

# Jones Pairs

by

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A thesis  
presented to the University of Waterloo  
in fulfilment of the  
thesis requirement for the degree of  
Doctor of Philosophy  
in  
Combinatorics and Optimization

Waterloo, Ontario, Canada, 2001

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# Abstract

Motivated by Jones' braid group representations constructed from spin models, we define a *Jones pair* to be a pair of  $n \times n$  matrices  $(A, B)$  such that the endomorphisms  $X_A$  and  $\Delta_B$  form a representation of a braid group. When  $A$  and  $B$  are type-II matrices, we call  $(A, B)$  an *invertible Jones pair*. We develop the theory of Jones pairs in this thesis.

Our aim is to study the connections among association schemes, spin models and four-weight spin models using the viewpoint of Jones pairs. We use Nomura's method to construct a pair of algebras from the matrices  $(A, B)$ , which we call the *Nomura algebras* of  $(A, B)$ . These algebras become the central tool in this thesis. We explore their properties in Chapters 2 and 3.

In Chapter 4, we introduce Jones pairs. We prove the equivalence of four-weight spin models and invertible Jones pairs. We extend some existing concepts for four-weight spin models to Jones pairs. In Chapter 5, we provide new proofs for some well-known results on the Bose-Mesner algebras associated with spin models.

We document the main results of the thesis in Chapter 6. We prove that every four-weight spin model comes from a symmetric spin model (up to odd-gauge equivalence). We present four Bose-Mesner algebras associated to each four-weight spin model. We study the relations among these algebras. In particular, we provide a strategy to search for four-weight spin models. This strategy is analogous to the method given by Bannai, Bannai and Jaeger for finding spin models.

## Acknowledgements

First of all, I wish to thank Chris Godsil for being a great mentor. He has introduced me to the fascinating area of algebraic combinatorics and given me excellent guidance on my research. I am very grateful to the amount of time and energy he has spent on helping me grow as a mathematician. His advice on doing research, giving presentations, writing and teaching will always be helpful.

I feel very fortunate that I was a part of the Department of Combinatorics and Optimization. The professors and staffs have provided a comfortable learning environment. In particular, I would like to thank Ian Goulden and Adrian Lewis for being my lecturing requirements examiners. I am also grateful to Ian Goulden, Sylvia Hobart, Peter Hoffman and David Wagner for being in my examining committee.

Without the love and the unwavering support from my parents and my husband, I would never have accomplished this work. Lo-Dou, Mummy, Stewart, thank you for always believing in me.

I appreciate the kind words of encouragement and valuable help from Miguel Anjos, Lise Arseneau, Mike Best, Eva Chan, Daniel Chan, Jason Chen, Kevin Cheung, Michael Cheung, Philip Fong, Jackie Ho, Peter Hoffman, Julia Huang, Saša Jurišić, Florence Kong, Kai Kwok, Serge Kruk, Charles Lam, Annie Lee, Patrick Lau, Bill Martin, Roxana and Stanley Poon. Thank you very much, my friends.

The most important of all, I admire God for creating the beauty in Mathematics. I am grateful for having a chance to get a glimpse of this beauty.

To Hing. Fung and Stewart

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# Chapter 1

## Introduction

We first give an overview of this thesis. In Sections 1.2 and 1.3, we give the background materials on spin models and braids. Then we present a historical overview of the research on the association schemes attached to spin models.

### 1.1 Overview

The purpose of this thesis is to introduce Jones pairs and to extend the existing theory of association schemes attached to four-weight spin models.

We now define Jones pairs. Given two  $n \times n$  matrices  $M$  and  $N$ , their Schur product  $M \circ N$  is defined by

$$(M \circ N)_{i,j} = M_{i,j} N_{i,j},$$

for all  $i, j = 1, \dots, n$ . If  $C$  is an  $n \times n$  matrix, we define two endomorphisms of

$\mathbf{M}_n(\mathbb{C})$ ,  $X_C$  and  $\Delta_C$ , by

$$X_C(M) = CM \quad \text{and} \quad \Delta_C(M) = C \circ M.$$

for any  $n \times n$  matrix  $M$ . We say that the pair of  $n \times n$  matrices  $(A, B)$  is a *Jones pair* if  $X_A$  and  $\Delta_B$  are invertible, and they satisfy

$$\begin{aligned} X_A \Delta_B X_A &= \Delta_B X_A \Delta_B, \quad \text{and} \\ X_A \Delta_{B^T} X_A &= \Delta_{B^T} X_A \Delta_{B^T}. \end{aligned}$$

We will show in this thesis that Jones pairs give representations of braid groups. Moreover, we will see that spin models and four-weight spin models belong to a special class of Jones pairs, called the invertible Jones pairs. Consequently, we obtain a representation of braid group from every four-weight spin model. This fact was not known previously.

The connections between spin models and association schemes have been the focus of existing research. We have always found association schemes intriguing, mainly because of their connections to a vast number of combinatorial objects such as distance regular graphs, codes and designs. It follows naturally that we are interested in the results due to Jaeger [12] and Nomura [23] which say that every spin model belongs to the Bose-Mesner algebra of some association scheme. In [12], Jaeger asked for an intrinsic characterization of the association schemes whose Bose-Mesner algebras contain a spin model and this is the question which motivates the work in this thesis.

Nomura used the type-II condition of spin models to obtain the result mentioned

above. We say an  $n \times n$  matrix  $A$  satisfies the *type-II condition* if

$$\sum_{x=1}^n \frac{A_{i,x}}{A_{j,x}} = \delta_{ij}n, \quad \text{for all } i, j = 1, \dots, n$$

and it is called a *type-II matrix*. Spin models and the matrices in four-weight spin models are examples of type-II matrices. In [23], Nomura constructed a Bose-Mesner algebra of some association scheme from each type-II matrix. This algebra is called the Nomura algebra of the type-II matrix. He also showed that every spin model belongs to its Nomura algebra. In Chapter 2, we generalize his construction to a pair of algebras built from a pair of matrices  $(A, B)$ . We also study the properties of this pair of algebras. In this process, we find new and simpler proofs of many known results on the Nomura algebras of type-II matrices and spin models. We are now convinced that Jones pairs provide a natural setting for the study of the problems related to spin models and four-weight spin models.

Relying on the fact that every spin model belongs to its Nomura algebra, Bannai, Bannai and Jaeger [3] designed a strategy to search for spin models. However, the matrices in a four-weight spin model do not belong to their Nomura algebras. So Bannai, Bannai and Jaeger's method does not apply directly to four-weight spin models. In Chapter 6, we provide a construction of an  $4n \times 4n$  symmetric spin model from each  $n \times n$  four-weight spin model. This construction generalizes a construction due to Nomura in [22]. As a result, four-weight spin models are not very different from symmetric spin models. Moreover, we design a strategy to find four-weight spin models using this newly constructed  $4n \times 4n$  symmetric spin model.

In addition to the Nomura algebras obtained from the matrices in an  $n \times n$  four-weight spin models and the  $4n \times 4n$  symmetric spin model, we will construct two more Nomura algebras from the four-weight spin model. In Chapter 6, we will

see how these Nomura algebras are intricately related to each other.

## 1.2 Spin Models

Using a statistical mechanical model called a spin model, Jones [17] constructed invariants of unoriented links in the form of partition functions. His definition of a spin model is essentially a symmetric matrix that satisfies certain properties such that its partition function is invariant under the Reidemeister moves of link diagrams. The Potts model, which is a linear combination of the identity matrix and the matrix of all ones, is the simplest spin model. From the Potts model, we can obtain the infamous Jones polynomial.

In 1994, Kawagoe, Munemasa and Watatani [18] generalized the definition of spin models by removing the symmetry condition. We adopt their definition in this thesis: an  $n \times n$  matrix  $W$  is a *spin model* if there exists a non-zero scalar  $a$  and  $d = \pm\sqrt{n}$  such that

(I) For  $i = 1, \dots, n$ ,

$$W_{i,i} = a$$

and

$$\sum_{x=1}^n W_{i,x} = \sum_{x=1}^n W_{x,i} = da^{-1}.$$

(II) For all  $i, j = 1, \dots, n$ ,

$$\sum_{x=1}^n \frac{W_{i,x}}{W_{j,x}} = \delta_{ij}n,$$

where  $\delta_{ij}$  equals one when  $i = j$  and zero otherwise.

(III) For all  $i, j, k = 1, \dots, n$ ,

$$\sum_{x=1}^n \frac{W_{k,x} W_{x,i}}{W_{j,x}} = d \frac{W_{k,i}}{W_{j,i} W_{j,k}}.$$

They also showed that the partition functions give invariants of oriented links. So far, we only know three infinite families of spin models. The first family is the Potts models. The second one comes from finite Abelian groups. In [3], Bannai, Bannai and Jaeger built a spin model from each finite Abelian group. The third family consists of the symmetric and the non-symmetric Hadamard spin models, constructed by Jaeger and Nomura [21, 15].

It turns out that the type-II condition for spin models is more important than the other two conditions. In particular, the type-II condition of a spin model plays a key role in Nomura's construction of the Bose-Mesner algebra containing it. Moreover, unitary type-II matrices are important objects in the study of Von Neumann algebras [16].

In 1995, Bannai and Bannai [2] gave a further generalization by defining the four-weight spin models. In [13], Jaeger normalized the partition function of a four-weight spin model to give an invariant of oriented links. A *four-weight spin model* is a 5-tuple  $(W_1, W_2, W_3, W_4; d)$  with  $d^2 = n$  and a non-zero scalar  $a$  satisfying

(I) For all  $\alpha = 1, \dots, n$ ,

$$(W_3)_{\alpha,\alpha} = a^{-1}, \quad (W_1)_{\alpha,\alpha} = a,$$

and

$$\begin{aligned}\sum_{x=1}^n (W_2)_{\alpha,x} &= \sum_{x=1}^n (W_2)_{x,\alpha} = da^{-1}, \\ \sum_{x=1}^n (W_4)_{\alpha,x} &= \sum_{x=1}^n (W_4)_{x,\alpha} = da.\end{aligned}$$

(II) For all  $\alpha, \beta = 1, \dots, n$ ,

$$\begin{aligned}(W_1)_{\alpha,\beta}(W_3)_{\beta,\alpha} &= 1, & (W_2)_{\alpha,\beta}(W_4)_{\beta,\alpha} &= 1, \\ \sum_{x=1}^n (W_1)_{\alpha,x}(W_3)_{x,\beta} &= \delta_{\alpha\beta}n, \\ \sum_{x=1}^n (W_2)_{\alpha,x}(W_4)_{x,\beta} &= \delta_{\alpha\beta}n.\end{aligned}$$

(III) For all  $\alpha, \beta, \gamma = 1, \dots, n$ ,

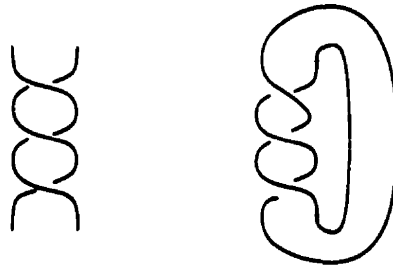
$$\begin{aligned}\sum_{x=1}^n (W_1)_{\alpha,x}(W_1)_{x,\beta}(W_4)_{\gamma,x} &= d(W_1)_{\alpha,\beta}(W_4)_{\gamma,\alpha}(W_4)_{\gamma,\beta}, \\ \sum_{x=1}^n (W_1)_{x,\alpha}(W_1)_{\beta,x}(W_4)_{x,\gamma} &= d(W_1)_{\beta,\alpha}(W_4)_{\alpha,\gamma}(W_4)_{\beta,\gamma}.\end{aligned}$$

In the same paper, Bannai and Bannai listed sixteen equations that are equivalent to the type-III conditions of four-weight spin models. Having to decide which equations are more suitable for a given problem complicates any analysis of four-weight spin models. One advantage of Jones pairs is that they save us from having to deal with these sixteen equations.

### 1.3 Braids

Jones pairs are designed to give representations of braid groups, so braids are naturally the topological object of interest here. In this section, we provide some essential background of braids and braid groups.

A *braid* on  $m$  strands is a set of  $m$  disjoint arcs in 3-space joining  $m$  points on a horizontal plane to  $m$  points on another horizontal plane directly below the first  $m$  points. Given a braid, we form the *closure of the braid* by joining the  $m$  points on top to the  $m$  points at the bottom as illustrated in the following figure. It is obvious that the closure of a braid is a link.



The following theorem due to Alexander [20] explains the connection between braids and links.

**Theorem 1.3.1** *Any link is isotopic to the closure of some braid.* □

When we stack two braids on  $m$  strands, we get a new braid, which we call the *product of the two braids*. This operation is associative. If we define  $\sigma_i$  and  $\sigma_i^{-1}$  as shown in the figure below, then it is easy to see that any braid on  $m$  strands is a product of  $\sigma_1, \dots, \sigma_{m-1}$ . Hence the inverse of a braid always exists. As a result, we get a group structure on the set of braids on  $m$  strands.



The *braid group on  $m$  strands*,  $\mathbf{B}_m$ , is generated by  $\sigma_1, \dots, \sigma_{m-1}$  subject to the following relations:

(a) For all  $|i - j| \geq 2$ .

$$\sigma_i \sigma_j = \sigma_j \sigma_i.$$

(b) For all  $i = 1, \dots, m - 2$ .

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

Given a spin model  $W$ , Jones [17] built a representation of the braid group  $\mathbf{B}_m$ . His construction uses the endomorphisms,  $X_W$  and  $\Delta_{W^{-1}}$ , of  $\mathbf{M}_n(\mathbb{C})$ . He showed that the trace of the element representing a braid equals the partition function invariant evaluated at the closure of the braid, up to normalization.

In summary, given a spin model  $W$ , we can obtain the same link invariant either using the partition function or the trace of a representation of the braid group. However Jones pointed out a puzzling distinction between the two approaches, which is that the type-II condition on  $W$  is needed for the first method, but not for the second. This distinction motivates our definition of Jones pairs in Chapter 4. In particular, we do not assume the type-II condition on the matrices in Jones pairs. In the same chapter, we will generalize Jones' construction to four-weight spin models.

## 1.4 Association Schemes

In 1992, Jaeger [10] made the first connection between association schemes and spin models. He showed that a spin model gives an evaluation of the Kauffman polynomial if and only if it belongs to a triply-regular formally self-dual two-class association scheme. Now two-class association schemes are equivalent to strongly regular graphs. Moreover, a triply-regular two-class association scheme corresponds to a strongly regular graph  $G$  with the property that the neighbourhoods of any vertex in  $G$  and its complement induce strongly regular graphs. From this result, we see strong combinatorial properties attached to spin models.

Four years later, Jaeger [12] obtained the surprising result that every spin model belongs to the Bose-Mesner algebra of a formally self-dual association scheme.

Jaeger's profound discoveries caught the interests of researchers in Algebraic Combinatorics. They produced a series of results on spin models and association schemes. Here, we document some other classical findings of this area.

Shortly after Jaeger announced that every spin model belongs to a formally self-dual Bose-Mesner algebra, Nomura [23] came up with a substantially simpler algebraic construction of a Bose-Mesner algebra from a spin model. In fact, Nomura showed that each type-II matrix gives rise to a Bose-Mesner algebra, now known as the Nomura algebra of the type-II matrix. Jaeger, Matsumoto and Nomura [14] examined further the Nomura algebras of type-II matrices, and concluded that a type-II matrix  $W$  belongs to its Nomura algebra if and only if it is a spin model up to scalar multiplication.

