

Arboricity and transversal problems on bounded degree graphs

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Statement of Contributions

This thesis contains published joint work with Penny Haxell, with Chapter 4 based on [106] and Chapter 5 based on [105]. Other chapters are based on solo-authored work [168, 169, 170].

Abstract

In this thesis, we investigate arboricity and transversal problems on graphs in a bounded degree setting. We pay particular attention to problems involving both the maximum degree and maximum density of the graphs.

The first topic is bounded degree arboricity. The arboricity of a multigraph G is the minimum number of forests required to cover its edge set. It is well understood through a theorem of Nash-Williams. For the problem of bounded degree arboricity, we seek to cover the edge set of a multigraph G by the minimum number of forests such that every vertex v has maximum degree at most $f(v)$ in every forest, for some fixed weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$. The case $f = 2$ is referred to as linear arboricity. The Linear Arboricity Conjecture, due to Akiyama, Exoo, and Harary, asserts that the linear arboricity of a simple graph G with maximum degree Δ is at most $\lceil (\Delta + 1)/2 \rceil$. Using tools from orientations of multigraphs, we prove that the Linear Arboricity Conjecture holds for multigraphs whose maximum degree is sufficiently large compared to a certain maximum density. This improves on previous results of this type, and moreover our methods extend to give a general upper bound on the bounded degree arboricity of multigraphs, as a maximum of a weighted maximum degree and maximum density of the multigraph. In addition, we disprove a conjecture of Truszczyński which proposes a more precise upper bound on the bounded degree arboricity of multigraphs, but we also show that the conjecture does hold for simple graphs with sufficiently large girth, and that it holds for all simple graphs asymptotically.

The second topic in this thesis is independent transversals in bounded degree graphs. Given a graph G and a partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of its vertex set, an independent transversal is a transversal of $\mathcal{P}$ that is independent in G . Haxell proved that there exists an independent transversal when G has maximum degree Δ and $|V_i| \geq 2\Delta$ for every i , and this condition has been shown to be best possible by Szabó and Tardos. Wanless and Wood proved the existence of an independent transversal when the maximum class average degree (a kind of maximum density) of $G, \mathcal{P}$ is D and $|V_i| \geq 4D$ for every i , and this condition has been shown to be asymptotically best possible by Groenland, Kaiser, Treffers, and Wales. Using topological methods, we find an interpolation between these two sufficient conditions, and moreover we give a construction showing that our result is best possible. The method for our tight construction, based on a simple lemma, is then further explored. We use our method to streamline tight constructions for Haxell's theorem, and we prove that our method in fact yields all possible extremal constructions. We also use it to give more streamlined counterexamples to a list coloring conjecture of Reed. Then we adapt the construction method to the setting of full rainbow matchings of multi-hypergraphs, which are independent transversals in their line graphs. Aharoni, Berger, and Meshulam proved that there exists a full rainbow matching in an r -uniform multigraph G with maximum degree Δ when every edge class has size $|E_i| \geq r\Delta$. We use our method to give tight constructions for this theorem. We also describe counterexamples to a color degree generalization of Galvin's Theorem on list edge-coloring bipartite multigraphs. Finally, we describe how our method applies to independent transversals in uniform hypergraphs and other related settings.

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Chapter 1

Introduction

In this thesis, we study covering and transversal problems on graphs in a bounded degree setting. Abstractly speaking, we are given a finite ground set X and a collection $\mathcal{C}$ of subsets of X . A covering problem asks for the minimum number of sets in $\mathcal{C}$ whose union is X . If we also fix a family $\mathcal{P}$ of subsets of X , then a transversal problem asks whether there exists a set in $\mathcal{C}$ containing one distinct element from each set in $\mathcal{P}$. In our settings, the collection $\mathcal{C}$ will usually form an *abstract simplicial complex* (or just *complex*), meaning that if $A \in \mathcal{C}$ and $B \subseteq A$, then $B \in \mathcal{C}$. Moreover, the family $\mathcal{P}$ will usually be a partition of the ground set X . Covering and transversal problems are well understood in some cases such as when $\mathcal{C}$ forms a matroid, but they are more difficult in the settings of bounded degree graphs that we study. In many of our covering and transversal results, both the maximum degree and the maximum density of the graphs involved will play a prominent role.

The main covering problem that we study is the problem of *bounded degree arboricity*, which is when $X = E(G)$ is the edge set of a multigraph G and $\mathcal{C}$ consists of the (edge subsets of) forests in which every vertex v satisfies a specified degree upper bound $d(v) \leq f(v)$. That is, it is a bounded degree variation of the classical arboricity problem of covering by forests, and we provide new approximations for bounded degree arboricity. The main transversal problem that we study is the problem of *independent transversals*, which is when $X = V(G)$ is the vertex set of a graph G , $\mathcal{C}$ consists of all independent sets of G , and $\mathcal{P}$ is some partition of the vertex set $V(G)$. A related problem that we also study is the problem of *full rainbow matchings*, which is instead when $X = E(G)$ is the edge set of a multi-hypergraph G , $\mathcal{C}$ consists of all matchings of G , and $\mathcal{P}$ is some partition of the edge set $E(G)$. Full rainbow matchings are the same as independent transversals in line graphs, but behave slightly differently to warrant their own study. Much of our study will be about optimal class size conditions for the existence of transversals when the (multi-hyper)graphs have a given maximum degree.

This thesis primarily consists of various related theorems, most of the proofs not being particularly long, and much of the emphasis is on the connections to different problems. Hence, this introductory chapter will be a rather thorough overview of the many results

proven in this thesis, together with the background and context. There will not be an overarching conclusion to this thesis, but there will be problems for future work presented at the end of each chapter.

1.1 Notation and terminology

In this section, we briefly describe the notation and terminology that will be used throughout this thesis.

A *multigraph* G is a pair (V, E) , where $V = V(G)$ is a finite set of elements called the *vertices* of G , and $E = E(G)$ is a finite multiset of unordered pairs $\{u, v\}$, usually denoted uv , of not necessarily distinct vertices of V , called the *edges* of G . A *loop* is an edge $\{v, v\}$ on a single vertex v , and two identical pairs in $E(G)$ are said to be *parallel edges*. A *simple graph* (or just a *graph*) is a multigraph with no parallel edges or loops. A vertex u is a *neighbor* or is *adjacent* to v if there is an edge $uv \in E(G)$, and the *neighborhood* $N(v) = N_G(v)$ is the set of all neighbors of v . For a vertex $v \in V(G)$, its *degree* $d(v) = d_G(v)$ is the number of times that v appears among the edges of G (in particular, loops at v are counted twice). If G has no loops, then $d_G(v)$ is the number of edges containing v , and if G also has no parallel edges, then $d_G(v) = |N_G(v)|$. The maximum degree of G is denoted by $\Delta(G)$. Some other fundamental concepts in graph theory that appear throughout our work are *paths*, *cycles*, *forests*, *trees*, *proper colorings*, *bipartite multigraphs*, *independent sets*, and *matchings*. The latter two are defined more generally for multi-hypergraphs below, and see [54] for definitions not covered here.

Part of our work will involve a directed version of multigraphs. A *directed multigraph* D is a pair (V, E) , where $V = V(D)$ is again a finite vertex set and $E = E(D)$ is a finite multiset of ordered pairs (u, v) of vertices from V , called the *arcs* of D . For the arc (u, v) , the vertex v is called the *head* and the vertex u is called the *tail* of the arc. Two arcs are said to be *parallel* if they have the same head and tail, and *anti-parallel* if the head of one is the tail of the other and vice versa. For a vertex v , its *indegree* $d^-(v) = d_D^-(v)$ is the number of arcs of D that have v as a head, and its *outdegree* $d^+(v) = d_D^+(v)$ is the number of arcs of D that have v as a tail. A *directed graph* (or *digraph*) is a directed multigraph with no directed loops or parallel edges. For a directed multigraph D , its *underlying undirected graph* G is the multigraph obtained from D by replacing every arc (u, v) by the edge $\{u, v\}$. In this case, we say that the directed multigraph D is an *orientation* of the multigraph G .

Part of our work will also involve hypergraphs. A *multi-hypergraph* G is a pair (V, E) , where $V = V(G)$ is again a finite vertex set and $E = E(G)$ is a finite multiset of subsets of vertices from V . (In contrast to multigraphs, we will assume in multi-hypergraphs that there are no repeated vertices in edges.) If there are no repeated edges in $E(G)$, then we will simply refer to G as a *hypergraph*. A multi-hypergraph G is *r -uniform* if every edge of G has size r , and in this case we will refer to G as an *r -graph*. An *r -graph* G is *r -partite* if there exists a partition of $V(G)$ into r parts $V_1, \dots, V_r$ such that every edge intersects every

part V_i in exactly one vertex. Thus, a loopless multigraph is the same as a 2-graph. Given a vertex subset $X \subseteq V(G)$, the (vertex-)induced subgraph $G[X]$ has vertex set X and all edges $\{u, v\} \in E(G)$ with $u, v \in X$. Given an edge subset $F \subseteq E(G)$, the edge-induced subhypergraph $G[F]$ has edge set F and vertex set $\bigcup F$ (so there are no isolated vertices). For a vertex v , its *degree* $d(v) = d_G(v)$ is the number of edges of G containing v . For two vertices u, v , their *codegree* is the number of edges of G containing both u and v . An *independent set* of G is a subset of its vertices that contains no edge as a subset. A *matching* of G is a set of pairwise disjoint edges.

Finally, much of our work on transversals will involve notions from combinatorial algebraic topology. Standard references for the algebraic topology we use are [94, 136]. An *abstract simplicial complex* (or just a *complex*) $\mathcal{C}$ is a collection of subsets (called *simplices*) of some ground set $V = V(\mathcal{C})$ (called the *vertex set*), such that if $A \in \mathcal{C}$ and $B \subseteq A$, then $B \in \mathcal{C}$. For a multi-hypergraph G , the collection $\mathcal{I}(G)$ of all independent sets of G forms a complex (called the *independence complex*), as does the collection $\mathcal{M}(G)$ of all matchings of G (called the *matching complex*). A *matroid* $\mathcal{M}$ is a complex with the following special augmentation property: If $A, B \in \mathcal{M}$ and $|A| < |B|$, then there exists $x \in B$ such that $A \cup \{x\} \in \mathcal{M}$. The simplices in a matroid $\mathcal{M}$ are usually called the *independent sets*, and see [140] for background on matroid theory. For a complex $\mathcal{C}$ and vertex subset $X \subseteq V(\mathcal{C})$, we denote the induced subcomplex $\mathcal{C}[X] = \{A \in \mathcal{C} : A \subseteq X\}$. Every complex $\mathcal{C}$ can be realized in some Euclidean space as a *geometric simplicial complex* $\|\mathcal{C}\|$ which is unique up to homeomorphism, and thus $\mathcal{C}$ can be analyzed as a topological space. The following topological notions are discussed in more detail in Appendix A, including some small explicit examples with illustration, and are also discussed in [5, 6, 13, 16]. A complex $\mathcal{C}$ is *homotopically k -connected* if the homotopy groups $\pi_j(\mathcal{C})$ are trivial for $-2 \leq j \leq k$. Likewise, a complex $\mathcal{C}$ is *homologically k -connected* if the reduced homology groups $\tilde{H}_j(\mathcal{C})$ with field coefficients are trivial for $-2 \leq j \leq k$. We define $\eta_\pi(\mathcal{C})$ as the largest integer $k \geq -2$ such that $\mathcal{C}$ is homotopically k -connected, plus two. Likewise, we define $\eta_H(\mathcal{C})$ as the largest integer $k \geq -2$ such that $\mathcal{C}$ is homologically k -connected, plus two. Intuitively speaking, these represent the largest dimension d for which $\mathcal{C}$ has no topological hole (of a homotopical or homological type) of dimension d or less. In general $0 \leq \eta_\pi(\mathcal{C}) \leq \eta_H(\mathcal{C})$, and a theorem of Hurewicz states that $\eta_\pi(\mathcal{C}) = \eta_H(\mathcal{C})$ whenever $\eta_\pi(\mathcal{C}) \geq 3$. There exist examples where $\eta_\pi(\mathcal{C}) < \eta_H(\mathcal{C})$ (see Example 2.38 in [94], in which we have $\eta_\pi = 2$ and $\eta_H = \infty$), but most of our complexes of interest have $\eta_\pi = \eta_H$. Both of these notions of connectedness have been used in the literature, so we discuss both of them in tandem.

1.2 Bounded degree arboricity

In this section, we describe our work on bounded degree arboricity. The work was inspired by the Goldberg-Seymour Conjecture for f -colorings. Given a multigraph G (which is allowed to have parallel edges and loops) and a weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 1}$ on its vertices, a *degree- f subgraph* is a spanning subgraph H of G such that every vertex v of G has degree $d(v)$

at most $f(v)$ in H . (Such a subgraph is also commonly called an f -bounded subgraph or an f -matching.) The f -chromatic index $\chi'_f(G)$ is the minimum number of degree- f subgraphs required to cover $E(G)$. In other words, it is the minimum number of colors required to edge-color G so that every color class is a degree- f subgraph, and such an edge-coloring is called an f -coloring.

The notion of an f -coloring was introduced by Hakimi and Kariv [90] as a way to generalize the case $f = 1$ of a *proper edge-coloring*, where $\chi'_1(G) = \chi'(G)$ is called the *chromatic index* of G . It is NP-hard to determine $\chi'(G)$ even if G is a 3-regular simple graph [109], so we cannot expect there to be a general efficient method to determine the f -chromatic index $\chi'_f(G)$. However, the *Goldberg-Seymour Conjecture for f -colorings*, due to Nakano, Nishizeki, and Saito [137], asserts that we should always be able to determine $\chi'_f(G)$ to within an additive error of 1. Define

$$\Delta_f(G) = \max_{v \in V(G)} \left\lceil \frac{d(v)}{f(v)} \right\rceil, \quad \mathcal{W}_f(G) = \max_{X \subseteq V(G), |X| \geq 2} \left\lceil \frac{|E(X)|}{\lfloor f(X)/2 \rfloor} \right\rceil,$$

where $f(X) = \sum_{v \in X} f(v)$. It is easy to see that $\chi'_f(G) \geq \max\{\Delta_f(G), \mathcal{W}_f(G)\}$. The Goldberg-Seymour Conjecture for f -colorings asserts the following.

Conjecture 1.1 (Goldberg-Seymour Conjecture for f -colorings [137]). *For every loopless multigraph G and weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 1}$, we have*

$$\chi'_f(G) \leq \max\{\Delta_f(G) + 1, \mathcal{W}_f(G)\}.$$

Conjecture 1.1 was inspired by the famous Goldberg-Seymour Conjecture due to Gupta [87], Goldberg [81], Andersen [26], and Seymour [153] for the case $f = 1$. When $f \geq 2$, we may allow loops in Conjecture 1.1. Hakimi and Kariv [90] proved that $\chi'_f(G) \leq \Delta_f(G) + 1$ if G is a simple graph, generalizing the classical upper bound of Vizing for the chromatic index [165]. They also proved a generalization of Shannon's upper bound [155]. Nakano, Nishizeki, and Saito [137] themselves approximated Conjecture 1.1 to within a factor of 9/8. Stiebitz, Scheide, Toft, and Favrholt [160] proved a fractional version of Conjecture 1.1, by connecting the problem to the description of the f -matching polytope. Much better approximations have been obtained for the case $f = 1$ of the chromatic index using the tool Tashkinov trees [46, 50, 151, 163], and a proof for this case was announced by Chen, Jing, and Zang [49]. A proof for Conjecture 1.1 has now also been announced [45].

Inspired by Conjecture 1.1, particularly the involvement of both the maximum degree $\Delta_f(G)$ and maximum density $\mathcal{W}_f(G)$ of the multigraph G , we work on the related problem of covering by bounded degree forests. Recall that a forest is a graph with no cycles. For a loopless multigraph G , its *arboricity* $a(G)$ is the minimum number of forests required to cover $E(G)$. Arboricity has the following maximum density formula due to Nash-Williams [139]:

$$a(G) = \max_{X \subseteq V(G), |X| \geq 2} \left\lceil \frac{|E(X)|}{|X| - 1} \right\rceil.$$

Moreover, arboricity can be computed in polynomial-time, and many of its aspects are well-understood through matroid theory (see [74]). On the other hand, *bounded degree arboricity* presents more challenges. Given a loopless multigraph G and a weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$ on the vertices of G , we define the *degree- f arboricity* $a_f(G)$ to be the minimum number of degree- f forests (forests in which every vertex v has degree at most $f(v)$) required to cover $E(G)$. The most well-known case of bounded degree arboricity is the case $f = 2$, where a degree-2 forest is called a *linear forest* and the degree-2 arboricity $a_2(G) = la(G)$ is called the *linear arboricity* of G . Linear arboricity was introduced by Harary [92], and it is NP-hard to determine the linear arboricity even of simple graphs with maximum degree four [141]. However, a conjecture called the *Linear Arboricity Conjecture*, due to Akiyama, Exoo, and Harary [19], states that the linear arboricity of any simple graph can be determined to within an additive error of one. Notice that the linear arboricity of any multigraph G satisfies $la(G) \geq \lceil \Delta(G)/2 \rceil$ because every linear forest covers at most 2 edges incident to any given vertex v of G . Moreover, $la(G) \geq a(G) = \lceil (\Delta(G) + 1)/2 \rceil$ if G is a $\Delta(G)$ -regular simple graph, where the equality follows from Nash-Williams' formula [139]. The Linear Arboricity Conjecture asserts the following.

Conjecture 1.2 (Linear Arboricity Conjecture [19]). *For every simple graph G , we have*

$$la(G) \leq \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil.$$

There has been a lot of effort dedicated toward establishing the Linear Arboricity Conjecture. It has been proven for special classes of graphs such as complete graphs [158], complete bipartite graphs [19], series-parallel graphs [176], and planar graphs [175, 177]. The conjecture has also been shown to hold for $\Delta(G) \in \{3, 4, 5, 6, 8, 10\}$ (see [19, 20, 63, 85], although the full proof for $\Delta(G) = 10$ does not seem to be publically available). It has been shown to hold for graphs with large girth [21, 22] and for all graphs asymptotically [21, 22, 25, 69, 123]. The best known asymptotic result, due to Lang and Postle [123], states that $la(G) = \Delta/2 + O(\Delta^{1/2} \log^4 \Delta)$ where $\Delta = \Delta(G) \rightarrow \infty$. The best known exact result for all graphs G , due to Guldan [86], states that $la(G) \leq \frac{3\Delta(G)}{5}$ when $\Delta(G)$ is even, and $la(G) \leq \frac{3\Delta(G)+2}{5}$ when $\Delta(G)$ is odd. McDiarmid and Reed [129] showed that the conjecture holds for random regular graphs with fixed degrees [129]. Glock, Kühn, and Osthus [79] showed that the linear arboricity of a random graph $G = G(n, p)$ is $\lceil \Delta(G)/2 \rceil$ for a large range of p , and they also verified the Linear Arboricity Conjecture for large and sufficiently dense regular graphs.

1.2.1 An orientation approach

In Chapter 2, we contribute new results on bounded degree arboricity, including linear arboricity, by connecting it to the problem of orientations of multigraphs with bounds on both indegrees and outdegrees. Using this orientation approach, we establish the Linear

Arboricity Conjecture for sparse graphs with sufficiently large maximum degree, improving on previous results of this form. This chapter is based on [168].

Our orientation framework is the following. Given a multigraph G and two weight functions $g, h : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ on its vertices, we say that G is (g, h) -upper-orientable if it has an orientation into a directed multigraph D so that every vertex v of D has indegree $d_D^-(v) \leq g(v)$ and outdegree $d_D^+(v) \leq h(v)$. We call such an orientation D a (g, h) -upper-orientation of G . Let $\chi'_{(g,h)}(G)$ denote the minimum number of (g, h) -upper-orientable subgraphs of G required to cover $E(G)$. We define the quantities

$$\Delta_{(g,h)}(G) = \max_{v \in V(G)} \left\lceil \frac{d(v)}{g(v) + h(v)} \right\rceil, \quad \mathcal{W}_{(g,h)}(G) = \max_{X \subseteq V(G), E(X) \neq \emptyset} \left\lceil \frac{|E(X)|}{\min\{g(X), h(X)\}} \right\rceil.$$

It is easy to show that $\chi'_{(g,h)}(G) \geq \max\{\Delta_{(g,h)}(G), \mathcal{W}_{(g,h)}(G)\}$. We prove the following exact result on $\chi'_{(g,h)}(G)$, which has some resemblance to the Goldberg-Seymour Conjecture for f -colorings. The term *valid pairs* below refers to weight functions (g, h) for which $\chi'_{(g,h)}(G) < \infty$ (see Section 2.1 for a cleaner definition).

Theorem 1.3. *For every multigraph G and valid pair of weight functions (g, h) , we have*

$$\chi'_{(g,h)}(G) = \max\{\Delta_{(g,h)}(G), \mathcal{W}_{(g,h)}(G)\}.$$

We will also discuss other covering and packing results related to Theorem 1.3 in Section 2.4. Our main direct application of Theorem 1.3 is an exact formula for *bounded degree pseudoarboricity*. A *pseudoforest* is a multigraph in which every component has at most one cycle (possibly a loop), and the *pseudoarboricity* $pa(G)$ of a multigraph G is the minimum number of pseudoforests required to cover $E(G)$. Pseudoarboricity has the following maximum density formula, due to Hakimi [89]:

$$pa(G) = \max_{X \subseteq V(G), |X| \geq 1} \left\lceil \frac{|E(X)|}{|X|} \right\rceil.$$

This pseudoarboricity formula is similar to the arboricity formula of Nash-Williams [139], and indeed both formulas are special cases of the Matroid Partition Theorem [60]. Given a multigraph G and a weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$ on the vertices of G , the *degree- f pseudoarboricity* $pa_f(G)$ is the minimum number of degree- f pseudoforests required to cover $E(G)$. Degree- f pseudoforests are the same as $(1, f-1)$ -upper-orientable multigraphs, so applying Theorem 1.3, we derive the following exact formula for degree- f pseudoarboricity when $f \geq 2$.

Theorem 1.4. *For every multigraph G and weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$, we have*

$$pa_f(G) = \max\{\Delta_f(G), pa(G)\}.$$

Then by approximating bounded degree forests via bounded degree pseudoforests, we use Theorem 1.4 to deduce the following upper bound on degree- f arboricity when $f \geq 2$.

Theorem 1.5. *For every loopless multigraph G and weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$, we have*

$$a_f(G) \leq \max \{ \Delta_f(G) + 1, 2pa(G) \}.$$

Compare Theorem 1.5 with the easy lower bound $a_f(G) \geq \max \{ \Delta_f(G), a(G) \}$. It is useful to note that $pa(G) \leq a(G) \leq 2pa(G)$ for all loopless multigraphs G , and $pa(G) \leq a(G) \leq pa(G) + 1$ for all simple graphs G (which are easy consequences of Nash-Williams' and Hakimi's formulas). In the case of linear arboricity $f = 2$, we slightly modify the proof of Theorem 1.5 to show that the Linear Arboricity Conjecture (Conjecture 1.2) holds when the maximum degree $\Delta(G)$ is sufficiently large compared to the pseudoarboricity $pa(G)$ of the multigraph.

Theorem 1.6. *For every loopless multigraph G , we have $la(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$ whenever $\Delta(G) \geq 4pa(G) - 2$.*

Theorem 1.6 improves on previous results that established the Linear Arboricity Conjecture when the maximum degree is somewhat larger than some maximum density (e.g., degeneracy [42, 47, 113], treewidth [162]). Moreover, our upper bound holds more generally for multigraphs. Interestingly, an improvement on Theorem 1.6 for simple graphs G has subsequently been established by Chen, Hao, and Yu [48], stating that the Linear Arboricity Conjecture holds when $\Delta(G) \geq 3pa(G) - 1$. They use the same basic approximation approach via bounded degree pseudoarboricity, which suggests that our work could be useful for further progress on the Linear Arboricity Conjecture and related problems. Our basic approach of approximating, solving, or understanding forest problems by proving easier results for pseudoforests has also played an important role in various other problems, such as more efficient arboricity algorithms [35, 74], the matroid secretary problem [157], Nine Dragon Tree conjectures [67, 84], and the Kohayakawa-Kreuter Conjecture [51].

1.2.2 Further approximations

We continue our work on bounded degree arboricity in Chapter 3. In this chapter, we disprove a conjecture of Truszczyński [164] for multigraphs but prove some partial results about it for simple graphs. This chapter is based on [170].

Comparing our Theorem 1.5 to the Goldberg-Seymour Conjecture for f -colorings (Conjecture 1.1) and the easy lower bound $a_f(G) \geq \max \{ \Delta_f(G), a(G) \}$, it seems natural to conjecture the following improved upper bound, which was a conjecture of Truszczyński [164] when f is constant.

Conjecture 1.7 (Truszczyński [164]). *For every loopless multigraph G and weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$, we have*

$$a_f(G) \leq \max \{ \Delta_f(G) + 1, a(G) \}.$$

More strongly, Truszczynski [164] conjectured that for every multigraph G and integer $t \geq 2$, we have $a_t(G) = \max\{\lceil \Delta(G)/t \rceil, a(G)\}$ unless $a(G) = \Delta(G)/t$, in which case we have $a_t(G) \in \{\Delta(G)/t, \Delta(G)/t + 1\}$. He established this conjecture when G is a complete multigraph with all edges having the same multiplicity, when G is a complete bipartite multigraph with all edges having the same multiplicity, when the underlying simple graph of G is a forest, and when $t \geq \Delta(G) - a(G) + 1$. Notice that in the particular case $f = 2$, Conjecture 1.7 asserts that the linear arboricity of every multigraph G satisfies

$$la(G) \leq \max \left\{ \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1, a(G) \right\}.$$

Since Nash-Williams' arboricity formula [139] implies that $a(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$ for every simple graph G , the above upper bound implies that $la(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil + 1$ for every simple graph G , which is almost the Linear Arboricity Conjecture. This shows that Conjecture 1.7 is close to a generalization of the Linear Arboricity Conjecture, both to multigraphs and to other weight functions f .

We show that Conjecture 1.7 is false for general multigraphs G and any fixed constant weight function $f = t$.

Theorem 1.8. *For every integer $t \geq 2$, there exists a constant $c_t > 1$ such that the following holds. For infinitely many integers $d \geq 2$, there exists a multigraph G such that $\max\{\Delta_t(G), a(G)\} = d$ and $a_t(G) \geq c_t d$.*

We will prove Theorem 1.8 by exhibiting multigraphs G_t that have large *fractional* degree- t arboricity. We will then obtain the graph G of the theorem by replacing every edge in G_t with many parallel edges, which will make $a_t(G)$ and $\max\{\Delta_t(G), a(G)\}$ grow arbitrarily far apart by a constant factor $c_t > 1$. Our examples will show that we can take the constant c_t in Theorem 1.8 to satisfy $c_t \geq (4t+7)/(4t+6)$ for every integer $t \geq 2$, with slight improvements $c_2 \geq 9/8, c_3 \geq 15/14, c_4 \geq 21/20$ for $t \in \{2, 3, 4\}$.

On the other hand, we believe that Conjecture 1.7 holds for all simple graphs G . We will prove some partial results in this direction. Notice that if G is a simple graph that is $\Delta(G)$ -regular, then $a(G) = \lceil (\Delta(G) + 1)/2 \rceil$ by Nash-Williams' formula [139], and then Conjecture 1.7 for simple graphs would be implied by the Linear Arboricity Conjecture. But for non-regular simple graphs G , a reduction to the Linear Arboricity Conjecture is not clear. We prove that Conjecture 1.7 nearly holds for simple graphs with large girth.

Theorem 1.9. *Let G be a simple graph, let $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$ be a weight function, and let $d = \max\{\Delta_f(G), a(G)\}$. If G has girth $g \geq 4d$, then $a_f(G) \leq d + 1$.*

We also prove that Conjecture 1.7 holds for all simple graphs asymptotically when the weight function f is bounded.

Theorem 1.10. *For every integer $t \geq 2$, there exists a real constant $c_t > 0$ such that for every simple graph G and weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$ with maximum value at most t , we have*

$$a_f(G) \leq d + c_t d^{3/4} (\log d)^{1/2},$$

where $d = \max\{\Delta_f(G), a(G)\}$.

The proofs of Theorem 1.9 and Theorem 1.10 will be extensions of Alon's [21, 22, 24] probabilistic proofs for the case of linear arboricity $f = 2$, and they will also involve the orientation and pseudoarboricity ideas from the previous chapter. As with Alon's proofs, it is more convenient to prove our results in the setting of directed graphs. We will first prove analogues of our large-girth and asymptotic results for directed graphs, involving the notion of a *degree- f branching* which is analogous to a degree- f forest, and from those we will deduce Theorem 1.9 and Theorem 1.10.

1.3 Transversals in bounded degree graphs

In this section, we describe our work on transversal problems in bounded degree graphs, focusing on independent transversals and full rainbow matchings. Our original motivation for studying transversals was their connection to coloring and list coloring problems, which still form a significant part of our work, but transversals are also a source of many other interesting problems.

Given a ground set X and a family $\mathcal{P} = \{X_1, \dots, X_n\}$ of (not necessarily distinct) subsets of X , a *transversal* (or a *system of distinct representatives*) is a subset $\{x_1, \dots, x_n\}$ of X such that $x_i \in X_i$ for every i . Hall's Theorem [91] states that there exists a transversal if and only if

$$\left| \bigcup_{i \in I} X_i \right| \geq |I| \quad \text{for all } I \subseteq [n],$$

and we can rephrase transversals equivalently in terms of matchings in bipartite multigraphs. In many cases, we seek transversals that also meet specified conditions. For example, if X is the ground set of a matroid $\mathcal{M}$ with rank function r , then Rado's Theorem [144] states that there exists a transversal that is also an independent set of $\mathcal{M}$ if and only if

$$r \left(\bigcup_{i \in I} X_i \right) \geq |I| \quad \text{for all } I \subseteq [n].$$

Our work focuses on transversals that are independent sets of a graph or matchings of a multi-hypergraph, particularly ones with bounded degree. For our results, we will restrict

our attention to families $\mathcal{P}$ that are partitions of the ground set, since this simplifies our discussion and many problems on general families can be reduced to the case of partitions.

Part of our work will involve a topological generalization of Hall’s and Rado’s Theorem to all abstract simplicial complexes. See Appendix A for more details on the definitions and notations. Recall that for a complex $\mathcal{C}$, we defined the topological connectedness parameters $\eta_\pi(\mathcal{C})$ and $\eta_H(\mathcal{C})$. Given a complex $\mathcal{C}$ and a partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of its vertex set $V(\mathcal{C})$, a *colorful simplex* is a transversal of $\mathcal{P}$ that is simplex in $\mathcal{C}$. Meshulam [131] proved the following topological Hall theorem for the existence of a colorful simplex, while earlier versions were proved by Aharoni and Haxell [15] and by Meshulam [130]¹. Here, η is used to denote either η_π or η_H .

Theorem 1.11 ([15, 130, 131]). *Let $\mathcal{C}$ be a complex, and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(\mathcal{C})$. If*

$$\eta\left(\mathcal{C}\left[\bigcup_{i \in I} V_i\right]\right) \geq |I| \quad \text{for all } I \subseteq [n],$$

then there exists a colorful simplex.

Theorem 1.11 allows us to deduce results on the existence of a colorful simplex using lower bounds on the connectedness of the complex. There is a more recent (homological) strengthening of Theorem 1.11 (see Theorem A.2), but Theorem 1.11 suffices for our work. Theorem 1.11 generalizes Rado’s Theorem, which is the special case when $\mathcal{C} = \mathcal{M}$ is a matroid (where $\eta(\mathcal{M}) = r(\mathcal{M})$ if $\mathcal{M}$ has no coloop, and $\eta(\mathcal{M}) = \infty$ otherwise; see [34]). Theorem 1.11 also generalizes Sperner’s Lemma, which is the special case when $\mathcal{C}$ is a triangulation of a simplex and $\mathcal{P}$ forms a “Sperner coloring”. In fact, for the homotopical case $\eta = \eta_\pi$ one can prove Theorem 1.11 by reducing it to Sperner’s Lemma (see [6, 15, 100]), which allows one to give algorithmically effective (though inefficient) triangulation proofs of the various applications of Theorem 1.11 that we will discuss. For brevity, our proofs will stick with the topological language.

1.3.1 Extremal and stability results for independent transversals

Our work on transversals starts in Chapter 4, where we investigate extremal and stability theorems for independent transversals in graphs involving both maximum degree and average degree conditions. This chapter is based on joint work with Haxell [106].

For a graph G and partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$, an *independent transversal* (IT) of G with respect to $\mathcal{P}$ is a transversal of $\mathcal{P}$ that is an independent set of G . In other words,

¹According to [130], Aharoni privately communicated that the methods of [15] implies the statement of Theorem 1.11 for matching complexes of multi-hypergraphs. No special properties of matching complexes were used in that proof, so this earlier version also essentially captures Theorem 1.11.

it is a colorful simplex in the *independence complex* $\mathcal{I}(G)$ of a graph G , which is the complex consisting of all independent sets of G . In the literature, an independent transversal has also been referred to as an *independent system of representatives* or a *rainbow independent set*. Independent transversals have many combinatorial applications, such as to SAT [99], linear arboricity [21, 22], hypergraph matchings [15], list coloring [96, 127], circular coloring [114], linear groups [39], resource allocation [27, 103], and many others (see [82] and the references therein). It is NP-complete to decide whether a given vertex-partitioned graph has an IT, but there has been a lot of effort in proving sufficient conditions for the existence of an IT. Many of these sufficient conditions require the classes of $\mathcal{P}$ to have size large enough with respect to certain parameters of G and $\mathcal{P}$. The techniques used to prove such results have included the topological approach mentioned above [5, 7, 13, 14, 15, 16, 130, 131], as well as purely combinatorial arguments [31, 95, 96, 102], arguments related to the Lovász Local Lemma and entropy compression [21, 22, 24, 58, 167, 166], and the Rödl nibble method [43, 78, 116, 127].

When no special information is known about the structure of the graph with respect to the vertex partition, the combinatorial and topological methods appear to work best, giving best possible results in many cases. One example is the following frequently applied theorem due to Haxell [95, 96].

Theorem 1.12 (Haxell [95, 96]). *Let G be a graph and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. If $|V_i| \geq t$ for every i and G has maximum degree $\Delta(G) \leq t/2$, then there exists an IT.*

The original proof of Theorem 1.12 was combinatorial, but topological methods based on Theorem 1.11 have also shown to yield the Theorem 1.12 [14, 130, 131]. Then Szabó and Tardos [161] proved that the condition of Theorem 1.12 is best possible, by showing that for every $d \geq 1$, the graph G which is the disjoint union of $2d - 1$ copies of the complete bipartite graph $K_{d,d}$ admits a vertex partition $\mathcal{P} = \{V_1, \dots, V_{2d}\}$ such that $|V_i| = 2d - 1$ for every i and there is no IT. Such constructions were previously obtained independently by Jin [110] and Yuster [178] when d is a power of 2, although their partitions were different from those of Szabó and Tardos [161] (a matter that we will address in Chapter 5).

When certain interactions between the graph G and the vertex partition $\mathcal{P}$ are known, in particular when there is some limit on the number of edges between any pair of classes or the number of edges incident to any particular class, then the other techniques mentioned above tend to give stronger results. One example of this, due to Wanless and Wood [167], considers the IT problem with respect to the *maximum class average degree*, which is the quantity

$$\text{mad}(G, \mathcal{P}) = \max_{U \in \mathcal{P}} \frac{1}{|U|} \sum_{v \in U} d_G(v).$$

This should not be confused with the *maximum average degree* of the graph G , which is the maximum density parameter $\text{mad}(G) = \max \left\{ \frac{2|E(X)|}{|X|} : X \subseteq V(G), X \neq \emptyset \right\}$.

Theorem 1.13 (Wanless, Wood [167]). *Let G be a graph and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. If $|V_i| \geq t$ for every i and $G, \mathcal{P}$ has maximum class average degree $\text{mad}(G, \mathcal{P}) \leq t/4$, then there exists an IT.*

Wanless and Wood [167] proved Theorem 1.13 using the method of Rosenfeld counting [147], showing that in fact there exist at least $(t/2)^n$ distinct IT's. In [58, 116], it was noted that Theorem 1.13 follows from the Local Cut Lemma of Bernshteyn [32], and in [166] the theorem was proved using an entropy compression algorithm that runs in expected polynomial time. In view of Theorem 1.12, it was an open problem to determine whether the bound of $t/4$ in Theorem 1.13 can be improved to $t/2$. This problem was later resolved by Groenland, Kaiser, Treffers, and Wales [83], who showed that the bound $t/4$ is asymptotically best possible as $t \rightarrow \infty$.

Since the conditions of Theorem 1.12 and Theorem 1.13 are each asymptotically best possible in their respective settings, it is natural to try to interpolate between the two conditions. Groenland, Kaiser, Treffers, and Wales [83] asked the following general question.

Question 1.14 (Groenland, Kaiser, Treffers, Wales [83]). *What can be said about the set of pairs $(\alpha, \beta) \in [0, 1]^2$ such that for sufficiently large t , there exists a graph G and a partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$ such that $|V_i| \geq t$ for every i , $\Delta(G) \leq \alpha t$, $\text{mad}(G, \mathcal{P}) \leq \beta t$, and there is no IT?*

Our first result on independent transversals is a complete answer to Question 1.14. We will say that a pair $(\alpha, \beta) \in \mathbb{R}_{\geq 0}^2$ is a *good pair* if the following holds for every integer $t \geq 1$: Whenever G is a graph and $\mathcal{P} = \{V_1, \dots, V_n\}$ is a partition of $V(G)$ such that $|V_i| \geq t$ for every i , G has maximum degree $\Delta(G) \leq \alpha t$, and $G, \mathcal{P}$ has maximum class average degree $\text{mad}(G, \mathcal{P}) \leq \beta t$, there exists an IT. We prove the following characterization of the set of good pairs.

Theorem 1.15. *The pair $(\alpha, \beta) \in \mathbb{R}_{\geq 0}^2$ is good if and only if one of the following holds:*

- (1) $\alpha \leq 1/2$,
- (2) $\beta \leq 1/4$,
- (3) $\beta \leq 2\alpha(1 - \alpha)$.

In particular, Theorem 1.15 answers positively another question of [83], about whether in a good pair (α, β) , if β approaches $1/4$ from above, then α can be at most $\alpha' = \frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.85355$ (the larger solution to $2\alpha'(1 - \alpha') = \frac{1}{4}$). The latter question was asked because the authors of [83] were able to modify their construction for $\beta \sim 1/4$ to also have maximum degree at most $\alpha't$, but were unable to decrease the maximum degree any further.

To prove that the condition of Theorem 1.15 is sufficient for being a good pair, we use the topological approach based on Theorem 1.11. We prove one direction of Theorem 1.15 by

first proving a general technical theorem about the structure of graphs G with a given upper bound on the connectedness $\eta(\mathcal{I}(G))$, namely Theorem 4.9. One of the consequences of Theorem 4.9 is the following result, which states that graphs with low topological connectedness must be fairly dense.

Theorem 1.16. *Let G be a graph with N vertices and maximum degree d . Suppose that $\eta(\mathcal{I}(G)) \leq k$. Then $|E(G)| \geq dN - d^2k$.*

Since an upper bound on $\text{mad}(G, \mathcal{P})$ easily implies an upper bound on the density $|E(G_I)|/|V(G_I)|$ for any $I \subseteq [n]$, where $G_I = G[\bigcup_{i \in I} V_i]$, from here it is a short step to combine Theorem 1.11 and Theorem 1.16 to prove that condition (1), (2), or (3) of Theorem 1.15 is sufficient for being a good pair. Next, we provide a construction demonstrating that these conditions are necessary.

Theorem 1.17. *For all $\alpha > \frac{1}{2}$, $\beta > \max\{\frac{1}{4}, 2\alpha(1 - \alpha)\}$, and sufficiently large t , there exist a graph G and partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$ such that $|V_i| \geq t$ for every i , $\Delta(G) \leq \alpha t$, $\text{mad}(G, \mathcal{P}) \leq \beta t$, and G has no IT with respect to $\mathcal{P}$.*

We will prove Theorem 1.17 by iteratively applying a simple lemma (Lemma 1.20 below) that provides a way of constructing large vertex-partitioned graphs with no IT starting from smaller ones. This lemma is quite versatile and will be the subject of Chapter 5 and Chapter 6, including some applications to list coloring and list edge-coloring.

Our other results in Chapter 4 are stability theorems about graphs with a given maximum degree and low topological connectedness or no IT. Theorem 4.4 (proved in Section 4.2) states that every graph G with maximum degree d and at least $2dk$ vertices has topological connectedness $\eta(\mathcal{I}(G)) \geq k$, with equality precisely when G is a disjoint union of k copies of $K_{d,d}$. We prove the following stability version of this result.

Theorem 1.18. *Let G be a graph with N vertices and maximum degree d , and let ϵ be such that $N(1 + \epsilon) = 2dk$. Suppose that $\eta(\mathcal{I}(G)) \leq k$. Then $\epsilon \geq 0$ and $2|E(G)| \geq dN(1 - \epsilon)$, and $V(G)$ has a partition into sets $\{X_i, Y_i, Z_i\}_{i=1}^k$ such that $\sum_{i=1}^k |Z_i| \leq \epsilon N$ and the following hold:*

- (i) $\left|E(G) \setminus \bigcup_{i=1}^k E(X_i, Y_i)\right| \leq 2\epsilon dN$,
- (ii) $\sum_{i=1}^k |X_i||Y_i| - \left|\bigcup_{i=1}^k E(X_i, Y_i)\right| \leq \epsilon dN/2$.

Thus, when $|V(G)|$ is close to $2dk$ but $\eta(\mathcal{I}(G)) \leq k$, there are two respects in which G is close in structure to the disjoint union J of complete bipartite graphs with vertex sets $\{(X_i, Y_i)\}_{i=1}^k$: at most a small proportion of the edges of G are not edges of J , and almost all the edges of J are edges of G . We say that the graph G is γ -approximated by J if $V(J) \subseteq V(G)$ and the symmetric difference of $E(G)$ and $E(J)$ has size at most $\gamma|E(G)|$. Combining Theorem 1.18 and Theorem 1.11, our other result is the following stability theorem about graphs with no IT's that almost satisfy Theorem 1.12.

Theorem 1.19. *Let $0 < \beta < 1$, let G be a graph, and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. If $|V_i| \geq t$ for every i , $d = \Delta(G) \leq (1 + \beta)t/2$, and G has no IT with respect to $\mathcal{P}$, then for some $I \subseteq [n]$, the graph $G_I = G[\bigcup_{i \in I} V_i]$ satisfies $|E(G_I)| > (1 - \beta)d|V(G_I)|/2$, and G_I is $5\beta/(1 - \beta)$ -approximated by a disjoint union J of $|I| - 1$ complete bipartite graphs.*

1.3.2 Constructions with no independent transversals

Our work on independent transversals in graphs continues in Chapter 5, where we apply an iterative method to construct extremal bounded degree graphs with no IT. This chapter is based on joint work with Haxell [105].

Our construction method will involve different iterative applications of the following simple lemma.

Lemma 1.20. *Let G and H be disjoint graphs, and let $\mathcal{P} = \{V_1, \dots, V_n\}$ and $\mathcal{Q} = \{W_1, \dots, W_m\}$ be partitions of $V(G)$ and $V(H)$, respectively. Let $\mathcal{R} = \{V'_1, \dots, V'_n, W_1, \dots, W_{m-1}\}$, where $V'_1 \supseteq V_1, \dots, V'_n \supseteq V_n$ are obtained by distributing each of the vertices of W_m into one of $V_1, \dots, V_n$ arbitrarily. If G has no IT with respect to $\mathcal{P}$ and H has no IT with respect to $\mathcal{Q}$, then $G \cup H$ has no IT with respect to $\mathcal{R}$.*

Proof. Assume for contradiction that $G \cup H$ has an IT $\{v_1, \dots, v_n, w_1, \dots, w_{m-1}\}$ with respect to $\mathcal{R}$, where $v_i \in V'_i$ for $1 \leq i \leq n$ and $w_j \in W_j$ for $1 \leq j \leq m - 1$. If $v_i \in V_i$ for every $1 \leq i \leq n$, then $\{v_1, \dots, v_n\}$ is an IT of G with respect to $\mathcal{P}$, a contradiction. So suppose instead that $v_j \in V'_j \cap W_m$ for some $1 \leq j \leq n$. Then $\{w_1, \dots, w_{m-1}, v_j\}$ is an IT of H with respect to $\mathcal{Q}$, again a contradiction. $\square$

Our applications of Lemma 1.20 in this chapter will focus on bounded degree graphs. Recall Theorem 1.12 of Haxell [95, 96], which is rephrased below for convenience.

Theorem 1.21 (Haxell [95, 96]). *Let G be a graph with maximum degree Δ , and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. If $|V_i| \geq 2\Delta$ for every i , then there exists an IT.*

Theorem 1.21 improved on Alon's [21, 24] earlier result that there exists an IT whenever $|V_i| \geq 2e\Delta$ for every i . Recall that Szabó and Tardos [161] subsequently showed that the class size condition in Theorem 1.21 cannot be further improved, by proving the following.

Theorem 1.22 (Szabó, Tardos [161]). *Let $\Delta \geq 1$ and let G be the disjoint union of $2\Delta - 1$ copies of the complete bipartite graph $K_{\Delta, \Delta}$. There exists a partition $\mathcal{P} = \{V_1, \dots, V_{2\Delta}\}$ of $V(G)$ such that $|V_i| = 2\Delta - 1$ for every i and there is no IT.*

Theorem 1.22 was previously proved by Jin [110] and by Yuster [178] independently when Δ is a power of 2. However, their partitions $\mathcal{P}$ were different from those of Szabó and Tardos [161]. Additionally, Szabó and Tardos [161] proved a more general version of Theorem 1.22 where the class sizes $|V_i|$ depend on the number of classes n as well as on the maximum degree

Δ (see Theorem 5.7). Together with a refinement of Theorem 1.12 proven in [102], this more general construction of Szabó and Tardos [161] solved a problem of Bollobás, Erdős, and Szemerédi [37], who were interested in the complementary, Turán-type problem of minimum degree conditions for the existence of a clique K_n in a balanced n -partite graph. In addition to the above mentioned constructions [110, 161, 178], many other non-trivial constructions of graphs with no IT have been described with different applications [16, 23, 36, 83]. However, before our work there has not been an explicit systematic procedure for constructing vertex-partitioned graphs with no IT.

Our first application of Lemma 1.20 is a more streamlined proof of Theorem 1.22 that unifies previous constructions. Notice that the complete bipartite graph $K_{\Delta,\Delta}$ has no IT with respect to its standard bipartition (the vertex partition into two independent sets of size Δ). We will show in Section 5.1 that we can reconstruct each of the partitions of Jin [110], Yuster [178], and Szabó and Tardos [161] for Theorem 1.22 effectively by applying Lemma 1.20 iteratively on $K_{\Delta,\Delta}$ in different ways. Our iterative method also yields many new constructions for Theorem 1.22 when $\Delta \geq 4$. One of our main new results is that in fact all possible constructions for Theorem 1.22 can be derived by iteratively applying Lemma 1.20 on $K_{\Delta,\Delta}$ with the standard bipartition.

Theorem 1.23. *All possible constructions $G, \mathcal{P}$ for Theorem 1.22 can be derived via Lemma 1.20, by iteratively adding $K_{\Delta,\Delta}$ with the standard bipartition.*

Besides constructions for Theorem 1.22, we will use Lemma 1.20 to find many new constructions of graphs with no IT that answer some open questions. Let G be a graph and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. Using topological methods, Aharoni, Holzman, Howard, and Sprüssel [16] proved that if G has maximum degree $\Delta \neq 2$, $|V_i| \geq 2\Delta - 1$ for every i , and G has no IT, then G necessarily contains the disjoint union of $2\Delta - 1$ copies of $K_{\Delta,\Delta}$. Therefore, Theorem 1.22 verifies that the disjoint union of $2\Delta - 1$ copies of $K_{\Delta,\Delta}$ is the unique minimal obstruction to the existence of an IT when the graph has maximum degree $\Delta \neq 2$ and the class sizes satisfy $|V_i| \geq 2\Delta - 1$. On the other hand, for the case of maximum degree $\Delta = 2$ and $|V_i| \geq 3$, the same authors proved that G necessarily contains the disjoint union of three cycles each of length 1 modulo 3. This result is related to the cycle-plus-triangles problem (see [13, 56, 70, 149]). We verify that, indeed, all disjoint unions of three cycles of lengths 1 modulo 3 are minimal obstructions to the existence of an IT when $\Delta = 2$ and $|V_i| \geq 3$ for every i .

Theorem 1.24. *Let $G = C_{\ell_1} \sqcup C_{\ell_2} \sqcup C_{\ell_3}$ be the disjoint union of three cycles with lengths $\ell_j \equiv 1 \pmod{3}$, $\ell_j \geq 4$ for $1 \leq j \leq 3$. There exists a partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$ such that $|V_i| = 3$ for every i and G has no IT.*

Although the class size lower bound 2Δ in Theorem 1.12 is best possible for all $\Delta \geq 1$, there has been interest in decreasing it in some contexts. These decrements are achieved by restricting either the graphs G or the types of vertex partitions $\mathcal{P}$ under consideration, so as to avoid structures present in the known constructions [110, 161, 178] for Theorem 1.22. We will present constructions in both of these directions.

First we focus on the case when we restrict the structure of the partition $\mathcal{P}$ around the graph G , by working in the setting of locally sparse graphs. The *local degree* of $G, \mathcal{P}$ is the maximum, over all $v \in V(G)$ and $V_i \in \mathcal{P}$ not containing v , of the number of edges between v and V_i . The *multiplicity* of G is the maximum, over all connected components C of G and $V_i \in \mathcal{P}$, of the number of vertices in $C \cap V_i$. Loh and Sudakov [127] proved that if G has maximum degree Δ , local degree $m = o(\Delta)$, and $|V_i| \geq \Delta + o(\Delta)$ for every i , then there exists an IT. Thus the 2Δ lower bound in Theorem 1.12 decreases significantly when we assume that the vertex-partitioned graph G is “locally sparse” (in the sense of having small local degree). Glock and Sudakov [78] and Kang and Kelly [116] independently strengthened Loh and Sudakov’s [127] result to the setting of maximum class average degree instead of maximum degree. To complement Loh and Sudakov’s [127] asymptotic result, we apply Lemma 1.20 to prove the following.

Theorem 1.25. *For all integers $\Delta \geq m \geq 1$ where Δ is a multiple of m , there exists a graph G and partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$ such that G has maximum degree Δ , local degree and multiplicity m , $|V_i| \geq \Delta + 2m - \left\lceil \frac{2m^2+m}{\Delta+m} \right\rceil$ for every $V_i \in \mathcal{P}$, and there is no IT.*

In particular, when $\Delta \geq 2m^2$, we can take the class sizes to satisfy $|V_i| \geq \Delta + 2m - 1$ for every i . Theorem 1.25 was previously proven when $m = 1$ by Bohman and Holzman [36], with a slightly simplified construction described in [16]. The construction of Bohman and Holzman [36] was actually a counterexample a list coloring conjecture of Reed [145], which is our next problem of interest.

Given a graph H and list assignment L for $V(H)$, an L -coloring is a vertex-coloring ϕ of H such that $\phi(x) \in L(x)$. An L -coloring is *proper* if each color class $\phi^{-1}(c)$ is an independent set of H . The problem of finding a proper L -coloring of H can easily be turned into the problem of finding an IT in an auxiliary graph G with vertex partition $\mathcal{P}$ (see Section 5.4 for details). This turns out to be useful for studying list colorings in a color degree setting. The *maximum color degree* of H, L is the maximum, over all colors c , of the maximum degree of the induced subgraph $H[\{x \in V(H) : c \in L(x)\}]$. The maximum color degree of H, L is the same as the maximum degree of G , and $G, \mathcal{P}$ has multiplicity $m = 1$ (in particular it is locally sparse). As a color degree generalization of the greedy upper bound on the list chromatic number, Reed [145] conjectured that if H, L has maximum color degree Δ and $|L(x)| \geq \Delta + 1$ for every $x \in V(H)$, then H has a proper L -coloring. Reed and Sudakov [146] proved that there exists a proper L -coloring under the assumption that $|L(x)| \geq \Delta + o(\Delta)$ for every $x \in V(H)$, and the asymptotic result of Loh and Sudakov [127] for locally sparse graphs mentioned above is a generalization of this. Aharoni, Berger, and Ziv [12] proved that Reed’s conjecture holds for chordal graphs. On the other hand, Bohman and Holzman [36] disproved Reed’s conjecture as follows.

Theorem 1.26 (Bohman, Holzman [36]). *For every $\Delta \geq 2$, there exists a graph H and list assignment L for $V(H)$ such that H, L has maximum color degree Δ , $|L(x)| = \Delta + 1$ for every $x \in V(H)$, and there is no proper L -coloring.*

For their proof of Theorem 1.26, Bohman and Holzman [36] exhibited appropriate vertex-partitioned graphs with no IT. We will use Lemma 1.20 to give a more streamlined proof of Theorem 1.26, where we use only $4(\Delta + 1)^2$ classes compared to the $2(\Delta + 1)(\Delta^2 + 1)$ classes used by Bohman and Holzman [36].

Our next results are counterexamples to some conjectures on IT's, specifically a conjecture of Aharoni, Alon, and Berger [5] and a question of Aharoni, Holzman, Howard, and Sprüssel [16]. First, Aharoni, Alon, and Berger [5] were interested in the existence of IT's in $K_{1,k}$ -free graphs, meaning graphs that do not contain the star $K_{1,k}$ as an induced subgraph. They conjectured that for every $K_{1,k}$ -free graph G with maximum degree Δ and partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$, if $|V_i| \geq \Delta + k - 1$ for every i , then there exists an IT. We will construct counterexamples to their conjecture for all $k \geq 3$. Next, recall that Aharoni, Holzman, Howard, and Sprüssel [16] proved that if G has maximum degree $\Delta > 2$, $|V_i| \geq 2\Delta - 1$ for every i , and there is no IT, then G contains the disjoint union of $2\Delta - 1$ copies of the complete bipartite graph $K_{\Delta,\Delta}$. These authors asked whether, additionally, there always exists a copy of $K_{\Delta,\Delta}$ in G that is contained in the union of two classes V_i, V_j . We give constructions showing that the answer is negative.

Despite the negative answer to the latter question of Aharoni, Holzman, Howard, and Sprüssel [16], we prove that a variation of their question does hold, namely when the graph G is exactly the disjoint union of $2\Delta - 1$ copies of $K_{\Delta,\Delta}$, as in Theorem 1.22 (Corollary 5.16). Actually we prove a more general theorem (Theorem 5.15), which roughly states that this kind of result holds more generally for disjoint unions of complete bipartite that are *class-minimal* with no IT. Our main interest in this variation is that it implies Theorem 1.23 above (and the more general Theorem 5.18). Our proof of Theorem 5.15 is a technical combinatorial argument.

1.3.3 Constructions with no full rainbow matchings

In Chapter 6, we will again apply our iterative method but this time in the setting of bounded degree multi-hypergraphs with no full rainbow matchings. This chapter is based on [169].

Let G be a multi-hypergraph, and let $\mathcal{P} = \{E_1, \dots, E_n\}$ be the set of color classes of a (not necessarily proper) edge-coloring of G , meaning that $\mathcal{P}$ is any partition of $E(G)$. A *rainbow matching* of G with respect to $\mathcal{P}$ is a matching of G that contains at most one edge from each color class E_i . A rainbow matching is *full* if it contains exactly one edge from each color class E_i . In other words, a full rainbow matching is a colorful simplex in the matching complex $\mathcal{M}(G)$, which is the collection of all matchings of G .

Work on large or full rainbow matchings in edge-colored multi-hypergraphs goes back to Euler's study of orthogonal Latin squares. A *Latin square of order n* is an $n \times n$ array consisting of n symbols such that each symbol appears exactly once in each row and in each column. A *partial transversal* of a Latin square of order n is a collection of entries of the array that do not share the same row, column, or symbol. A *transversal* is a partial

transversal of size n . Finding an orthogonal mate to a given Latin square is equivalent to finding a decomposition of that Latin square into transversals. However, the question of whether there exists even a single large partial transversal in a given Latin square is already an interesting question. The following celebrated conjecture of Ryser, Brualdi, and Stein [40, 148, 159] states that every Latin square has a partial transversal that misses at most one symbol.

Conjecture 1.27 (Ryser, Brualdi, Stein [40, 148, 159]). *For every integer $n \geq 1$, every Latin square of order n has a partial transversal of size $n - 1$. If n is odd, then every Latin square of order n has a transversal.*

A Latin square L of order n is equivalent to a proper edge-coloring of the complete bipartite graph $K_{n,n}$ with n colors: If the parts of $K_{n,n}$ are $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$, then the Latin square L corresponds to the proper edge-coloring of $K_{n,n}$ where the edge $x_i y_j$ receives the color $L_{i,j}$, for all $i, j \in \{1, \dots, n\}$. A partial transversal of L corresponds to a rainbow matching in this proper edge-coloring of $K_{n,n}$, and a transversal of L corresponds to a full rainbow matching in $K_{n,n}$. This translation of Latin square partial transversals into the language of rainbow matchings has motivated further conjectures on the existence of large rainbow matchings in edge-colored multigraphs. For example, as an attempt to generalize the Ryser-Brualdi-Stein Conjecture for even n to more general edge-colored multigraphs, Aharoni and Berger [7] conjectured that for any n , if a bipartite multigraph G is properly edge-colored into n color classes each of size at least $n + 1$, then there exists a full rainbow matching. Gao, Ramadurai, Wanless, and Wormald [76] conjectured that if a general multigraph G is properly edge-colored into n color classes each of size at least $n + 2$, then there exists a full rainbow matching. Both of these two conjectures have received a lot of attention with many partial and asymptotic results [28, 76, 119, 142] (see [135] for short proofs). Stronger asymptotic results have been proven for the Ryser-Brualdi-Stein Conjecture [93, 118], and a proof for large even n was recently announced by Montgomery [134].

Our work in Chapter 6 is on the different but related problem about the existence of full rainbow matchings in edge-colored multi-hypergraphs when we assume that the color class sizes are sufficiently large compared to the maximum degree of the multi-hypergraph, irrespective of the number of color classes. This perspective on rainbow matchings has received some attention in recent years [7, 10, 43, 53, 76], and it is motivated by the fact that this is the prevalent perspective for independent transversals as we saw above. We will construct extremal bounded degree multi-hypergraphs with no full rainbow matchings. This will involve the following translation of Lemma 1.20 from the language of independence complexes of graphs to the language of matching complexes of multi-hypergraphs. The proof is the same.

Lemma 1.28. *Let G and H be disjoint multi-hypergraphs, and let $\mathcal{P} = \{E_1, \dots, E_n\}$ and $\mathcal{Q} = \{F_1, \dots, F_m\}$ be color classes of edge-colorings of G and H , respectively. Let $\mathcal{R} = \{E'_1, \dots, E'_n, F_1, \dots, F_{m-1}\}$ be color classes of an edge-coloring of $G \cup H$, where $E'_1 \supseteq E_1, \dots, E'_n \supseteq E_n$ are obtained by distributing each of the edges of F_n into one of*

$E_1, \dots, E_n$ arbitrarily. If G has no full rainbow matching with respect to $\mathcal{P}$ and H has no full rainbow matching with respect to $\mathcal{Q}$, then $G \cup H$ has no full rainbow matching with respect to $\mathcal{R}$.

We will apply Lemma 1.28 similar to how we apply Lemma 1.20 for independent transversals. First, we focus on full rainbow matchings in general edge-colored r -graphs with a given maximum degree Δ . One sufficient condition for the existence of full rainbow matchings is the following theorem that easily follows from topological results of Aharoni, Berger, and Meshulam [11].

Theorem 1.29 (Aharoni, Berger, Meshulam [11]). *Let G be an r -graph with maximum degree Δ , and let $\mathcal{P} = \{E_1, \dots, E_n\}$ be the color classes of an edge-coloring of G . If $|E_i| \geq r\Delta$ for every i , then there exists a full rainbow matching.*

We will use Lemma 1.28 to prove that the bound $|E_i| \geq r\Delta$ in Theorem 1.29 is best possible for all $r \geq 1$ and $\Delta \geq 2$.

Theorem 1.30. *For all integers $r \geq 1$ and $\Delta \geq 2$, there exist an (r -partite) r -graph G with maximum degree Δ and an edge-coloring of G into color classes $\mathcal{P} = \{E_1, \dots, E_n\}$, such that $|E_i| \geq r\Delta - 1$ for every i and there is no full rainbow matching.*

For all integers $r \geq 2$ and $\Delta \geq 2$, we will give a large family of constructions for Theorem 1.30, some of which are r -partite and some of which are not. Interestingly, our r -graphs will be disjoint unions of blow-ups of *nets* from design theory, which are equivalent to sets of mutually orthogonal Latin squares. Previously, even for the case of multigraphs $r = 2$, Theorem 1.30 was only known when $\Delta = 2$, and the best constructions for larger values of Δ had color class sizes $|E_i| = 2\Delta - 2$ for every i . These latter constructions, due to Aharoni, Berger, Kotlar, and Ziv [10] and to Gao, Ramadurai, Wanless, and Wormald [76], disproved earlier conjectures by Aharoni and Berger [7]. We will also give some constructions on results related to Theorem 1.30. Specifically, we will give some optimal constructions when we fix the number of color classes n in addition to the uniformity r and the maximum degree Δ (Theorem 6.14). Also, we observe that the multi-hypergraphs that we construct for Theorem 1.30 have many parallel edges, and we will give some slightly less optimal multi-hypergraph constructions in which the none of the edges intersect too much (Theorem 6.15).

Our next results focus on full rainbow matchings when we assume that the edge-colorings are proper, i.e., that each color class E_i is a matching. This will have a similar flavor to work on independent transversals in locally sparse graphs. We notice that our constructions for Theorem 1.30 are far from being proper edge-colorings. Indeed, many of the color classes E_i induce subhypergraphs $G[E_i]$ whose maximum degree is within a constant factor of the maximum degree Δ of the entire r -graph G . When we restrict the maximum degree of each color class, optimal conditions for the existence of full rainbow matchings can often be drastically improved. For example, Delcourt and Postle [53] proved that there exists a full rainbow matching in an edge-colored r -graph G with maximum degree Δ whenever

$|E_i| \geq \Delta + o(\Delta)$ for every i , as long as the maximum degree of every color class $G[E_i]$ is $o(\Delta)$ and the maximum codegree of G is $o(\Delta)$. This is an improvement over the bound $|E_i| \geq r\Delta$ from Theorem 1.29 in this setting, and it generalizes previous results by Gao, Ramadurai, Wanless, and Wormald [76]. Delcourt and Postle [53] proved more strongly that under their hypotheses there exist Δ disjoint full rainbow matchings. Chakraborti and Loh [43] proved a slightly weaker version of their result for simple graphs but with an explicit error term.

