

Nonlinear Beam Behaviour with a Moving Mass

by

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Abstract

Dynamics of a flexible cantilever beam carrying a moving mass-spring is investigated. The system is an idealisation of an important class of problems characterised by nonlinear interaction between a continuously distributed mass and stiffness sub-system and a lumped mass and stiffness sub-system. Two models are developed, one with the beam undergoing small oscillations under the presence of kinematic nonlinearities arising from the coupled mass-beam behaviour and a second model for large oscillations where geometric nonlinearities are also considered.

A closed form solution is obtained using the perturbation method of multiple scales and a parametric analysis is conducted. System response is investigated under internal resonance conditions between the moving mass and the beam. A new technique, based on the finite element method is presented for numerical solution of nonlinear partial differential equations. Using this method the dynamics of the mass-beam system is investigated numerically for both the small oscillation and the large oscillation models. Motion of a cantilever beam without the moving mass, undergoing large oscillations is also analysed. Time-Frequency analysis is used to investigate the spectral behaviour of the system.

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Chapter 1

Introduction

One of the distinguishing features of this thesis is that it encompasses many different areas of nonlinear dynamics: modelling of large oscillations in continuously distributed systems, analytical solution of nonlinear ordinary differential equations using perturbation techniques, numerical solution of nonlinear integro-partial differential equations with algebraic constraints using finite elements, and dynamic analysis using time-frequency methods. The techniques developed in all these areas are quite essential for investigating dynamics of nonlinear systems. The perturbation methods are generally the only means available with potential to find a closed form solution for nonlinear systems. Although the solution is generally obtained using a reduced model with many approximations, it provides qualitative insight into the system behaviour and allows for parametric analysis. In addition, using the perturbation solution, amplitude and phase modulation can be investigated which occurs in nonlinear systems under internal resonance conditions (when the natural frequencies of the system are commensurate). A numerical solution on the

other hand provides quantitative results and is also a necessary step in performing spectral analysis. Spectral analysis involves finding the frequencies in the system using the well known discrete Fourier transform (power spectrum). This approach however only gives the averaged behaviour of the frequencies over the length of the time series for which the Fourier transform was performed. As will be shown in this work, nonlinear systems can exhibit changes in frequencies with time. To investigate this affect, time-frequency analysis methods are used which capture the changes in the frequency content of a signal with time.

In this thesis, significant contributions are made in each of the areas outlined earlier. A small and large oscillation model is developed for a flexible cantilever beam carrying a moving spring-mass sub-system. The small oscillation model includes only kinematic nonlinearities arising due to the velocity and acceleration coupling between the beam and the moving mass and the large oscillation model includes kinematic nonlinearities as well as geometric nonlinearities arising due to nonlinear strain displacement relationships. In this work kinematic nonlinearities refers to the nonlinearities due to coupling between different bodies. A closed form analytical solution is obtained using the perturbation method of multiple scales for the small oscillation model. Using this solution a parametric analysis is conducted to investigate system behaviour under internal resonance conditions between the mass and the beam. A numerical solution is obtained for the small oscillation model using the Galerkin method and an automatic ordinary differential equation solver. The results are compared with the perturbation solution. When the geometric nonlinearities are included in the model and large amplitude oscillations are

considered, the strategy used for numerical solution in the small oscillation model, using an automatic ordinary equation solver does not work. Reliable methods to solve the nonlinear partial differential equations of motion for the large amplitude oscillation model do not exist. The presence of the moving mass makes the problem much more complicated as the equations now become integro-partial differential. In this thesis, a new technique is presented to obtain numerical solution for such systems. Using this technique, results are presented for a cantilever beam and for the mass beam system undergoing large amplitude oscillations.

One of the first challenges in investigation of dynamics of a continuously distributed nonlinear systems is to solve the equations of motion. Considering solid mechanics and the theory of elasticity as the domain of the problem, the equations of motion are traditionally derived using specialised theories for the system elements. An example is the well known Euler-Bernoulli theory for beams. Classical techniques like the Hamilton's principle complete the derivation giving equations of motion which are in general nonlinear, integral, partial differential with algebraic constraints and some complex boundary conditions. These equations are often difficult to solve, numerically or otherwise; therefore, approximate methods are sought.

Approximate techniques have emerged from many founding theories and principles. They constitute both analytical and numerical methods. Some of them will be expounded upon later in this chapter. Most of these methods are simply strategies which may or may not work for a particular problem. New methods are often developed in the process of solving a certain problem when the existing methods fail, or are very difficult to apply. Some of these techniques can have more far

reaching applications than the original problem for which they were designed. In this light, this work presents a new technique for numerical solution of equations of motion of continuously modelled nonlinear systems.

The application of interest, a flexible cantilever beam carrying a moving spring-mass system historically originates in the earlier part of the nineteenth century. A literature survey found the earliest relevant published paper to be by Stokes [1]) in 1849 in the design of railway bridges and later in other transportation engineering structures. It has many other applications from gantry cranes carrying moving loads, robotic arms, to space structures. An extension of this problem is the interaction between axial flow and slender structures which is of considerable interest to researchers in fluid dynamics area (example Belanger et al. [2]). Some novel applications such as using the moving mass as a controller to suppress vibrations in the beam have also been proposed (Golnaraghi et al. [3, 4]).

The motivation for the present work emerges from Golnaraghi et al. [3, 4] where owing to the difficulties in modelling and analysis of continuously distributed nonlinear systems, a rather simplistic model was used for the beam (a rigid body with a rotational spring). Although this model captures the effect of kinematic nonlinearities, a model with continuously distributed properties is quite essential for this system because the oscillations of the moving mass depends on the localised dynamic behaviour of the beam. In a later study Khalily et al. [5] used an Euler-Bernoulli beam undergoing small oscillations with a moving mass. The numerical results obtained in [5] were not satisfactory, as unrealistically large initial values were required to show the coupling between the mass and the beam.

In the next few sections, foundation is laid for presenting this thesis along with a literature survey highlighting significant accomplishments. The general theme here is to provide a brief description of various methods used in the literature to compare or contrast their relationship with the present work. Many of the methods discussed in these sections are not used in this work but they are presented here as a review of the existing literature. Four main areas are considered. Firstly in Section 1.1 beam modelling for large oscillations is discussed and the strategy used for modelling the beam in the present work is highlighted. Secondly, in Section 1.2 analytical methods for nonlinear systems are considered. Methods for semi-discretization (reduction of partial differential equations to ordinary differential equations) are discussed. Methodologies to obtain a closed form analytical solution for a simplified model (only kinematic nonlinearities are included) are highlighted. Thirdly, Section 1.3 discusses the various numerical methods for solving equations of motion. The new method developed to investigate dynamics of continuously distributed nonlinear systems is outlined. Fourthly in Section 1.4 a literature survey is presented for the beam carrying the moving mass system and the accomplishments of the present work as related to this system are discussed.

1.1 Beam Modelling

Nonlinearities in continuously modelled elastic solids are of two types, geometric and material. When multiple bodies are involved, kinematic type nonlinearities can also arise due to inertial coupling. In this work only geometric and kinematic nonlinearities are considered. The beam is modelled using the Euler-Bernoulli as-

sumptions and therefore shear deformation effects and rotatory inertia terms are neglected. In addition, the beam is assumed to be inextensible.

Linear elasticity is characterised by a linear stress-strain relationship. Models obtained using the full three dimensional theory of linear elasticity employing displacement fields are seldom used because analytical solutions are rarely possible and numerical solutions pose significant challenges. Therefore, approximate theories are developed by making assumptions regarding the displacement field of the continuum. The displacement field describes the deformations in the continuum. If the deformations can be represented by fewer than three dependent variables (dependency on body fixed spatial coordinates and time for dynamic problems) then considerable simplifications can be achieved. For small planar oscillations the Euler-Bernoulli beam model is an example where the deformations are described in terms of only one dependent variable. When large oscillations are involved three issues arise. Firstly, finite strains have to be considered which leads to geometric nonlinearities in the model. Secondly, in simple beam theories like the Euler-Bernoulli theory, the displacement field cannot be explicitly described using one variable without making additional assumptions, even though the problem is still one dimensional. The reason for this is that displacements in the axial and lateral directions are related through a nonholonomic constraint; therefore one of the variables cannot be explicitly eliminated. The third issue is boundary conditions; for beams with certain boundary conditions like hinged-hinged or fixed-fixed, the axial extension could be significant and has to be accounted for. The axial extension appears as an additional variable in quadratic form in the strain-displacement

relationship. The way in which these issues are dealt with gives rise to different models of the beam.

Since the Euler-Bernoulli beam theory was developed, in the eighteenth century, large deflection analysis has been the focus of many research papers. Most of the early work in large deflection analysis of beams was directed towards obtaining static deflection curves for uniform beams. These publications include Barten [6] (a mistake in this paper was corrected later in [7]) and Bishopp and Drucker [8] where the deflection curve of a simple cantilever beam with a concentrated load acting at the free end was obtained in terms of elliptic integrals. An approximate series solution for the deflection curve of a cantilever beam carrying uniformly distributed load is presented by Rohde [9]. A review of these and other works from that era (before 1963) is given by Eisley [10] and a comprehensive treatment is available in the book by Frisch-Fay [11]. Some of the later works in static analysis are by Schimdt and Dadeppo [12, 13] and Bishopp [14]. In [12], an approximate solution for a cantilever beam is obtained using a method similar to that used in [9] and also post buckling behaviour of a column is investigated. In the publications cited earlier [6]-[9] and [14] the beam models are based on the Euler-Bernoulli assumption that plane cross sections remain plane and perpendicular to the neutral axis and also the beam is assumed to be axially inextensible. In [13] a theory is presented to account for shear deformation in the nonlinear model, thus relaxing the assumption that the plane cross sections should remain perpendicular to the neutral axis. Sinclair [15] presented a model for a cantilever beam where, in addition to shear deformation, axial extension was also incorporated.

There are two traditional approaches to beam modelling, one is the geometric method where the deformation field is based on geometrical assumptions like plane cross sections remain plane and perpendicular to the neutral axis and the other is based on a direct derivation from the theory of linear elasticity. In the second approach, the objective is to derive a model for the beam through a consistent reduction from the theory of elasticity. The references cited earlier, [6]–[14] use the geometrical method which is the easiest of the two techniques to implement. The second approach was used by Dokmeci [16] where a consistent method is developed to derive beam theories of various orders from the three-dimensional theory of thermo-elasto-dynamics (thermal effects are considered and the method is not limited to linear elasticity). In [16], Euler-Bernoulli beam theory and Timoshenko beam theory are derived as special cases.

In the dynamic analysis of beams undergoing large oscillations a number of papers have been published on hinged-hinged beams or fixed-fixed beams with immovable supports. In these models the main focus has been on the nonlinearities that arise due to the axial extension of the beam. One of the early papers is by Woinowsky-Krieger [17] in which a hinged-hinged beam was considered, where the only nonlinearity is due to axial extension of the beam. A closed form solution was obtained in terms of Jacobi-elliptic cosine function. The effect of rotatory inertia and axial elongation was considered by Eringen [18], where a perturbation solution was obtained in terms of Jacobi-elliptic functions for a simply supported beam with immovable supports. Some of the inconsistencies in this model were pointed out and a new simpler set of equations was presented by Woodall [19]. This model,

in essence, is an Euler-Bernoulli beam undergoing large oscillations. In [19], the equations of motion were derived by considering the equilibrium of various forces acting on a small beam element.

Large nonplanar torsional oscillations were first modelled by Crespo da Silva and Glynn [20, 21]. In [20] the beam can undergo flexure in two planes and torsion about the longitudinal axis. This model was later extended to include axial elongation [22]. The resulting equations from these models contain trigonometric nonlinearities which do not allow the use of perturbation methods for analysis. Therefore in the second parts of these papers [21, 23] reduced equations with polynomial nonlinearities were employed to investigate the beam response under external periodic excitation using a perturbation method. The reduction of general nonlinear equations of motion for beams with varying properties and carrying fixed masses is further discussed in [24].

In the present work the equations of motion for a cantilever beam undergoing large planar oscillations are derived using Hamilton's principle following the approach taken by Crespo da Silva and Glynn [20]. The nonlinear terms in the equations of motion are however not reduced to polynomial nonlinearities as done for perturbation analysis in [21]. The goal here is to obtain a numerical solution for the free vibration of the beam using a new technique to be presented. It can be shown that the equations obtained here are identical to those presented in Woodall [19]. However, the final form used here is different and possesses certain properties that are desirable for numerical solution. In Woodall [19], the equations of motion are not a direct representation of force-acceleration relationship but rather some

additional derivatives were taken to obtain a more compact form. In the present work, such a form is undesirable because it imposes higher continuity requirements on the solution space.

1.2 Analytical Methods

Despite the availability of faster and cheaper computers, analytical or semi-analytical methods have been considered as the only reliable means of investigating complex nonlinear dynamic behaviour in many systems. These methods may be divided into two groups; perturbation and non-perturbation methods. Both are approximate techniques for analysis. The main difference lies in how the dependent variables are approximated. Asymptotic expansions are used in perturbation methods whereas non-perturbation methods are based on space and time separation and stationary principles of classical mechanics. This distinction between the two techniques is not very strict as semi-discretization of the partial differential equations using non-perturbation methods is generally done before perturbation methods are applied. It should be mentioned that perturbation methods do not always give a closed form solution in terms of known functions and often numerical techniques have to be used in conjunction with perturbation methods. In the next few paragraphs some of the non-perturbation and perturbation techniques are discussed and the relevant earlier work is highlighted.

1.2.1 Non-Perturbation Methods

Non-perturbation methods include the method of weighted residuals, the Rayleigh-Ritz method and the Rayleigh's quotient method. In direct application of these methods the objective is to obtain nonlinear frequencies or an amplitude frequency relationship and nonlinear mode shapes. Non-perturbation techniques have been mostly used for simply supported beams or fixed-fixed beams (because the linear mode shapes of such beams are simple compared to other boundary conditions and are easier to work with). Also, a question which has been investigated in length, with regards to these boundary conditions, is how the axial extension as a result of immovable ends effects the frequencies. The axial extension appears as a quadratic nonlinearity in the strain-displacement relationship.

The method of weighted residuals is a general technique which can be regarded as a basis for the Galerkin method and its other variants. A description of this method can be found in Botha and Pinder [25], here a brief review is provided.

Let us begin by considering the following differential equation in the operator form:

$$\mathcal{L}u(x, t) - f(x, t) = 0 \quad (1.1)$$

where $\mathcal{L}$ is a differential operator, u is the dependent variable, x is the spatial coordinate, t is for time and f represents the algebraic terms. An approximation for u is sought in the following form:

$$\hat{u}(x, t) = \sum_{i=0}^N a_i(t) \phi_i(x) \quad (1.2)$$

where ϕ_i are assumed basis functions, and a_i are undetermined coefficient functions. Since N cannot be, for practical reasons infinite, $\hat{u}$ becomes an approximation for u . The ϕ_i can be chosen to satisfy all the essential (prescribed) boundary conditions (natural boundary conditions are incorporated into the equations of motion using integration by parts) in which case the index i normally starts from unity. Another possibility is to make ϕ_0 satisfy the boundary conditions while the remaining basis functions are zero at the boundary. The first approach is commonly used in analytical investigations whereas the second approach is widely used in finite element formulations. The approximate function $\hat{u}$ in general will not satisfy the differential equation (1.1), thus giving a nonzero residual R defined by:

$$R(x, a_i) = \mathcal{L}\hat{u} - f \quad (1.3)$$

The method of weighted residuals requires finding the undetermined coefficients a_i that will make the residual R zero in a weighted average sense by requiring:

$$\langle w_j, R \rangle = \int_{\Omega} w_j(x) R(x, a_i) d\Omega = 0 \quad j = 1 \dots N \quad (1.4)$$

where w_j , are weighting functions, Ω represents the spatial domain and the notation $\langle w_j, R \rangle$ denotes the weighted inner product. In the standard Galerkin's method the weighting functions w_j are chosen to be same as the basis functions ϕ_i . In otherwords the weighted inner product $\langle \phi_j, R \rangle$ is used to find the undetermined coefficient functions a_i . The N inner product relations give a system of N nonlinear ordinary differential equations for a_i . The reduction of partial differential equations

to ordinary differential equations in the time dependent variable will be referred to as the semi-discretization process.

Another technique similar to the standard Galerkin's method is the Bubnov-Galerkin method where the weighting functions are taken as variations in $\hat{u}$. Semi-discretized equations are therefore obtained using $(\delta\hat{u}, R)$, where δ is the variational operator. An example of application of this technique can be found in Beltzer [26]. Yet another variant of the standard Galerkin's method is the Ritz-Galerkin approach. This method was used by Srinivasan [27] to find the first mode frequency of a hinged-hinged beam with immovable ends. In the Ritz-Galerkin method, $\hat{u}$ is expressed as

$$\hat{u} = \sum_{i=0}^N a_i \phi_i(x) \sin(\omega t) \quad (1.5)$$

where periodic motion with frequency ω is assumed and the basis functions satisfy the essential boundary conditions. The unknown parameter a_i is obtained using the following equation:

$$\int_0^T \int_{\Omega} R(x_i, a_i) \delta\hat{u} = 0 \quad (1.6)$$

Considering $T = \frac{2\pi}{\omega}$ and assuming simple harmonic motion ($\ddot{u} = -\omega^2 u$, where $\ddot{}$ indicates differentiation with respect to time), the Ritz-Galerkin technique provides an easy way of obtaining an approximate amplitude frequency relationship for spatially nonlinear systems with no damping. In Srinivasan [27] the basis function ϕ was taken as the first mode of a linear beam and results were compared with an earlier work by Woinowsky-Krieger [17] where a solution was obtained in terms Jacobi elliptic cosine function. The results were found to match well.

The Rayleigh-Ritz method is another frequently used technique for discretiza-

tion of partial differential equations. In comparison with the Galerkin method, which operates directly on the partial differential equations, the Rayleigh-Ritz method uses energy functionals. The approximation (1.2) is substituted into the Lagrangian and variations are taken with respect to a_i giving a set of ordinary differential equations. Raju et al. [28] used this method to find the amplitude frequency relationship for a simply supported beam. They used a linearisation scheme for the nonlinear term that arises due to axial extension. The effect of axial displacement and linearisation of the strain displacement relationship on nonlinear frequencies of simply supported beams was discussed by Singh et al. [29]. Numerical differences in the nonlinear frequencies as a result of linearisation or neglecting the axial displacements were shown.

Rayleigh's quotient method can also be used to obtain natural frequencies. Details of this technique can be found in books on solid mechanics and linear algebra (for example [30, 31]). The advantage of this method is that the natural frequencies can be predicted with fair accuracy using shape functions which are not the true eigenfunctions. Such shape functions are generally based on heuristics or static deflection curves under various loading conditions. Low [32] obtained frequency formulae using different shape functions for various boundary conditions (simply supported, fixed-fixed and clamped-free) for linear beams. The applicability of these equations for large oscillations were not discussed in this paper, however, another paper by Low et al. [33] uses similar equations for a fixed-fixed beam but with Rayleigh's quotient for the nonlinear problem. In Low et al. [33] the estimated frequencies were also compared with experimental results and one of these shape

functions was found to predict the frequencies quite accurately. The nonlinearity considered in [17, 27, 28, 29, 33] is due to the axial extension of the mid-plane of the beam and also the motion is generally assumed to be harmonic.

In linear systems, normal mode decomposition is a standard method to solve dynamical problems analytically. The normal modes of continuous systems are simply the normalised eigenfunctions. The method for obtaining the normal modes of a linear continuous system can be found in Meirovitch [31]. The method of normal modes can be extended in a limited sense to nonlinear systems. A detailed exposition of the techniques involved in constructing nonlinear mode shapes can be found in Vakakis et al. [34]. Rosenberg [35, 36] has defined the normal modes for nonlinear discrete systems and Shaw and Pierre [37] developed a method to construct nonlinear normal modes for continuous systems. This method was applied by Hsieh et al. [38] to a cantilever beam. In linear systems the normal mode decomposition allows the system response to be expressed as superposition of simple harmonic oscillators whose frequencies are given by eigenvalues. Such a superposition principle does not hold for nonlinear modes. Instead an asymptotic expansion is obtained for the dependent variables which may be regarded as a nonlinear separation of variables [38]. Normal modes for linear systems exhibit certain other properties which may be extended to nonlinear systems. One such property is that normal mode motion occurs in an invariant subspace of the system solution space. This property was used in the construction of nonlinear normal modes in Hsieh et al. [38]. An illustration of this invariant property is that if an initial value corresponding to the normal mode is used (normal mode multiplied by a scalar), the motion occurs only in that

particular mode. Another assumption used in Hsieh et al. [38] is that the invariant subspace corresponding to the nonlinear normal mode is a two dimensional manifold (a curved surface) which is tangent to the linear normal mode space (planar surface) in the phase space representation, the point of tangency being the equilibrium position. This method of constructing the nonlinear normal modes is referred to as the invariant manifold method. As mentioned in [37] this approach does not work under internal resonance conditions. It is shown in Nayfeh and Nayfeh [39] that the application of this method is considerably simplified if complex variables are used for the phase space representation.

Benamar et al. [40] presented a method which uses the Rayleigh-Ritz approach to obtain nonlinear mode shapes for large oscillations of beams. This technique is based on harmonic motion assumption which cannot be used when kinematic nonlinearities are present.

1.2.2 Perturbation Methods

Perturbation methods have been successfully used in obtaining approximate analytical solutions for many nonlinear problems. The book by Nayfeh and Mook [41] illustrates the application of these methods. Using these techniques, approximate amplitude frequency relationships and nonlinear mode shapes can be determined for nonlinear continuously modelled systems. Perturbation methods can be applied in more than one way to nonlinear partial differential equations. One can assume a periodic motion of frequency ω in the approximation (1.2) and using the Rayleigh-Ritz approach obtain ordinary differential equations for the basis functions $\phi_i(x)$.

A perturbation method can then be applied to solve the spatial problem. This is fine in principle, however practically it is difficult or often impossible to obtain the spatial solution in a closed form for the nonlinear equations of a beam. Another method is to assume linear mode shapes and use Galerkin's method or the Rayleigh-Ritz method to reduce the partial differential equations to nonlinear ordinary differential equations. The nonlinear ordinary differential equations can then be solved using the perturbation method of multiple scales to obtain an amplitude frequency relationship. Still another approach is to apply the perturbation method directly to the partial differential equations and obtain both the nonlinear mode shape and nonlinear frequency directly.

One of the early works where a perturbation method was used to obtain an approximate amplitude frequency relationship was by Evensen [42] where a nonlinear vibrations of a fixed-fixed beam and a fixed-hinged beam were considered. The nonlinearity was due to axial extension of the beam. A regular perturbation method was applied directly to the partial differential equations of motion. The semi-discretization was achieved by using, as the basis function, the first mode shape of the corresponding linear beam. The amplitude frequency relationship was determined as a condition to eliminate the secular terms from the second order equations in the perturbation analysis.

Nayfeh et al. [43] consider an experimental pressure relief valve system and solve the equations of motion using the perturbation method of multiple scales and the method of averaging. The multiple scales technique was applied directly to the partial differential equations of motion and also to ordinary differential equations

obtained after discretization. It was shown that the two methods give different results, with the direct approach being more accurate. This is however not a general result as was illustrated in a later paper, Nayfeh and Nayfeh [39], where the perturbation method of multiple scales and the invariant manifold method was used to obtain normal nonlinear modes in beams. The invariant manifold method is similar to that used in Hsieh et al. [38]. The method of multiple scales was applied to semi-discretized equations and also directly to the partial differential equations. All these techniques were shown to give the same results. Another paper by Nayfeh et al. [44] applies the various perturbation and the invariant manifold methods discussed earlier to a cantilever beam, where again it was shown that the direct application of the method of multiple scales to the partial differential equations or its application to the discretized problem, both give the same results. Differences between the two approaches may arise when higher order approximations are used. It should be noted that the nonlinear mode shapes obtained using these techniques may appear in a closed form. However some of the parameters involved in these equations have to be determined using a numerical procedure and consequently these techniques become semi-analytic methods. The difficulty in obtaining a time series response using this approach remains to be determined, as none of the papers [37, 38, 39, 44] have presented such results.

A recent paper by Ozkaya et al. [45] considered a beam with a fixed mass and different combination of supports (simply supported, fixed clamped and rolling) at the ends. By applying the perturbation method of multiple scales directly to the partial differential equations linear modes shapes and natural frequencies are ob-

tained from the lowest order perturbation equations. Corrections to the lower order frequencies were obtained from the higher order perturbation equations. Forced oscillations were also considered assuming a uniformly distributed external periodic excitation.

The various methods and research discussed earlier was mostly in the context of problems with geometric nonlinearities. Now let us consider systems with kinematic nonlinearities. These nonlinearities occur due to inertial coupling of the beam with other bodies. These nonlinearities play a prominent role in the dynamic response of the system, more so than geometric nonlinearities, as the effect of kinematic nonlinearities is significant even for small oscillations. Under certain conditions when one of the frequencies becomes an integral multiple of other frequencies in the system, the phenomenon of internal resonance (IR) may occur. When the system parameters are close to internal resonance conditions the dynamic behaviour undergoes a remarkable change which is characterised by the motion undergoing distinctive beats. Understanding IR is therefore an important part in the study of nonlinear coupled systems. A distinction must be drawn between internal resonance due to linear coupling, nonlinear coupling and resonance due to external periodic forces. A number of papers have been published (for example [21, 23, 46, 47, 45]) investigating response of beams under periodic uniformly distributed loading. When in undamped linear systems, a periodic force is applied whose frequency is the same as one of the linear frequencies the amplitude of the system continually increases. However in nonlinear systems the nonlinearities can prevent the amplitude from increasing to such an extent. In addition, the nonlinear system can exhibit

sub-harmonic, and super-harmonic resonances and the jump phenomenon. Further information on these topics can be found in Nayfeh and Mook [41]. The beating phenomenon as a result of linear coupling is well known. It occurs when two fundamental frequencies of the system are close to each other or are exactly the same. For example a gyroscopic system was considered by Siddiqui and Golnaraghi in [48, 49] where internal resonance, due to both linear and nonlinear coupling, was shown. It was also illustrated that when there are internal resonance conditions arising from linear coupling there can also be a strong interaction between the resonant modes without the characteristic beating motion. The order of the nonlinearities affects the ratio of the frequencies needed for internal resonance. Systems with quadratic nonlinearities exhibit internal resonances when two fundamental frequencies in the system are in an even ratio and systems with cubic nonlinearities exhibit internal resonance when two fundamental frequencies are in an odd ratio. Some other works in nonlinear internal resonances in continuous systems are: Zavodney and Nayfeh [50], Pakdemirli and Nayfeh [51] and Anderson et al. [52]. In [50] a slender cantilever beam carrying a fixed lumped mass subjected to base excitation was considered. A beam supported by a spring-mass was treated in [51] and a base excited cantilever beam was considered in [52]. In all of [50]-[52] perturbation method of multiple scales was used.

In this thesis the perturbation method of multiple scales is used to investigate dynamics of a flexible cantilever beam carrying a moving spring-mass sub-system. In obtaining the perturbation solution only kinematic nonlinearities are considered. The method is applied to semi-discretized nonlinear ordinary differential equations

obtained using Galerkin's method or the Rayleigh-Ritz method (both techniques give the same equations). A closed form solution is obtained in terms of elliptic functions. Using the closed form solution a parametric study of the system is conducted. The focus of this analysis is on internal resonance behaviour between the moving mass and the beam. The perturbation results are compared with a numerical solution and some of the results of this analysis have been published in Siddiqui et al. [53].

1.3 Numerical Methods

In dynamical analysis, numerical methods can be applied directly to find the natural frequencies or to solve the equations of motion for the time response. Mode superposition in linear systems allows one to obtain the time response by solving an eigenvalue problem. However in nonlinear systems the principle of superposition does not apply. Therefore such an analysis is limited to finding the nonlinear frequencies. Eigenvalue problems in continuously distributed linear systems can be formulated using a number of techniques like the transfer matrix method (TMM) (Leckie and Pestel [54]), dynamic stiffness matrix method (DSM) (Pestel [55]) and finite element methods (FEM). Some of these techniques can be extended to estimate the first few frequencies in nonlinear systems. The approach generally involves assuming a periodic motion and extending the linear method by employing an iterative scheme.

1.3.1 Nonlinear Frequencies

Finite element formulations can be divided into three groups; displacement methods, force methods, and mixed methods according to the unknown nodal variables. In the displacement based formulations, nodal displacements, rotations and possibly their derivatives are considered as the unknown variables. In the force methods the nodal variables are forces and moments, whereas in the mixed methods both displacements and forces are used as the nodal variables. Mei [56, 57] proposed a displacement based finite element method to find the frequencies of beams [56] and plates [57] with immovable ends. The extensional effect was accounted through an axial force which was assumed to be constant over each element (Berger's Hypothesis). This formulation resulted in a linear stiffness matrix augmented with another matrix (geometrical stiffness matrix) which was evaluated using linear mode shapes. The frequencies were obtained by solving a problem similar to the conventional eigenvalue problem but with the additional stiffness matrix. Rao et al. [58] determined the nonlinear frequencies in beams with immovable supports using an iterative scheme. For the first step in the iteration, linear stiffness matrix is used and eigenvalues and eigenvectors are obtained. The solution corresponding to these eigenvectors is used for formulating the stiffness matrix in the next step. The process is carried out until the eigenvector estimates converge within some specified tolerance. Reddy and Singh [59] developed a mixed finite element formulation using deflection and moment as the two degrees of freedom at each node. Taking variations of a Reissner type functional, an eigenvalue problem was formulated which was solved using an iterative scheme similar to that used in Rao et al. [58]. In

using a direct method to find the frequencies, it should be noted that systems with kinematic nonlinearities can exhibit complex behaviour like amplitude, phase and frequency modulation. Fixed frequencies of the system do not reflect this behaviour and can be misleading. Furthermore the nonlinear frequencies and nonlinear mode shapes (obtained using perturbation techniques) are based on many approximations which are often heuristic. A reliable numerical solution of nonlinear partial differential equations that can capture the complex nonlinear behaviour is therefore highly desirable.

1.3.2 Time Response

In solid mechanics or dynamics, the solution of partial differential equations involves three steps; spatial discretization, temporal discretization and solution of algebraic equations which are generally nonlinear. With regards to spatial discretization some of the techniques like the Rayleigh-Ritz method and the Galerkin method have already been discussed in Section 1.2.1. Here the finite element method is considered.

Traditional Finite Elements

The finite element method refers to a wide range of techniques. Therefore a brief description of some of these methods is in order. First we have finite elements which are exact, in the sense that the stiffness matrix for the element is same as one would obtain using the strength of material approach. Examples include beam element with axial loads (truss elements) and flexure beam element for small

oscillations. As these stiffness matrices can be derived without using finite element interpolation functions, the method is not always considered as a finite element method; it is also known as matrix displacement method or direct stiffness method (see Dawe [60]). The spatial discretization is achieved by dividing the physical system into regions so that appropriate stiffness matrices can be applied (example, using different elements for regions of different loading conditions). This is different from traditional finite element method, where generally discretization is interpreted as interpolating the solution at selected points. Stiffness matrix for beam elements for bending, derived directly from the differential equations for deformation, using the strength of materials approach can be shown to be same as that obtained using finite elements employing cubic Hermitian polynomials [30]. The exact finite elements approach however does not always works, as the differential equations, even for static deflections, cannot always be solved to formulate the stiffness matrix.

The next level of finite elements are the ones obtained using the Rayleigh-Ritz method or the Galerkin method. As explained in the Section 1.2.1, the Rayleigh-Ritz method involves taking variations of a functional, whereas the Galerkin method applies directly to the partial differential equations, hence is more general. Nevertheless, for linear structural mechanics problems they both give the same results. The semi-discretization process reduces the partial differential equations to ordinary differential equations by expressing the dependent variables in terms of time dependent coefficient functions and space coordinate dependent basis functions. The underlying methods themselves do not provide the basis functions but they do impose certain continuity requirements on the solution space which restricts the

choice of the basis functions (for a rigorous treatment see Becker et al. [61]).

The finite element method can be interpreted as a technique which provides the basis functions for the Rayleigh-Ritz method. In fact, this view forms the foundation of most mathematical treatments in finite elements, even though the method is not always applied in this restrictive sense. Following this view, the finite element method can be considered as an interpolation technique for dependent variables at significant points (nodes) in the spatial domain. The interpolation is achieved through shape functions defined piecewise over small elements. If the shape functions are chosen such that the interpolant (solution) meets the continuity requirements imposed by the Rayleigh-Ritz method or the Galerkin method then the finite elements are referred to as being conforming. One advantage of using finite elements is that only the shape functions for the elements near the boundary have to satisfy the boundary conditions. This is in contrast with classical applications of the Rayleigh-Ritz method where all the basis functions are defined over the complete spatial domain and have to meet the boundary conditions. As a result complex boundaries can be handled more easily using the finite element approach.

The two finite element formulations discussed; exact and conformable elements, will be referred to as traditional finite elements. Obtaining conformable finite elements is not an easy task as it may require using higher degree polynomials and many degrees of freedom at each node. In displacement based finite element formulations, many degrees of freedom at the nodes requires using displacement and its derivatives as the nodal variables. This increases the number of ordinary differential equations for the coefficient functions. So it is not uncommon to use nonconform-

ing elements. The convergence of a finite element solution to the actual solution is well established in the case of conforming elements. This however is not a simple matter in case of nonconforming elements and may lead to problems in solving the ordinary differential equations for the time dependent coefficient functions.

The partial differential equations obtained using Hamilton's principle or the ordinary differential equations obtained using Rayleigh-Ritz method or the Galerkin method describe a continuous relationship between the dependent variables and the independent coordinates (space and time). The spatial discretization provides a means of interpolating the dependent variables at significant points in the spatial domain. Nevertheless the dependency of the variables on the spatial coordinates is still continuous (assuming conformable elements). The term discretization is used here because of its colloquial usage. In a stricter sense the process of reducing the partial differential to ordinary differential equations involves a smooth approximation of the solution over the spatial domain. It should be noted here, that in the present work the highest spatial derivative which appears in the ordinary differential equations is made continuous by appropriately choosing the shape functions and the number of degrees of freedom at each node. Convergence of this approximate solution to the actual solution of the nonlinear partial differential equations is well established as a result of the Rayleigh-Ritz method. Finite difference approximations for the time derivatives can be used to reduce the ordinary differential equations to nonlinear algebraic equations. At this stage the process truly becomes discretized as the finite difference approximations involve incremental jumps in the value of the time coordinate. This approach of reducing the partial differential

equations to nonlinear algebraic equations will be referred to as delayed discretization. This is in contrast with early discretization or premature discretization which is discussed next.

Conventional Nonlinear Finite Elements

The incremental finite element method is another technique widely used in nonlinear analysis. Problems with both geometric and material nonlinearities can be solved using this technique. Application of the incremental finite element method to systems with kinematic nonlinearities however have not been found in the literature. Dynamic problems also have been solved with only limited success (example Bathe et al. [62], Behdinan et al. [63]). In the traditional displacement based finite element method, the equations of motion are used in differential form before applying the finite element discretization, whereas in the incremental approach the equilibrium equations are obtained in terms of finite incremental displacements, and generally, after linearisation the finite element discretization is applied. Detailed exposition of this method can be found in the book by Bathe [64]. The incremental equations are generally obtained from the principle of virtual work using an incremental form of stress and strain measures. The principle of virtual work relates the stresses and strains in the material continuum to the applied forces. In small deformation analysis the volume of the continuum is not assumed to change during the application of the forces. Therefore the integrations in the principle of virtual work are performed over the original configuration. In large deformation analysis the integrations are carried out over the current configuration at $t + \Delta t$. However since the current configuration is not known, the stresses, strains and displacements

are referred to from a coordinate frame in the original configuration at time $t = 0$ or from the last known configuration at time t . In the first case the method is known as the total Lagrangian formulation in the second case the method is known as the updated Lagrangian formulation. The integrals are then computed in the referred configuration. The stress and strain measures are similarly transformed to the referred configuration. Different measures of stresses and strains are used for different types of nonlinearities, some of these measures can be found in Bathe [64]. Referring to different configurations obviously does not make the equations from the principle of virtual work solvable, it only changes the domain of the integration to a known configuration. Incremental decomposition of displacements, velocities, accelerations, stresses and strains have to be introduced. As an example if ${}^{t+\Delta t}_0 S_{ij}$ is a stress tensor at $t + \Delta t$ referred to in configuration at $t = 0$, the incremental decomposition would be ${}^{t+\Delta t}_0 S_{ij} = {}^t_0 S_{ij} + {}_0 \Delta S_{ij}$ where ${}_0 \Delta S_{ij}$ is the incremental change in the stress. The incremental formulation of the principle of virtual work thus gives equations relating the incremental displacements to incremental forces (after applying the constitutive law and expressing the stresses and strains and forces in terms of incremental displacements velocities and accelerations). These equations are generally nonlinear so they are first linearised and after substituting the finite element approximations for the displacements, velocities and accelerations a linear problem is formulated. However, since the original problem is nonlinear, a linearised solution may become unstable, so equilibrium iterations are often required. Equilibrium iterations are performed by using Taylor series expansion of the equilibrium equations about the previous time step (at time t).

A vast amount of literature exists on the incremental finite element method and its application. Here, only some of the research pertaining to fundamentals of the method and its application to large deformation beams is highlighted. Some of the references to early work can be found in Dupuis [65]. Bathe et al. [62] derived the incremental equations for finite element analysis from the principle of virtual work. Both the total Lagrangian formulation and the updated Lagrangian formulation were considered for problems with large displacements, large strains, and material nonlinearities. In a later work Bathe and Bolourchi [66] considered large displacement and large rotations of a three dimensional beam element. It was shown that the total Lagrangian method and the updated Lagrangian method both give the same stiffness matrices. The work done in [66] was for a static case. Reddy and Singh [59] used the principle of total potential energy to develop the incremental equations. This work was discussed earlier in Section 1.3.1.

Cardona and Geradin [67] considered the rotational effect due to large deformation of beams. The numerical results presented in Cardona and Geradin [67] were for the static deflection of a cantilever beam. Chen and Agar [68] developed a simpler finite element model than that used in Bathe and Bolourchi [66] by considering only geometric nonlinearities. The results presented in Chen and Agar [68] were for static deflection of a cantilever beam.

Iura and Atluri [69] used kinematic and rotating frames to develop a finite element formulation for Timoshenko beams in planar motion. Examples in [69] include the dynamics of flexible beam in free motion.

Behdinan et al. [63] used an updated Lagrangian formulation for static and dy-

dynamic analysis of beams. The main difference between the method used in [62] and [63] is that in [63] the nonlinear terms were carried out without linearisation. The discretized equations were then solved using modified Newton-Raphson's method. Dynamic analysis results shown in [63] include time response of beams with short time applied loads.

In the present work, the traditional finite element approach is used. Using Galerkin's method the nonlinear partial differential equations with time dependent Dirac-delta functions are reduced to nonlinear ordinary differential equations with Dirac-delta functions. Conforming finite element basis functions are used. The basis functions are chosen to be as simple as possible. They are required to satisfy the essential boundary conditions. The natural boundary conditions are incorporated in the Galerkin's formulation through integration by parts. Now the second step in the solution of partial differential equations, temporal discretization is discussed.

Temporal Discretization

Nonlinear ODEs can be solved using automatic solvers and direct time discretization using finite differences. Gear [70] gives a survey of automatic ODE solvers which can be applied to problems reduced from PDEs. The ODEs obtained through spatial discretization of PDEs are characterised by their natural frequencies ranging from small to very large values. Such equations are referred to as being stiff. Stiff ODE solvers are mostly based on an implicit formulation, which requires solving a system of nonlinear algebraic equations, often many times during each time step. When finite elements are used for discretization, the number of ODEs become very large and the step sizes required for integration become impractically small

therefore automatic ODE solvers are not used. On the other hand, in systems with only kinematic nonlinearities, the first few linear mode shapes can be used as basis functions in the Rayleigh-Ritz discretization and the resulting nonlinear ODEs can usually be solved using a stiff automatic ODE solver. This is one of the techniques used in this work for obtaining a numerical solution for the mass-beam system by neglecting the geometric nonlinearities. The automatic ODE solver used is based on an implicit formulation and a generalisation of the fourth order Runge-Kutta-Fehlberg method. To use an automatic ODE solver the equations must first be reduced to a first order system and when using an implicit method the Jacobian of the reduced first order equations must be computed. Finding this Jacobian analytically is a very difficult task, therefore it is computed numerically using central differences. A number of automatic ODE solvers were tried for the finite element formulation but without much success. Therefore direct discretization using finite differences is used.

Finite difference methods for temporal discretization are generally based on Taylor series. One method commonly used in structural dynamics is the Newmark's method [71]. Depending upon the parameters, this method is also referred to as constant average acceleration method or the trapezoidal rule. In the constant average acceleration scheme, the parameters are chosen such that the acceleration during a time interval is constant and equal to the average value of the acceleration at the ends of the interval. Successful solution of differential equations hinges upon the stability the integrating scheme. The step size for accurate integration theoretically has to be less than the time period corresponding to the highest frequency for

the system of equations. This time step, is usually impractical and much larger time steps are used in practice. As a result, error is introduced in the higher frequency response. If this error does not grow without bound the discretization scheme is referred to as being stable. If there is a certain time step above which the response becomes unstable the integration scheme is said to be conditionally stable.

The number of degrees of freedom in finite element analysis are usually large and correspondingly a large number of frequencies are involved in the solution (time series) obtained using these models. In general the higher frequencies obtained using the finite element methods are not very accurate and also they do not have any significant impact on the system response, but their mere presence makes the solution of the equations of motion very difficult to obtain. Therefore numerical dissipation is often used to suppress them. The average acceleration scheme does not involve any numerical dissipation. However, the more general Newmark's method can be used for numerical dissipation by appropriately setting the parameters. Other techniques include the Houbolt method [72], Wilson θ method [73] and the Hilber-Hughes-Taylor α method [74]. All these methods use a single step and can be applied directly to second order differential equations without reducing them to first order, as is normally required when automatic ODE solvers are used. Numerical dissipation decays the response amplitude even though there may not be any real damping in the system. It also affects the time periods by elongating them. The accuracy of the solution is therefore compromised for improved stability.

Stability properties of an integration scheme are generally based on linear second order differential equations with no damping. They are characterised by finding

the spectral radius (maximum eigenvalue) for the temporal integration operator as a function of step size. If the spectral radius is less than or equal to one, the discretization scheme is considered stable. Using this criteria, the average acceleration scheme can be shown to be unconditionally stable. Other unconditionally stable schemes are Houbolt method, Wilson θ method and the Hilber-Hughes-Taylor α method [74].

Heppler and Hansen [75] developed closed form equations for identifying regions of unconditional stability for many integral operators. System damping was also modelled in the analysis. It was also shown in [75], that a desirable property of Hilber-Hughes-Taylor α method; numerical dissipation in the higher modes does not exist when damping is present in the system, instead the numerical dissipation has the most significant effect on the lower modes.

Felippa and Park [76, 77] consider various computational aspects and errors involved in integration methods including linear multi-step methods. They show [77] that the average acceleration method can exhibit error growth for high frequencies. Cardona and Geradin [78] arrive at similar conclusions for systems with constraints. In fact, they refer to the trapezoidal rule as unconditionally unstable. The stability conclusions in [77, 78] were arrived at by considering a linearised problem similar to what is encountered in the incremental finite element method (use of tangent mass and stiffness matrices). Any abrupt changes in the tangent stiffness matrix or the force vector may lead to instabilities. These results however cannot be taken as general conclusions regarding the stability of the average acceleration method, as the stability also depends heavily on how these techniques

are implemented (computational path). Stability of the integration scheme in a nonlinear system depends upon a number of factors which include, consistency in the model, proper satisfaction of the boundary conditions (prescribed initial values must also satisfy the boundary conditions), conforming finite elements, temporal discretization, and consistency of the iterative scheme for solving the resulting nonlinear algebraic equations. All these factors are accounted for in developing a new strategy for solving the equations of motion in this work. Inconsistency in the model refers to dropping of nonlinear terms from the equations of motion, instead of a proper reduction of the functional from which the equations are derived. The initial value prescribed must also satisfy the boundary conditions, which is not a trivial task in nonlinear systems. Inconsistencies in the iterative scheme to solve the nonlinear algebraic equations can arise due to discontinuities in the Jacobian, as the extra derivative required in computing the Jacobian may not be included in the continuity criteria for the conforming elements.

Accuracy of the temporal discretization must also be considered before evaluating the suitability of a particular approach. The average acceleration method is well recognised to be accurate, in the sense that it does not cause amplitude reduction due to numerical dissipation, and the period elongation is minimal. Xie [79] compared the Wilson θ method, the Houbolt method, various Newmark family methods, including the average acceleration method, and the Runge-Kutta method for an ODE with hardening and softening type nonlinearities. The results show that the average acceleration method gives the most accurate response for hardening type nonlinearities and the Runge-Kutta method gives the best results for

softening type nonlinearities with the average acceleration method being very close.

The methods discussed earlier were based on linear finite differences. Some nonlinear finite difference schemes have also been proposed. Evans and Sanugi [80, 81] worked on a different class of integrators based on different “means”. The trapezoidal rule can be viewed as the arithmetic mean of Euler forward and backward formulae. The geometric mean is considered in [80] and the harmonic, logarithmic and a new mean were used to develop a nonlinear trapezoidal rule in [81]. Zienkiewicz et al. [82] use Galerkin’s method with shape functions for the time dependent variables. Xie [83] showed that such methods exhibit an unusual stability property in that there is no critical time step above which the method is unstable, instead there is a band for the time steps in which the method is unstable and outside this band method again becomes stable.

1.3.3 Nonlinear Algebraic Equations

Once a finite difference scheme is selected, the ordinary differential equations can be reduced to nonlinear algebraic equations. In the incremental finite element method, since the equations are linearised, the substitution of the finite differences can be done implicitly in a computer program. In the traditional approach, an explicit substitution is generally required, to reduce the ordinary differential equations to nonlinear algebraic equations. Conventionally the accelerations, and velocities are replaced with the finite difference approximations in terms of displacements. In this work a new approach is used where the displacements and velocities are replaced in terms of accelerations. This approach will be referred to as the acceleration formu-

lation. The conventional method of replacing the velocities and the accelerations in terms of displacements is referred to as the position formulation. In solving the nonlinear algebraic equations, the acceleration formulation has far superior convergence than the position formulation, as a result the numerical errors are small and it keeps the integration process stable. As will be illustrated in this thesis the acceleration formulation corresponds to a direct substitution of Taylor series expansions into the equations of motion, whereas the position formulation involves solving for velocities and accelerations from the Taylor series expansions (the right hand sides) and then substituting them in the equations of motion.

Solution techniques for nonlinear algebraic equations are generally based on Newton's method where an iterative problem is formulated using the Jacobian. The Jacobian for large systems is often difficult to compute and can actually be impossible to obtain analytically and numerical approximations may be required. For certain problems like a mass moving on a beam (the problem analysed in this work), the basis functions for the beam have an implicit dependency on the position of the mass. The extra derivative required in finding the Jacobian imposes an additional degree of smoothness requirement on the finite element approximation of the solution, which in turn requires increasing the number of degrees of freedom per node and using higher degree polynomials for the basis functions. In practice this is not done, and as a result in the convergence process the effect of some of higher order terms gets cancelled and may cause instability.

Another method for solving nonlinear algebraic equations is the substitution method (Ghouri and Touzat [84]) which does not require computing the Jacobian.

This method works by assuming the linear terms in the equations as the unknowns and, using previous time step values for the nonlinear terms, an improved estimate is obtained. An iterative scheme based on such a technique does not always converge, especially when large oscillations are involved. In fact for the systems analysed in this work the method failed to converge even for small oscillations.

A new technique to solve the nonlinear algebraic equations is presented. This method is similar to the substitution method but it also uses the nonlinear terms as unknowns in obtaining the new estimate. This iterative technique along with the acceleration formulation of the average acceleration method is used for solving the ordinary differential equations. The iterative method gives improved estimates of accelerations, and using the finite difference discretization of the average acceleration method, the position and velocities are obtained. The iterations are carried out until all the ordinary differential equations and the algebraic constraint equations are satisfied to a specified tolerance. In addition, a further step is taken to minimise error propagation. This step involves finding the accelerations from the ordinary differential equations of motion. This is a linear step as the equations of motion in dynamics are linear in accelerations. It is significant to note that in this process the iterations are continued until the ordinary differential equations are satisfied and not the nonlinear algebraic equations which are being used for the iterations.

1.4 Beam Carrying a Moving Mass

Investigation of the dynamics of a beam carrying a moving mass has been the focus of research for a number of years (the literature survey covers papers or monographs

published during 1849-1997). The problem arises in many applications. Examples include the motion of vehicles on bridges, cranes carrying moving loads, robotic arms and space structures. Some novel applications, like using the moving mass as a controller to suppress vibrations in the beam, have also been proposed (Golnaraghi [3, 4]). Historically the problem first arose in the design of railway bridges and later in other transportation engineering structures. There have been numerous investigations in this regard. Some of the early investigations were by Stokes [1], Ayre [85] and there are two well known monographs in this area by Inglis [86] and Hillerborg [87]. A more recent book by Fryba [88] includes analyses under different loading conditions. There were also some investigations into the effect of high speed moving forces on beams (e.g., Florence [89] and Steele [90]). These earlier studies neglected the inertial effect of the moving mass by considering it as a moving force and the solution techniques used were generally based on integral transformations or asymptotic expansions.

In transportation engineering applications the mass is usually assumed to be traversing the beam at a constant velocity and a simply supported or a clamped clamped beam is assumed. Nelson and Conover [91] consider the stability of a simply supported beam carrying moving masses with constant velocity using Floquet theory. Galerkin's method, with simply supported beam mode shapes, was used for discretization.

Success with analytical methods to obtain qualitative results has been limited. Stanisic et al. [92] used an asymptotic expansion method to obtain an approximate analytical solution. In transport engineering problems the traversing mass and a

continuous beam model results in a partial differential equation with inertial coupling terms which depend on the position of the mass. Due to these coupling terms the mode shapes of simple beams do not arise as eigenfunctions in the separation of variables method even if a linear model of the beam is assumed. Stanisic [93] developed a method to obtain mode shapes which account for the motion of the mass. Some other papers in this area are by Hayashikawa and Watanabe [94], Akin and Mofid [95], Cifuentes [96], Lin and Tretheway [97], Gbadeyan and Oni [98], Lee [99], Michaltsos et al. [100], and Henchi et al. [101].

Hayashikawa and Watanabe [94] developed a method similar to the dynamic stiffness approach to obtain natural frequencies and mode shapes and used it to obtain the response of multi-span beams with moving forces. Akin and Mofid [95] considered different boundary conditions (combinations of fixed and hinged ends), used linear beam mode shapes in a method similar to Galerkin's approach and employed a Runge-Kutta method for temporal discretization to obtain numerical solutions. The results were for very short times (maximum one time period).

Cifuentes [96] focussed on reducing the problem to a form such that standard finite element codes could be applied.

Lin and Tretheway [97] considered a moving mass with a spring and damper traversing the beam. The damping and the spring stiffness was assumed to be in the vertical direction. Finite elements and a Runge-Kutta method was used for obtaining the solution. The system used in this work was capable of exhibiting internal resonance. However no such results were analysed.

Gbadeyan and Oni [98] considered moving forces and masses over beams and

plates using integral transformations and asymptotic expansions. The beam and the plate was assumed to be of Rayleigh type (includes the effect of rotatory inertia).

Lee [99] analysed the problem of the moving mass separating from the beam by monitoring the contact forces. Michaltsos et al. [100] discussed the the effect of the moving mass and other parameters on the dynamic response of the beam. Henchi et al. [101] developed a dynamic stiffness matrix for the analysis beams with moving masses. A number of these papers also include parametric analysis of interest in transport engineering applications. These are however not discussed as it not the main focus of this work.