In 1997, Bannai, Bannai and Jaeger [3] found a necessary condition for a formally self-dual Bose-Mesner algebra to be the Nomura algebra of some spin model. They showed that the matrix of eigenvalues of the Nomura algebra of a spin model

must satisfy the modular invariance property. Unfortunately, we still do not have any way to tell whether a Bose-Mesner algebra is the Nomura algebra of some type-II matrix.

Much less is known about the connection between four-weight spin models and association schemes. The only result we know is due to Bannai, Guo and Huang [1, 9]. They proved that if  $(W_1, W_2, W_3, W_4; d)$  is a four-weight spin model, then the Nomura algebras of  $W_1$ ,  $W_2$ ,  $W_3$  and  $W_4$  coincide, and this algebra is formally self-dual.

## 1.5 Directory

In this section, we provide a layout of this thesis.

Chapter 2 lays the groundwork for subsequent chapters. In Sections 2.1 and 2.2, we introduce the Nomura algebras of two matrices  $(A, B)$  and the duality map. In Section 2.3, we present a useful tool called the Exchange Lemma. These three sections are joint work by Godsil, Munemasa and the author [6]. In Section 2.4, we examine effects on the Nomura algebras of  $(A, B)$  when  $A$  and  $B$  are multiplied by some monomial matrices. In Section 2.5, we show how the Nomura algebras of  $(A_1 \oplus A_2, B_1 \oplus B_2)$  are related to the Nomura algebras of  $(A_1, B_1)$  and  $(A_2, B_2)$ . The findings in these two sections generalize similar results from Jaeger, Matsumoto and Nomura [14].

Sections 3.1, 3.2 and 3.7 survey existing results about the type-II matrices and their Nomura algebras. We present new proofs using the tools developed in Chapter 2 and they are due to Godsil, Munemasa and the author. Sections 3.3 to 3.6 overview the standard theory of association schemes. In Section 3.8, we study the properties of the Nomura algebras of two type-II matrices. These properties are

used substantially in subsequent chapters. The results in this section are joint work by Godsil, Munemasa and the author.

We introduce one-sided Jones pairs and Jones pairs in Chapter 4. Based on Jones' construction, we build a braid group representation from a Jones pair. More importantly, we show that invertible Jones pairs are equivalent to four-weight spin models. In Sections 4.7 and 4.8, we extend Jaeger's notion of gauge equivalence to one-sided Jones pairs. The results in this chapter are joint work by Godsil, Munemasa and the author.

We examine spin models as Jones pairs in Chapter 5. In Section 5.2, we provide Jaeger, Matsumoto and Nomura's derivation of the modular invariance equation. In Section 5.3, we extend Curtin and Nomura's theorem to the strongly hyper-self-duality of the Terwilliger algebra of the Nomura algebra of a spin model. In Section 5.4, we provide a new and shorter proof of one direction of Jaeger's result on spin models from two class association schemes. The last section contains our new proof, using Jones pairs, to Jaeger and Nomura's construction of the symmetric and non-symmetric Hadamard spin models.

Chapter 6 contains the main results of this thesis. In Section 6.1, we construct a type-II matrix  $W$  from an invertible Jones pair  $(A, B)$ . This type-II matrix gives a formally dual pair of Bose-Mesner algebras. We determine the dimensions, the basis of Schur idempotents and the basis of principal idempotents for these algebras. In Section 6.2, we extend Nomura's construction [22] to build a pair of symmetric spin models,  $V$  and  $V'$ , from each four-weight spin model. We design an algorithm to exhaustively search for four-weight spin models, which is described in Section 6.3. Sections 6.4 to 6.6 document our observation of the relations of the Nomura algebras of  $A$ ,  $W$ ,  $V$  and  $V'$ . Section 6.7 documents what we know about the simplest class of Jones pairs. Finally, we discuss several directions of future work.

## Chapter 2

### Nomura Algebras

In [23], Nomura constructed  $n^2$  vectors from a symmetric type-II matrix  $A$ . He then considered the set of all matrices for which these vectors are eigenvectors, and showed that this set forms a Bose-Mesner algebra. This algebra is now called the Nomura algebra of  $A$ , and denoted by  $\mathcal{N}_A$ . Nomura went further and proved that if  $A$  is a spin model, then  $A$  belongs to its Nomura algebra. Thus Nomura algebras play a significant role in the theory of spin models.

During our investigation of Jones pairs, Godsil generalized Nomura's construction. Given a pair of matrices  $A$  and  $B$ , we consider the set of matrices of which for all  $i, j = 1, \dots, n$ ,

$$A\epsilon_i \circ B\epsilon_j$$

are eigenvectors. We call this set of matrices the Nomura algebra of  $(A, B)$ , and denote it by  $\mathcal{N}_{A,B}$ . In general, the Nomura algebra of  $(A, B)$  is not a Bose-Mesner algebra. However it will become clear in subsequent chapters that  $\mathcal{N}_{A,B}$  is a powerful tool in the study of Jones pairs.

Sections 2.1 and 2.2 introduce the Nomura algebra of  $(A, B)$  and a special function on  $\mathcal{N}_{A,B}$  called the duality map. The results in these two sections provide technical background for Chapters 3 to 6. They also appear in the paper [6] by Godsil, Munemasa and the author.

In Sections 2.4 and 2.5, we generalize several results due to Jaeger, Matsumoto and Nomura in [14], which are in turn extensions of Nomura's results on symmetric type-II matrices.

## 2.1 Nomura Algebra of Two Matrices

We introduce the Nomura algebras of a pair of  $n \times n$  matrices  $(A, B)$ , denoted by  $\mathcal{N}_{A,B}$  and  $\mathcal{N}'_{A,B}$ . We study the properties of these algebras and a map

$$\Theta_{A,B} : \mathcal{N}_{A,B} \rightarrow \mathcal{N}'_{A,B}$$

called the duality map. In this section, we resist the temptation of adding dispensable conditions on  $A$  and  $B$ .

For any pair of  $n \times n$  matrices  $A$  and  $B$ , their *Schur product*  $A \circ B$  is the entry-wise product of  $A$  and  $B$ . That is, for  $i, j = 1, \dots, n$ ,

$$(A \circ B)_{i,j} = A_{i,j} B_{i,j}.$$

The  $n \times n$  matrix of all ones, denoted by  $J$ , is the identity with respect to the Schur product. If all entries of  $A$  are non-zero, then the matrix  $A^{(-)}$  with

$$A_{i,j}^{(-)} = \frac{1}{A_{i,j}}$$

satisfies  $A \circ A^{(-)} = J$ . We say that  $A$  is *Schur invertible* and  $A^{(-)}$  is the *Schur inverse* of  $A$ . Note that  $M_n(\mathbb{C})$  forms a commutative algebra with respect to the Schur product.

Let  $\{\epsilon_1, \dots, \epsilon_n\}$  denote the standard basis for  $\mathbb{C}^n$ . Given a pair of  $n \times n$  matrices  $A$  and  $B$ , we obtain the following set of  $n^2$  vectors

$$\{A\epsilon_i \circ B\epsilon_j : i, j = 1, \dots, n\}.$$

The vector  $A\epsilon_i \circ B\epsilon_j$  equals the Schur product of the  $i$ -th column of  $A$  with the  $j$ -th column of  $B$ . In most cases we encounter, these  $n^2$  vectors are not distinct.

We define  $\mathcal{N}_{A,B}$  to be the set of  $n \times n$  matrices for which  $A\epsilon_i \circ B\epsilon_j$  is an eigenvector, for all  $i, j = 1, \dots, n$ .

**Lemma 2.1.1** *Let  $A$  and  $B$  be  $n \times n$  matrices. Then the set  $\mathcal{N}_{A,B}$  is a vector space which is closed under multiplication, and it contains the identity matrix.*

*Proof.* Let  $M, N \in \mathcal{N}_{A,B}$ . For  $i, j = 1, \dots, n$ , there exist  $m_{ij}$  and  $n_{ij}$  such that  $M(A\epsilon_i \circ B\epsilon_j) = m_{ij}(A\epsilon_i \circ B\epsilon_j)$  and  $N(A\epsilon_i \circ B\epsilon_j) = n_{ij}(A\epsilon_i \circ B\epsilon_j)$ . Now,

$$(MN)(A\epsilon_i \circ B\epsilon_j) = m_{ij}n_{ij}(A\epsilon_i \circ B\epsilon_j).$$

So we have  $MN \in \mathcal{N}_{A,B}$ . It is immediate that  $A\epsilon_i \circ B\epsilon_j$  is an eigenvector for  $I$ , for all  $i, j = 1, \dots, n$ . So  $I \in \mathcal{N}_{A,B}$ .  $\square$

We call  $\mathcal{N}_{A,B}$  the *Nomura algebra for the pair  $(A, B)$* . Now for each matrix  $M$  in  $\mathcal{N}_{A,B}$ , we use  $\Theta_{A,B}(M)$  to denote the  $n \times n$  matrix whose  $ij$ -entry is the eigenvalue of  $M$  with respect to the eigenvector  $A\epsilon_i \circ B\epsilon_j$ . That is,

$$M(A\epsilon_i \circ B\epsilon_j) = \Theta_{A,B}(M)_{ij}(A\epsilon_i \circ B\epsilon_j).$$

for  $i, j = 1, \dots, n$ . We call  $\Theta_{A,B}$  the *duality map*. We use  $\mathcal{N}'_{A,B}$  to denote the image of  $\mathcal{N}_{A,B}$  under  $\Theta_{A,B}$ . The following two lemmas tell us that  $\mathcal{N}'_{A,B}$  is an algebra with respect to the Schur product.

**Lemma 2.1.2** *Let  $A$  and  $B$  be  $n \times n$  matrices. Then for all  $M$  and  $N$  in  $\mathcal{N}'_{A,B}$ , we have*

$$\Theta_{A,B}(MN) = \Theta_{A,B}(M) \circ \Theta_{A,B}(N).$$

*Proof.* If  $M$  and  $N$  lie in  $\mathcal{N}_{A,B}$  then

$$\begin{aligned} (MN) \cdot A\epsilon_i \circ B\epsilon_j &= \Theta_{A,B}(M)_{i,j} \Theta_{A,B}(N)_{i,j} \cdot A\epsilon_i \circ B\epsilon_j \\ &= (\Theta_{A,B}(M) \circ \Theta_{A,B}(N))_{i,j} \cdot A\epsilon_i \circ B\epsilon_j. \end{aligned}$$

So the result follows.  $\square$

**Lemma 2.1.3** *Let  $A$  and  $B$  be  $n \times n$  matrices. Then  $\mathcal{N}'_{A,B}$  is a vector space in  $\mathbf{M}_n(\mathbb{C})$  which is closed under the Schur product, and it contains  $J$ .*

*Proof.* It follows from Lemma 2.1.2 that  $\mathcal{N}'_{A,B}$  is closed under the Schur product. Moreover, since  $I \cdot A\epsilon_i \circ B\epsilon_j = 1 \cdot A\epsilon_i \circ B\epsilon_j$ , we have  $\Theta_{A,B}(I) = J$ .  $\square$

The following standard result (Theorem 2.6.1 [5]) says that there is a basis of Schur idempotents for  $\mathcal{N}'_{A,B}$ .

**Lemma 2.1.4** *Let  $\mathcal{N} \subseteq \mathbf{M}_n(\mathbb{C})$  be a vector space that is closed under the Schur product. Assume further that  $J \in \mathcal{N}$ . Then  $\mathcal{N}$  has a basis of Schur idempotents.*

*Proof.* Choose  $M$  to be a matrix in  $\mathcal{N}$  with the maximal number of distinct non-zero entries. We can write  $M = \sum_{i=1}^d \alpha_i A_i$ , where  $\alpha_1, \dots, \alpha_d$  are distinct and

$A_1, \dots, A_d$  are 01-matrices that sum to  $J$  and satisfy  $A_i \circ A_j = \delta_{ij} A_i$ . Then the  $A_i$ 's are linearly independent. For any  $i = 1, \dots, d$ , the following product

$$\left( \prod_{j=1, j \neq i}^d \frac{1}{\alpha_i - \alpha_j} \right) (M - \alpha_1 J) \circ \dots \circ (M - \alpha_{i-1} J) \circ (M - \alpha_{i+1} J) \circ \dots \circ (M - \alpha_d J)$$

equals  $A_i$  and belongs to  $\mathcal{N}$ . So the dimension of  $\mathcal{N}$  is no less than  $d$ .

Suppose  $N \in \mathcal{N}$  is not a linear combination of  $A_1, \dots, A_d$ . Let  $N = \sum_{j=1}^k \beta_j B_j$  where  $\beta_1, \dots, \beta_k$  are distinct and  $B_1, \dots, B_k$  are 01-matrices that are mutually orthogonal with respect to the Schur product.

Since  $\{B_1, \dots, B_k\} \neq \{A_1, \dots, A_d\}$  and  $\mathcal{N}$  is closed under the Schur product,

$$\{A_i \circ B_j : i = 1, \dots, d; \quad j = 1, \dots, k\}$$

is a set of more than  $d$  Schur idempotents in  $\mathcal{N}$ . As a result, any linear combination of the matrices in this set with distinct non-zero coefficients contradicts the choice of  $M$ . We conclude that  $\{A_1, \dots, A_d\}$  spans  $\mathcal{N}$ .  $\square$

For example, when  $A = I$  and  $B = J$ , the set of eigenvectors is  $\{e_1, \dots, e_n\}$ . Therefore  $\mathcal{N}_{A,B}$  equals the set of  $n \times n$  diagonal matrices. Since for  $i, j = 1, \dots, n$ ,

$$Ie_i \circ Je_j = e_i,$$

we get  $\Theta_{I,J}(D) = DJ$  for any diagonal matrix  $D$ . The basis of Schur idempotents of  $\mathcal{N}'_{I,J}$  equals  $\{e_i \mathbf{1}^T : i = 1, \dots, n\}$ . In this case, both algebras have dimension  $n$ .

We can say more about these algebras if we assume  $A$  is invertible and  $B$  is Schur invertible. More importantly, we will see in Chapter 4 that these conditions

hold when  $(A, B)$  is a Jones pair.

**Theorem 2.1.5** *Let  $A$  be an invertible matrix and let  $B$  be a Schur invertible matrix. Then the duality map  $\Theta_{A,B}$  is an isomorphism from the algebra  $\mathcal{N}_{A,B}$  (with respect to matrix multiplication) to  $\mathcal{N}'_{A,B}$  (with respect to the Schur product). Moreover,  $\mathcal{N}_{A,B}$  is a commutative algebra.*

*Proof.* For each  $r$ , the set of vectors

$$\{Ae_i \circ Be_r : \text{for } i = 1, \dots, n\}$$

is linearly independent and hence it is a basis for  $\mathbb{C}^n$ . As a result, for every  $M$  in  $\mathcal{N}_{A,B}$ , each column of  $\Theta_{A,B}(M)$  contains all  $n$  eigenvalues of  $M$ . Now we conclude that  $\Theta_{A,B}(M) = \Theta_{A,B}(M')$  if and only if  $M = M'$ . So the map  $\Theta_{A,B}$  is a bijection. By Lemma 2.1.2, if  $M, N \in \mathcal{N}_{A,B}$ , then

$$\begin{aligned} \Theta_{A,B}(M.N) &= \Theta_{A,B}(M) \circ \Theta_{A,B}(N) \\ &= \Theta_{A,B}(N) \circ \Theta_{A,B}(M) \\ &= \Theta_{A,B}(N.M). \end{aligned}$$

It follows that  $MN = NM$ , for all  $M, N \in \mathcal{N}_{A,B}$ . □

## 2.2 The Duality Map $\Theta_{A,B}$

We now define two types of *endomorphisms* of  $\mathbf{M}_n(\mathbb{C})$ . They are used by Jones [17] to construct braid group representations from spin models. More importantly for us, Jones pairs are defined using these endomorphisms, so they show up everywhere in this thesis.