Now, Delcourt and Postle [53] also formulated two conjectures on full rainbow matchings in properly edge-colored multigraphs G with a given maximum degree Δ and color classes $\mathcal{P} = \{E_1, \dots, E_n\}$. Their first conjecture is that if G is a bipartite multigraph and $|E_i| \geq \Delta + 1$ for every i , then there exists a full rainbow matching. Their second conjecture is that if G is a general multigraph and $|E_i| \geq \Delta + 2$ for every i , then there exists a full rainbow matching. These conjectures were suggested as “sparse” analogues of conjectures of Aharoni and Berger [7] and of Gao, Ramadurai, Wanless, and Wormald [76]. We disprove both of their conjectures with the following theorem. The first statement of the theorem is a general result asserting the existence of properly edge-colored multigraphs with no full rainbow matchings in which the color classes all have size at least the chromatic index χ' . Applying it with the Shannon triangle, we get examples where $|E_i| \geq \lfloor 3\Delta/2 \rfloor$ for every i and there is no full rainbow matching, which strongly disprove the second conjecture of Delcourt and Postle [53]. The second and third statements demonstrate that their two conjectures are false even when $\chi' = \Delta$.

Theorem 1.31. *The following statements hold:*

- (1) *For every multigraph H with maximum degree Δ and chromatic index χ' , there exists an associated multigraph G with maximum degree Δ and chromatic index χ' and a proper edge-coloring of G into color classes $\mathcal{P} = \{E_1, \dots, E_n\}$, such that $|E_i| \geq \chi' - 1$ for every i and there is no full rainbow matching. If $|E(H)| \leq 2\Delta - 1$, then we can take $|E_i| \geq \chi'$ for every i .*
- (2) *For every $\Delta \geq 2$, there exist a bipartite graph G with maximum degree Δ and a proper edge-coloring of G into color classes $\mathcal{P} = \{E_1, \dots, E_n\}$, such that $|E_i| \geq \Delta + 1$ for every i and there is no full rainbow matching.*
- (3) *For every integer $\Delta \geq 3$ with $\Delta \equiv 3, 0 \pmod{4}$, there exist a multigraph G with maximum degree and chromatic index Δ , and a proper edge-coloring of G into color classes $\mathcal{P} = \{E_1, \dots, E_n\}$, such that $|E_i| \geq \Delta + 2$ for every i and there is no full rainbow matching. If $\Delta \in \{2^m - 1, 2^m\}$ for some integer $m \geq 2$, then we can take G to be a simple graph.*

Our final results in Chapter 6 is about applying results on full rainbow matchings to list edge-colorings of multi-hypergraphs. The celebrated List Edge-Coloring Conjecture (conjectured by various authors, e.g., [38]) asserts that for every multigraph H , its list chromatic index $\chi'_\ell(H)$ is equal to its chromatic index $\chi'(H)$. Famously, Galvin [75] proved

that the List Edge-Coloring Conjecture holds for all bipartite multigraphs H (in which case $\chi'(H) = \Delta(H)$), which in particular proved a conjecture of Dinitz on Latin squares.

Theorem 1.32 (Galvin [75]). *For every bipartite multigraph H , we have $\chi'_\ell(H) = \Delta(H)$.*

The List Edge-Coloring Conjecture has also been proven to hold for a few other classes of multigraphs such as complete graphs of even degree [88] and prime degree [150]. Kahn [112] proved that the List Edge-Coloring Conjecture holds asymptotically for all multigraphs H , and Kahn [111] also proved that for fixed uniformity $r \geq 2$, every r -graph H with maximum degree Δ and maximum codegree $o(\Delta)$ has list chromatic index $\chi'_\ell(H) = \Delta + o(\Delta)$.

Similar to list-vertex colorings and independent transversals, the problem of finding a proper list edge-coloring of a multi-hypergraph H can easily be translated into the problem of finding a full rainbow matching in an auxiliary edge-colored multi-hypergraph G , and this point of view is useful for studying list edge-colorings in a “color degree” setting. Given a multi-hypergraph H and list assignment L for $E(H)$, the *maximum color degree* of H, L is the maximum, over all colors c , of the maximum degree of the subhypergraph $H[E_c]$ induced by the edge multiset $E_c = \{e \in E(H) : c \in L(e)\}$. The *maximum color codegree* of H, L is defined the same way but with “maximum degree” replaced by “maximum codegree”. The maximum color degree of H, L is the same as the maximum degree of the auxiliary multi-hypergraph G , each list $L(e)$ corresponds to an edge-color class on G , and the associated edge-coloring of G is a proper edge-coloring.

In this fashion, Theorem 1.29 of Aharoni, Berger, and Meshulam [11] implies that if H, L has maximum color degree Δ and $|L(x)| \geq r\Delta$ for all $x \in V(H)$, then there exists a proper L -coloring. Likewise, the asymptotic theorem of Delcourt and Postle [53] implies that if H, L has maximum color degree Δ , maximum color codegree $o(\Delta)$, and $|L(x)| \geq \Delta + o(\Delta)$ for all $x \in V(H)$, then there exists a proper L -coloring. This latter result is a color degree generalization of Kahn’s [111] asymptotic result on the list chromatic index of r -graphs. It is then natural to ask whether Galvin’s Theorem [75] holds in the color degree setting. Our main result is that this is not the case.

Theorem 1.33. *For every integer $\Delta \geq 2$, there exist a bipartite graph H and list assignment L for $E(H)$ such that H, L has maximum color degree Δ , $|L(e)| = \Delta$ for every $e \in E(H)$, and there is no proper L -coloring.*

This result is motivated by the similar Theorem 1.26 of Bohman and Holzman [36], which disproved the list coloring conjecture of Reed [145].

1.3.4 Other constructions

In Chapter 7, we will discuss further aspects of our iterative transversal-free construction method. Most of the work here is unpublished.

First, we observe that our construction method based on Lemma 1.20 generalizes to all abstract simplicial complexes. Given two complexes $\mathcal{C}$ and $\mathcal{D}$ on disjoint ground sets, their *join* is the complex $\mathcal{C} * \mathcal{D} = \{A \cup B : A \in \mathcal{C}, B \in \mathcal{D}\}$. For example, if G and H are disjoint multi-hypergraphs, then their independence complexes satisfy $\mathcal{I}(G \cup H) = \mathcal{I}(G) * \mathcal{I}(H)$ and their matching complexes satisfy $\mathcal{M}(G \cup H) = \mathcal{M}(G) * \mathcal{M}(H)$. The notion of a join allows us to write Lemma 1.20 and Lemma 1.28 in terms of general complexes (see Lemma 7.2). We will discuss some further complexes of interest and use Lemma 7.2 to understand when they have no colorful simplices.

In Section 7.2, we will discuss independent transversals in matroids, which are well understood due to Rado’s Theorem [144]. We will also apply results on independent transversals to the problem of list coloring matroids based on an appropriate notion of “color degree”. Next, in Section 7.3 we will apply our construction method to independence complexes of r -uniform hypergraphs (or r -graphs for short). We will describe constructions of vertex-partitioned r -graphs with a given maximum degree and no independent transversal, and we conjecture that our constructions give tight bounds (Conjecture 7.10). On the other hand, independent transversals in r -graphs are better understood in the setting of maximum class average degree, due to results of Wanless and Wood [167] and tight constructions by Groenland, Kaiser, Treffers, and Wales [83]. As an attempt to interpolate between the maximum degree setting and the maximum class average degree setting, we conjecture a generalization of Theorem 1.15 on good pairs for r -graphs (Conjecture 7.13). We present tight constructions for this conjecture. We will conclude by briefly discussing two other transversal problems considered in the literature which are connected to independent transversals in uniform hypergraphs, namely H -free transversals in graphs and acyclic transversals in directed graphs.

Part I

Arboricity

Chapter 2

Bounded degree arboricity: An orientation approach

In this chapter, we find approximations on the bounded degree arboricity of multigraphs by connecting it to the problem of edge-coloring multigraphs into subgraphs with indegree and outdegree-restricted orientations. In the process, we will prove Theorem 1.3, Theorem 1.4, Theorem 1.5, and Theorem 1.6. Section 2.4 will be more expository, proving a general orientation theorem and some further corollaries on covering and packing. This chapter is based on [168].

2.1 Edge-coloring into orientable subgraphs

In this section, we prove Theorem 1.3 about edge-coloring multigraphs into subgraphs with indegree- and outdegree-bounded orientations, which is restated below.

Theorem 1.3. *For every multigraph G and valid pair of weight functions (g, h) , we have*

$$\chi'_{(g,h)}(G) = \max \{ \Delta_{(g,h)}(G), \mathcal{W}_{(g,h)}(G) \}.$$

To recall the definitions, let G be a multigraph and let $g, h : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ be two weight functions on its vertices. A spanning subgraph of G is (g, h) -upper-orientable if it has an orientation D such that every vertex v of D has indegree $d_D^-(v) \leq g(v)$ and outdegree $d_D^+(v) \leq h(v)$. Also recall that $\chi'_{(g,h)}(G)$ is the minimum number of colors required to edge-color G so that each color class is (g, h) -upper-orientable. To ensure that $\chi'_{(g,h)}(G) < \infty$, we will assume that the following conditions hold:

- $g(v) + h(v) \geq 1$ for all $v \in V(G)$,
- $g(u) + g(v) \geq 1$ for all adjacent $u, v \in V(G)$, and

- $h(u) + h(v) \geq 1$ for all adjacent $u, v \in V(G)$,

where v is considered adjacent to itself if there is a loop at v . The first condition ensures that every vertex u accepts either the head or the tail of any arc incident to it, the second condition ensures that every neighbor v of u accepts the head of an arc if u does not, and the third condition ensures that the neighbors of u accept the tail of an arc if u does not. We call (g, h) a *valid pair* of weight functions for G if they satisfy these three conditions. We defined the parameters

$$\Delta_{(g,h)}(G) = \max_{v \in V(G)} \left\lceil \frac{d(v)}{g(v) + h(v)} \right\rceil, \quad \mathcal{W}_{(g,h)}(G) = \max_{X \subseteq V(G), E(X) \neq \emptyset} \left\lceil \frac{|E(X)|}{\min\{g(X), h(X)\}} \right\rceil.$$

Then (g, h) being valid ensures that the denominators are never zero.

Our proof for Theorem 1.3 is elementary, following straightforwardly from known results and imitating a well-known proof of Petersen's 2-Factor Theorem (see [54]). We will defer one part of the proof to Section 2.4 and instead focus more on the ideas that lead to Theorem 1.3. The main steps are to prove Theorem 1.3 for the case $\chi'_{(g,h)}(G) = 1$, and then to reduce the general case to this case by exploiting a nice decomposition property of (g, h) -upper-orientable multigraphs. (It is possible to combine these steps and prove the theorem in one step, but this separated approach is slightly cleaner.)

The case $\chi'_{(g,h)}(G) = 1$ simply asks for a characterization of when a multigraph G is itself (g, h) -upper-orientable. This was already established in a little-known work of Entringer and Tolman [64], which we state below.

Theorem 2.1 (Entringer, Tolman [64]). *For a multigraph G and weight functions $g, h : V(G) \rightarrow \mathbb{Z}_{\geq 0}$, G is (g, h) -upper-orientable if and only if*

- (1) $d(v) \leq g(v) + h(v)$ for all $v \in V(G)$, and
- (2) $|E(X)| \leq \min\{g(X), h(X)\}$ for all $X \subseteq V(G)$.

In fact, Entringer and Tolman [64] proved a more general orientation theorem, in which there are also specified lower bounds on the indegrees and outdegrees of the vertices. We give a modern exposition of their proof, as well as some related covering and packing results, in Section 2.4. For a slightly less arduous proof of Theorem 2.1 itself, see Appendix A in [168].

Now, the key decomposition property of (g, h) -upper-orientable multigraphs comes from a reduction to matchings in bipartite multigraphs. To that end, we will use the following f -coloring generalization of König's edge-coloring theorem proved by Hakimi and Kariv [90]. This is easy to establish from König's edge-coloring theorem by a vertex splitting argument (see the proof of Lemma 2.27 in Section 2.4.2, or [160]).

Theorem 2.2 (Hakimi, Kariv [90]). *For every bipartite multigraph G and weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 1}$, we have $\chi'_f(G) = \Delta_f(G)$.*

The following lemma specifies the decomposition property of (g, h) -upper-orientable subgraphs that we will use.

Lemma 2.3. *For a multigraph G , a valid pair of weight functions $g, h : V(G) \rightarrow \mathbb{Z}_{\geq 0}$, and an integer $k \geq 1$, we have $\chi'_{(g,h)}(G) \leq k$ if and only if G is (kg, kh) -upper-orientable.*

Proof. First suppose that $\chi'_{(g,h)}(G) \leq k$. Consider an edge-coloring of G into k color classes that are each (g, h) -upper-orientable. Giving each of these k color classes a (g, h) -upper-orientation, we combine these orientations to obtain a (kg, kh) -orientation of G .

Conversely, suppose that G has a (kg, kh) -upper-orientation D . We construct an auxiliary bipartite graph H with parts A and B as follows: for every vertex v of D we put a copy v^+ in A and a copy v^- in B , and for every arc e of D with tail u and head v we put an edge in H between u^+ and v^- . Then in the multigraph H , vertex v^+ has degree $d_D^+(v) \leq kh(v)$ and vertex v^- has degree $d_D^-(v) \leq kg(v)$, for every $v \in V(D)$. Define the weight function $f : V(H) \rightarrow \mathbb{Z}_{\geq 0}$ by $f(v^+) = h(v)$ and $f(v^-) = g(v)$ for all $v \in V(D)$. By Theorem 2.2, we have

$$\chi_f(H) = \Delta_f(H) = \max \left\{ \max_{\substack{v \in V(D), \\ h(v) \geq 1}} \left\lceil \frac{d_D^+(v)}{h(v)} \right\rceil, \max_{\substack{v \in V(D), \\ g(v) \geq 1}} \left\lceil \frac{d_D^-(v)}{g(v)} \right\rceil \right\} \leq \max\{k, k\} = k.$$

Thus H can be edge-colored into k degree- f subgraphs. This edge-coloring of H corresponds to an edge-coloring of G into k (g, h) -upper-orientable, so $\chi'_{(g,h)}(G) \leq k$. $\square$

Now the proof of Theorem 1.3 is a straightforward combination of Theorem 2.1 and Lemma 2.3.

Proof of Theorem 1.3. By Lemma 2.3, $\chi'_{(g,h)}(G) \leq k$ if and only if G is (kg, kh) -upper-orientable. By Theorem 2.1, this is the case if and only if $d(v) \leq k(g(v) + h(v))$ for every $v \in V(G)$, and $|E(X)| \leq k \cdot \min\{g(X), h(X)\}$ for every $X \subseteq V(G)$. This is true if and only if

$$\Delta_{(g,h)}(G) = \max_{v \in V(G)} \left\lceil \frac{d(v)}{g(v) + h(v)} \right\rceil \leq k, \text{ and}$$

$$\mathcal{W}_{(g,h)}(G) = \max_{X \subseteq V(G), E(X) \neq \emptyset} \left\lceil \frac{|E(X)|}{\min\{g(X), h(X)\}} \right\rceil \leq k.$$

In summary, $\chi'_{(g,h)}(G) \leq k$ if and only if $\max\{\Delta_{(g,h)}(G), \mathcal{W}_{(g,h)}(G)\} \leq k$. Therefore, $\chi'_{(g,h)}(G) = \max\{\Delta_{(g,h)}(G), \mathcal{W}_{(g,h)}(G)\}$. $\square$

To conclude this section, we observe that the formula in Theorem 1.3 also holds more strongly in the setting of *list edge-coloring*. Recall from the Introduction that for a multigraph G and a list assignment $L = (L(e) : e \in E(G))$ for $E(G)$, an L -coloring is an edge-coloring

ϕ of G such that $\phi(e) \in L(e)$ for all $e \in E(G)$. Let $f : V(G) \rightarrow \mathbb{Z}_{\geq 1}$ be a weight function on the vertices of G . The quantity $\chi'_{f,\ell}(G)$ is the minimum k such that for any list assignment L for $E(G)$ with $|L(e)| \geq k$ for all $e \in E(G)$, there exists an L -coloring of G where every color class is a degree- f subgraph. Note that $\chi'_{f,\ell}(G) \geq \chi'_f(G)$. Galvin [75] famously proved that if G is a bipartite multigraph, then $\chi'_1(G) = \chi'_{1,\ell}(G)$. From this, it is easy to establish the following generalization of Galvin's Theorem using the same vertex splitting argument as for Theorem 2.2 (see [108]).

Theorem 2.4. *For every bipartite multigraph G and weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 1}$, we have $\chi'_{f,\ell}(G) = \chi'_f(G)$.*

Naturally, we define the list coloring analogue $\chi'_{(g,h),\ell}(G)$ of $\chi'_{(g,h)}(G)$ as the minimum k such that for every list assignment L for $E(G)$ with $|L(e)| \geq k$ for all $e \in E(G)$, there exists an L -coloring of G where every color class is a (g, h) -orientable subgraph of G . As usual, we have $\chi'_{(g,h),\ell}(G) \geq \chi'_{(g,h)}(G)$. Using Theorem 2.4 in place of Theorem 2.2, we derive the following.

Theorem 2.5. *For a multigraph G and valid pair of weight functions $g, h : V(G) \rightarrow \mathbb{Z}_{\geq 0}$, we have $\chi'_{(g,h),\ell}(G) = \chi'_{(g,h)}(G)$.*

Proof. Suppose that $\chi'_{(g,h)}(G) = k$. Let L be a list assignment for $E(G)$ with $|L(e)| = k$ for all $e \in E(G)$. Construct the same auxiliary bipartite multigraph H and weight function $f : V(H) \rightarrow \mathbb{Z}_{\geq 0}$ as in Lemma 2.3. Then by Theorem 2.4, we have $\chi'_{f,\ell}(H) = \chi_f(H) = k$. Letting L' denote list assignment for $E(H)$ corresponding to L , we get that there exists an L' -coloring of H where every color class is a degree- f subgraph. This corresponds to an L -coloring of G where every color class is (g, h) -upper-orientable, so $\chi'_{(g,h),\ell}(G) \leq k = \chi'_{(g,h)}(G)$. Therefore, $\chi'_{(g,h),\ell}(G) = \chi'_{(g,h)}(G)$. $\square$

2.2 Approximation of bounded degree arboricity

In this section, we apply Theorem 1.3 to bounded degree (pseudo)arboricity and linear arboricity. In particular, we prove Theorem 1.4, Theorem 1.5, and Theorem 1.6.

2.2.1 Bounded degree (pseudo)arboricity

One immediate application of Theorem 1.3 is an exact formula for the degree- f pseudoarboricity of a multigraph. Recall that for a multigraph G and weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$, the *degree- f pseudoarboricity* $pa_f(G)$ of G is the minimum number of degree- f pseudoforests required to cover $E(G)$. It is well-known that pseudoforests are exactly the $(1, \infty)$ -upper-orientable multigraphs. Applying Theorem 1.3 with $(g, h) = (1, \infty)$ yields

$$pa(G) = \max_{X \subseteq V(G), |X| \geq 1} \left\lceil \frac{|E(X)|}{|X|} \right\rceil,$$

which is Hakimi's formula [89] for the pseudoarboricity of a multigraph G . We extend this result to bounded degree pseudoarboricity via the following observation.

Observation 2.6. *For $f \geq 2$, degree- f pseudoforests are the same as $(1, f - 1)$ -orientable multigraphs.*

To see this, notice that every $(1, f - 1)$ -orientable multigraph is a degree- f subgraph, and since this multigraph is necessarily $(1, \infty)$ -orientable, it is also a pseudoforest. Conversely, given a degree- f pseudoforest, start by orienting the edges of the cycles to form directed cycles (this requires that $f \geq 2$). The unoriented edges now form disjoint trees. Root these trees at some vertex; if the tree is part of a cyclic component of the pseudoforest, then root that tree at the unique vertex in the cycle; otherwise, root the tree at a leaf. Then we orient the edges of each tree away from the root of that tree, and the result will be a $(1, f - 1)$ -orientation of the degree- f pseudoforest.

Applying Theorem 1.3 with $(g, h) = (1, f - 1)$ yields Theorem 1.4, which is restated below.

Theorem 1.4. *For every multigraph G and weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$, we have*

$$pa_f(G) = \max \{ \Delta_f(G), pa(G) \}.$$

We note that the formula for Theorem 1.4 does not work when f is allowed to take value 1 on at least two vertices. In this case, degree- f pseudoforests can no longer be characterized as $(1, f - 1)$ -orientable multigraphs. For example, if $V(G) = \{u, v\}$ and $E(G) = \{uv\}$ with $f(u) = f(v) = 1$, then G is not $(1, 0)$ -orientable despite being a degree-1 pseudoforest (i.e., a matching). And if $f = 1$, then the degree-1 pseudoarboricity of a multigraph is the same as the chromatic index, which is a more difficult problem.

Next, we notice that the formula for Theorem 1.4 also holds in the setting of list edge-coloring. Let $pa_{f,\ell}(G)$ denote the list coloring analogue of $pa_f(G)$. Applying Theorem 2.5 with $(g, h) = (1, f - 1)$ yields the following.

Theorem 2.7. *For every multigraph G and weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$, we have*

$$pa_{f,\ell}(G) = pa_f(G).$$

Now we discuss our approximation for bounded degree arboricity. Recall that Nash-Williams' Theorem [139] states that the arboricity of a loopless multigraph G is given by

$$a(G) = \max_{X \subseteq V(G), |X| \geq 2} \left\lceil \frac{|E(X)|}{|X| - 1} \right\rceil.$$

This is analogous to Hakimi's formula for pseudoarboricity [89], and indeed both are consequences of the Matroid Partition Theorem [60]. Recall that pseudoarboricity approximates arboricity as follows: $pa(G) \leq a(G) \leq 2pa(G)$ for every loopless multigraph G , and

$pa(G) \leq a(G) \leq pa(G)+1$ for every simple graph G . These can be proven via Nash-Williams' and Hakimi's formulas, or via combinatorial arguments [35, 74].

For bounded degree arboricity, recall that the degree- f arboricity $a_f(G)$ of a loopless multigraph G is the minimum number of degree- f forests required to cover $E(G)$. It satisfies the easy lower bound $a_f(G) \geq \max\{\Delta_f(G), a(G)\}$. To get an upper bound for $a_f(G)$, we use the idea that bounded degree pseudoforests approximate bounded degree forests. For an initial upper bound, since every degree- f pseudoforest can be edge-colored into 2 degree- f forests, Theorem 1.4 implies the upper bound $a_f(G) \leq 2pa_f(G) \leq \max\{2\Delta_f(G), 2pa(G)\}$. A simple modification of this argument yields a much improved upper bound on $a_f(G)$.

Observation 2.8. *Every loopless degree-2f pseudoforest can be edge-colored into 2 degree-f forests.*

To see this, start by two-coloring the edges in every cycle of the pseudoforest so that no cycle is monochromatic. This partial coloring is valid by the assumption that $f \geq 2$. The uncolored edges now form disjoint trees. We greedily color the edges of each tree in such a way to satisfy the degree constraint at each vertex for each of the two color classes. This is possible because the trees have no cycles. In the end, the degree constraints will be satisfied, and due to the first step there will not be any monochromatic cycle, proving the observation.

Combining Observation 2.8 with Theorem 1.4, we obtain Theorem 1.5, which we write here in more detailed form.

Theorem 2.9. *For every loopless multigraph G and function $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$, we have*

$$a_f(G) \leq 2pa_{2f}(G) = \max\{2\Delta_{2f}(G), 2pa(G)\} \leq \max\{\Delta_f(G) + 1, 2pa(G)\}.$$

We remark that the maximum density upper bound $2pa(G)$ in Theorem 2.9 is best possible as a function of $pa(G)$, at least when G is a general multigraph. Specifically, note that $a_f(G) = a(G) = 2pa(G)$ when G consists of an even number of parallel edges between two vertices. Still, the upper bound $a_f(G) \leq \max\{\Delta_f(G) + 1, 2pa(G)\}$ in Theorem 2.9 is sometimes far from the lower bound $a_f(G) \geq \max\{\Delta_f(G), a(G)\}$. In the next chapter, we will disprove a conjecture of Truszczyński [164] stating that $a_f(G) \leq \max\{\Delta_f(G) + 1, a(G)\}$ when $f = t$ is constant, which more closely resembles the Goldberg-Seymour Conjecture for f -colorings (Conjecture 1.1). Finally, we notice that the decomposition trick we applied above does not extend in an obvious way to the list edge-coloring analogue of the degree- f arboricity.

2.2.2 Linear arboricity

In this subsection, we focus on the most studied case of degree- f arboricity, namely the case of linear arboricity $f = 2$. Recall that the Linear Arboricity Conjecture, due to Akiyama, Exoo, and Harary [19], states the following.

Conjecture 2.10 (Linear Arboricity Conjecture [19]). *For every simple graph G , we have*

$$la(G) \leq \left\lceil \frac{\Delta(G) + 1}{2} \right\rceil.$$

There has been recent work on proving Conjecture 1.2 when the maximum degree is somewhat larger than some kind of maximum “density” of the graph, particularly degeneracy. A loopless multigraph G is k -degenerate if every subgraph of G has a vertex of degree at most k . Kainen [113] has proven that $la(G) \leq \lceil (\Delta(G) + k - 1)/2 \rceil$ if G is a k -degenerate simple graph, which implies the Linear Arboricity Conjecture when $k = 2$. Basavaraju, Bishnu, Francis, and Pattanayak [30] proved that $la(G) \leq \lceil (\Delta(G) + 1)/2 \rceil$ if G is a 3-degenerate simple graphs. Chen, Hao, and Yu [47] proved that $la(G) = \lceil \Delta(G)/2 \rceil$ if G is k -degenerate and $\Delta(G) \geq 2k^2 - k$. They also proved that $la(G) \leq \lceil (\Delta(G) + 1)/2 \rceil$ if G is a k -degenerate simple graph and $\Delta(G) \geq 2k^2 - 2k$. Tan and Wu [162] proved that $la(G) = \lceil \Delta(G)/2 \rceil$ if G has treewidth k and $\Delta(G) \geq 4k - 3$. It is easy to show that a graph with treewidth k is necessarily k -degenerate.

Theorem 1.6, restated below, improves on these partial results for the Linear Arboricity Conjecture as far as the upper bound $\lceil (\Delta(G) + 1)/2 \rceil$ is concerned.

Theorem 1.6. *For every loopless multigraph G , we have $la(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$ whenever $\Delta(G) \geq 4pa(G) - 2$.*

Before proving Theorem 1.6, we briefly explain why it improves on previous results. It is due to the fact that if G is a k -degenerate loopless multigraph, then we have $pa(G) \leq a(G) \leq k$. To see this, note that being k -degenerate is the same as having an acyclic orientation D such that every vertex has indegree at most k . Such an orientation is found by iteratively deleting a vertex v of degree at most k in G and orienting the deleted edges toward v in D . Let D be an acyclic orientation of G such that every vertex has indegree at most k . We color all the incoming arcs at each vertex of D a different color to obtain an edge-coloring of G with k colors such that each color class is a forest. Thus, $pa(G) \leq a(G) \leq k$, and the following result is a corollary of Theorem 1.6.

Corollary 2.11. *For every k -degenerate loopless multigraph G , we have $la(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$ whenever $\Delta(G) \geq 4k - 2$.*

Now we return to proving Theorem 1.6. It follows almost directly from Theorem 2.9, but we need to handle the case $\Delta(G) \equiv 1 \pmod{4}$ separately. The following lemma will be helpful for this case. This kind of result was also used in [63, 86].

Lemma 2.12. *For every loopless multigraph G , there exists a linear forest F such that every vertex of degree $\Delta(G)$ in G has degree at least 1 in F .*

Proof. Applying Theorem 1.3 with $g = h = 1$, we find that multigraph G can be edge-colored into $\lceil \Delta(G)/2 \rceil$ subgraphs that are $(1, 1)$ -orientable, and a $(1, 1)$ -orientable multigraph is just a disjoint union of paths and cycles. (What we wrote is just a restatement of Petersen's 2-Factor Theorem [54].) Let F' be any of the $(1, 1)$ -orientable color classes of the edge-coloring. Then every vertex v of degree $\Delta(G)$ in G has degree at least 1 in F' , and if v lies in a cycle of F' then it has degree at least 2 in F' . Let F be obtained from F' by removing one edge from each cycle in F' . Then F is a desired linear forest in G . $\square$

Proof of Theorem 1.6. Applying Theorem 2.9 with $f = 2$ shows that every loopless multigraph G satisfies

$$la(G) \leq 2pa_4(G) = \max \left\{ 2 \left\lceil \frac{\Delta(G)}{4} \right\rceil, 2pa(G) \right\}.$$

In particular, when $\Delta(G) \geq 4pa(G) - 3$, we have

$$la(G) \leq \begin{cases} \lceil \Delta(G)/2 \rceil & \text{if } \Delta(G) \equiv 0 \text{ or } 3 \pmod{4}, \\ \lceil \Delta(G)/2 \rceil + 1 & \text{if } \Delta(G) \equiv 1 \text{ or } 2 \pmod{4}. \end{cases}$$

When $\Delta(G) \equiv 2 \pmod{4}$, we have $\lceil \Delta(G)/2 \rceil + 1 = \lceil (\Delta(G) + 1)/2 \rceil$, so the proof is complete in all cases except $\Delta(G) \equiv 1 \pmod{4}$.

Assume that $\Delta(G) \equiv 1 \pmod{4}$. Let F be a linear forest in G from Lemma 2.12, and let $G' = G - E(F)$. Then $G = G' \cup F$ and $\Delta(G') \leq \Delta(G) - 1$, so that

$$\begin{aligned} la(G) &\leq la(G') + 1 \leq 2pa_4(G') + 1 = \max \left\{ 2 \left\lceil \frac{\Delta(G')}{4} \right\rceil + 1, 2pa(G') + 1 \right\} \\ &\leq \max \left\{ 2 \cdot \frac{\Delta(G) - 1}{4} + 1, 2pa(G) + 1 \right\} = \max \left\{ \frac{\Delta(G) + 1}{2}, 2pa(G) + 1 \right\}. \end{aligned}$$

When $\Delta(G) \geq 4pa(G) - 2$, since $\Delta(G) \equiv 1 \pmod{4}$ we have $\Delta(G) \geq 4pa(G) + 1$. This rearranges to $(\Delta(G) + 1)/2 \geq 2pa(G) + 1$, and by the above calculation this implies that $la(G) \leq \lceil (\Delta(G) + 1)/2 \rceil$, as required. $\square$

From the proof of Theorem 1.6, we see that we actually have the stronger result $la(G) = \lceil \Delta(G)/2 \rceil$ whenever $\Delta(G) \geq 4pa(G) - 2$ and $\Delta(G) \equiv 0, 3 \pmod{4}$.

A short time after the writing of this work first appeared [168], Chen, Hao, and Yu [48] extended some of the work here and got closer to the Linear Arboricity Conjecture. Inspired by our work, in place of our simple observation that every degree-4 pseudoforest can be edge-colored into 2 linear forests (see Observation 2.8), these authors proved the following result using a technical combinatorial argument.

Theorem 2.13 (Chen, Hao, Yu [48]). *If F_1 and F_2 are two degree-3 pseudoforests in a simple graph, then $F_1 \cup F_2$ can be edge-colored into 3 linear forests.*

The main consequence of Theorem 2.13 is the following improvement on Theorem 1.6 for simple graphs.

Theorem 2.14 (Chen, Hao, Yu [48]). *For every simple graph G , we have $la(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$ whenever $\Delta(G) \geq 3pa(G) - 1$.*

To conclude this section, we note that prior to the work presented here, there has not been as much attention paid to the linear arboricity of loopless multigraphs, nor to bounded degree arboricity beyond linear arboricity. One contribution to the linear arboricity of multigraphs was a paper of Ait-djafer [18], who generalized the Linear Arboricity Conjecture to $la(G) \leq \lceil (\Delta(G) + \mu(G))/2 \rceil$ where $\mu(G)$ is the edge-multiplicity of G . She verified this for $\mu(G) \geq \Delta(G) - 2$, as well as when $\Delta(G)$ is close to a power of 2 and $\mu(G)$ is close to $\Delta(G)/2$. Note that by Nash-Williams' Theorem [139], the arboricity $a(G)$ of a loopless multigraph G can be as high as $\lceil (\Delta(G) + \mu(G))/2 \rceil$, so this conjectured upper bound for $la(G)$ would be best possible in terms of $\Delta(G)$ and $\mu(G)$ alone. Our result, Theorem 1.6, implies shows that if G is a sufficiently sparse loopless multigraph, then we can improve this conjectured upper bound for linear arboricity to $la(G) \leq \lceil (\Delta(G) + 1)/2 \rceil$.

One contribution to bounded degree arboricity beyond linear arboricity was an upper bound for degree- t arboricity proved by Caro and Roditty [42]. Generalizing the linear arboricity upper bound of Kainen [113] stated above, Caro and Roditty proved that every k -degenerate simple graph G satisfies $a_t(G) \leq \left\lceil \frac{\Delta(G) + (t-1)k-1}{t} \right\rceil$, where $f = t \geq 2$ is taken to be constant. Applying Theorem 2.9 with $f = t$, we obtain that $a_t(G) \leq \max \left\{ \left\lceil \frac{\Delta(G)}{t} \right\rceil + 1, 2pa(G) \right\}$, which improves Caro and Roditty's bound at least when $\Delta(G) \geq (k+1)t+2$. Our result also applies more generally to when G is a loopless multigraph. Finally, as mentioned in the previous subsection, a conjecture of Truszczyński [164] states that for every $t \geq 2$, the degree- t arboricity of a multigraph G satisfies $a_t(G) \leq \max \left\{ \left\lceil \frac{\Delta(G)}{t} \right\rceil + 1, a(G) \right\}$. This is a strong generalization of the Linear Arboricity Conjecture both to multigraphs and to other degree bounds, and it would be an improvement on Theorem 2.9. In Chapter 3, we will show that this conjecture is false for general multigraphs, although we will also prove partial results providing evidence that this conjecture holds when G is a simple graph. Many of the techniques we use will be the same as in this chapter.

2.3 Approximation of f -chromatic index

In this section, we apply Theorem 1.3 to obtain results on proper edge-colorings and f -colorings. Some of the results will be similar to those we obtained for bounded degree arboricity, and although they are not the best known results, they demonstrate the potential for further applications of our work.

To start, we show that the following elementary result on the f -chromatic index $\chi'_f(G)$, due to Hakimi and Kariv [90], is a straightforward consequence of Theorem 1.3. Note that statement (a) is the same as Theorem 2.2.

Theorem 2.15 (Hakimi, Kariv [90]). *Let G be a multigraph and let $f : V(G) \rightarrow \mathbb{Z}_{\geq 1}$ be a weight function.*

- (a) *If G is bipartite, then $\chi'_f(G) = \Delta_f(G)$.*
- (b) *If $f(v)$ is even for all $v \in V(G)$, then $\chi'_f(G) = \Delta_f(G)$.*

Proof. For part (a), if A and B are the parts of G , then we apply Theorem 1.3 with the valid pair of weight functions $g, h : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ defined by $(g(v), h(v)) = (0, f(v))$ if $v \in A$, and $(g(v), h(v)) = (f(v), 0)$ if $v \in B$. Note that (g, h) -orientable subgraphs are the same as degree- f subgraphs. We have that $\Delta_{(g,h)}(G) = \Delta_f(G)$. Moreover, for $X \subseteq V(G)$ with $E(X) \neq \emptyset$, if say $h(X) = f(X \cap A) \leq f(X \cap B) = g(X)$ then

$$\frac{|E(X)|}{\min\{g(X), h(X)\}} = \frac{|E(X)|}{f(X \cap A)} \leq \frac{\sum_{v \in X \cap A} d(v)}{\sum_{v \in X \cap A} f(v)} \leq \max_{v \in X \cap A} \frac{d(v)}{f(v)} \leq \Delta_f(G),$$

and likewise if $g(X) \leq h(X)$. Thus $\mathcal{W}_{(g,h)}(G) \leq \Delta_f(G)$, and so

$$\chi'_f(G) = \chi'_{(g,h)}(G) = \max\{\Delta_{(f,g)}(G), \mathcal{W}_{(g,h)}\} = \Delta_f(G).$$

For part (b), we apply Theorem 1.3 with the valid pair of weight functions $g, h : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ defined by $(g(v), h(v)) = (f(v)/2, f(v)/2)$ for all $v \in V(G)$. Note that (g, h) -orientable subgraphs are the same as degree- f subgraphs. We have that $\Delta_{(g,h)}(G) = \Delta_f(G)$. Moreover, for $U \subseteq V(G)$ with $E(U) \neq \emptyset$, we have

$$\frac{|E(U)|}{\min\{g(U), h(U)\}} = \frac{|E(U)|}{f(U)/2} \leq \frac{\sum_{v \in U} d(v)}{\sum_{v \in U} f(v)} \leq \max_{v \in U} \frac{d(v)}{f(v)} \leq \Delta_f(G).$$

Thus $\mathcal{W}_{(g,h)}(G) \leq \Delta_f(G)$, and so

$$\chi'_f(G) = \chi'_{(g,h)}(G) = \max\{\Delta_{(f,g)}(G), \mathcal{W}_{(g,h)}\} = \Delta_f(G).$$

□

Of course, we used Theorem 2.15(a) to prove Theorem 1.3, and the above proof is not too different from Hakimi and Kariv's [90] original proof. Still, the above proof provides a more unified view of these exact results on f -colorings. Theorem 2.15 appears to be connected to the duality between bipartite matroids and Eulerian matroids in the binary setting [171].

As another application of Theorem 1.3, we provide a general upper bound on the f -chromatic index when $f \geq 2$. Let $g(v) = \lfloor f(v)/2 \rfloor$ and $h(v) = \lceil f(v)/2 \rceil$ for all $v \in V(G)$. Every (g, h) -orientable subgraph is indeed a degree- f subgraph, so Theorem 1.3 implies the following.

Theorem 2.16. *For every multigraph G and weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$, we have*

$$\chi'_f(G) \leq \max \left\{ \Delta_f(G), \max_{X \subseteq V(G), |X| \geq 2} \left\lceil \frac{|E(X)|}{\sum_{v \in X} \lfloor f(v)/2 \rfloor} \right\rceil \right\}.$$

Theorem 2.16 can be viewed as an “averaged” version of Theorem 2 of Hakimi and Kariv [90], and it is an improvement in most cases. This theorem is an approximation of the Goldberg-Seymour Conjecture for f -colorings (Conjecture 1.1), and as the values of f get larger, this approximation gets better, independently of the size of the multigraph: If $f(v) \geq k$ for all $v \in V(G)$, then the factor of approximation is about $k/(k-1)$. Of course, it does not come close to approximations achieved using the more precise tools of alternating trails and Tashkinov trees (see [137, 160]).

Now, similar to how we approximated bounded degree arboricity using bounded degree pseudoarboricity, we can get similar approximations for the chromatic index $\chi'(G)$ and f -chromatic index $\chi'_f(G)$. We observe that every degree-2 pseudoforest (i.e., every disjoint union of paths and cycles) has a proper edge-coloring using 3 colors, so Theorem 1.4 implies that $\chi'(G) \leq 3 \lceil \Delta(G)/2 \rceil$. This comes close to Shannon’s Theorem [155] for the chromatic index, $\chi'(G) \leq \lceil 3\Delta(G)/2 \rceil$, and they are in fact the same when $\Delta(G)$ is even. This approach to the chromatic index was described, for example, in [117] for the purposes of finding efficient parallelizable edge-coloring algorithms. Now, similar to Observation 2.8, we observe that every degree-3 pseudoforest has a proper edge-coloring using 3 colors. Combining this observation with Theorem 1.4 yields the following improved upper bound.

Theorem 2.17. *For every multigraph G , we have*

$$\chi'(G) \leq \max \left\{ 3 \left\lceil \frac{\Delta(G)}{3} \right\rceil, 3pa(G) \right\} \leq \max \{ \Delta(G) + 2, 3pa(G) \}.$$

In particular, we have that $\chi'(G) \leq \Delta(G) + 2$ whenever $\Delta(G) \geq 3pa(G) - 2$. Assuming that $\Delta(G) \geq 3pa(G) - 2$, this upper bound on $\chi'(G)$ almost matches Vizing’s Theorem [165] when G is a simple graph, and it improves on Vizing’s Theorem for multigraphs. More generally, we observe that every degree- $3f$ pseudoforest can be edge-colored into 3 degree- f subgraphs, which gives the following upper bound for the f -chromatic index $\chi'_f(G)$.

Theorem 2.18. *For every multigraph G , we have*

$$\chi'_f(G) \leq \max \{ 3\Delta_{3f}(G), 3pa(G) \} \leq \max \{ \Delta_f(G) + 2, 3pa(G) \}.$$

Of course, Theorem 2.17 and Theorem 2.18 are direct consequences of the Goldberg-Seymour Conjecture for f -colorings, and they already follow from known partial results on the conjecture (see [160]). Still, this section demonstrates that the methods of this chapter surrounding Theorem 1.3 are connected to the familiar problems of proper edge-colorings and f -colorings, and demonstrate that they may be useful beyond bounded degree arboricity.

2.4 Orientations, coverings, and packings

This section is expository. In the first subsection, we will give a modern exposition of the work of Entringer and Tolman [64], specifically their general result on orienting multigraphs with both upper and lower bounds on the indegrees and outdegrees of vertices. In particular, this will prove Theorem 2.1, as well as some other corollaries. In the second subsection, we will use these results to obtain corollaries on covering and packing orientable subgraphs, yielding results related to Theorem 1.3 which have not been described before.

2.4.1 Orientations with bounds on both indegrees and outdegrees

In this subsection, we prove the general orientation theorem of Entringer and Tolman [64], as well as some of its various corollaries. There are many other kinds of orientation theorems (see [152], Chapter 61), but these are the ones most relevant to our work on bounded degree arboricity.

Let G be a multigraph, allowing both parallel edges and loops. For $X, Y \subseteq V(G)$, let $E(X)$ denote the set of edges of G with both endpoints in X , let $E(X, Y)$ denote the set of edges with one endpoint in X and the other endpoint in Y , and let $E^+(X) = E(X) \cup E(X, V - X)$ denote the set of edges of G with at least one endpoint in X . For convenience, given weight functions $f_1, f_2 : V(G) \rightarrow \mathbb{Z}_{\geq 0}$, we define the vertex sets

$$\begin{aligned} V^-(f_1, f_2) &= \{v \in V(G) : d(v) < f_1(v) + f_2(v)\}, \\ V^+(f_1, f_2) &= \{v \in V(G) : d(v) \geq f_1(v) + f_2(v)\}. \end{aligned}$$

It actually does not matter exactly which of the two sets contains a given vertex v with $d(v) = f_1(v) + f_2(v)$, as long as the sets $V^-(f_1, f_2)$ and $V^+(f_1, f_2)$ are disjoint.

The following is the main general orientation theorem of Entringer and Tolman [64].

Theorem 2.19. *Let G be a multigraph, and let $g_\ell, g_u, h_\ell, h_u : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ be weight functions on the vertices of G . There exists an orientation D of G such that $g_\ell(v) \leq d_D^-(v) \leq g_u(v)$ and $h_\ell(v) \leq d_D^+(v) \leq h_u(v)$ for every vertex v , if and only if*

- (1) $g_\ell(v) + h_\ell(v) \leq d(v) \leq g_u(v) + h_u(v)$ for all $v \in V(G)$,
- (2) $g_u(X) - |E(X)| - h_\ell(Y) + |E^+(Y)| \geq |E(X, Y)|$ for all $X \subseteq V^+(g_u, h_\ell)$, $Y \subseteq V^-(g_u, h_\ell)$,
- (3) $h_u(X) - |E(X)| - g_\ell(Y) + |E^+(Y)| \geq |E(X, Y)|$ for all $X \subseteq V^+(g_\ell, h_u)$, $Y \subseteq V^-(g_\ell, h_u)$.

Proof. First we verify that conditions (1), (2), and (3) are necessary. Assume that an appropriate orientation D of G exists. Condition (1) follows immediately from the fact that

$d(v) = d_D^-(v) + d_D^+(v)$. To verify condition (2), notice that for $X \subseteq V^+(g_u, h_\ell)$, $Y \subseteq V^-(g_u, h_\ell)$ we have

$$\begin{aligned} |E(X)| &= \sum_{v \in X} d_D^-(v) - a \leq g_u(X) - a, \\ |E^+(Y)| &= \sum_{v \in Y} d_D^+(v) + b \geq h_\ell(Y) + b, \end{aligned}$$

where a is the number of arcs of D entering X , and b is the number of arcs of D entering Y . Thus,

$$g_u(X) - |E(X)| - h_\ell(Y) + |E^+(Y)| \geq a + b \geq |E(X, Y)|$$

The verification of condition (3) is analogous.

Now we show that conditions (1), (2), and (3) are sufficient. Choose an orientation D of G that maximizes the value of

$$F(D) = \sum_{v \in V(G)} \min\{0, d_D^-(v) - g_u(v), g_u(v) - d_D^-(v), d_D^+(v) - h_\ell(v), h_\ell(v) - d_D^+(v)\}$$

Then D is a desired orientation of G if and only if $F(D) = 0$. Suppose for contradiction that $F(D) < 0$. There are four possibilities.

Case 1: There exists a vertex $v \in V(D)$ such that $d_D^-(v) > g_u(v)$. Since $d(v) \leq g_u(v) + h_\ell(v)$, we have that $d_D^+(v) < h_\ell(v)$. Let $S \subseteq V(G)$ be the set of vertices w such that there exists a directed path in D from w to v . Also let $X = S \cap V^+(g_u, h_\ell)$ and $Y = S \cap V^-(g_u, h_\ell)$. Consider a vertex $w \in X$, so that $d(w) \geq g_u(w) + h_\ell(w)$. If $d_D^-(w) < g_u(w)$, then $d_D^+(w) > h_\ell(w)$, and then reversing a directed path from w to v results in an orientation D' with $F(D') > F(D)$, a contradiction. Thus, $d_D^-(w) \geq g_u(w)$ for all $w \in X$. For similar reasons, $d_D^+(w) \leq h_\ell(w)$ for all $w \in Y$. Hence,

$$\begin{aligned} |E(X)| &= \sum_{w \in X} d_D^-(w) - a \geq g_u(X) - a, \\ |E^+(Y)| &= \sum_{w \in Y} d_D^+(w) + b \leq h_\ell(Y) + b, \end{aligned}$$

where a is the number of arcs of D entering X , and b is the number of arcs of D entering Y . If $v \in X$, then the first inequality is strict, since $d_D^-(v) > g_u(v)$. If $v \in Y$, then the second inequality is strict, since $d(v) < g_u(v) + h_\ell(v)$ and $d_D^-(v) > g_u(v)$ imply that $d_D^+(v) < h_\ell(v)$. In either case, we get that

$$g_u(X) - |E(X)| - h_\ell(Y) + |E^+(Y)| < a + b.$$

By the definition of S , every arc of D entering X has its tail in Y , and every arc of D entering Y has its tail in X , so that $a + b = |E(X, Y)|$. Combining this latter equality with the previous inequality contradicts condition (2).

Case 2: There exists a vertex $v \in V(D)$ such that $d_D^-(v) < g_\ell(v)$. Since $d(v) \geq g_\ell(v) + h_\ell(v)$, we have that $d_D^+(v) > h_\ell(v)$. Let $S \subseteq V(G)$ be the set of vertices w such that there exists a directed path in D from w to v . Also let $X = S \cap V^+(g_\ell, h_u)$ and $Y = S \cap V^-(g_\ell, h_u)$. Consider a vertex $w \in X$, so that $d(w) \geq g_\ell(w) + h_u(w)$. If $d_D^+(w) < h_u(w)$, then $d_D^-(w) > g_\ell(w)$, and then reversing a directed path from w to v results in an orientation D' with $F(D') > F(D)$, a contradiction. Thus, $d_D^+(w) \geq h_u(w)$ for all $w \in X$. For similar reasons, $d_D^-(w) \leq g_\ell(w)$ for all $w \in Y$. Hence,

$$|E(X)| = \sum_{w \in X} d_D^+(w) - a \geq h_u(X) - a,$$

$$|E^+(Y)| = \sum_{w \in Y} d_D^-(w) + b \leq g_\ell(Y) + b,$$

where a is the number of arcs of D leaving X , and b is the number of arcs of D leaving Y . If $v \in X$, then the first inequality is strict, since $d(v) \geq g_\ell(v) + h_u(v)$ and $d_D^-(v) < g_\ell(v)$ imply that $d_D^+(v) > h_u(v)$. If $v \in Y$ then the second inequality is strict, since $d_D^-(v) < g_\ell(v)$. In either case, we get that

$$g_u(X) - |E(X)| - h_\ell(Y) + |E^+(Y)| < a + b.$$

By the definition of S , every arc of D leaving X has its head in Y , and every arc of D leaving Y has its head in X , so that $a + b = E(X, Y)$. Combining this latter equality with the previous inequality contradicts condition (3).

Case 3: There exists a vertex $v \in V(D)$ such that $d_D^+(v) > h_u(v)$. We resolve this case by reversing all the arcs of D , exchanging the roles of g_ℓ and h_ℓ , exchanging the roles of g_u and h_u , and applying Case 1.

Case 4: There exists a vertex $v \in V(D)$ such that $d_D^+(v) < h_\ell(v)$. We resolve this case by reversing all the arcs of D , exchanging the roles of g_ℓ and h_ℓ , exchanging the roles of g_u and h_u , and applying Case 2. $\square$

The proof of Theorem 2.19 provides a polynomial-time algorithm for finding an appropriate orientation of G or otherwise finding one of the necessary conditions that is violated. The statement of Theorem 2.19 greatly simplifies when we ignore some of the indegree and outdegree bounds. The following various corollaries are obtained from Theorem 2.19 by appropriately setting g_ℓ or h_ℓ to be 0, or setting g_u or h_u to be ∞ . Entringer and Tolman [64] themselves wrote a few more corollaries than described here.

First we derive Theorem 2.1, restated below, which requires only upper bounds on the indegrees and outdegrees.

Corollary 2.20 (Only upper bounds). *Let G be a multigraph, and let $g_u, h_u : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ be weight functions on the vertices of G . There exists an orientation D of G such that $d_D^-(v) \leq g_u(v)$ and $d_D^+(v) \leq h_u(v)$ for all $v \in V(D)$ if and only if*

- (1) $d(v) \leq g_u(v) + h_u(v)$ for all $v \in V(G)$,
(2) $|E(X)| \leq \min\{g_u(X), h_u(X)\}$ for all $X \subseteq V(G)$.

Next, we derive the following result about when we require only lower bounds on indegrees and outdegrees.

Corollary 2.21 (Only lower bounds). *Let G be a multigraph, and let $g_\ell, h_\ell : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ be weight functions on the vertices of G . There exists an orientation D of G such that $d_D^-(v) \geq g_\ell(v)$ and $d_D^+(v) \geq h_\ell(v)$ for all $v \in V(D)$ if and only if*

- (1) $d(v) \geq g_\ell(v) + h_\ell(v)$ for all $v \in V(G)$,
(2) $|E^+(X)| \geq \max\{g_\ell(X), h_\ell(X)\}$ for all $X \subseteq V(G)$.

The following corollary, where we require only indegree bounds, was given by Frank and Gyárfás [73].

Corollary 2.22 (Only indegree bounds). *Let G be a multigraph, and let $g_\ell, g_u : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ be weight functions on the vertices of G . There exists an orientation D of G such that $g_\ell(v) \leq d_D^-(v) \leq g_u(v)$ for all $v \in V(D)$ if and only if*

- (1) $|E(X)| \leq g_u(X)$ for all $X \subseteq V(G)$,
(2) $|E^+(X)| \geq g_\ell(X)$ for all $X \subseteq V(G)$.

Let us also mention the following corollary.

Corollary 2.23 (Indegree lower bound, outdegree upper bound). *Let G be a multigraph, and let $g_\ell, h_u : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ be weight functions on the vertices of G . There exists an orientation D of G such that $d_D^-(v) \geq g_\ell(v)$ and $d_D^+(v) \leq h_u(v)$ for all $v \in V(D)$ if and only if*

- (1) $d(v) \geq g_\ell(v)$ for all $v \in V(G)$,
(2) $h_u(X) - |E(X)| - g_\ell(Y) + |E^+(Y)| \geq E(X, Y)$ for all $X \subseteq V^+(g_\ell, h_u)$, $Y \subseteq V^-(g_\ell, h_u)$.

We note that each of the above corollaries implies the following well-known orientation result.

Corollary 2.24 (Existence of balanced orientations). *Every multigraph G admits a balanced orientation, i.e., an orientation D such that $|d_D^-(v) - d_D^+(v)| \leq 1$ for all $v \in V(D)$.*

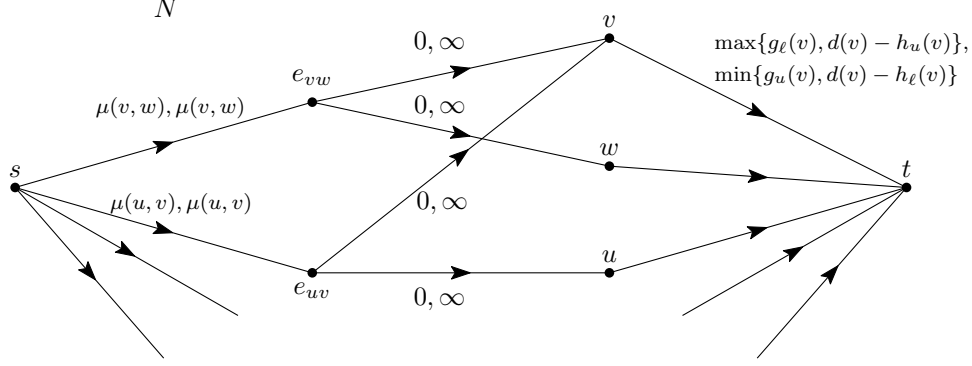


Figure 2.1: A network flow formulation for finding an orientation of a multigraph with lower and upper bounds on both indegrees and outdegrees. The values on the arcs represent the lower and upper capacities.

We obtain Corollary 2.24 from any of the above corollaries by setting $g_\ell(v) = h_\ell(v) = \left\lfloor \frac{d(v)}{2} \right\rfloor$ and $g_u(v) = h_u(v) = \left\lceil \frac{d(v)}{2} \right\rceil$ for all $v \in V(G)$. In the particular case when all the degrees are even, the resulting orientation satisfies $d_D^-(v) = d_D^+(v)$ for all $v \in V(D)$, so the orientation is Eulerian and can be used to find an Euler tour.

Next we mention the following two corollaries, which were originally proved by Hakimi [89] and which can alternatively be proved using Hall's theorem, network flows, or matroid intersection (see [152]).

Corollary 2.25 (Only indegree upper bound). *Let G be a multigraph, and let $g_u : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ be a weight function on the vertices of G . There exists an orientation D of G such that $d_D^-(v) \leq g_u(v)$ for all $v \in V(D)$ if and only if $|E(X)| \leq g_u(X)$ for all $X \subseteq V(G)$.*

Corollary 2.26 (Only indegree lower bound). *Let G be a multigraph, and let $g_\ell : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ be a weight function on the vertices of G . There exists an orientation D of G such that $d_D^-(v) \geq g_\ell(v)$ for all $v \in V(D)$ if and only if $|E^+(X)| \geq g_\ell(X)$ for all $X \subseteq V(G)$.*

Finally, we describe a network flow formulation for the problem of finding an orientation D of a multigraph G such that $g_\ell(v) \leq d_D^-(v) \leq g_u(v)$ and $h_\ell(v) \leq d_D^+(v) \leq h_u(v)$ for all $v \in V(G)$. See Figure 2.1 for illustration. The network N consists of a starting vertex s , a terminal vertex t , a copy of every vertex of G , and, for every pair of adjacent vertices v, w in G , a vertex labeled e_{vw} . Let $\mu_G(v, w)$ denote the multiplicity of the vertex pair v, w in G , which is the number of parallel edges between v and w . For every pair of adjacent vertices v, w in G , we add to N the arcs (s, e_{vw}) with both lower and upper capacity $\mu_G(v, w)$, and the arcs (e_{vw}, v) and (e_{vw}, w) both with lower capacity 0 and upper capacity ∞ . In addition, for every vertex v of G , we add to N the arc (v, t) with lower capacity $\max\{g_\ell(v), d(v) - h_u(v)\}$ and upper capacity $\min\{g_u(v), d(v) - h_\ell(v)\}$. Then from an integral feasible flow ϕ for N , if one exists, we construct an orientation D of G by orienting $\phi(e_{vw}, v)$ edges from w to v , and $\phi(e_{vw}, w)$ edges from v to w . The indegree of every vertex v will then satisfy

$\max\{g_\ell(v), d(v) - h_u(v)\} \leq d_D^-(v) \leq \min\{g_u(v), d(v) - h_\ell(v)\}$. In particular, $g_\ell(v) \leq d_D^-(v) \leq g_u(v)$ and $h_\ell(v) \leq d(v) - d_D^-(v) \leq h_u(v)$, and we have $d_D^+(v) = d(v) - d_D^-(v)$. Conversely, every feasible orientation D of G gives rise to such an integral feasible flow ϕ on N .

2.4.2 Covering and packing orientable subgraphs

In this subsection, we discuss covering and packing results associated with the orientation results in the previous subsection. These results do not appear to have been described before.

Let G be a multigraph, and let $g_\ell, g_u, h_\ell, h_u : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ be weight functions on the vertices of G . A spanning subgraph H of G is said to be $(g_\ell, g_u, h_\ell, h_u)$ -orientable if H has an orientation D such that $g_\ell(v) \leq d_D^-(v) \leq g_u(v)$ and $h_\ell(v) \leq d_D^+(v) \leq h_u(v)$ for all $v \in V(D)$. In this case we call D a $(g_\ell, g_u, h_\ell, h_u)$ -orientation of H . We wish to cover or pack the edge set G by $(g_\ell, g_u, h_\ell, h_u)$ -orientable subgraphs. Similar to Section 2.1, we reduce the problem to edge-colorings of bipartite multigraphs. This generalizes the strategy used in a well-known simple proof of Petersen's 2-Factor Theorem.

To simplify the discussion, we use the notion of an equitable edge-coloring. An edge-coloring of a multigraph G is called *equitable* if for any two color classes C_1 and C_2 , we have $|d_{C_1}(v) - d_{C_2}(v)| \leq 1$.

Lemma 2.27. ([90, 160, 173]) *For every bipartite multigraph G and integer $k \geq 1$, there exists an equitable edge-coloring of G with k colors.*

Proof. Given bipartite multigraph G and integer $k \geq 2$, let G_k be an auxiliary bipartite multigraph obtained as follows. For each vertex $v \in V(G)$, we write $d_G(v) = q_v k + r_v$ for some integers $q_v \geq 0$ and $0 \leq r_v \leq k - 1$. To obtain G_k , we split each vertex v of G into q_v vertices of degree k and one additional vertex of degree r_v . Then G_k is bipartite and $\Delta(G_k) \leq k$, with equality if $k \leq \Delta(G)$. By König's edge-coloring theorem, there exists a proper edge-coloring of G_k with at most k colors. This translates into an edge-coloring of G that is equitable. $\square$

We rephrase Lemma 2.27 in the following convenient language.

Lemma 2.28. *Let G be a bipartite multigraph, and let $f_\ell, f_u : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ be weight functions on the vertices of G . Suppose that integer $k \geq 1$ satisfies $k f_\ell(v) \leq d(v) \leq k f_u(v)$ for all $v \in V(G)$. Then there exists an edge-coloring of G with k colors such that for each color class C , we have $f_\ell(v) \leq d_C(v) \leq f_u(v)$.*

Proof. By Lemma 2.27, there exists an equitable edge-coloring of G using k colors. Then for every color class C in this edge-coloring, we have $d_C(v) \geq \left\lfloor \frac{d(v)}{k} \right\rfloor \geq f_\ell(v)$ and $d_C(v) \leq \left\lceil \frac{d(v)}{k} \right\rceil \leq f_u(v)$. $\square$

From Lemma 2.28, we deduce the following result stating that orientable subgraphs have a nice decomposition property.

Lemma 2.29. *Let G be a multigraph, let $g_\ell, g_u, h_\ell, h_u : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ be weight functions on the vertices of G , and let $k \geq 1$ be an integer. Then G can be edge-colored into k $(g_\ell, g_u, h_\ell, h_u)$ -orientable subgraphs if and only if G is $(kg_\ell, kg_u, kh_\ell, kh_u)$ -orientable.*

Proof. If G can be edge-colored into k $(g_\ell, g_u, h_\ell, h_u)$ -orientable subgraphs, then combining the $(g_\ell, g_u, h_\ell, h_u)$ -orientations on each of these color classes results in a $(kg_\ell, kg_u, kh_\ell, kh_u)$ -orientation of G . Conversely, suppose that G has a $(kg_\ell, kg_u, kh_\ell, kh_u)$ -orientation D . Construct an auxiliary bipartite multigraph H by first splitting each vertex v of D into two vertices v^- and v^+ , and then for every arc of D oriented from v to w we put an edge in H between v^+ and w^- . Then $kg_\ell(v) \leq d_H(v^-) \leq kg_u(v)$ and $kh_\ell(v) \leq d_H(v^+) \leq kh_u(v)$ for $v \in V(D)$. By Lemma 2.28, there exists an edge-coloring of H with k colors such that for every color class C and vertex $v \in V(D)$, we have $g_\ell(v) \leq d_C(v^-) \leq g_u(v)$ and $h_\ell(v) \leq d_C(v^+) \leq h_u(v)$. This edge-coloring of H translates into an edge-coloring of G into k $(g_\ell, g_u, h_\ell, h_u)$ -orientable subgraphs. $\square$

The following is the main theorem about edge-coloring multigraphs into orientable subgraphs. Recall that for two weight functions $f_1, f_2 : V(G) \rightarrow \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} V^-(f_1, f_2) &= \{v \in V(G) : d(v) < f_1(v) + f_2(v)\}, \\ V^+(f_1, f_2) &= \{v \in V(G) : d(v) \geq f_1(v) + f_2(v)\}. \end{aligned}$$

Theorem 2.30. *Let G be a multigraph, and let $g_\ell, g_u, h_\ell, h_u : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ be weight functions on the vertices of G . Then G can be edge-colored into k $(g_\ell, g_u, h_\ell, h_u)$ -orientable subgraphs if and only if*

- (1) $k(g_\ell(v) + h_\ell(v)) \leq d(v) \leq k(g_u(v) + h_u(v))$ for all $v \in V(G)$,
- (2) $kg_u(X) - |E(X)| - kh_\ell(Y) + |E^+(Y)| \geq |E(X, Y)|$
for all $X \subseteq V^+(g_u, h_\ell)$, $Y \subseteq V^-(g_u, h_\ell)$,
- (3) $kh_u(X) - |E(X)| - kg_\ell(Y) + |E^+(Y)| \geq |E(X, Y)|$
for all $X \subseteq V^+(g_\ell, h_u)$, $Y \subseteq V^-(g_\ell, h_u)$.