The works cited earlier dealt with a problem where the motion of the mass was prescribed, generally at a constant velocity and its effect on the beam response was studied. The beam models were considered linear and in most cases the kinematic nonlinearities due to the coupling between the mass and the beam were not fully modelled (Coriolis and centripetal accelerations were generally neglected). In this work the focus is on the nonlinear interaction between the mass and the beam and also on large oscillations. The system to be investigated is shown in Figure 1.1. It

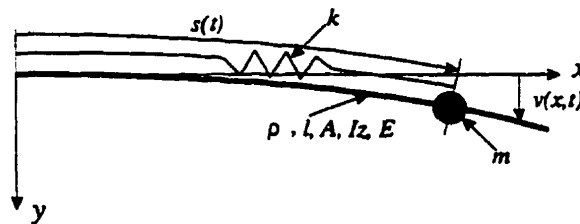


Figure 1.1: System Model

consists of a cantilever beam carrying a moving mass. A spring is attached to the

mass which serves two purposes: to prevent the mass from sliding off the beam and to allow an oscillatory motion for the moving mass. The equations of motion are a set of two coupled nonlinear partial differential equations where the coupling terms have to be evaluated at the position of the mass. Kinematic nonlinearities in the system arise due to the coupling between the mass and the beam and geometric nonlinearities arise due to the nonlinear stress-strain relationship for the beam. The mass-beam system considered here was investigated earlier by Khalily et al. [5] where numerical solutions were obtained using two mode shapes for the system. To account for the motion of the mass in the mode shapes, the method developed in [93] was used. The numerical results obtained using these mode shapes were not satisfactory, as unrealistically large initial values were required to show the coupling between the mass and the beam. Also the beam model used in [5] was linear.

In the present work, two models are used for the mass-beam system. In the first case, small oscillations are assumed and only the kinematic nonlinearities are considered. In the second case, large oscillations are investigated where both geometric and kinematic nonlinearities are modelled. A closed form analytical solution is obtained using the perturbation method of multiple scales. A parametric study is conducted to identify regions of strong coupling between the mass and the beam. The numerical solution is obtained using two methods; for the model with only kinematic nonlinearities, linear beam mode shapes along with an automatic ODE solver is used, and a second solution is obtained using traditional finite elements and the new method developed.

The kinematic nonlinearities play a dominant role in the system response. Under

certain conditions, when the frequency corresponding to the spring-mass sub-system is an integral multiple of the lowest frequency of the system, the phenomenon of internal resonance occurs. When the system parameters are close to internal resonance the dynamic behaviour undergoes a remarkable change characterised by the motion undergoing distinctive beats. The parametric study using the closed form solution obtained using the perturbation method identifies these regions. Internal resonance response is investigated at a 1:2 ratio of the frequency of the spring-mass sub-system and the lowest frequency of the entire system (1:2 IR). The effect of different ratios of the moving mass to the mass of the beam is also considered. The system is analysed by considering the equilibrium position of the spring-mass sub-system fixed at different locations on the beam. Under internal resonance conditions the system exhibits amplitude and phase modulation which corresponds to the characteristic beating motion.

The time response of the system shows the amplitude of the mass and the beam undergoing modulation due to internal resonance. To examine the evolution of this behaviour in the spectral domain, time frequency analysis techniques are used. The power spectrum, which is obtained by taking the discrete Fourier transform of the time series and computing the power (mean squared amplitude) at the various frequencies, gives the averaged behaviour for the length of time series. To investigate the local spectral behaviour of the system, a spectrograph is used. The spectrograph is obtained by finding the power spectrum of relatively small segments of data and the results are displayed on time frequency axes, with time corresponding to the centre of the data segment and the power is shown using a grey scaling for

the whole plot. Increasing the size of the data segment improves the spectral resolution but at the expense of time localisation. For smoother transitions the data segments are overlapped. To reduce leakage of the power from one frequency bin to another, the data is windowed using the Hann Window (for definition see [102]). The numerical, perturbation and the time frequency analysis results are compared and studied for a number of cases.

1.5 Thesis Outline

In Chapter 2 the equations of motion for small and large oscillations of the mass beam system are derived using Hamilton's principle. These are nonlinear integral partial differential equations with time dependent Dirac-delta functions. The Dirac-delta functions arise due to the moving mass. Using the Galerkin method the equations are reduced to a system of nonlinear ordinary differential equations with time dependent coefficients.

In Chapter 3, the basis functions for solving the nonlinear partial differential equations are derived. The eigenfunctions of a cantilever beam are discussed. A Finite element model based on fifth degree Hermitian polynomials is derived.

Chapter 4 deals with kinematic nonlinearities. The mass beam system for small oscillations is considered. A closed form analytical solution is derived using the perturbation method of multiple scales. A parametric study is conducted and time frequency analysis results are presented. Numerical solution is obtained using an automatic ODE solver. The numerical and the perturbation solution results are compared.

In Chapter 5, the focus is on geometric nonlinearities. A new solution strategy is presented to solve the nonlinear partial differential equations. The acceleration formulation for dynamical problems and a new method to solve the nonlinear algebraic equations as discussed in Section 1.3.3 is introduced. Results for a cantilever beam undergoing large oscillations obtained using this method are presented. The mass-beam system with both kinematic and geometric nonlinearities is simulated. Finally, in Chapter 6, conclusions for this thesis are presented.

Chapter 2

Modelling

The equations of motion of a flexible cantilever carrying a moving mass are developed using an energy approach. Two models are obtained, one for small oscillations and the other for large oscillations. The small motion model corresponds to linear strain displacement relationships, and the large motion model corresponds to nonlinear strain displacement relationships. For the large deformation model the beam is assumed to be inextensible. The inextensibility of the beam is represented by a nonholonomic constraint relating the deformation in the axial and transverse directions. This constraint is incorporated into the system Lagrangian through Lagrange multipliers. This approach was taken by Crespo da Silva and Glynn [20] for large nonplanar oscillations. The equations of motion for the beam obtained here can be shown to be identical to those derived by Woodall [19] except that the final form used in [19] involves taking additional derivatives of the equations of motion which are not desirable as they impose higher order continuity requirements on the solution space. Also in this work, a new systematic derivation is presented for the

equations of motion.

The system and the various parameters used in its modelling have already been shown in Figure 1.1. The beam parameters are length (l), area of cross section (A), mass density (ρ), second moment of the area about the z axis (I_z), and the modulus of elasticity (E). The moving mass (m) slides along the length of the beam. The position of the mass is measured by a curvilinear coordinate s . The mass is induced to move by an applied force. This is in contrast to the frequently used assumption of prescribed motion for the moving mass. In this thesis the applied force is assumed to be proportional to the displacement of the mass. Hence in effect a spring of stiffness k is attached to the mass. The spring serves two purposes, first to prevent the mass from sliding off the beam and second to establish internal resonance (IR). Internal resonance occurs when the stiffness of the spring is chosen to be such that the fundamental frequency of the spring-mass sub-system is an integral multiple of the lowest frequency of the beam. Under IR conditions the system undergoes a remarkable change characterised by distinctive beats and a continuous exchange of energy between the mass and the beam. Although not the focus of this work, it has been shown in Golnaraghi [3, 4] that the coupling between the mass and the beam becomes strong under IR conditions and the mass can be used to control or suppress vibrations in the beam. The effect of gravity on the system is neglected and it may be interpreted that the beam is vibrating in the horizontal plane.

2.1 Nondimensionalised Parameters

In this work only nondimensionalised parameters are used in the analysis. The following nondimensionalised parameters are defined:

$$\begin{aligned}
 \hat{s} &= \frac{s}{l} & \hat{x} &= \frac{x}{l} & \hat{y} &= \frac{y}{l} & \hat{u} &= \frac{u}{l} \\
 \hat{v} &= \frac{v}{l} & \hat{A} &= \frac{A}{l^2} & \hat{m} &= \frac{m}{\rho A l} & \hat{I}_z &= \frac{I_z}{A l^2} \\
 \hat{t} &= \frac{t}{\sqrt{\frac{\rho A l^4}{E I_z}}} & \omega^2 &= \frac{k \rho A l^4}{m E I_z} & \hat{\lambda} &= \frac{\lambda l^2}{E I_z}
 \end{aligned} \tag{2.1}$$

Subsequently we will be using only these parameters (2.1) and for convenience the (^)s are dropped. The λ in (2.1) is the Lagrange multiplier which is used in developing the large oscillation model.

2.2 Lagrangian

The Lagrangian $\mathcal{L}$ of the mass beam system is now formulated for the small and large oscillation models. It is defined as the difference between the kinetic T and the potential V energy of the system. The various energy terms for the small and the large oscillation model are developed in parallel so that the differences between them can be easily identified.

2.2.1 Displacement Field – Small Oscillations

To begin, a displacement field is assumed consistent with the Euler-Bernoulli beam model for small oscillations. The main feature of this model is that all the defor-

mations are measured using one variable $v(x, t)$. The dependent variable $v(x, t)$ measures the displacement of the reference axis as shown in Figure 2.1. Let u_x ,

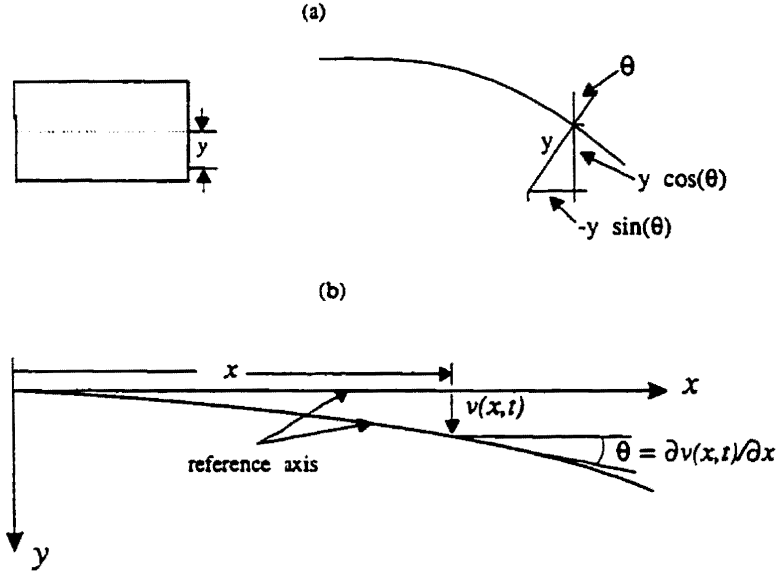


Figure 2.1: Deformation - Small oscillations (a) Displacement of a point located at a distance y below the reference axis, (b) Deflection of the reference axis.

u_y , and u_z denote the displacements in the undeformed x , y , and z , directions respectively. For small oscillations the following displacement field is assumed:

$$\begin{aligned} u_x &= -y \sin(\theta), \quad \sin(\theta) = \left(\frac{\partial}{\partial x} v(x, t) \right) \\ u_y &= v(x, t), \quad \cos(\theta) = 1 \\ u_z &= 0 \end{aligned} \tag{2.2}$$

where the small angle assumption $\cos(\theta) = 1$ is used. The equations of motion are obtained in terms of the dependent variable $v(x, t)$. However trigonometric

functions of θ are used in some of the equations for simplification purposes. It can be seen from Figure 2.1 that when θ is small, the additional vertical movement of any point located at a distance y below the neutral axis is also small, and is therefore neglected in (2.2).

2.2.2 Displacement Field – Large Oscillations

The displacement field for large oscillations is modelled using two dependent variables $u(x, t)$ and $v(x, t)$ as shown in Figure 2.2, where $u(x, t)$ measures the shortening of the beam in the horizontal direction, and $v(x, t)$ measures the vertical deflection of the reference axis. The variables $u(x, t)$ and $v(x, t)$ are not independent of each other but are related through a constraint equation arising due to the inextensibility of the beam. This inextensibility constraint is discussed later in Section 2.2.4. In Figure 2.2(b) the triangle indicating the differential deformations arises due to this inextensible beam assumption. For the large oscillation model, both horizontal and vertical displacements due to the rotation θ , of a point located at a distance y below the neutral axis are modelled (see Figure 2.1). The displacement field is therefore assumed as:

$$\begin{aligned} u_x &= u(x, t) - y \sin(\theta), \quad \sin(\theta) = \left(\frac{\partial}{\partial x} v(x, t) \right) \\ u_y &= v(x, t) - y(1 - \cos(\theta)), \quad \cos(\theta) = 1 + \left(\frac{\partial}{\partial x} u(x, t) \right) \\ u_z &= 0 \end{aligned} \tag{2.3}$$

where $\sin(\theta)$ and $\cos(\theta)$ are obtained using the deformation triangle in Figure 2.2(b).

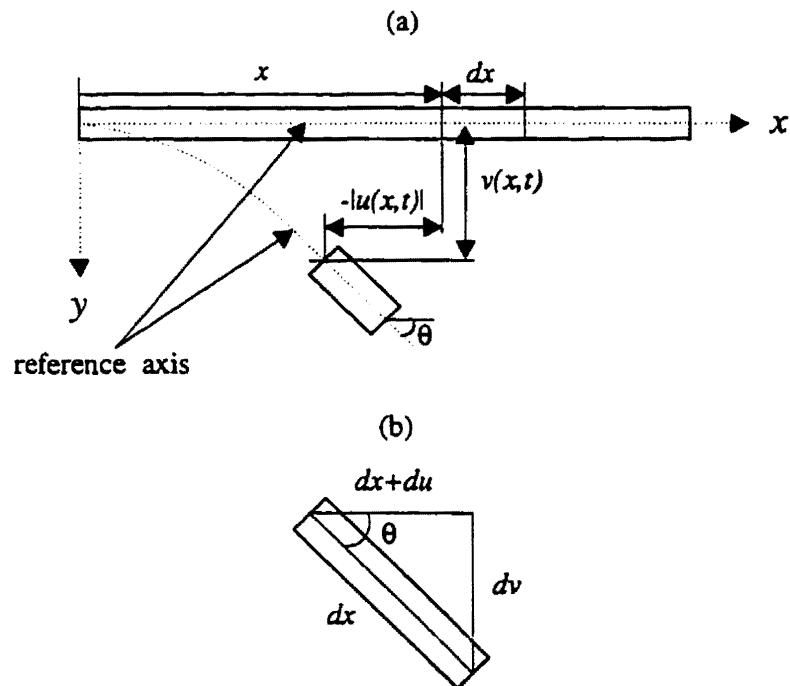


Figure 2.2: Deformation – Large oscillations (a) Deflection of the reference axis, (b) Deflection of a differential element.

2.2.3 Strains – Small Oscillations

When the deformations are small, infinitesimal strains (linear strain displacement relationship) can be assumed. They are obtained using the following tensor relationship [30]:

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (2.4)$$

where the index notation is used with $i, j = 1, 2, 3$ corresponding to x, y, z coordinates, respectively. The subscripts in ϵ_{ij} represent the plane (i) and the direction (j) in which these strains act. From (2.2) and (2.4) it follows that the only nonzero strain present is given by:

$$\begin{aligned} \epsilon_{xx} &= -y \frac{\partial \theta}{\partial x} \\ \frac{\partial \theta}{\partial x} &= \left(\frac{\partial^2}{\partial x^2} v(x, t) \right) \end{aligned} \quad (2.5)$$

2.2.4 Strains – Large Oscillations

When large oscillations are involved the nonlinear terms in the strain displacement relationships cannot be discarded. Therefore Green's finite strain tensor is used. It is given by [30]:

$$\epsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right) \quad (2.6)$$

where the repeated index k is summed over 1, 2 and 3. From (2.6) it follows that the strain on the x plane in the x direction is given by:

$$\epsilon_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2} \left(\frac{\partial u_x}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial u_y}{\partial x} \right)^2 \quad (2.7)$$

For the displacement field (2.3) it follows that all the other strains in (2.6) are zero.

The two dependent variables u and v are not independent of each other. The constraint equation relating the two is obtained using the inextensibility of the beam. Inextensibility implies that flexure does not causes any strain in the neutral axis ($y = 0$) in the x direction ($\epsilon_{xx} = 0$ at $y = 0$). The strains for small oscillations satisfy this requirement (see (2.5)). For large amplitude oscillations this inextensibility condition can be obtained from (2.7) by substituting the displacement field (2.3) and setting y to zero, thus giving:

$$\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial v}{\partial x} \right)^2 = 0 \quad (2.8)$$

Using the definitions of $\sin(\theta)$ and $\cos(\theta)$, (2.3), the constraint equation (2.8) can be written as follows:

$$\begin{aligned} \left(\frac{\partial v}{\partial x} \right)^2 + \left(1 + \frac{\partial u}{\partial x} \right)^2 &= 1 \\ \sin(\theta)^2 + \cos(\theta)^2 &= 1 \end{aligned} \quad (2.9)$$

Applying the trigonometric identity from (2.9) is the same as using the constraint equation. It is for this ease of simplification that the trigonometric terms in θ are used in developing the equations of motion.

Using the constraint equation (2.9), the displacement field (2.3) and the strain displacement relationship (2.7), the strains for large oscillations are obtained as:

$$\epsilon_{xx} = -y \frac{\partial \theta}{\partial x} + y^2 \left(\frac{\partial \theta}{\partial x} \right)^2$$

$$\frac{\partial \theta}{\partial x} = \frac{\partial^2 v}{\partial x^2} \cos(\theta) - \frac{\partial^2 u}{\partial x^2} \sin(\theta) \quad (2.10)$$

The constraint equation is incorporated in the Lagrangian through the Lagrange multipliers $\lambda(x, t)$. The corresponding term that would appear in the Lagrangian is given by:

$$C = \frac{1}{2} \int_0^1 \lambda(x, t) \left(\left(\frac{\partial v}{\partial x} \right)^2 + \left(1 + \frac{\partial u}{\partial x} \right)^2 - 1 \right) dx \quad (2.11)$$

2.2.5 Strain Energy

The energy function corresponding to the strains in linearly elastic bodies is given by:

$$U = \frac{1}{2} \int_{\mathcal{V}} \epsilon_{xx}^2 d\mathcal{V} \quad (2.12)$$

where $\mathcal{V}$ is the nondimensionalised volume, and also all the other terms are assumed to be nondimensional. It can be easily shown that when the nondimensionalised variables (2.1) are used, the strain energy U and all the other energy terms in the Lagrangian involve a multiplying factor $\frac{EI_x}{l}$ (has units of work). This factor appears as a constant multiplying the equations of motion. This constant is same as that would factor out when the nondimensionalised variables are substituted in the equations of the motion. Therefore it is ignored from all the energy terms.

For large oscillations the y^2 term in (2.10) leads to the fourth moment of area which is a very small term (compare the second moment $\frac{bh^3}{12}$ with the fourth moment $\frac{bh^5}{80}$ for rectangular beams) and is therefore dropped from further analysis. With this assumption and using the strain equations (2.5) or (2.10) the strain energy is

obtained as:

$$U = \frac{1}{2} \int_0^1 \left(\frac{\partial \theta}{\partial x} \right)^2 dx \quad (2.13)$$

In terms of θ , the strain energy for both small and large oscillations is given by the same expression (2.13), however in terms of the dependent variables u and v , the strain energy for both cases is quite different.

2.2.6 Kinetic Energy of the Beam – Small Oscillations

The velocity field is obtained by differentiating the displacement field (2.2) with respect to time, thus giving:

$$\begin{aligned} \dot{u}_x &= -y \cos(\theta) \frac{\partial \theta}{\partial t} \\ \dot{u}_y &= \frac{\partial v}{\partial t} \end{aligned} \quad (2.14)$$

For simplification the rotatory inertia terms are dropped from the velocity and the kinetic energy of the beam is thus obtained as:

$$T_b = \frac{1}{2} \int_0^1 \left(\frac{\partial v}{\partial t} \right)^2 dx \quad (2.15)$$

2.2.7 Kinetic Energy of the Beam – Large Oscillations

The velocity field for large oscillations is obtained by differentiating (2.3) as:

$$\begin{aligned} \dot{u}_x &= \frac{\partial u}{\partial t} - y \cos(\theta) \frac{\partial \theta}{\partial t} \\ \dot{u}_y &= \frac{\partial v}{\partial t} - y \sin(\theta) \frac{\partial \theta}{\partial t} \end{aligned} \quad (2.16)$$

The kinetic energy, after dropping the rotational velocity is given by:

$$T_b = \frac{1}{2} \int_0^l \left(\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 \right) dx \quad (2.17)$$

2.2.8 Kinetic Energy of the Mass – Small Oscillations

The velocity of the moving mass can be obtained from Figure 2.3 as follows:

$$\begin{aligned} \dot{u}_{mx} &= [\dot{u}_x + \dot{s} \cos(\theta)]_{x=s(t)} \\ \dot{u}_{my} &= [\dot{u}_y + \dot{s} \sin(\theta)]_{x=s(t)} \end{aligned} \quad (2.18)$$

In (2.18), $\dot{u}_{mx}$ and $\dot{u}_{my}$ are the inertial x and y components respectively of the velocity of the moving mass. They depend on the location of the mass on the beam. The kinetic energy of the moving mass obtained by neglecting the rotatory

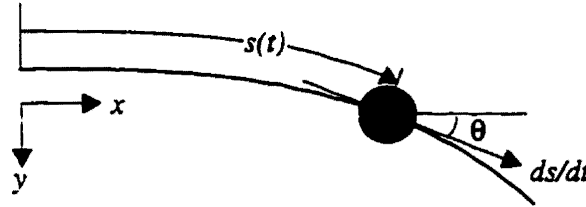


Figure 2.3: Velocity of the moving mass

inertia effect is given by:

$$T_m = \frac{1}{2} m \left[\dot{s}^2 + \left(\frac{\partial v}{\partial t} \right)^2 + 2 \dot{s} \sin(\theta) \frac{\partial v}{\partial t} \right]_{x=s(t)} \quad (2.19)$$

2.2.9 Kinetic Energy of the Mass – Large Oscillations

For large oscillations the velocity of the moving mass is still given by (2.18). Using the velocity field (2.16) and (2.18) the following expression is obtained for the kinetic energy of the moving mass:

$$T_m = \frac{1}{2}m \left[\dot{s}^2 + \left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 + 2\dot{s} \sin(\theta) \frac{\partial v}{\partial t} + 2\dot{s} \cos(\theta) \frac{\partial u}{\partial t} \right]_{x=s(t)} \quad (2.20)$$

2.2.10 Potential Energy due to the Spring

The potential energy due to the spring attached to the mass is given by:

$$V_m = \frac{1}{2}m\omega^2(s - s_e)^2 \quad (2.21)$$

where s_e is the equilibrium position of the moving mass and ω is the nondimensional frequency of the spring-mass sub-system. This energy term may appear a little unusual. It is because of factoring out $\frac{EI}{l}$ as discussed earlier in the first paragraph of Section 2.2.5.

2.2.11 Lagrangian – Small Oscillations

The Lagrangian for the small oscillations case is obtained from the various energy terms and can be written in the form:

$$\begin{aligned} \mathcal{L} &= T - V \\ &= T_m(s, \dot{s}, \dot{v}, v') + T_b(\dot{v}) - V_m(s) - U(v'') \end{aligned} \quad (2.22)$$

where T_m is the kinetic energy of the moving mass, T_b is the kinetic energy of the beam, V_m is the potential energy of the spring-mass sub-system and U is the strain energy of the beam. In (2.22) the dependent variables in each of the energy terms are shown to facilitate taking variations with respect these variables. Also in (2.22) $()'$ indicates differentiation with respect to x and $()'$ denotes differentiation with respect to time.

2.2.12 Lagrangian – Large Oscillations

Similarly the Lagrangian for the large oscillation model is written in the form:

$$\begin{aligned}\mathcal{L} &= T - V \\ &= T_m(s, \dot{s}, \dot{u}, \dot{v}, \theta) + T_b(\dot{u}, \dot{v}) - V_m(s) - U(\theta') - C(\lambda, u', v')\end{aligned}\quad (2.23)$$

where T_m is the kinetic energy of the moving mass, T_b is the kinetic energy of the beam, V_m is the potential energy of the moving mass, U is the strain energy of the beam and C represents the inextensibility constraint multiplied by the Lagrange multiplier λ .

2.3 Equations of Motion – Small Oscillations

The equations of motion are obtained using Hamilton's principle. This involves taking the first variation of the time integral of the Lagrangian:

$$\delta^{(1)} \int_{t_1}^{t_2} \mathcal{L} dt = 0 \quad (2.24)$$

where the Lagrangian for the small deformation model is given by (2.22). Taking the variations gives the following form for the equations of motion (see [30]):

s variation:

$$\left[\frac{\partial T_m}{\partial x} - \frac{\partial^2 T_m}{\partial t \partial \dot{s}} - \frac{\partial^2 T_m}{\partial x \partial \dot{s}} \frac{\partial s}{\partial t} \right]_{x=s(t)} - \frac{dV_m}{ds} = 0 \quad (2.25)$$

v variation:

$$\begin{aligned} & \left[-\frac{\partial^2 T_m}{\partial t \partial \dot{v}} - \frac{\partial^2 T_m}{\partial x \partial \dot{v}} \frac{\partial s}{\partial t} \right]_{x=s(t)} \\ & + \int_0^1 \left\{ -\frac{d}{dt} \left(\frac{\partial T_b}{\partial \dot{v}} \right) - \frac{d^2}{dx^2} \left(\frac{\partial U}{\partial v''} \right) \right\} dx = 0 \end{aligned} \quad (2.26)$$

which must be solved along with the following boundary conditions:

$$\begin{aligned} \left(\frac{\partial^2 U}{\partial x \partial v''} \right)_{x=1} &= 0 \\ \left(\frac{\partial U}{\partial v''} \right)_{x=1} &= 0 \end{aligned} \quad (2.27)$$

In (2.26) the integral has been taken out of the energy terms corresponding to the beam. Substituting the various energy terms ((2.13), (2.15), (2.19) and (2.21)) in (2.25) and (2.26), the following equations are obtained:

s variation:

$$\left(\frac{\partial^2}{\partial t^2} s(t) \right) + \omega^2 (s(t) - s_e) + \left(\frac{\partial^2}{\partial t^2} v(x, t) \right) \left(\frac{\partial}{\partial x} v(x, t) \right) \Big|_{x=s(t)} = 0 \quad (2.28)$$

v variation:

$$m f|_{x=s(t)} + \int_0^1 \left(\frac{\partial^4}{\partial x^4} v(x, t) + \frac{\partial^2}{\partial t^2} v(x, t) \right) dx = 0 \quad (2.29)$$

where f represents the nonlinear coupling terms between the mass and the beam, and is given by:

$$\begin{aligned} f = & \left(\frac{\partial^2}{\partial t^2} s(t) \right) \left(\frac{\partial}{\partial x} v(x, t) \right) + \left(\frac{\partial}{\partial t} s(t) \right)^2 \left(\frac{\partial^2}{\partial x^2} v(x, t) \right) \\ & + 2 \left(\frac{\partial}{\partial t} s(t) \right) \left(\frac{\partial^2}{\partial t \partial x} v(x, t) \right) + \left(\frac{\partial^2}{\partial t^2} v(x, t) \right) \end{aligned} \quad (2.30)$$

The first term in (2.30) comes from the acceleration of the mass in the x direction, the second term represents the centripetal acceleration, the third term represents the Coriolis acceleration and finally the fourth term is the acceleration of the beam in the y direction. All these acceleration terms form the nonlinear inertial coupling between the mass and the beam and they have to be evaluated at the position of the mass; a requirement which significantly increases the complexity of the problem. A notational detail which may be seen in (2.29)–(2.30) is that partial derivative notation is used for functions with only one independent variable (example $s(t)$), as this does not causes any mathematical inconsistency it will be used through out the work for convenience.

In addition to (2.28) and (2.29), the following natural boundary conditions must also be satisfied:

$$\begin{aligned} \left(\frac{\partial^3}{\partial x^3} v(x, t) \right) \Big|_{x=1} &= 0 \\ \left(\frac{\partial^2}{\partial x^2} v(x, t) \right) \Big|_{x=1} &= 0 \end{aligned} \quad (2.31)$$

At the fixed end, the following essential boundary conditions are enforced:

$$\begin{aligned} v(x, t)|_{x=0} &= 0 \\ \left(\frac{\partial}{\partial x} v(x, t) \right) \Big|_{x=0} &= 0 \end{aligned} \quad (2.32)$$

2.4 Equations of Motion – Large Oscillations

The equations of motion for the beam with large oscillations are obtained following a procedure similar to that used for small oscillations. In this case however the independent variables are s , u , v , and λ and correspondingly the following four equations are obtained.

s variation:

$$\left[\frac{\partial T_m}{\partial x} - \frac{\partial^2 T_m}{\partial t \partial s} - \frac{\partial^2 T_m}{\partial x \partial s} \frac{\partial s}{\partial t} \right]_{x=s(t)} - \frac{dV_m}{ds} = 0 \quad (2.33)$$

u variation:

$$\begin{aligned} & \left[-\frac{\partial^2 T_m}{\partial t \partial u} - \frac{\partial^2 T_m}{\partial x \partial u} \frac{\partial s}{\partial t} \right]_{x=s(t)} \\ & \int_0^1 -\frac{d}{dt} \left(\frac{\partial T_b}{\partial u} \right) - \frac{d}{dx} \left[\frac{d}{dx} \left(\frac{\partial U}{\partial \theta'} \right) (-\sin(\theta)) \right] + \frac{d}{dx} \left(\frac{\partial C}{\partial u'} \right) dx = 0 \end{aligned} \quad (2.34)$$

v variation:

$$\begin{aligned} & \left[-\frac{\partial^2 T_m}{\partial t \partial v} - \frac{\partial^2 T_m}{\partial x \partial v} \frac{\partial s}{\partial t} \right]_{x=s(t)} \\ & \int_0^1 -\frac{d}{dt} \left(\frac{\partial T_b}{\partial v} \right) - \frac{d}{dx} \left[\frac{d}{dx} \left(\frac{\partial U}{\partial \theta'} \right) \cos(\theta) \right] + \frac{d}{dx} \left(\frac{\partial C}{\partial v'} \right) dx = 0 \end{aligned} \quad (2.35)$$

λ variation:

$$\int_0^1 \frac{\partial C}{\partial \lambda} dx = 0 \quad (2.36)$$

The corresponding natural boundary conditions are given by:

$$\begin{aligned} \left[-\frac{\partial C}{\partial u'} + \frac{d}{dx} \left(\frac{\partial U}{\partial \theta'} \right) (-\sin(\theta)) \right]_{x=1} &= 0 \\ \left[-\frac{\partial U}{\partial \theta'} (-\sin(\theta)) \right]_{x=1} &= 0 \\ \left[-\frac{\partial C}{\partial v'} + \frac{d}{dx} \left(\frac{\partial U}{\partial \theta'} \right) \cos(\theta) \right]_{x=1} &= 0 \\ \left[-\frac{\partial U}{\partial \theta'} \cos(\theta) \right]_{x=1} &= 0 \end{aligned} \quad (2.37)$$

In (2.35) the integral has been taken out from the energy terms corresponding to the beam. Also in deriving these equations and the boundary conditions the variation in θ is replaced by:

$$\begin{aligned} \delta \theta &= -\frac{\partial v}{\partial x} \delta u' + \left(1 + \frac{\partial u}{\partial x} \right) \delta v' \\ &= -\sin(\theta) \delta u' + \cos(\theta) \delta v' \end{aligned} \quad (2.38)$$

In deriving (2.38) the trigonometric relationships between θ , u and v shown in the displacement field (2.3) are used. Using (2.11), (2.13), (2.17), (2.20) and (2.21) in the skeletal form (2.33)–(2.35), the following equations of motion are obtained:

s variation:

$$\frac{\partial^2 s}{\partial t^2} + \omega^2 (s - s_c) + \left[\frac{\partial^2 u}{\partial t^2} \cos(\theta) + \frac{\partial^2 v}{\partial t^2} \sin(\theta) \right]_{x=s(t)} = 0 \quad (2.39)$$

u variation:

$$m \left[\frac{\partial^2 s}{\partial t^2} \cos(\theta) + \frac{\partial^2 u}{\partial t^2} + 2 \frac{\partial s}{\partial t} \frac{\partial^2 u}{\partial t \partial x} + \left(\frac{\partial s}{\partial t} \right)^2 \frac{\partial^2 u}{\partial x^2} \right]_{x=s(t)} \\ \int_0^1 \left\{ \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x} \left[\lambda \cos(\theta) + \frac{\partial^2 \theta}{\partial x^2} \sin(\theta) \right] \right\} dx = 0 \quad (2.40)$$

v variation:

$$m \left[\frac{\partial^2 s}{\partial t^2} \sin(\theta) + \frac{\partial^2 v}{\partial t^2} + 2 \frac{\partial s}{\partial t} \frac{\partial^2 v}{\partial t \partial x} + \left(\frac{\partial s}{\partial t} \right)^2 \frac{\partial^2 v}{\partial x^2} \right]_{x=s(t)} \\ \int_0^1 \left\{ \frac{\partial^2 v}{\partial t^2} - \frac{\partial}{\partial x} \left[\lambda \sin(\theta) - \frac{\partial^2 \theta}{\partial x^2} \cos(\theta) \right] \right\} dx = 0 \quad (2.41)$$

λ variation:

$$\int_0^1 \left\{ \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial v}{\partial x} \right)^2 \right\} dx = 0 \quad (2.42)$$

The equations of motion for the large oscillation model (2.39)–(2.42) involve geometric nonlinearities (partial differential terms within the integrals) and kinematic nonlinearities (partial differential terms outside the integrals which have to be evaluated at the position of the moving mass) and a constraint equation with only spatial derivatives. The sources of the kinematic nonlinearities in (2.40) and (2.41) are same as in (2.29) with the components in the horizontal direction appearing in (2.40) and the components in the vertical direction appearing in (2.41). The geometric nonlinearities in (2.40) and (2.41) correspond to nonlinear stiffness in the beam. The terms outside the integrals can be brought within the integrals using the time dependent Dirac-delta functions and then the integrals can be removed.

This however is only a symbolic gesture without any significant advantage. Most of the references cited in Section 1.4 use this notation. In the present work however this approach is not used for the sake of mathematical clarity.

The natural boundary conditions at the free end of the cantilever beam are given by:

$$\begin{aligned}
 - \left[\lambda \cos(\theta) + \frac{\partial^2 \theta}{\partial x^2} \sin(\theta) \right]_{x=1} &= 0 \\
 - \sin(\theta) \frac{\partial \theta}{\partial x} \Big|_{x=1} &= 0 \\
 - \left[\lambda \sin(\theta) - \frac{\partial^2 \theta}{\partial x^2} \cos(\theta) \right]_{x=1} &= 0 \\
 - \cos(\theta) \frac{\partial \theta}{\partial x} \Big|_{x=1} &= 0
 \end{aligned} \tag{2.43}$$

The first and the third equation in (2.43) correspond to the shear force components in the horizontal and vertical directions, respectively, whereas the second and the fourth term correspond to the bending moment. In addition the following essential boundary conditions are enforced at the fixed end of the cantilever beam:

$$\begin{aligned}
 u(x, t) \Big|_{x=0} &= 0 \\
 \left(\frac{\partial}{\partial x} u(x, t) \right) \Big|_{x=0} &= 0 \\
 v(x, t) \Big|_{x=0} &= 0 \\
 \left(\frac{\partial}{\partial x} v(x, t) \right) \Big|_{x=0} &= 0 \\
 \lambda(x, t) \Big|_{x=0} &= 0
 \end{aligned} \tag{2.44}$$

The Lagrange multiplier $\lambda(x, t)$ is interpreted as the constraint force required to enforce the inextensibility condition. In (2.44), the essential boundary conditions for the derivatives of u are obtained from the constraint equation (2.9).

The terms involving θ in the equations of motion and the boundary conditions (2.39)–(2.43) can be replaced in terms of u and v using the relations in (2.3). This is however much more convenient when carried out during the spatial discretization process after incorporating the boundary conditions into the equations of motion using integration by parts. This process will be illustrated in the Chapter 3.

2.5 Initial Values

It is assumed that the mass-beam system is not subjected to external forces, in other words a free vibration problem is considered where the initial values are prescribed, and the time evolution is investigated. The following initial values are assumed:

$$\begin{aligned} s(0) &= s_0 & \left. \frac{\partial s(t)}{\partial t} \right|_{t=0} &= 0 \\ u(x, 0) &= u_0(x) & \left. \frac{\partial u(x, t)}{\partial t} \right|_{t=0} &= 0 \\ v(x, 0) &= v_0(x) & \left. \frac{\partial v(x, t)}{\partial t} \right|_{t=0} &= 0 \end{aligned} \tag{2.45}$$

where $u_0(x)$ and $v_0(x)$ represents the initial deflection curve of the beam, s_0 is the initial position of the moving mass, and the initial velocities are assumed to be zero.

The initial values for the beam must be selected such that the boundary condi-

tions are satisfied. In the present work the following two equations are used as the initial transverse deflection:

$$v_0(x) = \frac{1}{3}v_{t0}x^2(x^2 - 4x + 6) \quad (2.46)$$

$$v_0(x) = \frac{v_{t0}}{2} (\cosh(k_1x) - \cos(k_1x)) - \frac{\cos(k_1) + \cosh(k_1)}{\sin(k_1) + \sinh(k_1)} (\sinh(k_1x) - \sin(k_1x)) \quad (2.47)$$

where v_{t0} is a prescribed tip deflection. Equation (2.46) is the deflection curve for a simple cantilever beam carrying a uniformly distributed load, but instead of using the load intensity as the free parameter, the corresponding tip deflection is used. It can be easily shown by taking the second and third derivatives of that (2.46) satisfies all the boundary conditions for the small deformation model. Equation (2.47) is the scaled first mode shape of a simple cantilever beam ($k_1 = 1.8751$) which is also used as an initial value for the small oscillation model.

For the large oscillation model, (2.46) can be used as the initial value for the transverse deflection of the beam. The initial value for the axial variable u is obtained from the constraint equation (2.9) using:

$$\frac{\partial u}{\partial x} = -1 + \sqrt{1 - \left(\frac{\partial v}{\partial x}\right)^2} \quad (2.48)$$

where the positive root is chosen. The value of $u_0(x)$ is obtained from (2.48) using numerical integration. Since the initial deformation field satisfies the constraint equation it follows that the initial values of λ are zero. Figure 2.4 shows the initial values for u , v and its higher derivatives. It can be seen from this figure

that all the boundary conditions corresponding to u and v in (2.43) and (2.44) are satisfied. Satisfying the boundary conditions for the large deformation model using (2.47) is however much more difficult and therefore (2.47) is used only for the small oscillation model.

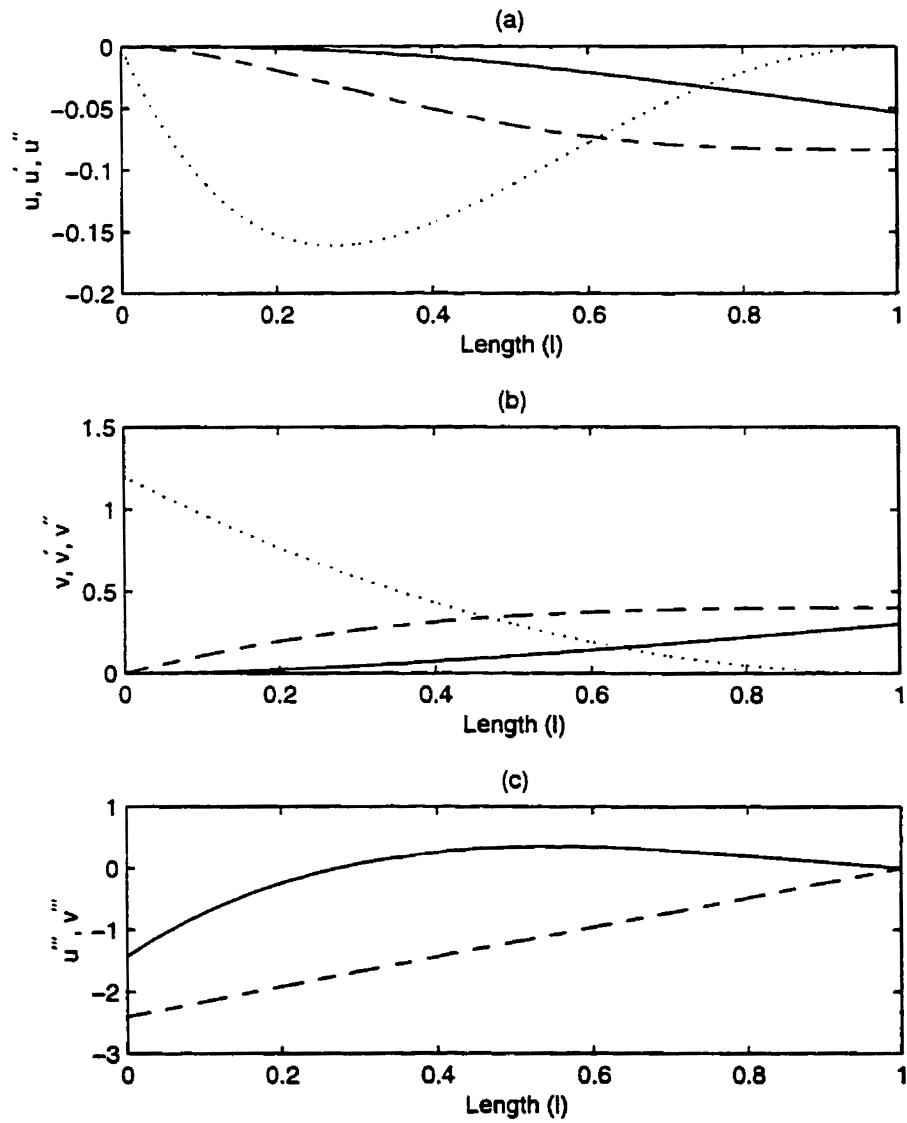


Figure 2.4: Initial values (a) $(-)$ u , $(--)$ u' , $(\cdots)$ u'' , (b) $(-)$ v , $(--)$ v' , $(\cdots)$ v'' , (c) $(-)$ u''' , $(--)$ v''' .

In this Chapter, equations of motion were developed for small and large oscillations of a flexible cantilever beam carrying a spring-mass sub-system. The equations for the small oscillation model (2.28) and (2.29), are integro-partial differential with kinematic nonlinearities arising due to the coupling between the mass and the beam and the equations for the large oscillation model (2.39)–(2.42) involve both kinematic and geometric nonlinearities, with the geometric nonlinearities arising from nonlinear strain displacement relationships. The focus of the next few chapters (Chapter 3, 4, and 5) is on obtaining analytical and numerical solutions to these equations. As a first step in this direction, in Chapter 3 equations (2.28), (2.29) and (2.39)–(2.42) are reduced to ordinary differential equations using spatial discretization. In Chapter 4, a closed form perturbation solution is obtained for the small oscillation model and a numerical solution is obtained using an automatic ordinary differential equation solver. In Chapter 5, a new method is developed to solve the equations of motion for the large oscillation model numerically.

Chapter 3

Spatial Discretization

The focus of this chapter is on reduction of nonlinear partial differential equations to ordinary differential equations using the Galerkin method. The resulting ordinary differential equations are in terms of time dependent coefficient functions and assumed basis functions. The convergence of the solution of the semi-discretized ordinary differential equations to the original partial differential equations depends on the basis functions and also on the number of basis functions used in approximating the variables. The basis functions are chosen such that the resulting approximation of the dependent variables satisfies the continuity requirements of the solution, as dictated by the equations of motion. For example in linear beams the highest spatial derivative is of fourth order, which implies that the solution must have continuous fourth derivative, but by integrating by parts the fourth order derivative can be reduced to second order, and also the natural boundary conditions can be incorporated into the equations of motion.

The basis functions can be defined over the length of the beam as in the classical

approach or piecewise over small elements as in the finite element method. In this work for the small oscillation model, where the beam is linear, the eigenfunctions corresponding to the beam are used as basis functions. It should be noted here that the eigenfunctions of a cantilever beam do not account for the effect of the moving mass and are not the eigenfunctions of the complete system. They are used here as basis functions in the Galerkin method, which is not the same as using them in the separation of variables approach commonly used in linear systems. In the Galerkin method the semi-discretized ordinary differential equations are obtained using the weighted residual approach and not through direct substitution of the approximations in the partial differential equations, as done in the separation of variables method.

For the large oscillation model the eigenfunctions of a cantilever beam are not used as basis functions; as, apart from the issue of convergence, the complexity of these basis functions makes the evaluation of the integrals in the equations of motion difficult. Therefore piecewise defined fifth degree Hermitian polynomials are used. These polynomials give an approximation which has continuous second derivatives, a requirement dictated by the equations of motion.

3.1 Symmetric Formulation – Small Oscillations

Following the Galerkin's approach [61] the equation of motion for the beam (2.29) is first multiplied by a weighting function $p(z)$ (also known as the test function) and using integration by parts, the natural boundary conditions (2.31) are incorporated

into the equations of motion, thus giving:

$$\frac{\partial^2 s}{\partial t^2} + \omega^2(s - s_e) + \left[\frac{\partial^2 v}{\partial t^2} \frac{\partial v}{\partial x} \right]_{x=s(t)} = 0 \quad (3.1)$$

$$\begin{aligned} m \left[\frac{\partial^2 v}{\partial t^2} p + \frac{\partial^2 s}{\partial t^2} \frac{\partial v}{\partial x} p + 2 \frac{\partial s}{\partial t} \frac{\partial^2 v}{\partial t \partial x} p + \left(\frac{\partial s}{\partial t} \right)^2 \frac{\partial^2 v}{\partial x^2} p \right]_{x=s(t)} \\ + \int_0^1 \left[\frac{\partial^2 v}{\partial x^2} \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 v}{\partial t^2} p \right] dx = 0 \end{aligned} \quad (3.2)$$

where equation (3.1) involving the mass motion is simply repeated from (2.28) as it does not require spatial discretization. This formulation (3.1)–(3.2) is referred to as being symmetric because the highest spatial derivative of the dependent variable $v(x, t)$ is of the same order as the highest derivative of the weighting function $p(x)$.

3.2 Symmetric Formulation – Large Oscillations

The symmetric formulation of the equations of motion for the large oscillations is similarly derived by multiplying (2.40)–(2.42) with the weighting function $p(x)$ and integrating by parts twice and substituting the boundary conditions (2.43) thus giving:

$$\frac{\partial^2 s}{\partial t^2} + \omega^2(s - s_e) + \left[\frac{\partial^2 u}{\partial t^2} \cos(\theta) + \frac{\partial^2 v}{\partial t^2} \sin(\theta) \right]_{x=s(t)} = 0 \quad (3.3)$$

$$m \left[\frac{\partial^2 u}{\partial t^2} p + \frac{\partial^2 s}{\partial t^2} p \cos(\theta) + 2 \frac{\partial s}{\partial t} \frac{\partial^2 u}{\partial t \partial x} p + \left(\frac{\partial s}{\partial t} \right)^2 \frac{\partial^2 u}{\partial x^2} p \right]_{x=s(t)}$$

$$+ \int_0^1 \left[\frac{\partial^2 u}{\partial t^2} p + \frac{\partial^2 p}{\partial x^2} + \frac{\partial p}{\partial x} \left(\lambda - \left(\frac{\partial \theta}{\partial x} \right)^2 \right) \cos(\theta) \right] dx = 0 \quad (3.4)$$

$$m \left[\frac{\partial^2 v}{\partial t^2} p + \frac{\partial^2 s}{\partial t^2} p \sin(\theta) + 2 \frac{\partial s}{\partial t} \frac{\partial^2 v}{\partial t \partial x} p + \left(\frac{\partial s}{\partial t} \right)^2 \frac{\partial^2 v}{\partial x^2} p \right]_{x=s(t)} \\ + \int_0^1 \left[\frac{\partial^2 v}{\partial t^2} p + \frac{\partial^2 p}{\partial x^2} \frac{\partial^2 v}{\partial x^2} + \frac{\partial p}{\partial x} \left(\lambda - \left(\frac{\partial \theta}{\partial x} \right)^2 \right) \sin(\theta) \right] dx = 0 \quad (3.5)$$

$$\int_0^1 \left[\frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial u}{\partial x} \right)^2 + \frac{1}{2} \left(\frac{\partial v}{\partial x} \right)^2 \right] dx = 0 \quad (3.6)$$

The term involving the spatial derivative of θ in (3.4) and (3.5) can be expressed in terms of u and v using the trigonometric relations given in (2.3) as follows:

$$\left(\frac{\partial \theta}{\partial x} \right)^2 = \left(\frac{\partial^2 u}{\partial x^2} \right)^2 + \left(\frac{\partial^2 v}{\partial x^2} \right)^2 \quad (3.7)$$

The Lagrange multiplier λ in the equations of motion (3.4) and (3.5) can be interpreted as being proportional to the constraint forces which arise due to the inextensibility of the beam. The difference between λ and $\left(\frac{\partial \theta}{\partial x} \right)^2$ occurring in (3.4) and (3.5) corresponds to difference between the constraint force and the nonlinear term in the beam stiffness.

3.3 Spatial Discretization – Small Oscillations

The Galerkin method is now applied to the equations of motion for the small oscillation model (3.1)–(3.2) using the following approximations:

$$\begin{aligned} v(x, t) &= \sum_i \alpha_i(t) \phi_i(x) \\ p(x) &= \sum_i p_i \phi_i(x) \end{aligned} \quad (3.8)$$

In (3.8), summation is carried out over the repeated index i . The symmetric formulation renders the continuity requirements for both p and v the same. Therefore the same basis functions ϕ_i may be selected for both p and v and the number of basis functions (maximum values of i) for both p and v are also assumed to be equal. The undetermined parameters p_i in (3.8) are arbitrary and therefore substituting (3.8) into the equations of motion (3.1)–(3.2) and setting p_i (i^{th} value) to one and all other p_j , $j \neq i$ to zero gives the i^{th} semi-discretized ordinary differential equation[61]. Using this process the following semi-discretized equations are obtained for the small oscillation model:

s equation:

$$\ddot{s} + \omega^2(s - s_e) + \left[\phi_i \phi'_j \right]_{x=s(t)} \{ \ddot{\alpha}_i \alpha_j \} = 0 \quad (3.9)$$

v equation:

$$\begin{aligned} m \left\{ \left[\phi_i \phi_j \right]_{x=s(t)} \{ \ddot{\alpha}_j \} + \ddot{s} \left[\phi_i \phi'_j \right]_{x=s(t)} \{ \alpha_j \} + 2\dot{s} \left[\phi_i \phi'_j \right]_{x=s(t)} \{ \dot{\alpha}_j \} \right. \\ \left. + \dot{s}^2 \left[\phi_i \phi''_j \right]_{x=s(t)} \{ \alpha_j \} \right\} + \left[\int_0^1 \phi_i \phi_j dx \right] \{ \ddot{\alpha}_j \} + \left[\int_0^1 \phi''_i \phi''_j dx \right] \{ \alpha_j \} = 0 \end{aligned} \quad (3.10)$$

In (3.9) and (3.10) index notation is used where the repeated indices imply summation over the index. To further clarify the index notation brackets are also used; $[\dots]$ denote a row matrix and $\{\dots\}$ denote a column matrix whereas $[\dots]$ denote a square matrix. The derivatives with respect to t and x are denoted by $(\dot{})$ and $()'$ respectively. Since the beam model is linear the nonlinearities are only due to kinematics, hence there are no nonlinear terms for the beam portion (integral matrices) of the equations of motion (3.10). The number of equations of motion depends on the number of basis functions ϕ_i used for the approximation. These basis functions are obtained later in Sections 3.5 and 3.6.

Equations (3.9) and (3.10) are ordinary differential equations with kinematic nonlinearities. These equations were also derived using the Rayleigh-Ritz method in Siddiqui et al. [53] for the small oscillation model. The only difference was found to be an additional term $\dot{s}^2[\phi_i'\phi_j']\{\alpha_j\}$ in the mass position dependent terms in equation (3.10). This additional term causes small differences in the beating period under internal resonance conditions and does not have any other significant impact. Therefore these results are not presented in this thesis. Galerkin's method was chosen here because it is more easy to apply to the large deflection model and therefore for comparison purposes the same method is used for the small oscillation model.

3.4 Spatial Discretization – Large Oscillations

For the large oscillation model the following approximations are assumed:

$$\begin{aligned}
 u(x, t) &= \sum_i \alpha_i(t) \phi_i(x) \\
 v(x, t) &= \sum_i \beta_i(t) \phi_i(x) \\
 \lambda(x, t) &= \sum_i \lambda_i(t) \phi_i(x) \\
 p(x) &= \sum_i p_i \phi_i(x)
 \end{aligned} \tag{3.11}$$

In the expansions (3.11), the same basis functions are used for all the four dependent variables $u(x, t)$, $v(x, t)$, $\lambda(x, t)$ and $p(x)$, the reason being that $u(x, t)$, $v(x, t)$, and $p(x)$ have the same continuity requirements (continuous second derivative) and λ is related to the second derivative of u and v . It may be noted here that in the equation of motion for axial vibrations of beams the highest spatial derivative is of second order and for transverse vibrations due to bending the highest spatial derivative is of fourth order and the use of different basis functions for axial and transverse vibrations may be appropriate in such cases. However for the large oscillation model considered here the axial deformations are due to the foreshortening effect and as such are not governed by the wave equation generally used in modelling axial vibrations.