Let  $C$  be an  $n \times n$  matrix. We define two endomorphisms of  $\mathbf{M}_n(\mathbb{C})$ :  $X_C$  and  $\Delta_C$ , as follows

$$X_C(M) = CM, \quad \Delta_C = C \circ M.$$

Now  $X_C$  is invertible if and only if  $C$  is invertible. Similarly,  $\Delta_C$  is invertible if and only if  $C$  is Schur invertible. If  $A$  is invertible and  $B$  is Schur invertible, we have

$$X_A^{-1} = X_{A^{-1}}, \quad \Delta_B^{-1} = \Delta_{B^{-1}}.$$

It is worth noting that  $\Delta_B$  and  $\Delta_C$  commute for all  $n \times n$  matrices  $B$  and  $C$ . Moreover, if  $D$  is a diagonal matrix, then  $X_D = \Delta_{DJ}$ . Thus  $X_D$  and  $\Delta_C$  commute.

Let  $M$  and  $N$  be two  $n \times n$  matrices. Then  $\text{tr}(M^T N) = \text{sum}(M \circ N)$  is a non-degenerate bilinear form on  $\mathbf{M}_n(\mathbb{C})$ . If  $Y$  is an endomorphism of  $\mathbf{M}_n(\mathbb{C})$ , we use  $Y^T$  to denote the adjoint of  $Y$  relative to this bilinear form. We call it the *transpose* of  $Y$ . It is straightforward to verify that

$$X_C^T = X_{C^T}, \quad \Delta_C^T = \Delta_C.$$

In Section 2.1,  $\Theta_{A,B}(R)$  is defined as a store of eigenvalues of  $R \in \mathcal{N}_{A,B}$ . The following theorem gives an equivalent definition of the duality map using the endomorphisms  $X_A$  and  $\Delta_B$ . This definition helps us identify when a matrix belongs to  $\mathcal{N}_{A,B}$  and it is used repeatedly in the rest of the thesis.

**Theorem 2.2.1** *Let  $A, B \in \mathbf{M}_n(\mathbb{C})$ . Then  $R \in \mathcal{N}_{A,B}$  and  $S = \Theta_{A,B}(R)$  if and*

only if

$$X_R \Delta_B X_A = \Delta_B X_A \Delta_S. \quad (2.1)$$

*Proof.* Let  $E_{ij} = e_i e_j^T$  be the  $n \times n$  matrix with one in its  $ij$ -entry and zero elsewhere. The set  $\{E_{ij} : i, j = 1, \dots, n\}$  is a basis for  $\mathbf{M}_n(\mathbb{C})$ . So (2.1) holds if and only if

$$X_R \Delta_B X_A(E_{ij}) = \Delta_B X_A \Delta_S(E_{ij}),$$

for all  $i, j = 1, \dots, n$ . Now the left-hand side equals

$$\begin{aligned} X_R \Delta_B X_A(e_i e_j^T) &= R(B \circ A(e_i e_j^T)) \\ &= R(Be_j \circ Ae_i) e_j^T. \end{aligned}$$

and the right-hand side equals

$$\begin{aligned} \Delta_B X_A \Delta_S(e_i e_j^T) &= B \circ (A(S \circ e_i e_j^T)) \\ &= S_{i,j} (Be_j \circ Ae_i) e_j^T. \end{aligned}$$

Both sides are equal if and only if

$$R(Be_j \circ Ae_i) = S_{i,j} (Be_j \circ Ae_i).$$

We conclude that the relation (2.1) holds if and only if  $S = \Theta_{A,B}(R)$ . □

## 2.3 The Exchange Lemma

The Exchange Lemma results from the trivial idea that the set

$$\{A\epsilon_i \circ B\epsilon_j : i, j = 1, \dots, n\}$$

does not change by swapping  $A$  and  $B$ . Therefore  $\Lambda'_{A,B} = \Lambda'_{B,A}$ . If  $S = \Theta_{A,B}(R)$  then

$$R B\epsilon_i \circ A\epsilon_j = S_{j,i} B\epsilon_i \circ A\epsilon_j,$$

which is the same as writing  $\Theta_{B,A}(R) = S^T$  and  $\Lambda'_{B,A} = \Lambda'_{A,B}{}^T$ . As a result, we have

$$X_R \Delta_B X_A = \Delta_B X_A \Delta_S$$

if and only if

$$X_R \Delta_A X_B = \Delta_A X_B \Delta_{S^T}.$$

The lemma below, called the *Exchange Lemma*, gives a more general form of the above equivalence. It was first discovered by Munemasa during our investigation of Jones pairs. This result is of much greater importance than it may first appear. It is applied at numerous places in this thesis.

**Lemma 2.3.1** *If  $A, B, C, Q, R, S \in \mathbf{M}_n(\mathbb{C})$  then*

$$X_A \Delta_B X_C = \Delta_Q X_R \Delta_S$$

if and only if

$$X_A \Delta_C X_B = \Delta_R X_Q \Delta_{S^T}$$

*Proof.* Pick any  $i, j \in \{1, \dots, n\}$ . Applying both sides of the first relation to  $E_{ij}$  gives

$$A (B\epsilon_j \circ C\epsilon_i) \epsilon_j^T = S_{i,j} (Q\epsilon_j \circ R\epsilon_i) \epsilon_i^T.$$

By multiplying each side by  $e_j \epsilon_i^T$ , we get

$$A (C\epsilon_i \circ B\epsilon_j) \epsilon_i^T = (S^T)_{j,i} (R\epsilon_i \circ Q\epsilon_j) \epsilon_i^T$$

which is the same as

$$X_A \Delta_C X_B (E_{ji}) = \Delta_R X_Q \Delta_{S^T} (E_{ji}).$$

So the result follows.  $\square$

Using the Exchange Lemma, we obtain several equivalent forms of (2.1).

**Corollary 2.3.2** *If  $A$  is invertible and  $B$  is Schur invertible, then the following are equivalent:*

- a.  $R \in \mathcal{N}_{A,B}$  and  $S = \Theta_{A,B}(R)$
- b.  $X_{B^T} \Delta_A X_{R^T} = \Delta_{S^T} X_{B^T} \Delta_A$
- c.  $\Delta_R X_{B(-)} \Delta_{B^T} = X_A \Delta_{A^{-1}} X_S$
- d.  $\Delta_{B^T} X_{B(-)^T} \Delta_R = X_{S^T} \Delta_{A^{-1}} X_{A^T}$

*Proof.* By Theorem 2.2.1 and the Exchange Lemma to (2.1), Condition (a) holds if and only if

$$X_R \Delta_A X_B = \Delta_A X_B \Delta_{S^T}.$$

Taking the transpose of both sides yields (b). So (a) and (b) are equivalent. Moreover we can write (2.1) as

$$\Delta_{B(-)} X_R \Delta_B = X_A \Delta_S X_{A^{-1}}.$$

Applying the Exchange Lemma gives (c). Lastly, we obtain (d) by taking the transpose of each side of (c). As a result, Conditions (a), (c) and (d) are equivalent.  $\square$

Assuming all diagonal entries of  $A$  are non-zero, we use Corollary 2.3.2 (b) to get an explicit formula of  $\Theta_{A,B}$ . All equivalent forms of Equation (2.1) give formulae of  $\Theta_{A,B}$ . We choose this particular one because we will use it in the proof of the modular invariance equation in Section 5.2.

**Lemma 2.3.3** *Let  $A$  and  $B$  be  $n \times n$  matrices, and let  $R \in \mathcal{N}_{A,B}$ . If  $A$  and  $A \circ I$  are invertible and  $B$  is Schur invertible, then*

$$\Theta_{A,B}(R) = B^{(-)} \circ ((A \circ I)^{-1} (A^T \circ R) B).$$

*Proof.* Let  $S = \Theta_{A,B}(R)$ . Applying Corollary 2.3.2 (b) to  $I$ , we get

$$B^T (A \circ (R^T I)) = S^T \circ (B^T (A \circ I)).$$

By taking the transpose of each side, we have

$$S \circ ((A^T \circ I)B) = (A^T \circ R)B.$$

Since  $A^T \circ I = A \circ I$  is a diagonal matrix, the left-hand side equals  $(A \circ I)(S \circ B)$ .

Hence we get

$$S = B^{(-)} \circ ((A \circ I)^{-1}(A^T \circ R)B).$$

□

## 2.4 Transformations of $(A, B)$

Two matrices  $A$  and  $C$  are *monomially equivalent* if  $C = MAN$ , where  $M$  and  $N$  are products of permutation matrices and diagonal matrices. When  $A$  is a type-II matrix, Jaeger, Matsumoto and Nomura examined the relation between the Nomura algebras of the pairs  $(A, A^{(-)})$  and  $(C, C^{(-)})$ , (see Proposition 2 and 3 in [14]). In this section, we extend their result to  $\mathcal{N}_{A,B}$  for any pair of  $n \times n$  matrices  $(A, B)$ .

As we will see in Sections 4.7 and 4.8, if  $(A_1, B_1)$  and  $(A_2, B_2)$  are gauge equivalent invertible Jones Pairs, then  $A_1$  and  $A_2$  are monomially equivalent, and so are  $B_1$  and  $B_2$ . So the following lemmas tell us how the Nomura algebras of gauge-equivalent invertible Jones pairs relate to each other.

**Lemma 2.4.1** *If  $D, E$  and  $F$  are invertible diagonal  $n \times n$  matrices, then*

$$\mathcal{N}_{DAE, D^{-1}BF} = \mathcal{N}_{A,B}, \quad \text{and} \quad \mathcal{N}'_{DAE, D^{-1}BF} = \mathcal{N}'_{A,B}.$$

*Proof.* Since  $D (B \circ C) = (DB \circ C)$ , we have

$$(DAE\epsilon_i) \circ (D^{-1}BF\epsilon_j) = (E)_{i,i}(F)_{j,j} A\epsilon_i \circ B\epsilon_j$$

which implies the lemma.  $\square$

**Lemma 2.4.2** *If  $P, Q$  and  $R$  are  $n \times n$  permutation matrices, then*

$$\mathcal{N}_{PAQ, PBR} = P \mathcal{N}_{A,B} P^{-1}, \quad \text{and} \quad \mathcal{N}'_{PAQ, PBR} = Q^{-1} \mathcal{N}'_{A,B} R.$$

*Proof.* Let  $M \in \mathcal{N}_{A,B}$ . Then for all  $i, j = 1, \dots, n$ ,

$$\begin{aligned} (PMP^{-1}) (PAQ\epsilon_i \circ PBR\epsilon_j) &= PMP^{-1}P (AQ\epsilon_i \circ BR\epsilon_j) \\ &= PM (AQ\epsilon_i \circ BR\epsilon_j). \end{aligned}$$

If  $Q\epsilon_i = \epsilon_{i'}$  and  $R\epsilon_j = \epsilon_{j'}$ , we get

$$\begin{aligned} (PMP^{-1}) (PAQ\epsilon_i \circ PBR\epsilon_j) &= PM (A\epsilon_{i'} \circ B\epsilon_{j'}) \\ &= \Theta_{A,B}(M)_{i',j'} (PAQ\epsilon_i \circ PBR\epsilon_j). \end{aligned}$$

So the first equality of the lemma follows. Moreover,  $\Theta_{A,B}(M)_{i',j'} = (Q^T \Theta_{A,B}(M) R)_{i,j}$ , and the second equality holds.  $\square$

## 2.5 Tensor Products

We use  $\otimes$  to denote both the Kronecker product of two matrices and the tensor product of two algebras. The next lemma generalizes Proposition 7 in [14], due to

Jaeger, Matsumoto and Nomura. In particular, we will use this lemma in Section 4.1 to show that the tensor product of two Jones pairs is also a Jones pair.

**Lemma 2.5.1** *Let  $A_1, B_1 \in M_m(\mathbb{C})$  and  $A_2, B_2 \in M_n(\mathbb{C})$ . If  $A_1, A_2$  are invertible, and  $B_1, B_2$  are Schur invertible, then*

$$\mathcal{N}_{A_1 \otimes A_2, B_1 \otimes B_2} = \mathcal{N}_{A_1, B_1} \otimes \mathcal{N}_{A_2, B_2},$$

and

$$\mathcal{N}'_{A_1 \otimes A_2, B_1 \otimes B_2} = \mathcal{N}'_{A_1, B_1} \otimes \mathcal{N}'_{A_2, B_2}.$$

*Proof.* Since  $\{e_i \otimes e_h : i = 1, \dots, m \text{ and } h = 1, \dots, n\}$  form a basis for  $\mathbb{C}^{mn}$ , we can write the eigenvectors for the matrices in  $\mathcal{N}_{A_1 \otimes A_2, B_1 \otimes B_2}$  as

$$\begin{aligned} & ((A_1 \otimes A_2)(e_i \otimes e_h)) \circ ((B_1 \otimes B_2)(e_j \otimes e_k)) \\ &= (A_1 e_i \otimes A_2 e_h) \circ (B_1 e_j \otimes B_2 e_k) \\ &= (A_1 e_i \circ B_1 e_j) \otimes (A_2 e_h \circ B_2 e_k), \end{aligned}$$

for all  $i, j = 1, \dots, m$  and  $h, k = 1, \dots, n$ . This implies if  $M \in \mathcal{N}_{A_1, B_1}$  and  $N \in \mathcal{N}_{A_2, B_2}$ , then  $M \otimes N \in \mathcal{N}_{A_1 \otimes A_2, B_1 \otimes B_2}$ . Consequently,

$$\mathcal{N}_{A_1, B_1} \otimes \mathcal{N}_{A_2, B_2} \subseteq \mathcal{N}_{A_1 \otimes A_2, B_1 \otimes B_2}.$$

Now  $A_1 \otimes A_2$  is invertible and  $B_1 \otimes B_2$  is Schur invertible. By Theorem 2.1.5, we have for  $i = 1, 2$ ,  $\dim \mathcal{N}_{A_i, B_i} = \dim \mathcal{N}'_{A_i, B_i}$  and  $\dim \mathcal{N}_{A_1 \otimes A_2, B_1 \otimes B_2} = \dim \mathcal{N}'_{A_1 \otimes A_2, B_1 \otimes B_2}$ .

It is sufficient to show that  $\mathcal{N}'_{A_1 \otimes A_2, B_1 \otimes B_2}$  and  $\mathcal{N}'_{A_1, B_1} \otimes \mathcal{N}'_{A_2, B_2}$  have the same dimension. If the Hermitian product of the two eigenvectors

$$((A_1 \otimes A_2)(e_{i_1} \otimes e_{h_1})) \circ ((B_1 \otimes B_2)(e_{j_1} \otimes e_{k_1})),$$

and

$$((A_1 \otimes A_2)(e_{i_2} \otimes e_{h_2})) \circ ((B_1 \otimes B_2)(e_{j_2} \otimes e_{k_2}))$$

is non-zero, then they belong to the same eigenspace of  $M$ , for all  $M$  in  $\mathcal{N}'_{A_1 \otimes A_2, B_1 \otimes B_2}$ .