Proof. This follows directly from combining Theorem 2.19 and Lemma 2.29. $\square$

The statement of Theorem 2.30 greatly simplifies if we ignore some of the degree bounds. Let G be a multigraph, and let $g, h : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ be weight functions on the vertices. In the first two corollaries below, we will implicitly assume that g, h satisfy the same properties as in Section 2.1 so that no denominators are zero, specifically:

- $g(v) + h(v) \geq 1$ for every non-isolated vertex $v \in V(G)$,
- $g(u) + g(v) \geq 1$ for any two adjacent vertices $u, v \in V(G)$,
- $h(u) + h(v) \geq 1$ for any two adjacent vertices $u, v \in V(G)$,

We refer to a $(0, g, 0, h)$ -orientable subgraph of G as (g, h) -upper-orientable, and we will refer to a (g, ∞, h, ∞) -orientable subgraph of G as (g, h) -lower-orientable. We derive the following results as direct corollaries of Theorem 2.30, again by appropriately setting g_ℓ or h_ℓ to be 0, or g_u or h_u to be ∞ .

First we derive Theorem 1.3, restated below, as a special case of Theorem 2.30.

Corollary 2.31 (Covering upper-orientable subgraphs). *Let $G = (V, E)$ be a multigraph, and let $g, h : V(G) \rightarrow \mathbb{Z}_{\geq 0}$ be weight functions on the vertices of G . The minimum number of (g, h) -upper-orientable subgraphs required to cover $E(G)$ is*

$$\max \{ \Delta_{(g,h)}(G), \mathcal{W}_{(g,h)}(G) \},$$

where

$$\Delta_{(g,h)}(G) = \max_{v \in V} \left\lceil \frac{d(v)}{g(v) + h(v)} \right\rceil, \quad \mathcal{W}_{(g,h)}(G) = \max_{X \subseteq V, E(X) \neq \emptyset} \left\lceil \frac{|E(X)|}{\min\{g(X), h(X)\}} \right\rceil.$$

Next, we write a similar corollary for the analogous problem of packing (g, h) -lower-orientable subgraphs.

Corollary 2.32 (Packing lower-orientable subgraphs). *Let $G = (V, E)$ be a multigraph, and let $g, h : V \rightarrow \mathbb{Z}_{\geq 0}$ be weight functions on the vertices of G . The maximum number of edge-disjoint (g, h) -lower-orientable subgraphs in $E(G)$ is*

$$\min \{ \delta_{(g,h)}(G), \mathcal{V}_{(g,h)}(G) \},$$

where

$$\delta_{(g,h)}(G) = \min_{v \in V} \left\lfloor \frac{d(v)}{g(v) + h(v)} \right\rfloor, \quad \mathcal{V}_{(g,h)}(G) = \min_{X \subset V} \left\lfloor \frac{|E| - |E(X)|}{\max\{g(V) - g(X), h(V) - h(X)\}} \right\rfloor.$$

We state the following corollary about edge-coloring into subgraphs with both lower and upper bounds on indegrees.

Corollary 2.33. *Let $G = (V, E)$ be a multigraph, and let $g_\ell, g_u : V \rightarrow \mathbb{Z}_{\geq 0}$ be weight functions on the vertices of G . Then G can be edge-colored into k subgraphs that each have an orientation D such that $g_\ell(v) \leq d_D^-(v) \leq g_u(v)$ for all $v \in V(D)$, if and only if*

$$\max_{X \subseteq V, X \neq \emptyset} \left\lceil \frac{|E(X)|}{g_u(X)} \right\rceil \leq k \leq \min_{X \subset V} \left\lfloor \frac{|E| - |E(X)|}{g_\ell(V) - g_\ell(X)} \right\rfloor$$

Finally, we remark that the (g, ∞) -upper-orientable subgraphs of a multigraph G form the independent sets of a matroid, namely the sparsity matroid with ground set $E(G)$ whose independent sets are subsets $F \subseteq E(G)$ such that $|F| \leq g(V(F))$, where $V(F)$ are the set of vertices of G incident to at least one edge in F . Notice that when $h = \infty$ in Corollary 2.31, the weighted maximum degree $\Delta_{(g, \infty)}(G)$ is 0 and the result simply states that the minimum number of (g, ∞) -upper-orientable subgraphs required to cover $E(G)$ is

$$\mathcal{W}_{(g, \infty)}(G) = \max_{X \subseteq V, E(X) \neq \emptyset} \left\lceil \frac{|E(X)|}{g(X)} \right\rceil,$$

which is a standard application of the Matroid Partition Theorem [60]. Likewise, when $h = 0$ in Corollary 2.32, the result simply states the maximum number of edge-disjoint $(g, 0)$ -lower-orientable subgraphs (which are spanning sets of the above matroid) in $E(G)$ is

$$\mathcal{V}_{(g, 0)}(G) = \min_{X \subseteq V} \left\lfloor \frac{|E| - |E(X)|}{g(V) - g(X)} \right\rfloor,$$

which is another application of the Matroid Partition Theorem [60]. Thus, the problems of covering and packing orientable subgraphs can be viewed as well-structured generalizations of covering and packing problems on certain sparsity matroids, where now the vertex degrees play as important of a role as the densities of the subgraphs.

2.5 Problems

We conclude this chapter with some problems for future investigation.

(1) In this chapter, we described a simple decomposition method based on bounded degree pseudoarboricity which demonstrates that the Linear Arboricity Conjecture holds for all multigraphs G such that $\Delta(G) \geq 4pa(G) - 2$ (Theorem 1.6). Using the same pseudoarboricity approach but with additional technical combinatorial arguments, Chen, Hao, and Yu [48] subsequently proved that the Linear Arboricity Conjecture holds for all simple graphs G such that $\Delta(G) \geq 3pa(G) - 1$ (Theorem 2.14). It would be interesting to see how much further we can push this pseudoarboricity approach toward the Linear Arboricity Conjecture. Note that our proof used the simple observation that every $(1, 3)$ -upper-orientable multigraph can be decomposed into two linear forests, while the proof of Chen, Hao, and Yu [48] primarily consisted of showing that every $(2, 4)$ -upper-orientable simple graph can be decomposed into three linear forests. Are there larger k for which we can show that every $(k - 1, k + 1)$ -upper-orientable simple graph can be decomposed into k linear forests? What about other values of k, ℓ such that every $(k - \ell, k + \ell)$ -upper-orientable simple graph can be decomposed into k linear forests? Could such an approach also be used to prove upper bounds for the more general degree- f arboricity, similar to Theorem 1.5?

(2) The decomposition method we used in this chapter is in some sense a variation of an older decomposition method for bounding the linear arboricity of graphs, used for example

in [63, 85]. In these works, they established the Linear Arboricity Conjecture for some fixed maximum degree, and then they used Petersen’s 2-Factor Theorem to decompose the graph into subgraphs with those maximum degrees or lower, applied their results to those subgraphs, and derived from that an upper bound on the linear arboricity of the original graph. Since Petersen’s 2-Factor Theorem is a special case of the main orientation theorem, Theorem 1.3, their approach is quite similar to the approach we took. It is worth investigating other possible decomposition approaches for attacking the Linear Arboricity Conjecture.

(3) The main application of the orientation results that was to bounded degree arboricity, which is a covering problem. However, we also wrote packing results in Section 2.4. There are some natural packing analogues of bounded degree arboricity that one could come up with. For example, one problem is finding the maximum number of edge-disjoint bounded degree spanning trees. This includes the famous problem of packing Hamilton paths. Unfortunately, our decomposition approach does not seem to be directly useful for this type of packing problem. The main issue is that we do not have guarantees that the orientable subgraphs in our decomposition are even connected. Perhaps there are some variations of our methods that could work better in the packing setting.

(4) In Section 2.4, we wrote relevant orientation results as well as associated covering and packing problem. However, there are some aspects that we have not yet investigated. In particular, we have not investigated a polyhedral description of the polytope $P_{(g_\ell, g_u, h_\ell, h_u)}(G)$ which is the convex hull of the indicator vectors of $(g_\ell, g_u, h_\ell, h_u)$ -orientable subgraphs of G . Such a polyhedral description would be helpful for the problem of finding a minimum weight $(g_\ell, g_u, h_\ell, h_u)$ -orientable subgraph in an edge-weighted multigraph G . Note that the simple 2-matching polytope is the same as $P_{(0,1,0,1)}(G)$, and the 2-factor polytope is the same as $P_{(1,1,1,1)}(G)$ (see [152]).

(5) Finally, we mention that one interesting well-studied problem related to bounded degree arboricity is the problem of minimum bounded degree spanning trees. In such a problem, given a graph G with edge weights $w : E(G) \rightarrow \mathbb{R}$ and a vertex weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 1}$, the goal is to find a spanning tree T of G with minimum weight $\sum_{e \in E(T)} w(e)$ among all f -bounded spanning trees, i.e., spanning trees satisfying the degree bounds $d_T(v) \leq f(v)$ for all $v \in V(G)$. Finding such a spanning tree or the minimum weight is an NP-hard problem, but there are approximation algorithms for this problem that find, in polynomial time, an $(f + 1)$ -bounded spanning tree T whose weight is at most that of a minimum f -bounded spanning tree [80, 125, 156]. Not much has been explored on the related problem of fractionally covering the edge set of a multigraph by f -bounded forests, which is related to the former problem via antiblocker duality. In particular, it would be interesting to look into polynomial time algorithms for finding a fractional covering by $(f + 1)$ -bounded forests with total weight at most that of a minimum fractional covering by f -bounded forests.

Chapter 3

Further approximations of bounded degree arboricity

In this chapter, we continue our work on bounded degree arboricity. We will prove Theorem 1.8 which disproves Conjecture 1.7 of Truszczyński [164] for multigraphs. Then we will prove Theorem 1.9 and Theorem 1.10, which support the possibility that his conjecture holds for simple graphs. This chapter is based on [170].

3.1 On a conjecture of Truszczyński

In this section, we prove Theorem 1.8 and thus disprove Conjecture 1.7 of Truszczyński [164] on bounded degree arboricity, which we restate below.

Conjecture 3.1 (Truszczyński [164]). *For every loopless multigraph G and weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$, we have*

$$a_f(G) \leq \max\{\Delta_f(G) + 1, a(G)\}.$$

Recall that Theorem 1.8 states the following.

Theorem 1.8. *For every integer $t \geq 2$, there exists a constant $c_t > 1$ such that the following holds. For infinitely many integers $d \geq 2$, there exists a multigraph G such that $\max\{\Delta_t(G), a(G)\} = d$ and $a_t(G) \geq c_t d$.*

Our proof of Theorem 1.8 involves finding multigraphs with large *fractional* degree- f arboricity. Let G be a multigraph and let $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$ be a weight function. Let $\mathcal{F}$ be the collection of edge-sets of degree- f forests in G . We have a variable y_F for every edge set $F \in \mathcal{F}$. The *fractional degree- f arboricity* $a_f^*(G)$ of a multigraph G is the optimal value of the following linear program:

$$\begin{aligned}
(P_0) \quad & \min \sum_{F \in \mathcal{F}} y_F \\
& \text{s.t.} \quad \sum_{F \in \mathcal{F}: e \in F} y_F \geq 1 \quad \forall e \in E(G), \\
& y_F \geq 0 \quad \forall F \in \mathcal{F}.
\end{aligned}$$

Notice that if we add the integrality constraints $y_F \in \{0, 1\}$ for all $F \in \mathcal{F}$, then a feasible solution of (P_0) corresponds to a collection of degree- f forests that covers all of $E(G)$. This implies that $a_f(G) \geq a_f^*(G)$.

We will work with a slightly simplified linear program. Let G' be the underlying simple graph of G , and let $\mathcal{F}'$ be the collection of edge-sets of degree- f forests of G' . For every edge $e = uv \in E(G')$, let $\mu_e = \mu_G(u, v)$ denote the number of parallel edges between vertices u and v in G . It is easy to show that (P_0) has the same optimal value as the following linear program.

$$\begin{aligned}
(P) \quad & \min \sum_{F \in \mathcal{F}'} y_F \\
& \text{s.t.} \quad \sum_{F \in \mathcal{F}': e \in F} y_F \geq \mu_e \quad \forall e \in E(G'), \\
& y_F \geq 0 \quad \forall F \in \mathcal{F}'.
\end{aligned}$$

The dual of (P) is the following, where x_e is the dual variable associated with edge $e \in E(G')$.

$$\begin{aligned}
(D) \quad & \max \sum_{e \in E(G')} \mu_e x_e \\
& \text{s.t.} \quad \sum_{e \in F} x_e \leq 1 \quad \forall F \in \mathcal{F}', \\
& x_e \geq 0 \quad \forall e \in E(G').
\end{aligned}$$

For the case $f = 2$, Feige, Ravi, and Singh [68] proved that the fractional linear arboricity of a Δ -regular simple graph G satisfies $la^*(G) = a_2^*(G) = \Delta/2 + O(\Delta^{1/2})$. Outside of their work, fractional linear arboricity has not been studied. To prove Theorem 1.8, we will use the following easy observation. For a multigraph G and integer $m \geq 1$, we let mG denote the multigraph obtained from G by replacing every edge of G by m parallel copies of that edge.

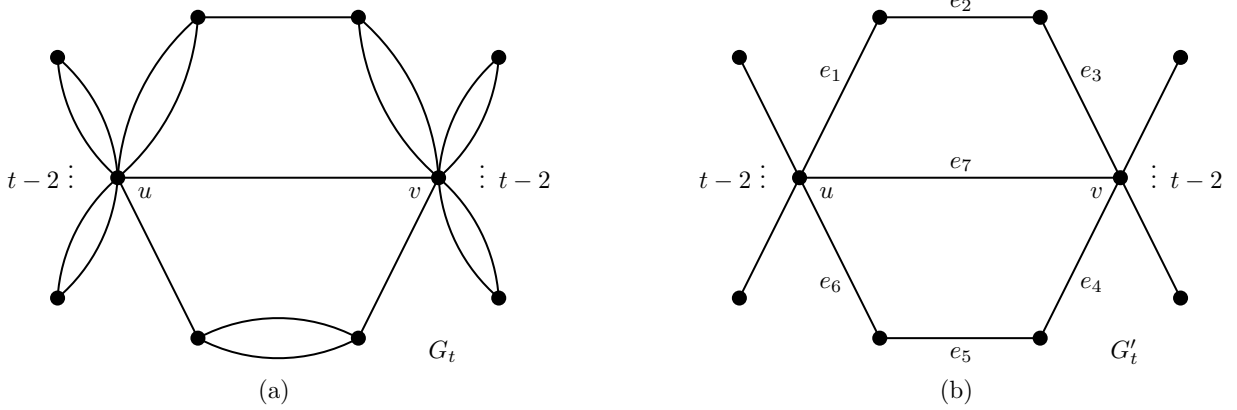


Figure 3.1: (a) The multigraph G_t . (b) The underlying simple graph G'_t .

Observation 3.2. For every multigraph G , function $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$, and integer $m \geq 1$, we have

$$a_f(mG) \geq a_f^*(mG) = m \cdot a_f^*(G).$$

We also observe that

$$\Delta_f(mG) \leq m \cdot \Delta_f(G), \quad a(mG) \leq m \cdot a(G)$$

If we can find a multigraph G such that $a_f^*(G) = c_f \cdot \max\{\Delta_f(G), a(G)\}$ and $c_f > 1$, then we have

$$a_f(mG) \geq m \cdot a_f^*(G) \geq m \cdot c_f \cdot \max\{\Delta_f(G), a(G)\} \geq c_f \cdot \max\{\Delta_f(mG), a(mG)\},$$

which is the required inequality for Theorem 1.8. Therefore our goal is to find, for every constant function $f = t \geq 2$, a multigraph $G = G_t$ and constant $c_t > 1$ such that

$$a_t^*(G) \geq c_t \cdot \max\{\Delta_t(G), a(G)\}.$$

An example of such a multigraph G_t is shown in Figure 3.1(a). It consists of a 6-cycle that alternates in one and two parallel edges, an edge connecting two antipodal vertices u and v of the cycle, and $t-2$ parallel pairs to new vertices attached to each of u and v . The underlying simple graph G'_t of G_t is shown in Figure 3.1(b), with some edges labeled $e_1, \dots, e_7$. Consider the following solution x for (D) with respect to G'_t :

$$x_e = \begin{cases} 2/(2t+3) & \text{if } e \in \{e_1, e_3, e_7\}, \\ 1/(2t+3) & \text{otherwise.} \end{cases}$$

We will show that x is feasible, using the following claim.

Claim 3.1. *In the simple graph G'_t , every degree- t forest has at most $2t + 1$ edges, and every degree- t forest containing each of e_1, e_3, e_7 at most $2t$ edges.*

Proof. The simple graph G'_t has $6 + 2(t - 2) = 2t + 2$ vertices, so every degree- t forest in G'_t has at most $2t + 1$ edges. Suppose that degree- t forest F of G'_t contains e_1, e_3, e_7 . Then F does not contain e_2 , but it may contain e_5 . Besides e_1 and e_7 , F contains at most $t - 2$ additional edges incident to u ; and besides e_3 and e_7 , F contains at most $t - 2$ additional edges incident to v . This implies that F has at most $3 + 1 + 2(t - 2) = 2t$ edges. $\square$

Fix a degree- t forest edge-set F in G'_t . If F contains at most two of e_1, e_3, e_7 , then by Claim 3.1 we have $|F| \leq 2t + 1$, so that

$$\sum_{e \in F} x_e \leq 2 \cdot \frac{2}{2t + 3} + (|F| - 2) \cdot \frac{1}{2t + 3} \leq 1.$$

If F contains each of e_1, e_3, e_7 , then by Claim 3.1 we have $|F| \leq 2t$, so that

$$\sum_{e \in F} x_e = 3 \cdot \frac{2}{2t + 3} + (|F| - 3) \cdot \frac{1}{2t + 3} \leq 1.$$

Since also $x \geq 0$, this proves that x is feasible for (D) .

We calculate objective value of x in (D) to be $(4t + 7)/(2t + 3)$. Thus, by weak duality and the equivalence of (P) and (P_0) , we have that $a_t^*(G_t) \geq (4t + 7)/(2t + 3)$. (One could prove that x is in fact an optimal solution for (D) , by exhibiting a feasible solution y for (P) with the same objective value, but this is not necessary for our proof.) On the other hand, we easily observe that $\Delta_t(G_t) = a(G_t) = 2$. Therefore,

$$a_t^*(G_t) \geq \frac{4t + 7}{2t + 3} = \frac{4t + 7}{4t + 6} \max\{\Delta_t(G_t), a(G_t)\},$$

which is the required inequality for proving Theorem 1.8 with $c_t \geq \frac{4t+7}{4t+6}$. (Note that $\Delta_t(mG_t) = a(mG_t) = 2m$ for every $m \geq 1$, so Theorem 1.8 applies for all even $d \geq 2$.)

For $t \leq 4$ we can improve this bound on the constant factor gap c_t by replacing G_t with the multigraph H_t shown in Figure 3.2(a). The underlying simple graph H'_t of H_t is shown in Figure 3.2(b). One can show that a feasible solution for (D) with respect to H'_t is $x_e = 1/(3t - 2)$ for every edge e , and that this x has objective value $(6t - 3)/(3t - 2)$. Thus $a_t^*(H_t) \geq (6t - 3)/(3t - 2)$ while $\Delta_t(H_t) = a(H_t) = 2$, and this gives the bound $c_t \geq (6t - 3)/(6t - 4)$. This bound improves the one from G_t for $t \leq 4$, giving $c_2 \geq 9/8$, $c_3 \geq 15/14$, and $c_4 \geq 21/20$. This finishes our constructions for Theorem 1.8.

We remark that the idea for constructing the multigraphs G_t and H_t above is to start with a suitable base graph and then to add certain gadgets to a subset of the vertices. These gadgets have the form of a number two parallel edges connecting to a new vertex. If the

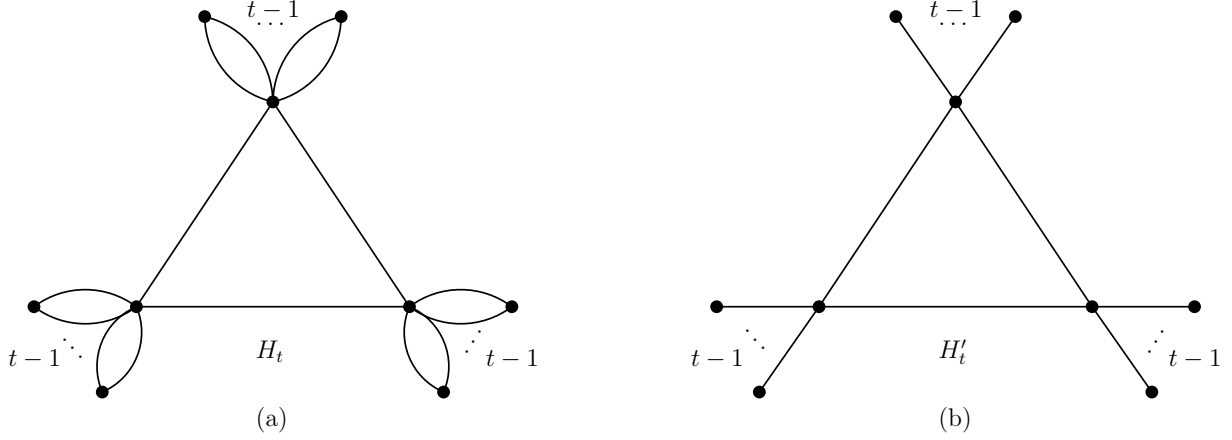


Figure 3.2: (a) The multigraph H_t . (b) The underlying simple graph H'_t .

base graph has large fractional linear arboricity, then by adding a certain number of gadgets we will create a graph with large fractional degree- t arboricity, for any given $t \geq 2$. Perhaps better gadgets can give better constants for Theorem 1.8.

To conclude this section, we remark that one can use the above approach of adding gadgets to show that it is NP-complete to decide whether a given multigraph G has degree- f arboricity $a_f(G) = 2$, for any fixed weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$, starting with the base case $f = 2$ due to Péroche [141]. Specifically, if G is an arbitrary multigraph and $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$ is an arbitrary weight function, we construct a multigraph H such that $la(G) \leq 2$ if and only if $a_{f|_{V(H)}}(H) \leq 2$. We obtain H by adding, for every vertex $v \in V(G)$, $f(v) - 2$ vertices $u_1, \dots, u_{f(v)-2}$ together with two parallel edges between v and u_i for each i . In other words, we are attaching $f(v) - 2$ copies of the gadget shown in Figure 3.3(a) to every vertex v of G . If $la(G) \leq 2$, then given an edge-coloring of G into two linear forests, we can extend it to an edge-coloring of H into two degree- $f|_{V(H)}$ forests by giving different colors to the edges in each gadget, so that $a_{f|_{V(H)}}(H) = 2$. If $a_{f|_{V(H)}}(H) \leq 2$, then given an edge-coloring of H into two degree- $f|_{V(H)}$ forests, the edges in each gadget must have different colors, so the two color classes restricted to G must be linear forests and thus $la(G) \leq 2$. Our multigraph H has at most $(t-1)|V(G)|$ vertices and at most $|E(G)| + 2(t-2)|V(G)|$ edges, so it can be constructed in polynomial time for fixed t . This finishes the proof when G is a multigraph.

If we want to show NP-completeness for simple graphs, we do the same construction but using the gadget shown in Figure 3.3(b) instead of Figure 3.3(a). The only properties required to make the above proof work is that the vertex v of the gadget has degree two, that the gadget can be edge-colored into two linear forests, that in every edge-coloring of the gadget into two forests the edges incident to v have different colors, and that the gadget has constant size. These properties are easily verified.



Figure 3.3: (a) The gadget for the case of multigraphs. (b) The gadget for the case of simple graphs.

3.2 A directed version of bounded degree arboricity

In this section, we introduce a directed version of degree- f arboricity and formulate a directed version of Conjecture 3.1. This reformulated conjecture (Conjecture 3.5 below) is a generalization of the Directed Linear Arboricity Conjecture due to Nakayama and Péroche [138]. This directed setting will be convenient in the next section for proving Theorem 1.9 and Theorem 1.10.

For a directed multigraph D (with no directed loops), let $\Delta^-(D)$ denote the maximum indegree $d^-(v)$ among vertices v of D , let $\Delta^+(D)$ denote the maximum outdegree, and let $\overline{D}$ denote the underlying undirected multigraph. Recall that two arcs are said to be *parallel* if they have the same head and tail, and *anti-parallel* if the head of one is the tail of the other and vice versa. The *arc-multiplicity* $\mu(D)$ is the maximum number of parallel arcs in D . We will call D a *directed graph* if $\mu(D) = 1$, that is, if D has no parallel arcs (but it may have anti-parallel arcs).

A directed graph B is a *branching* if every vertex v of B has indegree $d_B^-(v) \leq 1$ and its underlying undirected graph $\overline{B}$ is a forest. An *arborescence* is a branching whose underlying undirected graph is a tree. For a directed multigraph D , the *directed arboricity* $\vec{a}(D)$ of D is the minimum number of colors required to color the arcs of D so that every color class is a branching. Observe that $\vec{a}(D) \geq \Delta^-(D)$ and $\vec{a}(D) \geq a(\overline{D})$. Using Edmonds' celebrated theorem on packing arborescences [61, 128], Frank [72] proved the following formula for directed arboricity $\vec{a}(D)$.

Theorem 3.3 (Frank [72]). *For every directed multigraph D , we have*

$$\vec{a}(D) = \max \{ \Delta^-(D), a(\overline{D}) \}.$$

Frank also showed using Nash-Williams' Theorem [139] that every directed multigraph D satisfies $a(\overline{D}) \leq \Delta^-(D) + \mu(D)$, so that Theorem 3.3 implies that $\vec{a}(D) \leq \Delta^-(D) + \mu(D)$. In particular, if D is a directed graph then

$$\Delta^-(D) \leq \vec{a}(D) \leq \Delta^-(D) + 1.$$

Theorem 3.3 has some resemblance to Conjecture 3.1. We introduce a bounded degree version of branchings to connect the statements rigorously. Let D be a directed multigraph, and let $f : V(D) \rightarrow \mathbb{Z}_{\geq 2}$ be a weight function on the vertices. A directed subgraph B of D is a *degree- f branching* if it is a branching where every vertex v has outdegree $d_B^+(v) \leq f(v) - 1$. Notice that the underlying undirected graph $\overline{B}$ of a degree- f branching B is a degree- f forest. The *directed degree- f arboricity* $\vec{a}_f(D)$ of directed multigraph D is the minimum number of colors required to color the arcs of D so that every color class is a degree- f branching.

For the case $f = 2$, a degree-2 branching has been called a *directed linear forest*, and the directed degree-2 arboricity $\vec{a}_2(D) = \vec{la}(D)$ is called the *directed linear arboricity* of D . Noting that $\vec{la}(D) \geq \Delta^-(D)$ and $\vec{la}(D) \geq \Delta^+(D)$ for every directed multigraph D , Nakayama and Péroche [138] formulated the following *Directed Linear Arboricity Conjecture* for directed graphs D :

$$\vec{la}(D) \leq \max \{ \Delta^-(D), \Delta^+(D) \} + 1.$$

Nakayama and Péroche [138] proved that their conjecture holds if D is an acyclic directed graph, if $\Delta^-(D), \Delta^+(D) \leq 2$ and $|V(D)| \geq 4$, and if D is one of certain symmetric directed graphs G^* . (The directed graph G^* is obtained from the undirected graph G by replacing each edge by a pair of anti-parallel arcs.) On the other hand, He, Li, Bai, and Sun [107] later showed that their conjecture does not hold for the complete symmetric directed graphs K_3^* and K_5^* : $\Delta^-(K_n^*) = \Delta^+(K_n^*) = n - 1$ for all n but $\vec{la}(K_n^*) = n + 1$ for $n \in \{3, 5\}$. Still, these authors believe that these two directed graphs K_3^* and K_5^* are the only counterexamples, leading to the following updated version of Nakayama and Péroche's conjecture [138].

Conjecture 3.4 (Directed Linear Arboricity Conjecture [107, 138]). *For every directed graph D , we have*

$$\vec{la}(D) \leq \max \{ \Delta^-(D), \Delta^+(D) \} + 1,$$

except for $D = K_3^*, K_5^*$, in which case $\vec{la}(D) = \max \{ \Delta^-(D), \Delta^+(D) \} + 2$.

Now we consider a generalization of Conjecture 3.4 for general weight functions f and directed multigraphs D . As before, $\vec{a}_f(D) \geq \Delta^-(D)$ and $\vec{a}_f(D) \geq a(\overline{D})$. Now we also see that $\vec{a}_f(D) \geq \Delta_{f-1}^+(D) = \max_{v \in V(D)} \left\lceil \frac{d^+(v)}{f(v)-1} \right\rceil$. Based on preceding theorems and conjectures (ignoring Theorem 1.8), it is natural to conjecture the following.

Conjecture 3.5. *For every directed multigraph D , we have*

$$\vec{a}_f(D) \leq \max \{ \Delta^-(D), \Delta_{f-1}^+(D), a(\overline{D}) \} + 1.$$

Below we will observe that Conjecture 3.5 nearly implies Conjecture 3.1. Because Conjecture 3.1 is false for general multigraphs by Theorem 1.8, it follows that Conjecture 3.5 is too. However, Conjecture 3.5 could still be true for directed graphs (with no parallel arcs).

Since a directed graph D satisfies $\Delta^-(D) \leq a(\overline{D}) \leq \Delta^-(D) + 1$ as noted above, Conjecture 3.5 for directed graphs is basically the same as the statement that D satisfies

$$\vec{a}_f(D) \leq \max\{\Delta^-(D) + 1, \Delta_{f-1}^+(D)\} + 1.$$

Now, in the case $f = 2$, it has been observed that the Directed Linear Arboricity Conjecture (Conjecture 3.4) nearly implies the Linear Arboricity Conjecture (Conjecture 1.2) (see [22, 24]). The proof proceeds as follows. Given an undirected simple graph G , let D be a *balanced orientation* of G , meaning that every vertex v has both indegree and outdegree at most $\lceil d_G(v)/2 \rceil$ in D . (The existence of balanced orientations is an easy exercise from the theory of Euler tours or network flows, and it is also a special case of Theorem 3.6 below; see also Corollary 2.24.) Then for this orientation D we have $\Delta^-(D), \Delta^+(D) \leq \lceil \Delta(G)/2 \rceil$, and so by Conjecture 3.4 (for $D \neq K_3^*, K_5^*$) we have that $la(G) \leq la(D) \leq \max\{\Delta^-(D), \Delta^+(D)\} + 1 \leq \lceil \Delta(G)/2 \rceil + 1$, which is almost the Linear Arboricity Conjecture.

We make a similar reduction for more general weight functions f . To do this we will replace balanced orientations with the more general orientation theorem of Entringer and Tolman [64] that we used in Chapter 2 (see Theorem 2.1). We restate their theorem below for convenience.

Theorem 3.6 (Entringer, Tolman [64]). *For a multigraph G and weight functions $g, h : V(G) \rightarrow \mathbb{Z}_{\geq 0}$, G is (g, h) -upper-orientable if and only if*

- (1) $d(v) \leq g(v) + h(v)$ for all $v \in V(G)$, and
- (2) $|E(X)| \leq \min\{g(X), h(X)\}$ for all $X \subseteq V(G)$.

The following corollary of Theorem 3.6 is what enables our reduction.

Corollary 3.7. *Let G be a multigraph and let $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$ be a weight function. If $\max\{\Delta_f(G), a(G)\} \leq d$ then G has an orientation D such that $\max\{\Delta^-(D), \Delta_{f-1}^+(D), a(\overline{D})\} \leq d$.*

Proof. Define $g, h : V(G) \rightarrow \mathbb{Z}_{\geq 1}$ by $g(v) = d$ and $h(v) = d(f(v) - 1)$ for all $v \in V(G)$. Then $d_G(v) \leq d \cdot f(v) \leq g(v) + h(v)$ for all $v \in V(G)$, and $|E(X)| \leq d(|X| - 1) < d|X| = \min\{g(X), h(X)\}$ for all $X \subseteq V(G)$. By Theorem 3.6, G has an orientation D such that every vertex v has indegree at most d and outdegree at most $d(f(v) - 1)$. That is, $\Delta^-(D) \leq d$ and $\Delta_{f-1}^+(D) \leq d$. We also see that $a(\overline{D}) = a(G) \leq d$. $\square$

From here, it is easy to complete our reduction from undirected to directed. For any multigraph G , if $d = \max\{\Delta_f(G), a(G)\}$ and D is from Lemma 3.7, then Conjecture 3.5 implies that

$$a_f(G) \leq \vec{a}_f(D) \leq \max\{\Delta^-(D), \Delta_{f-1}^+(D), a(\overline{D})\} + 1 \leq d + 1 = \max\{\Delta_f(G), a(G)\} + 1,$$

which is almost the same as Conjecture 3.1. This is what we wanted to show.

To conclude this section, we remark that Conjecture 3.5 can be rewritten in terms of matroids. For a directed multigraph $D = (V, E)$, let

- M_1 be the partition matroid $(E, \mathcal{I}_1 = \{B \subseteq E : |B \cap \delta^-(v)| \leq 1 \text{ for all } v \in V\})$,
- M_2 be the partition matroid $(E, \mathcal{I}_2 = \{B \subseteq E : |B \cap \delta^+(v)| \leq f(v) - 1 \text{ for all } v \in V\})$,
- M_3 be the cycle matroid $(E, \mathcal{I}_3 = \{B \subseteq E : \overline{B} \text{ is a forest in } \overline{D}\})$,

where $\delta^-(v)$ and $\delta^+(v)$ are the set of incoming and outgoing arcs of the vertex v , respectively. Conjecture refdegree-f-branchings is simply the statement that

$$\beta(M_1, M_2, M_3) \leq \max\{\beta(M_1), \beta(M_2), \beta(M_3)\} + 1,$$

where $\beta(N_1, \dots, N_k)$ denotes the minimum number of common independent sets required to cover the common ground set of the loopless matroids $N_1, \dots, N_k$. It has been conjectured [6] that $\beta(N_1, N_2) \leq \max\{\beta(N_1) + 1, \beta(N_2)\}$ for any two loopless matroids N_1, N_2 on the same ground set, but we have shown that the natural extension of this conjecture to the matroid triple M_1, M_2, M_3 above does not always hold.

3.3 Bounded degree arboricity of simple graphs

We have proven that Conjecture 3.1 of Truszczynski [164] is false for general multigraphs. On the other hand, we believe that Conjecture 3.1 holds for simple graphs G . In this section, we will prove Theorem 1.9 and Theorem 1.10 supporting this belief. The former theorem states that the conjecture holds for simple graphs with large girth, and the latter theorem states that the conjecture holds for all simple graphs asymptotically when the weight function is bounded. Our proofs are extensions of Alon's proofs [22, 24] for linear arboricity, while also using some of the orientation ideas from Chapter 2.

3.3.1 Large girth

In this subsection, we prove that Conjecture 3.5 holds for directed graphs with large directed girth (Theorem 3.10). From this, we will deduce Theorem 1.9, which is restated below.

Theorem 1.9. *Let G be a simple graph, let $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$ be a weight function, and let $d = \max\{\Delta_f(G), a(G)\}$. If G has girth $g \geq 4d$, then $a_f(G) \leq d + 1$.*

The *directed girth* of a directed graph is the length of a shortest directed cycle. The idea of the proof of Theorem 3.10 is to first color the arcs of our directed graph into d directed

degree- f pseudoforests (which we know how to do optimally). Then we convert these d directed degree- f pseudoforests into $d + 1$ degree- f branchings by removing one edge from each monochromatic cycle, in such a way that the removed edges form a matching. The existence of such a collection of edges comes from the theory of *independent transversals*, or more accurately *full rainbow matchings*. (These are the subjects of the next few chapters, but we just need one result here.) It is possible to perform this large girth argument in the undirected setting, but the directed setting will be more convenient for obtaining asymptotic results in the next subsection.

For our proof, we will use Hakimi and Kariv's [90] generalization of König's edge-coloring theorem to the f -chromatic index, similar to how we used it in Section 2.1.

Theorem 3.8 (Hakimi, Kariv [90]). *For every bipartite multigraph G and weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 1}$, we have $\chi'_f(G) = \Delta_f(G)$.*

We will also use the following theorem on full rainbow matchings due to Aharoni, Berger, Meshulam [11]. This will play an important role later in Chapter 6.

Theorem 3.9 (Aharoni, Berger, Meshulam [11]). *Let G be a multigraph with maximum degree Δ , and let $E_1, \dots, E_n$ be pairwise disjoint subsets of $E(G)$. If $|E_i| \geq 2\Delta$ for every i , then there exists a matching $M \subseteq E(G)$ that contains one edge from each E_i .*

Now we prove our result on directed graphs with large directed girth.

Theorem 3.10. *Let D be a directed graph, let $f : V(D) \rightarrow \mathbb{Z}_{\geq 2}$ be a weight function, and let $d = \max\{\Delta^-(D), \Delta_{f-1}^+(D)\}$. If D has directed girth $g \geq 4d$, then $\vec{a}_f(D) \leq d + 1$.*

Proof. We construct an auxiliary bipartite graph H with parts A and B as follows. For every vertex v of D we put a copy v^+ in X and a copy v^- in B . For every arc e of D with tail u and head v we put an edge in H between u^+ and v^- . Then v^+ has degree $d_D^+(v)$ and v^- has degree $d_D^-(v)$ in H , for every $v \in V(D)$. Define the weight function $\tilde{f} : V(H) \rightarrow \mathbb{Z}_{\geq 1}$ by $\tilde{f}(v^+) = f(v) - 1$ and $\tilde{f}(v^-) = 1$ for all $v \in V(D)$. By Theorem 3.8,

$$\chi_{\tilde{f}}(H) = \Delta_{\tilde{f}}(H) = \max \left\{ \max_{v \in V(D)} \left\lceil \frac{d_D^+(v)}{f(v) - 1} \right\rceil, \max_{v \in V(D)} d_D^-(v) \right\} = \max \{ \Delta_{f-1}^+(D), \Delta^-(D) \} = d.$$

Thus H can be edge-colored into d degree- $\tilde{f}$ subgraphs. These subgraphs of H correspond to directed subgraphs $F_1, \dots, F_d$ of D with $d_{F_i}^-(v) \leq 1$ and $d_{F_i}^+(v) \leq f(v) - 1$ for all $1 \leq i \leq d$ and $v \in V(D)$. The F_i 's are thus directed degree- f pseudoforests, meaning that every vertex v of B_i has indegree at most one and the underlying undirected graph $\overline{F_i}$ is a degree- f pseudoforest.

Let D' be the directed subgraph of D that is the union of all the (monochromatic) directed cycles in $F_1, \dots, F_d$. Let G be the underlying simple graph of D' , and let $E_1, \dots, E_n$ be the partition of $E(G)$ corresponding to the monochromatic directed cycles in D . Then

for each vertex $v \in V(G)$, each of directed degree- f pseudoforests $F_1, \dots, F_d$ contributes at most 2 edges incident to v , so G has maximum degree $\Delta \leq 2d$. Then by the directed girth assumption, for every $i \in \{1, \dots, n\}$ we have $|E_i| \geq g \geq 4d \geq 2\Delta$. By Theorem 3.9, there exists a matching of G containing one edge from each E_i . But this means that there exists a (directed) matching M of D' containing one arc from every monochromatic directed cycle in D . Then $M, F_1 \setminus M, \dots, F_d \setminus M$ are all degree- f branchings, giving us an arc-coloring of D into $d + 1$ degree- f branchings. $\square$

Using Corollary 3.7, we now deduce Theorem 1.9 from Theorem 3.10, which is what we wanted to show.

Proof of Theorem 1.9. By Corollary 3.7, there exists an orientation D of G such that $\max\{\Delta^-(D), \Delta_{f-1}^+(D)\} \leq d$, and D has directed girth at least $4d \geq 4 \max\{\Delta^-(D), \Delta_{f-1}^+(D)\}$. By Theorem 3.10, we have $a_f(G) \leq \vec{a}_f(D) \leq \max\{\Delta^-(D), \Delta_{f-1}^+(D)\} + 1 \leq d + 1$. $\square$

Before moving on to asymptotics, we give some historical remarks which gives a preview of the next few chapters about transversals. For the case of linear arboricity $f = 2$, Alon originally proved these large-girth results under the girth condition $g \geq 100d$ [21], which he later improved to $g \geq 8ed$ [22]. He derived these girth conditions from earlier versions of Theorem 3.10 and Theorem 1.9, but he worked in the line graph of the original graph and looked for independent transversals of a vertex partition using the Lovász local lemma. Specifically, if $V_1, \dots, V_n$ are pairwise disjoint subsets of $V(G)$, then he showed that there exists an independent transversal under the conditions $|V_i| \geq 25\Delta(G)$ [21] and $|V_i| \geq 2e\Delta(G)$ [22]. Haxell [96] subsequently improved these class size condition to $|V_i| \geq 2\Delta(G)$. Haxell's bound is best possible for general graphs (by a construction of Szabó and Tardos [161]), but for the case of line graphs of simple graphs (which is more relevant for this discussion), Aharoni, Alon, and Berger [5] improved the class size condition to $|V_i| \geq \Delta(G) + 2$. In fact, this latter result is what we originally used to prove Theorem 3.10 in [170], but we later realized that it is slightly cleaner to work with the original graph instead of the line graph and to use Theorem 3.9 of Aharoni, Berger, and Meshulam [11] as the main transversal result. But the results are the same. Finally, more recent work of Delcourt and Postle [53] on full rainbow matchings implies that we can improve the girth condition in Theorem 3.10 and Theorem 1.9 to $g \geq (2 + o(1))d$, but we decided to present a more exact condition. What matters for the following subsection on asymptotics is just that the girth condition $g \geq \alpha d$ works for some constant α .

3.3.2 Asymptotics

In this subsection, we prove that Conjecture 3.5 holds asymptotically for all directed graphs D when the weight function f is bounded (Theorem 3.12). From this we will deduce Theorem 1.10, which is restated below.

Theorem 1.10. *For every integer $t \geq 2$, there exists a real constant $c_t > 0$ such that for every simple graph G and weight function $f : V(G) \rightarrow \mathbb{Z}_{\geq 2}$ with maximum value at most t , we have*

$$a_f(G) \leq d + c_t d^{3/4} (\log d)^{1/2},$$

where $d = \max\{\Delta_f(G), a(G)\}$.

Similar to Alon's [22, 24] proof for the case of linear arboricity $f = 2$, the proof proceeds by showing that every directed graph D can be decomposed into a specified number of directed subgraphs each with large directed girth and approximately the same maximum degree, and then we apply Theorem 3.10 to each of these directed subgraphs individually. We use the following slight modification of a lemma of Alon [22].

Lemma 3.11. *Let D be a directed graph and let $f : V(D) \rightarrow \mathbb{Z}_{\geq 2}$ be a weight function. Suppose that $d = \max\{\Delta^-(D), \Delta_{f-1}^+(D)\}$ is sufficiently large compared to the maximum value of f , and let $k \leq d^{9/10}$ be a positive integer. Then there is a k -coloring of $V(D)$ using the colors $0, 1, \dots, k-1$ with the following property: for every vertex v and color i , the numbers*

$$\begin{aligned} d^-(v, i) &= |\{u \in V(D) : (u, v) \in E(D) \text{ and } u \text{ is colored } i\}|, \\ d^+(v, i) &= |\{u \in V(D) : (v, u) \in E(D) \text{ and } u \text{ is colored } i\}| \end{aligned}$$

$$\text{satisfy } d^-(v, i), \frac{d^+(v, i)}{f(v)-1} \leq \frac{d}{k} + 3\sqrt{\frac{d \log d}{k}}.$$

Proof. Start by augmenting D to a directed graph D' , adding auxiliary vertices and arcs so that every vertex $v \in V(D)$ has indegree $d_{D'}^-(v) = d$ and outdegree $d_{D'}^+(v) = d(f(v) - 1)$ in D' (not caring about the indegrees and outdegrees of the added vertices). Consider a random coloring of the vertices of D' with the colors $0, 1, \dots, k-1$, where the color of every vertex is chosen uniformly at random. For every vertex $v \in V(D)$ and color i , let $A_{v,i}^-$ be the event that $d_{D'}^-(v, i) > d/k + 3\sqrt{(d \log d)/k}$. Observe that $d_{D'}^-(v, i)$ is a binomial random variable with mean d/k . By Chernoff's inequality (see Appendix A in [24]), we have that $\Pr[A_{v,i}^-] \leq 1/d^4$. Likewise, letting $A_{v,i}^+$ be the event that $d_{D'}^+(v, i)/(f(v) - 1) > d/k + 3\sqrt{(d \log d)/k}$, we have that $\Pr[A_{v,i}^+] \leq 1/d^{4(f(v)-1)} \leq 1/d^4$.

Each of the events $A_{v,i}^-, A_{v,i}^+$ is mutually independent of the events $A_{u,j}^-, A_{u,j}^+$ for all the vertices $u \in V(D)$ that do not have a common neighbor with v in $\overline{D}$. Thus each of $A_{v,i}^-, A_{v,i}^+$ is mutually independent of all but at most $k(td)^2$ of the events $A_{u,j}^-, A_{u,j}^+$, where t is the maximum output of f . Since $e(1/d^4)(k(td)^2 + 1) < 1$ for d sufficiently large compared to t , by the symmetric Lovász local lemma (see Chapter 5 in [24]) no event $A_{v,i}^-$ or $A_{v,i}^+$ occurs. Thus there is a coloring of $V(D')$ satisfying $d_{D'}^-(v, i), d_{D'}^+(v, i)/(f(v) - 1) \leq d/k + 3\sqrt{(d \log d)/k}$ for all $v \in V(D)$ and $0 \leq i \leq k-1$. Deleting the auxiliary vertices and arcs from D' , this gives a desired coloring of $V(D)$. $\square$

We now prove the following asymptotic version of Conjecture 3.5 for directed graphs.

Theorem 3.12. *For every integer $t \geq 2$, there exists a constant $c_t > 0$ such that for every directed graph D and weight function $f : V(D) \rightarrow \mathbb{Z}_{\geq 2}$ with maximum value at most t , we have*

$$\vec{a}_f(D) \leq d + c_t d^{3/4} (\log d)^{1/2},$$

where $d = \max\{\Delta^-(D), \Delta_{f-1}^+(D)\}$.

Proof. We may assume that d is sufficiently large compared to t wherever necessary. Pick a prime k satisfying $5d^{1/2} \leq k \leq 10d^{1/2}$. By Lemma 3.11, there exists a k -coloring ϕ of $V(D)$ satisfying the stated inequalities. For each $0 \leq i \leq k-1$, let D_i be the spanning directed subgraph of D with arc set $E(D_i) = \{(u, v) \in E(D) : \phi(v) \equiv \phi(u) + i \pmod{k}\}$. The inequalities in Lemma 3.11 imply that $d_i = \max\{\Delta^-(D_i), \Delta_{f-1}^+(D_i)\} \leq d/k + 3\sqrt{(d \log d)/k}$ for every $0 \leq i \leq k-1$. Moreover, for $i \neq 0$ the length of every directed cycle in D_i is divisible by k , and thus D_i has directed girth $g_i \geq k \geq 4d_i$ (using that $k \geq 5d^{1/2}$ and d is sufficiently large). By Theorem 3.10, we deduce that $\vec{a}_f(D_i) \leq d_i + 1 \leq d/k + 3\sqrt{(d \log d)/k} + 1$ for every $1 \leq i \leq k-1$. For D_0 we only use the trivial inequality $\vec{a}_f(D_0) \leq 2d_0 \leq 2d/k + 6\sqrt{(d \log d)/k}$, obtained by first using Theorem 3.8 to arc-color D_0 into d_0 directed degree- f pseudoforests, and then trivially arc-coloring each of these directed degree- f pseudoforests into 2 degree- f branchings. These inequalities together with $5d^{1/2} \leq k \leq 10d^{1/2}$ give us that

$$\vec{a}_f(D) \leq (k-1) \left(\frac{d}{k} + 3\sqrt{\frac{d \log d}{k}} + 1 \right) + \left(\frac{2d}{k} + 6\sqrt{\frac{d \log d}{k}} \right) \leq d + c_t d^{3/4} (\log d)^{1/2},$$

for some constant c_t depending on t (since we assumed d is large compared to t). □

Applying Corollary 3.7, we now deduce Theorem 1.10 from Theorem 3.12.

Proof of Theorem 1.10. By Corollary 3.7, there exists an orientation D of G such that $\max\{\Delta^-(D), \Delta_{f-1}^+(D)\} \leq d$. Let $d_0 = \max\{\Delta^-(D), \Delta_{f-1}^+(D)\}$. Then by Theorem 3.12, there exists a real constant $c_t > 0$ such that $a_f(G) \leq \vec{a}_f(D) \leq d_0 + c_t d_0^{3/4} (\log d_0)^{1/2} \leq d + c_t d^{3/4} (\log d)^{1/2}$. □

We conclude this section with a few remarks. Alon [22] stated that the lower order term in Theorem 3.12 (and thus also in Theorem 1.10) can be improved to $c'd^{2/3}(\log d)^{1/3}$ when $f = 2$, by using “recursion” instead of a naive arc-coloring of D_0 , as well as changing some parameters. This kind of modification could perhaps also work for more general vertex weight functions f , but we do not know the details. Alon, Teague, and Wormald [25] later recovered the same lower order term stated by Alon [22], in the undirected case, using a different and in some ways simpler method. Instead of using a large girth result like Theorem 3.10 or Theorem 1.9 above, these authors use a variant of the classical and easier result that the

complete graph K_{2n} can be decomposed into n Hamilton paths. However, their proof method does not appear to generalize as readily as Alon's for general weight functions f .

Alon's [21] original asymptotic proof of the Linear Arboricity Conjecture involved coloring the edges of the graph rather than the vertices as we did above. This original approach resulted in a worse error term, but Alon [21] noted without proof that this approach also gives the desired asymptotics more generally for multigraphs G with bounded edge-multiplicity $\mu(G)$. One can check that this approach also generalizes well to other weight functions f on the vertices, when one works in the directed setting. That is, the asymptotic upper bound $a_f(G) \leq (1 + o(1)) \max\{\Delta_f(G), a(G)\}$ that we proved above holds more generally for multigraphs G with bounded edge-multiplicity.

3.4 Problems

We conclude this chapter with some problems for future work.

(1) The main problem that this chapter leaves open is the following: For $t \geq 2$, what is the optimal constant $c_t > 1$ such that every multigraph G has degree- t arboricity satisfying $a_t(G) \leq (c_t + o(1)) \max\{\Delta_t(G), a(G)\}$? Theorem 1.5 proven in Chapter 2 implies that $c_t \leq 2$, while our constructions from Section 3.1 imply that $c_t \geq (4t + 7)/(4t + 6)$ for every integer $t \geq 2$, with slight improvements $c_2 \geq 9/8, c_3 \geq 15/14, c_4 \geq 21/20$ for $t \in \{2, 3, 4\}$. In particular, can c_t be bounded away from 1 as $t \rightarrow \infty$? We may also consider this problem for other bounded weight functions f , but already the constant case and even the case $t = 2$ is interesting. Of course, if we restrict to simple graphs G , then Theorem 1.10 implies that the optimal constant is $c_f = 1$.

(2) A related problem is how far apart the bounded degree arboricity $a_f(G)$ and the fractional bounded degree arboricity $a_f^*(G)$ can be. In Section 3.1, we only used the inequality $a_f(G) \geq a_f^*(G)$ and the fact that $a_f^*(G)$ and $\max\{\Delta_f(G), a(G)\}$ can be arbitrarily far apart so as to conclude that $a_f(G)$ and $\max\{\Delta_f(G), a(G)\}$ can be arbitrarily far apart. But we do not know whether $a_f(G)$ and $a_f^*(G)$ can be arbitrarily far apart. For comparison, it is well known that ordinary arboricity satisfies $a(G) = \lceil a^*(G) \rceil$. If $a_f(G)$ and $a_f^*(G)$ can be arbitrarily far apart, we may also ask problem (1) with $a_t(G)$ replaced by $a_t^*(G)$.

(3) We may consider the above problems in the setting of directed multigraphs. Specifically, using notation from Section 3.2, what is the optimal constant $c_t > 0$ such that the directed degree- t arboricity satisfies $\vec{a}_t(D) \leq (c_t + o(1)) \max\{\Delta^-(D), \Delta_{t-1}^+(D), a(\overline{D})\}$? How far apart can $\vec{a}_f(D)$ and its fractional version $\vec{a}_f^*(D)$ be? Work on the undirected case gives partial results on these questions, but perhaps the gaps can be even larger in the directed setting.

Part II

Transversals

Chapter 4

Extremal and stability results for independent transversals

In this chapter, we study optimal conditions for the existence of independent transversals in graphs, in terms of both maximum degree and average degree parameters. In particular, we will prove Theorem 1.15 and Theorem 1.17. We will also prove stability theorems for independent transversals in bounded degree graphs, namely Theorem 1.18 and Theorem 1.19. This chapter is based on joint work with Haxell [106].

4.1 Existence of independent transversals

In this section, we give an exposition of the topological approach for proving the existence of an independent transversal (IT) in a graph. This will prepare us for the proofs of the main results of this chapter. See Appendix A for more details on the various topological tools discussed here.

Let G be a graph and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. Recall that $\mathcal{I}(G)$ is the abstract simplicial complex consisting of all independent sets of G , called *independence complex* of G . Then an IT in G with respect to $\mathcal{P}$ is the same as a colorful simplex in $\mathcal{I}(G)$ with respect to $\mathcal{P}$. Recall the topological connectedness parameters $\eta_\pi(\mathcal{C}) = \max\{k : \pi_k(\mathcal{C}) = 0\} + 2$ and $\eta_H(\mathcal{C}) = \max\{k : \tilde{H}_k(\mathcal{C}) = 0\} + 2$, where $\pi_k(\mathcal{C})$ denotes the k^{th} homotopy group of $\mathcal{C}$, and $\tilde{H}_k(\mathcal{C})$ denotes the k^{th} reduced homology group with field coefficients. In this chapter, it will not matter whether we use $\eta_\pi(\mathcal{C})$ or $\eta_H(\mathcal{C})$, and we will denote either of them by $\eta(\mathcal{C})$. Recall that Theorem 1.11 is a topological Hall theorem for the existence of a colorful simplex. We rewrite here for independence complexes, where for a subset $I \subseteq [n]$, we denote the induced subgraph $G_I = G[\bigcup_{i \in I} V_i]$.

Theorem 4.1. *Let G be a graph and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. If*

$$\eta(\mathcal{I}(G_I)) \geq |I| \quad \text{for all } I \subseteq [n],$$

then there exists an IT.

Theorem 4.1 allows us to obtain sufficient conditions for independent transversals from lower bounds on connectedness. Conversely, the construction of graphs with no independent transversals is often aided by finding graphs G with a low value of $\eta(\mathcal{I}(G))$. Understanding when $\eta(\mathcal{I}(G))$ is small will be the main motivation for the upcoming Theorem 4.9, which will form the basis of our work in this chapter. The following lemma lays out some straightforward facts about topological connectedness (see Lemma A.3 in Appendix A).

Lemma 4.2. (a) Every graph G has $\eta(\mathcal{I}(G)) \geq 0$, with equality if and only if G is empty.

(b) If G has an isolated vertex, then $\eta(\mathcal{I}(G)) = \infty$.

(c) If G and H are disjoint graphs, then $\eta(\mathcal{I}(G \cup H)) \geq \eta(\mathcal{I}(G)) + \eta(\mathcal{I}(H))$.

For a graph G and edge $e \in E(G)$, we denote by $G - e$ the graph formed by deleting e . We write $G * e$ for the graph formed by *exploding* e , which is obtained from G by removing both endpoints of e together with all of their neighbours. In other words, if x and y are the endpoints of e , then $G * e = G - (N(x) \cup N(y))$. Our main tool for obtaining lower bounds for $\eta(\mathcal{I}(G))$ will be the following theorem of Meshulam [131] (see also [13] and Theorem A.5 in Appendix A.)

Theorem 4.3 (Meshulam [131]). For every graph G and edge $e \in E(G)$, we have

$$\eta(\mathcal{I}(G)) \geq \min\{\eta(\mathcal{I}(G - e)), \eta(\mathcal{I}(G * e)) + 1\}$$

We say that an edge e of a graph G is *deletable* if $\eta(\mathcal{I}(G - e)) \leq \eta(\mathcal{I}(G))$. We say that e is *explodable* if $\eta(\mathcal{I}(G * e)) \leq \eta(\mathcal{I}(G)) - 1$. By Theorem 4.3, every edge of G is either deletable or explodable. A graph G is called *reduced* if no edge is deletable (hence every edge is explodable). A subgraph $G' \subseteq G$ is called a *reduction* of G if G' is reduced, $V(G') = V(G)$, and $\eta(\mathcal{I}(G')) \leq \eta(\mathcal{I}(G))$. One may obtain a reduction of a graph G by iteratively deleting deletable edges until there are none left.

Recall that Theorem 1.12 of Haxell [95, 96] states that there exists an IT if G has maximum degree d and $|V_i| \geq 2d$ for every i . The topological proof of Theorem 1.12 is based on the lower bound $\eta(\mathcal{I}(G)) \geq \frac{|V(G)|}{2d}$ from [130], where $d = \Delta(G)$ is the maximum degree of G . As an introduction to working with the connectedness parameter $\eta(\mathcal{I}(G))$, we give a proof of this bound along with a characterization of the graphs G for which the bound is tight.

Theorem 4.4. Let G be a graph with maximum degree $d \geq 1$. If $|V(G)| \geq 2dk$, then $\eta(\mathcal{I}(G)) \geq k$. Further, if $\eta(\mathcal{I}(G)) = k$, then G is the disjoint union of k copies of $K_{d,d}$.

Proof. We use induction on k . The result is immediate for $k = 0$ by Lemma 4.2. The inequality holds for $k = 1$ since $|V(G)| \geq 1$ implies $\eta(\mathcal{I}(G)) \geq 1$ by Lemma 4.2. For the case

of equality when $k = 1$, suppose that $|V(G)| \geq 2d$ and $\eta(\mathcal{I}(G)) = 1$. Having $\eta(\mathcal{I}(G)) < 2$ means that $\mathcal{I}(G)$ is not path-connected. This is equivalent to saying that the complement of G is disconnected, since a topological path between two vertices in $\mathcal{I}(G)$ corresponds to a graph-theoretical path between those vertices in the complement of G . This then implies that G has a complete cut, that is, G contains a complete bipartite graph on all of its vertices. Since $|V(G)| \geq 2d$ and $\Delta(G) = d$, the only possibility is that G is the complete bipartite graph $K_{d,d}$. This proves the base case $k = 1$.

Assume that $k \geq 2$ and that the conclusion of the theorem holds for all values smaller than k . Now we apply Theorem 4.3. By iteratively deleting deletable edges from G , we may assume that G is a reduced graph, so that every edge of G is explodable. Let x be a vertex of minimum degree in G . Then $d(x) > 0$ by Lemma 4.2, so we may choose an edge e of G incident to x . Let $G' = G * e$, and let d' denote the maximum degree of G' . Then $d' \leq d$, and we may calculate that $|V(G')| \geq 2dk - (d(x) + d) \geq 2d(k - 1) \geq 2d'(k - 1)$. Hence, by the induction hypothesis we conclude that $\eta(\mathcal{I}(G')) \geq k - 1$, and if $\eta(\mathcal{I}(G')) = k - 1$ then G' is the union of $k - 1$ disjoint copies of $K_{d',d'}$. Since xy is explodable in G , it follows that $\eta(\mathcal{I}(G)) \geq \eta(\mathcal{I}(G')) + 1 \geq k$, proving the inequality.

For the case of equality, suppose that $\eta(\mathcal{I}(G)) = k$. Then equality holds everywhere in the previous paragraph, so that $d' = d$ and G' is the union of $k - 1$ disjoint copies of $K_{d,d}$. Since $\Delta(G) = d$ and G' is d -regular, the graph $H = G[N(x) \cup N(y)]$ is a connected component of G , and $G = G' \cup H$. This implies that $\eta(\mathcal{I}(H)) \leq \eta(\mathcal{I}(G)) - \eta(\mathcal{I}(G')) = 1$ and $|V(H)| = |V(G)| - |V(G')| \geq 2d$. From the base case $k = 1$, it follows that H is a $K_{d,d}$. Therefore, G is the union of k disjoint copies of $K_{d,d}$, finishing the induction. $\square$

Theorem 4.4 implies the following slightly stronger version of Theorem 1.12.

Corollary 4.5. *Let G be a graph with maximum degree d , and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. If*

$$\sum_{i \in I} |V_i| \geq 2d(|I| - 1) + 1 \quad \text{for all } I \subseteq [n],$$

then there exists an IT. In particular, if $|V_i| \geq 2d$ for every i , then there exists an IT.

Proof. For the first statement, for all $I \subseteq [n]$, by Theorem 4.4 we have

$$\eta(\mathcal{I}(G_I)) \geq \frac{|V(G_I)|}{2d} \geq \frac{2d(|I| - 1) + 1}{2d} > |I| - 1.$$

Since $\eta(\mathcal{I}(G_I))$ is an integer, we have $\eta(\mathcal{I}(G_I)) \geq |I|$. By Theorem 4.1, there exists an IT. $\square$

We remark that this kind of application of Theorem 4.3 for proving the existence of an IT has been done various times before [5, 13, 16, 101, 130, 131].

