this result allow us to easily compute examples of these objects, it also provides a framework for classifying them. The applications of this theorem are broad and we spend the majority of the thesis applying it to a variety of topological solitons—instantons and Euclidean, hyperbolic, and singular monopoles—using the ADHM and Nahm transforms. We also investigate the symmetry breaking of monopoles, part of their topological classification, as the main result allows us to compute novel examples of monopoles, as well as instantons, with higher rank structure groups.

Due to the length of the introduction, it has been split into various parts for ease of reading. As such, a layout of the remainder of the introduction is provided below. In Section 1.1, we introduce the main theorem of the thesis, Theorem 1.1.1. As this theorem itself is quite general, this section is written as an introduction for general audiences. Experts can focus on the statement of Theorem 1.1.1 and then skip to the rest of the chapter as they see fit. The current work fits in the vast tapestry of works in the literature examining solitons with various symmetries from various angles. To see how this work fits in the literature, read Sections 1.2, 1.3, and 1.4. In Section 1.2, we examine the literature regarding hyperbolic monopoles. In Section 1.3, we examine the literature regarding the symmetry breaking of monopoles. In Section 1.4, we examine the literature regarding instantons. In Section 1.5, we discuss the layout of the remainder of the thesis as well as the main results herein.

1.1 General background and main theorem

A complete solution to a given equation is the ultimate goal of a mathematician. Unfortunately, finding such a general solution is rarely attainable. Indeed, only the simplest partial differential equations have been completely solved, and entire courses are spent learning about these methods.

Given this unfortunate reality, as a whole, mathematics is about simplifying a problem, solving it, and seeing what that solution tells us. Often, when approached with an equation, the first tool in the mathematician's toolkit is to examine the problem assuming some symmetry. For instance, when solving a partial differential equation, symmetry is used to reduce the problem to an ordinary differential equation, where we at least have a chance at solving the problem. This idea is the heart of this thesis. We are tasked with finding solutions to certain geometrically motivated partial differential equations. As is often the case, we are unable to find the general solution. However, through the use of symmetry, we are able to create a whole zoo of solutions, where previously, solutions were relatively few and far between. While this in and of itself has its merits—indeed, we can use these solutions to approximate others—throughout this thesis, we will develop a framework for classifying all of these symmetric solutions.

Symmetries are synonymous with groups. Others have also examined the same objects that we study in the subsequent chapters with a variety of symmetries. However, they either study finite groups, abelian groups, or they find isolated examples. In this text, we study continuous symmetries: connected Lie groups. Note that our groups are also typically compact and whenever they are not, we are able to bypass this issue. The reason for examining this kind of symmetry is two-fold. Firstly, Lie groups are smooth and we can use this smoothness to differentiate equations, obtaining linear equations from non-linear ones. Doing so will greatly simplify our search for solutions. In fact, as mentioned above, these linear equations provide a framework for classifying these symmetric solutions. Secondly, connected, compact Lie groups correspond directly with their Lie algebras. That is, the exponential map is surjective onto a connected, compact Lie group. This fact allows us to reduce our problem to the realm of linear algebra, specifically representation theory.

Broadly speaking, gauge theory involves two ingredients: objects (solutions to some equation) and a gauge action (a group action preserving solutions to the equation). The underlying theme is that we only care about the object up to the gauge action. Denoting the set of objects by $\mathcal{X}$ and the gauge group $\mathcal{G}$, in gauge theory we only care about the moduli space $\mathcal{X}/\mathcal{G}$. The classical example of gauge theory is electrodynamics. Here, the objects are one-forms representing electromagnetic potentials and the gauge group is the

group of smooth maps $g: \mathbb{R}^4 \rightarrow \mathrm{U}(1)$ acting on the one-forms via $g.A := gAg^{-1} - (dg)g^{-1}$. In this case, we are interested in the curvature form $F_A = dA + A \wedge A$, as it encodes the electric and magnetic fields of the system. We see that $F_{g.A} = F_A$, so these fields are unaffected by the action of the gauge group (gauge transformations). As we can only measure the electric and magnetic fields, not the one-forms, we only care about the one-forms up to the gauge action. As $\mathrm{U}(1)$ is abelian, electrodynamics is an abelian gauge theory. In this work, we are interested in non-abelian gauge theories.

Gauge theory has been extremely important to the development of modern physics: the Standard Model is a non-abelian gauge theory. Additionally, gauge theory has generated many mathematical discoveries. For instance, gauge theory led to the construction of Donaldson invariants for four manifolds and novel examples of hyperkähler manifolds through examining these moduli spaces.

Returning briefly to our discussion on symmetry, suppose that we are trying to find examples of some gauge theoretic object $\mathcal{X}$ with gauge group $\mathcal{G}$. We wish to impose a group of symmetries $\mathcal{S}$ on our objects to help simplify our search. Forgetting the gauge group, the naive method involves enforcing symmetry on $\mathcal{X}$ directly. Said another way, $\mathcal{S}$ acts on $\mathcal{X}$ and we look for fixed points of this action. For the objects we study, this method provides limited results, often nothing but the most basic examples. As we only care about elements in the moduli space $\mathcal{X}/\mathcal{G}$, we want the symmetry action to commute with the gauge action, as it then descends to an action on the moduli space. In particular, this means that objects in the same equivalence class have the same symmetries.

We then search for fixed points of the symmetry action on the moduli space. An object A in $\mathcal{X}$ whose equivalence class $[A]$ in $\mathcal{X}/\mathcal{G}$ is fixed by $\mathcal{S}$ is considered symmetric as acting by $\mathcal{S}$ produces a new object that only differs from A by a gauge transformation. That is, as far as we are concerned, the object is unaffected by $\mathcal{S}$.

Studying fixed points on the moduli space means two things. Firstly, we have access to many more examples of these objects. Secondly, finding all of the symmetric objects is much harder. Indeed, we are not just looking at objects $A \in \mathcal{X}$ such that for all $s \in \mathcal{S}$, $s.A = A$. Instead, we are looking at objects $A \in \mathcal{X}$ such that for all $s \in \mathcal{S}$, there exists some $g \in \mathcal{G}$ such that $s.A = g.A$. That is, objects where the symmetry action is undone by the gauge action, introducing many new variables to the problem.

Let $\mathcal{X}$ be a smooth manifold and $\mathcal{G}$ a Lie group acting smoothly on $\mathcal{X}$. Due to the smoothness of Lie group actions, we obtain Lie algebra actions. If we have a right (left) group action, we obtain a Lie algebra (anti-)homomorphism taking value in $\mathfrak{X}(\mathcal{X})$. Given $y \in \mathrm{Lie}(\mathcal{G})$ and $A \in \mathcal{X}$, the Lie algebra action is given by $y.A := \left. \frac{d}{dt} \right|_{t=0} \exp(ty).A \in T_A\mathcal{X}$.

The main results of this thesis are based on the following theorem, which tells us

that if we satisfy the proper technical conditions, then we can differentiate the equations of symmetry. Thus, we can use linear algebra to find every example of our symmetric objects. As compact Lie groups can be realized as matrix Lie groups, the Lie algebra homomorphisms give us representations of Lie algebras, so we reduce the problem to the realm of representation theory.

Theorem 1.1.1. *Let $\mathcal{X}$ be a smooth manifold, $\mathcal{G}$ a compact Lie group, and $\mathcal{S}$ a compact, connected Lie group. Suppose that $\mathcal{G}$ and $\mathcal{S}$ act smoothly on $\mathcal{X}$ on the left and the two actions commute. We have that $[A] \in \mathcal{X}/\mathcal{G}$ is fixed by $\mathcal{S}$ if and only if there is some Lie algebra homomorphism $\rho: \text{Lie}(\mathcal{S}) \rightarrow \text{Lie}(\mathcal{G})$ such that, for all $x \in \text{Lie}(\mathcal{S})$,*

$$x.A + \rho(x).A = 0. \quad (1.1)$$

See Appendix A for the proof of this theorem. Also, note that $x.A$ and $\rho(x).A$ both belong to $T_A\mathcal{X}$ and thus can be added together to obtain zero. Additionally, if $A \in \mathcal{X}$ satisfies (1.1) with homomorphism ρ , then for any $g \in \mathcal{G}$, $g.A$ satisfies (1.1) with ρ replaced by $\rho \circ \text{Ad}_{g^{-1}}$. Moreover, note that in this theorem, we assume that we are dealing with left group actions. Similar results hold if we deal with right group actions or some mixture of the two.

Throughout this thesis, this result is applied to a variety of situations, although its applications extend well beyond those presented here. We will see that we are able to apply this result to situations where compactness is not originally assumed. In particular, in Chapter 3, we apply this result to a situation where, originally, we have a non-compact symmetry group. This application is possible after realizing that, up to some equivalence of symmetries and some results regarding the possible symmetries that a specific object can actually possess, we can reduce a non-compact symmetry group to its maximal compact subgroup. Also, in Chapter 3, we show that thoughtful handling can allow us to bypass issues arising from non-compact gauge groups. Additionally, in Theorem 3.2.1, we prove a similar result in the case where $\mathcal{S} = \mathbb{R}$. Moreover, through careful treatment, in Section 2.1, we also apply this result to situations where it appears that the symmetry and gauge actions do not commute.

We apply the above result to find symmetric examples of several topological solitons: instantons as well as hyperbolic, Euclidean, and singular monopoles. Solitons are particle-like solutions to a differential equation. The first description of a soliton was in 1834 by John Scott Russell when he observed a solitary wave. Solutions of the KdV equation, which models solitary waves, are called solitons, the “on” suffix indicating their particle-like nature. The soliton is a localized solution which does not decay with time and is unchanged, except for a phase translation, after interacting with another soliton.

In the 1960s and 1970s, physicists and mathematicians began studying other classical field equations, interpreting some solutions as classical particles for the theory. Particles arising through this method differ from elementary particles arising from quantizing the theory. These particles are called topological solitons, as they are distinct from the vacuum at the topological level. This helps distinguish these particles from elementary particles. Indeed, if quantum excitations about the vacuum are continuous deformations of the field, then such excitations are indistinguishable from the vacuum at the topological level. Thus, elementary particles like photons have no topological structure. Topological solitons differ in that they cannot be continuously deformed to the vacuum. This property is important, as many topological solitons possess much more energy than elementary particles; the fact that these particles are topologically distinct from the vacuum means that they do not decay into some combination of elementary particles.

Topological solitons provide us with examples of gauge theories. These objects are solutions to the Euler–Lagrange equations for a given Lagrangian. Their gauge group is the group of symmetries that preserve this Lagrangian. This approach has yielded important physical models. For instance, the skyrmion is a topological soliton arising from the Skyrme model, which approximates the nucleon. Magnetic monopoles, or just monopoles, are topological solitons that model hypothetical particles providing a source of magnetism. The equations satisfied by these solitons arise as a dimensional reduction of the Yang–Mills equations, solutions to which are called instantons.

There are few known examples of such topological solitons. As such, we would like to apply our method to find examples of and provide a framework for classifying symmetric topological solitons. However, we encounter a problem: given a kind of topological soliton, the space of all such solitons and the corresponding gauge group are infinite-dimensional. Theorem 1.1.1 relies on finite dimensional smooth manifolds and compactness. Thankfully, there is something that we can use to help us: the Nahm transform, a generalization of the ADHM transform. In Chapters 2 and 3, we introduce both these transforms and the objects they relate in much more detail. For now, we omit technical details in favour of clarity.

The ADHM transform relates instantons (solutions to the Yang–Mills equations on $\mathbb{R}^4$ with finite action) with matrices satisfying certain non-linear constraints. The Nahm transform is a generalization of this result, relating solutions to these equations, often with infinite action, symmetric under some translation subgroup $\Lambda \subseteq \mathbb{R}^4$ to solutions on $(\mathbb{R}^4)^*$ symmetric under Λ^* , the group of linear functionals $f \in (\mathbb{R}^4)^*$ such that $f(\Lambda) \subseteq \mathbb{Z}$. This transform is often called a non-linear Fourier Transform. The ADHM transform is a special case of this transform, where $\Lambda = \{0\}$, relating instantons on $\mathbb{R}^4$ with solutions to the Yang–Mills equations on $(\mathbb{R}^4)^*$ symmetric under the whole of $(\mathbb{R}^4)^*$, hence the reduction

to matrices satisfying non-linear constraints. Monopoles are solutions to the Bogomolny equation on $\mathbb{R}^3$, which is a dimensional reduction of the Yang–Mills equations along one axis. The Nahm transform relates solutions to the Yang–Mills equations on $\mathbb{R}^4$ symmetric under $\mathbb{R}$, monopoles, to solutions on $\mathbb{R}^4$ symmetric under $\mathbb{R}^3$. The latter solutions are called Nahm data and are functions of one variable that satisfy the Nahm equations. Higgs bundles are solutions to the Yang–Mills equations symmetric under $\Lambda = \mathbb{R}^2$ and the Nahm transform takes Higgs bundles to Higgs bundles. There are other cases for the Nahm transform: when $\Lambda = \mathbb{Z}$, the Nahm transform relates calorons to a version of Nahm data on a circle. In general, $\Lambda \simeq \mathbb{Z}^d \times \mathbb{R}^m$, with $d + m \leq 4$.

In this thesis, we apply Theorem 1.1.1 to instantons and Euclidean monopoles using the ADHM transform and the Nahm transform relating monopoles and Nahm data. The ADHM transform allows us to take our set of objects to be matrices satisfying certain non-linear constraints and take our gauge group to be a matrix Lie group, acting on these matrices. The Nahm transform when $\Lambda = \mathbb{R}$ relates monopoles and Nahm data, matrix-valued functions of one variable satisfying the Nahm equations. Nahm data are acted on pointwise by a matrix Lie group, the gauge group. Thus, pointwise, we again can take our objects to be matrices and our gauge group to be a matrix Lie group acting on these matrices. We see that in both these cases, we can apply our result for various symmetries. Of course, we could also apply the result to examine other cases of the Nahm transform. These cases were merely chosen to illustrate the power of this result.

As mentioned above, others have studied the same objects as us with different symmetries. In doing so, they have been able to reduce their problem to representation theory as well. However, the novelty of our approach is two-fold. Firstly, the only other work in this vein looking at continuous symmetries examined a specific abelian symmetry. Our approach works for all Lie groups, not just abelian ones, so we find novel examples of objects. Secondly, and more importantly, we reduce the problem to representation theory through careful applications of Lie theory. Previous work for other groups have relied on clever but specific methods that only directly apply to their particular situation. The methods here apply broadly, as we see through our applications to various objects. However, these methods are applicable well beyond the objects presented here. Indeed, we require only two group actions on a smooth manifold: one gauge and one symmetry. While compactness is assumed in the result above, note that in Theorem 3.2.1 we are able to circumvent this requirement.

1.2 Hyperbolic monopoles

This section has appeared in the Journal of Geometry and Physics and is presented here with minor alterations [Lan24]. Let M be a 3-manifold, $E \rightarrow M$ a vector bundle over M with structure group $\mathrm{Sp}(n)$, equipped with a connection A of curvature F_A , and a section Φ of $\mathrm{End}(E)$, called the Higgs field. Let $\star$ denote the Hodge star. Monopoles are solutions to the Bogomolny equations $F_A = \star D_A \Phi$ with finite energy $\varepsilon = \frac{1}{2\pi} \int_M |F_A|^2 \mathrm{vol}_M$ and are a kind of topological soliton. If M has constant sectional curvature, then twistor methods can be used. When $M = \mathbb{R}^3$, we have Euclidean monopoles. When $M = H^3$, we have hyperbolic monopoles, which have not received as much study as their Euclidean counterparts. Indeed, much of the research on hyperbolic monopoles, including this work, has arisen in an effort to find hyperbolic analogues to results about Euclidean monopoles. When $M = S^3$, in fact whenever M is compact, both sides of the Bogomolny equation must vanish. This result follows from applying Stoke's Theorem to $\langle \Phi, F_A \rangle$ and utilizing the Bogomolny equations. As such, the monopoles are flat and this case is not as deep as the previous two. In addition to their deep mathematical structure, investigated using these twistor methods, hyperbolic monopoles are interesting due to their connection with monopoles in Anti de-Sitter space and Skyrmsions [AS05, MS90, Sut11].

The main difference between hyperbolic monopoles and their Euclidean counterparts is the idea of mass, the eigenvalues of the Higgs field at infinity. For Euclidean monopoles, as long as the limit of one eigenvalue is non-zero, we can always change the length scale such that the limit has modulus 1. For hyperbolic monopoles, there is already a length scale, determined by the curvature of the underlying hyperbolic space. Thus, when changing the length scale to modify the mass, we change the curvature of the space. Here and in Section 2.1, we consider hyperbolic space to have constant sectional curvature -1 (in particular, the ball model with radius one).

Another difference between the two types of monopoles is their motion. The motion of Euclidean monopoles is approximated by geodesic motion in the moduli space equipped with the natural L^2 metric [Man82, AH88, Stu94]. For hyperbolic monopoles the natural metric is infinite. Regardless of its possible relationship with monopole motion, it is important to study the moduli space. For instance, the moduli space of Euclidean monopoles and various submanifolds have provided important examples of hyperkähler manifolds [AH88, Dan93, Men23]. However, as the natural metric is infinite, another metric is required to study the moduli space of hyperbolic monopoles. Moreover, should one find a suitable L^2 metric on the moduli space approximating the motion of hyperbolic monopoles, the novel families of $\mathrm{Sp}(n)$ hyperbolic monopoles presented in Section 2.1.3.2 can be used to examine the motion of hyperbolic monopoles.

The study of hyperbolic monopoles started when Atiyah used the conformal equivalence between $S^4 \setminus S^2$ and $S^1 \times H^3$ to associate hyperbolic monopoles with integral mass with circle-invariant instantons on $\mathbb{R}^4$, leading to a correspondence between these monopoles and based rational maps [Ati84a, Ati84b]. The integral mass condition above ensures that the corresponding instanton on $S^4 \setminus S^2$ extends to all of S^4 ; otherwise, we get non-trivial holonomy around $S^2 \subseteq S^4$ [Ati84b]. Instantons are in turn related to ADHM data, matrices satisfying non-linear constraints, via the ADHM transform. More information on instantons, ADHM data, and the ADHM transform is given in Section 1.4 and Chapter 3.

In this work, we only investigate hyperbolic monopoles with integral mass, due to their correspondence with circle-invariant instantons. Nonetheless, we review the literature for arbitrary mass hyperbolic monopoles. Because the zero curvature limit of hyperbolic space is Euclidean space, Atiyah conjectured that hyperbolic monopoles correspond to Euclidean monopoles in the zero curvature limit [Ati84a, Ati84b]. Chakrabarti quickly gave explicit examples of hyperbolic monopoles, providing evidence to support Atiyah's curvature conjecture [Cha86]. Jarvis and Norbury would later confirm this conjecture in general [JN97]. Additionally, Nash used the JNR ansatz (a generalization of 't Hooft's ansatz for instantons) to give charge one spherically symmetric hyperbolic monopoles with arbitrary mass [Nas86]. This result was further improved by Sibner and Sibner, who used Taubes' gluing argument to show that hyperbolic monopoles exist with arbitrary charge and mass [SS12].

Given the similarities between Euclidean and hyperbolic monopoles, Braam and Austin sought to find a hyperbolic analogue to the Nahm transform, which is a correspondence between Euclidean monopoles and Nahm data, solutions to the Nahm equations, a system of ordinary differential equations. They succeeded, finding a correspondence between $SU(2)$ hyperbolic monopoles with integral mass and solutions to their discrete Nahm equations: matrix-valued difference equations [BA90]. They also discovered a difference between hyperbolic and Euclidean monopoles reminiscent of the AdS-CFT correspondence: $SU(2)$ hyperbolic monopoles with integral mass are determined by their boundary values at infinity. From this we got a new viewpoint for hyperbolic monopoles: holomorphic spheres, which are holomorphic embeddings of the Riemann sphere in projective space. The discrete Nahm equations were later generalized by Chan to produce $SU(N)$ hyperbolic monopoles [Cha18]. Just as in the Euclidean case, the discrete Nahm equations were shown to be integrable [War99].

At this point, we have multiple ways of looking at hyperbolic monopoles with integral mass: solutions to the Bogomolny equation, rational maps, spectral curves, discrete Nahm data, and holomorphic spheres. The first four are hyperbolic analogues to Euclidean monopoles, however, no analogue of holomorphic spheres exists for Eu-

clidean monopoles, as Euclidean monopoles are not determined by their value at infinity. As there are hyperbolic monopoles with non-integral mass, the study of hyperbolic monopoles turned to finding relationships between these viewpoints for arbitrary mass [MS96, MNS03, MS00, Nor01, Nor04, NR07]. However, while the holomorphic sphere viewpoint was generalized to arbitrary mass and charge, spectral curves were found in general for arbitrary mass for specifically charge one and two hyperbolic monopoles as well as for $SU(N)$ monopoles satisfying specific boundary conditions. Additionally, substantial work has been done analyzing the spectral curves of hyperbolic monopoles with charge three and four [NR07].

As many had focused on the theoretical aspects of hyperbolic monopoles, there were very few explicit examples of these objects, with or without integral mass. Using Jarvis' construction of Euclidean monopoles from rational maps, Ioannidou and Sutcliffe generated some spherically symmetric $SU(N)$ hyperbolic monopoles, for $N = 3, 4$ that descended to spherically symmetric Euclidean monopoles in the zero curvature limit [IS99]. Harland created spherically symmetric $SU(2)$ hyperbolic monopoles with arbitrary mass by first constructing $SO(3)$ -symmetric hyperbolic calorons (instantons on $S^1 \times H^3$) and taking a limit, shrinking the circle [Har08]. Oliveira found that given any non-parabolic, spherically symmetric metric on $\mathbb{R}^3$, there is a one-parameter family of spherically symmetric $SU(2)$ monopoles with arbitrary mass, all of which vanish at the origin [Oli14, Appendix A]. In particular, this result holds true for hyperbolic and Euclidean space. In contrast to this result, Proposition 2.1.44 gives us a spherically symmetric, hyperbolic $Sp(4)$ monopole that vanishes nowhere. Cockburn used the discrete Nahm equation to generate hyperbolic monopoles with integral mass and axial symmetry and deformed these monopoles to replace axial symmetry with dihedral symmetry [Coc14]. Franchetti and Maldonado found examples of hyperbolic monopoles constructed from solutions to the Helmholtz equation and from vortices [FM16, Mal17].

The circle action used by Braam and Austin to generate their discrete Nahm equation led to using the upper-half-space model of hyperbolic space when considering the corresponding monopoles. While axial symmetries are easy to see in this space, others are difficult, such as Platonic symmetries. To that end, Manton and Sutcliffe used a different circle action to end up with the ball model, where rotational symmetries are very easy to see [MS14]. They found a set of quaternionic ADHM data that always give $SU(2) \simeq Sp(1)$ hyperbolic monopoles with mass $1/2$, using the ADHM transform [MS14]. In Section 2.1, we generalize this set of data, finding a set of ADHM data that always gives $Sp(n)$ hyperbolic monopoles with mass $1/2$. In particular, we provide a framework to classify all spherically symmetric hyperbolic monopoles that have ADHM data in our set. To generate Platonic monopoles, Manton and Sutcliffe used the JNR ansatz, which would generate a

circle invariant instanton if all poles were on the boundary of hyperbolic space in $\mathbb{R}^4$. In addition, they used ADHM data that was previously used to find Platonic instantons to find Platonic monopoles. Using this approach, Bolognesi et al. found exact examples of hyperbolic magnetic bags [BHS15].

Given the importance of the JNR ansatz and Manton and Sutcliffe’s ADHM data to generating examples of hyperbolic monopoles, work was done to find formulae for computing the spectral curve and rational map of monopoles constructed from this data [BCS14, Sut21]. Additionally, work was done to compute the holomorphic sphere and energy density for monopoles constructed from the JNR ansatz [MN21].

In Section 2.1, we prove Theorems 2.1.17 and 2.1.22, which linearize the equations of symmetry. These results inspired Theorem 1.1.1. In proving these results, we reduce the problem of finding symmetric hyperbolic monopoles to a problem in representation theory. We also prove Theorem 2.1.39, which we use to generate many infinite families of spherically symmetric, hyperbolic monopoles with varying rank structure groups. For instance, we find three distinct spherically symmetric $\mathrm{Sp}(4)$ hyperbolic monopoles. In particular, in Proposition 2.1.44, we identify a spherically symmetric $\mathrm{Sp}(4)$ hyperbolic monopole that vanishes nowhere.

1.3 Symmetry Breaking

While monopoles with arbitrary symmetry breaking is not a new concept [GNO77], the main focus of monopole research has been monopoles with structure group $\mathrm{SU}(2) \simeq \mathrm{Sp}(1)$. For such monopoles, the symmetry is always broken to $\mathrm{U}(1)$. Some work has been done on $\mathrm{SU}(n)$ monopoles, both hyperbolic and Euclidean; however, much of the work on $\mathrm{SU}(n)$ monopoles was focused on those with maximal symmetry breaking. Despite the focus on maximal symmetry breaking, some work was done on monopoles with minimal symmetry breaking [Dan92, Dan93, DL93, DL97]. In particular, a specific moduli space of monopoles with minimal symmetry was examined in depth, providing an important example of a hyperkähler manifold [Dan93]. Except for work on these two extremal cases, maximal and minimal symmetry breaking, little was known about monopoles with arbitrary symmetry breaking.

Recently, there has been more interest in monopoles with arbitrary symmetry breaking. For instance, recent work has generalized the Nahm transform for $\mathrm{SU}(n)$ Euclidean monopoles with arbitrary symmetry breaking [CN22]. Previous work of mine with collaborators, which we discuss in Section 2.2, has used this Nahm transform to generate $\mathrm{SU}(n)$

Euclidean monopoles with continuous symmetries and neither maximal nor minimal symmetry breaking [CDL⁺22]. Additionally, it has been proven that the moduli space of monopoles with arbitrary symmetry breaking and a compact, simply-connected structure group is hyper-Kähler [Men23], solving a long-standing conjecture [MS03, Conjecture 3.3].

The previous examples of arbitrary symmetry breaking primarily involve monopoles with gauge group $SU(n)$, where symmetry breaking is very well understood. For general compact, connected groups, little is written about classifying symmetry breaking, outside of definitions for maximal symmetry breaking that are much more abstract than that given for the $SU(n)$ case. These other cases are important as my work in Section 2.1 involves providing a framework for classifying $Sp(n)$ hyperbolic monopoles with continuous symmetries. For $SU(n)$ monopoles, their symmetry breaking is easily obtained by counting the multiplicity of distinct eigenvalues of the Higgs field at infinity. In Theorem 2.3.37, we prove that the symmetry breaking of $Sp(n)$ monopoles is not quite as straightforward. Given the recent focus on arbitrary symmetry breaking for arbitrary gauge groups G , we have a need for understanding this phenomenon beyond $SU(n)$.

In Section 2.3, we prove Theorem 2.3.13 which identifies the symmetry breaking of a monopole as a free abelian group and determines its rank for any compact, connected Lie group. Theorems 2.3.31, 2.3.35, 2.3.37, and 2.3.40 provide a simple classification of symmetry breaking for all Lie groups with a classical, simple Lie algebra. We also outline a method for doing the same for the exceptional simple Lie algebras. In doing so, we provide a way to classify symmetry breaking for all compact, connected Lie groups.

1.4 Instantons

Let M be a 4-manifold, $E \rightarrow M$ a vector bundle over M with structure group $Sp(n)$, equipped with a connection $\mathbb{A}$ of curvature $F_{\mathbb{A}}$. Instantons are solutions to the self-dual equations $F_{\mathbb{A}} = \star F_{\mathbb{A}}$ with finite action $\frac{1}{2\pi} \int_M |F_{\mathbb{A}}|^2 \text{vol}_M$. Due to the integrability of Chern numbers, the action of an instanton, which is related to the second Chern number, is always an integer multiple of some positive constant. This multiple N is known as the instanton number or charge of the instanton. Note that instantons and anti-instantons, solutions to the anti-self-dual equations $F_{\mathbb{A}} = -\star F_{\mathbb{A}}$ with finite action, differ by a choice of orientation, so there is no real difference between the objects.

Of particular interest is when $M = \mathbb{R}^4$. Due to the conformal invariance of the Hodge star acting on two-forms in four dimensions, instantons on $M = \mathbb{R}^4$ are equivalent to those on a punctured four-sphere. Uhlenbeck proves that, in fact, instantons on $M = \mathbb{R}^4$ are equivalent to those on $M = S^4$ [Uhl82].

As the self-dual equations are a system of coupled, non-linear partial differential equations, instantons are extremely difficult to obtain. The 't Hooft ansatz is a way of constructing $SU(2)$ instantons by way of harmonic maps. This construction of an $SU(2)$ instanton of charge N involves choosing N distinct poles and N corresponding weights. Varying these choices produces a $(5N + 3)$ -dimensional family in the configuration space of $SU(2)$ instantons with charge N , as a constant choice of gauge adds an additional three degrees of freedom. The JNR ansatz is a generalization of 't Hooft's work, using the conformal group to extend the choice of poles and weights by one each [JNR77]. This method generates a $(5N + 7)$ -dimensional family in the configuration space of $SU(2)$ instantons with charge N ; only the ratios of the weights in the 't Hooft ansatz relative to this extra weight matter. For a deeper discussion on these ansatzes, see Whitehead's work [Whi22, Construction B.1.2 & Construction B.1.3]. Overall, the moduli space of $SU(2)$ instantons with charge N is $8N$ -dimensional. In fact, for $N = 1$ and $N = 2$, every $SU(2)$ instanton can be constructed using the JNR ansatz. Note that for these cases, the extra dimensions of the family of JNR instantons can be shown to be degenerate. However, for $N \geq 3$, the JNR approach does not construct every charge N instanton.

The ADHM transform is a powerful result relating instantons on $\mathbb{R}^4$ with ADHM data, matrices satisfying a non-linear constraint [AHDM78]. Unlike the JNR ansatz, the ADHM transform constructs every instanton, even for higher rank structure groups. While they are much easier to find than instantons, finding examples of ADHM data is still challenging, due to the non-linear constraints on the data. Previous work finding examples dealt with using representation theory to simplify the data. However, this work was only done for finite or abelian groups. The current work provides linear algebraic equations that one must first solve (in fact, we solve these equations in most cases). Solving these linear equations is not only very simple, it greatly simplifies the non-linear constraints.

Using the conformal equivalence of $S^4 \setminus S^2 \equiv H^3 \times S^1$, Atiyah realized that hyperbolic monopoles with integral mass correspond to instantons symmetric under a certain circle action [Ati84a, Ati84b]. The integral mass condition is required so that we can extend an instanton on $S^4 \setminus S^2$ to one on S^4 [Ati84b]. More detail about this correspondence is given in Section 3.2.2.

The Nahm transform provides a connection between Euclidean monopoles and Nahm data, solutions to the Nahm equations. Braam–Austin proved a connection between $SU(2)$ instantons symmetric under a circle action, hyperbolic monopoles, and solutions to a one-dimensional integrable lattice system: a discretization of the Nahm equations [BA90]. Prompted by this, Ward looked at $SU(2)$ instantons symmetric under the action of a torus, connecting them with axially symmetric hyperbolic monopoles [War16]. Ward found such symmetric instantons correspond to solutions to an integrable, two-dimensional lattice

version of Hitchin's equations.

As mentioned in Section 1.2, monopoles are topological solitons over a 3-manifold $\tilde{M}$. In addition to hyperbolic and Euclidean monopoles, for which $\tilde{M} = H^3, \mathbb{R}^3$, respectively, we have singular monopoles, for which there are some $p_1, \dots, p_l \in \mathbb{R}^3$ such that $\tilde{M} = \mathbb{R}^3 \setminus \{p_1, \dots, p_l\}$.

The relationship between instantons and singular monopoles was first described by Kronheimer, who set up a mini-twistor approach to study the moduli space in a similar fashion to Hitchin's work on Euclidean monopoles [Kro85, Hit82]. This work was explored further by Pauly, in the case of singular monopoles on Euclidean space as well as compact spaces, where all non-flat monopoles must be singular [Pau96, Pau98]. In particular, Pauly investigates the moduli spaces of these singular monopoles with structure group $SU(2)$. Additionally, when studying monopoles on S^3 , Pauly found that associated to each gauge equivalence class of monopoles defined on an open subset $\Omega \subseteq S^3$ is a holomorphic function on the complex two-dimensional space of geodesics in Ω [Pau01]. Royston et al. determine the dimension of the moduli space of singular monopoles with any simple Lie group as a structure group and maximal symmetry breaking [MRvdB14].

The relationship between singular monopoles and instantons utilizes the Hopf fibration $\pi: S^3 \rightarrow S^2$, which describes S^3 as a non-trivial principal bundle over S^2 with fibre S^1 . That is, for every $p \in S^2$, $\pi^{-1}(p) \simeq S^1$. We can extend the fibration linearly along rays emanating from the origin to a map from $\mathbb{R}^4 \setminus \{0\}$ to $\mathbb{R}^3 \setminus \{0\}$. Doing so, for all $p \in \mathbb{R}^3 \setminus \{0\}$ we have that its pre-image under the fibration is a circle. Singular monopoles are in a one-to-one correspondence with circle-invariant instantons on $\mathbb{R}^4 \setminus \{0\}$ [Pau96]. Of particular interest are those with Dirac type singularities: the singular monopoles whose corresponding instantons on $\mathbb{R}^4 \setminus \{0\}$ can be extended smoothly over 0 in some gauge, giving us an instanton on $\mathbb{R}^4$. This allows us to describe these monopoles using ADHM data. Mirroring the relationships between Euclidean monopoles and the Nahm equations on a finite interval as well as hyperbolic monopoles and the discretized Nahm equation, Cherkis–Kapustin describe singular monopoles in terms of solutions to the Nahm equations on a semi-infinite interval [CK98, CK99]. Later, Cherkis used the bow formalism to construct instanton configurations on the Taub-NUT space [Che09]. In particular, Cheshire bows correspond to singular monopoles [BC11]. While all singular monopoles are created using the Nahm formalism, this method is difficult to use beyond two singularities. In contrast, the Cheshire bow formalism does not encounter this challenge.

Skyrmions are a three-dimensional topological soliton describing nuclei. For more information regarding these objects, see Manton's recent book [Man22]. A skyrmion has a topological charge B representing the baryon number of the nucleus. For a given charge B , mini-

mal energy skyrmions often have a great deal of symmetry. In particular, the $B = 1, 2, 3, 4$ minimal skyrmions have spherical, axial, tetrahedral, and cubic symmetry, respectively. Atiyah–Manton created a method for approximating the minimal energy skyrmions with charge B by computing the holonomy of a $SU(2)$ instanton on $\mathbb{R}^4$ with charge n along lines parallel to a chosen axis [AM89]. This approach is quite effective, resulting in Skyrme fields with less than 1% more energy than the numerical minimal energy skyrmions. This approach only applied to massless pions. However, the pair later used a trick similar to Atiyah’s trick relating hyperbolic monopoles and circle-invariant instantons to approximate massive pions. Indeed, by approximating Euclidean skyrmions with massive pions by hyperbolic skyrmions with massless pions, Atiyah–Manton approximated these hyperbolic skyrmions by computing the holonomy of a $SU(2)$ instanton about circles [AS05]. The computations required to use Atiyah–Manton’s method have recently been simplified using the theory of parallel transport of an induced connection, reducing the problem from solving a differential equation to a finite-difference approximation [CHW21, Har23]. This method also bypasses issues related to gauge singularities. Harland–Sutcliffe refine this method to approximate skyrmions, by using an ultra-discrete approach [HS23]. Whereas Atiyah–Manton’s method required solving a differential equation, this new method requires only evaluating the ADHM data. In particular, they produce Skyrme fields with charge 1 or 2, finding that their fields have even less excess energy than those generated by Atiyah–Manton.

The existence of these symmetric minimal energy skyrmions and Atiyah–Manton’s method of approximating skyrmions via instantons inspired a search for instantons with Platonic symmetries. In the same work, Atiyah–Manton find an axial instanton with charge 2 [AM89]. Later, Leese–Manton found a charge 3 tetrahedral instanton and a charge 4 cubic instanton [LM94]. Singer–Sutcliffe realized that for finite symmetry groups, the symmetry of an instanton is always generated by representations of the symmetry group [SS99]. Using representation theory, they construct an icosahedral instanton with charge 7 and compute a Skyrme field via the holonomy of the instanton. Additionally, they found a family of ADHM data approximating the scattering of seven skyrmions. The tetrahedral instanton found by Leese–Manton is obtained by placing four poles of equal weight on the vertices of a regular tetrahedron and using the JNR approach. Prompted by Singer–Sutcliffe’s work on symmetric instantons using ADHM data, Houghton found ADHM data with tetrahedral symmetry [Hou99]. By varying the ADHM parameters, Houghton examined the 3-skyrmion vibration modes. Motivated by the existence of a charge 17 skyrmion with icosahedral symmetry, Sutcliffe found an instanton with the same symmetry and charge, predicting a whole range of fullerene instantons [Sut04]. Sutcliffe also outlined a procedure for constructing instantons with Platonic symmetry, where these

symmetries leave one axis invariant [Sut05]. As a demonstration, Sutcliffe found a charge 17 instanton with the structure of a truncated icosahedron. Inspired by the charge five, six, and eight minimal energy skyrmions, which have the symmetry of D_{2d} , D_{4d} , and D_{6d} , respectively, Cork–Halcrow found instantons with the same charge and the same symmetry [CH22]. All of the Skyrme fields generated by the above symmetric instantons have less than 2% excess energy of the corresponding numerical skyrmion [CH22]. Cork–Halcrow also found instantons with toral symmetry and instantons which looked like spinning tops, interpolating well-separated clusters and highly-symmetric configurations [CH22].

Moving beyond symmetric instantons inspired by symmetric skyrmions, Allen–Sutcliffe constructed instantons with the symmetry of various polytopes [AS13]. Specifically, they find $SU(2)$ instantons with the structure of a pentatope, hyperoctahedron, and the 24-cell with charges 4, 7, and 23, respectively. Just like the Platonic symmetry case, the JNR approach can be used to place equal weights at the vertices of a polytope to force the symmetry. However, there may be an instanton with the same symmetry but a lower charge, not of the JNR type, as this does not describe all instantons. Thus, ADHM data is needed. Allen–Sutcliffe use the ADHM method and representation theory to prove that their symmetric instantons have the lowest charges, excluding the basic charge one instanton.

The preceding work relied on the finiteness of the symmetry groups to reduce the problem to representation theory. In the current work, the ability to reduce the problem to representation theory uses a completely different approach: Lie theory. Moreover, previous work focused on instantons with isometric symmetries, but the conformal equivalence of the self-dual equation begs for us to search for instantons symmetric under conformal symmetries, like Manton–Sutcliffe’s conformal circle action [MS14]. The Lie theory approach used here allows us to consider instantons with conformal symmetries, thereby allowing us to fully study instantons symmetric under Manton–Sutcliffe’s conformal circle action (which correspond to hyperbolic monopoles). We can study instantons symmetric under other conformal symmetries, however, we see in Theorem 3.4.44 and Corollary 3.4.45 that every instanton symmetric under other kinds of continuous conformal symmetries is constructed from the basic charge one instanton, leaving those symmetric under Manton–Sutcliffe’s conformal circle action as the only ones of note.

Beckett takes a different approach, using the Equivariant Index Theorem and the representation theory of S^1 to examine instantons symmetric under a circle action, which correspond to singular monopoles [Bec20]. Whereas previous work relied on the finiteness of symmetry groups, Beckett was able to prove that for his specific circle action, the infinite symmetry group of a circle-invariant instanton is generated by representations of S^1 . Using this result, Beckett produced several examples of circle-invariant $SU(2)$ instantons

and examined the moduli spaces of such instantons for low charges. In this work, we answer Beckett’s call for a formula for constructing a singular monopole given circle-invariant ADHM data.

Whitehead used the same approach as Beckett, returning to finite symmetries [Whi22]. Doing so, Whitehead was able to prove the non-existence of instantons with certain charges and finite symmetry groups and construct novel symmetric instantons. In particular, Whitehead finds a dodecahedral and an icosidodecahedral $SU(2)$ instanton with charge 13 and 23, respectively. Additionally, Whitehead proves that there is no $SU(2)$ instanton with charge in $(1, 119)$ and the symmetry of the 600-cell, answering a specific case of the conjecture of Allen–Sutcliffe: the lowest charge $SU(2)$ instanton with the symmetry of a polytope, other than the basic charge one instanton, is the one obtained using the JNR ansatz if and only if the polytope has triangles as two-faces [AS13].

When investigating hyperbolic monopoles, Manton and Sutcliffe realized that many of the Platonic instantons previously identified possessed more symmetry than originally realized. Indeed, in addition to their Platonic symmetry, these instantons are symmetric under Manton–Sutcliffe’s conformal circle action [MS14]. Because of this, these instantons correspond to Platonic hyperbolic monopoles. Definition 2.1.1 generalized the set of data considered by Manton–Sutcliffe for higher rank structure groups. While they knew that their data did not describe every hyperbolic monopole with integral mass, they were unable to identify the exact constraints satisfied by all such data. In Section 3.5.2, we identify the exact constraints and study all spherically symmetric hyperbolic monopoles, generalizing my previous work in Section 2.1.3. The current work takes this a step further, providing a framework for classifying all instantons with every various kind of continuous symmetry.

As noted above, previous work on symmetric instantons focused mostly on $Sp(1) \simeq SU(2)$ instantons. However, there were a few works that focused on finding symmetric instantons with higher rank structure groups. Conversely to the previous work going from symmetric skyrmions to symmetric instantons, Gat searched for spherically symmetric $SU(3)$ Skyrme fields by finding spherically symmetric $SU(3)$ instantons and using Atiyah–Manton’s holonomy method [Gat91]. On the other hand, later work found spherically symmetric $SU(N)$ skyrmions using harmonic maps [IPZ99]. This inspired a search for the corresponding spherically symmetric $SU(N)$ instantons, which was also done using harmonic maps, as opposed to ADHM data, which is used in this work [Ioa00]. Not only does the current work focus on a different family of structure groups, it deals with all kinds of continuous symmetries and investigates all instantons with these symmetries.

While Singer–Sutcliffe’s work using representation theory utilized two representations of the given group, Beckett and Whitehead’s work only used a single representation. While

this still allowed for powerful results when dealing with symmetric $SU(2)$ instantons, the lack of a second representation makes obtaining results with higher rank structure groups more challenging. In this work, just as Singer–Sutcliffe, we are able to obtain and use two representations. Whereas Singer–Sutcliffe and their contemporaries focused on $SU(2)$ instantons, we show how the second representation is extremely useful when consider instantons with higher rank structure groups.

Because of the examples of instantons with finite (Platonic) and infinite (circle) symmetries, one could combine the methods for finite symmetries and the methods used here to simplify the search for examples of instantons with finite symmetries and higher rank structure groups. Additionally, like Ioannidou, we could apply Atiyah–Manton’s holonomy method of approximating skyrmions, using the simplifications provided by Cork, Harland, and Winyard, to continuously symmetric instantons generated using this work [Ioa00].

In Chapter 3, we examine instantons with all kinds of continuous conformal symmetries. Using Theorem 1.1.1, we prove theorems providing algebraic equations for the different kinds of symmetries. Then, we use these results to identify novel instantons with higher rank structure groups and a variety of symmetries. Additionally, we prove Theorem 3.4.44 and Corollary 3.4.45. These results tell us that we know all of the instantons with conformal spherical symmetry or full symmetry: those that are direct sums of the basic instanton.

1.5 Layout

The layout of this thesis is based on the evolution of my work on symmetric solitons. My initiation into the topic was an investigation into Euclidean monopoles with continuous symmetries [CDL⁺22]. This collaborative work began during my undergraduate degree and was mostly completed during my Master’s, hence its exclusion from this work (though I discuss on my improvements to the results in Section 2.2). In this work, we thought to differentiate the equation of symmetry for symmetric Nahm data and obtained a more specific and weaker version of the result above. Using this, we were able to generate novel examples of spherically symmetric Euclidean monopoles. Unfortunately, we were unable to verify that all such monopoles were created this way (though based on my later work, it turns out they are). After reading Sutcliffe’s work regarding the spectral curves of hyperbolic monopoles [Sut21], I thought that I may be able to do the same thing for these new objects. Based on a comment from Ben Webster, it was during this project that I realized that I could obtain a much stronger result: continuous symmetries are always generated by representations. After learning more about the ADHM transform in an effort to compute examples of symmetric hyperbolic monopoles, I realized that my result

has wider applications than I realized, as I could also apply it to instantons with various continuous symmetries. This began my largest project and resulted in Theorem 1.1.1. I include this testimony to inspire others to look for other ways to apply their work and to push their work to its limits, as you never know where it will take you.

The main results of this work are the theorems where we apply Theorem 1.1.1 to find linear equations completely describing various symmetric topological solitons as well as the theorems where we solve these equations in general. In doing so, we provide a framework for classifying all these symmetric topological solitons. In addition to these results, we have others to highlight. In particular, in Theorem 3.4.44 and Corollary 3.4.45, we prove that we already know every instanton with two particular kinds of conformal symmetry. Additionally, Theorem 2.3.13 provides a straightforward method for determining the symmetry breaking of a monopole with any compact, connected Lie group as a structure group. Moreover, we explicitly identify the symmetry breaking for all the Lie groups with a classical, simple Lie algebra and outline a method for doing the same for the exceptional simple Lie algebras.

In Chapter 2, we explore symmetric monopoles and their symmetry breaking. In particular, in Section 2.1, we investigate hyperbolic monopoles with continuous symmetries, which is where Theorem 1.1.1 was developed. We use this result to identify several novel hyperbolic monopoles with higher rank structure groups. This has appeared in the Journal of Geometry and Physics and is presented here with minor alterations. In Section 2.2, we also examine Euclidean monopoles with continuous symmetries. The bulk of the work leading to my paper with Charbonneau–Dayaprema–Nagy–Yu on Euclidean monopoles was completed prior to the start of my PhD, hence it is not included in this thesis [CDL⁺22]. Specifically, in this section, we only examine the application of Theorem 1.1.1 to these objects. In Section 2.3, we investigate the symmetry breaking of monopoles with any compact, connected Lie group as a structure group. Here, we both prove Theorem 2.3.13 and identify the symmetry breaking for all the Lie groups with a classical, simple Lie algebra.

In Chapter 3, we investigate instantons with continuous conformal symmetries. While the work done on hyperbolic monopoles in Chapter 2 falls under the umbrella of the work done here, any significant overlaps have been avoided. In Section 3.1, we introduce what it means means for ADHM data to be symmetric under conformal transformations, proving several lemmas important to later sections. The various kinds of continuous conformal symmetries that an instanton can possess are identified in Section C.2 of Appendix C. In Section 3.2, we investigate instantons with various kinds of circular symmetry. In particular, we look at the connection between such instantons and hyperbolic and singular monopoles. In Section 3.3, we investigate instantons with toral symmetry. In particular, we look at the connection between such instantons and hyperbolic and singular monopoles

with axial symmetry. In Section 3.4, we investigate instantons with the three kinds of spherical symmetry: simple, isoclinic, and conformal. We also discuss the relationship between these symmetric instantons and hyperbolic analogues to Higgs bundles and Nahm data. Additionally, we prove the aforementioned Theorem 3.4.44 and Corollary 3.4.45. In Section 3.5, we study instantons with isoclinic and conformal superspherical symmetry. In particular, we also look at the connection between such instantons and axially symmetric hyperbolic Higgs bundles as well as spherically symmetric hyperbolic and singular monopoles. In Section 3.6, we study instantons with rotational symmetry. Throughout these sections, we use Theorem 1.1.1 to identify several novel instantons with higher rank structure groups and different symmetries.

In Appendix A, we prove Theorem 1.1.1. In Appendix B, we prove that for instantons and monopoles symmetric under a group of isometries, without loss of generality, we may assume that these isometries belong to the orthogonal group. As we are interested in connected Lie groups, we may always focus on subgroups comprised of rotations. This result is part of the folklore of research in this area, but we provide a reference for the fact here. In Appendix C, we classify the connected Lie subgroups, up to conjugacy, that we use to study our symmetric solitons. In Appendix D, we study the representation theory needed for our results. In particular, we discuss real, complex, and quaternionic representations, and the relationships between them. Additionally, we discuss representations of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ and dual representations. In Appendix E, we use representation theory to identify exactly what spans the unique trivial summand of a specific representation. This result allows us to completely understand the construction of hyperbolic monopoles in Chapter 2. Note that these methods can be applied for similar summands in Chapter 3 as well. In Appendix F, we list Chevalley bases for the classical simple Lie algebras. In Appendix G, we prove that the isotropy subgroup obtained when investigating the symmetry breaking of a monopole need not split like its Lie algebra.

Chapter 2

Symmetric monopoles and their symmetry breaking

In this chapter, we explore symmetric monopoles and their symmetry breaking. In particular, in Section 2.1, we investigate hyperbolic monopoles with continuous symmetries, which is where Theorem 1.1.1 was developed. This has appeared in the Journal of Geometry and Physics and is presented here with minor alterations. In Section 2.2, we examine Euclidean monopoles with continuous symmetries. The bulk of the work leading to my paper with Charbonneau–Dayaprema–Nagy–Yu on Euclidean monopoles was completed prior to the start of my PhD, hence it is not included in this thesis [CDL⁺22]. Specifically, in this section, we only examine the application of Theorem 1.1.1 to these objects. In Section 2.3, we investigate the symmetry breaking of monopoles with any compact, connected Lie group as a structure group. Here, we prove a simple formula for computing the symmetry breaking of a monopole with any compact, connected Lie group as a symmetry group.

2.1 Hyperbolic monopoles

In this section, we apply Theorem 1.1.1 to hyperbolic monopoles. Let M be a 3-manifold, $E \rightarrow M$ a vector bundle over M with structure group $\mathrm{Sp}(n)$, equipped with a connection A , of curvature F_A , and a section Φ of $\mathrm{End}(E)$, called the Higgs field. Let $\star$ denote the Hodge star. Monopoles are solutions to the Bogomolny equations $F_A = \star D_A \Phi$, with finite energy $\varepsilon = \frac{1}{2\pi} \int_M |F_A|^2 \mathrm{vol}_M$. When $M = \mathbb{R}^3$, we have Euclidean monopoles and when $M = H^3$, we have hyperbolic monopoles.

Aityah used the conformal equivalence between $S^4 \setminus S^2$ and $S^1 \times H^3$ to associate hyperbolic monopoles with integral mass with circle-invariant instantons on $\mathbb{R}^4$. Given a vector bundle $\mathbb{E} \rightarrow \mathbb{M}$ over a 4-manifold $\mathbb{M}$, equipped with a connection $\mathbb{A}$, an instanton is a solution to the self-dual equations $\star F_{\mathbb{A}} = F_{\mathbb{A}}$, with finite action $\int_{\mathbb{M}} |F_{\mathbb{A}}|^2 \text{vol}_{\mathbb{M}}$. With regards to hyperbolic monopoles, we are only interested in instantons on $\mathbb{M} = \mathbb{R}^4$, equivalently instantons on $\mathbb{M} = S^4$ [Uhl82]. The integral mass condition above ensures that the corresponding instanton on $S^4 \setminus S^2$ extends to all of S^4 ; otherwise, we get non-trivial holonomy around $S^2 \subseteq S^4$ [Ati84b].

The ADHM transform provides a correspondence between instantons on $\mathbb{R}^4$ (equivalently S^4) and ADHM data $\mathcal{A}$, which we can take to be the set of quaternionic matrices $\hat{M} := \begin{bmatrix} L \\ M \end{bmatrix}$ such that M is symmetric and $\hat{M}$ satisfies the following non-linear constraint.

Let $x \in \mathbb{H} \simeq \mathbb{R}^4$ and define $\Delta(x) := \begin{bmatrix} L \\ M - Ix \end{bmatrix}$. Then $\Delta(x)^\dagger \Delta(x)$ must be symmetric and non-singular for all $x \in \mathbb{H}$ [AHDM78]. For more information on ADHM data and why we can reduce it to this form, see Section 3.1. In the current section, we look at a subset of $\mathcal{A}$ whose instantons are symmetric under a particular circle action, giving us hyperbolic monopoles.

In earlier work, which we discuss in Section 2.2, we created a Structure Theorem that generates spherically symmetric Euclidean monopoles [CDL⁺22]. However, this theorem has some hypotheses, meaning it is not known if it generates all such Euclidean monopoles. Just like others before, given the connection between Euclidean and hyperbolic monopoles, I sought an analogue of this theorem for the hyperbolic case, allowing me to add to the small, but growing list of examples of hyperbolic monopoles. By giving a more abstract proof of the Structure Theorem, I am able to remove the aforementioned hypotheses not only for the case of hyperbolic monopoles, but also Euclidean monopoles. This means that not only do the Structure Theorems provide novel examples of spherically symmetric monopoles with higher rank structure groups, but they generate all spherically symmetric Euclidean monopoles and all spherically symmetric hyperbolic monopoles that can be obtained from the subset of ADHM data I identify below.

The main results of this section are Theorem 2.1.17, Theorem 2.1.22, and Theorem 2.1.39, the Structure Theorem. The first two theorems are able to linearize the equations of symmetry by working with Lie algebras. They do so for axial and spherical symmetry, respectively. These theorems reduce the problem of finding symmetric hyperbolic monopoles to a problem in representation theory. The Structure Theorem outlines all solutions to the equation of spherical symmetry, solving a key step in finding spherically

symmetric hyperbolic monopoles. Using this theorem, we generate many infinite families of spherically symmetric, hyperbolic monopoles.

In Section 2.1.1, I introduce the ADHM data that we study, as well as what it means for it to be equivariant under a rotation. In Section 2.1.2, we investigate hyperbolic monopoles with axial symmetry. In Section 2.1.3, we investigate hyperbolic monopoles with spherical symmetry and prove the Structure Theorem, which greatly simplifies the construction of such monopoles. We then provide examples of spherically symmetric hyperbolic monopoles and discuss constraints on the structure group of these monopoles, providing a framework for the classification of spherically symmetric hyperbolic monopoles.

2.1.1 Monopole data

Recall that ADHM data corresponds to instantons on $\mathbb{R}^4$ (equivalently S^4) via the ADHM transform. Additionally, Atiyah's work gave a relationship between circle-invariant instantons and hyperbolic monopole with integral mass. In this section, following Manton and Sutcliffe, we investigate a set of ADHM data whose instantons are symmetric under a specific circle action, giving us hyperbolic monopoles [MS14, Section 4]. We then examine what it means for such data to be equivariant under a rotation.

The following is a generalization of the ADHM data that Manton and Sutcliffe used to generate Platonic $SU(2) \simeq Sp(1)$ hyperbolic monopoles [MS14, (4.9)-(4.11)]. This data produces hyperbolic monopoles.

Definition 2.1.1. Let $\mathcal{M}_{n,k}$ be the set of $(n+k) \times k$ quaternionic matrices $\hat{M} = \begin{bmatrix} L \\ M \end{bmatrix}$ such that

- $M = M_1i + M_2j + M_3k \in \text{Mat}(k, k, \mathbb{H})$ is symmetric and M_1, M_2, M_3 are real;
- $L \in \text{Mat}(n, k, \mathbb{H})$ is such that $LL^\dagger$ is a positive definite matrix;
- $L^\dagger L - M^2 = I_k$;
- for $x \in \mathbb{H} \simeq \mathbb{R}^4$, let $\Delta(x) := \begin{bmatrix} L \\ M - I_k x \end{bmatrix}$, then $\Delta(x)^\dagger \Delta(x)$ is non-singular for all $x \in \mathbb{H}$.

In particular, the hyperbolic monopoles corresponding to data in $\mathcal{M}_{n,k}$ has structure group $\mathrm{Sp}(n)$ and instanton number k , which is the second Chern number of the instanton.

Note 2.1.2. *Throughout this thesis, k indicates both the instanton number and the standard quaternion ($k = ij$) and its use at any given time is contextual.*

When $n = 1$, $LL^\dagger$ is a 1×1 matrix. Moreover, it is a non-negative real number. Thus, the condition that $LL^\dagger$ is positive definite just becomes L is non-zero, just as in the definition of Manton and Sutcliffe [MS14, (4.9)-(4.11)]. Such a condition is natural, as requiring $LL^\dagger$ to be positive-definite only rules out those instantons which are trivial embeddings of smaller rank instantons. That is, those whose connection matrices have the form $A_\nu(x) = \tilde{A}_\nu(x) \oplus 0$. The final condition is required to use the ADHM transform.

Note 2.1.3. *The condition $LL^\dagger$ being positive definite implies that $n \leq k$, as the rows of L must be linearly independent.*

Manton and Sutcliffe's definition of $\mathcal{M}_{n,k}$ (where $n = 1$) includes a left eigenvalue μ [MS14, (4.11)]. The existence of μ is guaranteed by the first three conditions for $\mathcal{M}_{n,k}$.

Lemma 2.1.4. *Given $\hat{M}$ satisfying the first three conditions of $\mathcal{M}_{n,k}$. Then there exists a unique $\mu \in \mathfrak{sp}(n)$ such that $\mu L = LM$. Specifically, $\mu := LML^\dagger(LL^\dagger)^{-1}$. Moreover, $LL^\dagger - \mu^2 = I_n$.*

Proof. Using $L^\dagger L - M^2 = I_k$, we can see that $[L^\dagger L, M] = 0$ and $[LML^\dagger, LL^\dagger] = 0$.

Let $\mu := LML^\dagger(LL^\dagger)^{-1}$, we see that

$$\begin{aligned}\mu^\dagger &= (LL^\dagger)^{-1}L(-M)L^\dagger = -LML^\dagger(LL^\dagger)^{-1} = -\mu, \quad \text{and} \\ \mu L &= (LL^\dagger)^{-1}LML^\dagger L = (LL^\dagger)^{-1}LL^\dagger LM = LM.\end{aligned}$$

That μ is unique comes from the invertibility of $LL^\dagger$. For the last identity, note that $LM^2 = \mu^2 L$. As $L^\dagger L - M^2 = I_k$, we multiply by L on the left and $L^\dagger$ on the right to get

$$(LL^\dagger - \mu^2 - I_n)LL^\dagger = 0.$$

As $LL^\dagger$ is invertible, we obtain the desired identity. $\square$

After identifying $\mathbb{R}^4 \simeq \mathbb{H}$, Manton and Sutcliffe introduce the following circle action on $\mathbb{R}^4$:

$$x \mapsto \left(\cos\left(\frac{\alpha}{2}\right)x + \sin\left(\frac{\alpha}{2}\right) \right) \left(-\sin\left(\frac{\alpha}{2}\right)x + \cos\left(\frac{\alpha}{2}\right) \right)^{-1}. \quad (2.1)$$

Note that while this circle action is not well-defined on $\mathbb{R}^4$, we can extend it to a well-defined circle action on $S^4 = \mathbb{R}^4 \cup \{\infty\}$. Indeed, $x \mapsto \begin{bmatrix} \cos(\frac{\alpha}{2}) & \sin(\frac{\alpha}{2}) \\ -\sin(\frac{\alpha}{2}) & \cos(\frac{\alpha}{2}) \end{bmatrix} \cdot x$ as defined in (3.1.3). Moreover, note that instantons on $\mathbb{R}^4$ correspond to instantons on S^4 [Uhl82].

The following was proven for the $n = 1$ case [MS14, Section 4], though the proof works for arbitrary n .

Proposition 2.1.5. *All $\hat{M} \in \mathcal{M}_{n,k}$ correspond to instantons symmetric under this circle action. Hence, all of our ADHM data corresponds to hyperbolic monopoles.*

Using the conformal equivalence of $\mathbb{R}^4 \setminus \mathbb{R}^2 \equiv S^1 \times H^3$, the model of hyperbolic space produced by this circle action is the ball model: $H^3 = \{X = X_1i + X_2j + X_3k \in \mathfrak{sp}(1) \mid R^2 := X_1^2 + X_2^2 + X_3^2 < 1\}$ with metric given by

$$g_{ij}(X) = \frac{4}{(1 - R^2)^2} \delta_{ij}.$$

Given $\hat{M} \in \mathcal{M}_{n,k}$, the $\mathrm{Sp}(n)$ monopole (Φ, A) with instanton number k is constructed as follows [MS14, Section 4]. Let $X := X_1i + X_2j + X_3k \in H^3$ and recall $\Delta(X) := \begin{bmatrix} L \\ M - I_k X \end{bmatrix}$.

The kernel of $\Delta(X)^\dagger$ is n -dimensional. Let $\psi(X)$ be a $(n+k) \times n$ quaternionic matrix whose columns give an orthonormal basis for the kernel of $\Delta(X)^\dagger$. That is, $\psi(X)^\dagger \Delta(X) = 0$ and $\psi(X)^\dagger \psi(X) = I_n$. Then the Higgs field is given by [MS14, (4.21)]

$$\Phi(X) = \frac{1}{2} \psi(X)^\dagger \begin{bmatrix} -\mu & L \\ -L^\dagger & M \end{bmatrix} \psi(X). \quad (2.2)$$

The connection $A = A_1 dX_1 + A_2 dX_2 + A_3 dX_3$ is given by, for $i \in \{1, 2, 3\}$ [MS14, (4.18)],

$$A_i(X) = \psi(X)^\dagger \partial_i \psi(X). \quad (2.3)$$

Lemma 2.1.6. *If $\hat{M}$ satisfied the first three conditions of $\mathcal{M}_{n,k}$ in Definition 2.1.1 and satisfies the final condition for all points in $\overline{H^3} = \{X = X_1i + X_2j + X_3k \in \mathfrak{sp}(1) \mid R^2 \leq 1\}$, then $\hat{M} \in \mathcal{M}_{n,k}$. That is, we can relax the final condition to just points in $\overline{H^3}$.*

Proof. In the proof of Proposition 2.1.5, Manton and Sutcliffe use the following relationship between points x in $\mathbb{R}^4 \simeq \mathbb{H}$ and points $X \in \overline{H^3}$. Using toroidal coordinates (X, χ) introduced by Manton and Sutcliffe [MS14, (4.4)], we have

$$x = \frac{2X + (1 - R^2) \sin \chi}{1 + R^2 + (1 - R^2) \cos \chi}. \quad (2.4)$$

The circle action is the rotation $\chi \mapsto \chi + \alpha$. This map $(X, \chi) \mapsto x$ is surjective, and Manton and Sutcliffe show that $\Delta(x) = U\Delta(X)$ for some $U \in \mathrm{Sp}(n+k)$ [MS14, (4.19)], so $\Delta(x)^\dagger \Delta(x) = \Delta(X)^\dagger \Delta(X)$. Hence, if $\Delta(X)^\dagger \Delta(X)$ is non-singular for all $X \in \overline{H^3}$, then the final condition for $\mathcal{M}_{n,k}$ is satisfied. $\square$

2.1.1.1 Rotating monopole data

It is well known that if a non-flat monopole is symmetric under some group of isometries, then a change of coordinates takes the family to a subgroup of $\mathrm{O}(3)$. Indeed, for a proof, see Corollary B.0.8 in Appendix B. Otherwise, the energy of the monopole is infinite. As we are interested in investigating monopoles with continuous symmetries—those monopoles whose group of symmetries is a connected Lie group—we are only interested in subgroups of $\mathrm{SO}(3)$, that is rotations.

Now that we understand the relationship between hyperbolic monopoles and elements in $\mathcal{M}_{n,k}$, we need to understand what it means for an element of $\mathcal{M}_{n,k}$ to be equivariant under a rotation.

Definition 2.1.7. We define the **gauge** action of $\mathrm{Sp}(n) \times \mathrm{O}(k)$ on $\mathcal{M}_{n,k}$ as follows. For $q \in \mathrm{Sp}(n)$, $Q \in \mathrm{O}(k)$, and $\hat{M} \in \mathcal{M}_{n,k}$, let

$$(q, Q) \cdot \hat{M} := \begin{bmatrix} q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} L \\ M \end{bmatrix} Q^T. \quad (2.5)$$

Definition 2.1.8. We define the **rotation** action of $\mathrm{Sp}(1)$ on $\mathcal{M}_{n,k}$ as follows. For $p \in \mathrm{Sp}(1)$ and $\hat{M} \in \mathcal{M}_{n,k}$, let

$$p \cdot \hat{M} := \begin{bmatrix} pLp^\dagger \\ pMp^\dagger \end{bmatrix} = p\hat{M}p^\dagger. \quad (2.6)$$

The above actions are named as such to reflect what is happening to the corresponding monopoles.

Proposition 2.1.9. Let $\hat{M} \in \mathcal{M}_{n,k}$.

- (1) For $(q, Q) \in \mathrm{Sp}(n) \times \mathrm{O}(k)$, the monopoles corresponding to $\hat{M}$ and $(q, Q) \cdot \hat{M}$ are identical.

- (2) For $p \in \mathrm{Sp}(1)$, let $R_p \in \mathrm{SO}(3)$ be the image of p under the double cover map $\mathrm{Sp}(1) \rightarrow \mathrm{SO}(3)$. Then the monopole corresponding to $p.\hat{M}$ is the pull-back of the monopole corresponding to $\hat{M}$ by R_p^T , the inverse of R_p .

Proof. (1) Let $\hat{M} \in \mathcal{M}_{n,k}$ and $(q, Q) \in \mathrm{Sp}(n) \times \mathrm{O}(k)$. Consider the monopole (Φ, A) associated to $\hat{M}$ and the monopole (Φ', A') associated to $\hat{M}' := (q, Q).\hat{M}$.

Let $X \in H^3$ and consider $\psi(X)$ satisfying $\psi(X)^\dagger \Delta(X) = 0$ and $\psi(X)^\dagger \psi(X) = I$. Let $\psi'(X) := \begin{bmatrix} q & 0 \\ 0 & Q \end{bmatrix} \psi(X)$. We see that

$$\begin{aligned} \psi'(X)^\dagger \Delta'(X) &= \psi(X)^\dagger \begin{bmatrix} q^\dagger & 0 \\ 0 & Q^T \end{bmatrix} \begin{bmatrix} qLQ^T \\ QMQ^T - I_k X \end{bmatrix} = \psi(X)^\dagger \Delta(X) Q^T = 0, \quad \text{and} \\ \psi'(X)^\dagger \psi'(X) &= \psi(X)^\dagger \begin{bmatrix} q^\dagger & 0 \\ 0 & Q^T \end{bmatrix} \begin{bmatrix} q & 0 \\ 0 & Q \end{bmatrix} \psi(X) = \psi(X)^\dagger \psi(X) = I_n. \end{aligned}$$

Thus, we can use $\psi'(X)$ to compute the monopole (Φ', A') . By the definition of μ , under the gauge transformation, μ maps to $q\mu q^\dagger$. Hence,

$$\begin{aligned} A'_i(X) &= \psi'(X)^\dagger \partial_i \psi'(X) = A_i(X), \quad \text{and} \\ \Phi'(X) &= \frac{1}{2} \psi'(X)^\dagger \begin{bmatrix} -q\mu q^\dagger & qLQ^T \\ -QL^\dagger q^\dagger & QMQ^T \end{bmatrix} \psi'(X) = \Phi(X). \end{aligned}$$

Therefore, this action does not change the monopole at all.

- (2) Let $\hat{M} \in \mathcal{M}_{n,k}$ and $p \in \mathrm{Sp}(1)$. Consider the monopole (Φ, A) associated to $\hat{M}$ and the monopole (Φ', A') associated to $\hat{M}' := p.\hat{M}$.

Let $X \in H^3$ and consider $\psi(X)$ satisfying $\psi(X)^\dagger \Delta(X) = 0$ and $\psi(X)^\dagger \psi(X) = I_n$. Let $\psi'(X) := p\psi(p^\dagger X p)$. We see that

$$\begin{aligned} \psi'(X)^\dagger \Delta'(X) &= \psi(p^\dagger X p)^\dagger p^\dagger \begin{bmatrix} pLp^\dagger \\ pMp^\dagger - I_k X \end{bmatrix} = \psi(p^\dagger X p)^\dagger \Delta(p^\dagger X p) p^\dagger = 0, \quad \text{and} \\ \psi'(X)^\dagger \psi'(X) &= \psi(p^\dagger X p)^\dagger p^\dagger p \psi(p^\dagger X p) = I_n. \end{aligned}$$

Thus, we can use $\psi'(X)$ to compute the monopole.

By the definition of μ , under the rotation transformation, μ maps to $p\mu p^\dagger$. Note that the double cover $\mathrm{Sp}(1) \rightarrow \mathrm{SO}(3)$ takes $p \mapsto R_p$, which acts on H^3 via $R_p X = pXp^\dagger$. The inverse R_p^T of R_p , acts on $\mathbb{R}^3$ via $R_p^T X = p^\dagger X p$. We see that

$$A'_i(X) = \psi'(X)^\dagger \partial_i \psi'(X) = \psi(p^\dagger X p)^\dagger \partial_i \psi(p^\dagger X p).$$

Thus, we have that for $l \in \{1, 2, 3\}$,

$$\begin{aligned} A'_i(X) dX_i(\partial_l) &= \partial_i (R_p^T X)_j \psi(R_p^\dagger X)^\dagger (\partial_j \psi) (R_p^T X) dX_i(\partial_l) \\ &= \partial_l ((R_p^T)_{jk} X_k) A_j(R_p^T X) \\ &= (R_p)_{lj} A_j(R_p^T X). \end{aligned}$$

Similarly, we have

$$\begin{aligned} (R_p^T)^* A_i(X) dX_i(\partial_l) &= A_i(R_p^T X) dX_i(dR_p^T \partial_l) \\ &= A_i(R_p^T X) dX_i((R_p^T)_{jl} \partial_j) \\ &= (R_p)_{lj} A_j(R_p^T X). \end{aligned}$$

Hence, $A' = (R_p^T)^* A$. Additionally,

$$\Phi'(X) = \frac{1}{2} \psi'(X)^\dagger \begin{bmatrix} -p\mu p^\dagger & pLp^\dagger \\ -pL^\dagger p^\dagger & pMp^\dagger \end{bmatrix} \psi'(X) = \Phi(p^\dagger X p) = ((R_p^T)^* \Phi)(X).$$

Therefore, we see that this action just rotates the monopole (in the opposite direction). $\square$

Note 2.1.10. *The gauge action does not only give a gauge-equivalent monopole, it gives the exact same monopole. Thus, while it does not correspond to a gauge transformation on the monopole, it represents a gauge freedom on the ADHM data.*

There is another gauge freedom we have, which comes from our choice of $\psi(x)$ when constructing the monopole. Indeed, taking any smooth $\mathrm{Sp}(n)$ -valued function g , we see that if $\psi(x)^\dagger \Delta(x) = 0$ and $\psi(x)^\dagger \psi(x) = I_n$, then so too does $\psi'(x) := \psi(x)g(x)$. Under this choice, the monopole changes as

$$\begin{aligned} \Phi(X) &\mapsto g(X)^\dagger \Phi(X) g(X), \quad \text{and} \\ A_i(X) &\mapsto g(X)^\dagger A_i(X) g(X) + g(X)^\dagger \partial_i g(X). \end{aligned}$$

That is, we get a gauge-equivalent monopole. However, we note that multiplying on the right by g corresponds to a choice of the orthonormal basis of $\ker \Delta(x)^\dagger$ and has no effect on $\hat{M}$.

Note 2.1.11. *The rotation action descends to an action on $\mathcal{M}_{n,k}/(\mathrm{Sp}(n) \times \mathrm{O}(k))$ given by $p.[\hat{M}] := [p.\hat{M}]$. Indeed,*

$$p.(q, Q).\hat{M} = (pqp^\dagger, Q).p.\hat{M}.$$

In particular, we see that the gauge and rotation actions do not commute on the L part of the actions, but they do commute on the M part.

Definition 2.1.12. *Let $\hat{M} \in \mathcal{M}_{n,k}$ and $p \in \mathrm{Sp}(1)$. We say that $\hat{M}$ is **equivariant** under p if $[\hat{M}]$ is fixed under the action of p . That is, there is some $(q, Q) \in \mathrm{Sp}(n) \times \mathrm{O}(k)$ such that $p.\hat{M} = (q, Q).\hat{M}$. This means that rotation by p has no effect on the monopole, as it is just the result of gauging by (q, Q) .*

It turns out that M determines L , up to a $\mathrm{Sp}(n)$ factor. The choice of this factor is just a choice of gauge, so has no effect on the monopole.

Lemma 2.1.13. *Suppose that $\hat{M} \in \mathcal{M}_{n,k}$. Then L is uniquely determined by M , up to multiplication by some $q \in \mathrm{Sp}(n)$. Conversely, multiplying L by any $q \in \mathrm{Sp}(n)$ gives another element of $\mathcal{M}_{n,k}$. Moreover, all such data produces the same monopole.*

Proof. Suppose that $\begin{bmatrix} L \\ M \end{bmatrix}, \begin{bmatrix} \tilde{L} \\ M \end{bmatrix} \in \mathcal{M}_{n,k}$. We see that $L^\dagger L - M^2 = I_k = \tilde{L}^\dagger \tilde{L} - M^2$. Hence $\tilde{L} L^\dagger L = \tilde{L} \tilde{L}^\dagger \tilde{L}$. As $\tilde{L} \tilde{L}^\dagger$ is invertible, $\tilde{L} = (\tilde{L} \tilde{L}^\dagger)^{-1} \tilde{L} L^\dagger L$. Let $q := (\tilde{L} \tilde{L}^\dagger)^{-1} \tilde{L} L^\dagger$ so $\tilde{L} = qL$. We see that

$$qq^\dagger = (\tilde{L} \tilde{L}^\dagger)^{-1} \tilde{L} L^\dagger L \tilde{L}^\dagger (\tilde{L} \tilde{L}^\dagger)^{-1} = (\tilde{L} \tilde{L}^\dagger)^{-1} \tilde{L} \tilde{L}^\dagger \tilde{L} \tilde{L}^\dagger (\tilde{L} \tilde{L}^\dagger)^{-1} = I_n.$$

Conversely, suppose that $\hat{M} \in \mathcal{M}_{n,k}$ and $q \in \mathrm{Sp}(n)$. Then $\begin{bmatrix} qL \\ M \end{bmatrix} = (q, I_k) \cdot \begin{bmatrix} L \\ M \end{bmatrix}$.

Therefore, the two matrices are related via a gauge transformation, so they are both in $\mathcal{M}_{n,k}$ and they correspond to the same monopole. $\square$

Just as M determines L , if we satisfy the M part of equivariance, then Q determines q such that we have full equivariance.

Lemma 2.1.14. *Let $\hat{M} \in \mathcal{M}_{n,k}$. Suppose that there exists $p \in \mathrm{Sp}(1)$ and $Q \in \mathrm{O}(k)$ such that $QM Q^T = pMp^\dagger$. Then there is a unique $q \in \mathrm{Sp}(n)$ such that $qLQ^T = pLp^\dagger$, which is given by $q := p(LL^\dagger)^{-1}Lp^\dagger QL^\dagger$. That is, $\hat{M}$ is equivariant under p .*

Proof. That q is unique comes from the invertibility of $LL^\dagger$. Indeed, suppose q and $\tilde{q}$ satisfy the desired equation. Then $(q - \tilde{q})LQ^T = 0$. The invertibility of Q and $LL^\dagger$ give $q = \tilde{q}$.

Define q as above. We show that $q \in \text{Sp}(n)$ and satisfies the desired equation. As $\hat{M} \in \mathcal{M}_{n,k}$, $L^\dagger L - M^2 = I_k$. Thus,

$$\begin{aligned}
qLQ^T &= p(LL^\dagger)^{-1}Lp^\dagger Q(I_k + M^2)Q^T \\
&= p(LL^\dagger)^{-1}Lp^\dagger(I_k + QMQ^TQM^2Q^T) \\
&= p(LL^\dagger)^{-1}Lp^\dagger(I_k + pM^2p^\dagger) \\
&= pLp^\dagger, \quad \text{and} \\
qq^\dagger &= p(LL^\dagger)^{-1}Lp^\dagger Q(I_k + M^2)Q^T pL^\dagger(LL^\dagger)^{-1}p^\dagger \\
&= p(LL^\dagger)^{-1}Lp^\dagger(I_k + pM^2p^\dagger)pL^\dagger(LL^\dagger)^{-1}p^\dagger \\
&= I_n,
\end{aligned}$$

proving existence. □

In light of the previous lemmas, we see that when considering monopoles equivariant under some rotation, we need only consider the M part.

Definition 2.1.15. Let $H_{\hat{M}} \subseteq \text{Sp}(1)$ be the set of unit quaternions under which $\hat{M}$ is equivariant. Because we are dealing with group actions, $H_{\hat{M}}$ is a subgroup of $\text{Sp}(1)$.

The only non-zero, connected Lie subgroups of $\text{Sp}(1)$ are S^1 and $\text{Sp}(1)$ itself. As proven in Section C.1 in Appendix C, when examining monopoles with continuous symmetries, we need only consider one circle symmetry. This follows from the fact that the maximal tori of $\text{Sp}(1)$ are S^1 and all maximal tori are conjugate. In particular, they are all conjugate to

$$H_z := \{\sin(\theta/2)k + \cos(\theta/2) \mid \theta \in [0, 4\pi)\} \subseteq \text{Sp}(1).$$

The set H_z is the double cover of the set of rotations about the z -axis. Each maximal torus of $\text{Sp}(1)$ is the double cover of the set of rotations about a fixed axis.

Definition 2.1.16. When $H_{\hat{M}}$ contains a subgroup conjugate to H_z , we say that $\hat{M}$ is *axially symmetric*. When $H_{\hat{M}} = \text{Sp}(1)$, we say that $\hat{M}$ is *spherically symmetric*.

2.1.2 Axial symmetry

In this section, we determine when a monopole is axially symmetric.

Theorem 2.1.17. *Let $\hat{M} \in \mathcal{M}_{n,k}$ and $v := \frac{k}{2} \in \mathfrak{sp}(1)$. Note that $\hat{M}$ is said to be axially symmetric about the z -axis if it is equivariant under every element of H_z . Then $\hat{M}$ is axially symmetric about the z -axis if and only if there exists $Y \in \mathfrak{so}(k)$ such that*

$$[M, Y] = [v, M]. \quad (2.7)$$

Definition 2.1.18. *The matrix $Y \in \mathfrak{so}(k)$ is called the **generator** of the axial symmetry for $\hat{M}$.*

Note 2.1.19. *Expanding $M = M_1i + M_2j + M_3k$, (2.7) is equivalent to*

$$M_1 = [M_2, Y], \quad M_2 = [Y, M_1], \quad \text{and} \quad 0 = [Y, M_3]. \quad (2.8)$$

Note 2.1.20. *For axial symmetry around a general axis in H^3 , note that this axis is given by a unit vector $\hat{u} \in \partial H^3$ and let $v := \frac{\hat{u}}{2} \in \mathfrak{sp}(1)$. Then $\hat{M}$ is spherically symmetric about $\hat{u}$ if and only if there exists $Y \in \mathfrak{so}(k)$ such that (2.7) holds.*

Proof. We follow the proof of Theorem 1.1.1, making modifications when relevant. Let $\mathcal{X}$ be the smooth manifold comprised of all $(n+k) \times k$ quaternionic matrices. The rotation and gauge actions can easily be expanded from $\mathcal{M}_{n,k}$ to smooth actions on $\mathcal{X}$. Just as with $\mathcal{M}_{n,k}$, the rotation action on $\mathcal{X}$ descends to an action on $\mathcal{X}/(\mathrm{Sp}(n) \times \mathrm{O}(k))$.

Suppose that $\hat{M} \in \mathcal{M}_{n,k}$ is axially symmetric about the z -axis, so $[\hat{M}]$ is fixed by $S^1 \simeq H_z \subseteq \mathrm{Sp}(1)$. Let $S \subseteq S^1 \times \mathrm{O}(k)$ be the stabilizer group of $\hat{M}$ restricted to axial rotations. That is

$$S = \{(p, Q) \mid pQMQ^T p^\dagger = M\}. \quad (2.9)$$

Lemma 2.1.14 tells us that if $pQMQ^T p^\dagger = M$, then $p.(p^\dagger(LL^\dagger)^{-1}LpQL^\dagger, Q).\hat{M} = \hat{M}$. With that in mind, we can write S as

$$S = \{(p, Q) \mid p.(p^\dagger(LL^\dagger)^{-1}LpQL^\dagger, Q).\hat{M} = \hat{M}\}.$$

As $\hat{M}$ is axially symmetric, clearly the map $\pi_1|_S: S \subseteq S^1 \times \mathrm{O}(k) \rightarrow S^1$ is surjective. Note that S^1 is compact and its universal cover $\mathbb{R}$ is simply-connected.

Just in the proof of Theorem 1.1.1, we have that S is a compact Lie group. We then obtain a Lie group homomorphism $\Psi: \mathbb{R} \rightarrow S$ such that $\pi_1|_S \circ \Psi = \pi$.

Pushing forward at the identity, we find that as $\pi_1|_S \circ \Psi = \pi$, there is some Lie algebra homomorphism $\rho: \mathbb{R} \rightarrow \mathfrak{so}(k)$ such that $\Psi\left(\frac{\theta}{2}\right) = \left(e^{v\theta}, e^{\rho\left(\frac{\theta}{2}\right)}\right)$. Note that as ρ is a Lie algebra representation whose image is contained in $\mathfrak{so}(k)$, it is a real Lie algebra representation.

As $\Psi\left(\frac{\theta}{2}\right) \in S$, for all $\theta \in \mathbb{R}$, we have that

$$e^{v\theta} \cdot \left(e^{-v\theta} (LL^\dagger)^{-1} L e^{v\theta} e^{\frac{\rho(\theta)}{2}} L^\dagger, e^{\frac{\rho(\theta)}{2}}\right) \cdot \hat{M} = \hat{M}.$$

Moving the first factor to the other side, we differentiate and evaluate at $\theta = 0$, obtaining

$$\begin{bmatrix} -vL + (LL^\dagger)^{-1}L\left(v + \rho\left(\frac{1}{2}\right)\right)L^\dagger L - L\rho\left(\frac{1}{2}\right) \\ \left[\rho\left(\frac{1}{2}\right), M\right] \end{bmatrix} = \begin{bmatrix} [-v, L] \\ [-v, M] \end{bmatrix}.$$

Focusing on the bottom row, let $Y := \rho\left(\frac{1}{2}\right) \in \mathfrak{so}(k)$. Then, we see that $[v, M] = [M, Y]$.

Conversely, suppose the equations are true for some $Y \in \mathfrak{so}(k)$. Let $Q(\theta) := \exp(-\theta Y) \in \text{SO}(k)$, $\theta \in \mathbb{R}$. Consider $A(\theta) := Q(\theta)^T p(\theta) M p(\theta)^\dagger Q(\theta)$. As real matrices and quaternions commute, we see

$$A'(\theta) = Q(\theta)^T p(\theta) \left([Y, M] + \left[\frac{k}{2}, M \right] \right) p(\theta)^\dagger Q(\theta) = 0.$$

As A is constant, $A(\theta) = A(0) = M$, so $Q(\theta) M Q(\theta)^T = p(\theta) M p(\theta)^\dagger$. By Lemma 2.1.14, $\hat{M}$ axially symmetric about the z -axis. $\square$

For axially symmetric monopoles, we do not need to check the final condition of $\mathcal{M}_{n,k}$ in Definition 2.1.1 at every point in $\overline{H^3}$.

Lemma 2.1.21. *Suppose that $\hat{M}$ satisfies (2.7) for some $Y \in \mathfrak{so}(k)$ as well as the first three conditions of $\mathcal{M}_{n,k}$ in Definition 2.1.1. If the final condition is satisfied at all $X = X_2 j + X_3 k \in \overline{H^3}$ with $X_2 \geq 0$, then $\hat{M} \in \mathcal{M}_{n,k}$.*

Proof. Let $Z = Z_1 i + Z_2 j + Z_3 k \in \overline{H^3}$. Recall $v := \frac{k}{2} \in \mathfrak{sp}(1)$. Then there is some $p \in \{e^{v\theta} \mid \theta \in \mathbb{R}\} \simeq S^1$ such that $p Z p^\dagger = X_2 j + X_3 k =: X$ with $X_2 \in [0, 1]$. As $\hat{M}$ is axially symmetric, there is a pair $(q, Q) \in \text{Sp}(n) \times \text{O}(k)$ such that $p \cdot (q, Q) \cdot \hat{M} = \hat{M}$. Thus,

$$\Delta(X) = \begin{bmatrix} L \\ M - I_k X \end{bmatrix} = p \begin{bmatrix} q & 0 \\ 0 & Q \end{bmatrix} \Delta(Z) Q^T p^\dagger.$$

Hence, $\Delta(X)^\dagger \Delta(X) = p Q \Delta(Z)^\dagger \Delta(Z) Q^T p^\dagger$. As $\Delta(X)^\dagger \Delta(X)$ is non-singular, so too is $\Delta(Z)^\dagger \Delta(Z)$. $\square$

2.1.3 Spherical symmetry

In this section, we determine when a monopole is spherically symmetric, prove the Structure Theorem, which greatly simplifies the construction of such monopoles, and examine novel examples of hyperbolic monopoles. We also find a constraint on the structure groups of spherically symmetric monopoles, providing a framework for classifying these monopoles. We start by proving an analogue to Theorem 2.1.17.

Theorem 2.1.22. *Let $\hat{M} \in \mathcal{M}_{n,k}$. Recall that $\hat{M}$ is spherically symmetric when it is equivariant under all $p \in \mathrm{Sp}(1)$. Then $\hat{M}$ is spherically symmetric if and only if there exists real representation $(\mathbb{R}^k, \rho)$ with $\rho: \mathfrak{sp}(1) \rightarrow \mathfrak{so}(k)$, such that for all $v \in \mathfrak{sp}(1)$,*

$$[M, \rho(v)] = [v, M]. \quad (2.10)$$

Definition 2.1.23. *We call ρ the **generating representation** of the spherical symmetry of $\hat{M}$.*

Note 2.1.24. *Let $v_1 := \frac{i}{2}, v_2 := \frac{j}{2}, v_3 := \frac{k}{2} \in \mathfrak{sp}(1)$. The statement above is equivalent to finding a triple $Y_1, Y_2, Y_3 \in \mathfrak{so}(k)$ inducing a representation of $\mathfrak{sp}(1)$, such that for $l \in \{1, 2, 3\}$,*

$$[M, Y_l] = [v_l, M]. \quad (2.11)$$

Expanding $M = M_1i + M_2j + M_3k$, (2.11) are equivalent to, for all $i, j \in \{1, 2, 3\}$,

$$[Y_a, M_b] = \sum_{c=1}^3 \epsilon_{abc} M_c.$$

Proof. We follow the proof of Theorem 1.1.1, making modifications when relevant. Suppose that $\hat{M} \in \mathcal{M}_{n,k}$ is spherically symmetric, so $[\hat{M}]$ is fixed by $\mathrm{Sp}(1)$. Let $S \subseteq \mathrm{Sp}(1) \times \mathrm{O}(k)$ be the stabilizer group of $\hat{M}$. As $\hat{M}$ is spherically symmetric, clearly the map $\pi_1|_S: S \subseteq \mathrm{Sp}(1) \times \mathrm{O}(k) \rightarrow \mathrm{Sp}(1)$ is surjective. Note that $\mathrm{Sp}(1)$ is simply-connected and compact.

Just as in the axial case, S is a compact Lie group and we obtain a Lie group homomorphism $\Psi: \mathrm{Sp}(1) \rightarrow S$ such that $\pi_1|_S \circ \Psi = \mathrm{id}_{\mathrm{Sp}(1)}$.

Similar to the axial case, we know that there is a Lie algebra homomorphism $\rho: \mathfrak{sp}(1) \rightarrow \mathfrak{so}(k)$ such that for all $v \in \mathfrak{sp}(1)$, $\Psi(e^v) = (e^v, e^{\rho(v)})$. As ρ is a Lie algebra homomorphism whose image consists of real matrices, ρ gives us a real Lie algebra representation.

As $\Psi(e^{\theta v}) \in S$, for all $\theta \in \mathbb{R}$ and $v \in \mathfrak{sp}(1)$, by Lemma 2.1.14, we have that

$$e^{\theta v} \cdot (e^{-\theta v} (LL^\dagger)^{-1} L e^{\theta v} e^{\theta \rho(v)} L^\dagger, e^{\theta \rho(v)}) \cdot \hat{M} = \hat{M}.$$

Moving the first factor to the other side, we differentiate and evaluate at $\theta = 0$, obtaining

$$\begin{bmatrix} -vL + (LL^\dagger)^{-1}L(v + \rho(v))L^\dagger L - L\rho(v) \\ [\rho(v), M] \end{bmatrix} = \begin{bmatrix} [-v, L] \\ [-v, M] \end{bmatrix}. \quad (2.12)$$

We examine the top row in Section 2.1.3.3. For the current proof, we note that the bottom row gives us (2.10).

Conversely, suppose that the equations hold for some real representation $\rho: \mathfrak{sp}(1) \rightarrow \mathfrak{so}(k)$. Let $p \in \mathrm{Sp}(1)$. As the exponential map $\exp: \mathfrak{sp}(1) \rightarrow \mathrm{Sp}(1)$ is surjective, there is some $v \in \mathfrak{sp}(1)$ such that $p = \exp(v)$. Let $p(\theta) := \exp(\theta v)$ and $Q(\theta) := \exp(-\theta \rho(v))$.

We show that $A(\theta) := p(\theta)^\dagger Q(\theta) M Q(\theta)^T p(\theta)$ is constant. Indeed,

$$A'(\theta) = -p(\theta)^\dagger Q(\theta) ([v, M] + [\rho(v), M]) Q(\theta)^T p(\theta) = 0.$$

Hence, A is constant, so

$$p^\dagger Q(1) M Q(1)^T p = A(1) = A(0) = M.$$

Therefore, $Q(1) M Q(1)^T = p M p^\dagger$. Then by Lemma 2.1.14, we have that $\hat{M}$ is p -equivariant. As p was arbitrary, we have that $\hat{M}$ is spherically symmetric. $\square$

For spherically symmetric monopoles, we only have to check the final condition of $\mathcal{M}_{n,k}$ in Definition 2.1.1 along a ray from the origin of $\overline{H^3}$.

Lemma 2.1.25. *Suppose that $\hat{M}$ satisfies (2.11) for some $Y_1, Y_2, Y_3 \in \mathfrak{so}(k)$ as well as the first three conditions of $\mathcal{M}_{n,k}$ in Definition 2.1.1. If the final condition is satisfied at $X = X_3 k \in \overline{H^3}$, for all $X_3 \in [0, 1]$, then $\hat{M} \in \mathcal{M}_{n,k}$.*

Proof. We follow the same proof as Lemma 2.1.21, but note for any $Z = Z_1 i + Z_2 j + Z_3 k \in \overline{H^3}$, there is some $p \in \mathrm{Sp}(1)$ such that $p Z p^\dagger = X_3 k =: X$ with $X_3 \in [0, 1]$. $\square$

2.1.3.1 The structure of spherical symmetry

Theorem 2.1.22 tells us how to search for spherically symmetric monopoles: use a real k -representation of $\mathfrak{sp}(1)$ to narrow down the possible M . We are then only left with finding L that gives us $\hat{M} \in \mathcal{M}_{n,k}$. We now investigate what representations generate spherically symmetric monopoles and what the corresponding ADHM data looks like.

Definition 2.1.26. Let $\text{ad}: \mathfrak{sp}(1) \rightarrow \mathfrak{gl}(\mathfrak{sp}(1))$ be the Lie algebra homomorphism $\text{ad}(x)(z) := [x, z]$. Denote the adjoint representation of $\mathfrak{sp}(1)$ by $(\mathfrak{sp}(1), \text{ad})$ and let $V := \mathbb{R}^k$. Given a real representation (V, ρ) of $\mathfrak{sp}(1)$, we define the induced real representation

$$(\hat{V}, \hat{\rho}) := (V, \rho) \otimes_{\mathbb{R}} (V^*, \rho^*) \otimes_{\mathbb{R}} (\mathfrak{sp}(1), \text{ad}). \quad (2.13)$$

Unraveling how $\hat{\rho}$ acts on $\hat{V} = \text{Mat}(k, k, \mathfrak{sp}(1))$, let $v \in \mathfrak{sp}(1)$ and $A \in \text{Mat}(k, k, \mathfrak{sp}(1))$. Then

$$\hat{\rho}(v)(A) = [\rho(v), A] + [v, A].$$

Note 2.1.27. By Proposition D.2.9 in Appendix D, all representations of $\mathfrak{sp}(1)$ are self-dual.

The definition of $(\hat{V}, \hat{\rho})$ is well-motivated. Firstly, note that $M \in \hat{V}$. Secondly, given the connection between the action of $\hat{\rho}(v)$ and (2.10), we immediately obtain the following corollary.

Corollary 2.1.28. Let $\hat{M} \in \mathcal{M}_{n,k}$. Then $\hat{M}$ is spherically symmetric if and only if there is some real k -representation (V, ρ) such that $\hat{\rho}(v)(M) = 0$, for all $v \in \mathfrak{sp}(1)$.

Note 2.1.29. Suppose $\hat{M} \in \mathcal{M}_{n,k}$ is spherically symmetric. Corollary 2.1.28 tells us that there is some real representation (V, ρ) such that $\text{span}(M) \subseteq \hat{V}$ is an invariant subspace, which is acted on trivially.

As $\mathfrak{sp}(1)$ is semi-simple, the representation $(\hat{V}, \hat{\rho})$ decomposes into irreducible representations. We explore the case $M = 0$ below, so suppose $M \neq 0$. As $\text{span}(M)$ is a one-dimensional invariant subspace, acted on trivially, $(\text{span}(M), 0)$ is a summand of the representation. So the decomposition of $(\hat{V}, \hat{\rho})$ must contain trivial summands. Moreover, M is in the direct sum of these trivial summands. Trivially, if $M = 0$, then it is in the direct sum of trivial summands as well.

Proposition 2.1.30 (The $M = 0$ case). Suppose $M = 0_k$ and $\hat{M} \in \mathcal{M}_{n,k}$. Then $n = k$ and $L \in \text{Sp}(k)$. Furthermore, gauging by $L^\dagger$ gives us $L = I_k$. Such a $\hat{M}$ is spherically symmetric.

Moreover, for all $X = X_1 i + X_2 j + X_3 k \in H^3$, let $R := \sqrt{X_1^2 + X_2^2 + X_3^2}$. Also, let the norm of an element $A \in \mathfrak{sp}(k)$ be given by $|A|^2 := -\text{Tr}(A^2)$. Then we have that the corresponding monopole's Higgs field satisfies $|\Phi(X)| = \sqrt{k} \frac{R}{1+R^2}$. Also, up to gauge, we have that

$$\Phi(X) = \frac{X}{R^2 + 1} I_k.$$

Thus, the corresponding monopole has generalized minimal symmetry breaking, as given by Definition 2.3.17.

Proof. Suppose $M = 0$, so $\mu = 0$. Recall from Note 2.1.3 that $n \leq k$. As $\text{rank}(L^\dagger L) \leq n$ and $L^\dagger L = I_k + M^2 = I_k$, we see that $n = k$. As $L^\dagger L = I_k$, $L \in \text{Sp}(k)$. We can gauge the data without changing the monopole, so we may gauge by $L^\dagger$, to obtain $\tilde{L} := L^\dagger L = I_k$. Also, note that $M = 0$ satisfies the spherical symmetry conditions for any representation ρ .

We now wish to construct the corresponding monopole. Note that $\Delta(X) = \begin{bmatrix} I_k \\ -XI_k \end{bmatrix}$.

We see that $\psi(X) := \frac{1}{\sqrt{R^2+1}} \begin{bmatrix} -XI_k \\ I_k \end{bmatrix}$ satisfies $\psi(X)^\dagger \Delta(X) = 0$ and $\psi(X)^\dagger \psi(X) = I_k$.

Using this $\psi(X)$, by (2.2) we have, up to gauge,

$$\Phi(X) = \frac{1}{2(R^2+1)} \begin{bmatrix} XI_k & I_k \end{bmatrix} \begin{bmatrix} 0 & I_k \\ -I_k & 0 \end{bmatrix} \begin{bmatrix} -XI_k \\ I_k \end{bmatrix} = \frac{X}{R^2+1} I_k.$$

We see that such a Higgs field has norm $\sqrt{k} \frac{R}{1+R^2}$. Also, by Theorem 2.3.37, this monopole has generalized minimal symmetry breaking. $\square$

Note 2.1.31. Proposition 2.1.30 tells us that $\text{Sp}(k)$ monopoles with $M = 0$ have ADHM data in $\mathcal{M}_{k,k}$ and are reducible, being constructed by applying a homomorphism $\text{Sp}(1) \rightarrow \text{Sp}(k)$ to the basic spherically symmetric instanton number one $\text{Sp}(1)$ monopole.

As mentioned in Section D.1.3 of Appendix D, the irreducible real representations of $\mathfrak{sp}(1)$ are indexed by the dimension of the vector space acted upon; every irreducible real k -representation of $\mathfrak{sp}(1)$ is isomorphic. However, irreducible real representations only exist for $k \in \mathbb{N}_+$ odd or divisible by four. These representations are denoted by $(\mathbb{R}^k, \varrho_k)$. The tensor product of real representations is simplified by using complex representations, so we shall look at these too.

Like the irreducible real representations, irreducible complex representations of $\mathfrak{sp}(1)$ are indexed by the dimension of the vector space acted upon. Unlike the irreducible real representations, a unique irreducible complex representation exists for all $k \in \mathbb{N}_+$, up to isomorphism and is denoted by (V_k, ρ_k) . Note that the dimension of the vector space acted on k and the highest weight of the representation w are related via $w = \frac{k-1}{2}$.

The irreducible complex and real representations are related as follows. For k odd or divisible by four, ϱ_k can be complexified and made to act on $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^k \simeq \mathbb{C}^k$ as $\text{id}_{\mathbb{C}} \otimes \varrho_k$. The complexified representation $(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^k, \text{id}_{\mathbb{C}} \otimes \varrho_k)$ is a complex representation. For k odd, as complex representations, $(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^k, \text{id}_{\mathbb{C}} \otimes \varrho_k) \simeq (V_k, \rho_k)$. For k divisible by four,

as complex representations, $(\mathbb{C} \otimes_{\mathbb{R}} \mathbb{R}^k, \text{id}_{\mathbb{C}} \otimes_{\mathbb{R}} \varrho_k) \simeq (V_{k/2}, \rho_{k/2})^{\oplus 2}$. Thus, for k odd, the matrices that induce (V_k, ρ_k) can be chosen such that they are real matrices.

In particular, we note the following representations. The adjoint representation $(\mathfrak{sp}(1), \text{ad})$ is the irreducible 3-dimensional representation (V_3, ρ_3) . Let $F: \mathfrak{sp}(1) \rightarrow \mathfrak{su}(2)$ be the Lie algebra isomorphism given by

$$F(ai + bj + ck) := \begin{bmatrix} ai & b + ci \\ -b + ci & -ai \end{bmatrix}.$$

The fundamental representation $(\mathbb{C}^2, F)$ is the irreducible 2-representation (V_2, ρ_2) . Finally, the trivial representation $(\mathbb{C}, 0)$ is the irreducible 1-representation (V_1, ρ_1) .

As mentioned earlier, as $\mathfrak{sp}(1)$ is a semi-simple Lie algebra, all representations of it decompose as the direct sum of irreducible representations. The Clebsch–Gordon Decomposition tells us how to decompose tensor products of complex representations: for $m \geq n \geq 1$,

$$(V_m, \rho_m) \otimes_{\mathbb{C}} (V_n, \rho_n) \simeq \bigoplus_{k=1}^n (V_{m+n+1-2k}, \rho_{m+n+1-2k}).$$

As discussed earlier, given a spherically symmetric monopole with ADHM data $\hat{M} \in \mathcal{M}_{n,k}$, M is found in the trivial summands of $(\hat{V}, \hat{\rho})$. The following lemmas tell us where these trivial summands are, given a decomposition of $(V, \hat{\rho})$. First, we see where the trivial summands are when decomposed as a complex representation, providing us with the information needed for the real decomposition.

Lemma 2.1.32. *Given $m \geq n \geq 1$, we have that $(V_m, \rho_m) \otimes_{\mathbb{C}} (V_n, \rho_n) \otimes_{\mathbb{C}} (V_3, \rho_3)$ has a single trivial summand when $m = n \geq 2$ or $m = n + 2$. Otherwise, it does not have any such summands.*

Proof. We first see what representations give a trivial summand when taken as a tensor product with (V_3, ρ_3) . We see

$$(V_3, \rho_3) \otimes_{\mathbb{C}} (V_l, \rho_l) \simeq \begin{cases} (V_3, \rho_3) & \text{if } l = 1 \\ (V_4, \rho_4) \oplus (V_2, \rho_2) & \text{if } l = 2 \\ (V_{l+2}, \rho_{l+2}) \oplus (V_l, \rho_l) \oplus (V_{l-2}, \rho_{l-2}) & \text{if } l \geq 3 \end{cases}.$$

Therefore, we see that $(V_3, \rho_3) \otimes_{\mathbb{C}} (V_l, \rho_l)$ has a single trivial summand if and only if $l = 3$. Thus, $(V_m, \rho_m) \otimes_{\mathbb{C}} (V_n, \rho_n) \otimes_{\mathbb{C}} (V_3, \rho_3)$ has a (V_1, ρ_1) summand if and only if $(V_m, \rho_m) \otimes_{\mathbb{C}} (V_n, \rho_n)$ has a (V_3, ρ_3) summand.

As we have

$$(V_m, \rho_m) \otimes_{\mathbb{C}} (V_n, \rho_n) \simeq \bigoplus_{k=1}^n (V_{m+n+1-2k}, \rho_{m+n+1-2k}),$$

we see that there is a single (V_3, ρ_3) summand exactly when $m + n + 1 - 2k = 3$ for some $k \in \{1, \dots, n\}$. Thus, $m+n$ must be even, so they must have the same parity. Furthermore, as $m + n + 1 - 2k \geq m + 1 - n$, we must have $m + 1 - n \in \{1, 3\}$ (if it is two, then m and n have different parity). The condition $m + 1 - n \in \{1, 3\}$ is satisfied if and only if $m = n$ or $m = n + 2$.

However, as $m + n + 1 - 2k \leq m + n - 1$, we must have $m + n - 1 \geq 3$. Thus, $m + n \geq 4$. We see that if $m = n + 2$, then $m + n \geq 4$, but if $m = n$, then we must have $m = n \geq 2$. In each case, we get a single (V_3, ρ_3) summand, so the final product has a single (V_1, ρ_1) summand. Otherwise, we get no such summand. $\square$

We first investigate the trivial summand present in $(V_{n+2}, \rho_{n+2}) \otimes_{\mathbb{C}} (V_n^*, \rho_n^*) \otimes_{\mathbb{C}} (V_3, \rho_3)$.

Definition 2.1.33. *Let $n \in \mathbb{N}_+$ and let B^n be the generator of the unique trivial summand in the tensor product $(V_{n+2}, \rho_{n+2}) \otimes_{\mathbb{C}} (V_n^*, \rho_n^*) \otimes_{\mathbb{C}} (V_3, \rho_3)$. This generator is well-defined, up to a $\mathbb{C}^*$ factor, the choice of a scale leaves a $U(1)$ factor. The generator can be viewed as an $\mathfrak{sp}(1)$ -invariant triple (B_1^n, B_2^n, B_3^n) of complex $(n+2) \times n$ matrices.*

In previous work, we prove the following, giving us some useful identities for these maps [CDL⁺22, Theorem 4.8].