Thus

$$\Theta_{A_1 \otimes A_2, B_1 \otimes B_2}(M)_{((i_1, h_1), (j_1, k_1))} = \Theta_{A_1 \otimes A_2, B_1 \otimes B_2}(M)_{((i_2, h_2), (j_2, k_2))}.$$

But the Hermitian product of those two eigenvectors equals

$$\langle A_1 e_{i_1} \otimes B_1 e_{j_1}, A_1 e_{i_2} \otimes B_1 e_{j_2} \rangle \langle A_2 e_{h_1} \otimes B_2 e_{k_1}, A_2 e_{h_2} \otimes B_2 e_{k_2} \rangle,$$

and so the  $((i_1, h_1), (j_1, k_1))$  and  $((i_2, h_2), (j_2, k_2))$  entries of a Schur idempotent of  $\mathcal{N}'_{A_1 \otimes A_2, B_1 \otimes B_2}$  equal to one if and only if there exist a Schur idempotent  $F$  in  $\mathcal{N}'_{A_1, B_1}$  and a Schur idempotent  $G$  in  $\mathcal{N}'_{A_2, B_2}$  such that

$$F_{i_1, j_1} = F_{i_2, j_2} = 1, \quad \text{and} \quad G_{h_1, k_1} = G_{h_2, k_2} = 1.$$

In other words, the set of matrices  $F_r \otimes G_s$ , for all Schur idempotents  $F_r$  of  $\mathcal{N}'_{A_1, B_1}$  and all Schur idempotents  $G_s$  of  $\mathcal{N}'_{A_2, B_2}$ , forms a basis of Schur idempotents for  $\mathcal{N}'_{A_1 \otimes A_2, B_1 \otimes B_2}$ . So the result follows.  $\square$

## Chapter 3

# Type-II Matrices and Nomura Algebras

Jaeger, Matsumoto and Nomura showed in [14] that the type-II condition of a spin model is sufficient for the existence of a Bose-Mesner algebra containing the spin model. Further, they proved that a type-II matrix belongs to its Nomura algebra if and only if it is a spin model up to scalar multiplication. If a matrix satisfies the type-II condition of a spin model, we call it a type-II matrix.

Since the type-II condition plays a crucial role in the connection of spin models to Bose-Mesner algebras, we feel obliged to give a detailed treatment to type-II matrices and their Nomura algebras, see Sections 3.1, 3.2 and 3.7. Most results in these sections are originally due to Jaeger, Matsumoto and Nomura in [14]. However, we present new proofs, using the tools developed in the previous chapter, given by Godsil, Munemasa and the author in [6]. Sections 3.3 to 3.6 consist of the standard theory of Bose-Mesner algebras and association schemes. Section 3.8 examines the Nomura algebra of a pair of type-II matrices  $A$  and  $B$ . This section gives the foundation for the theory of invertible Jones pairs in Chapters 4 and 6.

The results in Section 3.8 also appear in [6].

### 3.1 Type-II Matrices

If an  $n \times n$  Schur-invertible matrix  $A$  satisfies

$$AA^{(-)T} = nI, \quad (3.1)$$

we call it a *type-II matrix*. In other words,  $A$  is a type-II matrix if and only if

$$A^{-1} = n^{-1}A^{(-)T},$$

or equivalently, for  $i, j = 1, \dots, n$ ,

$$\sum_{k=1}^n \frac{A_{i,k}}{A_{j,k}} = \delta_{i,j}n \quad \text{and} \quad \sum_{k=1}^n \frac{A_{k,i}}{A_{k,j}} = \delta_{i,j}n.$$

Note that Equation (3.1) is equivalent to the definition of type-II matrix in Section 1.1. Also note that  $A$  is a type-II matrix if and only if  $A^T$  is also type II.

Suppose  $A$  is a type-II matrix. If  $D_1$  and  $D_2$  are invertible diagonal matrices, then

$$(D_1AD_2)(D_1AD_2)^{(-)T} = (D_1AD_2)(D_2^{-1}A^{(-)T}D_1^{-1}) = nI.$$

Therefore  $D_1AD_2$  is also a type-II matrix. Similarly, if  $P_1$  and  $P_2$  are permutation matrices, then

$$(P_1AP_2)(P_1AP_2)^{(-)T} = (P_1AP_2)(P_2^T A^{(-)T} P_1^T) = nI.$$

and  $P_1AP_2$  is a type-II matrix. Note that  $A$ ,  $P_1AP_2$  and  $D_1AD_2$  are monomially equivalent. The relation between the Nomura algebras of two monomially equivalent matrices is discussed in Section 2.4.

**Example 3.1.1** We list all type-II matrices of orders two to five, up to monomially equivalence. For details, please see [14] and [24].

a.  $n = 2$ : 
$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

b.  $n = 3$ : 
$$\begin{pmatrix} 1 & 1 & \omega \\ \omega & 1 & 1 \\ 1 & \omega & 1 \end{pmatrix}, \text{ where } \omega \text{ is a cube root of unity.}$$

c.  $n = 4$ : 
$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \lambda & -\lambda \\ 1 & -1 & -\lambda & \lambda \end{pmatrix}, \text{ for any non-zero complex number } \lambda.$$

d.  $n = 5$ : for  $\eta$  a fifth-root of unity.

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \eta & \eta^2 & \eta^3 & \eta^4 \\ 1 & \eta^2 & \eta^4 & \eta & \eta^3 \\ 1 & \eta^3 & \eta & \eta^4 & \eta^2 \\ 1 & \eta^4 & \eta^3 & \eta^2 & \eta \end{pmatrix} :$$

and

$$\left( \frac{-5 - \sqrt{5}}{2} \right) I + J; \quad \left( \frac{-5 + \sqrt{5}}{2} \right) I + J.$$

The Kronecker product of two type-II matrices is also type II. So there exist infinitely many type-II matrices. Some significant examples are spin models and four-weight spin models. Furthermore, if a matrix is unitary and all its entries are roots of unity, then it is type II. These matrices are objects of interest in the theory of Von Neumann algebras [16].

### 3.2 Nomura Algebras of a Type-II Matrix

Godsil [8] observed the following condition on  $\mathcal{N}_{A,A^{(-)}}$  that is equivalent to  $A$  being type II.

**Lemma 3.2.1** *Let  $A$  be both Schur invertible and invertible. Then  $A$  is type II if and only if  $J \in \mathcal{N}_{A,A^{(-)}}$ . Moreover,  $\Theta_{A,A^{(-)}}(J) = nI$ .*

*Proof.* Assume  $J \in \mathcal{N}_{A,A^{(-)}}$ . For each  $j = 1, \dots, n$ , the set

$$\{Ae_1 \circ A^{(-)}e_j, \dots, Ae_n \circ A^{(-)}e_j\}$$

is a basis for  $\mathbb{C}^n$  consisting eigenvectors of  $J$ . Now  $Ae_j \circ A^{(-)}e_j$  equals  $\mathbf{1}$ , the vector of all ones, which is the only eigenvector of  $J$  with non-zero eigenvalue. So we conclude that

$$J Ae_i \circ A^{(-)}e_j = \delta_{i,j} n Ae_i \circ A^{(-)}e_j.$$

That is, for all  $i, j = 1, \dots, n$ ,

$$\sum_{k=1}^n \frac{A_{k,i}}{A_{k,j}} = \delta_{i,j} n.$$

and  $A$  is type II.

The converse is straightforward.  $\square$

Following existing conventions, we use  $\mathcal{N}_A$ ,  $\mathcal{N}'_A$  and  $\Theta_A$  to stand for  $\mathcal{N}_{A,A^{(-)}}$ ,  $\mathcal{N}'_{A,A^{(-)}}$  and  $\Theta_{A,A^{(-)}}$  respectively.

**Theorem 3.2.2** *Let  $A$  be a type-II matrix. If  $R \in \mathcal{N}_A$  then  $\Theta_A(R) \in \mathcal{N}_{A^T}$  and  $\Theta_{A^T}(\Theta_A(R)) = nR^T$ .*

*Proof.* Letting  $B = A^{(-)}$  in Corollary 2.3.2 (c), we have  $R \in \mathcal{N}_A$  and  $S = \Theta_A(R)$  if and only if

$$\Delta_R X_A \Delta_{A^{(-)T}} = X_A \Delta_{A^{(-)}} X_S,$$

which is the same as

$$\Delta_{(A^{(-)})^{(-)}} X_{A^{(-)}} \Delta_R = X_S \Delta_{A^T} X_{A^{(-)}}.$$

Since  $A^{-1} = n^{-1} A^{(-)T}$ , we can rewrite the above as

$$n \Delta_{A^T} X_{A^{(-)T}} \Delta_R = X_S \Delta_{A^T} X_{A^{(-)T}}.$$

Applying the Exchange Lemma, we get

$$\Delta_{A^{(-)T}} X_{A^T} \Delta_{nR^T} = X_S \Delta_{A^{(-)T}} X_{A^T}.$$

By Theorem 2.2.1, we conclude that  $S$  belongs to  $\mathcal{N}_{A^T}$  and  $\Theta_{A^T}(S) = nR^T$ .  $\square$

**Corollary 3.2.3** *If  $A$  is a type-II matrix, then  $\mathcal{N}'_A = \mathcal{N}_{A^T}$  and  $\mathcal{N}'_{A^T} = \mathcal{N}_A$ . Moreover,  $R \in \mathcal{N}_A$  if and only if  $R^T \in \mathcal{N}_A$ .*

*Proof.* Applying Theorem 3.2.2 to the type-II matrix  $A$ , we get  $\mathcal{N}'_A \subseteq \mathcal{N}_{A^T}$ . Since  $A^T$  is also type II, so  $\mathcal{N}'_{A^T} \subseteq \mathcal{N}_A$ . By Theorem 2.1.5,

$$\dim \mathcal{N}_A = \dim \mathcal{N}''_A, \quad \text{and} \quad \dim \mathcal{N}_{A^T} = \dim \mathcal{N}''_{A^T}.$$

As a result, we get  $\mathcal{N}'_A = \mathcal{N}_{A^T}$  and  $\mathcal{N}'_{A^T} = \mathcal{N}_A$ .

The second part of the corollary follows from Theorem 3.2.2 and the fact that  $\mathcal{N}'_{A^T} = \mathcal{N}_A$ .  $\square$

We deduce from Theorem 2.1.5 and this corollary that  $\mathcal{N}_A$  is a commutative algebra (with respect to matrix multiplication) containing  $I$  and  $J$  which is closed under the Schur product and the transpose. An algebra with all these properties is called a *Bose-Mesner algebra*. We introduce the theory of Bose-Mesner algebras in Section 3.3.

**Theorem 3.2.4** *If  $A$  is a type-II matrix, then both  $\mathcal{N}_A$  and  $\mathcal{N}'_A$  are Bose-Mesner algebras.*  $\square$

**Lemma 3.2.5** *If  $A$  is a type-II matrix and if  $R_1, R_2 \in \mathcal{N}_A$  then*

$$\Theta_A(R_1 R_2) = \Theta_A(R_1) \circ \Theta_A(R_2)$$

and

$$\Theta_A(R_1 \circ R_2) = n^{-1} \Theta_A(R_1) \Theta_A(R_2).$$

*Proof.* The first equation follows directly from Lemma 2.1.2. From the equality  $\mathcal{N}_A = \mathcal{N}'_{A^T}$ , there exist  $S_1$  and  $S_2$  in  $\mathcal{N}_{A^T}$  such that  $\Theta_{A^T}(S_i) = R_i$ , for  $i = 1, 2$ .

Then by Theorem 3.2.2

$$\begin{aligned}
 \Theta_A(R_1 \circ R_2) &= \Theta_A(\Theta_{A^T}(S_1) \circ \Theta_{A^T}(S_2)) \\
 &= \Theta_A(\Theta_{A^T}(S_1 S_2)) \\
 &= n S_2^T S_1^T.
 \end{aligned}$$

The second equation follows from Theorem 3.2.2 and the commutativity of  $\mathcal{N}_{A^T}$ .

□

We have shown that the duality map  $\Theta_A$  interchanges the Schur product and the matrix multiplication. Moreover, the following lemma shows that  $\Theta_A$  and the transpose map commute.

**Lemma 3.2.6** *Let  $A$  be a type-II matrix. If  $R \in \mathcal{N}_A$ , then  $\Theta_A(R^T) = \Theta_A(R)^T$ .*

*Proof.* Let  $S = \Theta_A(R)$ . By setting  $B = A^{(-)}$  in Corollary 2.3.2 (b), we get

$$X_{A^{(-)T}} \Delta_A X_{R^T} = \Delta_{S^T} X_{A^{(-)T}} \Delta_A,$$

which can be rewritten as

$$X_{R^T} \Delta_{A^{(-)}} X_{(A^{(-)T})^{-1}} = \Delta_{A^{(-)}} X_{(A^{(-)T})^{-1}} \Delta_{S^T}.$$

Since  $A^{(-)T} = n A^{-1}$ , we have

$$X_{R^T} \Delta_{A^{(-)}} X_A = \Delta_{A^{(-)}} X_A \Delta_{S^T}.$$

So by Theorem 2.2.1, we get  $\Theta_A(R^T) = \Theta_A(R)^T$ .

□

### 3.3 Association Schemes and Bose-Mesner Algebras

A *Bose-Mesner algebra* is a finite dimensional vector space of  $n \times n$  matrices that is closed under the transpose, the Schur product and the matrix multiplication. It is commutative with respect to the matrix multiplication and it contains  $I$  and  $J$ . The Nomura algebras of type-II matrices are Bose-Mesner algebras. As we see in this section, Bose-Mesner algebras are equivalent to association schemes. Association schemes can be viewed as partitions of the complete graph on  $n$  vertices into directed graphs that satisfy some regular conditions. Sections 3.4 to 3.6 serve as an introduction to the theory of association schemes. For further information on association schemes, please refer to [5].

In this thesis, we choose to give the definition of association schemes in terms of matrices. An *association scheme on  $n$  elements with  $d$  classes* is a set of  $n \times n$  01-matrices  $\mathcal{A} = \{A_0, \dots, A_d\}$  that satisfies the following conditions:

- a.  $A_0 = I$
- b.  $\sum_{i=0}^d A_i = J$
- c.  $A_i^T = A_{i'}$ , for some  $i' \in \{0, \dots, d\}$
- d. There exist non-negative integers  $p_{ij}^k$  such that for all  $i, j = 0, \dots, d$ .

$$A_i A_j = \sum_{k=0}^d p_{ij}^k A_k.$$

- e.  $A_i A_j = A_j A_i$ , for all  $i, j = 0, \dots, d$ .

If  $A_i^T = A_i$  for  $i = 0, \dots, d$ , we say that  $\mathcal{A}$  is a symmetric association scheme. Let  $\mathcal{B}$  be the span of  $\{A_0, \dots, A_d\}$ . Condition (b) says that  $A_i \circ A_j = \delta_{i,j} A_i$  for all  $i, j = 1, \dots, n$ . Hence  $\mathcal{B}$  is closed under the Schur product. Conditions (d) and (e) tell us that  $\mathcal{B}$  is closed under and commutative with respect to the matrix multiplication. Now the set  $\{A_0, \dots, A_d\}$  is a basis of Schur idempotents for  $\mathcal{B}$ . With the first three conditions, we know that  $\mathcal{B}$  is also closed under the transpose and it contains  $I$  and  $J$ . Consequently, we get a Bose-Mesner algebra of dimension  $d + 1$  for each association scheme with  $d$  classes.

Conversely, by Lemma 2.1.4, we know that any Bose-Mesner algebra of dimension  $m$  has a basis of Schur idempotents  $A_0, \dots, A_{m-1}$ . It is standard result that the properties of a Bose-Mesner algebra enforce conditions (a) to (e) to hold for  $A_0, \dots, A_{m-1}$ .

### 3.4 Examples of Association Schemes

We list examples of association schemes that often appear in the context of spin models and four-weight spin models.