Now we present some additional results about connectedness and IT's that will also be useful in this chapter as well as Chapter 5. Theorem 4.4 tells us that the extremal graphs for the lower bound $\eta(\mathcal{I}(G)) \geq \frac{|V(G)|}{2d}$ are disjoint unions of copies of $K_{d,d}$. Now we wish to use this to understand the extremal graphs for the existence of IT's, as in Corollary 4.5. To do that, we need a refinement of Theorem 4.4 that tells us how much $\eta(\mathcal{I}(G))$ increases in the absence of $K_{d,d}$. For that purpose, Aharoni, Holzman, Howard, and Sprüssel [16] proved the following.

Theorem 4.6 (Aharoni, Holzman, Howard, Sprüssel [16]). *Let G be a graph with maximum degree $d > 2$. If G does not contain $K_{d,d}$, then*

$$\eta(\mathcal{I}(G)) \geq \frac{|V(G)|}{2d-1}.$$

The proof of Theorem 4.6 in [16] is similar to the above proof of Theorem 4.4, just with some extra tricks. For a rough summary, Aharoni, Holzman, Howard, and Sprüssel [16] observe that for a graph G that does not contain $K_{d,d}$, the sequence of edges that are deleted or exploded can be chosen so that the first explosion step may remove $2d$ vertices, but all subsequent explosion steps remove at most $2d-1$ vertices. The lower bound of Theorem 4.6 holds immediately if any fewer vertices are removed at some explosion step. Otherwise, the authors show that G must contain $K_{d,d-1}$ as a subgraph, and that this further implies that the edges could have been chosen so that the second explosion removes at most $d+1$ edges, which is less than $2d-1$ for $d > 2$, a contradiction. Using Theorem 4.6, the authors [16] proved the following theorem about extremal graphs for IT's.

Theorem 4.7 (Aharoni, Holzman, Howard, Sprüssel [16]). *Let G be a graph with maximum degree $d > 2$, and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. If $|V_i| \geq 2d-1$ for every i but there is no IT, then G contains $2d-1$ disjoint copies of $K_{d,d}$.*

Proof. For $I \subseteq [n]$, let k_I denote the number of copies of $K_{d,d}$ contained in G_I . Then we can write G_I as the disjoint union $H \sqcup K_{d,d} \sqcup \dots \sqcup K_{d,d}$, where there are k_I copies of $K_{d,d}$ and H is $K_{d,d}$ -free. By Lemma 4.2 and Theorem 4.6,

$$\begin{aligned} \eta(\mathcal{I}(G_I)) &\geq \eta(\mathcal{I}(H)) + k_I \cdot \eta(\mathcal{I}(K_{d,d})) \geq \frac{|V(G_I)| - 2dk_I}{2d-1} + k_I = \frac{|V(G_I)| - k_I}{2d-1} \\ &\geq \frac{(2d-1)|I| - k_I}{2d-1} = |I| - \frac{k_I}{2d-1}. \end{aligned}$$

If G contains fewer than $2d-1$ disjoint copies of $K_{d,d}$, then $k_I < 2d-1$, and thus $\eta(\mathcal{I}(G_I)) > |I| - 1$. Since $\eta(\mathcal{I}(G_I))$ is an integer, we have $\eta(\mathcal{I}(G_I)) \geq |I|$. By Theorem 4.1, there exists an IT. $\square$

Now, recall that Szabó and Tardos [161] showed that the graph G which is the disjoint union of $2d-1$ copies of $K_{d,d}$ admits a vertex partition $\mathcal{P} = \{V_1, \dots, V_{2d}\}$ such that $|V_i| =$

$2d - 1$ for every i and there is no IT. Jin [110] and Yuster [178] previously gave constructions when d is a power of 2, but their partitions were different from those of Szabó and Tardos [161]. From Theorem 4.7, we see that it is not a coincidence that all these constructions involved the same underlying graph G (the disjoint union of $2d - 1$ copies of $K_{d,d}$), since it is the minimal graph that can admit such a partition for $d \neq 2$. But note that Theorem 4.7 does not apply to the case of maximum degree $d = 2$. It turns out that there exist additional minimal graphs in the case $d = 2$ that admit the said vertex partitions with no IT. Aharoni, Holzman, Howard, and Sprüssel [16] proved the following.

Theorem 4.8 (Aharoni, Holzman, Howard, Sprüssel [16]). *Let G be a graph with maximum degree $d = 2$, and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. If $|V_i| \geq 3$ for every i but there is no IT, then G contains 3 disjoint cycles each of length $1 \pmod{3}$.*

The above authors proved Theorem 4.8 similar to how they deduced Theorem 4.7 from Theorem 4.6, but this time relying on connectedness formulas for paths and cycles from previous work (Claim 3.3 of Meshulam [131]): $\eta(\mathcal{I}(P_n)) = \lceil n/3 \rceil$ if $n \equiv 0, 2 \pmod{3}$, $\eta(\mathcal{I}(P_n)) = \infty$ if $n \equiv 1 \pmod{3}$, and $\eta(\mathcal{I}(C_n)) = \lceil n/3 \rceil$ for all n (where $\lceil \alpha \rceil$ denotes the rounding of α to the nearest integer). These hold for both η_π and η_H . See Figure A.1 for a visualization of the independence complexes of C_n and P_n for small values of n .

Theorem 4.7 and Theorem 4.8 together suggest a Brooks-type theorem for IT's. Recall that Brooks' theorem for the chromatic number of a graph states that the only minimal graph with maximum degree d and chromatic number at least $d + 1$ is the complete graph K_{d+1} , unless $d = 2$ in which case the minimal graphs are all the odd cycles. Analogously, the results above suggest that the only minimal graph G with maximum degree d that admits a vertex partition $\mathcal{P} = \{V_1, \dots, V_n\}$, such that $|V_i| \geq 2d - 1$ for every i and there is no IT, is the disjoint union of $2d - 1$ copies of $K_{d,d}$, unless $d = 2$, in which case the minimal graphs are disjoint unions of three cycles each of length $1 \pmod{3}$. We will verify this Brooks-type theorem in Chapter 5.

There are many other results about maximum degree conditions for IT's that have been proven using similar topological approaches as described here. For example, Aharoni, Berger, and Ziv [12] (see also [13, 131]) proved that if G is a chordal graph with maximum degree Δ and $|V_i| \geq d + 1$ for every i , then there exists an IT. They used this to verify a list coloring conjecture of Reed [145] for chordal graphs, a conjecture that is the subject of Section 5.4. In addition, Aharoni, Alon, and Berger [5] proved that if G is an (induced) $K_{1,k}$ -free graph with maximum degree Δ and $|V_i| \geq \frac{(2k-3)d+k-1}{k-1}$ for every i , then there exists an IT. If G is the line graph of a $(k - 1)$ -uniform linear hypergraph, they improved this to condition to $|V_i| \geq d + k - 1$. These are some of the ways that the condition $|V_i| \geq 2d$ in Theorem 1.12 can be improved assuming additional structure on the graph G .

4.2 Graphs with low topological connectedness

In this section, we describe our main technical result, Theorem 4.9, on the structure of graphs whose independence complexes have low topological connectedness. We will use this result to prove Theorem 1.16, and in the next sections we will apply these results to prove Theorem 1.15, Theorem 1.18, and Theorem 1.19.

The following is Theorem 4.9. It is helpful to keep in mind that the partition in the statement is not related to any given partition $\mathcal{P}$ into classes as we have been discussing in the previous sections, but is rather a different partition that comes about as a consequence of the upper bound on $\eta(\mathcal{I}(G))$.

Theorem 4.9. *Let G be a graph with $\eta(\mathcal{I}(G)) < \ell$. Then $V(G)$ has a partition into sets $\{X_i, Y_i, Z_i\}_{i=1}^k$ for some $k < \ell$, such that*

- (a) *for all i , there exist $x_i \in X_i$ and $y_i \in Y_i$ with $Y_i \cup Z_i \subseteq N(x_i)$ and $X_i \cup Z_i \subseteq N(y_i)$,*
- (b) $4 \sum_{i=1}^k |X_i||Y_i| + \sum_{i=1}^k (|X_i| + |Y_i|)|Z_i| \leq 2|E(G)| + 2 \sum_{i=1}^k |E(X_i, Y_i)|.$

Proof. We construct a sequence of subgraphs $G_0, H_0, G_1, H_1, \dots, G_k, H_k$ with $G_0 = G$ and H_k having no edges, such that H_i is a reduction of G_i and G_i is obtained from H_{i-1} by exploding an edge $x_i y_i$ that kills the smallest number of vertices. Note that every edge of each H_i is explodable, and that k explosions are performed in total. Since $\eta(\mathcal{I}(G)) < \ell$, Theorem 4.3 implies that $k < \ell$.

For $1 \leq i \leq k$, let $Z_i = N(x_i) \cap N(y_i)$, $X_i = N(y_i) \setminus Z_i$, and $Y_i = N(x_i) \setminus Z_i$. Note that $\{X_i, Y_i, Z_i\}_{i=1}^k$ is a partition of $V(G)$, that $x_i \in X_i$ and $y_i \in Y_i$, and that $Y_i \cup Z_i \subseteq N(x_i)$ and $X_i \cup Z_i \subseteq N(y_i)$. We classify the edges of G as follows. If an edge joins two vertices in distinct sets among $\{X_i, Y_i, Z_i\}$ for some i , then we label it as *black*. All other edges are labeled as *pink*. Further, we assign a direction to each pink edge that joins some $X_i \cup Y_i \cup Z_i$ to $X_j \cup Y_j \cup Z_j$ with $i < j$ by orienting it from its i end to its j end.

Fix $1 \leq i \leq k$. Consider any vertex $w \in Y_i$ ($\subseteq N(x_i)$). By the choice of the edge $x_i y_i$, the vertex w has at least $|X_i|$ neighbours outside of $N(x_i)$ in H_{i-1} . The edges joining w to X_i are black. Denoting their number by $b_{X_i}(w)$, this implies that at least $|X_i| - b_{X_i}(w)$ edges incident to w in H_{i-1} are pink out-edges. Similarly, for every vertex $w \in X_i$, denoting the number of edges joining to w to Y_i by $b_{Y_i}(w)$, at least $|Y_i| - b_{Y_i}(w)$ edges incident to w in H_{i-1} are pink out-edges. Now consider any vertex $w \in Z_i$ ($= N(x_i) \cap N(y_i)$). Let $b_{X_i}(w)$ and $b_{Y_i}(w)$ denote the black degree of w into X_i and Y_i , respectively. Again by the choice of the edge $x_i y_i$, the vertex w has at least $|X_i|$ neighbours outside of $N(x_i)$ and at least $|Y_i|$ neighbours outside of $N(y_i)$. This tells us that w has pink outdegree at least $|X_i| - b_{X_i}(w)$ and at least $|Y_i| - b_{Y_i}(w)$, and so in particular at least $(|X_i| + |Y_i| - b_{X_i}(w) - b_{Y_i}(w))/2$.

We get that the total pink outdegree of vertices in $X_i \cup Y_i \cup Z_i$ is at least

$$\sum_{w \in X_i} (|Y_i| - b_{Y_i}(w)) + \sum_{w \in Y_i} (|X_i| - b_{X_i}(w)) + \sum_{w \in Z_i} (|X_i| + |Y_i| - b_{X_i}(w) - b_{Y_i}(w))/2.$$

Note that $\sum_{w \in X_i} b_{Y_i}(w) = \sum_{w \in Y_i} b_{X_i}(w) = |E(X_i, Y_i)|$ and that $\sum_{w \in Z_i} b_{X_i}(w) + b_{Y_i}(w) = |E(X_i \cup Y_i, Z_i)|$. Therefore, summing over $1 \leq i \leq k$, the total pink outdegree of vertices of G is at least

$$\sum_{i=1}^k (2|X_i||Y_i| - 2|E(X_i, Y_i)| + (|X_i| + |Y_i|)|Z_i|/2 - |E(X_i \cup Y_i, Z_i)|/2).$$

Next we give an upper bound on the total pink indegree of vertices of G . As before, each vertex $w \in Y_i$ has at least $|X_i|$ neighbours outside of $N(x_i)$ in H_{i-1} , and thus it has at most $d(w) - |X_i|$ neighbours in $\bigcup_{j < i} X_j \cup Y_j \cup Z_j$, since these vertices are not present in H_{i-1} . Likewise, each vertex $w \in X_i$ has at most $d(w) - |Y_i|$ neighbours in $\bigcup_{j < i} X_j \cup Y_j \cup Z_j$. Finally, each vertex $w \in Z_i$ has at least $(|X_i| + |Y_i| - b_{X_i}(w) - b_{Y_i}(w))/2$ pink out-neighbours, plus $b_{X_i}(w) + b_{Y_i}(w)$ neighbours in $X_i \cup Y_i$. Thus, its pink indegree is at most $d(w) - (|X_i| + |Y_i| + b_{X_i}(w) + b_{Y_i}(w))/2$. Therefore, the total pink indegree is at most

$$\begin{aligned} \sum_{i=1}^k \left(\sum_{w \in Y_i} (d(w) - |X_i|) + \sum_{w \in X_i} (d(w) - |Y_i|) + \sum_{w \in Z_i} (d(w) - \frac{|X_i| + |Y_i| + b_{X_i}(w) + b_{Y_i}(w)}{2}) \right) \\ = 2|E(G)| - \sum_{i=1}^k 2|X_i||Y_i| - \sum_{i=1}^k (|X_i| + |Y_i|)|Z_i|/2 - \sum_{i=1}^k |E(X_i \cup Y_i, Z_i)|/2. \end{aligned}$$

Combining the lower bound on the total pink outdegree and upper bound on the total pink indegree, we find that

$$\begin{aligned} \sum_{i=1}^k (2|X_i||Y_i| - 2|E(X_i, Y_i)| + (|X_i| + |Y_i|)|Z_i|/2 - |E(X_i \cup Y_i, Z_i)|/2) \\ \leq 2|E(G)| - \sum_{i=1}^k 2|X_i||Y_i| - \sum_{i=1}^k (|X_i| + |Y_i|)|Z_i|/2 - \sum_{i=1}^k |E(X_i \cup Y_i, Z_i)|/2. \end{aligned}$$

Then some rearranging gives that

$$4 \sum_{i=1}^k |X_i||Y_i| + \sum_{i=1}^k (|X_i| + |Y_i|)|Z_i| \leq 2|E(G)| + 2 \sum_{i=1}^k |E(X_i, Y_i)|,$$

as required. $\square$

For some intuition on the statement of Theorem 4.9, in (a) we have special pairs of vertices x_i, y_i whose neighbourhoods we will use to partition the vertex set $V(G)$ into sets X_i, Y_i, Z_i . In (b) we give an inequality that implies an upper bound on the quantity $\sum_{i=1}^k |X_i||Y_i| - \sum_{i=1}^k |E(X_i, Y_i)|$, which measures how far the pairs X_i, Y_i are away from inducing disjoint complete bipartite subgraphs of G . Later, when we apply our assumptions on the maximum

degree of G as well as on the number of vertices, we will deduce that the sets Z_i have small size overall, and that the collection of bipartite subgraphs induced by the pairs X_i, Y_i are not too many edges away from all being complete bipartite. That is how we will get our density and stability results.

We call a partition $\{X_i, Y_i, Z_i\}_{i=1}^k$ of $V(G)$ a *basic partition* if it satisfies conclusions (a), (b) of Theorem 4.9. Next we establish another technical result about basic partitions that will be useful for us.

Lemma 4.10. *Let G be a graph with N vertices with maximum degree d . Suppose that $\{X_i, Y_i, Z_i\}_{i=1}^k$ is a basic partition of $V(G)$. Then*

$$2|E(G)| \geq 2Q + 2dN - 2d^2k + \sum_{i=1}^k |Z_i|(2d - |X_i| - |Y_i| - 2|Z_i|)$$

where $Q = \sum_{i=1}^k |X_i||Y_i| - \sum_{i=1}^k |E(X_i, Y_i)|$.

Proof. For every i , we set $r_i = |X_i| + |Y_i|$ and assume without loss of generality that $|X_i| \geq |Y_i|$. Observe that, for any positive integers x, r with $r - x \leq x < r$, we have that $x(r - x) > (x + 1)(r - x - 1)$. Hence with these conditions the expression $x(r - x)$ is smallest when x is as large as possible. By property (a) of the basic partition we know that $|X_i| \leq d - |Z_i|$, and this implies that

$$|X_i||Y_i| \geq (d - |Z_i|)(r_i + |Z_i| - d) = dr_i + 2d|Z_i| - d^2 - |Z_i|r_i - |Z_i|^2.$$

Recalling $Q = \sum_{i=1}^k |X_i||Y_i| - \sum_{i=1}^k |E(X_i, Y_i)|$ we obtain from property (b) that

$$\begin{aligned} 2|E(G)| &\geq 2Q + 2 \sum_{i=1}^k |X_i||Y_i| + \sum_{i=1}^k (|X_i| + |Y_i|)|Z_i| \\ &\geq 2Q + 2 \sum_{i=1}^k (dr_i + 2d|Z_i| - d^2 - |Z_i|r_i - |Z_i|^2) + \sum_{i=1}^k |Z_i|r_i \\ &= 2Q + 2d \sum_{i=1}^k (r_i + |Z_i|) - 2d^2k + 2d \sum_{i=1}^k |Z_i| - \sum_{i=1}^k |Z_i|r_i - 2 \sum_{i=1}^k |Z_i|^2 \\ &= 2Q + 2dN - 2d^2k + \sum_{i=1}^k |Z_i|(2d - r_i - 2|Z_i|), \end{aligned}$$

where in the last line we use the fact that $\{X_i, Y_i, Z_i\}_{i=1}^k$ is a partition of $V(G)$. This finishes the proof. $\square$

Finally, we deduce Theorem 1.16, which will also be used in our applications.

Theorem 1.16. *Let G be a graph with N vertices and maximum degree d . Suppose that $\eta(\mathcal{I}(G)) \leq k$. Then $|E(G)| \geq dN - d^2k$.*

Proof. Let $s \leq k$ be such that $\{X_i, Y_i, Z_i\}_{i=1}^s$ is a basic partition of $V(G)$, which exists by Theorem 4.9. We apply Lemma 4.10, first noting that $Q \geq 0$. By property (a) of the basic partition we find that $|X_i| + |Y_i| + 2|Z_i| \leq 2d$, and hence the terms $(2d - |X_i| - |Y_i| - 2|Z_i|)$ in the conclusion of Lemma 4.10 are all nonnegative. Hence $2|E(G)| \geq 2dN - 2d^2s \geq 2dN - 2d^2k$, completing the proof. $\square$

4.3 Both maximum degree and average degree conditions

In this section, we prove Theorem 1.15 (restated below) about the existence of IT's in terms of both maximum degree and maximum class average degree. Recall that $(\alpha, \beta) \in \mathbb{R}_{>0}^2$ is a *good pair* if the following holds for every integer $t \geq 1$: whenever G is a graph and $\mathcal{P} = \{V_1, \dots, V_n\}$ is a partition of $V(G)$, such that $|V_i| \geq t$ for every i , G has maximum degree $\Delta(G) \leq \alpha t$, and $G, \mathcal{P}$ has maximum class average degree $\text{mad}(G, \mathcal{P}) \leq \beta t$, then there exists an IT.

Theorem 1.15. *The pair $(\alpha, \beta) \in \mathbb{R}_{>0}^2$ is good if and only if one of the following holds:*

- (1) $\alpha \leq 1/2$,
- (2) $\beta \leq 1/4$,
- (3) $\beta \leq 2\alpha(1 - \alpha)$.

This is an interpolation between the maximum degree condition of Theorem 1.12 ($|V_i| \geq t$ and $\Delta(G) \leq t/2$) and the average degree condition in Theorem 1.13 ($|V_i| \geq t$ and $\text{mad}(G, \mathcal{P}) \leq t/4$) for the existence of an IT. We now prove Theorem 1.15 here assuming Theorem 1.17 (the latter providing tight constructions), deferring the proof of Theorem 1.17 to Section 4.5.

Proof of Theorem 1.15. If (1) holds ($\alpha \leq 1/2$), then for every $t \geq 1$ we see that G and $\mathcal{P}$ satisfy the conditions of Theorem 1.12, and hence there exists an IT. If (2) holds ($\beta \leq 1/4$), then $G, \mathcal{P}$ satisfy the assumptions of Theorem 1.13, and hence again there exists an IT.

Assume that (3) holds ($\beta \leq 2\alpha(1 - \alpha)$). By the previous paragraph we may assume that $\alpha > 1/2$ and $\beta > 1/4$, and hence also that $\alpha < 1$. Let $t \geq 1$, graph G , and partition $\mathcal{P} = \{V_1, \dots, V_n\}$ be given so that $\Delta(G) \leq \alpha t$. Suppose on the contrary that G has no IT with respect to $\mathcal{P}$. Then by Theorem 4.1 there exists $I \subseteq [n]$ such that the induced subgraph $G_I = G[\bigcup_{i \in I} V_i]$ satisfies $\eta(\mathcal{I}(G_I)) \leq |I| - 1$. Define α' so that $\Delta(G_I) = \alpha't$. Then

$\alpha' \leq \Delta(G)/t \leq \alpha < 1$. Also, by Theorem 4.4, we have $\eta(\mathcal{I}(G_I)) \geq \frac{|V(G_I)|}{2\Delta(G_I)} \geq \frac{t|I|}{2\alpha't} = \frac{|I|}{2\alpha'}$, from which we conclude that $\alpha' \geq \frac{|I|}{2(|I|-1)} > \frac{1}{2}$.

Observe that $2|E(G_I)| \leq \text{mad}(G, \mathcal{P})|V(G_I)| \leq \beta t|V(G_I)|$. Thus, applying Theorem 1.16 to G_I , we get that

$$\begin{aligned} \beta t|V(G_I)| &\geq 2|E(G_I)| \geq 2(\alpha't)|V(G_I)| - 2(\alpha't)^2(|I| - 1) \\ &> 2\alpha't(|V(G_I)| - \alpha't|I|) \geq 2\alpha't(|V(G_I)| - \alpha'|V(G_I)|) \\ &= 2\alpha'(1 - \alpha')t|V(G_I)|, \end{aligned}$$

which yields that $\beta > 2\alpha'(1 - \alpha')$. Since $\frac{1}{2} < \alpha' \leq \alpha < 1$, we have $2\alpha'(1 - \alpha') > 2\alpha(1 - \alpha)$, and therefore $\beta > 2\alpha(1 - \alpha)$. This contradicts assumption (3) and hence proves that (α, β) is a good pair.

Finally suppose that none of (1), (2), or (3) hold. Then $\alpha > \frac{1}{2}$, $\beta > \frac{1}{4}$, and $\beta > 2\alpha(1 - \alpha)$. By Theorem 1.17, there exist t , G , and $\mathcal{P}$ as in the definition of a good pair, but such that G has no IT with respect to $\mathcal{P}$. This shows that (α, β) is not a good pair. $\square$

To conclude this section, we describe another related consequence of Theorem 1.16. Instead of the maximum class average degree

$$\text{mad}(G, \mathcal{P}) = \max_{U \in \mathcal{P}} \frac{1}{|U|} \sum_{v \in U} d_G(v),$$

we consider just the *maximum average degree* of G , which is the quantity

$$\text{mad}(G) = \max_{U \subseteq V(G), U \neq \emptyset} \frac{1}{|U|} \sum_{v \in U} d_{G[U]}(v) = \max_{U \subseteq V(G), U \neq \emptyset} \frac{2|E(U)|}{|U|}.$$

Theorem 1.16 implies that if G has maximum degree and maximum average degree γ , then

$$\eta(\mathcal{I}(G)) \geq \frac{d|V(G)| - |E(G)|}{d^2} \geq \frac{(2d - \gamma)|V(G)|}{2d^2}.$$

Similar to the proof of Corollary 4.5 from the bound $\eta(\mathcal{I}(G)) \geq \frac{|V(G)|}{2d}$, we deduce the following.

Theorem 4.11. *Let G be a graph with maximum degree d and maximum average degree γ , and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. If $|V_i| \geq \frac{2d^2}{2d - \gamma}$ for every i , then there exists an IT.*

Since $\gamma \leq d$ in general, this generalizes Theorem 1.12. In Section 4.5, we will describe how the class size condition in Theorem 4.11 is basically best possible for all appropriate choices of d and γ , with more rigorous detail given in Section 5.2.

4.4 Stability for independent transversals

In this section, we use our main technical result (Theorem 4.9) to prove Theorem 1.18 and Theorem 1.19. These latter two results are stability theorems for graphs with a given maximum degree d that have low topological connectedness or have no IT, asserting that they must be close to the disjoint union of many copies of $K_{d,d}$. First we prove Theorem 1.18, restated below.

Theorem 1.18. *Let G be a graph with N vertices and maximum degree d , and let ϵ be such that $N(1 + \epsilon) = 2dk$. Suppose that $\eta(\mathcal{I}(G)) \leq k$. Then $\epsilon \geq 0$ and $2|E(G)| \geq dN(1 - \epsilon)$, and $V(G)$ has a partition into sets $\{X_i, Y_i, Z_i\}_{i=1}^k$ such that $\sum_{i=1}^k |Z_i| \leq \epsilon N$ and the following hold:*

- (i) $|E(G) \setminus \bigcup_{i=1}^k E(X_i, Y_i)| \leq 2\epsilon dN$,
- (ii) $\sum_{i=1}^k |X_i||Y_i| - \left| \bigcup_{i=1}^k E(X_i, Y_i) \right| \leq \epsilon dN/2$.

Proof. Let G be a graph with $\eta(\mathcal{I}(G)) \leq k$, and set $N = |V(G)|$, $d = \Delta(G)$, and define ϵ by $N(1 + \epsilon) = 2dk$. Then $\epsilon \geq 0$ by Theorem 4.4. Theorem 1.16 tells us that $|E(G)| \geq dN - d^2k = dN - d(N + \epsilon N)/2$ and hence $2|E(G)| \geq dN - \epsilon dN$, as claimed.

By Theorem 4.9, there exists a basic partition $\{X_i, Y_i, Z_i\}_{i=1}^s$ of $V(G)$ for some $s \leq k$. If $s < k$ then define $X_i = Y_i = Z_i = \emptyset$ for $s + 1 \leq i \leq k$. Set $z = \sum_{i=1}^s |Z_i| = \sum_{i=1}^k |Z_i|$. Using property (a) of our basic partition we get that

$$z = \sum_{i=1}^s (|X_i| + |Y_i| + 2|Z_i|) - \sum_{i=1}^s (|X_i| + |Y_i| + |Z_i|) \leq 2ds - N \leq 2dk - N.$$

Since $N(1 + \epsilon) = 2dk$, we get that $z \leq \epsilon N$.

Next, we aim to establish conclusion (i) of Theorem 4.9. We assume without loss of generality that $|X_i| \geq |Y_i|$. Setting $r_i = |X_i| + |Y_i|$ and proceeding exactly as in the proof of Lemma 4.10, we may conclude for each $i \leq s$ that

$$|X_i||Y_i| \geq (d - |Z_i|)(r_i + |Z_i| - d) = dr_i + 2d|Z_i| - d^2 - |Z_i|r_i - |Z_i|^2.$$

Setting $m = |E(G)| - \sum_{i=1}^s |E(X_i, Y_i)|$, we find from property (b) of the basic partition that

$$\begin{aligned} 4|E(G)| - 2m &\geq 4 \sum_{i=1}^s (dr_i + 2d|Z_i| - d^2 - |Z_i|r_i - |Z_i|^2) + \sum_{i=1}^s |Z_i|r_i \\ &= 4d \sum_{i=1}^s (r_i + |Z_i|) - 4d^2s + 4d \sum_{i=1}^s |Z_i| - 3 \sum_{i=1}^s |Z_i|r_i - 4 \sum_{i=1}^s |Z_i|^2 \\ &= 4dN - 4d^2s + \sum_{i=1}^s |Z_i|(4d - 3r_i - 4|Z_i|), \end{aligned}$$

where in the last line we again use the fact that $\{X_i, Y_i, Z_i\}_{i=1}^s$ is a partition of $V(G)$.

By property (a) of the basic partition, we know that $r_i \leq 2d - 2|Z_i|$ for each i , from which we get that

$$\begin{aligned}
4|E(G)| - 2m &\geq 4dN - 4d^2s + \sum_{i=1}^s |Z_i|(4d - 6d + 2|Z_i|) \\
&= 4dN - 4d^2s + \sum_{i=1}^s |Z_i|(2|Z_i| - 2d) \\
&= 4dN - 4d^2s + 2 \sum_{i=1}^s |Z_i|^2 - 2d \sum_{i=1}^s |Z_i| \\
&\geq 4dN - 4d^2s + \frac{2}{s} \left(\sum_{i=1}^s |Z_i| \right)^2 - 2d \sum_{i=1}^s |Z_i| \\
&= 4dN - 4d^2s + \frac{2z^2}{s} - 2dz.
\end{aligned}$$

Therefore,

$$m \leq 2|E(G)| - dN - dN + 2d^2s - \frac{z^2}{s} + dz.$$

Since $d = \Delta(G)$ we know that $2|E(G)| \leq dN$. Using also the facts that $s \leq k$, $z \leq \epsilon N$, and $N(1 + \epsilon) = 2dk$, we conclude that

$$\left| E(G) \setminus \bigcup_{i=1}^s E(X_i, Y_i) \right| = m \leq 2d^2s - dN + dz \leq 2d^2k - dN + \epsilon dN = 2\epsilon dN,$$

which is the statement of conclusion (i) since $X_i = Y_i = \emptyset$ for $s + 1 \leq i \leq k$.

For conclusion (ii), we know by Lemma 4.10 that

$$2|E(G)| \geq 2Q + 2dN - 2d^2s + \sum_{i=1}^s |Z_i|(2d - |X_i| - |Y_i| - 2|Z_i|),$$

where $Q = \sum_{i=1}^s |X_i||Y_i| - \sum_{i=1}^s |E(X_i, Y_i)|$. As before, the quantity $(2d - |X_i| - |Y_i| - 2|Z_i|)$ is nonnegative by property (a) of basic partitions, and so

$$2Q \leq 2|E(G)| - 2dN + 2d^2s = (2|E(G)| - dN) + (2d^2s - dN) \leq 0 + 2d^2k - dN = \epsilon dN,$$

which is the statement of conclusion (ii). □

Now we prove Theorem 1.19, restated below.

Theorem 1.19. *Let $0 < \beta < 1$, let G be a graph, and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. If $|V_i| \geq t$ for every i , $d = \Delta(G) \leq (1 + \beta)t/2$, and G has no IT with respect to $\mathcal{P}$, then for some $I \subseteq [n]$, the graph $G_I = G[\bigcup_{i \in I} V_i]$ satisfies $|E(G_I)| > (1 - \beta)d|V(G_I)|/2$, and G_I is $5\beta/(1 - \beta)$ -approximated by a disjoint union J of $|I| - 1$ complete bipartite graphs.*

Proof. Since G has no IT with respect to $\mathcal{P}$, by Theorem 4.1, there exists $I \subseteq [n]$ such that $\eta(\mathcal{I}(G_I)) \leq k$, where $k = |I| - 1$. Set $N = |V(G_I)|$ and define ϵ by $N(1 + \epsilon) = 2dk$. Since $|V_i| \geq t$ for every i , we have that $N \geq t|I|$, so that

$$\epsilon = \frac{2dk}{N} - 1 \leq \frac{2(1 + \beta)t(|I| - 1)}{2t|I|} - 1 < 1 + \beta - 1 = \beta.$$

By Theorem 1.18, we know that $|E(G_I)| \geq (1 - \epsilon)dN/2 > (1 - \beta)dN/2$.

Let $\{X_i, Y_i, Z_i\}_{i=1}^k$ be the partition of $V(G_I)$ guaranteed by Theorem 4.9, and let K be the union of the k complete bipartite graphs induced by $\{X_i, Y_i\}_{i=1}^k$. Then by Theorem 1.18, the symmetric difference of $E(G_I)$ and K has size at most

$$2\epsilon dN + \epsilon dN/2 = 5\epsilon dN/2 \leq 5\frac{\epsilon}{1 - \epsilon}|E(G_I)|.$$

Hence, G_I is $5\epsilon/(1 - \epsilon)$ -approximated by K . Noting that $5\epsilon/(1 - \epsilon) < 5\beta/(1 - \beta)$, this completes the proof. $\square$

We remark that it is possible to show that if $\eta(\mathcal{I}(G))$ is small then in fact G contains many disjoint complete bipartite subgraphs, not just “near” complete bipartite subgraphs as in Theorem 1.18 (see Theorem 19 in [106]). The statement is technical and does not provide an immediate application for us, so we leave it out from this writing.

4.5 Tight construction

In this section, we prove Theorem 1.17, which gives tight constructions showing that conditions (1), (2), and (3) of Theorem 1.15 on good pairs are all necessary. We restate Theorem 1.17 below.

Theorem 1.17. *For all $\alpha > \frac{1}{2}$, $\beta > \max\{\frac{1}{4}, 2\alpha(1 - \alpha)\}$, and sufficiently large t , there exist a graph G and partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$ such that $|V_i| \geq t$ for every i , $\Delta(G) \leq \alpha t$, $\text{mad}(G, \mathcal{P}) \leq \beta t$, and G has no IT with respect to $\mathcal{P}$.*

For the proof, we will say that a pair $(\alpha, \beta) \in \mathbb{R}_{>0}^2$ is *relevant* if it satisfies the conditions of Theorem 1.17, i.e., if $\alpha > \frac{1}{2}$ and $\beta > \max\{\frac{1}{4}, 2\alpha(1 - \alpha)\}$. Our aim is to construct, for every relevant pair (α, β) and every sufficiently large t , a graph G and a partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$ such that

- (a) $|V_i| \geq t$ for every $i \in [n]$;
- (b) $\Delta(G) \leq \alpha t$;
- (c) $\text{mad}(G, \mathcal{P}) \leq \beta t$; and
- (d) G has no IT with respect to $\mathcal{P}$.

Szabó and Tardos [161] proved Theorem 1.17 when $\alpha > \frac{1}{2}$ and $\beta \geq \alpha$. In fact, the maximum degree in their construction is $d = \lceil \frac{t+1}{2} \rceil$ for all $t \geq 1$. See Section 5.1 for more details on their construction. Groenland, Kaiser, Treffers, and Wales [83] proved Theorem 1.17 when $\alpha \geq 1$ and $\beta > \frac{1}{4}$. These authors also slightly modified their construction to prove Theorem 1.17 when $\alpha > \frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.85355$ and $\beta > \frac{1}{4}$. As explained in the remarks at the end of this section, a basic extension of their construction works actually for all $\alpha \geq \frac{2}{3}$ and $\beta > \max\{\frac{1}{4}, 2\alpha(1-\alpha)\}$, but their construction requires modifications to work for all $\alpha > \frac{1}{2}$ and $\beta > \max\{\frac{1}{4}, 2\alpha(1-\alpha)\}$.

At a high level, our construction for Theorem 1.17 will be an interpolation (or, more accurately, a combination) of the constructions of Groenland, Kaiser, Treffers, and Wales [83] and of Szabó and Tardos [161]. Our construction will be streamlined through the use of the following simple lemma (a restatement of Lemma 1.20). This lemma will be the main subject of Chapter 5 and Chapter 6, and indeed the construction we describe here will feel more natural after reading parts of those chapters.

Lemma 4.12. *Let G and H be disjoint graphs, and let $\mathcal{P} = \{V_1, \dots, V_n\}$ and $\mathcal{Q} = \{W_1, \dots, W_m\}$ be partitions of $V(G)$ and $V(H)$, respectively. Let $\mathcal{R} = \{V'_1, \dots, V'_n, W_1, \dots, W_{m-1}\}$, where $V'_1 \supseteq V_1, \dots, V'_n \supseteq V_n$ are obtained by distributing each of the vertices of W_m into one of $V_1, \dots, V_n$ arbitrarily. If G has no IT with respect to $\mathcal{P}$ and H has no IT with respect to $\mathcal{Q}$, then $G \cup H$ has no IT with respect to $\mathcal{R}$.*

The basic ingredient for our construction will be applying Lemma 4.12 iteratively on complete bipartite graphs $K_{A,B}$, which have no IT with respect to their standard bipartition $\mathcal{P} = \{A, B\}$. The construction will proceed in two main phases, summarized as follows:

- I. For all $\alpha \geq \beta > \frac{1}{4}$ and sufficiently large t , we will construct a graph G' and vertex partition $\mathcal{P}'$ satisfying properties (b), (c), and (d), and with property (a) replaced by
 - (a') Every class $U \in \mathcal{P}$ with $|U| < t$ has only vertices of degree 1 and satisfies $|U| = \lfloor \alpha t \rfloor$.

This construction will be obtained by taking the disjoint union of a large collection of stars K_{1,d_j} of various sizes.

- II. Assuming now that $\frac{1}{2} < \alpha < 1$ and $\beta > \max\{\frac{1}{4}, 2\alpha(1-\alpha)\}$, we will take the construction $G', \mathcal{P}'$ from the first phase and enlarge the classes of size $\lfloor \alpha t \rfloor < t$ into classes of size t , while maintaining properties (b), (c), and (d) of relevant pairs. This construction will be done by adding to G' various disjoint copies of the complete bipartite graph $K_{t-\lfloor \alpha t \rfloor+1, \lfloor \alpha t \rfloor}$.

Phase I. A construction $G', \mathcal{P}'$ for this phase was given by Groenland, Kaiser, Treffers, and Wales [83], who used it to prove that $(1, \frac{1}{4} + \epsilon)$ is a relevant pair for every $\epsilon > 0$ and thus demonstrate that Theorem 1.13 is asymptotically best possible. For completeness, we show how to derive their construction using Lemma 4.12. The construction sounds technical, but the idea is a relatively simple. See Figure 4.1 for illustration.

Let (α, β) be any pair with $\alpha \geq \beta > 1/4$. We fix $t \geq 1$, and a sequence $d_1 < d_2 < \dots < d_k = \min\{t, \lfloor \alpha t \rfloor\}$ of increasing positive integers, with the values of t and elements of the sequence chosen as our argument progresses. We inductively define graphs $G_1, G_2, \dots, G_k$ and partitions $\mathcal{P}_1, \mathcal{P}_2, \dots, \mathcal{P}_k$ such that each G_i has no IT with respect to $\mathcal{P}_i$. Each G_{i+1} is obtained from G_i by adding a collection of components, each of which is a copy of the star $K_{1, d_{i+1}}$. We refer to the singleton set formed by the center vertex of $K_{1, d_{i+1}}$ as its *A-side*, and its set of d_{i+1} leaves as its *B-side*. We start with G_1 being the disjoint union of t copies of K_{1, d_1} . The partition $\mathcal{P}_1$ has as classes each of the *B-side* classes (each of size d_1), plus one more class of size t (the *initial* class) that is the union of all the *A-side* classes. It is easy to see that G_1 has no IT with respect to $\mathcal{P}_1$, a fact that can also be derived by applying Lemma 4.12 successively to K_{1, d_1} .

Set $d_0 = 0$, $G_0 = \emptyset$, and $\mathcal{P}_0 = \emptyset$. Suppose $1 \leq j < k$ and that we have defined G_j and $\mathcal{P}_j$ such that the following hold.

- (i) G_j has no IT with respect to $\mathcal{P}_j$.
- (ii) Any class U of $\mathcal{P}_j$ of size different from t satisfies $U \cap V(G_{j-1}) = \emptyset$ and $|U| = d_j$, and each $u \in U$ has degree 1 in G_j . These are called the *terminal* classes of $\mathcal{P}_j$.
- (iii) Any class U of $\mathcal{P}_j$ with $|U| = t$ that contains a vertex of $V(G_j) \setminus V(G_{j-1})$ has average degree $\frac{1}{t}(d_{j-1} + (t - d_{j-1})d_j)$ in G_j with respect to $\mathcal{P}_j$.

We define G_{j+1} and $\mathcal{P}_{j+1}$ as follows. We will essentially be adding a new “layer” of star components $K_{1, d_{j+1}}$. By (ii) we know that all classes of $\mathcal{P}_j$ have size t except the terminal classes of $\mathcal{P}_j$, each of which has size d_j . Fix one terminal class X , and let $X_1 = X$. With Lemma 4.12 in mind, for each $i = 1, \dots, t - d_j$ in sequence, we add to the current graph a copy of $K_{1, d_{j+1}}$, distribute all vertices of X_i into the *A-side* of $K_{1, d_{j+1}}$ to form a new class X_{i+1} of size $|X_i| + 1$, and let the *B-side* of $K_{1, d_{j+1}}$ form one more new class. (This *B-side* of $K_{1, d_{j+1}}$ will become a terminal class of the partition $\mathcal{P}_{j+1}$.) For each i , Lemma 4.12 guarantees that the new graph has no IT with respect to the new partition. Repeating this step until i reaches $t - d_j$, we obtain $|X_{t-d_j+1}| = t$, so this finishes the process of enlarging X to size t . We repeat this procedure for every terminal class X of $\mathcal{P}_j$, creating a new layer of star components $K_{1, d_{j+1}}$, whose *B-sides* are the terminal classes of the new partition $\mathcal{P}_{j+1}$. We denote the new graph by G_{j+1} .

Next we verify properties (i), (ii), (iii) for $(G_{j+1}, \mathcal{P}_{j+1})$. By repeated applications of Lemma 4.12 as indicated, we know that G_{j+1} has no IT with respect to $\mathcal{P}_{j+1}$. Hence (i) holds for $(G_{j+1}, \mathcal{P}_{j+1})$. To check (ii), observe that by (ii) for $(G_j, \mathcal{P}_j)$ and by our construction,

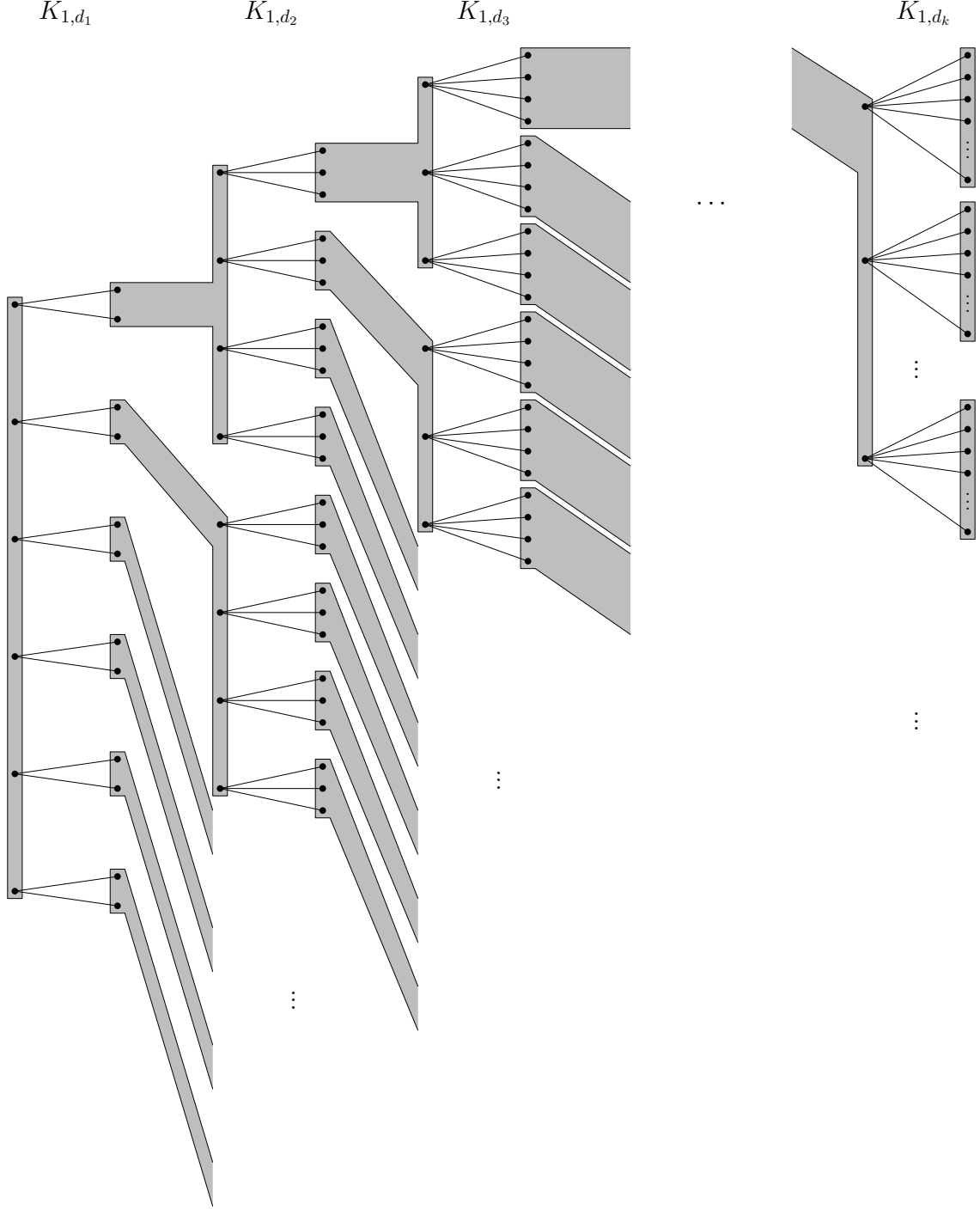


Figure 4.1: A construction $G', \mathcal{P}'$ for Phase I. Each shaded region represents a vertex class of $\mathcal{P}'$. We have a sequence $d_1 < d_2 < \dots < d_k = \min\{t, \lfloor \alpha t \rfloor\}$ of positive integers, and at step j , we effectively add a “layer” of stars K_{1,d_j} and applying Lemma 4.12 repeatedly to get non-terminal classes of size t . The sequence of integers d_j will be chosen so that $G', \mathcal{P}'$ has maximum class average degree at most βt , for any fixed $\beta > \frac{1}{4}$.

the only classes that are not now of size t are the B -side classes of the newly added layer of copies of $K_{1,d_{j+1}}$, which all have size d_{j+1} , and their vertices all have degree 1. Thus $(G_{j+1}, \mathcal{P}_{j+1})$ satisfies (ii). To verify (iii), the classes U of $\mathcal{P}_{j+1}$ with $|U| = t$ that contain a vertex of $V(G_{j+1}) \setminus V(G_j)$ are precisely those that were previously a terminal class X of $\mathcal{P}_j$. By property (ii) of $(G_j, \mathcal{P}_j)$ we know that $|X| = d_j$ and each vertex of X has degree 1 in G_j . Our construction adds $t - d_j$ vertices to X , each of degree d_{j+1} , to obtain the class U of size t . Hence the average degree of U is $\frac{1}{t}(d_j + (t - d_j)d_{j+1})$, verifying (iii) for $(G_{j+1}, \mathcal{P}_{j+1})$.

Let $G' = G_k$ and $\mathcal{P}' = \mathcal{P}_k$. Observe that G' has maximum degree $d_k = \min\{t, \lfloor \alpha t \rfloor\} \leq \alpha t$, so property (b) is satisfied. Property (d) is given by (i). To ensure that Property (c) is satisfied, we use the following lemma from [83].

Lemma 4.13 (Groenland, Kaiser, Treffers, Wales [83]). *For every $\beta > \frac{1}{4}$, there exists $t_0 \geq 1$ such that for all $t \geq t_0$, there exists a sequence $0 = d_0 < d_1 < d_2 < \dots < d_k = \min\{t, \lfloor \alpha t \rfloor\}$ of integers such that for all $j \geq 0$,*

$$\frac{1}{t}(d_j + (t - d_j)d_{j+1}) \leq \beta t.$$

Proof. Let $d_1 = \lfloor \beta t \rfloor$, and let $d_j = d_{j-1} + 1$ for every $j \in \{2, \dots, d_k - d_1 + 1\}$. Then we get

$$\frac{1}{t}(d_j + (t - d_j)d_{j+1}) = \frac{1}{t}(d_j + (t - d_j)(d_j + 1)) = \frac{1}{t}(t + d_j(t - d_j)).$$

Since $d_j \leq d_k = \min\{t, \lfloor \alpha t \rfloor\} \leq t$, this expression is at most $\frac{1}{t}\left(t + \frac{t^2}{4}\right) = 1 + \frac{t}{4} \leq \beta t$ for sufficiently large t , since $\beta > \frac{1}{4}$. $\square$

We choose t_0 and a sequence such that Lemma 4.13 holds for our given β , and we apply our construction with this sequence and with $t \geq t_0$. Any terminal class of $\mathcal{P}' = \mathcal{P}_k$ has average degree 1 by (ii) with $j = k$. All remaining classes in $\mathcal{P}'$ have size t and are of the type in (iii) for exactly one value of j , and hence by (iii) and Lemma 4.13 each has average degree at most βt . Hence (c) holds for $(G', \mathcal{P}')$.

Finally, to check (a') we again note that any class U that has size less than t is a terminal class of $\mathcal{P}_k$, and hence satisfies (ii). Then necessarily U has size $d_k = \lfloor \alpha t \rfloor < t$, and by (ii) consists of vertices of degree one. This verifies (a'), and thus completes Phase I.

Phase II. In this phase, we will use Lemma 4.12 to enlarge the classes of $\mathcal{P}'$ of size $\lfloor \alpha t \rfloor$ into classes of size t , by adding a suitable number of copies of $K = K_{t - \lfloor \alpha t \rfloor + 1, \lfloor \alpha t \rfloor}$ to G' and suitably extending the current partition $\mathcal{P}'$. We note that the strategy we use in this phase is similar to an approach that we will use in Section 5.1 to derive a construction of Szabó and Tardos [161], which showed that the class size condition in Theorem 1.12 is best possible.

Let (α, β) be a relevant pair, i.e., $\alpha > \frac{1}{2}$ and $\beta > \max\{\frac{1}{4}, 2\alpha(1 - \alpha)\}$, and let t be large enough such that $(G', \mathcal{P}')$ satisfying properties (a'), (b), (c), and (d) exists, as given in Phase

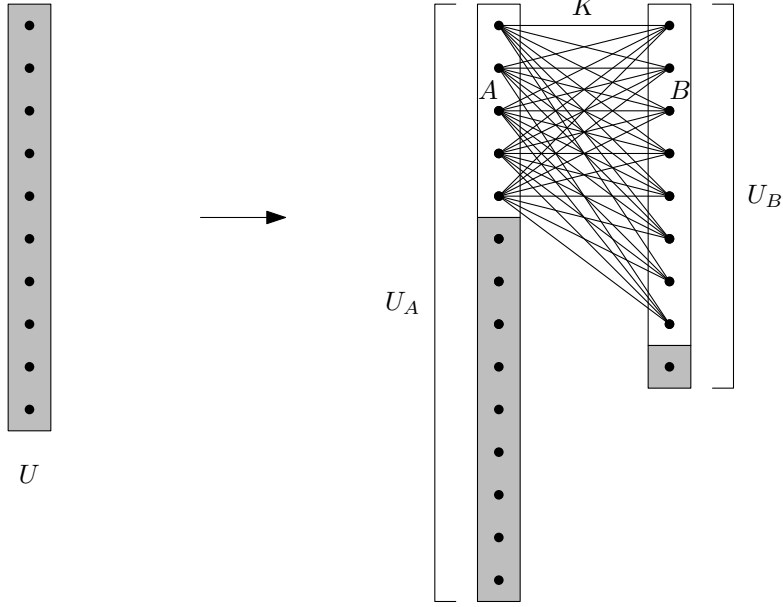


Figure 4.2: A step in Phase II. Adding a copy of K and applying Lemma 4.12, a deficient class U gets replaced by class U_A of size t and class U_B of size $|U| + 1$. This step is repeated until all classes have size t .

I. We refer to the classes of $(G', \mathcal{P}')$ of size less than t as being *deficient*. If the set of deficient classes in $(G', \mathcal{P}')$ is empty, then $(G', \mathcal{P}')$ itself proves Theorem 1.17 for (α, β) , so we may assume the contrary from now on. In particular, by (a') we may assume that $\alpha < 1$.

Throughout this process, the current graph G and partition $\mathcal{P}$ will be *valid*, meaning that $(G, \mathcal{P})$ satisfies properties (b), (c), and (d) together with

(a'') Every deficient class U satisfies $|U| \geq \lfloor \alpha t \rfloor$, and $d(u) \leq t - \lfloor \alpha t \rfloor + 1$ for every $u \in U$.

Observe that the initial pair $(G', \mathcal{P}')$ is valid.

The main construction step is as follows (see Figure 4.2 for illustration). Given a valid pair $(G, \mathcal{P})$, if no classes are deficient then $(G, \mathcal{P})$ provides a proof of Theorem 1.17 and we stop. Otherwise, we take a deficient class U , add a disjoint copy of K to G , distribute $\lfloor \alpha t \rfloor - 1$ vertices of U to the A -side of K to form a new class U_A , and distribute the remaining $|U| - \lfloor \alpha t \rfloor + 1$ vertices of U to the B -side of K , forming class U_B . Note that this process removes U , adds one new class U_A of size t and one new class U_B of size $|U| + 1$. All other classes remain unchanged. Thus the resulting pair $(G^+, \mathcal{P}^+)$ satisfies (a''), and Lemma 4.12 tells us that (d) is also satisfied, i.e., G^+ has no IT with respect to $\mathcal{P}^+$. We verify that $(G^+, \mathcal{P}^+)$ also satisfies properties (b) and (c), so that $(G^+, \mathcal{P}^+)$ is valid and the construction may continue until all classes have size t .

Property (b) requires that $\Delta(G^+) \leq \alpha t$. This holds because $(G, \mathcal{P})$ satisfies (b), and $\Delta(K) = \max\{\lfloor \alpha t \rfloor, t - \lfloor \alpha t \rfloor + 1\} = \lfloor \alpha t \rfloor$ for large enough t . Property (c) requires that

$\text{mad}(G^+, \mathcal{P}^+) \leq \beta t$. To verify this, note that since $(G, \mathcal{P})$ satisfies (a''), the new class U_A has average degree at most

$$\begin{aligned} \frac{1}{t}((t - \lfloor \alpha t \rfloor + 1)\lfloor \alpha t \rfloor + (\lfloor \alpha t \rfloor - 1)(t - \lfloor \alpha t \rfloor + 1)) &= \frac{1}{t}(2\lfloor \alpha t \rfloor - 1)(t - \lfloor \alpha t \rfloor + 1) \\ &< \frac{2\alpha t}{t}(t - \alpha t + 2). \end{aligned}$$

This quantity is $2\alpha(1 - \alpha)t + 4\alpha$, which for large enough t is at most βt , since $\beta > 2\alpha(1 - \alpha)$, as required. Each vertex in the other new class U_B has degree exactly $t - \lfloor \alpha t \rfloor + 1$ (if it came from K) or at most $t - \lfloor \alpha t \rfloor + 1$ by (a'') (if it came from U). Since $t - \lfloor \alpha t \rfloor + 1 < t - \alpha t + 2 < 2\alpha(1 - \alpha)t + 2$, which is at most βt for large enough t , this verifies (c). This completes Phase II, and thus the proof of Theorem 1.17.

We conclude this section with a few remarks. For the case when $\alpha > \frac{1}{2} + \frac{1}{2\sqrt{2}}$ and $\beta > \frac{1}{4}$, Groenland, Kaiser, Treffers, and Wales [83] proved Theorem 1.17 as follows. Recall that the graph G' with partition $\mathcal{P}'$ from Phase I satisfies properties (a'), (b), (c), and (d). In their version of Phase II, for each deficient class U of $\mathcal{P}'$, a single complete bipartite graph $K' = K_{\lfloor \beta' t / \alpha \rfloor, \lfloor \alpha t \rfloor}$ is added, where $\beta' = \max\{\frac{1}{4}, 2\alpha(1 - \alpha)\}$, and the vertices of U are distributed into the two classes of K' so that the size of each class becomes at least t . This is possible because $(t - \frac{\beta' t}{\alpha}) + (t - \alpha t) \leq (1 - 2(1 - \alpha))t + (1 - \alpha)t = \alpha t$, and $|U| = \lfloor \alpha t \rfloor$. As before, Lemma 4.12 ensures that the new graph has no IT with respect to the new partition. The maximum class average degree does indeed stay below βt after this distribution. However, for their strategy to work for a wider range of (α, β) than assumed above, we would need that $\beta' t / \alpha \leq \alpha t$ to ensure $\Delta(K') \leq \alpha t$ (property (b)). Hence this proves Theorem 1.17 in general only when $\alpha \geq \frac{2}{3}$. This is why we needed to add copies of $K = K_{t - \lfloor \alpha t \rfloor + 1, \lfloor \alpha t \rfloor}$ as opposed to $K_{\lfloor \beta' t / \alpha \rfloor, \lfloor \alpha t \rfloor}$ in Phase II.

The above construction of Groenland, Kaiser, Treffers, and Wales [83] when $\alpha > \frac{1}{2} + \frac{1}{2\sqrt{2}}$ and $\beta > \frac{1}{4}$ shows that the construction given in this section is far from unique. Here is another way to see this. In Phase II of the construction, given a deficient block U , we added $\lfloor \alpha t \rfloor - 1$ vertices of U to the A -side of $K = K_{t - \lfloor \alpha t \rfloor + 1, \lfloor \alpha t \rfloor}$ and the remaining $|U| - \lfloor \alpha t \rfloor + 1$ vertices of U to the B -side of K . We could instead fix any positive integer C and, in each step, add $\lfloor \alpha t \rfloor - C$ vertices of U to the A -side of $K' = K_{t - \lfloor \alpha t \rfloor + C, \lfloor \alpha t \rfloor}$ and the remaining $|U| - \lfloor \alpha t \rfloor + C$ vertices of U to the B -side of K' . This would require our minimum value for t to be larger, but as a trade-off would give constructions with fewer classes. The bipartite component K' and its number of copies would also be different. However, we know by Theorem 1.19 that when α is close to $\frac{1}{2}$, the constructed graph with no IT should have many copies of “near” $K_{t/2, t/2}$ ’s, so in any case we would get many components that are close in structure to complete bipartite graphs. Since we need “near” $K_{t/2, t/2}$ ’s when α is close to $\frac{1}{2}$, this also helps explain our choice of adding copies of $K = K_{t - \lfloor \alpha t \rfloor + 1, \lfloor \alpha t \rfloor}$ in Phase II.

Finally, we note that the construction given in Phase II can also be used to show that the class size condition in Theorem 4.11 is close to being best possible. Basically, we successively

apply Lemma 4.12 by adding copies of $K_{d,d'}$ where $d' = \left\lfloor \frac{d\gamma}{2d-\gamma} \right\rfloor$ (while ignoring Phase I). The result after applying a similar procedure to Phase II is a graph with maximum degree d , maximum average degree at most γ , classes of size $\left\lfloor \frac{2d^2}{2d-\gamma} \right\rfloor - 1$ and no IT. See Section 5.2 for more rigorous details.

4.6 Problems

To conclude this chapter, we discuss some open problems for further investigation.

(1) One problem is getting a more precise stability theorem than Theorem 1.19 for IT's in bounded degree graphs. Recall that this theorem states roughly that for a graph G and vertex partition $\mathcal{P} = \{V_1, \dots, V_n\}$, if $|V_i| \geq t$ for every i , $d = \Delta(G) \geq (1 + \beta)t/2$, and there is no IT, then some subgraph of G is close to being d -regular and is close to being a disjoint union of complete bipartite graphs. However, it does not tell us precisely how large this subgraph is guaranteed to be. In view of Theorem 4.7, we would like to say that there exists a subgraph that is close to being the disjoint union of say $2d/(1 + \beta) - 1$ copies of $K_{d,d}$. In the setting of Theorem 4.7, we took advantage of $K_{d,d}$ being d -regular, so that the graph can be decomposed into disjoint union of $K_{d,d}$'s and a $K_{d,d}$ -free subgraph, and then the topological connectedness is more easily analyzed. We do not have this kind of simplification in the stability setting, and we would need to more carefully analyze the effect of adding edges on the topological connectedness to get the type of stability result we seek.

(2) This chapter's stability theorems focused on the maximum degree setting, surrounding Theorem 1.12. It would also be interesting to investigate extremal and stability theorems in the average degree setting, surrounding Theorem 1.13. The constructions showing that this theorem is asymptotically tight (Phase I in Section 4.5, previously given in [83]) consisted of large collections of stars K_{1,d_j} of various sizes. It is possible to reduce the number of connected components by using slightly larger imbalanced complete bipartite graphs $K_{a,b}$. We ask whether there exist asymptotically tight constructions that do not contain large disjoint unions of imbalanced complete bipartite graphs. If not, then it is also worth investigating appropriate stability theorems.

(3) Groenland, Kaiser, Treffers, and Wales [83] formulated some other open problems that we did not investigate in this chapter, including considerations of "local degree", avoiding complete bipartite graphs or 4-cycles, and fixing the number of classes. They also consider the more general setting of independent transversals in uniform hypergraphs. We will investigate this latter problem in detail in Section 7.3. In particular, we will formulate Conjecture 7.13 which attempts to extend Theorem 1.15 on good pairs.

Chapter 5

Constructing graphs with no independent transversals

In this chapter, we introduce and extensively apply an iterative method for constructing vertex-partitioned graphs with no independent transversals. We already used this method in Section 4.5 to prove Theorem 1.17, which gave tight constructions for Theorem 1.15), and the method will also be central to Chapter 6. The construction method is centered around the following simple lemma (a restatement of Lemma 1.20).

Lemma 5.1. *Let G and H be disjoint graphs, and let $\mathcal{P} = \{V_1, \dots, V_n\}$ and $\mathcal{Q} = \{W_1, \dots, W_m\}$ be partitions of $V(G)$ and $V(H)$, respectively. Let $\mathcal{R} = \{V'_1, \dots, V'_n, W_1, \dots, W_{m-1}\}$, where $V'_1 \supseteq V_1, \dots, V'_n \supseteq V_n$ are obtained by distributing each of the vertices of W_m into one of $V_1, \dots, V_n$ arbitrarily. If G has no IT with respect to $\mathcal{P}$ and H has no IT with respect to $\mathcal{Q}$, then $G \cup H$ has no IT with respect to $\mathcal{R}$.*

In addition to some other results, we will use Lemma 5.1 to (re)prove Theorem 1.22 and Theorem 1.26, and to prove Theorem 1.23, Theorem 1.24, and Theorem 1.25. This chapter is based on joint work with Haxell [105].

5.1 Constructions for bounded degree graphs

In this section, we will demonstrate the usefulness of Lemma 5.1 by (re)proving Theorem 1.22 of Szabó and Tardos [161], and by proving Theorem 1.24. These two theorems provide tight constructions for Theorem 1.21 of Haxell [95, 96], which is restated below.