Following the same approach used for the small oscillation model (see Section 3.3), the following equations are obtained.

s equation:

$$\ddot{s} + \omega^2(s - s_e) + [\phi_i]_{x=s(t)} \{\ddot{\alpha}_i\} + [\phi_i \phi'_j]_{x=s(t)} \{\ddot{\alpha}_i \alpha_j + \ddot{\beta}_i \beta_j\} = 0 \quad (3.12)$$

u equation:

$$\begin{aligned} m \left\{ \ddot{s} \{\phi_i\}_{x=s(t)} + [\phi_i \phi_j]_{x=s(t)} \{\ddot{\alpha}_j\} + [\phi_i \phi'_j]_{x=s(t)} \{\ddot{s} \{\alpha_j\} + 2\dot{s} \{\dot{\alpha}_j\}\} \right. \\ \left. + \dot{s}^2 [\phi_i \phi''_j]_{x=s(t)} \{\alpha_j\} \right\} + \int_0^1 \left[[\phi_i \phi_j] \{\ddot{\alpha}_j\} + [\phi''_i \phi''_j] \{\alpha_j\} + [\phi'_i \phi_j] \{\lambda_j\} \right. \\ \left. + \{\phi'_i\} [\phi_j \phi'_k] \{\lambda_j \alpha_k\} - \{\phi'_i\} [\phi''_j \phi''_k] \{\{\alpha_j \alpha_k\} + \{\beta_j \beta_k\}\} \right. \\ \left. - \{\phi'_i\} [\phi'_j \phi''_k \phi''_l] \{\{\alpha_j \alpha_k \alpha_l\} + \{\alpha_j \beta_k \beta_l\}\} \right] dx = 0 \quad (3.13) \end{aligned}$$

v equation:

$$\begin{aligned} m \left\{ [\phi_i \phi_j]_{x=s(t)} \{\ddot{\beta}_j\} + [\phi_i \phi'_j]_{x=s(t)} \{\ddot{s} \{\beta_j\} + 2\dot{s} \{\dot{\beta}_j\}\} + \dot{s}^2 [\phi_i \phi''_j]_{x=s(t)} \{\beta_j\} \right\} \\ + \int_0^1 \left[[\phi_i \phi_j] \{\ddot{\beta}_j\} + [\phi''_i \phi''_j] \{\beta_j\} + \{\phi'_i\} [\phi_j \phi'_k] \{\lambda_j \beta_k\} \right. \\ \left. - \{\phi'_i\} [\phi'_j \phi''_k \phi''_l] \{\{\beta_j \alpha_k \alpha_l\} + \{\beta_j \beta_k \beta_l\}\} \right] dx = 0 \quad (3.14) \end{aligned}$$

λ equation:

$$\int_0^1 \left[[\phi_i \phi'_j] \{\alpha_j\} + \frac{1}{2} \{\phi_i\} [\phi'_j \phi'_k] \{\{\alpha_j \alpha_k\} + \{\beta_j \beta_k\}\} \right] dx = 0 \quad (3.15)$$

The equation of motion for the mass (3.12) is similar in form to that obtained for the small oscillation model (3.9) with quadratic nonlinearities forming the coupling between the mass and the beam. The equations of motion for the beam variables

(3.13) and (3.14) contain both quadratic and cubic nonlinear terms corresponding to the geometric nonlinearities with integrals over three and four dimensional matrices as indicated by the indices i, j, k and i, j, k, l , respectively. The constraint equation (3.15) does not have any time derivatives. Assuming the basis functions are known, the system of equations (3.12)–(3.15) consists of nonlinear algebraic integro-differential equations. The basis functions are considered next.

3.5 Eigenfunctions of a Cantilever Beam

As mentioned earlier, the small oscillation model is solved using the eigenfunctions of a simple cantilever beam as basis functions. The basis functions in Galerkin method are generally chosen to be as simple as possible but a desirable property is that these functions should be orthogonal to facilitate solving the initial value problem by producing well conditioned system matrices. The eigenfunctions for the cantilever beam are then a natural choice and one would expect fairly accurate results with relatively few basis functions. Increasing the number of basis functions increases the accuracy of the solution. It should be noted that the eigenfunctions of a cantilever beam do not account for the motion of mass however they are used here as basis functions in the Galerkin's method (see the discussion in the second paragraph at the beginning of this chapter). Stanisic [93] developed a method to improve the eigenfunctions to account for the motion of the mass and Khalily et al. [5] used two of these improved eigenfunctions in the Rayleigh-Ritz method and obtained a numerical solution to the coupled motion of a beam with a forced mass. The complexity of these improved shape functions does not justify their use as

basis functions in Galerkin method or the Rayleigh-Ritz method. It is shown in Chapter 4 that by considering four cantilever beam eigenfunctions one can solve this problem with good accuracy.

The cantilever beam eigenfunctions for the nondimensionalized parameters used here are given by [30]:

$$\phi_i = \cosh(k_i x) - \cos(k_i x) - \frac{\cos(k_i) + \cosh(k_i)}{\sin(k_i) + \sinh(k_i)} (\sinh(k_i x) - \sin(k_i x)) \quad (3.16)$$

where for the first four modes, k_i have the values

$$k_1 = 1.875104069, k_2 = 4.694091133, k_3 = 7.854757438, \text{ and } k_4 = 10.995547073$$

The four mode shapes are shown in Figure 3.1. The mass, and the stiffness ma-

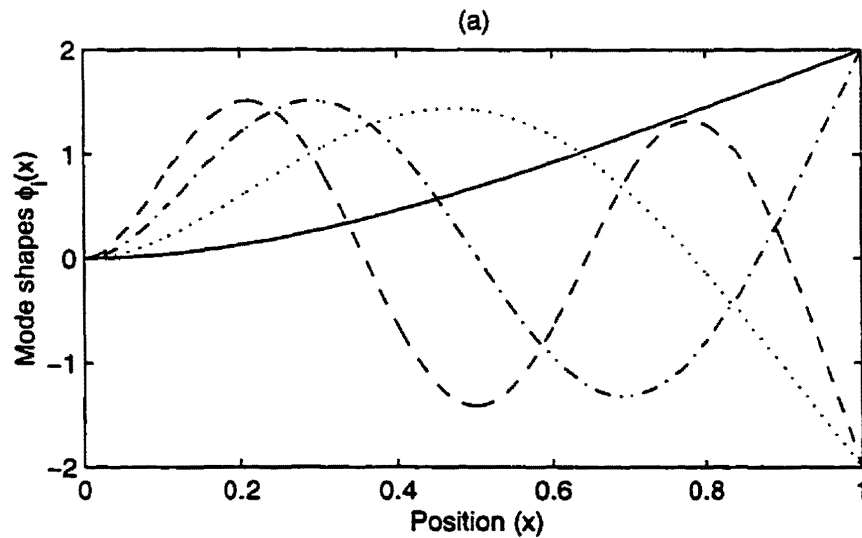


Figure 3.1: Cantilever beam mode shapes, $(-)$ ϕ_1 , $(-.)$ ϕ_2 , $(\dots)$ ϕ_3 and $(- -)$ ϕ_4

trix in (3.10) require integration of products of basis functions and their second derivatives over the length of the beam. These integrations are carried out analytically using the symbolic manipulation program MAPLE. The following mass and stiffness matrices are obtained for the small oscillation model:

$$[M_{ij}] = \left[\int_0^1 \phi_i \phi_j dx \right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (3.17)$$

$$[K_{ij}] = \left[\int_0^1 \phi_i'' \phi_j'' dx \right] \\ = \begin{bmatrix} 12.36236340 & 0 & 0 & 0 \\ 0 & 485.5188211 & 0 & 0 \\ 0 & 0 & 3806.546524 & 0 \\ 0 & 0 & 0 & 14617.25610 \end{bmatrix} \quad (3.18)$$

The other matrix terms in (3.10) depend on the position of the mass, therefore they are assembled online as discussed later in Section 3.6.2.

3.6 Basis Functions Using Finite Elements

In a well formulated problem the minimum degree of continuity for the interpolating polynomial is dictated by the smoothness requirement of the variational formulation. It can be seen from the symmetric formulation of the equations of motion for the small oscillation model (3.9) and (3.10) and the large oscillation model

(3.12)–(3.15), the highest spatial derivative of the dependent variables is at most second order. Therefore if the approximations in the Galerkin method (3.8) and (3.11) are C^2 continuous (continuous second spatial derivative), the derivatives in the equations of motion will be well defined.

If only the beam portion of the equations of motion of small oscillation model are considered, it can be seen from (3.10) that the smoothness requirement is imposed by the stiffness matrix $([\int_0^1 \phi_i'' \phi_j'' dx])$, which requires that the basis functions be such that the integral of the square of the second derivative exists. If the basis functions are at least C^1 continuous (i.e. the first derivative is continuous over the length of the beam), then by properly matching the values of their first derivative (slopes) at each the node, the second derivatives (curvature) can be made square integrable over the length of the beam. However with C^1 continuity the mass position dependent cubic nonlinear terms (see (3.10)) would have finite jumps in their finite element approximations. Also, numerical results show that with C^2 continuity higher frequencies are predicted more accurately. So in this work fifth degree Hermitian polynomials are used to obtain C^2 continuous solutions for both the small oscillation and the large oscillation model.

3.6.1 Fifth Degree Hermitian Polynomials - C^2 Continuity

The key variables like the deflection, slope or the curvature of the beam are obtained by interpolating the values of these quantities at certain significant points called nodes. The nodes are taken at the edges of an element and sometimes at intermediate points. The interpolant function is then obtained by combining piecewise

polynomials defined over each element.

In the finite element method the beam is divided into a number of small elements as shown in Figure 3.2. To obtain C^2 continuity fifth degree Hermitian polynomials

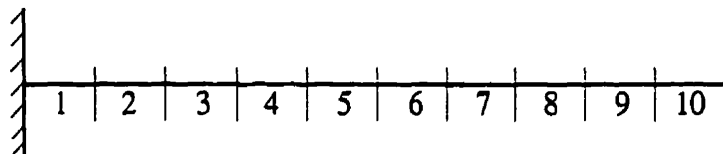


Figure 3.2: Beam Discretization

(also known as second order Hermitian polynomials) are used. These are a set of six polynomials of fifth degree. To illustrate their application as finite element basis functions, consider the discretization of the beam into N elements. The elements are numbered from 1 to N . For each element consider two local nodes one at each end. The global nodes are numbered sequentially from 0 to N . The transverse deflection of the beam $v(x, t)$ is then represented in terms of $3(N+1)$ basis functions as follows:

$$v(x, t) = \sum_{i=0}^N \alpha_i^0(t) \phi_i^0(x) + \sum_{i=0}^N \alpha_i^1(t) \phi_i^1(x) + \sum_{i=0}^N \alpha_i^2(t) \phi_i^2(x) \quad (3.19)$$

In (3.19), $\phi_i^0(x)$, $\phi_i^1(x)$ and $\phi_i^2(x)$ are basis functions corresponding to node i , where i ranges from 0 to N . The approximation (3.19) interpolates the deflection $v(x, t)$ at node i , denoted by $\alpha_i^0(t)$, the first spatial derivative of $v(x, t)$ (slope) at node i , denoted by $\alpha_i^1(t)$, and the second spatial derivative of $v(x, t)$ (curvature) at node i , denoted by $\alpha_i^2(t)$. The basis functions are chosen such that they satisfy the

following properties:

$$\begin{aligned}
 \phi_i^0(x_j) &= \delta_{ij}, & \frac{d\phi_i^0(x_j)}{dx} &= 0, & \frac{d^2\phi_i^0(x_j)}{dx^2} &= 0 \\
 \phi_i^1(x_j) &= 0, & \frac{d\phi_i^1(x_j)}{dx} &= \delta_{ij}, & \frac{d^2\phi_i^1(x_j)}{dx^2} &= 0 \\
 \phi_i^2(x_j) &= 0, & \frac{d\phi_i^2(x_j)}{dx} &= 0, & \frac{d^2\phi_i^2(x_j)}{dx^2} &= \delta_{ij}
 \end{aligned} \tag{3.20}$$

where δ_{ij} is the Kronecker-delta, also x_j denotes the position of the global node and subscripts i and j identify the global nodes. From (3.19) and (3.20) it follows that at node i :

$$\begin{aligned}
 v(x_i, t) &= v_i = \alpha_i^0(t) \\
 \frac{\partial v(x, t)}{\partial x} \Big|_{x=x_i} &= v'_i = \alpha_i^1(t) \\
 \frac{\partial^2 v(x, t)}{\partial x^2} \Big|_{x=x_i} &= v''_i = \alpha_i^2(t)
 \end{aligned} \tag{3.21}$$

So at each node i there are three degrees of freedom, the deflection $\alpha_i^0(t)$, the slope $\alpha_i^1(t)$ and the curvature $\alpha_i^2(t)$.

The properties of the Hermitian polynomials (3.20) imply that each basis function is nonzero over two adjacent elements with ϕ_i^0 , ϕ_i^1 , and ϕ_i^2 equal to unity at the joining node. Therefore it is more convenient to represent these basis functions as a combination of two functions where each function is defined over one element:

$$\begin{aligned}
 \phi_i^0(x_j) &= \begin{cases} \psi_2^0(x) & x_{i-1} \leq x \leq x_i \\ \psi_1^0(x) & x_i \leq x \leq x_{i+1} \end{cases} \\
 \phi_i^1(x_j) &= \begin{cases} \psi_2^1(x) & x_{i-1} \leq x \leq x_i \\ \psi_1^1(x) & x_i \leq x \leq x_{i+1} \end{cases}
 \end{aligned}$$

$$\phi_i^2(x_j) = \begin{cases} \psi_2^2(x) & x_{i-1} \leq x \leq x_i \\ \psi_1^2(x) & x_i \leq x \leq x_{i+1} \end{cases} \quad (3.22)$$

In (3.22) the ψ 's are referred to as the elemental shape functions or the elemental basis functions. The same set of element basis functions are used for all elements. The properties (3.20) when applied to the elemental shape functions ψ 's give the following result:

$$\begin{bmatrix} \psi_1^0(x_{i-1}) & \psi_1^0(x_i) & \frac{d\psi_1^0(x_{i-1})}{dx} & \frac{d\psi_1^0(x_i)}{dx} & \frac{d^2\psi_1^0(x_{i-1})}{dx^2} & \frac{d^2\psi_1^0(x_i)}{dx^2} \\ \psi_2^0(x_{i-1}) & \psi_2^0(x_i) & \frac{d\psi_2^0(x_{i-1})}{dx} & \frac{d\psi_2^0(x_i)}{dx} & \frac{d^2\psi_2^0(x_{i-1})}{dx^2} & \frac{d^2\psi_2^0(x_i)}{dx^2} \\ \psi_1^1(x_{i-1}) & \psi_1^1(x_i) & \frac{d\psi_1^1(x_{i-1})}{dx} & \frac{d\psi_1^1(x_i)}{dx} & \frac{d^2\psi_1^1(x_{i-1})}{dx^2} & \frac{d^2\psi_1^1(x_i)}{dx^2} \\ \psi_2^1(x_{i-1}) & \psi_2^1(x_i) & \frac{d\psi_2^1(x_{i-1})}{dx} & \frac{d\psi_2^1(x_i)}{dx} & \frac{d^2\psi_2^1(x_{i-1})}{dx^2} & \frac{d^2\psi_2^1(x_i)}{dx^2} \\ \psi_1^2(x_{i-1}) & \psi_1^2(x_i) & \frac{d\psi_1^2(x_{i-1})}{dx} & \frac{d\psi_1^2(x_i)}{dx} & \frac{d^2\psi_1^2(x_{i-1})}{dx^2} & \frac{d^2\psi_1^2(x_i)}{dx^2} \\ \psi_2^2(x_{i-1}) & \psi_2^2(x_i) & \frac{d\psi_2^2(x_{i-1})}{dx} & \frac{d\psi_2^2(x_i)}{dx} & \frac{d^2\psi_2^2(x_{i-1})}{dx^2} & \frac{d^2\psi_2^2(x_i)}{dx^2} \end{bmatrix} = I \quad (3.23)$$

where I is an identity matrix. For each shape function ψ_1^0 , ψ_2^0 , ψ_1^1 , ψ_2^1 , ψ_1^2 , and ψ_2^2 equation (3.23) provides six conditions. Using these six conditions, fifth degree Hermitian polynomials can be found. In general, finite element analysis is carried out in a coordinate frame local to a master element. The results are then assembled in global matrices. In this work however, as a number of terms in the equations of motion have to be evaluated at the current position of the moving mass, it becomes necessary to do the analysis in a global coordinate frame. Using (3.23) the following

polynomials are obtained:

$$\begin{aligned}
\psi_1^0 &= -6 \frac{x^5}{l_e^5} + 15 \frac{(2x_{i-1} + l_e)x^4}{l_e^5} - 10 \frac{(6x_{i-1}^2 + 6x_{i-1}l_e + l_e^2)x^3}{l_e^5} \\
&+ 30 \frac{(x_{i-1} + l_e)x_{i-1}(2x_{i-1} + l_e)x^2}{l_e^5} - 30 \frac{x_{i-1}^2(x_{i-1} + l_e)^2x}{l_e^5} \\
&+ \frac{(x_{i-1} + l_e)^3(6x_{i-1}^2 - 3x_{i-1}l_e + l_e^2)}{l_e^5} \\
\psi_1^1 &= -3 \frac{x^5}{l_e^4} + \frac{(15x_{i-1} + 8l_e)x^4}{l_e^4} - 2 \frac{(15x_{i-1}^2 + 16x_{i-1}l_e + 3l_e^2)x^3}{l_e^4} \\
&+ 6 \frac{(5x_{i-1} + 3l_e)x_{i-1}(x_{i-1} + l_e)x^2}{l_e^4} + \frac{(3x_{i-1} + l_e)(-5x_{i-1} + l_e)(x_{i-1} + l_e)^2x}{l_e^4} \\
&- \frac{(-3x_{i-1} + l_e)x_{i-1}(x_{i-1} + l_e)^3}{l_e^4} \\
\psi_1^2 &= -1/2 \frac{x^5}{l_e^3} + 1/2 \frac{(5x_{i-1} + 3l_e)x^4}{l_e^3} - 1/2 \frac{(10x_{i-1}^2 + 12x_{i-1}l_e + 3l_e^2)x^3}{l_e^3} \\
&+ 1/2 \frac{(x_{i-1} + l_e)(10x_{i-1}^2 + 8x_{i-1}l_e + l_e^2)x^2}{l_e^3} \\
&+ 1/2 \frac{x_{i-1}^2(x_{i-1} + l_e)^3}{l_e^3} - 1/2 \frac{(x_{i-1} + l_e)^2x_{i-1}(5x_{i-1} + 2l_e)x}{l_e^3} \\
\psi_2^0 &= 6 \frac{x^5}{l_e^5} - 15 \frac{(2x_{i-1} + l_e)x^4}{l_e^5} + 10 \frac{(6x_{i-1}^2 + 6x_{i-1}l_e + l_e^2)x^3}{l_e^5} \\
&- 30 \frac{(x_{i-1} + l_e)x_{i-1}(2x_{i-1} + l_e)x^2}{l_e^5} + 30 \frac{x_{i-1}^2(x_{i-1} + l_e)^2x}{l_e^5} \\
&- \frac{x_{i-1}^3(6x_{i-1}^2 + 15x_{i-1}l_e + 10l_e^2)}{l_e^5} \\
\psi_2^1 &= -3 \frac{x^5}{l_e^4} + \frac{(15x_{i-1} + 7l_e)x^4}{l_e^4} - 2 \frac{(15x_{i-1}^2 + 14x_{i-1}l_e + 2l_e^2)x^3}{l_e^4} \\
&+ 6 \frac{(5x_{i-1} + 2l_e)x_{i-1}(x_{i-1} + l_e)x^2}{l_e^4} - \frac{(3x_{i-1} + 2l_e)(5x_{i-1} + 6l_e)x_{i-1}^2x}{l_e^4} \\
&+ \frac{(3x_{i-1} + 4l_e)x_{i-1}^3(x_{i-1} + l_e)}{l_e^4}
\end{aligned}$$

$$\begin{aligned}
\psi_2^2 = & 1/2 \frac{x^5}{l_e^3} - 1/2 \frac{(5x_{i-1} + 2l_e)x^4}{l_e^3} + 1/2 \frac{(10x_{i-1}^2 + 8x_{i-1}l_e + l_e^2)x^3}{l_e^3} \\
& - 1/2 \frac{x_{i-1}(10x_{i-1}^2 + 12x_{i-1}l_e + 3l_e^2)x^2}{l_e^3} + 1/2 \frac{x_{i-1}^2(x_{i-1} + l_e)(5x_{i-1} + 3l_e)x}{l_e^3} \\
& - 1/2 \frac{x_{i-1}^3(x_{i-1} + l_e)^2}{l_e^3}
\end{aligned} \tag{3.24}$$

where $l_e = x_i - x_{i-1}$ is the length of an element. Figure 3.3 shows the fifth degree Hermitian polynomials and Figure 3.4 shows the first derivatives and Figure 3.5 shows the second derivatives of the Hermitian polynomials. Using these figures the properties of the Hermitian polynomials (3.23) can be verified.

Figure 3.6 shows how any given function can be interpolated by its nodal values. In this case the fourth mode of a cantilever beam is obtained using fifth degree Hermitian polynomials which interpolate the deflection and the slope (first spatial derivative) and curvature (second spatial derivative) at each node. In this case ten elements are used each of length 0.1. In Figure 3.6(a) six shape functions corresponding to fifth degree Hermitian polynomials are shown (difficult to see but they do exist) which interpolate the nodal variables and obtain the approximation to the fourth mode of a cantilever beam. The interpolated function as well as the actual function are both shown in the Figure, but as can be seen they are indistinguishable indicating the accuracy of the interpolation. For clarity in Figure 3.6(b) only the shape functions which interpolate the function values are shown.

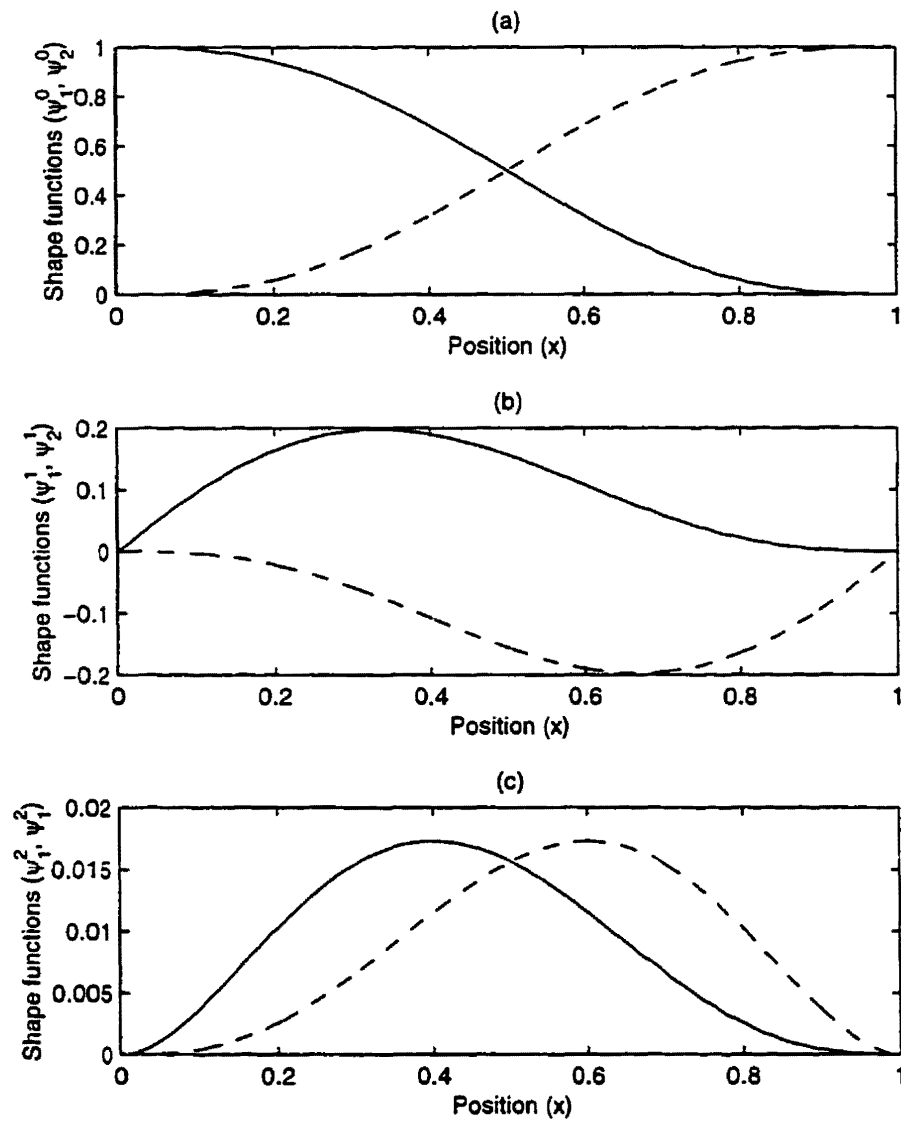


Figure 3.3: Fifth degree Hermitian polynomials (a) $(-)\psi_1^0$ and $(- -)\psi_2^0$, (b) $(-)\psi_1^1$ and $(- -)\psi_2^1$ and (c) $(-)\psi_1^2$ and $(- -)\psi_2^2$

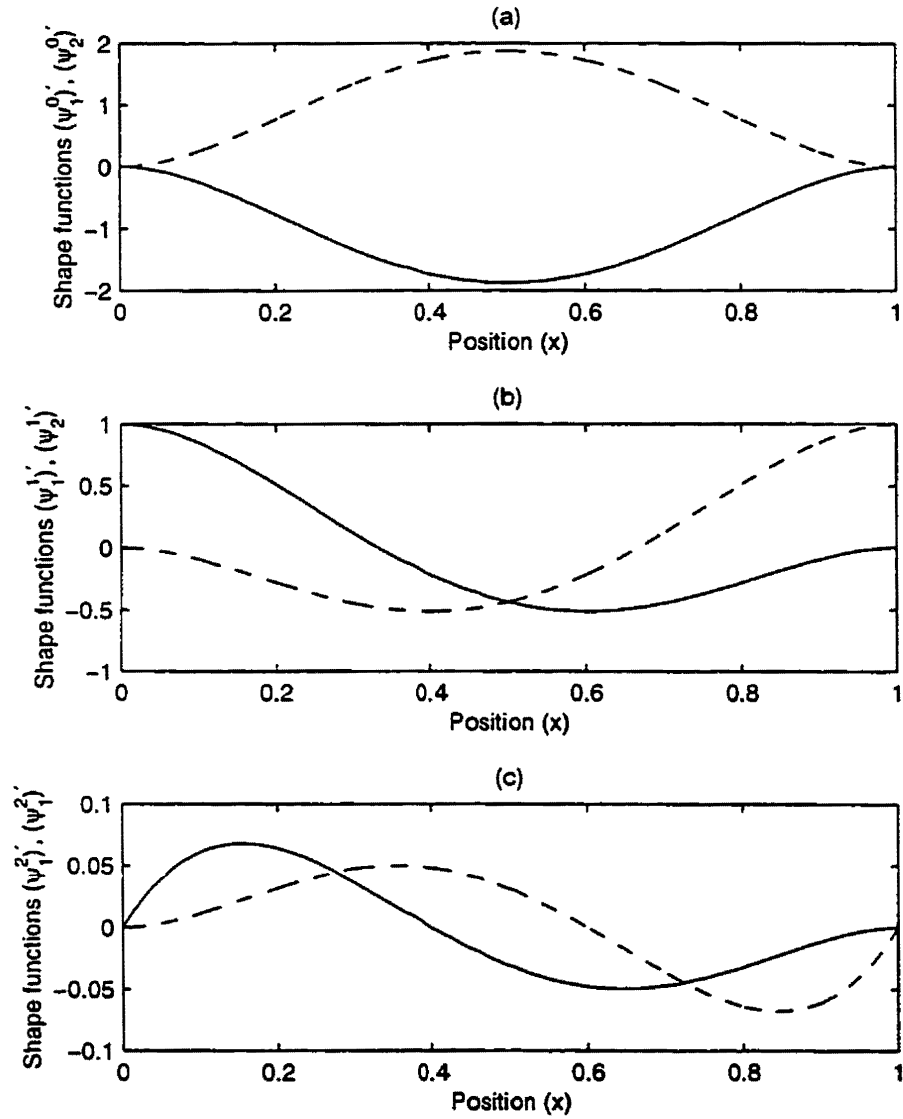


Figure 3.4: Fifth degree Hermite polynomials—first spatial derivatives (a) $(-)\frac{d\psi_1^0}{dx}$ and $(- -)\frac{d\psi_2^0}{dx}$, (b) $(-)\frac{d\psi_1^1}{dx}$ and $(- -)\frac{d\psi_2^1}{dx}$ and (c) $(-)\frac{d\psi_1^2}{dx}$ and $(- -)\frac{d\psi_2^2}{dx}$

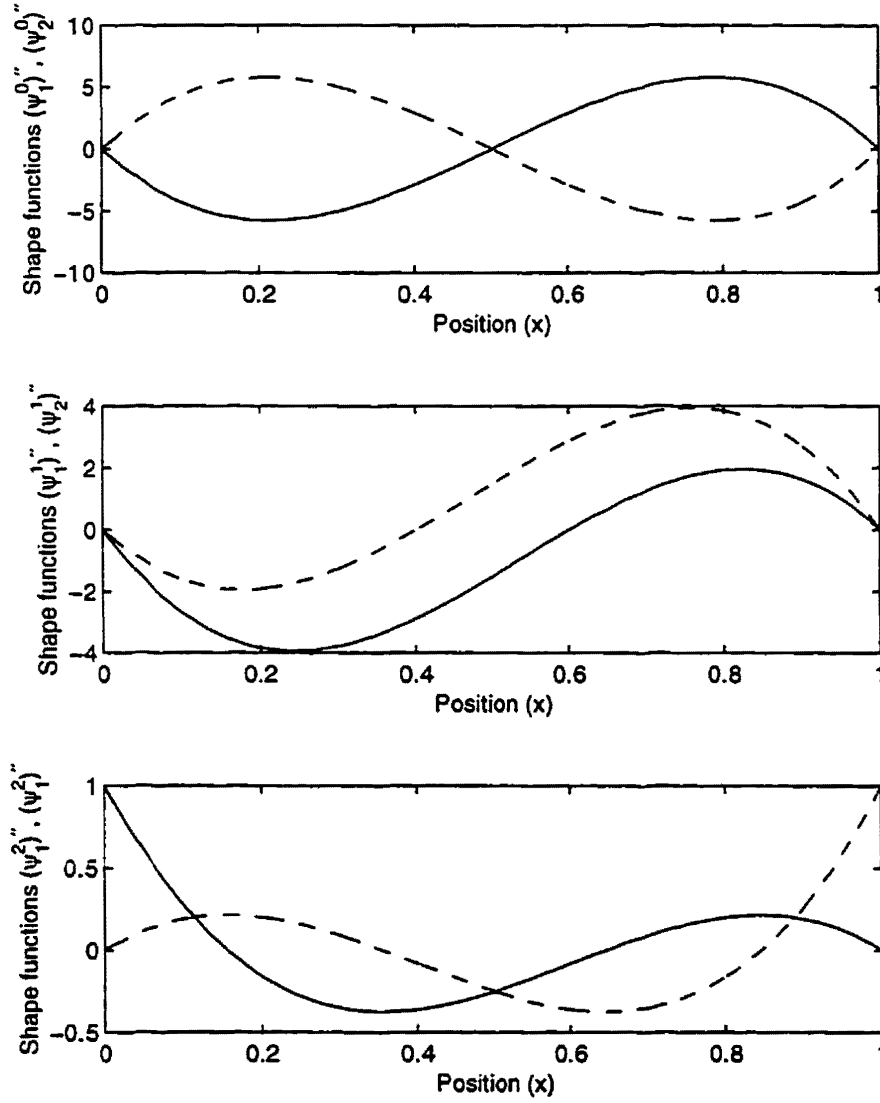


Figure 3.5: Fifth degree Hermite polynomials—second spatial derivatives (a) $(-)\frac{d^2\psi_1^0}{dx^2}$ and $(- -)\frac{d^2\psi_2^0}{dx^2}$, (b) $(-)\frac{d^2\psi_2^1}{dx^2}$ and $(- -)\frac{d^2\psi_3^1}{dx^2}$ and (c) $(-)\frac{d^2\psi_3^2}{dx^2}$ and $(- -)\frac{d^2\psi_4^2}{dx^2}$

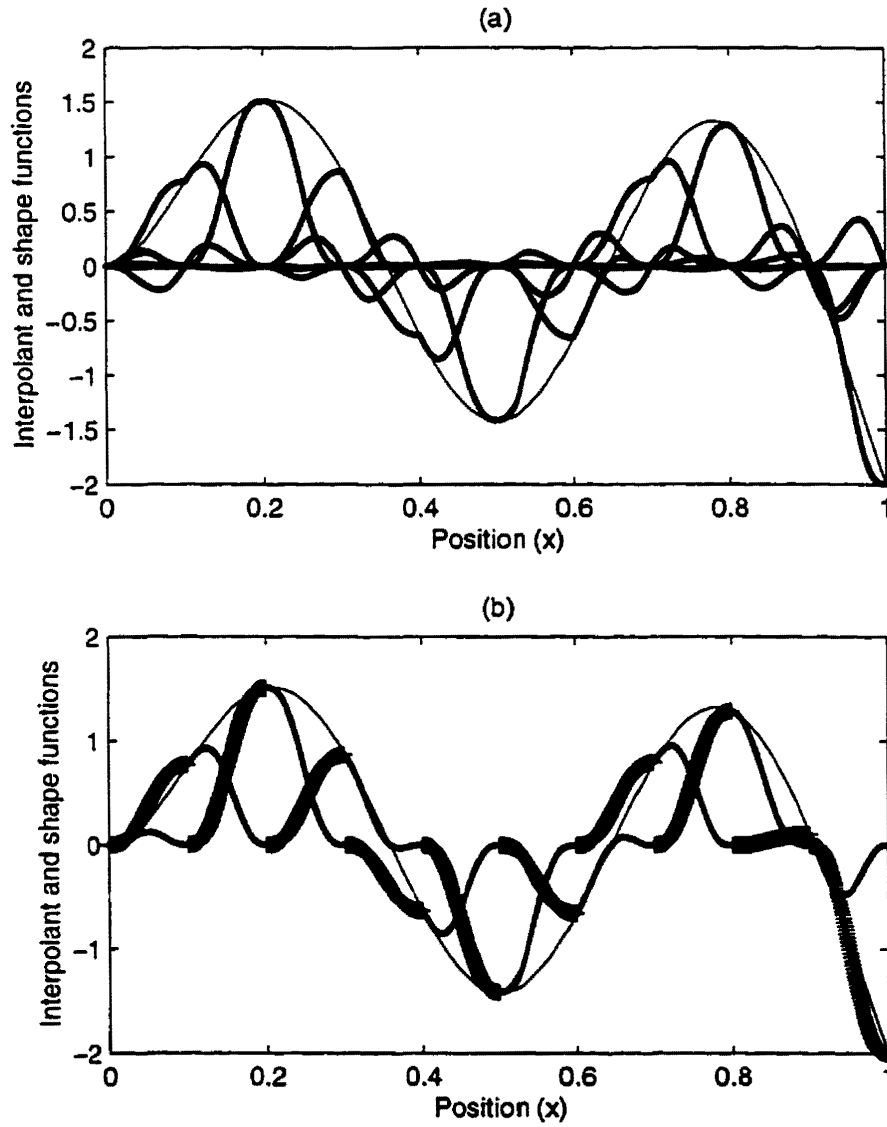


Figure 3.6: Interpolation using fifth degree Hermitian polynomials, (a) Six Hermitian polynomials interpolating the fourth mode of a simple cantilever beam, (b) $(\cdot)\psi_1^0$, $(+)\psi_1^1$. A clearer version of (a) where only two of the six Hermitian polynomials are shown.

3.6.2 Assembly of Matrices

Using the basis functions developed in the previous two sections the matrices in the discretized equations of motion can be assembled. For the small oscillation model with only kinematic nonlinearities the integral matrices are constant and hence need to be assembled only once. To assemble these matrices the usual finite element approach is used which involves adding the elemental matrices in their proper positions in the global matrices. Further details on this technique can be found in any finite element book (for example Becker et al. [61]). The mass position dependent matrices in the equations of motion are assembled by evaluating them at the position of the mass and they have to be updated as the motion of the mass evolves. Therefore the first step in assembling these matrices is to find the element on which the mass is currently positioned, evaluate the elemental matrix, and place the matrix at the proper position in the corresponding global matrix. As the mass moves both the elemental matrix and its position in the global matrix changes and hence they have to be constantly updated.

For the large oscillation model, the integral terms include three and four dimensional matrices. The integral elements of these matrices are computed analytically using the symbolic manipulation program MAPLE. The four and the three dimensional matrices are reduced to two dimensional or one dimensional matrices by multiplying with the required vectors which change with time. All the numerical computations are carried out using “C” functions developed specifically to solve this problem.

In this chapter spatial discretization was carried out of the partial differential

equations describing the dynamics of the system for the large and small oscillation models. The basis functions for use in the Galerkin method were obtained. Now the focus is on solving the nonlinear ordinary differential equations (3.9) and (3.10) for the small oscillation model and equations (3.12)–(3.15) for the large oscillation model. In the next chapter dynamics of the mass beam system for small oscillations is investigated analytically and numerically. The large oscillation model is the focus of Chapter 5.

Chapter 4

Kinematic Nonlinearities

In this chapter, dynamics of the mass beam system are investigated using the small deflection model. The semi-discretized ordinary differential equations developed in Chapter 3 ((3.9) and (3.10)) are further reduced for small oscillations about equilibrium positions using Taylor series expansion and retaining only quadratic nonlinearities. A closed form analytical solution is obtained for this reduced system using the perturbation method of multiple scales. A numerical solution for (3.9) and (3.10) is obtained using an automatic ODE solver and four cantilever beam mode shapes as basis functions. The numerical results are compared with the perturbation solution. Spectral analysis of the system response is carried out using time-frequency methods. The dynamics of the system are investigated for different ratios of the moving mass and the mass of the beam under internal resonance conditions. The system is shown to exhibit amplitude and phase modulation.

4.1 Small Motions about the Equilibrium Positions

In order to obtain a solution using the perturbation method of multiple scales, the equations of motion (3.9) and (3.10) are further simplified by expanding them about their equilibrium position. It can be seen from (3.9), that the equilibrium position for the mass is s_e and for the beam is $\alpha_i = 0$. Using the Taylor series expansion of (3.9) and (3.10) about the equilibrium position and including terms up to the second derivative the following equations are obtained for small motions about the equilibrium positions:

$$\begin{aligned}
 \ddot{s} + \omega^2 s + [\phi_i \phi_j]_{x=s_e} \{\ddot{\alpha}_i \alpha_j\} &= 0 \\
 \left[\left[\int_0^1 \phi_i \phi_j dx \right] + m [\phi_i \phi_j]_{x=s_e} \right] \{\ddot{\alpha}_j\} + \left[\int_0^1 \phi_i'' \phi_j'' dx \right] \{\alpha_j\} + m \ddot{s} [\phi_i \phi_i']_{x=s_e} \{\alpha_j\} \\
 + 2m \dot{s} [\phi_i \phi_i']_{x=s_e} \{\dot{\alpha}_j\} + m s [\phi_i \phi_j' + \phi_i' \phi_j]_{x=s_e} \{\ddot{\alpha}_j\} &= 0
 \end{aligned} \tag{4.1}$$

In (4.1), for convenience, the same variables have been used to represent the motion about the equilibrium positions as in the original equations (3.9) and (3.10).

4.2 Perturbation Analysis

A perturbation method is used to obtain some qualitative insight into the behaviour of the system, especially the parametric behaviour. The equations of motion, expanded about the equilibrium position, (4.1) are used along with the first mode of

a cantilever beam as the basis function.

Considering the first mode of a simple cantilever beam, denoted by ϕ_1 , as the basis function, the equations of motion (4.1) are reduced to:

$$\begin{aligned}\ddot{s} + \omega^2 s + c_1 \ddot{\alpha}_1 \alpha_1 &= 0 \\ \ddot{\alpha}_1 + \omega_1^2 \alpha_1 + mc_2 \ddot{s} \alpha_1 + 2mc_2 \dot{s} \dot{\alpha}_1 + 2mc_2 s \ddot{\alpha}_1 &= 0\end{aligned}\quad (4.2)$$

where the constants c_1 , c_2 , and ω_1 are defined according to:

$$\begin{aligned}c_1 &= \phi_1 \phi_1' |_{x=s_e} & c_2 &= \frac{\phi_1 \phi_1' |_{x=s_e}}{\int_0^1 (\phi_1)^2 dx + m(\phi_1)^2 |_{x=s_e}} \\ (\omega_1)^2 &= \frac{\int_0^1 (\phi_1'')^2 dx}{\int_0^1 (\phi_1)^2 dx + m(\phi_1)^2 |_{x=s_e}}\end{aligned}\quad (4.3)$$

Equations (4.2) are solved using the method of multiple scales by using a two term expansion. In application of this technique, the methodology presented in Nayfeh and Mook [41] is followed. Begin by defining two time scales T_0 and T_1 as:

$$\begin{aligned}T_0 &= t \\ T_1 &= \epsilon t\end{aligned}\quad (4.4)$$

where ϵ is a scaling parameter. If the nonlinear terms are neglected in (4.3), the system would be two uncoupled linear oscillators with frequencies ω and ω_1 . This would be the primary motion on time scale T_0 . The nonlinearities are expected to have a smaller effect and that effect will be on the slower time scale T_1 . Using the

chain rule, derivatives with respect to t can be written in terms of T_0 and T_1 as:

$$\begin{aligned}\frac{\partial}{\partial t} &= \frac{\partial}{\partial T_0} + \epsilon \frac{\partial}{\partial T_1} \\ \frac{\partial^2}{\partial t^2} &= \frac{\partial^2}{\partial T_0^2} + 2\epsilon \frac{\partial^2}{\partial T_0 \partial T_1}\end{aligned}\tag{4.5}$$

where the higher order terms are neglected. The next step is to assume an asymptotic series solution for s and α_1 . In this case a two term expansion is assumed as per:

$$\begin{aligned}s(t) &= \epsilon s_1(T_0, T_1) + \epsilon^2 s_2(T_0, T_1) \\ \alpha_1(t) &= \epsilon u_1(T_0, T_1) + \epsilon^2 u_2(T_0, T_1)\end{aligned}\tag{4.6}$$

where $s_1(T_0, T_1)$ and $s_2(T_0, T_1)$ are the ϵ and ϵ^2 order solutions respectively for the moving mass position, and $u_1(T_0, T_1)$ and $u_2(T_0, T_1)$ are the ϵ and ϵ^2 order solutions respectively for the beam deflection. It should also be noted that ϵ is not a physical parameter of the system and the perturbation method does not provide any means for obtaining its value. It is a parameter introduced as a device to track the nonlinearities and to allow the perturbation method to work in obtaining a solution. The closed form solution obtained using the perturbation method corresponds to the ϵ order terms s_1 and u_1 . To compare this solution with a numerical solution, the initial values of s_1 (denoted by s_{10}) and u_1 (denoted by u_{10}) are set equal to the initial values of s (denoted by s_0) and α_1 (denoted by α_{10}) respectively, thus assuming the value of ϵ equal to one.

Using (4.5) and (4.6), equations (4.2) are simplified and the ϵ and ϵ^2 terms are

collected and setting them to zero gives:

ϵ order equations:

$$\begin{aligned}\frac{\partial^2 s_1}{\partial T_0^2} + \omega^2 s_1 &= 0 \\ \frac{\partial^2 u_1}{\partial T_0^2} + \omega_1^2 u_1 &= 0\end{aligned}\tag{4.7}$$

ϵ^2 order equations:

$$\begin{aligned}\frac{\partial^2 s_2}{\partial T_0^2} + \omega^2 s_2 &= -2 \frac{\partial^2 s_1}{\partial T_0 \partial T_1} - c_1 \frac{\partial^2 u_1}{\partial T_0 \partial T_1} u_1 \\ \frac{\partial^2 u_2}{\partial T_0^2} + \omega_1^2 u_2 &= -2 \frac{\partial^2 u_1}{\partial T_0 \partial T_1} - mc_2 \frac{\partial^2 s_1}{\partial T_0^2} u_1 \\ &\quad - 2mc_2 \frac{\partial s_1}{\partial T_0} \frac{\partial u_1}{\partial T_0} - 2mc_2 s_1 \frac{\partial^2 u_1}{\partial T_0^2}\end{aligned}\tag{4.8}$$

The solution to (4.7) is:

$$\begin{aligned}s_1 &= P_1(T_1)e^{i\omega T_0} + \bar{P}_1(T_1)e^{-i\omega T_0} \\ u_1 &= P_2(T_1)e^{i\omega_1 T_0} + \bar{P}_2(T_1)e^{-i\omega_1 T_0}\end{aligned}\tag{4.9}$$

where P_1 and P_2 are complex variables that are, in general, functions of the slower time scales. The over-bars in (4.9) denote the complex conjugate. Substituting the first order solution (4.9) into the right hand sides of (4.8), the following equations are obtained:

$$rhs3 = -2i \frac{\partial P_1}{\partial T_1} \omega e^{i\omega T_0} + c_1 P_2^2 \omega_1^2 e^{2i\omega_1 T_0} + c_1 P_2 \bar{P}_2 \omega_1^2$$

$$\begin{aligned}
& +2i\frac{\partial \bar{P}_1}{\partial T_1}\omega e^{-i\omega T_0} + c_1\bar{P}_2^2\omega_1^2 e^{-2i\omega_1 T_0} + c_1 P_2 \bar{P}_2 \omega_1^2 \\
rhs4 = & -2i\frac{\partial P_2}{\partial T_1}\omega_1 e^{i\omega_1 T_0} + mc_2(2\omega_1^2 + \omega^2 + 2\omega\omega_1)P_1 P_2 e^{i(\omega_1+\omega)T_0} \\
& +mc_2(2\omega_1^2 + \omega^2 - 2\omega\omega_1)\bar{P}_1 P_2 e^{i(\omega_1-\omega)T_0} \\
& +2i\frac{\partial \bar{P}_2}{\partial T_1}\omega_1 e^{-i\omega_1 T_0} + mc_2(2\omega_1^2 + \omega^2 + 2\omega\omega_1)\bar{P}_1 \bar{P}_2 e^{-i(\omega_1+\omega)T_0} \\
& +mc_2(2\omega_1^2 + \omega^2 - 2\omega\omega_1)P_1 \bar{P}_2 e^{-i(\omega_1-\omega)T_0}
\end{aligned} \tag{4.10}$$

where $rhs3$ and $rhs4$ are used to denote the right hand sides of (4.8). The homogeneous solution of (4.8) contains the terms $e^{i\omega T_0}$ and $e^{i\omega_1 T_0}$ which also appear in the right hand sides of (4.8) (as can be seen from (4.10)). This implies, that the solution of (4.8) would contain secular terms. To eliminate the secular terms, the coefficients of $e^{i\omega T_0}$ and $e^{i\omega_1 T_0}$ in (4.10) are set to zero resulting in:

$$\begin{aligned}
\frac{\partial P_1}{\partial T_1} &= 0 \\
\frac{\partial P_2}{\partial T_1} &= 0
\end{aligned} \tag{4.11}$$

From (4.11), it follows that P_1 and P_2 are constants and the ϵ order solution (4.9), is linear and depends only on the fast time scale T_0 . This solution is not of much significance because under these conditions the coupling between the moving mass and beam is very weak, and consequently it is not pursued further. Instead, we investigate the possibility of modal coupling as a result of internal resonance (IR).

Consider the following relationship between the frequency of the moving mass

ω and the first frequency of the beam ω_1 :

$$\omega = 2\omega_1 + \epsilon\sigma \quad (4.12)$$

where σ is a small detuning parameter. When σ is zero, we have a perfect 1:2 ratio between the first two natural frequencies of the system. This case is referred to as 1:2 IR. Under internal resonance conditions, the secular terms are no longer eliminated by (4.11), because now we have more terms contributing to the secular terms as indicated by the following relationships:

$$\begin{aligned} e^{2\omega_1 T_0} &= e^{\omega T_0} e^{-\sigma T_1} \\ e^{-(\omega - \omega_1) T_0} &= e^{\omega_1 T_0} e^{\sigma T_1} \end{aligned} \quad (4.13)$$

Using (4.13) in (4.10), it can be shown that the elimination of the secular terms now requires the following conditions to be satisfied:

$$\begin{aligned} -2i\omega \frac{\partial P_1}{\partial T_1} + c_1 P_2^2 \omega_1^2 e^{-\sigma T_1} &= 0 \\ -2i\omega_1 \frac{\partial P_2}{\partial T_1} + mc_2(2\omega_1^2 + \omega^2 - 2\omega\omega_1) P_1 \bar{P}_2 e^{\sigma T_1} &= 0 \end{aligned} \quad (4.14)$$

The complex variables P_1 and P_2 are converted to polar form using the relations

$$\begin{aligned} P_1(T_1) &= \frac{1}{2} p_1(T_1) e^{i\varphi_1(T_1)} \\ P_2(T_1) &= \frac{1}{2} p_2(T_1) e^{i\varphi_2(T_1)} \end{aligned} \quad (4.15)$$

and equations (4.14), after simplification, can now be written as,

$$\begin{aligned}
\frac{\partial p_1}{\partial T_1} &= \frac{1}{4} \frac{c_1 p_2^2 \omega_1^2}{\omega} \sin(2\varphi_2 - \varphi_1 - \sigma T_1) \\
p_1 \frac{\partial \varphi_1}{\partial T_1} &= -\frac{1}{4} \frac{c_1 p_2^2 \omega_1^2}{\omega} \cos(2\varphi_2 - \varphi_1 - \sigma T_1) \\
\frac{\partial p_2}{\partial T_1} &= -\frac{1}{4} \frac{m c_2 (\omega^2 - 2\omega\omega_1 + 2\omega_1^2)}{\omega_1} p_1 p_2 \sin(2\varphi_2 - \varphi_1 - \sigma T_1) \\
p_2 \frac{\partial \varphi_2}{\partial T_1} &= -\frac{1}{4} \frac{m c_2 (\omega^2 - 2\omega\omega_1 + 2\omega_1^2)}{\omega_1} p_1 p_2 \cos(2\varphi_2 - \varphi_1 - \sigma T_1) \quad (4.16)
\end{aligned}$$

In (4.16), p_1 and p_2 are the modal amplitudes and φ_1 and φ_2 are the corresponding phases.