Proposition 2.1.34. *Given $n \in \mathbb{N}_+$, let $Y_i^+ := \rho_{n+2}(v_i) \in \mathfrak{su}(n+2)$ and $Y_i^- := \rho_n(v_i) \in \mathfrak{su}(n)$. After potential rescaling, reducing the factor to $U(1)$, we have the following identities*

(for all $i, j \in \{1, 2, 3\}$, where it applies)

$$Y_i^+ B_j^n - B_j^n Y_i^- = \sum_{l=1}^3 \epsilon_{ijl} B_l^n, \quad (2.14a)$$

$$\sum_{l=1}^3 B_l^n (B_l^n)^\dagger = I_{n+2}, \quad (2.14b)$$

$$\sum_{l=1}^3 (B_l^n)^\dagger B_l^n = \frac{n+2}{n} I_n, \quad (2.14c)$$

$$B_i^n (B_j^n)^\dagger - B_j^n (B_i^n)^\dagger = -\frac{2}{n+1} \sum_{l=1}^3 \epsilon_{ijl} Y_l^+, \quad (2.14d)$$

$$(B_i^n)^\dagger B_j^n - (B_j^n)^\dagger B_i^n = \frac{2(n+2)}{n(n+1)} \sum_{l=1}^3 \epsilon_{ijl} Y_l^-, \quad (2.14e)$$

$$Y_i^+ B_j^n - Y_j^+ B_i^n = \frac{n+3}{2} \sum_{l=1}^3 \epsilon_{ijl} B_l^n, \quad \text{and} \quad (2.14f)$$

$$B_i^n Y_j^- - B_j^n Y_i^- = -\frac{n-1}{2} \sum_{l=1}^3 \epsilon_{ijl} B_l^n. \quad (2.14g)$$

Above we determined where the trivial summands of $(\hat{V}, \hat{\rho})$ would occur if we decomposed it as a complex representation. We use this information to do the same for the real decomposition.

Lemma 2.1.35. *Given $m \geq n \geq 1$, we have that $(\mathbb{R}^m, \varrho_m) \otimes_{\mathbb{R}} (\mathbb{R}^n, \varrho_n) \otimes_{\mathbb{R}} (\mathbb{R}^3, \varrho_3)$ has a trivial summand when $m = n \geq 2$, both odd, or $m = n + 2$, both odd, and four trivial summands when $m = n \geq 4$, both divisible by four, or $m = n + 4$, both divisible by four. Otherwise, there are no trivial summands.*

Proof. Note that if m and n are both odd, then the complexification of the real representation is isomorphic, as a complex representation, to $(V_m, \rho_m) \otimes_{\mathbb{C}} (V_n, \rho_n) \otimes_{\mathbb{C}} (V_3, \rho_3)$. By Lemma 2.1.32, we know that this tensor product of complex representations has a single trivial summand when $m = n \geq 2$ or $m = n + 2$ and none otherwise. Furthermore, as this tensor product only contains odd dimensional summands, the representation is real, meaning that the real decomposition is the same as the complex. Thus, we have proven the first part of the lemma.

Suppose that m and n are both divisible by four. Then, as a complex representation, the complexification of the real representation is isomorphic to $(V_{m/2}, \rho_{m/2})^{\oplus 2} \otimes_{\mathbb{C}} (V_{n/2}, \rho_{n/2})^{\oplus 2} \otimes_{\mathbb{C}} (V_3, \rho_3)$. By Lemma 2.1.32, we know that this has four trivial summands when $\frac{m}{2} = \frac{n}{2} \geq 2$ or $\frac{m}{2} = \frac{n}{2} + 2$ and none otherwise. Just as above, this decomposition contains only odd dimensional summands, so the representation is real and the real and complex decompositions coincide. Thus, we have proven the second part of the lemma.

Finally, suppose that one of n and m is divisible by four and the other is odd. Then, decomposing the complexification of the real representation as a complex representation and expanding the tensor product, we find that the tensor product contains only even dimensional summands, meaning there are no trivial summands, proving the lemma. $\square$

Using Lemma 2.1.35, we investigate the trivial summands of $(\mathbb{R}^m, \varrho_m) \otimes_{\mathbb{R}} (\mathbb{R}^n, \varrho_n) \otimes_{\mathbb{R}} (\mathbb{R}^3, \varrho_3)$, starting with the case $m \neq n$.

Lemma 2.1.36. *Given $n \in \mathbb{N}_+$ odd, let $Y_i^+ := \varrho_{n+2}(v_i) \in \mathfrak{so}(n+2)$ and $Y_i^- := \varrho_n(v_i) \in \mathfrak{so}(n)$. Then there is a unique choice of $U(1)$ factor, up to a choice of sign, such that the B_i^n matrices are real. That is, given this choice of factor, B^n spans the unique trivial summand of $(\mathbb{R}^{n+2}, \varrho_{n+2}) \otimes_{\mathbb{R}} ((\mathbb{R}^n)^*, \varrho_n^*) \otimes_{\mathbb{R}} (\mathbb{R}^3, \varrho_3)$*

Given $n \in \mathbb{N}_+$ divisible by four, let $Y_i^+ := \varrho_{n+4}(v_i) \in \mathfrak{so}(n+4)$ and $Y_i^- := \varrho_n(v_i) \in \mathfrak{so}(n)$. Let $y_i^+ := \rho_{(n+4)/2}(v_i) \in \mathfrak{su}(\frac{n+4}{2})$ and $y_i^- := \rho_{n/2}(v_i) \in \mathfrak{su}(\frac{n}{2})$. As the complexification of $(\mathbb{R}^{n+4}, \varrho_{n+4})$ is isomorphic to $(V_{(n+4)/2}, \rho_{(n+4)/2})^{\oplus 2}$ as complex representations, and similarly for $(\mathbb{R}^n, \varrho_n)$, there exists $U_{\pm} \in \mathrm{SU}(\frac{n+2 \pm 2}{2})$ such that

$$Y_i^{\pm} = U_{\pm}^{\dagger} \begin{bmatrix} y_i^{\pm} & 0 \\ 0 & y_i^{\pm} \end{bmatrix} U_{\pm}.$$

We can find four linearly independent triples of real matrices spanning the four trivial summands of $(\mathbb{R}^m, \varrho_m) \otimes_{\mathbb{R}} (\mathbb{R}^n, \varrho_n) \otimes_{\mathbb{R}} (\mathbb{R}^3, \varrho_3)$. For some choices of $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, these linearly independent triples of real matrices are given by

$$\left(U_{+}^{\dagger} \begin{bmatrix} \alpha B_i^{\frac{n}{2}} & \beta B_i^{\frac{n}{2}} \\ \gamma B_i^{\frac{n}{2}} & \delta B_i^{\frac{n}{2}} \end{bmatrix} U_{-} \right)_{i=1}^3. \quad (2.15)$$

Proof. Suppose that n is odd. Note that $Y_i^{\pm} \in \mathfrak{su}(n+1 \pm 1)$. We are looking for a non-zero triple (C_1^n, C_2^n, C_3^n) of real matrices that satisfy (2.14a), replacing B_i^n by C_i^n . Such a triple exists. Indeed, from Lemma 2.1.35, we see that the tensor product $(\mathbb{R}^{n+2}, \varrho_{n+2}) \otimes_{\mathbb{R}}$

$((\mathbb{R}^n)^*, \varrho_n^*) \otimes_{\mathbb{R}} (\mathbb{R}^3, \varrho_3)$ contains a single trivial summand. Any element that spans this real summand satisfies (2.14a). Hence, any such element is in the span of B^n , as the solution space to (2.14a) is a one dimensional complex vector space. Therefore, the $U(1)$ factor that determines B_i^n can be chosen such that the B_i^n matrices are all real.

Suppose instead that n is divisible by four. Note that for $\alpha, \beta, \gamma, \delta \in \mathbb{C}$,

$$Y_i^+ U_+^\dagger \begin{bmatrix} \alpha B_j^{\frac{n}{2}} & \beta B_j^{\frac{n}{2}} \\ \gamma B_j^{\frac{n}{2}} & \delta B_j^{\frac{n}{2}} \end{bmatrix} U_- - U_+^\dagger \begin{bmatrix} \alpha B_j^{\frac{n}{2}} & \beta B_j^{\frac{n}{2}} \\ \gamma B_j^{\frac{n}{2}} & \delta B_j^{\frac{n}{2}} \end{bmatrix} U_- Y_i^- = \sum_{l=1}^3 \epsilon_{ijl} U_+^\dagger \begin{bmatrix} \alpha B_l^{\frac{n}{2}} & \beta B_l^{\frac{n}{2}} \\ \gamma B_l^{\frac{n}{2}} & \delta B_l^{\frac{n}{2}} \end{bmatrix} U_-. \quad (2.16)$$

By Lemma 2.1.35, there are four linearly independent triples of real matrices satisfying (2.14a). That the solution space is four dimensional comes from counting the trivial summands of $(\mathbb{R}^{n+4}, \varrho_{n+4}) \otimes_{\mathbb{R}} ((\mathbb{R}^n)^*, \varrho_n^*) \otimes_{\mathbb{R}} (\mathbb{R}^3, \varrho_3)$. After complexifying this tensor product, we see that it is isomorphic, as a complex representation, to $(V_{(n+4)/2}, \rho_{(n+4)/2})^{\oplus 2} \otimes_{\mathbb{C}} (V_{n/2}^*, \rho_{n/2}^*)^{\oplus 2} \otimes_{\mathbb{C}} (V_3, \rho_3)$. Given (2.16), we see that the direct sum of the four trivial summands of the above complex tensor product are given by

$$E := \left\{ \left(U_+^\dagger \begin{bmatrix} \alpha B_i^{\frac{n}{2}} & \beta B_i^{\frac{n}{2}} \\ \gamma B_i^{\frac{n}{2}} & \delta B_i^{\frac{n}{2}} \end{bmatrix} U_- \right)^3 \middle| \alpha, \beta, \gamma, \delta \in \mathbb{C} \right\}. \quad (2.17)$$

As the four linearly independent triples of real matrices above satisfy the same equation as the matrices in E , they belong to E . Thus, for some choices of the complex parameters in E , we can find four linearly independent triples of real matrices. $\square$

Note 2.1.37. *In the n divisible by four case above, we cannot say what relationship between the complex parameters $\alpha, \beta, \gamma, \delta$ gives a triple of real matrices without being given specific $U_\pm$. Indeed, suppose the triple $Y_i^\pm$ generating $(\mathbb{R}^{n+2\pm 2}, \varrho_{n+2\pm 2})$, the triple $y_i^\pm$ generating $(V_{(n+2\pm 2)/2}, \rho_{(n+2\pm 2)/2})$, and $U_\pm$ satisfy*

$$Y_i^\pm = U_\pm^\dagger \begin{bmatrix} y_i^\pm & 0 \\ 0 & y_i^\pm \end{bmatrix} U_\pm.$$

These relationships are still satisfied when replacing $U_\pm$ by $\tilde{U}_\pm := \lambda_\pm U_\pm$ for any $\lambda_\pm \in \{\pm 1, \pm i\}$. While these $U(1)$ factors cancel to create the $Y_i^\pm$ matrices from the $y_i^\pm$, they do not cancel when searching for our triples of real matrices, affecting the relationships between the $\alpha, \beta, \gamma, \delta$.

We now investigate the rest of the trivial summands of $(\hat{V}, \hat{\rho})$.

Lemma 2.1.38. *Given $n \in \mathbb{N}_+$, let $Y_i := \varrho_n(v_i) \in \mathfrak{so}(n)$. If n is odd, the trivial summand of $(\mathbb{R}^n, \varrho_n) \otimes_{\mathbb{R}} ((\mathbb{R}^n)^*, \varrho_n^*) \otimes_{\mathbb{R}} (\mathbb{R}^3, \varrho_3)$ is spanned by the triple (Y_1, Y_2, Y_3) .*

If n is divisible by four, Lemma 2.1.35 tells us that there are four trivial summands of $(\mathbb{R}^n, \varrho_n) \otimes_{\mathbb{R}} ((\mathbb{R}^n)^*, \varrho_n^*) \otimes_{\mathbb{R}} (\mathbb{R}^3, \varrho_3)$. Let $y_i := \rho_{n/2}(v_i) \in \mathfrak{su}(\frac{n}{2})$ induce the irreducible complex $\frac{n}{2}$ -representation. Let $U \in \mathrm{SU}(\frac{n}{2})$ such that

$$Y_i = U^\dagger \begin{bmatrix} y_i & 0 \\ 0 & y_i \end{bmatrix} U.$$

The trivial summands are spanned by the following triples of real matrices

$$(Y_i)_{i=1}^3, \left(Y_i U^\dagger \begin{bmatrix} iy_i & 0 \\ 0 & -iy_i \end{bmatrix} U \right)_{i=1}^3, \left(Y_i U^\dagger \begin{bmatrix} 0 & y_i \\ -y_i & 0 \end{bmatrix} U \right)_{i=1}^3, \left(Y_i U^\dagger \begin{bmatrix} 0 & iy_i \\ iy_i & 0 \end{bmatrix} U \right)_{i=1}^3. \quad (2.18)$$

Proof. Suppose that X is a real matrix commuting with Y_1, Y_2, Y_3 . Then

$$[Y_i, Y_j X] = [Y_i, Y_j] X = \sum_{l=1}^3 \epsilon_{ijl} Y_l X.$$

Thus, the triple $(Y_i X)$ is a trivial summand of $(\mathbb{R}^n, \varrho_n) \otimes_{\mathbb{R}} ((\mathbb{R}^n)^*, \varrho_n^*) \otimes_{\mathbb{R}} (\mathbb{R}^3, \varrho_3)$. Below, we identify exactly what matrices commute with real irreducible representations. These provide us with exactly enough matrices to span the rest of the trivial summands of $(\hat{V}, \hat{\rho})$.

Let $\mathfrak{g}$ be a Lie algebra and (V, ρ) an irreducible representation, over a field $\mathbb{F}$. An endomorphism of the representation is a morphism $f: V \rightarrow V$ with f a linear map such that for all $x \in \mathfrak{g}$ and $v \in V$,

$$\rho(x)(f(v)) = f(\rho(x)(v)).$$

Schur's Lemma tells us that the only endomorphisms are the zero morphism or an isomorphism [BTD85, Ch. II, Lemma 1.10]. This gives the endomorphism ring of the representation the structure of a division algebra over $\mathbb{F}$.

Let f be an endomorphism. As f and $\rho(x)$ are endomorphism, we can associate them to matrices. As matrices, $[f, \rho(x)] = 0$ for all $x \in \mathfrak{g}$. That is, the endomorphism ring is exactly the set of matrices that commute with the representation, and we will think of endomorphisms as these matrices going forward. Let $(\tilde{V}, \tilde{\rho}) := (V, \rho) \otimes_{\mathbb{F}} (V^*, \rho^*)$. Note that for $v \in \mathfrak{sp}(1)$, $\tilde{\rho}(v)$ acts on $f \in V \otimes_{\mathbb{F}} V^*$ as $\tilde{\rho}(v)(f) = [\rho(v), f]$. Thus, f is in the

endomorphism ring if and only if f is in a trivial summand of $(V, \rho) \otimes_{\mathbb{F}} (V^*, \rho^*)$. We are interested in the case $\mathfrak{g} = \mathfrak{sp}(1)$ and $\mathbb{F} = \mathbb{R}$.

The only real division algebras are $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$. In the case n is odd, we have that $(\mathbb{R}^n, \varrho_n) \otimes_{\mathbb{R}} ((\mathbb{R}^n)^*, \varrho_n^*)$ contains a single trivial summand, as the complexification of $(\mathbb{R}^n, \varrho_n)$ is irreducible as a complex representation. Thus, the endomorphism ring is isomorphic to $\mathbb{R}$. As the isomorphism sends I_k to 1, the endomorphism ring is just the span of the identity matrix.

In the case n is divisible by four, we have that $(\mathbb{R}^n, \varrho_n) \otimes_{\mathbb{R}} ((\mathbb{R}^n)^*, \varrho_n^*)$ contains four trivial summands, as the complexification of $(\mathbb{R}^n, \varrho_n)$ is isomorphic to $(V_{n/2}, \rho_{n/2})^{\oplus 2}$. Thus, the endomorphism ring is isomorphic to $\mathbb{H}$. We show that if X is a $n \times n$ real matrix commuting with $\varrho_n(x)$ for all $x \in \mathfrak{sp}(1)$, then $X - \frac{\text{Tr}(X)}{n} I_n \in \mathfrak{so}(n)$.

We have that there is some ring isomorphism $\psi: \text{End}(V, \rho) \rightarrow \mathbb{H}$. Let $\psi_0(X)$ be the real part of $\psi(X)$. Then $\psi(X - \psi_0(X)I_n) \in \mathfrak{sp}(1)$, so

$$\psi((X - \psi_0(X)I_n)^2) = \psi(X - \psi_0(X)I_n)^2 = -|\psi(X - \psi_0(X)I_n)|^2 \leq 0.$$

Let $\alpha := |\psi(X - \psi_0(X)I_n)|^2 \geq 0$. Then

$$(X - \psi_0(X)I_n)^2 = -\alpha I_n.$$

We know that the ring of $n \times n$ real matrices decomposes as

$$\text{Mat}(n, n, \mathbb{R}) \simeq \text{Span}_{\mathbb{R}}(I_n) \oplus \bigwedge^2 \mathbb{R} \oplus \text{Sym}_0(\mathbb{R}^n).$$

Thus, given X , there is a unique antisymmetric matrix A , a unique traceless, symmetric matrix B , and a unique $\beta \in \mathbb{R}$ such that $X - \psi_0(X)I_n = \beta I_n + A + B$. As $\psi(X - \psi_0(X)I_n) \in \mathfrak{sp}(1)$, we know that $\beta = 0$, as A and B give us something in $\mathfrak{sp}(1)$. Thus, $X - \psi_0(X)I_n = A + B$.

Note that $[Y_i, X] = 0$ for all i . Thus, $[Y_i, A + B] = 0$ for all i , so $[Y_i, A] = -[Y_i, B]$. Note that as A and Y_i are anti-symmetric, the left-hand-side is anti-symmetric. As Y_i is anti-symmetric and B is symmetric, the right-hand-side is symmetric. But the only symmetric and anti-symmetric matrix is the zero matrix. Therefore, $[Y_i, A] = 0 = [Y_i, B]$. Thus, A and B correspond to endomorphisms of $(\mathbb{R}^n, \varrho_n)$ themselves, just like X . We show that B must vanish.

Thus, just as we did above for $X - \psi_0(X)I_n$, $B^2 = -\gamma I_n$ for some $\gamma \geq 0$. Note that $\psi_0(B) = 0$ as B is traceless. As B is symmetric, for all $v \in V$, $v^T B^2 v = |Bv|^2 \geq 0$, so B^2

is positive semi-definite. Note that $-\gamma I_n$ is negative semi-definite. Hence, both sides must vanish, so $B^2 = 0$. Hence, $0 = v^T B^2 v = |Bv|^2$, so $B = 0$. Therefore, $X = \psi_0(X)I_n + A$. Taking the trace, we know that $\psi_0(X) = \frac{\text{Tr}(X)}{n}$.

We now show exactly what X looks like. Note that for all $a, b, c, d \in \mathbb{C}$ and $i \in \{1, 2, 3\}$

$$\left[U^\dagger \begin{bmatrix} aI_{n/2} & bI_{n/2} \\ cI_{n/2} & dI_{n/2} \end{bmatrix} U, Y_i \right] = 0.$$

As the space of such matrices is four-dimensional, some choice of these parameters gives us $X = U^\dagger \begin{bmatrix} aI_{n/2} & bI_{n/2} \\ cI_{n/2} & dI_{n/2} \end{bmatrix} U$. From above, we know that $X - \frac{\text{Tr}(X)}{n}I_n \in \mathfrak{so}(n)$. Hence,

$$X - \frac{\text{Tr}(X)}{n}I_n = U^\dagger \begin{bmatrix} \left(a - \frac{\text{Tr}(X)}{n}\right)I_{n/2} & bI_{n/2} \\ cI_{n/2} & \left(d - \frac{\text{Tr}(X)}{n}\right)I_{n/2} \end{bmatrix} U \in \mathfrak{so}(n) \subseteq \mathfrak{su}(n).$$

As this matrix is to be traceless, we see that $\frac{n}{2}(a+d) = \text{Tr}(X)$. Additionally, as the matrix is skew-adjoint,

$$\begin{bmatrix} \left(a^\dagger - \frac{\text{Tr}(X)}{n}\right)I_{n/2} & c^\dagger I_{n/2} \\ b^\dagger I_{n/2} & \left(d^\dagger - \frac{\text{Tr}(X)}{n}\right)I_{n/2} \end{bmatrix} + \begin{bmatrix} \left(a - \frac{\text{Tr}(X)}{n}\right)I_{n/2} & bI_{n/2} \\ cI_{n/2} & \left(d - \frac{\text{Tr}(X)}{n}\right)I_{n/2} \end{bmatrix} = 0.$$

Hence, we see that $c = -b^\dagger$, $a^\dagger + a = 2\frac{\text{Tr}(X)}{n}$, and $d^\dagger + d = 2\frac{\text{Tr}(X)}{n}$. Combining these equations, we see that $\text{Re}(a) = \text{Re}(d) = \frac{\text{Tr}(X)}{n}$ and $\text{Im}(a) = -\text{Im}(d)$. Therefore, we have that $d = a^\dagger$. Thus,

$$X = U^\dagger \begin{bmatrix} aI_{n/2} & bI_{n/2} \\ -b^\dagger I_{n/2} & a^\dagger I_{n/2} \end{bmatrix} U.$$

Hence, X has the desired form. Moreover, we see that we have two complex parameters, equivalently four real parameters, so all such matrices of this form are real and commute with the representation. $\square$

Now that we know exactly what spans the trivial summands of $(\hat{V}, \hat{\rho})$, the following theorem tells us exactly what form M takes if $\hat{M}$ is spherically symmetric.

Theorem 2.1.39 (Structure Theorem). *Let $\hat{M} \in \mathcal{M}_{n,k}$ be spherically symmetric. Theorem 2.1.22 tells us that the monopole corresponding with $\hat{M}$ is generated by a real representation (V, ρ) of $\mathfrak{sp}(1)$, which we can decompose as*

$$(V, \rho) \simeq \bigoplus_{a=1}^m (\mathbb{R}^{n_a}, \varrho_{n_a}). \quad (2.19)$$

Without loss of generality, we may assume that $n_a \geq n_{a+1}$, for all $a \in \{1, \dots, m-1\}$.

Let $Y_{i,a} := \varrho_{n_a}(v_i) \in \mathfrak{so}(n_a)$ and

$$Y_i := \text{diag}(Y_{i,1}, \dots, Y_{i,m}) \in \mathfrak{so}(k).$$

Then Y_i induces (V, ρ) . For n_a is divisible by four, let $y_{i,a} := \rho_{\frac{n_a}{2}}(v_i) \in \mathfrak{su}(\frac{n_a}{2})$ induce the irreducible $\frac{n_a}{2}$ complex representation. Then there is some $U_{n_a} \in \text{SU}(n_a)$ such that

$$Y_{i,a} = U_{n_a}^\dagger \begin{bmatrix} y_{i,a} & 0 \\ 0 & y_{i,a} \end{bmatrix} U_{n_a}.$$

Using the decomposition of (V, ρ) given in (2.19), we have $(M_i)_{ab} \in \mathbb{R}^{n_a} \otimes_{\mathbb{R}} (\mathbb{R}^{n_b})^*$ such that,

$$M_i = \begin{bmatrix} (M_i)_{11} & \cdots & (M_i)_{1m} \\ \vdots & \ddots & \vdots \\ (M_i)_{m1} & \cdots & (M_i)_{mm} \end{bmatrix}.$$

Then, up to a ρ -invariant gauge and for all $a, b \in \{1, \dots, m\}$ we have that

(1) if $a = b$ with n_a divisible by four, then $\exists \kappa_{a,1}, \kappa_{a,2}, \kappa_{a,3} \in \mathbb{R}$ such that

$$(M_i)_{aa} = Y_{i,a} U_{n_a}^\dagger \begin{bmatrix} \kappa_{a,1} i I_{\frac{n_a}{2}} & (\kappa_{a,2} + i \kappa_{a,3}) I_{\frac{n_a}{2}} \\ (-\kappa_{a,2} + i \kappa_{a,3}) I_{\frac{n_a}{2}} & -\kappa_{a,1} i I_{\frac{n_a}{2}} \end{bmatrix} U_{n_a};$$

(2) if $a < b$ and $n_a = n_b$ is divisible by four, then $\exists \kappa_{a,b,0}, \kappa_{a,b,1}, \kappa_{a,b,2}, \kappa_{a,b,3} \in \mathbb{R}$ such that

$$\begin{aligned} (M_i)_{ab} &= Y_{i,a} U_{n_a}^\dagger \begin{bmatrix} (\kappa_{a,b,0} + \kappa_{a,b,1} i) I_{\frac{n_a}{2}} & (\kappa_{a,b,2} + i \kappa_{a,b,3}) I_{\frac{n_a}{2}} \\ (-\kappa_{a,b,2} + i \kappa_{a,b,3}) I_{\frac{n_a}{2}} & (\kappa_{a,b,0} - \kappa_{a,b,1} i) I_{\frac{n_a}{2}} \end{bmatrix} U_{n_a}, \quad \text{and} \\ (M_i)_{ba} &= Y_{i,a} U_{n_a}^\dagger \begin{bmatrix} (-\kappa_{a,b,0} + \kappa_{a,b,1} i) I_{\frac{n_a}{2}} & (\kappa_{a,b,2} + i \kappa_{a,b,3}) I_{\frac{n_a}{2}} \\ (-\kappa_{a,b,2} + i \kappa_{a,b,3}) I_{\frac{n_a}{2}} & (-\kappa_{a,b,0} - \kappa_{a,b,1} i) I_{\frac{n_a}{2}} \end{bmatrix} U_{n_a}; \end{aligned}$$

(3) if $n_a = n_b + 4$ are both divisible by four, then $\exists \kappa_{a,b,0}, \kappa_{a,b,1}, \kappa_{a,b,2}, \kappa_{a,b,3} \in \mathbb{C}$ such that

$$(M_i)_{ab} = U_{n_a}^\dagger \begin{bmatrix} \kappa_{a,b,0} B_i^{\frac{n_b}{2}} & \kappa_{a,b,1} B_i^{\frac{n_b}{2}} \\ \kappa_{a,b,2} B_i^{\frac{n_b}{2}} & \kappa_{a,b,3} B_i^{\frac{n_b}{2}} \end{bmatrix} U_{n_b}, \quad \text{and}$$

$$(M_i)_{ba} = U_{n_b}^\dagger \begin{bmatrix} \overline{\kappa_{a,b,0}} \left(B_i^{\frac{n_b}{2}}\right)^\dagger & \overline{\kappa_{a,b,2}} \left(B_i^{\frac{n_b}{2}}\right)^\dagger \\ \overline{\kappa_{a,b,1}} \left(B_i^{\frac{n_b}{2}}\right)^\dagger & \overline{\kappa_{a,b,3}} \left(B_i^{\frac{n_b}{2}}\right)^\dagger \end{bmatrix} U_{n_a},$$

and these blocks are real;

(4) if $a < b$ with $n_a = n_b$ odd, then $\exists \lambda_{a,b} \in \mathbb{R}$ such that

$$(M_i)_{ab} = \lambda_{a,b} Y_{i,a}, \quad \text{and} \quad (M_i)_{ba} = -\lambda_{a,b} Y_{i,a};$$

(5) if $n_a = n_b + 2$ are both odd, then $\exists \lambda_{a,b} \in \mathbb{R}$ such that

$$(M_i)_{ab} = \lambda_{a,b} B_i^{n_b}, \quad \text{and} \quad (M_i)_{ba} = \lambda_{a,b} (B_i^{n_b})^T,$$

where the $U(1)$ factor for $B_i^{n_b}$ is chosen so the matrices are real;

(6) otherwise, $(M_i)_{ab} = 0$.

Conversely, if M has the above form for some real k -representation (V, ρ) , then $\hat{M}$ is spherically symmetric

Proof. Recall that as $\hat{M}$ is spherically symmetric, M must be in the direct sum of trivial summands of $(\hat{V}, \hat{\rho})$. From Lemma 2.1.35, we know the number of trivial summands N of $(\hat{V}, \hat{\rho})$ is

$$N = \#\{(a, b) \mid n_a = n_b \geq 2, \ 2 \nmid n_a\} + \#\{(a, b) \mid |n_a - n_b| = 2, \ 2 \nmid n_a\} \\ + 4\#\{(a, b) \mid n_a = n_b \geq 4, \ 4 \mid n_a\} + 4\#\{(a, b) \mid |n_a - n_b| = 4, \ 4 \mid n_a\}.$$

If we find an ansatz in $\hat{V}$ satisfying (2.11) with N free parameters, then it is the most general one and M is given by a choice of these parameters. Lemma 2.1.36 and Lemma 2.1.38 tell us what to choose as an ansatz. Let $(A_1, A_2, A_3) \in \hat{V}$ be written in the previous decomposition with blocks $(A_i)_{ab} \in \mathbb{R}^{n_a} \otimes_{\mathbb{R}} (\mathbb{R}^{n_b})^*$ as

$$A_i = \begin{bmatrix} (A_i)_{11} & \cdots & (A_i)_{1m} \\ \vdots & \ddots & \vdots \\ (A_i)_{m1} & \cdots & (A_i)_{mm} \end{bmatrix}.$$

To satisfy (2.11), we set the following:

- (1) if $n_a = n_b$ is divisible by four, then $\exists \kappa_{a,b,0}, \kappa_{a,b,1}, \kappa_{a,b,2}, \kappa_{a,b,3} \in \mathbb{R}$ such that

$$(A_i)_{ab} = Y_{i,a} U_{n_a}^\dagger \begin{bmatrix} (\kappa_{a,b,0} + \kappa_{a,b,1}i)I_{\frac{n_a}{2}} & (\kappa_{a,b,2} + i\kappa_{a,b,3})I_{\frac{n_a}{2}} \\ (-\kappa_{a,b,2} + i\kappa_{a,b,3})I_{\frac{n_a}{2}} & (\kappa_{a,b,0} - \kappa_{a,b,1}i)I_{\frac{n_a}{2}} \end{bmatrix} U_{n_a};$$

see (2.18),

- (2) if $n_a = n_b + 4$ are both divisible by four, then $\exists \kappa_{a,b,0}, \kappa_{a,b,1}, \kappa_{a,b,2}, \kappa_{a,b,3} \in \mathbb{C}$ such that

$$(A_i)_{ab} = U_{n_a}^\dagger \begin{bmatrix} \kappa_{a,b,0} B_i^{\frac{n_b}{2}} & \kappa_{a,b,1} B_i^{\frac{n_b}{2}} \\ \kappa_{a,b,2} B_i^{\frac{n_b}{2}} & \kappa_{a,b,3} B_i^{\frac{n_b}{2}} \end{bmatrix} U_{n_b},$$

and these blocks are real; see (2.15),

- (3) if $n_b = n_a + 4$ are both divisible by four, then $\exists \kappa_{a,b,0}, \kappa_{a,b,1}, \kappa_{a,b,2}, \kappa_{a,b,3} \in \mathbb{C}$ such that

$$(A_i)_{ab} = U_{n_a}^\dagger \begin{bmatrix} \kappa_{a,b,0} \left(B_i^{\frac{n_b}{2}}\right)^\dagger & \kappa_{a,b,1} \left(B_i^{\frac{n_b}{2}}\right)^\dagger \\ \kappa_{a,b,2} \left(B_i^{\frac{n_b}{2}}\right)^\dagger & \kappa_{a,b,3} \left(B_i^{\frac{n_b}{2}}\right)^\dagger \end{bmatrix} U_{n_b},$$

and these blocks are real;

- (4) if $n_a = n_b$ is odd, then $\exists \lambda_{a,b} \in \mathbb{R}$ such that

$$(A_i)_{ab} = \lambda_{a,b} Y_{i,a};$$

- (5) if $n_a = n_b + 2$ are both odd, then $\exists \lambda_{a,b} \in \mathbb{R}$ such that

$$(A_i)_{ab} = \lambda_{a,b} B_i^{n_b},$$

where the $U(1)$ factor for $B_i^{n_b}$ is chosen so the matrices are real;

- (6) if $n_b = n_a + 2$ are both odd, then $\exists \lambda_{a,b} \in \mathbb{R}$ such that

$$(A_i)_{ab} = \lambda_{a,b} (B_i^{n_b})^T,$$

where the $U(1)$ factor for $B_i^{n_b}$ is chosen so the matrices are real;

(7) otherwise, $(A_i)_{ab} = 0$.

Note that if $n_a = n_b = 1$, then $\lambda_{ab}Y_{i,a} = 0$, so we don't count these parameters. Thus, we have exactly N free parameters. Moreover, we see that this ansatz satisfies the spherically symmetric equations (2.11). That is, substituting all the cases for $(A_i)_{ab}$, we see that

$$[Y_i, A_j] = [Y_{i,a}(A_j)_{ab} - (A_j)_{ab}Y_{i,b}]_{1 \leq a, b \leq m} = \sum_{l=1}^3 \epsilon_{ijl} A_l. \quad (2.20)$$

As mentioned above, for some choices of the N parameters defining A_i , we get the M_i . As the M_i satisfy some constraints, we get constraints on the parameters.

As the M_i are symmetric and real, we know that $M_i^\dagger = M_i$. Hence, $(M_i)_{aa} = (M_i)_{aa}^T$ and $(M_i)_{aa} = (M_i)_{aa}^\dagger$. If n_a is odd, then $(M_i)_{aa} = \lambda_{a,a}Y_{i,a}$ is symmetric, meaning $\lambda_{a,a} = 0$. Similarly, if n_a is divisible by four, then

$$(M_i)_{aa} = Y_{i,a}U_{n_a}^\dagger \begin{bmatrix} (\kappa_{a,a,0} + \kappa_{a,a,1}i)I_{\frac{n_a}{2}} & (\kappa_{a,a,2} + i\kappa_{a,a,3})I_{\frac{n_a}{2}} \\ (-\kappa_{a,a,2} + i\kappa_{a,a,3})I_{\frac{n_a}{2}} & (\kappa_{a,a,0} - \kappa_{a,a,1}i)I_{\frac{n_a}{2}} \end{bmatrix} U_{n_a}$$

is Hermitian, meaning $\kappa_{a,a,0} = 0$.

If $b < a$ with $n_a = n_b$, then $(M_i)_{ba} = (M_i)_{ab}^T = (M_i)_{ab}^\dagger$. If n_a is odd, then

$$\lambda_{b,a}Y_{i,a} = (M_i)_{ba} = (M_i)_{ab}^T = \lambda_{a,b}Y_{i,a}^T = -\lambda_{a,b}Y_{i,a},$$

so $\lambda_{b,a} = -\lambda_{a,b}$. If n_a is divisible by four, then

$$\begin{aligned} Y_{i,a}U_{n_a}^\dagger \begin{bmatrix} (\kappa_{b,a,0} + \kappa_{b,a,1}i)I_{\frac{n_a}{2}} & (\kappa_{b,a,2} + i\kappa_{b,a,3})I_{\frac{n_a}{2}} \\ (-\kappa_{b,a,2} + i\kappa_{b,a,3})I_{\frac{n_a}{2}} & (\kappa_{b,a,0} - \kappa_{b,a,1}i)I_{\frac{n_a}{2}} \end{bmatrix} U_{n_a} &= (M_i)_{ba} \\ &= (M_i)_{ab}^\dagger = Y_{i,a}U_{n_a}^\dagger \begin{bmatrix} (-\kappa_{a,b,0} + \kappa_{a,b,1}i)I_{\frac{n_a}{2}} & (\kappa_{a,b,2} + i\kappa_{a,b,3})I_{\frac{n_a}{2}} \\ (-\kappa_{a,b,2} + i\kappa_{a,b,3})I_{\frac{n_a}{2}} & (-\kappa_{a,b,0} - \kappa_{a,b,1}i)I_{\frac{n_a}{2}} \end{bmatrix} U_{n_a} \end{aligned}$$

We multiply by $Y_{i,a}$, sum, and use $\sum_{i=1}^3 Y_{i,a}^2 = -\frac{n_a^2-4}{16}I_{n_a}$ to see that $\kappa_{b,a,0} = -\kappa_{a,b,0}$, $\kappa_{b,a,i} = \kappa_{a,b,i}$ for $i = 1, 2, 3$.

If $n_b = n_a + 2$ and n_a is odd, then $(M_i)_{ba} = (M_i)_{ab}^T$, so

$$\lambda_{b,a}B_i^{n_b} = (M_i)_{ba} = (M_i)_{ab}^T = \lambda_{a,b}B_i^{n_b},$$

so $\lambda_{b,a} = \lambda_{a,b}$. If $n_b = n_a + 4$ and n_a is divisible by four, then $(M_i)_{ba} = (M_i)_{ab}^\dagger$, so

$$U_{n_b}^\dagger \begin{bmatrix} \kappa_{b,a,0} B_i^{\frac{n_b}{2}} & \kappa_{b,a,1} B_i^{\frac{n_b}{2}} \\ \kappa_{b,a,2} B_i^{\frac{n_b}{2}} & \kappa_{b,a,3} B_i^{\frac{n_b}{2}} \end{bmatrix} U_{n_a} = (M_i)_{ba} = (M_i)_{ab}^\dagger = U_{n_b}^\dagger \begin{bmatrix} \overline{\kappa_{a,b,0}} B_i^{\frac{n_b}{2}} & \overline{\kappa_{a,b,2}} B_i^{\frac{n_b}{2}} \\ \overline{\kappa_{a,b,1}} B_i^{\frac{n_b}{2}} & \overline{\kappa_{a,b,3}} B_i^{\frac{n_b}{2}} \end{bmatrix} U_{n_a}$$

Hence, $\kappa_{b,a,0} = \overline{\kappa_{a,b,0}}$, $\kappa_{b,a,1} = \overline{\kappa_{a,b,2}}$, $\kappa_{b,a,2} = \overline{\kappa_{a,b,1}}$, and $\kappa_{b,a,3} = \overline{\kappa_{a,b,3}}$. In summary, we see that $\hat{M}_i$ is of the desired form.

We see that if M has the form in the statement of the theorem for some real k -representation (V, ρ) , then M satisfies the spherical symmetry conditions, so $\hat{M}$ is spherically symmetric, by Theorem 2.1.22. $\square$

2.1.3.2 Novel examples of spherically symmetric monopoles

Here we use the Structure Theorem to construct novel examples of spherically symmetric hyperbolic monopoles. We start with irreducible representations.

Proposition 2.1.40. *Suppose (V, ρ) is an irreducible real k -representation that generates a spherically symmetric hyperbolic monopole with $\hat{M} \in \mathcal{M}_{n,k}$.*

- (1) *If k is odd, then $M = 0$, which we investigated in Proposition 2.1.30.*
- (2) *If k is divisible by four, let Y_i and U have the same meaning as in the Structure Theorem. Then there is some $\kappa \geq 0$ such that M_i is gauge equivalent to*

$$\kappa Y_i U^\dagger \begin{bmatrix} iI_{\frac{k}{2}} & 0 \\ 0 & -iI_{\frac{k}{2}} \end{bmatrix} U.$$

Having identified all M generated by irreducible real representations, we construct a family of spherically symmetric hyperbolic monopoles. If k is divisible by four, let $0 < \kappa < \frac{4}{k+2}$ and let

$$\beta := \sqrt{\frac{16 - (k^2 + 4)\kappa^2 \pm \sqrt{(16 - (k^2 + 4)\kappa^2)^2 - 16k^2\kappa^4}}{2k^2}}, \quad \alpha := -\frac{\beta^2 + \kappa^2}{2\beta}, \quad \text{and} \quad (2.21)$$

$$L := \alpha I_k + \beta(Y_1 i + Y_2 j + Y_3 k).$$

Note that as $0 < \kappa < \frac{4}{k+2}$, $\beta > 0$, so L is well-defined. Using the M_i from the second point above, we have $\hat{M} \in \mathcal{M}_{k,k}$ and the corresponding $\text{Sp}(k)$ monopole is spherically symmetric.

Proof. (1) The Structure Theorem tells us that $M_i = 0$.

(2) The Structure Theorem tells us that there are some $\kappa_1, \kappa_2, \kappa_3 \in \mathbb{R}$ such that

$$M_i = Y_i U^\dagger \begin{bmatrix} \kappa_1 i I_{k/2} & (\kappa_2 + i\kappa_3) I_{k/2} \\ (-\kappa_2 + i\kappa_3) I_{k/2} & -\kappa_1 i I_{k/2} \end{bmatrix} U.$$

We now gauge the data as follows.

Let $\kappa := \sqrt{\kappa_1^2 + \kappa_2^2 + \kappa_3^2} \geq 0$. If $\kappa_2 = \kappa_3 = 0$ and $\kappa_1 \geq 0$, then we have the desired form. If $\kappa_2 = \kappa_3 = 0$ and $\kappa_1 < 0$, then we take $Q := U^\dagger \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} U$. From Lemma 2.1.38, we know that $Q \in \mathfrak{so}(k)$. In fact, we see that $QQ^T = QQ^\dagger = I_k$, so $Q \in O(k)$ and $QM_i Q^T$ has the desired form, as

$$QM_i Q^T = -\kappa_1 Y_i U^\dagger \begin{bmatrix} i I_{\frac{k}{2}} & 0 \\ 0 & -i I_{\frac{k}{2}} \end{bmatrix} U.$$

Otherwise, let

$$Q := U^\dagger \begin{bmatrix} \frac{i}{\kappa_2 + i\kappa_3} \sqrt{\frac{(\kappa + \kappa_1)(\kappa_2^2 + \kappa_3^2)}{2\kappa}} & \sqrt{\frac{\kappa_2^2 + \kappa_3^2}{2\kappa(\kappa + \kappa_1)}} \\ -\sqrt{\frac{\kappa_2^2 + \kappa_3^2}{2\kappa(\kappa + \kappa_1)}} & -\frac{i}{\kappa_2 - i\kappa_3} \sqrt{\frac{(\kappa + \kappa_1)(\kappa_2^2 + \kappa_3^2)}{2\kappa}} \end{bmatrix} U.$$

By Lemma 2.1.38, we know that $Q \in O(k)$ and Q commutes with the Y_i . Hence,

$$QM_i Q^T = \kappa Y_i U^\dagger \begin{bmatrix} i I_{\frac{k}{2}} & 0 \\ 0 & -i I_{\frac{k}{2}} \end{bmatrix} U.$$

Therefore, up to gauge, we may assume that $\kappa_2 = \kappa_3 = 0$ and $\kappa_1 \geq 0$.

Now we prove the final part of the proposition. Suppose that we have $\hat{M}$ as in the statement of the proposition. We show that $\hat{M} \in \mathcal{M}_{k,k}$. Such data is spherically symmetric by the Structure Theorem. Then

$$L^\dagger L - M^2 = \left(\alpha^2 + (\beta^2 + \kappa^2) \frac{k^2 - 4}{16} \right) I_k + (2\alpha\beta + \beta^2 + \kappa^2)(Y_1 i + Y_2 j + Y_3 k).$$

Simplifying, we see $L^\dagger L - M^2 = I_k$. We just need to verify that $LL^\dagger$ and $\Delta(X_3 k)^\dagger \Delta(X_3 k)$ are positive definite, the latter for all $X_3 \in [0, 1]$.

Note that

$$\Delta(X_3 k)^\dagger \Delta(X_3 k) = (1 + X_3^2)I_k - 2X_3 \kappa Y_3 U^\dagger \begin{bmatrix} iI_{\frac{k}{2}} & 0 \\ 0 & -iI_{\frac{k}{2}} \end{bmatrix} U.$$

Suppose that this matrix is not positive definite for some $X_3 \in [0, 1]$. Then there is some unit vector v such that $\Delta(X_3 k)^\dagger \Delta(X_3 k)v = 0$. Hence,

$$(1 + X_3^2)v = 2X_3 \kappa Y_3 U^\dagger \begin{bmatrix} iI_{\frac{k}{2}} & 0 \\ 0 & -iI_{\frac{k}{2}} \end{bmatrix} Uv.$$

Multiply each side on its left by its conjugate transpose, obtaining

$$(1 + X_3^2)^2 = -4X_3^2 \kappa^2 v^\dagger Y_3^2 v = 4X_3^2 \kappa^2 |Y_3 v|^2.$$

As the Y_i induce the irreducible real k -representation, whose complexification is isomorphic as a complex representation to $\left(V_{\frac{k}{2}}, \rho_{\frac{k}{2}}\right)^{\oplus 2}$, the largest modulus of an eigenvalue of Y_3 is $\frac{k-2}{4}$. Hence, $|Y_3 v|^2 \leq \left(\frac{k-2}{4}\right)^2$. As $0 < \kappa < \frac{4}{k+2}$,

$$(1 + X_3^2)^2 \leq 4X_3^2 \kappa^2 \frac{(k-2)^2}{16} < \left(2X_3 \frac{k-2}{k+2}\right)^2.$$

Thus, $1 + X_3^2 < 2X_3 \frac{k-2}{k+2}$. But this inequality is not satisfied, as $X_3^2 - 2X_3 \frac{k-2}{k+2} + 1$ has no real roots, contradiction! Thus, $\Delta(X_3 k)^\dagger \Delta(X_3 k)$ is positive definite for all $X_3 \in [0, 1]$.

Finally, suppose that $LL^\dagger = L^2$ is not positive definite. Then there is some unit vector v such that

$$0 = L^2 v = \left(\alpha^2 + \beta^2 \frac{k^2 - 4}{16}\right) v + (2\alpha\beta + \beta^2)(Y_1 i + Y_2 j + Y_3 k)v.$$

As $L^2 - M^2 = I_k$, we can rewrite this as

$$\left(1 - \kappa^2 \frac{k^2 - 4}{16}\right) v = \kappa^2 (Y_1 i + Y_2 j + Y_3 k)v. \quad (2.22)$$

Multiplying both sides on the left by their conjugate transpose, we get

$$\left(1 - \kappa^2 \frac{k^2 - 4}{16}\right)^2 = \kappa^4 v^\dagger (Y_1 i + Y_2 j + Y_3 k)^2 v = \kappa^4 v^\dagger \left(\frac{k^2 - 4}{16} I + Y_1 i + Y_2 j + Y_3 k\right) v.$$

Substituting (2.22), we see

$$\left(1 - \kappa^2 \frac{k^2 - 4}{16}\right)^2 = \kappa^4 \frac{k^2 - 4}{16} + \kappa^2 \left(1 - \kappa^2 \frac{k^2 - 4}{16}\right).$$

Simplifying, we have

$$\frac{(16 - (k + 2)^2 \kappa^2)(16 - (k - 2)^2 \kappa^2)}{256} = 0.$$

But this equation has no roots in $(0, \frac{4}{k+2})$, contradiction! Thus, $\hat{M} \in \mathcal{M}_{k,k}$ corresponds with a spherically symmetric $\mathrm{Sp}(k)$ monopole. $\square$

Note 2.1.41. *We have found a family of spherically symmetric $\mathrm{Sp}(k)$ monopoles, with ADHM data in $\mathcal{M}_{k,k}$, when k is divisible by four. However, there may be other monopoles with ADHM in $\mathcal{M}_{n,k}$ (for some $n < k$) whose corresponding ADHM data shares the M portion of the data with the aforementioned family.*

Now that we have investigated irreducible representations, let us move on to some reducible ones.

Proposition 2.1.42. *Let $n \in \mathbb{N}_+$ be odd. Suppose $(V, \rho) \simeq (\mathbb{R}^{n+2}, \varrho_{n+2}) \oplus (\mathbb{R}^n, \varrho_n)$ generates a spherically symmetric hyperbolic monopole with ADHM data $\hat{M} \in \mathcal{M}_{m, 2n+2}$ for some $m \in \mathbb{N}_+$. Up to gauge there is some $a \geq 0$ such that, choosing the $\mathrm{U}(1)$ factor to give real B_i^n matrices, we have*

$$M_i = a \begin{bmatrix} 0 & B_i^n \\ (B_i^n)^T & 0 \end{bmatrix}.$$

Having identified all M generated by (V, ρ) , we construct families of spherically symmetric hyperbolic monopoles, using M . For $n > 1$ odd and $0 < a < \sqrt{\frac{n+1}{2(n+2)}}$, let

$$\begin{aligned} \beta &:= \frac{n+1 \pm \sqrt{(n+1)(n+1-2(n+2)a^2)}}{(n+1)(n+2)}, & \alpha &:= -\frac{\beta^2 + \frac{2}{n+1}a^2}{2\beta}, \\ \delta &:= \frac{n+1 \pm \sqrt{(n+1)(n+1-2(n+2)a^2)}}{n(n+1)}, & \gamma &:= \frac{\frac{2(n+2)}{n(n+1)}a^2 - \delta^2}{2\delta}, \quad \text{and} \\ L &:= \begin{bmatrix} \alpha I_{n+2} & 0 \\ 0 & \gamma I_n \end{bmatrix} + \begin{bmatrix} \beta(Y_{1,1}i + Y_{2,1}j + Y_{3,1}k) & 0 \\ 0 & \delta(Y_{1,2}i + Y_{2,2}j + Y_{3,2}k) \end{bmatrix} \end{aligned} \quad (2.23)$$

Note that as $0 < a < \sqrt{\frac{n+1}{2(n+2)}}$, $\beta, \delta > 0$, so L is well-defined. We have $\hat{M} \in \mathcal{M}_{2n+2, 2n+2}$ and its corresponding $\mathrm{Sp}(2n+2)$ monopole is spherically symmetric.

In addition to the above families of spherically symmetric monopoles, the following $\hat{M} \in \mathcal{M}_{2,4}$ corresponds with a spherically symmetric $\mathrm{Sp}(2)$ monopole:

$$\hat{M} := \frac{1}{\sqrt{3}} \begin{bmatrix} \sqrt{2} & -\frac{i}{\sqrt{2}} & \frac{j}{\sqrt{2}} & 0 \\ 0 & \sqrt{\frac{3}{2}} & -k\sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & k \\ 0 & 0 & 0 & j \\ 0 & 0 & 0 & i \\ k & j & i & 0 \end{bmatrix}. \quad (2.24)$$

Proof. The Structure Theorem tells us that for a spherically symmetric monopole generated by (V, ρ) , there is a constant $\lambda_{1,2} \in \mathbb{R}$ such that the ADHM data $\hat{M} \in \mathcal{M}_{m, 2n+2}$ satisfies

$$M_i = \begin{bmatrix} 0 & \lambda_{1,2} B_i^n \\ \lambda_{1,2} (B_i^n)^T & 0 \end{bmatrix},$$

where the $\mathrm{U}(1)$ factor for the B_i^1 is chosen so they are all real matrices. Denoting $a := \lambda_{1,2}$, M has the form given in the statement of the proposition. Furthermore, any data with M_i of this form is spherically symmetric by the Structure Theorem.

To see that, up to gauge, we can take $a \geq 0$, consider $Q := \begin{bmatrix} I_{n+2} & 0 \\ 0 & -I_n \end{bmatrix} \in \mathrm{O}(2n+2)$.

We see that $QM_iQ^T = -M_i$.

The $\hat{M}$ in parts two and three of the proposition have M given as above. We show that in both cases, if $\hat{M}$ satisfies the first three conditions in Definition 2.1.1, then the final condition is satisfied. Assuming the first three conditions are satisfied, by Lemma 2.1.25, we need only check the final condition at $X = X_3 k$ for $X_3 \in [0, 1]$. For such X , we find

$$\Delta(X)^\dagger \Delta(X) = (1 + X_3^2)I - 2X_3 M_3.$$

Suppose that $\Delta(X)^\dagger \Delta(X)$ is not positive definite for some $X_3 \in [0, 1]$. Then there is some unit vector v such that $\Delta(X)^\dagger \Delta(X)v = 0$. Thus,

$$(1 + X_3^2)v = 2X_3 M_3 v.$$

Multiplying both sides on the left by their conjugate transpose, we find

$$(1 + X_3^2)^2 = 4X_3^2 v^\dagger M_3^2 v \leq 4X_3^2 v^\dagger \left(\sum_{i=1}^3 M_i^2 \right) v.$$

Substituting M_i and using Proposition 2.1.34, we see that

$$(1 + X_3^2)^2 \leq 4X_3^2 a^2 v^\dagger \begin{bmatrix} I_{n+2} & 0 \\ 0 & \frac{n+2}{n} I_n \end{bmatrix} v.$$

The largest eigenvalue of $\begin{bmatrix} I_{n+2} & 0 \\ 0 & \frac{n+2}{n} I_n \end{bmatrix}$ is $\frac{n+2}{n}$, so as $0 < a < \sqrt{\frac{n+1}{2(n+2)}}$,

$$(1 + X_3^2)^2 \leq 4X_3^2 a^2 \frac{n+2}{n} < 4X_3^2 \frac{n+1}{2n}.$$

As $n \geq 1$, we have $\frac{n+1}{2n} \leq 1$, so $(1 + X_3^2)^2 < 4X_3^2$. That is, $(1 - X_3)^2 < 0$, contradiction! Therefore, $\Delta(X)^\dagger \Delta(X)$ is positive definite.

Now we prove the second part of the proposition. Suppose $0 < a < \sqrt{\frac{n+1}{2(n+2)}}$. Let $\hat{M}$ be as in the statement. Let $Y^{n+2} := Y_{1,1}i + Y_{2,1}j + Y_{3,1}k$ and similarly for Y^n . Then

$$\begin{aligned} L^\dagger L - M^2 = \text{diag} & \left(\left(\alpha^2 + \beta^2 \frac{(n+2)^2 - 1}{4} + a^2 \right) I_{n+2} + \left(2\alpha\beta + \beta^2 + \frac{2}{n+1} a^2 \right) Y^{n+2}, \right. \\ & \left. \left(\gamma^2 + \delta^2 \frac{n^2 - 1}{4} + \frac{n+2}{n} a^2 \right) I_n + \left(2\gamma\delta + \delta^2 - \frac{2(n+2)}{n(n+1)} a^2 \right) Y^n \right). \end{aligned}$$

Substituting our expressions, we see that $L^\dagger L - M^2 = I_{2n+2}$. Finally, we show that $LL^\dagger = L^2$ is positive definite. Suppose not, then there is some unit vector v such that $L^2 v = 0$. As $L^2 = L^\dagger L = I_{2n+2} + M^2$, we see that

$$\begin{bmatrix} (1 - a^2)I_{n+2} & 0 \\ 0 & (1 - \frac{n+2}{n}a^2)I_n \end{bmatrix} v = -a^2 \begin{bmatrix} -\frac{2}{n+1}Y^{n+2} & 0 \\ 0 & \frac{2(n+2)}{n(n+1)}Y^n \end{bmatrix} v. \quad (2.25)$$

Multiply both sides on the left by $\begin{bmatrix} (1 - a^2)I_{n+2} & 0 \\ 0 & (1 - \frac{n+2}{n}a^2)I_n \end{bmatrix}$. Note that this matrix commutes with the matrix on the right-hand side. Thus, we can use (2.25) to obtain

$$\begin{bmatrix} (1 - a^2)^2 I_{n+2} & 0 \\ 0 & (1 - \frac{n+2}{n}a^2)^2 I_n \end{bmatrix} v = a^4 \begin{bmatrix} -\frac{2}{n+1}Y^{n+2} & 0 \\ 0 & \frac{2(n+2)}{n(n+1)}Y^n \end{bmatrix}^2 v.$$

Simplifying the right-hand side, we note that

$$\begin{aligned}\frac{4}{(n+1)^2}(Y^{n+2})^2 &= \frac{(n+2)^2-1}{(n+1)^2}I_{n+2} + \frac{4}{(n+1)^2}Y^{n+2}, \quad \text{and} \\ \frac{4(n+2)^2}{n^2(n+1)^2}(Y^n)^2 &= \frac{(n+2)^2(n^2-1)}{n^2(n+1)^2}I_n + \frac{4(n+2)^2}{n^2(n+1)^2}Y^n.\end{aligned}$$

Thus, we use (2.25) to find

$$\begin{aligned}\begin{bmatrix} (1-a^2)^2 I_{n+2} & 0 \\ 0 & (1-\frac{n+2}{n}a^2)^2 I_n \end{bmatrix} v &= a^4 \begin{bmatrix} \frac{(n+2)^2-1}{(n+1)^2} I_{n+2} & 0 \\ 0 & \frac{(n^2-1)(n+2)^2}{n^2(n+1)^2} I_n \end{bmatrix} v \\ &\quad - a^2 \begin{bmatrix} -(1-a^2)\frac{2}{n+1} I_{n+2} & 0 \\ 0 & (1-\frac{n+2}{n}a^2)\frac{2(n+2)}{n(n+1)} I_n \end{bmatrix} v\end{aligned}$$

Thus, we find

$$\frac{n+1-2(n+2)a^2}{n+1}v = 0.$$

But on $0 < a < \sqrt{\frac{n+1}{2(n+2)}}$, $\frac{n+1-2(n+2)a^2}{n+1} \neq 0$, contradiction! Therefore, $\hat{M} \in \mathcal{M}_{2n+2, 2n+2}$ corresponds with a spherically symmetric $\text{Sp}(2n+2)$ monopole.

Now we prove the final part of the proposition. Consider the additional $\hat{M}$ given in the statement. Let $a := \frac{1}{\sqrt{3}}$ and $n := 1$. The following matrices induce $(\mathbb{R}^3, \varrho_3) \oplus (\mathbb{R}^1, \varrho_1)$:

$$Y_i := \begin{bmatrix} y_i & 0 \\ 0 & 0 \end{bmatrix}, \text{ where}$$

$$y_1 := \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad y_2 := \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad y_3 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

We compute the B_i matrices for this case in Proposition E.0.6 in Appendix E and find that up to a sign,

$$B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

As above, consider $M_i = a \begin{bmatrix} 0 & B_i^1 \\ (B_i^1)^T & 0 \end{bmatrix}$. Using these generators, we see that

$$M = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 0 & 0 & k \\ 0 & 0 & 0 & j \\ 0 & 0 & 0 & i \\ k & j & i & 0 \end{bmatrix}.$$

Hence, our $\hat{M}$ is spherically symmetric, assuming it belongs to $\mathcal{M}_{2,4}$, which we now verify; indeed, $L^\dagger L - M^2 = I_4$, $LL^\dagger = I_2$ and $LL^\dagger = I_2$. Therefore, we have $\hat{M} \in \mathcal{M}_{2,4}$ corresponds with a spherically symmetric $\text{Sp}(2)$ monopole. $\square$

Note 2.1.43. We compute the $\text{Sp}(2)$ hyperbolic monopole corresponding to the additional ADHM data above in (2.24). While the hyperbolic monopole corresponding to the aforementioned ADHM data is spherically symmetric, in a given gauge, the Higgs field varies on spheres of constant radius. However, up to gauge, the Higgs monopole depends only on the radius r from the origin. Hence, along any ray emanating from the origin, up to gauge, the Higgs field Φ is given by

$$\Phi(r) := \begin{bmatrix} \frac{ir}{r^2+1} & 0 \\ 0 & -\frac{ir(r^4+6r^2+1)}{(r^2+1)(3r^4+2r^2+3)} \end{bmatrix}. \quad (2.26)$$

Note that the eigenvalues of $\Phi(r)$ as $r \rightarrow 1$ are $\pm \frac{i}{2}$. Thus, by Theorem 2.3.37, this monopole has generalized minimal symmetry breaking, as given by Definition 2.3.17. Then we have

$$|\Phi(r)|^2 = \frac{2(5r^8 + 12r^6 + 30r^4 + 12r^2 + 5)r^2}{(3r^4 + 2r^2 + 3)^2(r^2 + 1)^2}. \quad (2.27)$$

The energy density of any hyperbolic monopole is given by $\varepsilon = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j |\Phi|^2)$. In this case,

$$\varepsilon = \frac{(1 - r^2)^4}{(1 + r^2)^4} \cdot \frac{135r^{16} + 840r^{14} + 5252r^{12} + 13304r^{10} + 18282r^8 + 13304r^6 + 5252r^4 + 840r^2 + 135}{(3r^4 + 2r^2 + 3)^4}. \quad (2.28)$$

That $|\Phi(r)|^2$ and ε are both rational functions comprised of palindromic polynomials follows from Proposition 3.2.16.

In Figure 2.1, we see the norm squared of the Higgs field as well as the energy density of the monopole. From the figure, we see that the Higgs field only vanishes at the origin and the monopole looks like a point-particle, in that the energy density has a global maximum at the origin.

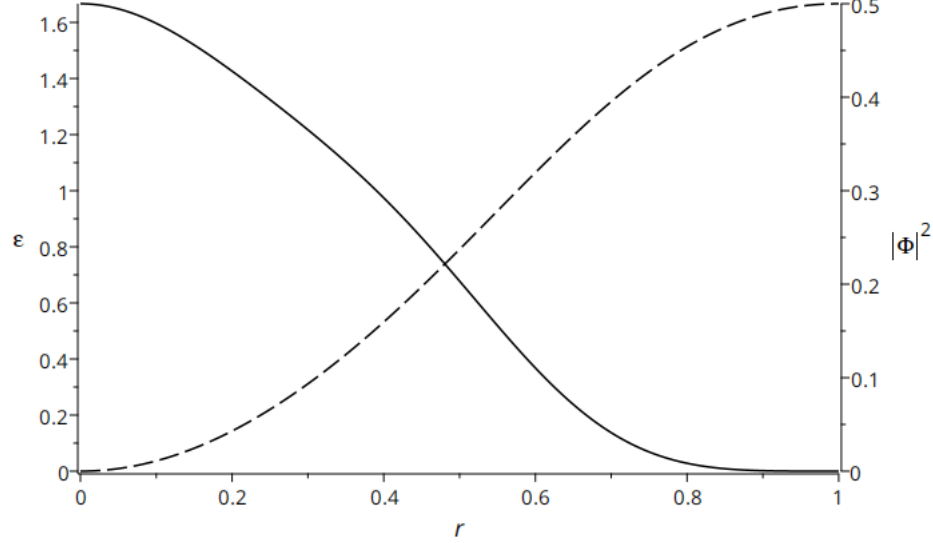


Figure 2.1: The norm squared of the Higgs field and the energy density for the additional ADHM data provided in Proposition 2.1.42. The solid line is ε with the left vertical axis and the dashed line is $|\Phi|^2$ with the right vertical axis.

Proposition 2.1.44. *Let $n \in \mathbb{N}_+$ be odd. Suppose $(V, \rho) \simeq (\mathbb{R}^n, \varrho_n) \oplus (\mathbb{R}^n, \varrho_n)$ generates a spherically symmetric hyperbolic monopole with ADHM data $\hat{M} \in \mathcal{M}_{m, 2n}$ for some $m \in \mathbb{N}_+$. Let $y_i := \varrho_n(v_i) \in \mathfrak{so}(n)$. Up to gauge there is some $a \geq 0$ such that*

$$M_i = a \begin{bmatrix} 0 & y_i \\ -y_i & 0 \end{bmatrix}.$$

Note that if $n = 1$, then $M = 0$, which we have already covered.

Having identified all M generated by (V, ρ) , we construct families of spherically symmetric hyperbolic monopoles, using M . For $n > 1$ odd and $0 < a < \frac{2}{n+1}$, let $Y := y_1 i + y_2 j + y_3 k$ and let

$$\beta := \sqrt{\frac{4 - a^2(n^2 + 1) \pm \sqrt{16 + (n^2 - 1)^2 a^4 - 8(n^2 + 1)a^2}}{2n^2}}, \quad \alpha := -\frac{\beta^2 + a^2}{2\beta}, \quad \text{and} \quad (2.29)$$

$$L := \alpha I_{2n} + \beta \begin{bmatrix} Y & 0 \\ 0 & Y \end{bmatrix}.$$

Note that as $0 < a < \frac{2}{n+1}$, $\beta > 0$, so L is well-defined. We have $\hat{M} \in \mathcal{M}_{2n,2n}$ and the corresponding $\mathrm{Sp}(2n)$ monopole is spherically symmetric.

In addition to the above families of spherically symmetric monopoles, the following $\hat{M} \in \mathcal{M}_{4,6}$ corresponds with a spherically symmetric $\mathrm{Sp}(4)$ monopole:

$$Y := \begin{bmatrix} 0 & i & -j \\ -i & 0 & k \\ j & -k & 0 \end{bmatrix}, \quad l := \begin{bmatrix} \sqrt{2} & -\frac{i}{\sqrt{2}} & \frac{j}{\sqrt{2}} \\ 0 & \sqrt{\frac{3}{2}} & -k\sqrt{\frac{3}{2}} \end{bmatrix}, \quad \text{and} \quad \hat{M} := \frac{1}{2} \begin{bmatrix} l & 0 \\ 0 & l \\ 0 & Y \\ -Y & 0 \end{bmatrix}. \quad (2.30)$$

Proof. The Structure Theorem tells us that for a spherically symmetric monopole generated by (V, ρ) , there is a constant $\lambda_{1,2} \in \mathbb{R}$ such that the ADHM data $\hat{M} \in \mathcal{M}_{m,2n}$ satisfies

$$M_i = \begin{bmatrix} 0 & \lambda_{12}y_i \\ -\lambda_{12}y_i & 0 \end{bmatrix}.$$

Denoting $a := \lambda_{12} \in \mathbb{R}$, M has the form given in the statement of the proposition. Furthermore, any data with M_i of this form is spherically symmetric by the Structure Theorem.

To see that, up to gauge, we can take $a \geq 0$, consider $Q := \begin{bmatrix} I_n & 0 \\ 0 & -I_n \end{bmatrix} \in \mathrm{O}(2n)$. We see that $QM_iQ^T = -M_i$.

The $\hat{M}$ in parts two and three of the proposition have M given as above. We show that in both cases, if $\hat{M}$ satisfies the first three conditions in Definition 2.1.1, then the final condition is satisfied. Assuming the first three conditions are satisfied, by Lemma 2.1.25, we need only check the final condition at $X = X_3k$ for $X_3 \in [0, 1]$. For such X , we find

$$\Delta(X)^\dagger \Delta(X) = (1 + X_3^2)I - 2X_3M_3.$$

Suppose that $\Delta(X)^\dagger \Delta(X)$ is not positive definite for some $X_3 \in [0, 1]$. Then there is some unit vector v such that $\Delta(X)^\dagger \Delta(X)v = 0$. Thus,

$$(1 + X_3^2)v = 2X_3M_3v.$$

Multiplying both sides by their conjugate transpose, we find

$$(1 + X_3^2)^2 = 4X_3^2 v^\dagger M_3^2 v \leq 4X_3^2 v^\dagger \left(\sum_{i=1}^3 M_i^2 \right) v.$$

Substituting M_i , we see that as $0 < a < \frac{2}{n+1}$,

$$(1 + X_3^2)^2 \leq X_3^2 a^2 (n^2 - 1) < 4X_3^2 \frac{n-1}{n+1} < 4X_3^2.$$

Hence, $(1 - X_3^2)^2 < 0$, contradiction! Thus, $\Delta(X)^\dagger \Delta(X)$ is positive definite.

Now we prove the second part of the proposition. Suppose $0 < a < \frac{2}{n+1}$. Let $\hat{M}$ be as in the statement. Then

$$L^\dagger L - M^2 = \left(\alpha^2 + (a^2 + \beta^2) \frac{n^2 - 1}{4} \right) I_{2n} + (2\alpha\beta + \beta^2 + a^2) \begin{bmatrix} Y & 0 \\ 0 & Y \end{bmatrix}.$$

Substituting our expressions, we see that $L^\dagger L - M^2 = I_{2n}$. Suppose that $LL^\dagger = L^2$ is not positive definite. Then there is some unit vector v such that $L^2 v = 0$. As $L^2 = L^\dagger L = I_{2n} + M^2$, we see that

$$\left(1 - a^2 \frac{n^2 - 1}{4} \right) v = a^2 \begin{bmatrix} Y & 0 \\ 0 & Y \end{bmatrix} v. \quad (2.31)$$

Multiplying by $\left(1 - a^2 \frac{n^2 - 1}{4} \right)$ and using (2.31), we see that

$$\left(1 - a^2 \frac{n^2 - 1}{4} \right)^2 v = a^2 \begin{bmatrix} Y & 0 \\ 0 & Y \end{bmatrix} \left(1 - a^2 \frac{n^2 - 1}{4} \right) v = a^4 \begin{bmatrix} Y & 0 \\ 0 & Y \end{bmatrix}^2 v.$$

Note that $Y^2 = \frac{n^2 - 1}{4} I_n + Y$. Simplifying,

$$\left(1 + \frac{(n^2 - 1)^2 a^4}{16} - \frac{a^2(n^2 + 1)}{2} \right) v = 0.$$

But on $(0, \frac{2}{n+1})$ $1 + \frac{(n^2 - 1)^2 a^4}{16} - \frac{a^2(n^2 + 1)}{2} \neq 0$. Contradiction! Therefore, $\hat{M} \in \mathcal{M}_{2n, 2n}$ corresponds with a spherically symmetric $\text{Sp}(2n)$ monopole.

Now we prove the final part of the proposition. Consider the additional $\hat{M}$ given in the statement. Let $a := \frac{1}{2}$ and $n := 3$. Recall from the proof of Proposition 2.1.42 that the following matrices induce $(\mathbb{R}^3, \varrho_3)$:

$$y_1 := \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad y_2 := \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad y_3 := \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

As above, consider $M_i = a \begin{bmatrix} 0 & y_i \\ -y_i & 0 \end{bmatrix}$. Using these generators, we see that $Y := y_1 i + y_2 j + y_3 k$ and M are given by

$$Y = \begin{bmatrix} 0 & i & -j \\ -i & 0 & k \\ j & -k & 0 \end{bmatrix} \quad \text{and} \quad M = \frac{1}{2} \begin{bmatrix} 0 & Y \\ -Y & 0 \end{bmatrix}.$$

Hence, our $\hat{M}$ is spherically symmetric, assuming it belongs to $\mathcal{M}_{4,6}$, which we now verify; indeed, $L^\dagger L - M^2 = I_6$ and $LL^\dagger = \frac{3}{4}I_4$. Therefore, we have $\hat{M} \in \mathcal{M}_{4,6}$ corresponds with a spherically symmetric $\text{Sp}(4)$ monopole. $\square$

Note 2.1.45. *We compute the $\text{Sp}(4)$ hyperbolic monopole corresponding to the additional ADHM data above in (2.30). The spherically symmetric, hyperbolic monopole corresponding to the aforementioned ADHM data has Higgs field Φ satisfying*

$$|\Phi|^2 = \frac{r^{12} + 9r^{10} + 33r^8 + 58r^6 + 33r^4 + 9r^2 + 1}{4(r^4 + r^2 + 1)^2(r^2 + 1)^2}. \quad (2.32)$$

The energy density of this monopole is given by

$$\varepsilon = \frac{3(1 - r^2)^4}{8(1 + r^2)^4} \cdot \frac{5r^{16} + 70r^{14} + 381r^{12} + 942r^{10} + 1260r^8 + 942r^6 + 381r^4 + 70r^2 + 5}{(r^4 + r^2 + 1)^4}. \quad (2.33)$$

That $|\Phi(r)|^2$ and ε are both rational functions comprised of palindromic polynomials follows from Proposition 3.2.16.

In Figure 2.2, we see the norm squared of the Higgs field as well as the energy density of the monopole. From the figure, we see that the Higgs field never vanishes, in contrast to the spherically symmetric $\text{Sp}(1)$ hyperbolic monopole from Proposition 2.1.30 and more broadly, the spherically symmetric $\text{Sp}(1)$ monopoles identified by Oliveira [Oli14, Appendix A]. Additionally, the monopole looks like a shell with a non-zero size, in that the energy density has a global maximum away from the origin.

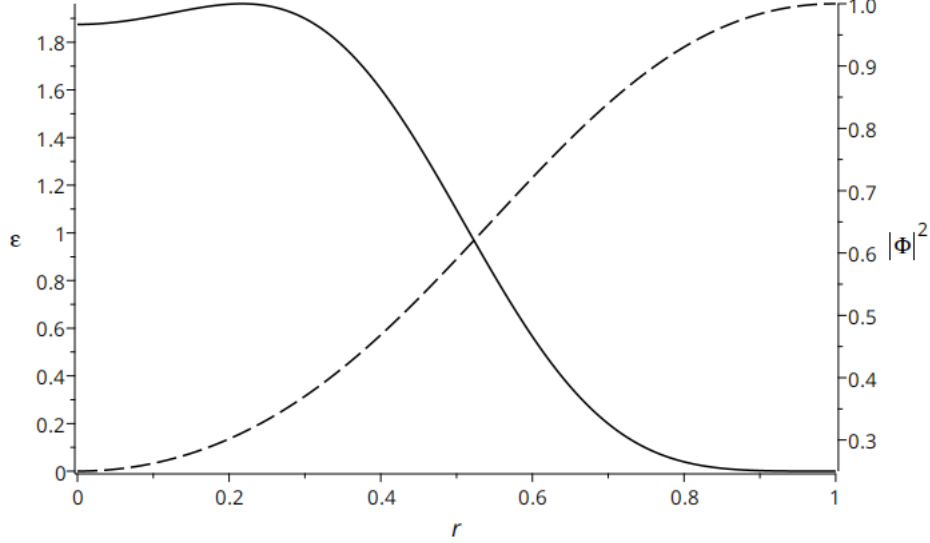


Figure 2.2: The norm squared of the Higgs field and the energy density for the additional ADHM data provided in Proposition 2.1.44. The solid line is ε with the left vertical axis and the dashed line is $|\Phi|^2$ with the right vertical axis.

We also find that as $r \rightarrow 1$,

$$\Phi \rightarrow \begin{bmatrix} \frac{17i}{50} & \frac{12}{175}\sqrt{14} & \frac{8}{3325}\sqrt{3990} & \frac{6i}{475}\sqrt{285} \\ -\frac{12}{175}\sqrt{14} & \frac{31i}{350} & -\frac{48i}{3325}\sqrt{285} & \frac{18}{3325}\sqrt{3990} \\ -\frac{8}{3325}\sqrt{3990} & -\frac{48i}{3325}\sqrt{285} & \frac{473i}{1330} & \frac{36}{665}\sqrt{14} \\ \frac{6i}{475}\sqrt{285} & -\frac{18}{3325}\sqrt{3990} & -\frac{36}{665}\sqrt{14} & \frac{41i}{190} \end{bmatrix}.$$

This matrix has eigenvalues $\frac{i}{2}, -\frac{i}{2}$ with multiplicity 3 and 1, respectively. By Theorem 2.3.37, this monopole has generalized minimal symmetry breaking, as given by Definition 2.3.17.

2.1.3.3 Constrained structure groups

In this section, we prove that there is a constraint on the structure groups of spherically symmetric monopoles generated by a representation of $\mathfrak{sp}(1)$.

Given a real representation (V, ρ) of $\mathfrak{sp}(1)$, the Structure Theorem helps us find a vector space in which the M component of ADHM data corresponding to spherically symmetric

monopoles lies. Lemma 2.1.13 tells us that in $\mathcal{M}_{n,k}$, L is uniquely determined by M , up to multiplication by a $\mathrm{Sp}(n)$ factor, which does not affect the corresponding monopole. Thus, for a given n , the Structure Theorem helps us find the unique family of spherically symmetric monopoles generated by (V, ρ) . However, if n is allowed to vary, the same vector space may give rise to monopoles with different structure groups. For example, see Proposition 2.1.42 or Proposition 2.1.44. In these examples, the exact M matrices differ when looking at different structure groups, but they belong to the same vector space. Nevertheless, it may be that the exact same M matrix can give rise to monopoles with different structure groups. We find a constraint on the structure group, identifying which values of n permit a spherically symmetric monopole generated by (V, ρ) .

In Theorem 2.1.22, we obtain a representation (V, ρ) of $\mathfrak{sp}(1)$ from a spherically symmetric monopole by focusing on the bottom of (2.12). This representation induces another, $(\hat{V}, \hat{\rho})$, which we use to determine what the M part of the ADHM data of a spherically symmetric monopole looks like. Theorem 2.1.22 gives us a second representation by focusing on the top of (2.12).

Lemma 2.1.46. *Suppose a real representation (V, ρ) generates a spherically symmetric monopole, with ADHM data $\hat{M} \in \mathcal{M}_{n,k}$. Let $Y_i := \rho(v_i)$ and $y_i := (LL^\dagger)^{-1}L(v_i I_k + Y_i)L^\dagger \in \mathfrak{sp}(n)$. Then y_1, y_2, y_3 induce a quaternionic representation of $\mathfrak{sp}(1)$ which we denote by (W, λ) , where $W := \mathbb{H}^n$ and λ is the linear map taking $v_i \mapsto y_i$.*

Proof. By Theorem 2.1.22, as the monopole with ADHM data $\hat{M} \in \mathcal{M}_{n,k}$ is generated by (V, ρ) , we have that $[v_i I_k + Y_i, M] = 0$. Hence, $[v_i I_k + Y_i, L^\dagger L] = 0$, as $L^\dagger L = I_k + M^2$. Also,

$$LL^\dagger L(v_i I_k + Y_i)L^\dagger = L(v_i I_k + Y_i)L^\dagger LL^\dagger,$$

so $LL^\dagger$ commutes with $L(v_i I_k + Y_i)L^\dagger$. Let y_i be as above. Then $y_i^\dagger = -y_i$ and as $v_i I_k$ and Y_j commute,

$$[y_i, y_j] = (LL^\dagger)^{-1}L[(v_i I_k + Y_i), (v_j I_k + Y_j)]L^\dagger = \sum_{l=1}^3 \epsilon_{ijl} y_l.$$

As the $y_i \in \mathfrak{sp}(n)$, they induce a quaternionic representation. $\square$

We use (W, λ) to determine what $\mathrm{Sp}(n)$ structure groups are possible, given (V, ρ) . Using $W = \mathbb{H}^n \simeq \mathbb{C}^{2n}$, we can restrict the scalars from $\mathbb{H}$ to $\mathbb{C}$. In doing so, the generators $y_i \in \mathfrak{sp}(n)$ correspond to elements of $\mathfrak{su}(2n)$. Thus, the induced complex representation is a $2n$ -representation. As mentioned in Section D.1.3 of Appendix D, the complex representation obtained by restricting the scalars of an irreducible, quaternionic representation is isomorphic to either (V_n, ρ_n) for some n even, or $(V_n, \rho_n)^{\oplus 2}$ for some n odd.

Definition 2.1.47. Let $\iota: \mathfrak{sp}(1) \rightarrow \mathbb{H}$ be the inclusion map. Then $(\mathbb{H}, \iota)$ is the fundamental representation. After restricting the scalars of the fundamental representation, the complex representation is isomorphic to (V_2, ρ_2) . Given representations (V, ρ) and (W, λ) of $\mathfrak{sp}(1)$, real and quaternionic, respectively, we define the real representation

$$(\hat{W}, \hat{\lambda}) := ((W, \lambda) \otimes_{\mathbb{R}} (V^*, \rho^*)) \otimes_{\mathbb{H}} (\mathbb{H}^*, \iota^*) \simeq (W, \lambda) \otimes_{\mathbb{H}} ((V^*, \rho^*) \otimes_{\mathbb{R}} (\mathbb{H}^*, \iota^*)).$$

Let $n := \dim_{\mathbb{H}}(W)$ and $k := \dim_{\mathbb{R}}(V)$. Unraveling how $\hat{\lambda}$ acts on $\hat{W} = \text{Mat}(n, k, \mathbb{H})$, let $v \in \mathfrak{sp}(1)$ and $A \in \hat{W}$. Then

$$\hat{\lambda}(v)(A) = \lambda(v)A - A\rho(v) - Av.$$

The definition of $(\hat{W}, \hat{\lambda})$ is well-motivated. First, note that $L \in \hat{W}$. Moreover, the following corollary tells us exactly where L lives in $\hat{W}$.

Corollary 2.1.48. Let $\hat{M} \in \mathcal{M}_{n,k}$ be spherically symmetric. Let (V, ρ) , (W, λ) and $(\hat{W}, \hat{\lambda})$ be as above. Then $\hat{\lambda}(v)(L) = 0$ for all $v \in \mathfrak{sp}(1)$.

Proof. As $\hat{M}$ is spherically symmetric, we have (2.12). Recall that the $Y_i := \rho(v_i)$. Then we see that for $v = \sum_{i=1}^3 a_i v_i \in \mathfrak{sp}(1)$,

$$\hat{\lambda}(v)(L) = \sum_{i=1}^3 a_i (\lambda(v_i)L - L\rho(v_i) - Lv_i) = 0. \quad \square$$

Corollary 2.1.49. Let $\hat{M} \in \mathcal{M}_{n,k}$ be spherically symmetric. Let $(\hat{W}, \hat{\lambda})$ be as above. Then $(\hat{W}, \hat{\lambda})$ has trivial summands and L lives in the direct sum of them.

Proof. As $L \neq 0$, we know $\text{span}(L) \subseteq \hat{W}$ is an invariant 1-dimensional subspace that is acted on trivially, so $(\text{span}(L), 0)$ is a trivial summand of the representation $(\hat{W}, \hat{\lambda})$. $\square$

This constraint on the representation (W, λ) narrows down the possibilities for the structure group. For instance, if we take $(V, \rho) = (\mathbb{R}^m, \varrho_m) \oplus (\mathbb{R}^{m+2}, \varrho_{m+2})$ with $m > 1$ odd, then $(\hat{W}, \hat{\lambda})$ only has a trivial summand when the complex representation obtained by restricting the scalars of (W, λ) has a (V_{m-1}, ρ_{m-1}) , (V_{m+1}, ρ_{m+1}) , or (V_{m+3}, ρ_{m+3}) summand. Hence, we must have $2n \geq m - 1$, giving us a lower bound on n . By Note 2.1.3, we also have an upper bound, $n \leq 2m + 2$.

For example, taking $m = 7$, we see that the complex representation obtained by restricting the scalars of (W, λ) must have a (V_6, ρ_6) , (V_8, ρ_8) , or (V_{10}, ρ_{10}) summand. Thus,

$n \geq 3$, so there is no spherically symmetric $\mathrm{Sp}(1)$ or $\mathrm{Sp}(2)$ hyperbolic monopole generated by $(\mathbb{R}^7, \varrho_7) \oplus (\mathbb{R}^9, \varrho_9)$ with ADHM data in $\mathcal{M}_{n,16}$ for some $n \in \mathbb{N}_+$.

We can use this constraint to extract information from low rank (V, ρ) . Consider the $n = 1$ case of Proposition 2.1.42. That is $(V, \rho) = (\mathbb{R}^3, \varrho_3) \oplus (\mathbb{R}^1, \varrho_1)$. This representation generates spherically symmetric monopoles: one family with structure group $\mathrm{Sp}(4)$ and another with structure group $\mathrm{Sp}(2)$. It does not generate a spherically symmetric monopole with a lower rank structure group.

Proposition 2.1.50. *Consider the $n = 1$ case of Proposition 2.1.42. That is $(V, \rho) = (\mathbb{R}^3, \varrho_3) \oplus (\mathbb{R}^1, \varrho_1)$. This representation does not generate a spherically symmetric $\mathrm{Sp}(1)$ hyperbolic monopole with ADHM data in $\mathcal{M}_{1,4}$.*

Proof. Suppose (V, ρ) generates some spherically symmetric $\mathrm{Sp}(1)$ monopole with ADHM data $\hat{M} \in \mathcal{M}_{1,4}$. We know that $(\hat{W}, \hat{\lambda})$ has a trivial summand. Consider $(V^*, \rho^*) \otimes_{\mathbb{R}} (\mathbb{H}^*, \iota^*)$. Restricting the scalars, this representation is isomorphic, as a complex representation, to $(V_4, \rho_4) \oplus (V_2, \rho_2)^{\oplus 2}$.

As $\hat{M}$ has $\mathrm{Sp}(1)$ as a structure group, after restricting the scalars of (W, λ) , we get a complex 2-representation. In order for $(\hat{W}, \hat{\lambda})$ to have a trivial summand and for the previously mentioned complex representation to be a 2-representation, we must have $(W, \lambda) \simeq (\mathbb{H}, \iota)$, the fundamental representation. Hence, there is some $p \in \mathrm{Sp}(1)$ such that $y_i = pv_i p^\dagger$ and

$$pv_i p^\dagger L - LY_i - Lv_i = 0.$$

Taking $\tilde{L} := p^\dagger L$, we see $[v_i, \tilde{L}] = \tilde{L} Y_i$. Such a choice has no effect on the monopole.

Using the Y_i matrices from the proof of Proposition 2.1.42 and solving these equations, we find there is some $c, d \in \mathbb{R}$ such that $\tilde{L} = \begin{bmatrix} ck & cj & ci & d \end{bmatrix}$. Recall that with these Y_i generators, we have

$$M = a \begin{bmatrix} 0 & 0 & 0 & k \\ 0 & 0 & 0 & j \\ 0 & 0 & 0 & i \\ k & j & i & 0 \end{bmatrix}.$$

Hence,

$$I_4 = L^\dagger L - M^2 = \tilde{L}^\dagger \tilde{L} - M^2 = \begin{bmatrix} a^2 + c^2 & (a^2 + c^2)i & -(a^2 + c^2)j & -cdk \\ (a^2 + c^2)i & (a^2 + c^2) & (a^2 + c^2)k & -cdj \\ (a^2 + c^2)j & -(a^2 + c^2)k & (a^2 + c^2) & -cdi \\ cdk & cdj & cdi & d^2 + 3a^2 \end{bmatrix}.$$

Hence, $a^2 + c^2 = 0$ and $a^2 + c^2 = 1$. Contradiction! Thus, (V, ρ) does not generate any spherically symmetric $\mathrm{Sp}(1)$ hyperbolic monopoles with ADHM data in $\mathcal{M}_{1,4}$. $\square$

Proposition 2.1.51. *Consider the $n = 4$ case in Proposition 2.1.40. That is $(V, \rho) = (\mathbb{R}^4, \varrho_4)$. This representation does not generate a spherically symmetric $\mathrm{Sp}(1)$ or $\mathrm{Sp}(2)$ hyperbolic monopole with ADHM data in $\mathcal{M}_{n,4}$, where $n = 1, 2$, respectively.*

Proof. We know that $(\hat{W}, \hat{\lambda})$ must have a trivial summand. Consider $(V^*, \rho^*) \otimes_{\mathbb{R}} (\mathbb{H}^*, \iota^*)$. Restricting the scalars, this representation is isomorphic, as a complex representation, to $(V_3, \rho_3)^{\oplus 2} \oplus (V_1, \rho_1)^{\oplus 2}$.

Suppose that (V, ρ) generates a spherically symmetric $\mathrm{Sp}(1)$ hyperbolic monopole with ADHM data $\hat{M} \in \mathcal{M}_{1,4}$. Then the complex representation obtained by restricting the scalars of (W, λ) is a 2-representation. In order for $(\hat{W}, \hat{\lambda})$ to have a trivial summand and for the previously mentioned complex representation to be a 2-representation, we must have that the complex representation obtained by restricting the scalars of (W, λ) is isomorphic to $(V_1, \rho_1)^{\oplus 2}$. Hence, $y_i = 0$, so $L(Y_i + v_i I_4) = 0$.

The following matrices induce the irreducible real 4-representation:

$$Y_1 = \frac{1}{2} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad Y_2 = \frac{1}{2} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad Y_3 = \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Indeed, the Casimir operator is $C = -\sum_{i=1}^3 Y_i^2 = \frac{3}{4} I_4$.

Using these matrices, we solve the equations $L(Y_i + v_i I_4) = 0$, finding there is some $A \in \mathbb{H}$ such that $L = A \begin{bmatrix} 1 & -i & -j & -k \end{bmatrix}$. In Proposition 2.1.40, we showed that there are some $\kappa_1, \kappa_2, \kappa_3 \in \mathbb{R}$ and $U \in \mathrm{SU}(4)$ such that

$$M_i = Y_i U^\dagger \begin{bmatrix} \kappa_1 i I_{k/2} & (\kappa_2 + i \kappa_3) I_{k/2} \\ (-\kappa_2 + i \kappa_3) I_{k/2} & -\kappa_1 i I_{k/2} \end{bmatrix} U.$$

Using the fact that the matrix multiplied by Y_i above commutes with all Y_i , we see that

$$I_4 = L^\dagger L - M^2 = |A|^2 \begin{bmatrix} 1 & -i & -j & -k \\ i & 1 & -k & j \\ j & k & 1 & -i \\ k & -j & i & 1 \end{bmatrix} - \frac{\kappa_1^2 + \kappa_2^2 + \kappa_3^2}{4} \begin{bmatrix} -3 & 2i & 2j & 2k \\ -2i & -3 & 2k & -2j \\ -2j & -2k & -3 & 2i \\ -2k & 2j & -2i & -3 \end{bmatrix}.$$

Hence, $|A|^2 + \frac{\kappa_1^2 + \kappa_2^2 + \kappa_3^2}{2} = 0$, so $A = 0$, meaning $L = 0$. Contradiction! Thus, (V, ρ) does not generate any spherically symmetric $\mathrm{Sp}(1)$ hyperbolic monopoles with ADHM data in $\mathcal{M}_{1,4}$.

Suppose that (V, ρ) generates a spherically symmetric $\mathrm{Sp}(2)$ hyperbolic monopole with ADHM data $\hat{M} \in \mathcal{M}_{2,4}$. Then the complex representation obtained by restricting the scalars of (W, λ) is a 4-representation. In order for $(\hat{W}, \hat{\lambda})$ to have a trivial summand and for the previously mentioned complex representation to be a 4-representation, we must have the complex representation obtained by restricting the scalars of (W, λ) to be isomorphic to $(V_1, \rho_1)^{\oplus 4}$, $(V_2, \rho_2) \oplus (V_1, \rho_1)^{\oplus 2}$, or $(V_3, \rho_3) \oplus (V_1, \rho_1)$. Note that we can ignore the final case, as this can not come from restricting the scalars of an irreducible quaternionic representation; the complex representations obtained by restricting the scalars of irreducible quaternionic representations are isomorphic as complex representations to (V_i, ρ_i) with i even and $(V_i, \rho_i)^{\oplus 2}$ with i odd.

Consider the former case. We again have $y_i = 0$, so from above, we see that there are some $A, B \in \mathbb{H}$ such that $L = \begin{bmatrix} A & -Ai & -Aj & -Ak \\ B & -Bi & -Bj & -Bk \end{bmatrix}$. Then from $I_4 = L^\dagger L - M^2$, we obtain $|A|^2 + |B|^2 + \frac{\kappa_1^2 + \kappa_2^2 + \kappa_3^2}{2} = 0$, which means $L = 0$. Thus, we must have the middle case.

We know that L lives in the direct sum of the trivial summands of $(\hat{W}, \hat{\lambda})$. Thus, we know that L lives in the $(V_1, \rho_1)^{\oplus 2}$ part of the complex representation obtained by restricting the scalars of (W, λ) . Thus, there is some $q \in \mathrm{Sp}(2)$ such that $y_i := q \begin{bmatrix} v_i & 0 \\ 0 & 0 \end{bmatrix} q^\dagger$.

Let $\tilde{L} := q^\dagger L$. We see that

$$\begin{bmatrix} v_i & 0 \\ 0 & 0 \end{bmatrix} \tilde{L} - \tilde{L} Y_i - \tilde{L} v_i = 0.$$

As L lives in the trivial summand of $(\hat{W}, \hat{\lambda})$, we know that

$$\tilde{L} = \begin{bmatrix} 0 \\ \tilde{l} \end{bmatrix}.$$

Hence, $\tilde{l}(v_i I_k + Y_i) = 0$. From the above, then $\tilde{l} = 0$, so $L = 0$. Contradiction! Hence, (V, ρ) does not generate any spherically symmetric $\mathrm{Sp}(2)$ hyperbolic monopoles with ADHM data in $\mathcal{M}_{2,4}$. $\square$

2.2 Euclidean monopoles

In this section, we apply Theorem 1.1.1 to Euclidean monopoles. In particular, we discuss how Theorem 1.1.1 allows us to remove technical assumptions present in previous work, which was largely completed prior to the start of my PhD [CDL⁺22]. Let $E \rightarrow \mathbb{R}^3$ be a vector bundle over $\mathbb{R}^3$ with structure group $\mathrm{SU}(n)$, equipped with a connection A , of curvature F_A , and a section Φ of $\mathrm{End}(E)$, called the Higgs field. Let $\star$ denote the Hodge star. As explained in Section 2.1, Euclidean monopoles are solutions to the Bogomolny equations $F_A = \star D_A \Phi$, with finite energy $\varepsilon = \frac{1}{2\pi} \int_{\mathbb{R}^3} |F_A|^2 \mathrm{vol}_{\mathbb{R}^3}$.

Euclidean monopoles share many properties with their hyperbolic counterparts. However, whereas hyperbolic monopoles correspond to circle-invariant ADHM data via the ADHM transform, Euclidean monopoles correspond to Nahm data via the Nahm transform. Nahm data are triples $\mathcal{T} := (T_1, T_2, T_3)$ of $\mathfrak{su}(n)$ -valued functions over an open subset of $\mathbb{R}$ satisfying the Nahm equations:

$$\begin{aligned} \frac{d}{dt} T_1(t) &= [T_2(t), T_3(t)], \\ \frac{d}{dt} T_2(t) &= [T_3(t), T_1(t)], \\ \frac{d}{dt} T_3(t) &= [T_1(t), T_2(t)]. \end{aligned} \tag{2.2.1}$$

Just as with ADHM data, Nahm data is still challenging to produce, due to the non-linearity of the Nahm equations. If the Nahm data has a simple pole at one of the endpoints of the interval, then the residues of the Nahm data at that point gives a representation of $\mathfrak{so}(3)$ [CDL⁺22, Lemma 2.1].

Just as with the ADHM data given in Definition 2.1.1, there are two group actions we can define. While we define these actions on matrices, they descend to actions on Nahm data, acting pointwise.

Definition 2.2.1. Let $\mathcal{M}_n := \mathrm{Mat}(n, n, \mathbb{C})^{\oplus 3}$. We define the **gauge** action of $\mathrm{SU}(n)$ on $\mathcal{M}_n$ as follows. For $U \in \mathrm{SU}(n)$ and $\mathcal{P} \in \mathcal{M}_n$, let ${}_U \mathcal{P} := ({}_U P_1, {}_U P_2, {}_U P_3)$ be given by

$${}_U P_i := U P_i U^\dagger, \quad \forall i \in \{1, 2, 3\}. \tag{2.2.2}$$

We define the **rotation** action of $\mathrm{SO}(3)$ on $\mathcal{M}_n$ as follows. For $A \in \mathrm{SO}(3)$ and $\mathcal{P} \in \mathcal{M}_n$, let ${}^A \mathcal{P} := ({}^A P_1, {}^A P_2, {}^A P_3)$ be given by

$${}^A P_i := \sum_{j=1}^3 A_{ij} P_j, \quad \forall i \in \{1, 2, 3\}. \tag{2.2.3}$$

These actions are named as such to reflect what is happening to the corresponding monopoles. Just as with hyperbolic monopoles, by Corollary B.0.8 in Appendix B, as we are studying continuous symmetries, we need only investigate monopoles with rotational symmetry. Notice also that the actions commute, so the rotation action descends to an action on the moduli space $\mathcal{M}_n/\mathrm{SU}(n)$.

Definition 2.2.2. *We say that $\mathcal{P} \in \mathcal{M}_n$ is **equivariant** under $A \in \mathrm{SO}(3)$ if $[\mathcal{P}]$ is fixed under the action of A . That is, there is some $U \in \mathrm{SU}(n)$ such that ${}^A_U \mathcal{P} = \mathcal{P}$, meaning that rotation by A has no effect on $\mathcal{P}$.*

Given $\mathcal{P} \in \mathcal{M}_n$, we denote the set of $A \in \mathrm{SO}(3)$ under which $\mathcal{P}$ is equivariant by $H_{\mathcal{P}} \subseteq \mathrm{SO}(3)$. As we are dealing with group actions, $H_{\mathcal{P}}$ is a subgroup.

If $\mathcal{T}$ is Nahm data and $\mathcal{T}(t_0) \in \mathcal{M}_n$ is equivariant under some $A \in \mathrm{SO}(3)$, then $\mathcal{T}(t)$ is equivariant under A for all t in the domain of $\mathcal{T}$ [CDL⁺22, Corollary 2.6]. In fact, the gauge transformation undoing the rotation is the same at every point. This phenomenon tells us that the equivariance of Nahm data can be obtained by looking at a single point. As such, we do exactly that. Moreover, in this case, we can look at $H_{\mathcal{T}}$, without specifying a point.

Similar to the case of hyperbolic monopoles, the only non-zero connected Lie subgroups of $\mathrm{SO}(3)$ are S^1 and $\mathrm{SO}(3)$. As proven in Section C.1 in Appendix C, when examining monopoles with continuous symmetries, we need only consider one circle symmetry. This follows from the fact that the maximal tori of $\mathrm{Sp}(1)$ are S^1 and all maximal tori are conjugate. In particular, they are all conjugate to

$$H_z := \{\tilde{A} \oplus 1 \mid \tilde{A} \in \mathrm{SO}(2)\} \subseteq \mathrm{SO}(3).$$

The set H_z is the set of rotations about the z -axis. Each maximal torus of $\mathrm{SO}(3)$ is the set of rotations about a fixed axis.

Definition 2.2.3. *When $H_{\mathcal{T}}$ contains a subgroup conjugate to S^1 , we say that $\mathcal{T}$ is **axially symmetric**. When $H_{\mathcal{T}} = \mathrm{SO}(3)$, we say that $\mathcal{T}$ is **spherically symmetric**.*

In previous work, we use this machinery to find linear equations for axial and spherical symmetry [CDL⁺22, Theorems 3.1 & 4.1]. In that work, in order to obtain representations, we had to make assumptions about differentiability and the existence of a representation. Given Theorem 1.1.1, we can remove these assumptions. We fill in the gaps for the spherical symmetry case, but the axial case is similar. Ultimately, we obtain the following results.

Theorem 2.2.4. *Let $\mathcal{T}$ be Nahm data taking value in $\mathfrak{su}(n)^{\oplus 3} \subseteq \mathcal{M}_n$. Note that $\mathcal{T}$ is said to be axially symmetric about the z -axis if it is equivariant under every element of H_z . Then $\mathcal{T}$ is axially symmetric about the z -axis if and only if there is some $Y \in \mathfrak{su}(n)$ such that at every point t in the domain of $\mathcal{T}$, we have*

$$T_1(t) = [T_2(t), Y], \quad (2.2.4)$$

$$T_2(t) = [Y, T_1(t)], \quad (2.2.5)$$

$$0 = [T_3(t), Y]. \quad (2.2.6)$$

Definition 2.2.5. *The matrix $Y \in \mathfrak{su}(n)$ is called the **generator** of axial symmetry for $\mathcal{T}$.*

Note that symmetry about another axis can be obtained by rotating the Nahm data. Let $v_1, v_2, v_3 \in \mathfrak{so}(3)$ be given by $(v_i)_{jk} = -\epsilon_{ijk}$, the infinitesimal generator of rotation about the x , y , and z -axes, respectively.

Theorem 2.2.6. *Let $\mathcal{T}$ be Nahm data taking value in $\mathfrak{su}(n)^{\oplus 3} \subseteq \mathcal{M}_n$. Note that $\mathcal{T}$ is said to be spherically symmetric if it is equivariant under every element of $\mathrm{SO}(3)$. Then $\mathcal{T}$ is spherically symmetric if and only if there is some representation $(\mathbb{C}^n, \rho)$ with $\rho: \mathfrak{so}(3) \rightarrow \mathfrak{su}(n)$ such that for all $i, j \in \{1, 2, 3\}$ and every point t in the domain of $\mathcal{T}$, we have*

$$[\rho(v_i), T_j(t)] = \sum_{k=1}^3 \epsilon_{ijk} T_k(t). \quad (2.2.7)$$

Definition 2.2.7. *We call ρ the **generating representation** of the spherical symmetry of $\mathcal{T}$.*

Proof. Suppose that $\mathcal{T}$ is spherically symmetric. The converse direction follows as in the original proof [CDL⁺22, Theorem 4.1]. Let S be the stabilizer group

$$S := \{(A, U) \in \mathrm{SO}(3) \times \mathrm{SU}(n) \mid {}^A_U \mathcal{T} = \mathcal{T}\}. \quad (2.2.8)$$

Note that S does not depend on t [CDL⁺22, Corollary 2.6].

Just as in Theorem 1.1.1, we get a Lie group homomorphism $\Psi: \mathrm{Sp}(1) \rightarrow S$ such that $\pi_1|_S \circ \Psi = \pi$, where $\pi: \mathrm{Sp}(1) \rightarrow \mathrm{SO}(3)$ is the double cover. Additionally, there is some representation $\rho: \mathfrak{so}(3) \rightarrow \mathfrak{su}(n)$ such that for all $\theta \in \mathbb{R}$, $\Psi(e^{\theta v_i}) = (\pi(e^{\theta v_i}), e^{\rho(\theta v_i)}) \in S$. Thus, we have

$$\pi(e^{\theta v_i}) e^{\rho(\theta v_i)} \mathcal{T} = \mathcal{T}.$$

Taking the derivative and evaluating at $\theta = 0$, we obtain (2.2.7) at every point in the domain of $\mathcal{T}$. $\square$

2.3 Symmetry breaking

In this section, we examine the symmetry breaking of monopoles. The main results in this section are as follows. Theorem 2.3.13 identifies $\pi_2(G/H)$, which determines the symmetry breaking of a monopole, as a free abelian group and finds its rank for any compact, connected Lie group. Theorems 2.3.31, 2.3.35, 2.3.37, and 2.3.40 provide a simple classification of symmetry breaking for all Lie groups with a classical, simple Lie algebra. We also outline a method for doing the same for the exceptional simple Lie algebras. In doing so, we provide a way to classify symmetry breaking for all compact, connected Lie groups.

Let G be a compact, connected, real Lie group and M a connected, geodesically complete three-manifold with infinite injectivity radius at some point. Let $\star$ be the Hodge star on M .

Note that by the Hopf–Rinow Theorem, as M is geodesically complete, the exponential map at any point is defined on the whole tangent space. In particular, at the point with infinite injectivity radius, the exponential map is a global diffeomorphism. Examples of such manifolds are Euclidean and hyperbolic three-space. Note that for compact manifolds, the injectivity radius at any point is finite. However, for compact manifolds, monopoles are all flat, so we are uninterested in this case.

Let E be a vector bundle with structure group G . Let A be a connection on E with curvature F_A and covariant derivative D_A . Let Φ be a section of the endomorphism bundle $\text{End}(E)$, called the Higgs field. The pair (Φ, A) is a monopole if it satisfies the Bogomolny equations $F_A = \star D_A \Phi$ and has finite energy $\frac{1}{2\pi} \int_M |F_A|^2 \text{vol}_M < \infty$.

Given $g \in G$, let $\Psi_g: G \rightarrow G$ be the Lie group homomorphism $\Psi_g(h) := ghg^{-1}$. Denote the Lie algebra of G by $\mathfrak{g}$ and its centre by $\mathfrak{z}(\mathfrak{g})$. The adjoint action of g on $\mathfrak{g}$ is the map $\text{Ad}_g: \mathfrak{g} \rightarrow \mathfrak{g}$ given by $\text{Ad}_g := d_e \Psi_g$. If G is a matrix Lie group, the exponential map is the matrix exponential. As such, $\text{Ad}_g(x) = gxg^{-1}$.

Gauge transformations are equivariant bundle automorphisms of E , acting trivially on M . Such transformations affect Φ and A , but do not physically change the monopole. As M is diffeomorphic to $\mathbb{R}^3$, all bundles are trivial. Thus, gauge transformations can be thought of as smooth maps $g: M \rightarrow G$. By the Peter–Weyl Theorem, as G is a compact Lie group, we can view it as a subgroup of a matrix Lie group. Thus, for monopoles, these gauge transformations take $\Phi \mapsto g\Phi g^{-1}$ and $A \mapsto gAg^{-1} - (dg)g^{-1}$.

On $M = \mathbb{R}^3$ with the standard metric, it is conjectured that the finite energy condition implies an extra boundary condition: there exists $\Phi_\infty \in \mathfrak{g} \setminus \mathfrak{z}(\mathfrak{g})$ such that along every

oriented geodesic in $M = \mathbb{R}^3$, as you approach infinity, Φ approaches $\text{Ad}_g(\Phi_\infty)$ for some $g \in G$ [MS03, BC1]. We assume that our monopoles satisfy this boundary condition for any M . Note that if $\Phi_\infty \in \mathfrak{z}(\mathfrak{g})$, we have that the adjoint action of any $g \in G$ on Φ_∞ has no effect: $\text{Ad}_g(\Phi_\infty) = \Phi_\infty$. We see that if $\Phi_\infty \in \mathfrak{z}(\mathfrak{g})$, then our monopole is topologically trivial, so we ignore this case.