The simplest example is the *trivial scheme*  $\mathcal{A} = \{I, J - I\}$ . Its Bose-Mesner algebra contains the Potts model.

This family of association schemes has the most number of classes possible. Let  $X$  be a finite Abelian group and  $n = |X|$ . For each  $z \in X$ , define the  $n \times n$  01-matrix  $A_z$  by

$$(A_z)_{x,y} = \delta_{y-x,z}.$$

These  $n$  permutation matrices form an association scheme with  $n - 1$  classes, called

the Abelian group scheme of  $X$ . Bannai, Bannai and Jaeger showed in [3] that the Bose-Mesner algebra of an Abelian group scheme always contains a spin model.

The last examples are two association schemes with four classes constructed from a Hadamard matrix. Let  $H$  be an  $n \times n$  Hadamard matrix. Then

$$\begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix} \cdot \begin{pmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{pmatrix} \cdot \begin{pmatrix} J-I & J-I & 0 & 0 \\ J-I & J-I & 0 & 0 \\ 0 & 0 & J-I & J-I \\ 0 & 0 & J-I & J-I \end{pmatrix} \cdot \\ \begin{pmatrix} 0 & 0 & \frac{J+H}{2} & \frac{J-H}{2} \\ 0 & 0 & \frac{J-H}{2} & \frac{J+H}{2} \\ \frac{J+H^T}{2} & \frac{J-H^T}{2} & 0 & 0 \\ \frac{J-H^T}{2} & \frac{J+H^T}{2} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & \frac{J-H}{2} & \frac{J+H}{2} \\ 0 & 0 & \frac{J+H}{2} & \frac{J-H}{2} \\ \frac{J-H^T}{2} & \frac{J+H^T}{2} & 0 & 0 \\ \frac{J+H^T}{2} & \frac{J-H^T}{2} & 0 & 0 \end{pmatrix}$$

form a symmetric association scheme on  $4n$  elements with four classes.

Replacing the last two matrices above by

$$\begin{pmatrix} 0 & 0 & \frac{J+H}{2} & \frac{J-H}{2} \\ 0 & 0 & \frac{J-H}{2} & \frac{J+H}{2} \\ \frac{J-H^T}{2} & \frac{J+H^T}{2} & 0 & 0 \\ \frac{J+H^T}{2} & \frac{J-H^T}{2} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & \frac{J-H}{2} & \frac{J+H}{2} \\ 0 & 0 & \frac{J+H}{2} & \frac{J-H}{2} \\ \frac{J+H^T}{2} & \frac{J-H^T}{2} & 0 & 0 \\ \frac{J-H^T}{2} & \frac{J+H^T}{2} & 0 & 0 \end{pmatrix}.$$

we obtain a non-symmetric association scheme with four classes. In [15], Jaeger and Nomura constructed symmetric and non-symmetric Hadamard spin models. They are contained in the Bose-Mesner algebras of the above schemes, respectively.

### 3.5 Idempotents and Eigenvalues

Let  $\mathcal{B}$  be a Bose-Mesner algebra of an association scheme with  $d$  classes. Since the matrices in  $\mathcal{B}$  are normal and they commute with respect to the matrix multiplication, they are simultaneously diagonalizable. Therefore  $\mathcal{B}$  has a basis  $\{E_0, \dots, E_d\}$  such that  $E_i$  is the orthogonal projection onto the  $i$ -th common eigenspace of the matrices in  $\mathcal{B}$ . So we have

$$\sum_{k=0}^d E_k = I. \quad \text{and} \quad E_i E_j = \delta_{i,j} E_i.$$

for all  $i, j = 1, \dots, n$ . The  $E_i$ 's are called the *principal idempotents* of the association scheme.

By the definition of the principal idempotents, there exist complex numbers  $p_i(j)$ 's such that

$$A_i E_j = p_i(j) E_j.$$

for all  $i, j = 0, \dots, d$ . The numbers  $p_i(j)$ 's are the eigenvalues for the Schur idempotents. Define  $P$  to be the matrix whose  $ji$ -entry equals  $p_i(j)$ . We call  $P$  the *matrix of eigenvalues* of the association scheme. Now, for each  $i = 0, \dots, d$ , we can write

$$A_i = \sum_{j=0}^d P_{j,i} E_j.$$

Similarly, if we let  $Q = nP^{-1}$ , then for each  $i = 0, \dots, d$ ,

$$E_i = n^{-1} \sum_{j=0}^d Q_{j,i} A_j.$$

Note that  $E_i \circ A_j = n^{-1} Q_{j,i} A_j$ . The entries of  $Q$  act like the eigenvalues of  $E_i$  with respect to the Schur product. Hence we name  $Q$  the *matrix of dual eigenvalues* of the association scheme.

### 3.6 Dualities of Association Schemes

By Theorem 3.2.4, if  $A$  is a type-II matrix, then  $\mathcal{N}_A$  and  $\mathcal{N}_{A^T}$  are Bose-Mesner algebras. Moreover, the map  $\Theta_A : \mathcal{N}_A \rightarrow \mathcal{N}_{A^T}$  satisfies

$$\begin{aligned} \Theta_A(MN) &= \Theta_A(M) \circ \Theta_A(N), \quad \text{and} \\ \Theta_A(M \circ N) &= n^{-1} \Theta_A(M) \Theta_A(N). \end{aligned}$$

In general, a *duality* between two Bose-Mesner algebras  $\mathcal{B}_1$  and  $\mathcal{B}_2$  is an invertible linear map  $\Psi : \mathcal{B}_1 \rightarrow \mathcal{B}_2$  that satisfies

$$\Psi(MN) = \Psi(M) \circ \Psi(N), \quad \text{and} \quad \Psi(M \circ N) = n^{-1} \Psi(M) \Psi(N).$$

We say that the  $\mathcal{B}_1$  and  $\mathcal{B}_2$  (or their corresponding association schemes) are *formally dual* to each other. As we have seen in Section 3.2, Nomura's construction provide an abundant source of formally dual pairs of association schemes.

When  $\mathcal{B}_1 = \mathcal{B}_2$  and  $\Psi^2(M) = nM^T$ , we say that  $\mathcal{B}_1$  is *formally self-dual*. In this case,  $\Psi$  maps the basis of Schur idempotents of  $\mathcal{B}_1$  to its basis of principal idempotents. It is possible to order the Schur idempotents and the principal idempotents

so that the matrix of eigenvalues  $P$  is the matrix of  $\Psi$  with respect to the basis of Schur idempotents. If  $T$  is the matrix of the transpose with respect to the basis of Schur idempotents, then we have  $P^2 = nT$ .

The duality map and the matrix of eigenvalues of the Nomura algebra of a spin model are crucial in the derivation of the modular invariance equation. We present this in Section 5.2.

### 3.7 Infinite Families of Type-II Matrices

We record four families of type-II matrices in this section. Each of these type-II matrices is monomially equivalent to some spin model.

The first example is the simplest family of type-II matrices. Let  $A = tI + (J - I)$  for some non-zero scalar  $t$ . Then

$$\begin{aligned} AA^{(-)T} &= (tI + (J - I))(t^{-1}I + (J - I)) \\ &= (-t - t^{-1} + 2)I + (t + t^{-1} - 2 + n)J. \end{aligned}$$

Therefore  $A$  is type II if and only if  $t + t^{-1} - 2 + n = 0$ . For each  $n \geq 2$ , the solutions to the quadratic equation give two type-II matrices contained in the Bose-Mesner algebra of the trivial scheme on  $n$  elements.

Now for any distinct  $i$  and  $j$  not equal to 1.

$$\begin{aligned} \langle Ae_1 \circ A^{(-)}e_i, Ae_1 \circ A^{(-)}e_j \rangle &= t^2 + 2t^{-1} + n - 3 \\ &= -n(t + 1). \end{aligned}$$

which is non-zero when  $n \neq 4$ . That is, when  $n \neq 4$ , no two vectors from

$$(\mathcal{A}e_1 \circ \mathcal{A}^{(-)}e_2), \dots, (\mathcal{A}e_1 \circ \mathcal{A}^{(-)}e_n)$$

are orthogonal to each other. So, they all lie in the same eigenspace. Hence the matrices in  $\mathcal{N}'_{\mathcal{A}}$  have at most two eigenspaces, one spanned by the  $n - 1$  vectors listed above and the other spanned by the vector  $\mathcal{A}e_1 \circ \mathcal{A}^{(-)}e_1$ . This implies that all matrices in  $\mathcal{N}'_{\mathcal{A}}$  have at most two distinct entries and therefore  $\mathcal{N}'_{\mathcal{A}} = \text{span}(I, J)$ . By Corollary 3.2.3, we have  $\mathcal{N}'_{\mathcal{A}} = \mathcal{N}_{\mathcal{A}}\tau$ . But  $\mathcal{A}$  is symmetric, so we conclude that  $\mathcal{N}_{\mathcal{A}} = \mathcal{N}_{\mathcal{A}}\tau$  is the Bose-Mesner algebra of the trivial scheme.

When  $n = 4$ , we have  $\mathcal{A} = -I + (J - I)$  and it is shown in Section 5.3 of [14] that  $\mathcal{N}_{\mathcal{A}}$  is the Abelian group scheme for  $\mathbb{Z}_2 \times \mathbb{Z}_2$ . Finally, the *Potts model* is defined to be  $u^{-1}\mathcal{A}$  where  $u^4 = -t$ , and it is easy to see that  $\mathcal{A} \in \mathcal{N}_{\mathcal{A}}$ .

The second example comes from finite Abelian groups. It is documented in Section 5.1 of [14]. Let  $X$  be a finite Abelian group and let  $\mathcal{A}_X$  denotes its Abelian group scheme. For any  $x, y \in X$ , we have  $\mathcal{A}_x \mathcal{A}_y = \mathcal{A}_{x+y}$ . Therefore the entries of the matrix of eigenvalues satisfy  $P_{z,x} P_{z,y} = P_{z,x+y}$ . Thus each row of  $P$  is a character of  $X$  which implies  $P$  is a type-II matrix. It is shown in Section 5.1 of [14] that  $\mathcal{N}'_P = \mathcal{A}_X$ . However, in general,  $\mathcal{N}_P$  may not equal  $\mathcal{N}'_P$  and it may not contain  $P$ . Further, if  $X$  is a cyclic group of order  $n$ , we define  $W$  to have entries  $W_{x,y} = \omega^{(x-y)^2}$  for some  $n$ -th root of unity  $\omega$ . Then  $W$  is symmetric and  $W \in \mathcal{N}_W = \mathcal{A}_X$ . This matrix is a spin model.

An  $n \times n$   $\{1, -1\}$ -matrix  $H$  is called a *Hadamard matrix* if  $HH^T = nI$ . Since  $H^T = H^{(-)T}$ , Hadamard matrices form an infinite family of type-II matrices. Using easy counting argument, when  $n \geq 12$  and  $n \equiv 4 \pmod{8}$ ,  $\mathcal{N}_H$  is just the span of  $\{I, J\}$  (see Section 5.2 of [14]).

In [15] and [21], Jaeger and Nomura constructed two  $4n \times 4n$  spin models (hence type-II matrices) from each  $n \times n$  Hadamard matrix  $H$ . We will provide a new proof that they are spin models in Section 5.5. For now, we focus on their type-II property. Let  $A$  be an  $n \times n$  Potts model, that is,  $A = -u^3I + u^{-1}(J - I)$  with  $(u^2 + u^{-2})^2 = n$ . For each  $\epsilon \in \{1, -1\}$  and  $\omega$  being a fourth root of  $\epsilon$ , the matrix

$$W_\epsilon = \begin{pmatrix} A & A & \omega H & -\omega H \\ A & A & -\omega H & \omega H \\ \epsilon \omega H^T & -\epsilon \omega H^T & A & A \\ -\epsilon \omega H^T & \epsilon \omega H^T & A & A \end{pmatrix}$$

is a type-II matrix.

The Bose-Mesner algebras of the symmetric and non-symmetric association schemes in Section 3.4 contain  $W_1$  and  $W_{-1}$ , respectively. In general, the Nomura algebras of these type-II matrices are not equal to the Bose-Mesner algebras of the four-class association schemes described in Section 3.4. For instance, when  $n = 4$ ,  $u = -i$  and

$$H = \begin{pmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 \end{pmatrix},$$

$\mathcal{N}_{W_\epsilon}$  has dimension 16.

### 3.8 Nomura Algebras of Two Type-II Matrices

We examine the case where  $A$  and  $B$  are type-II matrices, and study the interactions among the algebras  $\mathcal{N}_{A,B}$ ,  $\mathcal{N}'_{A,B}$ ,  $\mathcal{N}_A$ ,  $\mathcal{N}_{A^T}$ ,  $\mathcal{N}_B$  and  $\mathcal{N}_{B^T}$ . This section lays the groundwork for Chapters 4 to 6.

**Theorem 3.8.1** *Let  $A$ ,  $B$  and  $C$  be  $n \times n$  type-II matrices. If  $F \in \mathcal{N}_{A,B}$  and  $G \in \mathcal{N}_{B^{(-)},C}$ , then  $F \circ G \in \mathcal{N}_{A,C}$  and*

$$\Theta_{A,C}(F \circ G) = n^{-1} \Theta_{A,B}(F) \Theta_{B^{(-)},C}(G).$$

*Proof.* Let  $F' = \Theta_{A,B}(F)$  and  $G' = \Theta_{B^{(-)},C}(G)$ . Applying Corollary 2.3.2 (d) to  $F' = \Theta_{A,B}(F)$ , we get

$$\Delta_{B^T} X_{B^{(-)T}} \Delta_F = X_{(F')^T} \Delta_{A^{-1}} X_{A^T},$$

which is equivalent to

$$\Delta_F X_{A^{-T}} \Delta_{(A^{-1})^{(-)}} = X_{(B^{(-)T})^{-1}} \Delta_{B^{(-)T}} X_{(F')^T}.$$

Since  $(A^{-1})^{(-)} = nA^T$ ,  $A^{-T} = n^{-1}A^{(-)}$  and  $B^{(-)T} = nB^{-1}$ , the above equation equals

$$\Delta_F X_{A^{(-)}} \Delta_{A^T} = X_B \Delta_{B^{-1}} X_{(F')^T}. \quad (3.2)$$

Similarly, applying Corollary 2.3.2 (d) to  $G' = \Theta_{B^{(-)},C}(G)$ , we have

$$\Delta_{C^T} X_{C^{(-)T}} \Delta_G = X_{(G')^T} \Delta_{(B^{(-)})^{-1}} X_{B^{(-)T}}.$$

which becomes

$$\Delta_C \tau X_{C(-)\tau} \Delta_G = X_{(G')\tau} \Delta_B \tau X_{B^{-1}}.$$

Multiply the left-hand side of Equation (3.2) by  $\Delta_C \tau X_{C(-)\tau} \Delta_G$  and multiply its right-hand side by  $X_{(G')\tau} \Delta_B \tau X_{B^{-1}}$ . This gives

$$\Delta_C \tau X_{C(-)\tau} \Delta_G \Delta_F X_{A(-)} \Delta_A \tau = X_{(G')\tau} \Delta_B \tau X_{B^{-1}} X_B \Delta_B^{-1} X_{(F')\tau}.$$

which simplifies to

$$\Delta_C \tau X_{C(-)\tau} \Delta_{F \circ G} X_{A(-)} \Delta_A \tau = n^{-1} X_{(F'G')\tau}.$$

Hence

$$\Delta_{F \circ G} X_{A(-)} \Delta_A \tau = X_C \Delta_C^{-1} X_{n^{-1}(F'G')\tau}.$$

By Corollary 2.3.2 (c), we get

$$\Theta_{C,A}(F \circ G) = n^{-1} (F'G')^T.$$

Therefore

$$\Theta_{A,C}(F \circ G) = n^{-1} F'G'.$$

□

Lemma 3.2.6 is a special case of the next lemma.