To determine the initial values for p_1 , p_2 , φ_1 and φ_2 equations (4.9) are first expressed in terms of trigonometric functions as

$$\begin{aligned}
s_1 &= p_1(T_1) \cos(\omega T_0 + \varphi_1(T_1)) \\
u_1 &= p_2(T_1) \cos(\omega_1 T_0 + \varphi_2(T_1)) \quad (4.17)
\end{aligned}$$

Using equations (4.17) and their derivatives, and taking the initial velocities as zero and setting ϵ to one, the initial values $p_1(0) = p_{10}$, $p_2(0) = p_{20}$, $\varphi_1(0) = \varphi_{10}$ and $\varphi_2(0) = \varphi_{20}$ are obtained by solving:

$$\begin{aligned}
s_{10} &= p_{10} \cos(\varphi_{10}) \\
\alpha_{10} &= p_{20} \cos(\varphi_{20}) \\
0 &= \nu_1 p_{20}^2 \sin(2\varphi_{20}) - \omega p_{10} \sin(\varphi_{10}) \\
0 &= \nu_2 p_{10} p_{20} \sin(\varphi_{10} - \varphi_{20}) - \omega_1 p_{20} \sin(\varphi_{20}) \quad (4.18)
\end{aligned}$$

The following solution of (4.18) is used as the initial values:

$$p_{10} = s_{10}, p_{20} = \alpha_{10}, \varphi_{10} = 0, \varphi_{20} = 0 \quad (4.19)$$

As shown in Nayfeh and Mook [41], nonlinear differential equations of the form (4.16) can be solved analytically using elliptic functions. For the most part (equations (4.20)–(4.27) and (4.30)–(4.39)) the approach presented in Nayfeh and Mook [41] for obtaining a closed form solution to (4.16), is followed. We proceed by defining γ as:

$$\gamma = 2\varphi_2 - \varphi_1 - \sigma T_1 \quad (4.20)$$

Using (4.20), equations (4.16) can be reduced to the following three nonlinear differential equations:

$$\begin{aligned} \frac{\partial p_1}{\partial T_1} &= \nu_1 p_2^2 \sin(\gamma) \\ \frac{\partial p_2}{\partial T_1} &= -\nu_2 p_1 p_2 \sin(\gamma) \\ p_1 \frac{\partial \gamma}{\partial T_1} &= (-2\nu_2 p_1^2 + \nu_1 p_2^2) \cos(\gamma) - \sigma p_1 \end{aligned} \quad (4.21)$$

where ν_1 and ν_2 are constants given by

$$\begin{aligned} \nu_1 &= \frac{c_1 \omega_1^2}{4\omega} \\ \nu_2 &= \frac{mc_2(\omega^2 - 2\omega\omega_1 + 2\omega_1^2)}{4\omega_1} \end{aligned} \quad (4.22)$$

Eliminating γ from the first and the second equation in (4.21) and integrating the

result gives:

$$\frac{1}{2}\nu_1 p_2^2 + \frac{1}{2}\nu_2 p_1^2 = G \quad (4.23)$$

where G is a constant of integration, to be determined using the initial values for p_1 and p_2 . Equation (4.23) is an expression of conservation of energy and shows energy being exchanged between p_1 and p_2 .

Using (4.23) both p_1 and p_2 can be expressed in terms of one variable ξ as:

$$\begin{aligned} \frac{1}{2}\nu_1 p_2^2 &= G\xi \\ \frac{1}{2}\nu_2 p_1^2 &= G(1 - \xi) \end{aligned} \quad (4.24)$$

Eliminating T_1 between the first and the last equation in (4.21) and rearranging the terms gives

$$\begin{aligned} -\nu_1 d(p_2^2 p_1 \cos(\gamma)) + \frac{\sigma}{2} d(p_1^2) &= 0 \\ -\nu_1 p_2^2 p_1 \cos(\gamma) + \frac{\sigma}{2} p_1^2 &= L \end{aligned} \quad (4.25)$$

where L is another integration constant and $d(\cdots)$ implies implicit differentiation.

Solving for $\cos(\gamma)$ from (4.25) gives:

$$\cos(\gamma) = \frac{\sigma p_1^2 - 2L}{2\nu_1 p_2^2 p_1} \quad (4.26)$$

and using $\sin^2(\gamma) = 1 - \cos^2(\gamma)$ to eliminate γ from the first equation in (4.21) and

expressing p_1 and p_2 in terms of ξ gives the following differential equation:

$$\begin{aligned} \frac{1}{8\nu_2 G} \left(\frac{\partial \xi}{\partial T_1} \right)^2 = & -\xi^3 - \frac{\sigma^2 - 8\nu_2 G}{8\nu_2 G} \xi^2 \\ & - \frac{\sigma(L\nu_2 - \sigma G)}{4G^2\nu_2} \xi - \frac{(L\nu_2 - \sigma G)^2}{8\nu_2 G^3} \end{aligned} \quad (4.27)$$

The problem is thus reduced to solving the single differential equation (4.27). The solution of ξ depends on the roots ξ_1 , ξ_2 and ξ_3 of the left hand side of (4.27). The roots are

$$\begin{aligned} \xi_{1,2} &= \frac{(2\nu_2 p_{10} - \sigma \pm \mathcal{D})(2\nu_2 p_{10} + \sigma)}{16G\nu_2} \\ \xi_3 &= \frac{\nu_1 p_{20}^2}{2G} \end{aligned} \quad (4.28)$$

where the discriminant ($\mathcal{D}$) is given by:

$$\mathcal{D} = \sqrt{(\sigma - 2\nu_2 p_{10})^2 + 16\nu_1 \nu_2 p_{20}^2} \quad (4.29)$$

In deriving (4.28) the constants G and L are evaluated using the initial values p_{10} , p_{20} , φ_{10} and φ_{20} . In (4.29), ν_1 can become negative for large values of σ (when σ is negative and has a magnitude greater than $2\omega_1$), however in such a case the system would be far away from 1:2 resonance and this perturbation analysis is not applicable. The discriminant (4.29) is therefore real under internal resonance conditions. When the initial value p_{10} and the detuning parameter σ are close to zero the roots ξ_1 and ξ_2 approach each other and ξ_3 approaches unity and when the other initial value p_{20} and σ are close to zero the difference between ξ_1 and ξ_2

approaches a maximum value and ξ_3 approaches zero. These cases are discussed further in Section 4.3.1.

Assuming the roots are ordered such that $\xi_1 < \xi_2 < \xi_3$ and writing (4.27) as,

$$\frac{1}{8\nu_2 G} \left(\frac{\partial}{\partial T_1} \xi \right)^2 = (\xi_3 - \xi)(\xi - \xi_2)(\xi - \xi_1) \quad (4.30)$$

the following transformation is applied to ξ :

$$\xi_3 - \xi = (\xi_3 - \xi_2) \sin^2(\chi) \quad (4.31)$$

Equation (4.30) then reduces to:

$$\begin{aligned} \frac{1}{2\nu_2 G} \left(\frac{\partial}{\partial T_1} \chi \right)^2 &= \xi_3 - \xi_1 - (\xi_3 - \xi_2) \sin^2(\chi) \\ &= (\xi_3 - \xi_1) \left(1 - \frac{(\xi_3 - \xi_2)}{(\xi_3 - \xi_1)} \sin^2(\chi) \right) \end{aligned} \quad (4.32)$$

Taking the square root of (4.32) gives:

$$\frac{1}{\sqrt{2\nu_2 G}} \frac{\partial \chi}{\partial T_1} = \pm \sqrt{\xi_3 - \xi_1} \left(1 - \eta^2 \sin^2(\chi) \right)^{\frac{1}{2}} \quad (4.33)$$

where η is given by:

$$\eta = \sqrt{\frac{(\xi_3 - \xi_2)}{(\xi_3 - \xi_1)}} \quad (4.34)$$

Integrating (4.33) gives:

$$\int_0^x \frac{1}{\sqrt{1 - \eta^2 \sin^2(\chi)}} d\chi = \pm \sqrt{2\nu_2 G(\xi_3 - \xi_1)} \int_{T_c}^{T_1} dT_1 \quad (4.35)$$

In (4.35), T_e corresponds to $\chi = 0$ or $\xi = \xi_3$. The left hand side of (4.35) is Legendre's elliptic integral of the first kind, or the inverse of the Jacobi elliptic function sn . In terms of sn , (4.35) can be written as:

$$\text{sn}^{-1}(\sin(\chi); \eta) = \kappa(T_1 - T_e) \quad (4.36)$$

where κ is given by:

$$\kappa = \pm \sqrt{2\nu_2 G(\xi_3 - \xi_1)} \quad (4.37)$$

From (4.36) it follows that

$$\sin(\chi) = \text{sn}(\kappa(T_1 - T_e); \eta) \quad (4.38)$$

In (4.38), η is the modulus of the elliptic function and its value affects the period of sn . Substituting (4.38) in (4.31), the solution for ξ is obtained as:

$$\xi = \xi_3 - (\xi_3 - \xi_2) \text{sn}^2(\kappa(T_1 - T_e); \eta) \quad (4.39)$$

Using (4.24) and (4.39) the values of p_1 and p_2 can be easily obtained, and then using (4.26), γ can be determined. However to find the phases φ_1 and φ_2 , another equation is needed and is obtained by eliminating $\cos(\gamma)$ from the second and the fourth equation in (4.16) and integrating the result, thus giving:

$$\nu_2 p_1^2 \varphi_1 - \nu_1 p_2^2 \varphi_2 = Q \quad (4.40)$$

where Q is a constant of integration. Substituting the initial values for φ_1 and φ_2 (see equation (4.19)) in (4.40) makes Q zero and the following relationship between φ_1 and φ_2 is obtained:

$$\varphi_2 = \frac{\nu_2 p_1^2}{\nu_1 p_2^2} \varphi_1 \quad (4.41)$$

Using (4.41) and (4.20), φ_1 and φ_2 can now be determined.

The beating period (τ_b) is another important characteristic of the response. It is defined as the time elapsed between two successive peaks in the amplitude of motion, it corresponds to half the period of p_1 or p_2 and hence to half the period of the Jacobi elliptic function sn . The half period of $\text{sn}(\kappa T_1 - T_e)$ is given by the following relationship where the integral is known as the complete Jacobi elliptic integral of the first kind:

$$\tau_b = \frac{2}{\kappa} \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - \eta^2 \sin(x)^2}} dx \quad (4.42)$$

Using the perturbation analysis, solution is obtained for the modal amplitudes p_1 , p_2 , and the phases φ_1 and φ_2 (on To summarise, the perturbation solution is obtained using the following algorithm:

Algorithm 4.1 *Algorithm to obtain the perturbation solution*

Step 1 Set the initial value p_{10} to the initial position of the mass s_{10} , and p_{20} to the first mode contribution of the tip deflection α_{10} . If the initial value for the beam (2.46) is used then $\alpha_{10} = 0.5067 v_{t0}$ (obtained using the orthogonality of the modes $\alpha_{i0} = \int_0^1 \phi_i(x) v_0(x) dx$ where $v_0(x)$ is the initial deflection curve

[30]) and if the initial value (2.47) is used then $\alpha_{10} = 0.5v_{t0}$, where v_{t0} is the tip deflection. Select a value for the detuning parameter σ . An algorithm is presented in the next section (Algorithm 4.2) for selecting a value for σ that is used to compare the perturbation and numerical solutions. Also set the value of the equilibrium position of the moving mass s_e .

Step 2 Compute the value of the first mode shape of a cantilever beam and its first and second derivatives at the equilibrium position (see equation (3.16)). Using these values obtain the constants c_1 , c_2 , and ω from (4.3). Also find the constants ν_1 , ν_2 , G , and L using equations (4.22), (4.23) and (4.25) respectively. Calculate the discriminant $\mathcal{D}$ (equation (4.29)), the roots ξ_1 , ξ_2 , and ξ_3 (equation (4.28)) and sort the roots such that ξ_1 , ξ_2 , and ξ_3 are in ascending order. Find κ using (4.37) and the modulus η using (4.34). Compute the beating period τ_b using (4.42). This involves finding the complete Jacobi elliptic integral of the first kind.

Step 3 Find the time T_e . This requires calculating ξ from the initial values of p_1 and p_2 using (4.24) and then finding $\sin(\chi)$ using (4.31) and finally obtaining T_e from (4.36) by substituting for T_1 , the initial time.

Step 4 Vary T_1 from the initial time to the desired final time T_f and obtain ξ from (4.39), modal amplitudes p_1 and p_2 from (4.24), γ from (4.26), and the phases φ_1 and φ_2 from (4.20) and (4.41), respectively.

4.3 Comparison Between Perturbation and Numerical Solutions

The results obtained using perturbation analysis are now compared with numerical simulation of the ODEs derived using Taylor series expansion (4.2). The various models and solution methodologies used in this work are identified in Table 4.1 and the designations shown are used for future reference. In this section comparisons are made between the perturbation solution (PM1) and the numerical solution (NM1). The parameters used for all the simulations presented in this section are

Model	Description
PM1	Perturbation solution, Small oscillation model, First mode shape of a cantilever beam as basis function.
NM1	Numerical solution, Small oscillation model, First mode shape of a cantilever beam as basis function, Automatic ODE solver.
NM2	Numerical solution, Small oscillation model, Four mode shapes of a cantilever beam as basis functions, Automatic ODE solver.
NM3	Numerical solution, Small oscillation model, 10 fifth degree piecewise defined Hermitian polynomials as basis functions, Acceleration formulation.
NM4	Numerical solution, Small oscillation model, 20 fifth degree piecewise defined Hermitian polynomials as basis functions, Acceleration formulation.
NM5	Numerical solution, Large oscillation model, 10 fifth degree piecewise defined Hermitian polynomials as basis functions, Acceleration formulation.

Table 4.1: Model designations

given in Table 4.2. First consider the case where the initial displacement of the

Parameter Set 1 $m = 1.0$, $s_e = 0.5$, ω_1 using 20 finite elements = 2.891228 ω_1 using one mode of a cantilever beam = 2.908776				
	Initial Values equation (2.47) $s_0 = 0.50001$, $v_{t0} = 0.1$		Initial Values equation (2.47) $s_0 = 0.53$, $v_{t0} = 0.000001$	
Figure	4.1, 4.2	4.19, 4.20	4.3, 4.4	4.21, 4.22
Model	PM1,NM1	PM1,NM2	PM1,NM1	PM1,NM2
σ	-0.0002	-0.0016	-0.0085	-0.0115
$\Delta t(\text{average})$	0.01797	0.00354	0.0094	0.00669
Spectrogram				
No. of Segments	16	64	32	64
Segment Size	256	256	256	256
Power Spectrum				
No. of FFTs		2		6
Segment Size		65536		65536

Table 4.2: Comparison between perturbation and numerical solutions-1

moving mass about its equilibrium position, s_0 is 0.00001, the initial beam tip deflection is $v_{t0} = 0.1$ the equilibrium position of the moving mass $s_e = 0.5$, and the nondimensionalized mass ratio $m = 1.0$. Figures 4.1 and 4.2 show the response for the moving mass and the beam respectively.

Figures 4.1(a) and 4.1(b) are the analytical results obtained using the perturbation analysis. It may be noted here that $p_{20} = 0.5v_{t0}$ (see step 1 in Algorithm 4.1 and equation (4.19)). Figure 4.1(c) is the numerical solution obtained from (4.2) by using a variable step stiff ODE solver based on the fourth order Rosenbrock method presented in [102]. This technique is based on an implicit formulation and is a generalisation of the Runge-Kutta-Fehlberg method that uses the parameters presented by Shampine (for details see [102]). Later on, the same ODE solver is

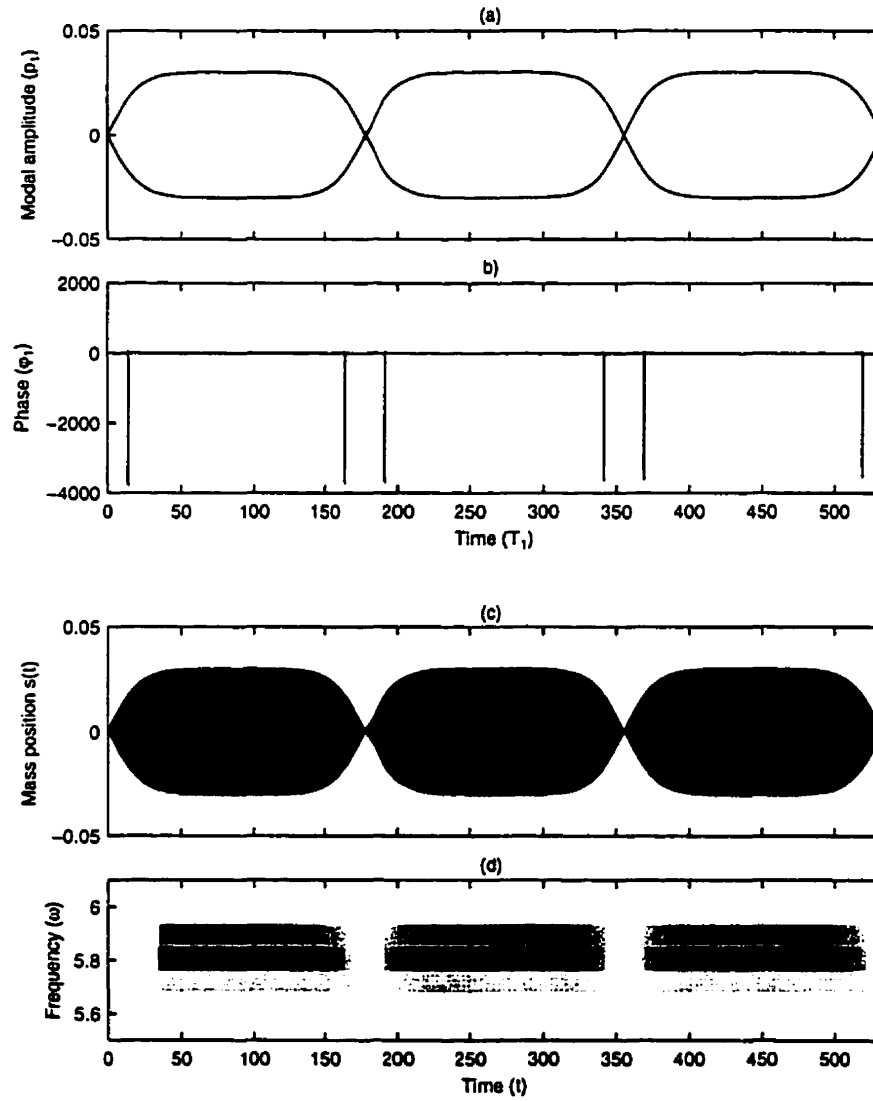


Figure 4.1: Mass response - 1:2 IR, $m = 1.0$, $s_e = 0.5$, $s_{10} = 0.00001$, and $v_{i0} = 0.1$. (a) and (b) Perturbation solution $\sigma = -0.0002$, (c) Numerical solution, and (d) Spectrogram.

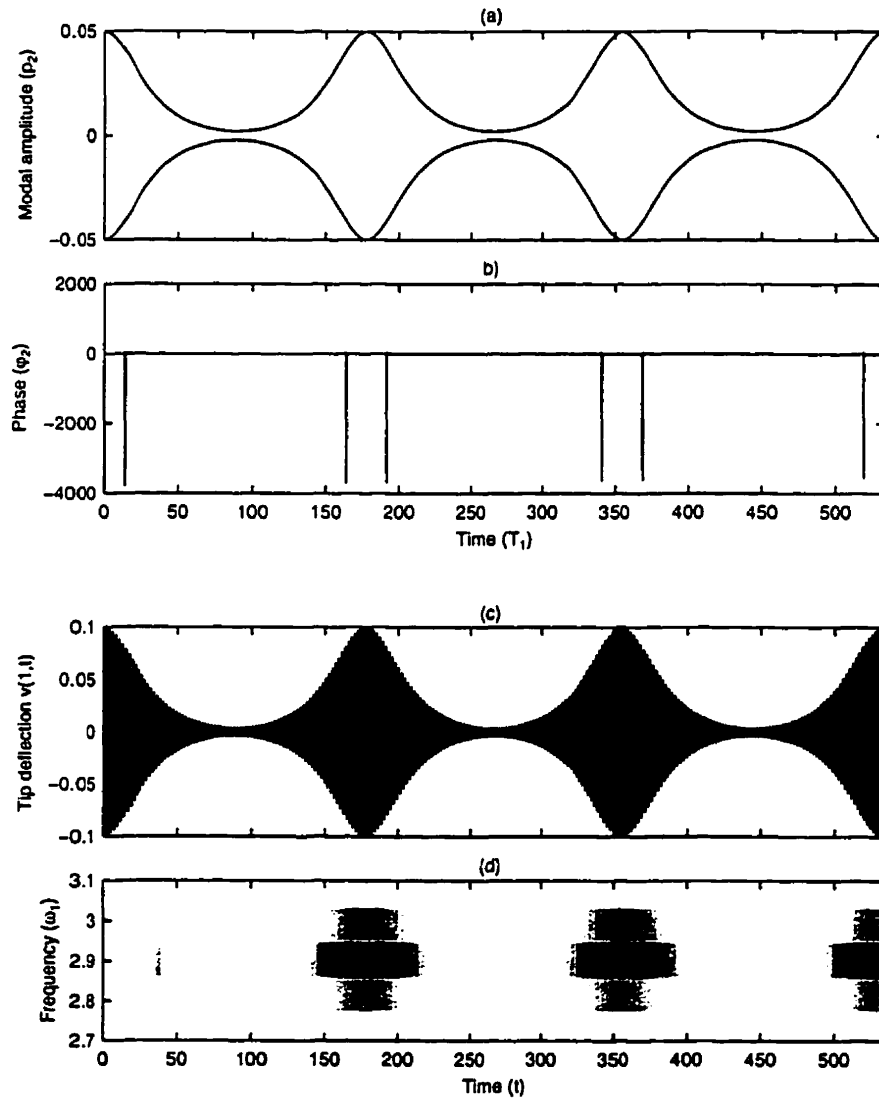


Figure 4.2: Tip deflection - 1:2 IR, $m = 1.0$, $s_e = 0.5$, $s_{10} = 0.00001$, and $v_{t0} = 0.1$. (a) and (b) Perturbation solution $\sigma = -0.0002$, (c) Numerical solution, and (d) Spectrogram.

used for solving the more general equations of motion (3.9) and (3.10) using four cantilever beam mode shapes. Figure 4.1(c) appears darkened because the oscillations are at a very high frequency and small time steps were used.

Figure 4.1(d) shows a spectrogram which is obtained by taking a small data window which consists of 16 segments of data with 256 data points per segment, of the time series and finding its one sided power spectral density (PSD), with the mean squared amplitude as the measure, using the Fast-Fourier transform (FFT) algorithm [102]. For a detailed treatment on time-frequency analyses see Cohen [103]. The data window is advanced along the time axis one segment at each step allowing overlapping of the previous data segments by the current data window. The spectrogram (Figure 4.1(d)) shows the variation in the PSD for the frequencies on the vertical axis with time on the horizontal axis. The change in the PSD is represented by the grey scaling with the darker regions representing higher values of PSD and lighter regions representing lower values of PSD. The empty space at the beginning of the graph is half the size of the data window. Note that the mean squared amplitude, the measure used for PSD, represents energy, and the spectrogram shows the variation in the energy content of the frequencies with time.

Figure 4.2 shows similar results obtained for the beam. The time responses in Figures 4.1(a), 4.2(a), 4.1(c) and 4.2(c) show the characteristic beating motion for the mass and the beam under internal resonance. The numerical solutions (Figure 4.1(c) and 4.2(c)) are obtained for $\omega = 2\omega_1$. Under such perfect resonance conditions, the perturbation analysis gives a solution where the amplitudes match closely with the numerical solution, but there are small differences in the beating

periods which may be attributed to neglecting the higher order terms in the perturbation analysis. In anticipation of comparison of the perturbation solution with the numerical solution obtained using the four modes of a cantilever beam and finite elements as basis functions, the approach used here is to match the perturbation solution with the numerical solution by choosing the detuning parameter σ . The value of σ then acts as means for comparing the various solutions. This approach is justified because the natural frequency of the beam ω_1 , obtained using (4.3) for the simplified model used in the perturbation analysis would in general be slightly different than ω_1 obtained using (3.9) and (3.10). The method for finding the value of σ is outlined in the following algorithm:

Algorithm 4.2 *Algorithm to obtain the detuning parameter σ .*

Step 1 Find the beating period (τ_b) for the numerical solution. This is accomplished by taking the Hilbert transform of the time series for the tip deflection of the beam, which gives an enveloping curve for the amplitude. Using the extremum values of this curve the beating periods can be found easily. The beating periods differ slightly from one beat to another, therefore an average value is selected.

Step 2 Using (4.42) the value of σ is determined numerically by varying σ and finding the beating period which matches T_b determined in Step 1.

For Figures 4.1 and 4.2 the value of σ was determined as -0.0002 which gives the same beating period as the numerical results.

It can be seen from (4.17), that p_1 and p_2 describe the change in the amplitude, and φ_1 and φ_2 could possibly effect the natural frequencies ω_1 and ω_2 , respectively.

This corresponds to both amplitude and frequency modulation. Figure 4.1(b) shows that the phases φ_1 and φ_2 remain constant, except that they approach singular positions just before and after the beam reaches a peak, or the mass amplitude becomes zero. A similar effect can be seen in the spectrogram, Figure 4.1(d), where the energy corresponding to the frequency ω_1 disappears. The energy is in fact transferred to the beam as can be seen from Figure 4.2(d).

Figures 4.3 and 4.4 are obtained for the same parameters as in Figure 4.1 and 4.2 but the initial values are now taken as $s_0 = 0.03$ and tip deflection $v_{t0} = 0.00001$. In this case the system is predominantly excited by the moving mass. The value of σ for the perturbation solution was found to be -0.0085 . The magnitude of the peaks for the phases φ_1 and φ_2 is not similar to that seen in Figures 4.1 and 4.2, but seem to increase with time.

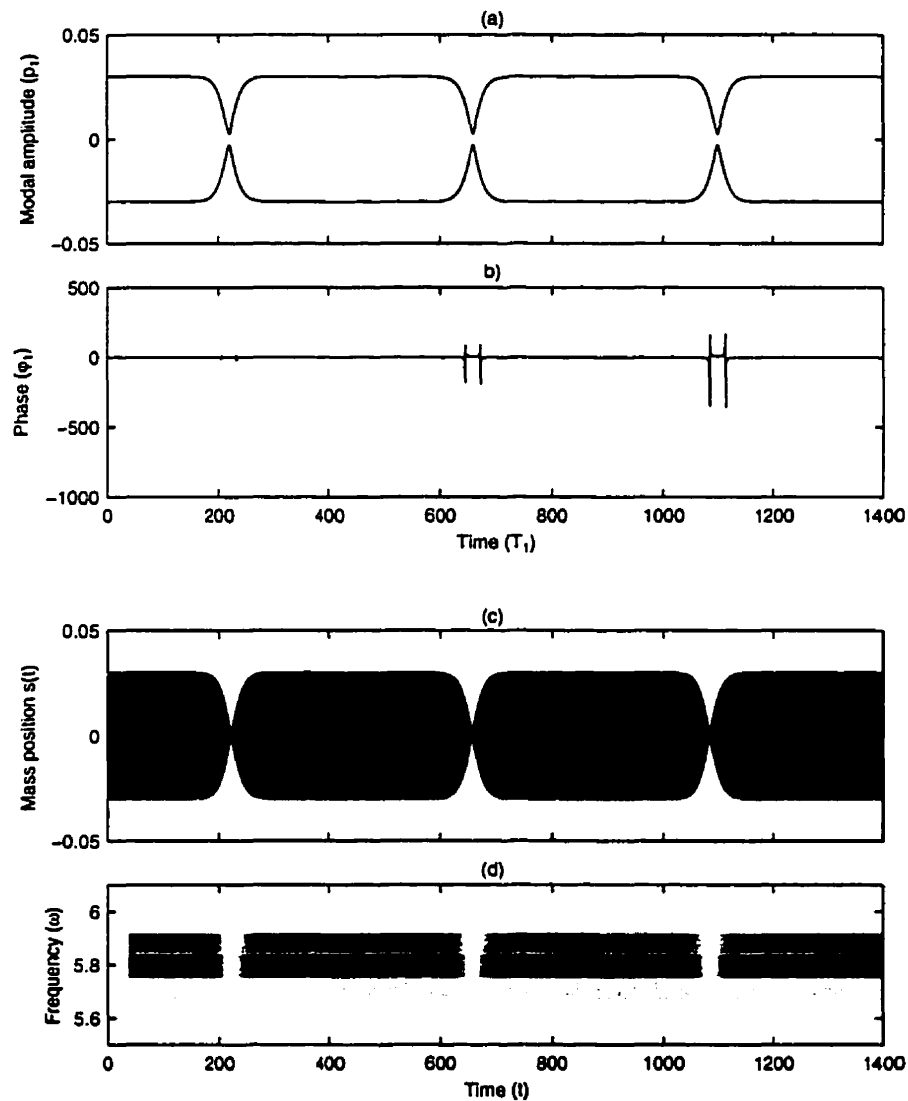


Figure 4.3: Mass response - 1:2 IR, $m = 1.0$, $s_e = 0.5$, $s_{10} = 0.03$, and $v_{10} = 0.00001$. (a) and (b) Perturbation solution $\sigma = -0.0085$, (c) Numerical solution, and (d) Spectrogram.

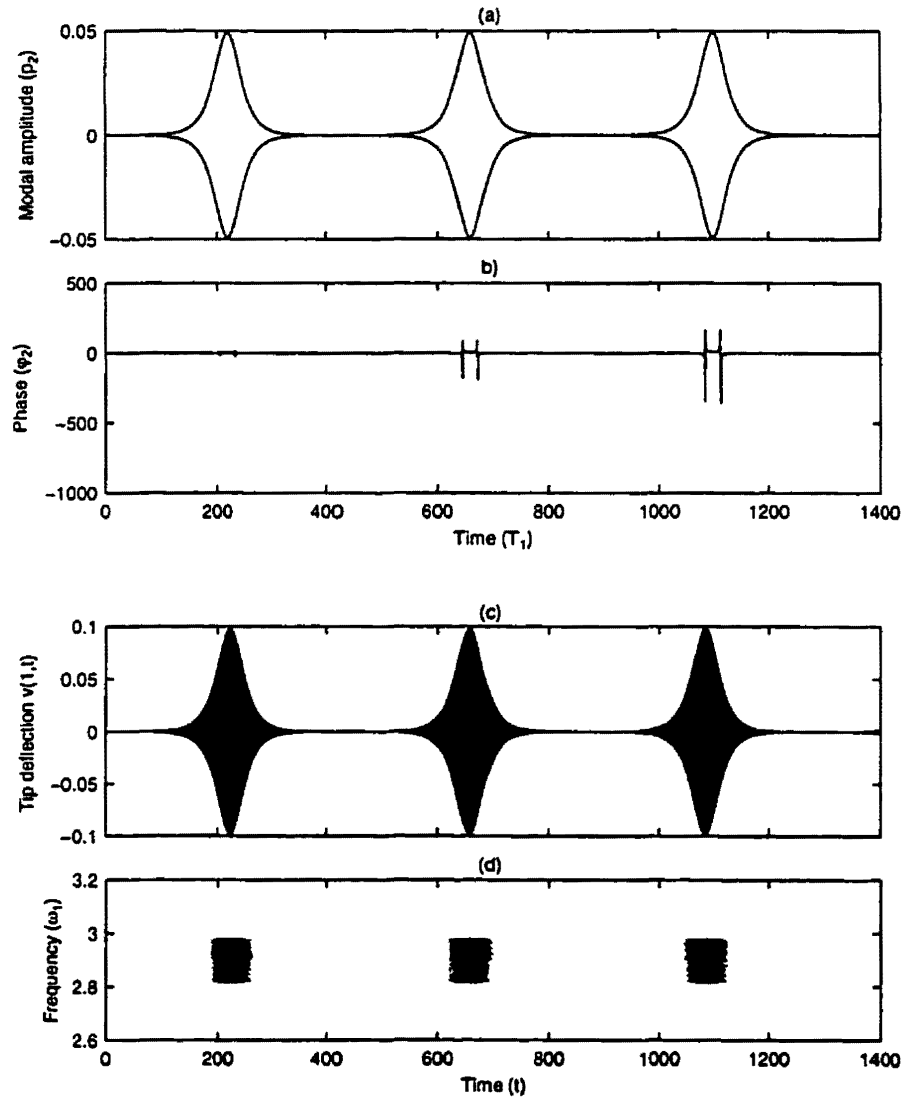


Figure 4.4: Tip deflection - 1:2 IR, $m = 1.0$, $s_e = 0.5$, $s_{10} = 0.03$, and $v_{t0} = 0.00001$. (a) and (b) Perturbation solution $\sigma = -0.0085$, (c) Numerical solution, and (d) Spectrogram.

4.3.1 Parametric Analysis

The response of the system depends on the roots ξ_1 , ξ_2 and ξ_3 and Figures 4.5 and 4.6 show the values of these roots for different m and σ . Figure 4.5 is obtained for initial values of mass position $s_0 = 0.00001$ and tip deflection $v_{t0} = 0.1$ and with the equilibrium position for the moving mass $s_e = 0.5$. Whereas Figure 4.6 is obtained for initial values $s_0 = 0.03$, $v_{t0} = 0.00001$ and equilibrium position $s_e = 0.5$. Similar to the circular sine function, the amplitude of the elliptic sine function sn also varies between -1 and 1. Therefore from (4.39) it follows that ξ oscillates between ξ_2 and ξ_3 . The farther apart the roots ξ_2 and ξ_3 are, the larger is the amplitude of ξ , which from equation (4.24) implies that more exchange of energy occurs between the moving mass and the beam. Both Figures 4.5 and 4.6 show that this region of strong coupling occurs when σ is close to zero and when σ moves away from zero the difference between the roots ξ_2 and ξ_3 decreases, finally becoming zero. As the ratio of the moving mass and the mass of the beam (m) increases, the range of σ for which the roots ξ_2 and ξ_3 are distinct also increases. This can be seen more clearly in Figure 4.6(c). When the roots ξ_2 and ξ_3 become equal, it follows from (4.39) that the solution for ξ becomes a constant, equal to one for the case considered in Figure 4.5 and equal to zero for the case considered in Figure 4.6.

The modulus η of the elliptic function sn gives some insight into the type of response. When $\eta = 0$ the elliptic function sn becomes the circular sine function and for $\eta = 1$, sn becomes $\tanh$. So it follows from equations (4.34), (4.39) and (4.24) that when ξ_2 is close to ξ_3 , p_1 and p_2 appear closer to sinusoidal functions,

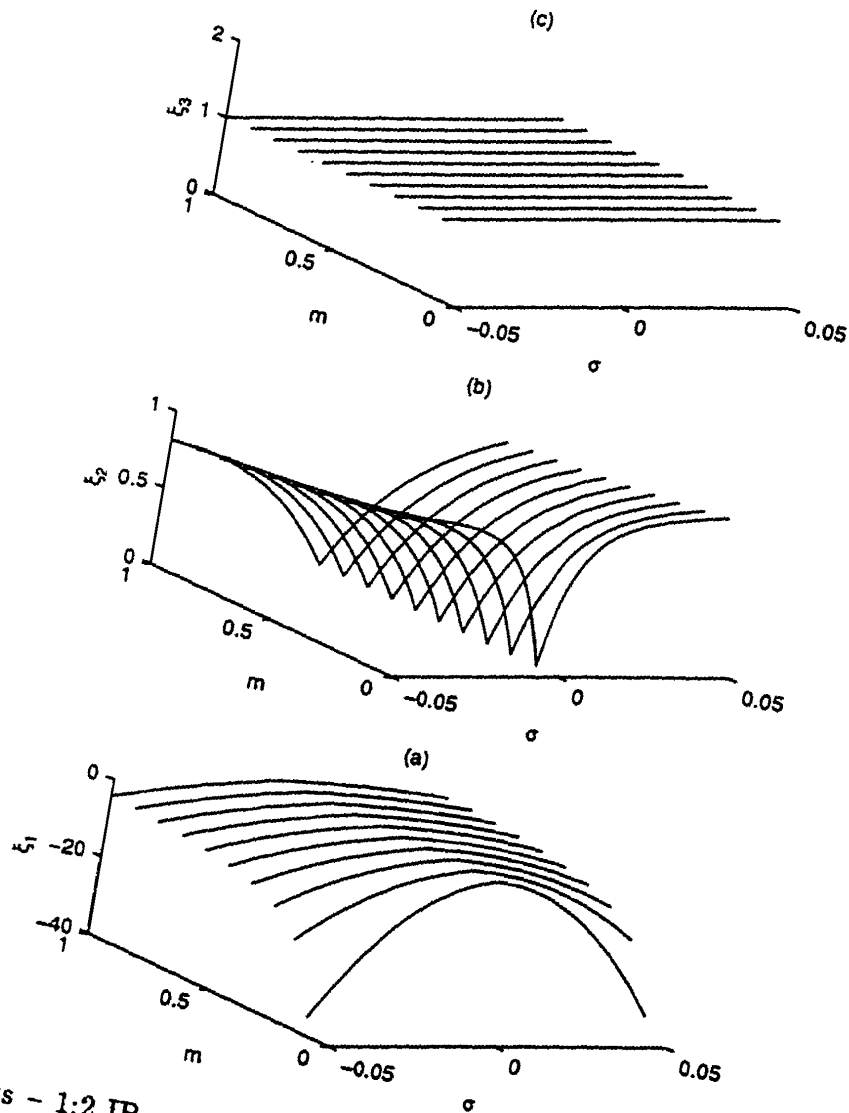


Figure 4.5: Roots - 1:2 IR, $s_e = 0.5$, $s_{10} = 0.00001$, and $v_{t0} = 0.1$. (a) ξ_1 , (b) ξ_2 , and (c) ξ_3 .

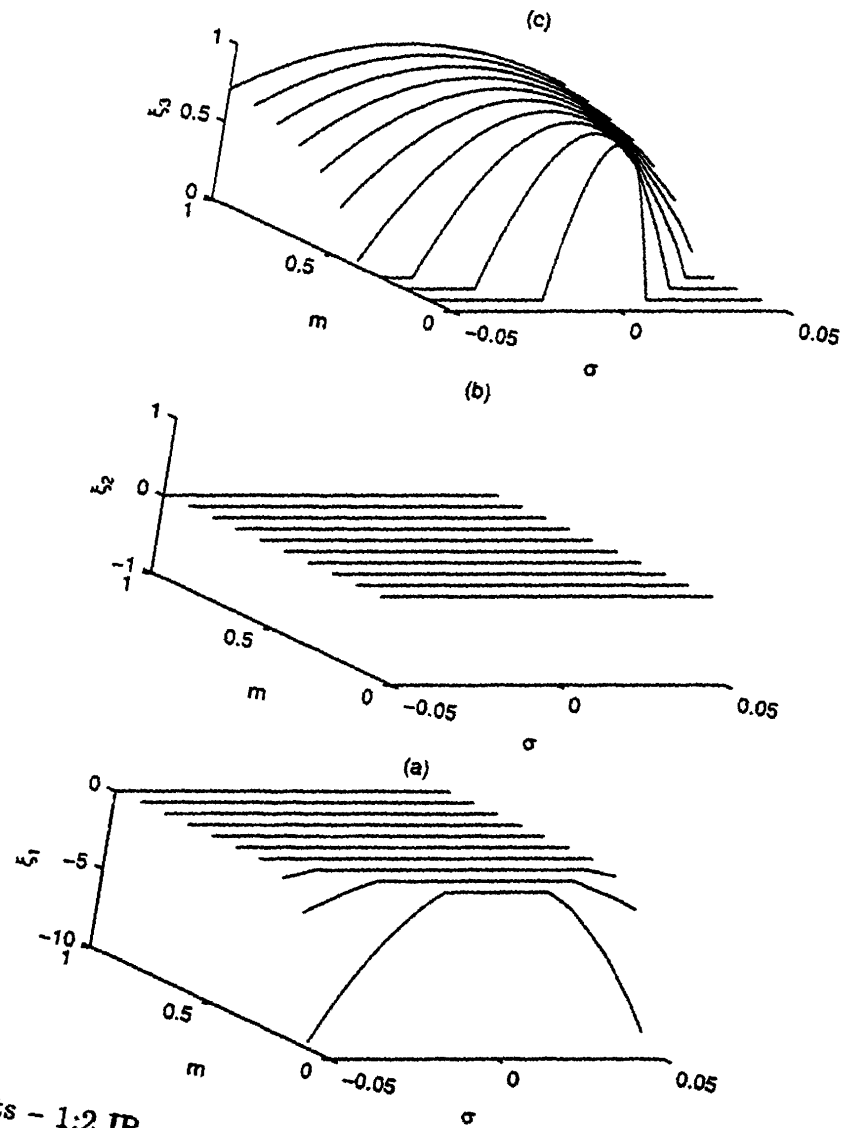


Figure 4.6: Roots - 1:2 IR, $s_e = 0.5$, $s_{10} = 0.03$, and $v_{10} = 0.00001$. (a) ξ_1 , (b) ξ_2 , and (c) ξ_3 .

and when ξ_2 and ξ_3 are numerically far apart from each other, p_1 and p_2 appear closer to hyperbolic functions (p_1 close to $\tanh$ and p_2 close to sech).

When ξ_1 and ξ_2 become equal, the modulus η becomes unity (see (4.34), the beating period τ_b becomes infinite (see 4.42), and in (4.39) the elliptic sine function sn becomes $\tanh$, thus once the energy is transferred from the mass to beam or vice versa it stays there as the period for p_1 and p_2 is infinite. This motion is however unstable and the smallest difference between ξ_1 and ξ_2 makes the beating period finite. In Figure 4.6 it appears that ξ_1 and ξ_2 are equal for a large range of σ , there is however a small difference between them which is due to the small nonzero initial value of the tip deflection v_{t0} . It may be noted here that when v_{t0} is zero, there is no coupling between the moving mass and the beam, but even a small nonzero value of v_{t0} results in large amplitude vibrations of the beam as was illustrated in Figures 4.3 and 4.4.

The maximum and minimum values of p_1 and p_2 are indicative of the exchange of energy between the mass and the beam. Since ξ oscillates between ξ_2 and ξ_3 , the maximum value being ξ_3 and the minimum value ξ_2 , using (4.24), the maximum and the minimum values of p_1 and p_2 are given by:

$$\begin{aligned} p_{1max} &= \sqrt{\frac{2G(1-\xi_2)}{\nu_2}} & p_{2max} &= \sqrt{\frac{2G\xi_3}{\nu_1}} \\ p_{1min} &= \sqrt{\frac{2G(1-\xi_3)}{\nu_2}} & p_{2min} &= \sqrt{\frac{2G\xi_2}{\nu_1}} \end{aligned} \quad (4.43)$$

Figure 4.7 shows p_{1max} and p_{2min} for various values of m and equilibrium position s_e . The initial value p_{10} is taken as 0.00001 and p_{20} as 0.05 (same as in Figures

4.1, 4.2, and 4.5). Figure 4.7 shows that when the equilibrium position of the moving mass is close to the fixed end of the beam, the value of p_{2min} is almost the same as its initial value p_{20} and also the corresponding maximum amplitude of the mass is close to zero indicating a weak coupling between the moving mass and the beam. As the equilibrium position is moved towards the free end of the beam, the exchange of energy between the mass and the beam increases as indicated by p_{2min} decreasing sharply and approaching zero and p_{1max} increasing correspondingly. The other maximum and minimum values p_{1min} and p_{2max} are not shown as they do not vary significantly. The value of p_{1min} remains close to its initial value p_{10} and the value of p_{2max} remains close to its initial value p_{20} . Figure 4.7 was obtained for a value of σ close to perfect 1:2 resonance ($\sigma = -0.0002$). Figure 4.7 also shows an expected result that a lighter moving mass (smaller value of m) oscillates with a larger amplitude than a larger moving mass (larger value of m). Figure 4.8 shows the behaviour of p_{1min} and p_{2max} for the case with initial values $p_{10} = 0.03$ and $p_{20} = 0.000005$ (same as in Figures 4.3, 4.4, and 4.6). The values of p_{1max} and p_{2min} are not shown as they remain same as their initial values. Figure 4.8 shows the maximum and minimum values of the amplitudes for the other case with initial values $p_{10} = 0.03$ and $p_{20} = 0.000005$. For this case the values of p_{1min} and p_{2max} are less sensitive to changes in s_e near the fixed end of the beam and as s_e moves away from the fixed end a sharp change in p_{1min} and p_{2max} is observed.

Figures 4.9 and 4.10 show the results when m and σ are varied. The same parameters used to obtain Figure 4.7 and Figure 4.8 were used in obtaining Figures 4.9 and 4.10 respectively. Figure 4.9(b) shows that the maximum amplitude of the

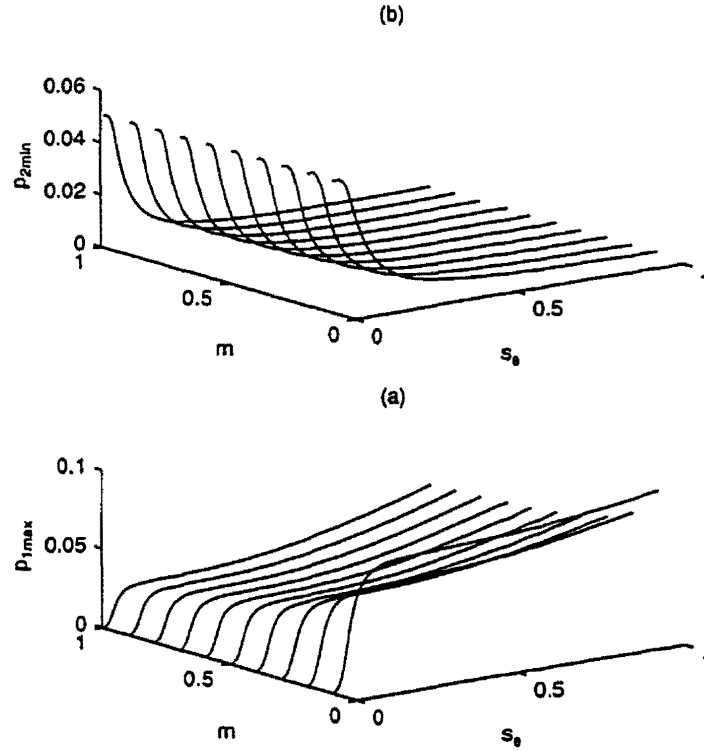


Figure 4.7: Maximum and minimum amplitude - 1:2 IR, $s_e = 0.5$, $\sigma = -0.0002$, $s_{10} = 0.00001$, and $v_{10} = 0.1$. (a) Maximum amplitude for the mass p_{1max} , (b) Minimum amplitude for the tip of the beam p_{2min} .

moving mass p_{1max} reaches a peak close to $\sigma = 0$. In this case the initial value of the amplitude of the moving mass p_{10} was taken as 0.00001. Similarly for the tip deflection of the beam which was given an initial value $p_{20} = 0.05$, Figure 4.9(a) shows that the minimum amplitude p_{2min} reaches a minima close to $\sigma = 0$. Figure 4.10 shows that for the smaller mass ratio $m = 0.1$, the range of σ for which the resonant effect is present is small and increases with m . This range of σ can be isolated in the figure by the sharp changes in the slope of the curves shown, this

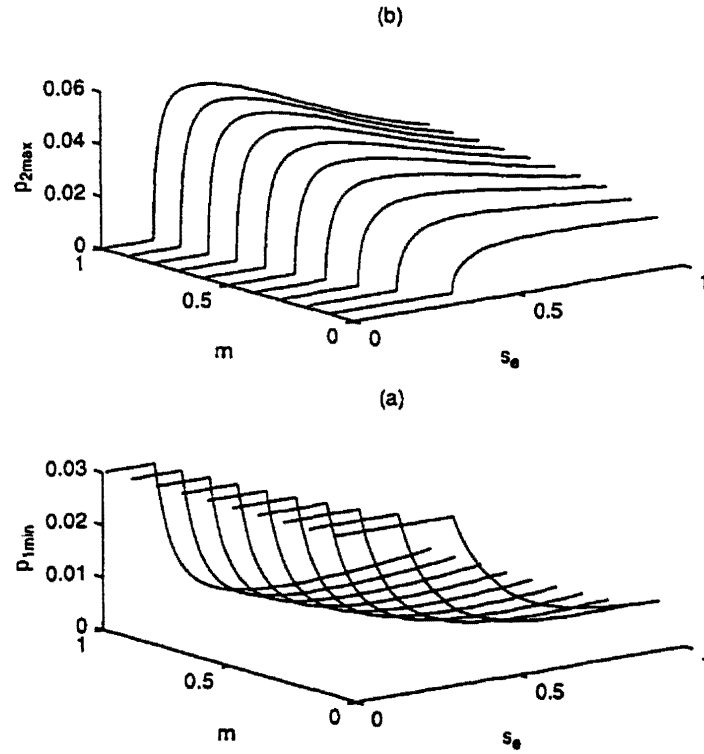


Figure 4.8: Maximum and minimum amplitude - 1:2 IR, $s_e = 0.5$, $\sigma = -0.0085$, $s_{10} = 0.03$, and $v_{t0} = 0.00001$. (a) Minimum amplitude for the mass p_{1min} , (b) Maximum amplitude for the tip of the beam p_{2max} .

behaviour is consistent with the region with distinct ξ_2 and ξ_3 shown earlier in Figure 4.6.

Figure 4.11 shows the beating period for different values of m and σ for the first case where the initial values are $p_{10} = 0.00001$ and $p_{20} = 0.05$. For σ close to zero the beating periods are large and decrease as σ increases. The maximum beating period occurs when ξ_1 and ξ_2 are close to each other or when the modulus η approaches unity (see (4.42)). In Figure 4.12, results are shown for the second

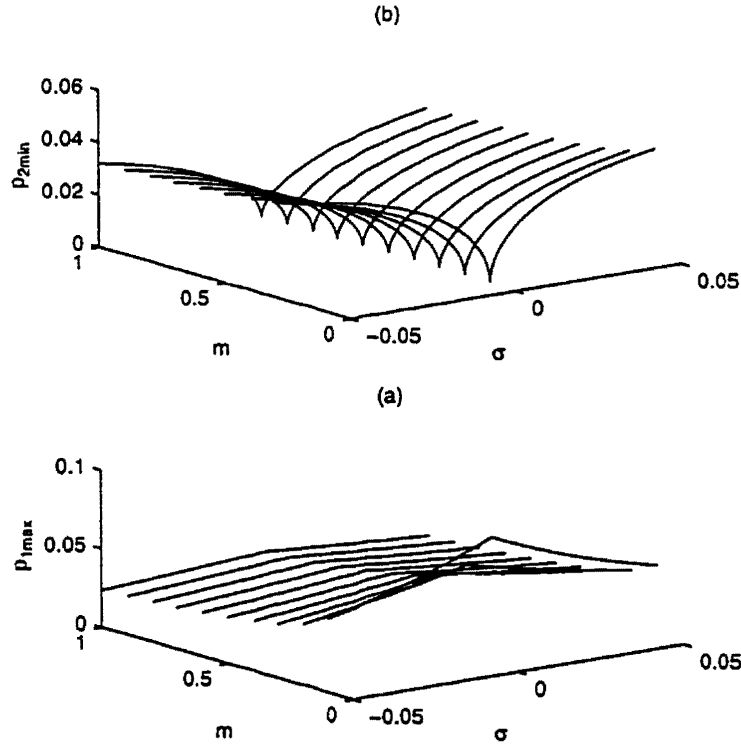


Figure 4.9: Maximum and minimum amplitude – 1:2 IR, $s_e = 0.5$, $s_{10} = 0.00001$, and $v_{t0} = 0.1$. (a) Maximum amplitude for the mass p_{1max} , (b) Minimum amplitude for the tip of the beam p_{2min} .

case where $p_{10} = 0.03$ and $p_{20} = 0.000005$. This figure is plotted in two parts (a) and (b) each with different range of values of m to improve the resolution. For the smaller mass ratio in Figure 4.12(a), two peaks in the beating period, which move apart as the mass ratio increases, are observed. From Figure 4.6 it can be seen that ξ_1 and ξ_2 are close to each other for a range of values of σ and the two peaks in Figure 4.12(a) correspond to the end points of this range. It can also be seen from Figure 4.6 that corresponding to the peaks in the beating period the values

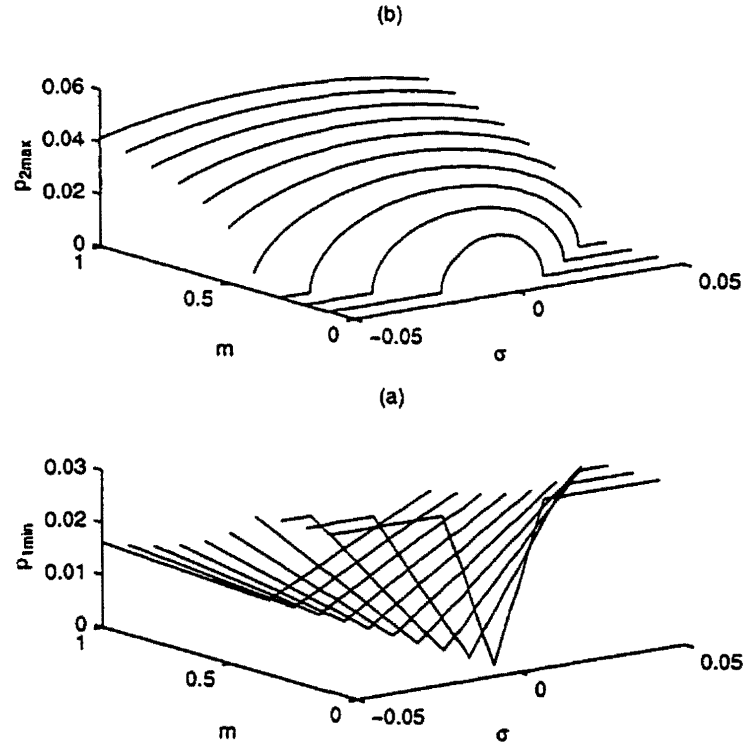


Figure 4.10: Maximum and minimum amplitude - 1:2 IR, $s_e = 0.5$, $\sigma = -0.0002$, $s_{10} = 0.03$, and $v_{10} = 0.00001$. (a) Minimum amplitude for the mass p_{1min} ; (b) Maximum amplitude for the tip of the beam p_{2max} .

of ξ_2 and ξ_3 are close to each other thus indicating a weak coupling between the beam and the moving mass. The beating period in Figure 4.12 thus behaves in an opposite manner to that observed in Figure 4.11 with the minimum value of τ_b being close to $\sigma = 0$.