Given this boundary condition, the Higgs field is said to spontaneously break the symmetry group G to the isotropy group $H \subseteq G$ given by

$$H := \{g \in G \mid \text{Ad}_g(\Phi_\infty) = \Phi_\infty\}. \quad (2.3.1)$$

This boundary condition supplies us with a topological classification of monopoles. Specifically, associated to every monopole is a gauge-invariant class in $\pi_2(G/H)$. Therefore, we can use this class to distinguish monopoles that are not related by gauge transformations. In addition, we can distinguish monopoles by studying $\pi_2(G/H)$ itself. The form this homotopy group takes is known as the symmetry breaking of a monopole.

In Section 2.3.1, we define a general notion of maximal and minimal symmetry breaking. In Section 2.3.2, we show that this notion of maximal symmetry breaking agrees with the myriad definitions found in the literature. In Section 2.3.3, we classify symmetry breaking for all Lie groups with a classical, simple Lie algebra. There, we find that the general definition of maximal symmetry breaking agrees with the concrete definition for $\text{SU}(n)$ monopoles. We shall also see that the usual definition of minimal symmetry breaking for $\text{SU}(n)$ monopoles is just a specific case of the general definition introduced here. We also outline a method for doing the same for the exceptional simple Lie algebras.

2.3.1 Classifying symmetry breaking

In this section, through identifying $\pi_2(G/H)$, we motivate and define a general notion of maximal and minimal symmetry breaking. Recall that G is a compact, connected, real Lie group with Lie algebra $\mathfrak{g}$. Given $\Phi_\infty \in \mathfrak{g} \setminus \mathfrak{z}(\mathfrak{g})$, H is the isotropy group of the adjoint action on Φ_∞ .

Lemma 2.3.1. *The group H is a compact, connected, real Lie group.*

Proof. Let $\text{Ad}_g: \mathfrak{g} \rightarrow \mathfrak{g}$ be the adjoint action of g on $\mathfrak{g}$. Consider the map $f: G \rightarrow \mathfrak{g}$ given by $f(g) := \text{Ad}_g(\Phi_\infty)$. Note that f is a smooth map, as it arises as the derivative of multiplication on a Lie group, which is smooth. As $H = f^{-1}(\Phi_\infty)$, we have that H is a closed subgroup of a compact, real Lie group. Thus, H itself is a compact, real Lie group.

Moreover, as G is a compact, connected Lie group, the isotropy group H of Φ_∞ is connected [DK00, Theorem 3.3.1]. $\square$

The Lie algebra $\mathfrak{h} := \text{Lie}(H)$ given by

$$\mathfrak{h} := \{x \in \mathfrak{g} \mid [x, \Phi_\infty] = 0\}. \quad (2.3.2)$$

Indeed, note that the derivative of Ad is $\text{ad}: \mathfrak{g} \rightarrow \text{Der}(\mathfrak{g})$ given by $\text{ad}(x)(y) = [x, y]$.

Before we identify $\pi_2(G/H)$, we must first understand why this group is important. To that end, we examine the various ways we can view G/H .

Definition 2.3.2. *The **adjoint orbit** $\text{Ad}(G)\Phi_\infty \subseteq \mathfrak{g}$ is the orbit of Φ_∞ under the adjoint action of G on $\mathfrak{g}$. The orbit is given by*

$$\text{Ad}(G)\Phi_\infty := \{\text{Ad}_g(\Phi_\infty) \mid g \in G\}. \quad (2.3.3)$$

As G is compact, it acts properly on $\mathfrak{g}$. As G also acts smoothly on $\mathfrak{g}$, the adjoint orbit is a submanifold of $\mathfrak{g}$, diffeomorphic to G/H .

Recall that the Higgs field is a section of $\text{End}(E)$. As M is diffeomorphic to $\mathbb{R}^3$, all real vector bundles over M , in particular E and $\text{End}(E)$, are trivial bundles. Thus, Φ can be thought of as a smooth map $M \rightarrow \mathfrak{g}$. Recall that along any oriented geodesic in M , as $r \rightarrow \infty$, we have that there is some $g \in G$ such that $\Phi \rightarrow \text{Ad}_g(\Phi_\infty)$. Let θ_1 and θ_2 be the standard coordinates on S^2 . Considering all oriented geodesics through a given point $p \in M$, we obtain a smooth map $\phi: S^2 \rightarrow \mathfrak{g}$ given by $\phi(\theta_1, \theta_2) := \lim_{r \rightarrow \infty} \Phi(r, \theta_1, \theta_2)$. By hypothesis, we have that $\phi: S^2 \rightarrow \text{Ad}(G)\Phi_\infty$.

Note that while the adjoint orbit $\text{Ad}(G)\Phi_\infty$ is gauge invariant, ϕ is not. However, we find something that is. In order to preserve the structure of our monopoles, our gauge transformations must preserve the boundary condition above. As such, along any oriented geodesic, a gauge transformation must have a limit as $r \rightarrow \infty$. Considering all oriented geodesics through a given point $p \in M$ and a gauge transformation g , we can consider $\tilde{g}: S^2 \rightarrow G$, obtained just as for ϕ above.

As G is a Lie group, $\pi_2(G) = 1$. Thus, any map $S^2 \rightarrow G$ is homotopy equivalent to the identity. As such, if (Φ, A) and (Φ', A') are two monopoles related via a gauge transformation, $\phi, \phi': S^2 \rightarrow \text{Ad}(G)\Phi_\infty$ are related by a map $\tilde{g}: S^2 \rightarrow G$. By the homotopic equivalence of $\tilde{g}$ and the identity, $[\phi] = [\phi']$. Thus, the class $[\phi] \in \pi_2(\text{Ad}(G)\Phi_\infty)$ is gauge invariant.

Definition 2.3.3. *The class $[\phi] \in \pi_2(\text{Ad}(G)\Phi_\infty)$ is the **degree** of the monopole.*

Monopoles are topologically classified by their asymptotic adjoint orbit $\text{Ad}(G)\Phi_\infty$ and their degree. Indeed, these parameters are part of the data used to stratify and investigate the moduli space of monopoles [MS03, Men23]. As such, we are interested in understanding $\text{Ad}(G)\Phi_\infty$ and $\pi_2(\text{Ad}(G)\Phi_\infty)$.

Note 2.3.4. *This view of G/H allows us to completely classify the various types of symmetry breaking. Some sources attempt to circumvent this approach by splitting the isotropy subgroup H into the product of a torus and a semi-simple Lie group [JT80, Exercise II.3.8; HW02, § 1; MS04, § 8.13]. They then use this splitting to find $\pi_2(G/H)$. However, this splitting does not always occur. In Appendix G, we consider the isotropy subgroup of $i \cdot \text{diag}(2, -1, -1) \in \mathfrak{su}(3)$. It is known that H is isomorphic, as a Lie group, to $(\text{U}(1) \times \text{SU}(2))/\{\pm(1, I)\}$ [Wei12, § 6.2.1]. In this appendix, Proposition G.0.1 tells us that this isotropy group does not split. As such, to understand $\pi_2(G/H)$, we must investigate the adjoint orbit using other methods.*

Before we continue to investigate $G/H \simeq \text{Ad}(G)\Phi_\infty$, we first introduce some terminology and facts from Lie theory.

Definition 2.3.5. *Given a Lie algebra $\mathfrak{g}$, the **centralizer** of an element $x \in \mathfrak{g}$ is the Lie subalgebra $C_{\mathfrak{g}}(x) := \{y \in \mathfrak{g} \mid [x, y] = 0\}$. Note that $\mathfrak{h} = C_{\mathfrak{g}}(\Phi_\infty)$.*

*Given a Lie group G , the **centre** $Z(G)$ of G is the set of all $g \in G$ such that $gh = hg$ for all $h \in G$. The **centre** $\mathfrak{z}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the set of all $x \in \mathfrak{g}$ such that $[x, y] = 0$ for all $y \in \mathfrak{g}$. Note that the Lie algebra of the centre of G is the centre of the Lie algebra of G .*

*Given a real Lie algebra $\mathfrak{g}$, let $\mathfrak{g}^{\mathbb{C}}$ be its **complexification**. For a complex, semi-simple Lie algebra, let $\text{rank}(\mathfrak{g})$ denote the rank of $\mathfrak{g}$. For a real, semi-simple Lie algebra, let $\text{rank}(\mathfrak{g})$ denote the rank of $\mathfrak{g}^{\mathbb{C}}$.*

Given $x \in \mathfrak{g}$, let $\mathfrak{g}^0(x) \subseteq \mathfrak{g}$ be the Lie subalgebra defined by

$$\mathfrak{g}^0(x) := \bigcup_{n \geq 0} \ker(\text{ad}(x)^n). \quad (2.3.4)$$

*We say that $x \in \mathfrak{g}$ is **regular** if the dimension of $\mathfrak{g}^0(x)$ is $\text{rank}(\mathfrak{g})$. An element $x \in \mathfrak{g}$ is regular if and only if $\mathfrak{g}^0(x)$ is a Cartan subalgebra [Bou08, Ch. VII § 3.3 Theorem 2]. In fact, every Cartan subalgebra is of the form $\mathfrak{g}^0(x)$ for some $x \in \mathfrak{g}$ regular [Bou08, Ch. VII § 3.3 Theorem 2].*

Note 2.3.6. *For a real Lie algebra $\mathfrak{g}$, for any $x \in \mathfrak{g}$, $C_{\mathfrak{g}}(x)^{\mathbb{C}} = C_{\mathfrak{g}^{\mathbb{C}}}(x)$. Additionally, $\mathfrak{z}(C_{\mathfrak{g}}(x))^{\mathbb{C}} = \mathfrak{z}(C_{\mathfrak{g}^{\mathbb{C}}}(x))^{\mathbb{C}}$.*

We outline some additional facts about regular elements of compact Lie algebras. Note that we consider a compact Lie algebra to mean the Lie algebra of a compact Lie group.

Lemma 2.3.7. *Let $\mathfrak{g}$ be a compact Lie algebra. We have that $x \in \mathfrak{g}$ is regular if and only if $C_{\mathfrak{g}}(x)$ is minimal. That is, if there is some $y \in \mathfrak{g}$ such that $C_{\mathfrak{g}}(y) \subseteq C_{\mathfrak{g}}(x)$, then $C_{\mathfrak{g}}(y) = C_{\mathfrak{g}}(x)$. Moreover, $x \in \mathfrak{g}$ is regular if and only if the dimension of $C_{\mathfrak{g}}(x)$ is $\text{rank}(\mathfrak{g})$.*

Proof. First we note that every centralizer contains a Cartan subalgebra. Indeed, consider $C_{\mathfrak{g}}(x)$. We have that x belongs to some Cartan subalgebra $\mathfrak{g}^0(y)$ for $y \in \mathfrak{g}$ regular. As the Cartan subalgebra is abelian, $\mathfrak{g}^0(y) \subseteq C_{\mathfrak{g}}(x)$.

Suppose that $x \in \mathfrak{g}$ is regular and suppose that there is some $y \in \mathfrak{g}$ such that $C_{\mathfrak{g}}(y) \subseteq C_{\mathfrak{g}}(x)$. As x is regular, $\mathfrak{g}^0(x)$ is a Cartan subalgebra. As $C_{\mathfrak{g}}(y) \subseteq C_{\mathfrak{g}}(x) \subseteq \mathfrak{g}^0(x)$ and $C_{\mathfrak{g}}(y)$ contains a Cartan subalgebra, we see that $C_{\mathfrak{g}}(y) = C_{\mathfrak{g}}(x) = \mathfrak{g}^0(x)$. Thus, $C_{\mathfrak{g}}(x)$ is minimal and its dimension is $\text{rank}(\mathfrak{g})$.

Conversely, suppose that $x \in \mathfrak{g}$ is such that $C_{\mathfrak{g}}(x)$ is minimal. There exists $y \in \mathfrak{g}$ regular such that $x \in \mathfrak{g}^0(y)$ and $\mathfrak{g}^0(y)$ is a Cartan subalgebra. Just as above, we have $\mathfrak{g}^0(y) \subseteq C_{\mathfrak{g}}(x)$. As $C_{\mathfrak{g}}(y) \subseteq \mathfrak{g}^0(y) \subseteq C_{\mathfrak{g}}(x)$, by minimality, $C_{\mathfrak{g}}(y) = \mathfrak{g}^0(y) = C_{\mathfrak{g}}(x)$. Thus, $C_{\mathfrak{g}}(x)$ is a Cartan subalgebra. Complexifying, we can change bases such that $\text{ad}(x)$ is diagonal. That is, there is some $U \in \text{Aut}(\mathfrak{g}^{\mathbb{C}})$ such that $U^{-1}\text{ad}(x)U$ is diagonal. Consider $z \in \mathfrak{g}^0(x)$. Then there is some $n \geq 0$ such that $\text{ad}(x)^n z = 0$. Hence, $(U^{-1}\text{ad}(x)U)^n U^{-1}z = 0$. As $U^{-1}\text{ad}(x)U$ is diagonal, we have that $U^{-1}\text{ad}(x)U U^{-1}z = 0$. That is, $z \in C_{\mathfrak{g}}(x)$. Thus, $\mathfrak{g}^0(x) = C_{\mathfrak{g}}(x)$ is a Cartan subalgebra, so x is regular.

Finally, suppose that the dimension of $C_{\mathfrak{g}}(x)$ is $\text{rank}(\mathfrak{g})$. As all centralizers contain Cartan subalgebras, we have that $C_{\mathfrak{g}}(x)$ is minimal, thus x is regular. $\square$

2.3.1.1 (Semi-)Simplifying adjoint orbits

In this section, we prove that studying adjoint orbits when G is compact and simply-connected is sufficient to understand the general case.

As G is a compact, connected Lie group, it has a compact, connected cover given by the product of a torus T with a simply-connected, compact Lie group J . Moreover, there is a finite central subgroup $Z \subseteq T \times J$ such that $G \simeq (T \times J)/Z$. Note that G and $T \times J$ have the same Lie algebra and act on it via their adjoint representations. It turns out that the adjoint orbit does not depend on the choice of Lie group with a given Lie algebra.

Lemma 2.3.8. *Let G and G' be Lie groups with the same Lie algebra $\mathfrak{g}$, up to isomorphism. For $x \in \mathfrak{g}$,*

$$\mathrm{Ad}(G)x = \mathrm{Ad}(G')x. \quad (2.3.5)$$

That is, the adjoint orbit is determined at the Lie algebra level.

Proof. Let $\tilde{G}$ be the universal cover of G and G' . We show that $\mathrm{Ad}(\tilde{G})x = \mathrm{Ad}(G)x$. As the same is true for G' , this fact is sufficient to prove our result.

Locally, G is isomorphic to $\tilde{G}$. Indeed, there is some discrete central subgroup $Z \subseteq \tilde{G}$ such that $G \simeq \tilde{G}/Z$. Thus, near the identity of $\tilde{G}$, we can identify $\tilde{G}$ with G . Denote the equivalence class of $g \in \tilde{G}$ by $[g] \in G$. For all $g \in \tilde{G}$ and t near zero, we have $g \cdot \exp(tx)g^{-1}$ is near the identity, so

$$\mathrm{Ad}_{[g]}x = \left. \frac{d}{dt} \right|_{t=0} [g \cdot \exp(tx)g^{-1}] = \left. \frac{d}{dt} \right|_{t=0} g \cdot \exp(tx)g^{-1} = \mathrm{Ad}_g x. \quad \square$$

As we are interested in looking at the adjoint orbit, it suffices to look at the adjoint orbit using $T \times J$. In fact, we can do better than this. As $\mathfrak{z}(\mathfrak{g})$ is abelian and $[\mathfrak{g}, \mathfrak{g}]$ semi-simple, we have that they are the Lie algebras of T and J , respectively, and $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$. Note that $\mathfrak{z}(\mathfrak{g})$ is abelian and $[\mathfrak{g}, \mathfrak{g}]$ is semi-simple. For $x \in \mathfrak{g}$, we may decompose it into its abelian and semi-simple parts as $x = (x_{\mathrm{ab}}, x_{\mathrm{ss}})$.

Proposition 2.3.9. *Let G be a compact, connected Lie group. Let T , J , and Z be as given above. Then for all $(x_{\mathrm{ab}}, x_{\mathrm{ss}}) \in \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, we have*

$$\mathrm{Ad}(G)(x_{\mathrm{ab}}, x_{\mathrm{ss}}) \simeq \mathrm{Ad}(J)x_{\mathrm{ss}}. \quad (2.3.6)$$

That is, the adjoint orbit only depends on the semi-simple part.

Proof. From Lemma 2.3.8, we know that $\mathrm{Ad}(G)(x_{\mathrm{ab}}, x_{\mathrm{ss}}) = \mathrm{Ad}(T \times J)(x_{\mathrm{ab}}, x_{\mathrm{ss}})$. However, we see that

$$\mathrm{Ad}(T \times J)(x_{\mathrm{ab}}, x_{\mathrm{ss}}) = \mathrm{Ad}(T)x_{\mathrm{ab}} \times \mathrm{Ad}(J)x_{\mathrm{ss}}.$$

Finally, as T is abelian, $\mathrm{Ad}(T)x_{\mathrm{ab}} = \{x_{\mathrm{ab}}\}$, giving us our result. $\square$

Hence, to understand adjoint orbits, we need only understand adjoint orbits using simply-connected, compact Lie groups. Note that such groups are semi-simple, as are their Lie algebras. Armed with this knowledge, we are now ready to investigate adjoint orbits in detail.

2.3.1.2 Examining homotopy groups of adjoint orbits

In this section, we find the homotopy groups of adjoint orbits, which we use to motivate our classification of symmetry breaking.

An adjoint orbit is also known as a generalized flag manifold. This perspective allows us to see the adjoint orbit as a complex manifold [MS03, § II.A]. Moreover, using the Bruhat decomposition, we gain further insight into the composition of the adjoint orbit. This point of view is obtained using standard results; however, we provide details for those unfamiliar with these techniques [DK00, Lemma 4.12.6; Kum02, Theorem 5.1.5].

Proposition 2.3.10. *For G compact and connected, the adjoint orbit G/H is a CW complex with cells only in even dimensions.*

Proof. Let T , J , and Z be as above. By Proposition 2.3.9, we have that for $(x_{\text{ab}}, x_{\text{ss}}) \in \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$,

$$\text{Ad}(G)(x_{\text{ab}}, x_{\text{ss}}) \simeq \text{Ad}(J)x_{\text{ss}}.$$

Thus, if we show that the result holds for a compact, simply-connected Lie group, then it holds for all compact, connected Lie groups. Hence, for the remainder of this proof, we assume that G is a compact, simply-connected Lie group.

Note that Φ_∞ belongs to some maximal abelian subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$, whose complexification $\mathfrak{t}^\mathbb{C} \subseteq \mathfrak{g}^\mathbb{C}$ is a Cartan subalgebra. Let R denote the set of roots with respect to $\mathfrak{t}^\mathbb{C}$. Let $R_s := \{\alpha_1, \dots, \alpha_{\text{rank}(\mathfrak{g})}\} \subseteq R$ denote a set of simple roots and denote the positive roots by R^+ and the negative roots by R^- . For each $\alpha \in R$, we let $\mathfrak{g}_\alpha^\mathbb{C} := \{x \in \mathfrak{g} \mid [y, x] = \alpha(y)x, \forall y \in \mathfrak{t}\}$. Then $\mathfrak{g}^\mathbb{C}$ decomposes as

$$\mathfrak{g}^\mathbb{C} = \mathfrak{t}^\mathbb{C} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha^\mathbb{C}. \quad (2.3.7)$$

We can similarly decompose $\mathfrak{g}$. For $\alpha \in R^+$, let

$$\mathfrak{g}_\alpha := (\mathfrak{g}_\alpha^\mathbb{C} \oplus \mathfrak{g}_{-\alpha}^\mathbb{C}) \cap \mathfrak{g} = \{x \in \mathfrak{g} \mid [y, [y, x]] = \alpha(y)^2 x \quad \forall y \in \mathfrak{t}\}. \quad (2.3.8)$$

Then we have

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha. \quad (2.3.9)$$

The fundamental Weyl chamber $C(R_s) \subseteq \mathfrak{t}$ is the subspace

$$C(R_s) := \{x \in \mathfrak{t} \mid \alpha_i(x) < 0 \quad \forall i \in \{1, \dots, \text{rank}(\mathfrak{g})\}\}. \quad (2.3.10)$$

The walls of the fundamental Weyl chamber are the hyperplanes $\ker(\alpha_i) \subseteq \mathfrak{t}$. Note that the simple roots can be chosen in such a way that $\Phi_\infty \in \overline{C(R_s)}$.

Suppose that Φ_∞ belongs to the walls $\ker(\alpha_{i_1}), \dots, \ker(\alpha_{i_k})$. Let $R(\Phi_\infty) \subseteq R$ denote

$$R(\Phi_\infty) := \{\alpha \in R \mid \alpha(\Phi_\infty) = 0\} = R \cap \text{span}_{\mathbb{Z}}(\alpha_{i_1}, \dots, \alpha_{i_k}). \quad (2.3.11)$$

Also, we denote $R^\pm(\Phi_\infty) := R(\Phi_\infty) \cap R^\pm$. Let $G_\infty \subseteq G$ be the unique connected Lie subgroup with Lie algebra $\mathfrak{g}_\infty \subseteq \mathfrak{g}$ given by

$$\mathfrak{g}_\infty := \mathfrak{t} \oplus \bigoplus_{\alpha \in R^+(\Phi_\infty)} \mathfrak{g}_\alpha. \quad (2.3.12)$$

Note that if $k = 0$, we have $\Phi_\infty \in C(R_s)$ and $\mathfrak{g}_\infty = \mathfrak{t}$, so G_∞ is a maximal torus. We show that $H \simeq G_\infty$.

As H and G_∞ are both connected Lie subgroups of G , we need only show that their Lie algebras coincide. We see that

$$\begin{aligned} \mathfrak{h} &= C_{\mathfrak{g}}(\Phi_\infty)^{\mathbb{C}} \cap \mathfrak{g} = \left(\mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in R(\Phi_\infty)} \mathfrak{g}_\alpha^{\mathbb{C}} \right) \cap \mathfrak{g} \\ &= \mathfrak{t} \oplus \bigoplus_{\alpha \in R^+(\Phi_\infty)} (\mathfrak{g}_\alpha^{\mathbb{C}} \oplus \mathfrak{g}_{-\alpha}^{\mathbb{C}}) \cap \mathfrak{g} = \mathfrak{g}_\infty. \end{aligned}$$

Thus, the Lie groups are equal.

Let $\mathfrak{b} \subseteq \mathfrak{g}^{\mathbb{C}}$ be the Borel subalgebra with respect to $\mathfrak{t}^{\mathbb{C}}$ and R_s . That is,

$$\mathfrak{b} := \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha^{\mathbb{C}}. \quad (2.3.13)$$

We also define the parabolic subalgebra $\mathfrak{p}_\infty \subseteq \mathfrak{g}^{\mathbb{C}}$ by

$$\mathfrak{p}_\infty := \mathfrak{t}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in R^+ \cup R^-(\Phi_\infty)} \mathfrak{g}_\alpha^{\mathbb{C}}. \quad (2.3.14)$$

Let $G^{\mathbb{C}}$ be the complexification of G . Let the Borel and parabolic subgroups $B, P_\infty \subseteq G^{\mathbb{C}}$ be the unique connected Lie subgroups with Lie algebras $\mathfrak{b}$ and $\mathfrak{p}_\infty$, respectively. Note that if $k = 0$, we have $\mathfrak{p}_\infty = \mathfrak{b}$, so $P_\infty = B$.

It turns out that as real manifolds [DK00, Lemma 4.12.6],

$$G/H \simeq G/G_\infty \simeq G^{\mathbb{C}}/P_\infty. \quad (2.3.15)$$

This relationship is important, as the Tits–Borel–Weil Theorem tells us that $G^{\mathbb{C}}/P_{\infty}$ is a smooth complex algebraic subvariety of $\mathbb{CP}(V)$ for some complex vector space V [DK00, Theorem 4.12.7].

Note that as G is a compact, connected, real Lie group, $G^{\mathbb{C}}$ is a complex, connected, reductive algebraic group. Thus, it possesses a Bruhat decomposition, which we now discuss.

Let $N_G(T)$ be the normalizer of T in G . The Weyl group $W := N_G(T)/T$ is a finite group that can also be thought of as being a group of linear transformations generated by reflections s_i across the hyperplane $\ker(\alpha_i)$, for $i \in \{1, \dots, \text{rank}(\mathfrak{g})\}$.

For each $w \in W$, let $n_w \in N_G(T)$ be a representative of w . The subspace $Bn_wB \subseteq G^{\mathbb{C}}$ is independent of the choice of n_w and is denoted BwB . For P_{∞} , let $W_{\infty} := \langle s_{i_1}, \dots, s_{i_k} \rangle \subseteq W$ be the subgroup generated by reflections across the hyperplanes that we are interested in. Similar to above, if $n_w \in N_G(T)$ is a representative for $w \in W_{\infty}$, then $n_w \in P_{\infty}$. Thus, the subset $BwP_{\infty} \subseteq G^{\mathbb{C}}$ depends only on the class of $[w] \in W/W_{\infty}$. Note that when $k = 0$, $W_{\infty} = 1$.

The Bruhat decomposition of $G^{\mathbb{C}}$ involves writing $G^{\mathbb{C}}$ as the following disjoint union [Kum02, Theorem 5.1.3].

$$G^{\mathbb{C}} = \bigsqcup_{[w] \in W/W_{\infty}} BwP_{\infty}.$$

Using this decomposition, we write $G^{\mathbb{C}}/P_{\infty}$ as [Kum02, Theorem 5.1.5]

$$G^{\mathbb{C}}/P_{\infty} = \bigsqcup_{[w] \in W/W_{\infty}} BwP_{\infty}/P_{\infty}. \quad (2.3.16)$$

The subsets BwP_{∞}/P_{∞} are known as Bruhat cells. For $w \in W$, let w' be the element of wW_{∞} such that its length

$$\text{length}(w') := \min(p \mid w' \text{ is the product of } p \text{ elements of } W)$$

is minimal. We have that the Bruhat cells are locally closed subvarieties of $G^{\mathbb{C}}/P_{\infty}$ isomorphic to $\mathbb{C}^{\text{length}(w')}$. Thus, $G^{\mathbb{C}}/P_{\infty}$ is a CW complex with cells of even real dimension. $\square$

This description of the adjoint orbit allows us to detail some of its properties.

Corollary 2.3.11. *Let G be a compact, connected Lie group. The adjoint orbit $\text{Ad}(G)\Phi_{\infty} \simeq G/H$ is compact, connected, and simply-connected. Moreover, the adjoint orbit only has non-vanishing homology groups in even degrees and these groups are free abelian groups. Finally, $\pi_2(G/H)$ is a free abelian group.*

Proof. As the adjoint orbit is the continuous image of G , a compact and connected Lie group, on $\mathfrak{g}$, it too is compact and connected. Additionally, by Proposition 2.3.10, there are no 1-cells in the adjoint orbit. Hence, any loop is contractible, so the adjoint orbit is simply-connected.

The cellular chain complex of the adjoint orbit looks like

$$\cdots \rightarrow \mathbb{Z}^{n_2} \rightarrow 0 \rightarrow \mathbb{Z}^{n_0} \rightarrow 0. \quad (2.3.17)$$

Note that $n_0 = 1$, as the adjoint orbit is connected. Thus, we see that in odd dimensions, the homology vanishes, and in even dimensions, the homology is a free abelian group.

Finally, as G/H is connected and simply-connected, it is 1-connected. By the Hurewicz Theorem, we have that $\pi_2(G/H) \simeq H_2(G/H)$, which is a free abelian group. $\square$

Unsurprisingly, as the choice of Lie group acting on the adjoint orbit does not matter, $\pi_2(G/H)$ is determined at the Lie algebra level.

Corollary 2.3.12. *For G a compact, simply-connected Lie group, $\pi_2(G/H) \simeq \mathbb{Z}^{\dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{h}))}$.*

Proof. Given the fibration $H \rightarrow G \rightarrow G/H$, we obtain the homotopy exact sequence

$$\cdots \rightarrow \pi_{n+1}(G/H) \rightarrow \pi_n(H) \rightarrow \pi_n(G) \rightarrow \pi_n(G/H) \rightarrow \cdots.$$

Note that as G is a simply-connected Lie group, $\pi_1(G) = \pi_2(G) = 1$. Thus, $\pi_2(G/H) \simeq \pi_1(H)$. As H is a connected, compact Lie group, $\pi_1(H)$ is a finitely generated abelian group, as it is a quotient of a finitely generated abelian group [BTD85, Theorem V.7.1]. Thus, from the homotopy exact sequence, we have the following exact sequence. From Corollary 2.3.11, we know that $\pi_2(G/H)$ is a free abelian group. Hence, $\pi_1(H)$ is as well, meaning $\pi_2(G/H) \simeq \mathbb{Z}^{\text{rank}(\pi_1(H))}$.

As H is a connected, compact Lie group, we have that there is some torus T , simply-connected, compact Lie group J , and finite central subgroup $Z \subseteq T \times J$ such that $H \simeq (T \times J)/Z$. Then, by the homotopy exact sequence for the fibration $Z \rightarrow T \times J \rightarrow H$, we have the short exact sequence

$$1 \rightarrow \pi_1(T \times J) \rightarrow \pi_1(H) \rightarrow \pi_0(Z) \rightarrow 1.$$

As J is simply-connected and Z is a finite group, we have

$$1 \rightarrow \mathbb{Z}^{\dim(T)} \rightarrow \pi_1(H) \rightarrow Z \rightarrow 1.$$

As $\pi_1(H)$ is a finitely generated abelian group, we see that the rank of $\pi_1(H)$ is precisely $\dim(T)$. Finally, as the Lie algebra of T is $\mathfrak{z}(\mathfrak{h})$, as $\pi_2(G/H) \simeq \mathbb{Z}^{\text{rank}(\pi_1(H))}$, we have that $\pi_2(G/H) \simeq \mathbb{Z}^{\dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{h}))}$. $\square$

We are now ready to tackle the general case.

Theorem 2.3.13. *For G a compact, connected Lie group,*

$$\pi_2(G/H) \simeq \mathbb{Z}^{\dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{h})) - \dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{g}))}. \quad (2.3.18)$$

Note 2.3.14. *If G is semi-simple, we have that $\dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{g})) = 0$, giving us the same result as Corollary 2.3.12.*

Proof. As G is a compact, connected Lie group, there is a torus T , simply-connected, compact Lie group J , and finite central subgroup $Z \subseteq T \times J$ such that $G \simeq (T \times J)/Z$. Using this relationship, we can decompose $\Phi_{\infty} = (x_{\text{ab}}, x_{\text{ss}}) \in \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$.

Let $K \subseteq J$ be the isotropy subgroup of x_{ss} under the adjoint action of J . Let $\mathfrak{k}$ be the Lie algebra of K and let $\mathfrak{j}$ be the Lie algebra of J . By Corollary 2.3.12, we have

$$\pi_2(\text{Ad}(J)x_{\text{ss}}) \simeq \mathbb{Z}^{\dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{k}))}.$$

Recall that H is the isotropy subgroup of Φ_{∞} under the adjoint action of G , with Lie algebra $\mathfrak{h} = C_{\mathfrak{g}}(\Phi_{\infty})$. As $\mathfrak{k} = C_{\mathfrak{j}}(x_{\text{ss}})$, we see that

$$\mathfrak{h} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{k}.$$

Hence, $\mathfrak{z}(\mathfrak{h}) = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{z}(\mathfrak{k})$. Thus,

$$\dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{h})) = \dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{g})) + \dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{k})).$$

Finally, as $G/H \simeq \text{Ad}(G)\Phi_{\infty} \simeq \text{Ad}(J)x_{\text{ss}} \simeq J/K$, we get our result. $\square$

Given Theorem 2.3.13, we now focus on understanding $\dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{h}))$. Just as above, we start with the semi-simple case before examining the general one.

Lemma 2.3.15. *Let $\mathfrak{g}$ be a compact, semi-simple Lie algebra. Given $x \in \mathfrak{g}$, we have*

$$0 \leq \dim_{\mathbb{R}}(\mathfrak{z}(C_{\mathfrak{g}}(x))) \leq \text{rank}(\mathfrak{g}). \quad (2.3.19)$$

Moreover, $\mathfrak{z}(C_{\mathfrak{g}}(x)) = 0$ if and only if $x = 0$.

Proof. We know that $\mathfrak{z}(C_{\mathfrak{g}}(x))$ is an abelian Lie subalgebra of $\mathfrak{g}$. As $\mathfrak{g}$ is semi-simple, x is semi-simple. Thus, there exists some regular element $y \in \mathfrak{g}^{\mathbb{C}}$ such that the Cartan

subalgebra $(\mathfrak{g}^{\mathbb{C}})^0(y)$ of $\mathfrak{g}^{\mathbb{C}}$ contains x . As $(\mathfrak{g}^{\mathbb{C}})^0(y)$ is abelian, all elements commute with x , so $(\mathfrak{g}^{\mathbb{C}})^0(y) \subseteq C_{\mathfrak{g}}(x)^{\mathbb{C}}$. Thus, $y \in (\mathfrak{g}^{\mathbb{C}})^0(y) \subseteq C_{\mathfrak{g}}(x)^{\mathbb{C}}$, so $\mathfrak{z}(C_{\mathfrak{g}}(x))^{\mathbb{C}} \subseteq (\mathfrak{g}^{\mathbb{C}})^0(y)$.

As a Cartan subalgebra, $(\mathfrak{g}^{\mathbb{C}})^0(y)$ has complex dimension $\text{rank}(\mathfrak{g})$, so

$$\dim_{\mathbb{R}}(\mathfrak{z}(C_{\mathfrak{g}}(x))) \leq \text{rank}(\mathfrak{g}).$$

Consider the second statement. Suppose that $x = 0$. Then $C_{\mathfrak{g}}(x) = \mathfrak{g}$. As $\mathfrak{g}$ is semi-simple, $\mathfrak{g}$ has no centre. Thus, $\mathfrak{z}(C_{\mathfrak{g}}(x)) = \mathfrak{z}(\mathfrak{g}) = 0$. Alternatively, suppose $x \neq 0$. We see that $\text{span}_{\mathbb{R}}(x) \subseteq \mathfrak{z}(C_{\mathfrak{g}}(x))$, so the dimension of the centre is at least one. $\square$

Proposition 2.3.16. *Let $\mathfrak{g}$ be a compact Lie algebra. Given $x \in \mathfrak{g}$, we have*

$$\dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{g})) \leq \dim_{\mathbb{R}}(\mathfrak{z}(C_{\mathfrak{g}}(x))) \leq \text{rank}(\mathfrak{g}). \quad (2.3.20)$$

Moreover, the lower bound is achieved if and only if $x \in \mathfrak{z}(\mathfrak{g})$.

Proof. As $\mathfrak{g}$ is a compact Lie algebra, we can decompose it as

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}].$$

Note that $[\mathfrak{g}, \mathfrak{g}]$ is a compact, semi-simple Lie algebra. Let $(x_{\text{ab}}, x_{\text{ss}}) \in \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$. We see that

$$C_{\mathfrak{g}}(x_{\text{ab}}, x_{\text{ss}}) = \mathfrak{z}(\mathfrak{g}) \oplus C_{[\mathfrak{g}, \mathfrak{g}]}(x_{\text{ss}}).$$

Hence,

$$\mathfrak{z}(C_{\mathfrak{g}}(x_{\text{ab}}, x_{\text{ss}})) = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{z}(C_{[\mathfrak{g}, \mathfrak{g}]}(x_{\text{ss}})).$$

By Lemma 2.3.15, we have

$$0 \leq \dim_{\mathbb{R}}(\mathfrak{z}(C_{[\mathfrak{g}, \mathfrak{g}]}(x_{\text{ss}}))) \leq \text{rank}([\mathfrak{g}, \mathfrak{g}]). \quad (2.3.21)$$

Note that $\text{rank}(\mathfrak{g}) = \dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{g})) + \text{rank}([\mathfrak{g}, \mathfrak{g}])$. Substituting into (2.3.21), we have

$$0 \leq \dim_{\mathbb{R}}(\mathfrak{z}(C_{\mathfrak{g}}(x_{\text{ab}}, x_{\text{ss}}))) - \dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{g})) \leq \text{rank}(\mathfrak{g}) - \dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{g})). \quad (2.3.22)$$

Rearranging, we obtain the desired inequality.

Note that the lower bound in (2.3.22) is an equality if and only if the lower bound of (2.3.21) is an equality. However, this only happens if and only if $x_{\text{ss}} = 0$. That is, if and only if $x \in \mathfrak{z}(\mathfrak{g})$. $\square$

In light of Theorem 2.3.13 and Proposition 2.3.16, as $\Phi_\infty \notin \mathfrak{z}(\mathfrak{g})$, we classify the extremal types of symmetry breaking.

Definition 2.3.17. *We say that Φ_∞ has **generalized minimal** symmetry breaking if*

$$\dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{h})) = \dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{g})) + 1. \quad (2.3.23)$$

*Additionally, we say that Φ_∞ has **maximal** symmetry breaking if*

$$\dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{h})) = \text{rank}(\mathfrak{g}). \quad (2.3.24)$$

In particular, if $\mathfrak{g}$ is semi-simple (equivalently G is), then $\mathfrak{z}(\mathfrak{g}) = 0$, so generalized minimal symmetry breaking occurs if $\mathfrak{z}(\mathfrak{h})$ is one-dimensional.

Note 2.3.18. *In Section 2.3.2, we see that this notion of maximal symmetry breaking is equivalent to several definitions in the literature. Additionally, in Section 2.3.3, we see that maximal symmetry breaking is exactly what is classically known as maximal symmetry breaking. Moreover, in Section 2.3.3, we see that if Φ_∞ has minimal symmetry breaking in the classical sense, then it has generalized minimal symmetry breaking. However, this is not an equivalence. Thus, we do not include “generalized” when discussing maximal symmetry breaking, but we do when discussing minimal symmetry breaking.*

There can be other kinds of symmetry breaking. These are just the extremal cases. Other kinds of symmetry breaking can only occur once $\text{rank}(\mathfrak{g}) - \dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{g}))$ is at least three. Moreover, when $\text{rank}(\mathfrak{g}) - \dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{g})) = 1$, maximal and minimal symmetry breaking are the same thing, just as in the $G = \text{SU}(2) \simeq \text{Sp}(1)$ case. In Section 2.3.3.1, we examine several examples of different symmetry breaking.

Before we discuss how this classification of symmetry breaking compares with the literature, the following result provides another method for computing the rank of $\pi_2(G/H)$.

Proposition 2.3.19. *Let $x \in \mathfrak{g}$ and let $\mathfrak{t}^{\mathbb{C}} \subseteq \mathfrak{g}^{\mathbb{C}}$ be a Cartan subalgebra containing x . Let R_s be the set of simple roots given $\mathfrak{t}^{\mathbb{C}} \subseteq \mathfrak{g}^{\mathbb{C}}$ and let $R_s(x) := \{\alpha \in R_s \mid \alpha(x) = 0\}$. Then*

$$\dim_{\mathbb{R}}(\mathfrak{z}(C_{\mathfrak{g}}(x))) + |R_s(x)| = \text{rank}(\mathfrak{g}). \quad (2.3.25)$$

Note 2.3.20. *In particular, Φ_∞ has maximal symmetry breaking if and only if $|R_s(\Phi_\infty)| = 0$ and it has generalized minimal symmetry breaking if and only if $|R_s(\Phi_\infty)| = \text{rank}(\mathfrak{g}) - \dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{g})) - 1$.*

Proof. From the proof of Lemma 2.3.15, we know that $\mathfrak{z}(C_{\mathfrak{g}}(x))^{\mathbb{C}} \subseteq \mathfrak{t}^{\mathbb{C}} \subseteq C_{\mathfrak{g}}(x)^{\mathbb{C}}$. We show that every simple root $\alpha \in R_s(x)$ removes one dimension from $\mathfrak{z}(C_{\mathfrak{g}}(x))^{\mathbb{C}}$.

Let $\alpha \in R_s(x)$ and consider $x_{\alpha} \in \mathfrak{g}_{\alpha}^{\mathbb{C}} \setminus \{0\}$. Consider the linear map $f_{\alpha}: \mathfrak{t}^{\mathbb{C}} \rightarrow \mathfrak{g}^{\mathbb{C}}$ given by $f_{\alpha}(y) := [y, x_{\alpha}] = \alpha(y)x_{\alpha}$. Note that f_{α} has a one-dimensional complex image, as $\alpha \neq 0$. Thus,

$$\dim_{\mathbb{C}}(\ker(f_{\alpha})) = \text{rank}(\mathfrak{g}) - 1.$$

Suppose that $\alpha, \beta \in R_s(x)$ such that $\ker(f_{\alpha}) = \ker(f_{\beta})$. As the kernels coincide, α and β are determined by their evaluation on a single element of $\mathfrak{t}^{\mathbb{C}}$. Hence, α and β are linearly dependent in $(\mathfrak{t}^{\mathbb{C}})^*$. As all simple roots are linearly independent, we must have $\alpha = \beta$. Hence, if $\alpha \neq \beta$, then

$$\dim_{\mathbb{C}}(\ker(f_{\alpha}) \cap \ker(f_{\beta})) = \text{rank}(\mathfrak{g}) - 2.$$

Note that for all $\alpha \in R_s(x)$, $x_{\alpha} \in C_{\mathfrak{g}}(x)^{\mathbb{C}}$. We show that $\mathfrak{z}(C_{\mathfrak{g}}(x))^{\mathbb{C}} = \bigcap_{\alpha \in R_s(x)} \ker(f_{\alpha})$.

Suppose that $y \in \mathfrak{z}(C_{\mathfrak{g}}(x))^{\mathbb{C}}$. As all $\alpha \in R_s(x)$ give $x_{\alpha} \in C_{\mathfrak{g}}(x)^{\mathbb{C}}$, we have that $[y, x_{\alpha}] = 0$, so $y \in \bigcap_{\alpha \in R_s(x)} \ker(f_{\alpha})$. Conversely, suppose that $y \in \bigcap_{\alpha \in R_s(x)} \ker(f_{\alpha}) \subseteq \mathfrak{t}^{\mathbb{C}}$. Consider $z \in C_{\mathfrak{g}}(x)^{\mathbb{C}}$. We have that there exists $y' \in \mathfrak{t}$ and $c_{\alpha} \in \mathbb{C}$ for every $\alpha \in R$ such that $z = y' + \sum_{\alpha \in R} c_{\alpha} x_{\alpha}$. As $[z, x] = 0$, we have that $c_{\alpha} = 0$ for all $\alpha \in R \setminus R(x)$. Thus, $z \in \mathfrak{t} \oplus \bigoplus_{\alpha \in R(x)} \mathfrak{g}_{\alpha}^{\mathbb{C}}$. As y is in the kernel of all the f_{α} for $\alpha \in R_s(x)$, y commutes with all x_{α} with $\alpha \in R(x)$. Thus, we see that $[y, z] = [y, y'] = 0$, as $y, y' \in \mathfrak{t}^{\mathbb{C}}$. Therefore, $y \in \mathfrak{z}(C_{\mathfrak{g}}(x))^{\mathbb{C}}$.

Finally, we see that

$$\dim_{\mathbb{C}}(\mathfrak{z}(C_{\mathfrak{g}}(x))^{\mathbb{C}}) = \dim_{\mathbb{C}} \left(\bigcap_{\alpha \in R_s(x)} \ker(f_{\alpha}) \right) = \text{rank}(\mathfrak{g}) - |R_s(x)|.$$

Counting the real dimensions of $\mathfrak{z}(C_{\mathfrak{g}}(x))$, we have our result. $\square$

The definitions of maximal and generalized minimal symmetry breaking in terms of the dimension of $\mathfrak{z}(\mathfrak{h})$ were chosen in lieu of the definitions based on $|R_s(\Phi_{\infty})|$ as they are motivated immediately by understanding the homotopy groups of the adjoint orbit.

The $\mathfrak{z}(\mathfrak{h})$ definitions were also chosen as they are straightforward to compute. Indeed, given $\Phi_{\infty} \in \mathfrak{g}$, find $\mathfrak{h} = C_{\mathfrak{g}}(\Phi_{\infty})$, all elements that commute with Φ_{∞} . Then find $\mathfrak{z}(\mathfrak{h})$, all elements of $\mathfrak{h}$ that commute with all the rest. The definitions based on $|R_s(\Phi_{\infty})|$ are more difficult to compute, but are useful for comparing to other definitions in the literature.

2.3.2 An equivalent definition

In the literature, there are several definitions for maximal symmetry breaking. In this section, we show that these are all equivalent to the definition introduced in Section 2.3.1, which was chosen to motivate the definition of generalized minimal symmetry breaking and for ease of computation. Note that when necessary, we extend the definitions to the general case of G compact and connected.

It turns out that our definition of maximal symmetry breaking is very familiar. Murray and Singer define maximal symmetry breaking as follows [MS03, § 1].

Lemma 2.3.21. *Let $\mathfrak{g}$ be a compact Lie algebra. We have that $\Phi_\infty \in \mathfrak{g}$ has maximal symmetry breaking if and only if it is regular (Murray and Singer's definition).*

Proof. Suppose that Φ_∞ is regular. Then $\mathfrak{h}^\mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}^\mathbb{C}$. Thus, $\mathfrak{h}$ is abelian. Hence,

$$\dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{h})) = \dim_{\mathbb{R}}(\mathfrak{h}) = \text{rank}(\mathfrak{g}).$$

Therefore, Φ_∞ has maximal symmetry breaking.

Suppose that Φ_∞ has maximal symmetry breaking. There exists some regular element $x \in \mathfrak{g}^\mathbb{C}$ such that $(\mathfrak{g}^\mathbb{C})^0(x)$ is a Cartan subalgebra of $\mathfrak{g}^\mathbb{C}$ containing Φ_∞ . As in the proof of Lemma 2.3.15, we have that $\mathfrak{z}(\mathfrak{h})^\mathbb{C} \subseteq (\mathfrak{g}^\mathbb{C})^0(x)$. As the two Lie subalgebras have the same complex dimension, we have $\mathfrak{z}(\mathfrak{h})^\mathbb{C} = (\mathfrak{g}^\mathbb{C})^0(x)$. Thus, $x \in \mathfrak{z}(\mathfrak{h})^\mathbb{C}$. Therefore, for all $y \in \mathfrak{h}^\mathbb{C}$, we have $[x, y] = 0$. Hence, $\mathfrak{h}^\mathbb{C} \subseteq (\mathfrak{g}^\mathbb{C})^0(x) = \mathfrak{z}(\mathfrak{h})^\mathbb{C}$. Thus, $\mathfrak{h} = \mathfrak{z}(\mathfrak{h})$, so

$$\dim_{\mathbb{R}}(\mathfrak{h}) = \dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{h})) = \text{rank}(\mathfrak{g}).$$

That is, Φ_∞ is regular. □

Using a similar method, we can prove the following corollary.

Corollary 2.3.22. *For a compact Lie algebra $\mathfrak{g}$, $x \in \mathfrak{g}$ is regular if and only if $C_{\mathfrak{g}}(x)$ is abelian.*

Proof. We saw in the proof of Lemma 2.3.21 that if $x \in \mathfrak{g}$ is regular, then $C_{\mathfrak{g}}(x)$ is abelian. Conversely, suppose that $C_{\mathfrak{g}}(x)$ is abelian. Then $\mathfrak{z}(C_{\mathfrak{g}}(x)) = C_{\mathfrak{g}}(x)$. We know that

$$\dim_{\mathbb{R}}(C_{\mathfrak{g}}(x)) \geq \text{rank}(\mathfrak{g})$$

and by Lemma 2.3.15,

$$\dim_{\mathbb{R}}(\mathfrak{z}(C_{\mathfrak{g}}(x))) \leq \text{rank}(\mathfrak{g}).$$

Note that in that proof, only the compactness of $\mathfrak{g}$ was used. Thus, $\dim_{\mathbb{R}}(C_{\mathfrak{g}}(x)) = \text{rank}(\mathfrak{g})$, so x is regular. □

Recall that an element of $\mathfrak{g}$ is regular when its centralizer is minimal. We have a similar result to Lemma 2.3.21 for generalized minimal symmetry breaking.

Lemma 2.3.23. *We have that $\Phi_\infty \in \mathfrak{g}$ has generalized minimal symmetry breaking if and only if $\mathfrak{h}$ is maximal. That is, $\mathfrak{h} \subsetneq \mathfrak{g}$ and if there is some $x \in \mathfrak{g}$ such that $\mathfrak{h} \subseteq C_{\mathfrak{g}}(x) \subseteq \mathfrak{g}$, then $C_{\mathfrak{g}}(x) = \mathfrak{h}$ or $C_{\mathfrak{g}}(x) = \mathfrak{g}$.*

Proof. Suppose that $\mathfrak{h}$ is maximal, in the above sense. Suppose, for the sake of contradiction, that Φ_∞ does not have generalized minimal symmetry breaking. That is, $\mathfrak{z}(\mathfrak{h})$ has dimension at least $\dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{g})) + 2$. Let $x \in \mathfrak{z}(\mathfrak{h}) \setminus (\text{span}_{\mathbb{R}}(\Phi_\infty) \oplus \mathfrak{z}(\mathfrak{g}))$. As in the proof of Lemma 2.3.15, there is some $z \in \mathfrak{g}^{\mathbb{C}}$ such that $(\mathfrak{g}^{\mathbb{C}})^0(z)$ is a Cartan subalgebra containing $\mathfrak{z}(\mathfrak{h})^{\mathbb{C}}$.

Note that $\mathfrak{g}^{\mathbb{C}}$ is a complex, compact Lie algebra with Cartan subalgebra $(\mathfrak{g}^{\mathbb{C}})^0(z)$. Let R be the set of roots with respect to $(\mathfrak{g}^{\mathbb{C}})^0(z) \subseteq \mathfrak{g}^{\mathbb{C}}$. Recall the decomposition

$$\mathfrak{g}^{\mathbb{C}} = (\mathfrak{g}^{\mathbb{C}})^0(z) \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_{\alpha}^{\mathbb{C}}.$$

Note that as $\mathfrak{g}$ is a compact Lie algebra, for all $y \in \mathfrak{g}$, $\text{ad}(y)$ is a skew-Hermitian matrix in any basis of $\mathfrak{g}^{\mathbb{C}}$. Thus, the eigenvalues of $\text{ad}(y)$ are all purely imaginary. Thus, for all $\alpha \in R$, $\alpha(\Phi_\infty), \alpha(x) \in i\mathbb{R}$. Indeed, recall that α was defined such that for $y \in \mathfrak{g}_{\alpha}^{\mathbb{C}}$, $[v, y] = \alpha(v)y$ for all $v \in (\mathfrak{g}^{\mathbb{C}})^0(z)$.

If $\alpha(\Phi_\infty) = 0$ for all $\alpha \in R$, then $C_{\mathfrak{g}}(\Phi_\infty) = \mathfrak{g}$, meaning $\Phi_\infty \in \mathfrak{z}(\mathfrak{g})$. As this is not the case, there exists some $\alpha \in R$ such that $\alpha(\Phi_\infty) \neq 0$. As $\frac{\alpha(x)}{\alpha(\Phi_\infty)} \in \mathbb{R}$, we find $\tilde{x} := x - \frac{\alpha(x)}{\alpha(\Phi_\infty)}\Phi_\infty \in \mathfrak{g}$.

Note that for all $y \in \mathfrak{h}$, we have $[\tilde{x}, y] = 0$. Hence, $\mathfrak{h} \subseteq C_{\mathfrak{g}}(\tilde{x})$. Additionally, consider $y \in \mathfrak{g}_{\alpha}^{\mathbb{C}} \setminus \{0\}$. We see that

$$[\tilde{x}, y] = \alpha(x)y - \frac{\alpha(x)}{\alpha(\Phi_\infty)}\alpha(\Phi_\infty)y = 0.$$

As $y \in \mathfrak{g}^{\mathbb{C}}$, write $y = y_0 + iy_1$, where $y_0, y_1 \in \mathfrak{g}$. As $\tilde{x} \in \mathfrak{g}$, we have $[\tilde{x}, y_i] = 0$ for $i = 0, 1$. As $y \neq 0$, we have that for some $i \in \{0, 1\}$, $y_i \neq 0$ and $y_i \in C_{\mathfrak{g}}(\tilde{x})$.

Note that $[\Phi_\infty, y] = \alpha(\Phi_\infty)y$. As $\alpha(\Phi_\infty) \neq 0$, we have that $y_0, y_1 \notin \mathfrak{h}$. Hence,

$$\mathfrak{h} \subsetneq C_{\mathfrak{g}}(\tilde{x}) \subseteq \mathfrak{g}.$$

As $\mathfrak{h}$ is maximal, $C_{\mathfrak{g}}(\tilde{x}) = \mathfrak{g}$. Hence, $\tilde{x} \in \mathfrak{z}(\mathfrak{g})$, so $x \in \text{span}_{\mathbb{R}}(\Phi_{\infty}) \oplus \mathfrak{z}(\mathfrak{g})$. Contradiction! Therefore, Φ_{∞} has generalized minimal symmetry breaking.

Conversely, suppose that Φ_{∞} has generalized minimal symmetry breaking. As $\Phi_{\infty} \notin \mathfrak{z}(\mathfrak{g})$, we have that $\mathfrak{h} \subsetneq \mathfrak{g}$. Consider some $x \in \mathfrak{g}$ such that $\mathfrak{h} \subseteq C_{\mathfrak{g}}(x) \subseteq \mathfrak{g}$. As $x \in \mathfrak{z}(C_{\mathfrak{g}}(x))$, we have $x \in \mathfrak{z}(\mathfrak{h})$. As Φ_{∞} has generalized minimal symmetry breaking, we have that $\mathfrak{z}(\mathfrak{h})$ is $(\dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{g})) + 1)$ -dimensional. Specifically, $\mathfrak{z}(\mathfrak{h}) = \text{span}_{\mathbb{R}}(\Phi_{\infty}) \oplus \mathfrak{z}(\mathfrak{g})$. Thus, there exists $a \in \mathbb{R}$ and $y \in \mathfrak{z}(\mathfrak{g})$ such that $x = a\Phi_{\infty} + y$. If $a = 0$, then $C_{\mathfrak{g}}(x) = \mathfrak{g}$. If $a \neq 0$, then $C_{\mathfrak{g}}(x) = \mathfrak{h}$. Therefore, $\mathfrak{h}$ is minimal, in the above sense. $\square$

Houghton and Weinberg define maximal symmetry breaking as follows [HW02, § 1]. As $\mathfrak{g}$ is a compact Lie algebra, there exists some Cartan subalgebra $\mathfrak{t}^{\mathbb{C}} \subseteq \mathfrak{g}^{\mathbb{C}}$ containing Φ_{∞} . Let $H_1, \dots, H_{\text{rank}(\mathfrak{g})} \in \mathfrak{t}^{\mathbb{C}}$ span $\mathfrak{t}^{\mathbb{C}}$. As $\Phi_{\infty} \in \mathfrak{t}^{\mathbb{C}}$, there is some vector $v \in \mathbb{C}^{\text{rank}(\mathfrak{g})}$ such that $\Phi_{\infty} = \sum_{i=1}^{\text{rank}(\mathfrak{g})} v_i H_i$.

Let R denote the set of roots of $\mathfrak{g}$, with respect to $\mathfrak{t}^{\mathbb{C}}$. A root $\alpha \in R$ is associated to a vector $w^{\alpha} \in \mathbb{C}^{\text{rank}(\mathfrak{g})}$ given by $w_i^{\alpha} := \alpha(H_i)$.

Lemma 2.3.24. *Let $\mathfrak{g}$ be a compact Lie algebra. Using the vectors $v, w^{\alpha} \in \mathbb{C}^{\text{rank}(\mathfrak{g})}$ defined above and the usual inner product on $\mathbb{C}^{\text{rank}(\mathfrak{g})}$, Φ_{∞} has maximal symmetry breaking if and only if $v \cdot w^{\alpha} \neq 0$ for all $\alpha \in R$ (Houghton and Weinberg's definition).*

Proof. Note that a change of basis gives us $H_1 = \Phi_{\infty}$. In this case, we have $v \cdot w^{\alpha} = \alpha(\Phi_{\infty})$. As the inner product is invariant under coordinate changes, we may assume that $v \cdot w^{\alpha} = \alpha(\Phi_{\infty})$.

By Proposition 2.3.19, we know that Φ_{∞} has maximal symmetry breaking if and only if $|R_s(\Phi_{\infty})| = 0$. However, this happens if and only if $|R(\Phi_{\infty})| = 0$. That is, if $\alpha(\Phi_{\infty}) \neq 0$ for all $\alpha \in R$. $\square$

Nagy and Oliveira provide the following definition for maximal symmetry breaking [NO20, Hypothesis 3.3].

Lemma 2.3.25. *Let $\mathfrak{g}$ be a compact Lie algebra. We have that Φ_{∞} has maximal symmetry breaking if and only if Φ_{∞} is in the interior of a Weyl chamber (Nagy and Oliveira's definition).*

Proof. Recall that Φ_{∞} is in the interior of the fundamental Weyl chamber if and only if it does not belong to any $\ker(\alpha_i)$, where α_i are simple roots. This happens exactly when $|R_s(\Phi_{\infty})| = 0$. By Proposition 2.3.19, this happens if and only if Φ_{∞} has maximal symmetry breaking. $\square$

Hurtubise defines maximal symmetry breaking as follows [Hur89, Introduction].

Lemma 2.3.26. *Let G be a compact, connected Lie group with Lie algebra $\mathfrak{g}$. We have that $\Phi_\infty \in \mathfrak{g}$ has maximal symmetry breaking if and only if there exists some maximal torus $T \subseteq G$ such that $\text{Ad}(G)\Phi_\infty \simeq G/T$, as manifolds (Hurtubise's definition).*

Proof. Suppose that Φ_∞ has maximal symmetry breaking. By Lemma 2.3.21, Φ_∞ is regular. Hence, by Corollary 2.3.22, $\mathfrak{h}$ is abelian of dimension $\text{rank}(\mathfrak{g})$. As H is a compact, connected subgroup of G and $\mathfrak{h}$ is a maximal abelian subalgebra, H is a maximal torus and $\text{Ad}(G)\Phi_\infty \simeq G/H$.

Conversely, suppose that the adjoint orbit is the quotient of G by a maximal torus T . Then

$$G/H \simeq \text{Ad}(G)\Phi_\infty \simeq G/T.$$

As T and H are closed, the Homogeneous Space Construction Theorem tells us that

$$\dim_{\mathbb{R}}(G) - \dim_{\mathbb{R}}(H) = \dim_{\mathbb{R}}(G) - \dim_{\mathbb{R}}(T).$$

Hence, $\dim_{\mathbb{R}}(H) = \dim_{\mathbb{R}}(T) = \text{rank}(\mathfrak{g})$. Thus, Φ_∞ has maximal symmetry breaking. $\square$

We summarize the previous results in the following theorem.

Theorem 2.3.27. *Let $\mathfrak{g}$ be a compact Lie algebra and $\Phi_\infty \in \mathfrak{g}$. Then all the definitions in Table 2.1 are equivalent descriptions of when Φ_∞ has maximal symmetry breaking. Moreover, all the definitions in Table 2.2 are equivalent descriptions of when Φ_∞ has generalized minimal symmetry breaking.*

2.3.3 Symmetry Breaking for Simple Groups

In this section, we outline the classification of symmetry breaking for all compact, connected Lie groups. We then investigate the classical Lie groups in more detail, before mentioning the exceptional simple Lie groups. In particular, we see that the general notion of maximal symmetry breaking matches the classical notion for the $\text{SU}(n)$ case. However, the classical notion of minimal symmetry breaking is just one case of generalized minimal symmetry breaking in the $\text{SU}(n)$ case.

To begin, recall that for G compact and connected, there exists a torus T , a simply-connected, compact Lie group J , and a finite central subgroup $Z \subseteq T \times J$ such that

Source	Definition
Definition 2.3.17	$\dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{h})) = \text{rank}(\mathfrak{g})$
Note 2.3.20	$ R_s(\Phi_{\infty}) = 0$
[MS03, § 1]	$\Phi_{\infty} \in \mathfrak{g}$ is regular
Corollary 2.3.22	$\mathfrak{h}$ is abelian
[HW02, § 1]	Every inner product of Φ_{∞} and a root is non-zero
[NO20, Hypothesis 3.3]	Φ_{∞} is in the interior of a Weyl chamber
[Hur89, Introduction]	The adjoint orbit is diffeomorphic to G/T for some maximal torus $T \subseteq G$

Table 2.1: A list of the equivalent definitions of maximal symmetry breaking.

Source	Definition
Definition 2.3.17	$\dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{h})) = \dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{g})) + 1$
Note 2.3.20	$ R_s(\Phi_{\infty}) = \text{rank}(\mathfrak{g}) - \dim_{\mathbb{R}}(\mathfrak{z}(\mathfrak{g})) - 1$
Lemma 2.3.23	$\mathfrak{h}$ is maximal

Table 2.2: A list of the equivalent definitions of generalized minimal symmetry breaking.

$G \simeq (T \times J)/Z$. Additionally, we can decompose $\mathfrak{g}$ into abelian and semi-simple parts as $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$. The proof of Proposition 2.3.16 tells us that

$$C_{\mathfrak{g}}(x_{\text{ab}}, x_{\text{ss}}) = \mathfrak{z}(\mathfrak{g}) \oplus C_{[\mathfrak{g}, \mathfrak{g}]}(x_{\text{ss}}).$$

Therefore, the symmetry breaking of $(x_{\text{ab}}, x_{\text{ss}})$ in $\mathfrak{g}$ depends entirely on the symmetry breaking of x_{ss} in $[\mathfrak{g}, \mathfrak{g}]$. Hence, as symmetry breaking is determined at the Lie algebra level, we need only focus on classifying the symmetry breaking of elements of semi-simple Lie algebras. In fact, the following lemma tells us that we can focus on just simple Lie algebras.

Lemma 2.3.28. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_n$ be a decomposition of $\mathfrak{g}$, a semi-simple Lie algebra, into simple Lie algebras. We then write $\Phi_{\infty} = (\Phi_1, \dots, \Phi_n)$ according to this decomposition. Let R_s^i be the set of simple roots of $\mathfrak{g}_i$ for all $i \in \{1, \dots, n\}$. Then we have the disjoint union*

$$R_s(\Phi_{\infty}) = \bigsqcup_{i=1}^n R_s^i(\Phi_i). \quad (2.3.26)$$

In particular, this means

$$|R_s(\Phi_\infty)| = \sum_{i=1}^n |R_s^i(\Phi_i)|. \quad (2.3.27)$$

Proof. We have that the set of simple roots of $\mathfrak{g}$ is the disjoint union $R_s = \bigsqcup_{i=1}^n R_s^i$. For each $\alpha \in R_s$, there exists $i \in \{1, \dots, n\}$ such that $\alpha \in R_s^i$. We have that $\alpha \in R_s(\Phi_\infty)$ if and only if $\alpha \in R_s^i(\Phi_i)$. Thus, we have the desired disjoint union. $\square$

In particular, we can discuss maximal and minimal symmetry breaking in terms of the decomposition of Φ_∞ .

Corollary 2.3.29. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_n$ be a decomposition of $\mathfrak{g}$, a semi-simple Lie algebra, into simple Lie algebras. We then write $\Phi_\infty = (\Phi_1, \dots, \Phi_n)$ according to this decomposition. We have Φ_∞ has maximal symmetry breaking if and only if all $\Phi_1, \dots, \Phi_n$ have maximal symmetry breaking. Moreover, Φ_∞ has generalized minimal symmetry breaking if and only if all but one of $\Phi_1, \dots, \Phi_n$ are zero and the non-zero Φ_i has generalized minimal symmetry breaking.*

Proof. From Lemma 2.3.28, we have

$$|R_s(\Phi_\infty)| = \sum_{i=1}^n |R_s^i(\Phi_i)|.$$

By Proposition 2.3.19, we know that Φ_∞ has maximal symmetry breaking if and only if $|R_s(\Phi_\infty)| = 0$. But this happens if and only if each $|R_s^i(\Phi_i)| = 0$. That is, if and only if each Φ_i has maximal symmetry breaking.

Similarly, Φ_∞ has generalized minimal symmetry breaking if and only if $|R_s(\Phi_\infty)| = \text{rank}(\mathfrak{g}) - 1$. As $\text{rank}(\mathfrak{g}) = \sum_{i=1}^n \text{rank}(\mathfrak{g}_i)$ and $0 \leq |R_s^i(\Phi_i)| \leq \text{rank}(\mathfrak{g}_i)$, we have that $|R_s(\Phi_\infty)| = \text{rank}(\mathfrak{g}) - 1$ if and only if there is one $i \in \{1, \dots, n\}$ such that $|R_s^i(\Phi_i)| = \text{rank}(\mathfrak{g}_i) - 1$ and the rest satisfy $|R_s^j(\Phi_j)| = \text{rank}(\mathfrak{g}_j)$. Finally, as $|R_s^j(\Phi_j)| = \text{rank}(\mathfrak{g}_j)$ if and only if $\Phi_j = 0$, we have our result. $\square$

We now focus on classifying symmetry breaking for each simple Lie algebra. Let $\mathfrak{g}$ be a simple Lie algebra with Cartan matrix $A(\mathfrak{g})$. We know that Φ_∞ belongs to a Cartan subalgebra of $\mathfrak{g}^\mathbb{C}$. Let $r := \text{rank}(\mathfrak{g})$ and let $H_1, \dots, H_r \in \mathfrak{g}^\mathbb{C}$ be part of a Chevalley basis for $\mathfrak{g}^\mathbb{C}$, generating this Cartan subalgebra. Then there exists $c = [c_1 \ \dots \ c_r]^T \in \mathbb{C}^r$ such that $\Phi_\infty = \sum_{i=1}^r c_i H_i$. Note that we cannot necessarily change basis as in the proof of Lemma 2.3.24.

Lemma 2.3.30. *We have that $|R_s(\Phi_\infty)|$ is precisely the number of zero components of $A(\mathfrak{g})^T c \in \mathbb{C}^r$.*

Proof. Let $R_s = (\alpha_1, \dots, \alpha_r)$. With respect to this order of simple roots and Cartan subalgebra, we have that $A(\mathfrak{g})_{ji} = \alpha_i(H_j)$. Thus, for $\alpha_i \in R_s$, we have that

$$\alpha_i(\Phi_\infty) = \sum_{j=1}^r c_j \alpha_i(H_j) = A(\mathfrak{g})_{ji} c_j = (A(\mathfrak{g})^T c)_i.$$

Thus, $\alpha_i \in R_s(\Phi_\infty)$ precisely when $(A(\mathfrak{g})^T c)_i = 0$. □

2.3.3.1 Classical simple Lie algebras

In this section, we discuss symmetry breaking for the classical simple Lie algebras. We start with $\mathfrak{g} = \mathfrak{su}(n)$, which has Dynkin diagram A_{n-1} .

Theorem 2.3.31. *Let $\Phi_\infty \in \mathfrak{su}(n)$. Let N be the number of distinct eigenvalues of Φ_∞ . Then $\pi_2(G/H) \simeq \mathbb{Z}^{N-1}$. In particular, Φ_∞ has maximal symmetry breaking if and only if all eigenvalues of Φ_∞ are distinct. Additionally, Φ_∞ has generalized minimal symmetry breaking if and only if Φ_∞ has two distinct eigenvalues.*

Note 2.3.32. *If Φ_∞ has generalized minimal symmetry breaking and one of the two distinct eigenvalues has a multiplicity of one, then Φ_∞ has minimal symmetry breaking in the classical sense.*

Proof. Recall that for $n = 2$, we have $A(\mathfrak{su}(2)) = [2]$. Thus, we see that $|R_s(\Phi_\infty)| = 0$ if and only if $\Phi_\infty \neq 0$. As the eigenvalues of Φ_∞ have to sum to zero, this happens if and only if the eigenvalues of Φ_∞ are distinct. Thus, the number of distinct eigenvalues of Φ_∞ is $N = 2$. Note that $\pi_2(G/H) \simeq \mathbb{Z}$, as $|R_s(\Phi_\infty)| = 0$. Additionally, as $\text{rank}(\mathfrak{su}(2)) = 1$ and $\mathfrak{z}(\mathfrak{su}(2)) = 0$, there is no distinction between maximal and generalized minimal symmetry breaking.

For $n > 2$, we have

$$A(\mathfrak{su}(n)) = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}. \quad (2.3.28)$$

Proposition F.0.2 in Appendix F tells us that the matrices

$$H_1 := \text{diag}(1, -1, 0, \dots, 0), \dots, H_{n-1} := \text{diag}(0, \dots, 0, 1, -1),$$

span a Cartan subalgebra of $\mathfrak{su}(n)^{\mathbb{C}}$ and are part of a Chevalley basis.

As $\Phi_{\infty} \in \mathfrak{su}(n)$ is a normal matrix, there exists $U \in \text{SU}(n)$ such that $U\Phi_{\infty}U^{\dagger}$ is diagonal, with the same eigenvalues as Φ_{∞} . Note that as $\text{SU}(n)$ is a matrix Lie group, with Lie algebra $\mathfrak{su}(n)$, the adjoint representation is just conjugation by elements of $\text{SU}(n)$. Thus, $U\Phi_{\infty}U^{\dagger}$ and Φ_{∞} are in the same adjoint orbit. As the adjoint orbit is independent of the Lie group acting on the Lie algebra, without loss of generality, we may assume that Φ_{∞} belongs to the span of the H_i .

As each $iH_i \in \mathfrak{su}(n)$, we have that there exists $c_1, \dots, c_{n-1} \in \mathbb{R}$ such that $\Phi_{\infty} = \sum_{i=1}^{n-1} ic_i H_i$. In this case,

$$\Phi_{\infty} = i \cdot \text{diag}(c_1, c_2 - c_1, \dots, c_{n-1} - c_{n-2}, -c_{n-1}). \quad (2.3.29)$$

Thus, we see the eigenvalues of Φ_{∞} .

Similar to above, conjugating by elements of $\text{SU}(n)$, we can swap rows. Thus, there is some point in the adjoint orbit of Φ_{∞} that is of the same form as Φ_{∞} but with

$$c_1 \geq c_2 - c_1 \geq \dots \geq c_{n-1} - c_{n-2} \geq -c_{n-1}. \quad (2.3.30)$$

Without loss of generality, we assume that Φ_{∞} satisfies these inequalities. We see that these inequalities become equalities each time $A(\mathfrak{su}(n))^T c$ has a zero component.

Let N be the number of distinct eigenvalues of Φ_{∞} . The number of zeros of $A(\mathfrak{su}(n))^T c$ is $n - N$. This gives us our result for $\pi_2(G/H)$. In particular, Φ_{∞} has maximal symmetry breaking if and only if $A(\mathfrak{su}(n))^T c$ has no zero components, meaning $N = n$. Additionally, we have that Φ_{∞} has generalized minimal symmetry breaking if and only if $A(\mathfrak{su}(n))^T c$ has only one non-zero component, meaning $N = 2$. $\square$

Example 2.3.33. *The matrix $i \cdot \text{diag}(1, 0, -1) \in \mathfrak{su}(3)$ has maximal symmetry breaking, and $i \cdot \text{diag}(1, 1, -2) \in \mathfrak{su}(3)$ has both minimal and generalized minimal symmetry breaking. However, $i \cdot \text{diag}(1, 1, 0, -2) \in \mathfrak{su}(4)$ has neither maximal nor generalized minimal symmetry breaking. As $\text{rank}(\mathfrak{su}(4)) = 3$, this is the lowest rank where we can simultaneously have non-maximal and non-generalized minimal symmetry breaking.*

Example 2.3.34. *In previous work with collaborators, which we discuss in Section 2.2, I classify all spherically symmetry Euclidean $\text{SU}(n)$ monopoles. I then compute some explicit*

examples of such monopoles. They have the following asymptotic values:

$$\begin{aligned} i \cdot \text{diag}(1, 1, 1, -1, -1, -1) &\in \mathfrak{su}(6), \\ i \cdot \text{diag}(1, 1, -1, -1) &\in \mathfrak{su}(4), \quad \text{and} \\ \frac{i}{5} \cdot \text{diag}(4, 4, 4, -6, -6) &\in \mathfrak{su}(5). \end{aligned} \tag{2.3.31}$$

Note that while none have maximal nor minimal symmetry breaking, they all have generalized minimal symmetry breaking [CDL⁺22, § 5.2].

Now we consider $\mathfrak{g} = \mathfrak{so}(2m+1)$, with $m \geq 1$, which has Dynkin diagram B_m . The eigenvalues of an element of $\mathfrak{so}(2m+1)$ are imaginary. Moreover, other than an extra eigenvalue of zero, they occur in pairs of the form $\pm\lambda i$ for some $\lambda \in \mathbb{R}$.

Theorem 2.3.35. *Let $\Phi_\infty \in \mathfrak{so}(2m+1)$, with $m \geq 1$. Let $\pm\lambda_1 i, \dots, \pm\lambda_m i$, and $\lambda_{m+1} := 0$ be the eigenvalues of Φ_∞ . Let N be the number of distinct values of λ_i^2 . Then $\pi_2(G/H) \simeq \mathbb{Z}^{N-1}$. In particular, Φ_∞ has maximal symmetry breaking if all λ_i^2 are distinct. Additionally, Φ_∞ has generalized minimal symmetry breaking if and only if there is some $\lambda \neq 0$ such that at least one $\lambda_i^2 = \lambda^2$ and the rest of the λ_i^2 are either λ^2 or zero.*

Proof. Note that for $m = 1$, we have $A(\mathfrak{so}(3)) = \begin{bmatrix} 2 \end{bmatrix}$. Thus, we see that $|R_s(\Phi_\infty)| = 0$ if and only if $\Phi_\infty \neq 0$. As the eigenvalues of Φ_∞ are $\pm\lambda_1 i$ and $\lambda_2 := 0$, this happens precisely when $N = 2$, where N is the number of distinct λ_i^2 . Note $\pi_2(G/H) \simeq \mathbb{Z}$, as $|R_s(\Phi_\infty)| = 0$. Additionally, as $\text{rank}(\mathfrak{so}(3)) = 1$ and $\mathfrak{z}(\mathfrak{so}(3)) = 0$, there is no distinction between maximal and generalized minimal symmetry breaking.

For $m > 1$, we have

$$A(\mathfrak{so}(2m+1)) = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -2 & 2 \end{bmatrix}. \tag{2.3.32}$$

Let $X := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathfrak{so}(2)$. Proposition F.0.3 in Appendix F tells us that the matrices

$$\begin{aligned} H_1 &:= i \cdot \text{diag}(X, -X, 0, \dots, 0), \\ &\vdots \\ H_{m-1} &:= i \cdot \text{diag}(0, \dots, 0, X, -X, 0), \\ H_m &:= 2i \cdot \text{diag}(0, \dots, 0, X, 0), \end{aligned}$$

span a Cartan subalgebra of $\mathfrak{so}(2m+1)^\mathbb{C}$ and are part of a Chevalley basis.

There exists $U \in \text{SO}(2m+1)$ such that $U\Phi_\infty U^T$ belongs to the span of the H_i , with the same eigenvalues as Φ_∞ . Note that as $\text{SO}(2m+1)$ is a matrix Lie group, with Lie algebra $\mathfrak{so}(2m+1)$, the adjoint representation is just conjugation by elements of $\text{SO}(2m+1)$. Thus, $U\Phi_\infty U^T$ and Φ_∞ are in the same adjoint orbit. As the adjoint orbit is independent of the Lie group acting on the Lie algebra, without loss of generality, we may assume that Φ_∞ belongs to the span of the H_i .

As each $iH_i \in \mathfrak{so}(2m+1)$, we have that there exists $c_1, \dots, c_{n-1} \in \mathbb{R}$ such that $\Phi_\infty = -\sum_{i=1}^{n-1} ic_i H_i$. In this case,

$$\Phi_\infty = \text{diag}(c_1 X, (c_2 - c_1)X, \dots, (c_{m-1} - c_{m-2})X, (2c_m - c_{m-1})X, 0). \quad (2.3.33)$$

Thus, Φ_∞ has eigenvalues $\pm c_1 i, \pm(c_2 - c_1)i, \dots, \pm(c_{m-1} - c_{m-2})i, \pm(2c_m - c_{m-1})i$, and 0. That is, $\lambda_1 := c_1, \lambda_2 := c_2 - c_1, \dots, \lambda_{m-1} := c_{m-1} - c_{m-2}, \lambda_m := 2c_m - c_{m-1}$, and $\lambda_{m+1} := 0$.

Similar to above, conjugating by elements of $\text{SO}(2m+1)$, we can swap rows. Thus, there is some point in the adjoint orbit of Φ_∞ that is of the same form as Φ_∞ , but with

$$c_1 \geq c_2 - c_1 \geq \dots \geq c_{m-1} - c_{m-2} \geq 2c_m - c_{m-1} \geq 0. \quad (2.3.34)$$

Without loss of generality, we assume that Φ_∞ satisfies these inequalities. We see that these inequalities become equalities each time $A(\mathfrak{so}(2m+1))^T c$ has a zero component.

Let N be the number of distinct values of λ_i^2 . The number of zeros of $A(\mathfrak{so}(2m+1))^T c$ is $m - N + 1$. This gives our result for $\pi_2(G/H)$. In particular, Φ_∞ has maximal symmetry breaking if and only if $A(\mathfrak{so}(2m+1))^T c$ has no zero components, meaning $N = m + 1$. Additionally, Φ_∞ has generalized minimal symmetry breaking if and only if $A(\mathfrak{so}(2m+1))^T c$ has only one non-zero component, meaning $N = 2$. $\square$

Example 2.3.36. All non-zero matrices in $\mathfrak{so}(3)$ have maximal and generalized minimal symmetry breaking. Note that $\text{rank}(\mathfrak{so}(3)) = 1$. The matrix $\text{diag}(X, 2X, 0) \in \mathfrak{so}(5)$ has maximal symmetry breaking and $\text{diag}(X, X, 0), \text{diag}(X, 0_3) \in \mathfrak{so}(5)$ have generalized minimal symmetry breaking. For $\mathfrak{g} = \mathfrak{so}(2m+1)$, $m = 3$ is the lowest rank where we can simultaneously have non-maximal and non-generalized minimal symmetry breaking.

Now we consider $\mathfrak{g} = \mathfrak{sp}(n)$, which has Dynkin diagram C_n . A right eigenvalue of an $n \times n$ quaternionic matrix X is a quaternion $\lambda \in \mathbb{H}$ such that there is some $v \in \mathbb{H}^n \setminus \{0\}$ satisfying $Xv = v\lambda$.

Theorem 2.3.37. Let $\Phi_\infty \in \mathfrak{sp}(n)$. Let $\alpha_1, \dots, \alpha_n$ be the modulus of the right eigenvalues of Φ_∞ . Let N be the number of distinct, non-zero values of α_i . Then $\pi_2(G/H) \simeq \mathbb{Z}^N$. In particular, Φ_∞ has maximal symmetry breaking if and only if all α_i^2 are non-zero and distinct. Additionally, Φ_∞ has generalized minimal symmetry breaking if and only if there is some $\alpha > 0$ such that at least one $\alpha_i = \alpha$ and the rest of the α_j are either α or zero.

Proof. Note that for $n = 1$, we have $A(\mathfrak{sp}(1)) = \begin{bmatrix} 2 \end{bmatrix}$. Thus, we see that $|R_s(\Phi_\infty)| = 0$ if and only if $\Phi_\infty \neq 0$, that is $N = 1$, where N is the number of distinct, non-zero α_i . Note $\pi_2(G/H) \simeq \mathbb{Z}$, as $|R_s(\Phi_\infty)| = 0$. Additionally, as $\text{rank}(\mathfrak{sp}(1)) = 1$ and $\mathfrak{z}(\mathfrak{sp}(1)) = 0$, there is no distinction between maximal and generalized minimal symmetry breaking.

For $n > 1$, we have

$$A(\mathfrak{sp}(n)) = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & -1 & 0 \\ \vdots & \ddots & \ddots & 2 & -2 \\ 0 & \cdots & 0 & -1 & 2 \end{bmatrix}. \quad (2.3.35)$$

Proposition F.0.4 in Appendix F tells us that the matrices

$$\begin{aligned} H_1 &:= i \cdot \text{diag}(1, -1, 0, \dots, 0) \otimes i, \\ &\vdots \\ H_{n-1} &:= i \cdot \text{diag}(0, \dots, 0, 1, -1) \otimes i, \\ H_n &:= i \cdot \text{diag}(0, \dots, 0, 1) \otimes i, \end{aligned}$$

span a Cartan subalgebra of $\mathfrak{sp}(n)^\mathbb{C}$ and are part of a Chevalley basis.

As $\Phi_\infty \in \mathfrak{sp}(n)$, there exists $U \in \mathrm{Sp}(n)$ such that $U\Phi_\infty U^\dagger$ is diagonal [Zha97, Corollary 6.2]. Specifically, note that as $U\Phi_\infty U^\dagger \in \mathfrak{sp}(n)$, the right eigenvalues of $U\Phi_\infty U^\dagger$ belong to $\mathfrak{sp}(1)$. Moreover, through conjugating by elements of $\mathrm{Sp}(1)$, we may assume that the right eigenvalues of $U\Phi_\infty U^\dagger$ are in $i\mathbb{R}$.

While $U\Phi_\infty U^\dagger$ does not have the same right eigenvalues as Φ_∞ , their eigenvalues are the same up to conjugation by elements of $\mathrm{Sp}(1)$. That is, the modulus of the eigenvalues are the same. Additionally, as $\mathrm{Sp}(n)$ is a matrix Lie group, with Lie algebra $\mathfrak{sp}(n)$, the adjoint representation is just conjugation by elements of $\mathrm{Sp}(n)$. Thus, $U\Phi_\infty U^\dagger$ and Φ_∞ are in the same adjoint orbit. As the adjoint orbit is independent of the Lie group acting on the Lie algebra, without loss of generality, we may assume that Φ_∞ belongs to the real span of the $(1 \otimes i)H_i$.

Thus, there exists $c_1, \dots, c_n \in \mathbb{R}$ such that $\Phi_\infty = -\sum_{i=1}^n c_i(1 \otimes i)H_i$. In this case,

$$\Phi_\infty = i \cdot \mathrm{diag}(c_1, c_2 - c_1, \dots, c_{n-1} - c_{n-2}, c_n - c_{n-1}). \quad (2.3.36)$$

Thus, we can read off the eigenvalues of Φ_∞ . In particular, we have $\alpha_1 := |c_1|$, $\alpha_2 := |c_2 - c_1|$, $\dots$, and $\alpha_n := |c_n - c_{n-1}|$.

Similar to above, conjugating by elements of $\mathrm{Sp}(n)$, we can swap rows and rotate elements on the diagonal. Thus, there is some point in the adjoint orbit of Φ_∞ that is of the same form as Φ_∞ but with

$$c_1 \geq c_2 - c_1 \geq \dots \geq c_{n-1} - c_{n-2} \geq c_n - c_{n-1} \geq 0. \quad (2.3.37)$$

Without loss of generality, we assume that Φ_∞ satisfies these inequalities. We see that these inequalities become equalities each time $A(\mathfrak{sp}(n))^T c$ has a zero component.

Let N be the number of distinct, non-zero values of α_i . The number of zeros of $A(\mathfrak{sp}(n))^T c$ is $n - N$. This gives our result for $\pi_2(G/H)$. In particular, Φ_∞ has maximal symmetry breaking if and only if $A(\mathfrak{sp}(n))^T c$ has no zero components, meaning $N = n$. Additionally, Φ_∞ has generalized minimal symmetry breaking if and only if $A(\mathfrak{sp}(n))^T c$ has only one non-zero component, meaning $N = 1$. $\square$

Example 2.3.38. *The matrices $i \cdot \mathrm{diag}(1, \pm 1)$, $i \cdot \mathrm{diag}(1, 0) \in \mathfrak{sp}(2)$ have generalized minimal symmetry breaking and $i \cdot \mathrm{diag}(1, 2) \in \mathfrak{sp}(2)$ has maximal symmetry breaking. However, $i \cdot \mathrm{diag}(0, 1, 2) \in \mathfrak{sp}(3)$ has neither maximal nor generalized minimal symmetry breaking. As $\mathrm{rank}(\mathfrak{sp}(3)) = 3$, this is the lowest rank where we can simultaneously have non-maximal and non-generalized minimal symmetry breaking.*

Example 2.3.39. In Proposition 2.1.30, Note 2.1.43, and Note 2.1.45, I compute some explicit examples of spherically symmetric hyperbolic monopoles. They have the following asymptotic values:

$$\begin{aligned} \frac{i}{2}I_k &\in \mathfrak{sp}(k), \\ \frac{i}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} &\in \mathfrak{sp}(2), \\ \begin{bmatrix} \frac{17i}{50} & \frac{12}{175}\sqrt{14} & \frac{8}{3325}\sqrt{3990} & \frac{6i}{475}\sqrt{285} \\ -\frac{12}{175}\sqrt{14} & \frac{31i}{350} & -\frac{48i}{3325}\sqrt{285} & \frac{18}{3325}\sqrt{3990} \\ -\frac{8}{3325}\sqrt{3990} & -\frac{48i}{3325}\sqrt{285} & \frac{473i}{1330} & \frac{36}{665}\sqrt{14} \\ \frac{6i}{475}\sqrt{285} & -\frac{18}{3325}\sqrt{3990} & -\frac{36}{665}\sqrt{14} & \frac{41i}{190} \end{bmatrix} &\in \mathfrak{sp}(4). \end{aligned}$$

These all have eigenvalues in $\{\pm i/2\}$. Thus, they all have generalized minimal symmetry breaking.

Now we consider $\mathfrak{g} = \mathfrak{so}(2m)$, with $m \geq 3$, which has Dynkin diagram D_m . The eigenvalues of an element of $\mathfrak{so}(2m)$ are imaginary.

Theorem 2.3.40. Let $\Phi_\infty \in \mathfrak{so}(2m)$, with $m \geq 2$. Let $\pm\lambda_1 i, \dots, \pm\lambda_m i$ be the eigenvalues of Φ_∞ . Let N be the number of distinct λ_i^2 and let μ be one if zero is an eigenvalue of Φ_∞ with multiplicity greater than two and zero otherwise. Then $\pi_2(G/H) \simeq \mathbb{Z}^{N-\mu}$. In particular, Φ_∞ has maximal symmetry breaking if and only if all λ_i^2 are distinct. Additionally, Φ_∞ has generalized minimal symmetry breaking if and only if there is some $\lambda \neq 0$ such that at least one $\lambda_i^2 = \lambda^2$ and either the rest of the $\lambda_j^2 = \lambda^2$ or at least two $\lambda_j^2 = 0$ and the rest are λ^2 .

Note 2.3.41. We do not consider $\mathfrak{so}(2)$ as it is not even semi-simple, it is abelian. However, we consider $\mathfrak{so}(4)$, which is not simple, for completeness.

Proof. As $\mathfrak{so}(4) \simeq \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, we have

$$A(\mathfrak{so}(4)) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}. \quad (2.3.38)$$

Let $X := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathfrak{so}(2)$. Proposition F.0.5 in Appendix F tells us that the matrices

$$H_1 := i \cdot \text{diag}(X, X), \quad \text{and} \quad H_2 := i \cdot \text{diag}(X, -X),$$

span a Cartan subalgebra of $\mathfrak{so}(4)^\mathbb{C}$ and are part of a Chevalley basis.

There exists $U \in \mathrm{SO}(4)$ such that $U\Phi_\infty U^T$ belongs to the span of the H_i , with the same eigenvalues as Φ_∞ . Note that as $\mathrm{SO}(4)$ is a matrix Lie group, with Lie algebra $\mathfrak{so}(4)$, the adjoint representation is just conjugation by elements of $\mathrm{SO}(4)$. Thus, $U\Phi_\infty U^T$ and Φ_∞ are in the same adjoint orbit. As the adjoint orbit is independent of the Lie group acting on the Lie algebra, without loss of generality, we may assume that Φ_∞ belongs to the span of the H_i .

As $iH_i \in \mathfrak{so}(4)$, there exists $c_1, c_2 \in \mathbb{R}$ such that

$$\Phi_\infty = - \sum_{i=1}^2 ic_i H_i = \mathrm{diag}((c_1 + c_2)X, (c_1 - c_2)X). \quad (2.3.39)$$

Therefore, the eigenvalues of Φ_∞ are $\pm(c_1 + c_2)i$ and $\pm(c_1 - c_2)i$.

We see that $A(\mathfrak{so}(4))^T c$ has a zero exactly when c does. Note that as $\mathrm{rank}(\mathfrak{so}(4)) = 2$, Φ_∞ has either maximal or generalized minimal symmetry breaking. Moreover, we cannot have $\mu = 1$ in this case, where μ is as defined in the statement. Thus, we need only investigate the extremal types of symmetry breaking for $m = 2$.

We note that Φ_∞ has maximal symmetry breaking if and only if c_1 and c_2 are non-zero. We see that $(c_1 + c_2)^2 = (c_1 - c_2)^2$ if and only if $c_1 c_2 = 0$. Therefore, Φ_∞ has maximal symmetry breaking if and only if all the eigenvalues of Φ_∞ are distinct.

We see that Φ_∞ has generalized minimal symmetry breaking if $c_1 = 0$ and $c_2 \neq 0$ or vice versa. In either case, we have that this happens exactly when the eigenvalues of Φ_∞ are $\pm ic_i \neq 0$, each with multiplicity two.

For $m \geq 3$, we have

$$A(\mathfrak{so}(2m)) = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & 2 & -1 & -1 \\ 0 & \cdots & 0 & -1 & 2 & 0 \\ 0 & \cdots & 0 & -1 & 0 & 2 \end{bmatrix}. \quad (2.3.40)$$

Proposition F.0.5 in Appendix F tells us that the matrices

$$\begin{aligned} H_1 &:= i \cdot \text{diag}(X, -X, 0, \dots, 0), \\ &\vdots \\ H_{m-2} &:= i \cdot \text{diag}(0, \dots, 0, X, -X, 0), \\ H_{m-1} &:= i \cdot \text{diag}(0, \dots, 0, X, -X), \\ H_m &:= i \cdot \text{diag}(0, \dots, 0, X, X), \end{aligned}$$

span a Cartan subalgebra of $\mathfrak{so}(2m)^\mathbb{C}$ and are part of a Chevalley basis.

Just as in the case $m = 2$, without loss of generality, we may assume that Φ_∞ belongs to the span of the H_i . As each $iH_i \in \mathfrak{so}(2m)$, we have that there exists $c_1, \dots, c_m \in \mathbb{R}$ such that $\Phi_\infty = -\sum_{i=1}^m i c_i H_i$. In this case,

$$\begin{aligned} \Phi_\infty = \text{diag}(c_1 X, (c_2 - c_1) X, \dots, (c_{m-2} - c_{m-3}) X, \\ (c_{m-1} + c_m - c_{m-2}) X, (c_m - c_{m-1}) X). \end{aligned} \quad (2.3.41)$$

Thus, we can read off the eigenvalues of Φ_∞ , finding $\lambda_1 := c_1$, $\lambda_2 := c_2 - c_1$, $\dots$, $\lambda_{m-2} := c_{m-2} - c_{m-3}$, $\lambda_{m-1} := c_{m-1} + c_m - c_{m-2}$, and $\lambda_m := c_m - c_{m-1}$.

Similar to above, conjugating by elements of $\text{SO}(2m)$, we can swap rows. Thus, there is some point in the adjoint orbit of Φ_∞ that is of the same form as Φ_∞ , but with

$$c_1 \geq c_2 - c_1 \geq \dots \geq c_{m-2} - c_{m-3} \geq c_{m-1} + c_m - c_{m-2} \geq c_m - c_{m-1}, 0. \quad (2.3.42)$$

Note that we cannot change whether the final eigenvalue is positive or negative, as we are conjugating by elements of $\text{SO}(2m)$. We can do it in the $\text{SO}(2m+1)$ case as the final coordinate gives us the freedom to change the determinant of the matrix. However, we can gain one more inequality. Suppose that $c_m - c_{m-1} \leq c_{m-2} - c_m - c_{m-1} \leq 0$. Then,

$$c_{m-1} - c_m \geq c_{m-1} + c_m - c_{m-2} \geq c_m - c_{m-1}. \quad (2.3.43)$$

Flipping the signs of the final two X coefficients and swapping rows, which is possible when conjugating by $\text{SO}(2m)$ matrices, without loss of generality, we may assume that

$$\begin{aligned} c_1 &\geq c_2 - c_1 \geq \dots \geq c_{m-2} - c_{m-3} \\ &\geq c_{m-1} + c_m - c_{m-2} \geq c_m - c_{m-1} \geq c_{m-2} - c_m - c_{m-1}. \end{aligned} \quad (2.3.44)$$

Without loss of generality, we assume that Φ_∞ satisfies these inequalities. We see that these inequalities become equalities each time $A(\mathfrak{so}(2m))^T c$ has a zero component.

Let N be the number of distinct values of λ_i^2 . Let μ be given as in the statement. We investigate the two possible values of μ . First, suppose that $\mu = 1$. That is, zero is an eigenvalue of Φ_∞ with multiplicity greater than two. As such, two λ_i must be zero. From (2.3.42), we see that since at least two λ_i must be zero, one of them must be $c_{m_1} + c_m - c_{m-2}$. From (2.3.44), we see that $c_m - c_{m-1} = 0$ as well. We then see that the number of zeros of $A(\mathfrak{so}(2m))^T c$ is $m - N + \mu$.

Alternatively, suppose that $\mu = 0$. That is, if zero is an eigenvalue of Φ_∞ , then it has multiplicity at most two. As such, at most one λ_i is zero. We note that $c_{m-1} + c_m - c_{m-2} > 0$, as we showed above that if it were zero, then $c_m - c_{m-1}$ would be zero as well. Thus, we cannot have the final two inequalities in (2.3.44) be equalities simultaneously. We then see that the number of zeros of $A(\mathfrak{so}(2m))^T c$ is $m - N = m - N + \mu$.

Combining cases, we see that the number of zeros of $A(\mathfrak{so}(2m))^T c$ is always $m - N + \mu$. This gives the result for $\pi_2(G/H)$. In particular, Φ_∞ has maximal symmetry breaking if and only if $A(\mathfrak{so}(2m))^T c$ has no zero components, meaning $N = m + \mu$. Note that we cannot have maximal symmetry breaking and $\mu = 1$, as $N \leq m$. Thus, we have maximal symmetry breaking if and only if we have m distinct λ_i^2 .

Additionally, Φ_∞ has generalized minimal symmetry breaking if and only if $A(\mathfrak{so}(2m))^T c$ has only one non-zero component, meaning $N = \mu + 1$. If $\mu = 1$, then we must have two distinct λ_i^2 and one must be zero. Moreover, the multiplicity of zero as an eigenvalue must be greater than two. Alternatively, if $\mu = 0$, then we must have one distinct λ_i^2 . Moreover, this must not be zero, as then $\Phi_\infty = 0$. $\square$

Example 2.3.42. *The matrix $\text{diag}(0_2, X, 2X) \in \mathfrak{so}(6)$ has maximal symmetry breaking, whereas $\text{diag}(X, X, X), \text{diag}(X, 0_2, 0_2) \in \mathfrak{so}(6)$ have generalized minimal symmetry breaking. The matrices $\text{diag}(X, X, 2X), \text{diag}(X, X, 0_2) \in \mathfrak{so}(6)$ have neither maximal nor generalized minimal symmetry breaking. As $\text{rank}(\mathfrak{so}(6)) = 3$, this is the lowest rank where we can simultaneously have non-maximal and non-generalized minimal symmetry breaking.*

2.3.3.2 Exceptional simple Lie algebras

As adjoint orbits in these Lie algebras are not classified by their eigenvalues, we cannot find simple analogues to the results for the classical simple Lie algebras. Thus, in this section, we outline the procedure for classifying symmetry breaking for the exceptional simple Lie algebras. We then list the Cartan matrices for these Lie algebras.

Given $\Phi_\infty \in \mathfrak{g}$, find a Cartan subalgebra of $\mathfrak{g}^\mathbb{C}$ containing Φ_∞ . Then find a basis $H_1, \dots, H_{\text{rank}(\mathfrak{g})} \in \mathfrak{g}^\mathbb{C}$ that is part of a Chevalley basis. Writing $\Phi_\infty = \sum_{i=1}^{\text{rank}(\mathfrak{g})} c_i H_i$ for

some $c_i \in \mathbb{C}$, denote the vector with components c_i by $c \in \mathbb{C}^{\text{rank}(\mathfrak{g})}$. Then evaluate $A(\mathfrak{g})^T c$. The number of zeros in this vector is $|R_s(\Phi_\infty)|$, which tells you the type of symmetry breaking for the monopole. Alternatively, you can find $\mathfrak{h}$, the set of all $x \in \mathfrak{g}$ such that $[x, \Phi_\infty] = 0$. Then determine its centre $\mathfrak{z}(\mathfrak{h})$. The dimension of this centre also tells you the type of symmetry breaking for the monopole. This result is particularly useful for $\mathfrak{g}_2$ monopoles for instance, as $\mathfrak{g}_2 \subseteq \mathfrak{so}(7)$ is known, making computing $\mathfrak{h}$ straightforward.

We have for $n \in \{6, 7, 8\}$,

$$A(\mathfrak{e}_n) = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ -1 & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 & 0 & 0 \\ 0 & \cdots & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & \cdots & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & \cdots & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & \cdots & 0 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}; \quad (2.3.45)$$

$$A(\mathfrak{f}_4) = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}; \quad (2.3.46)$$

$$A(\mathfrak{g}_2) = \begin{bmatrix} 2 & -3 \\ -1 & 2 \end{bmatrix}. \quad (2.3.47)$$

Chapter 3

Instantons and their derivatives

This chapter focuses on applying Theorem 1.1.1 to instantons. While the work done on hyperbolic monopoles in Chapter 2 falls under the umbrella of the work done here, any significant overlaps have been avoided.

Let M be a 4-manifold, $E \rightarrow M$ a vector bundle over M with structure group $\mathrm{Sp}(n)$, equipped with a connection $\mathbb{A}$ of curvature $F_{\mathbb{A}}$. Let $\star$ denote the Hodge star. Recall that instantons are solutions to the self-dual equations $F_{\mathbb{A}} = \star F_{\mathbb{A}}$ with finite action $\frac{1}{2\pi} \int_M |F_{\mathbb{A}}|^2 \mathrm{vol}_M$. Due to the integrability of Chern numbers, the action of an instanton, which is related to the second Chern number, is always an integer multiple of some positive constant. This multiple N is known as the instanton number or charge of the instanton. Note that instantons and anti-instantons, solutions to the anti-self-dual equations $F_{\mathbb{A}} = -\star F_{\mathbb{A}}$ with finite action, differ by a choice of orientation, so there is no real difference between the objects.

Of particular interest is when $M = \mathbb{R}^4$. Due to the conformal invariance of the Hodge star acting on two-forms in four dimensions, instantons on $M = \mathbb{R}^4$ are equivalent to those on a punctured four-sphere. Uhlenbeck proves that, in fact, instantons on $M = \mathbb{R}^4$ are equivalent to those on $M = S^4$ [Uhl82].

Using Theorem 1.1.1, in this chapter, we prove theorems providing algebraic equations for the different kinds of symmetries. Then, we use these results to identify novel instantons with higher rank structure groups and a variety of symmetries. In addition to these results, the main results of this chapter are Theorem 3.4.44 and Corollary 3.4.45. These results tell us that we know all of the instantons with conformal spherical symmetry or full symmetry: those that are direct sums of the basic instanton.

In Section 3.1, we introduce what it means for ADHM data to be symmetric under conformal transformations, proving several lemmas important to later sections. The various kinds of continuous conformal symmetries that an instanton can possess are identified in Section C.2 of Appendix C. In Section 3.2, we investigate instantons with various kinds of circular symmetry. In particular, we look at the connection between such instantons and hyperbolic and singular monopoles. In Section 3.3, we investigate instantons with toral symmetry. In particular, we look at the connection between such instantons and hyperbolic and singular monopoles with axial symmetry. In Section 3.4, we investigate instantons with the three kinds of spherical symmetry: simple, isoclinic, and conformal. We also discuss the relationship between these symmetric instantons and hyperbolic analogues to Higgs bundles and Nahm data. Additionally, we prove the aforementioned Theorem 3.4.44 and Corollary 3.4.45. In Section 3.5, we study instantons with isoclinic and conformal superspherical symmetry. In particular, we also look at the connection between such instantons and axially symmetric hyperbolic Higgs bundles as well as spherically symmetric hyperbolic and singular monopoles. In Section 3.6, we study instantons with rotational symmetry.

3.1 ADHM data and conformal transformations

In this section, we introduce ADHM data and what it means for ADHM data to be symmetric under different conformal transformations.

3.1.1 ADHM data and the ADHM transform

In this section, we introduce ADHM data and the ADHM transform.

Definition 3.1.1. Let $n, k \in \mathbb{N}$. Let $(a, b) \in \text{Mat}(n + k, k, \mathbb{H})^{\oplus 2}$ be a pair of quaternionic matrices and for all $x \in \mathbb{H}$, let $\Delta(x) := a - bx$. A pair (a, b) is **ADHM data** if b has rank k and $\Delta(x)^\dagger \Delta(x)$ is a real, positive definite matrix for all $x \in \mathbb{H}$. Let $\mathcal{N}_{n,k}$ be the set of ADHM data for given n and k .

Note 3.1.2. The condition $\Delta(x)^\dagger \Delta(x)$ being positive definite for all $x \in \mathbb{H}$ does not imply that the rank of b is k . Indeed, consider

$$\Delta(x) := a - bx = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -x \end{bmatrix}.$$

We see that $\Delta(x)^\dagger \Delta(x) = \begin{bmatrix} 1 & 0 \\ 0 & 1 + |x|^2 \end{bmatrix}$ is real and positive definite for all $x \in \mathbb{H}$. However, we see that the rank of b is 1, not $k = 2$.

Given an $\mathrm{Sp}(n)$ instanton $\mathbb{A}$, the instanton number I is given by $I = \frac{1}{4\pi^2} \int_{\mathbb{R}^4} \mathrm{Tr}(F_{\mathbb{A}} \wedge F_{\mathbb{A}}) \mathrm{vol}_{\mathbb{R}^4}$, the second Chern number. Imposing b has rank k ensures that $I = k$. Suppose (a, b) is $(n+k) \times k$ ADHM data with $k' := \mathrm{rank}(b) < k$. Then the corresponding instanton can also be made from $(n+k') \times k'$ ADHM data (a', b') where $\mathrm{rank}(b') = k'$. Indeed, the above example generates an instanton with $I = 1 = k - 1$, but this instanton can also be generated by $\Delta'(x) = \begin{bmatrix} 1 \\ -x \end{bmatrix}$.

We now outline some properties of ADHM data [CFTG78, §2, §5.1].

Lemma 3.1.3. *Let $(a, b) \in \mathcal{N}_{n,k}$. Then $a^\dagger a$ and $b^\dagger b$ are real, symmetric, and positive definite matrices. Also, $a^\dagger b$ and $b^\dagger a$ are symmetric matrices.*

Proof. Suppose $(a, b) \in \mathcal{N}_{n,k}$. Note that $\Delta(0)^\dagger \Delta(0) = a^\dagger a$ is real, symmetric, and positive definite. Let $x \in \mathbb{H} \setminus \{0\}$. Expanding, we see

$$\Delta(x)^\dagger \Delta(x) = a^\dagger b - x^\dagger b^\dagger a - a^\dagger b x + x^\dagger b^\dagger b x. \quad (3.1.1)$$

Thus, we see that

$$\Delta(x)^\dagger \Delta(x) + \Delta(-x)^\dagger \Delta(-x) = 2(a^\dagger a + x^\dagger b^\dagger b x).$$

As $\Delta(x)^\dagger \Delta(x)$, $\Delta(-x)^\dagger \Delta(-x)$, and $a^\dagger a$ are real, then $x^\dagger b^\dagger b x$ is real. Thus, $|x|^4 b^\dagger b$ is real, so $b^\dagger b$ is real, hence symmetric.