**Lemma 3.8.2** *Let  $A$  and  $B$  be  $n \times n$  type-II matrices. If  $R \in \mathcal{N}_{A,B}$  then*

$$R^T \in \mathcal{N}_{A^{(-)}, B^{(-)}}, \quad \text{and } \Theta_{A^{(-)}, B^{(-)}}(R^T) = \Theta_{A,B}(R).$$

*Proof.* Let  $S = \Theta_{A,B}(R)$ . By Corollary 2.3.2 (b), we have

$$X_{B^T} \Delta_A X_{R^T} = \Delta_{S^T} X_{B^T} \Delta_A$$

which can be rewritten as

$$X_{R^T} \Delta_{A^{(-)}} X_{B^{-T}} = \Delta_{A^{(-)}} X_{B^{-T}} \Delta_{S^T}.$$

Now  $B^{-T} = n^{-1} B^{(-)}$  and using the Exchange Lemma, we obtain

$$X_{R^T} \Delta_{B^{(-)}} X_{A^{(-)}} = \Delta_{B^{(-)}} X_{A^{(-)}} \Delta_S.$$

So the result follows by Theorem 2.2.1.  $\square$

The next two theorems are easy consequences of Theorem 3.8.1. They describe some interactions among the maps  $\Theta_{A,B}$ ,  $\Theta_A$  and  $\Theta_B$ .

**Theorem 3.8.3** *Let  $A$  and  $B$  be  $n \times n$  type-II matrices. If  $F \in \mathcal{N}_A$ ,  $G \in \mathcal{N}_{A,B}$  and  $H \in \mathcal{N}_B$ , then  $F \circ G$ , and  $G \circ H$  belong to  $\mathcal{N}_{A,B}$  and*

$$\begin{aligned} \Theta_{A,B}(F \circ G) &= n^{-1} \Theta_A(F) \Theta_{A,B}(G) \\ \Theta_{A,B}(G \circ H) &= n^{-1} \Theta_{A,B}(G) \Theta_B(H)^T. \end{aligned}$$

*Proof.* The first equation results from applying Theorem 3.8.1 to the matrices  $(A, A^{(-)}, B)$ .

Applying the same theorem to  $(A, B, B)$ , we find that

$$\begin{aligned}\Theta_{A,B}(G \circ H) &= n^{-1} \Theta_{A,B}(G) \Theta_{B(-),B}(H) \\ &= n^{-1} \Theta_{A,B}(G) \Theta_B(H)^T,\end{aligned}$$

because  $\Theta_B(H) = \Theta_{B,B(-)}(H) = \Theta_{B(-),B}(H)^T$ .  $\square$

**Theorem 3.8.4** *Let  $A$  and  $B$  be  $n \times n$  type-II matrices. If  $F, G \in \mathcal{N}_{A,B}$ , then  $F \circ G^T \in \mathcal{N}_A \cap \mathcal{N}_B$  and*

$$\begin{aligned}\Theta_A(F \circ G^T) &= n^{-1} \Theta_{A,B}(F) \Theta_{A,B}(G)^T \\ \Theta_B(F \circ G^T) &= n^{-1} \Theta_{A,B}(F)^T \Theta_{A,B}(G).\end{aligned}$$

*Proof.* By Lemma 3.8.2, we know  $G^T$  belongs to  $\mathcal{N}_{B(-),A(-)} = \mathcal{N}_{A(-),B(-)}$ . By applying Theorem 3.8.1 to the matrices  $(A, B, A^{(-)})$ , we get

$$\Theta_{A,A(-)}(F \circ G^T) = n^{-1} \Theta_{A,B}(F) \Theta_{B(-),A(-)}(G^T).$$

Using Lemma 3.8.2 again, we see that

$$\Theta_{B(-),A(-)}(G^T) = \Theta_{B,A}(G) = \Theta_{A,B}(G)^T.$$

Hence the first equation holds.

We now apply Theorem 3.8.1 to the matrices  $(B, A, B^{(-)})$ , and obtain

$$\begin{aligned}\Theta_B(F \circ G^T) &= n^{-1} \Theta_{B,A}(F) \Theta_{A(-),B(-)}(G^T) \\ &= n^{-1} \Theta_{A,B}(F)^T \Theta_{A,B}(G).\end{aligned}$$

□

The following is an important consequence of Theorems 3.8.3 and 3.8.4. It implies that if  $\mathcal{N}_{A,B}$  contains a Schur-invertible matrix, then  $\mathcal{N}_{A,B}$ ,  $\mathcal{N}'_{A,B}$ ,  $\mathcal{N}_A$ ,  $\mathcal{N}_B$ ,  $\mathcal{N}_{A^T}$  and  $\mathcal{N}_{B^T}$  have the same dimension and  $\mathcal{N}_A = \mathcal{N}_B$ .

**Theorem 3.8.5** *Let  $A$  and  $B$  be  $n \times n$  type-II matrices. If  $\mathcal{N}_{A,B}$  contains a Schur-invertible matrix  $G$  and  $H = \Theta_{A,B}(G)$ , then*

- a.  $G \circ \mathcal{N}_A = \mathcal{N}_{A,B}$  and  $G^T \circ \mathcal{N}_{A,B} = \mathcal{N}_A$ .
- b.  $\mathcal{N}_{A^T}H = \mathcal{N}'_{A,B}$  and  $\mathcal{N}'_{A,B}H^T = \mathcal{N}_{A^T}$ .
- c.  $\mathcal{N}_B = \mathcal{N}_A$ .
- d.  $\mathcal{N}_{B^T} = H^{-1}\mathcal{N}_{A^T}H$ .

*Proof.* By Theorem 3.8.3, we have  $G \circ \mathcal{N}_A \subseteq \mathcal{N}_{A,B}$  and  $G \circ \mathcal{N}_B \subseteq \mathcal{N}_{A,B}$ . Since  $G$  is Schur invertible, the dimensions of  $\mathcal{N}_A$  and  $\mathcal{N}_B$  are less than or equal to  $\dim \mathcal{N}_{A,B}$ . Similarly, Theorem 3.8.4 implies  $G^T \circ \mathcal{N}_{A,B}$  is a subset of  $\mathcal{N}_A$  and  $\mathcal{N}_B$ . The Schur-invertibility of  $G^T$  implies that the dimension of  $\mathcal{N}_{A,B}$  is less than or equal to the dimensions of  $\mathcal{N}_A$  and  $\mathcal{N}_B$ . As a result, we have  $\mathcal{N}_A = \mathcal{N}_B = G^T \circ \mathcal{N}_{A,B}$  and  $G \circ \mathcal{N}_A = \mathcal{N}_{A,B}$ .

By the first equation of Theorem 3.8.3, we have  $\mathcal{N}_{A^T}H \subseteq \mathcal{N}'_{A,B}$ . Since we have  $\mathcal{N}_{A,B} = G \circ \mathcal{N}_A$  for any  $M \in \mathcal{N}_{A,B}$ , there exists  $M' \in \mathcal{N}_A$  such that  $M = M' \circ G$ . It follows from Theorem 3.8.3 that

$$\begin{aligned} \Theta_{A,B}(M) &= \Theta_{A,B}(M' \circ G) \\ &= n^{-1}\Theta_A(M')H \end{aligned}$$

As a result, we have  $\mathcal{N}'_{A,B} \subseteq \mathcal{N}_{A^T}H$  and the first part of (b) follows.

Similarly, the first part of Theorem 3.8.4 implies that  $\mathcal{N}'_{A,B}H^T \subseteq \mathcal{N}_{A^T}$ . Because  $\mathcal{N}_A = G^T \circ \mathcal{N}_{A,B}$ , for all  $N \in \mathcal{N}_A$ , there exists  $N' \in \mathcal{N}'_{A,B}$  such that  $N = N' \circ G^T$ . It follows from the first equation of Theorem 3.8.4 that

$$\begin{aligned}\Theta_A(N) &= \Theta_A(N' \circ G^T) \\ &= n^{-1}\Theta_{A,B}(N')H^T.\end{aligned}$$

So  $\mathcal{N}_{A^T} \subseteq \mathcal{N}'_{A,B}H^T$  and we have proved the rest of (b).

Using the same kind of argument, the second equations of Theorems 3.8.3 and 3.8.4 imply that  $\mathcal{N}'_{A,B} = H\mathcal{N}_{B^T}$  and  $\mathcal{N}_{B^T} = H^T\mathcal{N}'_{A,B}$ , respectively. Consequently,

$$\mathcal{N}_{B^T} = H^{-1}\mathcal{N}'_{A,B} = H^{-1}\mathcal{N}_{A^T}H.$$

□

Since  $\mathcal{N}_A$  is closed under the transpose, part (a) of this theorem tells us that if  $\mathcal{N}_{A,B}$  contains a symmetric Schur-invertible matrix, then  $\mathcal{N}'_{A,B}$  is also closed under the transpose.

# Chapter 4

## Jones Pairs

Given an  $n \times n$  symmetric spin model  $W$ , Jones [17] defined endomorphisms of  $\mathbb{C}^n \otimes \mathbb{C}^n$ ,  $X_W$  and  $\Delta_W(-)$ , that provide a braid group representation. In Section 3.5 of the same paper, Jones questioned the necessity of the type-II condition on  $W$  for the representation to give a link invariant. This question motivates us to consider Jones' braid group representation without assuming type-II condition. We extend Jones' idea to use endomorphisms  $X_A$  and  $\Delta_B$ , where  $B$  may not equal  $A^{(-)}$ . Jones pairs are defined in this process.

We devote this chapter to develop the theory of Jones pairs. We introduce Jones pairs and a weaker version, called one-sided Jones pairs, in Sections 4.1 and 4.2. In Section 4.3, we discuss the braid group representations obtained from Jones pairs. In the remaining sections, we focus on the effect of the invertibility of  $B$  in a one-sided Jones pair  $(A, B)$ . An important consequence is the equivalence of invertible Jones pairs and four-weight spin models. In the last section, we reproduce Jaeger's results on gauge equivalence with weaker assumptions. Except for the three corollaries in Section 4.5, due to the author, all results in this chapter are joint work by Godsil, Munemasa and the author [6].

## 4.1 One-Sided Jones Pairs

A pair of  $n \times n$  matrices  $(A, B)$  is a *one-sided Jones pair* if both  $X_A$  and  $\Delta_B$  are invertible and they satisfy

$$X_A \Delta_B X_A = \Delta_B X_A \Delta_B. \quad (4.1)$$

Equivalent to the definition in Section 1.1, we say that the pair  $(A, B)$  is a *Jones pair* if  $(A, B)$  and  $(A, B^T)$  are one-sided Jones pairs.

By Theorem 2.2.1, an invertible matrix  $A$  and a Schur-invertible matrix  $B$  form a one-sided Jones pair if and only if

$$A \in \mathcal{N}_{A,B} \quad \text{and} \quad \Theta_{A,B}(A) = B.$$

The pair  $(I, J)$  is an obvious example of one-sided Jones pair. It is in fact a Jones pair because  $J$  is symmetric.

Using the eigenvector approach, if  $A$  is invertible and  $B$  is Schur-invertible, then  $(A, B)$  is a one-sided Jones pair if and only if

$$A(Ae_i \circ Be_j) = B_{i,j}(Ae_i \circ Be_j),$$

for all  $i, j = 1, \dots, n$ . The  $k$ -th entry of both sides equal

$$\sum_{x=1}^n A_{k,x} A_{x,i} B_{x,j} = B_{i,j} A_{k,i} B_{k,j}. \quad (4.2)$$

Moreover, if  $(A, B)$  is a Jones pair, replacing  $B$  by  $B^T$  in Equation (4.2) gives

$$\sum_{x=1}^n A_{k,x} A_{x,i} B_{j,x} = B_{j,i} A_{k,i} B_{j,k}. \quad (4.3)$$

Let  $W$  be a spin model with loop variable  $d$ . Then the type-III condition of  $W$  is

$$\sum_{x=1}^n \frac{W_{k,x} W_{x,i}}{W_{j,x}} = d \frac{W_{k,i}}{W_{j,i} W_{j,k}}.$$

which is exactly Equation (4.2) with  $A = d^{-1}W$  and  $B = W^{(-)}$ . As a result, all spin models give one-sided Jones pairs.

Let  $(W_1, W_2, W_3, W_4; d)$  be a four-weight spin model. Then its type-III conditions are

$$\begin{aligned} \sum_{x=0}^n (W_1)_{k,x} (W_1)_{x,i} (W_4)_{x,j} &= d (W_4)_{i,j} (W_1)_{k,i} (W_4)_{k,j} \quad \text{and} \\ \sum_{x=0}^n (W_1)_{k,x} (W_1)_{x,i} (W_4)_{j,x} &= d (W_4)_{j,i} (W_1)_{k,i} (W_4)_{j,k}. \end{aligned}$$

These equations are the same as Equations (4.2) and (4.3) with  $A = d^{-1}W_1$  and  $B = W_4$ , respectively. So we obtain a Jones pair from every four-weight spin model.

In addition, we can build Jones pairs using Kronecker product. Suppose  $(A, B)$  and  $(A', B')$  are one-sided Jones pairs. By Lemma 2.5.1, we have

$$A \otimes A' \in \mathcal{N}_{A \otimes A', B \otimes B'}.$$

From the proof of this lemma, we see that the eigenvectors can be written as

$(A\epsilon_i \circ B\epsilon_j) \in (A'\epsilon_h \circ B'\epsilon_k)$ . It follows that

$$\Theta_{A \otimes A', B \otimes B'}(A \otimes A') = \Theta_{A, B}(A) \otimes \Theta_{A', B'}(A') = B \otimes B'.$$

Hence  $(A \otimes A', B \otimes B')$  is also a one-sided Jones pair. As a result, there exist infinitely many one-sided Jones pairs.

Suppose  $(A, B)$  is a one-sided Jones pair. Fix any  $j \in \{1, \dots, n\}$ , define  $D_j$  to be the diagonal matrix with its  $ii$ -entry equals  $B_{i,j}$ . Then  $(D_j J)_{h,k} = B_{h,j}$  and  $(D_j J)\epsilon_k = B\epsilon_j$ , for all  $k = 1, \dots, n$ . As a result, we get

$$\begin{aligned} A(A\epsilon_h \circ (D_j J)\epsilon_k) &= A(A\epsilon_h \circ B\epsilon_j) \\ &= B_{h,j}(A\epsilon_h \circ B\epsilon_j) \\ &= (D_j J)_{h,k}(A\epsilon_h \circ (D_j J)\epsilon_k) \end{aligned}$$

and  $(A, D_j J)$  is a one-sided Jones pair.

Now  $(I, J)$ ,  $(A, D_j J)$  and  $(A \otimes I, B \otimes J)$  are the only known examples of one-sided Jones pair with the second matrix being non-invertible. In Section 4.6, we see that the Jones pairs with the second matrix invertible are equivalent to four-weight spin models. As a result, we would be very excited to see any new example of one-sided Jones pairs with the second matrix non-invertible.

## 4.2 Properties of One-Sided Jones Pairs

This section lists some useful properties of one-sided Jones pairs.