Figures 4.13 and 4.14 show the change in the beating period as m and σ are varied. In these figures the s_e axis is broken into two parts to improve the resolution. From Figure 4.13 (obtained for $p_{10} = 0.00001$ and $p_{20} = 0.05$ and $s_e = 0.5$) it can

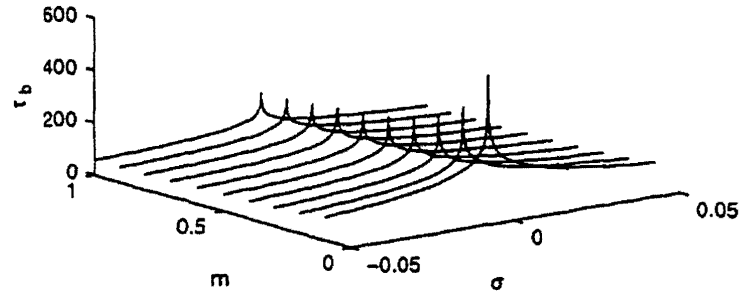


Figure 4.11: Beating period $\tau_b \sim 1:2$ IR, $s_e = 0.5$, $\sigma = -0.0085$, $s_{10} = 0.00001$, and $v_{t0} = 0.1$.

be seen that as the equilibrium position of the moving mass moves towards the free end, the beating period decreases. Figure 4.14 shows a similar result obtained for the other case ($p_{10} = 0.00001$ and $p_{20} = 0.05$, and $s_e = 0.5$) where the larger initial value is given to the mass. For this case, the beating period approaches a peak when s_e is closer to the fixed end of the beam, depending on the mass ratio, and decreases as s_e approaches the free end.

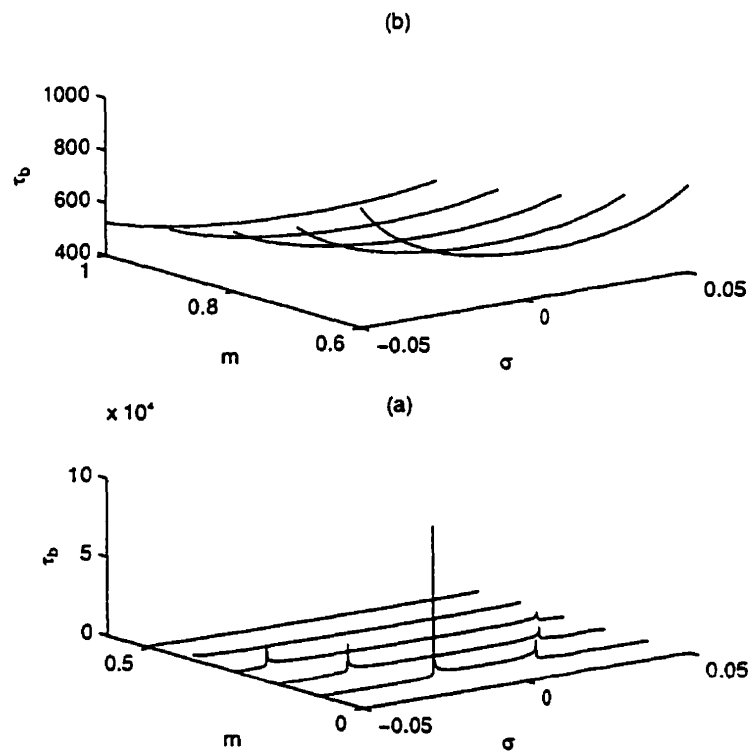


Figure 4.12: Beating period τ_b - 1:2 IR, $s_e = 0.5$, $s_{10} = 0.03$, and $v_{10} = 0.00001$. (a) $m = 0.1 \dots 0.5$ and (b) $m = 0.6 \dots 1.0$.

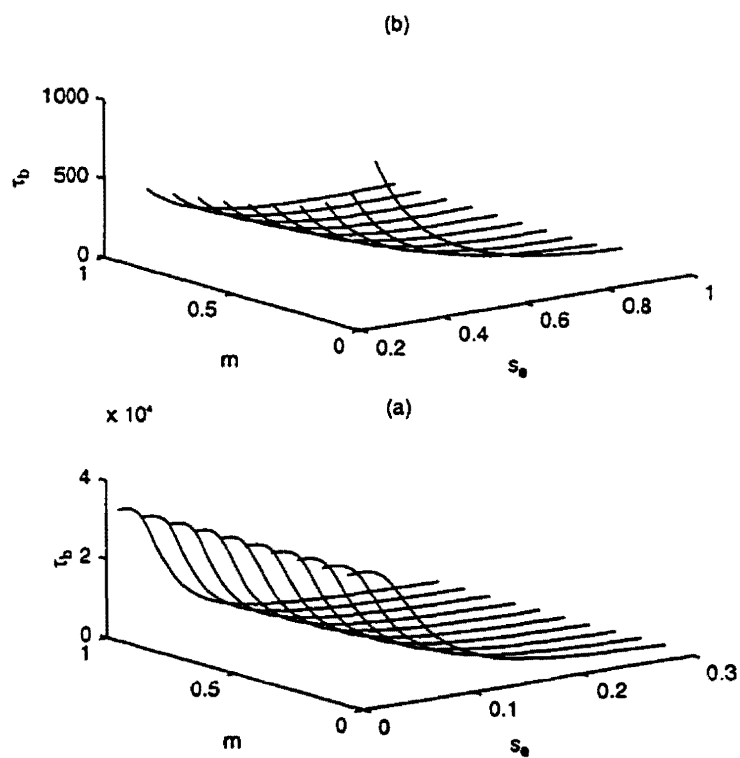


Figure 4.13: Beating period $\tau_b - 1:2$ IR, $s_e = 0.5$, $\sigma = -0.0002$, $s_{10} = 0.00001$, and $v_{t0} = 0.1$. (a) $s_e = 0.01 \dots 0.3$ and (b) $s_e = 0.3 \dots 0.9$.

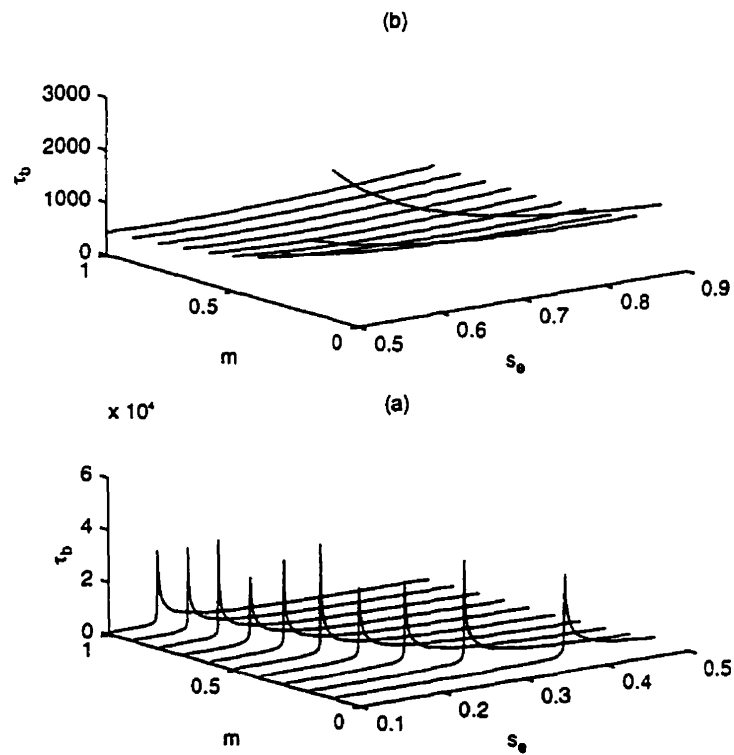


Figure 4.14: Beating period $\tau_b - 1:2$ IR, $s_e = 0.5$, $\sigma = -0.0085$, $s_{10} = 0.03$, and $v_{t0} = 0.00001$. (a) $s_e = 0.1 \dots 0.5$ and (b) $s_e = 0.5 \dots 0.9$.

4.4 Amplitude Modulation

In this section, the equations of motion (3.9) and (3.10) are solved numerically using the four cantilever beam mode shapes (3.16) as the basis functions. The automatic ODE solver discussed in Section 4.3 is used to obtain the solution and also time-frequency analysis is performed in the manner outlined earlier. The results are compared with the perturbation solution for the simplified model obtained in Section 4.3. To establish internal resonance, the fundamental frequency of the beam must be known. In the perturbation analysis for the simplified model, (4.3) was used to obtain the value of ω_1 . For equations (3.9) and (3.10), finite elements are employed to obtain the beam frequencies. The beam is divided into twenty equally spaced elements and at each node three degrees of freedom are considered, the deflection of the beam, the slope of the beam, and the curvature. Table 4.3 lists the first few frequencies for the mass beam system obtained using finite elements for the parameters used in the simulations. The natural frequency of the beam ω_1 , obtained using (4.3) is also shown in the table for comparison. Figures 4.15 and 4.16 show the results obtained for the mass and the beam, respectively, for $m = 1.0$, $s_e = 0.9$ and initial values $t_d = 0.1$ and $s_0 = 0.90001$. The parameters used for the simulations presented in this section are tabulated in Tables 4.2 and 4.4. Figure 4.15(a) shows the perturbation solution, Figure 4.15(b) shows the numerical solution, Figures 4.15(c)–(e) show the spectrogram where the higher frequencies are also included and Figure 4.15(f) shows the power spectrum. The power spectrum is obtained by applying the Hann window to the time series and using the FFT to obtain the spectrum. To reduce the variance, FFTs are obtained for a number

m	s_e	F.E.M. Frequencies	One mode Frequency
1.0	0.9	1.758689 19.537843 60.469989 120.762565	1.763542
1.0	0.5	2.891228 14.225427 61.681059 95.28855	2.908776

Table 4.3: Beam frequencies obtained using finite elements and the one mode equation.

of data segments (see Table 4.4 for the number of segments used) and the results are averaged. The power spectral density (PSD) is computed by taking the mean squared amplitude of the transformed data. This same format is used for all the simulation results presented in this section.

From the power spectrum, Figure 4.15(f), major peaks are observed at even multiples of ω_1 (eg. $2\omega_1, 4\omega_1, 6\omega_1 \dots$). As the peaks approach the second frequency of the beam (19.5378) the energy corresponding to these peaks increases slightly. This is much more clear in the beam plot, Figure 4.16(f), where the energy increases as the peaks approach the second and the third frequencies, 19.5378 and 60.470, respectively. For the beam, the major peaks in the power spectrum occur at odd multiples of ω_1 ($\omega_1, 3\omega_1, \dots$). The spectrograms (Figure 4.15(c)–(e) and Figure 4.16(c)–(e)) show the time variation of the frequencies. For the moving mass, the energy corresponding to the primary frequency ($2\omega_1$) becomes quite small when the amplitude reaches the minimum, and the energy in the harmonics of the moving

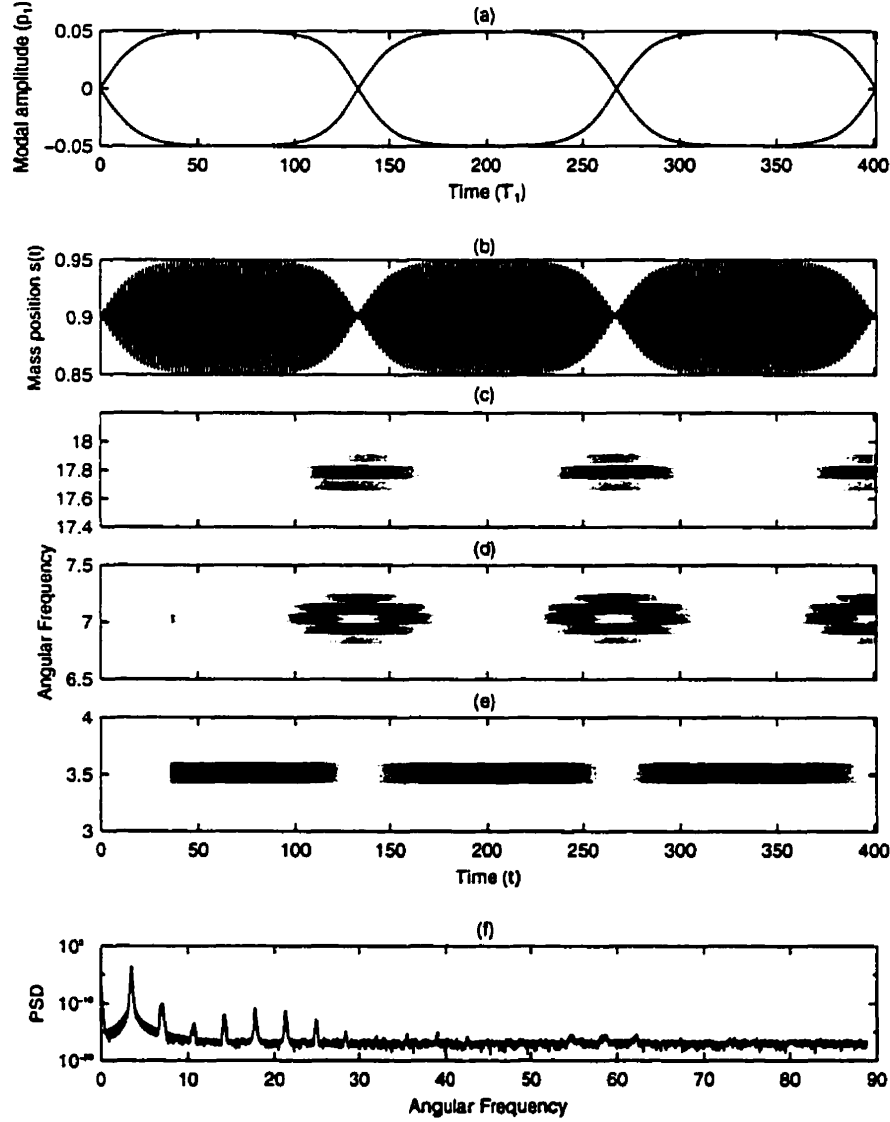


Figure 4.15: Mass response – 1:2 IR, $m = 1.0$, $s_e = 0.9$, $s_{10} = 0.90001$, and $v_{t0} = 0.1$. (a) Perturbation solution $\sigma = -0.0008$, (b) Numerical solution, (c)–(e) Spectrograms, and (f) Power spectrum.

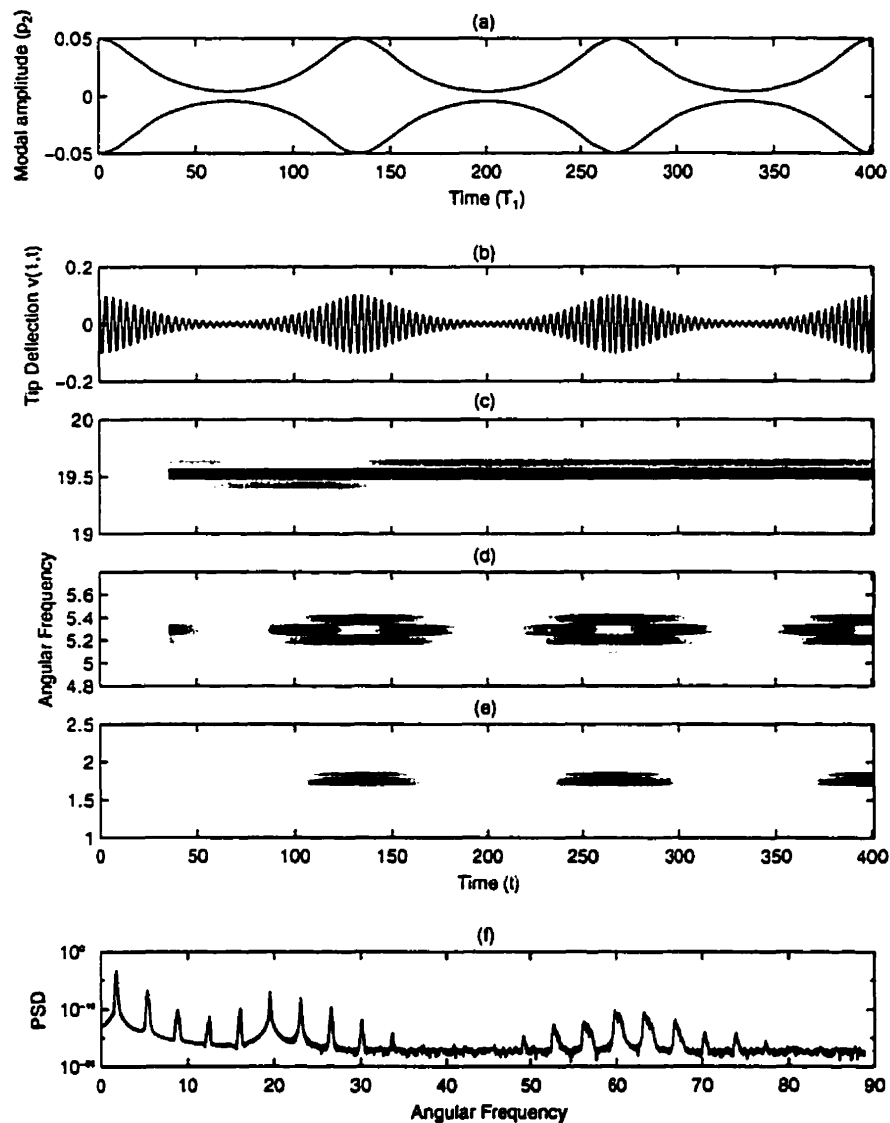


Figure 4.16: Tip deflection – 1:2 IR, $m = 1.0$, $s_e = 0.9$, $s_{10} = 0.90001$, and $v_{t0} = 0.1$. (a) Perturbation solution $\sigma = -0.0008$, (b) Numerical solution, (c)–(e) Spectrograms, and (f) Power spectrum.

Parameter Set 2		
$m = 1.0, s_e = 0.9, \omega_1$ using 20 finite elements = 1.758689		
ω_1 using one mode of a cantilever beam = 1.763542		
	Initial Values equation (2.47) $s_0 = 0.90001, v_{t0} = 0.1$	Initial Values equation (2.47) $s_0 = 0.95, v_{t0} = 0.00001$
Figure	4.15,4.16	4.17,4.18
Model	PM1,NM2	PM1,NM2
σ	-0.0008	-0.0097
Δt (average)	0.00442	0.00533
Spectrogram		
No. of Segments	64	64
Segment Size	256	256
Power Spectrum		
No. of FFTs	2	6
Segment Size	65536	65536

Table 4.4: Comparison between perturbation and numerical solutions-2

mass ($4\omega_1$ and $10\omega_1$ shown in Figures 4.15(c) and (d)) crests, indicating a change in the frequency content of the mass response when the amplitude is decreasing. The time variation of $4\omega_1$ (Figure 4.15(d)) shows dissipation of energy to the side bands when the amplitude of the moving mass reaches a minimum, however when such a frequency is close to a higher fundamental frequency of the beam, like $10\omega_1$ is to the second frequency of the beam, the dissipation of energy to the side bands does not occur. Such a behaviour was observed in all the simulations presented in this thesis. For the beam the beating pattern can be seen in ω_1 and $3\omega_1$. However the higher frequency $11\omega_1$, shown in Figure 4.16(c), does not show the beating pattern, as this frequency is close to the second frequency of the beam which is not in resonance.

Next, for the same parameters as in Figure 4.15 and 4.16, different initial conditions are used ($v_{t0} = 0.00001$ and $s_0 = 0.95$). Figure 4.17 and 4.18 show the results. Since in this case the system is predominantly excited by the moving mass, the energy corresponding to the higher frequencies of the beam is notably very low as can be seen by comparing Figure 4.16(f) and 4.18(f). The spectrograms for the moving mass show that initially the energy of the system is concentrated in the primary frequency of the moving mass $2\omega_1$ and as the amplitude starts decreasing some of the energy moves into the harmonics of the moving mass like $4\omega_1$ but predominantly the energy is transferred to the frequencies in the beam ω_1 , $3\omega_1$ and $5\omega_1$. The power spectrums for the moving mass and the beam are similar to those observed in Figures 4.15 and 4.16 except that the PSD is significantly lower for the higher frequencies.

In Figures 4.19 and 4.20 the equilibrium position of the moving spring-mass system s_e is changed to 0.5 and initial values to $v_{t0} = 0.1$ and $s_0 = 0.50001$. These results can be compared with Figures 4.1 and 4.2 which have the same parameters but are for the simplified model. Comparison shows that the beating period is smaller when four basis functions are used. The amplitude of the response however has remained the same.

In general, increasing the number of basis functions improves the accuracy of the solution obtained using the Galerkin method. The difference in the value of the beam frequency ω_1 for the two models (see Table 4.3) affects the value of the detuning parameter which is chosen to match the beating periods obtained using the perturbation solution and the numerical solution. The perturbation solution

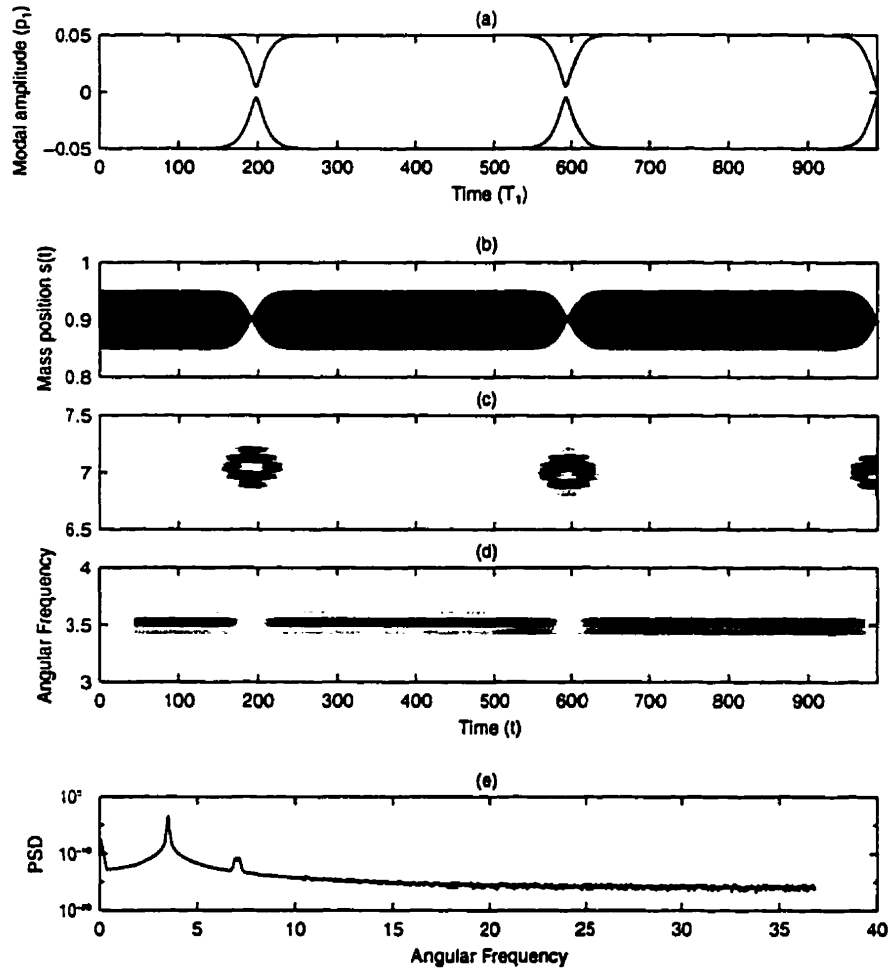


Figure 4.17: Mass response - 1:2 IR, $m = 1.0$, $s_e = 0.9$, $s_{10} = 0.95$, and $v_{t0} = 0.00001$. (a) Perturbation solution $\sigma = -0.0097$, (b) Numerical solution, (c) and (d) Spectrograms, and (e) Power spectrum.

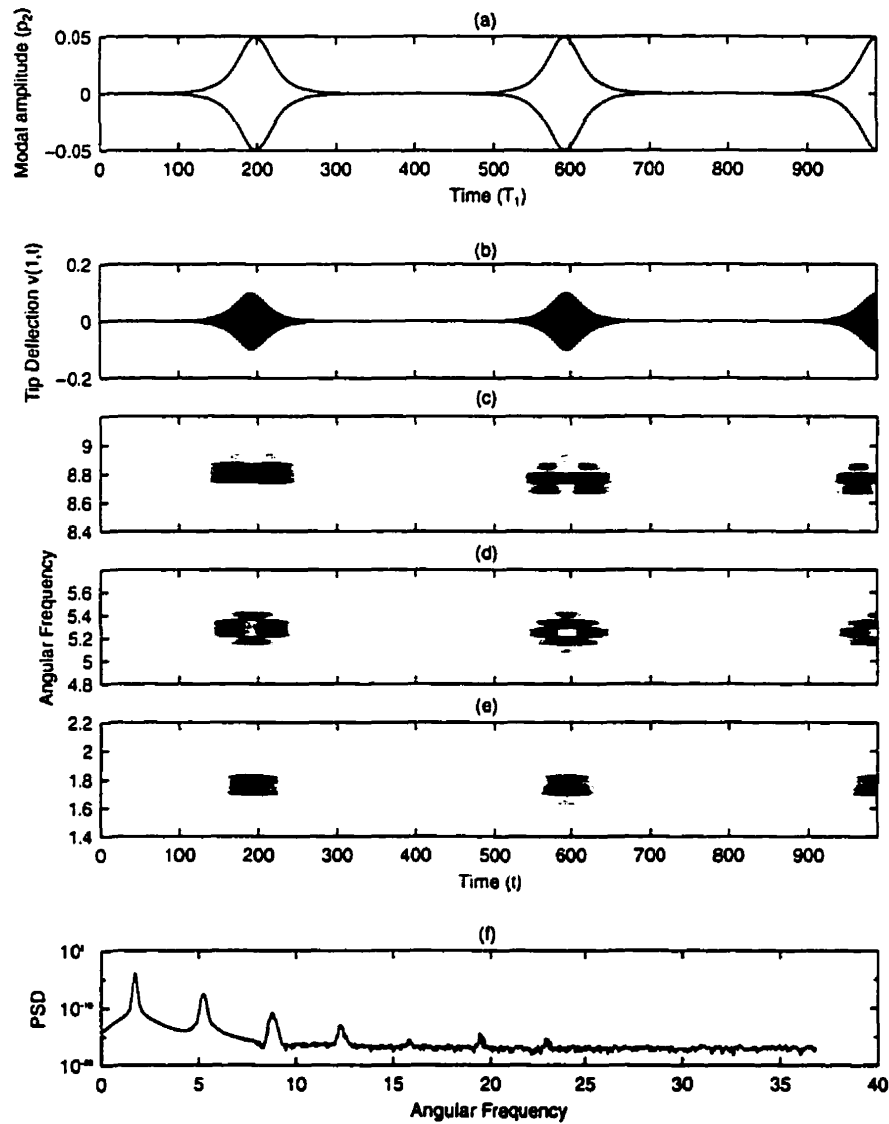


Figure 4.18: Tip deflection - 1:2 IR, $m = 1.0$, $s_e = 0.9$, $s_{10} = 0.95$, and $v_{t0} = 0.00001$. (a) Perturbation solution $\sigma = -0.0097$, (b) Numerical solution, (c)–(e) Spectrograms, and (f) Power spectrum.

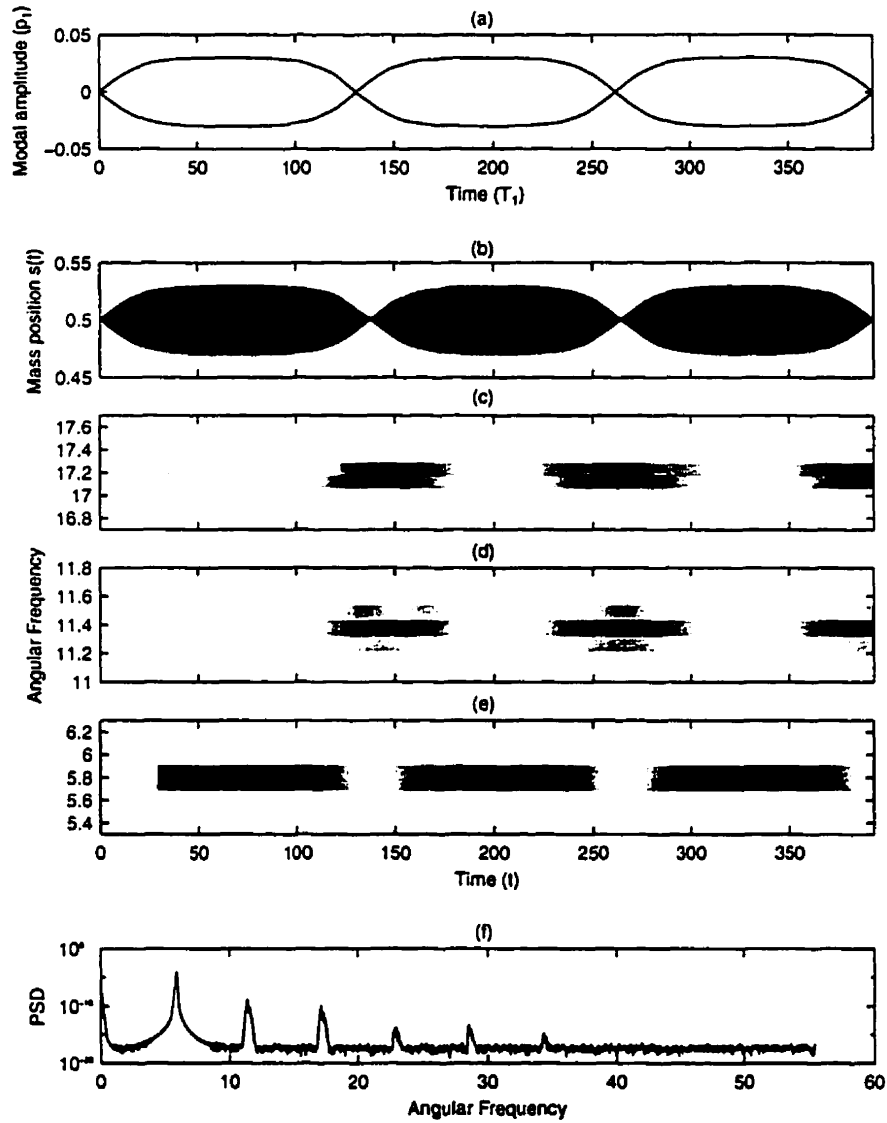


Figure 4.19: Mass response - 1:2 IR, $m = 1.0$, $s_e = 0.5$, $s_{10} = 0.50001$, and $v_{i0} = 0.1$. (a) Perturbation solution $\sigma = -0.0016$, (b) Numerical solution, (c)–(e) Spectrograms, and (f) Power spectrum.

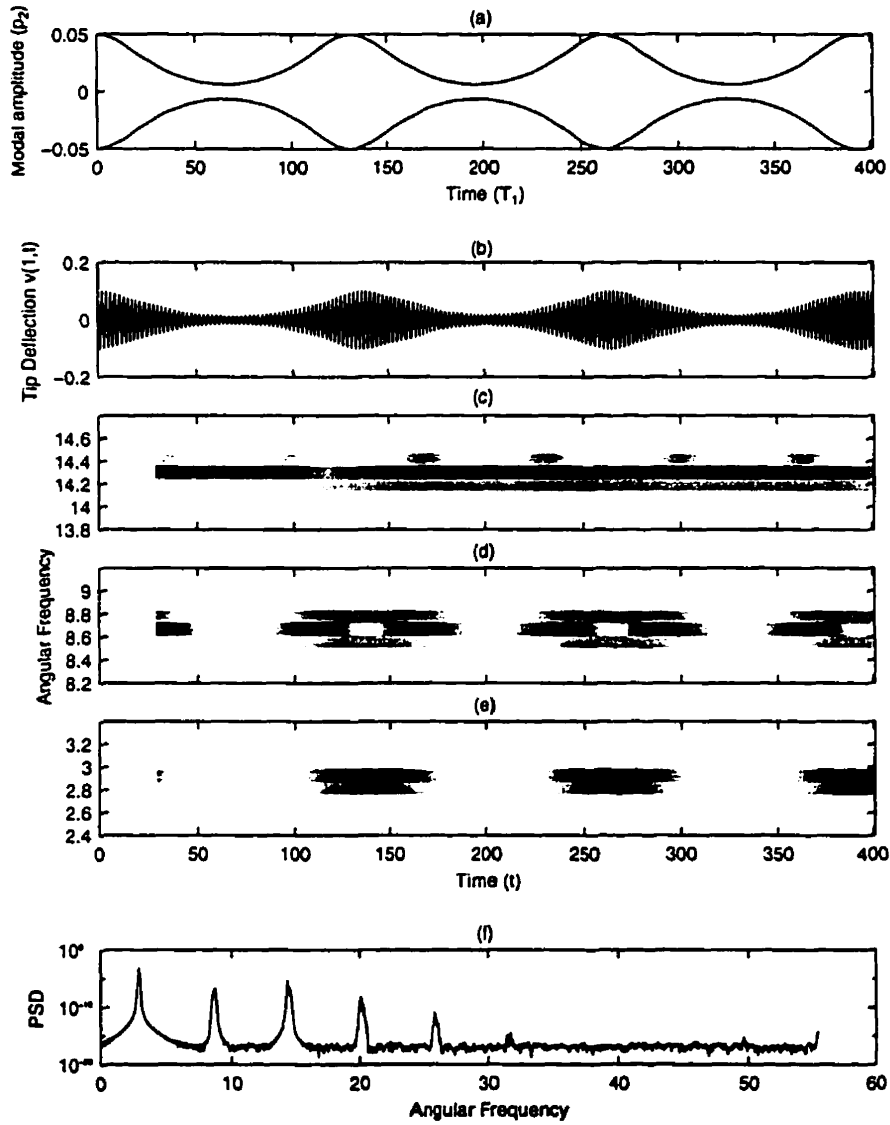


Figure 4.20: Tip deflection - 1:2 IR, $m = 1.0$, $s_e = 0.5$, $s_{10} = 0.50001$, and $v_{t0} = 0.1$. (a) Perturbation solution $\sigma = -0.0016$, (b) Numerical solution, (c)-(e) Spectrograms, and (f) Power spectrum.

in Figure 4.1 and 4.2 was obtained using $\sigma = -0.0002$, in this case (Figure 4.19 and 4.20) $\sigma = -0.0016$ was used. Comparing Figure 4.19(a) and 4.1(a) it can be seen that small changes in the detuning parameter do not have any significant effect on the maximum amplitude of the moving mass. However comparison of Figure 4.20(a) and 4.2(a) for the beam shows that the increase in the value of σ has increased the value of p_{2min} (the minimum amplitude of p_2). Although the numerical solutions Figure 4.19(b) and 4.20(b) were obtained for an exact 1:2 IR the minimum amplitude of p_2 as seen from Figure 4.20(b) matches well with the perturbation solution where the detuning parameter is involved. Note that in Figure 4.20(b) the amplitude of the tip deflection does not have a smooth appearance, this may be due to the presence of higher frequencies, also a numerical artifact as a result of higher frequencies cannot be ruled out.

Figures 4.21 and 4.22 show another interesting result where the parameters are same as in Figures 4.19 and 4.20 but with initial values $v_{i0} = 0.00001$ and $s_0 = 0.53$. These figures can be compared with Figures 4.3 and 4.4 as they are obtained using the same parameters. First it should be noted that, compared to other cases considered in this work, in this case small oscillations of the mass cause relatively large oscillations in the beam. So, the effect of the nonlinearities is more pronounced. The perturbation and the numerical results in this case don't match well. However when the simplified model is used, Figures 4.3 and 4.4, there is a good correspondence between the numerical and perturbation results, suggesting that the peculiar behaviour in Figures 4.21 and 4.22 is due to the inclusion of higher frequencies of the beam. The spectrograms indicate that the PSD at the higher

frequencies has some time dependency with large periods. If the initial value of the mass is increased the response was found to become chaotic with the beam undergoing large oscillations.

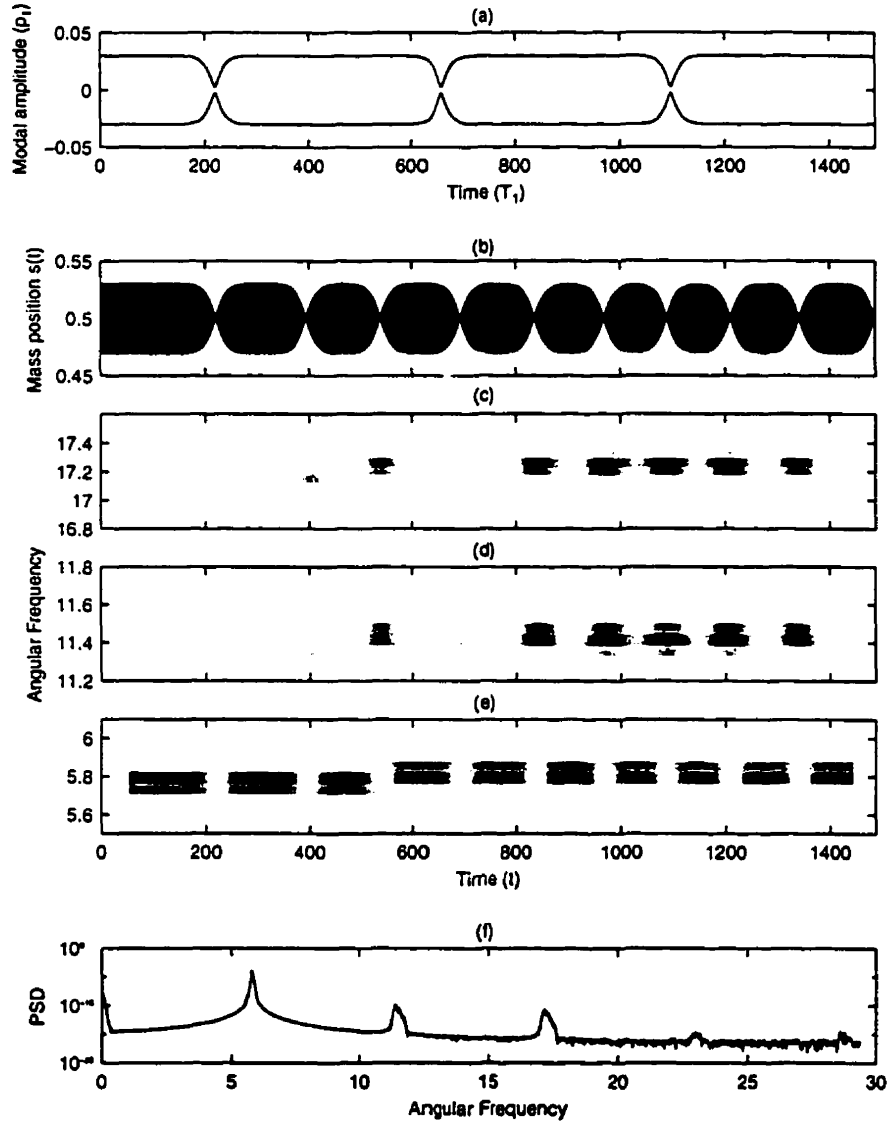


Figure 4.21: Mass response - 1:2 IR, $m = 1.0$, $s_e = 0.5$, $s_{10} = 0.53$, and $v_{t0} = 0.00001$. (a) Perturbation solution $\sigma = -0.0115$, (b) Numerical solution, (c)–(e) Spectrograms, and (f) Power spectrum.

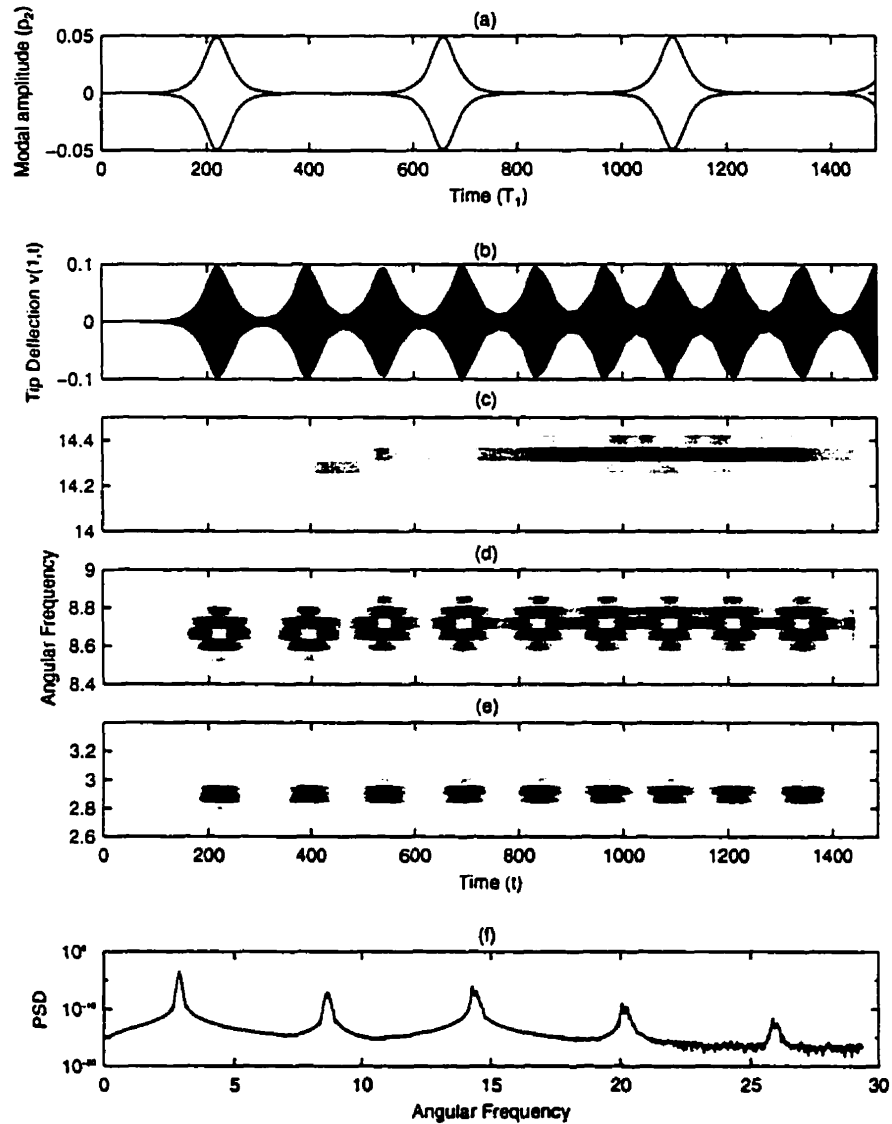


Figure 4.22: Tip deflection - 1:2 IR, $m = 1.0$, $s_e = 0.5$, $s_{10} = 0.53$, and $v_{t0} = 0.00001$. (a) Perturbation solution $\sigma = -0.0115$, (b) Numerical solution, (c)–(e) Spectrograms, and (f) Power spectrum.

Difficulty in obtaining a numerical solution for nonlinear systems is often one of the motivating reason for perturbation analysis. However perturbation methods do not always provide a closed form solution like the one obtained in this chapter and a combined analytic-numeric approach is often required. Comparison between closed form analytical and numerical solutions is therefore uncommon. In this work the perturbation solution for modal amplitudes is matched with the numerical solution by selecting a value for the detuning parameter σ . A numerical solution obtained under perfect 1:2 resonance conditions when compared with a perturbation solution under perfect 1:2 resonance conditions is expected to reflect differences as a result of the different models for the two cases and also due to neglecting the higher order terms in the perturbation solution. The approach used in this work was to quantify these differences using the detuning parameter σ . Comparison of the perturbation and the numerical solutions show that when the motion is predominantly bi-periodic (two fundamental frequencies, one ω_1 and the other beating frequency $\frac{\pi}{\tau_b}$ and their harmonics) the results match very well. Using the closed form solution an extensive parametric analysis was carried out which identifies regions of strong nonlinear coupling between the beam and the moving mass and gives the change in some of the important properties of the solution like the beating period, and the maximum and the minimum amplitudes with the detuning parameter, initial values and the equilibrium position of the moving mass on the beam.

In the next chapter the focus is on geometric nonlinearities, and solving equations of motion for the large oscillation model with both kinematic and geometric nonlinearities.

Chapter 5

Kinematic and Geometric Nonlinearities

In this chapter, a new method is presented to obtain numerical solution of equations of motion for nonlinear continuously modelled systems. Using this technique, results are presented for three types of problem: a system with only kinematic nonlinearities; beam carrying a moving mass – small oscillation model, a system with only geometric nonlinearities – large oscillation model of a cantilever beam, and a system with both kinematic and geometric nonlinearities – large oscillation model of a cantilever beam carrying a moving mass.

The solution strategy relies on a traditional finite element approach for spatial discretization. This was described in Chapter 3 where the nonlinear partial differential equations of motion were reduced to nonlinear ordinary differential equations. In this chapter the ordinary differential equations are reduced to nonlinear algebraic equations using finite difference approximations based on the average acceleration

method. A new formulation is used for the average acceleration method, wherein, using finite differences the displacements and velocities are expressed in terms of accelerations as opposed to the conventional method where the velocities and accelerations are expressed in terms of displacements. In this thesis this new formulation is called the acceleration formulation and the conventional approach is referred to as the displacement formulation.

The nonlinear algebraic equations obtained after the discretization are large in number (depending on the number of elements used) and are also stiff and hence difficult to solve. Conventional techniques based on Newton's method were tried, but were unsuccessful in obtaining the solution. One inherent problem with the algebraic equations obtained using finite elements and the use of Newton's method (or its variants) is that, in the finite element approach the solution is modelled to barely meet the continuity requirements of the differential equations and the extra derivative for the Jacobian required in the Newton's method introduces discontinuities which can prevent convergence in the Newton's method. Therefore in this work a new technique is proposed which does not require finding the Jacobian.

5.1 Average Acceleration Method

The average acceleration method is widely used in finite element analysis for temporal discretization of linear dynamic equations. In this work, however, a variant of this approach is proposed to solve nonlinear problems. Combined with the new iterative method to be presented in Section 5.6, this approach gives superior convergence than the conventional average acceleration method. The following finite

difference approximations for displacements and velocities are derived directly using Taylor series:

$$\begin{aligned}\dot{u}^{t+\Delta t} &= \dot{u}^t + \ddot{u}^t \Delta t \\ u^{t+\Delta t} &= u^t + \dot{u}^t \Delta t + \ddot{u}^t \frac{\Delta t^2}{2}\end{aligned}\tag{5.1}$$

where u represents the time dependent variables, α_i , β_i , and λ_i , Δt is a small increment in time and the superscripts identify the time at which the term is evaluated. In the average acceleration method, instead of just using (5.1) the following additional assumption is made for the accelerations [71] which makes (5.1) implicit (terms on both sides of the equation depend on $t + \Delta t$), thus giving stability for larger step sizes (see Newmark [71]):

$$\ddot{u}^t \Rightarrow \frac{\ddot{u}^{t+\Delta t} + \ddot{u}^t}{2}\tag{5.2}$$

where “ $\Rightarrow$ ” implies replacement. Substituting (5.2) into (5.1) gives the following equations:

$$\begin{aligned}\dot{u}^{t+\Delta t} &= \dot{u}^t + a_1 \ddot{u}^t + a_2 \ddot{u}^{t+\Delta t} \\ &= c1u(\dot{u}^t, \ddot{u}^t) + a_2 \ddot{u}^{t+\Delta t} \\ u^{t+\Delta t} &= u^t + a_3 \dot{u}^t + a_4 \ddot{u}^t + a_5 \ddot{u}^{t+\Delta t} \\ &= c2u(u^t, \dot{u}^t, \ddot{u}^t) + a_5 \ddot{u}^{t+\Delta t}\end{aligned}\tag{5.3}$$

where the parameters a_i are given by:

$$\begin{aligned} a_1 &= \frac{\Delta t}{2} & a_2 &= \frac{\Delta t}{2} & a_3 &= \Delta t \\ a_4 &= \frac{\Delta t^2}{4} & a_5 &= \frac{\Delta t^2}{4} \end{aligned} \quad (5.4)$$

and $c1u$ and $c2u$ are used to represent the terms which have to be evaluated at time t and are given by :

$$\begin{aligned} c1u &= \dot{u}^t + a_1 \ddot{u}^t \\ c2u &= u^t + a_3 \dot{u}^t + a_4 \ddot{u}^t \end{aligned} \quad (5.5)$$

An explanation of the notation $c1u$ and $c2u$ is in order here. The first two letters $c1$ and $c2$ identify two quantities whose values are known at each iterative step, as they are evaluated at time t . The third letter u represents the variable being discretized (α_i , β_i , or γ_i). It can be seen from (5.4) that there are only three parameters which are unique, however, five parameters are used so that other finite difference approximations can be easily implemented in the computer programs developed (for example the more general Newmark's method [71] where the parameters allow for introducing numerical dissipation).

To use this method for temporal discretization, the ordinary differential equations of motion are considered at time $t + \Delta t$. Using (5.3) the ordinary differential equations are reduced to a system of nonlinear algebraic equations involving acceleration at $t + \Delta t$ and displacement, velocity, and acceleration at time t . This approach to discretization will be referred to as the acceleration formulation.

The conventional approach to using the acceleration method, the displacement

formulation, involves solving equations (5.3) for $\dot{u}^{t+\Delta t}$ and $\ddot{u}^{t+\Delta t}$ thus giving:

$$\begin{aligned}
 \ddot{u}^{t+\Delta t} &= -a_1 u^t - a_2 \dot{u}^t - a_3 \ddot{u}^t + a_1 u^{t+\Delta t} \\
 &= c1u(u^t, \dot{u}^t, \ddot{u}^t) + a_1 u^{t+\Delta t} \\
 \dot{u}^{t+\Delta t} &= -a_4 u^t - a_5 \dot{u}^t - a_6 \ddot{u}^t + a_4 u^{t+\Delta t} \\
 &= c2u(u^t, \dot{u}^t, \ddot{u}^t) + a_4 u^{t+\Delta t}
 \end{aligned} \tag{5.6}$$

where the a_i are given by:

$$\begin{aligned}
 a_1 &= \frac{4}{\Delta t^2} & a_2 &= \frac{4}{\Delta t} & a_3 &= 1 \\
 a_4 &= \frac{2}{\Delta t} & a_5 &= 1 & a_6 &= 0
 \end{aligned} \tag{5.7}$$

and $c1u$ and $c2u$ are given by

$$\begin{aligned}
 c1u &= -a_1 u^t - a_2 \dot{u}^t - a_3 \ddot{u}^t \\
 c2u &= -a_4 u^t - a_5 \dot{u}^t - a_6 \ddot{u}^t
 \end{aligned} \tag{5.8}$$

Note that despite their different definitions, the same variables $c1u$, $c2u$, and $a_1 \dots a_6$ are used for both the displacement formulation and the acceleration formulation due to their similar functionality in the discretized equations of motion. Using (5.6), ordinary differential equations can be reduced to nonlinear algebraic equations in terms of displacements. In the next few sections (Sections 5.2–5.5), discretized equations are obtained for the small and the large oscillation model using the conventional displacement formulation and the new acceleration formu-

lation. The results obtained using these two approaches are compared in Section 5.9.