Theorem 1.21 (Haxell [95, 96]). *Let G be a graph with maximum degree Δ , and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. If $|V_i| \geq 2\Delta$ for every i , then there exists an IT.*

Together with Theorem 5.2 and Theorem 5.3 below (both proved in [16] and discussed in Section 4.1), Theorem 1.22 and Theorem 1.24 solve the following extremal problem: Find

all the minimal graphs G with maximum degree at most Δ that admit some vertex partition $\mathcal{P}$ into classes each of size at least $2\Delta - 1$ and have no IT. These graphs G are the disjoint union of $2\Delta - 1$ copies of $K_{\Delta,\Delta}$ when $\Delta \neq 2$, and every disjoint union of three cycles of each of length 1 modulo 3 when $\Delta = 2$.

5.1.1 General maximum degree

In this subsection, we show how to prove Theorem 1.22 of Szabó and Tardos [161], which is restated below.

Theorem 1.22 (Szabó, Tardos [161]). *Let $\Delta \geq 1$ and let G be the disjoint union of $2\Delta - 1$ copies of the complete bipartite graph $K_{\Delta,\Delta}$. There exists a partition $\mathcal{P} = \{V_1, \dots, V_{2\Delta}\}$ of $V(G)$ such that $|V_i| = 2\Delta - 1$ for every i and there is no IT.*

Theorem 1.22 was previously proved by Jin [110] and by Yuster [178] independently when Δ is a power of 2, although their partitions differed from those of Szabó and Tardos [161]. We will show that all of their constructions can be derived by iteratively applying Lemma 5.1 on the complete bipartite graph $K_{\Delta,\Delta}$ with the standard bipartition (the partition into two independent sets of size Δ). Later, in Section 5.6, we will prove Theorem 1.23, which states that in fact all possible partitions for Theorem 1.22 can be derived by iteratively applying Lemma 5.1 on $K_{\Delta,\Delta}$ with the standard bipartition. In this section, we will be more concrete and first show how to derive the vertex partitions of Szabó and Tardos [161], and then those of Jin [110] and Yuster [178].

Let G_0 be the complete bipartite graph $K_{\Delta,\Delta}$, and let $\mathcal{P}_0 = \{V_1, V_2\}$ be the standard bipartition of G_0 . Note that $|V_1| = |V_2| = \Delta$ and that $K_{\Delta,\Delta}$ has no IT with respect to $\mathcal{P}_0$. Our goal is to enlarge both V_1 and V_2 into classes of size $2\Delta - 1$ while preserving the property of the graph having no IT. We only show how to enlarge V_1 , since enlarging V_2 can be done symmetrically afterwards. Throughout this procedure, we will have a distinguished “small” vertex class U_k , starting with $U_0 = V_1$, that continually increases in size as we add copies of $K_{\Delta,\Delta}$, while all other vertex classes (apart from V_2) always have size $2\Delta - 1$. Eventually U_k will also have size $2\Delta - 1$, which finishes the required construction.

Let $k \geq 1$, and let G_{k-1} be the disjoint union of k copies of $K_{\Delta,\Delta}$. Assume we have a partition $\mathcal{P}_{k-1}$ of $V(G_{k-1})$ such that: there exists a class $U_{k-1} \in \mathcal{P}_{k-1}$ of size $|U_{k-1}| = \Delta + k - 1$, and $|U| = 2\Delta - 1$ for every $U \in \mathcal{P}_{k-1} \setminus \{V_2, U_{k-1}\}$. Let $G_k = G_{k-1} \sqcup K_{\Delta,\Delta}$, and let $\mathcal{P}' = \{V'_1, V'_2\}$ be the standard bipartition of this newly added copy of $K_{\Delta,\Delta}$. We let $\mathcal{P}_k$ be the partition of $V(G_k)$ obtained by first taking the union of the partitions $\mathcal{P}_{k-1}$ and $\mathcal{P}'$, and then distributing the vertices of U_{k-1} into the classes V'_1 and V'_2 . Specifically, we distribute $\Delta - 1$ vertices of U_{k-1} into V'_2 to make this modified class have size $2\Delta - 1$, and we distribute the remaining k vertices of U_{k-1} into V'_1 to make this modified class, which we now call U_k , have size $\Delta + k$. Then G_k has no IT with respect to $\mathcal{P}_k$ by Lemma 5.1, so the procedure may continue. Eventually when $k = \Delta - 1$, we have that all classes of $\mathcal{P}_{\Delta-1}$ (apart from V_2)

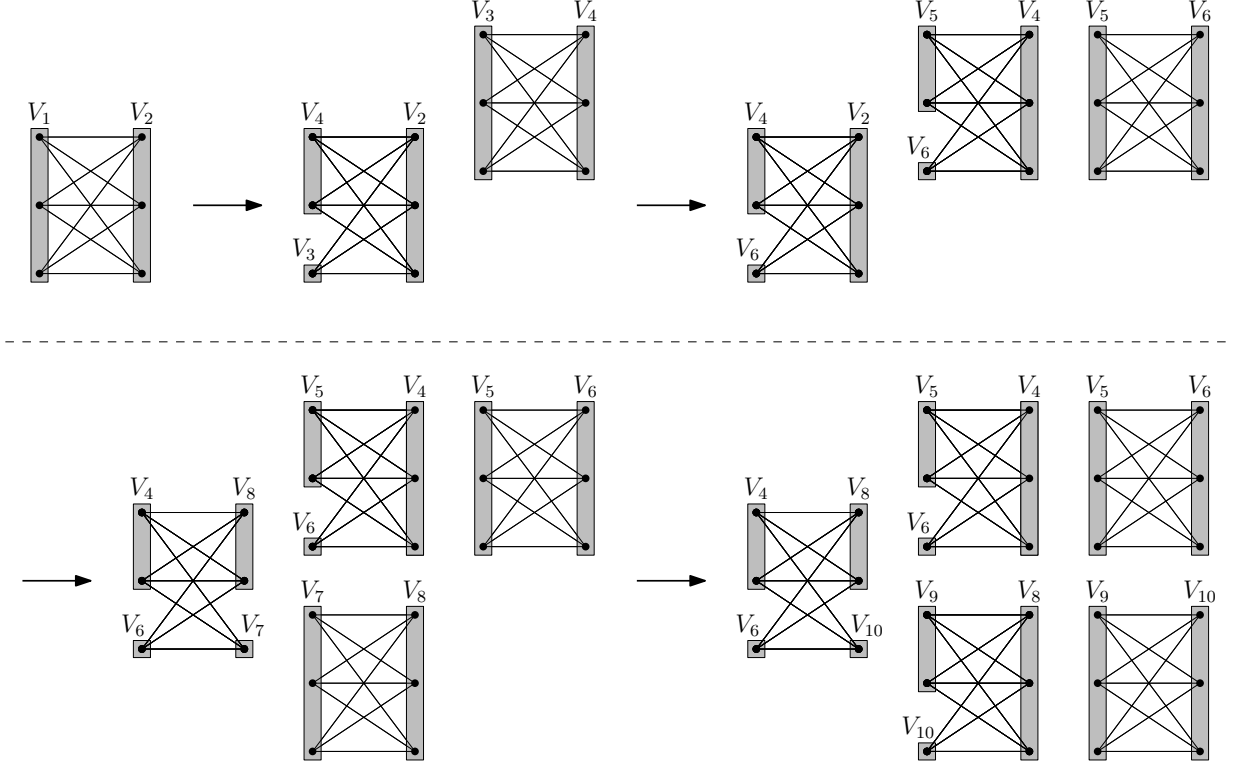


Figure 5.1: Deriving the Szabó-Tardos construction for $\Delta = 3$ by repeatedly applying Lemma 5.1. We start with $K_{3,3}$ and the standard bipartition $\{V_1, V_2\}$. In the first step, we add a new copy of $K_{3,3}$ and its standard bipartition $\{V_3, V_4\}$, and then we distribute one vertex of V_1 into V_3 and the other two vertices of V_1 into V_4 . In the second step, we add a new copy of $K_{3,3}$ with its standard bipartition $\{V_5, V_6\}$, and then we distribute two vertices of V_3 into V_5 and the other two vertices of V_3 into V_6 as shown. The remaining two steps are analogous.

have size $2\Delta - 1$, so we may stop. Enlarging V_2 in symmetric fashion, by adding another $\Delta - 1$ copies of $K_{\Delta,\Delta}$, leads to a partition $\mathcal{P}$ satisfying Theorem 1.22.

The partition of Szabó and Tardos [161] can be obtained exactly from this procedure by choosing, at each step k , the k vertices of U_{k-1} that get distributed into V'_1 to include one vertex from each of the k copies of $K_{\Delta,\Delta}$ in G_{k-1} . See Figure 5.1 for an illustration of this procedure when $\Delta = 3$. Eventually when $k = d - 1$, the distinguished class U_{d-1} forms a special “leftover” class in the construction Szabó and Tardos (which they label as V_{q+1}). We get their additional leftover class (which they label as V_{2q+2}) after doing the symmetric procedure on V_2 .

For $\Delta \geq 4$, the partition of Szabó and Tardos is not the only possible result of the above procedure. Indeed, Szabó and Tardos [161] themselves observed that when $\Delta \geq 4$ is a power of 2, their partitions $\mathcal{P}$ for Theorem 1.22 differ from those of Jin [110] and Yuster [178], despite all of them being on the same graph G , namely the disjoint union of $2\Delta - 1$ copies of $K_{\Delta,\Delta}$. Due to the somewhat arbitrary nature in which we can distribute vertices according

to Lemma 5.1, the fact that there exist such different partitions of the same graph G with no IT is not surprising.

When Δ is a power of 2, the partitions of Yuster [178] can be derived using the following slight modification of the above procedure. Instead of distributing vertices so as to always have one distinguished “small” class, at each step we distribute the vertices into V'_1 and V'_2 in equitable fashion, with half of them going to V'_1 and half of them going to V'_2 . The class of G_{k-1} that we choose to distribute to V'_1 and V'_2 at each step is any of the current smallest ones. The partitions of Jin [110] are special cases of those of Yuster. Specifically, one performs this described procedure in such a way as to eventually make one copy of $K_{\Delta,\Delta}$ “colorful”, containing one vertex from each of the 2Δ classes. Thus we can also derive Jin’s partitions using Lemma 5.1.

It is easy to see that there are many other schemes that result in partitions $\mathcal{P}$ satisfying Theorem 1.22. Moreover, we do not have to stop at $2\Delta - 1$ copies of $K_{\Delta,\Delta}$. Indeed, we may continually add further copies of $K_{\Delta,\Delta}$, and there will by necessity be some classes of size greater than $2\Delta - 1$ after this. However, if the number of $K_{\Delta,\Delta}$ components is k , then the resulting graph has $2\Delta k$ vertices partitioned into $k + 1$ classes, so in particular there will always be a class of size at most $2\Delta - 1$, in accordance with Theorem 1.21.

5.1.2 Maximum degree two

In this subsection, we prove Theorem 1.24, which is restated below.

Theorem 1.24. *Let $G = C_{\ell_1} \sqcup C_{\ell_2} \sqcup C_{\ell_3}$ be the disjoint union of three cycles with lengths $\ell_j \equiv 1 \pmod{3}$, $\ell_j \geq 4$ for $1 \leq j \leq 3$. There exists a partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$ such that $|V_i| = 3$ for every i and G has no IT.*

For some context, recall that Aharoni, Holzman, Howard, and Sprüssel [16] proved the following Brooks-type theorem (see Theorem 4.7).

Theorem 5.2. *Let G be a graph with maximum degree $\Delta \neq 2$, and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. If $|V_i| \geq 2\Delta - 1$ for every i but G has no IT, then G contains the disjoint union of $2\Delta - 1$ copies of $K_{\Delta,\Delta}$.*

Thus, it is not a coincidence that for all previous constructions [110, 161, 178] achieving class sizes $|V_i| \geq 2\Delta - 1$ and no IT, the underlying graph G is the disjoint union of $2\Delta - 1$ copies of $K_{\Delta,\Delta}$. This is despite different authors describing different vertex partitions $\mathcal{P}$. As with Brooks’ theorem, the case of maximum degree $\Delta = 2$ is slightly more complicated and interesting than the case $\Delta \neq 2$. When G is one cycle and $|V_i| = 3$ for every i , the existence of an IT was a conjecture of Du, Hsu, and Hwang [56]. Erdős [65] popularized a stronger 3-colorability version of the conjecture, and this became known as the “cycle-plus-triangles problem”. The conjecture was proved by Fleischner and Stiebitz [70] and by Sachs [149]. For more general graphs with maximum degree 2, Theorem 1.22 shows that when G contains the

disjoint union of three 4-cycles C_4 , the condition $|V_i| \geq 3$ does not guarantee the existence of an IT. But in contrast with Theorem 5.2, three disjoint 4-cycles is not the only obstruction to the existence of an IT when $\Delta = 2$. Aharoni, Holzman, Howard, and Sprüssel [16] also proved the following result suggesting that other cycle lengths are possible obstructions to the existence of an IT (see Theorem 4.8).

Theorem 5.3. *Let G be a graph with maximum degree $d = 2$, and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. If $|V_i| \geq 3$ for every i but G has no IT, then G contains the disjoint union of three cycles each of length $1 \pmod{3}$.*

To complement Theorem 5.3, Aharoni, Holzman, Howard, and Sprüssel [16] showed that for any $\ell \equiv 1 \pmod{3}$, $\ell \geq 4$, there exist $k \geq 1$ and a partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of the vertex set of the graph $G = C_\ell \sqcup \dots \sqcup C_\ell$ (k cycles of length ℓ) such that $|V_i| = 3$ for every i and G has no IT. They showed that one can take $k = \frac{\ell}{2} + 1$ if ℓ is even, and $k = \ell + 2$ if ℓ is odd. We now prove Theorem 1.24, which states that in fact one can always take $k = 3$, and that the cycle lengths do not have to be the same. Our proof will use the following lemma that was also proven and utilized by Aharoni, Holzman, Howard, and Sprüssel [16] in their constructions.

Lemma 5.4 (Aharoni, Holzman, Howard, Sprüssel [16]). *Let $\ell = 3n + 1$ with $n \geq 1$. There exists a partition $\mathcal{P} = \{V_1, \dots, V_{n+1}\}$ of the vertices of the ℓ -cycle C_ℓ such that $|V_i| = 3$ for $1 \leq i \leq n - 1$, $|V_n| = |V_{n+1}| = 2$, and there is no IT.*

Proof of Theorem 1.24. Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ be partitions of the vertices of the cycles $C_{\ell_1}, C_{\ell_2}, C_{\ell_3}$ given by Lemma 5.4. Let $\{X_1, X_2\} \subseteq \mathcal{P}_1$, $\{Y_1, Y_2\} \subseteq \mathcal{P}_2$, $\{Z_1, Z_2\} \subseteq \mathcal{P}_3$ be the classes of size 2 in each partition. Now let $\mathcal{P}'$ be the partition of the vertices of $G' = C_{\ell_1} \sqcup C_{\ell_2}$ obtained by taking the union of $\mathcal{P}_1$ and $\mathcal{P}_2$ while distributing one vertex of X_1 into Y_1 and the other vertex of X_1 into Y_2 . By Lemma 5.1, G' has no IT with respect to $\mathcal{P}'$. Finally, let $\mathcal{P}$ be the partition of the vertices of $G = G' \sqcup C_{\ell_3}$ obtained by taking the union of $\mathcal{P}'$ and $\mathcal{P}_3$ while distributing one vertex of X_2 into Z_1 and the other vertex of X_2 into Z_2 . By Lemma 5.1, G has no IT with respect to $\mathcal{P}$, and we have $|V_i| = 3$ for every $V_i \in \mathcal{P}$. $\square$

Aharoni, Holzman, Howard, and Sprüssel [16] proved Lemma 5.4 by describing the partition $\mathcal{P}$ explicitly. Here we give an alternative proof of Lemma 5.4 via an iterative approach similar to applications of Lemma 5.1. One could use this method to find a larger class of partitions $\mathcal{P}$ satisfying Lemma 5.4 than those described by the said authors. Our main tool is the following simple lemma.

Lemma 5.5. *Let G be a graph and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. Suppose that G has no IT with respect to $\mathcal{P}$. Let $uv \in E(G)$ be an edge with $u \in V_i$, $v \in V_j$, and $i \neq j$. Let $k \in \{1, \dots, n\} \setminus \{i, j\}$, and let F be a set of edges (possibly some in G) connecting every vertex in V_k to either u or v . Then the graph $H = G - uv \cup F$ has no IT with respect to $\mathcal{P}$.*

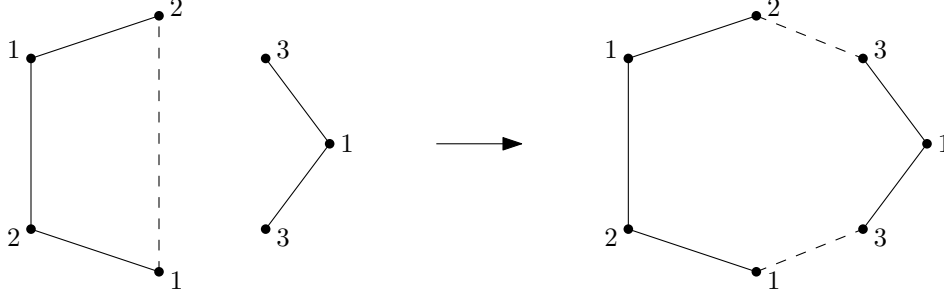


Figure 5.2: The disjoint union of a 4-cycle C_4 and a $K_{1,2}$ (with no IT) being transformed into a 7-cycle C_7 (with no IT). The label i next to each vertex indicates which class V_i the vertex lies in.

Proof. Assume for contradiction that H has an IT $\{v_1, \dots, v_n\}$ with respect to $\mathcal{P}$, where $v_\ell \in V_\ell$ for $1 \leq \ell \leq n$. Since G has no IT, we must have that $v_i = u$ and $v_j = v$. But either uv_k or vv_k lies in F , which implies that $\{v_1, \dots, v_n\}$ is not independent in H , a contradiction. $\square$

Proof of Lemma 5.4. We use induction on n . For $n = 1$, we take $\mathcal{P}$ to be the standard bipartition of the 4-cycle C_4 , which satisfies the requirements. Let $n \geq 2$, and let $\mathcal{P}' = \{V_1, \dots, V_n\}$ be a partition for the cycle C_{3n-2} given by the induction hypothesis. Let $\mathcal{Q} = \{U = \{u_1\}, V_{n+1} = \{v_1, v_2\}\}$ be the standard bipartition of the path $K_{1,2}$. Let $G = C_{3n-2} \sqcup K_{1,2}$, and let $\mathcal{P} = \{V_1, \dots, V_{n-2}, V_{n-1} \cup \{u_1\}, V_n, V_{n+1}\}$. By Lemma 5.1, G has no IT with respect to $\mathcal{P}$. Let $e = uv$ be any edge of C_{3n-2} , and let $F = \{uv_1, vv_2\}$. Then $G - uv \cup F$ is the $(3n+1)$ -cycle C_{3n+1} , and by Lemma 5.5 C_{3n+1} has no IT with respect to $\mathcal{P}$. See Figure 5.2 for illustration when $n = 2$. The classes in $\mathcal{P}$ have the required sizes, so this finishes the proof. $\square$

5.2 Disjoint unions of general graphs

In this section, we prove a general theorem about what happens when we successively apply Lemma 5.1 to one graph and vertex partition with no IT. This is motivated by the fact that in Section 5.1 we applied Lemma 5.1 to the same graph, namely $K_{\Delta, \Delta}$. This result will be useful in the next two chapters. After proving the main theorem, we describe some immediate applications of interest. In particular, we will explain this theorem's relevance to improvements on Theorem 1.21 of Haxell [95, 96] when the number of classes n is small. We will also show that the maximum degree and maximum average degree condition in Theorem 4.11 is close to being tight.

Theorem 5.6. *Let G be a graph and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$, such that G has no IT with respect to $\mathcal{P}$. Suppose that integer $q \geq 1$ satisfies*

$$\sum_{i \in I} |V_i| \geq q(|I| - 1) + 1 \quad \text{for all } I \subseteq [n].$$

Then there exist an integer $k \geq 1$ and a partition $\mathcal{Q} = \{U_1, \dots, U_{k(n-1)+1}\}$ of the graph $G^k = G \sqcup \dots \sqcup G$ (where G appears k times in the disjoint union), such that $|V_i| \geq q$ for every i and there is no IT. Specifically, if $J = \{j : |V_j| < q\} \neq \emptyset$ and $V_J = \bigcup_{j \in J} V_j$, then we can take any

$$k \geq 1 + \sum_{j \in J} \left\lceil \frac{q - |V_j|}{|V_j| - q(|J| - 1)} \right\rceil.$$

Proof. We start with the graph G and partition $\mathcal{P}$. Let $J = \{j : |V_j| < q\}$. If $J = \emptyset$, then G and $\mathcal{P}$ satisfy the conclusion of the theorem with $k = 1$. Otherwise, we will successively enlarge each class of $\mathcal{P}$ of size less than q into a class of size at least q , by iteratively adding new copies of $G, \mathcal{P}$ and applying Lemma 5.1. Let $G^1 = G$ and $\mathcal{P}^1 = \mathcal{P}$, and let $i \in J$ minimize the cardinality $|V_i|$.

Assume that we have defined $G^{\ell-1}, \mathcal{P}^{\ell-1}$ with no IT for some $\ell \geq 2$, and assume by induction that every class of $\mathcal{P}^{\ell-1}$ has size at least $|V_i|$. Now we construct $G^\ell, \mathcal{P}^\ell$. We fix any class $U \in \mathcal{P}^{\ell-1}$ of size less than q , and then we add a copy of $G, \mathcal{P}$ to the current graph while distributing the vertices of U into the copy of the classes $\{V_j : j \in J\}$ in this added copy of $\mathcal{P}$. We distribute $q - |V_j|$ vertices of U to V_j for each $j \in J \setminus \{i\}$ to create classes of size q , and we distribute the remaining vertices of U to V_i to form another, possibly smaller, class U' . It is possible to distribute vertices this way because by the inequality $|U| \geq |V_i|$ and our assumption on q , we have that

$$|U| - \sum_{j \in J \setminus \{i\}} (q - |V_j|) = |U| - |V_i| + \left(\sum_{j \in J} |V_j| - q(|J| - 1) \right) \geq 1,$$

meaning there is still at least one vertex in U available to distribute to V_i after distributing to the other classes indexed by J . Moreover, the size of the class U' will be

$$|U'| = |V_i| + \left(|U| - \sum_{j \in J \setminus \{i\}} (q - |V_j|) \right) = |U| + \left(\sum_{j \in J} |V_j| - q(|J| - 1) \right) \geq |U| + 1$$

Thus going from $G^{\ell-1}, \mathcal{P}^{\ell-1}$ to $G^\ell, \mathcal{P}^\ell$, we create at most one new class U' in $\mathcal{P}^\ell$ of size less than q , and the size of U' is at least one more than the size of the class U from $\mathcal{P}^{\ell-1}$. Therefore, after adding sufficiently many copies of $G, \mathcal{P}$, all the classes in the eventual vertex-partitioned graph $G^k, \mathcal{P}^k$ will have size at least q , and it will have no IT by Lemma 5.1.

To get the stated lower bound for the number k of copies of G we need to add, notice that at each step we are increasing the size of the “smaller” class U by $|V_J| - q(|J| - 1)$. Thus, for each class V_j , $j \in J$, we need a minimum of $\left\lceil \frac{q - |V_j|}{|V_j| - q(|J| - 1)} \right\rceil$ copies of $G, \mathcal{P}$ to enlarge V_j into classes all of size at least q . Together with the original copy of $G, \mathcal{P}$ that we started with, the sum of these quantities gives the stated lower bound for k .

To see that all larger k work, we simply allow the above procedure to continue for $\ell \geq k$. Assuming that $\ell \geq k+1$ and that we have defined $G^{\ell-1}, \mathcal{P}^{\ell-1}$, we may add it a copy of $G, \mathcal{P}$, choose any class $U \in \mathcal{P}^{\ell-1}$, and distribute its vertices into the classes of the added copy of $\mathcal{P}$ so that all resulting classes have size at least q , which is possible because

$$\sum_{j \in J} (q - |V_j|) = q - \left(\sum_{j \in J} |V_j| - q(|J| - 1) \right) \leq q - 1 < |U|.$$

Finally, to show that we wrote the correct the number of classes in $\mathcal{P}^k$, observe that the initial partition $\mathcal{P}^1$ has n classes, and after each step we add $n - 1$ new classes to the partition. Thus, we will end up with $n + (k - 1)(n - 1) = k(n - 1) + 1$ classes in $\mathcal{P}^k$. $\square$

Before discussing applications of Theorem 5.6, we give a few remarks. First, one can view Theorem 5.6 as an “anti-Hall theorem”. It asserts that if $|\bigcup_{i \in I} V_i|$ is sufficiently large for every $I \subseteq [n]$ but there is no IT, then we can use the vertex-partitioned graph to create another vertex-partitioned graph with a desired property, such as having all its class sizes be at least a certain size q .

Next, we observe that the condition on q in Theorem 5.6 is best possible if the construction is achieved by iteratively applying Lemma 5.1 the way we did in the proof. Specifically, suppose that $G^\ell, \mathcal{P}^\ell$ is obtained by iteratively applying Lemma 5.1 to $G, \mathcal{P}$, with G^ℓ being the disjoint union of ℓ copies of G , and suppose that $\sum_{i \in I} |V_i| \leq (q + 1)(|I| - 1)$ for some $I \subseteq [n]$. Then there will be ℓ copies of the vertex set $\bigcup_{i \in I} V_i$ partitioned into at least $|I| + (\ell - 1)(|I| - 1) = \ell(|I| - 1) + 1$ classes, so there will always be some class of size at most

$$\frac{\ell \sum_{i \in I} |V_i|}{\ell(|I| - 1) + 1} < \frac{\sum_{i \in I} |V_i|}{|I| - 1} \leq q + 1.$$

Finally, we note that there are some straightforward ways to generalize the statement of Theorem 5.6. For one generalization, we could have started with any vertex-partitioned graph $H, \mathcal{Q}$ with no IT, and added sufficiently many copies of $G, \mathcal{P}$ to still end up with classes all of size at least q and no IT. The proof is basically the same as above, except that we preprocess $H, \mathcal{Q}$ by adding some copies of $G, \mathcal{P}$ to ensure that the initial classes have size at least the size of the smallest class of $\mathcal{P}$. For another generalization, we could have a general collection of vertex-partitioned graphs $(G_1, \mathcal{P}_1), \dots, (G_k, \mathcal{P}_k)$ each with no IT and satisfying the hypothesis of Theorem 5.6, and then show that taking a disjoint union of a certain number of each of these graphs, we can find a vertex partitioned graph with classes of size at least q and no IT. This generalization is motivated by our construction for three cycles of lengths 1 modulo 3 in Section 5.1.2. However, the exact quantities of the number of copies required from each graph is more complicated in this general setting.

The rest of this section is dedicated to presenting applications of Theorem 5.6. Our first application is about optimal conditions for the existence of IT's depending on both the maximum degree Δ and the number of partition classes n . If $G = K_{\Delta, \Delta}$ and $\mathcal{P}$ is its

standard bipartition, then we can take $q = 2\Delta - 1$ and $k = 2\Delta - 1$ in Theorem 5.6. This means that the disjoint union of $k = 2\Delta - 1$ copies of $K_{\Delta,\Delta}$ admits a partition into classes of size $2\Delta - 1$ with no IT. Thus, Theorem 5.6 generalizes Theorem 1.22. More generally, for any $1 \leq q \leq 2\Delta - 1$, Theorem 5.6 implies that we can achieve classes of size at least q with no IT on the disjoint union of $k = 2 \left\lceil \frac{\Delta}{2\Delta - q} \right\rceil - 1$ copies of $K_{\Delta,\Delta}$, with $n = k + 1$ partition classes. Rearranging to write n in terms of q , we derive the following more general version of Theorem 1.22 described by Szabó and Tardos [161].

Theorem 5.7 (Szabó, Tardos [161]). *Let $\Delta \geq 1$, let $n \geq 2$ be even, and let $q = \left\lfloor \frac{2\Delta(n-1)}{n} \right\rfloor$. Let G be the disjoint $n - 1$ copies of $K_{\Delta,\Delta}$. There exists a partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$ such that $|V_i| \geq q$ for every i and there is no IT.*

Theorem 5.7 was the general result that Szabó and Tardos [161] used to solve one direction of an extremal problem of Bollobás, Erdős, and Szemerédi [37], about minimum degree conditions for the existence of a clique K_n in a balanced n -partite graph. The other direction was solved by Haxell and Szabó [102], who proved the following refinement of Theorem 1.21.

Theorem 5.8 (Haxell, Szabó [102]). *Let G be a graph with maximum degree Δ , and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$, where $n \geq 2$. If n is even and $|V_i| \geq \left\lfloor \frac{2\Delta(n-1)}{n} \right\rfloor + 1$ for every i , then there exists an IT. If n is odd and $|V_i| \geq \left\lfloor \frac{2\Delta(n-2)}{n-1} \right\rfloor + 1$ for every i , then there exists an IT.*

The proof of Theorem 5.8 for even n is an easy consequence of results from Section 4.1 and was noted by Alon [23]. Specifically, if $|V_i| \geq \left\lfloor \frac{2\Delta(n-1)}{n} \right\rfloor + 1$ for every i , then for all $I \subseteq [n]$, we have $\sum_{i \in I} |V_i| > \frac{2\Delta(n-1)}{n} |I| \geq 2\Delta(|I| - 1)$. By Corollary 4.5, there exists an IT. For odd n , the proof of Haxell and Szabó [102] is a more involved combinatorial argument that we will adapt for a different purpose in Section 5.6. Another common way to write these results is the following: The smallest integer $\Delta = \Delta(n, q)$ for which there exists a graph G with maximum degree Δ and a partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$ with n classes of size at least q and no IT is

$$\Delta(n, q) = \begin{cases} \left\lceil \frac{qn}{2(n-1)} \right\rceil & \text{if } n \text{ is even,} \\ \left\lceil \frac{q(n-1)}{2(n-2)} \right\rceil & \text{if } n \text{ is odd.} \end{cases}$$

Now, Lemma 5.1 gives us some intuition as to why there are different behaviors depending on whether the number n of classes is even or odd. Specifically, if the goal is to apply Lemma 5.1 to get a given class size with no IT using the fewest number of copies of $K_{\Delta,\Delta}$, then we will necessarily end up with an even number of classes, since each addition of $K_{\Delta,\Delta}$ can only enlarge at most one class and we must perform this enlargement on all classes starting from an even number of them (two of them).

For another application of Theorem 5.6, we address the optimality of Theorem 4.11 from Chapter 3, which is restated below.

Theorem 5.9. *Let G be a graph with maximum degree Δ and maximum average degree γ , and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. If $|V_i| \geq \frac{2\Delta^2}{2\Delta - \gamma}$ for every i , then there exists an IT.*

We show that the class size condition in Theorem 5.9 is close to being best possible. First consider applying Theorem 5.6 to the complete bipartite graph $K_{a,b}$ with the standard bipartition. We get that the disjoint union of $a + b - 1$ copies of $K_{a,b}$ admits a partition $\mathcal{P} = \{V_1, \dots, V_{a+b}\}$ such that $|V_i| \geq a + b - 1$ for every i and there is no IT. We apply this result when $a = \Delta$ and $b = \Delta'$, where $\Delta' = \left\lfloor \frac{\Delta\gamma}{2\Delta - \gamma} \right\rfloor$ (the fraction in the floor is the solution ℓ to $\frac{2\Delta\ell}{\Delta + \ell} = \gamma$). From above, we get that the disjoint union G of $k = \Delta + \Delta' - 1$ copies of $K_{\Delta, \Delta'}$ admits a vertex partition $\mathcal{P} = \{V_1, \dots, V_n\}$ such that $|V_i| \geq \Delta + \Delta' - 1$ for every i and there is no IT. Notice that G has maximum degree Δ , maximum average degree $\frac{2\Delta\Delta'}{\Delta + \Delta'} \leq \gamma$, and the class sizes all satisfy $|V_i| \geq \Delta + \Delta' - 1 = \left\lfloor \frac{2\Delta^2}{2\Delta - \gamma} \right\rfloor - 1$. This comes close to the condition in Theorem 5.9. The construction presented here is in fact basically the same as the Phase II construction of Section 4.5 (ignoring Phase I).

For one more application of Theorem 5.6, let $K_r(m)$ denote the complete r -partite graph with m vertices in each part. The disjoint union of $r - 1$ copies of $K_r(m)$ has no IT with respect to a standard r -partition, consisting of r independent sets of size $(r - 1)m$. Applying Theorem 5.6 to this graph yields the following.

Theorem 5.10. *Let $r \geq 2, m \geq 1$. Let G be the disjoint union of $(r - 1) \left(1 + r \left\lceil \frac{m-1}{r-1} \right\rceil\right)$ copies of $K_r(m)$, which has maximum degree $\Delta = (r - 1)m$. There exists a partition $\mathcal{P} = \{V_1, \dots, V_s\}$ of $V(G)$ such that $|V_i| \geq rm - 1 = \Delta + m - 1$ for every i and there is no IT.*

5.3 Constructions for locally sparse graphs

In this section, we prove Theorem 1.25 about vertex-partitioned graphs with no IT in the setting of locally sparse graphs. It is restated below.

Theorem 1.25. *For all integers $\Delta \geq m \geq 1$ where Δ is a multiple of m , there exists a graph G and partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$ such that G has maximum degree Δ , local degree and multiplicity m , $|V_i| \geq \Delta + 2m - \left\lceil \frac{2m^2 + m}{\Delta + m} \right\rceil$ for every $V_i \in \mathcal{P}$, and there is no IT.*

To recall terminology, let G be a graph and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. The *local degree* of $G, \mathcal{P}$ is the maximum, over all $v \in V(G)$ and $V_i \in \mathcal{P}$ not containing v , of the number of edges between v and V_i . If the classes V_i are independent sets of G (as they often are), then the local degree is the same as the maximum of $\Delta(G[V_i \cup V_j])$ over all

distinct classes $V_i, V_j \in \mathcal{P}$. Next, the *multiplicity* of $G, \mathcal{P}$ is defined to be the maximum, over all connected components C of G and $V_i \in \mathcal{P}$, of the number of vertices in $C \cap V_i$. Notice that the local degree of $G, \mathcal{P}$ is always at most its multiplicity. We refer to vertex-partitioned graphs whose local degree grows much more slowly than the maximum degree as *locally sparse*. Theorem 1.25 complements the following asymptotic result of Loh and Sudakov [127].

Theorem 5.11 (Loh, Sudakov [127]). *Let G be a graph with maximum degree Δ , and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. If $G, \mathcal{P}$ has local degree $o(\Delta)$ and $|V_i| \geq \Delta + o(\Delta)$ for every i , then there exists an IT.*

We remark that there are some more recent strengthenings of Theorem 5.11. Glock and Sudakov [78] and Kang and Kelly [116] independently strengthened Theorem 5.11 to the *maximum class average degree* setting instead of the maximum degree setting. In addition, Chakraborti and Tran [44] proved that under the hypotheses of Theorem 5.11, one can find $\Delta - o(\Delta)$ pairwise disjoint IT's.

Before proving Theorem 1.25, we describe an initial construction. We use complete multipartite graphs as in Theorem 5.10. Fix integers $m \geq 1$ and $r \geq 2$, and let $K_r(m)$ denote the complete r -partite graph with m vertices in each part. Let G_0 be the disjoint union of $r - 1$ copies of $K_r(m)$, and let the partition $\mathcal{P}_0 = \{V_1, \dots, V_r\}$ be a standard r -partition of G_0 , consisting of r independent sets of size $(r - 1)m$. Letting $\Delta = (r - 1)m$ denote the maximum degree of G_0 , we have that $|V_i| = (r - 1)m = \Delta$ for every i , the local degree and multiplicity of $G_0, \mathcal{P}_0$ are both m , and there is no IT with respect to $\mathcal{P}_0$. Theorem 5.10 implies that after applying Lemma 5.1 sufficiently many times on $G_0, \mathcal{P}_0$, we get a graph G_1 with maximum degree Δ and a partition $\mathcal{P}_1 = \{V'_1, \dots, V'_n\}$ of $V(G_1)$ such that $G_1, \mathcal{P}_1$ has local degree and multiplicity m , $|V'_i| = \lfloor \frac{r\Delta-1}{r-1} \rfloor = \Delta + m - 1$ for every i , and there is no IT. To get an improved construction for Theorem 1.25, the idea is to start with a vertex-partitioned graph with large class sizes that is not locally sparse, and then to add copies of the locally sparse $G_0, \mathcal{P}_0$ while applying Lemma 5.1 to make the result locally sparse. This idea will also be used in the next section.

Theorem 1.25. *For all integers $\Delta \geq m \geq 1$ where Δ is a multiple of m , there exists a graph G and partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$ such that G has maximum degree Δ , local degree and multiplicity m , $|V_i| \geq \Delta + 2m - \left\lceil \frac{2m^2+m}{\Delta+m} \right\rceil$ for every $V_i \in \mathcal{P}$, and there is no IT.*

Proof. Assume that $\Delta = (r - 1)m$, where $r \geq 2$. We start with any graph H and partition $\mathcal{Q}$ coming from Theorem 1.22, say the Szabó-Tardos construction, in which H is the disjoint union of $2\Delta - 1$ copies of $K_{\Delta, \Delta}$. Then $|U| = 2\Delta - 1$ for every $U \in \mathcal{Q}$ and there is no IT. This graph has local degree and multiplicity Δ , which may be larger than m . We fix this by applying Lemma 5.1 again. Fix a class $U \in \mathcal{Q}$. We add a copy of the vertex-partitioned graph $G_0, \mathcal{P}_0$ above, and then distribute vertices of U into the r classes of $\mathcal{P}_0$. We do this

equitably so that each of the r classes of $\mathcal{P}_0$ gets enlarged by at least

$$\left\lfloor \frac{2\Delta - 1}{r} \right\rfloor = \left\lfloor \frac{2\Delta - 1}{\Delta/m + 1} \right\rfloor = \left\lfloor \frac{2m(\Delta + m) - 2m^2 - m}{\Delta + m} \right\rfloor = 2m - \left\lceil \frac{2m^2 + m}{\Delta + m} \right\rceil.$$

Moreover, we do this in such a way that at most m vertices of any connected component of H get distributed to any class of $\mathcal{P}_0$. This is possible by a greedy process, since at most Δ vertices of U lie in any connected component of H , and $\lceil \frac{\Delta}{r} \rceil = \left\lceil \frac{(r-1)m}{r} \right\rceil \leq m$. After doing this procedure iteratively for every $U \in \mathcal{Q}$, we get a graph G and partition $\mathcal{P} = \{V_1, \dots, V_n\}$ that has local degree and multiplicity m , $|V_i| \geq \Delta + 2m - \left\lceil \frac{2m^2 + m}{\Delta + m} \right\rceil$ for every i , and there is no IT. $\square$

We remark that slightly smaller constructions for Theorem 1.25 are possible. Note that the number of classes in the graph constructed in the above proof is $c(H) \cdot r$, where $c(H)$ denotes the number of classes of the starting graph H . Instead of the Szabó-Tardos construction, we could have started with any vertex-partitioned graph H with no IT where every class has size at least $\left\lfloor \frac{2\Delta - 1}{r} \right\rfloor r = \left(2\Delta - \left\lceil \frac{2m^2 + m}{\Delta + m} \right\rceil\right) \left(\frac{\Delta}{m} + 1\right)$, and where every component has at most $\Delta + 1$ vertices from any class. Following the argument in Section 5.1, one could form a smaller such graph H by taking the disjoint union of possibly fewer than $2\Delta - 1$ copies of $K_{\Delta, \Delta}$, and also possibly deleting some vertices. For example, when $m = 1$ we only need classes of size at least $\Delta + 1$, and we can achieve these class sizes on the graph $H = K_{2,2} \sqcup K_{\Delta, \Delta} \sqcup K_{\Delta, \Delta}$ with 4 classes. Following the proof of Theorem 1.25 on this latter graph H yields a locally sparse graph (with $m = 1$) constructed by Aharoni, Holzman, Howard, and Sprüssel [16].

In the particular case when $\Delta \geq 2m^2$, Theorem 1.25 states that we can take the class sizes to satisfy $|V_i| \geq \Delta + 2m - 1$ for every i . Theorem 1.25 was previously proven when $m = 1$ by Bohman and Holzman [36], who used their construction to disprove a list coloring conjecture of Reed [145]. This connection to list coloring is the subject of the next section.

5.4 List colorings

In this section, we discuss the relationship between list colorings of graphs and independent transversals in graphs. We use this relationship to give a streamlined proof of Theorem 1.26 of Bohman and Holzman [36], restated below, which disproved a list coloring conjecture of Reed [145].

Theorem 1.26 (Bohman, Holzman [36]). *For every $\Delta \geq 2$, there exists a graph H and list assignment L for $V(H)$ such that H, L has maximum color degree Δ , $|L(x)| = \Delta + 1$ for every $x \in V(H)$, and there is no proper L -coloring.*

Recall the following terminology. Given a graph H and a list assignment $L = (L(x) : x \in V(H))$ for $V(H)$, where each $L(x)$ is a set of colors, an L -coloring is a map ϕ on $V(H)$ such that $\phi(x) \in L(x)$ for every vertex $x \in V(H)$. The L -coloring ϕ is *proper* if $\phi^{-1}(c)$ is an independent set of H for every color $c \in \bigcup_{x \in V(H)} L(x)$. The problem of finding a proper L -coloring of H can be turned into a problem of finding an IT in an auxiliary graph G with respect to some vertex partition $\mathcal{P}$. Specifically, the graph G has vertex set $\{(x, c) : x \in V(H), c \in L(x)\}$, and the pairs (x, c) and (y, c') form an edge in G whenever $xy \in E(H)$ and $c = c'$. The partition $\mathcal{P}$ of $V(G)$ is $\{V_x : x \in V(H)\}$ where $V_x = \{(x, c) : c \in L(x)\}$. More intuitively, we construct G from H, L by taking the disjoint union, over all colors c , of the subgraphs of H induced by vertices x with the color c in their lists $L(x)$, and we construct a class V_x of $\mathcal{P}$ by putting into V_x all vertices of G corresponding to the same vertex x of H . Then every proper L -coloring ϕ of H corresponds to a unique IT $\{(x, \phi(x)) : x \in V(H)\}$ of G with respect to $\mathcal{P}$. We say that the graph G with vertex partition $\mathcal{P}$ is the *list cover graph* arising from graph H and list assignment L . This kind translation of proper list colorings into independent transversals was described by Haxell [96], but analogues of it were known before in other contexts (e.g., [172] in the context of matroids).

Given a vertex $x \in V(H)$ and color $c \in L(x)$, the pair (x, c) has *color degree* equal to $|\{(y, c) : xy \in E(H), c \in L(y)\}|$. The *maximum color degree* of H, L is the maximum of the color degrees over all vertex-color pairs (x, c) . In other words, it is the maximum degree of the list cover graph G . Thus, by Theorem 1.21 of Haxell [95, 96], if H, L has maximum color degree Δ and $|L(x)| \geq 2\Delta$ for every vertex $x \in V(H)$, then H has a proper L -coloring. Unlike Theorem 1.21, this lower bound on the list sizes is not best possible. Lemma 5.13 implies that every list cover graph has local degree one, so by Theorem 5.11 of Loh and Sudakov [127] we get the following condition for the existence of a proper L -coloring, which was previously proved by Reed and Sudakov [146].

Theorem 5.12 ([127, 146]). *Let H be a graph and let L be a list assignment for $V(G)$. If H, L has maximum color degree Δ and $|L(x)| \geq \Delta + o(\Delta)$ for every $x \in V(H)$, then there exists a proper L -coloring.*

Reed [145] conjectured that a proper L -coloring exists whenever $|L(x)| \geq \Delta + 1$ for every $x \in V(H)$. This lower bound of $\Delta + 1$ is necessary, since for example there is no proper L -coloring if H is the complete graph $K_{\Delta+1}$ and L is the uniform list assignment $L(x) = \{1, \dots, \Delta\}$ for every $x \in V(H)$. Aharoni, Berger, and Ziv [12] proved Reed's conjecture for chordal graphs using topological methods. Surprisingly, Bohman and Holzman [36] found counterexamples to Reed's conjecture.

Theorem 1.26 (Bohman, Holzman [36]). *For every $\Delta \geq 2$, there exists a graph H and list assignment L for $V(H)$ such that H, L has maximum color degree Δ , $|L(x)| = \Delta + 1$ for every $x \in V(H)$, and there is no proper L -coloring.*

Our goal now is to provide smaller and more streamlined constructions for Theorem 1.26, by iteratively applying Lemma 5.1. Bohman and Holzman [36] proved Theorem 1.26

by constructing appropriate vertex-partitioned graphs with no IT, and we will do the same. To proceed, we need a proper understanding of when a vertex-partitioned graph $G, \mathcal{P}$ is the list cover graph of some graph H and list assignment L . The following characterization of list cover graphs is implicit in Bohman and Holzman [36], and we give a short proof for completeness.

Lemma 5.13. *A graph G with partition $\mathcal{P}$ of $V(G)$ is a list cover graph if and only if*

- (a) *for every connected component C of G , every class of $\mathcal{P}$ contains at most one vertex of C (i.e., $G, \mathcal{P}$ has multiplicity 1); and*
- (b) *for every two connected components C, C' of G , two classes V_i, V_j of $\mathcal{P}$, and vertices $v \in C \cap V_i, w \in C \cap V_j, v' \in C' \cap V_i, w' \in C' \cap V_j$, v and w are adjacent in C if and only if v' and w' are adjacent in C' .*

Proof. Suppose that G with vertex partition $\mathcal{P}$ is a list cover graph arising from graph H and list assignment L . Then every connected component C of G has vertex set $\{(x, c) : x \in U\}$ for some color c and vertex subset $U \subseteq V(H)$. Then for $V_x = \{(x, c) : c \in L(x)\} \in \mathcal{P}$, we have $V_x \cap C = \{(x, c)\}$ if $c \in L(x)$, and $V_x \cap C = \emptyset$ otherwise, so that condition (a) holds. Moreover, for every two connected components C, C' of G , two classes V_x, V_y of $\mathcal{P}$, and vertices $(x, c) \in C \cap V_x, (y, c) \in C \cap V_y, (x, c') \in C' \cap V_x, (y, c') \in C' \cap V_y$, we have that (x, c) and (y, c) are adjacent in C if and only if $xy \in E(H)$, which is the case if and only if (x, c') and (y, c') are adjacent in C' . Thus condition (b) also holds.

Suppose conversely that (a) and (b) hold. Let H be the graph with vertex set $\mathcal{P}$, where $V_i, V_j \in \mathcal{P}$ form an edge if and only if there exist vertices $v \in V_i, w \in V_j$ adjacent in G . Associate every connected component C of G with a distinct color c , and let L be the list assignment for $V(H)$ where $L(V_i)$ is the set of colors c associated with components C where $C \cap V_i \neq \emptyset$. Then one can easily verify that $G, \mathcal{P}$ is the list-cover graph arising from H, L . $\square$

Thus, proving Theorem 5.14 is the same as proving the following result.

Theorem 5.14. *For every $\Delta \geq 2$, there exists a graph G and a partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$ such that G has maximum degree Δ , $|V_i| = \Delta + 1$ for every i , conditions (a) and (b) of Lemma 5.13 hold, and there is no IT.*

Proof. Fix $\Delta \geq 2$. We start with any construction for Theorem 1.25 for the case $m = 1$, so that each class has size at least $\Delta + 1$. For convenience, we delete some vertices to make all the classes have size exactly $\Delta + 1$. Call the constructed graph H and partition $\mathcal{Q}$. We have that $|U| = \Delta + 1$ for every $U \in \mathcal{Q}$, and we satisfy condition (a) of Lemma 5.13 with H and $\mathcal{Q}$, but not condition (b). To fix this, we again apply Lemma 5.1, adding additional collections of complete graphs. Let G_0 be the disjoint union of Δ copies of the complete graph $K_{\Delta+1}$, and let $\mathcal{P}_0$ be a partition of G_0 into $\Delta + 1$ independent sets. Then G_0 has no IT with respect to $\mathcal{P}_0$. For every class $U \in \mathcal{Q}$ in succession, we add a copy of G_0 while distributing the $\Delta + 1$

vertices of U into the $\Delta + 1$ classes of $\mathcal{P}_0$, with each class of $\mathcal{P}_0$ receiving exactly one vertex. Doing this for each $U \in \mathcal{Q}$, we derive a graph G and vertex partition $\mathcal{P}$ with no IT, and every class of $\mathcal{P}$ has size $\Delta + 1$.

Observe that $G, \mathcal{P}$ still satisfy condition (a) of Lemma 5.13. We claim that $G, \mathcal{P}$ now also satisfies condition (b). To see this, let C and C' be distinct components of G . If C and C' both lie in an added copy of G_0 , then C and C' are cliques whose intersecting classes are either the same or disjoint, so condition (b) holds for C and C' . Next, suppose that the component C lies in H and component C' lies in some added copy of G_0 . Let U be the class of $\mathcal{Q}$ whose vertices were distributed into the classes intersecting C' when we added a copy of G_0 . Since H has multiplicity 1, there is at most one vertex in $U \cap C$, and thus there is at most one class of $\mathcal{P}$ intersecting both C and C' . Hence condition (b) holds in this case as well. Finally, suppose that C and C' both lie in H . In this case, there are no classes of $\mathcal{P}$ that intersect both C and C' : If there were such a class, then it would have been constructed by taking a class U of $\mathcal{Q}$ intersecting both C and C' , and distributing its vertices into the classes of $\mathcal{P}_0$ in such a way that the vertex in $U \cap C$ and the vertex in $U \cap C'$ end up in the same class of $\mathcal{P}$. But this is impossible because every class of $\mathcal{P}_0$ receives a unique vertex of U by construction. Therefore, condition (b) holds for all C and C' . $\square$

The construction we described in the proof of Theorem 5.14 is similar in structure to that of Bohman and Holzman [36], who used $2(\Delta + 1)(\Delta^2 + 1)$ classes. If we use the Szabó-Tardos construction [161] (Theorem 1.22) as the starting point in the proof of Theorem 1.25, then the output of our proof of Theorem 1.25 has $2\Delta(\Delta + 1)$ classes, and the output of our proof of Theorem 5.14 has $2\Delta(\Delta + 1)^2$ classes. We can get a smaller construction with $4(\Delta + 1)^2$ by following the proof of Theorem 1.25, but using the graph $K_{2,2} \sqcup K_{\Delta,\Delta} \sqcup K_{\Delta,\Delta}$ with 4 classes of size $\Delta + 1$ as the starting point instead of the Szabó-Tardos construction [161]. Our efforts have been unsuccessful in achieving constructions where all the classes have size at least $\Delta + 2$, a problem posed in [36, 146].

We comment on what occurs when we relax condition (a) or (b) in Lemma 5.13. In the case where we are only required to satisfy condition (a), this is the setting for $m = 1$ in Theorem 1.25. Even in this relaxed setting, we are unable to surpass the known lower bound $\Delta + 2$ by Bohman and Holzman [36] on the class sizes required to guarantee an IT in a graph with multiplicity one (a problem that was posed in [16]). On the other hand, if we are only required to satisfy condition (b), then the situation is quite different. For example, in [41] it is shown that there exist graphs G and partitions $\mathcal{P}$ of $V(G)$ such that condition (b) holds, $|V_i| = 2\Delta - 1$ for every $V_i \in \mathcal{P}$, and there is no IT. Thus, much of the difference between list cover graphs and general graphs lies in condition (a). However, if we require that condition (b) hold and we relax the local degree or multiplicity to being a fixed number $m \geq 1$, then the best we can currently achieve in general are class sizes $|V_i| \geq \Delta + m$ whenever Δ is a multiple of m and $\Delta > m$, obtained for example by “blowing up” any construction for Theorem 5.14, replacing every vertex by m independent vertices and replacing every edge by the complete bipartite graph $K_{m,m}$.

Finally, we remark that there are variations of list coloring for which independent transversals are also relevant. Ordinary list colorings correspond to IT's in quite specific types of cover graphs (as described in Lemma 5.13), but a variation of list coloring known as *conflict list coloring* corresponds to IT's in general graphs. This notion was originally defined in [59, 71] (see also [57, 58]), and a similar notion for hypergraphs was defined in [167]. Given a graph H and a list assignment L for $V(H)$, every edge $e = xy \in E(H)$ is assigned a list $K(x, y)$ of ordered pairs of colors $(c, c') \in L(x) \times L(y)$, called *conflicts*. We wish to find whether there exists an L -coloring ϕ of H such that no edge is in conflict, that is, there is no edge $xy \in E(H)$ and conflict $(c, c') \in K(x, y)$ such that $\phi(x) = c$ and $\phi(y) = c'$. Similar to ordinary list coloring, we can turn this into a problem on independent transversals in an associated *cover graph*. Let G be the graph with vertex set $\{(x, c) : x \in V(H), c \in L(x)\}$, where the pairs (x, c) and (y, c') form an edge if $xy \in E(H)$ and $(c, c') \in K(x, y)$, and let $\mathcal{P}$ be the partition of $V(G)$ into the vertex sets $V_x = \{(x, c) : c \in L(x)\}$. Then every conflict-free L -coloring of H corresponds to an IT of G with respect to $\mathcal{P}$. Conversely, every independent transversal problem can be turned into a conflict list coloring problem.

Many well-known list coloring variations are restrictions on the definition of conflict list colorings. For example, in the famous setting of *correspondence colorings* [59] (also called *DP-colorings*), the conflicts $K(x, y)$ are required to form a matching, and the cover graphs are exactly those with local degree 1. In the setting of *cooperative list colorings* [16], or equivalently *adaptable list colorings* [122], for every edge $xy \in E(H)$ every color $c \in L(x)$ is in conflict with at most one other color in $L(y)$, and the cover graphs are exactly those with multiplicity 1 (called *layered-ISR systems* in [16]). The locally sparse graphs constructed in Theorem 1.25 are thus relevant for these problems as well.

5.5 Counterexamples to some conjectures

In this section, we describe counterexamples to two statements on IT's in graphs. One is a conjecture of Aharoni, Alon, and Berger [5] on IT's in $K_{1,k}$ -free graphs, and the other is a question of Aharoni, Holzman, Howard, and Sprüssel [16] on the structure of extremal vertex-partitioned graphs with no IT.

First we address the problem of Aharoni, Alon, and Berger [5]. They conjectured that for every (induced) $K_{1,k}$ -free graph G with maximum degree Δ and partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$, if $|V_i| \geq \Delta + k - 1$ for every i then there exists an IT. They proved using topological methods that there exists an IT if $|V_i| \geq \frac{(2k-3)\Delta+k-1}{k-1}$ for every i . They also proved that their conjecture holds if G is the line graph of a $(k-1)$ -uniform linear hypergraph. We give a construction disproving their conjecture for all $k \geq 3$. We start with any construction $G_0, \mathcal{P}$ for Theorem 1.22 of Szabó and Tardos [161], consisting of $2m-1$ copies of $K_{m,m}$ where m is a multiple of $k-1$. To both parts of each complete bipartite component of G_0 , we add $k-1$ pairwise disjoint clique edge sets of size $\frac{m}{k-1}$ on the vertex sets. The resulting graph G is $K_{1,k}$ -free with maximum degree $\Delta = m + \frac{m}{k-1} - 1 = \frac{km}{k-1} - 1$, each class of $\mathcal{P}$ has size

$|V_i| = 2m - 1 = \frac{(2k-2)\Delta + k - 2}{k}$, and G has no IT with respect to $\mathcal{P}$. For $k \geq 3$ and $\Delta > k + \frac{2}{k-2}$, these class sizes $|V_i|$ are greater than $\Delta + k - 1$, which disproves the conjecture.

We add some remarks about the conjecture. The original conjecture of Aharoni, Alon, and Berger (Conjecture 2.8 in [5]) was actually about the topological connectedness of the independence complexes of $K_{1,k}$ -free graphs, but their original conjecture implies the above conjecture about IT's, via the methods described in Section 4.1. Their conjecture is that if G is a $K_{1,k}$ -free graph with maximum degree Δ , then the connectedness of the independence complex of G satisfies $\eta(\mathcal{I}(G)) \geq \frac{|V(G)|}{\Delta + k - 1}$. We can disprove this connectedness conjecture more directly. Let m be a multiple of $k - 1$, and let G be the complete bipartite graph $K_{m,m}$ together with $k - 1$ cliques of size $\frac{m}{k-1}$ added to both parts of $K_{m,m}$. Then the independence complex of G is disconnected, i.e., $\eta(\mathcal{I}(G)) = 1$, but the number of vertices of G is $|V(G)| = 2m = \frac{(2k-2)\Delta + 2k - 2}{k}$, which is greater than $\Delta + k - 1$ when $k \geq 3$ and $\Delta > k - 1$, contrary to the conjecture. We have shown that $\eta(\mathcal{I}(G)) \leq \frac{k|V(G)|}{(2k-2)\Delta + k - 2}$ for some $K_{1,k}$ -free graph G , whereas Aharoni, Alon, and Berger [5] proved that $\eta(\mathcal{I}(G)) \geq \frac{(k-1)|V(G)|}{(2k-3)\Delta + k - 1}$ for all $K_{1,k}$ -free graphs G . We believe that the right-hand side of the former inequality is closer to the optimal lower bound on $\eta(\mathcal{I}(G))$. We can show this to be the case when $\eta(\mathcal{I}(G)) = 1$. In this case, G must contain a complete bipartite graph on all its vertices, say with parts A and B . Then we must add edges on this $K_{A,B}$ so that $G[A]$ and $G[B]$ have no independent set of size k . Among all graphs on a given number of vertices with no independent set of size k , it is an easy consequence of Turán's theorem that the complement of the Turán graph (i.e., a disjoint union of equitably sized cliques) has the minimum possible maximum degree. Thus, G must have maximum degree at least that of the graph obtained by adding equitably sized cliques on both sides of a complete bipartite graph, which is close to our construction.

We also remark that the graphs with no IT we considered above are line graphs of $(k - 1)$ -uniform multi-hypergraphs with no full rainbow matchings that we will describe in Section 6.2. Indeed, this connection is how we discovered these counterexamples. These graphs were also described in [5] but for a different problem. Thus, while the conjecture Aharoni, Alon, and Berger [5] holds for line graphs of $(k - 1)$ -uniform linear hypergraphs, it does not generally hold for line graphs of $(k - 1)$ -uniform multi-hypergraphs.

Next, we address a question of Aharoni, Holzman, Howard, and Sprüssel [16]. Recall that these authors showed that for $\Delta \neq 2$, if G has maximum degree Δ and vertex partition $\mathcal{P} = \{V_1, \dots, V_n\}$ has $|V_i| \geq 2\Delta - 1$ and there is no IT, then G contains the disjoint union of $2\Delta - 1$ copies of $K_{\Delta,\Delta}$ (Theorem 5.2). These authors asked whether, additionally, there exists a $K_{\Delta,\Delta}$ component contained in the union of two classes V_i, V_j . We give a construction that demonstrates a negative answer to their question. On the other hand, Theorem 5.15 in Section 5.6 implies that their question does hold when the graph G is exactly the disjoint union of $2\Delta - 1$ copies of $K_{\Delta,\Delta}$, as in Theorem 1.22 (see Corollary 5.16). This explains why all previous constructions [110, 161, 178] for Theorem 1.22 contain such a special $K_{\Delta,\Delta}$.

Using Lemma 5.1, it is straightforward to produce counterexamples to the question of Aharoni, Holzman, Howard, and Sprüssel [16]. For example, start with the Szabó-Tardos construction $G_0, \mathcal{P}_0$ for Theorem 1.22, which has $2\Delta - 1$ copies of $K_{\Delta,\Delta}$. Fix any class $U \in \mathcal{P}_0$,

and note that there is exactly one component K of G_0 that intersects U at Δ vertices. We add a copy of the complete bipartite graph $K_{\Delta, \Delta-1}$ with the standard bipartition $\{A, B\}$, where $|A| = \Delta$ and $|B| = \Delta - 1$. We distribute $\Delta - 1$ vertices of $U \cap K$ into A , and distribute the remaining Δ vertices of U into B . We do this for each of the 2Δ classes of $\mathcal{P}_0$ successively. Let $G, \mathcal{P}$ be the final vertex-partitioned graph. Then G has no IT with respect to $\mathcal{P}$ by Lemma 5.1, every class $V_i \in \mathcal{P}$ has size $|V_i| = 2\Delta - 1$, and there is no $K_{\Delta, \Delta}$ component of G contained in the union of two classes: every $K_{\Delta, \Delta}$ component in G lies in G_0 , and each of them is now separated into at least four classes.

In the described counterexample, although there is no $K_{\Delta, \Delta}$ component contained in the union of two classes, there are many $K_{\Delta, \Delta-1}$ components contained in the union of two classes. Indeed, it is a feature of our construction method that whenever we apply Lemma 5.1 by adding a complete bipartite component with the standard bipartition, this added complete bipartite component will still be contained in the union of two classes. We could artificially break this property in our example by adding a new class U of size 2Δ , and attaching each vertex of U to a unique $K_{\Delta, \Delta-1}$ component to make it a $K_{\Delta, \Delta}$ component. In this new example, every class has size at least $2\Delta - 1$, there is no IT, and no component of the graph is contained in the union of two classes. Of course, by adding the class U we violate the class-minimality condition (a) required to apply Theorem 5.15. See Section 5.6.

5.6 On disjoint unions of complete bipartite graphs

In the last section, we answered negatively a question of Aharoni, Holzman, Howard, and Sprüssel [16], who asked whether for every graph G with maximum degree $\Delta \neq 2$ and partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$ such that $|V_i| \geq 2\Delta - 1$ for every i and there is no IT, there exists a component $K_{\Delta, \Delta}$ in G contained in the union of two classes of $\mathcal{P}$. In this section, we show that a variation of their question does hold, namely Theorem 5.15 below. Our main motivation for this variation is that it implies Theorem 1.23, which states that every construction $G, \mathcal{P}$ for Theorem 1.22 of Szabó and Tardos [161] can be derived by iteratively applying Lemma 5.1 on $K_{\Delta, \Delta}$.

Let G be a graph and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. We say that G is *class-minimal with no IT* if G has no IT with respect to $\mathcal{P}$ but $G - V_i$ has an IT with respect to $\mathcal{P} \setminus \{V_i\}$, for every $i \in [n]$. We will prove that the following variation of the question of Aharoni, Holzman, Howard, and Sprüssel [16] does hold.