Furthermore, we have that the ranks of $b^\dagger b$ and b are equal, so $b^\dagger b$ is a $k \times k$ matrix with rank k . That is, positive definite. Similarly, $a^\dagger a$ and a have the same rank, so a must have rank k .

Finally, from (3.1.1), we see that $x^\dagger b^\dagger a + a^\dagger b x$ is symmetric for all $x \in \mathbb{H}$. Taking $x = 1, i, j, k$ we see that $b^\dagger a + a^\dagger b$, $-ib^\dagger a + a^\dagger bi$, $-jb^\dagger a + a^\dagger bj$, and $-kb^\dagger a + a^\dagger bk$ are all symmetric. Multiplying $b^\dagger a + a^\dagger b$ on the left by i, j, k we see that $ib^\dagger a + ia^\dagger b$, $jb^\dagger a + ja^\dagger b$, and $kb^\dagger a + ka^\dagger b$ are all symmetric. Adding symmetric matrices, we see that $ia^\dagger b + a^\dagger bi$, $ja^\dagger b + a^\dagger bj$, and $ka^\dagger b + a^\dagger bk$ are symmetric.

As $ia^\dagger b + a^\dagger bi$ is symmetric, we know for all $l, m \in \{1, \dots, k\}$ we have

$$i(a^\dagger b)_{lm} + (a^\dagger b)_{lm}i = i(a^\dagger b)_{ml} + (a^\dagger b)_{ml}i.$$

Rearranging, we see i and $(a^\dagger b)_{ml} - (a^\dagger b)_{lm}$ anti-commute. Thus, $(a^\dagger b)_{ml} - (a^\dagger b)_{lm}$ must be in the span of j and k . We can do the same thing with $ja^\dagger b + a^\dagger bj$ and $ka^\dagger b + a^\dagger bk$ to find that $(a^\dagger b)_{ml} - (a^\dagger b)_{lm} = 0$. That is, $a^\dagger b$ is symmetric, so $b^\dagger a$ is as well. $\square$

The ADHM transform from ADHM data to $\mathrm{Sp}(n)$ instantons is given as follows.

Theorem 3.1.4 (ADHM data to instantons). *Let $(a, b) \in \mathcal{N}_{n,k}$ and $x \in \mathbb{H}$. The kernel of $\Delta(x)^\dagger$ is n -dimensional, so there is a $V(x) \in \mathrm{Mat}(n+k, n, \mathbb{H})$ such that $V(x)^\dagger \Delta(x) = 0$ and $V(x)^\dagger V(x) = I_n$. Moreover, V can be chosen to be smooth. The connection defined by $\mathbb{A}_\mu(x) := V(x)^\dagger \partial_\mu V(x)$ is an instanton.*

I omit the proof of this theorem, the converse transform, and the completeness of the transform, as they are covered in great detail by others [CFTG78, CG84, CWS78].

Note 3.1.5. *There is a freedom in the choice of $V(x)$ above. Indeed, given a smooth function $g: \mathbb{H} \rightarrow \mathrm{Sp}(n)$, consider $V'(x) := V(x)g(x)$. Then $V'(x)$ spans the kernel of $\Delta(x)^\dagger$. The corresponding instanton is a gauge transformation of the instanton generated using V . Thus, the choice of $V(x)$ does not matter to the final instanton.*

3.1.2 Conformal transformations

In this section, we examine the conformal transformations on $\mathbb{R}^4$, equivalently S^4 . Moreover, we examine their effect on ADHM data and their corresponding instantons. For more information on conformal transformations on $\mathbb{R}^4$, see Wilker's work [Wil93].

Definition 3.1.6. *As in Wilker's work on Möbius groups, we adopt the convention that conjugation by 0 is the identity [Wil93, § 3]. The **conformal group** $\mathrm{Conf}(\mathbb{R}^4)$ is isomorphic to $\mathrm{SL}(2, \mathbb{H})$, the group of two by two quaternionic matrices given by*

$$\mathrm{SL}(2, \mathbb{H}) = \left\{ \begin{bmatrix} A & B \\ C & D \end{bmatrix} \mid |AC^{-1}DC - BC| = 1 \right\}. \quad (3.1.2)$$

Note that C can be zero, in which case the condition is $|AD| = 1$. This group acts on $S^4 \simeq \mathbb{R}^4 \cup \{\infty\} \simeq \mathbb{H} \cup \{\infty\}$ as

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot x := \begin{cases} (Ax + B)(Cx + D)^{-1} & \text{if } Cx + D \neq 0 \\ AC^{-1} & \text{if } x = \infty \text{ and } C \neq 0 \\ \infty & \text{otherwise} \end{cases}. \quad (3.1.3)$$

We need to understand how this group acts on ADHM data, given by Definition 3.1.1.

Definition 3.1.7. Let $(a, b) \in \mathcal{N}_{n,k}$ and $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{SL}(2, \mathbb{H})$. There is a smooth right Lie group action

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot (a, b) := (aD - bB, bA - aC). \quad (3.1.4)$$

This action is called the **conformal action** on $\mathcal{N}_{n,k}$.

We have a second action acting on the ADHM data.

Definition 3.1.8. The gauge group $\mathrm{Sp}(n+k) \times \mathrm{GL}(k, \mathbb{R})$ acts on $\mathcal{N}_{n,k}$ as

$$(Q, K) \cdot (a, b) := (QaK^{-1}, QbK^{-1}). \quad (3.1.5)$$

This action is called the **gauge action** and is a smooth left Lie group action.

These actions are named to reflect their effect on the corresponding instantons.

Lemma 3.1.9. Let $(a, b) \in \mathcal{N}_{n,k}$.

(1) For $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{SL}(2, \mathbb{H})$, let $(a', b') \in \mathcal{N}_{n,k}$ be given by $(a', b') := \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot (a, b)$. The corresponding instantons $\mathbb{A}$ and $\mathbb{A}'$ are related via the same conformal transformation:

$$\mathbb{A}' = \begin{bmatrix} A & B \\ C & D \end{bmatrix}^* \mathbb{A}. \quad (3.1.6)$$

(2) For $(Q, K) \in \mathrm{Sp}(n+k) \times \mathrm{GL}(k, \mathbb{R})$, the instantons corresponding to (a, b) and $(Q, K) \cdot (a, b)$ are identical.

(3) The gauge and conformal actions on $\mathcal{N}_{n,k}$ commute.

Proof. (1) Let $\Delta(x) := a - bx$ and $\Delta'(x) := a' - b'x$. Let $V(x) \in \mathrm{Mat}(n+k, n, \mathbb{H})$ such that $V(x)^\dagger \Delta(x) = 0$ and $V(x)^\dagger V(x) = I_n$. Then define $V'(x) := V \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot x \right)$. As

$\Delta'(x) = \Delta \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} .x \right) (Cx + D)$, we see that $V'(x)^\dagger \Delta'(x) = 0$ and $V'(x)^\dagger V'(x) = I_n$.

Therefore, we have

$$\mathbb{A}'_\mu(x) = V \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} .x \right)^\dagger \partial_\mu V \left(\begin{bmatrix} A & B \\ C & D \end{bmatrix} .x \right).$$

Hence, just as in Proposition 2.1.9, we have our result.

- (2) Let $(a, b) \in \mathcal{N}_{n,k}$ and for some $(Q, K) \in \mathrm{Sp}(n+k) \times \mathrm{GL}(k, \mathbb{R})$, let $(a', b') := (Q, K).(a, b)$. Let $V(x) \in \mathrm{Mat}(n+k, n, \mathbb{H})$ such that $V(x)^\dagger \Delta(x) = 0$ and $V(x)^\dagger V(x) = I_n$. Then define $V'(x) := QV(x)$. We see that

$$V'(x)^\dagger \Delta'(x) = V(x)^\dagger Q^\dagger (a' - b'x) = V(x)^\dagger \Delta(x) K^{-1} = 0.$$

Furthermore, we see that $V'(x)^\dagger V'(x) = I_n$. Therefore, we can use V and V' to construct the corresponding instantons.

Let $\mathbb{A}$ and $\mathbb{A}'$ be the instantons corresponding to (a, b) and (a', b') , respectively. Then

$$\mathbb{A}'_\mu(x) = V(x)^\dagger Q^\dagger \partial_\mu Q V(x) = \mathbb{A}_\mu(x).$$

Hence, $\mathbb{A}' = \mathbb{A}$.

- (3) This fact follows from the definition of the actions. □

We are only interested in instantons up to gauge. That is, we are interested in equivalence classes of instantons in the moduli space of instantons, given by $\mathcal{N}_{n,k}/(\mathrm{Sp}(n+k) \times \mathrm{GL}(k, \mathbb{R}))$. As the actions commute, the conformal action descends to an action on the moduli space. When searching for symmetric instantons, we are looking at fixed points of this action.

Every instanton has a standard form, induced by the gauge action.

Lemma 3.1.10. *To each $(a, b) \in \mathcal{N}_{n,k}$, there is a unique pair $(\hat{M}, U) \in \mathcal{N}_{n,k}$ given by $U := \begin{bmatrix} 0 \\ I_k \end{bmatrix}$ and $\hat{M} := \begin{bmatrix} L \\ M \end{bmatrix}$ whose lower block M is symmetric, such that $[(\hat{M}, U)] = [(a, b)]$.*

Definition 3.1.11. *The collection of $\hat{M}$ such that $(\hat{M}, U) \in \mathcal{N}_{n,k}$ is denoted $\tilde{\mathcal{N}}_{n,k}$ and instantons of this form are said to be in **standard form**.*

Proof. Let $(a, b) \in \mathcal{N}_{n,k}$. Given $(Q, K) \in \mathrm{Sp}(n+k) \times \mathrm{GL}(k, \mathbb{R})$, consider $(a', b') := (Q, K).(a, b)$. Under this transformation, we see that $(b')^\dagger b' = K^T b^\dagger b K$. By Lemma 3.1.3, we know that $b^\dagger b$ is a real, symmetric, and positive definite matrix. Thus, it is orthogonally diagonalizable with real eigenvalues. That is, there is some $J \in \mathrm{O}(k)$ and $\lambda_1, \dots, \lambda_k > 0$ such that $J^T b^\dagger b J = \mathrm{diag}(\lambda_1, \dots, \lambda_k)$. Let $D := \mathrm{diag}(1/\sqrt{\lambda_1}, \dots, 1/\sqrt{\lambda_k})$. Letting $K := JD \in \mathrm{GL}(k, \mathbb{R})$ and $Q = I_{n+k}$, we note that $K^T b^\dagger b K = I_k$. Thus, (a, b) is gauge equivalent to some ADHM data with $(b')^\dagger b' = I_k$.

Now, suppose that $(a, b) \in \mathcal{N}_{n,k}$ such that $b^\dagger b = I_k$. Let $\tilde{Q}$ is a $n \times (n+k)$ quaternionic matrix satisfying $\tilde{Q}\tilde{Q}^\dagger = I_n$ and $\tilde{Q}b = 0$. These two equations impose $n(n+k)$ conditions on $\tilde{Q}$, so such a $\tilde{Q}$ exists. Then let $Q := \begin{bmatrix} \tilde{Q} \\ b^\dagger \end{bmatrix} \in \mathrm{Sp}(n+k)$ and $Qb = \begin{bmatrix} 0 \\ I_k \end{bmatrix}$. Letting $K = I_k$, we see that (a, b) is gauge equivalent to ADHM data with $b = \begin{bmatrix} 0 \\ I_k \end{bmatrix}$.

Finally, suppose that $(a, b) \in \mathcal{N}_{n,k}$ such that $b = \begin{bmatrix} 0 \\ I_k \end{bmatrix}$. Write $a = \begin{bmatrix} L \\ M \end{bmatrix}$. By Lemma 3.1.3, we know that $M = b^\dagger a$ is symmetric. Combining results, we have that any ADHM is equivalent to an element of $\tilde{\mathcal{N}}_{n,k}$. $\square$

The gauge action does not physically alter the instanton, so if a conformal action can be undone by a gauge action, then the conformal action physically does not alter the instanton.

Definition 3.1.12. *Given $\tilde{A} \in \mathrm{SL}(2, \mathbb{H})$, ADHM data $\hat{M} \in \tilde{\mathcal{N}}_{n,k}$ is said to be $\tilde{A}$ -equivariant if $\tilde{A}.[(\hat{M}, U)] = [(\hat{M}, U)]$, that is, if there exists some $(Q, K) \in \mathrm{Sp}(n+k) \times \mathrm{GL}(k, \mathbb{R})$ such that $(Q, K).(\hat{M}, U) = \tilde{A}.(\hat{M}, U)$. Note that we can extend this concept to $\mathcal{N}_{n,k}$.*

Note that, since the gauge and conformal actions commute, if two instantons are related by a gauge transformation, then they are both equivariant under the same conformal transformations.

In order to use Theorem 1.1.1, we need the group of symmetries to be compact. Right now, our symmetry group is $\mathrm{SL}(2, \mathbb{H})$, which is not compact. Just as we focus on orthogonal transformations when discussing symmetries of monopoles, we wish to reduce our symmetry group to a compact one. To that end, we investigate the symmetries of the basic instanton.

Proposition 3.1.13. *Consider the basic instanton $\hat{M} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \in \tilde{\mathcal{N}}_{1,1}$. The subgroup $H_{\hat{M}} \subseteq \mathrm{SL}(2, \mathbb{H})$ of symmetries of the instanton is exactly $\mathrm{Sp}(2)$.*

Proof. Suppose that $(\hat{M}, U)$ is equivariant under $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{SL}(2, \mathbb{H})$. Then there exists $Q \in \mathrm{Sp}(2)$ and $K \in \mathrm{GL}(1, \mathbb{R})$ such that $(Q, K) \cdot \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot (\hat{M}, U) = (\hat{M}, U)$. Thus,

$$\begin{aligned} \begin{bmatrix} D \\ -B \end{bmatrix} K^{-1} &= Q^\dagger \begin{bmatrix} 1 \\ 0 \end{bmatrix}; \\ \begin{bmatrix} -C \\ A \end{bmatrix} K^{-1} &= Q^\dagger \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Solving these equations, we see that $Q = \frac{1}{K} \begin{bmatrix} D^\dagger & -B^\dagger \\ -C^\dagger & A^\dagger \end{bmatrix}$. As $Q \in \mathrm{Sp}(2)$, we see that

$$I_2 = QQ^\dagger = \frac{1}{K^2} \begin{bmatrix} |D|^2 + |B|^2 & -D^\dagger C - B^\dagger A \\ -C^\dagger D - A^\dagger B & |A|^2 + |C|^2 \end{bmatrix}. \quad (3.1.7)$$

As $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{SL}(2, \mathbb{H})$, we have that $|AC^{-1}DC - BC| = 1$. If $C = 0$, we have that $|AD| = 1$. Furthermore, we have from the above identities that $A^\dagger B = 0$ and $|A|^2 = K^2$. Thus, $B = 0$, so $|D|^2 = K^2$. Hence, $1 = |AD| = K^2$.

If $C \neq 0$, using (3.1.7), we see that as $|AC^{-1}DC - BC| = 1$,

$$1 = \frac{|A|^2 + |C|^2}{|C|} |B| = K^2 \frac{|B|}{|C|}.$$

That is, $K^2 |B| = |C|$. Hence,

$$I_2 = Q^\dagger Q = \frac{1}{K^2} \begin{bmatrix} |D|^2 + |C|^2 & -DB^\dagger - CA^\dagger \\ -BD^\dagger - AC^\dagger & |A|^2 + |B|^2 \end{bmatrix}.$$

Thus, we see that $|A|^2 + |B|^2 = K^2 = |A|^2 + |C|^2$, so $|B| = |C|$. Thus, $K^2 = 1$.

In either case, $K^2 = 1$. Thus, we see that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^\dagger = \begin{bmatrix} |A|^2 + |B|^2 & AC^\dagger + BD^\dagger \\ CA^\dagger + DB^\dagger & |C|^2 + |D|^2 \end{bmatrix} = I_2.$$

That is, $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(2)$.

Conversely, given $\begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(2)$, we define $Q := \begin{bmatrix} D^\dagger & -B^\dagger \\ -C^\dagger & A^\dagger \end{bmatrix} \in \mathrm{Sp}(2)$. Then

$$(Q, 1) \cdot \begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot (\hat{M}, U) = (\hat{M}, U).$$

As the transformation was arbitrary, $H_{\hat{M}} = \mathrm{Sp}(2)$. $\square$

Proposition 3.1.13 tells us that the symmetry group of the basic instanton is $\mathrm{Sp}(2)$. Thus, instantons related by a conformal transformation $\tilde{A} \in \mathrm{SL}(2, \mathbb{H})$ have a symmetry group of the form $\tilde{A}^{-1}\mathrm{Sp}(2)\tilde{A}$. However, it is not known if similar results hold for all non-flat instantons. Based on this result, we make the following conjecture.

Conjecture 3.1.14. *Suppose that $\mathbb{A}$ is a non-flat instanton and define $H_{\mathbb{A}} \subseteq \mathrm{SL}(2, \mathbb{H})$ to be its group of symmetries. Then there is some $\tilde{A} \in \mathrm{SL}(2, \mathbb{H})$ such that $\tilde{A}H_{\mathbb{A}}\tilde{A}^{-1} \subseteq \mathrm{Sp}(2)$.*

Assuming this conjecture, when discussing symmetric instantons, we need only examine subgroups of $\mathrm{Sp}(2)$. In Appendix C, we classify all these subgroups.

Note 3.1.15. *The validity of this conjecture has no impact on the following results outside of Lemma 3.2.12, on which nothing else relies. If it is false, then there are other kinds of symmetric instantons and perhaps there are other kinds of circle actions relating instantons and hyperbolic monopoles.*

In an effort to ignore uninteresting instantons, we focus on the following subset of ADHM data.

Definition 3.1.16. *Let $\mathcal{M}_{n,k}$ be the set of $\hat{M} \in \tilde{\mathcal{N}}_{n,k}$ such that*

- M (a $k \times k$ quaternionic matrix) is symmetric,

- L ($a n \times k$ quaternionic matrix) is such that $LL^\dagger$ is a positive definite matrix (we already know it is positive semi-definite),
- $R := \Delta(0)^\dagger \Delta(0) = L^\dagger L + M^\dagger M$ is real and non-singular,
- let $\Delta(x) := \begin{bmatrix} L \\ M - I_k x \end{bmatrix}$, for all $x \in \mathbb{H}$, then $\Delta(x)^\dagger \Delta(x)$ is non-singular for all $x \in \mathbb{H}$.

Note 3.1.17. Given the form of R , it is real and non-singular if and only if it is symmetric and non-singular. Additionally, if R is real, the $\Delta(x)^\dagger \Delta(x)$ is automatically real.

Recall Note 2.1.2, which tells us that k indicates both the instanton number and the standard quaternion ($k = ij$) and its use at any given time is contextual.

The condition $LL^\dagger$ is positive definite forces $n \leq k$ and is a natural condition for finding novel instantons.

Proposition 3.1.18. Suppose that $\hat{M}$ satisfies all of the above conditions of $\mathcal{M}_{n,k}$ in Definition 3.1.16, except $LL^\dagger$ is not positive definite. Then there is some L' such that up to gauge, $L = \begin{bmatrix} L' \\ 0 \end{bmatrix}$. We can do this for more rows until $L'L'^\dagger$ is a positive definite $n' \times n'$ matrix. Then $\hat{M}' := \begin{bmatrix} L' \\ M \end{bmatrix} \in \mathcal{M}_{n',k}$. Letting $\mathbb{A}'$ and $\mathbb{A}$ be the instantons corresponding to $\hat{M}'$ and $\hat{M}$, respectively, we have that $\mathbb{A}_\mu = \mathbb{A}'_\mu \oplus 0$. Thus, we have an instanton $\tilde{A}$ embedded trivially as an instanton A with a higher rank structure group. We ignore these as they are reducible.

Proof. By hypothesis, there is a vector $v \neq 0$ such that $|L^\dagger v|^2 = v^\dagger LL^\dagger v = 0$. Choose an orthonormal basis $\{e_1, \dots, e_{n-1}, v/|v|\}$. Let $q^\dagger = \begin{bmatrix} e_1 & \cdots & e_{n-1} & v/|v| \end{bmatrix}$. Thus, $q \in \text{Sp}(n)$ and as $v^\dagger L = 0$,

$$qL = \begin{bmatrix} e_1^\dagger L \\ \vdots \\ e_{n-1}^\dagger L \\ 0 \end{bmatrix}.$$

Thus, we have a zero row in qL . We can do more rows by repeating this process.

Considering the second statement, suppose $L = \begin{bmatrix} L' \\ 0 \end{bmatrix}$, with $L'L^\dagger$ positive definite.

Suppose further that $\tilde{V}(x) = \begin{bmatrix} \psi'(x) \\ v'(x) \end{bmatrix}$ is the data that gives us $\mathbb{A}'_\mu$ for $\hat{M}'$. Then let

$$V(x) := \begin{bmatrix} \psi'(x) & 0 \\ 0 & I_m \\ v'(x) & 0 \end{bmatrix}. \text{ We see that } V(x) \text{ is normalized and satisfies } V(x)^\dagger \Delta(x) = 0.$$

Therefore, we see that $\mathbb{A}_\mu(x) = \mathbb{A}'_\mu(x) \oplus 0$. Thus, the instanton is indeed just an embedding of one with a smaller rank structure group. $\square$

Unlike the monopole case, there are a variety of connected Lie subgroups of $\mathrm{Sp}(2)$.

Definition 3.1.19. *Following Conjecture 3.1.14, for $\hat{M} \in \mathcal{M}_{n,k}$, let $H_{\hat{M}} \subseteq \mathrm{Sp}(2)$ be the set of conformal transformations under which $\hat{M}$ is equivariant. As we are dealing with group actions, $H_{\hat{M}}$ is a subgroup. We are interested in the case that $H_{\hat{M}}$ is a connected Lie subgroup. In Section C.2 in Appendix C, we classify all the different subgroups H of interest.*

When $H_{\hat{M}}$ contains a subgroup conjugate to one of the groups listed in Table C.1, we say that $\hat{M}$ has the corresponding symmetry given in the table.

3.1.3 A herd of lemmas

In this section, we prove some lemmas that prove to be quite useful in our investigation of symmetric instantons.

In order to use Theorem 1.1.1, we need the gauge group to be compact. Our gauge group is $\mathrm{Sp}(n+k) \times \mathrm{GL}(k, \mathbb{R})$, which is not compact. However, just like how we could ignore part of the equivariance data for hyperbolic monopoles, it turns out that when investigating $\tilde{A}$ -equivariance, K is determined uniquely by Q and $\tilde{A}$. Thus, we can ignore the non-compact part of the gauge group, leaving us with a compact one.

Lemma 3.1.20. *Let $\hat{M} \in \mathcal{M}_{n,k}$ be $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ -equivariant. More specifically, let $(Q, K) \in \mathrm{Sp}(n+k) \times \mathrm{GL}(k, \mathbb{R})$ be a gauge transformation realizing this equivariance, that is*

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \cdot (Q, K) \cdot (\hat{M}, U) = (\hat{M}, U).$$

Then K is determined by

$$K = U^T Q(UA - \hat{M}C). \quad (3.1.8)$$

Proof. The equivariance implies $Q(UA - \hat{M}C)K^{-1} = U$. The result follows. $\square$

Note 3.1.21. We see that K depends smoothly on Q and $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$.

Above, we reduced the gauge group from $\mathrm{Sp}(n+k) \times \mathrm{GL}(k, \mathbb{R})$ to $\mathrm{Sp}(n+k)$, which is compact. The following result tells us that when dealing with isometries, we may choose to reduce the gauge group in a different way, reducing it to $\mathrm{O}(k)$, which is much smaller than $\mathrm{Sp}(n+k)$.

Given that we focus on symmetries contained in $\mathrm{Sp}(2)$ and Definition 3.1.6 tells how this group acts on S^4 , we see that the only isometries on $\mathbb{R}^4$ in this group belong to the subgroup $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \subseteq \mathrm{Sp}(2)$. Note that $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \simeq \mathrm{Spin}(4)$, which is the double cover of $\mathrm{SO}(4)$.

Lemma 3.1.22. Let $\hat{M} \in \mathcal{M}_{n,k}$. Let $\mathrm{diag}(a, b) \in \mathrm{Sp}(1) \times \mathrm{Sp}(1) \subseteq \mathrm{Sp}(2)$ such that $\hat{M}$ is $\mathrm{diag}(a, b)$ -equivariant, specifically, let $(Q, K) \in \mathrm{Sp}(n+k) \times \mathrm{GL}(k, \mathbb{R})$ be such that

$$\mathrm{diag}(a, b).(Q, K).(\hat{M}, U) = (\hat{M}, U).$$

Then, $K \in \mathrm{O}(k)$ and there is some $q \in \mathrm{Sp}(n)$ such that $Q = \mathrm{diag}(q, a^\dagger K)$.

Proof. Suppose that $\mathrm{diag}(a, b).(Q, K).(\hat{M}, U) = (\hat{M}, U)$. Then $QUaK^{-1} = U$. Simplifying, we have that Q must have the form given in the statement. That $K \in \mathrm{O}(k)$ is because $Q \in \mathrm{Sp}(n+k)$. $\square$

When discussing hyperbolic monopoles, we had a different definition for equivariance, Definition 2.1.12. When discussing rotations of the form $x \mapsto p x p^\dagger$ for some $p \in \mathrm{Sp}(1)$, these definitions are equivalent. We continue to use Definition 3.1.12, as it allows us to consider additional transformations.

Lemma 3.1.23. We have that $\hat{M}$ is p -equivariant, in the sense given in Definition 2.1.12, if and only if $(\hat{M}, U)$ is $\mathrm{diag}(p^\dagger, p^\dagger)$ -equivariant, in the sense given in Definition 3.1.12.

Note 3.1.24. We were able to use Definition 2.1.12 for hyperbolic monopoles as we only considered rotations of $\mathfrak{sp}(1)$. The other definition also allowed us to ignore the U part of ADHM data in standard form.

Proof. Indeed, suppose that $\hat{M}$ is p -equivariant as in Definition 2.1.12. Then there is some $(q, K) \in \mathrm{Sp}(n) \times \mathrm{O}(k)$ such that

$$\begin{bmatrix} q & 0 \\ 0 & K \end{bmatrix} \hat{M} K^{-1} = p \hat{M} p^\dagger.$$

Let $Q := \mathrm{diag}(p^\dagger q, p^\dagger K) \in \mathrm{Sp}(n+k)$. We see that

$$(Q, K) \cdot (\hat{M}, U) = \begin{bmatrix} p^\dagger & 0 \\ 0 & p^\dagger \end{bmatrix} \cdot (\hat{M}, U).$$

Conversely, if $\hat{M}$ is $\mathrm{diag}(p^\dagger, p^\dagger)$ -equivariant, as in Definition 3.1.12, then by Lemma 3.1.22, $K \in \mathrm{O}(k)$ and there is some $q \in \mathrm{Sp}(n)$ such that Q is of the form $Q = \mathrm{diag}(q, p^\dagger K)$. Thus, we have that

$$\begin{bmatrix} pq & 0 \\ 0 & K \end{bmatrix} \hat{M} K^{-1} = p \hat{M} p^\dagger.$$

That is, $\hat{M}$ is p -equivariant, as in Definition 2.1.12. □

Lemma 3.1.20 tells us that Q and the conformal transformation in $\mathrm{Sp}(2)$ determine K . Furthermore, Lemma 3.1.22 tells us that if the conformal transformation is actually an isometry, then $K \in \mathrm{O}(k)$ and Q must be block diagonal. For such transformations, K determines Q . This result is similar to Lemma 2.1.14 in Section 2.1.

Lemma 3.1.25. *Suppose $\hat{M} \in \mathcal{M}_{n,k}$, $\mathrm{diag}(a, b) \in \mathrm{Sp}(2)$, and suppose there is some $K \in \mathrm{O}(k)$ such that $a^\dagger K M K^T = M b^\dagger$. If $[K, R] = 0$, then there exists a unique $q \in \mathrm{Sp}(n)$ such that $q L K^T = L b^\dagger$. That is, $\hat{M}$ is equivariant under $\mathrm{diag}(a, b)$. Specifically,*

$$(\mathrm{diag}(L b^\dagger K L^\dagger (L L^\dagger)^{-1}, a^\dagger K), K) \cdot \mathrm{diag}(a, b) \cdot (\hat{M}, U) = (\hat{M}, U). \quad (3.1.9)$$

Proof. If q and $\tilde{q}$ satisfy the constraint, then $(q - \tilde{q}) L K^T = 0$. As $L L^\dagger$ is positive definite, then $q = \tilde{q}$, so q is unique.

Consider $q := Lb^\dagger KL^\dagger(LL^\dagger)^{-1}$. We see that

$$\begin{aligned}
q^\dagger q &= (LL^\dagger)^{-1} LK^T bL^\dagger Lb^\dagger KL^\dagger (LL^\dagger)^{-1} \\
&= (LL^\dagger)^{-1} LK^T b(R - M^\dagger M) b^\dagger KL^\dagger (LL^\dagger)^{-1} \\
&= (LL^\dagger)^{-1} LK^T (R - bM^\dagger a^\dagger aMb^\dagger) KL^\dagger (LL^\dagger)^{-1} \\
&= (LL^\dagger)^{-1} LK^T (R - KM^\dagger MK^T) KL^\dagger (LL^\dagger)^{-1} \\
&= (LL^\dagger)^{-1} L(R - M^\dagger M) L^\dagger (LL^\dagger)^{-1} \\
&= (LL^\dagger)^{-1} LL^\dagger LL^\dagger (LL^\dagger)^{-1} \\
&= I_n.
\end{aligned}$$

Hence, $q \in \text{Sp}(n)$.

We see that

$$\begin{aligned}
Lb^\dagger KL^\dagger LL^\dagger &= Lb^\dagger K(R - M^\dagger M) L^\dagger \\
&= L(Rb^\dagger K - b^\dagger KM^\dagger K^T KMK^T K) L^\dagger \\
&= L(Rb^\dagger - b^\dagger bM^\dagger a^\dagger aMb^\dagger) KL^\dagger \\
&= L(R - M^\dagger M) b^\dagger KL^\dagger \\
&= LL^\dagger Lb^\dagger KL^\dagger,
\end{aligned}$$

so $[Lb^\dagger KL^\dagger, LL^\dagger] = 0$. Therefore, $q = (LL^\dagger)^{-1} Lb^\dagger KL^\dagger$, so as $a^\dagger KMK^T = Mb^\dagger$ and $[K, R] = 0$, we have

$$\begin{aligned}
qLK^T &= (LL^\dagger)^{-1} Lb^\dagger KL^\dagger LK^T \\
&= (LL^\dagger)^{-1} Lb^\dagger K(R - M^\dagger M) K^T \\
&= (LL^\dagger)^{-1} Lb^\dagger (R - bM^\dagger a^\dagger aMb^\dagger) \\
&= (LL^\dagger)^{-1} L(R - M^\dagger M) b^\dagger \\
&= (LL^\dagger)^{-1} LL^\dagger Lb^\dagger \\
&= Lb^\dagger.
\end{aligned}$$

Taking $Q := \text{diag}(q, a^\dagger K) \in \text{Sp}(2)$, we have that $(Q, K) \cdot \text{diag}(a, b) \cdot (\hat{M}, U) = (\hat{M}, U)$. That is, $\hat{M}$ is $\text{diag}(a, b)$ -equivariant. $\square$

Note 3.1.26. We see that q is smooth in K and the element in $\text{Sp}(1) \times \text{Sp}(1)$.

The following result allows us to narrow the search for instantons equivariant under isometries.

Corollary 3.1.27. *Suppose $\hat{M} \in \mathcal{M}_{n,k}$ is $\text{diag}(a, b)$ -equivariant. That is, there is some $(Q, K) \in \text{Sp}(n+k) \times \text{GL}(k, \mathbb{R})$ such that $(Q, K) \cdot \text{diag}(a, b) \cdot (\hat{M}, U) = (\hat{M}, U)$. Then $K \in \text{O}(k)$, $[R, K] = 0$, and $Q = \text{diag}(Lb^\dagger K L^\dagger (LL^\dagger)^{-1}, a^\dagger K)$.*

Proof. Suppose that $(Q, K) \in \text{Sp}(n+k) \times \text{GL}(k, \mathbb{R})$ such that $(Q, K) \cdot \text{diag}(a, b) \cdot (\hat{M}, U) = (\hat{M}, U)$. By Lemma 3.1.22, $K \in \text{O}(k)$ and there is some $q \in \text{Sp}(n)$ such that $Q = \text{diag}(q, a^\dagger K)$.

We see that

$$K R K^T = K L^\dagger q^\dagger q L K^T + K M^\dagger K^T a a^\dagger K M K^T.$$

Since $(Q, K) \cdot (\hat{M}, U) = \text{diag}(a^\dagger, b^\dagger) \cdot (\hat{M}, U)$ and R is real, we have

$$K R K^T = b R b^\dagger = R.$$

Hence, $[K, R] = 0$. Then, by Lemma 3.1.25, we have that $q = L b^\dagger K L^\dagger (L L^\dagger)^{-1}$. $\square$

In the subsequent sections, we investigate equivariant instantons. In doing so, we find that the continuous symmetries of instantons are generated by Lie algebra representations. We see that instantons generated by isomorphic representations are gauge equivalent, so we need only investigate a single representation from every isomorphism class. Moreover, it does not matter which representative we choose.

Lemma 3.1.28. *Let $\mathfrak{g} \subseteq \mathfrak{sp}(2)$ be a Lie subalgebra and $\hat{M} \in \mathcal{M}_{n,k}$. Let $\rho, \tilde{\rho}: \mathfrak{g} \rightarrow \mathfrak{sp}(n+k)$ and $\lambda, \tilde{\lambda}: \mathfrak{g} \rightarrow \mathfrak{so}(k)$ be pairs of isomorphic representations of $\mathfrak{g}$. That is, there is some $P \in \text{Sp}(n+k)$ and $\Lambda \in \text{O}(k)$ such that for all $x \in \mathfrak{g}$, $\tilde{\rho}(x) = P^\dagger \rho(x) P$ and $\tilde{\lambda}(x) = \Lambda^T \lambda(x) \Lambda$.*

Let $(\hat{M}', U') := (P, \Lambda) \cdot (\hat{M}, U) \in \mathcal{N}_{n,k}$. Then, for all $x \in \mathfrak{g}$, we have that

$$\left(e^{\tilde{\rho}(x)}, e^{\tilde{\lambda}(x)} \right) \cdot e^x \cdot (\hat{M}, U) = (\hat{M}, U), \quad (3.1.10)$$

if and only if

$$\left(e^{\rho(x)}, e^{\lambda(x)} \right) \cdot e^x \cdot (\hat{M}', U') = (\hat{M}', U'). \quad (3.1.11)$$

Proof. Consider (3.1.10). Using the relationship between the representations, we see that

$$\left(P^\dagger e^{\rho(x)} P, \Lambda^T e^{\lambda(x)} \Lambda \right) \cdot e^x \cdot (\hat{M}, U) = (\hat{M}, U).$$

As the actions commute, we see that upon simplification we obtain (3.1.11). Thus, the equations are equivalent. $\square$

3.1.4 A discontinuous group action

In this section, we discuss inversion, a discontinuous group action. While we do not make use of this action elsewhere in this thesis, it is presented for readers interested in other kinds of symmetry.

Definition 3.1.29. Let $\mathbb{Z}_2$ act on $\mathcal{M}_{n,k}$ as follows: for $\hat{M} \in \mathcal{M}$, define

$${}^1\hat{M} := \hat{M}, \quad {}^{-1}\hat{M} := \begin{bmatrix} L \\ -M \end{bmatrix}. \quad (3.1.12)$$

This action is called **inversion**.

Proposition 3.1.30. If $\mathbb{A}(x)$ is the instanton associated to $\hat{M} \in \mathcal{M}_{n,k}$, then $-\mathbb{A}(-x)$ is the instanton associated to ${}^{-1}\hat{M}$.

Proof. We know there is a $(n+k) \times n$ quaternionic matrix $V(x) := \begin{bmatrix} \psi(x) \\ W(x) \end{bmatrix}$ such that $V(x)^\dagger \Delta(x) = 0$ and $V(x)^\dagger V(x) = I_n$. Expanding, we see that

$$\psi(x)^\dagger L + W(x)^\dagger (M - I_k x) = 0; \quad \psi(x)^\dagger \psi(x) + W(x)^\dagger W(x) = I_n.$$

Let $\tilde{\Delta}(x) := \begin{bmatrix} L \\ -M - x I_k \end{bmatrix}$ and $\tilde{\mathbb{A}}$ be the instanton associated to ${}^{-1}\hat{M}$. Let $\tilde{V}(x) := \begin{bmatrix} -\psi(-x) \\ W(-x) \end{bmatrix}$. Then we see that as the above equations hold for all $x \in \mathbb{R}^4$,

$$\begin{aligned} \tilde{V}(x)^\dagger \tilde{\Delta}(x) &= \begin{bmatrix} -\psi(-x)^\dagger & W(-x)^\dagger \end{bmatrix} \begin{bmatrix} L \\ -M - I_k x \end{bmatrix} \\ &= -(\psi(-x)^\dagger L + W(-x)^\dagger (M + I_k x)) = 0 \\ \tilde{V}(x)^\dagger \tilde{V}(x) &= \psi(-x)^\dagger \psi(-x) + W(-x)^\dagger W(-x) = I_n. \end{aligned}$$

We have that $\mathbb{A}_\mu(x) = V(x)^\dagger \partial_\mu V(x)$. Hence,

$$\mathbb{A}_\mu(x) = \psi(x)^\dagger \partial_\mu \psi(x) + W(x)^\dagger \partial_\mu W(x).$$

Thus, we see that

$$\tilde{\mathbb{A}}_\mu(x) = \psi(-x)^\dagger \partial_\mu (\psi(-x)) + W(-x)^\dagger \partial_\mu (W(-x)) = -\mathbb{A}_\mu(-x),$$

by the chain rule. □

3.2 Circular symmetry

In this section, we find an equation describing all instantons with circular t -symmetry, as given in Table C.1 as well as below. We then discuss the connections between such instantons and hyperbolic and singular monopoles.

First, we reintroduce the notion of circular t -symmetry. For $t \in [0, 1]$, let $R_t := \{\text{diag}(e^{i\theta}, e^{ti\theta}) \mid \theta \in \mathbb{R}\}$. An instanton is said to have circular t -symmetry if it is equivariant under all the transformations of R_t . Note that for $t \in \mathbb{Q} \cap [0, 1]$, $R_t \simeq S^1$ and for $t \in [0, 1] \setminus \mathbb{Q}$, $R_t \simeq \mathbb{R}$.

Note that unlike other uses of Theorem 1.1.1, when $t \notin \mathbb{Q}$, $R_t \simeq \mathbb{R}$, which is simply-connected but not compact. Thankfully, the fact that R_t is one-dimensional helps us overcome this difficulty. However, due to this lack of compactness, we must diverge slightly from the proof of Theorem 1.1.1 to prove the following result.

Theorem 3.2.1. *Let $\hat{M} \in \mathcal{M}_{n,k}$ and let $t \in [0, 1]$. Then $\hat{M}$ has circular t -symmetry if and only if there exists $\rho \in \mathfrak{so}(k)$ such that*

$$tMi - iM + [\rho, M] = 0; \quad (3.2.1)$$

$$[\rho, R] = 0. \quad (3.2.2)$$

Definition 3.2.2. *The matrix ρ is called the **generator** of the circular t -symmetry of $\hat{M}$.*

Note 3.2.3. *All connected Lie subgroups of $\text{Sp}(2)$ with Lie algebra $\mathbb{R}$ are conjugate to some R_t . That is, of the form $AR_tA^\dagger$, for some $A \in \text{Sp}(2)$. Instantons equivariant under this group are of the form $A^\dagger \cdot (\hat{M}, U)$, with $\hat{M}$ equivariant under R_t .*

Proof. Let $\mathcal{X}$ be the smooth manifold comprised of all pairs of $(n+k) \times k$ quaternionic matrices. The gauge and conformal actions can easily be expanded from $\mathcal{M}_{n,k}$ to smooth actions on $\mathcal{X}$. Just as with $\mathcal{M}_{n,k}$, the conformal action on $\mathcal{X}$ descends to an action on $\mathcal{X}/(\text{Sp}(n+k) \times \text{GL}(k, \mathbb{R}))$.

Suppose that $\hat{M}$ has circular t -symmetry. Let $S \subseteq R_t \times \text{O}(k)$ be the stabilizer group of $(\hat{M}, U)$ restricted to rotations in R_t . That is

$$\begin{aligned} S := \{(\text{diag}(e^{i\theta}, e^{ti\theta}), K) \mid (\text{diag}(Le^{-ti\theta}KL^\dagger(LL^\dagger)^{-1}, e^{-i\theta}K), K) \cdot \text{diag}(e^{i\theta}, e^{ti\theta}) \cdot (\hat{M}, U) \\ = (\hat{M}, U)\}. \end{aligned} \quad (3.2.3)$$

Indeed, if $(Q, K) \in \text{Sp}(n+k) \times \text{GL}(k, \mathbb{R})$ such that $\text{diag}(e^{i\theta}, e^{ti\theta}) \cdot (Q, K) \cdot (\hat{M}, U) = (\hat{M}, U)$, then Corollary 3.1.27 tells us that $K \in \text{O}(k)$, $[R, K] = 0$, and Q is given by the diagonal

matrix in (3.2.3). Therefore, the pairs in S encapsulate all the circular t -symmetry of the instanton. That S is a group follows from this.

As $\hat{M}$ has circular t -symmetry, the map $\pi_1|_S: S \rightarrow R_t$ is surjective. Our goal is to find a smooth map $\mathbb{R} \rightarrow S$ that, when composed with $\pi_1|_S$, gives us the covering map $\pi: \mathbb{R} \rightarrow R_t$. This map allows us to differentiate the equation for equivariance and arrive at (3.2.1).

We start by showing that S is a Lie group. Let $f: R_t \times O(k) \rightarrow \mathcal{X}$ be the smooth map

$$f(\text{diag}(e^{i\theta}, e^{ti\theta}), K) := (\text{diag}(Le^{-ti\theta}KL^\dagger(LL^\dagger)^{-1}, e^{-i\theta}K), K) \cdot \text{diag}(e^{i\theta}, e^{ti\theta}) \cdot (\hat{M}, U).$$

Then $S = f^{-1}(\hat{M}, U)$, meaning S is closed. By the Closed-subgroup Theorem, as S is a closed subgroup of a Lie group, S is a Lie group [Lee12, Theorem 20.12].

Let e be the identity of $R_t \times O(k)$. Consider the Lie algebra homomorphism $\phi: \text{Lie}(S) \rightarrow \mathbb{R}$ given by $\phi := d_e(\pi_1|_S)$. Note that $\ker(\phi) \subseteq \text{Lie}(S)$ is an ideal of $\text{Lie}(S)$. Suppose that $\ker(\phi) = \text{Lie}(S)$. Then as $\text{im}(\pi_1|_S) = R_t$ is closed, by the isomorphism theorems,

$$0 \simeq \text{Lie}(S)/\ker(\phi) \simeq \text{im}(\phi) = \text{Lie}(\text{im}(\pi_1|_S)) = \mathbb{R}.$$

Contradiction! Thus, there exists $\xi := (\xi_0, \xi_1) \in \text{Lie}(S)$ such that $\xi_0 \neq 0$. That is, $\xi \notin \ker(\phi)$. Let $\psi: \mathbb{R} \rightarrow \langle \xi \rangle \subseteq \text{Lie}(S)$ be the Lie algebra homomorphism defined by $\psi(\alpha) := \alpha\xi$. Note that ξ spans the image of ϕ , so it is unique up to a $\mathbb{R}^*$ factor.

Let $Y \subseteq S$ be the unique connected Lie subgroup corresponding to the Lie algebra $\langle \xi \rangle \subseteq \text{Lie}(S)$, whose existence is guaranteed by the Subgroups-subalgebras Theorem [Lee12, Theorem 19.26]. As $\mathbb{R}$, the universal cover of R_t , is simply-connected, the Homomorphisms Theorem tells us that there is a unique Lie group homomorphism $\Psi: \mathbb{R} \rightarrow Y$ such that $d_0\Psi = \psi$ [Lee12, Theorem 20.19]. Consider the map $\pi_1|_S \circ \Psi: \mathbb{R} \rightarrow R_t$. We know that $d_0(\pi_1|_S \circ \Psi) = \phi \circ \psi$. Note that $\phi(\psi(\alpha)) = \alpha\phi(\xi)$. As $\phi(\xi) \neq 0$, $\phi \circ \psi$ is a Lie algebra isomorphism. Call the inverse of this map g , which we use to find the smooth map we desire.

By the Homomorphisms Theorem, we know that there is a unique Lie group homomorphism $G: \mathbb{R} \rightarrow \mathbb{R}$ such that $d_0G = g$ [Lee12, Theorem 20.19]. We show that $\Psi \circ G$ is the map that we are searching for. Indeed, we have that

$$d_0(\pi_1|_S \circ \Psi \circ G) = \phi \circ \psi \circ g = \text{id}_{\mathbb{R}}.$$

But $\pi: \mathbb{R} \rightarrow R_t$ is a Lie group homomorphism whose pushforward at the identity is the identity. By the Homomorphisms Theorem, the two maps must be equal, so $\pi_1|_S \circ \Psi \circ G = \pi$ [Lee12, Theorem 20.19].

As $\psi \circ g: \mathbb{R} \rightarrow \text{Lie}(S)$ is the right inverse to $\phi = d_e \pi_1|_S$, we know that $\psi \circ g(x) = (x, \rho x)$, for some $\rho \in \mathfrak{so}(k)$.

For all $\theta \in \mathbb{R}$, we have $\Psi(G(\theta)) \in Y \subseteq S$. Noting that $\pi_1|_S \circ \Psi \circ G = \pi$, we see that

$$(\text{diag}(Le^{-ti\theta}e^{\rho\theta}L^\dagger(LL^\dagger)^{-1}, e^{-i\theta}e^{\rho\theta}), e^{\rho\theta}).\text{diag}(e^{i\theta}, e^{ti\theta}).(\hat{M}, U) = (\hat{M}, U).$$

Differentiating and evaluating at $\theta = 0$, we obtain

$$\begin{bmatrix} -tLiL^\dagger(LL^\dagger)^{-1}L + L\rho L^\dagger(LL^\dagger)^{-1}L - L\rho + tLi \\ tMi - iM + [\rho, M] \end{bmatrix} = 0. \quad (3.2.4)$$

Focusing on the bottom row, we see that we get (3.2.1). Furthermore, as mentioned above, Corollary 3.1.27 tells us that $[R, e^{\rho\theta}] = 0$, for all $\theta \in \mathbb{R}$. Differentiating and evaluating at $\theta = 0$, we have $[R, \rho] = 0$.

Conversely, suppose that the equations hold for some $\rho \in \mathfrak{so}(k)$. Let $\text{diag}(e^{i\theta_0}, e^{ti\theta_0}) \in R_t$. Let $K(\theta) := e^{\rho\theta} \in O(k)$. We show that $A(\theta) := e^{-i\theta}K(\theta)MK(\theta)^Te^{ti\theta}$ is constant. Indeed,

$$A'(\theta) = e^{-i\theta}K(\theta)(-iM + [\rho, M] + tMi)K(\theta)^Te^{ti\theta} = 0.$$

Hence, A is constant, so

$$A(\theta_0) = A(0) = M.$$

As $[\rho, R] = 0$, $[K(\theta), R] = 0$, so by Lemma 3.1.25, we have that $\hat{M}$ is $\text{diag}(e^{i\theta_0}, e^{ti\theta_0})$ -equivariant. As θ_0 was arbitrary, we have circular t -symmetry. $\square$

When searching for instantons with circular t -symmetry, we need not check the final condition of $\mathcal{M}_{n,k}$ in Definition 3.1.16 everywhere, as Lemma 3.2.4 establishes. Note the similarity to Lemma 2.1.21.

Lemma 3.2.4. *Suppose that $\hat{M}$ satisfies (3.2.1) and (3.2.2) for some $t \in [0, 1]$ and $\rho \in \mathfrak{so}(k)$ as well as the first three conditions of Definition 3.1.16. If the final condition is satisfied at all $x = x_0 + x_1i + x_2j \in \mathbb{H}$ with $x_2 \geq 0$, then $\hat{M} \in \mathcal{M}_{n,k}$.*

Proof. Let $z = z_0 + z_1i + z_2j + z_3k \in \mathbb{H}$. Note that $\text{diag}(e^{i\theta}, e^{ti\theta})$ acts on z as $z \mapsto e^{i\theta}ze^{-ti\theta}$. Specifically, it takes $z \mapsto (z_0 + z_1i)e^{(1-t)i\theta} + (z_2 + z_3i)e^{(1+t)i\theta}j$. Thus, there is some $\theta \in \mathbb{R}$ such that $x := e^{i\theta}ze^{-ti\theta}$ has no k component and the j component is non-negative.

As $\hat{M}$ has circular t -symmetry, there is a pair $(Q, K) \in \mathrm{Sp}(n+k) \times \mathrm{O}(k)$ such that $(Q, K) \cdot \mathrm{diag}(e^{i\theta}, e^{ti\theta}) \cdot (\hat{M}, U) = (\hat{M}, U)$. Recall that K is necessarily an orthogonal matrix by Corollary 3.1.27. Thus,

$$\Delta(x) = (\hat{M}e^{ti\theta} - Ue^{i\theta}z)e^{-ti\theta} = Q^\dagger \Delta(z) K e^{-ti\theta}.$$

Hence, $\Delta(x)^\dagger \Delta(x) = e^{ti\theta} K^T \Delta(z)^\dagger \Delta(z) K e^{-ti\theta}$. As $\Delta(x)^\dagger \Delta(x)$ is real, we have

$$\Delta(z)^\dagger \Delta(z) = K \Delta(x)^\dagger \Delta(x) K^T.$$

As $\Delta(x)^\dagger \Delta(x)$ is non-singular, it follows that so too is $\Delta(z)^\dagger \Delta(z)$. $\square$

3.2.1 Structure of circular symmetry

In light of Theorem 3.2.1, we know that we can find instantons with circular t -symmetry given a matrix $\rho \in \mathfrak{so}(k)$. Note that such a matrix generates a real representation of $\mathbb{R}$. Starting with such a real representation, we can narrow down the possible M , so we are only left with finding L such that we have an instanton, which necessarily has circular t -symmetry.

By proving that the group action of S^1 on $\mathbb{H}$ given by $e^{i\theta} \cdot x := x e^{i\theta}$, which is equivalent to the action of R_0 on $\mathbb{H}$, lifts to an action on the spinor bundle, Beckett proved that for instantons with circular 0-symmetry, their symmetry is generated by representations of S^1 [Bec20, Proposition 4.2.1]. Using Lie theory, we prove this result for circular t -symmetry, for all $t \in \mathbb{Q} \cap [0, 1]$.

Note that a representation $\rho: \mathbb{R} \rightarrow \mathfrak{so}(k)$ corresponds to a representation of S^1 if and only if $\exp(\rho(2\pi)) = I_k$.

Proposition 3.2.5. *Let $t = \frac{a}{b} \in \mathbb{Q} \cap [0, 1]$ and let $\hat{M} \in \mathcal{M}_{n,k}$. Then $\hat{M}$ has circular t -symmetry if and only if there exists a real representation $\rho: \mathbb{R} \rightarrow \mathfrak{so}(k)$, corresponding to a representation of S^1 , such that for all $\theta \in \mathbb{R}$,*

$$a\theta Mi - b\theta iM + \frac{1}{2}[\rho(\theta), M] = 0; \tag{3.2.5}$$

$$[\rho(\theta), R] = 0. \tag{3.2.6}$$

Definition 3.2.6. *We call ρ the **generating representation** of the circular t -symmetry of $\hat{M}$.*

Proof. If such a ρ exists, by Theorem 3.2.1, taking $\tilde{\rho} := \frac{1}{2b}\rho(1)$, we have circular t -symmetry. Conversely, suppose that $\hat{M}$ has circular t -symmetry. By Theorem 3.2.1, there exists $\rho \in \mathfrak{so}(k)$ satisfying (3.2.1) and (3.2.2).

As $\rho \in \mathfrak{so}(k)$, there exists $U \in O(k)$ such that

$$\rho = U \operatorname{diag}(\rho_1, \dots, \rho_m) U^T,$$

with each ρ_l being either zero or $\begin{bmatrix} 0 & a_l \\ -a_l & 0 \end{bmatrix}$ for some $a_l \in \mathbb{R}$. We can write M according to the decomposition of ρ as

$$M = U \begin{bmatrix} M_{11} & \cdots & M_{1m} \\ \vdots & \ddots & \vdots \\ M_{1m}^T & \cdots & M_{mm} \end{bmatrix} U^T.$$

Consider (3.2.1). We can multiply on the right by U and the left by U^T to get

$$\left[\operatorname{diag}(\rho_1, \dots, \rho_m), \begin{bmatrix} M_{11} & \cdots & M_{1m} \\ \vdots & \ddots & \vdots \\ M_{1m}^T & \cdots & M_{mm} \end{bmatrix} \right] = \begin{bmatrix} iM_{11} - tM_{11}i & \cdots & iM_{1m} - tM_{1m}i \\ \vdots & \ddots & \vdots \\ iM_{1m}^T - tM_{1m}^T i & \cdots & iM_{mm} - tM_{mm}i \end{bmatrix}.$$

The on-diagonal components, which are symmetric, must satisfy $[\rho_l, M_{ll}] = iM_{ll} - tM_{ll}i$. If $t \neq 1$ and $|2a_l \pm t| \neq 1$, then $M_{ll} = 0$. If $t = 1$ and if $a_l \notin \{-1, 0, 1\}$, then $M_{ll} = \alpha_l I_2$ for some $\alpha_l \in \mathbb{C}$. However, note that such a M_{ll} satisfies the required equation for any value of a_l . The off-diagonal components must satisfy $\rho_l M_{lp} - M_{lp} \rho_p = iM_{lp} - tM_{lp}i$. If $|a_l \pm a_p| \neq 1 \pm t$, then $M_{lp} = 0$. It is important to note that the off-diagonal M components do not depend on the exact values of the a_l and a_p , but rather the relationships between them.

Using the above relationships, we construct a new matrix ρ' of the same form as ρ . We find $a'_l \in \mathbb{R}$ and define ρ'_l to have the same form as the corresponding ρ_l , swapping a_l with a'_l . These a'_l are chosen such that $\rho' := U \operatorname{diag}(\rho'_1, \dots, \rho'_m) U^T$ still satisfies (3.2.1) and (3.2.2), but a multiple of it generates a representation of $\mathbb{R}$ corresponding to a representation of S^1 .

Define the relation $\sim_0$ on $\{1, \dots, m\}$ by $l \sim_0 p$ if and only if $M_{lp} \neq 0$. This generates an equivalence relation, which we denote by $\sim$. Consider $[l] \in \{1, \dots, m\} / \sim$. For such a

class, if $M_{ll} = 0$, then we are free to set a'_l to anything. In particular, we may choose $a'_l = 0$. If $t \neq 1$ and $M_{ll} \neq 0$, then from above, we know that $|2a_l \pm t| = 1$, so we define $a'_l := a_l$. If $t = 1$ and M_{ll} is proportional to the identity, then we may choose $a'_l = 0$. In any case, we have that $2ba'_l$ is an integer. Regardless of the value of t or if M_{ll} vanishes, for $p \in [l]$, the relationship between a_p and a_l is determined by the form of M_{lp} if $M_{lp} \neq 0$. From above, we see that $|a_l \pm a_p| = 1 \pm t$. As we $2ba'_l \in \mathbb{Z}$, we have that if a'_p and a'_l satisfy the same relationship as a_p and a_l , then $a'_p \in \mathbb{Q}$. Specifically, we have $2ba'_p \in \mathbb{Z}$. Finally, as $p \in [l]$, if $M_{lp} = 0$, there is some $q \in [l]$ such that $M_{lq} \neq 0$. Then there is some string of elements in $[l]$ connecting p and l by non-zero off-diagonal components. From above, we see that for each element in this string, $2ba'_q \in \mathbb{Z}$ and the same is true for p in particular. Therefore, $2ba'_p \in \mathbb{Z}$ for all $p \in [l]$, including l . Note that by construction, ρ' satisfies (3.2.1). That ρ' satisfies (3.2.2) follows from Corollary 3.1.27.

Doing this for each equivalence class, we define $\tilde{\rho}: \mathbb{R} \rightarrow \mathfrak{so}(k)$ by $\tilde{\rho}(\theta) := 2b\theta\rho'$. From the work above, we note that $\exp(\tilde{\rho}(2\pi)) = I_k$. Furthermore, by construction, $\tilde{\rho}$ satisfies (3.2.5) and (3.2.6). $\square$

Proposition 3.2.5 tells us that we can use representation theory for S^1 to better understand instantons with circular t -symmetry, for $t \in \mathbb{Q}$. When $t \notin \mathbb{Q}$, we do not get representations of S^1 . Though, this phenomenon is to be expected, as $R_t \simeq \mathbb{R}$ when $t \notin \mathbb{Q}$.

Proposition 3.2.7. *Let $t \in [0, 1] \setminus \mathbb{Q}$ and suppose that $\hat{M} \in \mathcal{M}_{n,k}$ has circular t -symmetry. This symmetry does not always come from a representation of S^1 .*

Proof. Given $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $\lambda \in \mathbb{R} \setminus \{0\}$, consider

$$\hat{M} := \begin{bmatrix} \lambda & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \\ \beta j & 0 & 0 & 0 \\ 0 & -\beta j & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix}.$$

As $R := \hat{M}^\dagger \hat{M} = \text{diag}(|\beta|^2, |\beta|^2, |\alpha|^2, |\alpha|^2) + \lambda^2 I_4$ and $\Delta(x)^\dagger \Delta(x) = \text{diag}(|\beta j - x|^2, |\beta j + x|^2, |\alpha - x|^2, |\alpha + x|^2) + \lambda^2 I_4$, we have that $\hat{M} \in \mathcal{M}_{n,k}$.

Consider $\rho \in \mathfrak{so}(4)$. We have that ρ generates the circular t -symmetry of $\hat{M}$ if and only if

$$\rho = \begin{bmatrix} 0 & \frac{1+t}{2} & 0 & 0 \\ -\frac{1+t}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-t}{2} \\ 0 & 0 & -\frac{1-t}{2} & 0 \end{bmatrix}.$$

Suppose ρ does generate the circular t -symmetry. As t is irrational, $\exp(\theta\rho) = I_4$ if and only if $\theta = 0$. Hence, ρ does not correspond to a representation of S^1 . $\square$

Above, we see that circular t -symmetry is generated by a Lie algebra representation $(\mathbb{R}^k, \rho)$, corresponding to a representation of R_t . This representation was obtained by examining the bottom of (3.2.4). By focusing instead on the top of that equation, we obtain an induced representation.

Lemma 3.2.8. *Suppose that $\hat{M} \in \mathcal{M}_{n,k}$ has circular t -symmetry generated by $\rho \in \mathfrak{so}(k)$. That is, ρ satisfies (3.2.1) and (3.2.2). Let $y := L(\rho - ti)L^\dagger(LL^\dagger)^{-1} \in \mathfrak{sp}(n)$.*

Proof. By Theorem 3.2.1, we know that $tMi - iM + [\rho, M] = 0$. Using this equation, we see

$$\begin{aligned} M^\dagger M(\rho - ti) &= M^\dagger[M, \rho] + M^\dagger\rho M - tM^\dagger Mi \\ &= M^\dagger\rho M - M^\dagger iM \\ &= [M^\dagger, \rho]M + \rho M^\dagger M - M^\dagger iM \\ &= (\rho - ti)M^\dagger M. \end{aligned}$$

Hence, $[M^\dagger M, \rho - ti] = 0$. Then, as $[\rho, R] = 0$, by Theorem 3.2.1, and R is real, $[\rho - ti, R] = 0$, so $[L^\dagger L, \rho - ti] = 0$. Thus, we see that $[LL^\dagger, L(\rho - ti)L^\dagger] = 0$. Therefore, $y \in \mathfrak{sp}(n)$. $\square$

The following result relates y , ρ , and L .

Corollary 3.2.9. *If $\hat{M} \in \mathcal{M}_{n,k}$ has circular t -symmetry generated by $\rho \in \mathfrak{so}(k)$ and $y \in \mathfrak{sp}(n)$ is as defined above, then $yL - L\rho + tLi = 0$.*

Proof. This follows from the top of (3.2.4). $\square$

However, just as with ρ , when $t \in \mathbb{Q}$, we can improve on this result.

Proposition 3.2.10. *Suppose that $\hat{M} \in \mathcal{M}_{n,k}$ has circular t -symmetry, with $t = \frac{a}{b} \in \mathbb{Q}$, generated by a Lie algebra representation $(\mathbb{R}^k, \rho)$, corresponding to a representation of S^1 . That is, ρ satisfies (3.2.5) and (3.2.6). Let $\lambda(\theta) := L(\rho(\theta) - 2ai\theta)L^\dagger(LL^\dagger)^{-1} \in \mathfrak{sp}(n)$. We have $(\mathbb{H}^n, \lambda)$ is a Lie algebra representation of $\mathbb{R}$ corresponding to a representation of S^1 .*

Proof. From the work above, as $\lambda(\theta) = 2b\theta y$, we have $\lambda(\theta) \in \mathfrak{sp}(n)$. It remains to show that $e^{\lambda(2\pi)} = I_n$. Indeed, due to the commutativity of $LL^\dagger$ and $L(\rho(\theta) - 2a\theta i)L^\dagger$, we have that

$$e^{\lambda(\theta)} = Le^{\rho(\theta) - 2a\theta i}L^\dagger(LL^\dagger)^{-1}.$$

Taking $\theta = 2\pi$, we see that as ρ and i commute,

$$e^{\lambda(2\pi)} = Le^{\rho(2\pi)}e^{-4a\pi i}L^\dagger(LL^\dagger)^{-1} = I_n.$$

Therefore, $(\mathbb{H}^n, \lambda)$ corresponds to a representation of S^1 . $\square$

We know that this induced representation satisfies the following equation.

Corollary 3.2.11. *If $\hat{M} \in \mathcal{M}_{n,k}$ has circular t -symmetry generated by $(\mathbb{R}^k, \rho)$, then $\lambda(\theta)L - L\rho(\theta) + 2a\theta Li = 0$.*

Proof. This follows from multiplying the top of (3.2.4) by $2b\theta$. $\square$

3.2.2 Hyperbolic monopoles

In this section, we discuss the connection between hyperbolic monopoles and circle-invariant instantons. In particular, we discuss how hyperbolic monopoles with integral weight correspond to circular 1-symmetric ADHM data and how to transform this ADHM data into a hyperbolic monopole. This section is an extension of my previous work in Chapter 2 to all hyperbolic monopoles with integral weight.

Let $E \rightarrow H^3$ be a vector bundle over H^3 with structure group $\mathrm{Sp}(n)$, equipped with a connection A , of curvature F_A , and a section Φ of $\mathrm{End}(E)$, called the Higgs field. Let $\star$ denote the Hodge star. Hyperbolic monopoles are solutions to the Bogomolny equations $F_A = \star D_A \Phi$, with finite energy $\varepsilon = \frac{1}{2\pi} \int_{H^3} |F_A|^2 \mathrm{vol}_{H^3}$.

It is well known that the Bogomolny equations are a dimensional reduction of the self-dual equations. That is, monopoles are related to instantons that are independent of one coordinate. When this coordinate is one of the standard Cartesian coordinates, the instanton becomes translation invariant and either has no action (and is therefore flat and

of no interest) or has infinite action. Atiyah realized that if this coordinate corresponded to that of a circle, then the compactness of the circle implies that the instanton has finite action. Then, by utilizing the conformal equivalence $S^4 \setminus S^2 \equiv H^3 \times S^1$, he deduced that circle-invariant instantons correspond to hyperbolic monopoles [Ati84a, Ati84b]. While Atiyah used a particular circle action, we see below that this conclusion is not true for every circle action.

Atiyah's idea is that given a circle action on S^4 , remove some $S^2 \subseteq S^4$ invariant under the action and use coordinates (x, y, z, θ) to describe $S^4 \setminus S^2$. The coordinates (x, y, z) correspond to coordinates on H^3 and θ corresponds to the coordinate on S^1 . Specifically, θ is generated by the circle action. Then, an instanton that is equivariant under the circle action is independent of the θ coordinate and can be written as a hyperbolic monopole. Conversely, given a hyperbolic monopole, we can construct a circle-invariant instanton on $S^4 \setminus S^2$. In order to extend the obtained instanton to all of S^4 , the monopole must have integral mass [Ati84b].

It turns out that only one conjugacy class of circle actions gives rise to hyperbolic monopoles.

Lemma 3.2.12. *A circle action relates hyperbolic monopoles with integral mass and instantons invariant under this action if and only if the circle action is conjugate to R_1 .*

Note 3.2.13. *Both Braam–Austin and Manton–Sutcliffe's conformal circle actions are conjugate to R_1 . Thus, they are equivalent. However, they lead to different models of hyperbolic space, allowing us to view monopoles differently.*

Proof. The isometry group of H^3 is six-dimensional. Thus, adding rotation on S^1 , we have a seven-dimensional group acting on $H^3 \times S^1$ as isometries. As $S^4 \setminus S^2 \equiv H^3 \times S^1$, these isometries correspond to conformal maps on $S^4 \setminus S^2$, which we can complete, obtaining conformal maps on S^4 .

By Conjecture 3.1.14, we may assume that the circle action is conjugate to an action in $\mathrm{Sp}(2)$. This action is generated by an element in $\mathfrak{sp}(2)$. As this Lie algebra is compact, the adjoint orbit of an element of the Lie algebra has a non-empty intersection with any Cartan subalgebra. In particular, this means that up to conjugation, the circle action is generated by an element in $\{\mathrm{diag}(ai, bi) \mid a, b \in \mathbb{R}\}$. In particular, up to conjugation, we may assume that the circle action is generated by $\mathrm{diag}(i, ti) \in \mathfrak{sp}(2)$ for some $t \in [0, 1]$.

If a circle action generates this conformal equivalence, then there is a seven-dimensional Lie subalgebra $\mathfrak{l}$ of $\mathrm{Lie}(\mathrm{SL}(2, \mathbb{H}))$ that commutes with $\mathrm{diag}(i, ti)$ because the circle action corresponds to rotating S^1 in $H^3 \times S^1$. Note that $\mathrm{Lie}(\mathrm{SL}(2, \mathbb{H}))$ is the subset of $\mathrm{Mat}(2, 2, \mathbb{H})$

with purely imaginary trace. When $t = 0$ or $t \in (0, 1)$, the subalgebra of elements that commute with $\text{diag}(i, ti)$ is five-dimensional or three-dimensional, respectively. In either case, the space is not large enough to contain $\mathfrak{l}$. However, when $t = 1$, this space is seven-dimensional, as required. $\square$

Note 3.2.14. *That only one conjugacy class corresponds to hyperbolic monopoles is to be expected. All hyperbolic monopoles with integral mass arise from this construction. However, the circle action used to generate $S^4 \setminus S^2 \equiv H^3 \times S^1$ should always generate all hyperbolic monopoles with integral mass. As it stands, only circle actions conjugate to R_1 generate hyperbolic monopoles. Being related by conjugation, the hyperbolic monopoles created by each action are all related. If multiple conjugacy classes generated hyperbolic monopoles, then there would be no guarantee that the monopoles would be related.*

In later sections, we investigate symmetric hyperbolic monopoles more deeply. However, we see that such monopoles must all arise from instantons equivariant under R_1 , or a group conjugate to it.

The model of hyperbolic 3-space obtained from the action of R_1 is the half-space model. It is obtained as follows. As given above, the fixed two-sphere of the R_1 action is $\mathbb{C} \cup \{\infty\}$. On $S^4 \setminus (\mathbb{C} \cup \{\infty\})$, equivalently $\mathbb{H} \setminus \mathbb{C}$, we use coordinates (x_0, x_1, r, θ) , with $r > 0$, which corresponds to $x_0 + x_1 i + r \cos \theta j + r \sin \theta k$. Note that the action of $e^{i\phi/2}$ sends (x_0, x_1, r, θ) to $(x_0, x_1, r, \theta + \phi)$. Thus, (x_0, x_1, r) are coordinates for H^3 and θ is the coordinate for S^1 .

In these new coordinates, the metric for $\mathbb{H} \setminus \mathbb{C}$ is $dx_0^2 + dx_1^2 + dr^2 + r^2 d\theta^2$. As $r > 0$, we can divide by r^2 to get the metric

$$\frac{dx_0^2 + dx_1^2 + dr^2}{r^2} + d\theta^2.$$

This metric is exactly the metric for $H^3 \times S^1$, using the half-space model.

The ADHM transform relates instantons and ADHM data. Recall that given ADHM data $(a, b) \in \mathcal{M}$, for all $x \in \mathbb{H}$, we define $\Delta(x) := a - bx$. We then find $V(x) \in \text{Mat}(n + k, n, \mathbb{H})$ such that $V(x)^\dagger \Delta(x) = 0$ and $V(x)^\dagger V(x) = I_n$. The instanton is obtained via $\mathbb{A}_\mu(x) := V(x)^\dagger \partial_\mu V(x)$. By changing coordinates to X and θ , where X is the coordinate in H^3 and θ is the S^1 coordinate, as the instanton has circular 1-symmetry, there is some gauge in which the instanton is independent of θ . Thus, the instanton is a function of X . In these coordinates, $\mathbb{A} = \mathbb{A}_0 dx_0 + \mathbb{A}_1 dx_1 + \mathbb{A}_r dr + \mathbb{A}_\theta d\theta$. Letting $\Phi := \mathbb{A}_\theta$ and A be the remainder of $\mathbb{A}$, the pair (Φ, A) is the corresponding hyperbolic monopole.

This process is outlined in the case of Manton–Sutcliffe’s conformal circle action [MS14, § 4]. We now complete this process for our R_1 circle action. For other R_1 circle actions, similar results can be found.

Proposition 3.2.15. *Suppose that $\hat{M} \in \mathcal{M}_{n,k}$ has circular 1-symmetry, with respect to usual R_1 group, generated by $\rho \in \mathfrak{so}(k)$. Let X denote the coordinates of H^3 . Suppose that we complete the ADHM procedure for H^3 , finding $V(X)$ satisfying $V(X)^\dagger \Delta(X) = 0$ and $V(X)^\dagger V(X) = I_n$. The corresponding hyperbolic monopole (Φ, A) is given, up to gauge, by*

$$\Phi(X) = \frac{1}{2} V(X)^\dagger \text{diag}(L(i - \rho)L^\dagger(LL^\dagger)^{-1}, i - \rho)V(X); \quad (3.2.7)$$

$$A_l(X) = V(X)^\dagger \partial_l V(X). \quad (3.2.8)$$

The ∂_l denotes partial differentiation with respect to the l th coordinate of H^3 .

Proof. As $\mathbb{H} \setminus \mathbb{C} \simeq H^3 \times S^1$, any $x \in \mathbb{H} \setminus \mathbb{C}$ can be uniquely written as

$$x = \text{diag}(e^{i\theta/2}, e^{i\theta/2}).X = e^{i\theta/2}(x_0 + ix_1 + rj)e^{-i\theta/2}.$$

Note the factor of two means that given x , we get a unique $\theta \in [0, 2\pi]$ and $X \in H^3$.

Let $Q(\theta) := \text{diag}(Le^{-i\theta/2}e^{\rho\theta/2}L^\dagger(LL^\dagger)^{-1}, e^{-i\theta/2}e^{\rho\theta/2}) \in \text{Sp}(n+k)$ and $K(\theta) := e^{\rho\theta/2} \in \text{SO}(k)$. Note that by the proof of Lemma 3.1.25, we have that $[LL^\dagger, Le^{-i\theta/2}e^{\rho\theta/2}L^\dagger]$. Then, per (3.2.3), we have

$$(Q(\theta), K(\theta)).\text{diag}(e^{i\theta/2}, e^{i\theta/2}).(\hat{M}, U) = (\hat{M}, U). \quad (3.2.9)$$

Therefore,

$$\begin{aligned} \Delta(x) &= \Delta(e^{i\theta/2}(x_0 + ix_1 + rj)e^{-i\theta/2}) \\ &= \hat{M} - Ue^{i\theta/2}(x_0 + ix_1 + rj)e^{-i\theta/2} \\ &= (\hat{M}e^{i\theta/2} - Ue^{i\theta/2}X)e^{-i\theta/2} \\ &= Q(\theta)^\dagger \Delta(X)K(\theta)e^{-i\theta/2}. \end{aligned}$$

Extending V to all of $\mathbb{H} \setminus \mathbb{C}$ by $V(x) := Q(\theta)^\dagger V(X)$, we note that $V(x)^\dagger \Delta(x) = 0$ and $V(x)^\dagger V(x) = I_n$. Thus, we can use it to create our instanton and thus our monopole.

The θ component of the instanton $\mathbb{A}$ corresponding to $(\hat{M}, U)$ is given by

$$\mathbb{A}_\theta(x) = V(x)^\dagger \partial_\theta V(x) = V(X)^\dagger Q(\theta) \partial_\theta Q(\theta)^\dagger V(X).$$

Simplifying, we find that

$$\mathbb{A}_\theta(x) = \frac{1}{2} V(X)^\dagger \begin{bmatrix} (LL^\dagger)^{-1} Le^{-i\theta/2} e^{\rho\theta/2} L^\dagger Le^{-\rho\theta/2} e^{i\theta/2} (i - \rho) L^\dagger (LL^\dagger)^{-1} & 0 \\ 0 & i - \rho \end{bmatrix} V(X).$$

Recall that $R := L^\dagger L + M^\dagger M$ commutes with ρ , by Theorem 3.2.1, and thus $e^{\rho\theta}$. Furthermore, as R is real, it commutes with quaternions. By (3.2.9), we have that

$$e^{-i\theta/2} e^{\rho\theta/2} M e^{i\theta/2} e^{-\rho\theta/2} = M.$$

From this, we see that

$$\begin{aligned} e^{-i\theta/2} e^{\rho\theta/2} L^\dagger L e^{-\rho\theta/2} e^{i\theta/2} &= e^{-i\theta/2} e^{\rho\theta/2} (R - M^\dagger M) e^{-\rho\theta/2} e^{i\theta/2} \\ &= R - M^\dagger e^{\rho\theta/2} e^{-i\theta/2} M e^{-\rho\theta/2} e^{i\theta/2} = R - M^\dagger M = L^\dagger L. \end{aligned}$$

Returning to the computation of $\mathbb{A}_\theta(x)$, we have

$$\mathbb{A}_\theta(x) = \frac{1}{2} V(X)^\dagger \text{diag}(L(i - \rho)L^\dagger(LL^\dagger)^{-1}, i - \rho)V(X).$$

As in the procedure described above, we take $\Phi(X) := \mathbb{A}_\theta(X)$. Note that $\mathbb{A}_\theta$ is independent of θ .

We have that the other coordinates of $\mathbb{A}$ are given by

$$\mathbb{A}_l(x) = V(x)^\dagger \partial_l V(x) = V(X)^\dagger \partial_l V(X).$$

Similar to Φ , we take $A_l(X) := \mathbb{A}_l(X)$ and note that in this gauge, $\mathbb{A}$ is independent of θ . Thus, our monopole has the desired form. $\square$

Suppose that (Φ, A) is a hyperbolic monopole with integral mass. Both Φ and A are defined on H^3 . The connection between hyperbolic monopoles with integral mass and instantons with circular 1-symmetry grants such hyperbolic monopoles an additional symmetry. Such monopoles correspond to an instanton $\mathbb{A}$ with circular 1-symmetry via $\mathbb{A} = \Phi d\theta + A$. The integral mass condition implies that the instanton $\mathbb{A}$ is defined on the whole of S^4 . Thus, Φ and A are also defined on the whole of S^4 . Moreover, given an embedding of H^3 into S^4 , the ball model for instance, the definitions on S^4 agree with the original definition on H^3 .

Proposition 3.2.16. *Consider the ball model $H^3 = \{X = X_1i + X_2j + X_3k \in \mathfrak{sp}(1) \mid R^2 := X_1^2 + X_2^2 + X_3^2 < 1\}$ with curvature -1 . Suppose we have a hyperbolic monopole with integral mass and Higgs field Φ . For all $X \in H^3 \setminus \{0\}$, the gauge-invariant quantity $|\Phi|$ satisfies*

$$|\Phi(X)| = \left| \Phi \left(\frac{X}{R^2} \right) \right|. \quad (3.2.10)$$

Proof. Such hyperbolic monopoles correspond to instantons with circular 1-symmetry. In particular, as we are using the ball model, these monopoles correspond directly with instantons symmetric under Manton–Sutcliffe’s conformal circle action [MS14, (4.4)]. Given a point $0 \neq X \in H^3 \subseteq \mathbb{H}$, the orbit $S^1 \cdot X$ under this circle action is given by

$$S^1 \cdot X = \left\{ \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \cdot X \mid \theta \in \mathbb{R} \right\}.$$

In particular, when $\theta = \frac{\pi}{2}$, we see that this orbit contains the point X/R^2 .

We know that there is some gauge in which the instanton $\mathbb{A} := \Phi d\theta + A$ is independent of θ . Then, as X and X/R^2 are in the same orbit, we have that $\mathbb{A}(X) = \mathbb{A}\left(\frac{X}{R^2}\right)$. Recall from above that as we have a hyperbolic monopole with integral mass, Φ is defined on the whole of S^4 . Matching $d\theta$ terms, we have that $\Phi(X) = \Phi\left(\frac{X}{R^2}\right)$. This equality is gauge dependent. However, we have that the gauge-invariant quantities $|\Phi(X)|$ and $|\Phi\left(\frac{X}{R^2}\right)|$ are equal in any gauge. $\square$

The additional symmetry of hyperbolic monopoles with integral mass proven in Proposition 3.2.16 is only possible due to the connection between hyperbolic monopoles with integral mass and instantons with circular 1-symmetry.

Note 3.2.17. In Notes 2.1.43 and 2.1.45, we examine the norm squared of the Higgs field of a $\mathrm{Sp}(2)$ and $\mathrm{Sp}(4)$ hyperbolic monopole, respectively. Both functions are rational, comprised of palindromic polynomials. Additionally, for each rational function, the numerator and denominator have the same degree. The palindromic nature of these polynomials follows from Proposition 3.2.16 and the matching degrees of the numerators and denominators. The respective energy densities of these monopoles are also rational functions of palindromic polynomials. Both of these hyperbolic monopoles have integral mass. The same behaviour is not guaranteed for hyperbolic monopoles with non-integral mass.

Indeed, consider the family of spherically symmetric $\mathrm{Sp}(1)$ hyperbolic monopoles given by Chakrabarti and Nash [Cha86, Nas86]. Given $C > 1$, the norm of the Higgs field is given by

$$|\Phi|(r) = \frac{C}{2} \cdot \frac{\left(\frac{1+r}{1-r}\right)^{2C} + 1}{\left(\frac{1+r}{1-r}\right)^{2C} - 1} - \frac{r^2 + 1}{4r}.$$

Given a value C , the mass of the monopole is $\frac{C-1}{2}$.

When C is an integer, we see that $|\Phi|(r)$ satisfies Proposition 3.2.16. However, when $C = \frac{3}{2}$, we see that $|\Phi|(r) = \frac{5r-r^3}{4r^2+12}$, which does not satisfy Proposition 3.2.16. Moreover, when $2C$ is not an integer, the norm and its square are not rational functions of polynomials and not even defined for $r > 1$.

3.2.3 Singular monopoles

In this section, we discuss the connection between singular monopoles and circle-invariant instantons, though a different circle action from the one relating hyperbolic monopoles and instantons. In particular, we discuss how a class of singular monopoles correspond to circular 0-symmetric ADHM data and how to transform this data into a singular monopole.

Let $E \rightarrow \mathbb{R}^3 \setminus \{0\}$ be a vector bundle with structure group $\mathrm{Sp}(n)$, equipped with a connection A , of curvature F_A , and a section Φ of $\mathrm{End}(E)$, called the Higgs field, just as with hyperbolic and Euclidean monopoles. Singular monopoles are solutions to the Bogomolny equations, with finite energy.

The relationship between instantons and singular monopoles was first described by Kronheimer and explored further by Pauly [Kro85, Pau96, Pau98]. This relationship utilizes the Hopf fibration.

Definition 3.2.18. *The Hopf map $\pi: S^3 \rightarrow S^2$ is given by*

$$\pi(x, y, z, w) := (x^2 + y^2 - z^2 - w^2, 2(yz - xw), 2(xz + yw)). \quad (3.2.11)$$

We may rewrite the Hopf map as $\pi: \mathbb{H} \rightarrow \mathfrak{sp}(1)$, given by $\pi(x) := x^\dagger i x$. Note that this matches the previous definition on $S^3 \subseteq \mathbb{H}$.

The Hopf fibration describes S^3 as a non-trivial principal bundle over S^2 with fibre S^1 . That is, for every $p \in S^2$, $\pi^{-1}(p) \simeq S^1$. The same is true for all $p \in \mathbb{R}^3 \setminus \{0\}$. Additionally, $\pi^{-1}(0) = \{0\}$. Indeed, consider the R_0 action on $\mathbb{H}$, taking $x \mapsto \theta.x := e^{i\theta}x$. We see that $\pi(\theta.x) = \pi(x)$, so the R_0 action is transitive and free away from the origin. Note that π is not invariant under any other R_t action.

Singular monopoles are in a one-to-one correspondence with circle-invariant instantons on $\mathbb{R}^4 \setminus \{0\}$ [Pau96].

Definition 3.2.19. *As we are studying instantons over all of $\mathbb{R}^4$, we are interested in singular monopoles arising from instantons that, under some gauge transformation, extend smoothly over the origin. Such objects are called **monopoles with Dirac type singularities**. More information on such monopoles can be found in Pauly's works, where they are called good singular monopoles [Pau96, Pau98].*

Pauly's idea is that given a singular monopole, we can pull-back the vector bundle, Higgs field, and connection to $\mathbb{H} \setminus \{0\}$. Then, by using a one-form that is invariant under the R_0 action, we use the monopole to construct a circle-invariant instanton. Conversely, given a circle-invariant instanton, we can use π to uniquely define a monopole

on $\mathbb{R}^3 \setminus \{0\}$. The exact process has been proven by others, but we demonstrate it here for completeness, modifying the process in order to use the usual orientations on three and four dimensional Euclidean space as well as relating monopoles directly to instantons [Pau98, Proposition 2; Bec20, Lemma 4.1.7]. Additionally, we provide a formula for computing the monopole directly from the ADHM data.

Proposition 3.2.20. *Suppose that $\hat{M} \in \mathcal{M}_{n,k}$ has circular 0-symmetry, with respect to usual R_0 group, generated by $\rho \in \mathfrak{so}(k)$. Let X denote the coordinates of $\mathfrak{sp}(1) \setminus \{0\}$. Locally, we can identify any point in $\mathbb{H} \setminus \{0\}$ by $X \in \mathfrak{sp}(1) \setminus \{0\}$ and $\theta \in S^1$. Suppose that we have local sections $\mathfrak{sp}(1) \setminus \{0\} \rightarrow \mathbb{H} \setminus \{0\}$ taking $X \mapsto x_X$. Further, suppose we find $V(x_X)$ satisfying $V(x_X)^\dagger \Delta(x_X) = 0$ and $V(x_X)^\dagger V(x_X) = I_n$. The corresponding singular monopole (Φ, A) is given, up to gauge, by*

$$\Phi(X) = \frac{1}{2|X|} V(x_X)^\dagger \text{diag}(-L\rho L^\dagger (LL^\dagger)^{-1}, i - \rho) V(x_X); \quad (3.2.12)$$

$$A_l(X) = V(x_X)^\dagger \left(\pi^* \frac{\partial}{\partial X_l} \right) V(x_X). \quad (3.2.13)$$

The ∂_l denotes partial differentiation with respect to the l th coordinate of $\mathfrak{sp}(1) \setminus \{0\}$.

Proof. Let $\frac{\partial}{\partial \theta}$ be the vector field on $\mathbb{H} \setminus \{0\}$ given by

$$\frac{\partial}{\partial \theta} \Big|_x := \frac{d}{d\theta} \Big|_{\theta=0} e^{i\theta} x = -x_1 \frac{\partial}{\partial x_0} + x_0 \frac{\partial}{\partial x_1} - x_3 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3}. \quad (3.2.14)$$

Then let

$$\xi := 2(-x_1 dx_0 + x_0 dx_1 - x_3 dx_2 + x_2 dx_3). \quad (3.2.15)$$

We note that given any vector field V on $\mathbb{H} \setminus \{0\}$, $\xi(V) = 2 \langle V, \frac{\partial}{\partial \theta} \rangle$. In particular, $\xi(\frac{\partial}{\partial \theta}) = 2|x|^2$.

Given any vector field W on $\mathfrak{sp}(1) \setminus \{0\}$, denote by π^*W the unique vector field on $\mathbb{H} \setminus \{0\}$ such that $d\pi(\pi^*W) = W$ and $\xi(\pi^*W) = 0$. That is, π^*W is the unique lift of W orthogonal to the circle orbit.

The relationship between the instanton $\mathbb{A}$ on $\mathbb{H} \setminus \{0\}$ and the monopole (Φ, A) on $\mathfrak{sp}(1) \setminus \{0\}$ is given as follows. Given a monopole, we have

$$\mathbb{A} = \pi^*A + \pi^*\Phi \otimes \xi. \quad (3.2.16)$$

Given an instanton, let s be a section and W a vector field on $\mathfrak{sp}(1) \setminus \{0\}$; we have that the monopole is uniquely determined by

$$\begin{aligned}\pi^*(\Phi s) &= \frac{1}{2|x|^2} \mathbb{A}_\theta \pi^* s; \\ \pi^*(A(W)s) &= \mathbb{A}(\pi^* W) \pi^* s.\end{aligned}\tag{3.2.17}$$

That Φ is linear with respect to functions follows because the derivative of $\pi^* s$ with respect to θ vanishes.

Given the relationship between $\mathbb{A}$ and (Φ, A) , we see that the curvature of $\mathbb{A}$ is given by

$$F_{\mathbb{A}} = \pi^* F_A + \pi^* D_A \Phi \wedge \xi + \pi^* \Phi \otimes d\xi.$$

From the definition of ξ , we see that with respect to the standard orientation on $\mathbb{H} \setminus \{0\}$, we have that $d\xi$ is self-dual. That is, $\star d\xi = d\xi$. Denoting the anti-self-dual part of a two-form B by $B^- := \frac{B - \star B}{2}$, we have that

$$F_{\mathbb{A}}^- = (\pi^* F_A + \pi^* D_A \Phi \wedge \xi)^-.$$

For any one-form ω on $\mathfrak{sp}(1) \setminus \{0\}$, we can compute that $\star \pi^*(\star \omega) = -\xi \wedge \pi^* \omega$. As we are looking at the anti-self-dual part of the curvature, we have

$$F_{\mathbb{A}}^- = (\pi^*(F_A - \star D_A \Phi))^-.$$

If (Φ, A) is a monopole, then $\mathbb{A}$ is an instanton. The finite action condition is satisfied as the monopole has finite energy and we are integrating with respect to the coordinates on $\mathfrak{sp}(1) \setminus \{0\}$ and S^1 .

Conversely, if $\mathbb{A}$ is an instanton, (Φ, A) is a monopole. Indeed, $F_{\mathbb{A}}^- = 0$, so $\pi^*(F_A - \star D_A \Phi)$ is self-dual. Note that for any two-form B on $\mathfrak{sp}(1) \setminus \{0\}$, we have $\star B$ is a one-form. Thus, $\star \pi^* B = -\xi \wedge \pi^* \star B$. As $\pi^*(F_A - \star D_A \Phi)$ is self-dual, this means that

$$\pi^*(F_A - \star D_A \Phi) = \star \pi^*(F_A - \star D_A \Phi) = -\xi \wedge \pi^* \star (F_A - \star D_A \Phi).$$

However, ξ is orthogonal to all pullbacks via π , so $\pi^*(F_A - \star D_A \Phi) = 0$, meaning that (Φ, A) is a monopole. The finite energy condition is satisfied as $\mathbb{A}$ has finite action.

We can use this relationship between instantons and singular monopoles to compute the monopole in terms of the ADHM data. Note that any $x \in \mathbb{H} \setminus \{0\}$ can be uniquely written as

a pair $(\theta, X) \in S^1 \times (\mathfrak{sp}(1) \setminus \{0\})$ as $x = e^{i\theta} x_X$. Let $Q(\theta) := \text{diag}(Le^{\rho\theta}L^\dagger(LL^\dagger)^{-1}, e^{-i\theta}e^{\rho\theta}) \in \text{Sp}(n+k)$ and $K(\theta) := e^{\rho\theta} \in \text{SO}(k)$. Per (3.2.3), we have

$$(Q(\theta), K(\theta)).\text{diag}(e^{i\theta}, 1).(\hat{M}, U) = (\hat{M}, U).$$

Therefore, similar to the proof of Proposition 3.2.15, we have that $\Delta(x) = Q(\theta)^\dagger \Delta(X) K(\theta)$.

Extending V to all of $\mathbb{H} \setminus \{0\}$ by $V(x) := Q(\theta)^\dagger V(x_X)$, we note that we can use V to create our instanton and thus our monopole.

The θ component of the instanton $\mathbb{A}$ corresponding to $(\hat{M}, U)$ is given by

$$\mathbb{A}_\theta(x) = V(x_X)^\dagger Q(\theta) \partial_\theta Q(\theta)^\dagger V(x_X).$$

Simplifying, using the commutivity proven in Proposition 3.2.15, we have

$$\mathbb{A}_\theta(x) = V(x_X)^\dagger \text{diag}(-L\rho L^\dagger(LL^\dagger)^{-1}, i - \rho) V(x_X).$$

Recalling that as $X = \pi(x_X)$, so $|X| = |x_X|^2$, we define Φ via $\Phi(X) := \frac{1}{2|X|} \mathbb{A}_\theta(x_X)$, noting that Φ does not depend on θ .

Similarly, we find that

$$\mathbb{A} \left(\pi^* \frac{\partial}{\partial X_l} \right) (x) = V(x_X)^\dagger \left(\pi^* \frac{\partial}{\partial X_l} \right) V(x_X).$$

Thus, we can define $A_l(X) := \mathbb{A} \left(\pi^* \frac{\partial}{\partial X_l} \right) (x_X)$, noting that $A_l(X)$ does not depend on θ . $\square$

Note 3.2.21. *If we find $V(x)$ for all $x \in \mathbb{H} \setminus \{0\}$, we can construct $A_l(X)$ via*

$$A_1(X) := \frac{1}{2|X|} (x_0 \mathbb{A}_0(x_X) + x_1 \mathbb{A}_1(x_X) - x_2 \mathbb{A}_2(x_X) - x_3 \mathbb{A}_3(x_X)); \quad (3.2.18)$$

$$A_2(X) := \frac{1}{2|X|} (-x_3 \mathbb{A}_0(x_X) + x_2 \mathbb{A}_1(x_X) + x_1 \mathbb{A}_2(x_X) - x_0 \mathbb{A}_3(x_X)); \quad (3.2.19)$$

$$A_3(X) := \frac{1}{2|X|} (x_2 \mathbb{A}_0(x_X) + x_3 \mathbb{A}_1(x_X) + x_0 \mathbb{A}_2(x_X) + x_1 \mathbb{A}_3(x_X)). \quad (3.2.20)$$

This follows from finding the lifts of $\frac{\partial}{\partial X_l}$.

The following lemma follows immediately from the definition of the Hopf map, providing a relationship between symmetric singular monopoles and symmetric instantons.

Lemma 3.2.22. *We have that for $\theta \in \mathbb{R}$, $x \in \mathbb{H}$, and $p \in \mathrm{Sp}(1)$,*

$$\pi(e^{i\theta}xp^\dagger) = p\pi(x)p^\dagger. \quad (3.2.21)$$

If $p \mapsto R_p$ denotes the double cover $\mathrm{Sp}(1) \rightarrow \mathrm{SO}(3)$, then this tells us that under π , the action of $(e^{i\theta}, p) \in \mathrm{Sp}(2)$ becomes R_p .

In later sections, we investigate symmetric singular monopoles more deeply. However, we see that such monopoles must arise from instantons equivariant under R_0 , or a group conjugate to it.

3.3 Toral symmetry

In this section, we find an equation describing all instantons with toral symmetry, as given in Table C.1 as well as below. We then discuss the connection between such instantons and axially symmetric hyperbolic and singular monopoles.

First, we reintroduce the notion of toral symmetry. An instanton is said to have toral symmetry if it is equivariant under $\mathrm{diag}(e^{i\phi_1}, e^{i\phi_2})$ for all $\phi_1, \phi_2 \in \mathbb{R}$.

Theorem 3.3.1. *Let $\hat{M} \in \mathcal{M}_{n,k}$. Then $\hat{M}$ has toral symmetry if and only if there exists $\rho_1, \rho_2 \in \mathfrak{so}(k)$ such that $[\rho_1, \rho_2] = 0$ and*

$$iM = [\rho_1, M], \quad -Mi = [\rho_2, M], \quad (3.3.1)$$

$$[\rho_1, R] = 0, \quad [\rho_2, R] = 0. \quad (3.3.2)$$

Definition 3.3.2. *We call ρ_1, ρ_2 the **generators** of the toral symmetry of $\hat{M}$.*

Note 3.3.3. *All connected Lie subgroups of $\mathrm{Sp}(2)$ with Lie algebra $\mathbb{R} \oplus \mathbb{R}$ are conjugate to $S^1 \times S^1$. That is, there is some $A \in \mathrm{Sp}(2)$ such that the Lie group is of the form $A(S^1 \times S^1)A^\dagger$. Instantons equivariant under this group are of the form $A^\dagger(\hat{M}, U)$, where $\hat{M}$ is equivariant under $S^1 \times S^1$.*

Proof. We follow the proof of Theorem 1.1.1, making modifications when relevant. Let $\mathcal{X}$ be the smooth manifold comprised of all pairs of $(n+k) \times k$ quaternionic matrices. The gauge and conformal actions can easily be expanded from $\mathcal{M}_{n,k}$ to smooth actions on $\mathcal{X}$. Just as with $\mathcal{M}_{n,k}$, the conformal action on $\mathcal{X}$ descends to an action on $\mathcal{X}/(\mathrm{Sp}(n+k) \times \mathrm{GL}(k, \mathbb{R}))$.

Suppose that $\hat{M}$ has toral symmetry. Let $S \subseteq S^1 \times S^1 \times O(k)$ be the stabilizer group of $(\hat{M}, U)$ restricted to rotations in $S^1 \times S^1$. That is

$$S := \{(\text{diag}(e^{i\theta_1}, e^{i\theta_2}), K) \mid (\text{diag}(Le^{-i\theta_2}KL^\dagger(LL^\dagger)^{-1}, e^{-i\theta_1}K), K). \\ \text{diag}(e^{i\theta_1}, e^{i\theta_2}).(\hat{M}, U) = (\hat{M}, U)\}. \quad (3.3.3)$$

Indeed, if $(Q, K) \in \text{Sp}(n+k) \times \text{GL}(k, \mathbb{R})$ such that $\text{diag}(e^{i\theta_1}, e^{i\theta_2}).(Q, K).(\hat{M}, U) = (\hat{M}, U)$, then Corollary 3.1.27 tells us that $K \in O(k)$, $[R, K] = 0$, and Q is given by the diagonal matrix in (3.3.3). Therefore, the pairs in S encapsulate all the toral symmetry of the instanton. That S is a group follows from this.

Just as in Theorem 1.1.1, we have that S is a compact Lie group and we obtain a Lie group homomorphism $\Psi: \mathbb{R} \oplus \mathbb{R} \rightarrow S$ such that $\pi|_S \circ \Psi = \pi$, where π_1 treats $S^1 \times S^1$ as one component and $\pi: \mathbb{R} \oplus \mathbb{R} \rightarrow S^1 \times S^1$ is the covering map.

Additionally, there is a Lie algebra homomorphism $\rho: \mathbb{R} \oplus \mathbb{R} \rightarrow \mathfrak{so}(k)$ such that for all $t_1, t_2 \in \mathbb{R}$, we have $\Psi(t_1, t_2) = (e^{it_1}, e^{it_2}, e^{\rho(t_1, t_2)})$. As ρ is a Lie algebra homomorphism whose image consists of real matrices, it gives us a real Lie algebra representation. As $\mathbb{R} \oplus \mathbb{R}$ is abelian, there exists $\rho_1, \rho_2 \in \mathfrak{so}(k)$ such that $[\rho_1, \rho_2] = 0$ and $\rho(x_1, x_2) = \rho_1 x_1 + \rho_2 x_2$.

For all $\theta \in \mathbb{R}$, we have $\Psi(\theta, 0) \in S$, giving us

$$(\text{diag}(Le^{\rho_1\theta}L^\dagger(LL^\dagger)^{-1}, e^{-i\theta}e^{\rho_1\theta}), e^{i\theta}).(\hat{M}, U) = (\hat{M}, U).$$

Differentiating and evaluating at $\theta = 0$, we obtain

$$\begin{bmatrix} L\rho_1L^\dagger(LL^\dagger)^{-1}L - L\rho_1 \\ -iM + [\rho_1, M] \end{bmatrix} = 0. \quad (3.3.4)$$

For all $\theta \in \mathbb{R}$, we have $\Psi(0, \theta) \in S$, giving us

$$(\text{diag}(Le^{-i\theta}e^{\rho_2\theta}L^\dagger(LL^\dagger)^{-1}, e^{\rho_2\theta}), \text{diag}(1, e^{i\theta})).(\hat{M}, U) = (\hat{M}, U).$$

Differentiating and evaluating at $\theta = 0$, we obtain

$$\begin{bmatrix} -LiL^\dagger(LL^\dagger)^{-1}L + L\rho_2L^\dagger(LL^\dagger)^{-1}L + Li - L\rho_2 \\ [\rho_2, M] + Mi \end{bmatrix} = 0. \quad (3.3.5)$$

Focusing on the bottom rows of (3.3.4) and (3.3.5), we see that $iM = [\rho_1, M]$ and $-Mi = [\rho_2, M]$. Furthermore, as mentioned above, Corollary 3.1.27 tells us that $[R, e^{\rho_1\theta}] = 0$, for

all $\theta \in \mathbb{R}$. Differentiating and evaluating at $\theta = 0$ for both $l = 1, 2$, we have $[R, \rho_1] = 0$ and $[R, \rho_2] = 0$.

Conversely, suppose that the equations hold for some $\rho_1, \rho_2 \in \mathfrak{so}(k)$ with $[\rho_1, \rho_2] = 0$. Consider $\text{diag}(e^{it_1}, e^{it_2}) \in S^1 \times S^1$. Let $K(\theta) := e^{(\rho_1 t_1 + \rho_2 t_2)\theta} \in O(k)$. We show that

$$A(\theta) := e^{-it_1\theta} K(\theta) M K(\theta)^T e^{it_2\theta},$$

is constant. Indeed,

$$A'(\theta) = e^{-it_1\theta} K(\theta) (-it_1 M + t_1 [\rho_1, M] + t_2 [\rho_2, M] + t_2 M i) K(\theta)^T e^{it_2\theta} = 0.$$

Hence, A is constant, so

$$e^{-t_1} K(1) M K(1)^T e^{it_2} = A(1) = A(0) = M.$$

As $[\rho_1, R] = 0 = [\rho_2, R]$, $[K(\theta), R] = 0$, so by Lemma 3.1.25, we have that $\hat{M}$ is $\text{diag}(e^{it_1}, e^{it_2})$ -equivariant. As t_1 and t_2 were arbitrary, we have toral symmetry. $\square$

Just as with circular t -symmetry, we need not check the final condition of $\mathcal{M}_{n,k}$ in Definition 3.1.16 everywhere.

Lemma 3.3.4. *Suppose that $\hat{M}$ satisfies (3.3.1) and (3.3.2) for some $\rho_1, \rho_2 \in \mathfrak{so}(k)$ satisfying $[\rho_1, \rho_2] = 0$, as well as the first three conditions of Definition 3.1.16. If the final condition is satisfied at all $x = x_0 + x_2 j \in \mathbb{H}$ with $x_0, x_2 \geq 0$, then $\hat{M} \in \mathcal{M}_{n,k}$.*

Proof. Let $z = z_0 + z_1 i + z_2 j + z_3 k \in \mathbb{H}$. Note that $\text{diag}(e^{i\theta}, e^{i\phi})$ acts on z as $z \mapsto e^{i\theta} z e^{-i\phi}$. Specifically, it takes $z \mapsto (z_0 + z_1 i) e^{i(\theta-\phi)} + (z_2 + z_3 i) e^{i(\theta+\phi)} j$. Thus, there is some $\theta, \phi \in \mathbb{R}$ such that $x := e^{i\theta} z e^{-i\phi}$ has no k or i component and the real and j component are non-negative. The rest follows as in the proof of Lemma 3.2.4. $\square$

3.3.1 Structure of toral symmetry

In light of Theorem 3.3.1, we know that we can find instantons with toral symmetry given $\rho_1, \rho_2 \in \mathfrak{so}(k)$. Note that such matrices generate a real representation of $\mathbb{R} \oplus \mathbb{R}$. Starting with such a real representation, we can narrow down the possible M , so we are only left with finding L such that we have an instanton, which necessarily has toral symmetry.

Much like the case of circular t -symmetry, when t is rational, we can do better than representations of $\mathbb{R} \oplus \mathbb{R}$. Note that a representation $\rho: \mathbb{R} \oplus \mathbb{R} \rightarrow \mathfrak{so}(k)$ corresponds to a representation of $S^1 \times S^1$ if and only if $\exp(\rho(2\pi, 0)) = I_k = \exp(\rho(0, 2\pi))$.

Proposition 3.3.5. *Let $\hat{M} \in \mathcal{M}_{n,k}$. Then $\hat{M}$ has toral symmetry if and only if there exists a real representation $\rho: \mathbb{R} \oplus \mathbb{R} \rightarrow \mathfrak{so}(k)$, corresponding to a representation of $S^1 \times S^1$, such that for all $t_1, t_2 \in \mathbb{R}$,*

$$t_1 iM - t_2 Mi = \frac{1}{2}[\rho(t_1, t_2), M], \quad (3.3.6)$$

$$[\rho(t_1, t_2), R] = 0. \quad (3.3.7)$$

Definition 3.3.6. *We call ρ the **generating representation** of the toral symmetry of $\hat{M}$.*

Proof. If such a ρ exists, by Theorem 3.3.1, we have toral symmetry generated by the pair $\frac{1}{2}\rho(1, 0)$ and $\frac{1}{2}\rho(0, 1)$. Conversely, suppose that $\hat{M}$ has toral symmetry. By Theorem 3.3.1, there exists $\rho_1, \rho_2 \in \mathfrak{so}(k)$ satisfying $[\rho_1, \rho_2] = 0$, (3.3.1), and (3.3.2).

Note that ρ_1 and ρ_2 generate circular 0-symmetry in $(\hat{M}, U)$. However, ρ_1 generates the usual symmetry, whereas ρ_2 generates a conjugate circular 0-symmetry.