5.2 Small Oscillations–Acceleration Formulation

For the small oscillation model, the discretized equations of motion using the acceleration formulation are obtained by substituting the approximation (5.3) for all the dependent variables in the ordinary differential equations (3.9) and (3.10), thus giving the following equations.

$$\begin{aligned}
 & (1 + a_5 \omega^2) \ddot{s} + \omega^2 (c_2 s - s_c) \\
 & + \left([\phi_i \phi'_j] \{ \{ \ddot{\alpha}_i \} \{ a_5 \{ \ddot{\alpha}_j \} + \{ c_2 \alpha_j \} \} \} \right)_{x=a_5 \ddot{s} + c_2 s} = 0 \\
 & m \{ \ddot{s} \{ \phi_i \} + [\phi_i \phi_j] \{ \ddot{\alpha}_j \} \\
 & + [\phi_i \phi'_j] \{ \ddot{s} \{ a_5 \{ \ddot{\alpha}_j \} + \{ c_2 \alpha_j \} \} + 2(a_2 \ddot{s} + c_1 s) \{ a_2 \{ \ddot{\alpha}_j \} + \{ c_1 \alpha_j \} \} \} \\
 & + (a_2 \ddot{s} + c_1 s)^2 [\phi_i \phi''_j] \{ a_5 \{ \ddot{\alpha}_j \} + \{ c_2 \alpha_j \} \} \}_{x=a_5 \ddot{s} + c_2 s} \\
 & + \int_0^1 [\phi_i \phi_j] \{ \ddot{\alpha}_j \} + [\phi''_i \phi''_j] \{ a_5 \{ \ddot{\alpha}_j \} + \{ c_2 \alpha_j \} \} dx = 0 \quad (5.9)
 \end{aligned}$$

In (5.9) the time dependent variables $\ddot{s}$, and $\ddot{\alpha}_i$ are evaluated at $t + \Delta t$. The variables $c_1 s$, $c_2 s$, $c_1 \alpha_i$, and $c_2 \alpha_i$ (a subscripted notation identifying the dependent variables is not used to avoid multiple levels of subscripts), using the definition (5.5) are given by:

$$\begin{aligned}
 c_1 s &= \dot{s}^t + a_1 \ddot{s}^t \\
 c_2 s &= s^t + a_3 \dot{s}^t + a_4 \ddot{s}^t \quad (5.10)
 \end{aligned}$$

$$\begin{aligned}
c1\alpha_i &= \dot{\alpha}_i^t + a_1\ddot{\alpha}_i^t \\
c2\alpha_i &= \alpha_i^t + a_3\dot{\alpha}_i^t + a_4\ddot{\alpha}_i^t
\end{aligned} \tag{5.11}$$

5.3 Large Oscillations–Acceleration Formulation

The discretized equations for the large oscillation model using the acceleration formulation are obtained by substituting (5.3) for the dependent variables in the equations of motion (3.12)–(3.15), thus giving the following nonlinear algebraic equations:

$$\begin{aligned}
& (1 + a_5\omega^2)\ddot{s} + \omega^2(c2s - s_e) + ([\phi_i] \{\ddot{\alpha}_i\} \\
& + [\phi_i\phi_j'] \{ \{\ddot{\alpha}_i\} \{a_5 \{\ddot{\alpha}_j\} + \{c2\alpha_j\} \} + \{\ddot{\beta}_i\} \{a_5 \{\ddot{\beta}_j\} + \{c2\beta_j\} \} \})_{x=a_5\ddot{s}+c2s} = 0 \\
& m \{ \ddot{s} \{ \phi_i \} + [\phi_i\phi_j] \{ \ddot{\alpha}_j \} \\
& + [\phi_i\phi_j'] \{ \ddot{s} \{ a_5 \{ \ddot{\alpha}_j \} + \{ c2\alpha_j \} \} + 2(a_2\ddot{s} + c1s) \{ a_2 \{ \ddot{\alpha}_j \} + \{ c1\alpha_j \} \} \} \\
& + (a_2\ddot{s} + c1s)^2 [\phi_i\phi_j''] \{ a_5 \{ \ddot{\alpha}_j \} + \{ c2\alpha_j \} \} \}_{x=a_5\ddot{s}+c2s} \\
& + \int_0^1 [\phi_i\phi_j] \{ \ddot{\alpha}_j \} + [\phi_i''\phi_j''] \{ a_5 \{ \ddot{\alpha}_j \} + \{ c2\alpha_j \} \} + [\phi_i'\phi_j] \{ \lambda_j \} \\
& + \{ \phi_i' \} [\phi_j\phi_k'] \{ \{ \lambda_j \} \{ a_5 \{ \ddot{\alpha}_k \} + \{ c2\alpha_k \} \} \} - \{ \phi_i' \} [\phi_j''\phi_k''] \\
& \{ \{ a_5 \{ \ddot{\alpha}_j \} + \{ c2\alpha_j \} \} \{ a_5 \{ \ddot{\alpha}_k \} + \{ c2\alpha_k \} \} \\
& + \{ a_5 \{ \ddot{\beta}_j \} + \{ c2\beta_j \} \} \{ a_5 \{ \ddot{\beta}_k \} + \{ c2\beta_k \} \} \} \\
& - \{ \phi_i' \} [\phi_j'\phi_k''\phi_l''] \{ \{ a_5 \{ \ddot{\alpha}_j \} + \{ c2\alpha_j \} \} \{ a_5 \{ \ddot{\alpha}_k \} + \{ c2\alpha_k \} \} \{ a_5 \{ \ddot{\alpha}_l \} + \{ c2\alpha_l \} \} \\
& + \{ a_5 \{ \ddot{\alpha}_j \} + \{ c2\alpha_j \} \} \{ a_5 \{ \ddot{\beta}_k \} + \{ c2\beta_k \} \} \{ a_5 \{ \ddot{\beta}_l \} + \{ c2\beta_l \} \} \} dx = 0 \\
& m \{ [\phi_i\phi_j] \{ \ddot{\beta}_j \}
\end{aligned}$$

$$\begin{aligned}
& + [\phi_i \phi'_j] \left\{ \ddot{s} \{a_5 \{\ddot{\beta}_j\} + \{c2\beta_j\}\} + 2(a_2 \ddot{s} + c1s) \{a_2 \{\ddot{\beta}_j\} + \{c1\beta_j\}\} \right. \\
& \quad \left. + (a_2 \ddot{s} + c1s)^2 [\phi_i \phi''_j] \{a_5 \{\ddot{\beta}_j\} + \{c2\beta_j\}\} \right\}_{x=a_5 \ddot{s} + c2s} \\
& \quad + \int_0^1 [\phi_i \phi_j] \{\ddot{\beta}_j\} + [\phi''_i \phi''_j] \{a_5 \{\ddot{\beta}_j\} + \{c2\beta_j\}\} \\
& \quad + \{\phi'_i\} [\phi_j \phi'_k] \left\{ \{\lambda_j\} \{a_5 \{\ddot{\beta}_k\} + \{c2\beta_k\}\} \right\} \\
& - \{\phi'_i\} [\phi'_j \phi''_k \phi''_l] \left\{ \{a_5 \{\ddot{\beta}_j\} + \{c2\beta_j\}\} \{a_5 \{\ddot{\beta}_k\} + \{c2\beta_k\}\} \{a_5 \{\ddot{\beta}_l\} + \{c2\beta_l\}\} \right. \\
& \quad \left. + \{a_5 \{\ddot{\beta}_j\} + \{c2\beta_j\}\} \{a_5 \{\ddot{\alpha}_k\} + \{c2\alpha_k\}\} \{a_5 \{\ddot{\alpha}_l\} + \{c2\alpha_l\}\} \right\} dx = 0 \\
& \quad \int_0^1 [\phi_i \phi'_j] \{a_5 \{\ddot{\alpha}_j\} + \{c2\alpha_j\}\} + \frac{1}{2} \{\phi_i\} [\phi'_j \phi'_k] \\
& \quad \left\{ \{a_5 \{\ddot{\alpha}_j\} + \{c2\alpha_j\}\} \{a_5 \{\ddot{\alpha}_k\} + \{c2\alpha_k\}\} \right. \\
& \quad \left. + \{a_5 \{\ddot{\beta}_j\} + \{c2\beta_j\}\} \{a_5 \{\ddot{\beta}_k\} + \{c2\beta_k\}\} \right\} dx = 0
\end{aligned} \tag{5.12}$$

In (5.12), $c1s$, $c2s$, $c1\alpha_i$ and $c2\alpha_i$ are same as defined in (5.10) and (5.11) and $c1\beta_i$ and $c2\beta_i$ are given by:

$$\begin{aligned}
c1\beta_i &= \dot{\beta}_i^t + a_1 \ddot{\beta}_i^t \\
c2\beta_i &= \beta_i^t + a_3 \dot{\beta}_i^t + a_4 \ddot{\beta}_i^t
\end{aligned} \tag{5.13}$$

Equation (5.12) consists of four sets of nonlinear algebraic equations. The actual number of equations in (5.12) depends upon the number of elements used in the finite element discretization. For N elements, using fifth degree Hermitian polynomials (three degrees of freedom per node) this constitutes $9(N+1)+1$ equations, where the elimination of the equations corresponding to the zeroth and the N^{th} node to enforce the essential boundary conditions are not accounted for. At node 0

the essential boundary conditions (2.44) all have zero values therefore the equations in (5.12) corresponding to these boundary conditions are not considered. At node N there are no essential boundary conditions but because of the fifth degree Hermitian polynomials the second derivatives of u and v exists as nodal variables; since the natural boundary conditions require these values to be zero, the corresponding equations in (5.12) are also discarded. This same approach is used for applying the boundary conditions to all the beam models considered in this work.

5.4 Small Oscillations–Displacement Formulation

Similar to the acceleration formulation, the equations of motion are now discretized using the displacement formulation. For the small oscillation model, substituting (5.6) in (3.9) and (3.10) gives the following nonlinear equations in terms of displacements which are evaluated at time $t + \Delta t$:

$$\begin{aligned}
 & s(a_1 + \omega^2) + c_1 s - \omega^2 s_e \\
 & + \left([\phi_i \phi'_j] \{ \{ a_1 \{ \alpha_i \} + \{ c_1 \alpha_i \} \} \{ \alpha_j \} \} \right)_{x=s(t)} = 0 \\
 & m \{ (a_1 s + c_1 s) \{ \phi_i \} + [\phi_i \phi_j] \{ a_1 \{ \alpha_j \} + \{ c_1 \alpha_j \} \} \\
 & + [\phi_i \phi'_j] \{ (a_1 s + c_1 s) \{ \alpha_j \} + 2(a_4 s + c_2 s) \{ a_4 \{ \alpha_j \} + \{ c_2 \alpha_j \} \} + (a_1 s + c_1 s) \{ \alpha_j \} \} \\
 & (a_4 s + c_2 s)^2 [\phi_i \phi''_j] \{ \alpha_j \} \}_{x=s(t)} \\
 & + \int_0^1 [\phi_i \phi_j] \{ a_1 \{ \alpha_j \} + \{ c_1 \alpha_j \} \} + [\phi''_i \phi''_j] \{ \alpha_j \} dx = 0
 \end{aligned} \tag{5.14}$$

In (5.14), $c1s$, $c2s$, $c1\alpha_i$ and $c2\alpha_i$ are given by:

$$\begin{aligned} c1s &= -a_1 s^t - a_2 \dot{s}^t - a_3 \ddot{s}^t \\ c2s &= -a_4 s^t - a_5 \dot{s}^t - a_6 \ddot{s}^t \end{aligned} \quad (5.15)$$

$$\begin{aligned} c1\alpha_i &= -a_1 \alpha_i^t - a_2 \dot{\alpha}_i^t - a_3 \ddot{\alpha}_i^t \\ c2\alpha_i &= -a_4 \alpha_i^t - a_5 \dot{\alpha}_i^t - a_6 \ddot{\alpha}_i^t \end{aligned} \quad (5.16)$$

5.5 Large Oscillations–Displacement Formulation

The discretized nonlinear algebraic equations for this case are obtained by substituting (5.6) in (3.12)–(3.15) thus giving the following equations:

$$\begin{aligned} & s(a_1 + \omega^2) + c1s - \omega^2 s_e + ([\phi_i] \{a_1 \{\alpha_i\} + \{c1\alpha_i\}\} \\ & + [\phi_i \phi_j'] \{ \{a_1 \{\alpha_i\} + \{c1\alpha_i\}\} \{\alpha_j\} + \{a_1 \{\beta_i\} + \{c1\beta_i\}\} \{\beta_j\} \})_{x=s(t)} = 0 \\ & m \{ (a_1 s + c1s) \{\phi_i\} + [\phi_i \phi_j] \{a_1 \{\alpha_j\} + \{c1\alpha_j\}\} \\ & + [\phi_i \phi_j'] \{ (a_1 s + c1s) \{\alpha_j\} + 2(a_4 s + c2s) \{a_4 \{\alpha_j\} + \{c2\alpha_j\}\} \} \\ & (a_4 s + c2s)^2 [\phi_i \phi_j''] \{\alpha_j\} \}_{x=s(t)} \\ & + \int_0^1 [\phi_i \phi_j] \{a_1 \{\alpha_j\} + \{c1\alpha_j\}\} + [\phi_i'' \phi_j''] \{\alpha_j\} + [\phi_i' \phi_j] \{\lambda_j\} \\ & + \{\phi_i'\} [\phi_j \phi_k'] \{\lambda_j \alpha_k\} - \{\phi_i'\} [\phi_j'' \phi_k''] \{ \{\alpha_j \alpha_k\} + \{\beta_j \beta_k\} \} \\ & - \{\phi_i'\} [\phi_j' \phi_k'''] \{ \{\alpha_j \alpha_k \alpha_l\} + \{\alpha_j \beta_k \beta_l\} \} dx = 0 \\ & m \{ [\phi_i \phi_j] \{a_1 \{\beta_j\} + \{c1\beta_j\}\} \} \end{aligned}$$

$$\begin{aligned}
& + [\phi_i \phi'_j] \{ (a_1 s + c_1 s) \{\beta_j\} + 2(a_4 s + c_2 s) \{a_4 \{\beta_j\} + \{c_2 \beta_j\}\} \\
& \quad (a_4 s + c_2 s)^2 [\phi_i \phi''_j] \{\beta_j\} \}_{x=s(t)} \\
& + \int_0^1 [\phi_i \phi_j] \{a_1 \{\beta_j\} + \{c_1 \beta_j\}\} + [\phi''_i \phi''_j] \{\beta_j\} \\
& \quad + \{\phi'_i\} [\phi_j \phi'_k] \{\lambda_j \beta_k\} \\
& \quad - \{\phi'_i\} [\phi'_j \phi''_k \phi''_l] \{\{\beta_j \beta_k \beta_l\} + \{\beta_j \alpha_k \alpha_l\}\} dx = 0 \\
& \int_0^1 [\phi_i \phi'_j] \{\alpha_j\} + \frac{1}{2} \{\phi_i\} [\phi'_j \phi'_k] \{\{\alpha_j \alpha_k\} + \{\beta_j \beta_k\}\} dx = 0 \quad (5.17)
\end{aligned}$$

In (5.17), $c_1 s$, $c_2 s$, $c_1 \alpha_i$, $c_2 \alpha_i$ are same as in (5.15) and (5.16), and $c_1 \beta_i$ and $c_2 \beta_i$ are given by:

$$\begin{aligned}
c_1 \beta_i &= -a_1 \beta_i^t - a_2 \dot{\beta}_i^t - a_3 \ddot{\beta}_i^t \\
c_2 \beta_i &= -a_4 \beta_i^t - a_5 \dot{\beta}_i^t - a_6 \ddot{\beta}_i^t \quad (5.18)
\end{aligned}$$

5.6 Solution of Nonlinear Algebraic equations

The standard approach in solving systems of nonlinear algebraic equations is to use the Newton's method or its variants. Newton's method is difficult to apply when the number of equations are large as it requires computing the Jacobian. Obtaining the Jacobian analytically can be difficult in certain problems and requires a lot of computational effort. Another problem associated with finding the Jacobian which is very relevant in the beam carrying a moving mass problem is that we try to keep the basis functions simple and often barely meet the minimum smoothness requirements of the problem. In such cases, taking an extra derivative

in computing the Jacobian could make it discontinuous. In the present work, a sophisticated implementation of the Newton's method (supports line searching and back tracking)[102] was tried for the large oscillation model but without success (could not converge to the roots of the discretized equations of motion). Therefore in this work, a new technique is presented to solve the nonlinear algebraic equations of motion (5.9)–(5.17).

5.6.1 Iterative Scheme Formulation

The solution method is based on direct iteration of the nonlinear algebraic equations of motion (5.9)–(5.17) and does not involve finding the Jacobian explicitly. To illustrate this technique the nonlinear algebraic equations are represented in the following form:

$$[A]_n^{t+\Delta t} \{u\}_{n+1}^{t+\Delta t} + \{N\}_n^{t+\Delta t} = \{0\} \quad (5.19)$$

where u_i (elements of $\{u\}$) represents the accelerations in the acceleration formulation and displacements in the displacement formulation. The matrix $[A]$ is evaluated using the values of u_i at the n^{th} iteration step, then by solving the linear system (5.19) improved estimates of u_i are obtained. The key for convergence of this iterative scheme lies in the method for formulating the linear problem (5.19) from the nonlinear algebraic equations. This process involves extracting the linear variables u_i not only from the linear terms in the discretized algebraic equations but also from bilinear and multilinear terms (terms linear in any u_i). These terms, linear in u_i are written in the $[A]\{u\}$ form in (5.19) and the remaining nonlinear terms and constants are written in the $\{N\}$ matrix. Bilinear terms (or multilinear)

are accounted for twice (or as many times as the number of linear variables) in the $[A]$ matrix and the terms which have been multiplied accounted for are subtracted in the $\{N\}$ matrix to keep the equations the same.

The algorithm to obtain the numerical solution is presented in Section 5.6.2, here the formulation of the iterative scheme (5.19) is illustrated. First consider the geometric nonlinearities (terms within the integrals) in the second equation in (5.12). Each of these nonlinear terms are expanded as follows to fit the form (5.19):

$$\begin{aligned} \phi'_i \phi_j \phi'_k \lambda_j (a_5 \ddot{\alpha}_k + c2\alpha_k) = \\ \lambda_j \phi'_i \phi_j \phi'_k (a_5 \ddot{\alpha}_k + c2\alpha_k) + \ddot{\alpha}_j a_5 \phi'_i \phi'_j \phi_k \lambda_k - \phi'_i \phi_j \phi'_k a_5 \lambda_j \ddot{\alpha}_k \end{aligned} \quad (5.20)$$

$$\begin{aligned} \phi'_i \phi''_j \phi''_k (a_5 \ddot{\alpha}_j + c2\alpha_j) (a_5 \ddot{\alpha}_k + c2\alpha_k) = \\ \ddot{\alpha}_j 2a_5 \phi'_i \phi''_j \phi''_k ((1 - \delta_{jk}) \ddot{\alpha}_k + c1\alpha_k) \\ + \phi'_i \phi''_j \phi''_k (\delta_{jk} a_5^2 \ddot{\alpha}_j \ddot{\alpha}_k - (1 - \delta_{jk}) a_5^2 \ddot{\alpha}_j \ddot{\alpha}_k + c2\alpha_j \alpha_k) \end{aligned} \quad (5.21)$$

$$\begin{aligned} \phi'_i \phi'_j \phi''_k \phi''_l (a_5 \ddot{\alpha}_j + c2\alpha_j) (a_5 \ddot{\beta}_k + c2\beta_k) (a_5 \ddot{\beta}_l + c2\beta_l) = \\ \ddot{\alpha}_j a_5 \phi'_i \phi'_j \phi''_k \phi''_l (a_5 \ddot{\beta}_k + c2\beta_k) (a_5 \ddot{\beta}_l + c2\beta_l) \\ + \ddot{\beta}_j 2a_5 \phi'_i \phi''_j \phi''_k \phi'_l ((1 - \delta_{jk}) \ddot{\beta}_k + c2\beta_k) (a_5 \ddot{\alpha}_l + c2\alpha_l) \\ + \phi'_i \phi'_j \phi''_k \phi''_l \left(-(1 - \delta_{kl}) 2a_5^3 \ddot{\alpha}_j \ddot{\beta}_k \ddot{\beta}_l - (1 - \delta_{kl}) a_5^2 c2\alpha_j \ddot{\beta}_k \ddot{\beta}_l \right. \\ \left. + \delta_{kl} a_5^2 c2\alpha_j \ddot{\beta}_k \ddot{\beta}_l - 2a_5^2 \ddot{\alpha}_j \ddot{\beta}_k c2\beta_l + c2\alpha_j c2\beta_k c2\beta_l \right) \end{aligned} \quad (5.22)$$

$$\begin{aligned}
& \phi'_i \phi'_j \phi''_k \phi''_l (a_5 \ddot{\alpha}_j + c2\alpha_j)(a_5 \ddot{\alpha}_k + c2\alpha_k)(a_5 \ddot{\alpha}_l + c2\alpha_l) = \\
& \ddot{\alpha}_j a_5 \phi'_i \phi'_j \phi''_k \phi''_l (a_5(1 - \delta_{jk}) \ddot{\alpha}_k + c2\alpha_k)(a_5(1 - \delta_{jl}) \ddot{\alpha}_l + c2\alpha_l) \\
& + \ddot{\alpha}_j 2a_5 \phi'_i \phi'_j \phi''_k \phi''_l ((1 - \delta_{jk}) a_5 \ddot{\alpha}_k + c2\alpha_k)((1 - \delta_{jl}) a_5 \ddot{\alpha}_l + c2\alpha_l) \\
& + \phi'_i \phi'_j \phi''_k \phi''_l \left(-(1 - \delta_{jk})(1 - \delta_{jl}) 2a_5^3 \ddot{\alpha}_j \ddot{\alpha}_k \ddot{\alpha}_l + \delta_{jk} \delta_{jl} a_5^3 \ddot{\alpha}_j \ddot{\alpha}_k \ddot{\alpha}_l \right. \\
& \left. - (1 - \delta_{jk}) 2a_5^2 \ddot{\alpha}_j \ddot{\alpha}_k c2\alpha_l + \delta_{jk} 2a_5^2 \ddot{\alpha}_j \ddot{\alpha}_k c2\alpha_l \right. \\
& \left. - (1 - \delta_{kl}) a_5^2 c2\alpha_j \ddot{\alpha}_k \ddot{\alpha}_l + \delta_{kl} a_5^2 c2\alpha_j \ddot{\alpha}_k \ddot{\alpha}_l + c2\alpha_j c2\alpha_k c2\alpha_l \right) \quad (5.23)
\end{aligned}$$

In equations (5.20)–(5.23) the terms with bold type face $\ddot{\alpha}_j$, $\ddot{\beta}_j$, and $\ddot{\lambda}_j$ are written in the $[A]\{u\}$ form in (5.19) with $\ddot{\alpha}_j$, $\ddot{\beta}_j$, and $\ddot{\lambda}_j$ going in the u matrix and the remaining terms in (5.20)–(5.23) going in the column matrix N . It should be noted here that in this formulation the Lagrange multipliers λ_i are treated with accelerations (time derivatives of λ_i do not exist in the equations of motion) in the acceleration formulation and with displacements in the displacement formulation. Also as it can be seen from (5.20)–(5.23), index summation notation is implied and the Kronecker delta is used to select or eliminate nonlinear terms from the products of type $\alpha_i \alpha_j \alpha_k$ and $\alpha_i \alpha_j \alpha_k \alpha_l$. Similar expansions are used for the other terms in (5.9)–(5.17). The mass position dependent terms make the extraction of the linear terms in s or $\bar{s}$ from the basis functions ϕ_i and their products difficult. Therefore for these terms the iterative scheme cannot be fully implemented. The following equations illustrate the extraction of the linear terms from the nonlinear mass position dependent terms in the first and second equations of (5.12):

$$\phi_i \phi'_j \ddot{\alpha}_i (a_5 \ddot{\alpha}_j + c2\alpha_j) = \ddot{\alpha}_i \phi_i \phi'_j (a_5 \ddot{\alpha}_j + c2\alpha_j) \quad (5.24)$$

$$\begin{aligned}
\phi_i \phi_j' \ddot{s} (a_5 \ddot{\alpha}_j + c_2 \alpha_j) = \\
\ddot{s} \phi_i \phi_j' (a_5 \ddot{\alpha}_j + c_2 \alpha_j) + \ddot{\alpha}_j a_5 \ddot{s} \phi_i \phi_j' - \phi_i \phi_j' a_5 \ddot{s} \ddot{\alpha}_j
\end{aligned} \tag{5.25}$$

$$\begin{aligned}
2(a_2 \ddot{s} + c_1 s) \phi_i \phi_j' (a_2 \ddot{\alpha}_j + c_1 \alpha_j) = \\
\ddot{s} \phi_i \phi_j' 2a_2 (a_2 \ddot{\alpha}_j + c_1 \alpha_j) + \ddot{\alpha}_j \phi_i \phi_j' (2c_1 s c_1 \alpha_j - 2a_2^2 \ddot{s} \ddot{\alpha}_j) \\
+ \phi_i \phi_j' 2(c_1 s c_1 \alpha_j - a_2^2 \ddot{s} \ddot{\alpha}_j)
\end{aligned} \tag{5.26}$$

$$\begin{aligned}
\phi_i \phi_j'' (a_2 \ddot{s} + c_1 s)^2 (a_5 \ddot{\alpha}_j + c_2 \alpha_j) = \\
\ddot{s} 2a_2 c_1 s \phi_i \phi_j'' (a_5 \ddot{\alpha}_j + c_2 \alpha_j) + \ddot{\alpha}_j a_5 (a_2 \ddot{s} + c_1 s)^2 \phi_i \phi_j'' \ddot{\alpha}_j \\
+ \phi_i \phi_j'' ((a_2^2 \ddot{s}^2 + c_1 s^2) c_2 \alpha_j - 2a_2 a_5 c_1 s \ddot{s} \ddot{\alpha}_j)
\end{aligned} \tag{5.27}$$

The equations corresponding to (5.19) for the small and the large oscillation models, using the acceleration and the displacement formulations are presented in Appendix A.

5.6.2 Solution Algorithm

A unique feature of the solution strategy developed in this work is that the iterations in (5.19) are carried out until the semi-discretized ordinary differential equations (equations (3.9) and (3.10) for the small oscillation model and (3.12)–(3.15) for the large oscillation model) are satisfied and not the discretized nonlinear algebraic equations (equations (5.9) and (5.14) for the small oscillation model acceleration formulation and displacement formulation, respectively and (5.12) and (5.17) for

the large oscillation model acceleration formulation and displacement formulation, respectively). Therefore the question of the approximation (convergence) of the solution of the discretized nonlinear algebraic equations to the semi-discretized ordinary differential equations does not arise, as the solution obtained always satisfies the ordinary differential equations within the allowed tolerance. This however does not imply that the errors due to temporal discretization are totally eliminated; they are however significantly reduced. To illustrate this point; the iterative scheme (5.19) gives the values of accelerations at $t + \Delta t$ in the acceleration formulation and displacements at $t + \Delta t$ in the displacement formulation. The remaining state variables (displacement and velocity in the acceleration formulation and velocity and acceleration in the displacement formulation) are obtained from the discretization scheme (equations (5.3) for the acceleration formulation and (5.6) for the displacement formulation). Therefore at each time step, the displacement, velocity and acceleration satisfies the discretization scheme as well as the ordinary differential equations. This provides an automatic check on whether or not the time step is too large as the iterative scheme would then simply not converge. To further minimise propagation of error due to temporal discretization, after the solution has converged to the ordinary differential equations, the accelerations are determined from the ordinary differential equations by solving a linear system (as the equations of motion are linear in accelerations and the Lagrange multiplier λ_i). This last step performed using LU decomposition [102] is of much higher accuracy (related to precision of the algorithm and machine characteristics) compared to the tolerance allowed in satisfying the ordinary differential equations.

An algorithm for the implementation of the solution method developed is outlined below:

Algorithm 5.1 *Algorithm to obtain the numerical solution*

- Step 1** Set up the initial displacements and velocities in u^t , and $\dot{u}^t$ using (2.45) and (2.47). Note that the displacements for the beam refer to all the nodal variables $\alpha_i, \alpha'_i, \alpha''_i, \beta_i, \beta'_i, \beta''_i$ and for the displacement formulation it also includes λ_i, λ'_i , and λ''_i . In the acceleration formulation the λ_i, λ'_i , and λ''_i are stored in the acceleration vector $\ddot{u}^t$.
- Step 2** Find the initial acceleration $\ddot{u}^t$ using the semi-discretized equations of motion (equations (3.9), (3.10) for the small oscillation model and (3.12)-(3.14) for the large oscillation model). This step is performed using the LU decomposition [102].
- Step 3** Find $c1u$ and $c2u$ as defined in the discretization scheme (5.5) or (5.8). Also set $u^{t+\Delta t}$, $\dot{u}^{t+\Delta t}$, and $\ddot{u}^{t+\Delta t}$ equal to u^t , $\dot{u}^t$, and $\ddot{u}^t$, respectively.
- Step 4** Obtain an improved estimate for $\ddot{u}^{t+\Delta t}$ for the acceleration formulation and $u^{t+\Delta t}$ for the displacement formulation by solving the system of linear algebraic equations outlined in (5.19). LU decomposition method is used for this step.
- Step 5** Obtain the displacements and velocities at $t + \Delta t$ if the acceleration formulation is used (using equations (5.3)) and velocities and accelerations at $t + \Delta t$ if the displacement formulation is used (using equations (5.6)).

Step 6 Substitute $u^{t+\Delta t}$, $\dot{u}^{t+\Delta t}$ and $\ddot{u}^{t+\Delta t}$ in the ordinary differential equations of motion (equations (3.9), (3.10) for small oscillation model and (3.12)-(3.15) for the large oscillation model. Denoting this vector by R , the Euclidian norm $\|R\|$ (distance from the origin squared) is tested against the allowed tolerance. If it does not meet the tolerance go to step 4. If it meets the tolerance go to step 7.

Step 7 Find the acceleration $\ddot{u}^{t+\Delta t}$ using the semi-discretized equations of motion (equations (3.9), (3.10) for small oscillation model and (3.12)-(3.14) for the large oscillation model) using the values of $\dot{u}^{t+\Delta t}$, $\ddot{u}^{t+\Delta t}$. Set u^t , $\dot{u}^t$, and $\ddot{u}^t$ equal to $u^{t+\Delta t}$, $\dot{u}^{t+\Delta t}$, and $\ddot{u}^{t+\Delta t}$, respectively and go to step 3 until the desired length of the time series is obtained.

5.7 Numerical Simulations and Analysis

Employing the solution strategy presented in Section 5.6.2, numerical results are obtained for the small and the large oscillation models of the system. To compare the numerical results with a known analytical solution, first a linear Euler-Bernoulli beam is considered. Then the small oscillation model for the mass-beam system is analysed using the new finite element based solution strategy employing the acceleration formulation (Algorithm 5.1). Results for this case are compared with the automatic ODE solver approach used in Chapter 4. This analysis is limited to the small oscillation model, as the automatic ODE solver approach works only for this case. For convenience the computational path employing finite element basis

functions for spatial discretization, the acceleration formulation for the temporal discretization and solution of the resulting nonlinear algebraic equations using the new iterative technique developed will be referred to as the acceleration formulation and similarly the computational path employing spatial discretization using the finite element basis functions and temporal discretization using the displacement formulation and solution of the nonlinear algebraic equations using the new iterative technique will be referred to as the displacement formulation. The advantages of using the acceleration formulation over the displacement formulation are also pointed out. The difference between the linear and the nonlinear beam models are analysed by considering the beam alone without the moving mass and finally, results are presented for the mass beam system undergoing large oscillations.

5.7.1 Analytical and Numerical Solution – Linear Beam

In this section, a linear Euler-Bernoulli cantilever beam is considered. An analytical solution based on linear mode shapes is compared with the numerical solution obtained using the acceleration formulation and the displacement formulation. The main goal of this analysis is to determine the effect of using different numbers of elements for the spatial discretization and different step sizes for temporal discretization and observe how the results compare with the analytical solution. A linear beam model is used, as a reliable analytical solution is available only for this case. The analytical solution is based on four modes of a cantilever beam and is

System: Cantilever Beam			
	Initial Values equation (2.46) $v_{t0} = 0.1$		Initial Values equation (2.46) $v_{t0} = 0.3$
Figure	5.1	5.9–5.12	5.13–5.16
Model	NM3,NM4	NM4,NM5	NM4,NM5
Δt	0.005	0.005	0.005
Spectrogram			
No. of Segments		32	32
Segment Size		512	512
Power Spectrum			
No. of FFTs	6	6	6
Segment Size	16384	16384	16384

Table 5.1: Simulation parameters – Cantilever beam

given by the following equation (refer to [30] for derivation):

$$v(1, t) = 2A_1 \cos(\omega_1 t) - 2A_2 \cos(\omega_2 t) + 2A_3 \cos(\omega_3 t) - 2A_4 \cos(\omega_4 t) \quad (5.28)$$

where nondimensionalized parameters are assumed. Equation (5.28) is the solution for the tip of the beam. The first four frequencies are $\omega_1 = 3.51602$, $\omega_2 = 22.0345$, $\omega_3 = 61.6972$, and $\omega_4 = 120.9019$. Using (2.46) as the initial value, the constants A_i are given by $A_1 = 0.5067v_{t0}$, $A_2 = 0.0072v_{t0}$, $A_3 = 0.0005v_{t0}$ and $A_4 = 0.0001v_{t0}$. Figure 5.1 compares the power spectrums obtained using the analytical and numerical solutions. Figure 5.1(a) shows the comparison between the analytical and the numerical solution obtained using the acceleration formulation and ten finite elements. The parameters for these simulations are tabulated in Table 5.1.

Figure 5.1 shows that the frequencies obtained using the analytical and the

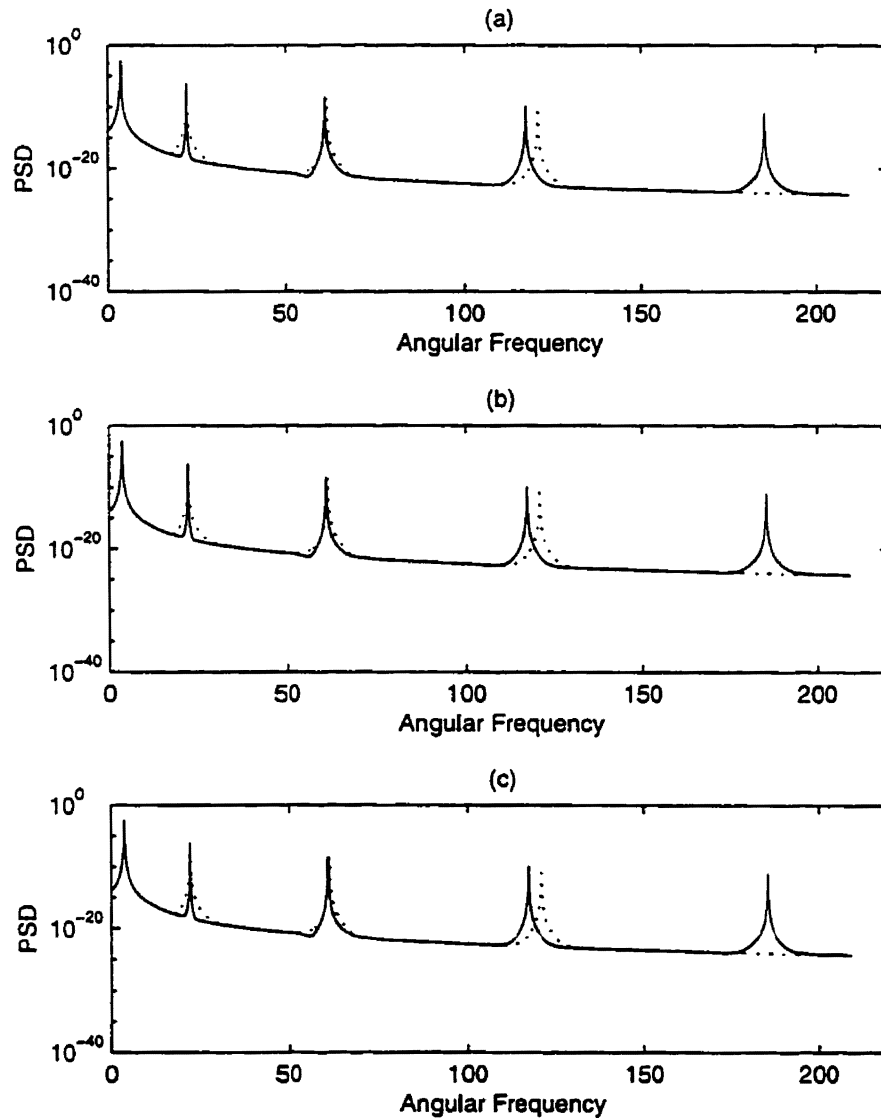


Figure 5.1: Comparison of power spectrums for the analytical and the numerical solutions of the linear beam, $v_{t0} = 0.1$ (a) ($\cdots$) Analytical solution, ($-$) Numerical solution using the acceleration formulation with ten elements, (b) ($\cdots$) Analytical solution, ($-$) Numerical solution using the displacement formulation with ten elements, (c) ($\cdots$) Analytical solution, ($-$) Numerical solution using the acceleration formulation with twenty elements.

numerical method match very well up to the third mode frequency but there is a small difference in the fourth mode frequency. The error in the fourth mode frequency and other higher frequencies can be reduced by decreasing the step size at the expense of computational effort. However these higher frequencies do not effect the dynamic behaviour of the system significantly and as long these errors do not propagate and make the integration process difficult it is justified to use a step size which gives accurate results only for the first few frequencies. In all the simulations presented in this chapter, using the finite element approach a step size of 0.003 or 0.005 is used. In general the step size of 0.003 is used for the mass-beam system whereas 0.005 is used when only the beam is considered. Also using these step sizes it was found that the number of iterations required for the convergence of the iterative scheme are minimal (one iteration for the linear beam model, two iterations for the nonlinear beam model).

Figure 5.1(b) compares the analytical solution and the numerical solution obtained using the displacement formulation with ten elements, and Figure 5.1(c) compares the analytical solution and the numerical solution obtained using the acceleration formulation with twenty elements. It can be seen from these figures that there is no significant difference as a result of increasing the number of elements for the first five frequencies shown, small differences were however observed in the higher frequency range. It must be mentioned here that the power spectrums presented in this thesis do not show the full frequency spectrum, only a small portion of the spectrum is plotted to show the most significant frequencies clearly. It can also be noted from Figure 5.1 that there is no peak for the fifth mode in the analyti-

cal solution as in the numerical solution because only four frequencies are modelled analytically. The time response for these simulations are not shown as there are no discernible differences between them.

5.7.2 Automatic ODE Solver and Acceleration Formulation Method – Kinematic Nonlinearities

In this section, comparison is made between the results obtained using the acceleration formulation method and the automatic ODE solver approach used in Chapter 4. The small oscillation model for the mass-beam system is used, as the automatic ODE solver approach was found to work only for this case. The difference in the system model between the two methods is that in the automatic ODE solver approach the four mode shapes of a simple cantilever beam are used as the basis functions whereas in the finite element approach piecewise defined fifth degree Hermitian polynomials are used as the basis functions. Due to different basis functions, it is expected that the fundamental frequencies for these two cases would also differ slightly. However to establish internal resonance the value of the fundamental frequency of system as obtained using twenty finite elements (see Section 4.4 for further details) is used for both cases. As in Chapter 4, the results are also compared with the perturbation solution by selecting a detuning parameter to match the numerical results (see Section 4.3 for additional information).

Figures 5.2 and 5.3 are obtained using the acceleration formulation and Figures 5.4 and 5.5 are obtained using the automatic ODE solver approach for the same set of parameters. The various parameters for these sets of simulations are tabulated

Parameter Set 1				
$m = 1.0$				
See Tables 4.2 and 4.4 for more simulations using similar parameters.				
	Initial Values equation (2.46) $s_0 = 0.90001, v_{t0} = 0.1$		Initial Values equation (2.46) $s_0 = 0.53, v_{t0} = 0.000001$	
Figure	5.2, 5.3, 5.6	5.4, 5.5, 5.6	5.7	5.8
Model	PM1,NM3	PM1,NM2	NM3	NM2
s_e	0.9	0.9	0.5	0.5
ω_1 (FEM N=20)	1.758689	1.758689	2.891228	2.891228
ω_1 (One mode)	1.763542	1.763542	2.908776	2.908776
σ	-0.0098	-0.0009		
Δt	0.003	0.00296	0.003	0.00666
Spectrogram				
No. of Segments	64	64		
Segment Size	256	256		
Power Spectrum				
No. of FFTs	6	6		
Segment Size	32768	32768		

Table 5.2: Comparison between automatic ODE solver and FEM approach

in Table 5.2. Comparison of the time responses (for example Figure 5.2(b) and Figure 5.4(b)) shows different beating periods for the two cases. This difference is due to small changes in the fundamental frequency between the two models, which is also reflected in the values of the detuning parameter σ (equals to -0.0098 for the finite element approach and -0.0009 for the automatic ODE solver approach). It has been established using parametric analysis (see Figure 4.11) that small changes in σ near perfect internal resonance conditions can cause large changes in the beating periods.

Figures 5.2–5.5 (c)–(e) show the time frequency analysis results. There is no

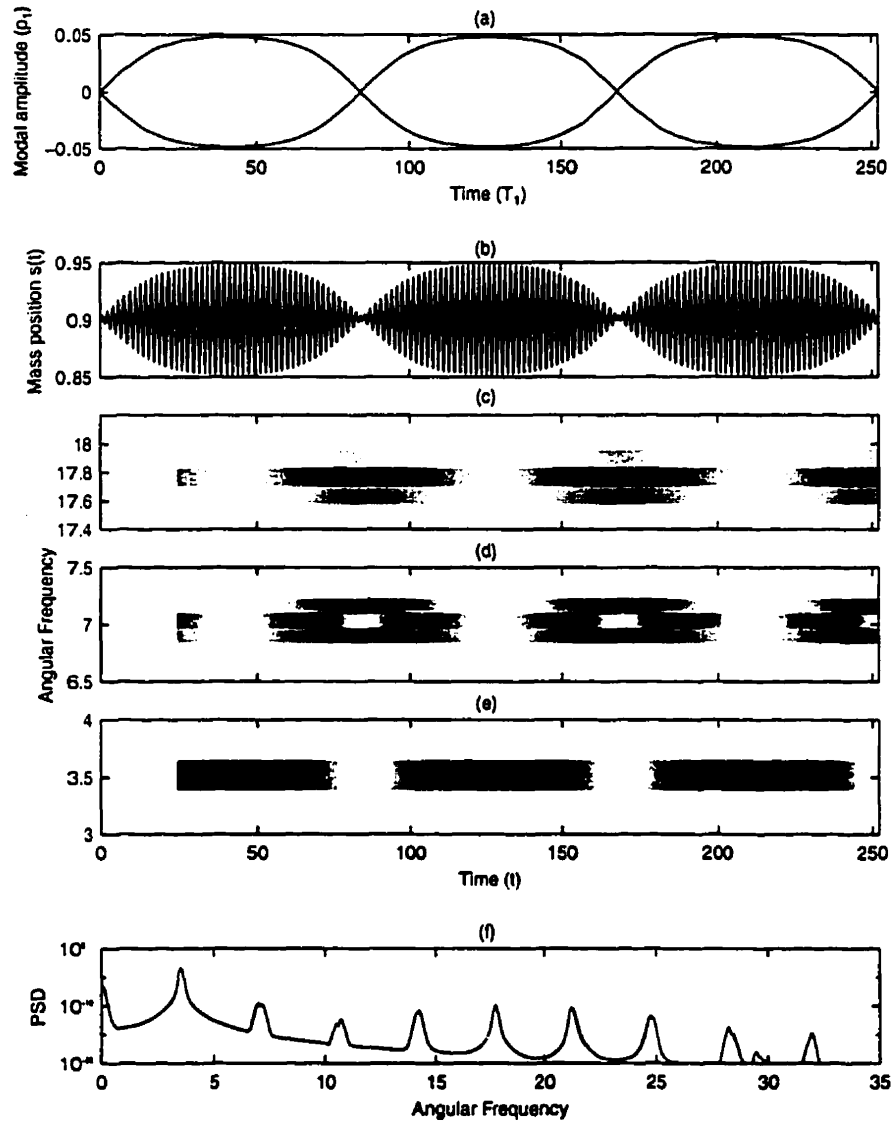


Figure 5.2: Mass response – 1:2 IR, $m = 1.0$, $s_e = 0.9$, $s_{10} = 0.90001$, and $v_{t0} = 0.1$. (a) Perturbation solution $\sigma = -0.0098$, (b) Numerical solution, (c)–(e) Spectrograms, and (f) Power spectrum.

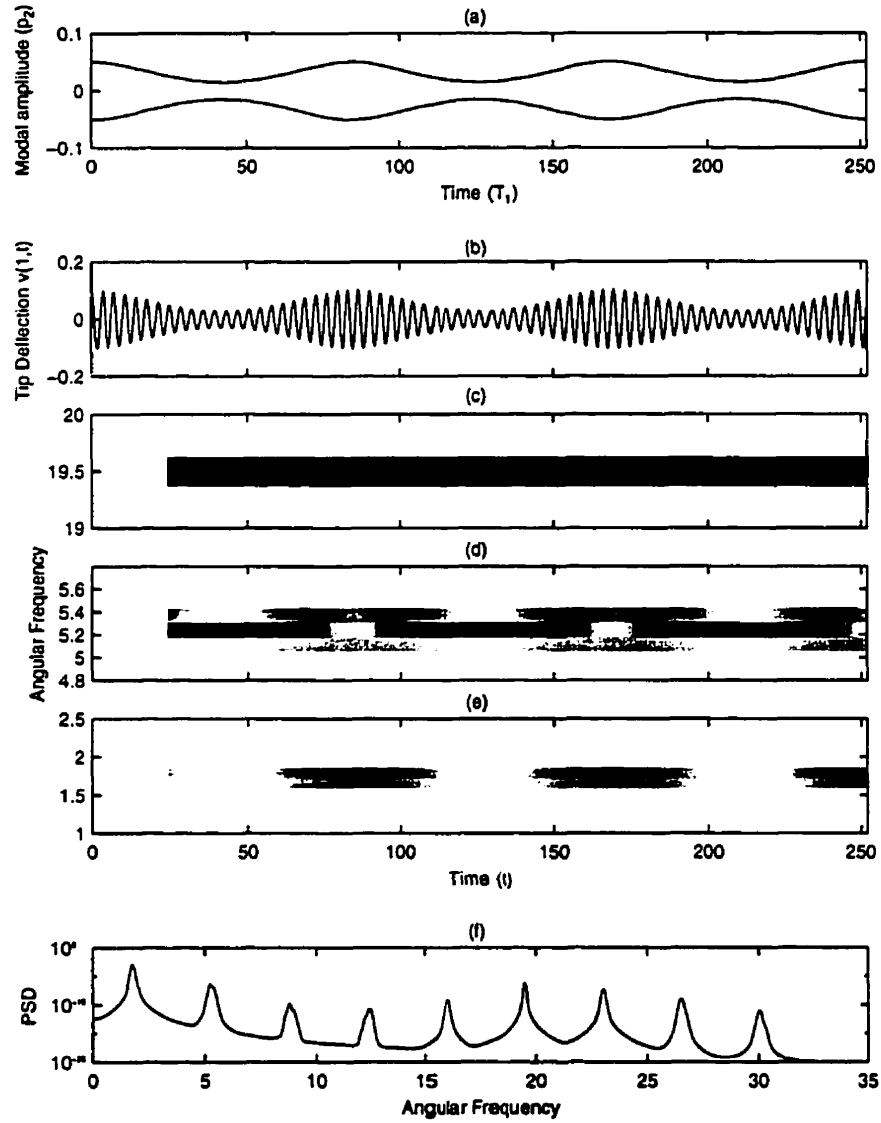


Figure 5.3: Tip deflection - 1:2 IR, $m = 1.0$, $s_e = 0.9$, $s_{10} = 0.90001$, and $v_{t0} = 0.1$. (a) Perturbation solution $\sigma = -0.0098$, (b) Numerical solution, (c)–(e) Spectrograms, and (f) Power spectrum.

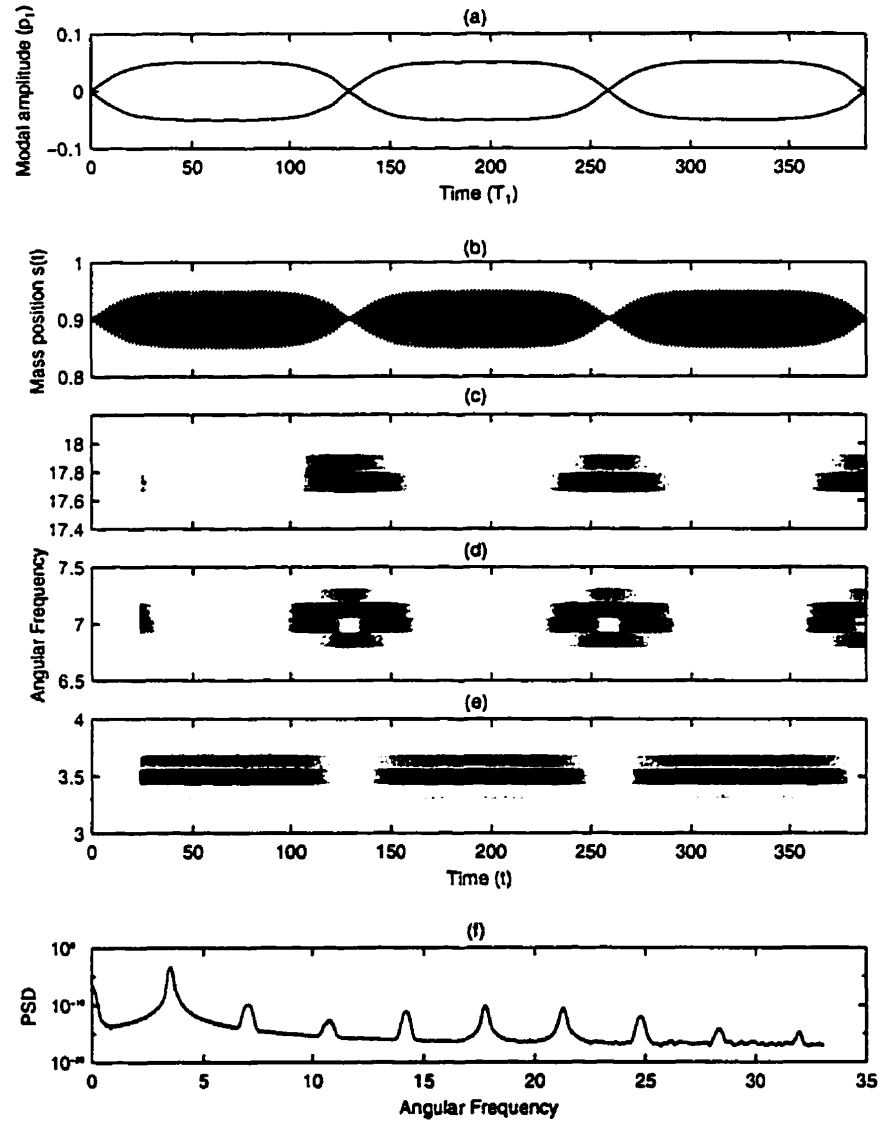


Figure 5.4: Mass response - 1:2 IR, $m = 1.0$, $s_e = 0.9$, $s_{10} = 0.90001$, and $v_{t0} = 0.1$. (a) Perturbation solution $\sigma = -0.0009$, (b) Numerical solution, (c)–(e) Spectrograms, and (f) Power spectrum.