Theorem 5.15. *Let G be a graph and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$, where $n \geq 2$. Suppose that*

- (a) *G is class-minimal with no IT,*
- (b) *G is the disjoint union of complete bipartite graphs, and*
- (c) *the number of components of G is $n - 1$.*

Then there exist a component $K_{A,B}$ of G and classes V_i, V_j of $\mathcal{P}$ such that $A \subseteq V_i$ and $B \subseteq V_j$.

The proof of Theorem 5.15 is somewhat technical and deferred to Subsection 5.6.2. Here, we note one immediate consequence of Theorem 5.15, namely that it implies a positive answer for the question of Aharoni, Holzman, Howard, and Sprüssel [16] when the graph G is exactly the disjoint union of $2\Delta - 1$ copies of $K_{\Delta,\Delta}$, as in Theorem 1.22.

Corollary 5.16. *For every construction $G, \mathcal{P}$ for Theorem 1.22, there exists a component $K_{\Delta,\Delta}$ of G that is contained in the union of two classes of $\mathcal{P}$.*

Proof. Conditions (b) and (c) of Theorem 5.15 are satisfied by hypothesis. Condition (a) of Theorem 5.15 is satisfied by Theorem 5.2 when $\Delta \neq 2$, and by Theorem 5.3 when $\Delta = 2$. $\square$

5.6.1 Characterization of minimal graphs

In this subsection, we show how Theorem 5.15 implies Theorem 1.23. More generally, we prove Theorem 5.18, which states that all possible vertex-partitioned graphs $G, \mathcal{P}$ satisfying conditions (a), (b), (c) of Theorem 5.15 can be derived by iteratively applying Lemma 5.1.

Graphs that are class-minimal with no IT were used in previous works [13, 31, 99] to give combinatorial proofs of Theorem 1.21 and to obtain information about dominating sets in such graphs. In the lemma below, we observe that Lemma 5.1 preserves the minimality properties of the vertex-partitioned graphs G and H involved.

Lemma 5.17. *Let graphs $G, H, G \cup H$ and vertex partitions $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ be as in Lemma 5.1 (not assuming they have no IT's). Then with respect to the appropriate partitions,*

- (i) *if G and H are both class-minimal with no IT, then so is $G \cup H$;*
- (ii) *if $G \cup H$ and H are both class-minimal with no IT, then so is G ;*
- (iii) *if $G \cup H$ and G are both class-minimal with no IT, then so is H .*

Proof. We only prove (iii), as the proofs of (i) and (ii) are similar. Recall that $\mathcal{P} = \{V_1, \dots, V_n\}$, $\mathcal{Q} = \{W_1, \dots, W_m\}$, and $\mathcal{R} = \{V'_1, \dots, V'_n, W_1, \dots, W_{m-1}\}$. Suppose for contradiction that H has an IT $\{w_1, \dots, w_m\}$ with respect to $\mathcal{Q}$, where $w_j \in W_j$ for every $1 \leq j \leq m$. Suppose that $w_s \in V'_i$, where $1 \leq i \leq n$. Since G is class-minimal with no IT, there exists an IT $\{v_1, \dots, v_{i-1}, v_{i+1}, \dots, v_n\}$ of $G - V_i$ with respect to $\mathcal{P} \setminus \{V_i\}$. Then $\{v_1, \dots, v_{i-1}, w_j, v_{i+1}, \dots, v_n, w_1, \dots, w_{m-1}\}$ is an IT of $G \cup H$ with respect to $\mathcal{R}$, a contradiction. To prove the class-minimality of H , first note that by the class-minimality of $G \cup H$, for every $1 \leq j \leq m-1$, $(G \cup H) - W_j$ has an IT $\{v_1, \dots, v_n, w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_{m-1}\}$ with respect to $\mathcal{R} \setminus \{W_j\}$. Since G has no IT with respect to $\mathcal{P}$, we have that $v_i \in W_m$ for some $1 \leq i \leq n$. Then $\{w_1, \dots, w_{j-1}, w_{j+1}, \dots, w_{m-1}, v_i\}$ is an IT of $H - W_j$ with respect to $\mathcal{Q} \setminus \{W_j\}$, as required. To check for W_m , note again by the class-minimality of

$G \cup H$, $(G \cup H) - V'_1$ has an IT $\{v_2, \dots, v_n, w_1, \dots, w_{m-1}\}$ with respect to $\mathcal{R} \setminus \{V'_1\}$. Then $\{w_1, \dots, w_{m-1}\}$ is an IT of $H - W_m$ with respect to $\mathcal{Q} \setminus \{W_m\}$. Therefore, H is class-minimal with no IT. $\square$

Since $K_{\Delta, \Delta}$ with its standard bipartition $\mathcal{P}$ is class-minimal with no IT, Lemma 5.17 implies that all the constructions we presented for Theorem 1.22 are class-minimal with no IT. (This fact also follows from Theorem 5.2.) All of the other constructions in this chapter are also class-minimal with no IT. Now we use Theorem 5.15 to prove the main result of this subsection, stated below.

Theorem 5.18. *Let G be a graph and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$, where $n \geq 2$. Suppose that*

- (a) *G is class-minimal with no IT,*
- (b) *G is the disjoint union of complete bipartite graphs, and*
- (c) *the number of components of G is $n - 1$.*

Then $G, \mathcal{P}$ can be constructed via Lemma 5.1, by iteratively adding complete bipartite graphs with the standard bipartition.

Proof. We use induction on n . The case $n = 2$ is just our assumed starting point: G is a complete bipartite graph and $\mathcal{P}$ is its standard bipartition. Suppose that $n \geq 3$. By Theorem 5.15, there exist a component $K_{A,B}$ of G and classes $V_{n-1}, V_n \in \mathcal{P}$ (after relabeling) such that $A \subseteq V_{n-1}$ and $B \subseteq V_n$. Let $G' = G - (A \cup B)$ and let $\mathcal{P}' = \{V_1, \dots, V_{n-2}, V'_{n-1}\}$, where $V'_{n-1} = (V_{n-1} \cup V_n) - (A \cup B)$.

Note that $V'_{n-1} \neq \emptyset$, as otherwise $A = V_{n-1}$ and $B = V_n$, and since $K_{A,B}$ has no IT with respect to $\{V_{n-1}, V_n\}$, this contradicts the assumption that G is class-minimal with no IT. Thus, $\mathcal{P}'$ is a partition of $V(G')$. Observe that $G = G' \cup K_{A,B}$, and that $\mathcal{P}$ is obtained from $\mathcal{P}'$ and $\{A, B\}$ by distributing every vertex of V'_{n-1} into either A or B . That is, G can be constructed from G' and $K_{A,B}$ by applying Lemma 5.1. Since G and $K_{A,B}$ are both class-minimal with no IT, by Lemma 5.17(iii) G' is also class-minimal with no IT. Moreover, G' is the disjoint union of $n - 2$ complete bipartite graphs. By induction, it follows that $G', \mathcal{P}'$ can be constructed via Lemma 5.1, by iteratively adding complete bipartite graphs with the standard bipartition. We conclude that the same is true for $G, \mathcal{P}$. $\square$

Now we derive Theorem 1.23 as an immediate consequence of Theorem 5.18, which was our main goal.

Theorem 1.23. *All possible constructions $G, \mathcal{P}$ for Theorem 1.22 can be derived via Lemma 1.20, by iteratively adding $K_{\Delta, \Delta}$ with the standard bipartition.*

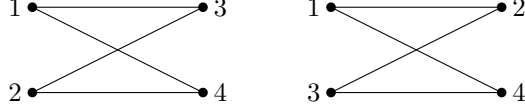


Figure 5.3: An example of a vertex-partitioned graph satisfying conditions (a) and (b) of Theorem 5.15 and Theorem 5.18 but not satisfying (c) or the conclusion of either theorem.

Proof. Conditions (b) and (c) of Theorem 5.18 are satisfied by hypothesis. Condition (a) of Theorem 5.15 is satisfied by Theorem 5.2 when $\Delta \neq 2$, and by Theorem 5.3 when $\Delta = 2$. $\square$

We add some remarks. First, we observe that the converse of Theorem 5.18 also holds: If $G, \mathcal{P}$ is constructed via Lemma 5.1 by iteratively adding complete bipartite graphs with the standard bipartition, then conditions (a), (b), (c) of Theorem 5.15 are satisfied. This is proved by applying Lemma 5.17(i) iteratively. Also, we note that condition (c) of Theorem 5.15 and Theorem 5.18 is necessary. Figure 5.3 shows an example of a vertex-partitioned graph where conditions (a) and (b) are satisfied but not (c), and the graph does not satisfy the conclusion of either theorem. Conditions (a) and (b) of the theorems are more easily observed to be necessary.

5.6.2 Proof of structural result

In this subsection, we prove Theorem 5.15. Our technical proof adapts combinatorial methods from [102, 104]. We will follow the same argument as [102] for the construction of feasible pairs, but the sequence of claims after Claim 5.1 are more specific to our setting.

Let graph G and vertex partition $\mathcal{P} = \{V_1, \dots, V_n\}$ satisfy conditions (a), (b), (c) of Theorem 5.15. A *partial* independent transversal of G with respect to $\mathcal{P}$ is an independent set of G containing at most one vertex from each class $V_i \in \mathcal{P}$. Let $\mathcal{T}$ be the set of all partial independent transversals of G . For a partial independent transversal $T \in \mathcal{T}$ and a vertex $w \in V(G) \setminus T$, let $C(w, T)$ denote the vertex set of the component of $G[\{w\} \cup T]$ containing w . Then $G[C(w, T)]$ is a star with center w . A class V_i is *active* for a subset $I \subseteq V(G)$ if $V_i \cap I \neq \emptyset$. Let $S(I)$ denote the set of active classes for I . The *class graph* $\mathcal{G}_I$ of I is obtained from $G[I]$ by contracting, for each $V_i \in S(I)$, the vertices of $V_i \cap I$ into a single vertex. Thus the vertex set of $\mathcal{G}_I$ is $S(I)$. A *non-trivial* star is a star with at least two vertices. We say that a pair (I, T) is *feasible* if

- (i) $I \subseteq V(G)$ and $T \in \mathcal{T}$ is a partial independent transversal of maximum size;
- (ii) $G[I]$ is a forest whose components are the $|W|$ non-trivial vertex-disjoint stars $C(w, T)$, for $w \in W = I \setminus T$; and
- (iii) the class graph $\mathcal{G}_I$ is a tree on vertex set $S(I)$.

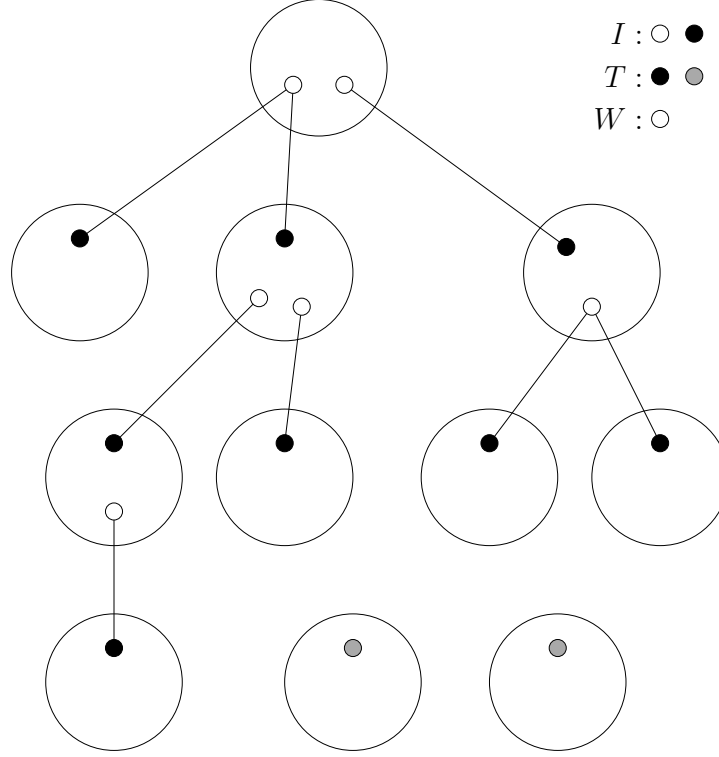


Figure 5.4: A feasible pair (I, T) .

See Figure 5.4 for an illustration of a feasible pair. Note that $(\emptyset, T)$ is a feasible pair for any $T \in \mathcal{T}$. Our goal is to construct a feasible pair (I, T) that is an *induced matching configuration* (IMC), meaning that $G[I]$ is a perfect matching and $\mathcal{G}_I$ is a tree on n vertices (meaning that every class of $\mathcal{P}$ is active). See the left side of Figure 5.5 for an illustration of an IMC. IMC's are the main structures that allow us to prove Theorem 5.15.

To construct an IMC, we follow the same algorithm as in [102], Section 2. Start by fixing a class $V_i \in \mathcal{P}$. By condition (a) of Theorem 5.15, there exists a maximum partial independent transversal $T_0 \in \mathcal{T}$ that intersects every class of $\mathcal{P}$ except V_i . Our initial pair is $(\emptyset, T_0)$. Suppose that we currently have the pair (I, T) . If $I = \emptyset$, we let w be any vertex in V_i . Otherwise, suppose there exists a vertex $w \in \bigcup_{V_j \in S(I)} V_j$ (in one of the active classes) that is not in I and is not adjacent to any vertex in I . Let $T' \in \mathcal{T}$ be a maximum partial independent transversal that, subject to the conditions that T' agrees with T inside $S(I)$ and also satisfies $(T' \setminus I) \cap N_G(I) = \emptyset$, minimizes the degree $\deg_{T'}(w)$ of w into T' . Then we update the pair (I, T) to $(I \cup C(w, T'), T')$. We iterate until there are no more choices of vertices w as above. Note that I grows by at least one vertex at each iteration, so the procedure terminates. The following claim is proved in [102]. We reprove it for completeness.

Claim 5.1. *A feasible pair (I, T) is maintained throughout the above procedure.*

Proof. First recall that $(\emptyset, T_1)$ is a feasible pair. Assuming the feasibility of a pair (I, T) as

well as all previous pairs, we verify that the pair $(I', T') = (I \cup C(w, T'), T')$ is also feasible.

For $I = \emptyset$, we need to show that $(C(w, T'), T')$ is a feasible pair, having chosen $w \in V_1$. Conditions (i) and (iii) are immediate. For condition (ii), note that if $C(w, T')$ is a single vertex, then $T' \cup \{w\}$ is an IT of G with respect to $\mathcal{P}$, contradicting that G has no IT.

Suppose that $I \neq \emptyset$. Condition (i) for (I', T') is immediate. For condition (ii), first note that $C(v, T') = C(v, T)$ for $v \in W = I \setminus T$. Since (I, T) is feasible, these sets are pairwise disjoint and have at least two vertices. The last added set $C(w, T')$ is disjoint from any $C(v, T')$ because w is independent of I and T' agrees with T on the active classes for I .

To show that $C(w, T')$ has at least two vertices, suppose not. That is, suppose that w has degree zero into T' . Then $w \notin V_i$, as otherwise $T' \cup \{w\}$ would be an IT of G with respect to $\mathcal{P}$, contradicting that G has no IT. Let V_i be the class of $\mathcal{P}$ containing w , and let v be the unique vertex in $V_j \cap T'$. Then V_j is an active class by our choice of w , and because T and T' agree on active class, we have $v \in V_j \cap T$. Then $v \in I \cap T$ by construction, and thus v has exactly one neighbor in $G[I]$ by (ii). Call this neighbor $u \in W$, let (I_u, T_u) be the feasible pair right before we considered the vertex u in the procedure, and let (I'_u, T'_u) be the feasible pair right after. Then $T'' = T' - \{v\} \cup \{w\} \in \mathcal{T}$ is a maximum partial independent transversal that agrees with T' outside V_i , and thus it agrees with T_u on $S(I_u)$ by induction. Moreover, $(T'' \setminus I_u) \cap N_G(I_u) = (T'_u \setminus I_u) \cap N_G(I_u) = \emptyset$ by this agreement, and $\deg_{T''}(u) = \deg_{T'}(u) - 1 = \deg_{T'_u}(u) - 1$ by induction. Therefore, at the iteration when we considered the vertex u , the choice $T'_u \in \mathcal{T}$ did not minimize the degree $\deg_{T'_u}(u)$, which gives the desired contradiction.

For condition (iii), notice that w is in an active class for I and its neighbors in T' are in non-active class, so $\mathcal{G}_{I'}$ is obtained from $\mathcal{G}_I$ by simply adding $\deg_{T'}(w)$ leaves. $\square$

Now let (I, T) be the feasible pair once the procedure terminates. We prove a series of claims showing that (I, T) is an IMC. Let $U(I) = \bigcup_{V_j \in S(I)} V_j$ be the set of vertices that lie in an active class for I .

Claim 5.2. *Every vertex in $U(I)$ lies in a component of G intersecting I .*

Proof. If $v \in U(I)$ and v is in a component not intersecting I , then $v \notin I$ and v is independent of I , so we could have continued our procedure with the vertex v , contradicting that (I, T) is the feasible pair once the procedure terminated. $\square$

Claim 5.3. *Every vertex in $T \setminus I$ lies in a component of G not intersecting I .*

Proof. Suppose that vertex $v \in T \setminus I$ is in a component of G intersecting I . Because every component of G is a complete bipartite graph (condition (b) of Theorem 5.15), v is adjacent to a vertex in either $I \cap T$ or $I \setminus T$ (using condition (ii) of feasible pairs), so in particular v is adjacent to a vertex in I . Then $(T \setminus I) \cap N_G(I) \neq \emptyset$, which contradicts our assumption on how $T \in \mathcal{T}$ was chosen. $\square$

Claim 5.4. $G[U(I)]$ has no IT with respect to $S(I)$.

Proof. Suppose that $G[U(I)]$ has an IT T' with respect to $S(I)$. Then $T' \cup (T \setminus I)$ is a transversal of $\mathcal{P}$, and it is independent in G by Claim 5.2 and Claim 5.3. Thus $T' \cup (T \setminus I)$ is an IT of G with respect to $\mathcal{P}$, contradicting condition (a) of Theorem 5.15. $\square$

Claim 5.5. We have $S(I) = \mathcal{P}$, and thus $U(I) = V(G)$ and $T \subseteq I$.

Proof. Because G is class-minimal with no IT by condition (a) of Theorem 5.15, Claim 5.4 implies that $S(I) = \mathcal{P}$. $\square$

Claim 5.6. Every vertex $w \in I \setminus T$ has degree $\deg_T(w) = 1$ into T .

Proof. By Claim 5.2 and Claim 5.5, every component of G intersects I , and by construction every such component contains exactly one vertex from $I \setminus T$. Therefore,

$$n - 1 = |T| = \sum_{w \in I \setminus T} \deg_T(w) \geq \sum_{w \in I \setminus T} 1 = |I \setminus T| = \# \text{ components of } G = n - 1,$$

where $\deg_T(w) \geq 1$ by condition (ii) of feasible pairs, and the last equality is by condition (c) of Theorem 5.15. Hence, $\deg_T(w) = 1$ for every $w \in I \setminus T$. $\square$

Claim 5.7. (I, T) is an IMC.

Proof. By condition (ii) of feasible pairs and Claim 5.6, every component of $G[I]$ is a 2-vertex star, i.e., a single edge, meaning that $G[I]$ is a perfect matching. By Claim 5.5, every class of $\mathcal{P}$ is active and thus $\mathcal{G}_I$ is a tree on vertex set $\mathcal{P}$. $\square$

We have shown that we can construct an IMC (I, T) with $V_i \cap T = \emptyset$ for any initial class $V_i \in \mathcal{P}$. Now we observe additional facts that we will utilize in our proof of Theorem 5.15.

Claim 5.8. Every vertex of G has exactly one neighbor in I .

Proof. By Claim 5.2 and Claim 5.5, every vertex $v \in V(G)$ lies in a component intersecting I . Since $G[I]$ is a perfect matching (by Claim 5.7) and every component of G is a complete bipartite graph (by condition (b) of Theorem 5.15), it follows that v is adjacent to exactly one vertex in I . $\square$

For the remaining part of the proof, we need some additional terminology. Let (I, T) be an IMC with $V_i \cap T = \emptyset$. We view the class graph $\mathcal{G}_I$ as a rooted tree whose root is V_i with respect to T , and every other class is a descendant of V_i . For a class V_j and vertex $w \in V_j \cap (I \setminus T)$, the w -child of V_j in $\mathcal{G}_I$ is the class V_k containing the unique neighbor of w in T . We say that a class V_k is a w -descendant of V_j if there is a sequence of classes $V_j = U_1, U_2, \dots, U_{\ell-1}, U_\ell = V_k$ together with a sequence of vertices $w = w_1, w_2, \dots, w_{\ell-1}, w_\ell = v_\ell$ such that $w_m \in U_m \cap (I \setminus T)$, $v_m \in U_m \cap T$, and v_m is the unique neighbor of w_{m-1} in T ,

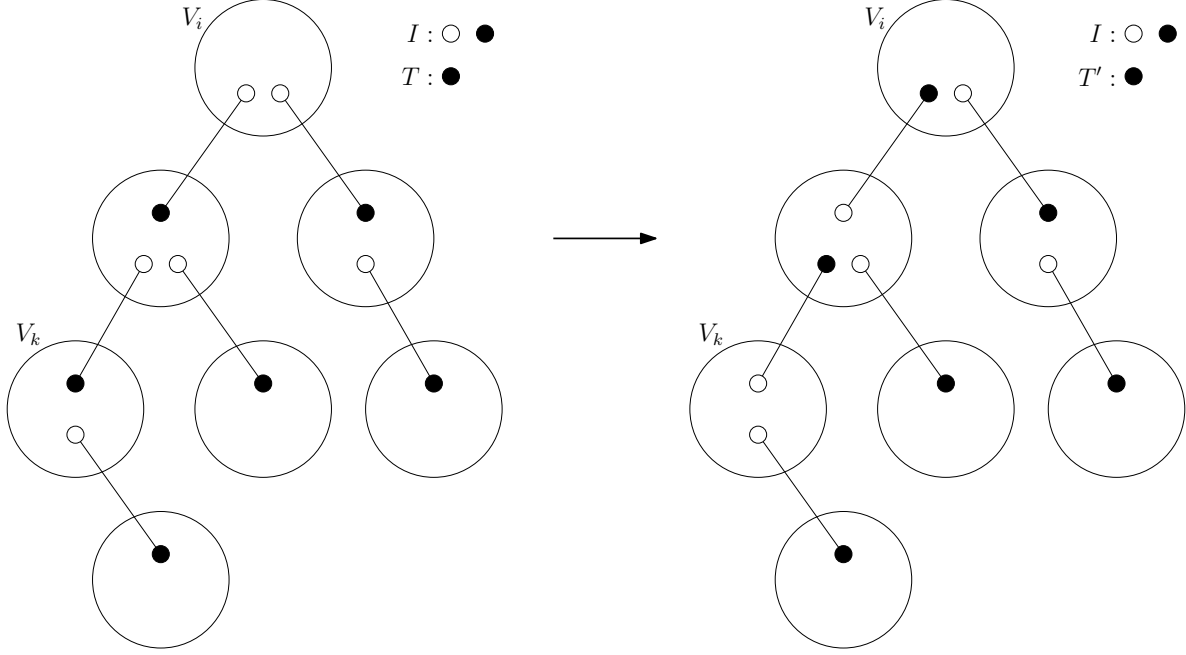


Figure 5.5: Switching along the path from V_i to V_k to obtain T' from T in the IMC (I, T) .

for every m . In particular, each U_m is the w_{m-1} -child of U_{m-1} . The sequence of classes $U_1, \dots, U_\ell$ is the unique path from V_j to V_k in the class graph $\mathcal{G}_I$.

When $V_j = V_i$ is the root of $\mathcal{G}_I$ with respect to T , the set $T' = T \cup \{w_1, \dots, w_{\ell-1}\} - \{v_2, \dots, v_\ell\}$ is another maximum partial independent transversal. This is an application of the fact that $G[I]$ is a perfect matching. Moreover, (I, T') is an IMC with root V_k (as $V_k \cap T' = \emptyset$). We say that T' is obtained from T by *switching* along the path from V_i to V_k in $\mathcal{G}_I$. See Figure 5.5 for illustration.

Claim 5.9. *For every vertex $w \in I \setminus T$, say $w \in V_j$, every neighbor of w in G lies in a class that is a w -descendant of V_j in $\mathcal{G}_I$.*

Proof. Suppose for contradiction that w has a neighbor x in a class V_k that is not a w -descendant of V_j in $\mathcal{G}_I$. Let T' be the maximum partial independent transversal obtained from T by switching along the path from V_i to V_k in $\mathcal{G}_I$, so that $T' \cap V_k = \emptyset$. Because V_k is not a w -descendant of V_j , we have that $w \in I \setminus T'$. Then $T' \cup \{x\}$ is independent in G because $w \in I \setminus T'$ and w is the unique neighbor of x in I (applying Claim 5.8). Since $x \in V_k$, it follows that $T' \cup \{x\}$ is an IT of G , a contradiction. $\square$

We now prove the following lemma, from which we easily derive Theorem 5.15.

Lemma 5.19. *Let graph G and partition $\mathcal{P} = \{V_1, \dots, V_n\}$ satisfy conditions (a), (b), (c) of Theorem 5.15. Then for every class V_i , there exists a component $K_{A,B}$ of G such that $A \subseteq V_i$.*

Proof. Let V_i be any class of $\mathcal{P}$. We wish to find a vertex $v \in V(G)$ such that $N_G(v) \subseteq V_i$. If we succeed, then because every component of G is a complete bipartite graph (by condition (b)), it follows that $N_G(v)$ is a part A of some complete bipartite component $K_{A,B}$ of G , and then $A \subseteq V_i$ as required.

Applying conditions (a), (b), (c) as we have done, let (I, T) be an IMC such that $V_i \cap T = \emptyset$, i.e., V_i is a root of the class graph $\mathcal{G}_I$ with respect to T . First suppose that V_i is a leaf of $\mathcal{G}_I$ (allowing the root to be considered a leaf). Let V_j be the unique neighbor of V_i in $\mathcal{G}_I$. Let $T' = T \cup \{w\} - \{v\}$ be obtained from T by switching along the path from V_i to V_j , where $w \in V_i \cap (I \setminus T)$ and $v \in V_j \cap T$. Now $\mathcal{G}_I$ has root V_j with respect to T' . Then the class V_i is the unique v -descendant of V_j in $\mathcal{G}_I$, so by Claim 5.9 we have $N_G(v) \subseteq V_i$, as we wanted.

Now suppose that V_i is not a leaf of $\mathcal{G}_I$. Let $w \in V_i \cap (I \setminus T)$, let V_j be the w -child of V_i , and let $v \in V_j \cap T$ be the neighbor of w in T . If $N_G(v) \subseteq V_i$, then we are done. Otherwise, let $x \in N_G(v) \setminus V_i$. Note that $x \notin V_j$, as otherwise $T - \{v\} \cup \{w, x\}$ is an IT of G with respect to $\mathcal{P}$ (using condition (b)), which contradicts condition (a) that there is no IT. Let $I' = I - \{w\} \cup \{x\}$. We claim that (I', T) is another IMC. To see this, first observe that v is the unique neighbor of x in I by Claim 5.8, which implies that $G[I']$ is still a perfect matching. Also, let T'' be obtained from T by switching along the single-edge path from V_i to V_j in $\mathcal{G}_I$. Then in the IMC (I, T'') , the vertex x lies in a class that is a v -descendant of V_j , by Claim 5.9. From this we conclude that the new class graph $\mathcal{G}_{I'}$ is still a tree on n vertices, rather than the disjoint union of a unicyclic component and a tree. We deduce that (I', T) is still an IMC.

Now, V_i has degree one less in $\mathcal{G}_{I'}$ than it had in $\mathcal{G}_I$. Therefore, applying the above procedure iteratively, we produce a sequence of IMC's where the degree of V_i decreases by one at each step. Thus before or at the base case where V_i is a leaf, we will find a desired vertex $v \in V(G)$ such that $N_G(v) \subseteq V_i$. $\square$

Proof of Theorem 5.15. By Lemma 5.19, each of the n classes in $\mathcal{P}$ covers one side of one out of $n - 1$ complete bipartite components of G . Therefore, by the pigeonhole principle, there exist two distinct class V_i and V_j and a single component $K_{A,B}$ such that $A \subseteq V_i$ and $B \subseteq V_j$. $\square$

5.7 Problems

We conclude this chapter with some open problems for future work.

(1) We gave constructions of locally sparse graphs with no IT in Theorem 1.25, but the optimal class size conditions are still unknown. Given integers $\Delta, m \geq 1$, let $f(\Delta, m)$ denote the minimum integer $t \geq 1$ such that a graph G has an IT with respect to a partition $\mathcal{P} = \{V_1, \dots, V_n\}$ whenever G has maximum degree Δ , $G, \mathcal{P}$ has local degree m , and $|V_i| \geq t$ for every i . Let $g(d, m)$ denote the same with “local degree” replaced by “multiplicity”. We have $f(d, m) \geq g(d, m)$. Theorem 1.25 gives an explicit lower bound on $g(\Delta, m)$, while

Loh and Sudakov [127] proved that $f(\Delta, o(\Delta)) = \Delta + o(\Delta)$. Besides the hard problems of determining $f(\Delta, m)$ and $g(\Delta, m)$, even the question of whether $f(\Delta, 1) \leq \Delta + c$ for some universal constant c is still open. We also ask how significantly $f(\Delta, m)$ and $g(\Delta, m)$ differ from each other, particularly when $m = o(\Delta)$.

(2) We can consider the analogous problem for list cover graphs. Let $g(\Delta)$ be the minimum integer $t \geq 1$ such that a graph G has an IT with respect to a partition $\mathcal{P} = \{V_1, \dots, V_n\}$ whenever $G, \mathcal{P}$ is a list cover graph with maximum degree Δ and $|V_i| \geq t$ for every i . Then $g(\Delta) \leq g(\Delta, 1) = \Delta + o(\Delta)$, while Theorem 5.14 says that $g(\Delta) \geq \Delta + 2$ for every $\Delta \geq 2$. Although we have simplified the original construction of Bohman and Holzman [36] for Theorem 5.14, we were unsuccessful in improving their lower bound $\Delta + 2$. As above, it is open even whether $g(\Delta) \leq \Delta + c$ for some universal constant c .

(3) In Section 5.5, we disproved a conjecture of Aharoni, Alon, and Berger [5] about the topological connectedness of $K_{1,k}$ -free graphs (and thus the existence of IT's in such graphs). We showed that $\eta(\mathcal{I}(G)) \leq \frac{k|V(G)|}{(2k-2)\Delta+k-2}$ for some $K_{1,k}$ -free graphs G . On the other hand, the above authors proved that $\eta(\mathcal{I}(G)) \geq \frac{(k-1)|V(G)|}{(2k-3)\Delta+k-1}$ for all $K_{1,k}$ -free graphs G . We explained that our lower bound is close to being tight when $\eta(\mathcal{I}(G)) = 1$, but the optimal bound is still open in general.

(4) In Section 5.5, we answered negatively a question of Aharoni, Holzman, Howard, and Sprüssel [16], who asked whether $|V_i| \geq 2\Delta - 1$ for every i and no IT guarantees the existence of a $K_{\Delta,\Delta}$ component contained in the union of two classes V_i, V_j . We gave an example where there is no $K_{\Delta,\Delta}$ component contained in the union of two classes, but our construction still had many copies of $K_{\Delta-1,\Delta}$ contained in the union of two classes. This leaves open the question of whether there necessarily always exists a large complete bipartite subgraph (containing say $\Omega(\Delta^2)$ edges) contained in the union of two classes. This is related to a question of Loh and Sudakov [127], who asked whether their asymptotic condition in Theorem 5.11 holds under the weaker locally sparse condition that the number of edges between any two classes is $o(\Delta^2)$.

(5) We proved Theorem 1.23 stating that every construction $G, \mathcal{P}$ for Theorem 1.22 (where the graphs are disjoint unions of $2\Delta - 1$ copies of $K_{\Delta,\Delta}$) can be derived by iteratively applying Lemma 5.1. We actually proved this in the more general setting of Theorem 5.18. It would be interesting to prove similar universality statements in other settings. For example:

- Does a similar statement hold for more general disjoint unions of complete multipartite graphs?
- Can every construction $G, \mathcal{P}$ for Theorem 1.24 (where the graphs are disjoint unions of 3 cycles each of length 1 (mod 3)) be obtained by iteratively applying Lemma 5.5 as well as Lemma 5.1, as in Section 5.1.2?
- Does the conclusion of Theorem 5.18 hold if we replace conditions (b) and (c) of the theorem by the condition (b') that G is *edge-minimal with no IT*, meaning that $G, \mathcal{P}$ has no IT but $G - e, \mathcal{P}$ has an IT for every edge $e \in E(G)$?

Chapter 6

Constructing multi-hypergraphs with no full rainbow matchings

In this chapter, we adapt our iterative construction method from Chapter 5 to construct edge-colored multi-hypergraphs with no full rainbow matchings. Many of the results in this chapter are analogous to results in Chapter 5. The main lemma used throughout this chapter is the following (a restatement of Lemma 1.28).

Lemma 6.1. *Let G and H be disjoint multi-hypergraphs, and let $\mathcal{P} = \{E_1, \dots, E_n\}$ and $\mathcal{Q} = \{F_1, \dots, F_m\}$ be color classes of edge-colorings of G and H , respectively. Let $\mathcal{R} = \{E'_1, \dots, E'_n, F_1, \dots, F_{m-1}\}$ be color classes of an edge-coloring of $G \cup H$, where $E'_1 \supseteq E_1, \dots, E'_n \supseteq E_n$ are obtained by distributing each of the edges of F_n into one of $E_1, \dots, E_n$ arbitrarily. If G has no full rainbow matching with respect to $\mathcal{P}$ and H has no full rainbow matching with respect to $\mathcal{Q}$, then $G \cup H$ has no full rainbow matching with respect to $\mathcal{R}$.*

We will use Lemma 6.1 to prove Theorem 1.29, Theorem 1.31, and Theorem 1.33. This chapter is based on [169].

6.1 Existence of full rainbow matchings

In this section, we briefly discuss the background on Theorem 1.29, which is restated below, as well as some related results.

Theorem 1.29 (Aharoni, Berger, Meshulam [11]). *Let G be an r -graph with maximum degree Δ , and let $\mathcal{P} = \{E_1, \dots, E_n\}$ be the color classes of an edge-coloring of G . If $|E_i| \geq r\Delta$ for every i , then there exists a full rainbow matching.*

As with independent transversals in graphs, the proof of Theorem 1.29 due to Aharoni, Berger, and Meshulam [11] uses the topological approach. Recall that a complex $\mathcal{C}$ is *homologically k -connected* if the reduced homology group $\tilde{H}_j(\mathcal{C})$ with field coefficients vanishes

for all $-1 \leq j \leq k$. The homological connectedness $\eta_H(\mathcal{C})$ is the largest integer k for which $\mathcal{C}$ is homologically k -connected, plus two. For a multi-hypergraph G , the *matching complex* $\mathcal{M}(G)$ is the collection of all matchings of G . Similar to how an independent transversal was a colorful simplex in the *independence complex* $\mathcal{I}(G)$, a full rainbow matching is a colorful simplex in the matching complex. The *line graph* of multi-hypergraph G is the graph H with vertex set $V(H) = E(G)$ and edge set $E(H)$ consisting of all pairs $\{e, e'\} \subseteq E(G)$ such that $e \cap e' \neq \emptyset$. A matching of G corresponds to an independent set of H , and a full rainbow matching of G corresponds to an independent transversal of H .

Recall Theorem 1.11 from [15, 130, 131], which is a topological Hall theorem for the existence of a colorful simplex and which is restated below for matching complexes. For $I \subseteq [n]$, we let G_I denote the edge-induced subhypergraph $G[\bigcup_{i \in I} E_i]$.

Theorem 6.2 ([15, 130, 131]). *Let G be a multi-hypergraph, and let $\mathcal{P} = \{E_1, \dots, E_n\}$ be a partition of $E(G)$. If*

$$\eta_H(\mathcal{M}(G_I)) \geq |I| \quad \text{for all } I \subseteq [n],$$

then there exists a full rainbow matching.

As with independent transversals in graphs, we deduce results about full rainbow matchings by using lower bounds on the connectedness of the matching complex of a multi-hypergraph. While we may simply apply results from Chapter 4 on independence complexes to matching complexes via line graphs, that turns out not to lead to optimal results. Instead, we use the following result of Aharoni, Berger, and Meshulam [11].

Theorem 6.3 (Aharoni, Berger, Meshulam [11]). *For every r -graph G with maximum degree Δ , we have*

$$\eta_H(\mathcal{M}(G)) \geq \frac{|E(G)|}{r\Delta}.$$

Aharoni, Berger, and Meshulam [11] do not state Theorem 6.3 explicitly, but it easily follows from their results. Here, we quickly observe that it implies the following slight stronger version of Theorem 1.29.

Corollary 6.4. *Let G be an r -graph with maximum degree Δ , and let $\mathcal{P} = \{E_1, \dots, E_n\}$ be color classes of an edge-coloring of G . If*

$$\sum_{i \in I} |E_i| \geq r\Delta(|I| - 1) + 1 \quad \text{for all } I \subseteq [n],$$

then there exists a full rainbow matching. In particular, if $|E_i| \geq r\Delta$ for every i , then there exists a full rainbow matching.

Proof. By Theorem 6.3, we have that

$$\eta_H(\mathcal{M}(G_I)) \geq \frac{|E(G_I)|}{r\Delta} \geq \frac{1}{r\Delta}(r\Delta(|I| - 1) + 1) > |I| - 1.$$

Since $\eta_H(\mathcal{M}(G_I))$ is integral, we have $\eta(\mathcal{M}(G_I)) \geq |I|$. By Theorem 6.2, there exists a full rainbow matching. $\square$

We remark that Aharoni, Berger, and Meshulam [11] actually proved a slightly stronger result than Theorem 6.3. Given a multi-hypergraph G , the *matching number* $\nu(G)$ is the cardinality of a largest matching in G . The *fractional matching number* $\nu^*(G)$ is the maximum value of $\sum_{e \in E(G)} x(e)$ over all *fractional matchings* $x : E(G) \rightarrow \mathbb{R}$, which are weight functions that satisfy $x(e) \geq 0$ for all $e \in E(G)$, and $\sum_{e \ni v} x(e) \leq 1$ for all $v \in V(G)$. Note that $\nu^*(G) \geq \nu(G)$. By doing further manipulations of vector representations of graphs, Aharoni, Berger, and Meshulam [11] proved the following.

Theorem 6.5 (Aharoni, Berger, Meshulam [11]). *For every r -graph G , we have*

$$\eta_H(\mathcal{M}(G)) \geq \frac{\nu^*(G)}{r}.$$

Observe that Theorem 6.5 implies Theorem 6.3 because $\nu^*(G) \geq \frac{|E(G)|}{\Delta}$, as $x = \frac{1}{\Delta}$ is always a fractional matching of G . From Theorem 6.5 the authors deduced the following Hall-type theorem for full rainbow matchings, using the same proof as Corollary 6.4.

Theorem 6.6 (Aharoni, Berger, Meshulam [11]). *Let G be an r -graph, and let $\mathcal{P} = \{E_1, \dots, E_n\}$ be color classes of an edge-coloring of G . If*

$$\nu^* \left(G \left[\bigcup_{i \in I} E_i \right] \right) > r(|I| - 1) \quad \text{for all } I \subseteq [n],$$

then there exists a full rainbow matching.

Theorem 6.6 verified a conjecture of Aharoni and Kessler [17]. Previously, Aharoni and Haxell [15] proved the following weaker version of Theorem 6.6.

Theorem 6.7 (Aharoni, Haxell [15]). *Let G be an r -graph, and let $\mathcal{P} = \{E_1, \dots, E_n\}$ be color classes of an edge-coloring of G . If*

$$\nu \left(G \left[\bigcup_{i \in I} E_i \right] \right) > r(|I| - 1) \quad \text{for all } I \subseteq [n],$$

then there exists a full rainbow matching.

Theorem 6.7 verified a conjecture of Aharoni [3, 17], and it was significant in part because it quickly led Aharoni [4] to prove another infamous conjecture of Ryser for uniformity $r = 3$. The original proof of Theorem 6.7 was a triangulation argument reducing to Sperner's lemma, but Meshulam [131] demonstrated that Theorem 6.7 can also be proved using Theorem 4.3.

The proof of Theorem 6.5 by Aharoni, Berger, and Meshulam [11] is somewhat technical. In summary, they first prove their main result, Theorem A.6, which gives a lower bound on the homological connectedness of the independence complex of a graph depending on the maximum Laplacian eigenvalue of the graph. This is the most technical part, involving the simplicial Hodge theorem and a local-to-global method of Garland [77]. Then they apply this main result in various ways, including to line graphs of r -graphs. For these applications, they define a notion of a *vector representation* of a graph, which is similar to geometric representations of graphs used to define the Lovász theta function. They connect vector representations to Laplacian eigenvalues, thus to homological connectedness, and then they find appropriate vector representations of line graphs of r -graphs to prove Theorem 6.5. We remark that their method also yields the following generalized version of Theorem 1.29 for not necessarily uniform multi-hypergraphs, which was originally stated in [10].

Theorem 6.8 ([10]). *Let G be a multi-hypergraph, let $\mathcal{P} = \{E_1, \dots, E_n\}$ be color classes of an edge-coloring of G , and let $\tilde{\Delta}(G) = \max_{e \in E(G)} \sum_{v \in e} d_G(v)$. If $|E_i| \geq \tilde{\Delta}(G)$ for every i , then there exists a full rainbow matching.*

Our construction method can be used to give tight constructions for other cases of Theorem 6.8, but our work sticks to the uniform setting for simplicity.

Finally, we remark that theorems and conjectures about full rainbow matchings in multi-hypergraphs can be reformulated as theorems and conjectures about matchings in bipartite multi-hypergraphs. Indeed, many of the results discussed in this chapter were originally described in such a language, and this language is convenient for some applications (see [4, 7, 53, 76]). A *bipartite multi-hypergraph* is a multi-hypergraph H whose vertex set $V(H)$ can be partitioned into two sets A and B , such that every edge of H intersects A in exactly one vertex. The goal is to find an *A-perfect matching*, which is a matching of H that covers every vertex of A . Full rainbow matchings in an r -graph G are equivalent to A -perfect matchings in a bipartite $(r + 1)$ -graph H , as follows. If G is an r -graph and $\mathcal{P} = \{E_1, \dots, E_n\}$ are the color classes of an edge-coloring of G , then let H be the $(r + 1)$ -graph with bipartition $A = \{a_1, \dots, a_n\}$, $B = V(G)$ and with edge multiset $\{e \cup \{a_i\} : e \in E_i\}$. A full rainbow matching of G with respect to $\mathcal{P}$ translates into an A -perfect matching of H . Conversely, if H is a bipartite $(r + 1)$ -graph with bipartition $A = \{a_1, \dots, a_n\}$, B , then defining edge multisets $E_i = \{e - \{a_i\} : e \in E(H), e \cap A = \{a_i\}\}$, let G be the r -graph with vertex set B and edge multiset $\bigcup_{i=1}^n E_i$, and let $\mathcal{P} = \{E_1, \dots, E_n\}$. Then an A -perfect matching of H translates into a full rainbow matching of G with respect to $\mathcal{P}$.

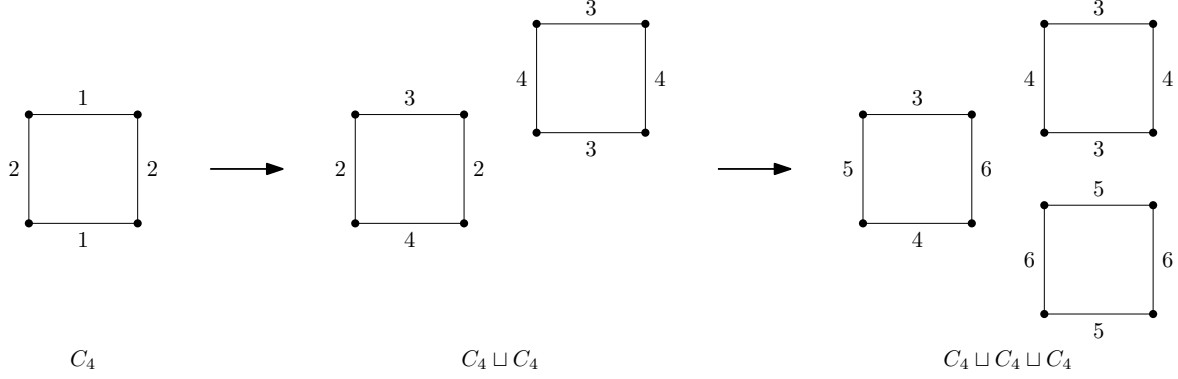


Figure 6.1: An iterative application of Lemma 6.1 to the 4-cycle graph C_4 with two color classes coming from a proper edge-coloring of C_4 . At the beginning the two color classes have size 2, and after adding a two more copies of C_4 the resulting 4 color classes have size 3. Throughout the procedure, there is no full rainbow matching.

6.2 Constructions for edge-colored multi-hypergraphs

In this section, we prove Theorem 1.30, thus giving tight constructions for Theorem 1.29 of Aharoni, Berger, and Meshulam [11]. We will also describe some related constructions in more restricted settings.

Throughout this section, we will be applying Lemma 6.1 iteratively on a fixed multi-hypergraph with no full rainbow matching, until we achieve the required class sizes. This is similar to what we did with independent transversals in Section 5.1. The following “anti-Hall theorem” quantifies how large we can eventually make the class sizes. It is a translation of Theorem 5.6 to the language of full rainbow matchings.

Theorem 6.9. *Let H be a multi-hypergraph and let $\mathcal{Q} = \{F_1, \dots, F_m\}$ be color classes of an edge-coloring of H , such that H has no full rainbow matching with respect to $\mathcal{Q}$. Suppose that integer $q \geq 1$ satisfies*

$$\sum_{i \in I} |F_i| \geq q(|I| - 1) + 1 \quad \text{for all } I \subseteq [m].$$

Then there exist an integer $k \geq 1$ and an edge-coloring of the multi-hypergraph $H^k = H \sqcup \dots \sqcup H$ (where H appears k times in the disjoint union) into color classes $\mathcal{P} = \{E_1, \dots, E_{k(m-1)+1}\}$, such that $|E_i| \geq q$ for every i and there is no full rainbow matching. Specifically, if $J = \{j : |F_j| < q\} \neq \emptyset$ and $F_J = \bigcup_{j \in J} F_j$, then we can take any

$$k \geq 1 + \sum_{j \in J} \left\lceil \frac{q - |F_j|}{|F_J| - q(|J| - 1)} \right\rceil.$$

The proof of Theorem 6.9 is the same as the proof of Theorem 5.6, rephrased in terms of full rainbow matchings. Figure 6.1 gives an illustration of how to iteratively apply Lemma

6.1. In this example, we apply Lemma 6.1 twice in succession to the 4-cycle C_4 that is properly edge-colored with 2 colors. The initial graph C_4 has 2 color classes each of size 2, and the final edge-colored graph $C_4 \sqcup C_4 \sqcup C_4$ has 4 color classes each of size 3. Notice that the line graph of this final edge-colored graph is an example of a vertex-partitioned graph with no independent transversal that has been described various times in the literature [105, 110, 161, 178], and it came up in Chapter 5.

6.2.1 General constructions

In this subsection, we prove Theorem 1.30, which is restated below.

Theorem 1.30. *For all integers $r \geq 1$ and $\Delta \geq 2$, there exist an $(r$ -partite) r -graph G with maximum degree Δ and an edge-coloring of G into color classes $\mathcal{P} = \{E_1, \dots, E_n\}$, such that $|E_i| \geq r\Delta - 1$ for every i and there is no full rainbow matching.*

Our constructions for Theorem 1.30 will be disjoint unions of blow-ups of nets, which are equivalent to sets of mutually orthogonal Latin squares. We briefly review nets and mutually orthogonal Latin squares below, as far as they are relevant for our constructions. See [66] for a more extensive overview. Our primary motivation for using nets is that their line graphs are balanced complete multipartite graphs, and as we saw in the case of independent transversals in Chapter 5, these often provide extremal constructions for transversal problems.

Fix $r \geq 1$ and $s \geq 2$. An s -net of order r , or an (r, s) -net for short, is an r -graph $H_{r,s}$ with r^2 vertices and rs edges, such that the edges $E(H_{r,s})$ can be partitioned into s perfect matchings $F_1, \dots, F_s$ each of size r , and $|e \cap f| = 1$ for all $e \in F_i$ and $f \in F_j$ with $i \neq j$. The matchings $F_1, \dots, F_s$ are called the *parallel classes* of $H_{r,s}$. For example, an $(r, 2)$ -net $H_{r,2}$ is an $r \times r$ grid, with one parallel class F_1 being r “horizontal” edges and the other parallel class F_2 being r “vertical” edges. For $r \geq 2$, the maximum integer s for which there exists some (r, s) -net is always at most $r + 1$, and if such an $(r, r + 1)$ -net exists then it is called an *affine plane of order r* . An affine plane of order r exists whenever r is a prime power, as it uniquely corresponds to a projective plane of order r .

Two Latin squares L^1 and L^2 of order r are *orthogonal* if for every pair $(x, y) \in [r] \times [r]$, there is exactly one pair $(i, j) \in [r] \times [r]$ such that $M_{ij}^1 = x$ and $M_{ij}^2 = y$. A set of Latin squares $L^1, \dots, L^n$ is said to be *mutually orthogonal* if every pair of distinct Latin squares in this set are orthogonal. Then an (r, s) -net $H_{r,s}$ is equivalent to a set of $s - 2$ mutually orthogonal Latin squares $L^1, \dots, L^{s-2}$. To see this, we start with an $(r, 2)$ -net $H_{r,2}$ with parallel classes $F_1 = \{e_1, \dots, e_r\}$ and $F_2 = \{f_1, \dots, f_r\}$, and we let v_{ij} denote the unique vertex in $e_i \cap f_j$. Assuming for $2 \leq t \leq s - 1$ that we have constructed a (r, t) -net $H_{r,t}$ with parallel classes $F_1, \dots, F_t$, we construct the r -graph $H_{r,t+1}$ by adding a parallel class F_{t+1} with edges $\{v_{ij} : L_{ij}^{t-1} = k\}$, for $1 \leq k \leq r$. The mutual orthogonality of the Latin squares $L^1, \dots, L^{t-1}$ ensures that $H_{r,t+1}$ is an $(r, t + 1)$ -net with parallel classes $F_1, \dots, F_{t+1}$. Moreover, we can reverse this process and derive the $s - 2$ mutually orthogonal Latin squares $L^1, \dots, L^{s-2}$ from the (r, s) -net $H_{r,s}$.

For the Latin squares $L^1, \dots, L^{s-2}$ above, notice that for $2 \leq t \leq s-1$, the Latin square L^{t-1} corresponds to an r -partition $V_1, \dots, V_r$ of the vertices of the r -graph $H_{r,t}$ that shows that $H_{r,t}$ is an r -partite r -graph, namely $V_k = \{v_{ij} : L_{ij}^{t-1} = k\}$ for $1 \leq k \leq r$. Conversely, any proper r -partition $V_1, \dots, V_r$ of the vertices of $H_{r,t}$ is a decomposition of each of the Latin squares $L^1, \dots, L^{t-2}$ into transversals, and this is equivalent to a Latin square orthogonal to each of $L^1, \dots, L^{t-2}$. Therefore, the r -graph $H_{r,s}$ is not r -partite if and only if the corresponding set of mutually orthogonal Latin squares $L^1, \dots, L^{s-2}$ is maximal. In this case, we will say that $H_{r,s}$ is a *maximal* net.

Now we construct examples for Theorem 1.30. Example 6.10 below gives our most basic construction that works for all integers $r \geq 1$ and $\Delta \geq 2$, but we will present a larger family of examples together with it.

Fix integers $r \geq 1$ and $s \geq 2$, and suppose that there exists an (r, s) -net $H_{r,s}$ with parallel classes $F_1, \dots, F_s$. (Note that an (r, s) -net exists for all $r \geq 1$ and $s \in \{2, 3\}$.) Naturally, we view $F_1, \dots, F_s$ as the color classes of a proper edge-coloring of $H_{r,s}$. Then every rainbow matching in $H_{r,s}$ has size at most one, since any two edges from distinct color classes intersect in exactly one vertex. Given a sequence of integers $a_1, \dots, a_s \geq 1$, let $H_{r,s}(a_1, \dots, a_s)$ denote the r -graph obtained from $H_{r,s}$ by replacing, for $1 \leq i \leq s$, every edge in the parallel class F_i by a_i parallel edges. Let $G_{r,s}(a_1, \dots, a_s)$ be the disjoint union of $s-1$ copies of $H_{r,s}(a_1, \dots, a_s)$, and for $1 \leq i \leq s$, let F'_i be the set edges of $G_{r,s}(a_1, \dots, a_s)$ that are copies of edges from F_i . Then $F'_1, \dots, F'_s$ are color classes of an edge-coloring of $G_{r,s}(a_1, \dots, a_s)$, and moreover there is no full rainbow matching, since there are s color classes on $s-1$ components, and in a given component any two edges from distinct color classes intersect in exactly one vertex.

Note that the r -graph $G_{r,s}(a_1, \dots, a_s)$ has maximum degree $\sum_{i=1}^s a_i$, and that the color class sizes are $|F'_i| = r(s-1)a_i$ for $1 \leq i \leq s$. We wish to apply Theorem 6.9 to $G_{r,s}(a_1, \dots, a_s)$ to obtain an r -graph with maximum degree Δ and color classes $E_1, \dots, E_n$, such that $|E_i| \geq r\Delta - 1$ for every i and there is no full rainbow matching. For this to work, we require that the sequence of integers $a_1, \dots, a_s \geq 1$ satisfy the following conditions:

- (1) the equality $\sum_{i=1}^s a_i = \Delta$, and
- (2) the inequalities $\sum_{i \in I} r(s-1)a_i > (r\Delta - 1)(|I| - 1)$ for all $I \subseteq [s]$.

We present various examples that satisfy these conditions, leaving out the algebraic verifications. We let $G_{r,s}$ denote the final r -graph obtained in each case.

Example 6.10. One example that works for all $r \geq 1$ and $\Delta \geq 2$ (and thus proves Theorem 1.30) is taking $s = 2$ and any two integers $a_1, a_2 \geq 1$ such that $a_1 + a_2 = \Delta$. In this case, Theorem 6.9 implies that if $G_{r,2}$ is the disjoint union of $k = r\Delta - 1$ copies of $G_{r,2}(a_1, a_2) = H_{r,2}(a_1, a_2)$, then there exists an edge-coloring of $G_{r,2}$ into $m = r\Delta$ color classes $E_1, \dots, E_{r\Delta}$ such that $|E_i| \geq r\Delta - 1$ for every i and there is no full rainbow matching. This example

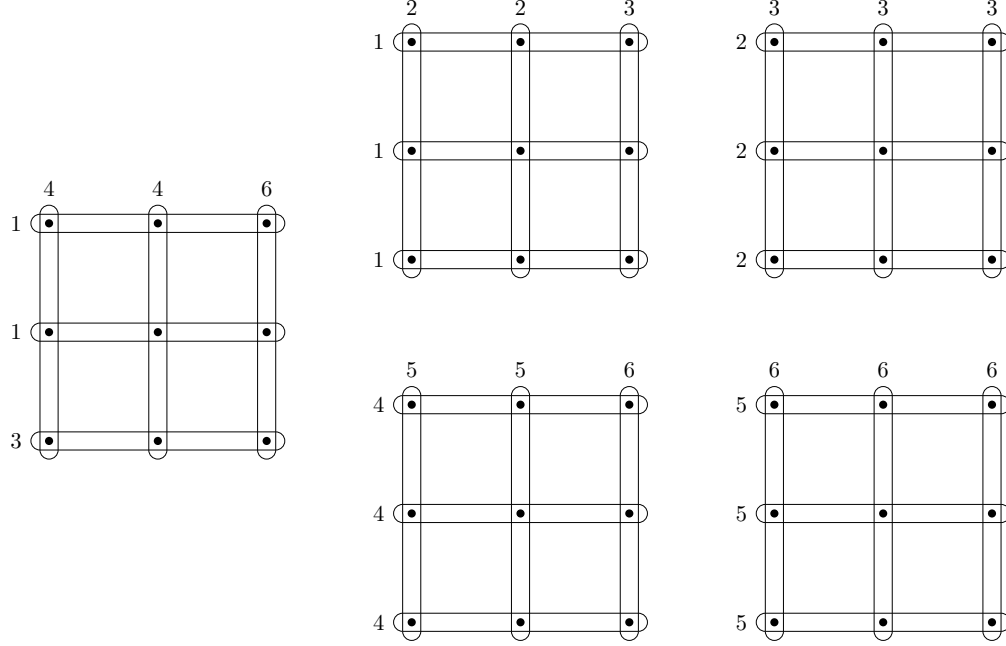


Figure 6.2: The disjoint union of five copies of the $(3,2)$ -net, i.e., the 3×3 grid, edge-colored into color classes each of size $3 \cdot 2 - 1 = 5$ with no full rainbow matching.

is always r -partite because there always exists a Latin square of order r . Note that for $r = \Delta = 2$, the graph $G_{2,2}$ is the disjoint union three 4-cycles presented from Figure 6.1. For $r = 3$ and $\Delta = 2$, the 3-graph $G_{3,2}$ and an appropriate edge-coloring is shown in Figure 6.2.

Example 6.11. For $r \geq 1$ and $s \geq 2$, suppose that an (r, s) -net $H_{r,s}$ exists. Then whenever Δ is a positive multiple of s , we can take $a_1 = \dots = a_s = \frac{\Delta}{s}$. In this case, Theorem 6.9 states that if $G_{r,s}$ is the disjoint union of $k = 1 + s \left\lceil \frac{r\Delta - s}{s(s-1)} \right\rceil$ copies of $G_{r,s}(\frac{\Delta}{s}, \dots, \frac{\Delta}{s})$, then there exists an edge-coloring of $G_{r,s}$ into $n = s + s(s-1) \left\lceil \frac{r\Delta - s}{s(s-1)} \right\rceil$ color classes $E_1, \dots, E_n$, such that $|E_i| \geq r\Delta - 1$ for every i and there is no full rainbow matching. This example $G_{r,s}$ is r -partite if and only if the net $H_{r,s}$ is not maximal.

Example 6.12. For $r \geq 1$ and $s \geq 2$, assume that an (r, s) -net $H_{r,s}$ exists. The following two examples again are either r -partite or not r -partite depending on whether the net $H_{r,s}$ is maximal.

The first example is that if $\Delta \equiv \ell \pmod{s}$, $1 \leq \ell \leq s-1$, and $\Delta > (s-1)(s-\ell) - \frac{s(s-1-\ell)}{r\ell}$, then we can take $a_1 = \dots = a_\ell = \left\lceil \frac{\Delta}{s} \right\rceil$ and $a_{\ell+1} = \dots = a_s = \left\lfloor \frac{\Delta}{s} \right\rfloor$. If $\ell = 0$ and $\Delta \geq s$, then this sequence $a_1, \dots, a_s$ is the same as in Example 6.11.

The second example is that if $\Delta - 1 \equiv \ell \pmod{s-1}$, $0 \leq \ell \leq s-2$, and $\Delta > (\ell+1)(s-1)$, then we can take $a_1 = \dots = a_{s-1} = \left\lfloor \frac{\Delta-1}{s-1} \right\rfloor$ and $a_s = \Delta - (s-1) \left\lfloor \frac{\Delta-1}{s-1} \right\rfloor$.

There are many other integer sequences $a_1, \dots, a_s \geq 1$ that satisfy conditions (1) and (2). Even more constructions are possible, for example, by taking disjoint unions of different

r -graphs described above instead of taking disjoint unions of the same one. This approach would involve carefully applying Lemma 6.1 iteratively instead of just applying Theorem 6.9 directly like we did above.

We observe that our large family of extremal constructions for Theorem 1.30 contrasts with extremal constructions for independent transversals in graphs. Recall that Haxell's Theorem [96] states that for every graph G with maximum degree Δ and vertex partition $\mathcal{P} = \{V_1, \dots, V_n\}$, if $|V_i| \geq 2\Delta$ for every i then there exists an independent transversal. We saw in Chapter 5 that we can use the analogue of Lemma 6.1 for independent transversals to show that the disjoint union of $2\Delta - 1$ copies of $K_{\Delta, \Delta}$ admits a partition $\mathcal{P} = \{V_1, \dots, V_{2\Delta}\}$ into classes of size $2\Delta - 1$ with no independent transversal, generalizing constructions by Jin [110], Yuster [178], and Szabó and Tardos [161]. Indeed, we showed that all such partitions can be derived by iteratively applying the lemma (see Theorem 1.23). Moreover, Aharoni, Holzman, Howard, and Sprüssel [16] proved that for $\Delta > 2$, the disjoint union of $2\Delta - 1$ copies of $K_{\Delta, \Delta}$ is the unique minimal graph with maximum degree Δ that admits a partition into classes of size at least $2\Delta - 1$ with no independent transversal. In contrast, our large family of examples for Theorem 1.30 indicates that there are many minimal r -graphs that admit an edge-coloring into color classes $\mathcal{P} = \{E_1, \dots, E_n\}$ such that $|E_i| \geq r\Delta - 1$ for every i and there is no full rainbow matching. It is also interesting that both r -partite and non- r -partite r -graphs can be used to achieve color class sizes $|E_i| \geq r\Delta - 1$ with no full rainbow matching.

On the other hand, we ask whether there are constructions for Theorem 1.30 that do not involve blow-ups of nets. The answer is yes when $r = \Delta = 2$. Recall from Chapter 5 that Aharoni, Holzman, Howard, and Sprüssel [16] proved the unique minimal graphs with maximum degree $\Delta = 2$, that admit a partition into classes of size 3 with no independent transversal, are disjoint unions of three cycles each of length 1 modulo 3. We showed in Section 5.1 that appropriate constructions exist on all such disjoint unions. Translating this construction to full rainbow matchings, we see that for any disjoint union of three cycles each of length 1 modulo 3, there exists an edge-coloring of this graph into color classes $\mathcal{P} = \{E_1, \dots, E_n\}$ such that $|E_i| = 3$ for every i and there is no full rainbow matching. This construction generalizes the disjoint union of three 4-cycles from Example 6.10, which was illustrated in Figure 6.1. Currently we do not know of alternative constructions for other values of $r \geq 2$ and $\Delta \geq 2$.