As ρ_1 and ρ_2 are diagonalizable and commute, they are simultaneously diagonalizable. Furthermore, noting that the eigenvalues of a matrix in $\mathfrak{so}(k)$ come in pairs $\pm\lambda i$ for $\lambda \in \mathbb{R}$ or single zeros and $[\rho_1, \rho_2] = 0$, there is some $U \in O(k)$ such that

$$\rho_1 = U \text{diag}(\rho_{11}, \dots, \rho_{1m}) U^T \quad \text{and} \quad \rho_2 = U \text{diag}(\rho_{21}, \dots, \rho_{2m}) U^T,$$

where ρ_{1j} is either $\begin{bmatrix} 0 & a_{1j} \\ -a_{1j} & 0 \end{bmatrix}$ for some $a_{1j} \in \mathbb{R}$ or $\rho_{1j} = 0$. Additionally, ρ_{2j} is either

zero or there exists some $a_{2j}, b_{2j} \in \mathbb{R}$ such that $\rho_{2j} = \begin{bmatrix} ib_{2j} & a_{2j} \\ -a_{2j} & ib_{2j} \end{bmatrix}$. This follows from the simultaneous diagonalization of the matrices.

Just as in the proof of Proposition 3.2.5, we can choose the a_{1j} such that $2\rho_1$ generates a representation of S^1 . Importantly, we note that changing the a_{1j} does not affect the commutivity of ρ_1 and ρ_2 . Similarly, we can choose a $U' \in O(k)$ such that the forms of ρ_1 and ρ_2 are swapped. We can then repeat a similar analysis and modify ρ_2 such that $2\rho_2$ generates a representation of S^1 .

As the matrices commute and each generates a representation of S^1 , $\tilde{\rho}: \mathbb{R} \oplus \mathbb{R} \rightarrow \mathfrak{so}(k)$ taking $\tilde{\rho}(t_1, t_2) := 2t_1\rho_1 + 2t_2\rho_2$ satisfies (3.3.6) and generates a representation of $S^1 \times S^1$. That $\tilde{\rho}$ satisfies (3.3.7) follows from Corollary 3.1.27. $\square$

Proposition 3.3.5 tells us that we can use the representation theory of $S^1 \times S^1$ to better understand instantons with toral symmetry. While toral symmetry contains every circular t -symmetry, and not every such symmetry is generated by a representation of S^1 , it is enough that circular t -symmetry, with t rational, can always be generated by a representation of S^1 .

Above, we see that toral symmetry is generated by a representation $(\mathbb{R}^k, \rho)$, corresponding to a representation of $S^1 \times S^1$. This representation was obtained by examining the bottom of (3.3.4) and (3.3.5). By focusing instead on the top of those equations, we obtain an induced representation.

Lemma 3.3.7. *Suppose that $\hat{M} \in \mathcal{M}_{n,k}$ has toral symmetry generated by a representation $(\mathbb{R}^k, \rho)$, which corresponds to a representation of $S^1 \times S^1$. Let $\lambda: \mathbb{R} \oplus \mathbb{R} \rightarrow \mathfrak{sp}(n)$ be defined by $\lambda(t_1, t_2) := L(\rho(t_1, t_2) - 2t_2i)L^\dagger(LL^\dagger)^{-1}$. We have $(\mathbb{H}^n, \lambda)$ is a representation of $\mathbb{R} \oplus \mathbb{R}$, corresponding to a representation of $S^1 \times S^1$.*

Proof. By Proposition 3.3.5, we know that for all $t_1, t_2 \in \mathbb{R}$, (3.3.6) and (3.3.7) hold. Using the former,

$$\begin{aligned} M^\dagger M (\rho(t_1, t_2) - 2t_2i) &= M^\dagger [M, \rho(t_1, t_2)] + M^\dagger \rho(t_1, t_2)M - 2t_2 M^\dagger M i \\ &= M^\dagger \rho(t_1, t_2)M - 2t_1 M^\dagger i M \\ &= [M^\dagger, \rho(t_1, t_2)]M + \rho(t_1, t_2)M^\dagger M - 2t_1 M^\dagger i M \\ &= (\rho(t_1, t_2) - 2t_2i) M^\dagger M. \end{aligned}$$

Hence, $[M^\dagger M, \rho(t_1, t_2) - 2t_2i] = 0$. Then, as $[\rho(t_1, t_2), R] = 0$, by (3.3.7), and R is real, $[\rho(t_1, t_2) - 2t_2i, R] = 0$, so $[L^\dagger L, \rho(t_1, t_2) - 2t_2i] = 0$. Thus, we see that

$$[LL^\dagger, L(\rho(t_1, t_2) - 2t_2i)L^\dagger] = 0.$$

Therefore, $\lambda(t_1, t_2) \in \mathfrak{sp}(n)$.

Just as in the circular t -symmetry case, when $t \in \mathbb{Q}$, we have that

$$e^{\lambda(t_1, t_2)} = L e^{\rho(t_1, t_2) - 2t_2i} L^\dagger (LL^\dagger)^{-1}.$$

As ρ and i commute, we have that as $(\mathbb{R}^k, \rho)$ corresponds to a representation of $S^1 \times S^1$, $e^{\lambda(2\pi, 0)} = I_n = e^{\lambda(0, 2\pi)}$. Therefore, $(\mathbb{H}^n, \lambda)$ corresponds to a representation of $S^1 \times S^1$. $\square$

We know that this induced representation satisfies the following equation.

Corollary 3.3.8. *If $\hat{M} \in \mathcal{M}_{n,k}$ has toral symmetry generated by $(\mathbb{R}^k, \rho)$, then $\lambda(t_1, t_2)L - L\rho(t_1, t_2) + 2t_2Li = 0$, for all $t_1, t_2 \in \mathbb{R}$.*

Proof. This follows from the top components of (3.3.4) and (3.3.5). $\square$

3.3.2 Axially symmetric singular monopoles

In this section, we discuss the connections between toral symmetric instantons and axially symmetric singular monopoles. Just like hyperbolic monopoles, axially symmetric singular monopoles are monopoles symmetric under all rotations about one of their axes.

Proposition 3.3.9. *A singular $\mathrm{Sp}(n)$ -monopole with Dirac type singularities is axially symmetric if and only if its ADHM data has toral symmetry.*

Proof. Recall the correspondence between a singular monopole (Φ, A) and an instanton $\mathbb{A}$ given by (3.2.15). Suppose that we have an axially symmetric singular monopole with Dirac type singularities. Let $p \mapsto R_p$ be the double cover $\mathrm{Sp}(1) \rightarrow \mathrm{SO}(3)$. Then there is some $v \in \mathfrak{sp}(1)$ such that the monopole is equivariant under $R_{e^{v\phi}}$ for all $\phi \in \mathbb{R}$. Then there is some gauge transformation g such that $g \cdot \Phi = R_{e^{v\phi}}^* \Phi$ and $g \cdot A = R_{e^{v\phi}}^* A$. Let $f_{\theta, \phi}: \mathbb{H} \setminus \{0\} \rightarrow \mathbb{H} \setminus \{0\}$ be the rotation $f_{\theta, \phi}(x) := e^{i\theta} x e^{-v\phi}$. It is straightforward to show that the one-form ξ defined in (3.2.15) is invariant under $f_{\theta, \phi}$. That is, $f_{\theta, \phi}^* \xi = \xi$. By Lemma 3.2.22, we have that $\pi \circ f_{\theta, \phi} = R_{e^{v\phi}} \circ \pi$. Then we have that

$$\begin{aligned} f_{\theta, \phi}^* \mathbb{A} &= f_{\theta, \phi}^* \circ \pi^* A + (f_{\theta, \phi}^* \circ \pi^* \Phi) \otimes \xi \\ &= \pi^* \circ R_{e^{v\phi}}^* A + (\pi^* \circ R_{e^{v\phi}}^* \Phi) \otimes \xi \\ &= \pi^*(g) \cdot \pi^* A + \pi^*(g) \cdot \Phi \otimes \xi = \pi^*(g) \cdot \mathbb{A}. \end{aligned} \tag{3.3.8}$$

Thus, $\mathbb{A}$ is equivariant under $f_{\theta, \phi}$.

Conversely, suppose that $\mathbb{A}$ is equivariant under $f_{\theta, \phi}$, as defined above. Then there is some gauge transformation g such that $f_{\theta, \phi}^* \mathbb{A} = g \cdot \mathbb{A}$. Hence,

$$\pi^*(R_{e^{v\phi}}^* A) + \pi^*(R_{e^{v\phi}}^* \Phi) \otimes \xi = f_{\theta, \phi}^* \mathbb{A} = g \cdot \mathbb{A} = g \cdot \pi^* A + g \cdot \pi^* \Phi \otimes \xi. \tag{3.3.9}$$

Evaluating on $\pi^* \partial_l$, recalling that $\xi(\pi^* \partial_l) = 0$ and $d\pi(\pi^* \partial_l) = \partial_l$ by definition, we can decompose the previous expression, finding that the monopole is equivariant under $R_{e^{v\phi}}$. Indeed, the gauge transformation descends as everything but g is already known to be circle-invariant.

Therefore, (Φ, A) is equivariant under all $R_{e^{v\phi}}$ if and only if $\mathbb{A}$ is equivariant under $\mathrm{diag}(e^{i\theta}, e^{v\phi})$ for all $\theta, \phi \in \mathbb{R}$. We see that this subgroup is conjugate to $S^1 \times S^1$, so we have an axially symmetric, singular monopole with Dirac type singularities if and only if its ADHM data has toral symmetry. $\square$

Although there are two commuting circle actions conjugate to R_0 in $S^1 \times S^1$, only one singular monopole can be created from an instanton with toral symmetry using the given Hopf map, as the other circle action does not preserve this map. The Hopf map must be changed in order for another circle action to create a singular monopole.

3.3.3 Axially symmetric hyperbolic monopoles

In this section, we discuss the connections between toral symmetric instantons and axially symmetric hyperbolic monopoles. An axially symmetric hyperbolic monopole is a monopole that is symmetric under rotations about one of its axes. As the subgroups rotating about axes are all conjugate, we focus on rotations about the z -axis.

Proposition 3.3.10. *A hyperbolic $\mathrm{Sp}(n)$ -monopole with integral mass is axially symmetric if and only if its ADHM data has toral symmetry.*

Proof. Through isometries, we may assume that the hyperbolic model is the half-space model and the monopole is symmetric about the z -axis. Because of the conformal equivalence of $S^4 \setminus S^2 \equiv H^3 \times S^1$, these isometries on hyperbolic space correspond to conformal maps on S^4 .

The action of R_1 induces the half-space model of hyperbolic space, with coordinates (x_0, x_1, r) with $r > 0$. The two-sphere removed is $\mathbb{C} \cup \{\infty\}$. Letting θ be the coordinate of S^1 , we have that a point $(x_0, x_1, r, \theta) \in H^3 \times S^1$ corresponds to a point $x_0 + x_1 i + r \cos \theta j + r \sin \theta k \in \mathbb{H}$.

We now look at how $S^1 \times S^1$ acts on these coordinates. Given $x \in \mathbb{H}$, write $x = x_0 + x_1 i + r \cos \theta j + r \sin \theta k$. Also, let $\mathrm{diag}(e^{i(\phi_1+\phi_2)}, e^{i(\phi_1-\phi_2)}) \in S^1 \times S^1$. This isometry takes $x \mapsto e^{i(\phi_1+\phi_2)} x e^{-i(\phi_1-\phi_2)}$. Simplifying, we see that x_0, x_1, r are transformed via

$$\begin{bmatrix} x_0 \\ x_1 \\ r \end{bmatrix} \mapsto \begin{bmatrix} \cos 2\phi_2 & -\sin 2\phi_2 & 0 \\ \sin 2\phi_2 & \cos 2\phi_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ r \end{bmatrix}.$$

Additionally, $\theta \mapsto \theta - 2\phi_1$. Thus, we see that this rotates H^3 about the z -axis by $2\phi_2$ and rotates S^1 by $-2\phi_1$.

Now that we understand the relationship between the toral action on S^4 and the corresponding action on $H^3 \times S^1$, we can proceed. Note the correspondence between a hyperbolic monopole with integral mass (Φ, A) and an instanton $\mathbb{A}$ is given by $\mathbb{A} = \pi_1^* A + \pi_1^* \Phi d\theta$, where $\pi_1: S^4 \setminus S^2 \rightarrow H^3$. As $\pi^* \Phi d\theta$ and $\pi^* A$ are orthogonal, we can proceed as in the proof of Proposition 3.3.9. Ultimately, we have a hyperbolic monopole with integral mass is axially symmetric if and only if its ADHM data has toral symmetry. $\square$

Unlike singular monopoles, as there are two commuting circle actions conjugate to R_1 in $S^1 \times S^1$, different hyperbolic monopoles can be constructed from one instanton with toral symmetry by using different circle actions conjugate to R_1 . In Proposition 3.5.24, we see that these monopoles need not be gauge equivalent.

3.4 Spherical symmetry

In this section, we find equations describing all instantons with the various kinds of spherical symmetry, as given in Table C.1. We separate the cases of simple, isoclinic, and conformal spherical symmetry. We also discuss the connections between these symmetric instantons and hyperbolic analogues of Higgs bundles and Nahm data. Finally, we discuss instantons with full symmetry.

Note 3.4.1. *All connected Lie subgroups of $\mathrm{Sp}(2)$ with Lie algebra $\mathfrak{sp}(1)$ are conjugate to the connected Lie subgroups $G_{3,1,1}$, $G_{4,1}$, or G_5 induced by $\mathfrak{h}_{3,1,1}$, $\mathfrak{h}_{4,1}$, and $\mathfrak{h}_5$, respectively. That is, there is some $A \in \mathrm{Sp}(2)$ and G one of the subgroups $G_{3,1,1}$, $G_{4,1}$, or G_5 such that the Lie group is of the form $AGA^\dagger$. Instantons equivariant under this group are of the form $A^\dagger \cdot (\hat{M}, U)$, where $\hat{M}$ is equivariant under G .*

Definition 3.4.2. *Let ε be the Levi-Civita symbol. Let $(v_1, v_2, v_3) := (i/2, j/2, k/2)$ be the **standard basis** of $\mathfrak{sp}(1)$. Note that $[v_l, v_m] = \sum_{p=1}^3 \varepsilon_{lmp} v_p$.*

3.4.1 Simple spherical symmetry

In this section, we find an equation describing all instantons with simple spherical symmetry, as given in Table C.1 as well as below. We also discuss the connections between these instantons and hyperbolic monopoles with no continuous symmetries and a hyperbolic analogue to Higgs bundles.

First, we reintroduce the notion of simple spherical symmetry. An instanton is said to have simple spherical symmetry if it is equivariant under $\mathrm{diag}(p, p)$ for all $p \in \mathrm{Sp}(1)$.

Theorem 3.4.3. *Let $\hat{M} \in \mathcal{M}_{n,k}$. Then $\hat{M}$ has simple spherical symmetry if and only if there exists a real representation $\rho: \mathfrak{sp}(1) \rightarrow \mathfrak{so}(k)$ such that for all $v \in \mathfrak{sp}(1)$,*

$$[v, M] + [\rho(v), M] = 0, \quad (3.4.1)$$

$$[\rho(v), R] = 0. \quad (3.4.2)$$

Definition 3.4.4. *We call ρ the **generating representation** of the simple spherical symmetry of $\hat{M}$.*

Proof. We follow the proof of Theorem 1.1.1, making modifications when relevant. Suppose that $\hat{M}$ has simple spherical symmetry. Let $S \subseteq \mathrm{Sp}(1) \times \mathrm{O}(k)$ be the stabilizer group of

$(\hat{M}, U)$ restricted to rotations in the diagonal embedding of $\mathrm{Sp}(1)$ into $\mathrm{Sp}(2)$. That is

$$S := \{(\mathrm{diag}(p, p), K) \mid (\mathrm{diag}(LpKL^\dagger(LL^\dagger)^{-1}, pK), K) \cdot \mathrm{diag}(p^\dagger, p^\dagger) \cdot (\hat{M}, U) = (\hat{M}, U)\}. \quad (3.4.3)$$

Indeed, if $(Q, K) \in \mathrm{Sp}(n+k) \times \mathrm{GL}(k, \mathbb{R})$ such that $\mathrm{diag}(p^\dagger, p^\dagger) \cdot (Q, K) \cdot (\hat{M}, U) = (\hat{M}, U)$, then Corollary 3.1.27 tells us that $K \in \mathrm{O}(k)$, $[R, K] = 0$, and $Q = \mathrm{diag}(LpKL^\dagger(LL^\dagger)^{-1}, pK)$. Therefore, the pairs in S encapsulate all the simple spherical symmetry of the instanton. That S is a group follows from this and that the conformal action is a right Lie group action. Indeed, it is easy to see that S contains the identity and is closed under inversion. To see that S is closed under multiplication, suppose that $(p_1 I_2, K_1), (p_2 I_2, K_2) \in S$. Then there exists unique $Q_1, Q_2 \in \mathrm{Sp}(n+k)$ such that for $l = 1, 2$,

$$(Q_l, K_l) \cdot p_l^\dagger I_2 \cdot (\hat{M}, U) = (\hat{M}, U).$$

Then we see that

$$(Q_1 Q_2, K_1 K_2) \cdot (p_1 p_2)^\dagger I_2 \cdot (\hat{M}, U) = (Q_1, K_1) \cdot (Q_2, K_2) \cdot p_1^\dagger I_2 \cdot p_2^\dagger I_2 \cdot (\hat{M}, U).$$

As the two actions commute, we see that $(Q_1 Q_2, K_1 K_2) \cdot (p_1 p_2)^\dagger I_2 \cdot (\hat{M}, U) = (\hat{M}, U)$. By Corollary 3.1.27, $Q_1 Q_2$ must have the correct form such that $(p_1 p_2 I_2, K_1 K_2) \in S$. That is, S is a group.

Just as in Theorem 1.1.1, S is a compact Lie group and we obtain a Lie group homomorphism $\Psi: \mathrm{Sp}(1) \rightarrow S$ such that $\pi_1|_S \circ \Psi = \mathrm{id}_{\mathrm{Sp}(1)}$.

Additionally, there is a Lie algebra homomorphism $\rho: \mathfrak{sp}(1) \rightarrow \mathfrak{so}(k)$ such that for all $v \in \mathfrak{sp}(1)$, $\Psi(e^v) = (\mathrm{diag}(e^v, e^v), e^{\rho(v)})$. As ρ is a Lie algebra homomorphism whose image consists of real matrices, it gives us a real Lie algebra representation.

As $\Psi(e^{\theta v}) \in S$ for all $\theta \in \mathbb{R}$ and $v \in \mathfrak{sp}(1)$, we have that

$$(\mathrm{diag}(Le^{\theta v} e^{\theta \rho(v)} L^\dagger (LL^\dagger)^{-1}, e^{\theta v} e^{\theta \rho(v)}), e^{\theta \rho(v)}) \cdot \mathrm{diag}(e^{-\theta v}, e^{-\theta v}) \cdot (\hat{M}, U) = (\hat{M}, U)$$

Differentiating and evaluating at $\theta = 0$, we obtain

$$\begin{bmatrix} LvL^\dagger(LL^\dagger)^{-1}L + L\rho(v)L^\dagger(LL^\dagger)^{-1}L - L\rho(v) - Lv \\ [v, M] + [\rho(v), M] \end{bmatrix} = 0. \quad (3.4.4)$$

In Note 3.4.22, we discuss the top row. For now, focusing on the bottom row, we see that $[v, M] + [\rho(v), M] = 0$. Furthermore, as mentioned above, Corollary 3.1.27 tells us that $[R, e^{\theta \rho(v)}] = 0$, for all $\theta \in \mathbb{R}$. Differentiating and evaluating at $\theta = 0$, we have $[R, \rho(v)] = 0$.

Conversely, suppose that the equations hold for some real representation $\rho: \mathfrak{sp}(1) \rightarrow \mathfrak{so}(k)$. Consider $p \in \mathrm{Sp}(1)$. As the exponential map $\exp: \mathfrak{sp}(1) \rightarrow \mathrm{Sp}(1)$ is surjective, there is some $v \in \mathfrak{sp}(1)$ such that $p = e^v$. Let $p(\theta) := e^{\theta v} \in \mathrm{Sp}(1)$ and $K(\theta) := e^{\theta \rho(v)} \in \mathrm{O}(k)$. We show that $A(\theta) := p(\theta)K(\theta)MK(\theta)^T p(\theta)^\dagger$ is constant. Indeed,

$$A'(\theta) = p(\theta)K(\theta)([v, M] + [\rho(v), M])K(\theta)^T p(\theta)^\dagger = 0.$$

Hence, A is constant, so

$$pK(1)MK(1)^T p^\dagger = A(1) = A(0) = M.$$

As $[\rho(v), R] = 0$, $[K(\theta), R] = 0$, so by Lemma 3.1.25, we have that $\hat{M}$ is $\mathrm{diag}(p^\dagger, p^\dagger)$ -equivariant. As p is arbitrary, we have simple spherical symmetry. $\square$

Just as with circular t -symmetry, we need not check the final condition of $\mathcal{M}_{n,k}$ in Definition 3.1.16 everywhere.

Lemma 3.4.5. *Suppose that $\hat{M}$ satisfies (3.4.1) and (3.4.2) for some real representation $(\mathbb{R}^k, \rho)$ as well as the first three conditions of Definition 3.1.16. If the final condition is satisfied at all $x = x_0 + x_1 i \in \mathbb{H}$ with $x_1 \geq 0$, then $\hat{M} \in \mathcal{M}_{n,k}$.*

Proof. Let $z = z_0 + z_1 i + z_2 j + z_3 k \in \mathbb{H}$. Note that $z - z_0 \in \mathfrak{sp}(1)$. Recall that for $p \in \mathrm{Sp}(1)$ and $x \in \mathfrak{sp}(1)$, $x \mapsto p x p^\dagger$ corresponds to rotating x by $R_p \in \mathrm{SO}(3)$. There exists some $p \in \mathrm{Sp}(1)$ and $x_1 \geq 0$ such that $p(z - z_0)p^\dagger = x_1 i$. We see that $p z p^\dagger = z_0 + x_1 i =: x$. The rest follows as in the proof of Lemma 3.2.4. $\square$

3.4.1.1 Structure of simple spherical symmetry

Theorem 3.4.3 tells us exactly how to search for instantons with simple spherical symmetry: use a real k -representation of $\mathfrak{sp}(1)$ to narrow down the possible M . We are only left with finding L such that $\hat{M} \in \mathcal{M}_{n,k}$. In this section, we investigate what representations generate instantons with simple spherical symmetry and what the corresponding ADHM data looks like.

Note 3.4.6. *In Section 2.1.3.1, we discuss the representation theory of $\mathfrak{sp}(1)$ and define the notation for its irreducible real and complex representations.*

Definition 3.4.7. Let $\text{ad}: \mathfrak{sp}(1) \rightarrow \mathfrak{gl}(\mathbb{H})$ be the Lie algebra homomorphism $\text{ad}(x)(z) := [x, z]$. Consider the real representation $(\mathbb{H}, \text{ad})$ and let $V := \mathbb{R}^k$. Given a real representation (V, ρ) of $\mathfrak{sp}(1)$, we define the induced real representation

$$(\hat{V}, \hat{\rho}) := (V, \rho) \otimes_{\mathbb{R}} (V^*, \rho^*) \otimes_{\mathbb{R}} (\mathbb{H}, \text{ad}). \quad (3.4.5)$$

Unraveling how $\hat{\rho}$ acts on $\hat{V} = \text{Mat}(k, k, \mathbb{H})$, let $v \in \mathfrak{sp}(1)$ and $A \in \text{Mat}(k, k, \mathbb{H})$. Then

$$\hat{\rho}(v)(A) = [\rho(v), A] + [v, A]. \quad (3.4.6)$$

Note 3.4.8. As $\mathfrak{sp}(1)$ has Dynkin diagram A_1 , by Proposition D.2.9 in Appendix D, all representations of $\mathfrak{sp}(1)$ are self-dual.

Note 3.4.9. Note that we can restrict the action of $\text{ad}(x)$ to $\mathfrak{sp}(1)$. Denote the adjoint representation of $\mathfrak{sp}(1)$ by $(\mathfrak{sp}(1), \text{ad})$ and the trivial representation by $(\mathbb{R}, 0)$. The real representation $(\mathbb{H}, \text{ad})$ decomposes as

$$(\mathbb{H}, \text{ad}) = (\mathfrak{sp}(1), \text{ad}) \oplus (\mathbb{R}, 0). \quad (3.4.7)$$

The definition of $(\hat{V}, \hat{\rho})$ is well-motivated. Firstly, note that $M \in \hat{V}$. Secondly, given the connection between the action of $\hat{\rho}(v)$ and (3.4.1), we immediately obtain the following corollary.

Corollary 3.4.10. Let $\hat{M} \in \mathcal{M}_{n,k}$. Then $\hat{M}$ has simple spherical symmetry if and only if there is some real k -representation (V, ρ) such that $\hat{\rho}(v)(M) = 0$ and $[\rho(v), R] = 0$ for all $v \in \mathfrak{sp}(1)$.

Note 3.4.11. If R is proportional to the identity, then it automatically commutes with everything, so we can ignore the commutators with R . Hence, such a condition does not appear anywhere in Section 2.1.

Note 3.4.12. Suppose that $\hat{M} \in \mathcal{M}_{n,k}$ has simple spherical symmetry. Corollary 3.4.10 tells us that there is some real representation (V, ρ) of $\mathfrak{sp}(1)$ such that $\text{span}(M) \subseteq \hat{V}$ is an invariant subspace, which is acted on trivially.

As $\mathfrak{sp}(1)$ is semi-simple, the representation $(\hat{V}, \hat{\rho})$ decomposes into irreducible representations. We explore the case $M = 0$ below, so suppose that $M \neq 0$. As $\text{span}(M)$ is a one-dimensional invariant subspace, acted on trivially, $(\text{span}(M), 0)$ is a summand of the representation $(\hat{V}, \hat{\rho})$. So the decomposition of $(\hat{V}, \hat{\rho})$ must contain trivial summands. Moreover, M is in the direct sum of these trivial summands. Trivially, if $M = 0$, then it is in the direct sum of the trivial summands as well.

Proposition 3.4.13 (The $M = 0$ case). *Suppose $M = 0_k$ and $\hat{M} \in \mathcal{M}_{n,k}$. Then $n = k$ and there is some gauge in which L is diagonal and positive-definite. Such a $\hat{M}$ has rotational symmetry.*

Let the norm of an element $A \in \mathfrak{sp}(k)$ be given by $|A|^2 := -\frac{1}{k}\text{Tr}(A^2)$. Then we have that the corresponding instanton is given, up to gauge, by

$$\mathbb{A} = \frac{xdx^\dagger - (dx)x^\dagger}{2} \cdot \text{diag} \left(\frac{1}{|x|^2 + \alpha_1^2}, \dots, \frac{1}{|x|^2 + \alpha_k^2} \right). \quad (3.4.8)$$

Note 3.4.14. *While such instantons have rotational symmetry, one class in the moduli space has full symmetry (the class with $L \in \text{Sp}(k)$). However, we recall that if all α_i are equal, then the symmetry group of the instanton is conjugate to $\text{Sp}(2)$. This serves as a reminder that if an instanton possesses a certain kind of symmetry, it may possess more.*

Proof. Suppose $M = 0$. We know that $n \leq k$. As $\text{rank}(L^\dagger L) \leq n$ and $R := L^\dagger L + M^\dagger M = L^\dagger L$ is non-singular, we see that $n = k$. As R is real and positive-definite, it has a unique symmetric square root $R^{1/2}$. Furthermore, as R is positive-definite, $R^{1/2}$ is non-singular. Gauging L by $R^{-1/2}L^\dagger$, we obtain $R^{1/2}$. As $R^{1/2}$ is real and symmetric, it is orthogonally diagonalizable. Hence, we can gauge our $\hat{M}$ by such a matrix so as to ensure that L is diagonal and positive-definite. That is, there are $\alpha_1, \dots, \alpha_k > 0$ such that $L = \text{diag}(\alpha_1, \dots, \alpha_k)$.

We now show that the corresponding instanton has rotational symmetry. Consider $\text{diag}(p, q) \in \text{Sp}(2)$. Let $(\hat{M}', U') := \text{diag}(p, q) \cdot (\hat{M}, U)$. We see that $\hat{M}' = \hat{M}q$ and $U' = Up$. Let $Q := \text{diag}(q^\dagger, p^\dagger) \otimes I_k \in \text{Sp}(2k)$. Simplifying, we see that $Q\hat{M}' = \hat{M}$ and $QU' = U$. That is,

$$(Q, I_k) \cdot \text{diag}(p, q) \cdot (\hat{M}, U) = (\hat{M}, U).$$

Hence, $\hat{M}$ is $\text{diag}(p, q)$ -equivariant. As the transformation was arbitrary, $\hat{M}$ has rotational symmetry.

Finally, we compute the instanton. Let $\{e_1, \dots, e_k\}$ be the standard basis for $\mathbb{H}^k$. For $l \in \{1, \dots, k\}$, let $v_l(x) := \frac{1}{\sqrt{\alpha_l^2 + |x|^2}} \begin{bmatrix} x^\dagger e_l \\ \alpha_l e_l \end{bmatrix}$. Then let $V(x) := \begin{bmatrix} v_1(x) & \dots & v_k(x) \end{bmatrix}$. We see that $V(x)^\dagger \Delta(x) = 0$ and $V(x)^\dagger V(x) = I_k$. Therefore, we can use V to construct our instanton. Simplifying $\mathbb{A} = V(x)^\dagger dV(x)$, we obtain the desired form for $\mathbb{A}$. $\square$

Note 3.4.15. *Proposition 3.4.13 tells us that $\text{Sp}(k)$ instantons with $M = 0$ are reducible, being formed from the class of $\text{Sp}(1)$ instantons with rotational symmetry and $M = 0$.*

Recall the discussion regarding irreducible representations of $\mathfrak{sp}(1)$ in Section 2.1.3.1, based on the results of Section D.1.3 in Appendix D. We are interested in the trivial summands of $(\hat{V}, \hat{\rho})$.

Lemma 3.4.16. *Given $m \geq n \geq 1$, consider $(V_m, \rho_m) \otimes_{\mathbb{C}} (V_n, \rho_n) \otimes_{\mathbb{C}} (V_3, \rho_3)$ and $(V_m, \rho_m) \otimes_{\mathbb{C}} (V_n, \rho_n) \otimes_{\mathbb{C}} (V_1, \rho_1)$. The former has a single trivial summand when $m = n \geq 2$ or $m = n+2$. The latter has a single trivial summand when $m = n \geq 1$. Otherwise, there are no trivial summands.*

Proof. The first statement follows from Lemma 2.1.32. The second statement follows from the Clebsch–Gordan decomposition, noting that taking the tensor product with (V_1, ρ_1) is just an isomorphism. $\square$

Given $A \in \text{Mat}(k, k, \mathbb{H})$, let $A_0 \in \text{Mat}(k, k, \mathbb{R})$ and $\vec{A} \in \text{Mat}(k, k, \mathfrak{sp}(1))$ such that $A = A_0 + \vec{A}$. From the decomposition of $(\mathbb{H}, \text{ad}) = (\mathfrak{sp}(1), \text{ad}) \oplus (\mathbb{R}, 0)$, we have that $\text{span}(M_0)$ is an invariant subspace of $(V, \rho) \otimes_{\mathbb{R}} (V^*, \rho^*) \otimes_{\mathbb{R}} (\mathbb{R}, 0)$ and $\text{span}(\vec{M})$ is an invariant subspace of $(V, \rho) \otimes_{\mathbb{R}} (V^*, \rho^*) \otimes_{\mathbb{R}} (\mathbb{R}^3, \varrho_3)$, each of which is acted on trivially. Therefore, the representation theory used to construct $\vec{M}$ is the same as that used for M in Section 2.1.3.

In Section 2.1.3.1, specifically from Definition 2.1.33 to Lemma 2.1.38, we investigate the trivial summands of $(V_m, \rho_m) \otimes_{\mathbb{C}} (V_n, \rho_n) \otimes_{\mathbb{C}} (V_3, \rho_3)$. We now investigate the remaining trivial summands.

Lemma 3.4.17. *Given $m \geq n \geq 1$, we have that $(\mathbb{R}^m, \varrho_m) \otimes_{\mathbb{R}} ((\mathbb{R}^n)^*, \varrho_n^*) \otimes_{\mathbb{R}} (\mathbb{R}, 0)$ has a single trivial summand when $m = n$, both odd, and four trivial summands when $m = n$, both divisible by four. Otherwise, there are no trivial summands.*

Proof. Note that if m and n are both odd, then the complexification of the real representation is isomorphic, as a complex representation, to $(V_m, \rho_m) \otimes_{\mathbb{C}} (V_n, \rho_n) \otimes_{\mathbb{C}} (V_1, \rho_1)$. By Lemma 3.4.16, we know that this tensor product has a single trivial summand when $m = n$ and none otherwise. Furthermore, as this tensor product only contains odd dimensional summands, the representation is real, meaning that the real decomposition is the same as the complex. Thus, we have proven the first part of the lemma.

Note that if m and n are both divisible by four, then the complexification of the real representation is isomorphic, as a complex representation, to $(V_{m/2}, \rho_{m/2})^{\oplus 2} \otimes_{\mathbb{C}} (V_{n/2}, \rho_{n/2})^{\oplus 2} \otimes_{\mathbb{C}} (V_1, \rho_1)$. By Lemma 3.4.16, we know that this has four trivial summands when $m = n$ and none otherwise. Just as above, this decomposition contains only odd dimensional summands, so the representation is real, meaning that the real decomposition is the same as the complex. Thus, we have proven the second part of the lemma.

Finally, suppose that one of m or n is odd and the other divisible by four. Then, decomposing the complexification of the real representation as a complex representation and expanding the tensor product, we find that the tensor product contains only even dimensional summands, meaning there are no trivial summands, proving the lemma. $\square$

We now identify the trivial summands discussed above.

Lemma 3.4.18. *Given $n \in \mathbb{N}_+$ odd, I_n spans the unique trivial summand of*

$$(\mathbb{R}^n, \varrho_n) \otimes_{\mathbb{R}} ((\mathbb{R}^n)^*, \varrho_n^*) \otimes_{\mathbb{R}} (\mathbb{R}, 0).$$

Given $n \in \mathbb{N}_+$ divisible by four, there are four trivial summands of

$$(\mathbb{R}^n, \varrho_n) \otimes_{\mathbb{R}} ((\mathbb{R}^n)^*, \varrho_n^*) \otimes_{\mathbb{R}} (\mathbb{R}, 0).$$

Let $Y_i := \varrho_n(v_i) \in \mathfrak{so}(n)$ induce the irreducible real n -representation and $y_i := \rho_{n/2}(v_i) \in \mathfrak{su}(\frac{n}{2})$ induce the irreducible complex $\frac{n}{2}$ -representation. Let $U \in \mathrm{SU}(\frac{n}{2})$ such that

$$Y_i = U^\dagger \begin{bmatrix} y_i & 0 \\ 0 & y_i \end{bmatrix} U.$$

The trivial summands are spanned by the following real matrices

$$I_n, \quad U^\dagger \begin{bmatrix} iI_{n/2} & 0 \\ 0 & -iI_{n/2} \end{bmatrix} U, \quad U^\dagger \begin{bmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{bmatrix} U, \quad U^\dagger \begin{bmatrix} 0 & iI_{n/2} \\ iI_{n/2} & 0 \end{bmatrix} U. \quad (3.4.9)$$

Note 3.4.19. *While the first matrix, I_n , in (3.4.9) is clearly symmetric, the remaining three matrices are antisymmetric. Indeed, as they are real, the transpose of these matrices agrees with their Hermitian conjugate, which we see results in multiplying the latter three matrices by -1 .*

Proof. A matrix A is acted on trivially by $(\mathbb{R}^n, \varrho_n) \otimes_{\mathbb{R}} ((\mathbb{R}^n)^*, \varrho_n^*) \otimes_{\mathbb{R}} (\mathbb{R}, 0)$ if and only if $[\varrho_n(v), A] = 0$ for all $v \in \mathfrak{sp}(1)$. The result follows from Lemma 2.1.38. $\square$

Note 3.4.20. *We have that $(R, 0)$ is a trivial summand of $(V, \rho) \otimes_{\mathbb{R}} (V^*, \rho^*)$, as R is acted on trivially by this representation. From the above work, we know exactly where R must live. Moreover, this constraint helps us identify L , as R requires the entirety of $\hat{M}$.*

Now that we know exactly what spans the trivial summands of $(\hat{V}, \hat{\rho})$, the following theorem tells us exactly what form M takes if $\hat{M}$ has simple spherical symmetry.

Theorem 3.4.21 (Simple Spherical Structure Theorem). *Let $\hat{M} \in \mathcal{M}_{n,k}$ have simple spherical symmetry. Theorem 3.4.3 tells us that $\hat{M}$ is generated by a real representation (V, ρ) of $\mathfrak{sp}(1)$, which we can decompose as*

$$(V, \rho) \simeq \bigoplus_{a=1}^m (\mathbb{R}^{n_a}, \varrho_{n_a}). \quad (3.4.10)$$

Without loss of generality, we may assume that $n_a \geq n_{a+1}$, for all $a \in \{1, \dots, m-1\}$.

Let $Y_{i,a} := \varrho_{n_a}(v_i) \in \mathfrak{so}(n_a)$ and

$$Y_i := \text{diag}(Y_{i,1}, \dots, Y_{i,m}) \in \mathfrak{so}(k).$$

Then Y_i induces (V, ρ) . For n_a is divisible by four, let $y_{i,a} := \rho_{\frac{n_a}{2}}(v_i) \in \mathfrak{su}(\frac{n_a}{2})$ induce the irreducible $\frac{n_a}{2}$ complex representation. Then there is some $U_{n_a} \in \text{SU}(n_a)$ such that

$$Y_{i,a} = U_{n_a}^\dagger \begin{bmatrix} y_{i,a} & 0 \\ 0 & y_{i,a} \end{bmatrix} U_{n_a}.$$

Using the decomposition of (V, ρ) given in (3.4.10), we have $(M_\mu)_{ab} \in \mathbb{R}^{n_a} \otimes_{\mathbb{R}} (\mathbb{R}^{n_b})^*$ such that,

$$M_\mu = \begin{bmatrix} (M_\mu)_{11} & \cdots & (M_\mu)_{1m} \\ \vdots & \ddots & \vdots \\ (M_\mu)_{m1} & \cdots & (M_\mu)_{mm} \end{bmatrix}.$$

Then, up to a ρ -invariant gauge and for all $a, b \in \{1, \dots, m\}$ we have that $\vec{M} = M_1 i + M_2 j + M_3 k$ is given by the Structure Theorem for hyperbolic monopoles, Theorem 2.1.39. Additionally, up to the same ρ -invariant gauge and for all $a, b \in \{1, \dots, m\}$, we have that

(1) if $a = b$, then $\exists \lambda_a \in \mathbb{R}$ such that

$$(M_0)_{aa} = \lambda_a I_{n_a};$$

(2) if $a < b$ and $n_a = n_b$ are odd, then $\exists \lambda_{a,b} \in \mathbb{R}$ such that

$$(M_0)_{ab} = \lambda_{a,b} I_{n_a}, \quad \text{and} \quad (M_0)_{ba} = \lambda_{a,b} I_{n_a};$$

(3) if $a < b$ and $n_a = n_b$ are divisible by four, then $\exists \kappa_{a,0}, \kappa_{a,1}, \kappa_{a,2}, \kappa_{a,3} \in \mathbb{R}$ such that

$$(M_0)_{ab} = U_{n_a}^\dagger \begin{bmatrix} (\kappa_{a,0} + \kappa_{a,1}i)I_{n_a/2} & (\kappa_{a,2} + \kappa_{a,3}i)I_{n_a/2} \\ (-\kappa_{a,2} + \kappa_{a,3}i)I_{n_a/2} & (\kappa_{a,0} - \kappa_{a,1}i)I_{n_a/2} \end{bmatrix} U_{n_a}, \quad \text{and} \\ (M_0)_{ba} = U_{n_a}^\dagger \begin{bmatrix} (\kappa_{a,0} - \kappa_{a,1}i)I_{n_a/2} & -\kappa_{a,2} - \kappa_{a,3}i)I_{n_a/2} \\ (\kappa_{a,2} - \kappa_{a,3}i)I_{n_a/2} & (\kappa_{a,0} + \kappa_{a,1}i)I_{n_a/2} \end{bmatrix} U_{n_a};$$

(4) otherwise, $(M_0)_{ab} = 0$.

Conversely, if M_0 has the above form, $\vec{M}$ has the form given in the Structure Theorem for hyperbolic monopoles, Theorem 2.1.39, for some real representation (V, ρ) , and $[\rho(v), R] = 0$ for all $v \in \mathfrak{sp}(1)$, then $\hat{M}$ has simple spherical symmetry.

Proof. The proof follows the same structure as the proof of the Structure Theorem for hyperbolic monopoles, Theorem 2.1.39. $\square$

Examples of ADHM data with simple spherical symmetry are given in Propositions 2.1.40, 2.1.42, and 2.1.44, though there they are viewed through the lens of hyperbolic monopoles. Additionally, in Section 3.5.2.1, we construct a novel example of ADHM data with conformal superspherical symmetry, meaning this data has simple spherical symmetry.

Note 3.4.22. We can consider the constraints on the structure group of instantons with simple spherical symmetry just as we did for hyperbolic monopoles. In this case, the top row of (3.4.4) gives us the same induced representation as in the hyperbolic case examined in Section 2.1.3.3. Therefore, we do not retread the same ground here.

3.4.1.2 Non-symmetric hyperbolic monopoles

In this section, we discuss the connection between instantons with simple spherical symmetry and hyperbolic monopoles with integral mass and no continuous symmetries.

Note that the simple spherical subgroup of $\mathrm{Sp}(2)$ contains R_1 as a subgroup, whereas the isoclinic and conformal spherical subgroups do not even contain a subgroup conjugate to R_1 . Hence, we can associate to an instanton with simple spherical symmetry a hyperbolic monopole with integral mass. Unlike toral symmetry, simple spherical symmetry does not impart extra symmetry to hyperbolic monopoles. The lack of additional symmetry is because the rest of the simple spherical symmetry does not commute with the R_1 subgroup.

Indeed, suppose that $\text{diag}(p, p)$ descends to a symmetry of the hyperbolic space. As the circle action only acts on the S^1 component of $H^3 \times S^1$, we have that at least on $S^4 \setminus S^2$, $\text{diag}(p, p)$ and the R_1 subgroup must commute. As $S^2 \subseteq S^4$ is a codimension two subspace, and the commutator is a smooth map, $\text{diag}(p, p)$ and R_1 must commute everywhere. Thus, we must have $[p, e^{i\theta}] = 0$ for all $\theta \in \mathbb{R}$. This only happens when $p \in \mathbb{C}$. Thus, the only part of the simple spherical group that descends to a symmetry of $H^3 \times S^1$ is the R_1 subgroup, which only acts on the S^1 component.

Note that an instanton with simple spherical symmetry can possess more symmetry, in which case the hyperbolic monopole may have a continuous symmetry. In Section 3.5.2.2, we discuss the connection between instantons with conformal superspherical symmetry and hyperbolic monopoles with different symmetries. If the instanton has simple spherical symmetry and does not have conformal superspherical symmetry, then the corresponding hyperbolic monopole has no continuous symmetries.

3.4.1.3 Hyperbolic Higgs bundles

In this section, we discuss the connection between instantons with simple spherical symmetry and a hyperbolic analogue to Higgs bundles.

Just as Euclidean monopoles, Higgs bundles are a dimensional reduction of instantons. However, while Euclidean monopoles are invariant under translation along one axis, Higgs bundles are invariant under translations along two axes. In Section 3.2.2, we cover the connection between hyperbolic monopoles and circle-invariant instantons. This connection is due to the conformal equivalence $S^4 \setminus S^2 \equiv H^3 \times S^1$. There is a whole family of these conformal equivalences. Indeed, for $d \in \{0, 1, 2\}$ we have $S^4 \setminus S^d \equiv H^{d+1} \times S^{4-d-1}$. Note that we can replace $S^4 \setminus S^d$ with $\mathbb{R}^4 \setminus \mathbb{R}^d$. The hyperbolic monopole case $d = 2$. Here, we investigate the case $d = 1$. In Section 3.4.2.5, we study the case $d = 0$.

While searching for multi-instanton solutions, Witten searched for $\text{Sp}(1)$ -instantons that have simple spherical symmetry, in our language [Wit77]. In doing so, Witten found that the symmetric $\text{Sp}(1)$ -instantons he was searching for correspond to vortices in the abelian Higgs model on the hyperbolic plane. Moreover, these vortices can be found by solving the Liouville equation. By embedding these vortices into three dimensions, Maldonado was able to generate hyperbolic monopoles [Mal17]. Here, we examine such symmetric instantons with higher rank structure group, where we do not necessarily obtain the abelian Higgs model.

Note that the fixed point set of the simple spherical action on $\mathbb{R}^4$ is $\mathbb{R}$. Moreover, for $x \notin \mathbb{R}$, we have that the orbit of x under the simple spherical action is S^2 . Indeed, given

$x = x_0 + \vec{x}$, where $x_0 \in \mathbb{R}$ and $\vec{x} \in \mathfrak{sp}(1)$, we note that the simple spherical action of $p \in \mathrm{Sp}(1)$ on x corresponds to fixing x_0 and rotating $\vec{x}$ by the element in $\mathrm{SO}(3)$ given by the double cover $\mathrm{Sp}(1) \rightarrow \mathrm{SO}(3)$.

On $\mathbb{R}^4 \setminus \mathbb{R}$, we can use the coordinates (x_0, r, θ, ϕ) , where the $\mathfrak{sp}(1)$ part is written in spherical coordinates. Doing so, the metric on $\mathbb{R}^4 \setminus \mathbb{R}$ is given by

$$ds^2 = dx_0^2 + dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

As $r \neq 0$ on this set, we can divide by r^2 , obtaining the metric of $H^2 \times S^2$. Specifically, we note that we obtain the upper-half space model of the hyperbolic plane.

Therefore, just as in the hyperbolic monopole case, an instanton with simple spherical symmetry corresponds to a hyperbolic Higgs bundle and vice versa.

3.4.2 Isoclinic spherical symmetry

In this section, we find an equation describing all instantons with isoclinic spherical symmetry, as given in Table C.1 as well as below. We also discuss the connections between these instantons and singular monopoles with no continuous symmetries and a hyperbolic analogue to Nahm data.

First, we reintroduce the notion of isoclinic spherical symmetry. An instanton is said to have isoclinic spherical symmetry if it is equivariant under $\mathrm{diag}(p, 1)$ for all $p \in \mathrm{Sp}(1)$.

Theorem 3.4.23. *Let $\hat{M} \in \mathcal{M}_{n,k}$. Then $\hat{M}$ has isoclinic spherical symmetry if and only if there exists a real representation $\rho: \mathfrak{sp}(1) \rightarrow \mathfrak{so}(k)$ such that for all $v \in \mathfrak{sp}(1)$,*

$$vM + [\rho(v), M] = 0, \tag{3.4.11}$$

$$[\rho(v), R] = 0. \tag{3.4.12}$$

Definition 3.4.24. *We call ρ the **generating representation** of the isoclinic spherical symmetry of $\hat{M}$.*

Proof. We follow the proof of Theorem 1.1.1, making modifications when relevant. Suppose that $\hat{M}$ has isoclinic spherical symmetry. Let $S \subseteq \mathrm{Sp}(1) \times \mathrm{O}(k)$ be the stabilizer group of $\hat{M}$ restricted to rotations in $\mathrm{Sp}(1) \times \{1\} \subseteq \mathrm{Sp}(2)$. That is

$$\begin{aligned} S := \{(\mathrm{diag}(p, 1), K) \mid (\mathrm{diag}(LKL^\dagger(LL^\dagger)^{-1}, pK), K) \cdot \mathrm{diag}(p^\dagger, 1) \cdot (\hat{M}, U) \\ = (\hat{M}, U)\}. \end{aligned} \tag{3.4.13}$$

Indeed, if $(Q, K) \in \mathrm{Sp}(n+k) \times \mathrm{GL}(k, \mathbb{R})$ such that $\mathrm{diag}(p^\dagger, 1).(Q, K).(\hat{M}, U) = (\hat{M}, U)$, then Corollary 3.1.27 tells us that $K \in \mathrm{O}(k)$, $[R, K] = 0$, and $Q = \mathrm{diag}(LKL^\dagger(LL^\dagger)^{-1}, pK)$. Therefore, the pairs in S encapsulate all the isoclinic spherical symmetry of the instanton. That S is a group follows from this and that the conformal action is a right Lie group action, just as in the proof of Theorem 3.4.3.

Just as in Theorem 1.1.1, S is a compact Lie group and we obtain a Lie group homomorphism $\Psi: \mathrm{Sp}(1) \rightarrow S$ such that $\pi_1|_S \circ \Psi = \mathrm{id}_{\mathrm{Sp}(1)}$.

Additionally, there is a Lie algebra homomorphism $\rho: \mathfrak{sp}(1) \rightarrow \mathfrak{so}(k)$ such that for all $v \in \mathfrak{sp}(1)$, $\Psi(e^v) = (\mathrm{diag}(e^v, 1), e^{\rho(v)})$. As ρ is a Lie algebra homomorphism whose image consists of real matrices, it gives us a real Lie algebra representation.

As $\Psi(e^{\theta v}) \in S$ for all $\theta \in \mathbb{R}$ and $v \in \mathfrak{sp}(1)$, we have that

$$(\mathrm{diag}(Le^{\theta\rho(v)}L^\dagger(LL^\dagger)^{-1}, e^{\theta v}e^{\theta\rho(v)}), e^{\theta\rho(v)}).\mathrm{diag}(e^{-\theta v}, 1).(\hat{M}, U) = (\hat{M}, U)$$

Differentiating and evaluating at $\theta = 0$, we obtain

$$\begin{bmatrix} L\rho(v)L^\dagger(LL^\dagger)^{-1}L - L\rho(v) \\ vM + [\rho(v), M] \end{bmatrix} = 0. \quad (3.4.14)$$

In Section 3.4.2.3, we study the top row. For now, focusing on the bottom row, we see that $vM + [\rho(v), M] = 0$. Furthermore, as mentioned above, Corollary 3.1.27 tells us that $[R, e^{\theta\rho(v)}] = 0$, for all $\theta \in \mathbb{R}$. Differentiating and evaluating at $\theta = 0$, we have $[R, \rho(v)] = 0$.

Conversely, suppose that the equations hold for some real representation $\rho: \mathfrak{sp}(1) \rightarrow \mathfrak{so}(k)$. Consider $p \in \mathrm{Sp}(1)$. As the exponential map $\exp: \mathfrak{sp}(1) \rightarrow \mathrm{Sp}(1)$ is surjective, there is some $v \in \mathfrak{sp}(1)$ such that $p = e^v$. Let $p(\theta) := e^{\theta v} \in \mathrm{Sp}(1)$ and $K(\theta) := e^{\theta\rho(v)} \in \mathrm{O}(k)$. We show that $A(\theta) := p(\theta)K(\theta)MK(\theta)^T$ is constant. Indeed,

$$A'(\theta) = p(\theta)K(\theta)(vM + [\rho(v), M])K(\theta)^T = 0.$$

Hence, A is constant, so

$$pK(1)MK(1)^T = A(1) = A(0) = M.$$

As $[\rho(v), R] = 0$, $[K(\theta), R] = 0$, so by Lemma 3.1.25, we have that $\hat{M}$ is $\mathrm{diag}(p^\dagger, 1)$ -equivariant. As p was arbitrary, we have isoclinic spherical symmetry. $\square$

Just as with circular t -symmetry, we need not check the final condition of $\mathcal{M}_{n,k}$ in Definition 3.1.16 everywhere.

Lemma 3.4.25. *Suppose that $\hat{M}$ satisfies (3.4.11) and (3.4.12) for some real representation $(\mathbb{R}^k, \rho)$ as well as the first three conditions of Definition 3.1.16. If the final condition is satisfied at all $x = x_0 \in \mathbb{H}$ with $x_0 \geq 0$, then $\hat{M} \in \mathcal{M}_{n,k}$.*

Proof. Let $z = z_0 + z_1i + z_2j + z_3k \in \mathbb{H}$. There is some $p \in \text{Sp}(1)$ such that $z = |z|p^\dagger$. Thus, $x := pz = |z|$. The rest follows as in the proof of Lemma 3.2.4. $\square$

3.4.2.1 Structure of isoclinic spherical symmetry

Theorem 3.4.23 tells us exactly how to search for instantons with isoclinic spherical symmetry. It follows the same story as the case of simple spherical symmetry, just with a different equation. In this section, we investigate what representations generate instantons with isoclinic spherical symmetry and what the corresponding ADHM data looks like.

Definition 3.4.26. *Let $\iota: \mathfrak{sp}(1) \rightarrow \mathfrak{gl}(\mathbb{H})$ be the inclusion map $\iota(v)(x) := vx$. Consider the real representation $(\mathbb{H}, \iota)$, where the scalars are restricted from $\mathbb{H}$ to $\mathbb{R}$, and let $V := \mathbb{R}^k$. Given a real representation (V, ρ) of $\mathfrak{sp}(1)$, we define the induced real representation*

$$(\hat{V}, \hat{\rho}) := (V, \rho) \otimes_{\mathbb{R}} (V^*, \rho^*) \otimes_{\mathbb{R}} (\mathbb{H}, \iota). \quad (3.4.15)$$

Unraveling how $\hat{\rho}$ acts on $\hat{V} = \text{Mat}(k, k, \mathbb{H})$, let $v \in \mathfrak{sp}(1)$ and $A \in \text{Mat}(k, k, \mathbb{H})$. Then

$$\hat{\rho}(v)(A) = [\rho(v), A] + vA. \quad (3.4.16)$$

Note 3.4.27. *By keeping $(\mathbb{H}, \iota)$ as a quaternionic representation, we could have considered $(\hat{V}, \hat{\rho})$ as a quaternionic representation. The choice to restrict scalars merely simplifies computation of ADHM data.*

The definition of $(\hat{V}, \hat{\rho})$ is well-motivated. Firstly, note that $M \in \hat{V}$. Secondly, given the connection between the action of $\hat{\rho}(v)$ and (3.4.14), we immediately obtain the following corollary.

Corollary 3.4.28. *Let $\hat{M} \in \mathcal{M}_{n,k}$. Then $\hat{M}$ has isoclinic spherical symmetry if and only if there is some real k -representation (V, ρ) such that $\hat{\rho}(v)(M) = 0$ and $[\rho(v), R] = 0$ for all $v \in \mathfrak{sp}(1)$.*

Note 3.4.29. *Just as in the simple spherical symmetry case, we have $(\text{span}(M), 0)$ is a summand of $(\hat{V}, \hat{\rho})$. As we have dealt with the case $M = 0$, we must have that $(\hat{V}, \hat{\rho})$ has trivial summands and M lives in their direct sum.*

Note 3.4.30. Restricting scalars to $\mathbb{R}$, $(\mathbb{H}, \iota)$ is isomorphic to $(\mathbb{R}^4, \varrho_4)$. Indeed, one can examine how $v \in \mathfrak{sp}(1)$ acts on the standard real basis of $\mathbb{H}$ and compute the Casimir operator C_ι of this representation, obtaining $C_\iota = \frac{3}{4}I_4$. Additionally, the complexification of $(\mathbb{R}^4, \varrho_4)$ is isomorphic to $(V_2, \rho_2)^{\oplus 2}$.

We now look at where the trivial summands of $(\hat{V}, \hat{\rho})$ are found. First, we examine the tensor product of the relevant complex representations.

Lemma 3.4.31. Given $m \geq n \geq 1$, consider $(V_m, \rho_m) \otimes_{\mathbb{C}} (V_n, \rho_n) \otimes_{\mathbb{C}} (V_2, \rho_2)$. This product has a single trivial summand when $m = n + 1$ and none otherwise.

Proof. Note that $(V_a, \rho_a) \otimes_{\mathbb{C}} (V_2, \rho_2)$ contains a single trivial summand when $a = 2$ and none otherwise. Thus, the complete tensor product contains a single trivial summand for each (V_2, ρ_2) summand of $(V_m, \rho_m) \otimes_{\mathbb{C}} (V_n, \rho_n)$. However, this product contains a single (V_2, ρ_2) summand when $m = n + 1$ and none otherwise. $\square$

Now we are equipped to examine the tensor product of the relevant real representations.

Lemma 3.4.32. Given $m \geq n \geq 1$, we have that $(\mathbb{R}^m, \varrho_m) \otimes_{\mathbb{R}} ((\mathbb{R}^n)^*, \varrho_n^*) \otimes_{\mathbb{R}} (\mathbb{R}^4, \varrho_4)$ has four trivial summands when m is divisible by four, n is odd, and $m = 2n \pm 2$. Otherwise, there are no trivial summands.

Proof. Suppose that m is divisible by four and n is odd. The complexification of the real representation is isomorphic, as a complex representation, to

$$(V_{m/2}, \rho_{m/2})^{\oplus 2} \otimes_{\mathbb{C}} (V_n, \rho_n) \otimes_{\mathbb{C}} (V_2, \rho_2)^{\oplus 2}.$$

By Lemma 3.4.31, we have that there are 4 trivial summands when $m = 2n \pm 2$ and none otherwise. Furthermore, as the tensor product only contains odd dimensional summands, the representation is real, meaning that the real decomposition is the same as the complex. Thus, we have proven the first part of the lemma.

Suppose that m and n are divisible by four. The complexification of the real representation is isomorphic, as a complex representation, to

$$(V_{m/2}, \rho_{m/2})^{\oplus 2} \otimes_{\mathbb{C}} (V_{n/2}, \rho_{n/2})^{\oplus 2} \otimes_{\mathbb{C}} (V_2, \rho_2)^{\oplus 2}.$$

As $m/2$ and $n/2$ are both even, by Lemma 3.4.31, this product has no trivial summands.

Suppose that m and n are both odd. The complexification of the real representation is isomorphic, as a complex representation, to

$$(V_m, \rho_m) \otimes_{\mathbb{C}} (V_n, \rho_n) \otimes_{\mathbb{C}} (V_2, \rho_2)^{\oplus 2}.$$

As m and n are both odd, by Lemma 3.4.31, this product has no trivial summands.

Finally, suppose m is odd and n is divisible by four. The complexification of the real representation is isomorphic, as a complex representation, to

$$(V_m, \rho_m) \otimes_{\mathbb{C}} (V_{n/2}, \rho_{n/2})^{\oplus 2} \otimes_{\mathbb{C}} (V_2, \rho_2)^{\oplus 2}.$$

By Lemma 3.4.31, we have that there are 4 trivial summands when $n = 2m \pm 2$. As $m \geq n \geq 1$, we have $m \geq 2m \pm 2 \geq 1$. This set of inequalities has no solution. Thus, there are no trivial summands, proving the lemma. $\square$

We now identify the trivial summands discussed above.

Definition 3.4.33. Let C^n be the unit length generator of the unique one-dimensional representation in $(V_{n+1}, \rho_{n+1}) \otimes_{\mathbb{C}} (V_n^*, \rho_n^*) \otimes_{\mathbb{C}} (V_2, \rho_2)$. This generator is well-defined, up to a $\mathbb{C}^*$ factor, though a choice of scale leaves a S^1 factor. This generator can be thought of as a $\mathfrak{sp}(1)$ -invariant pair (C_0^n, C_1^n) of complex $(n+1) \times n$ matrices.

These generators can be identified in much the same way as we compute the B^n matrices from Definition 2.1.33 in Appendix E.

Above, we defined what spans the trivial summands of $(\hat{V}, \hat{\rho})$ when decomposed as a complex representation. We use this to determine what spans the real representation.

Lemma 3.4.34. Given $n \in \mathbb{N}_+$ odd, let $Y_i^+ := \varrho_{2n+2}(v_i) \in \mathfrak{so}(2n+2)$ and $Y_i^- := \varrho_n(v_i) \in \mathfrak{so}(n)$. Let $y_i^+ := \rho_{n+1}(v_i) \in \mathfrak{su}(n+1)$. As the complexification of $(\mathbb{R}^{2n+2}, \varrho_{2n+2})$ is isomorphic to $(V_{n+1}, \rho_{n+1})^{\oplus 2}$, there exists $U_+ \in \mathrm{SU}(2n+2)$ such that

$$Y_i^+ = U_+^\dagger \begin{bmatrix} y_i^+ & 0 \\ 0 & y_i^+ \end{bmatrix} U_+.$$

We can find four linearly independent quadruples of real matrices spanning the four trivial summands of $(\mathbb{R}^{2n+2}, \varrho_{2n+2}) \otimes_{\mathbb{R}} ((\mathbb{R}^n)^*, \varrho_n^*) \otimes_{\mathbb{R}} (\mathbb{R}^4, \varrho_4)$. For some choices of $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, these linearly independent quadruples of matrices are given by

$$\left(U_+^\dagger \begin{bmatrix} \alpha C_0^n - i\gamma C_1^n \\ \beta C_0^n - i\delta C_1^n \end{bmatrix}, U_+^\dagger \begin{bmatrix} -i\alpha C_0^n + \gamma C_1^n \\ -i\beta C_0^n + \delta C_1^n \end{bmatrix}, U_+^\dagger \begin{bmatrix} -\alpha C_1^n - i\gamma C_0^n \\ -\beta C_1^n - i\delta C_0^n \end{bmatrix}, U_+^\dagger \begin{bmatrix} -i\alpha C_1^n - \gamma C_0^n \\ -i\beta C_1^n - \delta C_0^n \end{bmatrix} \right) \quad (3.4.17)$$

Given $n \in \mathbb{N}_+$ odd and at least three, let $Y_i^+ := \varrho_{2n-2}(v_i) \in \mathfrak{so}(2n-2)$ and $Y_i^- := \varrho_n(v_i) \in \mathfrak{so}(n)$. Let $y_i^+ := \rho_{n-1}(v_i) \in \mathfrak{su}(n-1)$. As the complexification of $(\mathbb{R}^{2n-2}, \varrho_{2n-2})$ is isomorphic to $(V_{n-1}, \rho_{n-1})^{\oplus 2}$, there exists $U_+ \in \mathrm{SU}(2n-2)$ such that

$$Y_i^+ = U_+^\dagger \begin{bmatrix} y_i^+ & 0 \\ 0 & y_i^+ \end{bmatrix} U_+.$$

We can find four linearly independent quadruples of real matrices spanning the four trivial summands of $(\mathbb{R}^{2n-2}, \varrho_{2n-2}) \otimes_{\mathbb{R}} ((\mathbb{R}^n)^*, \varrho_n^*) \otimes_{\mathbb{R}} (\mathbb{R}^4, \varrho_4)$. For some choices of $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, these linearly independent quadruples of matrices are given by

$$\begin{aligned} & \left(U_+^\dagger \begin{bmatrix} \alpha(C_1^{n-1})^\dagger + i\gamma(C_0^{n-1})^\dagger \\ \beta(C_1^{n-1})^\dagger + i\delta(C_1^{n-1})^\dagger \end{bmatrix}, U_+^\dagger \begin{bmatrix} -i\alpha(C_1^{n-1})^\dagger - \gamma(C_0^{n-1})^\dagger \\ -i\beta(C_1^{n-1})^\dagger - \delta(C_0^{n-1})^\dagger \end{bmatrix}, \right. \\ & \quad \left. U_+^\dagger \begin{bmatrix} \alpha(C_0^{n-1})^\dagger - i\gamma(C_1^{n-1})^\dagger \\ \beta(C_0^{n-1})^\dagger - i\delta(C_1^{n-1})^\dagger \end{bmatrix}, U_+^\dagger \begin{bmatrix} i\alpha(C_0^{n-1})^\dagger - \gamma(C_1^{n-1})^\dagger \\ i\beta(C_0^{n-1})^\dagger - \delta(C_1^{n-1})^\dagger \end{bmatrix} \right) \quad (3.4.18) \end{aligned}$$

Proof. Consider the standard Lie algebra isomorphism $\rho_2: \mathfrak{sp}(1) \rightarrow \mathfrak{su}(2)$ defined by

$$\rho_2 \left(\frac{i}{2} \right) := \frac{1}{2} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \rho_2 \left(\frac{j}{2} \right) := \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \rho_2 \left(\frac{k}{2} \right) := \frac{1}{2} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Note that $(\mathbb{C}^2, \rho_2)$ is the irreducible complex 2-representation of $\mathfrak{sp}(1)$. Additionally, consider the Lie algebra homomorphism $\varrho_4: \mathfrak{sp}(1) \rightarrow \mathfrak{so}(4)$ defined by

$$\begin{aligned} \varrho_4 \left(\frac{i}{2} \right) &:= \frac{1}{2} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \\ \varrho_4 \left(\frac{j}{2} \right) &:= \frac{1}{2} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \\ \varrho_4 \left(\frac{k}{2} \right) &:= \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \end{aligned}$$

Note that $(\mathbb{R}^4, \varrho_4)$ is the irreducible real 4-representation of $\mathfrak{sp}(1)$. Recall that the complexification of $(\mathbb{R}^4, \varrho_4)$ is isomorphic, as a complex representation, to $(V_2, \rho_2)^{\oplus 2}$. That is, there is some $U \in \mathrm{SU}(4)$ such that for all $v \in \mathfrak{sp}(1)$, $\varrho_4(v) = U^\dagger \mathrm{diag}(\rho_2(v), \rho_2(v))U$. Given the above definitions for ρ_2 and ϱ_4 , one such U is given by

$$U := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i & 0 & 0 \\ 0 & 0 & -1 & i \\ 0 & 0 & i & -1 \\ i & 1 & 0 & 0 \end{bmatrix}. \quad (3.4.19)$$

Consider the first part of the lemma. We are searching for quadruples $(D_\mu)_{\mu=0}^3$ such that for all $v \in \mathfrak{sp}(1)$,

$$(\varrho_{2n+2}(v) \otimes I_4) (D_\mu)_{\mu=0}^3 - (D_\mu)_{\mu=0}^3 (\varrho_n(v) \otimes I_4) + (I_{2n+2} \otimes \varrho_4) (D_\mu)_{\mu=0}^3 = 0. \quad (3.4.20)$$

We have that

$$\begin{aligned} \varrho_{2n+2}(v) &= U_+^\dagger \mathrm{diag}(\rho_{n+1}(v), \rho_{n+1}(v))U_+, \\ \varrho_n(v) &= \rho_n(v), \quad \text{and} \\ \varrho_4(v) &= U^\dagger \mathrm{diag}(\rho_2(v), \rho_2(v))U. \end{aligned}$$

Letting $(D'_\mu)_{\mu=0}^3 := (U_+ U_{\mu\nu} D_\nu)_{\mu=0}^3$, we have that

$$\begin{aligned} (\mathrm{diag}(\rho_{n+1}(v), \rho_{n+1}(v)) \otimes I_4) (D'_\mu)_{\mu=0}^3 - (D'_\mu)_{\mu=0}^3 (\rho_n(v) \otimes I_4) \\ + (I_{2n+2} \otimes \mathrm{diag}(\rho_2(v), \rho_2(v))) (D'_\mu)_{\mu=0}^3 = 0. \end{aligned} \quad (3.4.21)$$

Note that $(D'_\mu)_{\mu=0}^3$ is acted trivially upon by $(V_{n+1}, \rho_{n+1})^{\oplus 2} \otimes_{\mathbb{C}} (V_n^*, \rho_n^*) \otimes_{\mathbb{C}} (V_2, \rho_2)^{\oplus 2}$. Writing $D'_\mu := \begin{bmatrix} A_\mu \\ B_\mu \end{bmatrix}$, we see that (A_0, A_1) , (A_2, A_3) , (B_0, B_1) , and (B_2, B_3) are acted trivially upon by

$$(V_{n+1}, \rho_{n+1}) \otimes_{\mathbb{C}} (V_n^*, \rho_n^*) \otimes_{\mathbb{C}} (V_2, \rho_2).$$

That is, all of these pairs belong to the span of (C_0^n, C_1^n) . Hence, there is some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that

$$\begin{aligned} (A_0, A_1) &= \sqrt{2}(\alpha C_0^n, \alpha C_1^n), \\ (A_2, A_3) &= \sqrt{2}(\gamma C_0^n, \gamma C_1^n), \\ (B_0, B_1) &= \sqrt{2}(\beta C_0^n, \beta C_1^n), \quad \text{and} \\ (B_2, B_3) &= \sqrt{2}(\delta C_0^n, \delta C_1^n). \end{aligned}$$

Given the definition of D'_μ , we note that $D_\mu = U_+^\dagger (U^\dagger)_{\mu\nu} D'_\nu$. Substituting the values for D'_ν , we find that $(D_\mu)_{\mu=0}^3$ is given by (3.4.17). In particular, for some choices of $\alpha, \beta, \gamma, \delta$, we obtain four linearly independent quadruples of real matrices, as the space of quadruples of real matrices acted on trivially by $(\mathbb{R}^{2n+2}, \varrho_{2n+2}) \otimes_{\mathbb{R}} ((\mathbb{R}^n)^*, \varrho_n^*) \otimes_{\mathbb{R}} (\mathbb{R}^4, \varrho_4)$ is four dimensional.

Consider the second part of the lemma. We are searching for quadruples $(D_\mu)_{\mu=0}^3$ such that for all $v \in \mathfrak{sp}(1)$,

$$(\varrho_{2n-2}(v) \otimes I_4) (D_\mu)_{\mu=0}^3 - (D_\mu)_{\mu=0}^3 (\varrho_n(v) \otimes I_4) + (I_{2n-2} \otimes \varrho_4) (D_\mu)_{\mu=0}^3 = 0. \quad (3.4.22)$$

We have that

$$\begin{aligned} \varrho_{2n-2}(v) &= U_+^\dagger \text{diag}(\rho_{n-1}(v), \rho_{n-1}(v)) U_+, \\ \varrho_n(v) &= \rho_n(v), \quad \text{and} \\ \varrho_4(v) &= U^\dagger \text{diag}(\rho_2(v), \rho_2(v)) U. \end{aligned}$$

Letting $(D'_\mu)_{\mu=0}^3 := (U_+ U_{\mu\nu} D_\nu)_{\mu=0}^3$, we have that

$$\begin{aligned} (\text{diag}(\rho_{n-1}(v), \rho_{n-1}(v)) \otimes I_4) (D'_\mu)_{\mu=0}^3 - (D'_\mu)_{\mu=0}^3 (\rho_n(v) \otimes I_4) \\ + (I_{2n-2} \otimes \text{diag}(\rho_2(v), \rho_2(v))) (D'_\mu)_{\mu=0}^3 = 0. \end{aligned} \quad (3.4.23)$$

Note that $(D'_\mu)_{\mu=0}^3$ is acted trivially upon by $(V_{n-1}, \rho_{n-1})^{\oplus 2} \otimes_{\mathbb{C}} (V_n^*, \rho_n^*) \otimes_{\mathbb{C}} (V_2, \rho_2)^{\oplus 2}$. Writing $D'_\mu := \begin{bmatrix} A_\mu \\ B_\mu \end{bmatrix}$, we see that (A_0, A_1) , (A_2, A_3) , (B_0, B_1) , and (B_2, B_3) are acted trivially upon by

$$(V_{n-1}, \rho_{n-1}) \otimes_{\mathbb{C}} (V_n^*, \rho_n^*) \otimes_{\mathbb{C}} (V_2, \rho_2).$$

We know that this tensor product has a single trivial summand.

Recalling that (C_0^{m-1}, C_1^{m-1}) spans the unique trivial summand of

$$(V_n, \rho_n) \otimes_{\mathbb{C}} (V_{n-1}^*, \rho_{n-1}^*) \otimes_{\mathbb{C}} (V_2, \rho_2),$$

we find that $((C_1^{m-1})^\dagger, -(C_0^{m-1})^\dagger)$ spans the trivial summand of $(V_{n-1}, \rho_{n-1}) \otimes_{\mathbb{C}} (V_n^*, \rho_n^*) \otimes_{\mathbb{C}} (V_2, \rho_2)$. Hence, there is some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ such that

$$\begin{aligned} (A_0, A_1) &= \sqrt{2}(\alpha(C_1^{m-1})^\dagger, -\alpha(C_0^{m-1})^\dagger), \\ (A_2, A_3) &= \sqrt{2}(\gamma(C_1^{m-1})^\dagger, -\gamma(C_0^{m-1})^\dagger), \\ (B_0, B_1) &= \sqrt{2}(\beta(C_1^{m-1})^\dagger, -\beta(C_0^{m-1})^\dagger), \quad \text{and} \\ (B_2, B_3) &= \sqrt{2}(\delta(C_1^{m-1})^\dagger, -\delta(C_0^{m-1})^\dagger). \end{aligned}$$

Given the definition of D'_μ , we note that $D_\mu = U_+^\dagger (U^\dagger)_{\mu\nu} D'_\nu$. Substituting the values for D'_ν , we find that $(D_\mu)_{\mu=0}^3$ is given by (3.4.18). In particular, for some choices of $\alpha, \beta, \gamma, \delta$, we obtain four linearly independent quadruples of real matrices, as the space of quadruples of real matrices acted on trivially by $(\mathbb{R}^{2n+2}, \varrho_{2n+2}) \otimes_{\mathbb{R}} ((\mathbb{R}^n)^*, \varrho_n^*) \otimes_{\mathbb{R}} (\mathbb{R}^4, \varrho_4)$ is four dimensional. $\square$

Now that we know exactly what spans the trivial summands of $(\hat{V}, \hat{\rho})$, the following theorem tells us exactly what form M takes if $\hat{M}$ has isoclinic spherical symmetry.

Theorem 3.4.35 (Isoclinic Spherical Structure Theorem). *Let $\hat{M} \in \mathcal{M}_{n,k}$ have isoclinic spherical symmetry. Theorem 3.4.23 tells us that $\hat{M}$ is generated by a real representation (V, ρ) of $\mathfrak{sp}(1)$, which we can decompose as*

$$(V, \rho) \simeq \bigoplus_{a=1}^m (\mathbb{R}^{n_a}, \varrho_{n_a}). \quad (3.4.24)$$

Without loss of generality, we may assume that $n_a \geq n_{a+1}$, for all $a \in \{1, \dots, m-1\}$.

Let $Y_{i,a} := \varrho_{n_a}(v_i) \in \mathfrak{so}(n_a)$ and

$$Y_i := \text{diag}(Y_{i,1}, \dots, Y_{i,m}) \in \mathfrak{so}(k).$$

Then Y_i induces (V, ρ) . For n_a is divisible by four, let $y_{i,a} := \rho_{\frac{n_a}{2}}(v_i) \in \mathfrak{su}(\frac{n_a}{2})$ induce the irreducible $\frac{n_a}{2}$ complex representation. Then there is some $U_{n_a} \in \text{SU}(n_a)$ such that

$$Y_{i,a} = U_{n_a}^\dagger \begin{bmatrix} y_{i,a} & 0 \\ 0 & y_{i,a} \end{bmatrix} U_{n_a}.$$

Using the decomposition of (V, ρ) given in (3.4.24), we have $(M_\mu)_{ab} \in \mathbb{R}^{n_a} \otimes_{\mathbb{R}} (\mathbb{R}^{n_b})^*$ such that,

$$M_\mu = \begin{bmatrix} (M_\mu)_{11} & \cdots & (M_\mu)_{1m} \\ \vdots & \ddots & \vdots \\ (M_\mu)_{m1} & \cdots & (M_\mu)_{mm} \end{bmatrix}.$$

Then, up to a ρ -invariant gauge and for all $a, b \in \{1, \dots, m\}$ we have that,

- (1) if $a < b$ and $n_a = 2n_b + 2$, then $\exists \alpha_{a,b}, \beta_{a,b}, \gamma_{a,b}, \delta_{a,b} \in \mathbb{C}$ such that $(M_\mu)_{ab}$ take the form of (3.4.17) and are real and $(M_\mu)_{ba} = (M_\mu)_{ab}^T$;

- (2) if $a < b$ and $n_a = 2n_b - 2$, then $\exists \alpha_{a,b}, \beta_{a,b}, \gamma_{a,b}, \delta_{a,b} \in \mathbb{C}$ such that $(M_\mu)_{ab}$ take the form of (3.4.18) and are real and $(M_\mu)_{ba} = (M_\mu)_{ab}^T$;
- (3) otherwise, $(M_\mu)_{ab} = 0$.

Conversely, if M has the above form for some real representation (V, ρ) and $[\rho(v), R] = 0$ for all $v \in \mathfrak{sp}(1)$, then $\hat{M}$ has isoclinic spherical symmetry.

Proof. The proof follows the same structure as the proof of the Structure Theorem for hyperbolic monopoles, Theorem 2.1.39. $\square$

3.4.2.2 Novel example of isoclinic spherical symmetry

In this section, we identify novel examples of isoclinic spherical symmetry, using the Isoclinic Spherical Structure Theorem. The first representation that comes to mind to consider is $(\mathbb{R}^4, \varrho_4) \oplus (\mathbb{R}, 0)$. This representation generates an instanton with rotational symmetry and is discussed in Proposition 3.6.17. Thus, we consider the next simplest case.

Proposition 3.4.36. *Let $\lambda > 0$. Then define $\hat{M}$ by*

$$M := \lambda \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -i & j \\ 0 & 0 & 0 & 0 & i & 1 & -k \\ 0 & 0 & 0 & 0 & j & -k & -1 \\ 0 & 0 & 0 & 0 & k & j & i \\ 1 & i & j & k & 0 & 0 & 0 \\ -i & 1 & -k & j & 0 & 0 & 0 \\ j & -k & -1 & i & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad (3.4.25)$$

$$L := \lambda \begin{bmatrix} 3 & -i & -j & -k & 0 & 0 & 0 \\ 0 & 2\sqrt{2} & k\sqrt{2} & -j\sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{6} & i\sqrt{6} & 0 & 0 & 0 \end{bmatrix}.$$

We have $\hat{M} \in \mathcal{M}_{3,7}$ and it corresponds to a $\mathrm{Sp}(3)$ instanton with isoclinic spherical symmetry and instanton number 7.

Proof. Consider the representation $(V, \rho) := (\mathbb{R}^3, \varrho_3) \oplus (\mathbb{R}^4, \varrho_4)$. Seeking instantons with isoclinic spherical symmetry generated by (V, ρ) , the Isoclinic Spherical Structure Theorem tells us that M has to have the form given in the statement, up to a ρ -invariant gauge. Thus, if $\hat{M} \in \mathcal{M}_{3,7}$, then it has isoclinic spherical symmetry. To that end, we have $LL^\dagger = 12\lambda^2 I_3$ and $R = 12\lambda^2 I_4 \oplus 4\lambda^2 I_3$. Due to the symmetry of the instanton, we need only check the final condition for $x = x_0 \geq 0$. Doing so, we see that $\Delta(x)^\dagger \Delta(x)$ is given by

$$\begin{bmatrix} 12\lambda^2 + x_0^2 & 0 & 0 & 0 & -2x_0\lambda & 0 & 0 \\ 0 & 12\lambda^2 + x_0^2 & 0 & 0 & 0 & -2x_0\lambda & 0 \\ 0 & 0 & 12\lambda^2 + x_0^2 & 0 & 0 & 0 & 2x_0\lambda \\ 0 & 0 & 0 & 12\lambda^2 + x_0^2 & 0 & 0 & 0 \\ -2x_0\lambda & 0 & 0 & 0 & 4\lambda^2 + x_0^2 & 0 & 0 \\ 0 & -2x_0\lambda & 0 & 0 & 0 & 4\lambda^2 + x_0^2 & 0 \\ 0 & 0 & 2x_0\lambda & 0 & 0 & 0 & 4\lambda^2 + x_0^2 \end{bmatrix}.$$

The eigenvalues of $\Delta(x)^\dagger \Delta(x)$ are $12\lambda^2 + x_0^2$, $8\lambda^2 + x_0^2 \pm 2\lambda\sqrt{4\lambda^2 + x_0^2}$ with multiplicity 1, 3, and 3, respectively. Therefore, $\Delta(x)^\dagger \Delta(x)$ is positive-definite everywhere. $\square$

3.4.2.3 Constraint on isoclinic spherical symmetry

In Theorem 3.4.23, we obtain a representation (V, ρ) of $\mathfrak{sp}(1)$ from an instanton with isoclinic spherical symmetry, by focusing on the bottom of (3.4.14). This representation induces another $(\hat{V}, \hat{\rho})$, which we use to determine what an instanton with isoclinic spherical symmetry looks like. However, by focusing on the top of (3.4.14), Theorem 3.4.23 gives us a second representation.

Lemma 3.4.37. *Let $v_1 := i/2$, $v_2 := j/2$, and $v_3 := k/2$. Suppose $\hat{M} \in \mathcal{M}_{n,k}$ has isoclinic spherical symmetry generated by (V, ρ) , a real representation. Let $y_i := (LL^\dagger)^{-1} L\rho(v_i)L^\dagger \in \mathfrak{sp}(n)$. The y_i induce a representation of $\mathfrak{sp}(1)$, which we denote by (W, λ) , where $W := \mathbb{H}^n$ and λ is the linear map taking $v_i \mapsto y_i$.*

Proof. Let $i, j \in \{1, 2, 3\}$. By Theorem 3.4.23, we know that $[\rho(v_i), M] = v_i M$, so

$$\begin{aligned} M^\dagger M \rho(v_i) &= M^\dagger [M, \rho(v_i)] + M^\dagger \rho(v_i) M \\ &= -M^\dagger v_i M + [M^\dagger, \rho(v_i)] M + \rho(v_i) M^\dagger M \\ &= -M^\dagger v_i M + M^\dagger v_i M + \rho(v_i) M^\dagger M \\ &= \rho(v_i) M^\dagger M. \end{aligned}$$

Thus, $[M^\dagger M, \rho(v_i)] = 0$. Also, from Theorem 3.4.23, $[\rho(v_i), R] = 0$. Hence, $[L^\dagger L, \rho(v_i)] = 0$. Thus,

$$LL^\dagger L \rho(v_i) L^\dagger = L \rho(v_i) L^\dagger LL^\dagger.$$

Thus, $[LL^\dagger, L \rho(v_i) L^\dagger] = 0$, so $y_i \in \mathfrak{sp}(n)$. Therefore, we see

$$\begin{aligned} [y_i, y_j] &= (LL^\dagger)^{-1} L(\rho(v_i) L^\dagger L \rho(v_j) - \rho(v_j) L^\dagger L \rho(v_i)) L^\dagger (LL^\dagger)^{-1} \\ &= (LL^\dagger)^{-1} L \rho([v_i, v_j]) L^\dagger. \end{aligned}$$

Thus, the y_i satisfy the correct commutation relations. $\square$

We use (W, λ) to determine what $\mathrm{Sp}(n)$ structure groups are possible given ρ . Using $W = \mathbb{H}^n \simeq \mathbb{C}^{2n}$, we can restrict the scalars from $\mathbb{H}$ to $\mathbb{C}$. In doing so, the generators $y_i \in \mathfrak{sp}(n)$ correspond to elements of $\mathfrak{su}(2n)$. Thus, the induced complex representation is a $2n$ -representation. As mentioned in Section D.1.3 of Appendix D, the complex representation obtained by restricting the scalars of an irreducible, quaternionic representation is either isomorphic to (V_a, ρ_a) for some a even or $(V_a, \rho_a)^{\oplus 2}$ for some a odd.

Definition 3.4.38. *Given a real representation (V, ρ) of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$, with induced representation (W, λ) as defined above, we define the quaternionic representation*

$$(\hat{W}, \hat{\lambda}) := (W, \lambda) \otimes_{\mathbb{R}} (V^*, \rho^*). \quad (3.4.26)$$

Unraveling how $\hat{\lambda}$ acts on $\hat{W} = \mathrm{Mat}(n, k, \mathbb{H})$, let $v \in \mathfrak{sp}(1)$ and $A \in \hat{W}$. Then

$$\hat{\lambda}(v)(A) = \lambda(v)A - A\rho(v). \quad (3.4.27)$$

The definition of $(\hat{W}, \hat{\lambda})$ is well-motivated. First, note that $L \in \hat{W}$. Moreover, the following lemma tells us exactly where L lives in $\hat{W}$.

Lemma 3.4.39. *Let $\hat{M} \in \mathcal{M}_{n,k}$ have isoclinic spherical symmetry. Let (V, ρ) , (W, λ) and $(\hat{W}, \hat{\lambda})$ be as above. Then $\hat{\lambda}(v)(L) = 0$ for all $v \in \mathfrak{sp}(1)$.*

Proof. As $\hat{M}$ has isoclinic symmetry, we have (3.4.14). Then we see that for $v \in \mathfrak{sp}(1)$,

$$\hat{\lambda}(v)(L) = \lambda(v)L - L\rho(v) = 0. \quad \square$$

Corollary 3.4.40. *Let $\hat{M} \in \mathcal{M}_{n,k}$ have isoclinic spherical symmetry. Then $(\hat{W}, \hat{\lambda})$ must have trivial summands and L must live in the direct sum of these summands.*

Proof. As $L \neq 0$, we know $\text{span}(L) \subseteq \hat{W}$ is an invariant 1-dimensional subspace, so $(\text{span}(L), 0)$ is a trivial summand of the representation $(\hat{W}, \hat{\lambda})$. $\square$

Therefore, we know that $(\hat{W}, \hat{\lambda})$ must have trivial summands, which narrows the possibilities for the structure group. Just like in the hyperbolic monopole case, we have straightforward restrictions for high dimensional representations in addition to more subtle restrictions for low dimensional representations. For instance, in Proposition 3.4.36, we examine isoclinic symmetric instantons generated by $(V, \rho) \simeq (\mathbb{R}^4, \varrho_4) \oplus (\mathbb{R}^3, \varrho_3)$. This representation generated instantons with isoclinic spherical symmetry and structure group $\text{Sp}(3)$. However, this representation cannot generate instantons with isoclinic spherical symmetry and lower rank structure groups.

Proposition 3.4.41. *Consider Proposition 3.4.36, with $(V, \rho) \simeq (\mathbb{R}^4, \varrho_4) \oplus (\mathbb{R}^3, \varrho_3)$. This representation does not generate an instanton with isoclinic spherical symmetry and structure group $\text{Sp}(1)$ or $\text{Sp}(2)$.*

Proof. Suppose $(V, \rho) \simeq (\mathbb{R}^4, \varrho_4) \oplus (\mathbb{R}^3, \varrho_3)$ generates an instanton with isoclinic spherical symmetry. We know that $(\hat{W}, \hat{\lambda})$ has a trivial summand. If we have a $\text{Sp}(1)$ instanton, then we have that the representation obtained by restricting the scalars of (W, λ) is isomorphic, as a complex representation, to $(V_1, \rho_1)^{\oplus 2}$ or (V_2, ρ_2) . The former does not give $(\hat{W}, \hat{\lambda})$ any trivial summands. Thus we look to the latter, which gives two trivial complex summands.

If the representation obtained by restricting the scalars of (W, λ) is given by (V_2, ρ_2) , then there is some $q \in \text{Sp}(1)$ such that $y_i = qv_i q^\dagger$. Taking $\tilde{L} := q^\dagger L$ and dropping the tilde, we have

$$v_i L = L \rho(v_i).$$

Solving these equations, there is some $a \in \mathbb{H}$ such that $L = \begin{bmatrix} 1 & -i & -j & k & 0 & 0 & 0 \end{bmatrix} a$. But we know that as, up to a ρ -invariant gauge,

$$M = \lambda \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -i & j \\ 0 & 0 & 0 & 0 & i & 1 & -k \\ 0 & 0 & 0 & 0 & j & -k & -1 \\ 0 & 0 & 0 & 0 & k & j & i \\ 1 & i & j & k & 0 & 0 & 0 \\ -i & 1 & -k & j & 0 & 0 & 0 \\ j & -k & -1 & i & 0 & 0 & 0 \end{bmatrix},$$

we have $R := L^\dagger L + M^\dagger M$ is given by

$$R = a^\dagger \begin{bmatrix} 1 & -i & -j & k & 0 & 0 & 0 \\ i & 1 & -k & -j & 0 & 0 & 0 \\ j & k & 1 & i & 0 & 0 & 0 \\ -k & j & -i & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} a + \lambda^2 \begin{bmatrix} 3 & 3i & 3j & 3k & 0 & 0 & 0 \\ -3i & 3 & -3k & 3j & 0 & 0 & 0 \\ -3j & 3k & 3 & -3i & 0 & 0 & 0 \\ -3k & -3j & 3i & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

As we must have R real and commuting with ρ , we must have $a^\dagger i a = 3\lambda^2 i$, $a^\dagger j a = 3\lambda^2 j$, and $a^\dagger k a = -3\lambda^2 k$. Solving these, we have $a = 0$. But then $L = 0$, contradiction! Therefore, we cannot have a $\text{Sp}(1)$ instanton from this representation (V, ρ) .

Instead, if we have a $\text{Sp}(2)$ instanton, then we have that the representation obtained by restricting the scalars of (W, λ) is some combination of the previously discussed 2-representations or (V_4, ρ_4) . However, the only choices for the representation obtained by restricting the scalars of (W, λ) that create a trivial summand in $(\hat{W}, \hat{\lambda})$ include (V_2, ρ_2) . But we already showed that these representations do not generate an instanton with isoclinic spherical symmetry. Thus, this representation does not generate a $\text{Sp}(2)$ instanton with isoclinic spherical symmetry either. $\square$

3.4.2.4 Non-symmetric singular monopoles

In this section, we discuss the connection between instantons with isoclinic spherical symmetry and singular monopoles with Dirac type singularities and no continuous symmetries.

Note that the isoclinic spherical subgroup of $\text{Sp}(2)$ contains R_0 as a subgroup, whereas the simple and conformal spherical subgroups do not even contain a subgroup conjugate to R_0 . Hence, we can associate to an instanton with isoclinic spherical symmetry a singular monopole with Dirac type singularities. Unlike toral symmetry, isoclinic spherical symmetry does not impart extra symmetry to singular monopoles, as the rest of the isoclinic spherical symmetry does not commute with the R_0 subgroup. The lack of commuting actions follows from Lemma 3.2.22 and the work in Section 3.3.2, which tell us that any symmetry on the singular monopole corresponds to isoclinic symmetry of the form $\text{diag}(1, p)$, with $p \in \text{Sp}(1)$.

Note that an instanton with isoclinic spherical symmetry can possess more symmetry, in which case the singular monopole may have a continuous symmetry. In Section 3.5.1.1, we discuss the connection between instantons with isoclinic superspherical symmetry and singular monopoles with different symmetries. If the instanton has isoclinic spherical symmetry and does not have isoclinic superspherical symmetry, then the corresponding singular monopole has no continuous symmetries.

3.4.2.5 Hyperbolic Nahm data

In this section, we discuss the connection between instantons with isoclinic spherical symmetry and a hyperbolic analogue to Nahm data.

Just as Euclidean monopoles and Higgs bundles, Nahm data are a dimensional reduction of instantons. However, while Euclidean monopoles are invariant under translation along one axis and Higgs bundles are invariant under translations along two axes, Nahm data are invariant under translations along three axes. Recall from Section 3.4.1.3, for $d \in \{0, 1, 2\}$ we have the conformal equivalence $S^4 \setminus S^d \equiv H^{d+1} \times S^{4-d-1}$. Note that we can replace $S^4 \setminus S^d$ with $\mathbb{R}^4 \setminus \mathbb{R}^d$.

The fixed point set of the isoclinic spherical action on $\mathbb{R}^4$ is $\{0\}$. Moreover, for $x \neq 0$, we have that the orbit of x under the isoclinic spherical action is S^3 . Indeed, $\mathrm{Sp}(1) \simeq S^3$ acts freely on $\mathbb{R}^4 \setminus \{0\}$.

On $\mathbb{R}^4 \setminus \{0\}$, we can use four dimensional spherical coordinates (r, θ, ϕ, ψ) . Doing so, the metric on $\mathbb{R}^4 \setminus \{0\}$ is given by

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2 + \sin^2 \theta \sin^2 \phi d\psi^2).$$

As $r \neq 0$ on this set, we can divide by r^2 , obtaining the metric of $H^1 \times S^3$. Specifically, we note that we obtain the upper-half space model of H^1 .

Therefore, just as in the hyperbolic monopole and hyperbolic Higgs bundle cases, an instanton with isoclinic spherical symmetry corresponds to hyperbolic Nahm data and vice versa.

3.4.3 Conformal spherical symmetry

In this section, we find an equation describing all instantons with conformal spherical symmetry, as given in Table C.1 as well as below. We then prove that we already know all examples of such instantons. We also discuss instantons with full symmetry.

Lemma 3.4.42. *Given $\hat{M} \in \mathcal{M}_{n,k}$ and $A \in \mathrm{Sp}(2)$, let $(\hat{M}_A, U_A) := A.(\hat{M}, U)$. We have that $\hat{M}_A$ and U_A satisfy*

$$\begin{bmatrix} \hat{M}_A & U_A \end{bmatrix} = \begin{bmatrix} \hat{M} & U \end{bmatrix} I_k \otimes \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} A \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right). \quad (3.4.28)$$

Given $\Upsilon \in \mathfrak{sp}(2)$, consider $A(\theta) := e^{\theta\Upsilon}$. Let $\hat{M}_\Upsilon := \hat{M}'_A(0)$ and $U_\Upsilon := U'_A(0)$. Then

$$\begin{bmatrix} \hat{M}_\Upsilon & U_\Upsilon \end{bmatrix} = \begin{bmatrix} \hat{M} & U \end{bmatrix} I_k \otimes \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Upsilon \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right). \quad (3.4.29)$$

Note that $\hat{M}_{-\Upsilon} = -\hat{M}_\Upsilon$ and similar for U .

Proof. Expanding the right-hand side of (3.4.28), we see that it exactly matches the definitions of $\hat{M}_A$ and U_A . The second claim follows by differentiating (3.4.28). The final claim follows from comparing (3.4.29) for $-\Upsilon$ and Υ . $\square$

Recall that $\mathfrak{h}_5 \subseteq \mathfrak{sp}(2) \simeq \mathfrak{sp}(1)$. Let $\tau: \mathfrak{sp}(1) \rightarrow \mathfrak{h}_5$ be a Lie algebra isomorphism. An instanton has conformal spherical symmetry if and only if it is equivariant under the connected Lie group corresponding to $\mathfrak{h}_5 = \tau(\mathfrak{sp}(1)) \subseteq \mathfrak{sp}(2)$ listed in Table C.1.

Theorem 3.4.43. *Let $\hat{M} \in \mathcal{M}_{n,k}$. Then $\hat{M}$ has conformal spherical symmetry if and only if there exists a representation $\rho: \mathfrak{sp}(1) \rightarrow \mathfrak{sp}(n+k)$ such that for all $v \in \mathfrak{sp}(1)$,*

$$\rho(v)\hat{M} - \hat{M}_{\tau(v)} - \hat{M}U^T\rho(v)U + \hat{M}U^TU_{\tau(v)} = 0; \quad (3.4.30)$$

$$\rho(v)U - U_{\tau(v)} - UU^T\rho(v)U + UU^TU_{\tau(v)} = 0. \quad (3.4.31)$$

Additionally, we must have that ρ induces a real representation $\lambda: \mathfrak{sp}(1) \rightarrow \mathfrak{so}(k)$ given by

$$\lambda(v) := U^T\rho(v)U - U^TU_{\tau(v)}. \quad (3.4.32)$$

Proof. We follow the proof of Theorem 1.1.1, making modifications when relevant. Suppose that $\hat{M}$ has conformal spherical symmetry. Let $S \subseteq \mathrm{Sp}(1) \times \mathrm{Sp}(n+k)$ be the stabilizer group of $\hat{M}$. Here $\mathrm{Sp}(1)$ is the group $e^{\mathfrak{h}_5}$. Recall that for $A \in \mathrm{Sp}(2)$, $(\hat{M}_A, U_A) := A.(\hat{M}, U)$. Then

$$\begin{aligned} S &:= \{(A, Q) \mid K(A, Q) := U^TQU_{A^\dagger} \in \mathrm{GL}(k, \mathbb{R}), (Q, K(A, Q)).A^\dagger.(\hat{M}, U) \\ &= (\hat{M}, U)\}. \end{aligned} \quad (3.4.33)$$

Indeed, if $(Q, K) \in \mathrm{Sp}(n+k) \times \mathrm{GL}(k, \mathbb{R})$ such that $A^\dagger.(Q, K).(\hat{M}, U) = (\hat{M}, U)$, then Lemma 3.1.20 tells us that $K = U^T Q U_{A^\dagger}$. Note that as $(Q, K).(\hat{M}, U) = A.(\hat{M}, U)$, we know that $K^{-1} = U^T Q^\dagger U_A$. Therefore, the pairs in S encapsulate all the conformal spherical symmetry of the instanton. The condition $K(A, Q) \in \mathrm{GL}(k, \mathbb{R})$ makes proving S is a closed subgroup more challenging. However, we now prove that S is, in fact, a closed subgroup, hence a compact Lie subgroup. In doing so, we prove that $K: S \rightarrow \mathrm{GL}(k, \mathbb{R})$ is a Lie group homomorphism.

Let $f_1: \mathrm{Sp}(1) \times \mathrm{Sp}(n+k) \rightarrow \mathcal{X}$ be the smooth map

$$f_1(A, Q) := (Q^\dagger \hat{M} K(A, Q), Q^\dagger U K(A, Q)) - A^\dagger.(\hat{M}, U).$$

Additionally, let $f_2: \mathrm{Sp}(1) \times \mathrm{Sp}(n+k) \rightarrow \mathrm{Mat}(k, k, \mathbb{H})$ be the smooth map

$$f_2(A, Q) := K(A, Q) - \overline{K(A, Q)}.$$

It turns out that $S = f_1^{-1}(0) \cap f_2^{-1}(0)$. Indeed, the inclusion $S \subseteq f_1^{-1}(0) \cap f_2^{-1}(0)$ follows from the definition of S . Suppose that $(A, Q) \in f_1^{-1}(0) \cap f_2^{-1}(0)$. Then, in particular, $Q^\dagger U K(A, Q) = U_{A^\dagger}$, so $U K(A, Q) = Q U_{A^\dagger}$. As $U_{A^\dagger}$ has rank k and is a $(n+k) \times k$ quaternionic matrix, we know there exists a $k \times (n+k)$ quaternionic matrix W such that $W U_{A^\dagger} = I_k$. Thus, $W Q^\dagger U K(A, Q) = I_k$, so K is invertible. As $f_2(A, Q) = 0$, $K(A, Q)$ is real, so belongs in $\mathrm{GL}(k, \mathbb{R})$. Finally, as $f_1(A, Q) = 0$, $(A, Q) \in S$. Hence, S is closed. We now show that S is a group.

We see that $(I, I) \in S$. Additionally, suppose that $(A, Q) \in S$. Then

$$(Q, K(A, Q)).A^\dagger.(\hat{M}, U) = (\hat{M}, U).$$

Hence, $(Q^\dagger, K(A, Q)^{-1}).A.(\hat{M}, U) = (\hat{M}, U)$. As $Q U K(A, Q)^{-1} = U_A$,

$$K(A, Q)^{-1} = U^T Q^\dagger U_A = K(A^\dagger, Q^\dagger).$$

Hence, $K(A^\dagger, Q^\dagger) \in \mathrm{GL}(k, \mathbb{R})$ and $(Q^\dagger, K(A^\dagger, Q^\dagger)).A.(\hat{M}, U) = (\hat{M}, U)$, so $(A^\dagger, Q^\dagger) \in S$. Finally, suppose that $(A_1, Q_1), (A_2, Q_2) \in S$. Then as the conformal action is a right Lie group action,

$$\begin{aligned} (Q_1 Q_2, K(A_1, Q_1) K(A_2, Q_2)).(A_1 A_2)^\dagger.(\hat{M}, U) &= (Q_1, K(A_1, Q_1)).(Q_2, K(A_2, Q_2)) \\ &\quad .A_1^\dagger.A_2^\dagger.(\hat{M}, U) \\ &= (\hat{M}, U), \end{aligned}$$

as the conformal and gauge actions commute. Thus,

$$((Q_1 Q_2)^{-1}, (K(A_1, Q_1) K(A_2, Q_2))^{-1}).(\hat{M}, U) = (A_1 A_2)^\dagger.(\hat{M}, U).$$

Hence,

$$(Q_1 Q_2)^{-1} U K(A_1, Q_1) K(A_2, Q_2) = U_{(A_1 A_2)^\dagger},$$

so

$$K(A_1, Q_1) K(A_2, Q_2) = U^T Q_1 Q_2 U_{(A_1 A_2)^\dagger} = K(A_1 A_2, Q_1 Q_2).$$

Thus, $K(A_1 A_2, Q_1 Q_2) \in \mathrm{GL}(k, \mathbb{R})$ and $(Q_1 Q_2, K(A_1 A_2, Q_1 Q_2)).(A_1 A_2)^\dagger.(\hat{M}, U) = (\hat{M}, U)$. Hence, $(A_1 A_2, Q_1 Q_2) \in S$, so S is a group and K is a Lie group homomorphism.

Just as in Theorem 1.1.1, there is a Lie group homomorphism $\Psi: \mathrm{Sp}(1) \rightarrow S$ such that $\pi_1|_S \circ \Psi = \mathrm{id}_{\mathrm{Sp}(1)}$. Additionally, there is a Lie algebra homomorphism $\rho: \mathfrak{sp}(1) \rightarrow \mathfrak{sp}(n+k)$ such that for all $v \in \mathfrak{sp}(1)$, we have $\Psi(e^v) = (e^{\tau(v)}, e^{\rho(v)})$.

Recall the Lie group homomorphism $K: S \rightarrow \mathrm{GL}(k, \mathbb{R})$. As $\Psi: \mathrm{Sp}(1) \rightarrow S$, we have a Lie group homomorphism $K \circ \Psi: \mathrm{Sp}(1) \rightarrow \mathrm{GL}(k, \mathbb{R})$. As $\Psi(e^{\theta v}) = (e^{\theta \tau(v)}, e^{\theta \rho(v)})$, we have $K(\Psi(e^{\theta v})) = U^T e^{\theta \rho(v)} U_{e^{-\theta \tau(v)}}$. Then, as $\Psi: \mathrm{Sp}(1) \rightarrow S$,

$$(e^{\theta \rho(v)}, U^T e^{\theta \rho(v)} U_{e^{-\theta \tau(v)}}).e^{-\theta \tau(v)}.(\hat{M}, U) = (\hat{M}, U)$$

Using Lemma 3.4.42, we differentiate and evaluate at $\theta = 0$, obtaining

$$\begin{aligned} \rho(v) \hat{M} U^T U - \hat{M}_{\tau(v)} U^T U - \hat{M} U^T \rho(v) U + \hat{M} U^T U_{\tau(v)} &= 0; \\ \rho(v) U U^T U - U_{\tau(v)} U^T U - U U^T \rho(v) U + U U^T U_{\tau(v)} &= 0. \end{aligned}$$

Noting that $U^T U = I_k$, we obtain (3.4.30) and (3.4.31).

As $K \circ \Psi: \mathrm{Sp}(1) \rightarrow \mathrm{GL}(k, \mathbb{R})$ is a Lie group homomorphism, it gives rise to a Lie algebra homomorphism $\lambda: \mathfrak{sp}(1) \rightarrow \mathfrak{gl}(k)$. As $\mathrm{Sp}(1)$ is compact, all representations are unitary. In the Lie algebra realm, this means that all representations are skew-Hermitian, so $\lambda: \mathfrak{sp}(1) \rightarrow \mathfrak{so}(k)$. As $\mathfrak{so}(k)$ acts naturally on $\mathbb{R}^k$, $(\mathbb{R}^k, \lambda)$ is a real k -representation of $\mathfrak{sp}(1)$. In particular, as $K(\Psi(e^{\theta v})) = U^T e^{\theta \rho(v)} U_{e^{-\theta \tau(v)}}$, we have $\lambda(v) = U^T \rho(v) U - U^T U_{\tau(v)}$.