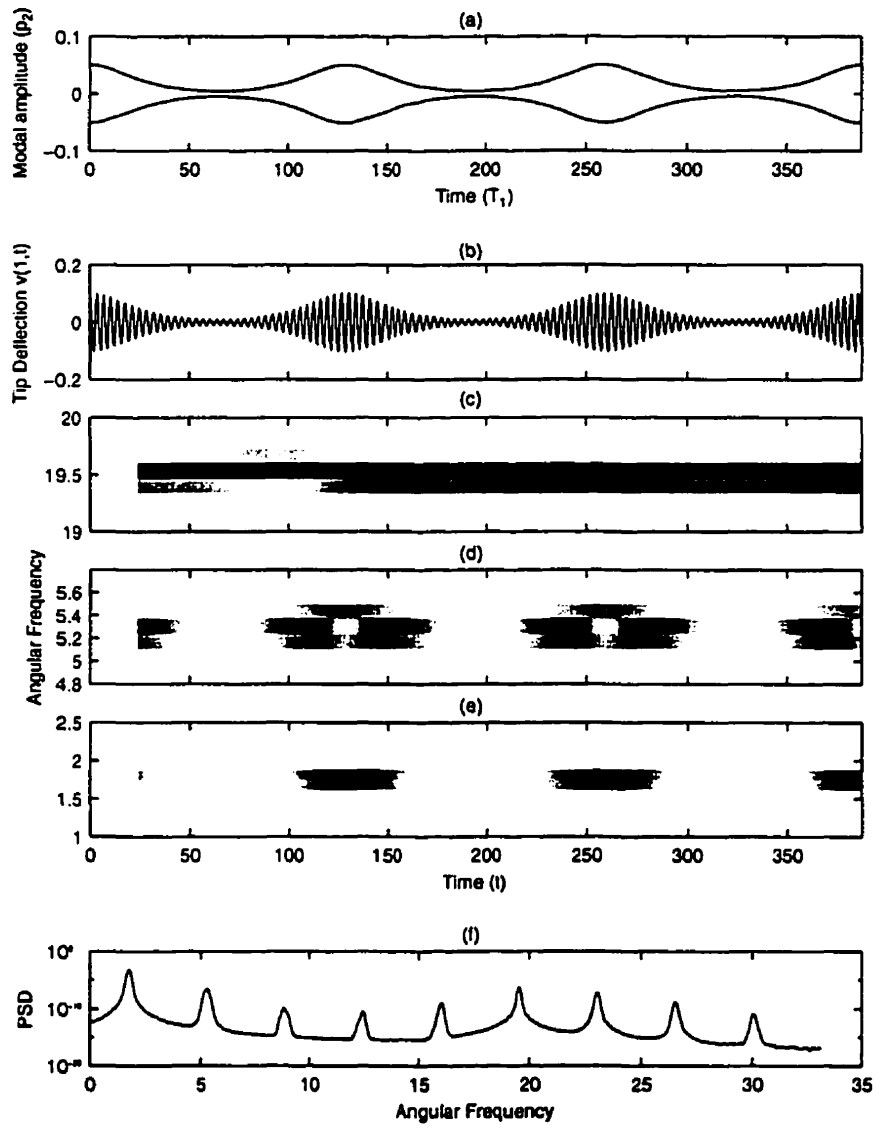


Figure 5.5: Tip deflection – 1:2 IR, $m = 1.0$, $s_e = 0.9$, $s_{10} = 0.90001$, and $v_{t0} = 0.1$. (a) Perturbation solution $\sigma = -0.0009$, (b) Numerical solution, (c)–(e) Spectrograms, and (f) Power spectrum.

significant difference between the automatic ODE solver approach and the finite element based method for the time variation of the frequencies. The power spectrums in Figures 5.2–5.5 (f) show that the major frequency peaks are at the same values for both the finite element based method and the automatic ODE solver approach. Figure 5.6 shows this more clearly, where the power spectrums are plotted on the same graph for ease of comparison.

At this point another comparison can be made, between Figures 4.15 and 4.16 and Figures 5.4 and 5.5. Both of these simulations are for the same parameters except that for the initial deflection of the beam, in Figures 4.15 and 4.16 equation (2.47) is used whereas in Figures 5.4 and 5.5 equation (2.46) is used. There are no significant differences in the responses obtained using the two different initial values. This is not surprising as in (2.46) the contribution to the higher modes is very small and in (2.47) the only contribution is to the first mode.

Figures 5.7 and 5.8 show another set of simulations obtained using the acceleration formulation method and the automatic ODE solver approach, respectively. The parameters in Figures 5.7 and 5.8 are also the same as those used in Figures 4.21 and 4.22 except for different initial condition for the beam (equation (2.46) is used in Figures 5.7 and 5.8 and equation (2.47) is used in Figures 4.21 and 4.22). The results obtained using the finite elements approach (see Figure 5.7(b)) show a smoother appearance of the peaks more consistent with the perturbation solution as compared to the automatic ODE solver approach. Figure 5.7(c) shows the number of iterations required at each time step in the integration process using the acceleration formulation. As the beam model is linear, only one iteration is

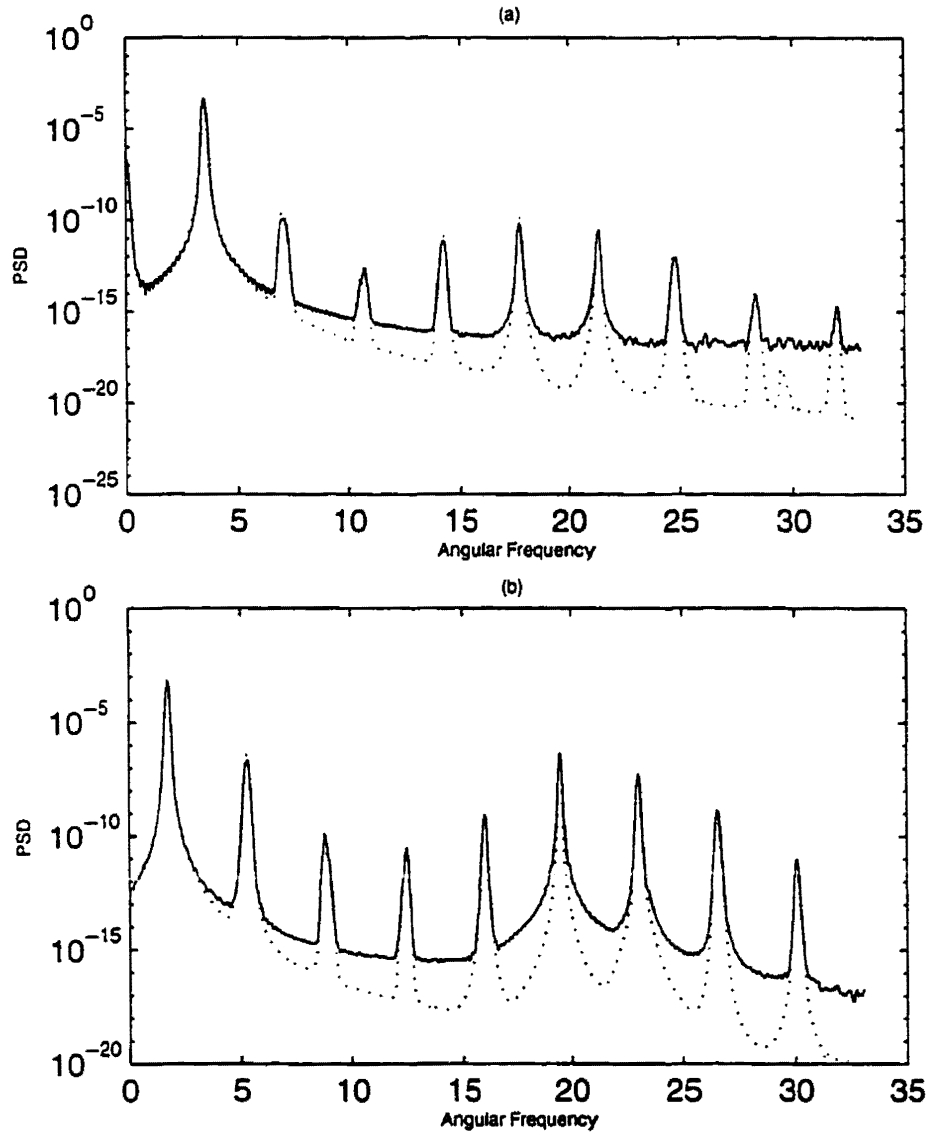


Figure 5.6: Comparison of power spectrums - 1:2 IR, $m = 1.0$, $s_e = 0.9$, $s_{10} = 0.90001$, and $v_{10} = 0.1$, (...) finite element approach, (—) automatic ODE solver approach (a) for the moving mass, (b) for the tip deflection.

required at each time step.

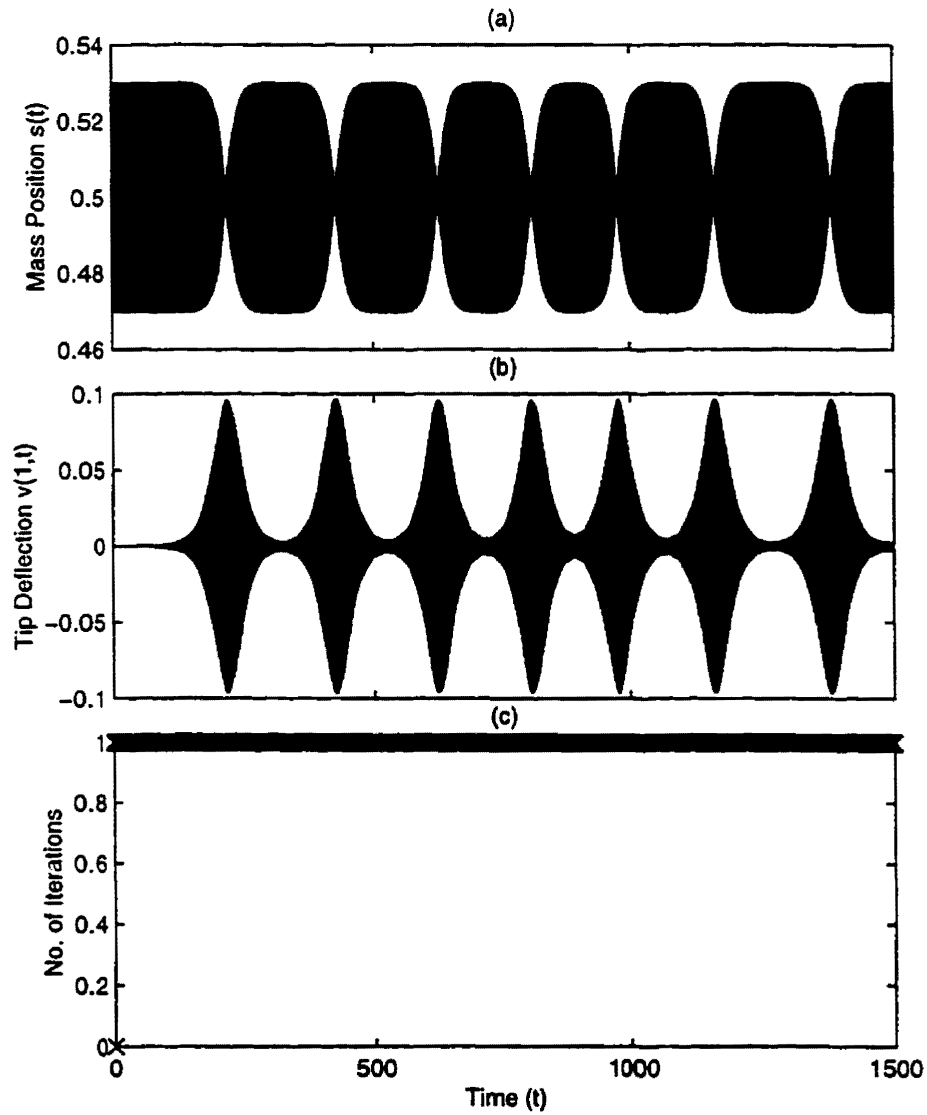


Figure 5.7: Numerical solution - 1:2IR, Acceleration formulation, $m = 1.0$, $s_e = 0.5$, $s_{10} = 0.53$, and $v_{k0} = 0.000001$. (a) Mass response, (b) Tip deflection, (c) Number of iterations at each time step.

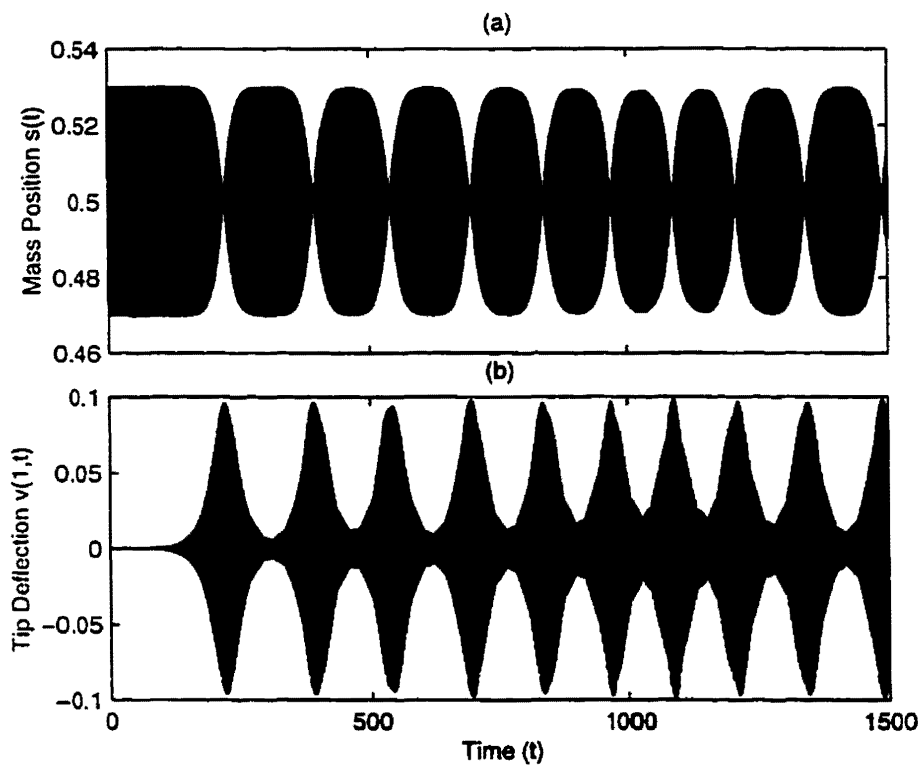


Figure 5.8: Numerical solution - 1:2IR, Automatic ODE solver, $m = 1.0$, $s_e = 0.5$, $s_{10} = 0.53$, and $v_{t0} = 0.000001$. (a) Mass response, (b) Tip deflection.

5.8 Comparison Between Linear and Nonlinear Beam Models

The focus of this section is to investigate the difference between the linear and the nonlinear beam response. Only the beam is considered without the moving spring-mass sub-system. Numerical simulations are obtained using the acceleration formulation method. Figures 5.9–5.12 show the results for the beam undergoing free vibrations given a small initial tip deflection (0.1) to both the linear and the nonlinear beam models. Later in Figures 5.13–5.16 the response is investigated by giving large initial tip deflection to both the linear and the nonlinear beam models. The various parameters for these figures are tabulated in Table 5.1.

Figure 5.9 shows the time response. The graphs are plotted only for a small length of time to see if there are any noticeable differences between the two cases but the actual simulation was carried out over a large time range ($t=0\ldots600$). The time response does not show any significant difference between the linear and the nonlinear models. Some small differences like flattening of some of the peaks for the nonlinear beam (Figure 5.9(b)) can be noted. The differences between the two models become more apparent in the power spectrums shown in Figure 5.10. The power spectrum for the nonlinear beam (5.14(b)) shows that in addition to the five frequency peaks which appear in the linear case, the nonlinear response has more frequencies which are however less significant (as indicated by the PSD) than those corresponding to the linear model. It should also be noted here that in the periodograms the full spectrum is not shown and only a small portion is plotted

for clarity.

Figure 5.11 shows the time frequency analysis results and Figure 5.12 shows the number of iterations required at each time step using the acceleration formulation. It can be seen from these figures that the frequencies are stationary with respect to time and for the linear case (Figure 5.12(a)) only one iteration is required at each time step whereas for the nonlinear case (Figure 5.12(b)) the solution converges in only two iterations. Thus the number of iterations required are minimal with a step size of 0.005. Increasing the step size increases the number of iterations required and can prevent the iteration scheme from converging (convergence of the solution of the nonlinear algebraic equations to semi-discretized ordinary differential equations see Section 5.6.2 for further discussion).

In Figures 5.9–5.12 a small initial tip deflection was given to both the small and large oscillation model based beams, now consider the other case where a large initial tip deflection is given to both models. Figures 5.13–5.16 show the results. As in Figure 5.9, the time response (Figure 5.13) shows small differences in the shape of the peaks for the two models. The difference in the time response becomes more apparent in the power spectrums (Figure 5.14). The first five frequency peaks shown for the linear response (Figure 5.14(a)) also appear in the nonlinear beam response (Figure 5.14(b)), along with additional frequencies which were not found in the linear case. It can also be noted from Figure 5.14(b) that one of these additional frequency peaks (third peak) in the nonlinear model response is more significant than the linear third mode frequency peak. The dependence of the frequencies on the initial values for the nonlinear model can be observed by comparing Figures

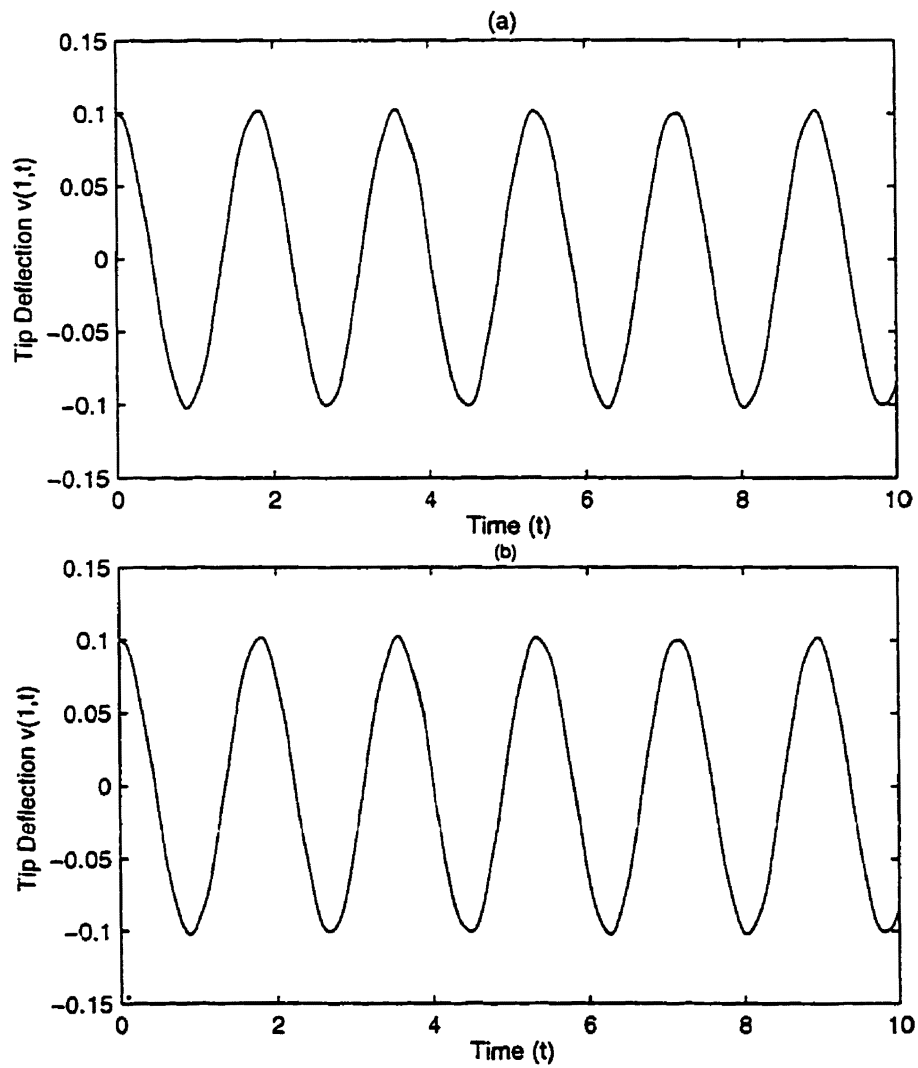


Figure 5.9: Tip deflection - Time response, $v_{t0} = 0.1$, (a) Small oscillation model, (b) Large oscillation model.

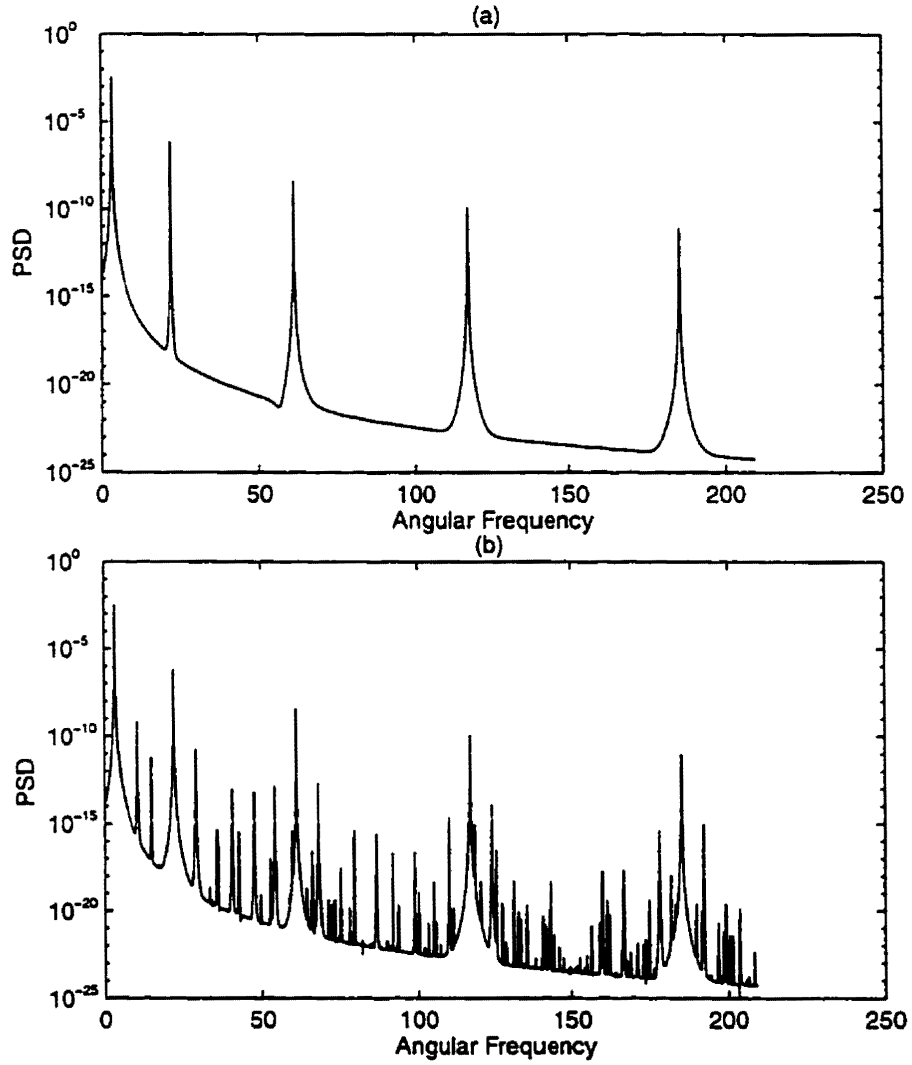


Figure 5.10: Tip deflection - Power spectrums, $v_{t0} = 0.1$, (a) Small oscillation model, (b) Large oscillation model.

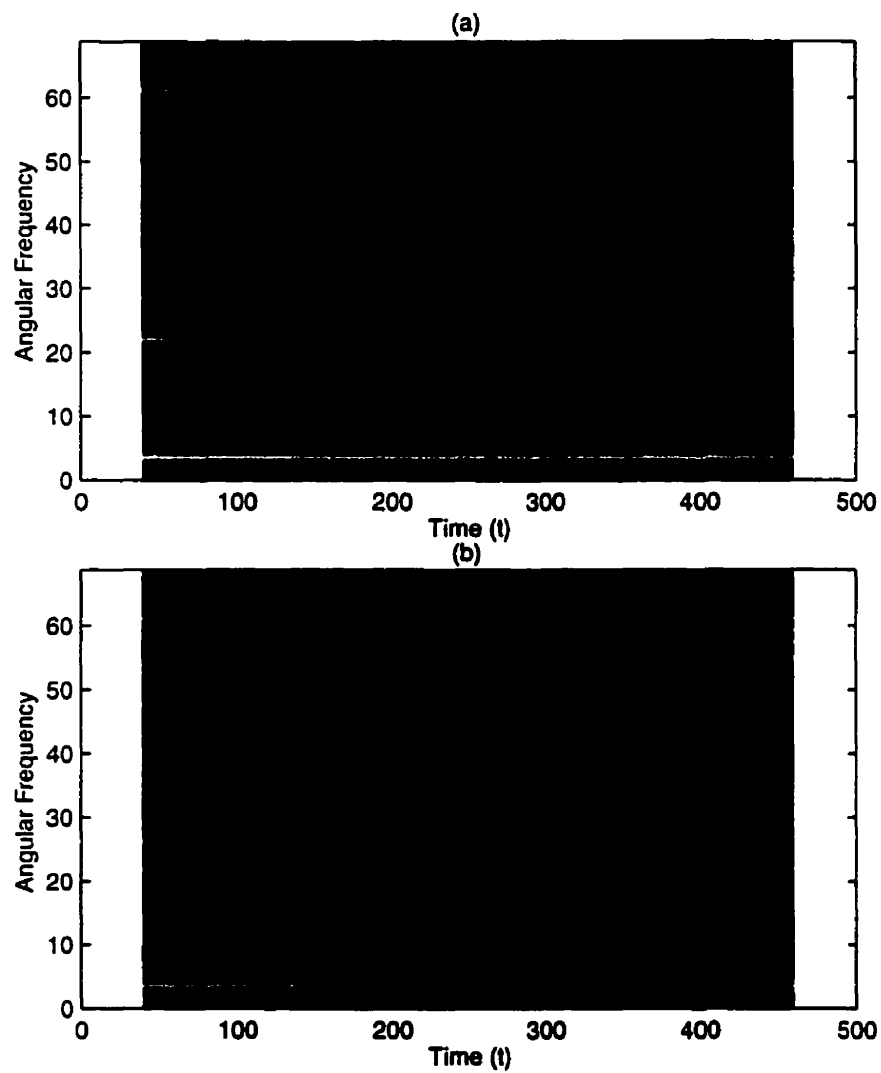


Figure 5.11: Tip deflection – Time frequency response, $\nu_{t0} = 0.1$, (a) Small oscillation model, (b) Large oscillation model.

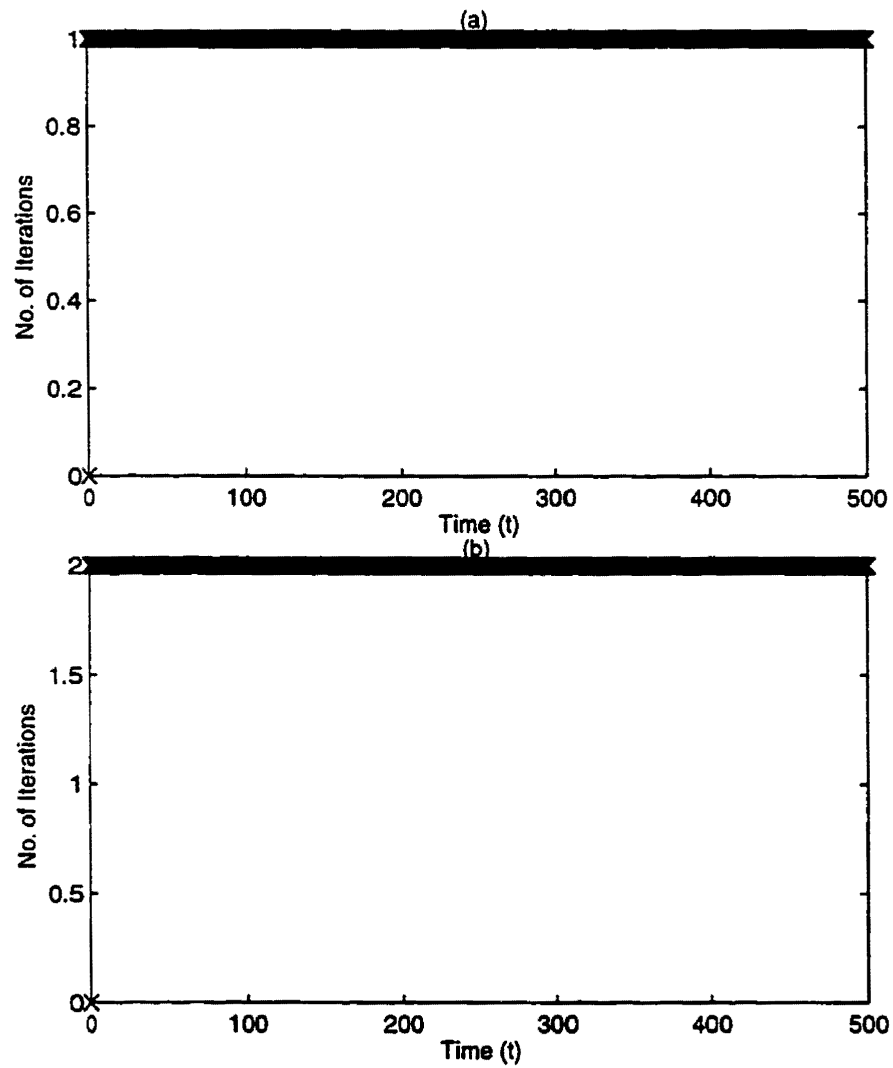


Figure 5.12: Number of iterations, $v_{t0} = 0.1$, (a) Small oscillation model, (b) Large oscillation model.

5.10(b) and 5.14(b). There are small differences in the frequencies between the small initial value and the large initial value cases for the nonlinear model and in addition there are more frequency peaks for the large initial value case and are at a higher power spectral density. No such dependence on the initial values exists for the linear beam model. Figure 5.15 shows the time frequency variation where the additional frequencies for the nonlinear model can be clearly seen and Figure 5.16 compares the number of iterations required at each time step. As can be seen from Figure 5.16 the number of iterations for the linear and the nonlinear beam undergoing large oscillations using the acceleration formulation are minimal (one for the linear beam and two for the nonlinear beam except for the first step which took three iterations).

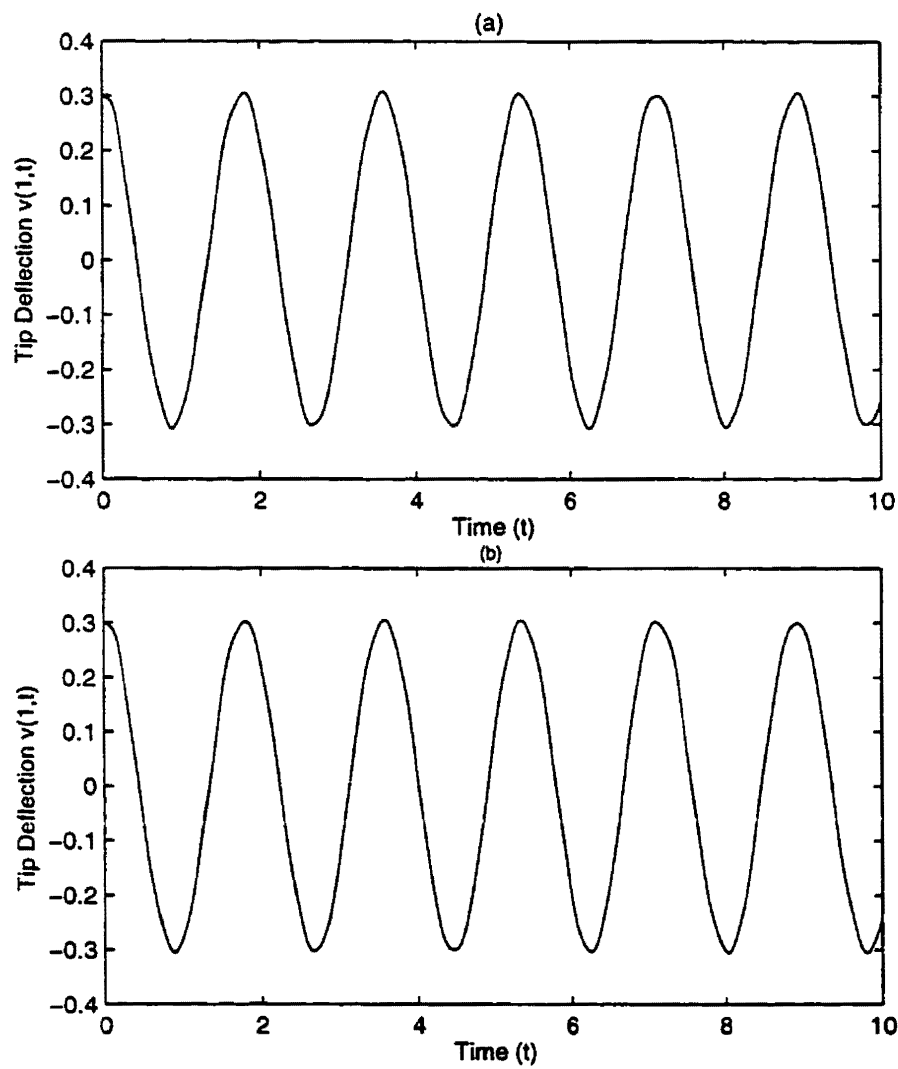


Figure 5.13: Tip deflection - Time response, $v_{t0} = 0.3$, (a) Small oscillation model, (b) Large oscillation model.

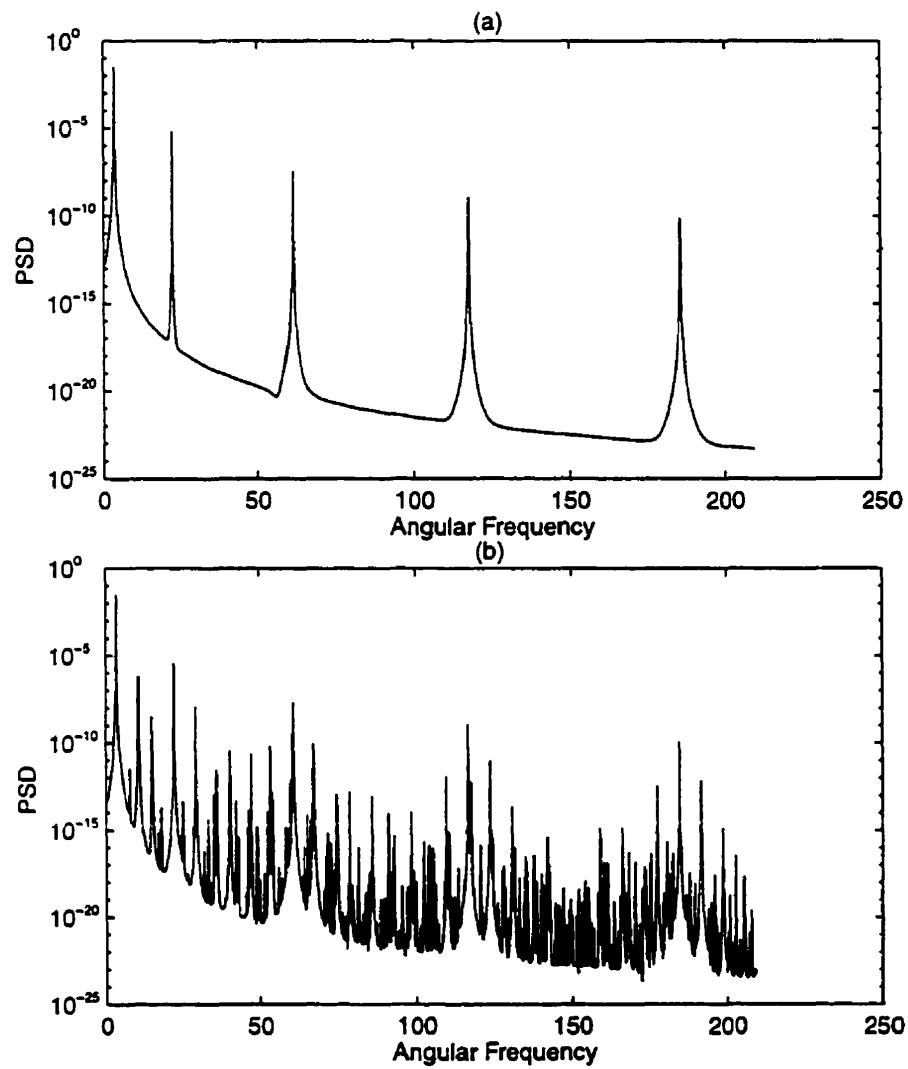


Figure 5.14: Tip deflection - Power spectrums, $v_{t0} = 0.3$, (a) Small oscillation model, (b) Large oscillation model.

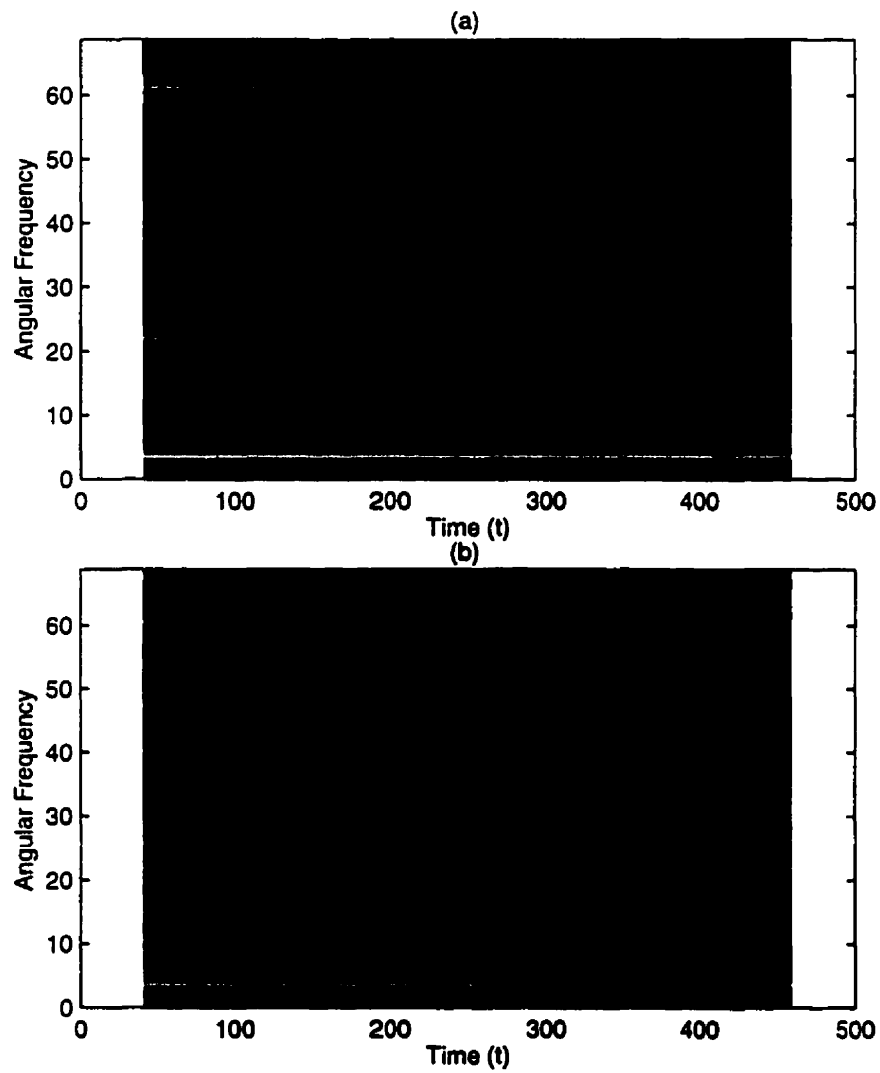


Figure 5.15: Tip deflection – Time frequency response, $v_{t0} = 0.3$, (a) Small oscillation model, (b) Large oscillation model.

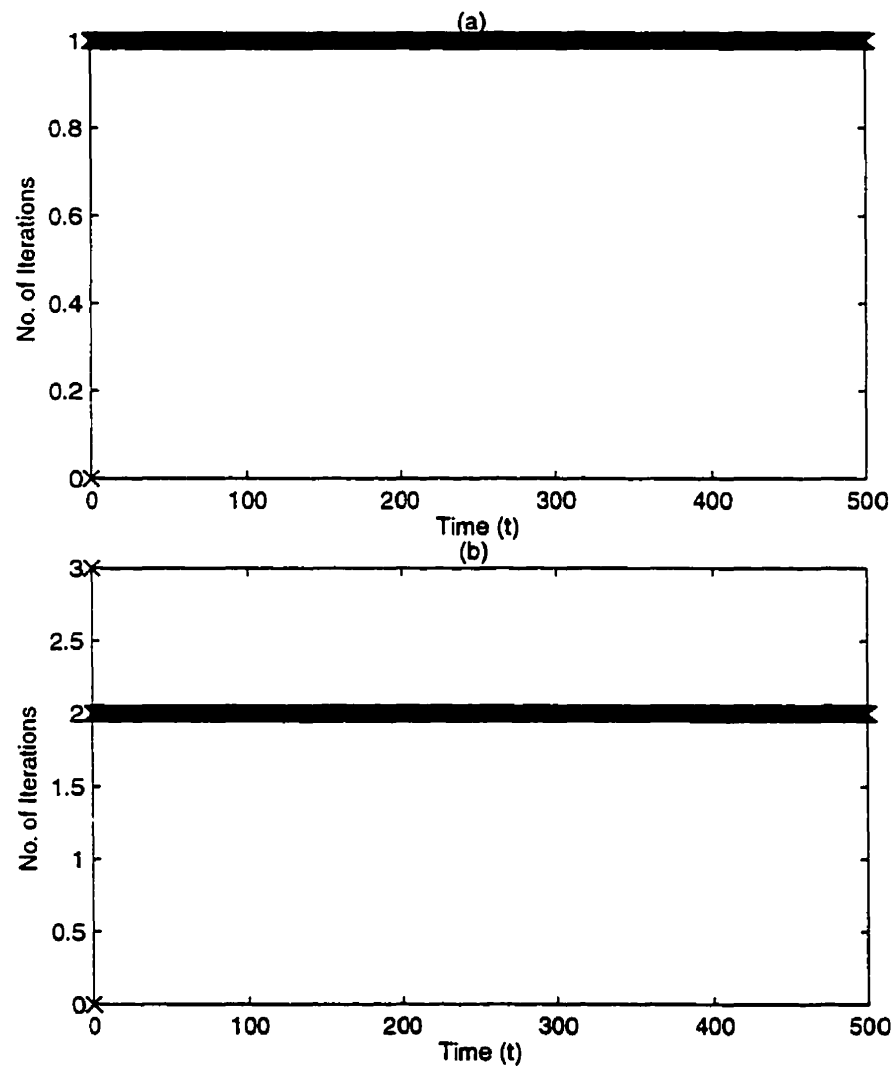


Figure 5.16: Number of iterations, $v_{t0} = 0.3$, (a) Small oscillation model, (b) Large oscillation model.

5.9 Displacement Formulation Versus Acceleration Formulation

As outlined in Algorithm 5.1 both the displacement and the acceleration formulation converge to the same semi-discretized ordinary differential equations. Therefore the solution using the two methods is identical. The difference between the techniques lies in how many iterations it takes for the initial estimate of the solution to converge. It was found that the displacement formulation takes many more iterations to converge than the acceleration formulation. Reducing the step size decreases the number of iterations required but increases the computational effort. It was also computationally prohibitive to obtain simulations for longer times ($t > 500$) for the large oscillation model with large initial values, like all the other simulations presented in this work. This reflects the superiority of the convergence of the acceleration formulation over the displacement formulation.

Figure 5.17 compares the number of iterations at each time step for the displacement and the acceleration formulation. The results are obtained for the large oscillation model of the beam with an initial tip deflection of 0.1. The same parameters were used in obtaining Figures 5.9(b)–5.12(b) using the acceleration formulation. The response obtained using the displacement formulation is not shown as it is identical to that obtained using the acceleration formulation (Figures 5.9–5.12). It is evident from Figure 5.17 that the displacement formulation requires many more iterations than the acceleration formulation.

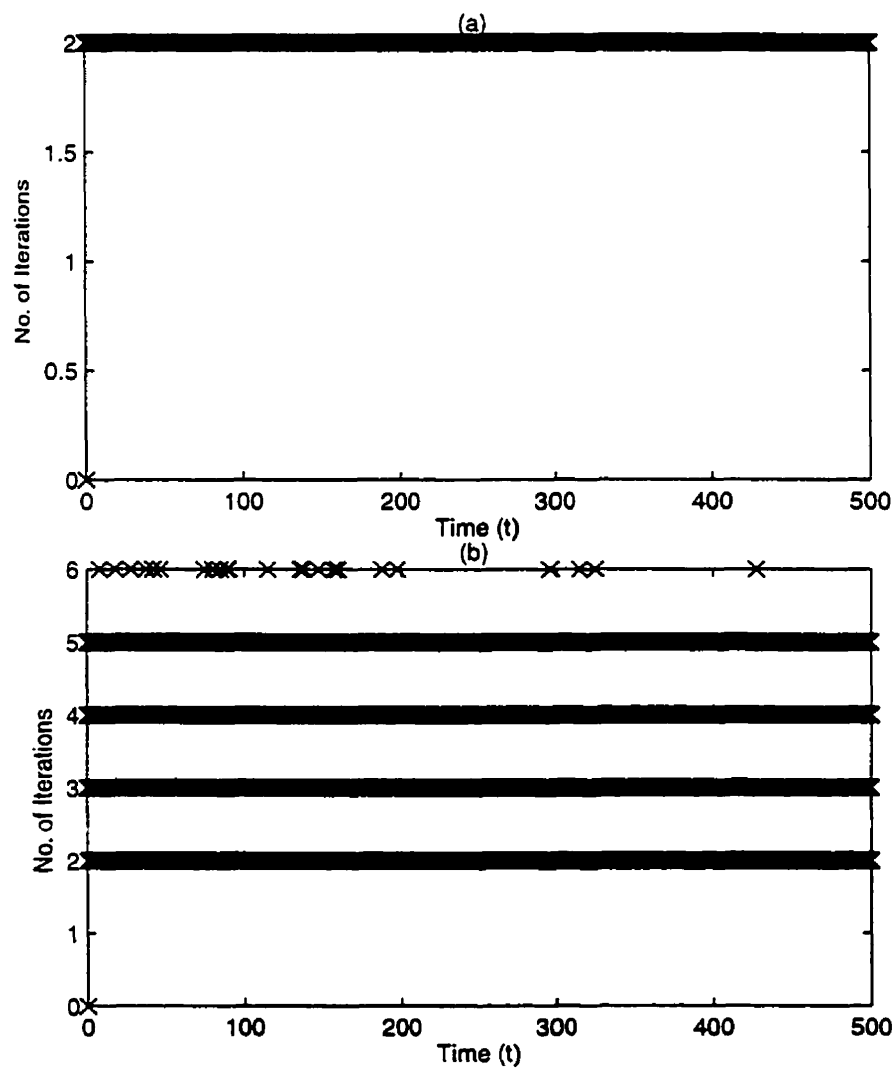


Figure 5.17: Number of iterations, $v_{t0} = 0.1$, (a) Acceleration formulation, (b) Displacement formulation.

5.10 Mass Beam System – Large Oscillations

The equations of motion for the cantilever beam carrying the spring-mass sub-system using the large oscillation model are now solved using the acceleration formulation technique. Internal resonance in the system is established using linear frequencies (these frequencies account for the equilibrium position of the moving mass on the beam) obtained using finite elements, as was the case in all the other simulations presented in this thesis. A better approach would be to develop nonlinear frequency amplitude relationships for the mass beam system (using the nonlinear beam model). The system can then be more accurately tuned. This is a fairly challenging task and a big project in itself. Hence this approach is deferred for future work. Therefore by using the linear frequencies, the nonlinear beam system cannot be perfectly tuned to exhibit 1:2 internal resonance especially for large oscillations.

The parameters for the simulations presented in section are tabulated in Table 5.4. Figure 5.18 shows the results obtained for an initial tip deflection of 0.1. The results are also compared with the perturbation solution, thus giving a detuning parameter of -0.0005. The detuning parameter for simulations obtained using different models and solution methods for this same set of parameters are tabulated in Table 5.3.

For all the cases listed in Table 5.3, the frequency of the spring-mass sub-system ω is assumed to be twice the fundamental frequency of the system as obtained using a linear finite element approach (outlined in Section 4.4) with twenty elements. The first entry in Table 5.3 is for the simplified model used in the perturbation analysis. The detuning parameter for this case is obtained analytically using using (4.12)

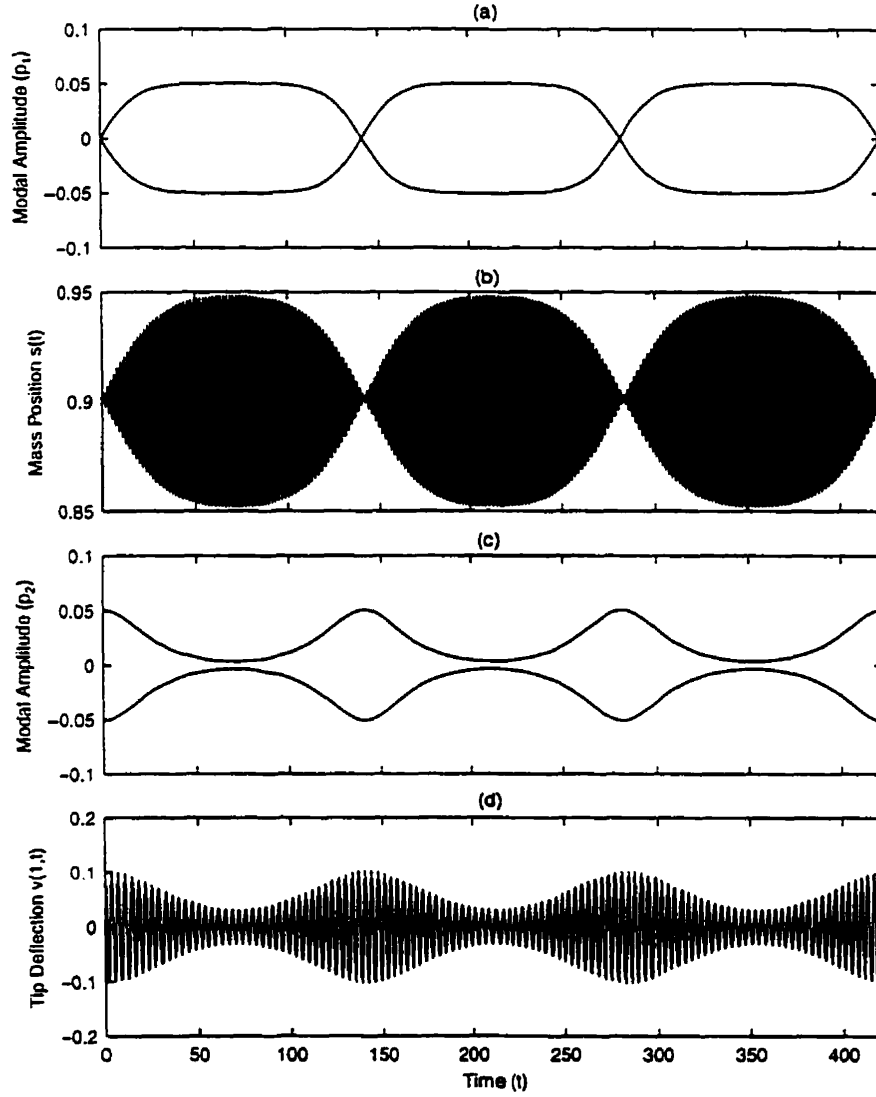


Figure 5.18: System response - 1:2 IR, $m = 1.0$, $s_e = 0.9$, $s_{10} = 0.90001$, and $v_{10} = 0.1$. (a) Perturbation solution for the moving mass $\sigma = -0.0005$, (b) Numerical solution for the moving mass, (c) Perturbation solution for the tip deflection $\sigma = -0.0005$, (d) Numerical solution for the tip deflection.