6.2.2 Constructions in other settings

In this subsection, we give constructions related to those for Theorem 1.30, but in more restricted settings. Specifically, we consider what happens when the number of color classes n is small, and when we assume that the multi-hypergraph is t -simple for some $1 \leq t \leq r$.

First we consider what happens when we bound the number of classes n . We observe that if the number of color classes n is somewhat less than $r\Delta$, then the sufficient condition $|E_i| \geq r\Delta$ for the existence of a full rainbow matching in Theorem 1.29 can be improved.

Theorem 6.13. *Let G be an r -graph with maximum degree Δ , and let $\mathcal{P} = \{E_1, \dots, E_n\}$ be color classes of an edge-coloring of G . If $|E_i| \geq \left\lfloor \frac{(n-1)r\Delta}{n} \right\rfloor + 1$ for every i , then there exists a full rainbow matching.*

Proof. For all $I \subseteq [n]$, we have

$$\sum_{i \in I} |E_i| \geq |I| \left(\left\lfloor \frac{(n-1)r\Delta}{n} \right\rfloor + 1 \right) > |I| \left(\frac{(n-1)r\Delta}{n} \right) \geq r\Delta(|I| - 1).$$

By Theorem 6.4, there exists a full rainbow matching. $\square$

Here, we prove that the condition $|E_i| \geq \left\lfloor \frac{(n-1)r\Delta}{n} \right\rfloor + 1$ in Theorem 6.13 is best possible for certain values of r , Δ , and n . This result is analogous to Theorem 5.7 on independent transversals in graphs.

Theorem 6.14. *For integers $r \geq 1$, $s \geq 2$, suppose that an (r, s) -net exists, $\Delta \equiv 0 \pmod{s}$, and $n \equiv s \pmod{s(s-1)}$. Then there exists an r -graph G with maximum degree Δ and an edge-coloring of G into color classes $\mathcal{P} = \{E_1, \dots, E_n\}$ such that $|E_i| \geq \left\lfloor \frac{(n-1)r\Delta}{n} \right\rfloor$ for every i and there is no full rainbow matching*

Proof. Like in Example 6.11, we apply Theorem 6.9 to the r -graph $G_{r,s}(\frac{\Delta}{s}, \dots, \frac{\Delta}{s})$ and the color classes $F'_1, \dots, F'_s$ each of size $|F'_i| = \frac{r(s-1)\Delta}{s}$, but this time when we apply Theorem 6.9, we take $q = \left\lfloor \frac{(n-1)r\Delta}{n} \right\rfloor$. We can write $n \geq \frac{r\Delta}{r\Delta - q} = n_0$. Theorem 6.9 then says that for some $k \geq 1$ we can find an edge-coloring of the disjoint union of k copies of $G_{r,s}(\frac{\Delta}{s}, \dots, \frac{\Delta}{s})$ with $k(s-1) + 1$ color classes $E_1, \dots, E_{k(s-1)+1}$ each of size at least q , such that there is no full rainbow matching. Specifically, we can take any integer $k \geq 1 + s \left\lceil \frac{q - \frac{r(s-1)\Delta}{s}}{r(s-1)\Delta - q(s-1)} \right\rceil = 1 + s \left\lceil \frac{qs - r(s-1)\Delta}{s(s-1)(r\Delta - q)} \right\rceil = 1 + s \left\lceil \frac{n_0 - s}{s(s-1)} \right\rceil$. In particular, we can take $k = 1 + s \left\lceil \frac{n-s}{s(s-1)} \right\rceil = 1 + \frac{n-s}{s-1} = \frac{n-1}{s-1}$, and then the number of color classes is $k(s-1) + 1 = n$, as required. $\square$

Next, we wish to find appropriate constructions where we restrict how much the edges are allowed to intersect each other. For $t \geq 1$, a multi-hypergraph is t -simple if the intersection of any two edges has size at most t . A 1-simple hypergraph is commonly referred to as a *linear hypergraph*. We prove the following.

Theorem 6.15. *For all integers $1 \leq t \leq r$ and $\Delta \geq 2$, there exist a t -simple, r -partite r -graph G with maximum degree Δ and an edge-coloring of G into color classes $E_1, \dots, E_n$, such that $|E_i| \geq r(\Delta - 1) + t - 1$ for every i and there is no full rainbow matching.*

Our construction for Theorem 6.15 is a modification of the r -graph $G_{r,2}$ from Example 6.10, which was a disjoint union of blow-ups of the $r \times r$ grid $H_{r,2}$. Fix $1 \leq t \leq r$. For $d \geq 1$,

the (r, t, d) -sunflower $S_{r,t,d}$ is the r -graph with $d(r - t) + t$ vertices and d edges, such that there exists a vertex subset $K \subseteq V(S_{r,t,d})$ of size t such that $e \cap f = K$ for any two distinct edges $e, f \in E(S_{r,t,d})$. The set K is called the *kernel* of the sunflower $S_{r,t,d}$. Note that $S_{r,t,d}$ is a t -simple, r -partite r -graph with maximum degree d . For $\Delta \geq 2$, let $H_{r,t,\Delta}$ be the t -simple, r -partite r -graph with maximum degree Δ constructed as follows. First we take the disjoint union of r copies $S_1, \dots, S_r$ of the sunflower $S_{r,t,\Delta-1}$, and say S_i has kernel $K_i = \{v_{i1}, \dots, v_{it}\}$ for $1 \leq i \leq r$. Then we add the edges $e_j = \{v_{1j}, \dots, v_{rj}\}$ for $1 \leq j \leq t$, which form a perfect matching on $K_1 \cup \dots \cup K_r$. Let $F_1 = \bigcup_{i \in [r]} E(S_i)$ and let $F_2 = \{e_1, \dots, e_t\}$. Then F_1, F_2 form the color classes of an edge-coloring of $H_{r,t,\Delta}$ with no full rainbow matching.

Proof of Theorem 6.9. We apply Theorem 6.9 to the multi-hypergraph $H_{r,t,\Delta}$ above with the color classes F_1, F_2 . Note that $|F_1| = r(\Delta - 1)$ and $|F_2| = t$. Then applying Theorem 6.9 with $q = r(\Delta - 1) + t - 1$, if $G_{r,t,\Delta}$ is the disjoint union of $k = r(\Delta - 1) + t - 1$ copies of $H_{r,t,\Delta}$, there exists an edge-coloring of $G_{r,t,\Delta}$ into $k + 1 = r(\Delta - 1) + t$ color classes $E_1, \dots, E_{r(\Delta-1)+t}$ such that $|E_i| \geq r(\Delta - 1) + t - 1$ for every i and there is no full rainbow matching. Since $G_{r,t,\Delta}$ is a t -simple, r -partite r -graph with maximum degree Δ , this completes the proof of Theorem 6.15. $\square$

We note that when $r = 2$ and $t = 1$, the construction in the above proof is the same as a double star construction by Gao, Ramadurai, Wanless, and Wormald [76]. We currently do not know of better constructions for t -simple r -graphs than those given in Theorem 6.15.

6.3 Constructions for properly edge-colored multigraphs

In this section, we prove Theorem 1.31 about properly edge-colored multigraphs with no full rainbow matchings, which is restated below.

Theorem 1.31. *The following statements hold:*

- (1) *For every multigraph H with maximum degree Δ and chromatic index χ' , there exists an associated multigraph G with maximum degree Δ and chromatic index χ' and a proper edge-coloring of G into color classes $\mathcal{P} = \{E_1, \dots, E_n\}$, such that $|E_i| \geq \chi' - 1$ for every i and there is no full rainbow matching. If $|E(H)| \leq 2\Delta - 1$, then we can take $|E_i| \geq \chi'$ for every i .*
- (2) *For every $\Delta \geq 2$, there exist a bipartite graph G with maximum degree Δ and a proper edge-coloring of G into color classes $\mathcal{P} = \{E_1, \dots, E_n\}$, such that $|E_i| \geq \Delta + 1$ for every i and there is no full rainbow matching.*
- (3) *For every integer $\Delta \geq 3$ with $\Delta \equiv 3, 0 \pmod{4}$, there exist a multigraph G with maximum degree and chromatic index Δ , and a proper edge-coloring of G into color classes $\mathcal{P} = \{E_1, \dots, E_n\}$, such that $|E_i| \geq \Delta + 2$ for every i and there is no full*

rainbow matching. If $\Delta \in \{2^m - 1, 2^m\}$ for some integer $m \geq 2$, then we can take G to be a simple graph.

Theorem 1.31 disproves conjectures of Delcourt and Postle [53], which are the following. Their first conjecture is that if G is a bipartite multigraph and $|E_i| \geq \Delta + 1$ for every i , then there exists a full rainbow matching. Their second conjecture is that if G is a general multigraph and $|E_i| \geq \Delta + 2$ for every i , then there exists a full rainbow matching. They suggested these conjectures as “sparse” analogues of conjectures of Aharoni and Berger [7] (Conjecture 6.18 below) and of Gao, Ramadurai, Wanless, and Wormald [76] (Conjecture 6.21 below). We note that they did prove the following asymptotic result, which is useful to keep in mind.

Theorem 6.16 (Delcourt, Postle [53]). *For all integers $r \geq 2$ and real $\beta > 0$, there exist an integer $\Delta_\beta > 0$ and a real $\alpha > 0$ such that the following holds for all $\Delta \geq \Delta_\beta$. Let G be an r -graph with maximum degree Δ , and let $\mathcal{P} = \{E_1, \dots, E_n\}$ be color classes of an edge-coloring of G , such that the maximum codegree of G is at most $\Delta^{1-\beta}$, every color class $G[E_i]$ has maximum degree at most $\Delta^{1-\beta}$, and $|E_i| \geq (1 + \Delta^{-\alpha})\Delta$ for every i . Then there exists a full rainbow matching.*

In less precise terms, Theorem 6.16 states that there exists a full rainbow matching in an edge-colored r -graph G with maximum degree Δ whenever $|E_i| \geq \Delta + o(\Delta)$ for every i , as long as the maximum degree of every color class $G[E_i]$ is $o(\Delta)$ and the maximum codegree of G is $o(\Delta)$.

Before proving Theorem 1.31, we will outline some non-optimal examples of properly edge-colored multigraphs with no full rainbow matchings that will be used in our constructions.

6.3.1 Preliminary constructions

In this subsection, we describe properly edge-colored graphs with no full rainbow matchings that will be used to prove Theorem 1.31. First, we note the following basic construction arising from the chromatic index.

Proposition 6.17. *Let H be a multigraph with chromatic index χ' , and let G be the disjoint union of $\chi' - 1$ copies of H . Assuming that the edge set of H is $\{e_1, \dots, e_m\}$, consider the proper edge-coloring of G into color classes $E_1, \dots, E_m$ where E_i consists of the copies of the edge e_i among the disjoint copies of H in G . Then there is no full rainbow matching.*

Proof. A full rainbow matching in G corresponds to a proper edge-coloring of H using $\chi' - 1$ colors: If $H_1, \dots, H_{\chi'-1}$ are the copies of H in G , then a full rainbow matching M of G corresponds to the proper edge-coloring of H where the edges of H whose copies lie in $M \cap E(H_j)$ are given the color j , for $j \in \{1, \dots, \chi' - 1\}$. Since there is no proper edge-coloring of H using $\chi' - 1$ colors, there is no full rainbow matching in G . $\square$

The connection to edge-coloring indicated by Proposition 6.17 is analogous to what we described with (list) vertex-colorings in Section 5.4, and we will return to (list) edge-colorings in Section 6.4. To proceed, we will utilize some known constructions from the literature. Recall that Aharoni and Berger [7] conjectured the following.

Conjecture 6.18 (Aharoni, Berger [7]). *Let G be a bipartite multigraph, and let $\mathcal{P} = \{E_1, \dots, E_n\}$ be color classes of a proper edge-coloring of G . If $|E_i| \geq n + 1$ for every i , then there exists a full rainbow matching.*

The condition $|E_i| \geq n + 1$ in Conjecture 6.18 cannot be improved for even $n \geq 2$, as shown by the following well-known construction, which we reprove for convenience.

Proposition 6.19. *For every even integer $n \geq 2$, there exists a proper edge-coloring of $G = K_{n,n}$ into n perfect matchings each of size n , such that there is no full rainbow matching.*

Proof. Label the vertices of each part of $K_{n,n}$ as $A = \{x_0, \dots, x_{n-1}\}$ and $B = \{y_0, \dots, y_{n-1}\}$. Taking the additive abelian group $\mathbb{Z}_n$ to be our color set, consider the edge-coloring of $K_{n,n}$ where edge $x_i y_j$ is given color $i + j$, for all $i, j \in \mathbb{Z}_n$. It is easy to see that this edge-coloring is proper. Suppose for contradiction that there exists a full rainbow matching M , where edge $x_{a_k} y_{b_k} \in M$ is given color $k \in \mathbb{Z}_n$. Then modulo n ,

$$\sum_{k \in \mathbb{Z}_n} a_k = \sum_{k \in \mathbb{Z}_n} b_k = \sum_{k \in \mathbb{Z}_n} k = \frac{n(n-1)}{2}$$

because the matching is perfect. Thus, $\sum_{k \in \mathbb{Z}_n} (a_k + b_k) = n(n-1) = 0$. But also $\sum_{k \in \mathbb{Z}_n} (a_k + b_k) = \sum_{k \in \mathbb{Z}_n} k = \frac{n(n-1)}{2}$ because the matching is full rainbow, and this is not 0 when n is even, a contradiction. $\square$

Proposition 6.19 equivalently states that for all even $n \geq 2$, there exists a Latin square of order n with no transversal. The proof above gives the example of the Cayley table of the additive group $\mathbb{Z}_n$. The Ryser-Brualdi-Stein Conjecture asserts that for odd n , there cannot be a proper edge-coloring of $K_{n,n}$ into n perfect matchings that has no full rainbow matching. However, one can still show that the condition $|E_i| \geq n + 1$ in Conjecture 6.18 would be best possible also for odd integers $n \geq 3$, by slightly modifying a multigraph construction of Barát and Wanless [29] (who extended a construction of Drisko [55]), as follows.

Proposition 6.20. *For every integer $n \geq 2$, there exist a bipartite multigraph G and a proper edge-coloring of G into n perfect matchings each of size n , such that there is no full rainbow matching.*

Proof. First suppose that $n \geq 2$ is even, say $n = 2m - 2$. Let C be the cycle of length $2m$ where every edge is replaced with $m - 1$ parallel edges. Consider a proper edge-coloring of C into $2m - 2$ perfect matchings, each of size m . Then every rainbow matching in C has size at most $m - 1$, since a matching of size m is a perfect matching of C , and there are at

most $m - 1$ colors found in any perfect matching. Let P be the multigraph where $2m - 2$ parallel edges connect two vertices. We construct G by taking the disjoint union of C and $m - 2$ copies of P . We properly edge-color each component of G into $n = 2m - 2$ perfect matchings, each with the same color set. Then G has no full rainbow matching with respect to this proper edge-coloring, since every rainbow matching C has size at most $m - 1$ and the remaining $m - 2$ components of G contribute at most one edge to a matching.

Now suppose that $n \geq 3$ is odd, say $n = 2m - 3$. Let C be the cycle of length $2m$ where the edges are replaced by $m - 1$ or $m - 2$ parallel edges in alternating fashion around the cycle. Consider a proper edge-coloring of G into $2m - 3$ perfect matchings, each of size m . Like above, C has no rainbow matching of size m . Let P be the multigraph consisting of $2m - 3$ parallel edges between two vertices. We construct G by taking the disjoint union of C and $m - 3$ copies of P . We properly edge-color each component of G into $n = 2m - 3$ perfect matchings, each with the same color set. Like above, G has no full rainbow matching with respect to this proper edge-coloring. $\square$

Next, recall that Gao, Ramadurai, Wanless, and Wormald [76] conjectured the following.

Conjecture 6.21 (Gao, Ramadurai, Wanless, Wormald [76]). *Let G be a multigraph, and let $\mathcal{P} = \{E_1, \dots, E_n\}$ be color classes of a proper edge-coloring of G . If $|E_i| \geq n + 2$ for every i , then there exists a full rainbow matching.*

The bound $|E_i| \geq n + 2$ in Conjecture 6.21 also cannot be improved for many values of n . For example, if $n = 3$ then take G to be the disjoint union of two copies of the complete graph K_4 that is properly edge-colored into 3 perfect matchings each of size 4, which has no full rainbow matching. (This is the construction of Example 6.11 when $r = 2$ and $s = \Delta = 3$.) One way to generalize this example to other values of n is to use, in place of K_4 , a construction of Barát, Gyárfás, and Sárközy [28] (see also [8]), as follows.

Proposition 6.22. *For every integer $n \geq 3$ with $n \equiv 3 \pmod{4}$, there exist a multigraph G and a proper edge-coloring of G into n perfect matchings each of size $n + 1$, such that there is no full rainbow matching.*

Proof. Let $n = 2m - 1$. Let C be the cycle of length $2m$ where every edge is replaced with $m - 1$ parallel edges. We start with a proper edge-coloring of C into $2m - 2$ perfect matchings, each of size m . Let $v_1, \dots, v_{2m}$ be the vertices of C in cyclic order. Let H be obtained from C by adding a perfect matching M all of whose edges $v_i v_j$ have $j - i$ even (e.g. $M = \{v_{4k-3} v_{4k-1}, v_{4k-2} v_{4k} : 1 \leq k \leq \frac{m}{2}\}$). We give the edges of M a new color, so that H is properly edge-colored into $2m - 1$ perfect matchings each of size m . We claim that every rainbow matching in H has size at most $m - 1$. This claim follows from the observations that every perfect matching of C has two edges of the same color, and that for any edge $e \in M$ there is no perfect matching of $C \cup \{e\}$ containing e . Now let G be the disjoint union of two copies of H each with this proper edge-coloring. Then G is properly edge-colored into $n = 2m - 1$ color classes each of size $n + 1 = 2m$, and there is no full rainbow matching. $\square$

The above construction adapted from Barát, Gyárfás, and Sárközy [28] has many parallel edges. We present a different generalization of the disjoint union of two copies of K_4 , a construction that has not previously appeared in the literature as far as we are aware.

Proposition 6.23. *For every integer $m \geq 2$, there exists a proper edge-coloring of the disjoint union G of two copies of the complete graph K_{2^m} into $2^m - 1$ perfect matchings each of size 2^m , such that there is no full rainbow matching.*

Proof. For the complete graph K_{2^m} , take its vertex set to be the additive elementary abelian group $\mathbb{Z}_2^m$. Take $\mathbb{Z}_2^m \setminus \{(0, \dots, 0)\}$ to be our color set, and consider the edge-coloring of K_{2^m} where for two vertices $x, y \in V(K_{2^m})$ we give the edge xy the color $x + y$. It is easy to see that this edge-coloring is proper. Letting $G = K_{2^m} \sqcup K_{2^m}$, assign this proper edge-coloring to both copies of K_{2^m} in G . Now suppose for contradiction that there exists a full rainbow matching $M = \{x_1y_1, \dots, x_{2^m-1}y_{2^m-1}\}$ in G , where $\ell = 2^{m-1}$. (We slightly abuse notation and consider $\mathbb{Z}_2^m$ as the vertex set of both components of G). Since the maximum size of a matching in either component of G is $\ell = 2^{m-1}$, there is one subset $M_1 = \{x_1y_1, \dots, x_\ell y_\ell\} \subset M$ of size ℓ whose edges lie in one component of G , and there is another subset $M_2 = M \setminus M_1 = \{x_{\ell+1}y_{\ell+1}, \dots, x_{2^m-1}y_{2^m-1}\}$ of size $\ell - 1$ whose edges lie in the other component of G . The matching M_1 is a perfect matching on its component, so $\sum_{i=1}^\ell (x_i + y_i) = 2^{m-1}(1, \dots, 1) = (0, \dots, 0)$. On the other hand, the matching M_2 covers all but two vertices on its component, and because these two uncovered vertices have a nonzero sum, we have $\sum_{i=\ell+1}^{2^m-1} (x_i + y_i) \neq (0, \dots, 0)$. It follows that $\sum_{i=1}^{2^m-1} (x_i + y_i) \neq (0, \dots, 0)$. But we also have that $\sum_{i=1}^{2^m-1} (x_i + y_i) = (0, \dots, 0)$ because the matching is full rainbow, a contradiction. $\square$

6.3.2 Applying the method

In this subsection, we prove Theorem 1.31. We will prove statements (1), (2), and (3) of the theorem separately as Theorem 6.24, Theorem 6.25, and Theorem 6.26 below.

The examples in the previous subsection already come quite close to the target class sizes. The goal now is to increase the size of each color class by at least one or two while maintaining the properties of being a proper edge-coloring and having no full rainbow matching. In some cases, it suffices to apply Theorem 6.9 to these properly edge-colored multigraphs, similar to what was done in Section 6.2. However, we will use a different strategy similar to what we did with locally sparse graphs with no independent transversals in Section 5.3. Roughly, we will start with an improperly edge-colored multigraph with large color classes, and then we will derive a properly edge-colored multigraph from it (with necessarily smaller color classes) by taking the disjoint union of it with one of the examples from the previous subsection and applying Lemma 6.1. First we prove statement (1) of Theorem 1.31, restated as follows.

Theorem 6.24. *Let H be a multigraph with maximum degree Δ , chromatic index χ' , and m edges, where $2 \leq \Delta \leq \chi' \leq m$. Then there exist a multigraph G with maximum degree Δ*

and chromatic index χ' , and a proper edge-coloring of G into color classes $\mathcal{P} = \{E_1, \dots, E_n\}$ that has no full rainbow matching, and such that $|E_i| \geq \chi'$ for every i if $m \leq 2\Delta - 1$, and $|E_i| \geq \chi' - 1$ for every i if $m \geq 2\Delta$.

Proof. Recall from Example 6.10 that the multigraph $G_{2,2}$ is the disjoint union of $2\Delta - 1$ subgraphs $H_1, \dots, H_{2\Delta-1}$ each of which is a blow-up of a 4-cycle, specifically a 4-cycle where in one of the pairs of non-adjacent edges we replace each edge by $\Delta - 1$ parallel edges. We saw that there is an edge-coloring of $G_{2,2}(1, \Delta - 1)$ into 2Δ color classes $F_1, \dots, F_{2\Delta}$ such that $|F_i| = 2\Delta - 1$ for every i and there is no full rainbow matching. Note that $G_{2,2}(1, \Delta - 1)$ is a bipartite multigraph with maximum degree and chromatic index Δ .

Now, let H be a multigraph from the hypothesis of the theorem statement with $2 \leq \Delta \leq \chi' \leq m$. Then let G' be the multigraph from Proposition 6.17 with color classes $\mathcal{P}' = \{E'_1, \dots, E'_m\}$, where $|E'_i| = \chi' - 1$ for every i . If $m \geq 2\Delta$, then $G', \mathcal{P}'$ is already the desired properly edge-colored multigraph. Assume instead that $m \leq 2\Delta - 1$. Then we start with the multigraph $G_{2,2}(1, \Delta - 1)$ and color classes $F_1, \dots, F_{2\Delta}$. Iteratively for each color class F_i , $i \in \{1, \dots, 2\Delta\}$, we apply Lemma 6.1 by adding a disjoint copy of $G', \mathcal{P}'$ and distributing each of the edges of F_i into one of the classes of $\mathcal{P}'$. Specifically, since F_i is a disjoint union of collections of parallel edges with maximum degree at most Δ and $|F_i| = 2\Delta - 1 \geq m$, we partition F_i into m nonempty matchings, and we distribute each of these matchings into a unique color class E'_i in $\mathcal{P}'$. After doing this for every color class F_i , the result is a multigraph G that is properly edge-colored into $n = 2\Delta m$ color classes $E_1, \dots, E_n$, with $|E_i| \geq \chi'$ for every i and there is no full rainbow matching. $\square$

Next, we prove statement (2) of Theorem 1.31, restated as follows.

Theorem 6.25. *For all $\Delta \geq 2$, there exist a bipartite simple graph G with maximum degree Δ and a proper edge-coloring of G into color classes $\mathcal{P} = \{E_1, \dots, E_n\}$, such that $|E_i| \geq \Delta + 1$ for every i and there is no full rainbow matching.*

Proof. Recall from the proof of Theorem 6.15 that the graph $G_{2,1,\Delta}$ is the disjoint union of $2\Delta - 2$ subgraphs $H_1, \dots, H_{2\Delta-2}$ each of which is a double star, that is, each H_i is a tree with two central adjacent vertices v_i and w_i , and v_i also connected to $\Delta - 1$ leaves with edges $e_{i,1}, \dots, e_{i,\Delta-1}$, and w_i also connected to $\Delta - 1$ leaves with edges $f_{i,1}, \dots, f_{i,\Delta-1}$. The color classes of $G_{2,1,\Delta}$ are $F_1, \dots, F_{2\Delta-1}$, where $F_{2\Delta-1} = \{v_1 w_1, \dots, v_{2\Delta-2} w_{2\Delta-2}\}$ and $F_i = \{e_{i,1}, \dots, e_{i,\Delta-1}, f_{i,1}, \dots, f_{i,\Delta-1}\}$ for $i \in \{1, \dots, 2\Delta - 2\}$. Then $|F_i| = 2\Delta - 2$ for every i , and there is no full rainbow matching. Note that $G_{2,1,\Delta}$ is a bipartite simple graph with maximum degree and chromatic index Δ .

We start with the graph $G_{2,1,\Delta}$ and color classes $F_1, \dots, F_{2\Delta-1}$. If Δ is even, then from Proposition 6.19 let $\mathcal{P}' = \{E'_1, \dots, E'_\Delta\}$ be the color classes of a proper edge-coloring of the complete bipartite graph $G' = K_{\Delta,\Delta}$ where $|E'_i| = \Delta$ for every i and there is no full rainbow matching. Iteratively for each of the color classes F_i , $i \in \{1, \dots, 2\Delta - 1\}$, we apply Lemma 6.1 by adding a disjoint copy of $G', \mathcal{P}'$ and distributing each of the edges of F_i into one of

the classes of $\mathcal{P}'$. Specifically, since each such F_i is the disjoint union of two copies of the star $K_{1,\Delta-1}$, we partition F_i into Δ nonempty matchings, and we distribute each of these matchings into a unique color class E'_i in $\mathcal{P}'$. After doing this for every class F_i , the result is a bipartite simple graph G that is properly edge-colored into $n = 1 + (2\Delta - 2)\Delta$ color classes $E_1, \dots, E_n$, such that $|E_i| \geq \Delta + 1$ for every i and there is no full rainbow matching.

If Δ is odd, we instead use the graph $G' = K_{\Delta-1,\Delta-1}$ and color classes $\mathcal{P}' = \{E'_1, \dots, E'_{\Delta-1}\}$ from Proposition 6.19, with $|E'_i| = \Delta - 1$ for every i . For each of the classes F_i , $i \in \{1, \dots, 2\Delta - 2\}$, of $G_{2,1,\Delta}$ we partition it into $\Delta - 1$ matchings each of size 2, and we distribute the edges of each of these matchings into a unique color class $E'_i \in \mathcal{P}'$. Then the sizes of the color classes E'_i increase from $\Delta - 1$ to $\Delta + 1$. $\square$

Finally we prove statement (3) of Theorem 1.31, restated as follows.

Theorem 6.26. *For every integer $\Delta \geq 3$ with $\Delta \equiv 3, 0 \pmod{4}$, there exist a multigraph G with maximum degree and chromatic index Δ , and a proper edge-coloring of G into color classes $\mathcal{P} = \{E_1, \dots, E_n\}$, such that $|E_i| \geq \Delta + 2$ for every i and there is no full rainbow matching. If $\Delta \in \{2^m - 1, 2^m\}$ for some integer $m \geq 2$, then we can take G to be a simple graph.*

Proof. The proof follows the same structure as the proof of Theorem 6.25, but in place of Proposition 6.19, we use Proposition 6.22 for the case of multigraphs, and Proposition 6.23 for the case of simple graphs. $\square$

6.4 List edge-colorings

In this section, we discuss a connection between list edge-colorings and full rainbow matchings. We will use this connection to prove Theorem 1.33, restated below.

Theorem 1.33. *For every integer $\Delta \geq 2$, there exist a bipartite graph H and list assignment L for $E(H)$ such that H, L has maximum color degree Δ , $|L(e)| = \Delta$ for every $e \in E(H)$, and there is no proper L -coloring.*

Theorem 1.33 demonstrates that a color degree generalization of Galvin's Theorem [75] does not hold. This result is motivated by the similar Theorem 1.26 of Bohman and Holzman [36], which disproved a list vertex-coloring of Reed [145] as discussed in Section 5.4.

Let H be a multi-hypergraph and let $L = (L(e) : e \in E(H))$ be a list assignment for $E(H)$. An L -coloring of H is a map ϕ on $E(H)$ such that $\phi(e) \in L(e)$ for every edge $e \in E(H)$. The L -coloring ϕ is *proper* if $\phi^{-1}(c)$ is a matching of H for every color $c \in \bigcup_{e \in E(H)} L(e)$. We can transform the problem of finding a proper L -coloring of a multi-hypergraph H into the problem of finding a full rainbow matching in an auxiliary properly

edge-colored multi-hypergraph G . This is an adaptation of a similar argument for list vertex-coloring that was discussed in Section 5.4 (see also [96]). For a color $c \in \bigcup_{e \in E(H)} L(e)$, we let $H_c = H[\{e \in E(H) : c \in L(e)\}]$, and we relabel the copy of edge $e \in E(H)$ in H_c by (e, c) . Let multi-hypergraph G be the disjoint union of the subhypergraphs H_c over all colors c . Assuming that H has edge set $E(H) = \{e_1, \dots, e_n\}$, consider the edge-coloring of G with color classes $\mathcal{P} = \{E_1, \dots, E_n\}$ defined by $E_i = \{(e_i, c) : c \in L(e_i)\}$ for every $i \in [n]$. Then every proper L -coloring ϕ of H corresponds to a unique full rainbow matching $\{(e, \phi(e)) : e \in E(H)\}$ in G .

We refer to the auxiliary multi-hypergraph G together with the set of color classes $\mathcal{P} = \{E_1, \dots, E_n\}$ as the *list edge-cover multi-hypergraph* of the original multi-hypergraph H with list assignment L . Recall that the *maximum color degree* of H, L is the maximum, over all colors c , of the maximum degree of the subhypergraph H_c . We see that this is the same as the maximum degree of the auxiliary multi-hypergraph G . Likewise, the *maximum color codegree* of H, L is the same as the maximum codegree of G . Based on the above translation of list edge-colorings to full rainbow matchings, we deduce the following result from Theorem 1.29 of Aharoni, Berger, and Meshulam [11].

Theorem 6.27. *Let H be an r -graph and let L be a list assignment for $E(H)$. If H, L has maximum color degree Δ and $|L(e)| \geq r\Delta$ for every $e \in E(H)$, then there exists a proper L -coloring of H .*

Also, notice that the color classes $\mathcal{P} = \{E_1, \dots, E_n\}$ form a proper edge-coloring of G , since edges in the same class E_i lie in different components of G . Thus, we also deduce the following result from Theorem 6.16 of Delcourt and Postle [53].

Theorem 6.28. *Let H be an r -graph and let L be a list assignment for $E(H)$. If H, L has maximum color degree Δ , maximum color codegree $o(\Delta)$, and $|L(e)| \geq \Delta + o(\Delta)$ for every $e \in E(H)$, then there exists a proper L -coloring of H .*

Notice that Theorem 6.28 is a color degree generalization of Kahn's [111] classical asymptotic result on the list chromatic index of r -graphs. Now, Galvin's Theorem [75] states that if bipartite multigraph H has maximum degree Δ and list assignment L for $E(H)$ such that $|L(e)| \geq \Delta$ for all $e \in E(H)$, then there exists a proper L -coloring. We now prove Theorem 1.33, demonstrating that a color degree generalization of Galvin's Theorem does not hold.

Our proof of Theorem 1.33 will be similar to how we proved Theorem 1.26 of Bohman and Holzman [36], which disproved a list coloring conjecture of Reed [145], in Section 5.4. Similar to that proof, we need an understanding of when a given multi-hypergraph G with color classes $\mathcal{P} = \{E_1, \dots, E_n\}$ is the list edge-cover multi-hypergraph of some other multi-hypergraph and list assignment. The following are two necessary conditions, analogous to Lemma 5.13 from Section 5.4:

- (a) For every connected component C and color class E_i , there is at most one edge in $E(C) \cap E_i$.

- (b) For every two distinct connected components C, C' and distinct color classes E_i, E_j , if there exist edges $e \in E(C) \cap E_i$, $f \in E(C) \cap E_j$, $e' \in E(C') \cap E_i$, $f' \in E(C') \cap E_j$, then $|e \cap f| = |e' \cap f'|$.

Conditions (a) and (b) are consequences of the following two facts: the connected components of G are all subhypergraphs of some multi-hypergraph H , and the edges of G lying in the same color class E_i correspond to the same edge in H . Unlike Lemma 5.13, these conditions are not always sufficient, but they are still useful to keep in mind for the proof.

Proceeding with the proof of Theorem 1.33, based on the reduction of list edge-colorings to full rainbow matchings, it suffices to prove the following theorem.

Theorem 6.29. *For every $\Delta \geq 2$, there exist a bipartite graph G and edge-color classes $E_0, \dots, E_n$ that is the list edge-cover graph of some bipartite graph, such that G has maximum degree Δ , $|E_i| = \Delta$ for every i , and there is no full rainbow matching.*

Our construction for Theorem 6.29, as usual, will involve iterative applications of Lemma 6.1. The steps are summarized as follows: First we will use Lemma 6.1 iteratively to construct a bipartite graph G_0 and edge-color classes $\mathcal{P}_0$ that is the list edge-cover graph of some bipartite graph, such that there is no full rainbow matching, and all but one of the color classes in $\mathcal{P}_0$ have size Δ . Then we will add multiple copies of $G_0, \mathcal{P}_0$ together and apply Lemma 6.1 to get a desired construction for Theorem 6.29. We note that slightly more straightforward constructions are possible for bipartite *multigraphs*, but the ideas are the same and the simple graph construction we present is more aesthetically pleasing.

First we describe the bipartite graph G_0 and the color classes $\mathcal{P}_0 = \{F_0, F_1, \dots, F_{\Delta^2}\}$, where $|F_0| = 1$ and $|F_i| = \Delta$ for $1 \leq i \leq \Delta^2$. The case $\Delta = 4$ is shown in Figure 6.3. The graph G_0 has two kinds of building blocks: one Δ -broom, and many disjoint unions of stars $K_{1,\Delta}$. A Δ -broom B is a tree with two central adjacent vertices v and w , one leaf u_0 adjacent to v , and $\Delta - 1$ leaves $u_1, \dots, u_{\Delta-1}$ adjacent to v . The graph G_0 has one copy of B , and we assign the edge vw the color 0, the edge u_0v the color 1, and the edge u_iw the color $1 + i \cdot \Delta$, for $i \in \{1, \dots, \Delta - 1\}$. Next, let forest S be the disjoint union of $\Delta - 1$ copies of the star $K_{1,\Delta}$. In addition to B , the graph G_0 has $\Delta + 1$ copies of S , say $S_0, \dots, S_\Delta$. In the copy S_0 , if its components are $K^1, \dots, K^{\Delta-1}$, then the Δ edges in K^i are given a unique color in $\{1 + i + j \cdot \Delta : j \in \{0, \dots, \Delta - 1\}\}$. In the copy S_i , for $i \in \{1, \dots, \Delta\}$, we properly edge-color each component with the colors $\{(i - 1) \cdot \Delta + j : j \in \{1, \dots, \Delta\}\}$. In summary, the graph G_0 is the disjoint union of the subgraphs $B, S_0, S_1, \dots, S_\Delta$ with the above edge-coloring. Letting F_i be the set of edges that were given color i , for $i \in \{0, 1, \dots, \Delta^2\}$, the resulting color classes $\mathcal{P}_0 = \{F_0, F_1, \dots, F_{\Delta^2}\}$ have the sizes stated above.

We claim that G_0 has no full rainbow matching with respect to $\mathcal{P}_0$. This can be shown directly, but here we explain how $G_0, \mathcal{P}_0$ can be derived by iteratively applying Lemma 6.1, which helps to motivate their construction. We start with the Δ -broom B above and the edge-color classes $\mathcal{Q} = \{F_0, F_1''\}$, where $F_0 = \{vw\}$ and $F_1'' = E(B) \setminus \{vw\}$. Notice that

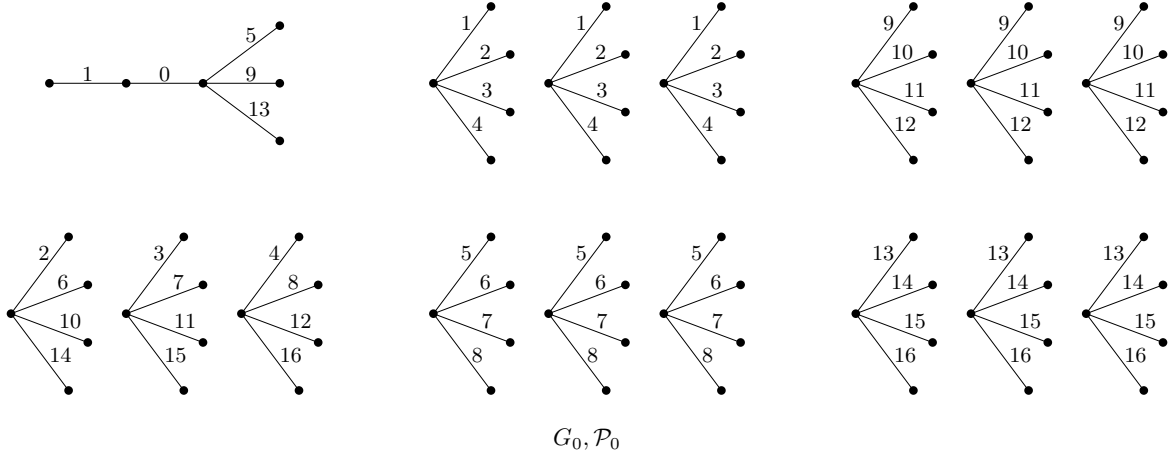


Figure 6.3: For maximum degree $\Delta = 4$, the list edge-cover graph G_0 and its edge-color classes $\mathcal{P}_0$ with no full rainbow matching. One color class has size 1, while the other color classes have size Δ .

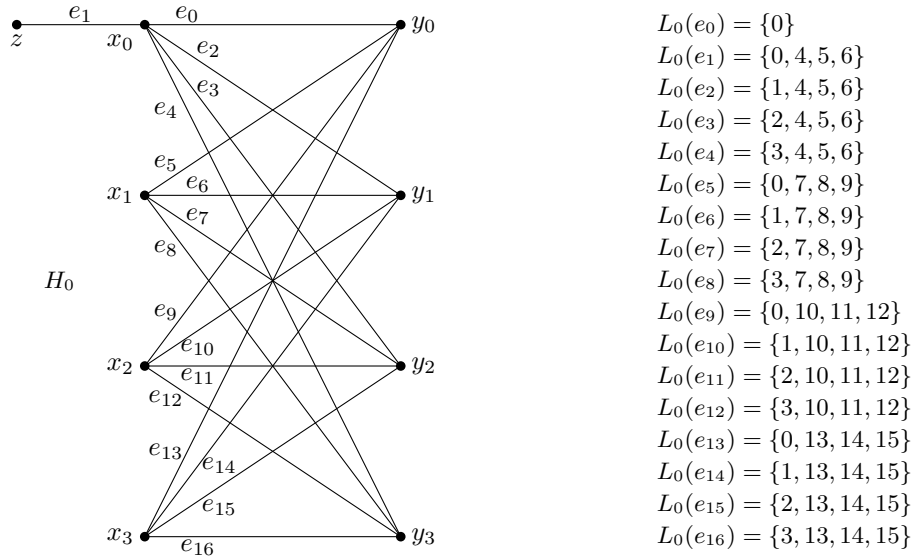


Figure 6.4: For maximum color degree $\Delta = 4$, the bipartite graph H_0 and list assignment L_0 for $E(H_0)$ with no proper L_0 -coloring. One list has size 1, while the other lists have size Δ . The colors in the list $L_0(e_i)$ correspond to the connected components in Figure 6.3 that contain an edge with color i .

$|F_0| = 1$, $|F_1''| = \Delta$, and that $B, \mathcal{Q}$ has no full rainbow matching. However, $B, \mathcal{Q}$ does not satisfy the necessary condition (a) of being a list edge-cover graph that we stated above. To fix that, we proceed by iteratively adding copies of $S, \mathcal{P}'$ and applying Lemma 6.1, where S is as above the disjoint union of $\Delta - 1$ copies of the star $K_{1,\Delta}$, and $\mathcal{P}'$ is the set of color classes of a proper edge-coloring of S using Δ colors. (Observe that $S, \mathcal{P}'$ has no full rainbow matching, and that it is the list edge-cover graph of the star $K_{1,\Delta}$ with the uniform list assignment $L(e) = \{1, \dots, \Delta - 1\}$, for $e \in E(K_{1,\Delta})$.) We add to $B, \mathcal{Q}$ one copy $S_0, \mathcal{P}'_0$ of the edge-colored graph $S, \mathcal{P}'$, and then we distribute each of the Δ edges of $F_1'' \in \mathcal{Q}$ into a unique class in $\mathcal{P}'_0$. The resulting edge-colored graph $B \sqcup S_0, \mathcal{Q}' = \{F_0, F_1', \dots, F_\Delta'\}$ has no full rainbow matching, $|F_0| = 1$, $|F_i'| = \Delta$ for $i \in \{1, \dots, \Delta\}$, and it satisfies the necessary condition (a) but not the necessary condition (b). Again, we fix that by adding copies of $S, \mathcal{P}'$. Iteratively for each color class F_i' , where $i \in \{1, \dots, \Delta\}$, we add to the current edge-colored graph a copy $S_i, \mathcal{P}'_i$ of $S, \mathcal{P}'$, and then we distribute each of the Δ edges of F_i' into a unique class in $\mathcal{P}'_i$. The result is the graph G_0 and color classes $\mathcal{P}_0 = \{F_0, F_1 \dots, F_{\Delta^2}\}$ that we described above.

Next, we claim that $G_0, \mathcal{P}_0$ is the list edge-cover graph of some bipartite simple graph H_0 with list assignment L_0 for $E(H_0)$. An illustration of H_0, L_0 when $\Delta = 4$ is shown in Figure 6.4. The graph H_0 is the complete bipartite graph $K_{\Delta,\Delta}$, together with an external vertex and an edge connecting this vertex to some vertex in $K_{\Delta,\Delta}$. Labeling the parts of $K_{\Delta,\Delta}$ as $\{x_0, \dots, x_{\Delta-1}\}$ and $\{y_0, \dots, y_{\Delta-1}\}$, we label the edge $x_i y_j$ as $e_{1+j+i\Delta}$ for $i, j \in \{0, \dots, \Delta-1\}$, except for the edge $x_0 y_0$ which we label as e_0 . We connect the vertex x_0 to an external vertex z and label the edge $x_0 z$ as e_1 . The list assignment L_0 for $E(H_0)$ is obtained by assigning a unique number to each connected component of G_0 , and letting the list $L_0(e_i)$ consist of those numbers whose corresponding connected component has an edge of color i , for $i \in \{0, 1, \dots, \Delta^2\}$. We then see that $G_0, \mathcal{P}_0$ is indeed the list edge-cover graph of H_0, L_0 : The broom B in G_0 comes from the edges $\{e_0\} \cup \{e_{1+i\Delta} : i \in \{0, \dots, \Delta-1\}\}$, the collection of stars S_0 in G_0 arises from the collection of stars in H_0 centered at $y_1, \dots, y_{\Delta-1}$, and each of the collection of stars S_i in G_0 arises from a star in H_0 centered at x_{i-1} , for $i \in \{1, \dots, \Delta\}$.

To complete our construction for Theorem 6.29, we only need to enlarge the color class F_0 of size 1 into a class of size at least Δ . We achieve this by iteratively applying Lemma 6.1 to G_0 . More directly, we take the disjoint union of Δ copies of $G_0, \mathcal{P}_0$ and then putting the Δ copies of e_0 into one common color class. Let G be the resulting bipartite graph and let $\mathcal{P} = \{E_0, \dots, E_n\}$ be the resulting color classes, where $n = \Delta^3$. Then we have that $|E_i| \geq \Delta$ for every i and there is no full rainbow matching. Moreover, $G, \mathcal{P}$ is still the list edge-cover graph of some bipartite graph H and list assignment L . Specifically, H, L is obtained by taking Δ disjoint copies of H_0, L_0 , then pasting together the disjoint copies of H along the copies of e_0 (identifying all the copies of x_0 , identifying all the copies of y_0 , and then collapsing all the copies of e_0 into a single edge that we still call e_0), and letting $L(e_0)$ consist of the Δ distinct colors from the copies of list $L_0(e_0)$. Note that the resulting graph H is still bipartite. Therefore, the edge-colored graph $G, \mathcal{P}$ gives the required construction for Theorem 6.29, and the associated bipartite graph H and list assignment L gives the required construction for Theorem 1.33.

To conclude this section, we summarize the various analogies between list edge-coloring results discussed in this section and list vertex-coloring results we discussed in Section 5.4. Recall that in the vertex setting, Reed [145] conjectured that if a graph H and list assignment L for $V(H)$ has maximum (vertex) color degree Δ , and $|L(x)| \geq \Delta + 1$ for every $x \in V(H)$, then there exists a proper L -coloring. This was suggested as a color degree generalization of the greedy upper bound on the list chromatic number, $\chi_\ell(H) \leq \Delta(H) + 1$. Theorem 6.27 above can be viewed analogously to Haxell's Theorem [96] that a proper L -coloring exists if $|L(x)| \geq 2\Delta$ for every $x \in V(H)$. And Theorem 6.28 can be viewed analogously to the theorem of Reed and Sudakov [146] that a proper L -coloring exists if $|L(x)| \geq (1 + o(1))\Delta$ for every $x \in V(H)$ (see also [127]). Bohman and Holzman [36] showed that Reed's conjecture is false, by finding examples where $|L(x)| \geq \Delta + 1$ for every $x \in V(H)$ and there is no proper L -coloring, and we saw in Section 5.4 that such examples can be constructed using Lemma 4.12. Likewise, we used Lemma 6.1 to show that a color degree generalization of Galvin's Theorem [75] is false, by finding examples where $|L(e)| \geq \Delta$ for every $e \in E(H)$ with H bipartite and there is no proper L -coloring.

6.5 Problems

We conclude this chapter with some open problems for future work.

(1) We view Theorem 1.29 on maximum degree conditions for full rainbow matchings in r -graphs as analogous to Theorem 1.12 on independent transversals in graphs, and we view the tightness result Theorem 1.30 as analogous to Theorem 1.22. To that end, we ask whether some of the results we have proved about independent transversals in Chapter 5 have analogues in the setting of full rainbow matchings. For example,

- Are there constructions for Theorem 1.30 that do not involve the disjoint union of many blow-ups of nets, besides the case $r = \Delta = 2$? This was discussed a bit in Section 6.2.
- Do there exist better constructions than those given in Theorem 6.15 for t -simple multi-hypergraphs, where the classes have size $|V_i| \geq r(\Delta - 1) + t - 1$ for every i .
- Can every “minimal construction” for Theorem 1.30 be obtained via iterative applications of Lemma 6.1 (analogous to Theorem 1.23 or Theorem 5.18 for independent transversals)?

Similar to independent transversals, to answer some of these questions it is worth investigating for what r -graphs the lower bound $\eta(\mathcal{M}(G)) \geq \frac{|E(G)|}{r\Delta}$ from Theorem 6.3 is tight, and when it can be improved.

(2) When it comes to rainbow matchings in properly edge-colored multigraphs, our investigations make it appear more relevant to look at the chromatic index instead of the

maximum degree for appropriate class size conditions. To that end, we ask the following. Let G be a multigraph with chromatic index χ' , and let $\mathcal{P} = \{E_1, \dots, E_n\}$ be the color classes of a proper edge-coloring of G . Is it true that if $|E_i| \geq \chi' + o(\chi')$ for every i , then there exists a full rainbow matching? Could the lower bound $\chi' + o(\chi')$ be replaced by $\chi' + c$ for some universal constant c ? Theorem 6.16 of Delcourt and Postle [53] implies that the asymptotic lower bound is true if we assume that the multigraph has edge-multiplicity $o(\Delta)$ (in which case $\chi' = \Delta + o(\Delta)$). Our construction for Theorem 6.26 implies that the result is not true with the constant $c = 2$, but we have not ruled out any $c \geq 3$. If we only consider bipartite graphs, then Theorem 6.25 implies that result does not hold with the constant $c = 1$, but we have not ruled out any value $c \geq 2$. We could also ask this kind of question for r -graphs, but as their chromatic index is largely not understood, we did not touch on them.

(3) Positive results on Problem (2) would lead to positive answers to the following question, about how the List Edge-Coloring Conjecture could extend to the “color” setting. For a multigraph H , list assignment L for $E(H)$, and a color c , recall that the subgraph H_c is the one induced by the edge multiset $E_c = \{e \in E(H) : c \in L(e)\}$. The *maximum color chromatic index* of H, L is the maximum, over all colors c , of the chromatic index of H_c . Let H be a multigraph, and let L be a list assignment for $E(H)$. Is it true that if H, L has maximum color chromatic index χ' and $|L(e)| \geq \chi' + o(\chi')$ for every $e \in E(H)$, then there exists a proper L -coloring of H ? Could the lower bound $\chi' + o(\chi')$ be replaced by $\chi' + c$ for some constant c ? Note that Theorem 1.33 implies that this result does not hold with constant $c = 0$ even for bipartite graphs, but we have not ruled out any $c \geq 1$. Answers for bipartite multigraphs H may be of particular interest. Again, we could ask this kind of question for general r -graphs.

Chapter 7

Other simplicial complexes with no transversals

In this chapter, we observe that the iterative construction method that was applied extensively in Chapter 4, Chapter 5, and Chapter 6 generalizes to all simplicial complexes. This widely expands its usefulness. We will discuss this generalization as well as related results in Section 7.1. Then we will discuss independent transversals in other complexes of interest. This includes matroids in Section 7.2, and we will then discuss independent transversals in uniform hypergraphs, H -free transversals in graphs, and acyclic transversals in directed graphs in Section 7.3. Most of the writing here is unpublished, and the emphasis is more on drawing connections than on proving particular results.

7.1 Construction method for simplicial complexes

Recall the following terminology from the Introduction. An *abstract simplicial complex* (or just a *complex*) $\mathcal{C}$ is a collection of subsets of some finite ground set $V = V(\mathcal{C})$, such that if $A \in \mathcal{C}$ and $B \subseteq A$, then $B \in \mathcal{C}$. The elements of V are the *vertices* of $\mathcal{C}$, and the sets in $\mathcal{C}$ are the *simplices* of $\mathcal{C}$. For a subset $X \subseteq V(\mathcal{C})$, we define the induced subcomplexes $\mathcal{C}[X] = \{A \in \mathcal{C} : A \subseteq X\}$ and $\mathcal{C} - X = \mathcal{C}[V - X]$. Our work in the last three chapters were versions of the following general problem: Given a complex $\mathcal{C}$ and a partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of its vertex set $V(\mathcal{C})$, when does there exist a simplex $T \in \mathcal{C}$ containing one vertex from each class V_i , called a *colorful simplex*? In Chapter 4 and Chapter 5, the complex of interest was the *independence complex* $\mathcal{I}(G)$ of a graph G , and a colorful simplex corresponds to an independent transversal. In Chapter 6, the complex of interest was the *matching complex* $\mathcal{M}(G)$ of a multi-hypergraph G , and a colorful simplex corresponds to a full rainbow matching. In both the setting of independent sets and matchings, the following topological Hall theorem from [15, 130, 131] was useful for deriving sufficient conditions for the existence of a colorful simplex, where η denotes either the homotopical or homological connectedness of the complex, plus two.

Theorem 1.11 ([15, 130, 131]). *Let $\mathcal{C}$ be a complex, and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(\mathcal{C})$. If*

$$\eta\left(\mathcal{C}\left[\bigcup_{i \in I} V_i\right]\right) \geq |I| \quad \text{for all } I \subseteq [n],$$

then there exists a colorful simplex.

Throughout our work, we were often concerned with the existence of colorful simplices when the class sizes $|V_i|$ are sufficiently large compared to some parameter of the complex $\mathcal{C}$. For example, recall that Haxell [95, 96] proved the existence of an independent transversal in a graph G with maximum degree Δ whenever $|V_i| \geq 2\Delta$ for every i (see Theorem 1.21). Also recall that Aharoni, Berger, and Meshulam [11] proved the existence of a full rainbow matching in an r -uniform multi-hypergraph G with maximum Δ whenever $|E_i| \geq r\Delta$ for every i (see Theorem 1.29). Both of these results can be deduced using Theorem 1.11 by analyzing the connectedness of the associated complexes.

For general complexes, we can formulate the following result. For a complex $\mathcal{C}$, define $\tilde{\eta}(\mathcal{C}) = \min\{\eta(\mathcal{C}), \max\{|A| : A \in \mathcal{C}\}\}$. The *expansion number* of $\mathcal{C}$ is the quantity

$$\Delta(\mathcal{C}) = \max_{X \subseteq V(\mathcal{C}), X \neq \emptyset} \left\lceil \frac{|X|}{\tilde{\eta}(\mathcal{C}[X])} \right\rceil.$$

This terminology is due to Aharoni and Berger [6]. The motivation for the expansion number is that it defines the best class size lower bound for the existence of a colorful simplex that we can achieve using the topological approach of Theorem 1.11, as follows.

Theorem 7.1. *Let $\mathcal{C}$ be a complex with expansion number Δ , and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(\mathcal{C})$. If*

$$\sum_{i \in I} |V_i| \geq \Delta(|I| - 1) + 1 \quad \text{for all } I \subseteq [n],$$

then there exists a colorful simplex. In particular, if $|V_i| \geq \Delta$ for every i , then there exists a colorful simplex.

Proof. For $I \subseteq [n]$, let $\mathcal{C}_I = \mathcal{C}\left[\bigcup_{i \in I} V_i\right]$. By definition of expansion number, we have

$$\eta(\mathcal{C}_I) \geq \frac{1}{\Delta} \left| \bigcup_{i \in I} V_i \right| \geq \frac{1}{\Delta} (\Delta(|I| - 1) + 1) > |I| - 1.$$

Since $\eta(\mathcal{C}_I)$ is integral, it follows that $\eta(\mathcal{C}_I) \geq |I|$. By Theorem 1.11, there exists a colorful simplex. $\square$

In general, taking the connectedness parameter η to be the homological η_H gives a better expansion number than taking η to be the homotopical η_π . Also, recent strengthenings of Theorem 7.1 such as Theorem A.2 lead to stronger notions of an expansion number as far as Theorem 7.1 is concerned. As we saw in Chapter 4, Chapter 5, and Chapter 6, the expansion number often led to near optimal class size conditions for colorful simplices. However, this was not always the case, particularly in the “locally sparse” settings. In addition, the expansion number is far from optimal when it comes to independence complexes of hypergraphs of higher uniformity, which we will discuss in Section 7.3.

Now we describe our method for constructing complexes with no colorful simplices, and how it relates to Theorem 7.1. Given two complexes $\mathcal{C}$ and $\mathcal{D}$ on disjoint vertex sets, their *join* is the complex $\mathcal{C} * \mathcal{D} = \{A \cup B : A \in \mathcal{C}, B \in \mathcal{D}\}$. For example, if G and H are disjoint graphs then the independence complexes satisfy $\mathcal{I}(G \cup H) = \mathcal{I}(G) * \mathcal{I}(H)$, and if G and H are disjoint multi-hypergraphs then matching complexes satisfy $\mathcal{M}(G \cup H) = \mathcal{M}(G) * \mathcal{M}(H)$. The following lemma generalizes Lemma 1.20 which was used for independent transversals, and Lemma 1.28 which was used for full rainbow matchings.

Lemma 7.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be disjoint complexes, and let $\mathcal{P} = \{V_1, \dots, V_n\}$ and $\mathcal{Q} = \{W_1, \dots, W_m\}$ be partitions of $V(\mathcal{C})$ and $V(\mathcal{D})$, respectively. Let $\mathcal{R} = \{V'_1, \dots, V'_n, W_1, \dots, W_{m-1}\}$, where $V'_1 \supseteq V_1, \dots, V'_n \supseteq V_n$ are obtained by distributing each of the vertices of W_m into one of $V_1, \dots, V_n$ arbitrarily. If $\mathcal{C}$ has no colorful simplex with respect to $\mathcal{P}$ and $\mathcal{D}$ has no colorful simplex with respect to $\mathcal{Q}$, then $\mathcal{C} * \mathcal{D}$ has no colorful simplex with respect to $\mathcal{R}$.*

Proof. Assume for contradiction that $\mathcal{C} * \mathcal{D}$ has a colorful simplex $\{v_1, \dots, v_n, w_1, \dots, w_{m-1}\}$ with respect to $\mathcal{R}$, where $v_i \in V'_i$ for $1 \leq i \leq n$ and $w_j \in W_j$ for $1 \leq j \leq m-1$. If $v_i \in V_i$ for every $1 \leq i \leq n$, then $\{v_1, \dots, v_n\}$ is a colorful simplex of $\mathcal{C}$ with respect to $\mathcal{P}$, a contradiction. So suppose instead that $v_j \in V'_j \cap W_m$ for some $1 \leq j \leq n$. Then $\{w_1, \dots, w_{m-1}, v_j\}$ is a colorful simplex of $\mathcal{D}$ with respect to $\mathcal{Q}$, again a contradiction. $\square$

In many settings, like some in Chapter 5 and Chapter 6, it is useful to apply Lemma 7.2 iteratively on the same complex with no colorful simplex. The following theorem tells us how large we can eventually make the partition classes when we do this iteratively. It is a generalization of Theorem 5.6 and Theorem 6.9 to all complexes, with the proof being the same as the proof we presented for Theorem 5.6 in Section 5.2.

Theorem 7.3. *Let $\mathcal{C}$ be a complex and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(\mathcal{C})$, such that $\mathcal{C}$ has no colorful simplex with respect to $\mathcal{P}$. Suppose that integer $q \geq 1$ satisfies*

$$\sum_{i \in I} |V_i| \geq q(|I| - 1) + 1 \quad \text{for all } I \subseteq [n].$$

Then there exist an integer $k \geq 1$ and a partition $\mathcal{Q} = \{U_1, \dots, U_{k(n-1)+1}\}$ of the complex $\mathcal{C}^k = \mathcal{C} * \dots * \mathcal{C}$ (where $\mathcal{C}$ appears k times in the join), such that $|U_i| \geq q$ for every i and

there is no colorful simplex. Specifically, if $J = \{j : |V_j| < q\} \neq \emptyset$ and $V_J = \bigcup_{j \in J} V_j$, then we can take any

$$k \geq 1 + \sum_{j \in J} \left\lceil \frac{q - |V_j|}{|V_J| - q(|J| - 1)} \right\rceil.$$

Notice the quantitative similarities of the statement of Theorem 7.3 and the statement of Theorem 7.1, even though their conclusions are qualitative opposites.

Here, we briefly discuss how our construction method relates to the topological methods. For two complexes $\mathcal{C}$ and $\mathcal{D}$, the homological connectedness of their join satisfies $\eta_H(\mathcal{C} * \mathcal{D}) = \eta_H(\mathcal{C}) + \eta_H(\mathcal{D})$ (see Lemma A.3; for η_π we have an inequality $\eta_\pi(\mathcal{C} * \mathcal{D}) \geq \eta_\pi(\mathcal{C}) + \eta_\pi(\mathcal{D})$). If $\mathcal{C}$ has no colorful simplex with respect to $\mathcal{P}$, and $\mathcal{D}$ has no colorful simplex with respect to $\mathcal{Q}$, then by Theorem 1.11, for some $I \subseteq [n]$ and $J \subseteq [m]$ we have $\eta_H(\mathcal{C}_I) \leq |I| - 1$ and $\eta_H(\mathcal{D}_J) \leq |J| - 1$, so that $\eta_H(\mathcal{C}_I * \mathcal{D}_J) \leq (|I| - 1) + (|J| - 1) \leq |I| + |J| - 2$. Thus, $\eta_H(\mathcal{C}_I * \mathcal{D}_J)$ differs slightly more from $|I| + |J|$ than $\eta_H(\mathcal{C}_I)$ did from $|I|$, or $\eta_H(\mathcal{D}_J)$ did from $|J|$. It stands to reason that there is a way to decrease the collective number of classes from $|I| + |J|$ to $|I| + |J| - 1$ while maintaining the property of having no colorful simplex. That is what we do in Lemma 7.2. Moreover, it is easy to show that for two disjoint complexes $\mathcal{C}$ and $\mathcal{D}$, the (homological) expansion number satisfies $\Delta(\mathcal{C} * \mathcal{D}) = \max\{\Delta(\mathcal{C}), \Delta(\mathcal{D})\}$. Thus, applications of Lemma 7.2 can be used to derive larger class sizes while maintaining a given expansion number. That is what we sought to do when we constructed multi-hypergraphs with a given maximum degree and no transversal. In some cases, applications of Lemma 7.2 yield all possible partitions of the join with no colorful simplex, such as independent transversals in certain disjoint unions of complete bipartite graphs (see Theorem 1.23 or Theorem 5.18).

We remark that there are nice ways to generalize the discussion above. For example, Theorem 1.11 remains true when $\mathcal{P}$ is a general family of subsets of the vertex set $V(\mathcal{C})$ instead of a partition. In turn, one can modify our main lemma, Lemma 7.2, to allow the construction of more general families with no colorful simplices (colorful meaning a transversal of the family). Specifically, we may assume that $\mathcal{P}$ and $\mathcal{Q}$ are arbitrary families instead of partitions, and instead of distributing each of the vertices of W_m into only one of $V_1, \dots, V_n$, we may distribute each of the vertices of W_m into each of an arbitrary subset of $V_1, \dots, V_n$. The result will still have no colorful simplex. In addition, Aharoni and Berger [6] described a broad generalization of Theorem 1.11 to a statement about the existence of a set in the intersection of a complex and a matroid on the same ground set, a statement that also generalizes the Matroid Intersection Theorem [62]. One of the corollaries of their theorem is the following.