Conversely, suppose that the equations hold for some $\rho: \mathfrak{sp}(1) \rightarrow \mathfrak{sp}(n+k)$ and ρ induces a real representation $\lambda: \mathfrak{sp}(1) \rightarrow \mathfrak{gl}(k)$ of the given form. Consider $A \in \mathrm{Sp}(1)$. As the exponential map $\exp: \mathfrak{sp}(1) \rightarrow \mathrm{Sp}(1)$ is surjective, there is some $v \in \mathfrak{sp}(1)$ such that $A = e^{\tau(v)}$. Let $A(\theta) := e^{\theta \tau(v)} \in \mathrm{Sp}(2)$, $Q(\theta) := e^{\theta \rho(v)} \in \mathrm{Sp}(n+k)$, and $K(\theta) := e^{\theta \lambda(v)} \in \mathrm{GL}(k, \mathbb{R})$. We show that $F(\theta) := (Q(\theta), K(\theta)).A(\theta)^\dagger.(\hat{M}, U) \in \mathcal{X}$ is constant. Expanding,

$$F(\theta) = (Q(\theta) \hat{M}_{A(\theta)^\dagger} K(\theta)^{-1}, Q(\theta) U_{A(\theta)^\dagger} K(\theta)^{-1}).$$

Before we differentiate F , we first recall that the group actions commute and we note that

$$\frac{d}{d\theta} \begin{bmatrix} \hat{M}_{A(\theta)^\dagger} & U_{A(\theta)^\dagger} \end{bmatrix} = - \begin{bmatrix} \hat{M} & U \end{bmatrix} I_k \otimes \left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tau(v) A(\theta)^\dagger \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right).$$

From this we see that

$$\frac{d}{d\theta} \begin{bmatrix} \hat{M}_{A(\theta)^\dagger} & U_{A(\theta)^\dagger} \end{bmatrix} = - \begin{bmatrix} (\hat{M}_{\tau(v)})_{A(\theta)^\dagger} & (U_{\tau(v)})_{A(\theta)^\dagger} \end{bmatrix}.$$

Now we are equipped to differentiate F , obtaining

$$F'(\theta) = (Q(\theta) \left(\rho(v)\hat{M} - \hat{M}_{\tau(v)} - \hat{M}\lambda(v) \right)_{A(\theta)^\dagger} K(\theta)^{-1}, \\ Q(\theta) (\rho(v)U - U_{\tau(v)} - U\lambda(v))_{A(\theta)^\dagger} K(\theta)^{-1}) = 0.$$

Hence, F is constant, so

$$(Q(1), K(1)).A^\dagger.(\hat{M}, U) = F(1) = F(0) = (\hat{M}, U).$$

Hence, $\hat{M}$ is $A^\dagger$ -equivariant. As $A \in \text{Sp}(1)$ was arbitrary, we have conformal spherical symmetry. $\square$

Using Theorem 3.4.43, we find that we already know all instantons with conformal spherical symmetry.

Theorem 3.4.44. *Let $\hat{M} \in \mathcal{M}_{n,k}$. Then $\hat{M}$ has conformal spherical symmetry if and only if $n = k$ and $(\hat{M}, U)$ is gauge equivalent to $\left(\begin{bmatrix} I_k \\ 0 \end{bmatrix}, U \right)$.*

Proof. We know that all instantons with ADHM data of this form have full symmetry and therefore have conformal spherical symmetry. It remains to show that all such instantons have ADHM data of this form.

Suppose that $\hat{M}$ has conformal spherical symmetry. Consider the Lie algebra isomorphism $\tau: \mathfrak{sp}(1) \rightarrow \mathfrak{h}_5$ given by

$$\tau(v_1) := \begin{bmatrix} i/2 & 0 \\ 0 & 3i/2 \end{bmatrix}, \quad \tau(v_2) := \begin{bmatrix} j & \sqrt{3}/2 \\ -\sqrt{3}/2 & 0 \end{bmatrix}, \quad \tau(v_3) := \begin{bmatrix} k & -\sqrt{3}i/2 \\ -\sqrt{3}i/2 & 0 \end{bmatrix}.$$

From Lemma 3.4.42, we have that

$$\begin{aligned} (\hat{M}_{\tau(v_1)}, U_{\tau(v_1)}) &= \left(\frac{3}{2}\hat{M}i, \frac{i}{2}U \right), \\ (\hat{M}_{\tau(v_2)}, U_{\tau(v_2)}) &= \left(-\frac{\sqrt{3}}{2}U, \frac{\sqrt{3}}{2}\hat{M} + Uj \right), \\ (\hat{M}_{\tau(v_3)}, U_{\tau(v_3)}) &= \left(\frac{\sqrt{3}}{2}Ui, \frac{\sqrt{3}}{2}\hat{M}i + Uk \right). \end{aligned}$$

By Theorem 3.4.43, we know there is some $\rho: \mathfrak{sp}(1) \rightarrow \mathfrak{sp}(n+k)$ such that $\lambda: \mathfrak{sp}(1) \rightarrow \mathfrak{so}(k)$, given by $\lambda(v) := U^T \rho(v) U - U^T U_{\tau(v)}$, is a real representation. Thus, we see that

$$-U^T \rho(v) U - (U_{\tau(v)})^\dagger U = \lambda(v)^\dagger = -\lambda(v) = -U^T \rho(v) U + U^T U_{\tau(v)}.$$

Evaluating at $v = v_2$, we see that

$$\frac{\sqrt{3}}{2} U^T \hat{M} + U^T U j + \frac{\sqrt{3}}{2} \hat{M}^\dagger U - j U^T U = 0.$$

As M is symmetric, we have that $M_0 = 0$. Evaluating at $v = v_3$, we see that

$$\frac{\sqrt{3}}{2} U^T \hat{M} i + U^T U k - \frac{\sqrt{3}}{2} i \hat{M}^\dagger U - k U^T U = 0.$$

Simplifying, noting that $M^\dagger = -M$, as $M_0 = 0$, we have that $M i + i M = 0$. Thus, $M = M_1 i$.

As $\hat{M}$ has conformal spherical symmetry, it is $\text{diag}(e^{i\theta/2}, e^{3i\theta/2})$ -equivariant for all $\theta \in \mathbb{R}$. Using the methods from Theorem 3.2.1, we have that there is some $\tilde{\rho} \in \mathfrak{so}(k)$ such that

$$3M i - i M + [\tilde{\rho}, M] = 0.$$

As M is purely imaginary, we can separate the real and imaginary parts of this equation, finding $M = 0$. By Proposition 3.4.13, we know that $n = k$ and L is positive definite and diagonal in some gauge.

Finally, we have that ρ satisfies (3.4.30) and (3.4.31). There exists $A, C \in \mathfrak{sp}(k)$ and $B \in \text{Mat}(k, k, \mathbb{H})$ such that $\rho(v_2) = \begin{bmatrix} A & B \\ -B^\dagger & C \end{bmatrix}$. Evaluating (3.4.31) at $v = v_2$, we see that

$$\begin{bmatrix} B \\ C \end{bmatrix} - \frac{\sqrt{3}}{2} \begin{bmatrix} L \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ I_k j \end{bmatrix} - \begin{bmatrix} 0 \\ \lambda(v_2) \end{bmatrix} = 0.$$

Thus, $B = \frac{\sqrt{3}}{2} L$. Evaluating (3.4.30) at $v = v_2$, we see that

$$\begin{bmatrix} AL \\ -B^\dagger L \end{bmatrix} + \frac{\sqrt{3}}{2} \begin{bmatrix} 0 \\ I_k \end{bmatrix} - \begin{bmatrix} L \lambda(v_2) \\ 0 \end{bmatrix} = 0.$$

Thus, $B^\dagger L = \frac{\sqrt{3}}{2} I_k$. As $B = \frac{\sqrt{3}}{2} L$, we have that $L^\dagger L = I_k$. Therefore, $L \in \text{Sp}(k)$, proving the theorem. $\square$

The only connected Lie subgroup of $\mathrm{Sp}(2)$ containing the conformal spherical subgroup is the group $\mathrm{Sp}(2)$ itself. Theorem 3.4.44 tells us about instantons with full symmetry as well. Note that an instanton has full symmetry if and only if it is equivariant under every element of $\mathrm{Sp}(2)$.

Corollary 3.4.45. *Let $\hat{M} \in \mathcal{M}_{n,k}$. Then $\hat{M}$ has full symmetry if and only if $n = k$ and $(\hat{M}, U)$ is gauge equivalent to $\left(\begin{bmatrix} I_k \\ 0 \end{bmatrix}, U\right)$.*

Proof. We know that such instantons have full symmetry. Conversely, if an instanton has full symmetry, then it has conformal spherical symmetry. Theorem 3.4.44 tells us the form that the ADHM data must take. $\square$

3.5 Superspherical symmetry

In this section, we find equations describing all instantons with two kinds of superspherical symmetry, as given in Table C.1. We separate the cases of isoclinic and conformal superspherical symmetry. We also discuss the connections between these symmetric instantons and symmetric hyperbolic monopoles, singular monopoles, and hyperbolic Higgs bundles.

Note 3.5.1. *All connected Lie subgroups of $\mathrm{Sp}(2)$ with Lie algebra $\mathfrak{sp}(1) \oplus \mathbb{R}$ are conjugate to the connected Lie subgroups $G_{3,1,1}$ or $G_{4,1}$ induced by $\mathfrak{p}_{3,1,1}$ and $\mathfrak{p}_{4,1}$, respectively. That is, there is some $A \in \mathrm{Sp}(2)$ and G one of the subgroups $G_{3,1,1}$ or $G_{4,1}$ such that the Lie group is of the form $AGA^\dagger$. Instantons equivariant under this group are of the form $A^\dagger \cdot (\hat{M}, U)$, where $\hat{M}$ is equivariant under G .*

3.5.1 Isoclinic superspherical symmetry

In this section, we find an equation describing all instantons with isoclinic superspherical symmetry, as given in Table C.1 as well as below. We also discuss the connection between these instantons and symmetric singular monopoles. Note that a representation $\rho: \mathfrak{sp}(1) \oplus \mathbb{R} \rightarrow \mathfrak{so}(k)$ corresponds to a representation of $\mathrm{Sp}(1) \times S^1$ if and only if $\exp(\rho(0, 2\pi)) = I_k$.

First, we reintroduce the notion of isoclinic superspherical symmetry. An instanton is said to have isoclinic superspherical symmetry if it is equivariant under $\mathrm{diag}(p, e^{it})$ for all $p \in \mathrm{Sp}(1)$ and $t \in \mathbb{R}$.

Theorem 3.5.2. *Let $\hat{M} \in \mathcal{M}_{n,k}$. Then $\hat{M}$ has isoclinic superspherical symmetry if and only if there exists a real representation $\rho: \mathfrak{sp}(1) \oplus \mathbb{R} \rightarrow \mathfrak{so}(k)$, corresponding to a representation of $\mathrm{Sp}(1) \times S^1$, such that for all $v \in \mathfrak{sp}(1)$ and $t \in \mathbb{R}$,*

$$vM + [\rho(v, t), M] - 2tMi = 0, \quad (3.5.1)$$

$$[\rho(v, t), R] = 0. \quad (3.5.2)$$

Definition 3.5.3. *We call ρ the **generating representation** of the isoclinic superspherical symmetry of $\hat{M}$.*

Proof. We follow the proof of Theorem 1.1.1, making modifications when relevant. Suppose that $\hat{M}$ has isoclinic superspherical symmetry. Let $S \subseteq \mathrm{Sp}(1) \times S^1 \times \mathrm{O}(k)$ be the stabilizer group of $(\hat{M}, U)$ restricted to rotations in $\mathrm{Sp}(1) \times S^1$. That is

$$S := \{(\mathrm{diag}(p, e^{i\theta}), K) \mid (\mathrm{diag}(Le^{2i\theta}KL^\dagger(LL^\dagger)^{-1}, pK), K) \cdot \mathrm{diag}(p^\dagger, e^{-2i\theta}) \cdot (\hat{M}, U) = (\hat{M}, U)\}. \quad (3.5.3)$$

Indeed, if $(Q, K) \in \mathrm{Sp}(n+k) \times \mathrm{GL}(k, \mathbb{R})$ such that $\mathrm{diag}(p^\dagger, e^{-2i\theta}) \cdot (Q, K) \cdot (\hat{M}, U) = (\hat{M}, U)$, then Corollary 3.1.27 tells us that $K \in \mathrm{O}(k)$, $[R, K] = 0$, and Q is given by the diagonal matrix in (3.5.3). Therefore, the pairs in S encapsulate all the isoclinic superspherical symmetry of the instanton. That S is a group follows from this and that the conformal action is a right Lie group action, just as in the proof of Theorem 3.4.3.

Just as in Theorem 1.1.1, S is a compact Lie group and we obtain a Lie group homomorphism $\Psi: \mathrm{Sp}(1) \times \mathbb{R} \rightarrow S$ such that $\pi_1|_S \circ \Psi = \pi$, where π_1 treats $\mathrm{Sp}(1) \times S^1$ as a single component and $\pi: \mathrm{Sp}(1) \times \mathbb{R} \rightarrow \mathrm{Sp}(1) \times S^1$ is the covering map.

Additionally, there is a Lie algebra homomorphism $\rho: \mathfrak{sp}(1) \oplus \mathbb{R} \rightarrow \mathfrak{so}(k)$ such that for all $v \in \mathfrak{sp}(1)$ and $t \in \mathbb{R}$, $\Psi(e^v, t) = (\mathrm{diag}(e^v, e^{it}), e^{\rho(v, t)})$. As ρ is a Lie algebra homomorphism whose image consists of real matrices, it gives us a real Lie algebra representation.

As $\Psi(e^{\theta v}, \theta t) \in S$ for all $v \in \mathfrak{sp}(1)$ and $t, \theta \in \mathbb{R}$, we have that

$$(\mathrm{diag}(Le^{2\theta it}e^{\theta\rho(v, t)}L^\dagger(LL^\dagger)^{-1}, e^{\theta v}e^{\theta\rho(v, t)}), e^{\theta\rho(v, t)}) \cdot \mathrm{diag}(e^{-\theta v}, e^{-2\theta it}) \cdot (\hat{M}, U) = (\hat{M}, U).$$

Differentiating and evaluating at $\theta = 0$, we obtain

$$\begin{bmatrix} 2tLiL^\dagger(LL^\dagger)^{-1}L + L\rho(v, t)L^\dagger(LL^\dagger)^{-1}L - L\rho(v, t) - 2tLi \\ vM + [\rho(v, t), M] - 2tMi \end{bmatrix} = 0. \quad (3.5.4)$$

Focusing on the bottom row, we see that $vM + [\rho(v, t), M] - 2tMi = 0$. Furthermore, as mentioned above, Corollary 3.1.27 tells us that $[R, e^{\theta\rho(v, t)}] = 0$, for all $\theta \in \mathbb{R}$. Differentiating and evaluating at $\theta = 0$, we have $[R, \rho(v, t)] = 0$.

Moreover, looking at the map $t \mapsto \rho(0, t)$, we can proceed as in Proposition 3.2.5 and Proposition 3.3.5 to find that without loss of generality, we may assume that ρ corresponds to a representation of $\mathrm{Sp}(1) \times S^1$.

Conversely, suppose that the equations hold for some real representation $\rho: \mathfrak{sp}(1) \oplus \mathbb{R} \rightarrow \mathfrak{so}(k)$. Consider $p \in \mathrm{Sp}(1)$ and $e^{2it_0} \in S^1$. As the exponential map $\exp: \mathfrak{sp}(1) \rightarrow \mathrm{Sp}(1)$ is surjective, there is some $v \in \mathfrak{sp}(1)$ such that $p = e^v$. Let $p(\theta) := e^{\theta v} \in \mathrm{Sp}(1)$ and $K(\theta) := e^{\theta\rho(v, t_0)} \in \mathrm{O}(k)$. We show that $A(\theta) := p(\theta)K(\theta)MK(\theta)^T e^{-2it_0\theta}$ is constant. Indeed,

$$A'(\theta) = p(\theta)K(\theta)(vM + [\rho(v, t_0), M] - 2t_0Mi)K(\theta)^T e^{-2it_0\theta} = 0.$$

Hence, A is constant, so

$$pK(1)MK(1)^T e^{-2it_0} = A(1) = A(0) = M.$$

As $[\rho(v, t_0), R] = 0$, $[K(\theta), R] = 0$, so by Lemma 3.1.25, we have that $\hat{M}$ is $\mathrm{diag}(p^\dagger, e^{-2it_0})$ -equivariant. As p and t_0 were arbitrary, we have isoclinic superspherical symmetry. $\square$

Above, we see that isoclinic superspherical symmetry is generated by a representation $(\mathbb{R}^k, \rho)$, corresponding to a representation of $\mathrm{Sp}(1) \times S^1$. This representation was obtained by examining the bottom of (3.5.4). By focusing instead on the top of this equation, we obtain an induced representation.

Lemma 3.5.4. *Suppose that $\hat{M} \in \mathcal{M}_{n,k}$ has isoclinic superspherical symmetry generated by a representation $(\mathbb{R}^k, \rho)$, which corresponds to a representation of $\mathrm{Sp}(1) \times S^1$. Let $\lambda: \mathfrak{sp}(1) \oplus \mathbb{R} \rightarrow \mathfrak{sp}(n)$ be defined by $\lambda(v, t) := L(\rho(v, t) + 2ti)L^\dagger(LL^\dagger)^{-1}$. We have $(\mathbb{H}^n, \lambda)$ is a representation of $\mathfrak{sp}(1) \oplus \mathbb{R}$, corresponding to a representation of $\mathrm{Sp}(1) \times S^1$.*

Proof. By Theorem 3.5.2, we know that for all $t \in \mathbb{R}$ and $v \in \mathfrak{sp}(1)$, (3.5.1) and (3.5.2) hold. Using the former,

$$\begin{aligned} M^\dagger M(\rho(v, t) + 2ti) &= M^\dagger[M, \rho(v, t)] + M^\dagger\rho(v, t)M + 2tM^\dagger Mi \\ &= M^\dagger\rho(v, t)M + M^\dagger vM \\ &= [M^\dagger, \rho(v, t)]M + \rho(v, t)M^\dagger M + M^\dagger vM \\ &= (\rho(v, t) + 2ti)M^\dagger M. \end{aligned}$$

Hence, $[M^\dagger M, \rho(v, t) + 2ti] = 0$. Then, as $[\rho(v, t), R] = 0$, by (3.5.2), and R is real, $[\rho(v, t) + 2ti, R] = 0$, so $[L^\dagger L, \rho(v, t) + 2ti] = 0$. Thus, we see that

$$[LL^\dagger, L(\rho(v, t) + 2ti)L^\dagger] = 0.$$

Therefore, $\lambda(v, t) \in \mathfrak{sp}(n)$.

Just as in the toral symmetry case, we have that

$$e^{\lambda(v, t)} = Le^{\rho(v, t) + 2ti}L^\dagger(LL^\dagger)^{-1}.$$

As ρ and i commute, we have that as $(\mathbb{R}^k, \rho)$ corresponds to a representation of $\mathrm{Sp}(1) \times S^1$, $e^{\lambda(0, 2\pi)} = I_n$. Therefore, $(\mathbb{H}^n, \lambda)$ corresponds to a representation of $\mathrm{Sp}(1) \times S^1$. $\square$

We know that this induced representation satisfies the following equation.

Corollary 3.5.5. *If $\hat{M} \in \mathcal{M}_{n, k}$ has isoclinic superspherical symmetry generated by $(\mathbb{R}^k, \rho)$, then $\lambda(v, t)L - L\rho(v, t) - 2tLi = 0$, for all $t \in \mathbb{R}$ and $v \in \mathfrak{sp}(1)$.*

Proof. This follows from the top component of (3.5.4). $\square$

Note 3.5.6. *Just as with circular t -symmetry, we need not check the final condition of $\mathcal{M}_{n, k}$ in Definition 3.1.16 everywhere. Indeed, if $\hat{M}$ satisfies (3.5.1) and (3.5.2) for some real representation $(\mathbb{R}^k, \rho)$, then it satisfies (3.4.11) and (3.4.12) for some real representation of $\mathfrak{sp}(1)$. Thus, just as in the isoclinic spherical symmetry case, we need only check that the final condition of $\mathcal{M}_{n, k}$ in Definition 3.1.16 is satisfied for all $x = x_0 \in \mathbb{H}$ with $x_0 \geq 0$.*

Note 3.5.7. *The Lie algebra of isoclinic superspherical symmetry is comprised of the Lie algebras for isoclinic spherical symmetry and the Lie algebra of another isoclinic circle 0-symmetry. Therefore, an instanton has isoclinic superspherical symmetry if and only if it has isoclinic spherical symmetry and it is symmetric under this additional circle action. This viewpoint is beneficial, as we can use the work done in Section 3.4.2 to narrow down our search for isoclinic superspherical symmetry.*

Corollary 3.5.8. *Let $\hat{M} \in \mathcal{M}_{n, k}$. Then $\hat{M}$ has isoclinic superspherical symmetry if and only if there is some $\rho: \mathfrak{sp}(1) \rightarrow \mathfrak{so}(k)$ and $\tau \in \mathfrak{so}(k)$ such that for all $v \in \mathfrak{sp}(1)$,*

$$vM + [\rho(v), M] = 0, \tag{3.5.5}$$

$$-Mi + [\tau, M] = 0, \tag{3.5.6}$$

$$[\rho(v), \tau] = [\rho(v), R] = [\tau, R] = 0. \tag{3.5.7}$$

Proof. This follows from Theorem 3.5.2, as we have merely decomposed the representation of $\mathfrak{sp}(1) \oplus \mathbb{R}$. $\square$

Proposition 3.6.17 provides us with an example of an instanton with rotational symmetry. Such an instanton also possesses isoclinic superspherical symmetry.

3.5.1.1 Spherically symmetric singular monopoles

In this section, we discuss the connections between instantons with isoclinic superspherical symmetry and spherically symmetric singular monopoles with Dirac type singularities. Note that the conformal superspherical subgroup contains no subgroup conjugate to R_0 , so it does not lead to any singular monopoles.

Theorem 3.5.9. *A $\mathrm{Sp}(n)$ singular monopole with Dirac type singularities is spherically symmetric if and only if its ADHM data has isoclinic superspherical symmetry.*

Proof. The proof follows from the proof of Proposition 3.3.9, changing from $e^{v\phi}$ to p . $\square$

Note 3.5.10. *We might expect a similar result regarding symmetric hyperbolic Nahm data. However, this does not happen. Indeed, recall that we use the conformal equivalence $\mathbb{R}^4 \setminus \{0\} \equiv H^1 \times S^3$ to relate instantons with isoclinic spherical symmetry to hyperbolic Nahm data. In particular, using four-dimensional spherical coordinates (r, θ, ϕ, ψ) on $\mathbb{R}^4 \setminus \{0\}$, the hyperbolic coordinate is r . However, r is invariant under the entire isoclinic superspherical action. Thus, the extra symmetry of the instanton does not impart any extra symmetry on the hyperbolic Nahm data.*

3.5.2 Conformal superspherical symmetry

In this section, we find an equation describing all instantons with conformal superspherical symmetry, as given in Table C.1 as well as below. We also discuss the connection between these instantons and symmetric hyperbolic monopoles and hyperbolic Higgs bundles.

First, we reintroduce the notion of conformal superspherical symmetry. An instanton is said to have conformal superspherical symmetry if it is equivariant under the connected Lie group corresponding to $\mathfrak{p}_{3,1,1} \subseteq \mathfrak{sp}(2)$ listed in Table C.1.

Theorem 3.5.11. Let $X := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathfrak{sp}(2)$. Let $\hat{M} \in \mathcal{M}_{n,k}$. Then $\hat{M}$ has conformal superspherical symmetry if and only if there exists a representation $\rho: \mathfrak{sp}(1) \oplus \mathbb{R} \rightarrow \mathfrak{sp}(n+k)$ such that for all $v \in \mathfrak{sp}(1)$ and $t \in \mathbb{R}$,

$$\rho(v, t)\hat{M} - \hat{M}_{vI_2+tX} - \hat{M}U^T\rho(v, t)U + \hat{M}U^TU_{vI_2+tX} = 0; \quad (3.5.8)$$

$$\rho(v, t)U - U_{vI_2+tX} - UU^T\rho(v, t)U + UU^TU_{vI_2+tX} = 0. \quad (3.5.9)$$

Additionally, we must have that ρ induces a real representation $\lambda: \mathfrak{sp}(1) \oplus \mathbb{R} \rightarrow \mathfrak{gl}(k)$ given by

$$\lambda(v, t) := U^T\rho(v, t)U - U^TU_{vI_2+tX}. \quad (3.5.10)$$

Definition 3.5.12. We call ρ the **generating representation** of the conformal superspherical symmetry of $\hat{M}$.

Note 3.5.13. In contrast to the cases of conformal spherical and full symmetry, λ is allowed to take values in $\mathfrak{gl}(k)$ outside of $\mathfrak{so}(k)$. This phenomenon is possible because the universal cover of this symmetry group is $\mathrm{Sp}(1) \times \mathbb{R}$, which is not compact.

Proof. We follow the proof of Theorem 3.4.43. Suppose that $\hat{M}$ has conformal superspherical symmetry. Let $S \subseteq ((\mathrm{Sp}(1) \times S^1)/\{\pm(1, 1)\}) \times \mathrm{Sp}(n+k)$ be the stabilizer group of $\hat{M}$. Recall that for $A \in \mathrm{Sp}(2)$, $(\hat{M}_A, U_A) := A.(\hat{M}, U)$. Note that $e^{\theta X} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$.

We have

$$S := \{([(p, e^{i\theta})], Q) \mid K([(p, e^{i\theta})], Q) := U^T Q U_{p^\dagger e^{-\theta X}} \in \mathrm{GL}(k, \mathbb{R}), \\ (Q, K([(p, e^{i\theta})], Q)).p^\dagger e^{-\theta X}.(\hat{M}, U) = (\hat{M}, U)\}. \quad (3.5.11)$$

Just as in Theorem 3.4.43, noting that $(\mathrm{Sp}(1) \times S^1)/\{\pm(1, 1)\}$ is compact, as per Proposition G.0.1, we have that S is a compact Lie subgroup, encapsulating all the conformal superspherical symmetry of the instanton.

Similar to Theorem 3.4.43, we obtain a Lie group homomorphism $\Psi: \mathrm{Sp}(1) \times \mathbb{R} \rightarrow S$ such that $\pi_1|_S \circ \Psi = \pi$, where $\pi: \mathrm{Sp}(1) \times \mathbb{R} \rightarrow (\mathrm{Sp}(1) \times S^1)/\{\pm(1, 1)\}$ is the covering map.

Additionally, there is a Lie algebra homomorphism $\rho: \mathfrak{sp}(1) \oplus \mathbb{R} \rightarrow \mathfrak{sp}(n+k)$ such that for all $v \in \mathfrak{sp}(1)$ and $t \in \mathbb{R}$, we have $\Psi(e^v, t) = (e^v e^{tX}, e^{\rho(v, t)})$.

As $\Psi(e^{\theta v}, \theta t) \in S$ for all $\theta, t \in \mathbb{R}$ and $v \in \mathfrak{sp}(1)$, we have that

$$(e^{\theta \rho(v, t)}, U^T e^{\theta \rho(v, t)} U_{e^{-\theta v} e^{-\theta t X}}).e^{-\theta v} e^{-\theta t X}.(\hat{M}, U) = (\hat{M}, U).$$

Using Lemma 3.4.42, we differentiate and evaluate at $\theta = 0$, obtaining

$$\begin{aligned}\rho(v, t)\hat{M}U^TU - \hat{M}_{vI_2+tX}U^TU - \hat{M}U^T\rho(v, t)U + \hat{M}U^TU_{vI_2+tX} &= 0; \\ \rho(v, t)UU^TU - U_{vI_2+tX}U^TU - UU^T\rho(v, t)U + UU^TU_{vI_2+tX} &= 0.\end{aligned}$$

Noting that $U^TU = I_k$, we obtain (3.5.8) and (3.5.9).

As $K \circ \Psi \circ G: \mathrm{Sp}(1) \times \mathbb{R} \rightarrow \mathrm{GL}(k, \mathbb{R})$ is a Lie group homomorphism, it gives rise to a Lie algebra homomorphism $\lambda: \mathfrak{sp}(1) \oplus \mathbb{R} \rightarrow \mathfrak{gl}(k)$. As $\mathfrak{gl}(k)$ acts naturally on $\mathbb{R}^k$, $(\mathbb{R}^k, \lambda)$ is a real k -representation of $\mathfrak{sp}(1) \oplus \mathbb{R}$. In particular, as $K(\Psi(G(e^{\theta v}e^{\theta tX}))) = U^Te^{\theta\rho(v, t)}U_{e^{-\theta v}e^{-\theta tX}}$, we have $\lambda(v, t) = U^T\rho(v, t)U - U^TU_{vI_2+tX}$.

Conversely, suppose that the equations hold for some $\rho: \mathfrak{sp}(1) \oplus \mathbb{R} \rightarrow \mathfrak{sp}(n+k)$ and ρ induces a real representation $\lambda: \mathfrak{sp}(1) \oplus \mathbb{R} \rightarrow \mathfrak{gl}(k)$ of the given form. Consider $[(p, e^{it})] \in (\mathrm{Sp}(1) \times S^1)/\{\pm(1, 1)\}$. As the exponential map $\exp: \mathfrak{sp}(1) \rightarrow \mathrm{Sp}(1)$ is surjective, there is some $v \in \mathfrak{sp}(1)$ such that $p = e^v$. Let $A(\theta) := e^{\theta v}e^{\theta tX} \in \mathrm{Sp}(2)$, $Q(\theta) := e^{\theta\rho(v, t)} \in \mathrm{Sp}(n+k)$, and $K(\theta) := e^{\theta\lambda(v, t)} \in \mathrm{GL}(k, \mathbb{R})$. We show that $F(\theta) := (Q(\theta), K(\theta)).A(\theta)^\dagger.(\hat{M}, U) \in \mathcal{X}$ is constant. Expanding,

$$F(\theta) = (Q(\theta)\hat{M}_{A(\theta)^\dagger}K(\theta)^{-1}, Q(\theta)U_{A(\theta)^\dagger}K(\theta)^{-1}).$$

Differentiating F , we obtain

$$\begin{aligned}F'(\theta) &= (Q(\theta) \left(\rho(v, t)\hat{M} - \hat{M}_{vI_2+tX} - \hat{M}\lambda(v, t) \right)_{A(\theta)^\dagger} K(\theta)^{-1}, \\ &\quad Q(\theta) (\rho(v, t)U - U_{vI_2+tX} - U\lambda(v, t))_{A(\theta)^\dagger} K(\theta)^{-1}) = 0.\end{aligned}$$

Hence, F is constant, so

$$(Q(1), K(1)).p^\dagger e^{-tX}.(\hat{M}, U) = F(1) = F(0) = (\hat{M}, U).$$

Hence, $\hat{M}$ is $p^\dagger e^{-tX}$ -equivariant. As p and t were arbitrary, we have conformal superspherical symmetry. $\square$

Note 3.5.14. *Just as with circular t -symmetry, we need not check the final condition of $\mathcal{M}_{n,k}$ in Definition 3.1.16 everywhere. Indeed, if $\hat{M}$ has conformal superspherical symmetry, then it has simple spherical symmetry. Thus, just as in the simple spherical symmetry case, we need only check that the final condition of $\mathcal{M}_{n,k}$ in Definition 3.1.16 is satisfied for all $x = x_0 + x_1 i \in \mathbb{H}$ with $x_1 \geq 0$.*

Note 3.5.15. *The Lie algebra of conformal superspherical symmetry is comprised of the Lie algebras for simple spherical symmetry and symmetry under Manton–Sutcliffe’s conformal circle action. Therefore, an instanton has conformal superspherical symmetry if and only if it has simple spherical symmetry and is symmetric under Manton–Sutcliffe’s conformal circle action. This viewpoint is beneficial as we can use the work done in Section 3.4.1 to narrow down our search for conformal superspherical symmetry.*

Corollary 3.5.16. *Let $\hat{M} \in \mathcal{M}_{n,k}$. Then $\hat{M}$ has conformal superspherical symmetry if and only if it has simple spherical symmetry and there exists $\rho \in \mathfrak{sp}(n+k)$ such that $U^T \rho U - U^T \hat{M}$ is real and*

$$\rho \hat{M} + U - \hat{M} U^T \rho U + \hat{M} U^T \hat{M} = 0; \quad (3.5.12)$$

$$\rho U - \hat{M} - U U^T \rho U + U U^T \hat{M} = 0. \quad (3.5.13)$$

Note 3.5.17. *Because the simple spherical symmetry and the additional circular 1-symmetry commute, ρ must commute with the matrix Q used in the simple spherical symmetry.*

Proof. Suppose $\hat{M} \in \mathcal{M}_{n,k}$. Let $\Upsilon := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \in \mathfrak{sp}(2)$ be the generator of Manton–Sutcliffe’s conformal circle action. Using Lemma 3.4.42, we see that $\hat{M}_\Upsilon = -U$ and $U_\Upsilon = \hat{M}$. Using methods from the proof of Theorem 3.4.43, we find that $\hat{M}$ is symmetric under Manton–Sutcliffe’s conformal circle action if and only if there exists $\rho \in \mathfrak{sp}(n+k)$ such that (3.5.12) and (3.5.13) hold and $U^T \rho U - U^T \hat{M}$ is real.

The preceding note tells us that an instanton has conformal superspherical symmetry if and only if it has simple spherical symmetry and is symmetric under Manton–Sutcliffe’s conformal circle action, proving the result. $\square$

3.5.2.1 Novel example of conformal superspherical symmetry

In this section, we identify a novel example of an instanton with conformal superspherical symmetry. Note that several examples were identified in Propositions 2.1.40, 2.1.42, and 2.1.44. In fact, the ADHM data for any spherically symmetric hyperbolic monopole in previous work has conformal superspherical symmetry.

More generally, the ADHM data for any hyperbolic monopole in previous work is symmetric under Manton–Sutcliffe’s conformal circle action, by design. We identify the condition that forces the ADHM data to take the form in Definition 2.1.1.

Proposition 3.5.18. *Consider $\hat{M} \in \mathcal{M}_{n,k}$. Then $\hat{M}$ is symmetric under Manton–Sutcliffe’s conformal circle action and $M_0 = 0$ if and only if $\hat{M}$ is in the generalization of Manton–Sutcliffe’s set, Definition 2.1.1. That is, M is pure and $R := L^\dagger L + M^\dagger M = I_k$.*

Proof. Manton–Sutcliffe prove that data in their set is symmetric under their conformal circle action [MS14, § 4]. We prove the converse. Suppose that $\hat{M}$ is symmetric under Manton–Sutcliffe’s conformal circle action and $M_0 = 0$.

From the proof of Corollary 3.5.16, we have that there is some $\rho \in \mathfrak{sp}(n+k)$ such that $U^T \rho U - U^T \hat{M}$ is real and (3.5.12) and (3.5.13) hold.

Write $\rho := \begin{bmatrix} X & Y \\ -Y^\dagger & Z \end{bmatrix}$, so $X \in \mathfrak{sp}(n)$ and $Z \in \mathfrak{sp}(k)$. We then see that $Z - M$ must be real. Thus, there exists $P \in \mathfrak{so}(k)$ such that $Z = M + P$. Substituting and simplifying (3.5.13), we see that $Y = L$. Substituting and simplifying (3.5.12), we see that

$$\begin{bmatrix} XL + LM - LZ + LM \\ -L^\dagger L + ZM + I_k - MZ + M^2 \end{bmatrix} = 0.$$

Note that as M is pure, $M^\dagger M = -M^2$, so the bottom row is $(I_k - R) + [Z, M] = 0$. As $Z = M + P$ and M is pure, $[Z, M]$ is pure. As $I_k - R$ is real, we have that $I_k - R = 0 = [Z, M]$. So, in particular, $R = I_k$. Therefore, the ADHM data is in the generalization of Manton–Sutcliffe’s set given by Definition 2.1.1. $\square$

The preceding proposition tells us that if we are to find ADHM symmetric under Manton–Sutcliffe’s conformal circle action but not in the previously considered set, it must have $M_0 \neq 0$. We find exactly such an instanton. This is not the first example of such an instanton, but it is the first with a higher rank structure group [MS14, (7.1)].

Proposition 3.5.19. *For $B \in (0, \frac{2}{3}\sqrt{6})$, let*

$$\begin{aligned} A &:= \frac{\sqrt{12 - 15B^2 + 12\sqrt{B^4 - B^2 + 1}}}{3}, \quad \text{and} \\ a &:= \frac{2B^2 - \sqrt{B^4 - B^2 + 1} - 1}{\sqrt{-3B^2 + 6 + 6\sqrt{B^4 - B^2 + 1}}}. \end{aligned} \tag{3.5.14}$$

Note that for all B in our range, $A > 0$ and $a \in \mathbb{R}$. Then consider

$$\hat{M} := \begin{bmatrix} A & 0 \\ 0 & B \\ 0 & a \\ a & 0 \end{bmatrix}. \quad (3.5.15)$$

We have that $\hat{M} \in \mathcal{M}_{2,2}$ and has conformal superspherical symmetry and instanton number 2. Moreover, when $B \neq 1$, the ADHM data is not in the generalization of Manton–Sutcliffe’s set, given by Definition 2.1.1.

Proof. First we show that $\hat{M}$ satisfies the equations for simple spherical symmetry. Indeed, consider $(\mathbb{R}^2, \rho) := (\mathbb{R}, 0)^{\oplus 2}$. As $a \in \mathbb{R}$ for all $B \in (0, \frac{2}{3}\sqrt{6})$, we have that M is a real matrix. As $\rho(v) = 0$ for all $v \in \mathfrak{sp}(1)$, we have that (3.4.1) and (3.4.2) are satisfied for all $v \in \mathfrak{sp}(1)$. Hence, if $\hat{M} \in \mathcal{M}_{2,2}$, then it has simple spherical symmetry.

Next we show that $\hat{M}$ satisfies the equations for symmetry under Manton–Sutcliffe’s conformal circle action. Let

$$\begin{aligned} b &:= -\frac{\sqrt{12 - 15B^2 + 12\sqrt{B^4 - B^2 + 1}}}{\sqrt{-3B^2 + 6 + 6\sqrt{B^4 - B^2 + 1}}}B, \quad \text{and} \\ c &:= -\frac{\sqrt{-3B^2 + 6 + 6\sqrt{B^4 - B^2 + 1}}}{3}. \end{aligned} \quad (3.5.16)$$

Then consider $\tilde{\rho} \in \mathfrak{sp}(4)$ given by

$$\tilde{\rho} := \begin{bmatrix} 0 & b & A & 0 \\ -b & 0 & 0 & B \\ -A & 0 & 0 & c \\ 0 & -B & -c & 0 \end{bmatrix}. \quad (3.5.17)$$

Then (3.5.12) and (3.5.13) are satisfied, along with the condition that $U^T \tilde{\rho} U - U^T \hat{M}$ is real. Hence, if $\hat{M} \in \mathcal{M}_{2,2}$, then it has conformal superspherical symmetry. In particular, it is symmetric under Manton–Sutcliffe’s conformal circle action.

Note that if $B \neq 1$, then $a \neq 0$. Thus, the ADHM data is not in the generalization of Manton–Sutcliffe’s set. It remains to show that $\hat{M} \in \mathcal{M}_{2,2}$. In particular, we note that M is symmetric and $LL^\dagger$ is always positive-definite. Additionally, $R = L^\dagger L + M^\dagger M$ is

real and non-singular. It remains to check the final condition in Definition 3.1.16. Due to the aforementioned symmetry, we need only check that this condition is satisfied for all $x = x_0 + x_1 i$ with $x_1 \geq 0$. The eigenvalues of $\Delta(x)^\dagger \Delta(x)$ for such x are

$$-\frac{2}{3}B^2 + |x|^2 + \frac{4}{3}\sqrt{B^4 - B^2 + 1} + \frac{1}{3} \\ \pm \frac{2}{3}\sqrt{(-4B^2 + 6x_0^2 + 2)\sqrt{B^4 - B^2 + 1} - (3B^2 + 3)x_0^2 + 5B^4 - 5B^2 + 2}.$$

Letting $I := -2B^2 + 3x_0^2 + 4\sqrt{B^4 - B^2 + 1} + 1 > 0$, we have that the eigenvalues can be written as

$$\frac{I + x_1^2 \pm \sqrt{I^2 - (9x_0^4 + 6x_0^2 + 9)}}{3} > 0.$$

Therefore, we see that $\Delta(x)^\dagger \Delta(x)$ is non-singular everywhere, so $\hat{M} \in \mathcal{M}_{2,2}$. $\square$

Note 3.5.20. *In the previous example, we have that $t \mapsto \lambda(0, t)$ does not correspond to a representation of S^1 when $B \neq 1$, as $\lambda(0, t)$ is not skew-symmetric. This seems to contradict the result in Proposition 3.2.5. However, this apparent contradiction is only because we are using a non-standard circle action, conjugate to the usual R_1 action. In particular, we are using a conformal circle action.*

The usual R_1 circle action and Manton–Sutcliffe’s are related, for all $\theta \in \mathbb{R}$, via

$$\frac{1}{2} \begin{bmatrix} -k & -j \\ i & 1 \end{bmatrix} \exp \left(\theta \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right) \begin{bmatrix} k & -i \\ j & 1 \end{bmatrix} = \exp \left(\theta \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \right).$$

So, taking the ADHM data $(\hat{M}, U) \in \mathcal{N}_{2,2}$ equivariant under Manton–Sutcliffe’s conformal circle action, consider the ADHM data $(\hat{M}', U') \in \mathcal{N}_{2,2}$ defined by

$$(\hat{M}', U') := \frac{1}{\sqrt{2}} \begin{bmatrix} k & -i \\ j & 1 \end{bmatrix} . (\hat{M}, U) = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} A & 0 \\ 0 & B \\ i & a \\ a & i \end{bmatrix}, \begin{bmatrix} -Aj & 0 \\ 0 & -Bj \\ k & -aj \\ -aj & k \end{bmatrix} \right).$$

Note that the symmetry under Manton–Sutcliffe’s conformal circle action in $\hat{M}$ corresponds to symmetry under the usual R_1 circle action in $(\hat{M}', U')$.

This data is not in the standard form. Let

$$Q := \begin{bmatrix} \sqrt{\frac{a^2+1}{A^2+a^2+1}} & 0 & -\frac{Ai}{\sqrt{(A^2+a^2+1)(a^2+1)}} & -\frac{Aa}{\sqrt{(A^2+a^2+1)(a^2+1)}} \\ 0 & \sqrt{\frac{a^2+1}{A^2+a^2+1}} & -\frac{Ba}{\sqrt{(B^2+a^2+1)(a^2+1)}} & -\frac{Bi}{\sqrt{(B^2+a^2+1)(a^2+1)}} \\ \frac{Aj}{\sqrt{A^2+a^2+1}} & 0 & -\frac{k}{\sqrt{A^2+a^2+1}} & \frac{aj}{\sqrt{A^2+a^2+1}} \\ 0 & \frac{Bj}{\sqrt{B^2+a^2+1}} & \frac{aj}{\sqrt{B^2+a^2+1}} & -\frac{k}{\sqrt{B^2+a^2+1}} \end{bmatrix} \in \text{Sp}(4) \quad \text{and}$$

$$K := \text{diag} \left(\sqrt{\frac{A^2+a^2+1}{2}}, \sqrt{\frac{B^2+a^2+1}{2}} \right) \in \text{GL}(2, \mathbb{R}).$$

Gauging by (Q, K) , we get something in standard form: $(\hat{M}'', U) := (Q, K).(\hat{M}', U') \in \mathcal{N}_{2,2}$.

Then, given $\rho = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \in \mathfrak{so}(2)$, we have that ρ generates the circular 1-symmetry if and only if $b = -1$. Thus, the symmetry is generated by a representation corresponding to one of S^1 .

3.5.2.2 Spherically symmetric hyperbolic monopoles

In this section, we discuss the connection between superspherical symmetric instantons and symmetric hyperbolic monopoles.

Note that both the conformal and isoclinic superspherical subgroups of $\text{Sp}(2)$ contain subgroups conjugate to R_1 . Hence, we can associate to an instanton with conformal or isoclinic superspherical symmetry a hyperbolic monopole with integral mass.

Just like instantons with toral symmetry, the conformal and isoclinic superspherical subgroups impart additional symmetry to the hyperbolic monopole. In the case of the isoclinic superspherical subgroup, there is a $S^1 \times S^1$ subgroup of R_1 circle actions. Thus, such monopoles are axially symmetric. As the rest of the isoclinic subgroup does not commute with either of the R_1 circle actions, it does not impart any additional symmetry to the hyperbolic monopole.

The case of conformal superspherical symmetry imparts additional symmetry, as one of the R_1 circle actions commutes with more than just an S^1 subgroup. However, note that if we choose a different R_1 circle action, then we obtain an hyperbolic monopole with axial symmetry, not spherical symmetry.

Proposition 3.5.21. *A hyperbolic $\mathrm{Sp}(n)$ -monopole with integral mass is spherically symmetric if and only if its ADHM data has conformal superspherical symmetry.*

Proof. Through isometries, we may assume that the hyperbolic model is the ball model. Because of the conformal equivalence of $S^4 \setminus S^2 \equiv H^3 \times S^1$, these isometries on hyperbolic space correspond to conformal maps on S^4 . Because of the conformal equivalence of $S^4 \setminus S^2 \equiv H^3 \times S^1$, these isometries on hyperbolic space correspond to conformal maps on S^4 .

The action of $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ induces the ball of hyperbolic space, with coordinates (x, y, z) satisfying $r := \sqrt{x^2 + y^2 + z^2} < 1$ [MS14, § 4], where $(x, y, z) \in H^3$ corresponds to a point $xi + yj + zk \in \mathfrak{sp}(1) \subseteq \mathbb{H}$. The two-sphere removed is the $S^2 \subseteq \mathfrak{sp}(1)$ comprised of all points $xi + yj + zk$ with $r = 1$.

An element of the conformal superspherical subgroup is of the form $p \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix}$.

Let $R_p \in \mathrm{SO}(3)$ be the rotation generated by $\mathrm{diag}(p, p)$ acting on $\mathfrak{sp}(1)$. The transformation sends (x, y, z, θ) to $(x', y', z', \theta - 2\phi)$, where $(x', y', z') := R_p(x, y, z)$ [MS14, § 4]. Thus, we see that p part of the transformation just rotates the H^3 part and the ϕ part rotates the S^1 part. Note that $p \mapsto R_p$ is the double cover $\mathrm{Sp}(1) \rightarrow \mathrm{SO}(3)$.

Therefore, just as in Proposition 3.3.10, a hyperbolic monopole with integral mass has spherical symmetry if and only if its corresponding ADHM data has conformal superspherical symmetry. $\square$

Note 3.5.22. *A hyperbolic monopole with integral mass corresponds with a circle-invariant instanton. Let $\mathfrak{g} \subseteq \mathfrak{sp}(2)$ be the Lie subalgebra generating the circle action. From the above proposition, we know that a hyperbolic monopole with integral mass is spherically symmetric if and only if $\mathfrak{g}$ commutes with a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{sp}(2)$ isomorphic to $\mathfrak{sp}(1)$.*

Note 3.5.23. *The conformal superspherical symmetry subgroup that corresponds to spherically symmetric hyperbolic monopoles using the Braam–Austin circle action has Lie algebra*

$$\left\langle \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \frac{1}{2} \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 0 & j \\ j & 0 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 0 & k \\ k & 0 \end{bmatrix} \right\rangle \simeq \mathbb{R} \oplus \mathfrak{sp}(1). \quad (3.5.18)$$

Toral symmetry is generated by two circle actions. We can take both circle actions to be conjugate to R_1 . Indeed, we have the commuting circle actions of $\mathrm{diag}(e^{i\theta}, e^{-i\theta})$ and $\mathrm{diag}(e^{i\theta}, e^{i\theta})$. Using either one of these circle actions, we can transform a toral symmetric instanton into an axially symmetric hyperbolic monopole. However, these monopoles need not be gauge equivalent.

Proposition 3.5.24. *Consider the basic instanton, with ADHM data $\hat{M} = \begin{bmatrix} 1 & 0 \end{bmatrix}^T \in \mathcal{M}_{1,1}$. In Proposition 3.1.13, we see that the group of symmetries of this instanton is $\mathrm{Sp}(2)$. The hyperbolic monopoles obtained using two commuting circle actions conjugate to R_1 are not gauge equivalent, they have different symmetries.*

Proof. Using Manton–Sutcliffe’s conformal circle action $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$, which is conjugate to R_1 , we obtain a spherically symmetric hyperbolic monopole, given by Proposition 2.1.30.

The circle action $\mathrm{diag}(e^{i\theta}, e^{i\theta})$ commutes with Manton–Sutcliffe’s and is exactly the R_1 circle action. If the hyperbolic monopole obtained using this action were gauge equivalent to the one from above, there would have to be a Lie subgroup of $\mathrm{Sp}(2)$ (the group of symmetries of the instanton), commuting with R_1 , with Lie algebra isomorphic to $\mathfrak{sp}(1)$. However, no such subgroup exists. Indeed, the Lie algebra isomorphic to $\mathfrak{sp}(1)$ would have to be a Lie subalgebra of $\mathfrak{sp}(2)$ commuting with $\mathrm{diag}(i, i)$, which generates the R_1 circle action and no such subalgebra exists. Therefore, the two hyperbolic monopoles have different symmetries and are therefore not gauge equivalent. $\square$

Note 3.5.25. *This gauge inequivalence is not a problem, as the choice of circle action makes no difference. Either choice obtains all axially symmetric hyperbolic monopoles with integral mass.*

3.5.2.3 Axially symmetric hyperbolic Higgs bundles

In this section, we discuss the connection between instantons with conformal superspherical symmetry and axially symmetric hyperbolic Higgs bundles.

Proposition 3.5.26. *A hyperbolic $\mathrm{Sp}(n)$ Higgs bundle is axially symmetric if and only if its ADHM data has conformal superspherical symmetry.*

Proof. Through isometries, we may assume that the hyperbolic model is the half-space model. Because of the conformal equivalence of $S^4 \setminus S^1 \equiv H^2 \times S^2$, these isometries correspond to conformal maps on S^4 .

The simple spherical action induces the half-space model of hyperbolic space, with coordinates (x_0, r) with $r > 0$. The circle removed is $\mathbb{R} \cup \{\infty\}$. Letting θ, ϕ be the coordinates of S^2 , we have that a point $(x_0, r, \theta, \phi) \in H^2 \times S^2$ corresponds to a point $x_0 + r \sin \theta \cos \phi i + r \sin \theta \sin \phi j + r \cos \theta k \in \mathbb{H}$.

We now look at how the conformal superspherical action acts on these coordinates. Given $x \in \mathbb{H}$, write it in the above coordinates. Let $[(p, e^{i\theta})] \in (\mathrm{Sp}(1) \times S^1)/\{\pm(1, 1)\}$. If $x \notin \mathbb{R}$, then this conformal transformation takes x to another element in $\mathbb{H}$. Specifically, $x \mapsto p(x \cos \theta + \sin \theta)(-x \sin \theta + \cos \theta)^{-1}p^\dagger$. Simplifying, we find that this transformation acts on H^2 taking

$$\begin{aligned} x_0 &\mapsto \frac{x_0 \cos(2\theta) + \frac{1-x_0^2-r^2}{2} \sin(2\theta)}{(\cos \theta - x_0 \sin \theta)^2 + r^2 \sin^2 \theta} \quad \text{and} \\ r &\mapsto \frac{r}{(\cos \theta - x_0 \sin \theta)^2 + r^2 \sin^2 \theta}. \end{aligned}$$

This corresponds to rotation of (x_0, r) by 2θ about the point $(0, 1)$. That is, it is an isometry.

Therefore, just as in Proposition 3.3.10, a hyperbolic Higgs bundle is axially symmetric if and only if its ADHM data has conformal superspherical symmetry. $\square$

3.6 Rotational symmetry

In this section, we find an equation describing all instantons with rotational symmetry, as given in Table C.1 as well as below.

First, we reintroduce the notion of rotational symmetry. An instanton is said to have rotational symmetry if it is equivariant under $\mathrm{diag}(p, q)$ for all $p, q \in \mathrm{Sp}(1)$.

Theorem 3.6.1. *Let $\hat{M} \in \mathcal{M}_{n,k}$. Then $\hat{M}$ has rotational symmetry if and only if there exists a real representation $\rho: \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \rightarrow \mathfrak{so}(k)$ such that for all $v, \omega \in \mathfrak{sp}(1)$,*

$$vM + [\rho(v, \omega), M] - M\omega = 0, \tag{3.6.1}$$

$$[\rho(v, \omega), R] = 0. \tag{3.6.2}$$

Definition 3.6.2. *We call ρ the **generating representation** of the rotational symmetry of $\hat{M}$.*

Note 3.6.3. *All connected Lie subgroups of $\mathrm{Sp}(2)$ with Lie algebra $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ are conjugate to $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$. That is, there is some $A \in \mathrm{Sp}(2)$ such that the Lie group is of the form $A(\mathrm{Sp}(1) \times \mathrm{Sp}(1))A^\dagger$. Instantons equivariant under this group are of the form $A^\dagger \cdot (\hat{M}, U)$, where $\hat{M}$ is equivariant under $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$.*

Proof. We follow the proof of Theorem 1.1.1, making modifications when relevant. Suppose that $\hat{M}$ has rotational symmetry. Let $S \subseteq \mathrm{Sp}(1) \times \mathrm{Sp}(1) \times \mathrm{O}(k)$ be the stabilizer group of $(\hat{M}, U)$ restricted to rotations in $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$. That is

$$S := \{(\mathrm{diag}(p, q), K) \mid (\mathrm{diag}(LqKL^\dagger(LL^\dagger)^{-1}, pK), K) \cdot \mathrm{diag}(p^\dagger, q^\dagger) \cdot (\hat{M}, U) = (\hat{M}, U)\}. \quad (3.6.3)$$

Indeed, if $(Q, K) \in \mathrm{Sp}(n+k) \times \mathrm{GL}(k, \mathbb{R})$ such that $\mathrm{diag}(p^\dagger, q^\dagger) \cdot (Q, K) \cdot (\hat{M}, U) = (\hat{M}, U)$, then Corollary 3.1.27 tells us that $K \in \mathrm{O}(k)$, $[R, K] = 0$, and $Q = \mathrm{diag}(LqKL^\dagger(LL^\dagger)^{-1}, pK)$. Therefore, the pairs in S encapsulate all the rotational symmetry of the instanton. That S is a group follows from this and that the conformal action is a right Lie group action, just as in the proof of Theorem 3.4.3.

Just as in Theorem 1.1.1, S is a compact Lie group and we obtain a Lie group homomorphism $\Psi: \mathrm{Sp}(1) \times \mathrm{Sp}(1) \rightarrow S$ such that $\pi_1|_S \circ \Psi = \mathrm{id}_{\mathrm{Sp}(1) \times \mathrm{Sp}(1)}$, where π_1 treats $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ as one component.

Additionally, there is a Lie algebra homomorphism $\rho: \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \rightarrow \mathfrak{so}(k)$ such that for all $v, \omega \in \mathfrak{sp}(1)$, $\Psi(e^v, e^\omega) = (\mathrm{diag}(e^v, e^\omega), e^{\rho(v, \omega)})$. As ρ is a Lie algebra homomorphism whose image consists of real matrices, it gives us a real Lie algebra representation.

As $\Psi(e^{\theta v}, e^{\theta \omega}) \in S$ for all $\theta \in \mathbb{R}$ and $v, \omega \in \mathfrak{sp}(1)$, we have that

$$(\mathrm{diag}(Le^{\theta \omega} e^{\theta \rho(v, \omega)} L^\dagger (LL^\dagger)^{-1}, e^{\theta v} e^{\theta \rho(v, \omega)}), e^{\theta \rho(v, \omega)}) \cdot \mathrm{diag}(e^{-\theta v}, e^{-\theta \omega}) \cdot (\hat{M}, U) = (\hat{M}, U).$$

Differentiating and evaluating at $\theta = 0$, we obtain

$$\begin{bmatrix} L\omega L^\dagger (LL^\dagger)^{-1} L + L\rho(v, \omega) L^\dagger (LL^\dagger)^{-1} L - L\rho(v, \omega) - L\omega \\ vM + [\rho(v, \omega), M] - M\omega \end{bmatrix} = 0. \quad (3.6.4)$$

In Section 3.6.3, we study the top row. For now, focusing on the bottom row, we see that $vM + [\rho(v, \omega), M] - M\omega = 0$. Furthermore, as mentioned above, Corollary 3.1.27 tells us that $[R, e^{\theta \rho(v, \omega)}] = 0$, for all $\theta \in \mathbb{R}$. Differentiating and evaluating at $\theta = 0$, we have $[R, \rho(v, \omega)] = 0$.

Conversely, suppose that the equations hold for some real representation $\rho: \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \rightarrow \mathfrak{so}(k)$. Consider $p, q \in \mathrm{Sp}(1)$. As the exponential map $\exp: \mathfrak{sp}(1) \rightarrow \mathrm{Sp}(1)$ is surjective, there are some $v, \omega \in \mathfrak{sp}(1)$ such that $p = e^v$ and $q = e^\omega$. Let $p(\theta) := e^{\theta v} \in \mathrm{Sp}(1)$, $q(\theta) := e^{\theta \omega} \in \mathrm{Sp}(1)$, and $K(\theta) := e^{\theta \rho(v, \omega)} \in \mathrm{O}(k)$. We show that $A(\theta) := p(\theta)K(\theta)MK(\theta)^T q(\theta)^\dagger$ is constant. Indeed,

$$A'(\theta) = p(\theta)K(\theta)(vM + [\rho(v, \omega), M] - M\omega)K(\theta)^T q(\theta)^\dagger = 0.$$

Hence, A is constant, so

$$pK(1)MK(1)^Tq^\dagger = A(1) = A(0) = M.$$

As $[\rho(v, \omega), R] = 0$, $[K(\theta), R] = 0$, so by Lemma 3.1.25, we have that $\hat{M}$ is $\text{diag}(p^\dagger, q^\dagger)$ -equivariant. As p and q were arbitrary, we have rotational symmetry. $\square$

Note 3.6.4. *All instantons with rotational symmetry correspond to axially symmetric hyperbolic monopoles because there is a $S^1 \times S^1$ subgroup of commuting circle actions in $\text{Sp}(1) \times \text{Sp}(1)$ conjugate to R_1 . For instance, the toral subgroup is contained in this group. The rest of $\text{Sp}(1) \times \text{Sp}(1)$ does not commute with either of these circle actions. Thus, the rest of the group does not impart any additional symmetry.*

While not all axially symmetric hyperbolic monopoles are described by instantons with rotational symmetry, the extra restrictions make it easier to find such monopoles.

Additionally, all instantons with rotational symmetry correspond to spherically symmetric singular monopoles with Dirac type singularities because there is an isoclinic superspherical subgroup of $\text{Sp}(1) \times \text{Sp}(1)$. The rest of $\text{Sp}(1) \times \text{Sp}(1)$ does not impart any additional symmetry.

While not all spherically symmetric singular monopoles are described by instantons with rotational symmetry, the extra restrictions make it easier to find such monopoles.

Unlike with the instantons with isoclinic superspherical symmetry, any choice of subgroup conjugate to R_0 leads to a spherically symmetric singular monopole.

Finally, any choice of an isoclinic spherical symmetry group leads to an axially symmetric hyperbolic Higgs bundle.

Note 3.6.5. *Just as with circular t -symmetry, we need not check the final condition of $\mathcal{M}_{n,k}$ in Definition 3.1.16 everywhere. Indeed, if $\hat{M}$ satisfies (3.6.1) and (3.6.2) for some real representation $(\mathbb{R}^k, \rho)$, then it satisfies (3.4.11) and (3.4.12) for some real representation of $\mathfrak{sp}(1)$. Thus, just as in the isoclinic spherical symmetry case, we need only check that the final condition of $\mathcal{M}_{n,k}$ in Definition 3.1.16 is satisfied for all $x = x_0 \in \mathbb{H}$ with $x_0 \geq 0$.*

3.6.1 Structure of rotational symmetry

In light of Theorem 3.6.1, we know that we can find instantons with rotational symmetry. That is, starting with a real representation of $\mathfrak{spin}(4)$, we can narrow down the possible M , so we are only left with finding L such that we have $\hat{M} \in \mathcal{M}_{n,k}$, which necessarily has rotational symmetry. In this section, we investigate what representations generate such instantons and what the corresponding ADHM data looks like.

Note 3.6.6. In Section D.3.1 of Appendix D, we investigate the representation theory of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. Therein, we note that for every $m, n \in \mathbb{N}$, there is a unique, up to isomorphism, irreducible complex mn -representation $(V_{m,n}, \rho_{m,n})$ with highest weight $\left[\frac{m-1}{2} \quad \frac{n-1}{2} \right]$.

Additionally, for every $m, n \in \mathbb{N}$, if $n \equiv m \pmod{2}$, then there is a unique, up to isomorphism, irreducible real mn -representation $(\mathbb{R}^{mn}, \varrho_{m,n})$. However, if $n \not\equiv m \pmod{2}$, then there is a unique, up to isomorphism, irreducible real $2mn$ -representation $(\mathbb{R}^{2mn}, \varrho_{m,n})$. In the former case, the complexification of this representation is isomorphic to $(V_{m,n}, \rho_{m,n})$ and in the latter, the complexification is isomorphic to $(V_{m,n}, \rho_{m,n})^{\oplus 2}$.

Definition 3.6.7. Recall the standard basis of $\mathfrak{sp}(1)$ given in Definition 3.4.2. Let $v_1, \dots, v_6$ be the **standard basis** for $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. That is, v_1, v_2, v_3 give the standard basis for $\mathfrak{sp}(1) \oplus 0$ and v_4, v_5, v_6 give the standard basis for $0 \oplus \mathfrak{sp}(1)$.

Definition 3.6.8. Let $V := \mathbb{R}^k$. Consider a k -representation (V, ρ) of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. Consider the real representation $(\mathbb{H}, \nu)$ of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ given by $\nu(v, \omega)(x) = vx - x\omega$. Define the induced real representation

$$(\hat{V}, \hat{\rho}) := (V, \rho) \otimes_{\mathbb{R}} (V^*, \rho^*) \otimes_{\mathbb{R}} (\mathbb{H}, \nu). \quad (3.6.5)$$

Note that $(\mathbb{H}, \nu) \simeq (\mathbb{R}^4, \varrho_{2,2})$. Explicitly, the action of $\hat{\rho}(v, \omega)$ on $A \in \hat{V} = \text{Mat}(k, k, \mathbb{H})$ is

$$\hat{\rho}(v, \omega)A = vA - A\omega + [\rho(v, \omega), A]. \quad (3.6.6)$$

Note 3.6.9. As $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \simeq \mathfrak{spin}(4)$ has Dynkin diagram D_2 , by Proposition D.2.12 in Appendix D, all representations of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ are self-dual.

The definition of $(\hat{V}, \hat{\rho})$ is well-motivated. Firstly, note that $M \in \hat{V}$. Secondly, given the connection between the action of $\hat{\rho}(v, \omega)$ and (3.6.1), we immediately obtain the following corollary.

Corollary 3.6.10. Let $\hat{M} \in \mathcal{M}$. Then $\hat{M}$ has rotational symmetry if and only if there is some real k -representation (V, ρ) of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ such that $[R, \rho(v, \omega)] = 0$ and $\hat{\rho}(v, \omega)(M) = 0$ for all $v, \omega \in \mathfrak{sp}(1)$.

Note 3.6.11. Suppose $\hat{M} \in \mathcal{M}$ has rotational symmetry. Corollary 3.6.10, tells us that there is some real representation (V, ρ) such that $\text{span}(M) \subseteq \hat{V}$ is an invariant subspace, which is acted on trivially.

As $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ is semi-simple, the representation $(\hat{V}, \hat{\rho})$ decomposes into irreducible representations. We have previously explored the $M = 0$ case, so suppose $M \neq 0$. As

$\text{span}(M)$ is a one-dimensional invariant subspace, acted on trivially, $(\text{span}(M), 0)$ is a summand of the representation. So the decomposition of $(\hat{V}, \hat{\rho})$ must contain trivial summands. Moreover, M is in the direct sum of these trivial summands. In fact, if $M = 0$, then it is in the direct sum of trivial summands as well.

In particular, we note the following representations. The trivial representation $(\mathbb{C}, 0)$ is the representation $(V_{1,1}, \rho_{1,1})$. The real representation $(\mathbb{H}, \nu)$, where $\nu(v, \omega)(x) := vx - x\omega$, is the representation $(\mathbb{R}^4, \varrho_{2,2})$, whose complexification is $(V_{2,2}, \rho_{2,2})$. Consider the representations $(\mathbb{H}, \iota_1)$ and $(\mathbb{H}, \iota_2)$, where $\iota_1(v, \omega)(x) := vx$ and $\iota_2(v, \omega)(x) := \omega x$. Restricting the scalars to $\mathbb{C}$, these are isomorphic to $(V_{2,1}, \rho_{2,1})$ and $(V_{1,2}, \rho_{1,2})$, respectively.

When searching for instantons with rotational symmetry, M is found in the trivial summands of $(\hat{V}, \hat{\rho})$. The following lemmas tell us where these trivial summands are, given a decomposition of $(\hat{V}, \hat{\rho})$. First, we see where the trivial summands are when decomposed as a complex representation, providing us with the information needed for the real decomposition.

Lemma 3.6.12. *Given $a \geq b \geq 1$ and $m, n \geq 1$, consider $(V_{a,m}, \rho_{a,m}) \otimes_{\mathbb{C}} (V_{b,n}, \rho_{b,n}) \otimes_{\mathbb{C}} (V_{2,2}, \rho_{2,2})$. The product has a single trivial summand when $a = b + 1$ and $|n - m| = 1$. Otherwise, there are no trivial summands.*

Proof. First we look at the product $(V_{c,d}, \rho_{c,d}) \otimes_{\mathbb{C}} (V_{2,2}, \rho_{2,2})$. We see that this product contains a single trivial component when $c = d = 2$ and otherwise contains none. Thus, the total product contains a single trivial component for every $(V_{2,2}, \rho_{2,2})$ summand in the decomposition of $(V_{a,m}, \rho_{a,m}) \otimes_{\mathbb{C}} (V_{b,n}, \rho_{b,n})$.

Examining the decomposition of $(V_{a,m}, \rho_{a,m}) \otimes_{\mathbb{C}} (V_{b,n}, \rho_{b,n})$, we see that this contains a single $(V_{2,2}, \rho_{2,2})$ summand when $a = b + 1$ and $|n - m| = 1$. Otherwise, it contains none, proving the lemma. $\square$

We first investigate the trivial summand present in $(V_{m+1,n\pm 1}, \rho_{m+1,n\pm 1}) \otimes_{\mathbb{C}} (V_{m,n}^*, \rho_{m,n}^*) \otimes_{\mathbb{C}} (V_{2,2}, \rho_{2,2})$.

Definition 3.6.13. *Let $B^{m,n,\pm}$ be the unit length generator of the unique one-dimensional representation in $(V_{m+1,n\pm 1}, \rho_{m+1,n\pm 1}) \otimes_{\mathbb{C}} (V_{m,n}^*, \rho_{m,n}^*) \otimes_{\mathbb{C}} (V_{2,2}, \rho_{2,2})$. This generator is well-defined, up to a $\mathbb{C}^*$ factor, though a choice of scale leaves a S^1 factor. This generator can be thought of as a $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ -invariant quadruple $(B_{\mu}^{m,n,\pm})_{\mu=0}^3$ of complex $(m+1)(n\pm 1) \times mn$ matrices.*

These generators can be identified in much the same way as we compute the B^n matrices from Definition 2.1.33 in Appendix E.

Above, we defined what spans the trivial summands of $(\hat{V}, \hat{\rho})$ when decomposed as a complex representation. We use this to determine what spans the real decomposition.

Let $N(c, d) := \frac{3-(-1)^{c+d}}{2}cd$. That is, $N(c, d) = cd$ if c and d have the same parity and $N(c, d) = 2cd$ otherwise.

Lemma 3.6.14. *Given $a \geq b \geq 1$ and $m, n \geq 1$, we have that*

$$(\mathbb{R}^{N(a,m)}, \varrho_{a,m}) \otimes_{\mathbb{R}} ((\mathbb{R}^{N(b,n)})^*, \varrho_{b,n}^*) \otimes_{\mathbb{R}} (\mathbb{R}^4, \varrho_{2,2})$$

possesses trivial summands when $a = b+1$ and $|n-m| = 1$. Otherwise, there are no trivial summands. More specifically, if a and m have the same parity, then there is a single trivial summand. If a and m have different parities, then there are four trivial summands.

Proof. Suppose that $a \neq b+1$ or $|n-m| \neq 1$. Regardless of the relative parities of a , b , m , and n , the complexification of the real representations does not contain any trivial summands. Henceforth, we assume that $a = b+1$ and $|n-m| = 1$.

Suppose that a and m , and hence b and n , have the same parity. Then the complexification of the real representation is isomorphic, as a complex representation, to $(V_{a,m}, \rho_{a,m}) \otimes_{\mathbb{C}} (V_{b,n}, \rho_{b,n}) \otimes_{\mathbb{C}} (V_{2,2}, \rho_{2,2})$. By Lemma 3.6.12, we know that this tensor product has a single trivial summand. Furthermore, as this tensor product only contains summands $(V_{c,d}, \rho_{c,d})$ with c and d having the same parity, the representation is real, meaning that the real decomposition is the same as the complex. Thus, we have proven the first part of the lemma.

Suppose that a and m , and hence b and n , have different parity. Then the complexification of the real representation is isomorphic, as a complex representation, to $(V_{a,m}, \rho_{a,m})^{\oplus 2} \otimes_{\mathbb{C}} (V_{b,n}, \rho_{b,n})^{\oplus 2} \otimes_{\mathbb{C}} (V_{2,2}, \rho_{2,2})$. By Lemma 3.6.12, we know that this tensor product has a single trivial summand. Furthermore, as this tensor product only contains summands $(V_{c,d}, \rho_{c,d})$ with c and d having the same parity, the representation is real, meaning that the real decomposition is the same as the complex. Thus, we have proven the second part of the lemma. $\square$

Using Lemma 3.6.14, we investigate the trivial summands of

$$(\mathbb{R}^{N(m+1,n\pm 1)}, \varrho_{m+1,n\pm 1}) \otimes_{\mathbb{R}} ((\mathbb{R}^{N(m,n)})^*, \varrho_{m,n}^*) \otimes_{\mathbb{R}} (\mathbb{R}^4, \varrho_{2,2}).$$

Lemma 3.6.15. *Let $m, n \in \mathbb{N}_+$. If m and n have the same parity, then there is a unique choice of S^1 factor, up to sign, such that the $B_i^{m,n,\pm}$ matrices are real. Then the quadruple spans the unique trivial summand of $(\mathbb{R}^{N(m+1,n\pm 1)}, \varrho_{m+1,n\pm 1}) \otimes_{\mathbb{R}} ((\mathbb{R}^{N(m,n)})^*, \varrho_{m,n}^*) \otimes_{\mathbb{R}} (\mathbb{R}^4, \varrho_{2,2})$.*

If m and n have different parity, then $N(m, n) = 2mn$ and $N(m+1, n\pm 1) = 2(m+1)(n\pm 1)$. Let $Y_i^+ := \varrho_{m+1,n\pm 1}(v_i) \in \mathfrak{so}(2(m+1)(n\pm 1))$ and $Y_i^- := \varrho_{m,n}(v_i) \in \mathfrak{so}(2mn)$. Let $y_i^+ := \rho_{m+1,n\pm 1}(v_i) \in \mathfrak{su}((m+1)(n\pm 1))$ and $y_i^- := \rho_{m,n}(v_i) \in \mathfrak{su}(mn)$. As the complexification of $(\mathbb{R}^{2(m+1)(n\pm 1)}, \varrho_{m+1,n\pm 1})$ is isomorphic to $(V_{m+1,n\pm 1}, \rho_{m+1,n\pm 1})^{\oplus 2}$ and similarly for $(\mathbb{R}^{2mn}, \varrho_{m,n})$, there exists $U_+ \in \mathrm{SU}((m+1)(n\pm 1))$ and $U_- \in \mathrm{SU}(mn)$ such that

$$Y_i^{\pm} = U_{\pm}^{\dagger} \begin{bmatrix} y_i^{\pm} & 0 \\ 0 & y_i^{\pm} \end{bmatrix} U_{\pm}.$$

We can find four linearly independent quadruples of real matrices spanning the four trivial summands of $(\mathbb{R}^{N(m+1,n\pm 1)}, \varrho_{m+1,n\pm 1}) \otimes_{\mathbb{R}} ((\mathbb{R}^{N(m,n)})^*, \varrho_{m,n}^*) \otimes_{\mathbb{R}} (\mathbb{R}^4, \varrho_{2,2})$. For some choices of $\alpha, \beta, \gamma, \delta \in \mathbb{C}$, these linearly independent quadruples of matrices are given by

$$\left(U_+^{\dagger} \begin{bmatrix} \alpha B_{\mu}^{m,n,\pm} & \beta B_{\mu}^{m,n,\pm} \\ \gamma B_{\mu}^{m,n,\pm} & \delta B_{\mu}^{m,n,\pm} \end{bmatrix} U_- \right)_{\mu=0}^3. \quad (3.6.7)$$

Proof. The proof follows from Lemma 2.1.36. □

Now that we know exactly what spans the trivial summands of $(\hat{V}, \hat{\rho})$, the following theorem tells us exactly what form M takes if $\hat{M}$ has rotational symmetry.

Theorem 3.6.16 (Rotational Structure Theorem). *Let $\hat{M} \in \mathcal{M}_{n,k}$ have rotational symmetry. Theorem 3.6.1 tells us that $\hat{M}$ is generated by a real representation (V, ρ) of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$, which we can decompose as*

$$(V, \rho) \simeq \bigoplus_{a=1}^k (\mathbb{R}^{N(m_a, n_a)}, \varrho_{m_a, n_a}). \quad (3.6.8)$$

Without loss of generality, we may assume that $m_a \geq m_{a+1}$, $\forall a \in \{1, \dots, I-1\}$.

Let $Y_{i,a} := \rho_{m_a, n_a}(v_i)$ and

$$Y_i := \mathrm{diag}(Y_{i,1}, \dots, Y_{i,k}).$$

Then Y_i induces (V, ρ) . When n_a and m_a have different parity, let $y_{i,a} := \rho_{m_a, n_a}(v_i) \in \mathfrak{su}(n_a m_a)$ induce the corresponding irreducible complex representation. Then there is some $U_{m_a, n_a} \in \mathrm{SU}(n_a m_a)$ such that

$$Y_{i,a} = U_{m_a, n_a}^\dagger \mathrm{diag}(y_{i,a}, y_{i,a}) U_{m_a, n_a}.$$

Using the decomposition of (V, ρ) given in (3.6.8), we have $(M_\mu)_{ab} \in \mathbb{R}^{N(m_a, n_a)} \otimes_{\mathbb{R}} (\mathbb{R}^{N(m_a, n_a)})^*$ such that

$$M_\mu = \begin{bmatrix} (M_\mu)_{11} & \cdots & (M_\mu)_{1k} \\ \vdots & \ddots & \vdots \\ (M_\mu)_{k1} & \cdots & (M_\mu)_{kk} \end{bmatrix}.$$

Then, up to a ρ -invariant gauge and for all $a, b \in \{1, \dots, I\}$ we have that

- (1) if $a < b$, $m_a = m_b + 1$, $n_a = n_b \pm 1$, and m_a and n_a have the same parity, then $\exists \lambda_{ab} \in \mathbb{R}$ such that

$$\begin{aligned} (M_\mu)_{ab} &= \lambda_{ab} B_\mu^{m_b, n_b, \pm}, \quad \text{and} \\ (M_\mu)_{ba} &= \lambda_{ab} (B_\mu^{m_b, n_b, \pm})^T, \end{aligned}$$

where the S^1 factor has been chosen so that the $B_\mu^{m_b, n_b, \pm}$ matrices are real;

- (2) if $a < b$, $m_a = m_b + 1$, $n_a = n_b \pm 1$, and m_a and n_a have different parity, then $\exists \kappa_{a,b,0}, \kappa_{a,b,1}, \kappa_{a,b,2}, \kappa_{a,b,3} \in \mathbb{C}$ such that

$$\begin{aligned} (M_\mu)_{ab} &= U_{m_a, n_a}^\dagger \begin{bmatrix} \kappa_{a,b,0} B_\mu^{m_b, n_b, \pm} & \kappa_{a,b,1} B_\mu^{m_b, n_b, \pm} \\ \kappa_{a,b,2} B_\mu^{m_b, n_b, \pm} & \kappa_{a,b,3} B_\mu^{m_b, n_b, \pm} \end{bmatrix} U_{m_b, n_b}, \quad \text{and} \\ (M_\mu)_{ba} &= U_{m_b, n_b}^\dagger \begin{bmatrix} \overline{\kappa_{a,b,0}} (B_\mu^{m_b, n_b, \pm})^\dagger & \overline{\kappa_{a,b,2}} (B_\mu^{m_b, n_b, \pm})^\dagger \\ \overline{\kappa_{a,b,1}} (B_\mu^{m_b, n_b, \pm})^\dagger & \overline{\kappa_{a,b,3}} (B_\mu^{m_b, n_b, \pm})^\dagger \end{bmatrix} U_{m_b, n_b}, \end{aligned}$$

and these blocks are real;

- (3) otherwise, $(M_\mu)_{ab} = 0$.

Conversely, if M has the above form for some real representation (V, ρ) , and for all $v_1, v_2 \in \mathfrak{sp}(1)$, $[\rho(v_1, v_2), R] = 0$, then $\hat{M}$ has rotational symmetry.

Proof. This proof follows the same structure as the proof of the Structure Theorem for hyperbolic monopoles, Theorem 2.1.39, noting that M is symmetric. $\square$

3.6.2 Novel example of rotational symmetry

In this section, we identify novel examples of rotational symmetry, using the Rotational Structure Theorem.

Proposition 3.6.17. *Let $\lambda > 0$. Then define $\hat{M}$ by*

$$M := \lambda \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & j \\ 0 & 0 & 0 & 0 & k \\ 1 & i & j & k & 0 \end{bmatrix} \quad \text{and} \quad (3.6.9)$$

$$L := \lambda \begin{bmatrix} \sqrt{3} & -\frac{i}{\sqrt{3}} & -\frac{j}{\sqrt{3}} & -\frac{k}{\sqrt{3}} & 0 \\ 0 & 2\sqrt{\frac{2}{3}} & k\sqrt{\frac{2}{3}} & -j\sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & \sqrt{2} & i\sqrt{2} & 0 \end{bmatrix}.$$

We have $\hat{M} \in \mathcal{M}_{3,5}$ and it is a $\mathrm{Sp}(3)$ instanton with rotational symmetry and instanton number 5.

Proof. Consider the representation $(V, \rho) := (\mathbb{H}, \nu) \oplus (\mathbb{R}, 0)$. Seeking instantons with rotational symmetry generated by (V, ρ) , the Rotational Structure Theorem tells us that M has to have the form given in the statement. Thus, if $\hat{M} \in \mathcal{M}_{3,5}$, then it has rotational symmetry. To that end, we have that $LL^\dagger = 4\lambda^2 I_3$. Additionally, $R = 4\lambda^2 I_5$. Due to the symmetry of the instanton, we need only check the final condition for $x = x_0 \geq 0$. Doing so, we see that

$$\Delta(x)^\dagger \Delta(x) = \begin{bmatrix} 4\lambda^2 + x_0^2 & 0 & 0 & 0 & -2x_0\lambda \\ 0 & 4\lambda^2 + x_0^2 & 0 & 0 & 0 \\ 0 & 0 & 4\lambda^2 + x_0^2 & 0 & 0 \\ 0 & 0 & 0 & 4\lambda^2 + x_0^2 & 0 \\ -2x_0\lambda & 0 & 0 & 0 & 4\lambda^2 + x_0^2 \end{bmatrix}.$$

This has eigenvalues $4\lambda^2 + x_0^2$, $4\lambda^2 \pm 2x_0\lambda + x_0^2$ with multiplicity 3, 1, and 1, respectively. Therefore, $\Delta(x)^\dagger \Delta(x)$ is positive-definite everywhere, proving the proposition. $\square$

3.6.3 Constraint on rotational symmetry

In Theorem 3.6.1, we obtain a representation (V, ρ) of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ from an instanton with rotational symmetry by focusing on the bottom of (3.6.4). This representation induces another $(\hat{V}, \hat{\rho})$, which we use to determine what an instanton with rotational symmetry looks like. However, by focusing on the top of (3.6.4), Theorem 3.6.1 gives us a second representation.

Lemma 3.6.18. *Let $\pi_2: \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \rightarrow \mathfrak{sp}(1)$ projection on the second component. Suppose $(\hat{M}, U) \in \mathcal{M}_{n,k}$ has rotational symmetry generated by (V, ρ) , a real representation. Let $y_i := (LL^\dagger)^{-1}L(\pi_2(v_i)I_k + \rho(v_i))L^\dagger \in \mathfrak{sp}(n)$. Then y_i induce a representation of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$, which we denote by (W, λ) , where $W := \mathbb{H}^n$ and λ is the linear map taking $v_i \mapsto y_i$.*

Proof. Let $i, j \in \{1, \dots, 6\}$. By Theorem 3.6.1, we know that $[\rho(v_i), M] = M\pi_2(v_i) - \pi_1(v_i)M$, so,

$$\begin{aligned} M^\dagger M(\pi_2(v_i) + \rho(v_i)) &= M^\dagger[M, \rho(v_i)] + M^\dagger \rho(v_i)M + M^\dagger M\pi_2(v_i) \\ &= M^\dagger \pi_1(v_i)M + [M^\dagger, \rho(v_i)]M + \rho(v_i)M^\dagger M \\ &= (\pi_2(v_i)I_k + \rho(v_i))M^\dagger M. \end{aligned}$$

Thus, $[M^\dagger M, \pi_2(v_i)I_k + \rho(v_i)] = 0$. Also from Theorem 3.6.1, $[\rho(v_i), R] = 0$. As R is real, $[R, \pi_2(v_i)] = 0$. Hence, $[L^\dagger L, \pi_2(v_i)I_k + \rho(v_i)] = 0$. Thus

$$LL^\dagger L(\rho(v_i) + \pi_2(v_i))L^\dagger = L(\rho(v_i) + \pi_2(v_i))L^\dagger LL^\dagger.$$

Hence, $[LL^\dagger, L(\rho(v_i) + \pi_2(v_i))L^\dagger] = 0$, so $y_i \in \mathfrak{sp}(n)$. Therefore, we see

$$\begin{aligned} [y_i, y_j] &= (LL^\dagger)^{-1}[L(\rho(v_i) + \pi_2(v_i))L^\dagger, L(\rho(v_j) + \pi_2(v_j))L^\dagger](LL^\dagger)^{-1} \\ &= (LL^\dagger)^{-1}L((\rho(v_i) + \pi_2(v_i))L^\dagger L(\rho(v_j) + \pi_2(v_j)) \\ &\quad - (\rho(v_j) + \pi_2(v_j))L^\dagger L(\rho(v_i) + \pi_2(v_i)))L^\dagger (LL^\dagger)^{-1} \\ &= (LL^\dagger)^{-1}L[(\rho(v_i) + \pi_2(v_i)), (\rho(v_j) + \pi_2(v_j))]L^\dagger \\ &= (LL^\dagger)^{-1}L(\rho([v_i, v_j]) + \pi_2([v_i, v_j]))L^\dagger. \end{aligned}$$

Thus, the y_i satisfy the correct commutation relations. $\square$

We use (W, λ) to determine what $\mathrm{Sp}(n)$ structure groups are possible given ρ . Using $W = \mathbb{H}^n \simeq \mathbb{C}^{2n}$, we can restrict the scalars from $\mathbb{H}$ to $\mathbb{C}$. In doing so, the generators $y_i \in$

$\mathfrak{sp}(n)$ correspond to elements of $\mathfrak{su}(2n)$. Thus, the induced complex representation is a $2n$ -representation. As mentioned in Section D.3.1 of Appendix D, the complex representation obtained by restricting the scalars of an irreducible, quaternionic representation is either isomorphic to $(V_{a,b}, \rho_{a,b})$ for some a and b with opposite parity or $(V_{a,b}, \rho_{a,b})^{\oplus 2}$ for some a and b with the same parity.

Definition 3.6.19. Recall the representation $(\mathbb{H}, \iota_2)$, where $\iota_2(v, \omega)(x) = \omega x$. Restricting the scalars to $\mathbb{C}$, this representation is isomorphic to $(V_{1,2}, \rho_{1,2})$.

Given a real representation (V, ρ) of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$, with induced representation (W, λ) as defined above, we define the real representation

$$(\hat{W}, \hat{\lambda}) := ((W, \lambda) \otimes_{\mathbb{R}} (V^*, \rho^*)) \otimes_{\mathbb{H}} (\mathbb{H}^*, \iota_2^*) = (W, \lambda) \otimes_{\mathbb{H}} ((V^*, \rho^*) \otimes_{\mathbb{R}} (\mathbb{H}^*, \iota_2^*)). \quad (3.6.10)$$

Unraveling how $\hat{\lambda}$ acts on $\hat{W} = \text{Mat}(n, k, \mathbb{H})$, let $v, \omega \in \mathfrak{sp}(1)$ and $A \in \hat{W}$. Then

$$\hat{\lambda}(v, \omega)(A) = \lambda(v, \omega)A - A\rho(v, \omega) - A\omega. \quad (3.6.11)$$

The definition of $(\hat{W}, \hat{\lambda})$ is well-motivated. First, note that $L \in \hat{W}$. Moreover, the following lemma tells us exactly where L lives in $\hat{W}$.

Lemma 3.6.20. Let $\hat{M} \in \mathcal{M}_{n,k}$ have rotational symmetry. Let (V, ρ) , (W, λ) and $(\hat{W}, \hat{\lambda})$ be as above. Then $\hat{\lambda}(v, \omega)(L) = 0$ for all $(v, \omega) \in \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$.

Proof. As $\hat{M}$ has rotational symmetry, we have (3.6.4). Then we see that for $v, \omega \in \mathfrak{sp}(1)$,

$$\hat{\lambda}(v, \omega)(L) = \lambda(v, \omega)L - L\rho(v, \omega) - L\omega = 0. \quad \square$$

Corollary 3.6.21. Let $\hat{M} \in \mathcal{M}_{n,k}$ have rotational symmetry. Then $(\hat{W}, \hat{\lambda})$ must have trivial summands and L must live in the direct sum of these summands.

Proof. As $L \neq 0$, we know $\text{span}(L) \subseteq \hat{W}$ is an invariant 1-dimensional subspace, so $(\text{span}(L), 0)$ is a trivial summand of the representation $(\hat{W}, \hat{\lambda})$. $\square$

Therefore, we know that $(\hat{W}, \hat{\lambda})$ must have trivial summands, which narrows the possibilities for the structure group. Just like in the hyperbolic monopole case, we have straightforward restrictions for high dimensional representations in addition to more subtle restrictions for low dimensional representations. For instance, in Proposition 3.6.17, we examine instantons with rotational symmetry generated by $(V, \rho) \simeq (\mathbb{R}^4, \varrho_{2,2}) \oplus (\mathbb{R}, 0)$. This representation generated instantons with rotational symmetry and structure group $\text{Sp}(3)$. However, this representation cannot generate instantons with rotational symmetry with lower rank structure groups.

Proposition 3.6.22. *Consider Proposition 3.6.17, with $(V, \rho) \simeq (\mathbb{R}^4, \varrho_{2,2}) \oplus (\mathbb{R}, 0)$. This representation does not generate an instanton with rotational symmetry and structure group $\mathrm{Sp}(1)$ or $\mathrm{Sp}(2)$.*

Proof. Suppose $(V, \rho) \simeq (\mathbb{R}^4, \varrho_{2,2}) \oplus (\mathbb{R}, 0)$ generates an instanton with rotational symmetry. We know that $(\hat{W}, \hat{\lambda})$ has a trivial summand. Consider $(V^*, \rho^*) \otimes_{\mathbb{R}} (\mathbb{H}^*, \iota^*)$. Restricting the scalars, this representation is isomorphic, as a complex representation, to $(V_{1,2}, \rho_{1,2}) \oplus (V_{2,1}, \rho_{2,1}) \oplus (V_{2,3}, \rho_{2,3})$.

If we have a $\mathrm{Sp}(1)$ instanton, then we have that, after restricting the scalars of (W, λ) , the representation is isomorphic, as a complex representation, to $(V_{1,1}, \rho_{1,1})^{\oplus 2}$, $(V_{1,2}, \rho_{1,2})$, or $(V_{2,1}, \rho_{2,1})$. The former does not give $(\hat{W}, \hat{\lambda})$ any trivial summands. Thus we look to the latter two, which do.

If the representation obtained by restricting the scalars of (W, λ) is given by $(V_{1,2}, \rho_{1,2})$, then there is some $q \in \mathrm{Sp}(1)$ such that $y_i = q\pi_2(v_i)q^\dagger$. Taking $\tilde{L} := q^\dagger L$ and dropping the tilde, we have

$$[\pi_2(v_i), L] = L\rho(v_i).$$

The equations when $i \in \{1, 2, 3\}$ imply that $L = \begin{bmatrix} 0 & 0 & 0 & 0 & d \end{bmatrix}$ for some $d \in \mathbb{H}$. The equations for $i \in \{4, 5, 6\}$ imply that $d \in \mathbb{R}$. But we know that as

$$M = a \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & j \\ 0 & 0 & 0 & 0 & k \\ 1 & i & j & k & 0 \end{bmatrix},$$

we have

$$R = L^\dagger L + M^\dagger M = d^2 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} + a^2 \begin{bmatrix} 1 & i & j & k & 0 \\ -i & 1 & -k & j & 0 \\ -j & k & 1 & -i & 0 \\ -k & -j & i & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

As R is real, $a = 0$. But then R is singular. Contradiction!

If the representation obtained by restricting the scalars of (W, λ) is given by $(V_{2,1}, \rho_{2,1})$ then there is some $q \in \text{Sp}(1)$ such that $y_i = q\pi_1(v_i)q^\dagger$. Gauging L as above, then we have

$$\pi_1(v_i)L - LY_i - L\pi_2(v_i) = 0.$$

The equations when $i \in \{1, 2, 3\}$ imply that there is some $A \in \mathbb{H}$ such that

$$L = \begin{bmatrix} 1 & i & j & k & 0 \end{bmatrix} A.$$

The remaining equations imply that $A \in \mathbb{R}$. Then we have

$$R = A^2 \begin{bmatrix} 1 & i & j & k & 0 \\ -i & 1 & -k & j & 0 \\ -j & k & 1 & -i & 0 \\ -k & -j & i & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + a^2 \begin{bmatrix} 1 & i & j & k & 0 \\ -i & 1 & -k & j & 0 \\ -j & k & 1 & -i & 0 \\ -k & -j & i & 1 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

As R is real, $A = a = 0$, so $L = 0$. Contradiction! Therefore, we cannot have a $\text{Sp}(1)$ instanton from this representation (V, ρ) .

Instead, if we have a $\text{Sp}(2)$ instanton, then we have that the representation obtained by restricting the scalars of (W, λ) is some combination of the previously discussed 2-representations or one of $(V_{4,1}, \rho_{4,1})$ or $(V_{1,4}, \rho_{1,4})$. However, the only choices for the representation obtained by restricting the scalars of (W, λ) that create a trivial summand in $(\hat{W}, \hat{\lambda})$ include $(V_{1,2}, \rho_{1,2})$ or $(V_{2,1}, \rho_{2,1})$. But we already showed that these representations do not generate an instanton with rotational symmetry. Thus, this representation does not generate a $\text{Sp}(2)$ instanton with rotational symmetry either. $\square$

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APPENDICES

Appendix A

Proof of Theorem 1.1.1

In this appendix, we prove Theorem 1.1.1, whose statement we repeat for readability. The proof of this theorem relies on some classical results from Lie theory. More information on this topic can be found in Lee's book [Lee12].

Let $\mathcal{X}$ be a smooth manifold and $\mathcal{G}$ a Lie group acting smoothly on $\mathcal{X}$. Due to the smoothness of Lie group actions, we obtain Lie algebra actions. If we are dealing with right (left) group actions, we obtain Lie algebra (anti-)homomorphisms taking value in $\mathfrak{X}(\mathcal{X})$. Given $y \in \text{Lie}(\mathcal{G})$ and $A \in \mathcal{X}$, the Lie algebra action is given by $y.A := \left. \frac{d}{dt} \right|_{t=0} \exp(ty).A \in T_A\mathcal{X}$.

Theorem 1.1.1. *Let $\mathcal{X}$ be a smooth manifold, $\mathcal{G}$ a compact Lie group, and $\mathcal{S}$ a compact, connected Lie group. Suppose that $\mathcal{G}$ and $\mathcal{S}$ act smoothly on $\mathcal{X}$ on the left and the two actions commute. We have that $[A] \in \mathcal{X}/\mathcal{G}$ is fixed by $\mathcal{S}$ if and only if there is some Lie algebra homomorphism $\rho: \text{Lie}(\mathcal{S}) \rightarrow \text{Lie}(\mathcal{G})$ such that, for all $x \in \text{Lie}(\mathcal{S})$,*

$$x.A + \rho(x).A = 0.$$

Proof. Suppose that $[A] \in \mathcal{X}/\mathcal{G}$ is fixed by $\mathcal{S}$. Let $S_A \subseteq \mathcal{S} \times \mathcal{G}$ be the stabilizer group of A , given by

$$S_A := \{(s, g) \mid s.g.A = A\}. \quad (\text{A.0.1})$$

Note that the identity belongs to S_A and the commutativity of the actions imply that S_A is a subgroup.

As $\mathcal{S}$ is connected, it has a simply-connected universal cover $\tilde{\mathcal{S}}$. Denote the covering map $\pi: \tilde{\mathcal{S}} \rightarrow \mathcal{S}$. As A is symmetric under $\mathcal{S}$, the projection map $\pi_1|_{S_A}: S_A \rightarrow \mathcal{S}$ is

surjective. Our goal is to find a smooth map $\tilde{\mathcal{S}} \rightarrow S_A$ that, when composed with $\pi_1|_{S_A}$, gives us the covering map π . This map allows us to differentiate the equation of symmetry.

We start by showing that S_A is a Lie group. Let $f: \mathcal{S} \times \mathcal{G} \rightarrow \mathcal{X}$ be the smooth map

$$f(s, g) := s.g.A.$$

Then $S_A = f^{-1}(A)$, meaning S_A is closed. By the Closed-subgroup Theorem, as S_A is a closed subgroup of a Lie group, S_A is a Lie group [Lee12, Theorem 20.12]. Moreover, as $\mathcal{S} \times \mathcal{G}$ is compact and S_A is closed, S_A is compact.

Consider the Lie algebra homomorphism $\phi := d_{(e_{\mathcal{S}}, e_{\mathcal{G}})}(\pi_1|_{S_A}): \text{Lie}(S_A) \rightarrow \text{Lie}(\mathcal{S})$. Note that $\ker(\phi) \subseteq \text{Lie}(S_A)$ is an ideal of $\text{Lie}(S_A)$. As S_A is compact, it has a bi-invariant metric corresponding to a $\text{Ad}(S_A)$ -invariant inner product on $\text{Lie}(S_A)$. Notably, this inner product satisfies

$$\langle [x, y], z \rangle = \langle x, [y, z] \rangle.$$

Let $L \subseteq \text{Lie}(S_A)$ be the orthogonal complement to $\ker(\phi)$. We show that L is an ideal and $L \oplus \ker(\phi) = \text{Lie}(S_A)$ as Lie algebras (the bracket between the two ideals vanishes). Suppose that $l \in L$ and $s \in \text{Lie}(S_A)$. We see that for all $x \in \ker(\phi)$, we have $[x, s] \in \ker(\phi)$, and thus

$$\langle x, [s, l] \rangle = \langle [x, s], l \rangle = 0.$$

Hence, $[s, l] \in L$, so L is an ideal. Finally, we see that if $x \in \ker(\phi)$ and $l \in L$, then as both $\ker(\phi)$ and L are ideals, $[l, x] \in L \cap \ker(\phi)$. Hence, $\|[l, x]\|^2 = 0$, as it is orthogonal to itself. Hence, $[l, x] = 0$, so $\text{Lie}(S_A) = L \oplus \ker(\phi)$.

By the isomorphism theorems, we know that as $\text{im}(\pi_1|_{S_A}) = \mathcal{S}$ is closed,

$$L \simeq \text{Lie}(S_A)/\ker(\phi) \simeq \text{im}(\phi) = \text{Lie}(\text{im}(\pi_1|_{S_A})) = \text{Lie}(\mathcal{S}).$$

Let $\psi: \text{Lie}(\mathcal{S}) \rightarrow L$ be the isomorphism.

Let $Y \subseteq S_A$ be the unique connected Lie subgroup corresponding to the Lie algebra $L \subseteq \text{Lie}(S_A)$, whose existence is guaranteed by the Subgroups-subalgebras Theorem [Lee12, Theorem 19.26]. As $\tilde{\mathcal{S}}$ is simply-connected, the Homomorphisms Theorem tells us that there is a unique Lie group homomorphism $\Psi: \tilde{\mathcal{S}} \rightarrow Y$ such that $d_{e_{\tilde{\mathcal{S}}}}\Psi = \psi$ [Lee12, Theorem 20.19]. Consider the map $\pi_1|_{S_A} \circ \Psi: \tilde{\mathcal{S}} \rightarrow \mathcal{S}$. We know that $d_{e_{\tilde{\mathcal{S}}}}(\pi_1|_{S_A} \circ \Psi) = \phi \circ \psi$. We show that this map is an isomorphism and we use it to find the smooth map we desire.

Indeed, if $\phi(\psi(x)) = 0$, then $\psi(x) \in L \cap \ker(\phi) = \{0\}$, so $x = 0$, as ψ is an isomorphism. Furthermore, consider $y \in \text{Lie}(\mathcal{S}) = \text{im}(\phi)$. Then there is some $x \in \text{Lie}(S_A)$ such that

$\phi(x) = y$. We can uniquely write $x = l + z$ for $l \in L$ and $z \in \ker(\phi)$. But then $\phi(l) = y$. As ψ is an isomorphism, there exists $w \in \text{Lie}(\mathcal{S})$ such that $\psi(w) = l$. Therefore, $\phi(\psi(w)) = y$, so $\phi \circ \psi$ is a Lie algebra isomorphism. Call the inverse of this map g .

By the Homomorphisms Theorem, we know that there is a unique Lie group homomorphism $G: \tilde{\mathcal{S}} \rightarrow \tilde{\mathcal{S}}$ such that $d_{e_{\tilde{\mathcal{S}}}}G = g$ [Lee12, Theorem 20.19]. We show that $\Psi \circ G$ is the map we are searching for. Indeed, we have

$$d_{e_{\tilde{\mathcal{S}}}}(\pi_1|_{S_A} \circ \Psi \circ G) = \phi \circ \psi \circ g = \text{id}_{\text{Lie}(\mathcal{S})}.$$

But $\pi: \tilde{\mathcal{S}} \rightarrow \mathcal{S}$ is a Lie group homomorphism whose pushforward at the identity is the identity. By the Homomorphisms Theorem, the two maps must be equal, so $\pi_1|_{S_A} \circ \Psi \circ G = \pi$ [Lee12, Theorem 20.19].

As $\psi \circ g$ is a Lie algebra homomorphism and $\phi \circ \psi \circ g = \text{id}_{\text{Lie}(\mathcal{S})}$, we have that $\psi \circ g(x) = (x, \rho(x))$ for some Lie algebra homomorphism $\rho: \text{Lie}(\mathcal{S}) \rightarrow \text{Lie}(\mathcal{G})$.

For all $x \in \text{Lie}(\mathcal{S})$, we have $\Psi(G(\exp(x))) \in Y \subseteq S_A$. Noting that $\pi_1|_{S_A} \circ \Psi \circ G = \pi$, we see that for all $x \in \text{Lie}(\mathcal{S})$ and $\theta \in \mathbb{R}$,

$$\exp(x\theta) \cdot \exp(\rho(x)\theta) \cdot A = A.$$

Differentiating with respect to θ and evaluating at $\theta = 0$, we obtain

$$x \cdot A + \rho(x) \cdot A = 0.$$

Conversely, suppose that for some Lie algebra homomorphism ρ , we have for all $x \in \text{Lie}(\mathcal{S})$, $x \cdot A + \rho(x) \cdot A = 0$. Consider $s \in \mathcal{S}$. As $\mathcal{S}$ is compact and connected, the exponential map is surjective. Hence, there is some $x \in \text{Lie}(\mathcal{S})$ such that $\exp(x) = s$. We show that

$$F(\theta) := \exp(x\theta) \cdot \exp(\rho(x)\theta) \cdot A,$$

is constant.

First, recall that the symmetry action is a smooth map $\alpha: \mathcal{S} \times \mathcal{X} \rightarrow \mathcal{X}$. We also use the notation $s \cdot A$ to denote $\alpha(s, A)$. Given $x \in \text{Lie}(\mathcal{S})$ and $A \in \mathcal{X}$, consider the curve $\gamma(t) := \exp(tx) \cdot A$ in $\mathcal{X}$. Note that $\gamma(0) = A \in \mathcal{X}$. We define $x \cdot A := \left. \frac{d}{dt} \right|_{t=0} \exp(tx) \cdot A \in T_A \mathcal{X}$. As the symmetry action is a left Lie group action, it induces a left Lie algebra action, a Lie algebra anti-homomorphism $\text{Lie}(\mathcal{G}) \rightarrow \mathfrak{X}(\mathcal{X})$.

Given $s \in \mathcal{S}$, define $\iota_s: \mathcal{X} \rightarrow \mathcal{S} \times \mathcal{X}$ by $\iota_s(A) := (s, A)$. As the symmetry action is a left Lie group action, for all $s, s' \in \mathcal{S}$ and $A \in \mathcal{X}$, we have $\alpha(e_{\mathcal{S}}, A) = A$ and

$$\alpha \circ \iota_s \circ \alpha \circ \iota_{s'}(A) = \alpha(s, \alpha(s', A)) = \alpha(ss', A) = \alpha \circ \iota_{ss'}(A). \quad (\text{A.0.2})$$

Note that for $s \in \mathcal{S}$, $\alpha \circ \iota_s: \mathcal{X} \rightarrow \mathcal{X}$. For all $A \in \mathcal{X}$, $\alpha \circ \iota_s(A) = \alpha(s, A) = s.A$, so we have $d_A(\alpha \circ \iota_s): T_A\mathcal{X} \rightarrow T_{s.A}\mathcal{X}$. Define the smooth map $\tilde{\alpha}: \mathcal{S} \times T\mathcal{X} \rightarrow T\mathcal{X}$ as follows. For $s \in \mathcal{S}$ and $v \in T_A\mathcal{X}$, $\tilde{\alpha}(s, v) := d_A(\alpha \circ \iota_s)(v) \in T_{s.A}\mathcal{X}$.