**Lemma 4.2.1** *If  $(A, B)$  is a one-sided Jones pair, then  $B^T J = \text{tr}(A)J$ . Furthermore, if  $(A, B)$  is a Jones pair, then the columns and the rows of  $B$  sum to  $\text{tr}(A)$ .*

*Proof.* Since  $A$  is invertible and  $B$  is Schur invertible, the set

$$\{A\epsilon_1 \circ B\epsilon_r, A\epsilon_2 \circ B\epsilon_r, \dots, A\epsilon_n \circ B\epsilon_r\}$$

is a basis of  $\mathbb{C}^n$ , for each  $r = 1, \dots, n$ . As a result, each column of  $B$  contains all eigenvalues of  $A$  and  $B^T J = \text{tr}(A)J$ .

Similarly, if  $(A, B^T)$  is also a one-sided Jones pair, then  $BJ = \text{tr}(A)J$ . So if  $(A, B)$  is a Jones pair, then  $B^T J = BJ = \text{tr}(A)J$ .  $\square$

**Lemma 4.2.2** *Suppose  $(A, B)$  is a one-sided Jones pair. So is*

- a.  $(A^T, B)$
- b.  $(A^{-1}, B^{(-)})$
- c.  $(D^{-1}AD, B)$ , for any invertible diagonal matrix  $D$
- d.  $(A, BP)$ , for any permutation matrix  $P$
- e.  $(PAP^{-1}, PBP^{-1})$ , for any permutation matrix  $P$
- f.  $(\lambda A, \lambda B)$ , for any non-zero complex number  $\lambda$

*Proof.*

- a. Taking the transpose of both sides of (4.1), we get  $X_{A^T} \Delta_B X_{A^T} = \Delta_B X_{A^T} \Delta_B$ .
- b. Since  $X_A^{-1} = X_{A^{-1}}$  and  $\Delta_B^{-1} = \Delta_{B^{(-)}}$ , inverting both sides of (4.1) gives  $X_{A^{-1}} \Delta_{B^{(-)}} X_{A^{-1}} = \Delta_{B^{(-)}} X_{A^{-1}} \Delta_{B^{(-)}}$ .

c. Note that for any diagonal matrix  $D$ ,  $X_D = \Delta_{DJ}$  and  $X_D \Delta_M = \Delta_M X_D$ . So

$$\begin{aligned}
 X_{D^{-1}AD} \Delta_B X_{D^{-1}AD} &= X_{D^{-1}} X_A (X_D \Delta_B X_{D^{-1}}) X_A X_D \\
 &= X_{D^{-1}} (X_A \Delta_B X_A) X_D \\
 &= X_{D^{-1}} \Delta_B X_A \Delta_B X_D \\
 &= \Delta_B (X_{D^{-1}} X_A X_D) \Delta_B \\
 &= \Delta_B X_{D^{-1}AD} \Delta_B.
 \end{aligned}$$

The third equality results from  $(A, B)$  being a one-sided Jones pair.

d. By Lemma 2.4.2, we have

$$A \in \mathcal{N}_{A,B} = \mathcal{N}_{A,BP}$$

and

$$\Theta_{A,BP}(A) = \Theta_{A,B}(A)P = BP.$$

e. Using the same lemma with  $Q = R = P^{-1}$ , we have

$$PAP^{-1} \in \mathcal{N}_{PAP^{-1}, PBP^{-1}}$$

and

$$\Theta_{PAP^{-1}, PBP^{-1}}(PAP^{-1}) = P\Theta_{A,B}(A)P^{-1} = PBP^{-1}.$$

f. Replacing  $(A, B)$  by  $(\lambda A, \lambda B)$  is equivalent to multiplying both sides of (4.1)

by  $\lambda^3$ .

□

If  $(A, B)$  is a one-sided Jones pair, then  $A$  is invertible and  $B$  is Schur invertible and  $B = \Theta_{A,B}(A)$ . So Corollary 2.3.2 provides three equivalent forms of Equation (4.1). We give one more useful reformulation of Equation (4.1) below when  $A$  is also Schur invertible and  $B$  is also invertible.

**Lemma 4.2.3** *If  $A$  and  $B$  are both invertible and Schur-invertible, then  $(A, B)$  is a one-sided Jones pair if and only if*

$$\Delta_{A(-)} X_A \Delta_A = X_B \Delta_B \tau X_{B^{-1}}.$$

*Proof.* Applying the Exchange Lemma to Equation (4.1) yields

$$X_A \Delta_A X_B = \Delta_A X_B \Delta_B \tau.$$

Since both  $A^{(-)}$  and  $B^{-1}$  exist, we get the relation in the lemma immediately. □

### 4.3 Braid Group Representations

Given a Jones pair  $(A, B)$ , we demonstrate Jones' method of constructing braid group representations from  $X_A$  and  $\Delta_B$ . The *braid group*  $\mathbf{B}_m$  on  $m$  strands is generated by  $\sigma_1, \dots, \sigma_{m-1}$  satisfying

- a. For all  $i = 1, \dots, m-2$ ,

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}. \quad (4.4)$$

b. For all  $|i - j| \geq 2$ .

$$\sigma_j \sigma_k = \sigma_k \sigma_j.$$

Let  $k = \lceil \frac{m}{2} \rceil$  and let  $V$  denote the vector space  $\mathbb{C}^n$ . Given a pair of  $n \times n$  matrices  $(A, B)$ , we map the generators of  $\mathbf{B}_m$  to the endomorphisms  $g_1, \dots, g_{m-1}$  of  $V^{\otimes k}$  as follows:

$$\begin{aligned} g_{2h-1}(e_{r_1} \otimes \dots \otimes e_{r_k}) &= e_{r_1} \otimes \dots \otimes (Ae_{r_h}) \otimes \dots \otimes e_{r_k}, \\ g_{2h}(e_{r_1} \otimes \dots \otimes e_{r_k}) &= B_{r_h, r_{h+1}}(e_{r_1} \otimes \dots \otimes e_{r_k}). \end{aligned}$$

When  $|i - j| \geq 2$ ,  $g_i$  and  $g_j$  act on different tensor factors of  $V^{\otimes k}$ , so they commute. Note that  $g_{2h-1}$ ,  $g_1$  and  $g_3$  have the same action on the  $h$ -th, 1-st and 2-nd tensor factors of  $V^{\otimes k}$ , respectively. Similarly, the action of  $g_{2h}$  on the  $h$ -th and the  $(h+1)$ -th tensor factors of  $V^{\otimes k}$  is the same as the action of  $g_2$  on the first and second tensor factors. Consequently, showing  $g_1 g_2 g_1 = g_2 g_1 g_2$  and  $g_2 g_3 g_2 = g_3 g_2 g_3$  is sufficient to prove that (4.4) holds for all  $i = 1, \dots, m-2$ .

**Lemma 4.3.1** *The relation  $g_1 g_2 g_1 = g_2 g_1 g_2$  holds if and only if  $(A, B)$  is a one-sided Jones pair.*

*Proof.* We use the isomorphism  $\phi : V \otimes V \rightarrow \mathbf{M}_n(\mathbb{C})$  which maps  $e_i \otimes e_j$  to  $e_i e_j^T$ . We get

$$\begin{aligned} \phi(g_1(e_i \otimes e_j)) &= \phi(Ae_i \otimes e_j) \\ &= Ae_i e_j^T \\ &= X_A(E_{ij}). \end{aligned}$$

So we have  $X_A = \phi g_1 \phi^{-1}$ . Similarly,

$$\phi(g_2(e_i \odot e_j)) = B_{i,j} e_i e_j^T = \Delta_B(E_{i,j})$$

and  $\Delta_B = \phi g_2 \phi^{-1}$ . Consequently,

$$\phi g_1 g_2 g_1 \phi^{-1} = X_A \Delta_B X_A$$

and

$$\phi g_2 g_1 g_2 \phi^{-1} = \Delta_B X_A \Delta_B.$$

So the result follows. □

Since  $\mathbf{B}_3$  has only two generators, every one-sided Jones pair gives a representation of  $\mathbf{B}_3$ .

**Lemma 4.3.2** *The relation  $g_2 g_3 g_2 = g_3 g_2 g_3$  holds if and only if  $(A, B^T)$  is a one-sided Jones pair.*

*Proof.* For any  $n \times n$  matrix  $C$ , let  $Y_C$  be the endomorphism of  $\mathbf{M}_n(\mathbb{C})$  defined as  $Y_C(M) = MC^T$ . Using  $\phi$  as above, we have

$$\phi(g_3(e_i \odot e_j)) = \phi(e_i \odot A e_j) = e_i (A e_j)^T = Y_A(E_{i,j})$$

and hence  $Y_A = \phi g_3 \phi^{-1}$ . As a result, the relation  $g_2 g_3 g_2 = g_3 g_2 g_3$  holds if and only if  $\Delta_B Y_A \Delta_B = Y_A \Delta_B Y_A$ . We can write this equation as

$$(B \circ (E_{i,j} A^T)) A^T = B \circ ((B \circ E_{i,j}) A^T).$$

for all  $i, j = 1, \dots, n$ . Taking the transpose of each side, we get

$$A(B^T \circ (AE_{i,j})) = B^T \circ (A(B^T \circ E_{i,j})),$$

which is equivalent to  $(A, B^T)$  being a one-sided Jones pair.  $\square$

**Corollary 4.3.3** *Every Jones pair gives a representation of  $\mathbf{B}_m$ .*

Suppose  $(A, B)$  is a Jones pair. Let  $g_1, \dots, g_{m-1}$  be the representation of  $\mathbf{B}_m$  described above. We proved in [6] that if  $A$  and  $B$  satisfy

$$\begin{aligned} A \circ I &= A^{-1} \circ I = \frac{1}{\sqrt{n}} I, \quad \text{and} \\ BJ &= B^{(-)} J = \sqrt{n} J, \end{aligned} \tag{4.5}$$

then for any  $h$  generated by  $g_1, \dots, g_{m-2}$ . We have

$$\begin{aligned} \text{tr}(hg_{m-1}) &= \frac{1}{n} \text{tr}(h) \text{tr}(A), \quad \text{and} \\ \text{tr}(hg_{m-1}^{-1}) &= \frac{1}{n} \text{tr}(h) \text{tr}(A)^{-1}. \end{aligned}$$

These conditions are sufficient for a Jones pair to give a link invariant in the form of the trace of the endomorphisms generated by  $g_1, \dots, g_{m-1}$ . In the next section, we will see that if  $B$  is invertible, then (4.5) holds.

## 4.4 Invertibility

By definition, if  $(A, B)$  is a one-sided Jones pair then  $A$  is invertible and  $B$  is Schur invertible. We call  $(A, B)$  an *invertible one-sided Jones pair* if  $A^{(-)}$  and  $B^{-1}$  also

exist.

In this section, we show that for a one-sided Jones pair  $(A, B)$ , the invertibility of  $B$  implies the Schur-invertibility of  $A$ . In fact, assuming  $B$  is invertible is a more stringent condition than one may expect at first. We prove that the invertibility of  $B$  implies the type-II condition on both  $A$  and  $B$ : This will be used in Section 4.6 to justify our assertion that invertible Jones pairs and four-weight spin models are equivalent concepts.

**Theorem 4.4.1** *Let  $(A, B)$  be a one-sided Jones pair. If  $B$  is invertible then both  $A$  and  $B$  are type-II matrices. Moreover,  $A$  has constant diagonal and  $B$  has constant row sums.*

*Proof.* Since  $A^{-1}$  has the same eigenvectors as  $A$  and their eigenvalues with respect to the same eigenvector are reciprocal to each other, we have  $\Theta_{A,B}(A^{-1}) = B^{(-)}$ . Now applying Corollary 2.3.2 (b) to  $\Theta_{A,B}(A^{-1}) = B^{(-)}$ , we get

$$X_{B^T} \Delta_A X_{A^{-T}} = \Delta_{B^{(-)T}} X_{B^T} \Delta_A.$$

Evaluating this equation at  $I$  yields

$$\begin{aligned} B^T(A \circ A^{-T}) &= B^{(-)T} \circ (B^T(A \circ I)) \\ &= (B^{(-)T} \circ B^T)(A \circ I) \\ &= J(A \circ I). \end{aligned}$$

Since  $(A, B)$  is a one-side Jones pair,  $B^T J = \text{tr}(A)J$ . So

$$A \circ A^{-T} = B^{-T} J(A \circ I) = \text{tr}(A)^{-1} J(A \circ I).$$

The sum of the  $i$ -th column of  $A \circ A^{-T}$  equals

$$\sum_{k=1}^n A_{k,i}(A^{-1})_{i,k} = 1.$$

The  $i$ -th column of  $\text{tr}(A)^{-1}J(A \circ I)$  sums to  $n \text{tr}(A)^{-1}A_{i,i}$ . We have  $A_{i,i} = n^{-1} \text{tr}(A)$  and  $A \circ I = n^{-1} \text{tr}(A)I$ . Therefore  $A \circ A^{-T} = n^{-1}J$  and it follows that  $A$  is a type-II matrix.

By Lemma 4.2.3, we have

$$\Delta_{A^{(-)}}X_A\Delta_A = X_B\Delta_{B^T}X_{B^{-1}}.$$

Evaluating both sides at  $I$  gives

$$A^{(-)} \circ (A(A \circ I)) = B(B^T \circ B^{-1})$$

which leads to

$$B^{-1}(A^{(-)} \circ A)(A \circ I) = B^T \circ B^{-1}.$$

and finally

$$\frac{\text{tr}(A)}{n}B^{-1}J = B^T \circ B^{-1}.$$

The sum of the  $i$ -th row of  $B^T \circ B^{-1}$  equals

$$\sum_{k=1}^n B_{k,i}(B^{-1})_{i,k} = 1.$$

So we have  $\text{tr}(A) \sum_{k=1}^n (B^{-1})_{i,k} = 1$ , or equivalently,  $BJ = \text{tr}(A)J$ . Consequently, we get

$$B^T \circ B^{-1} = n^{-1}J,$$

which is equivalent to the type-II condition on  $B$ .  $\square$

**Theorem 4.4.2** *Let  $(A, B)$  be a Jones pair. If  $A$  is Schur invertible then  $B$  is invertible.*

*Proof.* By Theorem 2.2.1,  $\Theta_{A,B}(A^{-1}) = B^{(-)}$  is equivalent to

$$X_{A^{-1}} \Delta_B X_A = \Delta_B X_A \Delta_{B^{(-)}}.$$

Applying the Exchange Lemma, we get

$$X_{A^{-1}} \Delta_A X_B = \Delta_A X_B \Delta_{B^{(-)T}}.$$

Evaluating both sides at  $J$  yields

$$A^{-1}(A \circ BJ) = A \circ (B(B^{(-)T} \circ J)).$$

Since  $(A, B^T)$  is a one-sided Jones pair, we have  $BJ = \text{tr}(A)J$ , by Lemma 4.2.1. So

$$\text{tr}(A) A^{-1}(A \circ J) = A \circ (BB^{(-)T}).$$

The left-hand side equals  $\text{tr}(A)I$ , so

$$BB^{(-)T} = \text{tr}(A) (A^{(-)} \circ I)$$

and since  $(A^{(-)} \circ I)^{-1} = A \circ I$ ,

$$B^{-1} = \text{tr}(A)^{-1} B^{(-)T} (A \circ I).$$

□

These two theorems tell us that if  $(A, B)$  is a Jones pair, then  $A$  is Schur invertible if and only if  $B$  is invertible. In this case, both  $A$  and  $B$  are type-II matrices with  $A \circ I = \text{tr}(A)n^{-1}I$  and  $BJ = \text{tr}(A)J = B^T J$ . So the Jones pair

$$\left( \frac{\sqrt{n}}{\text{tr}(A)} A, \frac{\sqrt{n}}{\text{tr}(A)} B \right)$$

satisfies the conditions in (4.5). Thus it gives a link invariant.