Theorem 7.4 (Aharoni, Berger [6]). *Let $\mathcal{C}$ be a complex and let $\mathcal{M}$ be a matroid, both on the same ground set V . If $\mathcal{C}$ has expansion number Δ and $\mathcal{M}$ has at least Δ disjoint bases, then there exists a simplex in $\mathcal{C}$ that is a basis of $\mathcal{M}$.*

Theorem 7.1 describes the special case when $\mathcal{M}$ is the partition matroid defined by the partition $\mathcal{P}$. As a broad problem, it would be interesting to understand more generally when

Theorem 7.4 gives a tight condition. Perhaps there is a natural generalization of Lemma 7.2 to more general pairs of complexes and matroids which could be helpful for this problem.

7.2 Independent transversals in matroids

In this section, we briefly discuss the problem of independent transversals in matroids, and a corollary on list coloring matroids. We observe similarities and differences from the problems of independent transversals in graphs and full rainbow matchings in multi-hypergraphs.

Recall that a *matroid* $\mathcal{M}$ is an abstract simplicial complex with the following special augmentation property: If $A, B \in \mathcal{M}$ and $|A| < |B|$, then there exists $x \in B$ such that $A \cup \{x\} \in \mathcal{M}$. We will denote the ground set of $\mathcal{M}$ by $E(\mathcal{M})$, and the sets in $\mathcal{M}$ are usually called the *independent sets* of $\mathcal{M}$. Matroids form a highly structured class of simplicial complexes and are useful in many graph theory and optimization problems (see [140] for background and terminology on matroid theory). Given a matroid $\mathcal{M}$ and a partition $\mathcal{P} = \{E_1, \dots, E_n\}$ of its ground set $E(\mathcal{M})$, an *independent transversal* is a transversal of $\mathcal{P}$ that is independent in $\mathcal{M}$. The topological join of two matroids $\mathcal{M}$ and $\mathcal{N}$ on disjoint ground sets is usually called their *direct sum* and denoted by $\mathcal{M} \oplus \mathcal{N}$.

As usual, letting $\eta(\mathcal{M})$ denote the topological connectedness of $\mathcal{M}$ plus two, it is well-known that for a matroid $\mathcal{M}$ with rank function r , we have $\eta(\mathcal{M}) = r(\mathcal{M})$ if $\mathcal{M}$ has no coloops, and $\eta(\mathcal{M}) = \infty$ otherwise (see [34]). Indeed, $\mathcal{M}$ is either contractible or homotopy equivalent to the wedge sum of $\tilde{\chi}(\mathcal{M})$ many $(r(\mathcal{M}) - 1)$ -dimensional spheres, where $\tilde{\chi}(\mathcal{M})$ is the reduced Euler characteristic of $\mathcal{M}$. Thus, Theorem 1.11 implies (the hard direction of) the following classical theorem of Rado [144].

Theorem 7.5 (Rado's Theorem [144]). *Let $\mathcal{M}$ be a matroid with rank function r , and let $\mathcal{P} = \{E_1, \dots, E_n\}$ be a partition of $E(\mathcal{M})$. There exists an independent transversal if and only if*

$$r\left(\bigcup_{i \in I} E_i\right) \geq |I| \quad \text{for all } I \subseteq [n].$$

Again, Rado's Theorem remains true if the partition $\mathcal{P}$ is instead a general family of sets in $\mathcal{M}$, and it is easy to show that these two versions are equivalent using parallel elements. Now we discuss the matroid analogue of theorems from previous chapters about class size conditions for transversals. The *covering number* $\beta(\mathcal{M})$ of a loopless matroid $\mathcal{M}$ is the minimum number of independent sets of $\mathcal{M}$ required to cover its ground set $E(\mathcal{M})$. By the Matroid Partition Theorem [60], a generalization of Nash-Williams' arboricity formula [139], the covering number has the following formula:

$$\beta(\mathcal{M}) = \max_{X \subseteq E(\mathcal{M}), X \neq \emptyset} \left\lceil \frac{|X|}{r(X)} \right\rceil.$$

Moreover, the maximum is attained at some flat X that is connected in $\mathcal{M}$. The covering number of a matroid can be found in polynomial time given an independence oracle or a rank oracle. We have the following corollary of Rado's Theorem.

Corollary 7.6. *Let $\mathcal{M}$ be a matroid with covering number β , and let $\mathcal{P} = \{E_1, \dots, E_n\}$ be a partition of $E(\mathcal{M})$. If $|E_i| \geq \beta$ for every i , then there exists an independent transversal.*

Proof. For all $I \subseteq [n]$, we have

$$r\left(\bigcup_{i \in I} E_i\right) \geq \frac{1}{\beta} \left|\bigcup_{i \in I} E_i\right| \geq \frac{1}{\beta} \cdot \beta |I| = |I|.$$

By Rado's Theorem, there exists an independent transversal. $\square$

Now we explain how the class size lower bound β in Corollary 7.6 is in some sense best possible, in a similar sense that the lower bound 2Δ for independent transversals in graphs was best possible in Theorem 1.21, and $r\Delta$ was best possible for full rainbow matchings in r -graphs in Theorem 1.29. Similar to the latter two cases, we look at direct sums of sufficiently many copies of a given matroid $\mathcal{M}$ with ground set $E = E(\mathcal{M})$. Consider any nonempty subset $X \subseteq E$ that maximizes $|X|/r_{\mathcal{M}}(X)$, from the formula of the covering number. Then $\beta = \lceil |X|/r_{\mathcal{M}}(X) \rceil$ and we can write $|X| = (\beta - 1)r_{\mathcal{M}}(X) + \ell$ for some $1 \leq \ell \leq r_{\mathcal{M}}(X)$. Consider the direct sum $\mathcal{N}_0 = \mathcal{M}[X] \oplus \dots \oplus \mathcal{M}[X]$ of $k \geq \lceil \frac{\beta-1}{\ell} \rceil$ copies of $\mathcal{M}[X]$, which has a ground set of size $|E(\mathcal{N}_0)| = (\beta - 1)k \cdot r_{\mathcal{M}}(X) + k \cdot \ell \geq (\beta - 1)(k \cdot r_{\mathcal{M}}(X) + 1)$, rank $k \cdot r_{\mathcal{M}}(X)$, and covering number β . Consider any partition $\mathcal{P}_0 = \{E_1, \dots, E_n\}$ of $E(\mathcal{N}_0)$ into $n = k \cdot r_{\mathcal{M}}(X) + 1$ sets each of size at least $\beta - 1$. Then there is no independent transversal of $\mathcal{N}_0$ with respect to $\mathcal{P}_0$ because the number of classes $n = k \cdot r_{\mathcal{M}}(X) + 1$ is greater than the rank $r(\mathcal{N}_0) = k \cdot r_{\mathcal{M}}(X)$, so any transversal of $\mathcal{P}_0$ is dependent in $\mathcal{N}_0$. If $X = E$, then we are done. Otherwise, let $\mathcal{N} = \mathcal{M} \oplus \dots \oplus \mathcal{M}$ be the direct sum of $k \geq \max\left\{\lceil \frac{\beta-1}{\ell} \rceil, \lceil \frac{\beta-1}{|E|-|X|} \rceil\right\}$ copies of $\mathcal{M}$, and let $\mathcal{P} = \{E_1, \dots, E_n, E_{n+1}\}$ be the partition of $E(\mathcal{M})$ where $E_1, \dots, E_n$ come from $\mathcal{P}_0$ and E_{n+1} consists of the remaining elements of $E(\mathcal{M})$. Then again $\mathcal{N}$ has covering number β , $|E_i| \geq \beta - 1$ for every i , and there is no independent transversal. This completes the construction.

Notice that, even though we could have, we did not apply Lemma 7.2 to get the above tight construction. This is because by Rado's Theorem, to ensure the non-existence of an independent transversal all we needed was a greater number of partition classes than the rank, and then it does not matter how the partition classes are arranged around the matroid. This contrasts with what we saw with independent transversals in graphs in Chapter 5 and full rainbow matchings in multi-hypergraphs in Chapter 6. In those cases, it really does matter how the partition classes are arranged around the complex, and we needed to apply Lemma 7.2 to get desirable complexes with no colorful simplices. This is one respect in which the transversal problem for matroids is better behaved than in independence complexes or matching complexes.

Differences between matroids and other complexes can also be observed in the context of list coloring. Let $\mathcal{M}$ be a loopless matroid, and let L be a list assignment for $E(\mathcal{M})$, which assigns a list of colors $L(x)$ to every element $x \in E(\mathcal{M})$. An L -coloring is a function $\phi : E(\mathcal{M}) \rightarrow \bigcup_{x \in E(\mathcal{M})} L(x)$ such that $\phi(x) \in L(x)$. An L -coloring ϕ is *proper* if each color class $\phi^{-1}(c)$ is an independent set of $\mathcal{M}$. The list covering number $\beta_\ell(\mathcal{M})$ is the minimum k such that for every list assignment L for $E(\mathcal{M})$ such that $|L(x)| \geq k$ for all $x \in E(\mathcal{M})$, there exists a proper L -coloring of $\mathcal{M}$. Seymour [154] proved that the covering number and list covering number of matroids are always the same: $\beta_\ell(\mathcal{M}) = \beta(\mathcal{M})$. Lason [124] generalized Seymour's theorem by allowing list sizes to vary, proving that the following two statements are equivalent for any list size function $\ell : E(\mathcal{M}) \rightarrow \mathbb{Z}_{\geq 1}$:

- (1) $\mathcal{M}$ has a proper L -coloring for the list assignment L where $L(x) = \{1, \dots, \ell(x)\}$ for all $x \in E(\mathcal{M})$;
- (2) $\mathcal{M}$ has a proper L -coloring for any list assignment L where $|L(x)| \geq \ell(x)$ for all $x \in E(\mathcal{M})$.

Both of their results are straightforward applications of the Matroid Partition Theorem [60].

Here, we present a different (but still straightforward) generalization of Seymour's theorem [154] that connects to our discussion in Section 5.4 and Section 6.4 about list coloring based on *maximum color degree*. Analogous to maximum color degree, we define the *maximum color covering number* of $\mathcal{M}$ as the maximum, over all colors $c \in \bigcup_{x \in E(\mathcal{M})} L(x)$, of the covering number of the restricted submatroid $\mathcal{M}[\{x \in E(\mathcal{M}) : c \in L(x)\}]$. Instead of assuming that list sizes are sufficiently large compared to a global parameter such as the covering number of the matroid, our list sizes will only depend more locally on how the colors are distributed among the lists. We prove the following as a simple consequence of Rado's Theorem (or more specifically Corollary 7.6).

Theorem 7.7. *Let $\mathcal{M}$ be a loopless matroid, and let L be a list assignment for $E(\mathcal{M})$. If $\mathcal{M}, L$ has maximum color covering number β and $|L(x)| \geq \beta$ for every $x \in E(\mathcal{M})$, then there exists a proper L -coloring of $\mathcal{M}$.*

Proof. For every color $c \in \bigcup_{x \in E(\mathcal{M})} L(x)$, define the restricted submatroid $\mathcal{M}_c = \mathcal{M}[\{x \in E(\mathcal{M}) : c \in L(x)\}]$. We relabel every element $x \in E(\mathcal{M}_c)$ by (x, c) , and we let $\mathcal{N}$ be the direct sum of the matroids $\mathcal{M}_c$ over all colors c . Let $\mathcal{P} = \{E_x : x \in \mathcal{M}\}$ be the partition of $E(\mathcal{N})$ where $E_x = \{(x, c) : c \in L(x)\}$. Then every proper L -coloring ϕ of $\mathcal{M}$ corresponds to an independent transversal $\{(x, \phi(x)) : x \in E(\mathcal{M})\}$ of $\mathcal{N}$ with respect to $\mathcal{P}$. Moreover, the maximum color covering number β of $\mathcal{M}, L$ is the same as the covering number of $\mathcal{N}$, and $|E_x| = |L(x)| \geq \beta$ for all $x \in \mathcal{M}$. By Corollary 7.6, there exists an independent transversal of $\mathcal{N}$ with respect to $\mathcal{P}$, and this corresponds to a proper L -coloring of $\mathcal{M}$. $\square$

We notice that Theorem 7.7 stands in contrast to what we saw in the cases of list vertex-coloring in Chapter 5 and of list edge-coloring in Chapter 6. In those cases, we observed

that color degree generalizations of classical list coloring bounds (the greedy upper bound in the vertex setting, Galvin's Theorem [75] in the edge setting) do not always hold. On the other hand, the counterexamples we constructed (Theorem 1.26 and Theorem 1.33) just barely disproved these possible generalizations. Matroids are more tightly structured, and Theorem 7.7 demonstrates that these kind of pathological list coloring behaviors do not exist. On the other hand, we note that there are related problems on transversals that are still difficult even for matroids. This includes the problem of decomposing a matroid into independent transversals, which is a source of famous open problems such as Rota's Basis Conjecture [174] and a conjecture of Aharoni and Berger [6].

7.3 Independent transversals in hypergraphs

In this section, we discuss independent transversals in uniform hypergraphs. This generalizes some of the discussion on independent transversals in graphs from Chapter 4 and Chapter 5. In particular, we will formulate Conjecture 7.13 which attempts to generalize Theorem 1.15 to hypergraphs of higher uniformity. We will also briefly mention other transversal problems, namely H -free transversals in graphs and acyclic transversals in directed graphs.

7.3.1 Uniform hypergraphs

Recall that a *hypergraph* G consists of a vertex set $V = V(G)$ and an edge set $E = E(G)$, where $E(G)$ is a collection of subsets of $V(G)$. The *degree* $d_G(v)$ of a vertex $v \in V(G)$ is the number of edges of G containing v . An *independent set* of G is a subset of its vertex set $V(G)$ that contains no edge of G , and the *independence complex* $\mathcal{I}(G)$ of G is the complex consisting of all independent sets of G . Given a hypergraph G and a partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$, an *independent transversal* of G with respect to $\mathcal{P}$ is a transversal of $\mathcal{P}$ that is independent in G . In other words, it is a colorful simplex in $\mathcal{I}(G)$. Notice that every complex $\mathcal{C}$ is the independence complex of some hypergraph, namely the hypergraph G with vertex set $V(G) = V(\mathcal{C})$ and edge set $E(G)$ consisting of all minimal non-faces of $V(\mathcal{C})$. In this subsection, we are primarily interested in independence complexes of *r -uniform hypergraphs* (abbreviated as *r -graphs*), which are hypergraphs where all edges have size r .

First we look at independent transversals in r -graphs in the maximum degree setting. Generalizing our previous use of balanced complete bipartite graphs for the graph case, as in Section 5.1, for the r -uniform case we will consider the hypergraph $H = K_r^r(m)$, which is the complete r -partite r -graph with m vertices in each part. Specifically, there is a partition of $V(H)$ into r vertex subsets $W_1, \dots, W_r$ and there is one edge $\{w_1, \dots, w_r\}$ for every choice of $w_1 \in W_1, \dots, w_r \in W_r$. Notice that H has no independent transversal with respect to $\mathcal{Q} = \{W_1, \dots, W_r\}$. Applying Theorem 7.3 to $\mathcal{I}(H)$, $\mathcal{Q}$ yields the following.

Theorem 7.8. *Let G be the disjoint union of $k = 1 + r \left\lceil \frac{1}{\ell} \left\lfloor \frac{m-1}{r-1} \right\rfloor \right\rceil$ copies of $K_r^r(m)$, where $\ell = m - (r-1) \left\lfloor \frac{m-1}{r-1} \right\rfloor$. There exists a partition $\mathcal{P} = \{V_1, \dots, V_{k(r-1)+1}\}$ of $V(G)$ such that $|V_i| \geq \left\lfloor \frac{rm-1}{r-1} \right\rfloor$ for every i and there is no independent transversal.*

The r -graph G in Theorem 7.8 has maximum degree $\Delta = m^{r-1}$, so the result states that we can achieve partition classes each of size at least $\left\lfloor \frac{r\Delta^{1/(r-1)}-1}{r-1} \right\rfloor$ and no independent transversal. We conjecture that our construction is always tight.

Conjecture 7.9. *Let G be an r -graph with maximum degree Δ and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. If $|V_i| \geq \frac{r}{r-1} \Delta^{1/(r-1)}$ for every i , then there exists an independent transversal.*

Conjecture 7.9 would generalize Theorem 1.21 of Haxell [95, 96] to all uniformities $r \geq 2$. We note that the topological approach for independent transversals in graphs works rather poorly for r -graphs when $r \geq 3$, at least when it comes Conjecture 7.9. Methods for estimating the connectedness of the independence complex of an r -graph G from [9, 120] yield that $\eta(\mathcal{I}(G)) \geq \frac{(r-1)|V(G)|}{r\Delta}$. Applying Theorem 1.11, this implies that there exists an independent transversal in G whenever $|V_i| \geq \frac{r}{r-1} \Delta$ for every i , and this is far from the condition stated in Conjecture 7.9 for $r \geq 3$. Moreover, we cannot expect much better general lower bounds for the connectedness $\eta(\mathcal{I}(G))$ of an r -graph G . For example, consider the r -uniform sunflower G with a core of size $r-1$, which is the hypergraph consisting of $\Delta + r - 1$ vertices $v_1, \dots, v_{\Delta+r-1}$ and Δ edges $e_i = \{v_1, \dots, v_{r-1}, v_i\}$, for $i \in \{r, r+1, \dots, \Delta + r - 1\}$. Then $\mathcal{I}(G)$ has the homotopy type of an $(r-2)$ -dimensional sphere, so that $\eta(\mathcal{I}(G)) = r-1 = \frac{(r-1)|V(G)|}{\Delta+r-1}$. Thus, we cannot have a general lower bound of the form $\eta(\mathcal{I}(G)) \geq \frac{|V(G)|}{f(\Delta)}$ where $f(\Delta)$ is sublinear in Δ for fixed r . Hence, directly applying the usual topological methods surrounding Theorem 1.11 does not help resolve Conjecture 7.9 for $r \geq 3$.

On the other hand, methods related to the Lovász Local Lemma have yielded results closer to Conjecture 7.9. For comparing such results, it is useful to restate Conjecture 7.9 as follows.

Conjecture 7.10 (Restatement of Conjecture 7.9). *Let G be an r -graph and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. If $|V_i| \geq t$ for every i and $\Delta(G) \leq (1 - \frac{1}{r})^{r-1} t^{r-1}$, then there exists an independent transversal.*

Recall that for a hypergraph G and partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$, the *maximum class average degree* is

$$\text{mad}(G, \mathcal{P}) = \max_{U \in \mathcal{P}} \frac{1}{|U|} \sum_{v \in U} d_G(v).$$

Notice that $\text{mad}(G, \mathcal{P}) \leq \Delta(G)$. A simple application of the Lovász Local Lemma, extending an argument of Alon [22, 24], shows that if $|V_i| \geq t$ for every i and $\text{mad}(G, \mathcal{P}) \leq \frac{1}{er} t^{r-1}$ then there exists an IT. Wanless and Wood [167] slightly improved this result of the Lovász Local Lemma for small r by proving the following.

Theorem 7.11 (Wanless, Wood [167]). *Let G be an r -graph, and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(G)$. If $|V_i| \geq t$ for every i and $\text{mad}(G, \mathcal{P}) \leq \frac{1}{r} \left(1 - \frac{1}{r}\right)^{r-1} t^{r-1}$, then there exists an independent transversal.*

Theorem 7.11 is the more general version of Theorem 1.13 that Wanless and Wood [167] proved using Rosenfeld counting. As a direct corollary, Theorem 7.11 implies the existence of an independent transversal when G has maximum degree $\Delta(G) \leq \frac{1}{r} \left(1 - \frac{1}{r}\right)^{r-1} t^{r-1}$, which comes just within a factor of r of Conjecture 7.10. On the other hand, Groenland, Kaiser, Treffers, and Wales [83] proved that the upper bound $\text{mad}(G, \mathcal{P}) \leq \frac{1}{r} \left(1 - \frac{1}{r}\right)^{r-1} t^{r-1}$ in Theorem 7.11 is asymptotically best possible for all $r \geq 2$. Their proof was an extension of the construction they gave the graph case $r = 2$, which we presented in Section 4.5. Below we will also give an asymptotically tight construction for r -graphs, specifically in Phase I.

Even though we have not resolved the problem of independent transversals in the maximum degree setting, for the remainder of this section we will discuss the more general problem of interpolating between maximum degree and maximum class average degree conditions for independent transversals in r -graphs. This is analogous to our work in Chapter 3. The following more general version of Question 1.14 was considered by Groenland, Kaiser, Treffers, Wales [83].

Question 7.12 (Groenland, Kaiser, Treffers, Wales [83]). *What can be said about the set of pairs $(\alpha, \beta) \in [0, 1]^2$ such that for sufficiently large t , there exists an r -graph G and a partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$ such that $\Delta(G) \leq \alpha t^{r-1}$, $\text{mad}(G, \mathcal{P}) \leq \beta t^{r-1}$, and there is no IT?*

Recall that we had fully resolved this problem in the graph case $r = 2$ with Theorem 1.15. Here, in the case of r -graphs, we will say that the pair $(\alpha, \beta) \in \mathbb{R}_{\geq 0}^2$ is an r -good pair if the following holds for every integer $t \geq 1$: Whenever G is an r -graph and $\mathcal{P} = \{V_1, \dots, V_n\}$ is a partition of $V(G)$ such that $|V_i| \geq t$ for every i , G has maximum degree $\Delta(G) \leq \alpha t^{r-1}$, and $G, \mathcal{P}$ has maximum class average $\text{mad}(G, \mathcal{P}) \leq \beta t^{r-1}$, then there exists an independent transversal. We conjecture the following extension of Theorem 1.15 for all $r \geq 2$.

Conjecture 7.13. *The pair $(\alpha, \beta) \in \mathbb{R}_{\geq 0}^2$ is r -good if and only if one of the following holds:*

- (1) $\alpha \leq \left(1 - \frac{1}{r}\right)^{r-1}$,
- (2) $\beta \leq \frac{1}{r} \left(1 - \frac{1}{r}\right)^{r-1}$,
- (3) $\beta \leq (r-1) \left(1 - \alpha^{1/(r-1)}\right) \left(r \alpha^{1/(r-1)} - r + 2\right) \alpha^{(r-2)/(r-1)}$.

We now present a construction showing that conditions (1), (2), and (3) of Conjecture 7.13 are necessary. This is an extension of our construction for the graph case in Section 4.5 (see Theorem 1.17). To generalize imbalanced complete bipartite graphs, let $K_{a,b}^r$ denote the complete r -partite r -graph where one part W_1 has size a and the other parts $W_2, \dots, W_r$

have size b . We refer to W_1 as the A -side $K_{a,b}^r$, and we refer to $W_2, \dots, W_r$ as the B -sides of $K_{a,b}^r$. Our constructions will imitate those in Section 4.5, but using complete r -partite graphs $K_{a,b}^r$ in place of complete bipartite graphs $K_{a,b}^2$. Again, note that $K_{a,b}^r$ has no independent transversal with respect to the standard r -partition $\{W_1, \dots, W_r\}$. Our construction will again run in two main phases.

Phase I. In this phase, for all $\alpha \geq \beta > \frac{1}{r} \left(1 - \frac{1}{r}\right)^{r-1}$ and sufficiently large t , we construct an r -graph G' and vertex partition $\mathcal{P}'$ with no independent transversal such that $\Delta(G') \leq \alpha t^{r-1}$, $\text{mad}(G', \mathcal{P}') \leq \beta t^{r-1}$, and every class $U \in \mathcal{P}'$ with $|U| < t$ has only vertices of degree $\lfloor \alpha^{1/(r-1)} t \rfloor^{r-2}$ and satisfies $|U| = \lfloor \alpha^{1/(r-1)} t \rfloor$. In particular when $\alpha = 1$, this gives an asymptotically tight construction for Theorem 7.11.

Similar to the graph case, we fix $t \geq 1$ and a sequence $d_1 < d_2 < \dots < d_k = \min\{t, \lfloor \alpha t \rfloor\}$ of positive integers. We inductively define r -graphs $G_1, \dots, G_{k-1}, G_k = G'$ and partitions $\mathcal{P}_1, \dots, \mathcal{P}_{k-1}, \mathcal{P}_k = \mathcal{P}'$ with no independent transversals. We start with G_1 being the disjoint union of t copies of K_{1,d_1}^r , and with $\mathcal{P}_1$ being the union of the standard r -partitions of the components, except merging into one class the A -side classes. For $j \geq 1$, we obtain G_{j+1} from G_j by adding, for each class $U \in \mathcal{P}_j$ of size $|U| < t$, exactly $t - |U|$ copies of $K_{1,d_{j+1}}^r$, and we obtain $\mathcal{P}_{j+1}$ from $\mathcal{P}_j$ by taking the union of $\mathcal{P}_j$ with the standard r -partition of each added component, while merging into one class the added copies of the A -sides along with U . Like in the graph case, we are effectively adding a layer of copies of $K_{1,d_{j+1}}^r$. As usual, one can check via Lemma 7.2 that G_{j+1} has no independent transversal with respect to $\mathcal{P}_{j+1}$. Also, G_{j+1} has maximum degree $\Delta(G_{j+1}) = d_{j+1}^{r-1} \leq \lfloor \alpha^{1/(r-1)} t \rfloor^{r-1} \leq \alpha t^{r-1}$, and the average degree of every new class in $\mathcal{P}_{j+1} \setminus \mathcal{P}_j$ is $\frac{1}{t}(d_j^{r-1} + (t - d_j)d_{j+1}^{r-1})$.

Just like in the graph case (see Lemma 4.13), we take our integer sequence to be $d_1 = \lfloor \beta t \rfloor$ and $d_j = d_{j-1} + 1$ for every $j \in \{2, \dots, d_k - d_1 + 1\}$. Then the average degree of every new class in $\mathcal{P}_{j+1}$ is

$$\frac{1}{t}(d_j^{r-1} + (t - d_j)d_{j+1}^{r-1}) = \frac{1}{t}(d_j^{r-1} + (t - d_j)(d_j + 1)^{r-1}) \leq \frac{1}{t}(d_j + 1)^{r-1}(1 + t - d_j).$$

As a function of d_j for fixed t , the right-hand side is maximized when $d_j = \left(1 - \frac{1}{r}\right)(t+1) - 1$, when it takes value

$$\frac{1}{t} \left(1 - \frac{1}{r}\right)^{r-1} (t+1)^{r-1} \left(\frac{1}{r}(t+1) + 1\right).$$

Since $\beta > \frac{1}{r} \left(1 - \frac{1}{r}\right)^{r-1}$, this quantity is at most βt^{r-1} for sufficiently large t . Since this upper bound holds for all $j \in \{1, \dots, k\}$, we conclude that $\text{mad}(G', \mathcal{P}') \leq \beta t^{r-1}$. All other required properties are satisfied, so this completes Phase I.

Phase II. In this phase, we assume that $\alpha > \left(1 - \frac{1}{r}\right)^{r-1}$ and

$$\beta > \max \left\{ \frac{1}{r} \left(1 - \frac{1}{r}\right)^{r-1}, (r-1) \left(1 - \alpha^{1/(r-1)}\right) \left(r \alpha^{1/(r-1)} - r + 2\right) \alpha^{(r-2)/(r-1)} \right\}.$$

If $\alpha \geq 1$, then $G', \mathcal{P}'$ from Phase I is a desired construction. Otherwise, we seek to enlarge the classes of $\mathcal{P}'$ of size $\lfloor \alpha t \rfloor < t$ into classes of size at least t . This will be done by taking disjoint unions of G' with copies of $K = K_{a,b}^r$, where $a = (r-1)(t - \lfloor \alpha^{1/(r-1)} t \rfloor) + 1$ and $b = \lfloor \alpha^{1/(r-1)} t \rfloor$. Note that $a + (r-1)b = (r-1)t + 1$, and one can check that $a < b$ and $a < t$ for sufficiently large t using that $\alpha > (1 - \frac{1}{r})^{r-1}$.

Throughout the procedure, we maintain a pair $(G, \mathcal{P})$ with the following properties: there is no independent transversal, $\Delta(G) \leq \alpha t^{r-1}$, $\text{mad}(G, \mathcal{P}) \leq \beta t^{r-1}$, and every class $U \in \mathcal{P}$ with $|U| < t$ satisfies $|U| \geq \lfloor \alpha^{1/(r-1)} t \rfloor$ and $d(u) \leq ab^{r-2}$ for all $u \in U$. We call such a pair $(G, \mathcal{P})$ a *valid pair*. At a given step, we take some class $U \in \mathcal{P}$ with $|U| < t$, add a disjoint copy of K to G , distribute $t - a$ vertices of U to the A -side of K to form a new class U_A , and distribute the remaining $|U| - t + a$ vertices of U to the B -sides of K to form $r-1$ new classes $U_{B,2}, \dots, U_{B,r}$ all of size at least $|U| + 1$. This is possible because $a < t$ and $b < |U| + 1$ and

$$(t - a) + (r-1)(|U| + 1 - b) \leq |U|,$$

with the latter inequality verified by substituting $a + b = (r-1)t + 1$ and applying that $|U| + 1 \leq t$. We repeat this procedure until all classes have size at least t .

We show that a valid pair $(G, \mathcal{P})$ is maintained. Suppose that we perform the above to derive the pair $(G^+, \mathcal{P}^+)$ from $(G, \mathcal{P})$. Then $(G^+, \mathcal{P}^+)$ has no independent transversal by Lemma 7.2, and we have $\Delta(G^+) \leq \alpha t^{r-1}$ because $\Delta(G) \leq \alpha t^{r-1}$ and $\Delta(K) \leq \max\{ab^{r-2}, b^{r-1}\} = b^{r-1} = \lfloor \alpha^{1/(r-1)} t \rfloor^{r-1} \leq \alpha t^{r-1}$ for large enough t . Now we verify that $\text{mad}(G^+, \mathcal{P}^+) \leq \beta t^{r-1}$. The new class U_A has average degree at most

$$\begin{aligned} & \frac{1}{t}(ab^{r-1} + (t-a)ab^{r-2}) \\ &= (r-1) \left(1 - \alpha^{1/(r-1)}\right) \left(r\alpha^{1/(r-1)} - r + 2\right) \alpha^{(r-2)/(r-1)} t^{r-1} + O(t^{r-2}), \end{aligned}$$

which is at most βt^{r-1} for sufficiently large t . Every vertex in one of the other new classes $U_{B,i}$ has degree exactly ab^{r-2} (if it came from K) or at most ab^{r-2} (if it came from U , by assumption). Thus, each $U_{B,i}$ has average degree at most

$$ab^{r-2} = (r-1) \left(1 - \alpha^{1/(r-1)}\right) \alpha^{(r-2)/(r-1)} t^{r-1} + O(t^{r-2}),$$

which is at most βt^{r-1} for sufficiently large t , using that $\alpha > (1 - \frac{1}{r})^{r-1}$. Since also $\text{mad}(G, \mathcal{P}) \leq \beta t^{r-1}$, it follows that $\text{mad}(G^+, \mathcal{P}^+) \leq \beta t^{r-1}$. Therefore, a valid pair is maintained throughout the stated procedure, and this completes Phase II and our construction.

We note that the value of b in Phase II was chosen so that $\Delta(K_{a,b}^r) = b^{r-1} \leq \alpha t^{r-1}$, and then the value of a was chosen so that $a + (r-1)b = (r-1)t + 1$. That way, (a slight generalization of) Theorem 7.3 implies that we can use $K_{a,b}^r$ with the standard r -partition to enlarge every class of a given vertex-partitioned graph with no independent transversal into a class of size at least t . These values of a and b are what led us to the rather complicated-looking condition (3) in Conjecture 7.13.

7.3.2 H -free transversals and acyclic transversals

In this subsection, we briefly discuss two other transversal problems from the literature that are related to independent transversals in uniform hypergraphs, namely H -free transversals in graphs and acyclic transversals in digraphs.

The first problem is H -free transversals in graphs. This problem was originally considered by Szabó and Tardos [161]. Here, H is a fixed k -regular graph, and given a graph G and partition $\mathcal{P} = \{V_1, \dots, V_n\}$, we seek a transversal T of $\mathcal{P}$ such that the induced subgraph $G[T]$ is H -free, meaning it does not contain a subgraph isomorphic to H . This time the parameter of interest is the maximum degree Δ of the graph G . Independent transversals correspond to the case $H = K_2$, and the general case is the hypergraph independent transversal problem in which $r = |V(H)|$ and the edges of the r -graph are those r -subsets of $V(G)$ that induce a subgraph containing an isomorphic copy of H . In this setting, the complete r -partite r -graph $K_r^r(m)$ can be associated with the graph $H(m)$, the m -fold *blow-up* of H , which is the graph obtained from H by replacing every vertex of H by an independent set of size m and replacing every edge by a copy of the complete bipartite graph $K_{m,m}$. Indeed, notice that $H(m)$ has no H -free transversal with respect to the partition where the classes are the blown up vertices of H . When Δ is a multiple of k , Theorem 7.3 applied to $H(\Delta/k)$ yields a graph G with maximum degree Δ and partition $\mathcal{P} = \{V_1, \dots, V_n\}$ such that $|V_i| \geq \left\lceil \frac{|V(H)|}{(|V(H)|-1)k} \Delta \right\rceil - 1$ for every i and there is no H -free transversal. This lower bound on the class sizes was also derived by Szabó and Tardos [161], who described a special case of our construction. Szabó and Tardos [161] conjectured that these class sizes are the best that can be achieved, and the case where H is a clique may be of particular interest.

Conjecture 7.14 (Szabó, Tardos [161]). *For a graph G with maximum degree Δ , partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(G)$, and regular graph H , if $|V_i| \geq \left\lceil \frac{|V(H)|}{(|V(H)|-1)k} \Delta \right\rceil$ for every i then there exists an H -free transversal.*

The second problem is *acyclic transversals* in digraphs. This problem was originally considered by Aharoni, Berger, and Kfir [9]. Given a digraph D (with no loops) and partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(D)$, we seek a transversal T of $\mathcal{P}$ such that the induced digraph $D[T]$ is *acyclic*, meaning it does not contain a directed cycle. Acyclic vertex sets are often considered as directed analogues of independent sets, and proper dicolorings (coloring vertices into acyclic vertex sets) are a widely studied subject. For our transversal problem, the parameter of interest is the maximum outdegree Δ^+ of the digraph D . (We can also restate the results in terms of the maximum indegree Δ^- , or in terms of $\Delta_{\min} = \min\{\Delta^-, \Delta^+\}$.) Independent transversals of a graph G are just acyclic transversals of the digraph D obtained from G by replacing every edge by two anti-parallel arcs. More interesting to study are digraphs with larger *digirth*, which is the length of a shortest directed cycle in the digraph. Aharoni, Berger, and Kfir [9] studied how topological methods apply to the problem of acyclic transversals, proving the following result.

Theorem 7.15 (Aharoni, Berger, Kfir [9]). *Let D be a digraph with maximum outdegree Δ^+ and digirth r , and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(D)$. If $|V_i| \geq \frac{r}{r-1}\Delta^+$, then there exists an acyclic transversal.*

Further, Aharoni, Berger, and Kfir [9] generalized an independent transversal construction of Jin [110] and Yuster [178] to show that the class size lower bound in Theorem 7.15 is best possible whenever $r \geq 2$ and Δ^+ is a power of r . Here, we observe that we can use Theorem 7.3 to produce tight constructions for all choices of $r \geq 2$ and $\Delta^+ \geq 1$. Notice that acyclic transversals in a digraph D can be translated into the independent transversal problem on the hypergraph with the same vertex set and with the edges being vertex sets that induce directed cycles of D . In this setting, the complete r -partite r -graph $K_r^r(m)$ can be associated with the digraph $\vec{C}_r(m)$, the m -fold *blow-up* of the directed cycle $\vec{C}_r$ of length r , which is the digraph obtained from $\vec{C}_r$ by replacing every vertex by an independent set of size m and replacing every arc e by a copy of the complete bipartite graph $K_{m,m}$ with all its edges oriented in the same direction as e . Notice that $\vec{C}_r(m)$ has no acyclic transversal with respect to the partition whose classes are the blown up vertices of $\vec{C}_r$. Applying Theorem 7.3 to $\vec{C}_r(m)$ with $\Delta^+ = m$ yields the following.

Theorem 7.16. *Let $\Delta^+ \geq 1$ and $r \geq 2$ be integers. Let $q = \left\lfloor \frac{r\Delta^+-1}{r-1} \right\rfloor$, and let D be the disjoint union of k copies of $\vec{C}_r(\Delta^+)$, where $k \geq 1 + r \left\lfloor \frac{q-\Delta^+}{\Delta^+-q(r-1)} \right\rfloor$. There exists a partition $\mathcal{P} = \{V_1, \dots, V_{k(r-1)+1}\}$ of $V(D)$ such that $|V_i| \geq q$ for every i and there is no acyclic transversal.*

To conclude this section, we mention that there are many other complexes that have been associated with graphs, hypergraphs, and other discrete structures, and we could consider the associated transversal problem on any of them. For each of these transversal problems there are several directions to explore. Besides optimal class size conditions for the existence of transversals, the following are some other possible directions to explore for a transversal problem, inspired by our various results on independent transversals in graphs and full rainbow matchings in multi-hypergraphs:

- Prove a Brooks-type theorem about for which (hyper)graphs the optimal class size conditions are tight (see Section 5.1).
- Improve class size conditions if we assume that the (hyper)graph has a special structure, such as being claw-free or chordal (see Section 5.5 and [5, 13]).
- Improve class size conditions if we assume that the partition has a special structure, such as being locally sparse (see Section 5.3 and [43, 78, 116, 127]).
- Find interesting list coloring results based on an appropriate analogue of color degree (see Section 5.4 and Section 6.4).

- Figure out whether the construction method produces all possible extremal constructions in some sense (see Section 5.6).
- Consider average degree conditions instead of, or in addition to, maximum degree conditions (see Chapter 4, particularly Section 4.3 and Section 4.5, and [83, 167]).
- Explore strong coloring results, which are about optimal class size conditions for the existence of a decomposition into transversals (see [13, 97, 98]).

Many of the problems formulated at the end of previous chapters are of these sorts.

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Appendices

Appendix A

Topological background for transversals

The goal of this appendix is to provide some more information about the topological tools that are used in this thesis, particularly those applied to independent transversals. Further background on algebraic topology can be found in [94, 136]. It is not necessary to understand all the topological definitions, as our work primarily uses the results stated here as black boxes and most of our contributions will be combinatorial arguments. Our writing here can be found in some form in various sources [5, 6, 13, 16, 99], and our emphasis is homological. For surveys with a more combinatorial emphasis, see [100, 101].

For the sake of clarity, we will repeat some of the concepts that are mentioned throughout this thesis. An *abstract simplicial complex* (which we will also refer to as just a *simplicial complex* or a *complex*) $\mathcal{C}$ is a collection of subsets of some finite ground set $V = V(\mathcal{C})$ such that if $A \in \mathcal{C}$ and $B \subseteq A$, then $B \in \mathcal{C}$. The elements of the ground set V are called the *vertices* of $\mathcal{C}$, and the sets in $\mathcal{C}$ are called the *simplices* or *faces* of $\mathcal{C}$. For $X \subseteq V(\mathcal{C})$, denote the induced subcomplex $\mathcal{C}[X] = \{A \in \mathcal{C} : A \subseteq X\}$. Recall for a graph G , the *independence complex* $\mathcal{I}(G)$ is the complex consisting of all independent sets of G . For a multi-hypergraph G , the *matching complex* $\mathcal{M}(G)$ is the complex consisting of all matchings of G .

A *geometric simplex* σ is the convex hull of affinely independent points in Euclidean space (called the *vertices* of σ). A *face* of σ is the convex hull of some subset of its vertices. A *geometric simplicial complex* $\mathcal{K}$ is a collection of geometric simplices in a fixed Euclidean space such that (1) if $\sigma \in \mathcal{K}$ and τ is a face of σ then $\tau \in \mathcal{K}$, and (2) if $\sigma, \sigma' \in \mathcal{K}$ then $\sigma \cap \sigma'$ is a face of both σ and σ' . The union of all the simplices of $\mathcal{K}$ is called the *body* of $\mathcal{K}$, and is denoted $\|\mathcal{K}\|$. We view the body as a topological space. Every complex $\mathcal{C}$ has a *geometric realization*, which is a geometric simplicial complex $\mathcal{K}$ in some Euclidean space such that the faces in $\mathcal{C}$ are precisely the vertex sets of the simplices in $\mathcal{K}$. Every geometric realization of $\mathcal{C}$ is unique up to homeomorphism, and we denote the topological space associated with the body as $\|\mathcal{C}\|$. See Figure A.1 for illustration of geometric realizations of independence complexes of small cycles and paths.

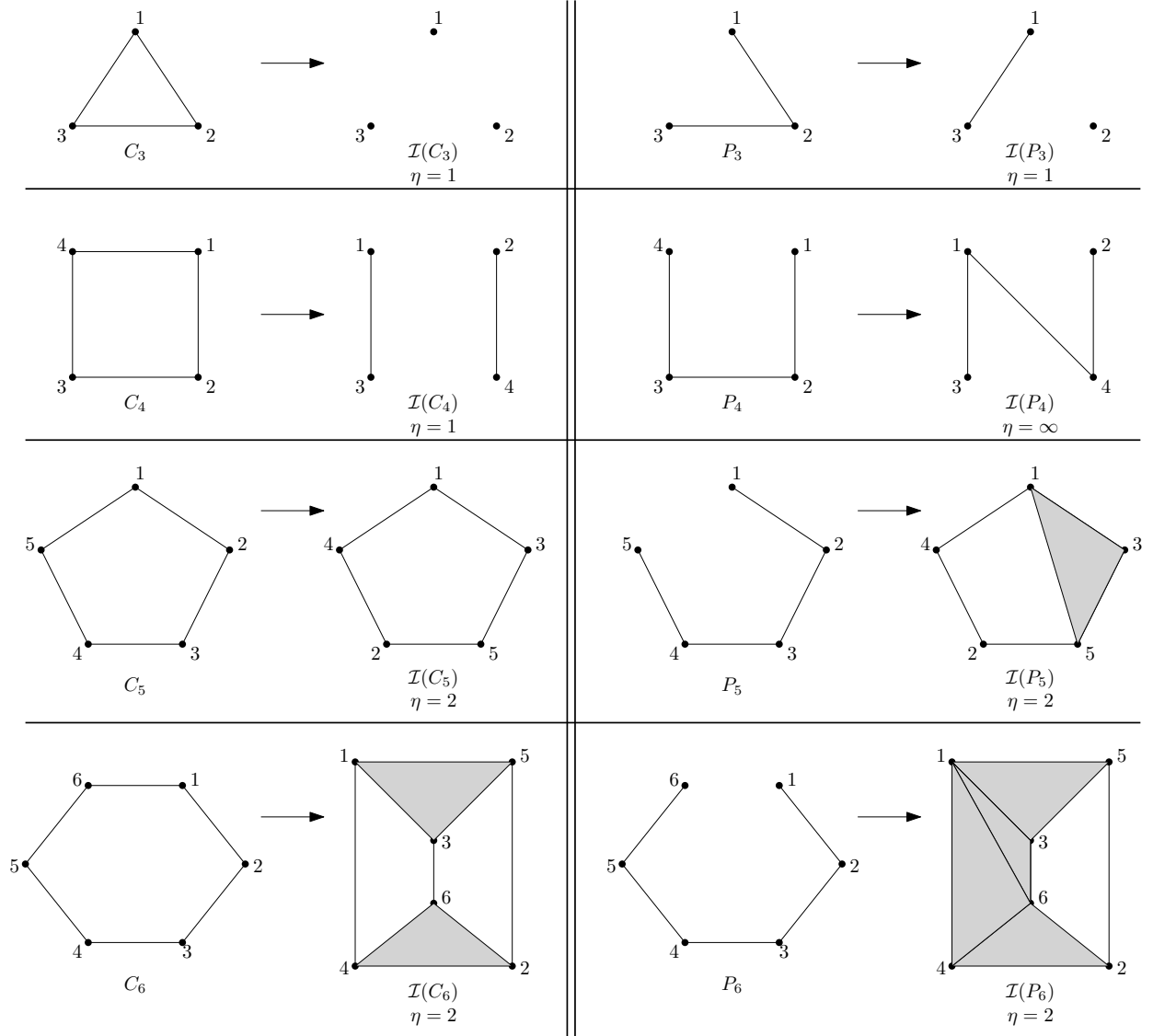


Figure A.1: The independence complexes of small cycles and paths, and their topological connect- edness η . In general, $\eta(\mathcal{I}(P_n)) = \lceil n/3 \rceil$ if $n \equiv 0, 2 \pmod{3}$, $\eta(\mathcal{I}(P_n)) = \infty$ if $n \equiv 1 \pmod{3}$, and $\eta(\mathcal{I}(C_n)) = \lceil n/3 \rceil$ for all n (where $\lceil \alpha \rceil$ denotes the rounding of α to the nearest integer), by Claim 3.3 of Meshulam [131].

Let B^{n+1} denote the n -dimensional unit ball and let S^n denote its boundary, the $(n-1)$ -dimensional unit sphere. A complex $\mathcal{C}$ is *homotopically k -connected* if for every $-1 \leq j \leq k$, every continuous map $f : S^j \rightarrow \|\mathcal{C}\|$ can be extended to a continuous map $\tilde{f} : B^{j+1} \rightarrow \|\mathcal{C}\|$. Equivalently, $\mathcal{C}$ is k -connected if the homotopy groups $\pi_j(\mathcal{C})$ are trivial for $-1 \leq j \leq k$. By convention, we take -1 -connectedness to mean being nonempty. Intuitively, a topological space being k -connected means it has no holes of dimensions k or less, with a hole being a sphere that cannot be continuously shrunk to a point. Further intuition is explained below. We define $\eta_\pi(\mathcal{C})$ to be the largest k such that $\mathcal{C}$ is k -connected, plus two:

$$\eta_\pi(\mathcal{C}) = \max\{k : \pi_k(\mathcal{C}) = 0\} + 2.$$

Analogously, a complex $\mathcal{C}$ is *homologically k -connected* (also called *k -acyclic*) if the reduced homology groups $\tilde{H}_j(\mathcal{C})$ with rational coefficients (or any field coefficients) vanish for $-1 \leq j \leq k$. Then we define

$$\eta_H(\mathcal{C}) = \max\{k : \tilde{H}_k(\mathcal{C}) = 0\} + 2.$$

We write $\eta(\mathcal{C})$ to mean either $\eta_\pi(\mathcal{C})$ or $\eta_H(\mathcal{C})$ when it does not matter for the context. In general $0 \leq \eta_\pi(\mathcal{C}) \leq \eta_H(\mathcal{C})$, and a theorem of Hurewicz (see [94]) implies that $\eta_\pi(\mathcal{C}) = \eta_H(\mathcal{C})$ whenever $\eta_\pi(\mathcal{C}) \geq 3$. The notation $\eta(\cdot)$ was introduced by Aharoni and Berger [6]. The addition of two in the definition helps to simplify many of the results formulated below. Combinatorially, it represents the number of vertices of a simplex used to fill in the holes. There are different benefits to working with η_π or η_H , and the literature uses both of them, so we included both of them in our discussion.

See Figure A.1 for the connectedness values of small cycles and paths. For concrete intuition on homotopical connectedness, by convention $\eta_\pi(\mathcal{C}) = 0$ is the same as $\mathcal{C}$ being empty. Next, $\eta_\pi(\mathcal{C}) = 1$ is the same as saying that the complex $\mathcal{C}$ is nonempty but not path-connected: If $\mathcal{C}$ has two components, one containing vertex u and one containing vertex v , then the continuous map $f : S^0 \rightarrow \|\mathcal{C}\|$ with $f(-1) = u$ and $f(1) = v$ cannot be extended to a continuous map $\tilde{f} : B^1 \rightarrow \|\mathcal{C}\|$. Next, $\eta_\pi(\mathcal{C}) = 2$ is the same as saying that the complex $\mathcal{C}$ is nonempty and path-connected, but not “simply-connected”: there exist loops in $\|\mathcal{C}\|$ (i.e., continuous maps $f : S^1 \rightarrow \|\mathcal{C}\|$) that cannot be contracted to a point (i.e., cannot be extended to continuous maps $\tilde{f} : B^2 \rightarrow \|\mathcal{C}\|$). This is what we mean by saying that $\mathcal{C}$ has a 1-dimensional hole. Similar statements hold for $\eta_\pi(\mathcal{C}) \geq 3$. By Whitehead’s Theorem (see [94]), $\eta_\pi(\mathcal{C}) = \infty$ is the same as saying that the complex $\mathcal{C}$ is contractible, i.e., $\mathcal{C}$ can be contracted (or is homotopy equivalent) to a point. This concrete intuition does not always hold for homological connectedness. In particular, there exist complexes $\mathcal{C}$ such that $\eta_\pi(\mathcal{C}) = 2$ but $\eta_H(\mathcal{C}) = \infty$ (see Example 2.38 in [94]). But for most of the complexes of our interest, η_π and η_H are the same.

The following topological Hall theorem from [15, 130, 131] is useful for many of the transversal problems in this thesis. Recall that for a complex $\mathcal{C}$ and partition $\mathcal{P} = \{V_1, \dots, V_n\}$ of $V(\mathcal{C})$, a *colorful simplex* of $\mathcal{C}$ with respect to $\mathcal{P}$ is a simplex of $\mathcal{C}$ that contains one vertex from each V_i .

Theorem A.1 ([15, 130, 131]). *Let $\mathcal{C}$ be a complex, and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(\mathcal{C})$. If*

$$\eta \left(\mathcal{C} \left[\bigcup_{i \in I} V_i \right] \right) \geq |I| \quad \text{for all } I \subseteq [n],$$

then there exists a colorful simplex.

An early version of Theorem A.1 when $\eta = \eta_\pi$ was proven in [15] (see also [14]). There, the statement was proved by reducing it to Sperner's Lemma, which is the special case when $\mathcal{C}$ is a triangulation of a simplex and $\mathcal{P}$ forms a ‘‘Sperner coloring’’ of $V(\mathcal{C})$. This kind of reduction leads to a constructive combinatorial proof of Theorem A.1. Another variant with $\eta = \eta_H$ was proven in [130]. Finally Meshulam [131] proved Theorem A.1 explicitly with $\eta = \eta_H$ (see Proposition 1.6 in [131]). He proved it as a corollary of a homological Nerve Theorem that he obtained using a spectral sequence argument, and this homological Nerve Theorem has also been applied to prove topological Helly theorems [115]. A more elementary proof of this Nerve Theorem using acyclic carriers was given in [132].

Recently, Theorem A.1 has been strengthened to prove the existence a colorful simplex under the weaker condition stated in the following theorem. This is due to [52, 132, 133].

Theorem A.2 ([52, 132, 133]). *Let $\mathcal{C}$ be a complex, and let $\mathcal{P} = \{V_1, \dots, V_n\}$ be a partition of $V(\mathcal{C})$. If*

$$\tilde{H}_{|I|-2} \left(\mathcal{C} \left[\bigcup_{i \in I} V_i \right] \right) = 0 \quad \text{for all } I \subseteq [n],$$

then there exists a colorful simplex.

In particular, see Proposition 2.6 in [52] for a short proof using $\mathbb{Z}_2$ coefficients. This is a stronger result than Theorem A.1, which requires that $\tilde{H}_j(\mathcal{C}[\bigcup_{i \in I} V_i]) = 0$ for every $-1 \leq j \leq |I| - 2$. Nevertheless, Theorem A.1 suffices for the work in this thesis. This is primarily because the independence complexes of the extremal graphs we will analyze are usually either contractible or homotopy equivalent to wedge sums of spheres of equal dimensions. In particular, their reduced homology groups will usually vanish in all but at most one dimension.

Based on Theorem A.1, to prove the existence of a colorful simplex we need effective lower bounds on the connectedness $\eta(\mathcal{C})$ of a given complex $\mathcal{C}$. The proofs we give in this section will focus on homological connectedness $\eta = \eta_H$ for independence complexes of graphs, but similar results hold for homotopical connectedness η_π . We will obtain these lower bounds by breaking down the complex into simpler sub-complexes. The *join* of two complexes $\mathcal{A}$ and $\mathcal{B}$ is the complex $\mathcal{A} * \mathcal{B} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}$. For example, if G and H are disjoint graphs, then we have $\mathcal{I}(G \cup H) = \mathcal{I}(G) * \mathcal{I}(H)$.

Lemma A.3. (a) Every graph G has $\eta_H(\mathcal{I}(G)) \geq 0$, with equality if and only if G is empty.

(b) If G has an isolated vertex, then $\eta_H(\mathcal{I}(G)) = \infty$.

(c) If G and H are disjoint graphs, then $\eta_H(\mathcal{I}(G \cup H)) = \eta_H(\mathcal{I}(G)) + \eta_H(\mathcal{I}(H))$.

Proof. Part (a) follows from the definition. For part (b), if G has v as an isolated vertex, then $\mathcal{I}(G) = \{v\} * \mathcal{I}(G - v)$ is a cone, which is contractible and hence all reduced homology groups of G vanish. Part (c) comes from the more general equality for complexes, $\eta_H(\mathcal{A} * \mathcal{B}) = \eta_H(\mathcal{A}) + \eta_H(\mathcal{B})$, which in turn follows from the formula $\tilde{H}_{n+1}(\mathcal{A} * \mathcal{B}) \cong \bigoplus_{p+q=n} (\tilde{H}_p(\mathcal{A}) \otimes \tilde{H}_q(\mathcal{B}))$ (a consequence of Künneth's Theorem, see [33]). $\square$

In Lemma A.3, when it comes to homotopical connectedness η_π the statements of part (a) and part (b) still hold, but for part (c) we only have the inequality $\eta_\pi(\mathcal{I}(G \cup H)) \geq \eta_\pi(\mathcal{I}(G)) + \eta_\pi(\mathcal{I}(H))$. This inequality is still sufficient for the applications we discuss.

For two complexes $\mathcal{A}$ and $\mathcal{B}$, the Mayer-Vietoris sequence is a naturally defined sequence of abelian group homomorphisms

$$\cdots \rightarrow \tilde{H}_{n+1}(\mathcal{A} \cup \mathcal{B}) \rightarrow \tilde{H}_n(\mathcal{A} \cap \mathcal{B}) \rightarrow \tilde{H}_n(\mathcal{A}) \oplus \tilde{H}_n(\mathcal{B}) \rightarrow \tilde{H}_n(\mathcal{A} \cup \mathcal{B}) \rightarrow \tilde{H}_{n-1}(\mathcal{A} \cap \mathcal{B}) \rightarrow \cdots$$

The Mayer-Vietoris Theorem states that this sequence is exact (see [94, 136]). In particular, if we have $\tilde{H}_n(\mathcal{A}) \oplus \tilde{H}_n(\mathcal{B}) = 0$ and $\tilde{H}_{n-1}(\mathcal{A} \cap \mathcal{B}) = 0$, then $\tilde{H}_n(\mathcal{A} \cup \mathcal{B}) = 0$. In particular, this gives us that

$$\eta_H(\mathcal{A} \cup \mathcal{B}) \geq \min\{\eta_H(\mathcal{A}), \eta_H(\mathcal{B}), \eta_H(\mathcal{A} \cap \mathcal{B}) + 1\}.$$

One application of this inequality is the following result stated in [6]. The *link* of a simplex A in a complex $\mathcal{C}$ is the complex $\text{link}_{\mathcal{C}}(A) = \{B \in \mathcal{C} : A \cap B = \emptyset, A \cup B \in \mathcal{C}\}$.

Theorem A.4. For every complex $\mathcal{C}$ and vertex $v \in V(\mathcal{C})$, we have

$$\eta_H(\mathcal{C}) \geq \min\{\eta_H(\mathcal{C} - v), \eta_H(\text{link}_{\mathcal{C}}(v)) + 1\}$$

Proof. Let $\mathcal{A} = \mathcal{C} - v$ and $\mathcal{B} = \text{link}_{\mathcal{C}}(v) * \{v\} = \{B \cup \{v\} : B \in \text{link}_{\mathcal{C}}(v)\}$ (where $*$ denotes the join operation). Then we have $\mathcal{A} \cap \mathcal{B} = \text{link}_{\mathcal{C}}(v)$ and $\mathcal{A} \cup \mathcal{B} = \mathcal{C}$. Since $\mathcal{B}$ is a cone, it is contractible and $\eta_H(\mathcal{B}) = \infty$. By the previous inequality, the statement follows. $\square$

In the particular case when $\mathcal{C} = \mathcal{I}(G)$ is the independence complex of a graph G , we have that $\text{link}_{\mathcal{I}(G)}(v) = \{A \subseteq V(G - v) : A \cup \{v\} \in \mathcal{I}(G)\} = \mathcal{I}(G - v - N(v))$, where $N(v)$ is the set of neighbors of v in G . Thus, Theorem A.4 states that for any graph G and vertex $v \in V(G)$, we have

$$\eta_H(\mathcal{I}(G)) \geq \min\{\eta_H(\mathcal{I}(G - v)), \eta_H(\mathcal{I}(G - v - N(v))) + 1\}.$$

Our work on independent transversals in this thesis involves a variation of the above inequality due to Meshulam [131]. As another application of the Mayer-Vietoris theorem, if $\tilde{H}_n(\mathcal{A} \cup \mathcal{B}) = 0$ and $\tilde{H}_n(\mathcal{A} \cap \mathcal{B}) = 0$, then $\tilde{H}_n(\mathcal{A}) \oplus \tilde{H}_n(\mathcal{B}) = 0$. In particular,

$$\eta(\mathcal{A}) \geq \min\{\eta(\mathcal{A} \cup \mathcal{B}), \eta(\mathcal{A} \cap \mathcal{B})\}.$$

We apply this inequality as follows. For a graph G and edge $e = uv \in E(G)$, the *explosion* of e in G is the graph $G * e = G - (N(u) \cup N(v))$. That is, $G * e$ is the graph obtained from G by deleting both endpoints of e and all of their neighbors.

Theorem A.5 (Meshulam [131]). *For every graph G and edge $e \in E(G)$, we have*

$$\eta_H(\mathcal{I}(G)) \geq \min\{\eta_H(\mathcal{I}(G - e)), \eta_H(\mathcal{I}(G * e)) + 1\}$$

Proof. For edge $e = uv$, let $\bar{e}$ denote the 1-dimensional simplex with vertices u and v . Let $\mathcal{A} = \mathcal{I}(G)$ and $\mathcal{B} = \mathcal{I}(G * e) * \bar{e}$. We notice that $\mathcal{A} \cup \mathcal{B} = \mathcal{I}(G - e)$. On the other hand, $\mathcal{A} \cap \mathcal{B} = \mathcal{I}(G * e) * \{u, v\}$, which is just the suspension of $\mathcal{I}(G * e)$. The suspension theorem (another application of the Mayer-Vietoris theorem) implies that $\tilde{H}_{j+1}(\mathcal{I}(G * e) * \{u, v\}) \cong \tilde{H}_j(\mathcal{I}(G * e))$ for all $j \geq -1$, so that $\eta_H(\mathcal{A} \cap \mathcal{B}) = \eta_H(\mathcal{I}(G * e) * \{u, v\}) = \eta_H(\mathcal{I}(G * e)) + 1$. By the previous inequality, the result follows. $\square$

All of the results we stated above also hold for η_π in place of η_H (see [6, 13, 100, 101]). Both Theorem A.4 and Theorem A.5 are useful for deriving lower bounds on the connectedness of independence complexes of graphs. By Theorem A.1, they will also be useful for finding sufficient conditions for the existence of IT's. Applications of these theorems are sometimes described in terms of a game, between players CON who wants to prove that η is large and NON who wants to make η small (see [5, 13, 16]). Theorem A.4 and Theorem A.5 often give the same lower bounds. Our proofs mostly utilize Theorem A.5 because it is easier to analyze in the general bounded degree setting. On the other hand, Theorem A.4 is more useful when studying graphs characterized by forbidden induced subgraphs (e.g., $K_{1,k}$ -free graphs in [5]).

We also mention the following lower bound on the homological connectedness of the independence complex of a graph, due to Aharoni, Berger, and Meshulam [11]. The *Laplacian matrix* $L = L(G)$ of a graph $G = (V, E)$ is the $V \times V$ positive semidefinite matrix defined by $L(u, v) = d_G(u)$ if $u = v$, $L(u, v) = -1$ if $uv \in E$, and $L(u, v) = 0$ otherwise. Let $\lambda(G)$ denote the largest eigenvalue of G .

Theorem A.6 (Aharoni, Berger, Meshulam [11]). *For every graph G , we have*

$$\eta_H(\mathcal{I}(G)) \geq \frac{|V(G)|}{\lambda(G)}.$$

The proof of Theorem A.6 is technical, involving simplicial Hodge theorem and a local-to-global method of Garland [77]. Theorem A.6 turns out to be useful for the problem of full

rainbow matchings in bounded degree multi-hypergraphs, which is the subject of Chapter 6. Other applications of Theorem A.6 are described in [5, 11].

We conclude this appendix by briefly discussing some differences between homotopical η_π and homological connectedness η_H . Homotopical proofs, which are often based on analyzing triangulations, are usually more combinatorially intuitive and algorithmically constructive, but they sometimes require technical considerations to be rigorous. In particular, the triangulations often need to be ensured to be piecewise-linear (PL), which can be tedious to verify. Homology-based proofs are often easier to perform without technicalities, but require a bit more background in algebraic topology, and they are usually algorithmically non-constructive.

We also discuss the complexity of computing connectedness and of finding a colorful simplex, which is also explained in [1, 2]. For computing homotopical connectedness η_π , it is undecidable to determine whether $\eta_\pi(\mathcal{C}) \geq 3$, even if $\mathcal{C} = \mathcal{I}(G)$ is the independence complex of a graph. If we know that $\eta_\pi(\mathcal{C}) \geq 3$, then by Hurewicz’s Theorem we know that $\eta_\pi(\mathcal{C}) = \eta_H(\mathcal{C})$, and computing $\eta_H(\mathcal{C})$ can be done via standard homology group calculations. Nevertheless, it has been shown to be NP-complete to determine whether $\eta_H(\mathcal{I}(G)) \geq k$ for any $k \geq 3$, even if G is a chordal graph [2]. For the problem of finding a colorful simplex, knowing that the connectedness is sufficiently high, there is no known efficient general procedure. Homological proofs of Theorem A.1 are ineffective algebraic arguments. Triangulation-based homotopic proofs are algorithmically effective, finding a colorful simplex via a reduction to Sperner’s lemma whose proof is an effective path-following argument. However, these reductions lead to inefficient algorithms. Efficient algorithms for independent transversals in bounded degree graphs are discussed in [82].