As α is a Lie group action of $\mathcal{S}$, so too is $\tilde{\alpha}$. We denote this group action by $s.v := \tilde{\alpha}(s, v)$. Note that $\alpha \circ \iota_{e_S} = \text{id}_{\mathcal{X}}$. Then we see that for all $s, s' \in \mathcal{S}$ and $v \in T_A\mathcal{X}$, by (A.0.2), we have

$$\begin{aligned} \tilde{\alpha}(e_S, v) &= d_A(\alpha \circ \iota_{e_S})(v) = d_A \text{id}_{\mathcal{X}}(v) = v \\ \tilde{\alpha}(s, \tilde{\alpha}(s', v)) &= d_{s'.A}(\alpha \circ \iota_s) \circ d_A(\alpha \circ \iota_{s'})(v) \\ &= d_A(\alpha \circ \iota_s \circ \alpha \circ \iota_{s'})(v) \\ &= d_A(\alpha \circ \iota_{ss'})(v) \\ &= \tilde{\alpha}(ss', v). \end{aligned}$$

We can do the same thing for the gauge group $\mathcal{G}$. Returning to $F(\theta)$, we note that for $x \in \text{Lie}(\mathcal{S})$, we have for any $\theta_0 \in \mathbb{R}$, $\exp(\theta x) = \exp(\theta_0 x) \cdot \exp((\theta - \theta_0)x)$. Thus, as the actions commute, we have

$$\begin{aligned} F'(\theta_0) &= \left. \frac{d}{d\theta} \right|_{\theta=\theta_0} \exp(\theta x) \cdot \exp(\theta \rho(x)) \cdot A \\ &= \left. \frac{d}{d\theta} \right|_{\theta=\theta_0} \exp(\theta_0 x) \cdot \exp(\theta_0 \rho(x)) \cdot \exp((\theta - \theta_0)x) \cdot \exp((\theta - \theta_0)\rho(x)) \cdot A. \end{aligned}$$

Letting $t := \theta - \theta_0$, we have from the induced actions defined above

$$\begin{aligned} F'(\theta_0) &= \exp(\theta_0 x) \cdot \exp(\theta_0 \rho(x)) \cdot \left(\left. \frac{d}{dt} \right|_{t=0} \exp(tx) \cdot \exp(t\rho(x)) \cdot A \right) \\ &= \exp(\theta_0 x) \cdot \exp(\theta_0 \rho(x)) \cdot (x.A + \rho(x).A) \\ &= 0. \end{aligned}$$

Therefore, $\exp(x) \cdot \exp(\rho(x)) \cdot A = F(1) = F(0) = A$. As $s = \exp(x)$ was arbitrary, $[A]$ is fixed by $\mathcal{S}$. $\square$

Appendix B

Invariant isometries

In this appendix, we prove that for hyperbolic and Euclidean monopoles and instantons, non-trivial solutions are only symmetric under rotations and reflections. In particular, if we are interested in connected Lie subgroups, we need only look at rotations. This appendix has appeared on arXiv [[Lan24](#), arXiv version 1].

Definition B.0.1. *A monopole or instanton is said to be **flat** if $F_A = 0$.*

From the Bogomolny equations, we see that a flat monopole satisfies $D_A \Phi = 0$ as well. As such, flat monopoles and instantons are very well understood and trivial. We are only concerned with non-trivial monopoles and instantons, that is the non-flat ones, where the curvature $F_A \neq 0$.

Definition B.0.2. *Let M be a complete, simply-connected Riemannian manifold with non-positive sectional curvature, the manifold on which our connection lives, and let G be a compact Lie group, the structure group of the connection. By the Peter–Weyl Theorem, we can view G as a subgroup of a matrix Lie group. A **gauge transformation** is a smooth map $g: M \rightarrow G$ taking*

$$g.\Phi := g\Phi g^{-1} \quad \text{and} \quad g.A := gAg^{-1} - (dg)g^{-1}. \quad (\text{B.0.1})$$

*We say that A is **equivariant** under an isometry $F \in \text{Isom}(M)$ if there is some gauge transformation g such that*

$$F^*A = g.A.$$

*If we are dealing with a monopole, we also need $F^*\Phi = g.\Phi$ for the pair (Φ, A) to be **equivariant**.*

First, we determine how F_A transforms under gauge transformations.

Lemma B.0.3. *If we gauge transform the connection with $g: M \rightarrow G$, then F_A transforms as gF_Ag^{-1} .*

Proof. Under the gauge transformation, $A \mapsto A' := gAg^{-1} - (dg)g^{-1}$. Note that as $g^{-1}g = I_n$, we have $dg^{-1} = -g^{-1}(dg)g^{-1}$. As $F_A = dA + A \wedge A$, we simply find $F_{A'} = gF_Ag^{-1}$. $\square$

Definition B.0.4. *Let $\mathcal{H} \subseteq \text{Isom}(M)$ be a subgroup. As M is complete, simply-connected, and has non-positive sectional curvature, the Cartan–Hadamard Theorem tells us that any two points $x, y \in M$ are connected by a unique minimizing geodesic γ_{xy} . Let $d(x, y)$ be the length of the segment of γ_{xy} between x to y . Given $x \in M$ and $r > 0$, denote the geodesic sphere of radius r by $B(x, r) := \{y \in M \mid d(x, y) < r\}$. Finally, denote $\text{Fix}(\mathcal{H}) := \{x \in M \mid f(x) = x, \forall f \in \mathcal{H}\}$ to be the fixed point set of $\mathcal{H}$.*

We now discuss the orbits of points in M induced by $\mathcal{H}$.

Lemma B.0.5. *Let $x \in M$. If $\mathcal{H} \cdot x$ is bounded, then every orbit is bounded. If $\mathcal{H} \cdot x$ is unbounded, then every orbit is unbounded.*

Proof. Suppose that $\mathcal{H} \cdot x \subseteq B(x, R)$. Let $y \in M$ and $f \in \mathcal{H}$. Then

$$d(f(y), y) \leq d(f(y), f(x)) + d(f(x), x) + d(x, y) < R + 2d(x, y).$$

Thus, $\mathcal{H} \cdot y \subseteq B(y, R + 2d(x, y))$ is bounded.

The second statement follows from the first, as if another orbit is bounded, then so too must $\mathcal{H} \cdot x$. $\square$

We now relate the orbits induced by $\mathcal{H}$ with the fixed point set of $\mathcal{H}$.

Proposition B.0.6. *Every orbit is bounded if and only if $\text{Fix}(\mathcal{H}) \neq \emptyset$.*

Proof. Suppose that $\text{Fix}(\mathcal{H}) \neq \emptyset$. Let $x \in \text{Fix}(\mathcal{H})$. Then $\mathcal{H} \cdot x = \{x\}$, which is bounded. Thus, every orbit is bounded.

Conversely, suppose that every orbit is bounded. We show that $\text{Fix}(\mathcal{H}) \neq \emptyset$. For $x \in M$, let

$$E_x := \{R > 0 \mid \exists y \in M: \mathcal{H} \cdot x \subseteq B(y, R)\}.$$

As $\mathcal{H} \cdot x$ is bounded, E_x is non-empty. As it is bounded from below by zero, $e_x := \inf(E_x) \geq 0$ exists.

Note that if $x \in \text{Fix}(\mathcal{H})$, then $e_x = 0$. In fact, $x \in \text{Fix}(\mathcal{H})$ if and only if $e_x = 0$, as we now prove. Suppose that some $e_x = 0$. Let $f \in \mathcal{H}$. For all $R > 0$, there is some $y \in M$ such that $\mathcal{H} \cdot x \subseteq B(y, \frac{R}{2})$. Hence,

$$d(f(x), x) \leq d(f(x), y) + d(x, y) < R.$$

As this inequality is true for all $R > 0$, then $d(f(x), x) = 0$, so $f(x) = x$. As f is arbitrary, $x \in \text{Fix}(\mathcal{H})$, as claimed.

To prove that $\text{Fix}(\mathcal{H}) \neq \emptyset$, we either find a $x \in M$ such that $e_x = 0$, meaning $x \in \text{Fix}(\mathcal{H})$, or we find a convergent sequence $\{x_i\}$ in H^3 where all $e_{x_i} > 0$ and $e_{x_i} \rightarrow 0$. We prove that the limit of this sequence belongs to $\text{Fix}(\mathcal{H})$.

Let $x_1 \in M$. If $e_{x_1} = 0$, then we are done. Otherwise, let $\epsilon > 0$ such that $\epsilon(2e_{x_1} + \epsilon) < e_{x_1}^2/16$, which is always possible as $e_{x_1} > 0$. Then there is some $R_1 \in [e_{x_1}, e_{x_1} + \epsilon)$ and $x_2 \in M$ such that $\mathcal{H} \cdot x_1 \subseteq B(x_2, R_1)$. If $e_{x_2} = 0$, then we are done. Otherwise, note that for all $f, g \in \mathcal{H}$, we have that

$$d(f(x_1), g(x_2)) = d(g^{-1}(f(x_1)), x_2) < R_1. \quad (\text{B.0.2})$$

Hence, $\mathcal{H} \cdot x_1 \subseteq B(g(x_2), R_1)$, so $\mathcal{H} \cdot x_1 \subseteq \bigcap_{g \in \mathcal{H}} B(g(x_2), R_1)$.

Let $f \in \mathcal{H}$ such that $f(x_2) \neq x_2$. If no such f exists, then $x_2 \in \text{Fix}(\mathcal{H})$. Let $y \in M$ be the midpoint of the minimizing geodesic segment connecting x_2 and $f(x_2)$, so $d(x_2, y) = \frac{d(x_2, f(x_2))}{2} = d(f(x_2), y)$. As $\mathcal{H} \cdot x_1 \subseteq B(x_2, R_1) \cap B(f(x_2), R_1)$, we have $d(x_2, f(x_2)) < 2R_1$.

Let $r := \sqrt{R_1^2 - \frac{d(x_2, f(x_2))^2}{4}} > 0$. We now show that $\mathcal{H} \cdot x_1 \subseteq B(y, r)$.

Consider a point $z \in B(x_2, R_1) \cap B(f(x_2), R_1)$. We construct two geodesic triangles by taking the minimizing geodesics between $f(x_2), x_2, y$, and z , as in Figure B.1. Let L be the length of the minimizing geodesic between y and z . Let θ be the angle between the minimizing geodesic from x_2 to y and the minimizing geodesic from y to z . As y was chosen as the midpoint of the minimizing geodesic from x_2 to $f(x_2)$, we know that the angle between the minimizing geodesic from y to $f(x_2)$ and the minimizing geodesic from y to z is $\pi - \theta$, as the geodesic is smooth. Finally, by definition, we know the lengths of the minimizing geodesics from z to x_2 and $f(x_2)$ are less than R_1 .

As we have a complete, simply-connected manifold with non-positive sectional curvature, we have that [DC92, §12 Lemma 3.1]

$$\begin{aligned} L^2 + \frac{d(x_2, f(x_2))^2}{4} - d(x_2, f(x_2)) \cos(\theta) &< R_1^2, \quad \text{and} \\ L^2 + \frac{d(x_2, f(x_2))^2}{4} - d(x_2, f(x_2)) \cos(\pi - \theta) &< R_1^2. \end{aligned}$$

Using $\cos(\pi - \theta) = -\cos(\theta)$, we add these inequalities to see that $L^2 < R_1^2 - \frac{d(x_2, f(x_2))^2}{4} = r^2$. Therefore, $z \in B(y, r)$, so $B(x_2, R_1) \cap B(f(x_2), R_1) \subseteq B(y, r)$. In particular, this means $\mathcal{H} \cdot x_1 \subseteq B(y, r)$.

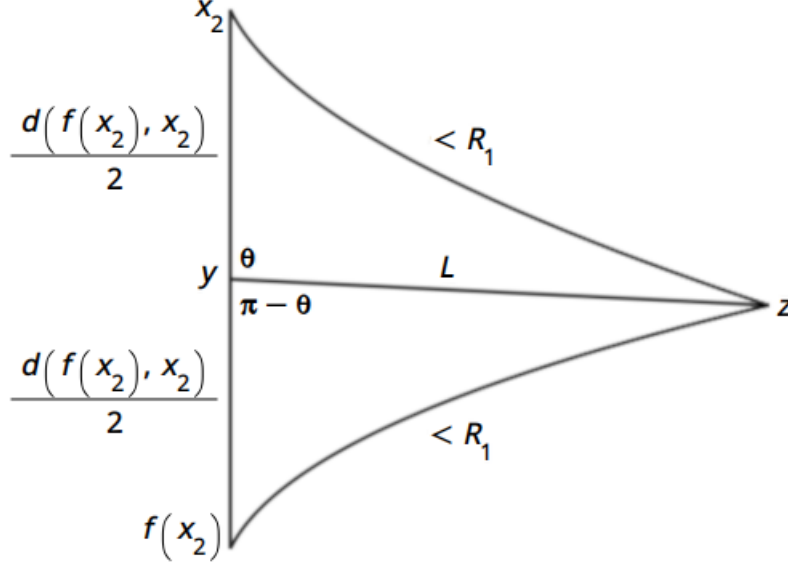


Figure B.1: Geodesic triangles between the points x_2 , $f(x_2)$, y , and some $z \in B(x_2, R_1) \cap B(f(x_2), R_1)$.

Hence, we know that $r \geq e_{x_1}$. Recall that $R_1 < e_{x_1} + \epsilon$. Hence,

$$\frac{d(x_2, f(x_2))^2}{4} = R_1^2 - r^2 < (e_{x_1} + \epsilon)^2 - e_{x_1}^2 = \epsilon(2e_{x_1} + \epsilon) < \frac{e_{x_1}^2}{16}.$$

Recall that f was arbitrary, amongst all those elements of $\mathcal{H}$ for which $f(x_2) \neq x_2$. As such, we see that for all $f \in \mathcal{H}$, $d(x_2, f(x_2)) = 0$ or $d(x_2, f(x_2)) < \frac{e_{x_1}}{2}$. Thus, $\mathcal{H} \cdot x_2 \subseteq B(x_2, \frac{e_{x_1}}{2})$, so $e_{x_2} \leq \frac{e_{x_1}}{2}$. Note that as $\epsilon(2e_{x_1} + \epsilon) < e_{x_1}^2/16$, we must have $\epsilon < e_{x_1}/2$, otherwise we violate the former inequality. Hence, $R_1 < e_{x_1} + \epsilon < 3e_{x_1}/2$, so $d(f(x_2), g(x_1)) < 3e_{x_1}/2$ for all $f, g \in \mathcal{H}$, as per (B.0.2).

We can repeat this process. If at any point $e_{x_i} = 0$, then we are done. Otherwise, we obtain a sequence $\{x_i\} \subseteq H^3$ such that all $e_{x_i} > 0$ and for all $f, g \in \mathcal{H}$ and $i \in \mathbb{N}_+$,

$$d(f(x_i), g(x_{i+1})) < \frac{3e_{x_i}}{2}, \quad \text{and} \quad e_{x_{i+1}} \leq \frac{e_{x_i}}{2}.$$

By induction, these inequalities imply for all $f, g \in \mathcal{H}$ and $i \in \mathbb{N}_+$,

$$d(f(x_i), g(x_{i+1})) < \frac{3e_{x_1}}{2^i}, \quad \text{and} \quad e_{x_{i+1}} \leq \frac{e_{x_1}}{2^i}.$$

Thus, we see that we have a convergent sequence $\{x_i\}$, whose limit we denote by $\hat{x}$. We show that $\hat{x}$ is a fixed point of $\mathcal{H}$.

Let $f \in \mathcal{H}$ and let $\epsilon > 0$. Then there is some $i \in \mathbb{N}_+$ such that

$$d(f(x_i), x_{i+1}) < \frac{\epsilon}{4}, \quad d(x_{i+1}, x_i) < \frac{\epsilon}{4}, \quad \text{and} \quad d(x_i, \hat{x}) < \frac{\epsilon}{4}.$$

Then

$$d(f(\hat{x}), \hat{x}) \leq d(f(\hat{x}), f(x_i)) + d(f(x_i), x_{i+1}) + d(x_{i+1}, x_i) + d(x_i, \hat{x}) < \epsilon.$$

As ϵ was arbitrary, $f(\hat{x}) = \hat{x}$, so $\hat{x} \in \text{Fix}(\mathcal{H})$. $\square$

Using Proposition B.0.6 and a condition on the integral of the curvature, we arrive at the following theorem.

Theorem B.0.7. *Consider a non-flat connection A with $\mathcal{H} \subseteq \text{Isom}(M)$ the set of isometries under which A is equivariant. Suppose there exists $r_0 > 0$ such that at every point $x \in M$, the injectivity radius $\iota(x) \geq r_0$. If, for some $p \in (0, \infty)$, $\int_M |F_A|^p \text{vol}_M$ must be finite, then $\text{Fix}(\mathcal{H}) \neq \emptyset$.*

Proof. Suppose that $\text{Fix}(\mathcal{H}) = \emptyset$. Then all orbits are unbounded, by Proposition B.0.6. As the connection is non-flat, the value of F_A at some $x_1 \in M$ is non-zero. Hence, there is some radius $0 < R \leq r_0$ and lower bound $\epsilon > 0$ such that $|F_A|$ is at least ϵ in $B(x_1, R)$. We use the unbounded orbit of x_1 to prove that the integral in the statement diverges, which is not permissible. Therefore, we must have $\text{Fix}(\mathcal{H}) \neq \emptyset$.

Let $i \in \mathbb{N}_+$. Suppose we have i points $x_1, \dots, x_i \in \mathcal{H} \cdot x_1$ such that $d(x_j, x_k) \geq 2R$ for all $j, k \in \{1, \dots, i\}$. Let $r := \max_{1 \leq j \leq i} (d(x_1, x_j)) \geq 2R$. As the orbit of x_1 is unbounded, there exists a point $x_{i+1} \in \mathcal{H} \cdot x_1$ such that $d(x_1, x_{i+1}) \geq r + 2R$. Let $j \in \{1, \dots, i\}$. Then

$$d(x_j, x_{i+1}) \geq |d(x_1, x_j) - d(x_1, x_{i+1})| = d(x_1, x_{i+1}) - d(x_1, x_j) \geq r + 2R - r = 2R..$$

By induction, we get an infinite sequence of points in the orbit of x_1 that are all separated by at least $2R$.

We now show that around each of these points, there is a ball of radius R in which $|F_A|$ is at least ϵ . Indeed, let $f \in \mathcal{H}$ and let $z \in B(f(x_1), R)$. We see that

$$d(x_1, f^{-1}(z)) = d(f(x_1), z) < R.$$

Hence, $f^{-1}(z) \in B(x_1, R)$. Thus, $|F_A(f^{-1}(z))| \geq \epsilon$. Note that

$$F_{(f^{-1})^*A} = d(f^{-1})^*A + (f^{-1})^*A \wedge (f^{-1})^*A = (f^{-1})^*F_A.$$

As the connection is equivariant under $\mathcal{H}$, we have that there is a gauge transformation $g: M \rightarrow G$ such that $g.A = (f^{-1})^*A$. By Lemma B.0.3, we see that

$$gF_Ag^{-1} = F_{g.A} = F_{(f^{-1})^*A} = (f^{-1})^*F_A.$$

As f^{-1} is an isometry, we have that

$$|F_A(z)| = |g(z)F_A(z)g(z)^{-1}| = |((f^{-1})^*F_A)(z)| = |F_A(f^{-1}(z))| \geq \epsilon.$$

Therefore, in $B(f(x_1), R)$, $|F_A|$ is at least ϵ .

Finally, we show that the integral in the statement diverges. For every $i \in \mathbb{N}_+$, we have

$$\int_{B(x_i, R)} |F_A|^p \text{vol}_M \geq \epsilon^p \text{vol}(B(x_i, R)).$$

As $R \leq r_0 \leq \iota(x_i)$ and the sectional curvature is non-positive, Bishop and Günther's Theorem tell us that $\text{vol}(B(x_i, R))$ is at least the volume of a Euclidean $\dim(M)$ -sphere of radius R [Cha06, Theorem III.4.2]. That is, the volume of the geodesic balls is bounded below by a positive constant. As the points x_i are at least $2R$ apart, the balls $B(x_i, R)$ never intersect. Thus, adding up the contribution from the various $B(x_i, R)$, we see that the energy or action is infinite. $\square$

Corollary B.0.8. *Consider a non-flat hyperbolic or Euclidean monopole of finite energy. Then the group $\mathcal{H}$ of isometries under which the monopole is equivariant is conjugate to a subgroup of $O(3)$.*

Consider a non-flat hyperbolic or Euclidean instanton of finite action. Then the group $\mathcal{H}$ of isometries under which the instanton is equivariant is conjugate to a subgroup of $O(4)$.

Proof. In all cases considered, the manifold M on which the object lives is complete, simply-connected, and has non-positive sectional curvature. Furthermore, the injectivity radius of hyperbolic and Euclidean space is infinite. Theorem B.0.7 tells us that $\text{Fix}(\mathcal{H}) \neq \emptyset$, as the energy and action are a multiple of $\int_M |F_A|^2 \text{vol}_M$ and must be finite.

Let $x \in \text{Fix}(\mathcal{H})$. Using an isometry F taking x to the origin (where we use the ball models of H^3 and H^4), we have that the origin is fixed by $F\mathcal{H}F^{-1}$, so $\mathcal{H} \subseteq F^{-1}\text{Isom}_0(M)F$. In all of the above cases, $\text{Isom}_0(M) \simeq O(\dim(M))$. $\square$

Note B.0.9. *Furthermore, as we are only interested in connected Lie subgroups, we only consider $\mathcal{H} \subseteq \text{SO}(\dim(M))$.*

Appendix C

Continuous subgroups of conformal transformations

Our ultimate goal is to classify monopoles and instantons with continuous symmetries. In particular, we are interested in Euclidean and hyperbolic monopoles and Euclidean instantons. In the case of monopoles, in [Appendix B](#), we show that we need only consider continuous rotations: connected Lie subgroups of $\mathrm{SO}(3)$. For instantons, we are interested in classifying those symmetric under continuous subgroups of conformal transformations. In this appendix, we classify the relevant connected Lie subgroups up to conjugacy, providing a list of symmetries that these objects can have.

C.1 Continuous rotations in three dimensions

In this section, we show that for monopoles symmetric under rotations, we only care about the angle of rotation, not the axis. This result follows from the fact that $S^1 \subseteq \mathrm{SO}(3)$ is a maximal torus and all maximal tori are conjugate. Thus, we need only check symmetry about one axis in addition to spherical symmetry. Recall the definition of equivariance given in [Definition B.0.2](#) in [Appendix B](#).

For monopoles, only the angle of rotation matters, the axis of rotation does not.

Proposition C.1.1. *Let M be $\mathbb{R}^3$ or H^3 . Let $R, R' \in \mathrm{SO}(3)$ be rotations by $\theta \in [0, \pi]$ about normal vectors $u, u' \in M$, respectively. Let (Φ, A) be a G -monopole equivariant under R . Then (Φ, A) corresponds to a monopole equivariant under R' .*

Note C.1.2. *Note that if R and R' rotate by $\theta \in (\pi, 2\pi)$, then their inverses rotate by $2\pi - \theta \in (0, \pi)$. Moreover, if a monopole is equivariant under R or R' then it is equivariant under its inverse as well.*

Proof. To start, note that R' and R are conjugate. Let $\{u, v, w\}$ be an orthonormal basis of M , oriented so that for all $a, b \in \mathbb{R}$,

$$R(av + bw) = (a \cos \theta - b \sin \theta)v + (a \sin \theta + b \cos \theta)w. \quad (\text{C.1.1})$$

Let $\{u', v', w'\}$ be an analogous orthonormal basis of M for R' . Let F be the change of basis matrix taking elements of the second basis to the corresponding element in the first basis. As F is a change of basis matrix for orthonormal bases, $F \in \text{O}(3)$

We have for all $a, b \in \mathbb{R}$

$$\begin{aligned} F^T R F (av' + bw') &= F^T R (av + bw) \\ &= F^T ((a \cos \theta - b \sin \theta)v + (a \sin \theta + b \cos \theta)w) \\ &= R'(av' + bw'). \end{aligned} \quad (\text{C.1.2})$$

Furthermore, as

$$F^T R F (u') = F^T R (u) = F^T (u) = u' = R'(u'), \quad (\text{C.1.3})$$

we have that $R' = F^T R F$ on all of M . Hence, $FR' = RF$.

Let (Φ, A) be a monopole equivariant under R and let $\Phi' := F^* \Phi$ and $A' := F^* A$. As $F \in \text{O}(3)$ is an isometry, we have that (Φ', A') is a monopole and we show that it is equivariant under R' .

As (Φ, A) is equivariant under R , there is some gauge transformation $g: M \rightarrow G$ such that $R^* \Phi = g \cdot \Phi$ and $R^* A = g \cdot A$. Then

$$\begin{aligned} R'^* \Phi' &= (FR')^* \Phi = (RF)^* \Phi = F^* R^* \Phi = F^* (g \cdot \Phi) = (F^* g) \cdot \Phi', \quad \text{and} \\ R'^* A' &= (FR')^* A = (RF)^* A = F^* R^* A = F^* (g \cdot A) = (F^* g) \cdot A'. \end{aligned}$$

Therefore, (Φ', A') is equivariant under R' . □

Note C.1.3. *A rotation matrix $R \in \text{SO}(3)$ rotating about u by θ has three eigenvalues: 1, $e^{i\theta}$, and $e^{-i\theta}$. Note that for any $F \in \text{O}(3)$, $F^T R F$ has the same eigenvalues. Thus, if we want $R' = F^T R F$, then R and R' must rotate by the same angle in $[0, \pi]$. Thus, when looking at monopoles equivariant under S^1 , we only have one case to consider: axial symmetry around the z -axis.*

If we are consider monopoles equivariant under more than just rotations about one axis, then we do care about the axis of additional rotations. This result tells us that we can always set one axis of rotation though.

C.2 Continuous conformal transformations

In this section, we classify the continuous subgroups of conformal transformations that we need consider when investigating symmetric instantons. If we were just studying instantons symmetric under isometries, by Corollary B.0.8 in Appendix B, we would just need to study instantons symmetric under rotations. However, the conformal invariance of the self-dual equations allows us to broaden our study to conformal symmetries.

Recall Conjecture 3.1.14, which states that up to conjugation by an element of $\mathrm{SL}(2, \mathbb{H})$, the group of symmetries of a non-flat instanton is a subgroup of $\mathrm{Sp}(2)$. Before we move on to classifying the Lie subalgebras of $\mathfrak{sp}(2)$, let us discuss rotations in $\mathbb{R}^4$. In four dimensions there are three main types of rotations.

Definition C.2.1. *Let $R \in \mathrm{SO}(4)$. Associated to R are two angles $\alpha, \beta \in [0, 2\pi)$ and orthogonal two-planes A and B such that $A \oplus B \simeq \mathbb{R}^4$. The planes A and B are invariant under the action of R . Specifically, on A and B , R rotates about the origin by α and β , respectively. Note that R is completely determined by its action on A and B .*

*If either of α or β is zero, then R is said to be a **simple** rotation. Without loss of generality, we may assume $\alpha = 0$. In this case, R fixes a two-plane, A . Every completely orthogonal two-plane B , that is where every line in B is orthogonal to every line in A , intersects A at a point P . Then R rotates elements in B by β about the point P .*

*If $\alpha = \beta > 0$, then R is said to be **isoclinic**. In this case, there are infinitely many invariant two-planes and every vector is rotated by R by $\alpha = \beta$. Otherwise, R is said to be a **double** rotation.*

Lemma C.2.2. *For every rotation $R \in \mathrm{SO}(4)$ there exists $p, q \in \mathrm{Sp}(1)$ such that $Rx = pxq^\dagger$ for all $x \in \mathbb{H} \simeq \mathbb{R}^4$. The rotation is isoclinic if and only if p or q is real. The rotation R is simple if and only if $\mathrm{Re}(p) = \mathrm{Re}(q)$. For instance, $x \mapsto -x$ can be constructed with $p = -1$ and $q = 1$ and this transformation is not a simple rotation. Indeed, this rotation has no fixed points other than zero.*

Proof. The double cover of $\mathrm{SO}(4)$ is $\mathrm{Spin}(4) \simeq \mathrm{Sp}(1) \times \mathrm{Sp}(1)$, via the map $\pi: \mathrm{Sp}(1) \times \mathrm{Sp}(1) \rightarrow \mathrm{SO}(4)$ which acts on $x \in \mathbb{R}^4 \simeq \mathbb{H}$ as $pxq^\dagger = \pi(p, q)x$.

First, we note that for $p \in \mathrm{Sp}(1)$ and $x \in \mathfrak{sp}(1)$, we have $pxp^\dagger \in \mathfrak{sp}(1)$. Thus, if we decompose $x \in \mathbb{H}$ as $x = x_0 + \vec{x}$, where $x_0 \in \mathbb{R}$ and $\vec{x} \in \mathfrak{sp}(1)$, we decompose $pxp^\dagger = x_0 + p\vec{x}p^\dagger$. Note that the double cover $\pi: \mathrm{Sp}(1) \rightarrow \mathrm{SO}(3)$ acts on $\mathbb{R}^3 \simeq \mathfrak{sp}(1)$ as $pxp^\dagger = \pi(p)x$. Thus, all rotations are of the form $pxp^\dagger$.

Suppose that R is simple. Then there is some $x \in \mathbb{H} \setminus \{0\}$ such that $pxq^\dagger = x$. Then $p = \frac{xqx^\dagger}{|x|^2}$. Using the decomposition from above, $p_0 = q_0$. Conversely, suppose that $p_0 = q_0$. Let $x \in \text{Sp}(1)$ such that $\vec{p} = x\vec{q}x^\dagger$. We see that

$$pxq^\dagger = (q_0 + x\vec{q}x^\dagger)x(q_0 - \vec{q}) = x. \quad (\text{C.2.1})$$

Thus, the rotation is simple.

Suppose that R is isoclinic. Then, for every $x \in \mathbb{H}$, we have that the angle between x and $pxq^\dagger$ is some $\cos \alpha$. That is, $(x^\dagger pxq^\dagger)_0 = |x|^2 \cos \alpha$ for all $x \in \mathbb{H}$. Suppose that p and q are not real. Let $x \in \text{Sp}(1)$ and $A > 0$ such that $\vec{p} = Ax\vec{q}x^\dagger$. Then

$$(x^\dagger pxq^\dagger)_0 = ((p_0 + A\vec{q})(q_0 - \vec{q}))_0 = p_0q_0 - A|\vec{q}|^2. \quad (\text{C.2.2})$$

However, if we choose $y \in \text{Sp}(1)$ such that $y^\dagger \vec{p}y$ and $\vec{q}$ are orthogonal, then $(y^\dagger \vec{p}y\vec{q})_0 = 0$. Hence, $(y^\dagger pyq^\dagger)_0 = p_0q_0$. Thus, we see that $A|\vec{q}|^2 = 0$. But $A > 0$, so $q \in \mathbb{R}$. But q is not real, contradiction! Hence, p or q is real.

Conversely, suppose that p is real. Then for all $x \in \mathbb{H}$, we have

$$(x^\dagger pxq^\dagger)_0 = pq_0|x|^2. \quad (\text{C.2.3})$$

Alternatively, if q is real, then for all $x \in \mathbb{H}$, we have

$$(x^\dagger pxq^\dagger)_0 = qp_0|x|^2. \quad (\text{C.2.4})$$

In either case, we have that the rotation is isoclinic. $\square$

C.2.1 Classifying Lie subalgebras of $\mathfrak{sp}(2)$

Before we classify the connected Lie subgroups up to conjugacy, we classify the Lie subalgebras of $\mathfrak{sp}(2)$.

Lemma C.2.3. *The non-trivial Lie subalgebras of $\mathfrak{sp}(2)$ are $\mathbb{R}$, $\mathbb{R} \oplus \mathbb{R}$, $\mathfrak{sp}(1)$, $\mathfrak{sp}(1) \oplus \mathbb{R}$, and $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$.*

Proof. As $\text{Sp}(2)$ is a simple, compact Lie group, its Lie algebra $\mathfrak{sp}(2)$ is simple and compact. Subalgebras of compact Lie algebras are compact, hence reductive. Let $\mathfrak{g} \subseteq \mathfrak{sp}(2)$ be a Lie subalgebra. Then $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$, where $\mathfrak{z}(\mathfrak{g})$ is abelian and $[\mathfrak{g}, \mathfrak{g}]$ is semi-simple.

Note that the rank of $\mathfrak{sp}(2)$ is two. Thus, $\text{rank}(\mathfrak{g}) \leq 2$, so $\mathfrak{z}(\mathfrak{g})$ is one of 0 , $\mathbb{R}$, or $\mathbb{R} \oplus \mathbb{R}$. If $\mathfrak{z}(\mathfrak{g}) = \mathbb{R} \oplus \mathbb{R}$, then the rank of $[\mathfrak{g}, \mathfrak{g}]$ is zero. As this Lie subalgebra is semi-simple, this means that $[\mathfrak{g}, \mathfrak{g}] = 0$. Hence, $\mathfrak{g} = \mathbb{R} \oplus \mathbb{R}$.

Suppose that $\mathfrak{z}(\mathfrak{g}) = \mathbb{R}$. Then the rank of $[\mathfrak{g}, \mathfrak{g}]$ is at most one. Just as above, if the rank of $[\mathfrak{g}, \mathfrak{g}]$ is zero, then $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) = \mathbb{R}$. However, if the rank of $[\mathfrak{g}, \mathfrak{g}]$ is one, then as it is semi-simple, $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{sp}(1)$. Thus, $\mathfrak{g} = \mathfrak{sp}(1) \oplus \mathbb{R}$.

Finally, suppose that $\mathfrak{z}(\mathfrak{g}) = 0$. That is, $\mathfrak{g}$ is semi-simple. Then the rank of $\mathfrak{g}$ is at most two. If the rank of $\mathfrak{g}$ is zero, then $\mathfrak{g} = 0$. If the rank of $\mathfrak{g}$ is one, then $\mathfrak{g} = \mathfrak{sp}(1)$. However, we have options should the rank of $\mathfrak{g}$ be two. The only semi-simple Lie algebras of rank two are $\mathfrak{sp}(2)$, $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$, $\mathfrak{su}(3)$, and $\mathfrak{g}_2$. However, we note that $\mathfrak{sp}(2)$ is 10-dimensional, so $\mathfrak{g} \subseteq \mathfrak{sp}(2)$ must be at most ten dimensional. As $\mathfrak{g}_2$ is 14-dimensional, we can exclude this possibility. It remains to show that $\mathfrak{su}(3)$ is not a Lie subalgebra of $\mathfrak{sp}(2)$.

Indeed, suppose for the sake of contradiction that $\mathfrak{su}(3) \simeq \mathfrak{g} \subseteq \mathfrak{sp}(2)$. As $\mathfrak{g}$ and $\mathfrak{sp}(2)$ both have rank two, a Cartan subalgebra of $\mathfrak{g}$ is a Cartan subalgebra of $\mathfrak{sp}(2)$. Thus, $\mathfrak{g}$ is a regular Lie subalgebra of $\mathfrak{sp}(2)$ [Dyn57]. Therefore, $\mathfrak{g}$ has a closed root subsystem of the root system of $\mathfrak{sp}(2)$. However, by looking at their root diagrams, we note that the root system of $\mathfrak{su}(3)$ is not a closed root subsystem of $\mathfrak{sp}(2)$. Contradiction! Therefore, $\mathfrak{su}(3)$ is not a Lie subalgebra of $\mathfrak{sp}(2)$. $\square$

Consider two Lie subgroups G and G' of $\text{Sp}(2)$ with isomorphic Lie algebras. Unlike the case of rotations in three dimensions, an instanton equivariant under G does not necessarily correspond to an instanton equivariant under G' . To study the different subgroups, we have the following lemma.

Lemma C.2.4. *Let $x \in \mathfrak{sp}(2)$. Then there is some $A \in \text{Sp}(2)$ such that $AxA^\dagger = \text{diag}(ai, bi)$. Moreover, A can be chosen such that $a \geq b \geq 0$.*

Proof. The first claim follows from the fact that a maximal torus of $\mathfrak{sp}(2)$ is given by $\{\text{diag}(ai, bi) \mid a, b \in \mathbb{R}\}$. As $\text{Sp}(2)$ is a matrix group, the adjoint action of $\text{Sp}(2)$ on $\mathfrak{sp}(2)$ is just conjugation. Then, as all adjoint orbits intersect maximal tori, we have the first claim.

For the second claim, if $x = \text{diag}(ai, bi)$, we can conjugate by $A = jI_2 \in \text{Sp}(2)$ to get $AxA^\dagger = \text{diag}(-ai, -bi)$. Additionally, conjugating by $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, we have that $AxA^\dagger = \text{diag}(bi, ai)$. Thus, we may assume that $a \geq b \geq 0$. $\square$

Now we are equipped to study the continuous conformal subgroups that we use to study symmetric instantons. From Lemma C.2.3, we only need to study Lie subgroups with Lie algebra of the form $\mathbb{R}$, $\mathbb{R} \oplus \mathbb{R}$, $\mathfrak{sp}(1)$, $\mathfrak{sp}(1) \oplus \mathbb{R}$, $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$, and $\mathfrak{sp}(2)$.

C.2.2 Classifying continuous conformal subgroups

In this section, we classify the different connected Lie subgroups of $\mathrm{Sp}(2)$ up to conjugacy that we use to study symmetric instantons.

Suppose we take two connected Lie subgroups whose Lie algebras are conjugate. This conjugacy lifts to the connected Lie subgroups, as the exponential map on $\mathrm{Sp}(2)$ is surjective. As we are dealing with Lie subalgebras of $\mathfrak{sp}(2)$, which naturally act on $\mathbb{H}^2$, these Lie algebras are conjugate if and only if they are isomorphic representations. Therefore, we need only classify Lie subalgebras up to conjugacy, or rather, up to isomorphic representations.

To start with, we consider subgroups with Lie algebra $\mathbb{R}$.

Proposition C.2.5. *The Lie subgroups of $\mathrm{Sp}(2)$ with Lie algebra $\mathbb{R}$ are conjugate to some member of the family of subgroups $\{R_t \mid 0 \leq t \leq 1\}$ with R_t given by*

$$R_t := \left\{ \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{ti\theta} \end{bmatrix} \mid \theta \in \mathbb{R} \right\}. \quad (\text{C.2.5})$$

Note C.2.6. *The subgroup R_t is compact if and only if t is rational. Thus, R_t is isomorphic to S^1 when t is rational and $\mathbb{R}$ otherwise. While R_t is not always compact, it turns out that we do not need compactness in this case when investigating symmetric instantons.*

Proof. Let $G \subseteq \mathrm{Sp}(2)$ be a Lie subgroup with Lie algebra $\mathbb{R}$ generated by $x \in \mathfrak{sp}(2)$. Thus, every element of G is of the form $e^{x\theta}$ for some $\theta \in \mathbb{R}$. By Lemma C.2.4, we can take x to be of the form $\mathrm{diag}(ai, bi)$ for some $a \geq b \geq 0$. If $a = 0$, then $G = \{I\}$, which does not have a Lie algebra isomorphic to $\mathbb{R}$. Hence, $a \neq 0$. Let $t := b/a$. Then $0 \leq t \leq 1$. Finally, we see that the Lie algebra generated by $\mathrm{diag}(ai, bi)$ is the same as that generated by $\mathrm{diag}(i, ti)$. $\square$

Now we move on to subgroups with Lie algebra $\mathbb{R} \oplus \mathbb{R}$.

Proposition C.2.7. *The Lie subgroups of $\mathrm{Sp}(2)$ with Lie algebra $\mathbb{R} \oplus \mathbb{R}$ are conjugate to the Lie subgroup given by*

$$\left\{ \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{i\phi} \end{bmatrix} \mid \theta, \phi \in \mathbb{R} \right\}. \quad (\text{C.2.6})$$

Note C.2.8. *This Lie subgroup is compact and connected. Therefore, it is a maximal torus, isomorphic to $S^1 \times S^1$.*

Proof. Note that $\mathbb{R} \oplus \mathbb{R} \subseteq \mathfrak{sp}(2)$ is a Cartan subalgebra. As all Cartan subalgebras are conjugate, we may take the Cartan subalgebra to be given by $\{\text{diag}(i\theta, i\phi) \mid \theta, \phi \in \mathbb{R}\}$. Exponentiating, we get our group. $\square$

Now that we move on to non-abelian Lie subalgebras, we can use representation theory to reduce the list of subgroups to investigate. Next, we study subgroups with Lie algebra $\mathfrak{sp}(1)$.

Proposition C.2.9. *The connected Lie subgroups of $\text{Sp}(2)$ with Lie algebra $\mathfrak{sp}(1)$ are conjugate to one of the unique connected Lie subgroups with the following Lie algebras:*

$$\begin{aligned} \mathfrak{h}_{3,1,1} &:= \left\langle \begin{bmatrix} i/2 & 0 \\ 0 & i/2 \end{bmatrix}, \begin{bmatrix} j/2 & 0 \\ 0 & j/2 \end{bmatrix}, \begin{bmatrix} k/2 & 0 \\ 0 & k/2 \end{bmatrix} \right\rangle; \\ \mathfrak{h}_{4,1} &:= \left\langle \begin{bmatrix} i/2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} j/2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} k/2 & 0 \\ 0 & 0 \end{bmatrix} \right\rangle; \\ \mathfrak{h}_5 &:= \left\langle \begin{bmatrix} i/2 & 0 \\ 0 & 3i/2 \end{bmatrix}, \begin{bmatrix} j & \sqrt{3}/2 \\ -\sqrt{3}/2 & 0 \end{bmatrix}, \begin{bmatrix} k & -\sqrt{3}i/2 \\ -\sqrt{3}i/2 & 0 \end{bmatrix} \right\rangle. \end{aligned} \tag{C.2.7}$$

Note C.2.10. *The subscripts of the above Lie algebras are related to the representations of $\mathfrak{sp}(1)$ they induce under the isomorphism $\mathfrak{sp}(2) \simeq \mathfrak{so}(5)$. The first two Lie algebras give rise to Lie subgroups composed only of isometries. In particular, the former is composed of only isoclinic rotations and the latter only simple rotations.*

Note C.2.11. *As $\mathfrak{sp}(1)$ is simple, all the Lie subgroups associated to the above Lie algebras are compact. In particular, all the Lie subgroups are isomorphic to $\text{Sp}(1)$.*

Proof. Recall that $\text{Sp}(2)$ acts on $S^4 \subseteq \mathbb{R}^5$. By extending the action, the group acts on $\mathbb{R}^5$. This action is induced by the isomorphism $\text{Sp}(2) \simeq \text{Spin}(5)$, meaning $\mathfrak{sp}(2) \simeq \mathfrak{so}(5)$. If $\mathfrak{g} \subseteq \mathfrak{sp}(2)$ is a Lie algebra isomorphic to $\mathfrak{sp}(1)$, then the isomorphism induces a real 5-representation of $\mathfrak{sp}(1)$.

Recall from Section D.1.3 that irreducible real representations $(\mathbb{R}^k, \varrho_k)$ only exist in odd dimensions as well as dimensions divisible by four. Moreover, all irreducible real representations of the same dimension are isomorphic, that is they are conjugate. The

only five-representations are $(\mathbb{R}^5, \varrho_5)$, $(\mathbb{R}^4, \varrho_4) \oplus (\mathbb{R}, \varrho_1)$, $(\mathbb{R}^3, \varrho_3) \oplus (\mathbb{R}, \varrho_1)^{\oplus 2}$, and $(\mathbb{R}, \varrho_1)^{\oplus 5}$. Note that the final representation is the trivial representation, taking all elements $v \in \mathfrak{sp}(1)$ to zero. As the real 5-representation we have comes from a Lie algebra isomorphism, the representation must be injective. Thus, it cannot be the final representation. As we only care about Lie algebras up to isomorphic representations, it now only remains to show that the Lie algebras in the statement correspond to the real representations above.

Consider the generators of the Chevalley bases of $\mathfrak{so}(5)$ and $\mathfrak{sp}(2)$ given in Appendix F. As these Lie algebras are isomorphic, mapping the Chevalley basis of $\mathfrak{so}(5)$ to that of $\mathfrak{sp}(2)$ and vice versa provides an isomorphism between the Lie algebras. In particular, this isomorphism maps one set of generators to another. However, we note that the generators in Appendix F are written with respect to the Cartan matrices

$\begin{bmatrix} 2 & -1 \\ -2 & 2 \end{bmatrix}$ for $\mathfrak{so}(5)$ and $\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix}$ for $\mathfrak{sp}(2)$. Thus, a Lie algebra isomorphism must swap the order of the generators. Hence,

$$\begin{aligned} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \in \mathfrak{sp}(2) &\leftrightarrow \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathfrak{so}(5); \\ \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix} \in \mathfrak{sp}(2) &\leftrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathfrak{so}(5). \end{aligned} \tag{C.2.8}$$

A representation (V, ρ) of $\mathfrak{sp}(1)$ is determined by $Y_1 := \rho(i/2)$, $Y_2 := \rho(j/2)$, and $Y_3 := \rho(k/2)$. Furthermore, the decomposition of the representation is determined by the eigenvalues of $Y_1 \otimes i$. In each Lie algebra in the statement, Y_1 is the first matrix listed. Using (C.2.8), we view $Y_1 \in \mathfrak{so}(5)$ in order to see its eigenvalues. Note that the eigenvalues of $Y_1 \otimes i$ for $(\mathbb{R}^k, \varrho_k)$ with k odd are $\{\frac{k-1}{2}, \frac{k-3}{2}, \dots, -\frac{k-1}{2}\}$, each with multiplicity one. When k is divisible by four, the eigenvalues of $Y_1 \otimes i$ for $(\mathbb{R}^k, \varrho_k)$ are $\{\frac{k-2}{4}, \frac{k-6}{4}, \dots, -\frac{k-2}{4}\}$, each with multiplicity two.

In the case $\mathfrak{h}_{3,1,1}$, we see that $Y_1 \otimes i$ is given by

$$Y_1 \otimes i = \begin{bmatrix} 0 & i & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (\text{C.2.9})$$

The eigenvalues of $Y_1 \otimes i$ in this case are ± 1 , with multiplicity one each, and 0 with multiplicity three. Thus, the representation is $(\mathbb{R}^3, \varrho_3) \oplus (\mathbb{R}, \varrho_1)^{\oplus 2}$.

In the case $\mathfrak{h}_{4,1}$, we see that $Y_1 \otimes i$ is given by

$$Y_1 \otimes i = \frac{1}{2} \begin{bmatrix} 0 & i & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i & 0 \\ 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (\text{C.2.10})$$

The eigenvalues of $Y_1 \otimes i$ in this case are $\pm 1/2$, with multiplicity two each, and 0 with multiplicity one. Thus, the representation is $(\mathbb{R}^4, \varrho_4) \oplus (\mathbb{R}, \varrho_1)$.

Finally, in the case $\mathfrak{h}_5$, we see that $Y_1 \otimes i$ is given by

$$Y_1 \otimes i = \begin{bmatrix} 0 & 2i & 0 & 0 & 0 \\ -2i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (\text{C.2.11})$$

The eigenvalues of $Y_1 \otimes i$ in this case are ± 2 , ± 1 , and 0, each with multiplicity one. Thus, the representation is $(\mathbb{R}^5, \varrho_5)$.

Suppose that $G \subseteq \text{Sp}(2)$ is a connected Lie group with Lie algebra $\mathfrak{sp}(1) \simeq \mathfrak{g} \subseteq \mathfrak{sp}(2)$. Then $\mathfrak{g}$ gives a representation of $\mathfrak{sp}(1)$ isomorphic to one of $\mathfrak{h}_{3,1,1}$, $\mathfrak{h}_{4,1}$, or $\mathfrak{h}_5$. As isomorphic representations are conjugate, we have that G is conjugate to the unique connected Lie group with the corresponding Lie algebra. $\square$

With these Lie subalgebras of $\mathfrak{sp}(2)$ isomorphic to $\mathfrak{sp}(1)$ in mind, we continue with the remaining Lie subgroups. Next we examine subgroups with Lie algebra $\mathfrak{sp}(1) \oplus \mathbb{R}$.

Proposition C.2.12. *The Lie subgroups of $\mathrm{Sp}(2)$ with Lie algebra $\mathfrak{sp}(1) \oplus \mathbb{R}$ are conjugate to one of the unique connected Lie subgroups with the following Lie algebras:*

$$\begin{aligned} \mathfrak{p}_{3,1,1} &:= \left\langle \begin{bmatrix} i/2 & 0 \\ 0 & i/2 \end{bmatrix}, \begin{bmatrix} j/2 & 0 \\ 0 & j/2 \end{bmatrix}, \begin{bmatrix} k/2 & 0 \\ 0 & k/2 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle; \\ \mathfrak{p}_{4,1} &:= \left\langle \begin{bmatrix} i/2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} j/2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} k/2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix} \right\rangle. \end{aligned} \quad (\text{C.2.12})$$

Note C.2.13. *Both of the Lie subgroups associated to the above Lie algebras are compact. In particular, the Lie subgroup associated with $\mathfrak{p}_{4,1}$ is isomorphic to $\mathrm{Sp}(1) \times S^1$. The Lie subgroup associated with $\mathfrak{p}_{3,1,1}$ is isomorphic to $(\mathrm{Sp}(1) \times S^1)/\{\pm(1,1)\}$, which is studied in Appendix G.*

Proof. Let $\mathfrak{g} \subseteq \mathfrak{sp}(2)$ be a Lie subalgebra isomorphic to $\mathfrak{sp}(1) \oplus \mathbb{R}$. Thus, $\mathfrak{g} \simeq \mathfrak{g}_0 \oplus \mathfrak{g}'$, where $\mathfrak{g}_0 \subseteq \mathfrak{sp}(2)$ is isomorphic to $\mathfrak{sp}(1)$ and $\mathfrak{g}' \subseteq \mathfrak{sp}(2)$ is isomorphic to $\mathbb{R}$. Additionally, $\mathfrak{g}_0$ and $\mathfrak{g}'$ must commute. We know from Proposition C.2.9, we know that $\mathfrak{g}_0$ is conjugate to one of $\mathfrak{h}_{3,1,1}$, $\mathfrak{h}_{4,1}$, or $\mathfrak{h}_5$. Let $\mathfrak{g}'$ be generated by $x \in \mathfrak{sp}(2)$. Then x must commute with all of $\mathfrak{g}_0$.

Suppose that $\mathfrak{g}_0$ is conjugate to $\mathfrak{h}_5$. One can show that the only element of $\mathfrak{sp}(2)$ commuting with $\mathfrak{h}_5$ is zero. But then $\mathfrak{g} \simeq \mathfrak{sp}(1)$, contradiction! Thus, $\mathfrak{g}_0$ cannot be isomorphic to $\mathfrak{h}_5$.

Suppose that $\mathfrak{g}_0$ is conjugate to $\mathfrak{h}_{3,1,1}$. One can show that the only elements commuting with $\mathfrak{h}_{3,1,1}$ are real multiples of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. As choosing any non-zero multiple induces the same Lie algebra, we see that we get $\mathfrak{p}_{3,1,1}$ as in the statement.

Finally, suppose that $\mathfrak{g}_0$ is conjugate to $\mathfrak{h}_{4,1}$. One can show that the only elements commuting with $\mathfrak{h}_{4,1}$ are given by $\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}$, for some $v \in \mathfrak{sp}(1)$. Up to conjugation, we may assume that v is a real multiple of i . As the choice of non-zero multiple induces the same Lie algebra, we see that we get $\mathfrak{p}_{4,1}$ as in the statement.

Just as in Proposition C.2.9, when dealing with connected Lie subgroups up to conjugacy, it is enough to investigate Lie subalgebras up to conjugacy. $\square$

We finally consider the subgroups with Lie algebra $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$, as the $\mathfrak{sp}(2)$ case is trivial. Indeed, only $\mathrm{Sp}(2)$ has a Lie algebra $\mathfrak{sp}(2)$. Note that $\mathrm{Sp}(2)$ is compact.

Proposition C.2.14. *The Lie subgroups of $\mathrm{Sp}(2)$ with Lie algebra $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ are conjugate to $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \subseteq \mathrm{Sp}(2)$, the subgroup comprised of diagonal matrices.*

Note C.2.15. *The subgroup $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ is compact.*

Proof. Let $\mathfrak{g} \subseteq \mathfrak{sp}(2)$ be a Lie subalgebra isomorphic to $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. Thus, $\mathfrak{g} \simeq \mathfrak{g}_1 \oplus \mathfrak{g}_2$, where both $\mathfrak{g}_1 \subseteq \mathfrak{sp}(2)$ and $\mathfrak{g}_2 \subseteq \mathfrak{sp}(2)$ are isomorphic to $\mathfrak{sp}(1)$. Additionally, $\mathfrak{g}_1$ and $\mathfrak{g}_2$ must commute. We know from Proposition C.2.12 that no non-zero elements of $\mathfrak{sp}(2)$ commute with $\mathfrak{h}_5$. Hence, neither $\mathfrak{g}_1$ nor $\mathfrak{g}_2$ are conjugate to $\mathfrak{h}_5$.

Suppose that $\mathfrak{g}_1$ is conjugate to $\mathfrak{h}_{3,1,1}$. We know that the only elements of $\mathfrak{sp}(2)$ commuting with $\mathfrak{h}_{3,1,1}$ are the real multiples of $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Thus, $\mathfrak{g}_2$ must be contained in the span of this matrix. However, $\mathfrak{g}_2$ is three-dimensional, contradiction!

Therefore, $\mathfrak{g}_1$ is conjugate to $\mathfrak{h}_{4,1}$. We know that the only elements of $\mathfrak{sp}(2)$ commuting with $\mathfrak{h}_{4,1}$ are the elements of the form $\begin{bmatrix} 0 & 0 \\ 0 & v \end{bmatrix}$ for $v \in \mathfrak{sp}(1)$. This space is three-dimensional, so $\mathfrak{g}_2$ is exactly this space. Thus, we have that $\mathfrak{g}$ is exactly given by

$$\mathfrak{g} = \left\{ \begin{bmatrix} v_1 & 0 \\ 0 & v_2 \end{bmatrix} \mid v_1, v_2 \in \mathfrak{sp}(1) \right\}. \quad (\text{C.2.13})$$

This space is exactly the Lie algebra of $\mathrm{Sp}(1) \times \mathrm{Sp}(1) \subseteq \mathrm{Sp}(2)$, the subgroup comprised of diagonal matrices.

Just as in the preceding propositions, when dealing with connected Lie subgroups up to conjugacy, it is enough to investigate Lie subalgebras up to conjugacy. $\square$

Table C.1 summarizes the Lie subgroups mentioned in the preceding propositions in addition to $\mathrm{Sp}(2)$, lists their corresponding Lie algebras, and names the corresponding symmetries. The preceding propositions tell us that when studying instantons equivariant under continuous subgroups of $\mathrm{Sp}(2)$, it suffices to study these subgroups.

G	Name of symmetry	Lie algebra
R_t , for $t \in [0, 1]$	Circular t	$\langle \text{diag}(i, it) \rangle$
$S^1 \times S^1$	Toral	$\langle \text{diag}(i, 0), \text{diag}(0, i) \rangle$
$\text{Sp}(1)$	Simple spherical	$\mathfrak{h}_{3,1,1}$
	Isoclinic spherical	$\mathfrak{h}_{4,1}$
	Conformal spherical	$\mathfrak{h}_5$
$(S^1 \times \text{Sp}(1))/\{\pm(1, 1)\}$	Conformal superspherical	$\mathfrak{p}_{3,1,1}$
$S^1 \times \text{Sp}(1)$	Isoclinic superspherical	$\mathfrak{p}_{4,1}$
$\text{Sp}(1) \times \text{Sp}(1)$	Rotational	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$
$\text{Sp}(2)$	Full	$\mathfrak{sp}(2)$

Table C.1: List of connected Lie subgroups G of $\text{Sp}(2)$, of non-zero dimension, with corresponding Lie algebras needed to classify all symmetric instantons. For $\mathfrak{h}_{3,1,1}$, $\mathfrak{h}_{4,1}$, and $\mathfrak{h}_5$, see Proposition C.2.9. For $\mathfrak{p}_{3,1,1}$ and $\mathfrak{p}_{4,1}$, see Proposition C.2.12.

Appendix D

Representation theory

In this appendix, we outline some lesser-known results from the representation theory of Lie algebras. We investigate what real and quaternionic representations are and how they relate to complex ones. Additionally, we examine the relationship between representations and their duals. Finally, we look at representations of direct sums of Lie algebras, classifying representations of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$.

D.1 Real and Quaternionic Representations

In this section, we follow Bröcker and Dieck, adapting results for compact Lie groups to Lie algebras, examining the relationships between real, complex, and quaternionic representations [BTD85, Ch. II, § 6].

D.1.1 Different types of representations

In this section, we define what it means to be a real, complex, or quaternionic representation as well as functors between them. We then examine the relationships between these functors.

Definition D.1.1. A *(complex) representation* of a Lie algebra $\mathfrak{g}$ is a pair (V, ρ) , where V is a complex vector space and $\rho: \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is a Lie algebra homomorphism. That is, ρ is linear and for all $x, y \in \mathfrak{g}$, ρ satisfies $\rho([x, y]) = [\rho(x), \rho(y)]$. If V is a real (resp. quaternionic) vector space and ρ is $\mathbb{R}$ -linear (resp. $\mathbb{H}$ -linear), then (V, ρ) is called a **real** (resp. **quaternionic**) *representation*.

Definition D.1.2. Let (V, ρ) be a representation of $\mathfrak{g}$. We say that an inner product $\langle \cdot, \cdot \rangle$ on V is $\mathfrak{g}$ -invariant if

$$\langle \rho(x)v, w \rangle + \langle v, \rho(x)w \rangle = 0.$$

Proposition D.1.3. If $\mathfrak{g}$ is the Lie algebra of a compact Lie group and (V, ρ) a representation of $\mathfrak{g}$, then there is a $\mathfrak{g}$ -invariant inner product on V .

Proof. This result is true because all representations of compact Lie groups have a G -invariant inner product on V . Thus, we consider the associated Lie algebra representation and find that the inner product is $\mathfrak{g}$ -invariant. However, we need the Lie group to be simply-connected for all representations of $\mathfrak{g}$ to arise this way. If $\mathfrak{g}$ is the Lie algebra of a compact Lie group, then it is the Lie algebra of the connected component of this Lie group containing the identity. Furthermore, it is the Lie algebra of the universal cover of this connected, compact Lie group, which is a compact, simply-connected Lie group. $\square$

Based on the above result, we only consider Lie algebras that are the Lie algebra of a compact Lie group.

Definition D.1.4. As we always have a $\mathfrak{g}$ -invariant inner product, we consider **unitary**, **orthogonal**, and **symplectic** representations (V, ρ) , where the image of ρ is contained in $\mathfrak{su}(k)$, $\mathfrak{so}(k)$, and $\mathfrak{sp}(k)$, respectively. For such Lie algebras, one can show that every complex, real, and quaternionic representation is a direct sum of irreducible representations; such representations are semi-simple.

Definition D.1.5. Let (V, ρ) be a complex representation. A **real (resp. quaternionic) structure** on (V, ρ) is a conjugate-linear equivariant map $J: V \rightarrow V$ such that $J^2 = \text{id}_V$ (resp. $J^2 = -\text{id}_V$). Explicitly, a conjugate-linear equivariant map is one where for all $v, w \in V$, $\alpha \in \mathbb{C}$, and $x \in \mathfrak{g}$, we have

$$J(v + w) = J(v) + J(w), \quad J(\alpha v) = \bar{\alpha}J(v), \quad J(\rho(x)v) = \rho(x)J(v).$$

A complex representation (V, ρ) is said to be of **real (resp. quaternionic) type** if it admits a real (resp. quaternionic) structure.

Note that a representation can be both quaternionic and real type. For example, consider $(\mathbb{H}, 0)$. Using $\mathbb{H} \simeq \mathbb{C}^2$, we take $v \mapsto \bar{v}$ and $v \mapsto \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \bar{v}$ to get a real and quaternionic structure, respectively.

We define some categories of representations and functors between them.

Definition D.1.6. Let $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and let $\text{Rep}(\mathfrak{g}, K)$ be the category whose objects are representations on K vector spaces and whose morphisms are K -linear $\mathfrak{g}$ equivariant maps.

Let $\text{Rep}_+(\mathfrak{g}, \mathbb{C})$ and $\text{Rep}_-(\mathfrak{g}, \mathbb{C})$ be the categories of complex representations with real and quaternionic structures, respectively. Morphisms in these categories are $\mathbb{C}$ -linear equivariant maps which commute with the structure maps.

Proposition D.1.7. The categories $\text{Rep}(\mathfrak{g}, \mathbb{R})$ and $\text{Rep}_+(\mathfrak{g}, \mathbb{C})$ are equivalent. That is, real representations are just complex representations with additional structures. Similarly, $\text{Rep}(\mathfrak{g}, \mathbb{H})$ and $\text{Rep}_-(\mathfrak{g}, \mathbb{C})$ are equivalent categories.

Proof. We construct functors $e_+ : \text{Rep}(\mathfrak{g}, \mathbb{R}) \rightarrow \text{Rep}_+(\mathfrak{g}, \mathbb{C})$ and $s_+ : \text{Rep}_+(\mathfrak{g}, \mathbb{C}) \rightarrow \text{Rep}(\mathfrak{g}, \mathbb{R})$ as follows. Given a real representation (U, ρ) , let $e_+(U, \rho) := (\mathbb{C} \otimes_{\mathbb{R}} U, \text{id}_{\mathbb{C}} \otimes \rho, J)$ with structure map $J(\alpha \otimes u) := \bar{\alpha} \otimes u$. Given a representation (V, ρ, J) , let $V_{\pm}$ be the ± 1 -eigenspaces of J . As $2v = (v + J(v)) + (v - J(v))$, we have $V = V_+ \oplus V_-$. Note that multiplication by i induces $V_+ \simeq V_-$. Let $s_+(V, \rho, J) := (V_+, \rho)$, where ρ is restricted to act on V_+ . Note that $\rho(x)|_{V_+} : V_+ \rightarrow V_+$. Indeed, if $v \in V_+$, then $J(v) = v$, so

$$\rho(x)v = \rho(x)J(v) = J(\rho(x)v),$$

so $\rho(x)v \in V_+$.

The compositions $e_+ \circ s_+$ and $s_+ \circ e_+$ are naturally equivalent to the identity. Indeed,

$$e_+ \circ s_+(V, \rho, \tilde{J}) = e_+(V_+, \rho) = (\mathbb{C} \otimes_{\mathbb{R}} V_+, \text{id}_{\mathbb{C}} \otimes \rho, J).$$

But this representation is equivalent to $(V, \rho, \tilde{J})$ via the linear isomorphism $\phi : \mathbb{C} \otimes_{\mathbb{R}} V_+ \rightarrow V$ taking $\phi(\alpha \otimes v) := \alpha v$. Note that

$$\begin{aligned} \phi((\text{id}_{\mathbb{C}} \otimes \rho)(x)(\alpha \otimes v)) &= \phi(\alpha \otimes \rho(x)v) = \alpha \rho(x)v = \rho(x)\phi(\alpha \otimes v); \\ \phi(J(\alpha \otimes v)) &= \phi(\bar{\alpha} \otimes v) = \bar{\alpha}v = \tilde{J}(\alpha v) = \tilde{J}(\phi(\alpha \otimes v)). \end{aligned}$$

Above we use that $v \in V_+$. Similarly,

$$s_+ \circ e_+(U, \rho) = s_+(\mathbb{C} \otimes_{\mathbb{R}} U, \text{id}_{\mathbb{C}} \otimes \rho, J) = ((\mathbb{C} \otimes_{\mathbb{R}} U)_+, \text{id}_{\mathbb{C}} \otimes \rho).$$

As $J(\alpha \otimes u) = \bar{\alpha} \otimes u$, $(\mathbb{C} \otimes_{\mathbb{R}} U)_+ = \{1 \otimes u \mid u \in U\}$. This representation is equivalent to (U, ρ) via the linear isomorphism $\psi : (\mathbb{C} \otimes_{\mathbb{R}} U)_+ \rightarrow U$ taking $\psi(1 \otimes u) := u$. Note that

$$\psi((\text{id}_{\mathbb{C}} \otimes \rho)(x)(1 \otimes u)) = \psi(1 \otimes \rho(x)u) = \rho(x)u = \rho(x)\psi(1 \otimes u).$$

We construct functors $e_-: \text{Rep}(\mathfrak{g}, \mathbb{H}) \rightarrow \text{Rep}_-(\mathfrak{g}, \mathbb{C})$ and $s_-: \text{Rep}_-(\mathfrak{g}, \mathbb{C}) \rightarrow \text{Rep}(\mathfrak{g}, \mathbb{H})$ as follows. Let (W, ρ) be a quaternionic representation. Choose a $\mathbb{H}$ -basis $\{e_1, \dots, e_n\}$ of W . Let $V_1 := \text{span}_{\mathbb{C}}(e_1, \dots, e_n)$ and $V_2 := \text{span}_{\mathbb{C}}(je_1, \dots, je_n)$. As a complex vector space, W is isomorphic to $V_1 \oplus V_2$. Let $\rho_{1ij}, \rho_{2ij} \in \mathbb{C}$ such that $\rho(x)e_i = \rho_{1ij}e_j + \rho_{2ij}je_j$. As ρ is $\mathbb{H}$ -linear, we have that $\rho(x)je_i = j\rho(x)e_i = \overline{\rho_{1ij}}je_j - \overline{\rho_{2ij}}e_j$. Written in the $\mathbb{C}$ -basis, $\rho(x)$ acts on $V_1 \oplus V_2$ as $\tilde{\rho}(x) := \begin{bmatrix} \rho_1 & -\overline{\rho_2} \\ \rho_2 & \overline{\rho_1} \end{bmatrix}$. Moreover, we can define J to be the action of j on $V_1 \oplus V_2$. Explicitly, $J: V_1 \oplus V_2 \rightarrow V_1 \oplus V_2$ takes $J(\alpha_i e_i, \beta_j je_j) := (-\overline{\beta_j} e_j, \overline{\alpha_i} je_i)$. Note that J is conjugate-linear and equivariant, satisfying $J^2 = -\text{id}_{V_1 \oplus V_2}$. Indeed, we see

$$\begin{aligned} J(\tilde{\rho}(x)(e_i, 0)) &= J(\rho_{1ij}e_j, \rho_{2ij}je_j) = (-\overline{\rho_{2ij}}e_j, \overline{\rho_{1ij}}je_j) = \tilde{\rho}(x)J(e_i, 0), \\ J(\tilde{\rho}(x)(0, je_i)) &= J(-\overline{\rho_{2ij}}e_j, \overline{\rho_{1ij}}je_j) = (-\rho_{1ij}e_i, -\rho_{2ij}je_j) = \tilde{\rho}(x)J(0, je_i). \end{aligned}$$

Then $e_-(W, \rho) := (V_1 \oplus V_2, \tilde{\rho}, J)$. Given a representation (V, ρ, J) , define $f: \mathbb{H} \times V \rightarrow V$ as follows. Given $p \in \mathbb{H}$, we can write it uniquely as $p = \alpha + \beta j$ for $\alpha, \beta \in \mathbb{C}$. Then let $p \cdot v \equiv f(p, v) := \alpha v + \beta J(v)$. Because J is conjugate-linear and $J^2 = -\text{id}_V$, we have that this gives V the structure of a quaternionic vector space. Note that using this quaternionic scalar multiplication, we have that as J is equivariant and $\rho(x)$ is $\mathbb{C}$ -linear,

$$\rho(x)p \cdot v = \rho(x)(\alpha v + \beta J(v)) = \alpha \rho(x)v + \beta J(\rho(x)v) = p \cdot \rho(x)v.$$

Thus, $\rho(x)$ is $\mathbb{H}$ -linear. Then let $s_-(V, \rho, J) := (V, \rho)$, where we view V as a quaternionic vector space and $\rho(x)$ as $\mathbb{H}$ -linear.

The compositions $e_- \circ s_-$ and $s_- \circ e_-$ are naturally equivalent to the identity. Indeed,

$$s_- \circ e_-(W, \rho) = s_-(V_1 \oplus V_2, \tilde{\rho}, J) = (V_1 \oplus V_2, \tilde{\rho}).$$

This representation is equivalent to (W, ρ) via the $\mathbb{H}$ -linear isomorphism $\phi: W \rightarrow V_1 \oplus V_2$ sending basis vectors to $\phi(e_i) := (e_i, 0)$. Note that $\phi(je_i) = j \cdot \phi(e_i) = J(e_i, 0) = (0, e_i)$. Hence, ϕ is indeed bijective and

$$\phi(pe_i) = p \cdot \phi(e_i) = (\alpha e_i, \beta e_i).$$

We see that

$$\tilde{\rho}(x)\phi(e_i) = \begin{bmatrix} \rho_1 & -\overline{\rho_2} \\ \rho_2 & \overline{\rho_1} \end{bmatrix} \begin{bmatrix} e_i \\ 0 \end{bmatrix} = (\rho_{1ij}e_j, \rho_{2ij}je_j) = \phi(\rho(x)e_i).$$

Similarly, we have

$$e_- \circ s_-(V, \rho, \tilde{J}) = e_-(V, \rho) = (V_1 \oplus V_2, \tilde{\rho}, J).$$

This representation is equivalent to $(V, \rho, \tilde{J})$ via the $\mathbb{C}$ -linear isomorphism $\phi: V \rightarrow V_1 \oplus V_2$ given as follows. Let $\{e_1, \dots, e_n\}$ be a $\mathbb{H}$ -basis for V . Then $\{e_1, \dots, e_n, \tilde{J}(e_1), \dots, \tilde{J}(e_n)\}$ is a $\mathbb{C}$ -basis for V . Let ϕ send basis vectors to $\phi(e_i) := (e_i, 0)$ and $\phi(\tilde{J}(e_i)) := (0, j \cdot e_i)$. We see that ϕ is bijective. Recall that ρ_{1ij} and ρ_{2ij} were defined such that $\rho(x)e_i = \rho_{1ij}e_j + \rho_{2ij}j \cdot e_j$. Note that $j \cdot e_j = \tilde{J}(e_j)$. Then we see

$$\begin{aligned}\tilde{\rho}(x)\phi(e_i) &= (\rho_{1ij}e_j, \rho_{2ij}j \cdot e_j) = \phi(\rho_{1ij}e_j + \rho_{2ij}\tilde{J}(e_j)) = \phi(\rho(x)e_i), \\ \tilde{\rho}(x)\phi(\tilde{J}(e_i)) &= (-\overline{\rho_{2ij}}e_j, \overline{\rho_{1ij}}j \cdot e_j) = \phi(\tilde{J}(\rho(x)e_i)) = \phi(\rho(x)\tilde{J}(e_i)), \\ J(\phi(e_i)) &= J(e_i, 0) = (0, j \cdot e_i) = \phi(\tilde{J}(e_i)), \\ J(\phi(\tilde{J}(e_i))) &= J(0, j \cdot e_i) = (-e_i, 0) = \phi(\tilde{J}(\tilde{J}(e_i))).\end{aligned}\quad \square$$

Note D.1.8. If (V, ρ, J) is a complex representation of quaternionic type, then $\dim_{\mathbb{C}}(V)$ is even. In Proposition D.1.7, we see that V has the structure of a $\mathbb{H}$ -vector space. Thus, we can find a $\mathbb{H}$ -basis $\{e_1, \dots, e_n\}$, which gives us a $\mathbb{C}$ -basis $\{e_1, \dots, e_n, je_1, \dots, je_n\}$. Thus, $\dim_{\mathbb{C}}(V) = 2n$.

We can also define functors between different types of representations.

Definition D.1.9. Forgetting about various parts of the representations, we have restriction maps

$$\begin{aligned}r_{\mathbb{R}}^{\mathbb{C}} &: \text{Rep}(\mathfrak{g}, \mathbb{C}) \rightarrow \text{Rep}(\mathfrak{g}, \mathbb{R}), \\ r_{\mathbb{C}}^{\mathbb{H}} &: \text{Rep}(\mathfrak{g}, \mathbb{H}) \rightarrow \text{Rep}(\mathfrak{g}, \mathbb{C}), \\ r_{\mathbb{R}}^{\mathbb{H}} &: \text{Rep}(\mathfrak{g}, \mathbb{H}) \rightarrow \text{Rep}(\mathfrak{g}, \mathbb{R}), \quad r_{\mathbb{R}}^{\mathbb{H}} = r_{\mathbb{R}}^{\mathbb{C}} \circ r_{\mathbb{C}}^{\mathbb{H}}, \\ r_+ &: \text{Rep}_+(\mathfrak{g}, \mathbb{C}) \rightarrow \text{Rep}(\mathfrak{g}, \mathbb{C}), \\ r_- &: \text{Rep}_-(\mathfrak{g}, \mathbb{C}) \rightarrow \text{Rep}(\mathfrak{g}, \mathbb{C}),\end{aligned}$$

defined as follows. Given a complex representation (V, ρ) , we may regard V as a real vector space and use the same ρ . Explicitly, this means to choose a $\mathbb{C}$ -basis $\{e_1, \dots, e_n\}$ for V and let $U_1 := \text{span}_{\mathbb{R}}(e_1, \dots, e_n)$ and $U_2 := \text{span}_{\mathbb{R}}(ie_1, \dots, ie_n)$. As a real vector space, V is isomorphic to $U_1 \oplus U_2$. Let $\rho_{1ij}, \rho_{2ij} \in \mathbb{R}$ such that $\rho(x)e_i = \rho_{1ij}e_j + \rho_{2ij}ie_j$. As $\rho(x)$ is $\mathbb{C}$ -linear, we have $\rho(x)ie_i = i\rho(x)e_i = -\rho_{2ij}e_j + \rho_{1ij}ie_j$. Written in the $\mathbb{R}$ -basis, $\rho(x)$ acts

on $U_1 \oplus U_2$ as $\tilde{\rho}(x) := \begin{bmatrix} \rho_1 & -\rho_2 \\ \rho_2 & \rho_1 \end{bmatrix}$. The functor $r_{\mathbb{R}}^{\mathbb{C}}$ takes (V, ρ) to $(U_1 \oplus U_2, \tilde{\rho}(x))$. Similarly,

we get $r_{\mathbb{C}}^{\mathbb{H}}$ and $r_{\mathbb{R}}^{\mathbb{H}}$. Additionally, given a representation of real or quaternionic type, we can forget about the structure map, giving us a complex representation. These are the functors $r_{\pm}$. Explicitly, the map $r_{\mathbb{C}}^{\mathbb{H}} = r_- \circ e_-$ and $r_{\mathbb{R}}^{\mathbb{H}} = r_{\mathbb{R}}^{\mathbb{C}} \circ r_{\mathbb{C}}^{\mathbb{H}}$.

We also have extension maps, obtained by extending the scalars on the vector space

$$\begin{aligned} e_{\mathbb{R}}^{\mathbb{C}} &: \text{Rep}(\mathfrak{g}, \mathbb{R}) \rightarrow \text{Rep}(\mathfrak{g}, \mathbb{C}), \\ e_{\mathbb{C}}^{\mathbb{H}} &: \text{Rep}(\mathfrak{g}, \mathbb{C}) \rightarrow \text{Rep}(\mathfrak{g}, \mathbb{H}), \\ e_{\mathbb{R}}^{\mathbb{H}} &: \text{Rep}(\mathfrak{g}, \mathbb{R}) \rightarrow \text{Rep}(\mathfrak{g}, \mathbb{H}), \quad e_{\mathbb{R}}^{\mathbb{H}} = e_{\mathbb{C}}^{\mathbb{H}} \circ e_{\mathbb{R}}^{\mathbb{C}}, \end{aligned}$$

defined as follows. Given a real representation (U, ρ) , we take $r_+ \circ e_+(U, \rho)$. The resulting representation is $e_{\mathbb{R}}^{\mathbb{C}}(U, \rho)$. We view $\mathbb{H}$ as a right $\mathbb{C}$ -module via right multiplication $\mathbb{H} \times \mathbb{C} \rightarrow \mathbb{H}$ taking $(w, v) \mapsto wv$. Then, given a complex representation (V, ρ) , we have $e_{\mathbb{C}}^{\mathbb{H}}(V, \rho) := (\mathbb{H} \otimes_{\mathbb{C}} V, \text{id}_{\mathbb{H}} \otimes \rho)$. Given a real representation (U, ρ) , we can do the same as above, taking $e_{\mathbb{R}}^{\mathbb{H}}(U, \rho) := (\mathbb{H} \otimes_{\mathbb{R}} U, \text{id}_{\mathbb{H}} \otimes \rho)$. The result is the same as the composition of the two previous maps.

Finally, we have a conjugation map,

$$c: \text{Rep}(\mathfrak{g}, \mathbb{C}) \rightarrow \text{Rep}(\mathfrak{g}, \mathbb{C}),$$

defined as follows. Let (V, ρ) be a complex representation. Let $\bar{V}$ be the vector space with the same additive structure as V but scalar multiplication given by $\alpha \cdot v := \bar{\alpha}v$. Then $\rho(x)$ acts on $\bar{V}$ as $\bar{\rho}(x)$. Let $\bar{\rho}(x) := \overline{\rho(x)}$ and $c(V, \rho) := (\bar{V}, \bar{\rho})$.

Proposition D.1.10. *The above functors satisfy the following relations*

$$\begin{aligned} r_{\mathbb{R}}^{\mathbb{C}} \circ e_{\mathbb{R}}^{\mathbb{C}}(U, \rho) &\simeq (U, \rho)^{\oplus 2}, & e_{\mathbb{R}}^{\mathbb{C}} \circ r_{\mathbb{R}}^{\mathbb{C}}(V, \rho) &\simeq (V, \rho) \oplus c(V, \rho), \\ r_{\mathbb{C}}^{\mathbb{H}} \circ e_{\mathbb{C}}^{\mathbb{H}}(V, \rho) &\simeq (V, \rho) \oplus c(V, \rho), & e_{\mathbb{C}}^{\mathbb{H}} \circ r_{\mathbb{C}}^{\mathbb{H}}(W, \rho) &\simeq (W, \rho)^{\oplus 2}, \\ c \circ e_{\mathbb{R}}^{\mathbb{C}}(U, \rho) &\simeq e_{\mathbb{R}}^{\mathbb{C}}(U, \rho), & r_{\mathbb{R}}^{\mathbb{C}} \circ c(V, \rho) &\simeq r_{\mathbb{R}}^{\mathbb{C}}(V, \rho), \\ c \circ r_{\mathbb{C}}^{\mathbb{H}}(W, \rho) &\simeq r_{\mathbb{C}}^{\mathbb{H}}(W, \rho), & e_{\mathbb{C}}^{\mathbb{H}} \circ c(V, \rho) &\simeq e_{\mathbb{C}}^{\mathbb{H}}(V, \rho), \\ r_+ \circ e_+(U, \rho) &\simeq e_{\mathbb{R}}^{\mathbb{C}}(U, \rho), & r_- \circ e_-(W, \rho) &\simeq r_{\mathbb{C}}^{\mathbb{H}}(W, \rho), \\ & & c^2(V, \rho) &\simeq (V, \rho). \end{aligned}$$

Proof. The first equivalency is obtained via the $\mathbb{R}$ -linear isomorphism $\phi: \mathbb{R} \otimes_{\mathbb{R}} U \oplus i\mathbb{R} \otimes_{\mathbb{R}} U \rightarrow U \oplus U$ taking $\phi(a \otimes u, ib \otimes v) := (au, bv)$. We see that as $\rho(x)$ maps from a real vector space to itself,

$$\begin{aligned} \phi \left(\begin{bmatrix} 1 \otimes \rho(x) & 0 \\ 0 & 1 \otimes \rho(x) \end{bmatrix} \begin{bmatrix} a \otimes u \\ ib \otimes v \end{bmatrix} \right) &= (a\rho(x)u, ib\rho(x)v) \\ &= (\rho \oplus \rho)(x)\phi(a \otimes u, ib \otimes v). \end{aligned}$$

The second equivalency is obtained via the $\mathbb{C}$ -linear isomorphism $\phi: V \oplus \bar{V} \rightarrow \mathbb{C} \otimes_{\mathbb{R}} (U_1 \oplus U_2)$ sending the basis elements of $V \oplus \bar{V}$ to

$$\begin{aligned}\phi(e_i, 0) &:= \frac{1 \otimes (e_i, 0) - i \otimes (0, ie_i)}{2}, \\ \phi(0, e_i) &:= \frac{1 \otimes (e_i, 0) + i \otimes (0, ie_i)}{2}.\end{aligned}$$

We see that

$$\begin{aligned}1 \otimes \begin{bmatrix} \rho_1 & -\rho_2 \\ \rho_2 & \rho_1 \end{bmatrix} \phi(e_i, 0) &= \frac{1 \otimes (\rho_{1ij}e_j, \rho_{2ij}ie_j) - i \otimes (-\rho_{2ij}e_j, \rho_{1ij}ie_j)}{2} = \phi((\rho \oplus \bar{\rho})(x)(e_i, 0)), \\ 1 \otimes \begin{bmatrix} \rho_1 & -\rho_2 \\ \rho_2 & \rho_1 \end{bmatrix} \phi(0, e_i) &= \frac{1 \otimes (\rho_{1ij}e_j, \rho_{2ij}ie_j) + i \otimes (-\rho_{2ij}e_j, \rho_{1ij}ie_j)}{2} = \phi((\rho \oplus \bar{\rho})(x)(0, e_i)).\end{aligned}$$

Note that in the final equality, we have that ϕ is $\mathbb{C}$ -linear with respect to multiplication on $V \oplus \bar{V}$, so $\phi(0, \rho(x)_{ij}e_j) = \rho(x)_{ij}\phi(0, e_i)$.

The third equivalency is obtained via the $\mathbb{C}$ -linear isomorphism $\phi: (\mathbb{C} \otimes_{\mathbb{C}} V) \oplus (j\mathbb{C} \otimes_{\mathbb{C}} V) \rightarrow V \oplus \bar{V}$ taking

$$\begin{aligned}\phi(1 \otimes e_i, 0) &:= (e_i, 0), \\ \phi(0, j \otimes e_i) &:= (0, e_i).\end{aligned}$$

We see

$$\begin{aligned}\begin{bmatrix} \rho(x) & 0 \\ 0 & \overline{\rho(x)} \end{bmatrix} \cdot \phi(1 \otimes e_i, 0) &= (\rho(x)e_i, 0) = \rho(x)_{ij}\phi(1 \otimes e_j, 0) \\ &= \phi\left(\begin{bmatrix} 1 \otimes \rho(x) & 0 \\ 0 & 1 \otimes \overline{\rho(x)} \end{bmatrix} \begin{bmatrix} 1 \otimes e_i \\ 0 \end{bmatrix}\right), \\ \begin{bmatrix} \rho(x) & 0 \\ 0 & \overline{\rho(x)} \end{bmatrix} \cdot \phi(0, j \otimes e_i) &= (0, \rho(x)e_i) = \overline{\rho(x)_{ij}} \cdot (0, e_j) \\ &= \phi\left(\begin{bmatrix} 1 \otimes \rho(x) & 0 \\ 0 & 1 \otimes \overline{\rho(x)} \end{bmatrix} \begin{bmatrix} 0 & j \otimes e_i \end{bmatrix}\right).\end{aligned}$$

The fourth equivalency is obtained via the $\mathbb{H}$ -linear isomorphism $\phi: W \oplus W \rightarrow \mathbb{H} \otimes_{\mathbb{C}} V$

$(V_1 \oplus V_2)$ sending the basis elements of $W \oplus W$ to

$$\begin{aligned}\phi(e_i, 0) &:= \frac{1 \otimes (e_i, 0) - j \otimes (0, je_i)}{2}, \\ \phi(0, e_i) &:= \frac{i \otimes (e_i, 0) + k \otimes (0, je_i)}{2}.\end{aligned}$$

We see that

$$\begin{aligned}1 \otimes \begin{bmatrix} \rho_1 & -\overline{\rho_2} \\ \rho_2 & \overline{\rho_1} \end{bmatrix} \phi(e_i, 0) &= \frac{1 \otimes (\rho_{1ij}e_j, \rho_{2ij}je_j) - j \otimes (-\overline{\rho_{2ij}}e_j, \overline{\rho_{1ij}}je_j)}{2} \\ &= \rho(x)_{ij} \frac{1 \otimes (e_j, 0) - j \otimes (0, je_j)}{2} = \phi((\rho \oplus \rho)(x)(e_i, 0)), \\ 1 \otimes \begin{bmatrix} \rho_1 & -\overline{\rho_2} \\ \rho_2 & \overline{\rho_1} \end{bmatrix} \phi(0, e_i) &= \frac{i \otimes (\rho_{1ij}e_j, \rho_{2ij}je_j) + k \otimes (-\overline{\rho_{2ij}}e_j, \overline{\rho_{1ij}}je_j)}{2} \\ &= \rho(x)_{ij} \frac{i \otimes (e_j, 0) + k \otimes (0, je_j)}{2} = \phi((\rho \oplus \rho)(x)(0, e_i)).\end{aligned}$$

Note that we use that $\mathbb{H} \otimes_{\mathbb{C}} (V_1 \oplus V_2)$ is a $\mathbb{H}$ -vector space, where we multiply on the $\mathbb{H}$ factor on the left. That is, $p(q \otimes (v, w)) = (pq) \otimes (v, w)$. Additionally, $\mathbb{H}$ is a right $\mathbb{C}$ -module, so if we have $\alpha \in \mathbb{C}$, we have $(q\alpha) \otimes (v, w) = q \otimes \alpha(v, w)$. Thus, we have $\alpha(j \otimes (v, w)) = (\alpha j) \otimes (v, w) = j \otimes \overline{\alpha}(v, w)$.

The fifth equivalency is obtained via the $\mathbb{C}$ -linear isomorphism $\phi: \mathbb{C} \otimes_{\mathbb{R}} U \rightarrow \overline{\mathbb{C} \otimes_{\mathbb{R}} U}$ taking $\phi(\alpha \otimes u) := \overline{\alpha} \otimes u$. Indeed, we see

$$\phi(i\alpha \otimes u) = -i\phi(\alpha \otimes u) = i \cdot \phi(\alpha \otimes u).$$

Furthermore, we see that

$$\overline{1 \otimes \rho(x)} \cdot \phi(\alpha \otimes u) = \overline{\alpha} \otimes \rho(x)u = \phi((1 \otimes \rho(x))(\alpha \otimes u)).$$

The sixth equivalency is obtained via the $\mathbb{R}$ -linear isomorphism $\phi: U \oplus iU \rightarrow U \oplus iU$ taking $\phi(u, iv) := (u, -iv)$. Let $\rho(x) = \rho_1 + i\rho_2$. Then $\overline{\rho}(x) = \rho_1 - i\rho_2$. Thus, the representation on the first $U \oplus iU$ is $\begin{bmatrix} \rho_1 & -\rho_2 \\ \rho_2 & \rho_1 \end{bmatrix}$ as it is constructed from ρ and the representation on the second one is $\begin{bmatrix} \rho_1 & \rho_2 \\ -\rho_2 & \rho_1 \end{bmatrix}$, as it is constructed from $\overline{\rho}$. Then we see

$$\begin{bmatrix} \rho_1 & \rho_2 \\ -\rho_2 & \rho_1 \end{bmatrix} \phi(u, iv) = (\rho_1 u - \rho_2 v, -i\rho_2 u - i\rho_1 v) = \phi\left(\begin{bmatrix} \rho_1 & -\rho_2 \\ \rho_2 & \rho_1 \end{bmatrix} \begin{bmatrix} u \\ iv \end{bmatrix}\right).$$

The seventh equivalency is obtained via the $\mathbb{C}$ -linear map $\phi: (\mathbb{C} \otimes_{\mathbb{C}} V) \oplus (j\mathbb{C} \otimes_{\mathbb{C}} V) \rightarrow \overline{\mathbb{C} \otimes_{\mathbb{C}} V} \oplus \overline{j\mathbb{C} \otimes_{\mathbb{C}} V}$ taking the basis vectors to

$$\phi(1 \otimes e_i, 0) := (0, -j \otimes e_i), \quad \text{and} \quad \phi(0, j \otimes e_i) := (1 \otimes e_i, 0).$$

We see

$$\begin{aligned} \begin{bmatrix} 1 \otimes \overline{\rho_1(x)} & -1 \otimes \rho_2(x) \\ 1 \otimes \overline{\rho_2(x)} & 1 \otimes \rho_1(x) \end{bmatrix} \cdot \phi(1 \otimes e_i, 0) &= \rho_{1ij} \cdot (0, -j \otimes e_j) + \rho_{2ij} \cdot (1 \otimes e_j, 0) \\ &= \phi \left(\begin{bmatrix} 1 \otimes \rho_1(x) & -1 \otimes \overline{\rho_2(x)} \\ 1 \otimes \rho_2(x) & 1 \otimes \overline{\rho_1(x)} \end{bmatrix} \begin{bmatrix} 1 \otimes e_i \\ 0 \end{bmatrix} \right), \\ \begin{bmatrix} 1 \otimes \overline{\rho_1(x)} & -1 \otimes \rho_2(x) \\ 1 \otimes \overline{\rho_2(x)} & 1 \otimes \rho_1(x) \end{bmatrix} \cdot \phi(0, j \otimes e_i) &= \overline{\rho_{1ij}} \cdot (1 \otimes e_j, 0) + \overline{\rho_{2ij}} \cdot (0, j \otimes e_j) \\ &= \phi \left(\begin{bmatrix} 1 \otimes \rho_1(x) & -1 \otimes \overline{\rho_2(x)} \\ 1 \otimes \rho_2(x) & 1 \otimes \overline{\rho_1(x)} \end{bmatrix} \begin{bmatrix} 0 \\ j \otimes e_i \end{bmatrix} \right). \end{aligned}$$

The eighth equivalency is obtained via the $\mathbb{H}$ -linear isomorphism $\phi: \mathbb{H} \otimes_{\mathbb{C}} V \rightarrow \mathbb{H} \otimes_{\mathbb{C}} \overline{V}$ taking basis vectors $\phi(1 \otimes e_i) := j \otimes e_i$. We see

$$(1 \otimes \overline{\rho(x)}) \cdot \phi(1 \otimes e_i) = j \otimes \overline{\rho(x)}_{ij} e_j = (\rho(x)_{ij} j) \otimes e_j = \phi((1 \otimes \rho(x))(1 \otimes e_i)).$$

The ninth and tenth equivalencies are just by definition.

The eleventh equivalency is obtained via the $\mathbb{C}$ -linear isomorphism $\phi: V \rightarrow \overline{\overline{V}}$ taking $\phi(v) := v$. We see that

$$\rho(x) \cdot \phi(v) = \rho(x)v = \phi(\rho(x)v). \quad \square$$

D.1.2 Irreducible representations of various types

In this section, we uncover the relationship between irreducible real, quaternionic, and complex representations. The main result is a classification of irreducible representations.

Definition D.1.11. Let $\text{Irr}(\mathfrak{g}, K)$ for $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ be the sets of irreducible representations over K . For each $L = \mathbb{R}, \mathbb{C}, \mathbb{H}$, we define a subset $\text{Irr}(\mathfrak{g}, K)_L \subseteq \text{Irr}(\mathfrak{g}, K)$. The definitions of these nine subsets are given in Table D.1.

$L =$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$(U, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{R})_L$	$e_{\mathbb{R}}^{\mathbb{C}}(U, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{R}}$	$(U, \rho) = r_{\mathbb{R}}^{\mathbb{C}}(V, \lambda),$ $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{C}}$	$(U, \rho) = r_{\mathbb{R}}^{\mathbb{C}}(V, \lambda),$ $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{H}}$
$(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_L$	$(V, \rho) \in \text{Rep}_+(\mathfrak{g}, \mathbb{C})$	$(V, \rho) \not\sim (\bar{V}, \bar{\rho})$	$(V, \rho) \in \text{Rep}_-(\mathfrak{g}, \mathbb{C})$
$(W, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{H})_L$	$(W, \rho) = e_{\mathbb{C}}^{\mathbb{H}}(V, \lambda),$ $(V, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{R}}$	$(W, \rho) = e_{\mathbb{C}}^{\mathbb{H}}(V, \lambda),$ $(V, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{C}}$	$r_{\mathbb{C}}^{\mathbb{H}}(W, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{H}}$

Table D.1: This table outlines the various types of irreducible representations of $\mathfrak{g}$. In words, $(U, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{R})_{\mathbb{R}}$ if and only if $e_{\mathbb{R}}^{\mathbb{C}}(U, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{R}}$, which happens if and only if this representation is irreducible and of real type. The other definitions are similar.

We call an irreducible representation (whether real, complex, or quaternionic) in a set with subscript $\mathbb{R}, \mathbb{C}, \mathbb{H}$ of **real, complex, or quaternionic type**, respectively. Thus, we extend our use of type from just complex representations.

We show the following

Theorem D.1.12. For $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$, we have that $\text{Irr}(\mathfrak{g}, K)$ is the disjoint union of its subsets $\text{Irr}(\mathfrak{g}, K)_{\mathbb{R}}, \text{Irr}(\mathfrak{g}, K)_{\mathbb{C}}, \text{Irr}(\mathfrak{g}, K)_{\mathbb{H}}$. Note that some of these subsets can be empty.

First, we need some lemmas to simplify our proof.

Lemma D.1.13. A complex representation (V, ρ) is of real (resp. quaternionic) type if and only if there exists a nonsingular symmetric (resp. skew-symmetric) $\mathfrak{g}$ -invariant bilinear form $B: V \times V \rightarrow \mathbb{C}$.

Proof. Suppose that such a $B: V \times V \rightarrow \mathbb{C}$ is given. Then $B(v, w) = \epsilon B(w, v)$ for $\epsilon = \pm 1$. Choose a $\mathfrak{g}$ -invariant Hermitian inner product $\langle \cdot, \cdot \rangle$ on V and define $f: V \rightarrow V$ by requiring $B(v, w) = \langle v, f(w) \rangle$ for all $v \in V$. Hence, f is conjugate-linear. Indeed, for all $v \in V$,

$$\langle v, \bar{\alpha} f(w) - f(\alpha w) \rangle = \alpha B(v, w) - B(v, \alpha w) = 0.$$

Hence, $f(\alpha w) = \bar{\alpha} f(w)$. Furthermore, we see that as $B(\rho(x)v, w) = -B(v, \rho(x)w)$ and $\langle \rho(x)v, w \rangle = -\langle v, \rho(x)w \rangle$,

$$\begin{aligned} \langle v, \rho(x)f(w) \rangle &= -\langle \rho(x)v, f(w) \rangle = -B(\rho(x)v, w) \\ &= B(v, \rho(x)w) = \langle v, f(\rho(x)w) \rangle. \end{aligned}$$

Thus, $f(\rho(x)w) = \rho(x)f(w)$, so f is equivariant. Finally, suppose $f(w) = 0$. Then for all $v \in V$,

$$B(v, w) = \langle v, f(w) \rangle = 0.$$

Thus, $w = 0$, so f is an isomorphism (between (V, ρ) and $(\bar{V}, \bar{\rho})$).

We see that

$$\langle v, f(w) \rangle = B(v, w) = \epsilon B(w, v) = \epsilon \langle w, f(v) \rangle = \epsilon \overline{\langle f(v), w \rangle}.$$

Hence,

$$\langle v, f^2(w) \rangle = \epsilon \overline{\langle f(v), f(w) \rangle} = \epsilon^2 \langle f^2(v), w \rangle = \langle f^2(v), w \rangle.$$

Therefore, we see that ϵf^2 is Hermitian and positive-definite. Indeed,

$$\langle v, \epsilon f^2(v) \rangle = \epsilon^2 \overline{\langle f(v), f(v) \rangle} \geq 0. \quad (\text{D.1.1})$$

If $\langle v, \epsilon f^2(v) \rangle = 0$, then we see that $f(v) = 0$, so $v = 0$. Therefore, V can be decomposed into the direct sum of eigenspaces V_λ of ϵf^2 . As ϵf^2 is positive-definite and Hermitian, we have all the eigenvalues are positive real numbers.

Let $v \in V_\lambda$. Then

$$\begin{aligned} \epsilon f^2(\rho(x)v) &= \rho(x)\epsilon f^2(v) = \lambda \rho(x)v, \\ \epsilon f^2(f(v)) &= f(\epsilon f^2(v)) = f(\lambda v) = \lambda f(v). \end{aligned}$$

Thus, the eigenspaces are $\mathfrak{g}$ -invariant and invariant under f . Define $h: V \rightarrow V$ by taking $v \in V_\lambda$ to $h(v) := \sqrt{\lambda}v$ and extend linearly (take a basis of each eigenspace to get a basis on V). We see that for all $v \in V_\lambda$, as $f(v) \in V_\lambda$ as well,

$$h(f(v)) = \sqrt{\lambda}f(v) = f(\sqrt{\lambda}v) = f(h(v)).$$

Furthermore, we see that for all $v \in V_\lambda$, as $\rho(x)v \in V_\lambda$ as well,

$$h(\rho(x)v) = \sqrt{\lambda}\rho(x)v = \rho(x)h(v).$$

Hence, h is equivariant. Additionally, for all $v \in V_\lambda$,

$$h^2(v) = \lambda v = \epsilon f^2(v).$$

Therefore, $h^2 = \epsilon f^2$.

Define $J: V \rightarrow V$ by $J := h \circ f^{-1}$. As f is conjugate-linear and h linear, we have J is conjugate-linear. Furthermore, we see from above that

$$J(\rho(x)v) = h(f^{-1}(\rho(x)v)) = h(\rho(x)f^{-1}(v)) = \rho(x)J(v).$$

Thus, J is equivariant. Finally, we see that

$$J^2 = h \circ f^{-1} \circ h \circ f^{-1} = h^2 \circ f^{-2} = \epsilon \cdot \text{id}_V.$$

Therefore, if $\epsilon = 1$, then we have a real structure and if $\epsilon = -1$, then we have a quaternionic structure.

Conversely, suppose V has a structure map $J: V \rightarrow V$ such that $J^2 = \epsilon \cdot \text{id}_V$. If $\epsilon = 1$, then $V \simeq \mathbb{C} \otimes_{\mathbb{R}} V_+$, where V_+ is the $+1$ -eigenspace of J . Any non-singular, $\mathfrak{g}$ -invariant, symmetric, $\mathbb{R}$ -bilinear form on V_+ can be extended to a $\mathbb{C}$ -bilinear form B on V , which is still non-singular, $\mathfrak{g}$ -invariant, and symmetric.

If $\epsilon = -1$, then consider V as a $\mathbb{H}$ -module, where the action of j being that of J . Then V carries a $\mathfrak{g}$ -invariant, symplectic inner product $\langle \cdot, \cdot \rangle$. Write

$$\langle u, v \rangle = H(u, v) + B(u, v)j,$$

where $H(u, v), B(u, v) \in \mathbb{C}$. As $\lambda \langle u, v \rangle = \langle \lambda u, v \rangle$ and $\langle u, \lambda v \rangle = \langle u, v \rangle \bar{\lambda}$ for $\lambda \in \mathbb{H}$, we see that for $\alpha \in \mathbb{C}$,

$$\begin{aligned} H(u, \alpha v + w) + B(u, \alpha v + w)j &= \langle u, \alpha v + w \rangle = \langle u, w \rangle + \langle u, v \rangle \bar{\alpha} \\ &= H(u, w) + H(u, v)\bar{\alpha} + B(u, w)j + B(u, v)j\bar{\alpha}. \end{aligned}$$

Noting that $j\bar{\alpha} = \alpha j$ and

$$\begin{aligned} H(\alpha u + v, w) + B(\alpha u + v, w)j &= \langle \alpha u + v, w \rangle = \alpha \langle u, w \rangle + \langle v, w \rangle \\ &= \alpha H(u, w) + H(v, w) + \alpha B(u, w)j + B(v, w)j, \end{aligned}$$

we see that B is $\mathbb{C}$ -bilinear.

As $\langle u, v \rangle = \overline{\langle v, u \rangle}$, we have

$$\begin{aligned} H(u, v) + B(u, v)j &= \langle u, v \rangle = \overline{\langle v, u \rangle} = \overline{H(v, u) + B(v, u)j} \\ &= \overline{H(v, u)} - B(v, u)j. \end{aligned}$$

Hence, B is skew-symmetric.

Suppose $B(u, v) = 0$ for all $u \in V$. Then $\langle u, v \rangle \in \mathbb{C}$ for all $u \in V$. As,

$$\langle ju, v \rangle = j\langle u, v \rangle,$$

we see that $\langle u, v \rangle = 0$ for all $u \in V$. As symplectic inner products are non-singular, we have $v = 0$, so B is non-singular. Finally, we see that

$$\begin{aligned} H(\rho(x)u, v) + B(\rho(x)u, v)j &= \langle \rho(x)u, v \rangle = -\langle u, \rho(x)v \rangle \\ &= -H(u, \rho(x)v) - B(u, \rho(x)v)j. \end{aligned}$$

Thus, B is $\mathfrak{g}$ -invariant. □

Definition D.1.14. We say a complex representation is **self-dual** if $(V, \rho) \simeq (\bar{V}, \bar{\rho})$.

Definition D.1.15. Suppose $\mathfrak{g}$ is a semi-simple Lie algebra with representation (V, ρ) . The following representation (V^*, ρ^*) is known as the **dual representation**. Here $V^* := \text{Hom}(V, \mathbb{C})$ and $\rho^*(x)f(v) := -f(\rho(x)v)$.

Lemma D.1.16. The pair (V^*, ρ^*) is indeed a representation.

Proof. We see that

$$\begin{aligned} ([\rho^*(x), \rho^*(y)]f)(v) &= -\rho^*(y)f(\rho(x)v) + \rho^*(x)f(\rho(y)v) = f([\rho(y), \rho(x)]v) \\ &= -f(\rho([x, y])v) = \rho^*([x, y])f(v) \end{aligned}$$

Therefore, we have a representation. □

Recall the definition of the representation $(\bar{V}, \bar{\rho})$ given in Definition [D.1.9](#).

Lemma D.1.17. Given an $\mathfrak{g}$ -invariant inner product on V , we have that $(V^*, \rho^*) \simeq (\bar{V}, \bar{\rho})$.

Proof. Consider $\phi: \bar{V} \rightarrow V^*$ taking $\phi(v) \mapsto \langle \cdot, v \rangle$. This map is a $\mathbb{C}$ -linear isomorphism. Furthermore, we see that

$$\begin{aligned} \phi(\bar{\rho}(x) \cdot v)(w) &= \phi(\rho(x)v)(w) = \langle w, \rho(x)v \rangle = -\langle \rho(x)w, v \rangle \\ &= -\phi(v)(\rho(x)w) = \rho^*(x)\phi(v)(w). \end{aligned}$$
□

Lemma D.1.18. Given a representation (V, ρ) , $((V^*)^*, (\rho^*)^*)$ is isomorphic to (V, ρ) .

Proof. Let $\{v_1, \dots, v_n\}$ be a basis for V and $\langle \cdot, \cdot \rangle$ an inner product on V . Construct $f: V \rightarrow (V^*)^*$ taking $v \mapsto (v^*)^*$. We see that for all $x \in \mathfrak{g}$ and $\phi \in V^*$ we have

$$\begin{aligned} (\rho^*)^*(x)f(v)(\phi) &= (\rho^*)^*(x)(v^*)^*(\phi) = -(v^*)^*(\rho^*(x)\phi) \\ &= -(\rho^*(x)\phi)(v) = \phi(\rho(x)v) = f(\rho(x)v)(\phi). \end{aligned}$$

Thus, the representations are isomorphic. $\square$

Lemma D.1.19. *Let $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})$ be self-dual. Then it is of real or quaternionic type, but not both.*

Proof. Let $B: V \rightarrow V^* \simeq \overline{V}$ give an equivalence $(V, \rho) \simeq (\overline{V}, \bar{\rho}) \simeq (V^*, \rho^*)$. That is, B is equivariant and a linear isomorphism. We can think of B as a non-singular, bilinear form on V . Consider $B^\pm := B \pm B^T$, where $B^T(u, v) := B(v, u)$. At least one of these is non-zero, or else $B = 0$.

Notice that

$$\begin{aligned} B^+(v, u) &= B(v, u) + B(u, v) = B^+(u, v), \\ B^-(v, u) &= B(v, u) - B(u, v) = -B^-(u, v). \end{aligned}$$

Thus, B^+ is symmetric and B^- is skew-symmetric. Furthermore, as B is equivariant, we have that $B(\rho(x)u) = \rho^*(x)B(u)$. Evaluating at v , we see

$$B(\rho(x)u, v) = -B(u, \rho(x)v).$$

Thus, B is $\mathfrak{g}$ -invariant.

Suppose B^+ is non-zero. Then we have a nonsingular, symmetric, bilinear, $\mathfrak{g}$ -invariant inner product on V . By Lemma D.1.13, (V, ρ) is of real type.

Suppose B^- is non-zero. Then we have a nonsingular, skew-symmetric, bilinear, $\mathfrak{g}$ -invariant inner product on V . By Lemma D.1.13, (V, ρ) is of quaternionic type.