	Model	Detuning parameter	Figures
1	Perturbation	-0.0097	(from Table 4.3)
2	Linear Beam Automatic ODE solver	-0.0008	4.15, 4.16
3	Linear Beam Acceleration Formulation	-0.0098	5.2, 5.3
4	Nonlinear Beam Acceleration Formulation	-0.0005	5.18

Table 5.3: Comparison of the detuning parameter σ using different models and solution methods.

and (4.3). The model which is closest to the finite element discretization used in obtaining the fundamental frequency of the system is case 3 (case 3 uses only ten finite elements whereas the fundamental frequency of the system was obtained using twenty elements) followed by case 2 (uses four cantilever beam basis functions) and then case 4 (nonlinear beam). It is remarkable that the detuning parameter obtained using the acceleration formulation (case 3) matches so closely with the perturbation model (case 1), and as the fundamental frequency of the various other models move away from the finite element model frequency used in establishing internal resonance, the value of σ decreases (moves away from the value of σ for case 1). The difference in the beating periods observed in Figures 4.15, 4.16, 5.2.5.3, and 5.18 corresponds to the difference in the beating periods obtained analytically by varying the detuning parameter (see Figure 4.11).

In obtaining Figure 5.18 the number of iterations required at each time step to meet a tolerance of $1e-06$ was found to be 2 (constant like in Figure 5.12(b)). Figure 5.19 compares the power spectrums obtained using the small oscillation model and

Parameter Set 2		
Model: NM5, $m = 1.0$		
	Initial Values equation (2.46) $s_0 = 0.90001, v_{t0} = 0.1$	Initial Values equation (2.46) $s_0 = 0.7, v_{t0} = 0.3$
Figure	5.18, 5.19	5.20–5.22
Δt	0.003	0.003
Spectrogram		
No. of Segments		16
Segment Size		256
Power Spectrum		
No. of FFTs	6	6
Segment Size	32768	32768

Table 5.4: Simulation parameters – Large oscillation model

the large oscillation model of the system. The small oscillation model results are same as in Figures 5.2(f) and 5.3(f) and the large oscillation model results are obtained using the time response shown in Figure 5.18. As can be seen from Figure 5.19 the difference in the frequencies using the two models for small initial values is negligible.

Figure 5.20 shows the response obtained using the large oscillation model and for a large initial value (initial tip deflection of 0.3). The time response shows the beating due to internal resonance, however since the fundamental frequency based on a linear model was used in tuning the system, it appears, as indicated by the amplitude of the beats for the beam (not becoming zero or close to zero) that the system is not perfectly tuned for 1:2 resonance. The number iterations at each time step (see Figure 5.20(c)) are again mostly two. Figure 5.22 shows the power spectrums obtained for the response. For the moving mass, major frequency peaks

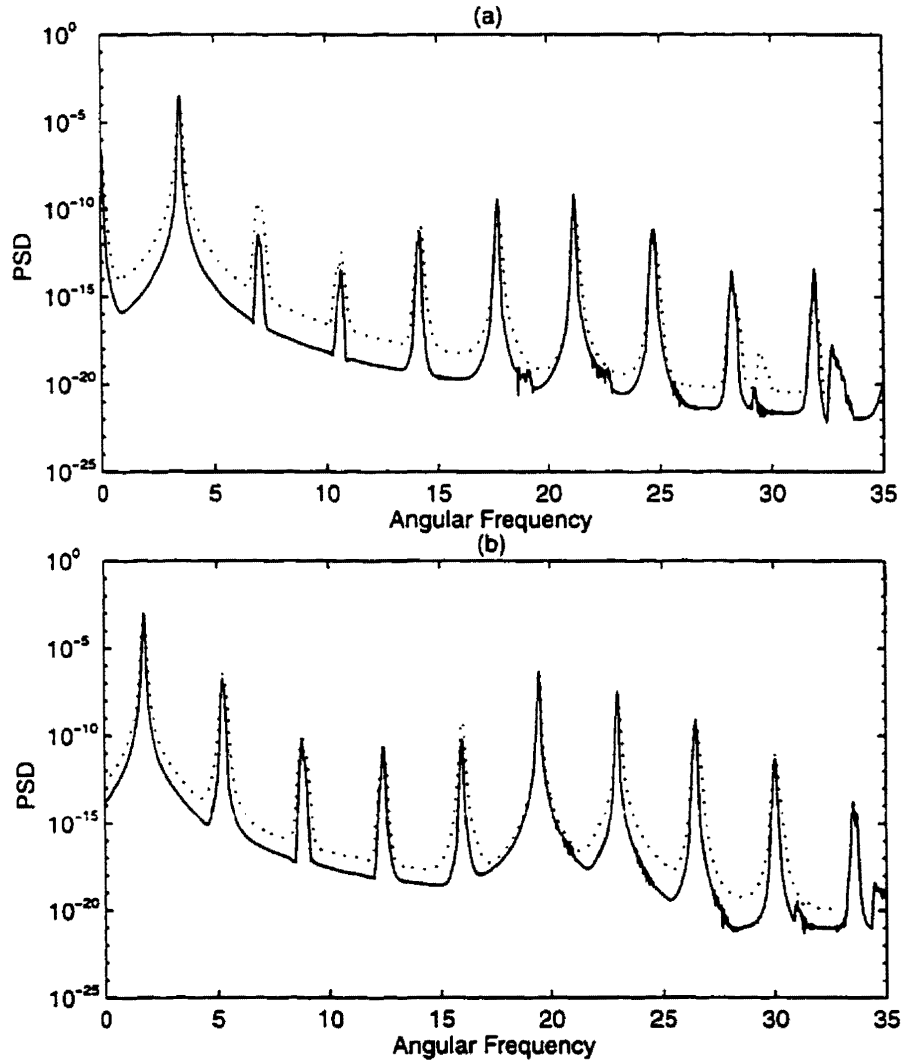


Figure 5.19: Power spectra, (...) small oscillation model with $v_{t0} = 0.1$ (—) large oscillation model with $v_{t0} = 0.1$ (a) Moving mass (b) Tip deflection.

are seen at even multiples of ω_1 ($2\omega_1, 4\omega_1, \dots$) and for the beam the major peaks can be at odd multiples of ω_1 ($\omega_1, 3\omega_1, \dots$). This distinction however is not clear for the higher harmonics as the energy in the side bands becomes as significant as the peaks corresponding to the harmonics of ω_1 . Figure 5.22 shows the time frequency analysis for the response where modulation can be observed in the higher frequencies. When the mass moves over the length of the beam the fundamental frequencies of the system change and Figure 5.22 shows that using time frequency analysis this affect can be captured very clearly for the higher frequencies.

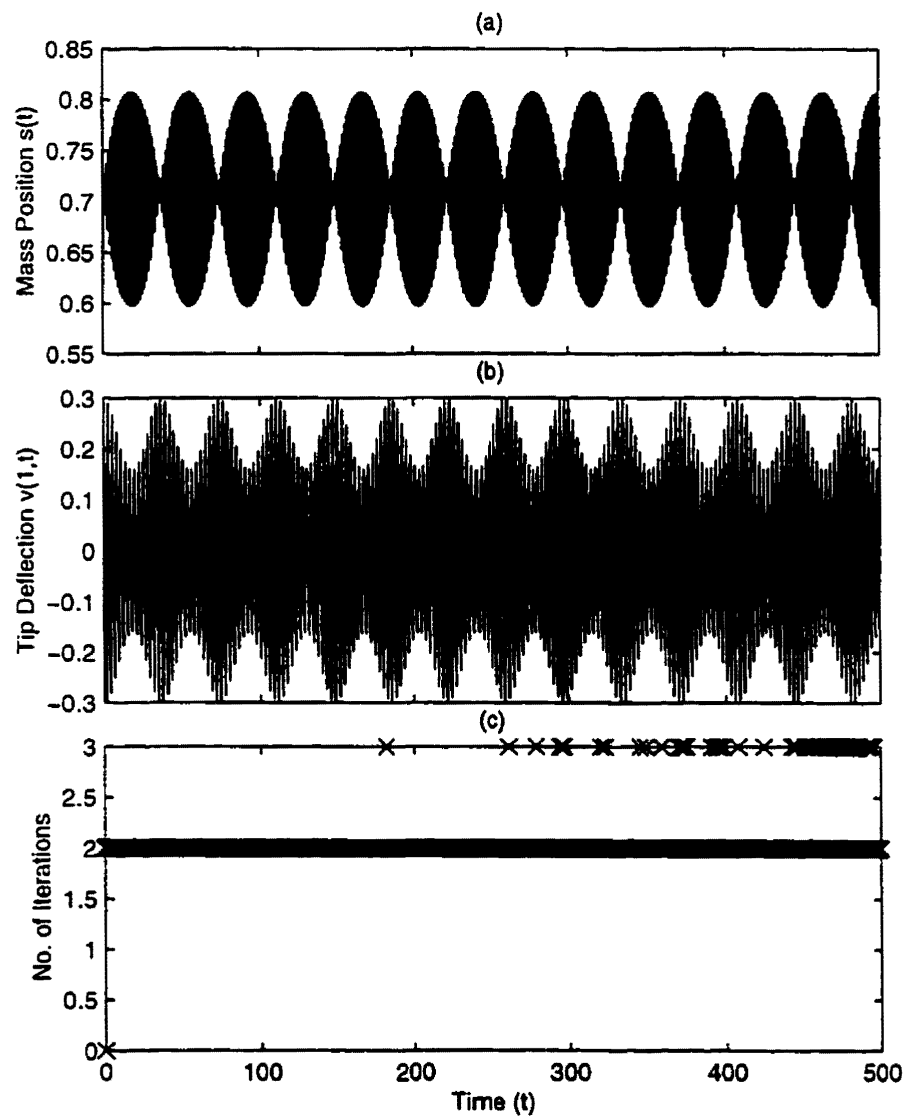


Figure 5.20: System response - 1:2 IR, $m = 1.0$, $s_e = 0.7$, $s_{10} = 0.70001$, and $v_{t0} = 0.3$. (a) Mass response, (b) Tip deflection, (c) Number of iterations at each time step.

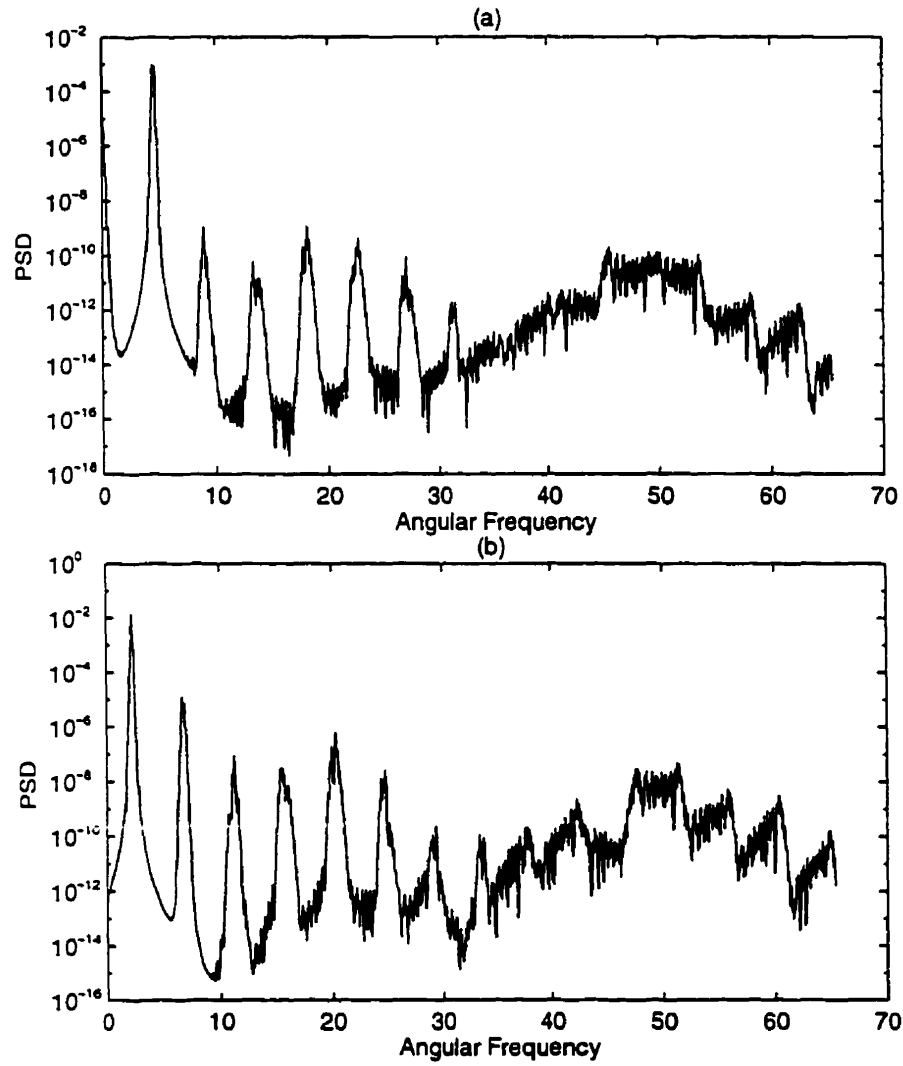


Figure 5.21: Power spectra - 1:2 IR, $m = 1.0$, $s_e = 0.7$, $s_{10} = 0.70001$, and $v_{t0} = 0.3$. (a) Moving mass, (b) Tip deflection.

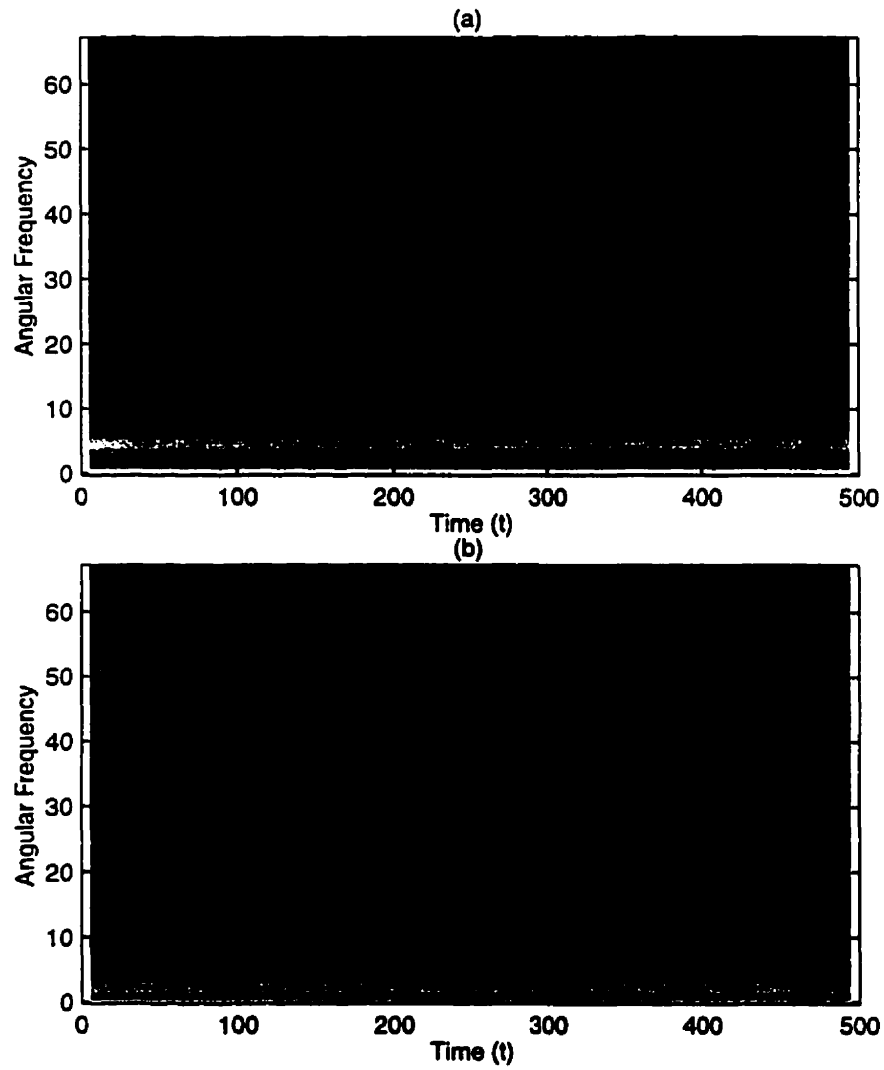


Figure 5.22: Spectrograms - 1:2 IR, $m = 1.0$, $s_e = 0.7$, $s_{10} = 0.70001$, and $v_{i0} = 0.3$.
(a) Moving mass, (b) Tip deflection.

In this chapter the new acceleration formulation method was used to obtain numerical results for the large oscillation model. The technique was evaluated by comparing the results with analytical, perturbation and numerical solutions obtained using the automatic ODE solver approach. In the next chapter a conclusion to this thesis is presented.

Chapter 6

Conclusion

This thesis may be divided into four parts: system modelling, analytical solution, numerical solution, and analysis of the results. Some concluding remarks on each of these areas are briefly presented.

6.1 System Modelling

In Section 1.4, a number of papers on beam carrying a moving mass problem have been cited. All of these papers deal with linear beam models and most of them consider a much simpler problem where the motion of the moving mass is assumed to be prescribed. In this work, however, a more general case has been considered where the motion of the mass introduces an additional degree of freedom in a free vibration problem. A small and large oscillation model was developed for the system. The small oscillation model included only kinematic nonlinearities due to the inertial coupling between the beam and the moving mass and the large oscillation model

accounted for kinematic nonlinearities as well as geometric nonlinearities arising due to nonlinear strain displacement relationships.

The equations of motion for the cantilever beam carrying a moving spring-mass for both the small oscillation model (equations (2.28)–(2.29) and the large oscillation model (2.39)–(2.42) were derived using the Hamilton's principle. The large oscillation model is characterised by nonlinear stiffness terms which appear in the corresponding equations of motion, whereas the stiffness in the small oscillation model is linear. Besides this difference, the acceleration terms (as a result of kinematic nonlinearities) have components tangent and normal to the bending curve in the large oscillation model equations, and in the small oscillation model only the normal components exists.

In deriving the equations for the large oscillation model the rotation of beam elements θ was used instead of spatial derivatives of u and v (see (2.3) for the relationships between θ, u and v). The variable θ was replaced in terms of u and v at a later stage after integration by parts for the symmetric formulation (see Section 3.2). This gave discretized equations which were symmetric in both u and v . If the integration by parts was performed without using θ then the discretized equation corresponding to the u variation (3.13) would not contain the stiffness matrix $[\phi_i'' \phi_j'']$, and the other nonlinear terms would also be not symmetric in the discretized variables α_i and β_i . Symmetricity of the equations of motion is desirable as the finite element discretization for both u and v assumes the same degree of continuity.

In addition to the mass beam system the large oscillation motion of a cantilever

beam without the moving mass was also investigated. The equations of motion for the beam were obtained by simply neglecting the mass position dependent terms in the equations of motion for the mass beam system.

6.2 Analytical Solution

A closed form perturbation solution was obtained for the small oscillation model where only kinematic nonlinearities were considered. The kinematic nonlinearities play a significant role, more so than geometric nonlinearities in the dynamic response of the system. This is evident from the simulation results presented.

The perturbation solution is based on some simplifying assumptions; the equations used are for small oscillations about the equilibrium positions and the higher order terms (beyond quadratic) in the Taylor series expansion were also dropped. Nevertheless the solution obtained gives qualitative results which were compared with the numerical solution. The perturbation analysis was also used in obtaining closed form equations for the phases and beating period (equations (4.40)–(4.42)).

Difficulty in obtaining a numerical solution for nonlinear systems is often one of the motivating reason for perturbation analysis. However, perturbation methods do not always provide a closed form solution and a combined analytic-numeric approach is generally required. Comparison between closed form analytical and numerical solutions is therefore uncommon. In this work the perturbation solution for modal amplitudes is matched with the numerical solution by selecting a value for the detuning parameter σ . A numerical solution obtained under perfect 1:2 resonance conditions when compared with a perturbation solution under perfect

1:2 resonance conditions is expected to reflect differences as a result of the different models for the two cases and also due to neglecting the higher order terms in the perturbation solution. The approach used in this work is to quantify these differences using the detuning parameter σ . The methodology to obtain the value of σ was presented in Algorithm 4.2 and the use of σ to compare different models was illustrated in Table 5.3 (also see the discussion in Section 5.10).

Using the closed form perturbation solution, an extensive parametric analysis was carried out which identifies regions of strong nonlinear coupling between the beam and the moving mass and gives the change in some of the important properties of the solution like the beating period, and the maximum and the minimum amplitudes with the detuning parameter, initial values and the equilibrium position of the moving mass on the beam.

6.3 Numerical Solution

A new finite element based method was developed for the numerical solution. The technique relied on the Galerkin's method for spatial discretization and used conformable finite element basis functions to give a C^2 continuous approximation for the solution. The basis functions used were fifth degree Hermitian polynomials with three degrees of freedom at each node. The use of conforming finite elements increased the number of equations to be solved and also made the resulting ordinary differential equations more "stiff". But as the success with this approach indicated, a well formulated problem using conforming finite elements gives results which can be relied upon, and also works well numerically as discontinuities in the finite

element approximation are not present.

Temporal discretization was carried out using the average acceleration method which was derived directly using Taylor series. As discussed in Section 1.3.2, more sophisticated methods exist where the aim is to dampen the effect of the higher frequencies so that the convergence properties of the numerical method can be improved. However with these techniques the accuracy is generally compromised. In this work, a new implementation of the average acceleration method was proposed which used a direct application of the Taylor series expansions employed in the average acceleration scheme to reduce the ordinary differential equations to nonlinear algebraic equations in terms of accelerations. This formulation was referred to as the acceleration formulation. The acceleration formulation coupled with a new iterative scheme developed for solving the resulting nonlinear algebraic equations gave a far superior convergence for an initial estimate of the solution when compared with the conventional method of discretization (displacement formulation).

The iterative scheme developed does not require computing the Jacobian as is normally the case in the Newton's method and its variants. In addition, the algorithm developed (Algorithm 5.1) for implementing this iterative scheme had an important feature wherein the iterations were carried out until the improved estimates converge to the semi-discretized nonlinear ordinary differential equations instead of the nonlinear algebraic equations. This prevents the temporal discretization errors from propagating as the solution evolves in time.

A number of analyses were carried out to evaluate the results obtained using the numerical method developed. The numerical solution was compared with the

analytical solution obtained using four modes of a cantilever beam. The results showed a very good match for the first three modes (Section 5.7.1) with a step size of 0.005. Decreasing the step size would improve the accuracy of the higher frequencies. In Chapter 5, a number of graphs were presented verifying the numerical results with the analytical solution obtained using the perturbation solution.

Another approach was also used for obtaining the numerical solution. In this case for the small oscillation model where the beam is linear, mode shapes of a simple cantilever beam were used as basis functions for spatial discretization and the resulting nonlinear ODEs were solved using an automatic ODE solver. Numerical results using this approach were compared with the perturbation solution in Chapter 4 and with the finite element based acceleration method in Chapter 5. There was also an attempt to solve the nonlinear ODEs for the small oscillation model obtained after spatial discretization using finite element basis functions using automatic ODE solvers. This was however unsuccessful and led to the development of the finite element based solution strategy.

To evaluate the performance of any numerical technique for nonlinear systems the simulations should be carried out for long time periods as the problems due to higher frequencies and error propagation generally do not effect the short time response. Therefore in this work all the simulations were carried out for large time periods. In addition damping was also ignored in the system model. Inclusion of damping only facilitates dissipation of errors in the higher frequencies and thus improving the convergence of the solution.

The equations of motion for the system using the large oscillation model are

quite difficult to solve numerically and the success of the acceleration formulation method in obtaining the solution and the reliability of the solution having been established using a perturbation method showed that the technique can be potentially applied to much larger and more complicated systems. As the method is based on explicit spatial and temporal discretization the resulting algebraic equations are very cumbersome and could be difficult to manage. However the symbolic manipulation softwares like MAPLE makes this task automatic. The development of the equations of motion and its discretization was all carried out on MAPLE. Even the various terms in the iterative equation (5.19) were obtained through programs written in MAPLE and then imported into 'C' programs where the simulations were carried out.

6.4 Analysis

The mass beam system exhibits internal resonance due to the nonlinear inertial coupling between the beam and the spring-mass subsystem. Under internal resonance conditions when the beam (or the mass) is given an initial displacement, the mass (or the beam) undergoes large oscillations with a characteristic beat from a very small initial value. When such a behaviour is desired it can be established, for example by tuning the stiffness of the moving mass through position feedback. An example of such an application is a vibration suppression scheme, where the goal is to transfer energy from the beam to the mass where it can be dissipated out of the system through the use of some damping strategy.

The time response obtained using the perturbation method or the numerical

method captures the amplitude modulation behaviour. Time frequency analysis was carried out to capture the spectral behaviour with time. The perturbation solution was compared with numerical solution obtained using different models and solution techniques. The results show that when the motion is predominantly bi-periodic (two fundamental frequencies, ω_1 and the beating frequency $\frac{\pi}{\tau_b}$ and their harmonics) the perturbation and the numerical solution results match very well with the differences being quantified by small value of the detuning parameter σ . There were however some other cases (Figures 4.3, 4.4, 4.21 and 4.22) where the differences were more significant and depended on the methodology used for obtaining the numerical solution. For these cases the solution obtained using finite element basis functions and acceleration formulation methodology gave the best results (see Section 5.10 and Table 5.3).

6.5 Future Work

The work done in this thesis on the mass-beam system mainly focussed on internal resonance behaviour which is characterised by amplitude modulation. There is however another very interesting phenomenon exhibited by the system when the spring-mass sub-system is allowed to traverse the beam instead of fixing the equilibrium position at a certain point. This leads to a combined amplitude and frequency modulation behaviour. Some preliminary results in this regard have already been obtained however were not included in this thesis.

Some other possible extensions of this work include; using the moving mass as a controller to suppress vibrations in the beam, development of equations for the

natural frequencies for the large oscillation model of the mass-beam system using perturbation methods, application of finite element methodology to more complex systems.

Appendix A

Formulation of the Iterative Problem

In this appendix the iterative problem to solve the discretized nonlinear algebraic equations (5.9)–(5.17), as outlined in Section 5.6.1 is formulated. The iterative equation (5.19) is written in the following form:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} + \begin{Bmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \quad (\text{A.1})$$

The terms in (A.1) are now given for the various mathematical models of the system.

A.1 Small Oscillations – Acceleration Formulation

The elements u_1 and u_2 in (A.1) for the small oscillation model using the acceleration formulation correspond to the variables $\bar{s}$ and $\bar{\alpha}_i$, respectively. The terms u_3 and u_4 do not exist for the small oscillation model and correspondingly the matrices in (A.1) are also smaller. Following the approach outlined in Section 5.6.1, the discretized equations of motion (5.9) are written in the matrix form (A.1) with the following elements:

$$\begin{aligned}
 a_{11} &= 1 + a_5 \omega^2 \\
 a_{12} &= \left[|\phi_i| + [\phi_i \phi'_j] \{a_5 \{\bar{\alpha}_j\} + \{c2\alpha_j\}\} \right]_{x=a_5 \bar{s} + c2s} \\
 n_1 &= \omega^2 (c2s - s_e) \\
 a_{21} &= m \left\{ \{\phi_i\} + [\phi_i \phi'_j] \left\{ (2a_2^2 + a_5) \{\bar{\alpha}_j\} + 2a_2 \{c1\alpha_j\} + \{c2\alpha_j\} \right\} \right. \\
 &\quad \left. + 2a_2 c1s [\phi_i \phi''_j] \{a_5 \bar{\alpha}_j\} + \{c2\alpha_j\} \right\}_{x=a_5 \bar{s} + c2s} \\
 a_{22} &= m \left[[\phi_i \phi_j] + ((2a_2^2 + a_5) \bar{s} + 2a_2 c1s) [\phi_i \phi'_j] \right. \\
 &\quad \left. + a_5 (a_2 \bar{s} + c1s)^2 [\phi_i \phi''_j] \right]_{x=a_5 \bar{s} + c2s} + \int_0^1 [\phi_i \phi_j] + a_5 [\phi''_i \phi''_j] dx \\
 n_2 &= m \left\{ [\phi_i \phi'_j] \{2c1s \{c1\alpha_j\} - (2a_2^2 + a_5) \bar{s} \{\bar{\alpha}_j\}\} \right. \\
 &\quad \left. + [\phi_i \phi''_j] \left\{ (a_2^2 \bar{s}^2 + c1s^2) \{c2\alpha_j\} - 2a_2 a_5 c1s \bar{s} \{\bar{\alpha}_j\} \right\} \right\}_{x=a_5 \bar{s} + c2s} \\
 &\quad + \int_0^1 [\phi''_i \phi''_j] \{c2\alpha_j\} dx
 \end{aligned} \tag{A.2}$$

A.2 Large Oscillations – Acceleration Formulation

For the large oscillation model and using the acceleration formulation the elements u_1 , u_2 , u_3 and u_4 correspond to variables $\bar{s}$, $\bar{\alpha}_i$, $\bar{\beta}_i$ and λ_i , respectively. The various matrix elements in (A.1) are obtained using the discretized equations (5.12) as follows:

$$a_{11} = 1 + a_5 \omega^2$$

$$a_{12} = \left[[\phi_i] + [\phi_i \phi'_j] \{a_5 \{\ddot{\alpha}_j\} + \{c2\alpha_j\}\} \right]_{x=a_5 \bar{s} + c2s}$$

$$a_{13} = \left[[\phi_i \phi'_j] \{a_5 \{\ddot{\beta}_j\} + \{c2\beta_j\}\} \right]_{x=a_5 \bar{s} + c2s}$$

$$a_{14} = [0]$$

$$n_1 = \omega^2 (c2s - s_e)$$

$$a_{21} = m \left\{ \{\phi_i\} + [\phi_i \phi'_j] \left\{ (2a_2^2 + a_5) \{\ddot{\alpha}_j\} + 2a_2 \{c1\alpha_j\} + \{c2\alpha_j\} \right\} \right. \\ \left. + 2a_2 c1s [\phi_i \phi''_j] \{a_5 \ddot{\alpha}_j\} + \{c2\alpha_j\} \right\}_{x=a_5 \bar{s} + c2s}$$

$$a_{22} = m \left[[\phi_i \phi_j] + \left((2a_2^2 + a_5) \bar{s} + 2a_2 c1s \right) [\phi_i \phi'_j] \right. \\ \left. + a_5 (a_2 \bar{s} + c1s)^2 [\phi_i \phi''_j] \right]_{x=a_5 \bar{s} + c2s} + \int_0^1 [\phi_i \phi_j] + a_5 [\phi'_i \phi'_j] \\ + a_5 [\phi'_i \phi'_j] [\phi_k] \{\lambda_k\} - 2a_5 [\phi'_i \phi'_j] [\phi''_k] \{ (1 - \delta_{jk}) a_5 \{\ddot{\alpha}_k\} + \{c2\alpha_k\} \} \\ - a_5 [\phi'_i \phi'_j] [\phi''_k \phi''_l] \{ \{ (1 - \delta_{jk}) a_5 \{\ddot{\alpha}_k\} + \{c2\alpha_k\} \} \{ (1 - \delta_{jl}) a_5 \{\ddot{\alpha}_l\} + \{c2\alpha_l\} \} \\ + \{a_5 \{\ddot{\beta}_k\} + \{c2\beta_k\} \} \{a_5 \{\ddot{\beta}_l\} + \{c2\beta_l\} \} \} - 2a_5 [\phi'_i \phi'_j] [\phi''_k \phi'_l] \\ \{ \{ (1 - \delta_{jk}) a_5 \{\ddot{\alpha}_k\} + \{c2\alpha_k\} \} \{ (1 - \delta_{jl}) a_5 \{\ddot{\alpha}_l\} + \{c2\alpha_l\} \} \} \} dx \\ a_{23} = \int_0^1 -2a_5 [\phi'_i \phi'_j] [\phi''_k] \{ (1 - \delta_{jk}) a_5 \{\ddot{\beta}_k\} + \{c2\beta_k\} \}$$

$$\begin{aligned}
& -2a_5 [\phi'_i \phi''_j] [\phi''_k \phi'_l] \left\{ \left\{ (1 - \delta_{jk}) a_5 \{\ddot{\beta}_k\} + \{c2\beta_k\} \right\} \{a_5 \{\ddot{\alpha}_l\} + c2\alpha_l\} \right\} dx \\
a_{24} &= \int_0^1 [\phi'_i \phi_j] + [\phi'_i \phi_j] [\phi'_k] \{a_5 \{\ddot{\alpha}_k\} + \{c2\alpha_k\}\} dx \\
n_2 &= m \left\{ [\phi_i \phi'_j] \{2c1s \{c1\alpha_j\} - (2a_2^2 + a_5) \ddot{s} \{\ddot{\alpha}_j\}\} \right. \\
&+ [\phi_i \phi''_j] \left\{ (a_2^2 \ddot{s}^2 + c1s^2) \{c2\alpha_j\} - 2a_2 a_5 c1s \ddot{s} \{\ddot{\alpha}_j\} \right\} \Big|_{x=a_5 \ddot{s} + c2s} \\
&+ \int_0^1 [\phi''_i \phi''_j] \{c2\alpha_j\} - a_5 \{\phi'_i\} [\phi'_j \phi_k] \{\ddot{\alpha}_j \lambda_k\} + \{\phi'_i\} [\phi''_j \phi''_k] \left\{ -\delta_{jk} a_5^2 \{\ddot{\alpha}_j \ddot{\alpha}_k\} \right. \\
&- \{c2\alpha_j c2\alpha_k\} - \delta_{jk} a_5^2 \{\ddot{\beta}_j \ddot{\beta}_k\} - \{c2\beta_j c2\beta_k\} + (1 - \delta_{jk}) a_5^2 \{\ddot{\alpha}_j \ddot{\alpha}_k\} \\
&+ (1 - \delta_{jk}) a_5^2 \{\ddot{\beta}_j \ddot{\beta}_k\} \Big\} - \{\phi'_i\} [\phi'_j \phi''_k \phi''_l] \left\{ -(1 - \delta_{kl}) 2a_5^3 \{\ddot{\alpha}_j \ddot{\beta}_k \ddot{\beta}_l\} \right. \\
&- (1 - \delta_{kl}) a_5^2 \{c2\alpha_j \ddot{\beta}_k \ddot{\beta}_l\} + \delta_{kl} a_5^2 \{c2\alpha_j \ddot{\beta}_k \ddot{\beta}_l\} - 2a_5^2 \{\ddot{\alpha}_j \ddot{\beta}_k c2\beta_l\} \\
&+ \{c2\alpha_j c2\beta_k c2\beta_l\} \Big\} - \{\phi'_i\} [\phi'_j \phi''_k \phi''_l] \left\{ -(1 - \delta_{jk}) (1 - \delta_{jl}) 2a_5^3 \{\ddot{\alpha}_j \ddot{\alpha}_k \ddot{\alpha}_l\} \right. \\
&+ \delta_{jk} \delta_{jl} a_5^3 \{\ddot{\alpha}_j \ddot{\alpha}_k \ddot{\alpha}_l\} - (1 - \delta_{jk}) 2a_5^3 \{\ddot{\alpha}_j \ddot{\alpha}_k c2\alpha_l\} \\
&+ \delta_{jk} 2a_5^3 \{\ddot{\alpha}_j \ddot{\alpha}_k c2\alpha_l\} + \delta_{kl} a_5^2 \{c2\alpha_j \ddot{\alpha}_k \ddot{\alpha}_l\} - (1 - \delta_{kl}) a_5^2 \{c2\alpha_j \ddot{\alpha}_k \ddot{\alpha}_l\} \\
&\left. + \{c2\alpha_j c2\alpha_k c2\alpha_l\} \right\} dx \\
a_{31} &= m \left\{ [\phi_i \phi'_j] \left\{ (2a_2^2 + a_5) \{\ddot{\beta}_j\} + 2a_2 \{c1\beta_j\} + \{c2\beta_j\} \right\} \right. \\
&+ 2a_2 c1s [\phi_i \phi''_j] \left\{ a_5 \{\ddot{\beta}_j\} + \{c2\beta_j\} \right\} \Big|_{x=a_5 \ddot{s} + c2s} \\
a_{32} &= \int_0^1 -2a_5 [\phi'_i \phi''_j] [\phi''_k \phi'_l] \left\{ \left\{ (1 - \delta_{jk}) a_5 \{\ddot{\alpha}_k\} + \{c2\alpha_k\} \right\} \{a_5 \{\ddot{\beta}_l\} + c2\beta_l\} \right\} dx \\
a_{33} &= m \left[[\phi_i \phi_j] + \left((2a_2^2 + a_5) \ddot{s} + 2a_2 c1s \right) [\phi_i \phi'_j] \right. \\
&+ a_5 (a_2 \ddot{s} + c1s)^2 [\phi_i \phi''_j] \Big|_{x=a_5 \ddot{s} + c2s} + \int_0^1 [\phi_i \phi_j] + a_5 [\phi'_i \phi''_j] \\
&+ a_5 [\phi'_i \phi'_j] [\phi_k] \{\lambda_k\} \\
&- a_5 [\phi'_i \phi'_j] [\phi''_k \phi''_l] \left\{ \left\{ (1 - \delta_{jk}) a_5 \{\ddot{\beta}_k\} + \{c2\beta_k\} \right\} \left\{ (1 - \delta_{jl}) a_5 \{\ddot{\beta}_l\} + \{c2\beta_l\} \right\} \right. \\
&+ \{a_5 \{\ddot{\alpha}_k\} + \{c2\alpha_k\}\} \{a_5 \{\ddot{\alpha}_l\} + \{c2\alpha_l\}\} \Big\} - 2a_5 [\phi'_i \phi''_j] [\phi''_k \phi'_l] \\
&\left. \left\{ \left\{ (1 - \delta_{jk}) a_5 \{\ddot{\beta}_k\} + \{c2\beta_k\} \right\} \left\{ (1 - \delta_{jl}) a_5 \{\ddot{\beta}_l\} + \{c2\beta_l\} \right\} \right\} \right] dx
\end{aligned}$$

$$\begin{aligned}
a_{34} &= \int_0^1 [\phi'_i \phi_j] [\phi'_k] \{a_5 \{\ddot{\beta}_k\} + \{c2\beta_k\}\} dx \\
n_3 &= m \{ [\phi_i \phi'_j] \{2c1s \{c1\beta_j\} - (2a_2^2 + a_5) \ddot{s} \{\ddot{\beta}_j\}\} \\
&\quad + [\phi_i \phi'_j] \{ (a_2^2 \ddot{s}^2 + c1s^2) \{c2\beta_j\} - 2a_2 a_5 c1s \ddot{s} \{\ddot{\beta}_j\} \} \}_{x=a_5 \ddot{s} + c2s} \\
&\quad + \int_0^1 [\phi''_i \phi''_j] \{c2\beta_j\} - a_5 \{\phi'_i\} [\phi'_j \phi_k] \{\ddot{\beta}_j \lambda_k\} \\
&\quad - \{\phi'_i\} [\phi'_j \phi''_k \phi'_l] \{ -(1 - \delta_{kl}) 2a_5^3 \{\ddot{\beta}_j \ddot{\alpha}_k \ddot{\alpha}_l\} - (1 - \delta_{kl}) a_5^2 \{c2\beta_j \ddot{\alpha}_k \ddot{\alpha}_l\} \\
&\quad + \delta_{kl} a_5^2 \{c2\beta_j \ddot{\alpha}_k \ddot{\alpha}_l\} - 2a_5^2 \{\ddot{\beta}_j \ddot{\alpha}_k c2\alpha_l\} + \{c2\beta_j c2\alpha_k c2\alpha_l\} \} \\
&\quad - \{\phi'_i\} [\phi'_j \phi''_k \phi'_l] \{ -(1 - \delta_{jk}) (1 - \delta_{jl}) 2a_5^3 \{\ddot{\beta}_j \ddot{\beta}_k \ddot{\beta}_l\} + \delta_{jk} \delta_{jl} a_5^3 \{\ddot{\beta}_j \ddot{\beta}_k \ddot{\beta}_l\} \\
&\quad - (1 - \delta_{jk}) 2a_5^3 \{\ddot{\beta}_j \ddot{\beta}_k c2\beta_l\} + \delta_{jk} 2a_5^3 \{\ddot{\beta}_j \ddot{\beta}_k c2\beta_l\} + \delta_{kl} a_5^2 \{c2\beta_j \ddot{\beta}_k \ddot{\beta}_l\} \\
&\quad - (1 - \delta_{kl}) a_5^2 \{c2\beta_j \ddot{\beta}_k \ddot{\beta}_l\} + \{c2\beta_j c2\beta_k c2\beta_l\} \} dx \\
a_{41} &= 0 \\
a_{42} &= \int_0^1 a_5 [\phi_i \phi'_j] + a_5 [\phi_i \phi'_j] [\phi'_k] \{ (1 - \delta_{jk}) a_5 \{\ddot{\alpha}_k\} + \{c2\alpha_k\} \} dx \\
a_{43} &= \int_0^1 a_5 [\phi_i \phi'_j] [\phi'_k] \{ (1 - \delta_{jk}) a_5 \{\beta_k\} + \{c2\beta_k\} \} dx \\
a_{44} &= 0 \\
n_4 &= \int_0^1 [\phi_i \phi'_j] + \frac{1}{2} \{\phi_i\} [\phi'_j \phi'_k] \{ \delta_{jk} a_5^2 \{\ddot{\alpha}_j \ddot{\alpha}_k\} - (1 - \delta_{jk}) a_5^2 \{\ddot{\alpha}_j \ddot{\alpha}_k\} \\
&\quad + \delta_{jk} a_5^2 \{\ddot{\beta}_j \ddot{\beta}_k\} - (1 - \delta_{jk}) \{\ddot{\beta}_j \ddot{\beta}_k\} + \{c2\alpha_j c2\alpha_k\} + \{c2\beta_j c2\beta_k\} \} dx \quad (A.3)
\end{aligned}$$

A.3 Small Oscillations – Displacement Formulation

For the small oscillation model the elements u_1 , and u_2 correspond to s and α_i , respectively. The other matrix elements in (A.1) are obtained using equations

(5.14) as follows:

$$\begin{aligned}
a_{11} &= a_1 + \omega^2 \\
a_{12} &= \left[a_1 [\phi_i] + [\phi_i \phi'_j]^T \{a_1 \{\alpha_i\} + \{c1\alpha_i\}\} \right]_{x=s(t)} \\
n_1 &= (c1s - \omega^2 s_e + [\phi_i] \{c1\alpha_i\})_{x=s(t)} \\
a_{21} &= m \{ a_1 \{\phi_i\} + [\phi_i \phi'_j] \{ (2a_4^2 + a_1) \{\alpha_j\} + 2a_4 \{c2\alpha_j\} \} \\
&\quad + 2a_4 c2s [\phi_i \phi''_j] \{\alpha_j\} \}_{x=s(t)} \\
a_{22} &= m [a_1 [\phi_i \phi_j] + ((2a_4^2 + a_1)s + 2a_4 c2s + c1s) [\phi_i \phi'_j] \\
&\quad + (a_4 s + c2s)^2 [\phi_i \phi''_j]]_{x=s(t)} + \int_0^1 a_1 [\phi_i \phi_j] + [\phi''_i \phi''_j] dx \\
n_2 &= m \{ c2s \{\phi_i\} + [\phi_i \phi_j] \{c1\alpha_j\} + [\phi_i \phi'_j] \{ 2c2s \{c2\alpha_j\} - (2a_4^2 + a_1)s \{\alpha_j\} \} \\
&\quad + [\phi_i \phi''_j] \{ -2a_4 c2s s \{\alpha_j\} \} \}_{x=s(t)} + \int_0^1 [\phi_i \phi_j] \{c1\alpha_j\} dx \tag{A.4}
\end{aligned}$$

A.4 Large Oscillations – Displacement Formulation

For the large oscillation model using the displacement formulation, the elements u_1 , u_2 , u_3 and u_4 correspond to s , α_i , β_i and λ_i , respectively. The other terms in (A.1) are obtained from the discretized nonlinear algebraic equation (5.17) as follows:

$$\begin{aligned}
a_{11} &= a_1 + \omega^2 \\
a_{12} &= \left[a_1 [\phi_i] + [\phi_i \phi'_j]^T \{a_1 \{\alpha_i\} + \{c1\alpha_i\}\} \right]_{x=s(t)} \\
a_{13} &= \left[[\phi_i \phi'_j]^T \{a_1 \{\beta_i\} + \{c1\beta_i\}\} \right]_{x=s(t)}
\end{aligned}$$

$$a_{14} = [0]$$

$$n_1 = (c_1 s - \omega^2 s_e + [\phi_i] \{c_1 \alpha_i\})_{x=s(t)}$$

$$a_{21} = m \left\{ a_1 \{ \phi_i \} + [\phi_i \phi_j'] \left\{ (2a_4^2 + a_1) \{ \alpha_j \} + 2a_4 \{ c_2 \alpha_j \} \right\} \right. \\ \left. + 2a_4 c_2 s [\phi_i \phi_j''] \{ \alpha_j \} \right\}_{x=s(t)}$$

$$a_{22} = m \left[a_1 [\phi_i \phi_j] + \left((2a_4^2 + a_1) s + 2a_4 c_2 s + c_1 s \right) [\phi_i \phi_j'] \right. \\ \left. + (a_4 s + c_2 s)^2 [\phi_i \phi_j''] \right]_{x=s(t)} + \int_0^1 a_1 [\phi_i \phi_j] + [\phi_i' \phi_j'] \\ + [\phi_i' \phi_j'] [\phi_k] \{ \lambda_k \} - 2 [\phi_i' \phi_j''] [\phi_k''] \{ (1 - \delta_{jk}) \{ \alpha_k \} \} \\ - [\phi_i' \phi_j'] [\phi_k'' \phi_l''] \{ \{ (1 - \delta_{jk}) (1 - \delta_{jl}) \{ \alpha_k \alpha_l \} \} + \{ \beta_k \beta_l \} \} \\ - 2 [\phi_i' \phi_j'] [\phi_k'' \phi_l'] \{ (1 - \delta_{jk}) (1 - \delta_{jl}) \{ \alpha_k \alpha_l \} \} \} dx$$

$$a_{23} = \int_0^1 -2 [\phi_i' \phi_j'] [\phi_k''] \{ (1 - \delta_{jk}) \{ \beta_k \} \} - 2 [\phi_i' \phi_j''] [\phi_k'' \phi_l'] \{ (1 - \delta_{jk}) \{ \beta_k \alpha_l \} \} dx$$

$$a_{24} = \int_0^1 [\phi_i' \phi_j] + [\phi_i' \phi_j'] [\phi_k'] \{ \alpha_k \} dx$$

$$n_2 = m \left\{ c_2 s \{ \phi_i \} + [\phi_i \phi_j] \{ c_1 \alpha_j \} + [\phi_i \phi_j'] \left\{ 2c_2 s \{ c_2 \alpha_j \} - (2a_4^2 + a_1) s \{ \alpha_j \} \right\} \right. \\ \left. + [\phi_i \phi_j''] \{ -2a_4 c_2 s s \{ \alpha_j \} \} \right\}_{x=s(t)} + \int_0^1 [\phi_i \phi_j] \{ c_1 \alpha_j \} - \{ \phi_i' \} [\phi_j' \phi_k] \{ \alpha_j \lambda_k \} \\ + \{ \phi_i' \} [\phi_j'' \phi_k''] \{ -\delta_{jk} \{ \alpha_j \alpha_k \} + (1 - \delta_{jk}) \{ \alpha_j \alpha_k \} \\ - \delta_{jk} \{ \beta_j \beta_k \} + (1 - \delta_{jk}) \{ \beta_j \beta_k \} \} - \{ \phi_i' \} [\phi_j' \phi_k'' \phi_l''] \{ -(1 - \delta_{kl}) 2 \{ \alpha_j \beta_k \beta_l \} \} \\ - \{ \phi_i' \} [\phi_j' \phi_k'' \phi_l'] \{ -(1 - \delta_{jk}) (1 - \delta_{jl}) 2 \{ \alpha_j \alpha_k \alpha_l \} + \delta_{jk} \delta_{jl} \{ \alpha_j \alpha_k \alpha_l \} \} \} dx$$

$$a_{31} = m \left\{ [\phi_i \phi_j'] \left\{ (2a_4^2 + a_1) \{ \beta_j \} + 2a_4 \{ c_2 \beta_j \} \right\} \right\}_{x=s(t)} + \int_0^1 2a_4 c_2 s [\phi_i \phi_j''] \{ \beta_j \} \} dx$$

$$a_{32} = \int_0^1 -2 [\phi_i' \phi_j''] [\phi_k'' \phi_l'] \{ (1 - \delta_{jk}) \{ \alpha_k \beta_l \} \} dx$$

$$a_{33} = m \left[a_1 [\phi_i \phi_j] + \left((2a_4^2 + a_1) s + 2a_4 c_2 s + c_1 s \right) [\phi_i \phi_j'] \right. \\ \left. + (a_4 s + c_2 s)^2 [\phi_i \phi_j''] \right]_{x=s(t)} + \int_0^1 a_1 [\phi_i \phi_j] + [\phi_i'' \phi_j''] + [\phi_i' \phi_j'] [\phi_k] \{ \lambda_k \} \\ - [\phi_i' \phi_j'] [\phi_k'' \phi_l''] \{ \{ (1 - \delta_{jk}) \{ \beta_k \} \} \{ (1 - \delta_{jl}) \{ \beta_l \} \} + \{ \alpha_k \} \{ \alpha_l \} \} \}$$

$$\begin{aligned}
& -2 \left[\phi'_i \phi''_j \right] \left[\phi''_k \phi'_l \right] \{ (1 - \delta_{jk}) \{ \beta_k \} \} \{ (1 - \delta_{jl}) \{ \beta_l \} \} \} dx \\
a_{34} &= \int_0^1 \left[\phi'_i \phi_j \right] \left[\phi'_k \right] \{ \beta_k \} dx \\
n_3 &= m \left\{ \left[\phi_i \phi_j \right] \{ c1\beta_j \} + \left[\phi_i \phi'_j \right] \left\{ 2c2s \{ c2\beta_j \} - (2a_4^2 + a_1) s \{ \beta_j \} \right\} \right. \\
& \quad + \left[\phi_i \phi''_j \right] \left\{ -2a_4 c2ss \{ \beta_j \} \right\} \Big|_{x=s(t)} + \int_0^1 \left[\phi_i \phi_j \right] \{ c1\beta_j \} - \{ \phi'_i \} \left[\phi'_j \phi_k \right] \{ \beta_j \lambda_k \} \\
& \quad - \{ \phi'_i \} \left[\phi'_j \phi''_k \phi''_l \right] \{ - (1 - \delta_{kl}) 2 \{ \beta_j \alpha_k \alpha_l \} \} \\
& \quad - \{ \phi'_i \} \left[\phi'_j \phi''_k \phi'_l \right] \{ - (1 - \delta_{jk}) (1 - \delta_{jl}) 2 \{ \beta_j \beta_k \beta_l \} + \delta_{jk} \delta_{jl} \{ \beta_j \beta_k \beta_l \} \} \} dx \\
a_{41} &= 0 \\
a_{42} &= \int_0^1 \left[\phi_i \phi'_j \right] + \left[\phi_i \phi'_j \right] \left[\phi'_k \right] \{ (1 - \delta_{jk}) \{ \alpha_k \} \} dx \\
a_{43} &= \int_0^1 \left[\phi_i \phi'_j \right] \left[\phi'_k \right] \{ (1 - \delta_{jk}) \{ \beta_k \} \} dx \\
a_{44} &= 0 \\
n_4 &= \int_0^1 \frac{1}{2} \{ \phi_i \} \left[\phi'_j \phi'_k \right] \{ \delta_{jk} \{ \alpha_j \alpha_k \} - (1 - \delta_{jk}) \{ \alpha_j \alpha_k \} \} \\
& \quad + \delta_{jk} \{ \beta_j \beta_k \} - (1 - \delta_{jk}) \{ \beta_j \beta_k \} \} dx
\end{aligned} \tag{A.5}$$

Equation (A.1) is solved using the iterative technique presented in Section 5.6.2.

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