Suppose (V, ρ) is of both types. Then there exists nonsingular, bilinear, $\mathfrak{g}$ -invariant inner products A, C on V , that are symmetric and skew-symmetric, respectively. Let $f: V \rightarrow V^*$ and $g: V \rightarrow V^*$ be given by $f(u)(v) := A(u, v)$ and $g(u)(v) := C(u, v)$. As A and C are nonsingular and bilinear, f and g are isomorphisms. Furthermore, we see

$$f(\rho(x)u)(v) = A(\rho(x)u, v) = -A(u, \rho(x)v) = \rho^*(x)f(u)(v).$$

Thus, f is equivariant, and g is similarly. By Schur's Lemma, any two isomorphisms are non-zero multiples of each other, so $A = \alpha C$ for some $\alpha \neq 0$. Hence, A and C are both symmetric and skew-symmetric, so they are both zero, contradiction! $\square$

We are now able to prove Theorem D.1.12.

Proof of Theorem D.1.12. We divide the proof into three parts, one for each $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Let $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})$. If the representation is of real or quaternionic type, then $J: V \rightarrow V$ is a structure map with $J^2 = \pm \text{id}_V$. In either case, J is conjugate-linear, so thinking of it as a map $V \rightarrow \bar{V}$, we see that it is linear. Furthermore, we have that J is equivariant, so

$$J(\rho(x)v) = \rho(x)J(v) = \bar{\rho}(x) \cdot J(v).$$

As $J^2 = \pm \text{id}$, we have that it is bijective. Therefore, as a map $V \rightarrow \bar{V}$, J is an isomorphism and it provides an equivalence $(V, \rho) \simeq (\bar{V}, \bar{\rho})$. Hence, if the representation is of real or quaternionic type, it cannot be of complex type.

Suppose (V, ρ) is self-dual. By Lemma D.1.19, we see that (V, ρ) is of real or quaternionic type, but not both. Otherwise, (V, ρ) is not self-dual and is of complex type.

Let $(U, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{R})$. Denote $e := e_{\mathbb{R}}^{\mathbb{C}}$ and $r := r_{\mathbb{R}}^{\mathbb{C}}$. As $e = r_+ \circ e_+$, we have $e(U, \rho)$ is of real type, as there is a real structure on $e(U, \rho)$. As such, if $e(U, \rho)$ is irreducible, then $e(U, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{R}}$, so $(U, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{R})_{\mathbb{R}}$. If $e(U, \rho)$ is not irreducible, let us decompose it in terms of irreducible summands as $e(U, \rho) = \bigoplus_{j=1}^t (V_j, \rho_j)$, where $(V_j, \rho_j) \in \text{Irr}(\mathfrak{g}, \mathbb{C})$ and $t \geq 2$. Then

$$(U, \rho) \oplus (U, \rho) = r \circ e(U, \rho) = \bigoplus_{j=1}^t r(V_j, \rho_j).$$

As $t \geq 2$ and (U, ρ) is irreducible, we have $t = 2$ and $r(V_j, \rho_j) = (U, \rho)$ for $j = 1, 2$. Then for $j = 1, 2$

$$(V_1, \rho_1) \oplus (V_2, \rho_2) = e(U, \rho) = e \circ r(V_j, \rho_j) = (V_j, \rho_j) \oplus (\bar{V}_j, \bar{\rho}_j).$$

Thus, we must have $(\bar{V}_1, \bar{\rho}_1) \simeq (V_2, \rho_2)$ and vice versa.

If (V_1, ρ_1) is not self-dual, then we see $(U, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{R})_{\mathbb{C}}$. Otherwise, we have a self-dual, irreducible, complex representation. Hence, (V_1, ρ_1) is of either real or quaternionic type. If it is of real type, then $(V_1, \rho_1) \in \text{Rep}_+(\mathfrak{g}, \mathbb{C})$ so $(X, \lambda) := s_+(V_1, \rho_1) \in \text{Rep}(\mathfrak{g}, \mathbb{R})$. But then $e_+(X, \lambda) = e_+ \circ s_+(V_1, \rho_1) \simeq (V_1, \rho_1)$, so

$$e(X, \lambda) = r_+ \circ e_+(X, \lambda) \simeq r_+(V_1, \rho_1) = (V_1, \rho_1).$$

Thus,

$$(U, \rho) = r(V_1, \rho_1) \simeq r \circ e(X, \lambda) = (X, \lambda) \oplus (X, \lambda).$$

But (U, ρ) is irreducible, contradiction! Thus, (V_1, ρ_1) is of quaternionic type and $(U, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{R})_{\mathbb{H}}$.

Note that (U, ρ) cannot be in the intersection of any two of these sets. If it is the restriction of an irreducible complex representation, that representation cannot be both not self-dual and of quaternionic type. Furthermore, if (U, ρ) is the restriction of an irreducible complex representation (V, λ) , then $e(U, \rho) = (V, \lambda) \oplus (\overline{V}, \overline{\lambda})$, which is not irreducible.

Finally, let $(W, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{H})$. Let $r := r_{\mathbb{C}}^{\mathbb{H}}$ and $e := e_{\mathbb{C}}^{\mathbb{H}}$. As $r = r_- \circ e_-$, $r(W, \rho)$ has a quaternionic structure. As such, if $r(W, \rho)$ is irreducible, then $r(W, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{H}}$, so $(W, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{H})_{\mathbb{H}}$. If $r(W, \rho)$ is not irreducible, let us decompose it in terms of irreducible summands as $r(W, \rho) = \bigoplus_{j=1}^t (V_j, \rho_j)$, where $(V_j, \rho_j) \in \text{Irr}(\mathfrak{g}, \mathbb{C})$ and $t \geq 2$. Then

$$(W, \rho) \oplus (W, \rho) = e \circ r(W, \rho) = \bigoplus_{j=1}^t e(V_j, \rho_j).$$

As $t \geq 2$ and (W, ρ) is irreducible, we have $t = 2$ and $e(V_j, \rho_j) = (W, \rho)$ for $j = 1, 2$. Then for $j = 1, 2$

$$(V_1, \rho_1) \oplus (V_2, \rho_2) = r(W, \rho) = r \circ e(V_j, \rho_j) = (V_j, \rho_j) \oplus (\overline{V_j}, \overline{\rho_j}).$$

Thus, we must have $(\overline{V_1}, \overline{\rho_1}) \simeq (V_2, \rho_2)$ and vice versa.

If (V_1, ρ_1) is not self-dual, then we see that $(W, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{H})_{\mathbb{C}}$. Otherwise, we have a self-dual, irreducible, complex representation. Hence, (V_1, ρ_1) is of either real or quaternionic type. If it is of quaternionic type, then $(V_1, \rho_1) \in \text{Rep}_-(\mathfrak{g}, \mathbb{C})$, so $(X, \lambda) := s_-(V_1, \rho_1) \in \text{Rep}(\mathfrak{g}, \mathbb{H})$. But then $e_-(X, \lambda) = e_- \circ s_-(V_1, \rho_1) \simeq (V_1, \rho_1)$, so

$$r(X, \lambda) = r_- \circ e_-(X, \lambda) \simeq r_-(V_1, \rho_1) = (V_1, \rho_1).$$

Thus,

$$(W, \rho) = e(V_1, \rho_1) \simeq e \circ r(X, \lambda) = (X, \lambda) \oplus (X, \lambda).$$

But (W, ρ) is irreducible, contradiction! Therefore, (V_1, ρ_1) is of real type and $(W, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{H})_{\mathbb{R}}$.

Note that (W, ρ) cannot be in the intersection of any two of these sets. If it is the extension of an irreducible complex representation, that representation cannot be both not self-dual and of real type. Furthermore, if (W, ρ) is the extension of an irreducible complex representation (V, λ) , then $r(W, \rho) = (V, \lambda) \oplus (\overline{V}, \overline{\lambda})$, which is not irreducible. $\square$

Proposition D.1.20. *We have the following relationships between irreducible representations.*

- (1) $(U, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{R})_{\mathbb{R}} \Rightarrow e_{\mathbb{R}}^{\mathbb{C}}(U, \rho) = (V, \lambda), \quad \text{for } (V, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{R}},$
- (2) $(U, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{R})_{\mathbb{C}} \Rightarrow e_{\mathbb{R}}^{\mathbb{C}}(U, \rho) = (V, \lambda) \oplus (\bar{V}, \bar{\lambda}), \quad \text{for } (V, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{C}},$
- (3) $(U, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{R})_{\mathbb{H}} \Rightarrow e_{\mathbb{R}}^{\mathbb{C}}(U, \rho) = (V, \lambda) \oplus (V, \lambda), \quad \text{for } (V, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{H}},$
- (4) $(W, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{H})_{\mathbb{R}} \Rightarrow r_{\mathbb{C}}^{\mathbb{H}}(W, \rho) = (V, \lambda) \oplus (V, \lambda), \quad \text{for } (V, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{R}},$
- (5) $(W, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{H})_{\mathbb{C}} \Rightarrow r_{\mathbb{C}}^{\mathbb{H}}(W, \rho) = (V, \lambda) \oplus (\bar{V}, \bar{\lambda}), \quad \text{for } (V, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{C}},$
- (6) $(W, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{H})_{\mathbb{H}} \Rightarrow r_{\mathbb{C}}^{\mathbb{H}}(W, \rho) = (V, \lambda), \quad \text{for } (V, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{H}},$
- (7) $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{R}} \Rightarrow r_{\mathbb{R}}^{\mathbb{C}}(V, \rho) = (U, \lambda) \oplus (U, \lambda), \quad \text{for } (U, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{R})_{\mathbb{R}},$
 $e_{\mathbb{C}}^{\mathbb{H}}(V, \rho) = (W, \lambda), \quad \text{for } (W, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{H})_{\mathbb{R}},$
- (8) $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{C}} \Rightarrow r_{\mathbb{R}}^{\mathbb{C}}(V, \rho) = (U, \lambda) = r_{\mathbb{R}}^{\mathbb{C}}(\bar{V}, \bar{\rho}), \quad \text{for } (U, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{R})_{\mathbb{C}},$
 $e_{\mathbb{C}}^{\mathbb{H}}(V, \rho) = (W, \lambda) = e_{\mathbb{C}}^{\mathbb{H}}(\bar{V}, \bar{\rho}), \quad \text{for } (W, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{H})_{\mathbb{C}},$
- (9) $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{H}} \Rightarrow r_{\mathbb{R}}^{\mathbb{C}}(V, \rho) = (U, \lambda), \quad \text{for } (U, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{R})_{\mathbb{H}},$
 $e_{\mathbb{C}}^{\mathbb{H}}(V, \rho) = (W, \lambda) \oplus (W, \lambda), \quad \text{for } (W, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{H})_{\mathbb{H}}.$

Proof. (1) This is the definition of being in $\text{Irr}(\mathfrak{g}, \mathbb{R})_{\mathbb{R}}$.

(2) By definition, we have $(U, \rho) = r_{\mathbb{R}}^{\mathbb{C}}(V, \lambda)$ for some $(V, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{C}}$. By Proposition D.1.10, $e_{\mathbb{R}}^{\mathbb{C}} \circ r_{\mathbb{R}}^{\mathbb{C}} \simeq 1 \oplus c$.

(3) By definition, we have $(U, \rho) = r_{\mathbb{R}}^{\mathbb{C}}(V, \lambda)$ for some $(V, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{H}}$. By Proposition D.1.10, $e_{\mathbb{R}}^{\mathbb{C}} \circ r_{\mathbb{R}}^{\mathbb{C}} \simeq 1 \oplus c$ and we note that representations of quaternionic type are self-dual.

(4) By definition, we have $(W, \rho) = e_{\mathbb{C}}^{\mathbb{H}}(V, \lambda)$ for some $(V, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{R}}$. By Proposition D.1.10, $r_{\mathbb{C}}^{\mathbb{H}} \circ e_{\mathbb{C}}^{\mathbb{H}} \simeq 1 \oplus c$ and we note that representations of real type are self-dual.

(5) By definition, we have $(W, \rho) = e_{\mathbb{C}}^{\mathbb{H}}(V, \lambda)$ for some $(V, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{C}}$. By Proposition D.1.10, $r_{\mathbb{C}}^{\mathbb{H}} \circ e_{\mathbb{C}}^{\mathbb{H}} \simeq 1 \oplus c$.

(6) This is the definition of being in $\text{Irr}(\mathfrak{g}, \mathbb{H})_{\mathbb{H}}$.

(7) Suppose $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{R}}$. Then it belongs to $\text{Rep}_+(\mathfrak{g}, \mathbb{C})$, so let $(U, \lambda) := s_+(V, \rho) \in \text{Rep}(\mathfrak{g}, \mathbb{R})$. We have that $e_+(U, \lambda) \simeq (V, \rho)$, so

$$e_{\mathbb{R}}^{\mathbb{C}}(U, \lambda) = r_+ \circ e_+(U, \lambda) \simeq r_+(V, \rho) = (V, \rho).$$

Thus,

$$r_{\mathbb{R}}^{\mathbb{C}}(V, \rho) \simeq r_{\mathbb{R}}^{\mathbb{C}} \circ e_{\mathbb{R}}^{\mathbb{C}}(U, \lambda) = (U, \lambda) \oplus (U, \lambda).$$

Let us decompose (U, λ) as $(U, \lambda) = \bigoplus_{j=1}^t (U_j, \lambda_j)$, where $t \geq 1$ and $(U_j, \lambda_j) \in \text{Irr}(\mathfrak{g}, \mathbb{R})$. We see that

$$(V, \rho) \simeq \bigoplus_{j=1}^t e_{\mathbb{R}}^{\mathbb{C}}(U_j, \lambda_j).$$

As (V, ρ) is irreducible, we have $t = 1$, so $(U, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{R})_{\mathbb{R}}$, proving the first part.

Let $(W, \lambda) := e_{\mathbb{C}}^{\mathbb{H}}(V, \rho)$. If we prove that (W, λ) is irreducible, then we have proven the result. Suppose not, then let us decompose it as $(W, \lambda) = \bigoplus_{j=1}^t (W_j, \lambda_j)$, where $t \geq 2$ and $(W_j, \lambda_j) \in \text{Irr}(\mathfrak{g}, \mathbb{H})$. We see that as (V, ρ) is of real type,

$$(V, \rho) \oplus (V, \rho) = r_{\mathbb{C}}^{\mathbb{H}} \circ e_{\mathbb{C}}^{\mathbb{H}}(V, \rho) = \bigoplus_{j=1}^t r_{\mathbb{C}}^{\mathbb{H}}(W_j, \lambda_j).$$

As $t \geq 2$ and (V, ρ) is irreducible, we see that $t = 2$ and $r_{\mathbb{C}}^{\mathbb{H}}(W_j, \lambda_j) = (V, \rho)$ for $j = 1, 2$. We know from above that if (W_j, ρ_j) is of real or complex type, then $(V, \rho) = r_{\mathbb{C}}^{\mathbb{H}}(W_j, \rho_j)$ is not irreducible, but it is. Thus, (W_j, ρ_j) must be of quaternionic type. But then $(V, \rho) = r_{\mathbb{C}}^{\mathbb{H}}(W_j, \rho_j)$ is of real and quaternionic type, which is not possible. Contradiction! Therefore, (W, λ) is irreducible.

- (8) Suppose $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{C}}$. Let $(U, \lambda) := r_{\mathbb{R}}^{\mathbb{C}}(V, \rho)$. If (U, ρ) is irreducible, then we have proven the result, as

$$r_{\mathbb{R}}^{\mathbb{C}}(\overline{V}, \overline{\rho}) = r_{\mathbb{R}}^{\mathbb{C}} \circ c(V, \rho) = r_{\mathbb{R}}^{\mathbb{C}}(V, \rho) = (U, \lambda).$$

Suppose not, then let us decompose it as $(U, \lambda) = \bigoplus_{j=1}^t (U_j, \lambda_j)$, where $t \geq 2$ and $(U_j, \lambda_j) \in \text{Irr}(\mathfrak{g}, \mathbb{R})$. We see that

$$(V, \rho) \oplus (\overline{V}, \overline{\rho}) = e_{\mathbb{R}}^{\mathbb{C}} \circ r_{\mathbb{R}}^{\mathbb{C}}(V, \rho) = \bigoplus_{j=1}^t e_{\mathbb{R}}^{\mathbb{C}}(U_j, \lambda_j).$$

As (V, ρ) and its dual are irreducible (see the Dual Representation document) and $t \geq 2$, we have $t = 2$, so without loss of generality, $e_{\mathbb{R}}^{\mathbb{C}}(U_1, \lambda_1) = (V, \rho)$ and $e_{\mathbb{R}}^{\mathbb{C}}(U_2, \lambda_2) = (\overline{V}, \overline{\rho})$. As (U_1, λ_1) is irreducible, if it is of complex or quaternionic type, then $(V, \rho) = e_{\mathbb{R}}^{\mathbb{C}}(U_1, \rho_1)$ is not irreducible, but it is. Thus, (U_1, ρ_1) must be of real type. But then $(V, \rho) = e_{\mathbb{R}}^{\mathbb{C}}(U_1, \rho_1)$ is of real and complex type, which is not possible. Contradiction! Therefore, (U, λ) is irreducible.

Let $(W, \lambda) := e_{\mathbb{C}}^{\mathbb{H}}(V, \rho)$. If (W, ρ) is irreducible, then we have proven the result, as

$$e_{\mathbb{C}}^{\mathbb{H}}(\overline{V}, \overline{\rho}) = e_{\mathbb{C}}^{\mathbb{H}} \circ c(V, \rho) = e_{\mathbb{C}}^{\mathbb{H}}(V, \rho) = (W, \lambda).$$

Suppose not, then let us decompose it as $(W, \lambda) = \bigoplus_{j=1}^t (W_j, \lambda_j)$, where $t \geq 2$ and $(W_j, \lambda_j) \in \text{Irr}(\mathfrak{g}, \mathbb{H})$. We see that

$$(V, \rho) \oplus (\bar{V}, \bar{\rho}) = r_{\mathbb{C}}^{\mathbb{H}} \circ e_{\mathbb{C}}^{\mathbb{H}}(V, \rho) = \bigoplus_{j=1}^t r_{\mathbb{C}}^{\mathbb{H}}(W_j, \lambda_j).$$

As (V, ρ) and its dual are irreducible (see the Dual Representation document) and $t \geq 2$, we have $t = 2$, so without loss of generality, $r_{\mathbb{C}}^{\mathbb{H}}(W_1, \lambda_1) = (V, \rho)$ and $r_{\mathbb{C}}^{\mathbb{H}}(W_2, \lambda_2) = (\bar{V}, \bar{\rho})$. As (W_1, λ_1) is irreducible, if it is of complex or real type, then $(V, \rho) = r_{\mathbb{C}}^{\mathbb{H}}(W_1, \rho_1)$ is not irreducible, but it is. Thus, (W_1, ρ_1) must be of quaternionic type. But then $(V, \rho) = r_{\mathbb{C}}^{\mathbb{H}}(U_1, \rho_1)$ is of quaternionic and complex type, which is not possible. Contradiction! Therefore, (W, λ) is irreducible.

- (9) Suppose $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{H}}$. Then it belongs to $\text{Rep}_{-}(\mathfrak{g}, \mathbb{C})$, so let $(W, \lambda) := s_{-}(V, \rho) \in \text{Rep}(\mathfrak{g}, \mathbb{H})$. We have that $e_{-}(W, \lambda) \simeq (V, \rho)$, so

$$r_{\mathbb{C}}^{\mathbb{H}}(W, \lambda) = r_{-} \circ e_{-}(W, \lambda) \simeq r_{-}(V, \rho) = (V, \rho).$$

Thus,

$$e_{\mathbb{C}}^{\mathbb{H}}(V, \rho) \simeq e_{\mathbb{C}}^{\mathbb{H}} \circ r_{\mathbb{C}}^{\mathbb{H}}(W, \lambda) = (W, \lambda) \oplus (W, \lambda).$$

Let us decompose (W, λ) as $(W, \lambda) = \bigoplus_{j=1}^t (W_j, \lambda_j)$, where $t \geq 1$ and $(W_j, \lambda_j) \in \text{Irr}(\mathfrak{g}, \mathbb{H})$. We see that

$$(V, \rho) \simeq \bigoplus_{j=1}^t r_{\mathbb{C}}^{\mathbb{H}}(W_j, \lambda_j).$$

As (V, ρ) is irreducible, we have $t = 1$, so $(W, \lambda) \in \text{Irr}(\mathfrak{g}, \mathbb{H})_{\mathbb{H}}$, proving the first part.

Let $(U, \lambda) := r_{\mathbb{R}}^{\mathbb{C}}(V, \rho)$. If we prove that (U, λ) is irreducible, then we have proven the result. Suppose not, then let us decompose it as $(U, \lambda) = \bigoplus_{j=1}^t (U_j, \lambda_j)$, where $t \geq 2$ and $(U_j, \lambda_j) \in \text{Irr}(\mathfrak{g}, \mathbb{R})$. We see that as (V, ρ) is of quaternionic type,

$$(V, \rho) \oplus (V, \rho) = e_{\mathbb{R}}^{\mathbb{C}} \circ r_{\mathbb{R}}^{\mathbb{C}}(V, \rho) = \bigoplus_{j=1}^t e_{\mathbb{R}}^{\mathbb{C}}(U_j, \lambda_j).$$

As $t \geq 2$ and (V, ρ) is irreducible, we see that $t = 2$ and $e_{\mathbb{R}}^{\mathbb{C}}(U_j, \lambda_j) = (V, \rho)$ for $j = 1, 2$. We know from above that if (U_j, ρ_j) is of quaternionic or complex type, then $(V, \rho) = e_{\mathbb{R}}^{\mathbb{C}}(W_j, \rho_j)$ is not irreducible, but it is. Thus, (U_j, ρ_j) must be of real type. But then $(V, \rho) = e_{\mathbb{R}}^{\mathbb{C}}(U_j, \rho_j)$ is of real and quaternionic type, which is not possible. Contradiction! Therefore, (U, λ) is irreducible. $\square$

Corollary D.1.21. *For all $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{R}}$, $(V, \rho) \oplus (\bar{V}, \bar{\rho})$ with $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{C}}$, and $(V, \rho)^{\oplus 2}$ with $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{H}}$, we can choose real matrices (of the same dimension) inducing irreducible real representations whose extensions are the above representations. Furthermore, all irreducible real representations are constructed this way.*

For all $(V, \rho)^{\oplus 2}$ with $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{R}}$, $(V, \rho) \oplus (\bar{V}, \bar{\rho})$ with $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{C}}$, and $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{H}}$, we can choose quaternionic matrices (of half the dimension) that give rise to irreducible real representations whose restrictions give the above representations. Furthermore, all irreducible quaternionic representations are constructed this way.

Proof. If we start with an irreducible real representation (U, ρ) , then, by Proposition D.1.20, its extension is of the form in the statement of the proposition. Similarly, if we start with an irreducible quaternionic representation (W, ρ) , then its restriction is of the form in the statement of the proposition. Thus, it remains to show that for every one of the above representations, we can find an irreducible real or quaternionic representation whose extension or restriction, respectively, gives back the representation.

If we take $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{R}}$, then as this representation is of real type and irreducible, let $(U, \lambda) := s_+(V, \rho)$. Consider

$$e_{\mathbb{R}}^{\mathbb{C}}(U, \lambda) = r_+ \circ e_+ \circ s_+(V, \rho) \simeq r_+(V, \rho) = (V, \rho).$$

Thus, (U, λ) must be irreducible and its extension is (V, ρ) .

If we take $(V, \rho) \oplus (\bar{V}, \bar{\rho})$ with $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{C}}$, then as (V, ρ) is of complex type and irreducible, let $(U, \lambda) := r_{\mathbb{R}}^{\mathbb{C}}(V, \rho)$. By Proposition D.1.20, we have that (U, λ) is irreducible (and of complex type). Furthermore, we have that

$$e_{\mathbb{R}}^{\mathbb{C}}(U, \lambda) = e_{\mathbb{R}}^{\mathbb{C}} \circ r_{\mathbb{R}}^{\mathbb{C}}(V, \rho) = (V, \rho) \oplus (\bar{V}, \bar{\rho}).$$

Hence, we have an irreducible real representation whose extension is $(V, \rho) \oplus (\bar{V}, \bar{\rho})$.

If we take $(V, \rho) \oplus (V, \rho)$ with $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{H}}$, then as (V, ρ) is of quaternionic type and irreducible, let $(U, \lambda) := r_{\mathbb{R}}^{\mathbb{C}}(V, \rho)$. By Proposition D.1.20, we have that (U, λ) is irreducible (and of quaternionic type). Furthermore, we have that

$$e_{\mathbb{R}}^{\mathbb{C}}(U, \lambda) = e_{\mathbb{R}}^{\mathbb{C}} \circ r_{\mathbb{R}}^{\mathbb{C}}(V, \rho) = (V, \rho) \oplus (V, \rho).$$

Hence, we have an irreducible real representation whose extension is $(V, \rho) \oplus (V, \rho)$.

If we take $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{H}}$, then as this representation is of quaternionic type and irreducible, let $(W, \lambda) := s_-(V, \rho)$. Consider

$$r_{\mathbb{C}}^{\mathbb{H}}(W, \lambda) = r_- \circ e_- \circ s_-(V, \rho) \simeq r_-(V, \rho) = (V, \rho).$$

Thus, (W, λ) must be irreducible and its restriction is (V, ρ) .

If we take $(V, \rho) \oplus (\bar{V}, \bar{\rho})$ with $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{C}}$, then as (V, ρ) is of complex type and irreducible, let $(W, \lambda) := e_{\mathbb{C}}^{\mathbb{H}}(V, \rho)$. By Proposition D.1.20, we have that (W, λ) is irreducible (and of complex type). Furthermore, we have that

$$r_{\mathbb{C}}^{\mathbb{H}}(W, \lambda) = r_{\mathbb{C}}^{\mathbb{H}} \circ e_{\mathbb{C}}^{\mathbb{H}}(V, \rho) = (V, \rho) \oplus (\bar{V}, \bar{\rho}).$$

Hence, we have an irreducible quaternionic representation whose restriction is $(V, \rho) \oplus (\bar{V}, \bar{\rho})$.

If we take $(V, \rho) \oplus (V, \rho)$ with $(V, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{C})_{\mathbb{R}}$, then as (V, ρ) is of real type and irreducible, let $(W, \lambda) := e_{\mathbb{C}}^{\mathbb{H}}(V, \rho)$. By Proposition D.1.20, we have that (W, λ) is irreducible (and of real type). Furthermore, we have that

$$r_{\mathbb{C}}^{\mathbb{H}}(W, \lambda) = r_{\mathbb{C}}^{\mathbb{H}} \circ e_{\mathbb{C}}^{\mathbb{H}}(V, \rho) = (V, \rho) \oplus (V, \rho).$$

Hence, we have an irreducible quaternionic representation whose restriction is $(V, \rho) \oplus (V, \rho)$. $\square$

Note that the previous corollary tells us exactly how to get the irreducible real and quaternionic representations from the complex ones.

The next proposition gives us a simple characterization of the sets $\text{Irr}(\mathfrak{g}, \mathbb{R})_L$. It tells us that the endomorphism algebra determines the type of an irreducible real representation.

Theorem D.1.22. *The endomorphism algebra $\text{Hom}_{\mathfrak{g}}(U, U)$ of $(U, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{R})$ is isomorphic to L if and only if $(U, \rho) \in \text{Irr}(\mathfrak{g}, \mathbb{R})_L$.*

Proof. Every $0 \neq \phi \in \text{Hom}_{\mathfrak{g}}(U, U)$ is invertible by Schur's Lemma. Hence, the endomorphism algebra is a division algebra over $\mathbb{R}$, so it is isomorphic to $\mathbb{R}, \mathbb{C}, \mathbb{H}$.

Suppose that $\text{Hom}_{\mathfrak{g}}(U, U) \simeq \mathbb{C}$. Then $\mathbb{C}$ acts on (U, ρ) so $(U, \rho) = r_{\mathbb{R}}^{\mathbb{C}}(V, \lambda)$ for some irreducible (V, λ) (if (V, λ) were not irreducible, then so too would be its restriction, (U, λ)). If (V, ρ) is of real type, then Proposition D.1.20 tells us that $(U, \rho) = r_{\mathbb{R}}^{\mathbb{C}}(V, \lambda)$ is not irreducible, but it is. Thus, (U, ρ) is of complex or quaternionic type. If it were of quaternionic type, then $e_{\mathbb{C}}^{\mathbb{H}}(V, \rho) = (W, \mu) \oplus (W, \mu)$ for some $(W, \mu) \in \text{Irr}(\mathfrak{g}, \mathbb{H})_{\mathbb{H}}$. But then, as (V, λ) is of quaternionic type

$$r_{\mathbb{C}}^{\mathbb{H}}(W, \mu) \oplus r_{\mathbb{C}}^{\mathbb{H}}(W, \mu) = r_{\mathbb{C}}^{\mathbb{H}} \circ e_{\mathbb{C}}^{\mathbb{H}}(V, \lambda) = (V, \lambda) \oplus (V, \lambda).$$

Hence, $(V, \lambda) = r_{\mathbb{C}}^{\mathbb{H}}(W, \mu)$, so $(U, \rho) = r_{\mathbb{R}}^{\mathbb{H}}(W, \mu)$. Thus, $\mathbb{H}$ is contained in $\text{Hom}_{\mathfrak{g}}(U, U)$, which it is not. Thus, (V, λ) is of complex type, so (U, ρ) is too.

Suppose that $\text{Hom}_{\mathfrak{g}}(U, U) \simeq \mathbb{H}$. Then $\mathbb{H}$ acts on (U, ρ) so $(U, \rho) = r_{\mathbb{R}}^{\mathbb{H}}(W, \mu)$, for some (W, μ) irreducible (for the same reason as above). From Proposition D.1.20, we see that as $r_{\mathbb{R}}^{\mathbb{H}} = r_{\mathbb{R}}^{\mathbb{C}} \circ r_{\mathbb{C}}^{\mathbb{H}}$, if (W, μ) is of real or complex type, then $(U, \rho) = r_{\mathbb{R}}^{\mathbb{H}}(W, \mu)$ is not irreducible, but it is. Thus, (W, μ) is of quaternionic type. Hence, $r_{\mathbb{C}}^{\mathbb{H}}(W, \mu)$ is of quaternionic type and irreducible, so $(U, \rho) = r_{\mathbb{R}}^{\mathbb{C}} \circ r_{\mathbb{C}}^{\mathbb{H}}(W, \mu)$ is also of quaternionic type.

Finally, suppose that $\text{Hom}_{\mathfrak{g}}(U, U) \simeq \mathbb{R}$. Then (U, ρ) is not of the form $r_{\mathbb{R}}^{\mathbb{C}}(V, \rho)$ for (V, ρ) irreducible and of complex or quaternionic type, otherwise $\mathbb{C} \subseteq \text{Hom}_{\mathfrak{g}}(U, U)$. Thus, we must have, by Theorem D.1.12, that (U, ρ) is of real type. $\square$

D.1.3 Classifying Representations

In this section, we determine which irreducible, complex representations of $\mathfrak{sp}(1)$ are of real, complex, and quaternionic type. We find that this Lie algebra admits no complex type representations, so all their representations are self-dual, agreeing with Proposition D.2.9.

The irreducible, complex representation (V_n, ρ_n) of $\mathfrak{su}(2)$ is explicitly given as follows.

Definition D.1.23. *Let V_n be the space of homogeneous polynomials in X, Y of degree $n - 1$. Note that $\dim(V_n) = n$. We view polynomials as functions on $\mathbb{C}^2$. Consider $\rho_n: \mathfrak{su}(2) \rightarrow \mathfrak{gl}(V_n)$ given by*

$$\rho_n(v) := \begin{bmatrix} X & Y \end{bmatrix} v \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix}.$$

We see that $\rho_n(v): V_n \rightarrow V_n$.

Lemma D.1.24. *The pair (V_n, ρ_n) is a representation of $\mathfrak{sp}(1)$.*

Proof. We see that ρ_n is linear, so we need only check it is a Lie algebra homomorphism. We see

$$[\rho_n(v), \rho_n(\tau)] = \left[\begin{bmatrix} X & Y \end{bmatrix} v \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix}, \begin{bmatrix} X & Y \end{bmatrix} \tau \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix} \right].$$

We see that

$$\begin{aligned}
\begin{bmatrix} X & Y \end{bmatrix} v \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix} \begin{bmatrix} X & Y \end{bmatrix} \tau \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix} &= \begin{bmatrix} X & Y \end{bmatrix} v \left(\begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix} \begin{bmatrix} X & Y \end{bmatrix} \right) \tau \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix} \\
&+ \begin{bmatrix} X & Y \end{bmatrix} v \begin{bmatrix} X \frac{\partial}{\partial X} & Y \frac{\partial}{\partial X} \\ X \frac{\partial}{\partial Y} & Y \frac{\partial}{\partial Y} \end{bmatrix} \tau \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix} \\
&= \begin{bmatrix} X & Y \end{bmatrix} v \tau \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix} \\
&+ \begin{bmatrix} X & Y \end{bmatrix} v \begin{bmatrix} X \frac{\partial}{\partial X} & Y \frac{\partial}{\partial X} \\ X \frac{\partial}{\partial Y} & Y \frac{\partial}{\partial Y} \end{bmatrix} \tau \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix}
\end{aligned}$$

Substituting, we see that

$$\begin{aligned}
[\rho_n(v), \rho_n(\tau)] &= \begin{bmatrix} X & Y \end{bmatrix} [v, \tau] \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix} \\
&+ \begin{bmatrix} X & Y \end{bmatrix} \left(v \begin{bmatrix} X \frac{\partial}{\partial X} & Y \frac{\partial}{\partial X} \\ X \frac{\partial}{\partial Y} & Y \frac{\partial}{\partial Y} \end{bmatrix} \tau - \tau \begin{bmatrix} X \frac{\partial}{\partial X} & Y \frac{\partial}{\partial X} \\ X \frac{\partial}{\partial Y} & Y \frac{\partial}{\partial Y} \end{bmatrix} v \right) \begin{bmatrix} \frac{\partial}{\partial X} \\ \frac{\partial}{\partial Y} \end{bmatrix}.
\end{aligned}$$

Computing the final term for arbitrary $v, \tau \in \mathfrak{su}(2)$, we see that it vanishes. Thus, we are left with

$$[\rho_n(v), \rho_n(\tau)] = \rho_n([v, \tau]).$$

That is, we have a representation. □

Note D.1.25. Using the basis of $\mathfrak{su}(2)$ given by $v_1 := \frac{1}{2} \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$, $v_2 := \frac{1}{2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, and

$v_3 := \frac{1}{2} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$, we find

$$\begin{aligned}
\rho_n(v_1) &= \frac{i}{2} X \frac{\partial}{\partial X} - \frac{i}{2} Y \frac{\partial}{\partial Y}, \\
\rho_n(v_2) &= \frac{1}{2} X \frac{\partial}{\partial Y} - \frac{1}{2} Y \frac{\partial}{\partial X}, \\
\rho_n(v_3) &= \frac{i}{2} X \frac{\partial}{\partial Y} + \frac{i}{2} Y \frac{\partial}{\partial X}.
\end{aligned}$$

Lemma D.1.26. *This representation is irreducible.*

Proof. Consider the basis $\{X^{n-1}, \dots, X^j Y^{n-1-j}, \dots, Y^{n-1}\}$ of V_n . We compute the Casimir operator in this basis and show that it is $\frac{n^2-1}{4}I_n$. Note that as $\mathfrak{sp}(1)$ is rank one, the Casimir operator determines the representation completely.

We see that for $j \in \{0, \dots, n-1\}$, we have

$$\begin{aligned}\rho_n(v_1)X^jY^{n-1-j} &= \frac{i}{2}(2j-n+1)X^jY^{n-1-j}, \\ \rho_n(v_2)X^jY^{n-1-j} &= \frac{1}{2}(n-1-j)X^{j+1}Y^{n-2-j} - \frac{1}{2}jX^{j-1}Y^{n-j}, \\ \rho_n(v_3)X^jY^{n-1-j} &= \frac{i}{2}(n-1-j)X^{j+1}Y^{n-2-j} + \frac{i}{2}jX^{j-1}Y^{n-j}.\end{aligned}$$

Note that the coefficients vanish when we would get elements not in our basis.

Written as matrices in the aforementioned basis,

$$\begin{aligned}\rho_n(v_1) &= \frac{i}{2} \begin{bmatrix} n-1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -n+1 \end{bmatrix}, \\ \rho_n(v_2) &= \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ -n+1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & n-1 \\ 0 & \cdots & 0 & -1 & 0 \end{bmatrix}, \\ \rho_n(v_3) &= \frac{i}{2} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ n-1 & 0 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & n-1 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.\end{aligned}$$

The Casimir operator $\phi = -\sum_{j=1}^3 \rho_n(v_j)^2$ is then exactly $\frac{n^2-1}{4}I_n$, the Casimir operator of the irreducible, complex n -dimensional representation. $\square$

Following Itzkowitz et al., we now find real and quaternionic structures for different values of n [IRS91].

Proposition D.1.27. *For n even, (V_n, ρ_n) is quaternionic type. For n odd, it is real type.*

Note D.1.28. *As all irreducible, complex representations are of real or quaternionic type, they are all self-dual. In particular, we have*

$$\begin{aligned} \text{Irr}(\mathfrak{sp}(1), \mathbb{C})_{\mathbb{R}} &= \{(V_n, \rho_n) \mid n \text{ odd}\}, \\ \text{Irr}(\mathfrak{sp}(1), \mathbb{C})_{\mathbb{C}} &= \emptyset, \\ \text{Irr}(\mathfrak{sp}(1), \mathbb{C})_{\mathbb{H}} &= \{(V_n, \rho_n) \mid n \text{ even}\}. \end{aligned}$$

Thus, by Corollary D.1.21, for n odd or divisible by four, there is a unique irreducible real n -representation $(\mathbb{R}^n, \varrho_n)$ whose complexification is (V_n, ρ_n) when n is odd or $(V_{n/2}, \rho_{n/2})^{\oplus 2}$ when n is divisible by four.

Moreover, when restricting the scalars of an irreducible quaternionic representation to $\mathbb{C}$, the complex representation is isomorphic to (V_n, ρ_n) for some n even or $(V_n, \rho_n)^{\oplus 2}$ for some n odd.

Proof. Define $J_n: V_n \rightarrow V_n$ as follows. Given $P(X, Y) = \sum_{j=0}^{n-1} a_j X^j Y^{n-1-j}$, let $\bar{P}(X, Y) := \sum_{j=0}^{n-1} \overline{a_j} X^j Y^{n-1-j}$. Then define $J_n(P(X, Y)) := \bar{P}(-Y, X) = \sum_{j=0}^{n-1} \overline{a_j} (-Y)^j X^{n-1-j}$. Note that J_n is conjugate-linear. We see that $J_n X^j Y^{n-1-j} = (-1)^j X^{n-1-j} Y^j$. Writing J_n in the same basis as the $\rho_n(v_i)$, we find

$$J_n = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \vdots \\ (-1)^{n-1} & 0 & \cdots & 0 \end{bmatrix}.$$

Recalling that J_n is conjugate-linear, we see that J_n commutes with the $\rho_n(v_i)$. Furthermore, we see that

$$J_n^2(P(X, Y)) = J_n(\bar{P}(-Y, X)) = P(-X, -Y) = (-1)^{n-1} P(X, Y),$$

recalling that P is a homogeneous degree $n-1$ polynomial. Therefore, if n is even, then J_n is a quaternionic structure and if n is odd, it is a real structure. $\square$

Consider Corollary D.1.21. As we have no complex type representations, we see that for (V_n, ρ_n) with n odd and $(V_n, \rho_n) \oplus (V_n, \rho_n)$ with n even, we can choose real matrices (of the same dimension) inducing irreducible real representations and these are all the irreducible real representations. Additionally, for (V_n, ρ_n) with n even and $(V_n, \rho_n) \oplus (V_n, \rho_n)$ with n odd, we can choose quaternionic matrices (of half the dimension) inducing irreducible quaternionic representations and these are all the irreducible quaternionic representations.

D.2 Dual Representations

In this section, we discover exactly what the dual of a representation of a Lie algebra is. That is, for irreducible representations, we find its highest weight. We then show that for many Lie algebras, all representations are self-dual: the dual of a representation is isomorphic to itself. Note that we are only interested in finite-dimensional representations, so we assume all representations have finite dimension. Furthermore, we are only interested in semi-simple Lie algebras, so they are all assumed to be as such.

D.2.1 Identifying the Dual Representation

In this section, we show that the dual of an irreducible representation is irreducible and identify its highest weight. Recall the definition of a dual representation given in Definition D.1.15.

Proposition D.2.1. *Suppose that (V, ρ) is irreducible. Then so too is (V^*, ρ^*) .*

Proof. Let $W \subseteq V^*$ be an invariant subspace. Let U be the linear subspace of V given by $U := \{v \in V \mid f(v) = 0, \forall f \in W\}$. Given $v \in U$, we see that for all $f \in W$,

$$f(\rho(x)v) = -\rho^*(x)f(v) = 0,$$

as $\rho^*(x)f \in W$, due to W being invariant. Thus, $\rho(x)v \in U$, so U is invariant.

Let $n := \dim_{\mathbb{C}}(V)$. As (V, ρ) is irreducible, $U = 0$ or $U = V$. If the latter is true, then let $f \in W$. For some $b_i \in \mathbb{C}$, we find $f = \sum_{i=1}^n b_i v_i^*$. But we know for all $v \in V$, $f(v) = 0$, so $f(v_i) = 0$ for all i . Hence, $f = 0$. That is, $W = 0$.

Suppose the former is true. Let $v_1 \in V \setminus \{0\}$. Then there exists $f_1 \in W$ such that $f_1(v_1) = 1$. Consider $\ker(f_1)$. This space is $(n-1)$ -dimensional, so we choose $v_2 \in \ker(f_1) \setminus \{0\}$. Then there is some $f_2 \in W$ such that $f_2(v_2) = 1$. Suppose there are some

$c_1, c_2 \in \mathbb{C}$ such that $c_1 f_1 + c_2 f_2 = 0$. Evaluating at v_2 implies $c_2 = 0$, so $c_1 = 0$. That is, f_1 and f_2 are linearly independent. Consider $\ker(f_1) \cap \ker(f_2)$. This space is $(n-2)$ -dimensional, so we can choose $v_3 \in (\ker(f_1) \cap \ker(f_2)) \setminus \{0\}$. There exists $f_3 \in W$ such that $f_3(v_3) = 1$. We can repeat until we get a linearly independent set of $f_1, \dots, f_n \in W$. Hence, $W = V^*$.

Thus, the only invariant subspaces of V^* are 0 and V^* itself, so we see that (V^*, ρ^*) is irreducible. $\square$

Proposition D.2.2. *Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$, a semi-simple Lie algebra, and $\lambda \in \mathfrak{h}^*$. Then λ is a weight of (V, ρ) if and only if $-\lambda$ is a weight of (V^*, ρ^*) , with the same multiplicity.*

Proof. Suppose λ is a weight of (V, ρ) . The weight space of a weight $\lambda: \mathfrak{h} \rightarrow \mathbb{C}$ is given by

$$V_\lambda := \{v \in V \mid \forall H \in \mathfrak{h}, \rho(H)v = \lambda(H)v\}.$$

Consider $(V^*)_{-\lambda} = \{f \in V^* \mid \forall H \in \mathfrak{h}, \rho^*(H)f = -\lambda(H)f\}$. Let $H \in \mathfrak{h}$ and $\{v_1, \dots, v_m\} \subseteq V_\lambda$ a basis. As $\mathfrak{g}$ is semi-simple, we have that V decomposes into a direct sum of its weight spaces. That is, $V \simeq \bigoplus_{\mu \in \mathfrak{h}^*} V_\mu$. Extend the basis $\{v_1, \dots, v_m\}$ of V_λ to a basis $\{v_1, \dots, v_m, w_1, \dots, w_{n-m}\} \subseteq V$ where the w_i belong to different weight spaces.

Define $f_i: V \rightarrow \mathbb{C}$ by the linear map taking $v_i \mapsto 1$, $v_j \mapsto 0$, for all $j \neq i$ and $w_k \mapsto 0$, for all k . Thus, we get a linearly independent set $\{f_1, \dots, f_m\} \in V^*$. We show that each $f_i \in (V^*)_{-\lambda}$.

Indeed, let $v \in V$. We can write $v = \sum_{j=1}^m \alpha_j v_j + \sum_{k=1}^{n-m} \beta_k w_k$. Then

$$\rho^*(H)f_i(v) = -f_i\left(\sum_{j=1}^m \alpha_j \rho(H)v_j + \sum_{k=1}^{n-m} \beta_k \rho(H)w_k\right).$$

As the $v_j \in V_\lambda$, we have $\rho(H)v_j = \lambda(H)v_j$. For the w_k , there is some weight $\mu_k \in \mathfrak{h}^*$ such that $\rho(H)w_k = \mu_k(H)w_k$. Then, we see

$$\rho^*(H)f_i(v) = -\lambda(H)f_i\left(\sum_{j=1}^m \alpha_j v_j\right) = -\lambda(H)f_i(v).$$

Therefore, $(V^*)_{-\lambda}$ has the same dimension as V_λ .

Conversely, suppose that $-\lambda$ is a weight of (V^*, ρ^*) . As $((V^*)^*, (\rho^*)^*) \simeq (V, \rho)$, we have that λ is a weight of (V, ρ) with the same multiplicity. $\square$

Now that we understand the weights of the dual representation, we need only identify the highest weight.

Definition D.2.3. *Given a root system Φ in E , a finite dimensional Euclidean vector space, the **Weyl group** W is the subgroup of the isometry group generated by reflections through the hyperplanes orthogonal to the roots. Hence, it is a finite reflection group. It turns out that this group is generated by reflections through the hyperplanes orthogonal to the simple roots, so we need only consider these roots.*

Let $\langle \cdot, \cdot \rangle$ be the inner product on E . The complement of the set of hyperplanes above is disconnected. Each connected component is called a **Weyl chamber**. If we have a set of simple roots Δ , then the **fundamental Weyl chamber** is the set of points $v \in E$ such that $(\alpha, v) > 0$ for all $\alpha \in \Delta$.

There is a well-known theorem relating the Weyl group and Weyl chambers.

Theorem D.2.4. *The Weyl group acts freely and transitively on the Weyl chambers.*

Definition D.2.5. *Based on the above, there is a unique $w_0 \in W$ mapping the fundamental Weyl chamber to its negative chamber and vice versa. Such a map sends $\Delta \mapsto -\Delta$ and vice versa. This element is called the **longest element** of the Weyl group.*

Proposition D.2.6. *Let (V, ρ) be an irreducible (complex) representation of $\mathfrak{g}$, with highest weight λ . Then $w_0(\lambda)$ is the lowest weight of V .*

Proof. Let $\{\alpha_1, \dots, \alpha_m\} = \Delta$. Note that if μ is a weight, then so too is $w_0(\mu)$, as an element of the Weyl group permutes the set of weights. Thus, we have for a weight μ , as λ is the highest weight, there exists $a_i \in \mathbb{Z}_{\geq 0}$ such that

$$\lambda - w_0(\mu) = \sum_{i=1}^m a_i \alpha_i.$$

Note that as w_0 is a reflection, $w_0^2 = \text{id}_E$. We see that

$$w_0(\lambda) - \mu = \sum_{i=1}^m a_i w_0(\alpha_i).$$

Note that as $\alpha_i \in \Delta$, we have $w_0(\alpha_i) \in -\Delta$. Hence, $-w_0(\alpha_i) \in \Delta$. As w_0 acts freely on the set of simple roots, we have that $\{-w_0(\alpha_1), \dots, -w_0(\alpha_m)\} = \Delta$. Therefore, we see that

$$w_0(\lambda) - \mu = \sum_{i=1}^m (-a_i)(-w_0(\alpha_i)).$$

As all $-a_i \leq 0$, we see that $w_0(\lambda)$ is the lowest weight. □

Corollary D.2.7. *Let λ be the highest weight of an irreducible representation (V, ρ) . Then $-w_0(\lambda)$ is the highest weight of (V^*, ρ^*) .*

Proof. Let $-\mu$ be a weight of (V^*, ρ^*) . By Proposition D.2.2, μ is a weight of (V, ρ) . By Proposition D.2.6, we see that

$$-w_0(\lambda) + \mu = - \sum_{i=1}^m (-a_i)(-w_0(\alpha_i)) = \sum_{i=1}^m a_i(-w_0(\alpha_i)).$$

As $\{-w_0(\alpha_i)\} = R^+$ and $a_i \geq 0$, we see that $-w_0(\lambda)$ is the highest weight of (V^*, ρ^*) . $\square$

D.2.2 Lie algebras with all self-dual representations

In this section, we compute the map w_0 for all semi-simple Lie algebras. First, we must note that an irreducible representation is determined by its highest weight $\sum_{i=1}^m a_i \alpha_i$, where $a_i \geq 0$ and α_i are simple roots. On the Dynkin diagram associated to the Lie algebra, simple roots correspond with the nodes. Thus, we can find a correspondence between irreducible Lie algebra representations and decorated Dynkin diagrams, where we adorn the Dynkin diagram with the a_i appearing above the corresponding node.

The element $-w_0: \Delta \rightarrow \Delta$ is an isometry, so it provides an automorphism on the Dynkin diagram. If this automorphism is the identity, then every representation is self-dual. If not, then we can see what the dual representation is, by applying this automorphism to the decorated Dynkin diagrams. Note that this means that for $B_n, C_n, E_7, E_8, F_4, G_2$, $-w_0 = \text{id}$, as there are no nontrivial diagram automorphisms. Nevertheless, we show this more concretely below.

Definition D.2.8. *Let α_i be a simple root, with **coroot** $\alpha_i^\vee := \frac{2\alpha_i}{\langle \alpha_i, \alpha_i \rangle}$. The **reflection about the root α_i** is given by*

$$r_i(x) := x - \langle \alpha_i^\vee, x \rangle \alpha_i. \quad (\text{D.2.1})$$

*The **fundamental roots** are the elements ϖ_i defined such that $\langle \alpha_i^\vee, \varpi_j \rangle = \delta_{ij}$. Note that*

$$r_i(\varpi_j) = \begin{cases} \varpi_j & i \neq j \\ \varpi_j - \alpha_j & i = j \end{cases}. \quad (\text{D.2.2})$$

We use the root systems given by Humphreys [Hum72, Chapter 12]. Below, let $\{e_1, \dots, e_m\}$ be the standard basis of $\mathbb{R}^m$ and let $\langle \cdot, \cdot \rangle$ be the usual inner product.

Proposition D.2.9. *For $A_1 = B_1 = C_1$, we have $-w_0 = \text{id}$, so all representations are self-dual. For A_n , $n \geq 2$, we have that $-w_0 \neq \text{id}$. In either case, the diagram automorphism flips the diagram horizontally (for A_1 this automorphism is the identity).*

Proof. Labelling the roots left to right as $\alpha_1, \dots, \alpha_n$, we define $\alpha_i := e_i - e_{i+1}$. Then we see that

$$r_i(e_j) = \begin{cases} e_j & j \neq i, i+1 \\ e_{i+1} & j = i \\ e_i & j = i+1 \end{cases}. \quad (\text{D.2.3})$$

As these reflections generate the Weyl group W , we see that $W \simeq S_{n+1}$, the permutation group on $n+1$ elements. Consider the permutation sending $e_i \mapsto e_{n+2-i}$ for $i = 1, \dots, n+1$. Such a permutation sends all $\alpha_i \mapsto -\alpha_{n+1-i}$. Hence, it takes $\Delta \mapsto -\Delta$ and vice versa. However, only one element of the Weyl group can do this, w_0 .

Therefore, for $n = 1$, $-w_0 = \text{id}$, as $-w_0(\alpha_1) = \alpha_1$. Thus, for A_1 , all representations are self-dual. However, for $n > 1$, we see that $-w_0 \neq \text{id}$ flips the diagram horizontally. $\square$

Proposition D.2.10. *For B_n , $n \geq 2$, we have $-w_0 = \text{id}$, so all representations are self-dual. That is, the diagram automorphism is the identity (as expected).*

Proof. Labelling the roots left to right as $\alpha_1, \dots, \alpha_n$, define $\alpha_i := e_i - e_{i+1}$ for $i \in \{1, \dots, n-1\}$ and $\alpha_n := e_n$ (the short root). Then we see that for $i \leq n-1$, we have

$$r_i(e_j) = \begin{cases} e_j & j \neq i, i+1 \\ e_{i+1} & j = i \\ e_i & j = i+1 \end{cases}. \quad (\text{D.2.4})$$

Additionally, we have

$$r_n(e_j) = \begin{cases} e_j & j \neq n \\ -e_n & j = n \end{cases}. \quad (\text{D.2.5})$$

Thus, W can permute any e_j and negate any number of them. Take the element sending $e_i \mapsto -e_i$ for all $i \in \{1, \dots, n\}$. This element sends all $\alpha_i \mapsto -\alpha_i$, hence taking $\Delta \mapsto -\Delta$ and vice versa. Therefore, this element is w_0 and we see $-w_0 = \text{id}$. $\square$

Proposition D.2.11. *For C_n , $n \geq 2$, we have $-w_0 = \text{id}$, so all representations are self-dual. That is, the diagram automorphism is the identity (as expected).*

Proof. Labelling the roots left to right as $\alpha_1, \dots, \alpha_n$, define $\alpha_i := e_i - e_{i+1}$ for $i \in \{1, \dots, n-1\}$ and $\alpha_n := 2e_n$ (the long root). Then we see that for $i \leq n-1$, we have

$$r_i(e_j) = \begin{cases} e_j & j \neq i, i+1 \\ e_{i+1} & j = i \\ e_i & j = i+1 \end{cases}. \quad (\text{D.2.6})$$

Additionally, we have

$$r_n(e_j) = \begin{cases} e_j & j \neq n \\ -e_n & j = n \end{cases}. \quad (\text{D.2.7})$$

Thus, W can permute any e_j and negate any number of them. Take the element sending $e_i \mapsto -e_i$ for all $i \in \{1, \dots, n\}$. This element sends all $\alpha_i \mapsto -\alpha_i$, hence taking $\Delta \mapsto -\Delta$ and vice versa. Therefore, this element is w_0 and we see $-w_0 = \text{id}$. $\square$

Proposition D.2.12. *For D_n , $n \geq 2$, we have $-w_0 = \text{id}$ if n is even, so all representations are self-dual. That is, the diagram automorphism is the identity. If n is odd, we have $-w_0 \neq \text{id}$. The diagram automorphism flips the diagram vertically.*

Proof. Labelling the roots left to right as $\alpha_1, \dots, \alpha_{n-2}$, then α_{n-1} the top root and α_n the bottom root, define $\alpha_i := e_i - e_{i+1}$ for $i \in \{1, \dots, n-1\}$ and $\alpha_n := e_{n-1} + e_n$. Then we see that for $i \leq n-1$, we have

$$r_i(e_j) = \begin{cases} e_j & j \neq i, i+1 \\ e_{i+1} & j = i \\ e_i & j = i+1 \end{cases}. \quad (\text{D.2.8})$$

Additionally, we have

$$r_n(e_j) = \begin{cases} e_j & j \neq n-1, n \\ -e_n & j = n-1 \\ -e_{n-1} & j = n \end{cases}. \quad (\text{D.2.9})$$

Thus, W can permute any e_j and negate any even number of them.

If n is even, take the element sending $e_i \mapsto -e_i$ for all $i \in \{1, \dots, n\}$. This element sends all $\alpha_i \mapsto -\alpha_i$, hence taking $\Delta \mapsto -\Delta$ and vice versa. Therefore, this element is w_0 and we see $-w_0 = \text{id}$.

If n is odd, take the element sending $e_i \mapsto -e_i$ for $i \in \{1, \dots, n-1\}$ and $e_n \mapsto e_n$. Then for all $i \in \{1, \dots, n-2\}$ we have $\alpha_i \mapsto -\alpha_i$. We also have $\alpha_{n-1} \mapsto -\alpha_n$ and $\alpha_n \mapsto -\alpha_{n-1}$.

Thus, $\Delta \mapsto -\Delta$ and vice versa, so this element is w_0 . Therefore, $-w_0 \neq \text{id}$ and we see the diagram automorphism flips the diagram vertically. $\square$

Proposition D.2.13. *For F_4 , we have $-w_0 = \text{id}$, so all representations are self-dual. That is, the diagram automorphism is the identity (as expected).*

Proof. Labelling the roots left to right as $\alpha_1, \dots, \alpha_4$, define $\alpha_1 := e_2 - e_3$, $\alpha_2 := e_3 - e_4$, $\alpha_3 := e_4$, and $\alpha_4 := \frac{e_1 - e_2 - e_3 - e_4}{2}$ (where α_3, α_4 are the short roots). One can compute that the following Weyl element, in the $\{e_1, e_2, e_3, e_4\}$ basis, is given by

$$(r_1 \circ r_3 \circ r_2 \circ r_4)^6 = -I_4. \quad (\text{D.2.10})$$

Thus, this element takes $\alpha_i \mapsto -\alpha_i$ for $i = 1, 2, 3, 4$, so it takes $\Delta \mapsto -\Delta$ and vice versa. Hence, this element is w_0 and $-w_0 = \text{id}$. $\square$

Proposition D.2.14. *For G_2 , we have $-w_0 = \text{id}$, so all representations are self-dual. That is, the diagram automorphism is the identity (as expected).*

Proof. Labelling the roots left to right as α_1 and α_2 , define $\alpha_1 := e_1 - e_2$ and $\alpha_2 := -2e_1 + e_2 + e_3$ (the long root). One can compute that the following Weyl element, in the $\{e_1, e_2, e_3\}$ basis, is given by

$$(r_1 \circ r_2)^3 = \frac{1}{3} \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}. \quad (\text{D.2.11})$$

Thus, this element takes $\alpha_i \mapsto -\alpha_i$ for $i = 1, 2$, so it takes $\Delta \mapsto -\Delta$ and vice versa. Hence, this element is w_0 and $-w_0 = \text{id}$. $\square$

Proposition D.2.15. *For E_7, E_8 , we have $-w_0 = \text{id}$, so all representations are self-dual. That is, the diagram automorphism is the identity (as expected). For E_6 , we have $-w_0 \neq \text{id}$. The diagram automorphism in this case flips the diagram horizontally.*

Proof. For E_n , labelling the roots left to right as $\alpha_1, \dots, \alpha_{n-1}$ and α_n as the remaining root not in the line, define $\alpha_1 := \frac{1}{2}(e_1 + e_8 - \sum_{i=2}^7 e_i)$, $\alpha_i := e_i - e_{i-1}$ for $i = 2, \dots, n-1$, and $\alpha_n := e_1 + e_2$.

Consider $n = 8$. One can compute the following Weyl element, in the $\{e_1, \dots, e_8\}$ basis, is given by

$$(r_1 \circ r_3 \circ r_5 \circ r_7 \circ r_2 \circ r_4 \circ r_6 \circ r_8)^{15} = -I_8. \quad (\text{D.2.12})$$

Thus, this element takes $\alpha_i \mapsto -\alpha_i$ for $i = 1, \dots, 8$, so it takes $\Delta \mapsto -\Delta$ and vice versa. Hence, this element is w_0 and $-w_0 = \text{id}$.

Consider $n = 7$. One can compute the following Weyl element, in the $\{e_1, \dots, e_8\}$ basis, is given by

$$(r_1 \circ r_3 \circ r_5 \circ r_2 \circ r_4 \circ r_6 \circ r_7)^9 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad (\text{D.2.13})$$

Thus, this element takes $\alpha_i \mapsto -\alpha_i$ for $i = 1, \dots, 7$, so it takes $\Delta \mapsto -\Delta$ and vice versa. Hence, this element is w_0 and $-w_0 = \text{id}$.

Consider $n = 6$. One can compute the following Weyl element, in the $\{e_1, \dots, e_8\}$ basis, is given by

$$(r_1 \circ r_3 \circ r_5 \circ r_2 \circ r_4 \circ r_6)^6 = \frac{1}{2} \begin{bmatrix} M & 0 \\ 0 & -M \end{bmatrix}, \quad (\text{D.2.14})$$

where

$$M := \begin{bmatrix} -1 & -1 & -1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & -1 \end{bmatrix}. \quad (\text{D.2.15})$$

Thus, this element takes $\alpha_i \mapsto -\alpha_{6-i}$ for $i = 1, \dots, 5$ and $\alpha_6 \mapsto -\alpha_6$ so it takes $\Delta \mapsto -\Delta$ and vice versa. Hence, this element is w_0 and $-w_0 \neq \text{id}$. In particular, we see the diagram automorphism flips the diagram horizontally. $\square$

D.3 Representations of Direct Sums

In this section, we classify irreducible finite dimensional representations of the direct sum of Lie algebras. In particular, we are interested in representations of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$, as these are used when discussing instantons with rotational symmetry in Section 3.6.

Definition D.3.1. Suppose $\mathfrak{g}_1, \mathfrak{g}_2$ are Lie algebras. Let (V_i, ρ_i) be a representation of $\mathfrak{g}_i$. Denote the following representation of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ by $(V_1, \rho_1) \boxtimes (V_2, \rho_2) := (V, \rho)$, where $V := V_1 \otimes V_2$ and $\rho(x, z) := \rho_1(x) \otimes \text{id}_{V_2} + \text{id}_{V_1} \otimes \rho_2(z)$.

When dealing with compact Lie groups G_1 and G_2 , it is known that a finite dimensional representation (W, ρ) of $G_1 \times G_2$ is irreducible if and only if $(W, \rho) \simeq (V_1, \rho_1) \boxtimes (V_2, \rho_2)$ for finite dimensional irreducible representations (V_i, ρ_i) of G_i [Sep07, Theorem 3.9].

Translating this to the language of Lie algebras, if $\mathfrak{g}_1, \mathfrak{g}_2$ are the Lie algebras of compact Lie groups, then a finite dimensional representation of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ is irreducible if and only if it is isomorphic to $(V_1, \rho_1) \boxtimes (V_2, \rho_2)$ for finite dimensional irreducible representations (V_i, ρ_i) of $\mathfrak{g}_i$.

The proof of this theorem uses character theory, which is beyond the scope of this thesis. However, character theory is not necessary for the case we are interested in: $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$.

Before we deal with our specific case, we first mention a useful property of $\boxtimes$.

Proposition D.3.2. Suppose that $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are Lie algebras. Let (V_i, ρ_i) and (W_i, λ_i) be representations of $\mathfrak{g}_i$. Then

$$\begin{aligned} ((V_1, \rho_1) \boxtimes (V_2, \rho_2)) \otimes ((W_1, \lambda_1) \boxtimes (W_2, \lambda_2)) \\ \simeq ((V_1, \rho_1) \otimes (W_1, \lambda_1)) \boxtimes ((V_2, \rho_2) \otimes (W_2, \lambda_2)) \end{aligned} \quad (\text{D.3.1})$$

Proof. Let (V, ρ) denote the representation on the left-hand side of the statement and (W, λ) the representation on the right-hand side. Note that $V \simeq W$. We see that

$$\begin{aligned} \rho(x, z) &:= \rho_1(x) \otimes \text{id}_{V_2} \otimes \text{id}_{W_1} \otimes \text{id}_{W_2} + \text{id}_{V_1} \otimes \rho_2(z) \otimes \text{id}_{W_1} \otimes \text{id}_{W_2} \\ &\quad + \text{id}_{V_1} \otimes \text{id}_{V_2} \otimes \lambda_1(x) \otimes \text{id}_{W_2} + \text{id}_{V_1} \otimes \text{id}_{V_2} \otimes \text{id}_{W_1} \otimes \lambda_2(z). \end{aligned}$$

We can similarly write λ . Under the isomorphism $\phi: V \rightarrow W$, we see that $\lambda \circ \phi = \phi \circ \rho$. Thus, the representations are isomorphic. $\square$

D.3.1 Irreducible representations of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$

In this section, we classify all irreducible representations of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$.

Definition D.3.3. Let $v_1, \dots, v_6 \in \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ denote the standard basis of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. That is, v_1, v_2, v_3 is the standard basis for $\mathfrak{sp}(1) \oplus 0$ and v_4, v_5, v_6 is the standard basis for $0 \oplus \mathfrak{sp}(1)$.

Let (V, ρ) be an irreducible complex representation of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. Complexify the Lie algebra and, similar to the case of classifying representations of $\mathfrak{sp}(1)$, define

$$L^\pm := i\rho(v_1) \mp \rho(v_2), \quad L_3 := i\rho(v_3), \quad K^\pm := i\rho(v_4) \mp \rho(v_5), \quad K_3 := i\rho(v_6). \quad (\text{D.3.2})$$

Then L_3 and K_3 span the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$.

Proposition D.3.4. The irreducible representations of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ are indexed by two integers. Given $m, n \in \mathbb{N}$, denote the irreducible representation with highest weight $\begin{bmatrix} \frac{m-1}{2} & \frac{n-1}{2} \end{bmatrix}$ by $(V_{m,n}, \rho_{m,n})$.

Note D.3.5. The representation $(V_{m,n}, \rho_{m,n})$ is a mn -dimensional complex representation. Indeed, we generate mn distinct eigenvectors of L_3 and K_3 (when looking at the pair of eigenvalues together).

Proof. Let $\lambda: \mathfrak{h} \rightarrow \mathbb{C}$ be the highest weight of the representation. Note that we can write λ as a row vector $\begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}$ acting on $\mathfrak{h}$ by writing elements of $\mathfrak{h}$ in the L_3, K_3 basis and multiplying the matrices to get a complex number.

Note that $[L^\pm, L_3] = \mp L^\pm$ and $[K^\pm, K_3] = \mp K^\pm$. Additionally, $[L^+, L^-] = 2L_3$ and $[K^+, K^-] = 2K_3$. Finally, all other commutators vanish. Thus, we see that the roots of the Lie algebra are $\begin{bmatrix} \pm 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & \pm 1 \end{bmatrix}$. We choose the positive roots to be $\begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \end{bmatrix}$. As λ is the highest weight of the representation, we have that the weight space $V_{\begin{bmatrix} \lambda_1 + m & \lambda_2 + n \end{bmatrix}} = 0$ for all $m, n \in \mathbb{Z}_{\geq 0}$, other than $m = n = 0$. Let $0 \neq v \in V_\lambda$. Hence, $L_3 v = \lambda_1 v$ and $K_3 v = \lambda_2 v$. Then we see that

$$\begin{aligned} L_3(L^+)^m(K^+)^n v &= (\lambda_1 + m)(L^+)^m(K^+)^n v, \\ K_3(L^+)^m(K^+)^n v &= (\lambda_2 + n)(L^+)^m(K^+)^n v. \end{aligned}$$

Thus, $(L^+)^m(K^+)^n v \in V_{\begin{bmatrix} \lambda_1 + m & \lambda_2 + n \end{bmatrix}} = 0$. Hence, in particular, we have that $L^+ v = 0 = K^+ v$ (taking $(m, n) = (1, 0), (0, 1)$).

Note that $(K^-)^n v \in V_{\begin{bmatrix} \lambda_1 & \lambda_2 - n \end{bmatrix}}$ and $(L^-)^m v \in V_{\begin{bmatrix} \lambda_1 - m & \lambda_2 \end{bmatrix}}$. As $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ is semi-simple, the direct sum of the weight spaces gives the representation space. As

the above weights are distinct, the eigenvectors are linearly independent. As we are dealing with finite dimensional representations, eventually there is some $m, n \in \mathbb{N}$ such that $(L^-)^{m-1}v, (K^-)^{n-1}v \neq 0$ but $(L^-)^mv = (K^-)^nv = 0$.

Using the commutation relations, one can show that

$$\begin{aligned} L^+(L^-)^mv &= m(2\lambda_1 - m + 1)(L^-)^{m-1}v, \\ K^+(K^-)^nv &= n(2\lambda_2 - n + 1)(K^-)^{n-1}v. \end{aligned}$$

However, these must vanish, as $(K^-)^nv = 0$ and $(L^-)^mv = 0$. Hence, we see that $\lambda_1 = \frac{m-1}{2}$ and $\lambda_2 = \frac{n-1}{2}$. $\square$

We show exactly what these irreducible representations look like (and also prove that they exist for all $m, n \in \mathbb{N}$).

The dimension of the centre of the universal enveloping algebra of a semi-simple Lie algebra equals the rank of the Lie algebra. Indeed, the Casimir operators are a basis for this space. In our case of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$, we have two Casimir operators.

Definition D.3.6. *The **Casimir operators** for a representation (V, ρ) of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ are*

$$C_1 := -\sum_{i=1}^3 \rho(v_i)^2, \quad C_2 := -\sum_{i=4}^6 \rho(v_i)^2. \quad (\text{D.3.3})$$

Note that the sum of these Casimir operators gives the usual quadratic Casimir operator for this Lie algebra.

It is straightforward to see that these operators commute with $\rho(v, \omega)$ for all $v, \omega \in \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$, as they come from the Casimir operator for each summand of the direct sum.

Lemma D.3.7. *Suppose we have an irreducible complex representation $(V_{m,n}, \rho_{m,n})$. As the Casimir operators commute with the representation, Schur's Lemma tells us that the Casimir operators are proportional to the identity. In particular, $C_1 = \frac{m^2-1}{4}I_{mn}$ and $C_2 = \frac{n^2-1}{4}I_{mn}$.*

Proof. Using the definitions of $L^\pm, K^\pm, L_3, K_3$, we see that $C_1 = L_3^2 + L_3 + L^-L^+$ and $C_2 = K_3^2 + K_3 + K^-K^+$.

Let $v \neq 0$ be a highest weight eigenvector. That is, $L_3v = \frac{m-1}{2}v$ and $K_3v = \frac{n-1}{2}v$. From above, we have that $L^+v = 0 = K^+v$. Using these identities, we see that $C_1 = \frac{m^2-1}{4}I_{mn}$ and $C_2 = \frac{n^2-1}{4}I_{mn}$. $\square$

Therefore, we see that the Casimir operators classify the irreducible representations. Moreover, by examining their eigenvalues, they classify all representations.

Proposition D.3.8. *The irreducible representation $(V_{m,n}, \rho_{m,n})$ exists for all $m, n \in \mathbb{N}$. Denote the irreducible, complex, a -dimensional representation of $\mathfrak{sp}(1)$ by (V_a, ρ_a) . Then $(V_{m,n}, \rho_{m,n}) \simeq (V_m, \rho_m) \boxtimes (V_n, \rho_n)$.*

Proof. Consider the representation $(V_m, \rho_m) \boxtimes (V_n, \rho_n)$ of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. We see that the Casimir operators of this representation are $C_1 = \frac{m^2-1}{4}I_{mn}$ and $C_2 = \frac{n^2-1}{4}I_{mn}$. Therefore, $(V_m, \rho_m) \boxtimes (V_n, \rho_n) \simeq (V_{m,n}, \rho_{m,n})$.

In particular, such representations are irreducible and moreover, all irreducible representations of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ can be constructed in this way. $\square$

Therefore, we have proven the original theorem in our case, without the use of character theory.

Note D.3.9. *This approach does not work in general as the Casimir operators cannot generally be written in such a clear way, as they may not just be given by the quadratic Casimir operators of the summands of the direct sum. That is, being able to write C_1 and C_2 in terms of $L^\pm, K^\pm, L_3, K_3$ allowed us to prove this result.*

As $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \simeq \mathfrak{spin}(4)$ has Dynkin diagram D_2 , by Proposition D.2.12, its representations are self-dual. Therefore, its irreducible representations are either real or quaternionic.

Proposition D.3.10. *When $m \equiv n \pmod{2}$, we have that $(V_{m,n}, \rho_{m,n})$ is a real representation. Otherwise, $(V_{m,n}, \rho_{m,n})$ is a quaternionic representation.*

Note D.3.11. *As all irreducible, complex representations are of real or quaternionic type, they are all self-dual. In particular, we have*

$$\begin{aligned} \text{Irr}(\mathfrak{spin}(4), \mathbb{C})_{\mathbb{R}} &= \{(V_{m,n}, \rho_{m,n}) \mid n \equiv m \pmod{2}\}, \\ \text{Irr}(\mathfrak{spin}(4), \mathbb{C})_{\mathbb{C}} &= \emptyset, \\ \text{Irr}(\mathfrak{spin}(4), \mathbb{C})_{\mathbb{H}} &= \{(V_{m,n}, \rho_{m,n}) \mid n \not\equiv m \pmod{2}\}. \end{aligned}$$

Thus, by Corollary D.1.21, for $n \equiv m \pmod{2}$, there is a unique irreducible real mn -representation $(\mathbb{R}^{mn}, \varrho_{m,n})$ whose complexification is $(V_{m,n}, \rho_{m,n})$. For $n \not\equiv m \pmod{2}$, there is a unique irreducible real $2mn$ -representation $(\mathbb{R}^{2mn}, \varrho_{m,n})$ whose complexification is $(V_{m,n}, \rho_{m,n})^{\oplus 2}$.

Moreover, when restricting the scalars of an irreducible quaternionic representation to $\mathbb{C}$, the complex representation is isomorphic to $(V_{m,n}, \rho_{m,n})$ for some $n \not\equiv m \pmod{2}$ or $(V_{m,n}, \rho_{m,n})^{\oplus 2}$ for some $n \equiv m \pmod{2}$.

Proof. Suppose that $m \equiv n \pmod{2}$. We know that $(V_{m,n}, \rho_{m,n}) \simeq (V_m, \rho_m) \boxtimes (V_n, \rho_n)$. As (V_m, ρ_m) and (V_n, ρ_n) are either both real or both quaternionic representations of $\mathfrak{sp}(1)$, we have that there are conjugate-linear equivariant maps $J_m: V_m \rightarrow V_m$ and $J_n: V_n \rightarrow V_n$ such that $J_m^2 = J_n^2 = (\pm 1)\text{id}$, where we have $+1$ if the m and n are odd and -1 if m and n are even.

Let $J_{m,n} := J_m \otimes J_n: V_{m,n} \rightarrow V_{m,n}$. We see that this map is conjugate-linear. Expanding the definition of $\rho_{m,n}$, we see that $J_{m,n}$ is equivariant. As $J_{m,n}^2 = (\pm 1)^2 \text{id} = \text{id}$, we have that $(V_{m,n}, \rho_{m,n})$ is a real representation. That the other representations are quaternionic follows as all irreducible representations of $\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$ must be one or the other. $\square$

In order to understand tensor products, we need to know how to decompose them.

Proposition D.3.12. *The tensor product of $(V_{a,b}, \rho_{a,b}) \otimes (V_{c,d}, \rho_{c,d})$ can be decomposed into a sum of irreducible representations as follows:*

$$(V_{a,b}, \rho_{a,b}) \otimes (V_{c,d}, \rho_{c,d}) \simeq \bigoplus_{i=1}^{\min(a,c)} \bigoplus_{j=1}^{\min(b,d)} (V_{a+c+1-2i, b+d+1-2j}, \rho_{a+c+1-2i, b+d+1-2j}). \quad (\text{D.3.4})$$

Proof. Recall that $(V_{a,b}, \rho_{a,b})$ is given by $(V_a, \rho_a) \boxtimes (V_b, \rho_b)$. Then by Proposition D.3.2,

$$(V_{a,b}, \rho_{a,b}) \otimes (V_{c,d}, \rho_{c,d}) \simeq ((V_a, \rho_a) \otimes (V_c, \rho_c)) \boxtimes ((V_b, \rho_b) \otimes (V_d, \rho_d)).$$

The result follows from the decomposition of tensor products of irreducible representations of $\mathfrak{sp}(1)$. $\square$

Appendix E

Computing B^n

Let $n \in \mathbb{N}_+$. Recall that we denote the unique n -dimensional irreducible complex representation of $\mathfrak{sp}(1)$ by (V_n, ρ_n) . Also, recall that B^n is defined to be the generator of the unique trivial summand of $(V_{n+2}, \rho_{n+2}) \otimes (V_n^*, \rho_n^*) \otimes (V_3, \rho_3)$ and can be viewed as an $\mathfrak{sp}(1)$ -invariant triple (B_1^n, B_2^n, B_3^n) of complex matrices. Moreover, we saw that since we have chosen a scale, B^n is unique up to a $U(1)$ factor. Additionally, recall that we defined $v_1 := i/2$, $v_2 := j/2$, $v_3 := k/2$ as well as $Y_i^+ := \rho_{n+2}(v_i)$ and $Y_i^- := \rho_n(v_i)$.

While these matrices are uniquely determined, it is useful to know exactly what they look like. In this appendix, we compute the B_i^n matrices. This appendix has appeared on arXiv [Lan24, arXiv version 1].

Note that (2.14a) tells us that, in particular,

$$B_2^n = Y_3^+ B_1^n - B_1^n Y_3^-, \quad \text{and} \quad B_3^n = B_1^n Y_2^- - Y_2^+ B_1^n. \quad (\text{E.0.1})$$

That is, the triple can be computed from just B_1^n .

Definition E.0.1. Suppose that $Y_i^\pm$ are defined as above. Let $S_\pm := iY_2^\pm + Y_3^\pm$ be the lowering operators for the representations, with corresponding raising operators $S_\pm^\dagger = iY_2^\pm - Y_3^\pm$. Let v^+ be a highest weight unit eigenvector for (V_{n+2}, ρ_{n+2}) and v^- a highest weight unit eigenvector for (V_n, ρ_n) , that is $iY_1^\pm v^\pm = \frac{n \pm 1}{2} v^\pm$.

From the representation theory of $\mathfrak{sp}(1)$, we know that $(v^\pm, \dots, S_\pm^{n \pm 1} v^\pm)$ is a basis of $V_{n+1 \pm 1}$, with each element $S_\pm^j v^\pm$ an eigenvector of $iY_i^\pm$ with eigenvalue $\frac{n \pm 1 - 2j}{2}$. The following lemma tells us how B_1^n acts on these vectors.

Lemma E.0.2. *For all $j \in \{0, \dots, n-1\}$, there is some $a_j \in \mathbb{C}$ such that $B_1^n S_-^j v^- = a_j S_+^{j+1} v^+$.*

Proof. From (2.14a), we know that $iY_1^+ B_1^n = B_1^n iY_1^-$. Furthermore, we know

$$\begin{aligned} [iY_1^\pm, S_\pm] &= [iY_1^\pm, iY_2^\pm + Y_3^\pm] = -S_\pm, \quad \text{and} \\ [iY_1^\pm, S_\pm^\dagger] &= [iY_1^\pm, iY_2^\pm - Y_3^\pm] = S_\pm^\dagger. \end{aligned}$$

Then, by induction, we see that

$$iY_1^+ B_1^n S_-^j v^- = B_1^n (iY_1^-) S_-^j v^- = \frac{n-1-2j}{2} B_1^n S_-^j v^-.$$

That is, $B_1^n S_-^j v^-$ is an eigenvector of iY_1^+ with eigenvalue $\frac{n-1-2j}{2}$. therefore, it is in the span of $S_+^{j+1} v^+$. $\square$

We can find a relationship between these a_j .

Lemma E.0.3. *For all $j \in \{0, \dots, n-1\}$, we have*

$$a_j = (n-j)a_{n-1}.$$

Proof. From (2.14a), we know that

$$S_+(B_2^n - iB_3^n) = (B_2^n - iB_3^n)S_-.$$

Using (E.0.1), we see that

$$B_2^n - iB_3^n = S_+ B_1^n - B_1^n S_-.$$

Combining the two previous equations, we get

$$S_+^2 B_1^n - 2S_+ B_1^n S_- + B_1^n S_-^2 = 0. \tag{E.0.2}$$

Applying (E.0.2) to $S_-^j v^-$ for $j \in \{0, \dots, n-3\}$, we see that

$$(a_j - 2a_{j+1} + a_{j+2})S_+^{j+3} v^+ = 0.$$

As $j+3 < n+2$, we have that $S_+^{j+3} v^+ \neq 0$, so $a_j = 2a_{j+1} - a_{j+2}$.

Furthermore, applying (E.0.2) to $S_-^{n-2} v^-$, we get

$$(a_{n-2} - 2a_{n-1})S_+^{n+1} v^+ = 0,$$

given that $S_-^n v^- = 0$. As $n+1 < n+2$, $S_+^{n+1} v^+ \neq 0$, so $a_{n-2} = 2a_{n-1}$.

We can solve this recurrence relation, obtaining $a_j = (n-j)a_{n-1}$ for all $j \in \{0, \dots, n-1\}$. $\square$

All that is left is to determine $|a_{n-1}|$. By Lemma E.0.3, we see that the choice of phase for a_{n-1} is equivalent to the choice of $U(1)$ factor for B^n .

First, note $[S_\pm^\dagger, S_\pm] = 2iY_1^\pm$. Then, by induction, we know that for all $j \in \{1, \dots, n \pm 1\}$, $S_\pm^\dagger S_\pm^j v^\pm = j(n+1-j \pm 1)S_\pm^{j-1} v^\pm$ and $S_\pm^\dagger v^\pm = 0$. So, for all $j \in \{0, \dots, n \pm 1\}$ we have

$$|S_\pm^j v^\pm|^2 = \prod_{k=0}^{j-1} (j-k)(n+1-j+k \pm 1) = \frac{j!(n \pm 1)!}{(n-j \pm 1)!}. \quad (\text{E.0.3})$$

We are now ready to compute the modulus of a_{n-1} .

Lemma E.0.4. *The modulus is given by $|a_{n-1}| = \frac{\sqrt{2}}{n(n+1)}$.*

Proof. Recall (E.0.1) and note that $Y_3^\pm = \frac{1}{2}(S_\pm - S_\pm^\dagger)$ and $Y_2^\pm = -\frac{i}{2}(S_\pm + S_\pm^\dagger)$. Then, we see that for all $j \in \{0, \dots, n-1\}$,

$$\begin{aligned} B_2^n S_-^j v^- &= Y_3^+ B_1^n S_-^j v^- - B_1^n Y_3^- S_-^j v^- \\ &= \frac{n-j}{2} a_{n-1} (S_+ - S_+^\dagger) S_+^{j+1} v^+ - \frac{1}{2} B_1^n (S_- - S_-^\dagger) S_-^j v^- \\ &= a_{n-1} \left(\frac{1}{2} S_+^{j+2} v^+ - \frac{(n-j)(n+1-j)}{2} S_+^j v^+ \right), \quad \text{and} \\ B_3^n S_-^j v^- &= B_1^n Y_2^- S_-^j v^- - Y_2^+ B_1^n S_-^j v^- \\ &= \frac{i(n-j)}{2} (S_+ + S_+^\dagger) a_{n-1} S_+^{j+1} v^+ - \frac{i}{2} B_1^n (S_- + S_-^\dagger) S_-^j v^- \\ &= i a_{n-1} \left(\frac{1}{2} S_+^{j+2} v^+ + \frac{(n-j)(n+1-j)}{2} S_+^j v^+ \right). \end{aligned}$$

Thus, we now know what B_2^n and B_3^n look like in the $\{v^\pm, \dots, S_\pm^{n \pm 1} v^\pm\}$ bases.

Multiplying (2.14c) on the right by v^- and on the left by $(v^-)^\dagger$, we use the previous equations when $j=0$, the orthogonality of the $S_+^j v^+$, and (E.0.3) to see that

$$\begin{aligned} \frac{n+2}{n} |v^-|^2 &= |a_0|^2 |S_+ v^+|^2 + \frac{|a_{n-1}|^2}{2} (|S_+^2 v^+|^2 + n^2(n+1)^2 |v^+|^2) \\ \frac{n+2}{n} &= |a_{n-1}|^2 \left(n^2(n+1) + n(n+1) + \frac{1}{2} n^2(n+1)^2 \right) \\ \frac{n+2}{n} &= |a_{n-1}|^2 \frac{n(n+1)^2(n+2)}{2}. \end{aligned}$$

Simplifying, we see $|a_{n-1}|^2 = \frac{2}{n^2(n+1)^2}$. □

We have seen how the B_i^n looks when written in the bases that diagonalize $Y_i^\pm$, however, we wish to see how they look when written in the same bases as the $Y_i^\pm$.

Proposition E.0.5. *Written in the same bases as the $Y_i^\pm$ matrices, we have that for $\theta \in \mathbb{R}$,*

$$B_1^n = \frac{\sqrt{2}e^{i\theta}}{n(n+1)} \begin{bmatrix} v^+ & \cdots & S_+^{n+1}v^+ \end{bmatrix} \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ n & \ddots & & \vdots \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} \frac{v^-}{|v^-|^2} & \cdots & \frac{S_-^{n-1}v^-}{|S_-^{n-1}v^-|^2} \end{bmatrix}^\dagger. \quad (\text{E.0.4})$$

Note that θ represents the $U(1)$ factor that we are free to choose.

Proof. By Lemma E.0.2, Lemma E.0.3, and Lemma E.0.4 we know that in the same bases as the $Y_i^\pm$ matrices, B_1^n is given by

$$B_1^n = \frac{\sqrt{2}e^{i\theta}}{n(n+1)} \begin{bmatrix} v^+ & \cdots & S_+^{n+1}v^+ \end{bmatrix} \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ n & \ddots & & \vdots \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \begin{bmatrix} v^- & \cdots & S_-^{n-1}v^- \end{bmatrix}^{-1}.$$

Note that

$$\begin{bmatrix} \frac{v^-}{|v^-|^2} & \cdots & \frac{S_-^{n-1}v^-}{|S_-^{n-1}v^-|^2} \end{bmatrix}^\dagger = \begin{bmatrix} v^- & \cdots & S_-^{n-1}v^- \end{bmatrix}^{-1},$$

because the $S_-^j v^-$ are eigenvectors of iY_1^i with different eigenvalues, so the $S_-^j v^-$ are orthogonal. $\square$

Proposition E.0.5 tells us that in order to write B_1^n , we need only find the highest weight unit eigenvectors v^+ and v^- of the representations (V_{n+2}, ρ_{n+2}) and (V_n, ρ_n) , respectively, and construct $S_\pm$. Then B_2^n and B_3^n are constructed from B_1^n .

To illustrate this process, we use it to construct B_1^1, B_2^1, B_3^1 .

Proposition E.0.6. *Consider the adjoint representation $(\mathbb{R}^3, \text{ad})$, which is the irreducible 3-dimensional representation. This representation can be induced by the matrices*

$$Y_1^+ = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad Y_2^+ = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}; \quad Y_3^+ = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Additionally, consider the trivial representation $(\mathbb{R}, 0)$, which is the irreducible 1-dimensional representation. The trivial representation can be induced by the matrices $Y_i^- = 0$.

For a choice of $\theta \in \mathbb{R}$, the B_i^1 matrices are given by

$$B_1^1 = e^{i\theta} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}; \quad B_2^1 = e^{i\theta} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}; \quad B_3^1 = e^{i\theta} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

Note that in agreeance with Lemma 2.1.36, we can choose θ such that all the B_i^1 are real matrices.

Proof. The Y_i^- matrices are all zero, as $\rho_1(v_i) = 0$. Consider the ordered basis (v_3, v_2, v_1) of $\mathfrak{sp}(1)$. In this basis $Y_i^+ = \text{ad}(v_i)$ has the above form.

A highest weight unit eigenvector for iY_1^+ is $v^+ = \frac{1}{\sqrt{2}} \begin{bmatrix} i & 1 & 0 \end{bmatrix}^T$. Indeed, it has a weight of 1, which is the highest weight for the representation. A highest weight unit eigenvector for $iY_1^- = 0$ is $v^- = 1$. Furthermore, we know $S_- = iY_2^- + Y_3^- = 0$ and

$$S_+ = iY_2^+ + Y_3^+ = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 1 \\ i & -1 & 0 \end{bmatrix}.$$

Therefore,

$$S_+ v^+ = \begin{bmatrix} 0 & 0 & -\sqrt{2} \end{bmatrix}^T; \quad S_+^2 v^+ = \sqrt{2} \begin{bmatrix} i & -1 & 0 \end{bmatrix}^T.$$

By (E.0.4), we have

$$B_1^1 = \frac{\sqrt{2}e^{i\theta}}{2} \begin{bmatrix} \frac{i}{\sqrt{2}} & 0 & i\sqrt{2} \\ \frac{1}{\sqrt{2}} & 0 & -\sqrt{2} \\ 0 & -\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \end{bmatrix}^\dagger = e^{i\theta} \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

Using $B_2^1 = Y_3^+ B_1^1 - B_1^1 Y_3^-$ and $B_3^1 = B_1^1 Y_2^- - Y_2^+ B_1^1$, we obtain the result. $\square$

Appendix F

Chevalley bases

In this appendix, we list Chevalley bases for $\mathfrak{su}(n)$, $\mathfrak{so}(2m+1)$, $\mathfrak{sp}(n)$, and $\mathfrak{so}(2m)$. Humphreys gives us a definition for a Chevalley basis which we summarize below [Hum72, Ch. VII § 25.2].

Definition F.0.1. *Given a simple complex Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, denote the simple roots by $\alpha_1, \dots, \alpha_r \in \mathfrak{h}^*$. The **Chevalley basis** is a basis of $\mathfrak{g}$ with structure constants provided by the Cartan matrix $A(\mathfrak{g})$, with components a_{ij} . Specifically, for each simple root α_i , we have some $h_i, x_i, y_i \in \mathfrak{g}$ such that for all $i, j \in \{1, \dots, r\}$ we have that $\alpha_j(h_i) = a_{ij}$ as well as*

$$\begin{aligned} [h_i, h_j] &= 0, & [x_i, y_i] &= h_i, & [h_i, x_j] &= a_{ij}x_j, & [h_i, y_j] &= -a_{ij}y_j, \\ [x_i, y_j] &= 0, & \forall i &\neq j. \end{aligned} \tag{F.0.1}$$

By considering $[x_i, x_j]$ and $[y_i, y_j]$ for $i \neq j$, $\{h_i, x_i, y_i\}_{i=1}^r$ generate the entirety of the Chevalley basis, which satisfies (F.0.1) as well as other relationships [Hum72, Ch. VII § 25.2].

In this thesis, we are only interested in the generators $\{h_i, x_i, y_i\}_{i=1}^r$ satisfying (F.0.1), as they determine the entire Chevalley basis.

Proposition F.0.2. *The Chevalley basis of the complexification $\mathfrak{su}(n)^{\mathbb{C}}$ is generated by the following elements. Let $X := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, Y := \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \in \mathfrak{su}(2)$. Then let*

$$\begin{aligned} h_1 &:= -i \cdot \text{diag}(1, -1, 0, \dots, 0) \otimes i, \\ x_1 &:= \frac{1}{2} \text{diag}(X, 0, \dots, 0) \otimes i + \frac{1}{2} \text{diag}(Y, 0, \dots, 0) \otimes 1, \\ y_1 &:= \frac{1}{2} \text{diag}(X, 0, \dots, 0) \otimes i - \frac{1}{2} \text{diag}(Y, 0, \dots, 0) \otimes 1 \\ &\vdots \\ h_{n-1} &:= -i \cdot \text{diag}(0, \dots, 0, 1, -1) \otimes i, \\ x_{n-1} &:= \frac{1}{2} \text{diag}(0, \dots, 0, X) \otimes i + \frac{1}{2} \text{diag}(0, \dots, 0, Y) \otimes 1, \\ y_{n-1} &:= \frac{1}{2} \text{diag}(0, \dots, 0, X) \otimes i - \frac{1}{2} \text{diag}(0, \dots, 0, Y) \otimes 1 \end{aligned}$$

Proof. The Cartan matrix for $\mathfrak{su}(n)$, the Cartan matrix for A_{n-1} , is given by (2.3.28) for $n \geq 3$ and $\begin{bmatrix} 2 \end{bmatrix}$ for $n = 2$. Given these matrices, the generators satisfy the correct commutation relations. $\square$

Proposition F.0.3. *The Chevalley basis of the complexification $\mathfrak{so}(2m+1)^\mathbb{C}$ is generated by the following elements. Let $X := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then let*

$$\begin{aligned}
 h_1 &:= -\text{diag}(X, -X, 0, \dots, 0) \otimes i, \\
 x_1 &:= \frac{1}{2} \left(\begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} \oplus 0 \right) \otimes i + \frac{1}{2} \left(\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \oplus 0 \right) \otimes 1, \\
 y_1 &:= \frac{1}{2} \left(\begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} \oplus 0 \right) \otimes i - \frac{1}{2} \left(\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \oplus 0 \right) \otimes 1, \\
 &\vdots \\
 h_{m-1} &:= -\text{diag}(0, \dots, 0, X, -X, 0_1) \otimes i, \\
 x_{m-1} &:= \frac{1}{2} \left(0 \oplus \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} \oplus 0_1 \right) \otimes i + \frac{1}{2} \left(0 \oplus \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \oplus 0_1 \right) \otimes 1, \\
 y_{m-1} &:= \frac{1}{2} \left(0 \oplus \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} \oplus 0_1 \right) \otimes i - \frac{1}{2} \left(0 \oplus \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \oplus 0_1 \right) \otimes 1, \\
 h_m &:= -2 \cdot \text{diag}(0, \dots, 0, X, 0_1) \otimes i, \\
 x_m &:= \left(0 \oplus \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \right) \otimes i + \left(0 \oplus \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right) \otimes 1, \\
 y_m &:= \left(0 \oplus \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \right) \otimes i - \left(0 \oplus \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \right) \otimes 1.
 \end{aligned}$$

Proof. The Cartan matrix for $\mathfrak{so}(2m+1)$, the Cartan matrix for B_m , is given by (2.3.32) for $m \geq 2$ and $\begin{bmatrix} 2 \end{bmatrix}$ for $m = 1$. Given these matrices, the generators satisfy the correct commutation relations. $\square$

Proposition F.0.4. *The Chevalley basis of the complexification $\mathfrak{sp}(n)^{\mathbb{C}}$ is generated by the following elements. Let $X := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, Y := \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} \in \mathfrak{sp}(2)$. Then let*

$$\begin{aligned}
 h_1 &:= -i \cdot \text{diag}(1, -1, 0, \dots, 0) \otimes i, \\
 x_1 &:= \frac{1}{2} \text{diag}(X, 0, \dots, 0) \otimes i + \frac{1}{2} \text{diag}(Y, 0, \dots, 0) \otimes 1, \\
 y_1 &:= \frac{1}{2} \text{diag}(X, 0, \dots, 0) \otimes i - \frac{1}{2} \text{diag}(Y, 0, \dots, 0) \otimes 1 \\
 &\vdots \\
 h_{n-1} &:= -i \cdot \text{diag}(0, \dots, 0, 1, -1) \otimes i, \\
 x_{n-1} &:= \frac{1}{2} \text{diag}(0, \dots, 0, X) \otimes i + \frac{1}{2} \text{diag}(0, \dots, 0, Y) \otimes 1, \\
 y_{n-1} &:= \frac{1}{2} \text{diag}(0, \dots, 0, X) \otimes i - \frac{1}{2} \text{diag}(0, \dots, 0, Y) \otimes 1, \\
 h_n &:= -i \cdot \text{diag}(0, \dots, 0, 1) \otimes i, \\
 x_n &:= \frac{j}{2} \cdot \text{diag}(0, \dots, 0, 1) \otimes i + \frac{k}{2} \cdot \text{diag}(0, \dots, 0, 1) \otimes 1, \\
 y_n &:= \frac{j}{2} \cdot \text{diag}(0, \dots, 0, 1) \otimes i - \frac{k}{2} \cdot \text{diag}(0, \dots, 0, 1) \otimes 1.
 \end{aligned}$$

Proof. The Cartan matrix for $\mathfrak{sp}(n)$, the Cartan matrix for C_n , is given by (2.3.35) for $n \geq 2$ and $\begin{bmatrix} 2 \end{bmatrix}$ for $n = 1$. Given these matrices, the generators satisfy the correct commutation relations. $\square$

Proposition F.0.5. *The Chevalley basis of the complexification $\mathfrak{so}(2m)^\mathbb{C}$, for $m \geq 2$, is generated by the following elements. Let $X := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, $Y := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $Z := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then let*

$$\begin{aligned}
 h_1 &:= -\text{diag}(X, -X, \dots, 0) \otimes i, \\
 x_1 &:= \frac{1}{2} \left(\begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} \oplus 0 \right) \otimes i + \frac{1}{2} \left(\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \oplus 0 \right) \otimes 1, \\
 y_1 &:= \frac{1}{2} \left(\begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} \oplus 0 \right) \otimes i - \frac{1}{2} \left(\begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \oplus 0 \right) \otimes 1, \\
 &\vdots \\
 h_{m-1} &:= -\text{diag}(0, \dots, 0, X, -X) \otimes i, \\
 x_{m-1} &:= \frac{1}{2} \left(0 \oplus \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} \right) \otimes i + \frac{1}{2} \left(0 \oplus \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \right) \otimes 1, \\
 y_{m-1} &:= \frac{1}{2} \left(0 \oplus \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix} \right) \otimes i - \frac{1}{2} \left(0 \oplus \begin{bmatrix} 0 & X \\ X & 0 \end{bmatrix} \right) \otimes 1, \\
 h_m &:= -\text{diag}(0, \dots, 0, X, X) \otimes i, \\
 x_m &:= \frac{1}{2} \left(0 \oplus \begin{bmatrix} 0 & Z \\ -Z & 0 \end{bmatrix} \right) \otimes i + \frac{1}{2} \left(0 \oplus \begin{bmatrix} 0 & Y \\ -Y & 0 \end{bmatrix} \right) \otimes 1, \\
 y_m &:= \frac{1}{2} \left(0 \oplus \begin{bmatrix} 0 & Z \\ -Z & 0 \end{bmatrix} \right) \otimes i - \frac{1}{2} \left(0 \oplus \begin{bmatrix} 0 & Y \\ -Y & 0 \end{bmatrix} \right) \otimes 1.
 \end{aligned}$$

Proof. The Cartan matrix for $\mathfrak{so}(2m)$, the Cartan matrix for D_m , is given by (2.3.40) for $m \geq 3$ and $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ for $m = 2$. Given these matrices, the generators satisfy the correct commutation relations. This basis satisfies the correct commutation relations, according to the Cartan matrix for $\mathfrak{so}(2m)$, the Cartan matrix for D_m . $\square$

Appendix G

The isotropy subgroup needs to stretch

In this appendix, we show that the isotropy subgroup need not always split. Suppose $\mathfrak{h}$ is a compact Lie algebra, so it splits into an abelian and a semi-simple part as $\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus [\mathfrak{h}, \mathfrak{h}]$. However, this does not mean that the compact isotropy group H splits into an abelian part and a semi-simple part, as the following result shows. In this appendix, we use notation and results from Section 2.3.1.

Proposition G.0.1. *Consider $G = \mathrm{SU}(3)$, a compact, simply-connected, simple Lie group. Let $\Phi_\infty = i \cdot \mathrm{diag}(2, -1, -1) \in \mathfrak{su}(3)$. Then H , the isotropy subgroup of Φ_∞ , is not isomorphic, as a Lie group, to $\mathrm{U}(1)^n \times K$ for some $n \geq 0$ and K a semi-simple Lie group.*

Proof. Let Φ_∞ be as given. We find that $\mathfrak{h} := C_{\mathfrak{su}(3)}(\Phi_\infty)$ is given by

$$\mathfrak{h} = \mathrm{span}_{\mathbb{R}}(\Phi_\infty) \oplus \mathrm{span}_{\mathbb{R}} \left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix} \right).$$

Here we have split $\mathfrak{h}$ into its abelian and semi-simple part. Notice that $\mathfrak{h} \simeq \mathbb{R} \oplus \mathfrak{su}(2)$. We start by identifying H as a Lie group.

Consider the Lie subgroups $T, J \subseteq G$ given by

$$\begin{aligned} T &:= \{\mathrm{diag}(e^{2it}, e^{-it}, e^{-it}) \mid t \in \mathbb{R}\}; \\ J &:= \{1 \oplus A \mid A \in \mathrm{SU}(2)\}. \end{aligned} \tag{G.0.1}$$

These are connected Lie subgroups of G with Lie algebras $\mathfrak{z}(\mathfrak{h}) \simeq \mathbb{R}$ and $[\mathfrak{h}, \mathfrak{h}] \simeq \mathfrak{su}(2)$, respectively. Note that T and J are isomorphic, as Lie groups, to $U(1)$ and $SU(2)$, respectively. Also, note that $T \cap J = \{\text{diag}(1, \pm 1, \pm 1)\} \simeq \mathbb{Z}_2$.

Consider $TJ \subseteq G$, a connected Lie subgroup of G with Lie algebra $\mathfrak{z}(\mathfrak{h}) \oplus [\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$. But H is the unique connected Lie subgroup of G with this Lie algebra. Thus, $H \simeq TJ$, as Lie groups.

We have that $T \times J$ acts smoothly on G as $(t, j).g := t j g$. As $T \times J$ is compact, this action is also proper. The orbit of $I \in G$ is exactly $TJ \subseteq G$. The isotropy subgroup of I under this action is the collection of pairs (t, j) such that $j = t^{-1} \in T \cap J \simeq \mathbb{Z}_2$. Thus, the isotropy subgroup is isomorphic, as a Lie group, to $T \cap J$. We have that $H \simeq TJ$ is diffeomorphic to $(T \times J)/T \cap J$. As H is the adjoint orbit of Φ_∞ , it is compact, meaning $(T \times J)/T \cap J$ is compact as well.

We can do even better than diffeomorphic. Let $f: T \times J \rightarrow G$ be the Lie group homomorphism $f(t, j) := t j$. We see that the image and kernel of f match the orbit and isotropy subgroup listed above. Therefore, the natural map $(T \times J)/T \cap J \rightarrow TJ$ is not only a diffeomorphism, but also a group isomorphism. Hence, as Lie groups,

$$(T \times J)/T \cap J \simeq TJ \simeq H. \quad (\text{G.0.2})$$

Now that we have identified H , almost splitting it, we show that we cannot split it. Suppose, for the sake of contradiction, that H is isomorphic, as a Lie group, to $U(1)^n \times K$ for some $n \geq 0$ and K a semi-simple Lie group. Then

$$\mathbb{R} \oplus \mathfrak{su}(2) \simeq \mathfrak{h} \simeq \mathbb{R}^{\oplus n} \oplus \mathfrak{k}. \quad (\text{G.0.3})$$

Thus, we have that $n = 1$ and $\mathfrak{k} = \mathfrak{su}(2)$, so K is isomorphic, as a Lie group, to $SU(2)$ or $SO(3)$.

Since G is simply-connected, $\pi_1(H) \simeq \pi_2(G/H) \simeq \mathbb{Z}$, as the dimension of the centre of H is one. We then see

$$\mathbb{Z} \simeq \pi_1(H) \simeq \pi_1(U(1) \times K) \simeq \mathbb{Z} \times \pi_1(K). \quad (\text{G.0.4})$$

Thus, K must be simply-connected, so $K \simeq SU(2)$. Hence, we have an isomorphism of Lie groups

$$(U(1) \times SU(2))/\{\pm(1, I)\} \simeq U(1) \times SU(2). \quad (\text{G.0.5})$$

Therefore, there is a surjective Lie group homomorphism $F: U(1) \times SU(2) \rightarrow U(1) \times SU(2)$ with kernel $\ker(F) = \{\pm(1, I)\}$.

Looking at the induced Lie algebra isomorphism $d_{(1,I)}F$, we see that the commutation relations imply that $d_{(1,I)}F(1,0) = (A,0)$ for some $A \in \mathbb{R}$ and $d_{(1,I)}F(0,x) = (0,p(x))$, for some Lie algebra isomorphism $p: \mathfrak{su}(2) \rightarrow \mathfrak{su}(2)$. This Lie algebra isomorphism gives rise to a Lie group isomorphism $P: \mathrm{SU}(2) \rightarrow \mathrm{SU}(2)$. As $\mathrm{U}(1) \times \mathrm{SU}(2)$ is compact and connected, the exponential map is surjective. Therefore, we have

$$F(e^{i\theta}, M) = (e^{Ai\theta}, P(M)). \quad (\text{G.0.6})$$

We see that the kernel of F is the set of points $(e^{i\theta}, I)$ such that $e^{Ai\theta} = 1$. Thus, we do not have the correct kernel, contradiction! Therefore, H does not split. $\square$