If  $(A, B)$  is a one-sided Jones pair, then it follows immediately from the definition of one-sided Jones pair that  $A \in \mathcal{N}_{A,B}$ . The following theorem investigates the opposite direction. It extends Jaeger, Matsumoto and Nomura's result [14] which says that if  $A \in \mathcal{N}_A$  then  $A$  is a spin model up to scalar multiplication.

**Theorem 4.4.3** *Let  $A$  and  $B$  be  $n \times n$  type-II matrices. Suppose  $A \circ I = aI$  and  $B^T J = bJ$  for some non-zero  $a, b \in \mathbb{C}$ . If  $A \in \mathcal{N}_{A,B}$  then  $(A, ab^{-1}nB)$  is a one-sided Jones pair.*

*Proof.* Since  $A$  and  $A^{-1}$  share the same set of eigenvectors, the matrix  $A$  belongs to  $\mathcal{N}_{A,B}$  if and only if  $A^{-1} = n^{-1}A^{(-)T}$  belongs to  $\mathcal{N}_{A,B}$ . Using the formula of  $\Theta_{A,B}$

in Lemma 2.3.3. we get

$$\begin{aligned}
 \Theta_{A,B}(A^{-1}) &= B^{(-)} \circ ((A \circ I)^{-1}(A^T \circ n^{-1}A^{(-)T})B) \\
 &= \frac{1}{n}B^{(-)} \circ ((a^{-1}I)JB) \\
 &= \frac{b}{an}B^{(-)} \circ J \\
 &= \frac{b}{an}B^{(-)}.
 \end{aligned}$$

Now an eigenvalue of  $A$  is the reciprocal of the eigenvalue of  $A^{-1}$  with respect to the same eigenvector. we conclude that

$$\Theta_{A,B}(A) = \frac{an}{b}B.$$

Hence

$$X_A \Delta_{\frac{an}{b}B} X_A = \Delta_{\frac{an}{b}B} X_A \Delta_{\frac{an}{b}B}.$$

□

## 4.5 Nomura Algebras of Invertible Jones Pairs

If  $(A, B)$  is an invertible one-sided Jones pair, then  $A$  and  $B$  are type II. From  $A$  and  $B$ , we construct several Nomura algebras such as  $\mathcal{N}_A$ ,  $\mathcal{N}_B$ ,  $\mathcal{N}_{A,B}$ ,  $\mathcal{N}_{A,B}^{\tau}$ , et cetera. In this section, we investigate the relations among these algebras.

**Theorem 4.5.1** *Let  $(A, B)$  be an invertible one-sided Jones pair. Then*

$$\mathcal{N}_A = \mathcal{N}_{A^{\tau}} = \mathcal{N}_B \quad \text{and} \quad \mathcal{N}_{A,B}^{\tau} = \mathcal{N}_{A^{\tau},B}.$$

*Proof.* Since  $A \in \mathcal{N}_{A,B}$  is Schur invertible, by Theorem 3.8.5 (c), we have

$$\mathcal{N}_A = \mathcal{N}_B.$$

By Lemma 4.2.2 (a),  $(A^T, B)$  is also a one-sided Jones pair. Applying Theorem 3.8.5 (c) on  $A^T \in \mathcal{N}_{A^T,B}$  leads to  $\mathcal{N}_B = \mathcal{N}_{A^T}$ .

Since both  $(A, B)$  and  $(A^T, B)$  are one-sided Jones pairs, we have

$$\Theta_{A,B}(A) = B = \Theta_{A^T,B}(A^T).$$

We deduce from Theorem 3.8.5 (b) that

$$\mathcal{N}'_{A,B} = \mathcal{N}_{A^T} B, \quad \text{and} \quad \mathcal{N}'_{A^T,B} = \mathcal{N}_A B.$$

But  $\mathcal{N}_A = \mathcal{N}_{A^T}$  and so  $\mathcal{N}'_{A,B} = \mathcal{N}'_{A^T,B}$ . □

Note that Theorem 3.8.4 applied to  $(A, B)$  yields

$$\Theta_A(A \circ A^T) = n^{-1} B B^T,$$

which indicates that  $\mathcal{N}_A$  is non-trivial in most cases.

In the rest of this section, we examine the relations among different Nomura algebras and their duality maps constructed from an invertible Jones pair.

**Theorem 4.5.2** *Let  $(A, B)$  be an invertible Jones pair. Then*

$$\mathcal{N}_A = \mathcal{N}_{A^T} = \mathcal{N}_B = \mathcal{N}_{B^T}.$$

*Moreover, the dualities  $\Theta_A, \Theta_B$  satisfy  $\Theta_B(F)^T = B^{-1} \Theta_A(F) B$  for all  $F \in \mathcal{N}_A$ .*

*Proof.* Since  $(A, B)$  is an invertible one-sided Jones pair, the previous theorem gives  $\mathcal{N}_A = \mathcal{N}_{A^T} = \mathcal{N}_B$ . Similarly, since  $(A, B^T)$  is also an invertible one-sided Jones pair, we get  $\mathcal{N}_A = \mathcal{N}_{B^T}$  using Theorem 3.8.5 (c). So the first part of the theorem holds. For any  $F \in \mathcal{N}_A = \mathcal{N}_B$ , Theorem 3.8.3 gives

$$\begin{aligned}\Theta_{A,B}(F \circ A) &= n^{-1} \Theta_A(F) B, \\ \Theta_{A,B}(A \circ F) &= n^{-1} B \Theta_B(F)^T.\end{aligned}$$

Hence we get  $\Theta_B(F)^T = B^{-1} \Theta_A(F) B$ . □

**Corollary 4.5.3** *If  $(A, B)$  is an invertible Jones pair, then*

$$\mathcal{N}_{A,B^T} = \mathcal{N}_{A,B} \quad \text{and} \quad \mathcal{N}'_{A,B^T} = B^{-1} \mathcal{N}'_{A,B} B^T.$$

*Proof.* Theorem 3.8.5 (a) applied to  $A$  in  $\mathcal{N}_{A,B}$  and  $A$  in  $\mathcal{N}_{A,B^T}$  gives

$$\mathcal{N}_{A,B} = A \circ \mathcal{N}_A = \mathcal{N}_{A,B^T}.$$

We now prove the second equality. Now both  $A$  and  $B$  are type-II matrices, by Corollary 3.2.3,  $\mathcal{N}'_A = \mathcal{N}_{A^T}$  and  $\mathcal{N}'_B = \mathcal{N}_{B^T}$ . So the second part of Theorem 4.5.2 says that

$$\mathcal{N}_{B^T} = B^{-1} \mathcal{N}_{A^T} B.$$

But  $(A, B)$  is an invertible Jones pair. So using the same theorem, we have  $\mathcal{N}_{A^T} = \mathcal{N}_{B^T}$ . Hence  $\mathcal{N}_{A^T} = B^{-1} \mathcal{N}_{A^T} B$ , or equivalently  $B \mathcal{N}_{A^T} = \mathcal{N}_{A^T} B$ . Applying

Theorem 3.8.5 (b) to  $\Theta_{A,B}(A) = B$ , we get

$$\mathcal{N}'_{A,B} = \mathcal{N}_{A^T} B = B \mathcal{N}_{A^T}.$$

Similarly, applying the same theorem to  $\Theta_{A,B^T}(A) = B^T$  yields

$$\begin{aligned} \mathcal{N}'_{A,B^T} &= \mathcal{N}_{A^T} B^T \\ &= B^{-1} \mathcal{N}'_{A,B} B^T. \end{aligned}$$

□

We will need the following lemma in the computation in Chapter 5.

**Lemma 4.5.4** *If  $(A, B)$  is an invertible Jones pair and  $R \in \mathcal{N}_{A,B}$ , then*

$$\Theta_{A,B}(R) B^T = \Theta_{A,B^T}(R) B.$$

*Proof.* By Theorem 3.8.4, for any  $R$  in  $\mathcal{N}_{A,B}$ , we have

$$\Theta_A(R \circ A^T) = n^{-1} \Theta_{A,B}(R) B^T.$$

Since  $\mathcal{N}_{A,B} = \mathcal{N}_{A,B^T}$ ,  $R$  also belongs to  $\mathcal{N}_{A,B^T}$ . The same theorem tells us that

$$\Theta_A(R \circ A^T) = n^{-1} \Theta_{A,B^T}(R) B.$$

Thus we have the equation in the lemma. □

## 4.6 Four-Weight Spin Models

Bannai and Bannai [2] generalized spin models to four-weight spin models, and showed that the partition functions of four-weight spin models provide invariants for oriented links. In this section, we show that four-weight spin models are equivalent to invertible Jones pairs.

As defined in Section 1.2, a *four-weight spin model* is a 5-tuple  $(W_1, W_2, W_3, W_4; d)$  with  $d = \pm\sqrt{n}$  satisfying

(a) There exists non-zero scalar  $a$  such that

$$\begin{aligned} W_1 \circ I &= aI, & W_3 \circ I &= a^{-1}I, \\ W_2 J &= W_2^T J = da^{-1}J, & W_4 J &= W_4^T J = daJ. \end{aligned}$$

(b) The matrices  $W_1, W_2, W_3$  and  $W_4$  are type II and

$$W_3 = W_1^{(-)T}, \quad W_2 = W_4^{(-)T}.$$

(c)

$$\sum_{x=1}^n (W_1)_{a,x} (W_1)_{x,b} (W_4)_{c,x} = d(W_1)_{a,b} (W_4)_{c,a} (W_4)_{c,b}, \quad (4.6)$$

$$\sum_{x=1}^n (W_1)_{x,a} (W_1)_{b,x} (W_4)_{x,c} = d(W_1)_{b,a} (W_4)_{a,c} (W_4)_{b,c}. \quad (4.7)$$

Using Theorems 4.4.1 and 4.4.2, it is almost immediate that invertible Jones pairs are the same as four-weight spin models.

**Theorem 4.6.1** *Let  $A, B \in M_n(\mathbb{C})$  and  $d^2 = n$ . Then the following are equivalent.*

- a.  $(A, B)$  is an invertible Jones pair.
- b.  $(dA, nB^{-1}, dA^{-1}, B; d)$  is a four-weight spin model.

*Proof.* If  $(dA, nB^{-1}, dA^{-1}, B; d)$  is a four-weight spin model, then as we have seen in Section 4.1,  $(A, B)$  is a Jones pair where  $A$  and  $B$  are both type II.

Conversely, suppose  $(A, B)$  is an invertible Jones pair. By Theorem 4.4.1, both  $A$  and  $B$  are type-II matrices. Moreover,  $A$  has constant diagonal and  $B$  has constant row sum and column sum. Let  $W_1 = dA$  and  $W_4 = B$ . Considering Equation (4.2) with  $k = b$ ,  $i = a$ , and  $j = c$ , we see that  $(A, B)$  is a one-sided Jones pair if and only if Condition (4.7) holds. Similarly, using the same equation,  $(A^T, B^T)$  is a one-sided Jones pair if and only if Condition (4.6) holds. By Lemma 4.2.2 (a), we find that  $(A^T, B^T)$  is a one-sided Jones pair if and only if  $(A, B^T)$  is also a one-sided Jones pair. As a result,  $(dA, nB^{-1}, dA^{-1}, B; d)$  is a four-weight spin model.  $\square$

We will see in Theorem 5.1.3 that  $W$  is a spin model if and only if  $(d^{-1}W, W^{(-)T})$  is an invertible Jones pair. Therefore it is equivalent to  $(W, W, W^{(-)T}, W^{(-)T}; d)$  being a four-weight spin model.

In [2], Bannai and Bannai studied three types of four-weight spin models: Jones type, pseudo-Jones type and Hadamard type. They correspond respectively to the following types of Jones pairs

- a.  $(A, dA^{(-)T})$ , where  $A$  is a spin model.
- b.  $(A, dA)$ ,
- c.  $(A, B)$ , where one of  $A, B$  is a Hadamard matrix.

Using  $n \times n$  Hadamard matrices satisfying certain conditions, Yamada constructed  $n^2 \times n^2$  four-weight spin models of pseudo-Jones type and symmetric

Hadamard type. Using tensor products, her construction gives infinite families of four-weight spin models of both types. For details, see [26].

In [1], Bannai extended work by Guo and proved that if  $(W_1, W_2, W_3, W_4; d)$  is a four-weight spin model, then for  $i = 1, \dots, 4$ ,

$$\mathcal{N}_{W_1} = \mathcal{N}_{W_2} = \mathcal{N}_{W_3}^T.$$

Note that by Theorem 4.6.1, the pair  $(d^{-1}W_1, W_4)$  is an invertible Jones pair. As shown in Theorem 4.5.1, we are able to obtain part of the above equations, assuming a weaker condition on  $(d^{-1}W_1, W_4)$ , that is, it is a one-sided Jones pair.

## 4.7 Odd Gauge Equivalence

Two four-weight spin models  $(W_1, W_2, W_3, W_4; d)$  and  $(W'_1, W'_2, W'_3, W'_4; d)$  are *gauge equivalent* if there exist an invertible diagonal matrix  $D$ , a permutation matrix  $P$  and a non-zero scalar  $c$  such that

$$\begin{aligned} W'_1 &= cDW_1D^{-1}, & W'_3 &= c^{-1}DW_3D^{-1}, \\ W'_2 &= c^{-1}P^{-1}W_2, & W'_4 &= cW_4P. \end{aligned}$$

Jaeger proved that gauge-equivalent spin models give the same invariant (see Proposition 11 in [13]). In the same paper, he showed that  $W'_2 = W_2$  and  $W'_4 = W_4$  if and only if  $W'_1 = DW_1D^{-1}$  and  $W'_3 = DW_3D^{-1}$  for some invertible diagonal matrix  $D$ . In this case, we say that the two four-weight spin models are related by an *odd gauge transformation*. He also proved that  $W'_1 = W_1$  and  $W'_3 = W_3$  if and only if  $W'_2 = P^{-1}W_2$  and  $W'_4 = W_4P$  for some permutation matrix  $P$ . The two four-weight spin models are said to be related by an *even gauge transformation*.

In Sections 4.7 and 4.8, we extend Jaeger's result on gauge equivalence of four-weight spin models to invertible one-sided Jones pairs. In particular, the last lemma in this section allows us to consider only the invertible Jones pairs with their first matrix symmetric. This will simplify the computation in Chapter 6 immensely.

We say that the invertible one-sided Jones pairs  $(A, B)$  and  $(C, B)$  are *odd gauge equivalent* if  $A = DC D^{-1}$ , for some invertible diagonal matrix  $D$ . In the following, we examine the odd gauge equivalence of one-sided invertible Jones pairs.

**Lemma 4.7.1** *Let  $A, C, M$  be Schur-invertible matrices. If  $X_A \Delta_M = \Delta_M X_C$ , then there exists invertible diagonal matrix  $D$  such that*

$$C^{(-)} \circ A = D J D^{-1}.$$

*Proof.* Since  $\Delta_J = X_I$  is the identity endomorphism of  $M_n(\mathbb{C})$ , we have

$$\Delta_J X_A \Delta_M = X_I \Delta_M X_C.$$

Applying the Exchange Lemma, we obtain

$$\begin{aligned} \Delta_A X_J \Delta_{M^T} &= X_I \Delta_C X_M \\ &= \Delta_C X_M \end{aligned}$$

which gives

$$\Delta_{C^{(-)} \circ A} X_J = X_M \Delta_{M^{(-)T}}.$$

Evaluating the left-hand side at  $E_{1,j} = \epsilon_1 \epsilon_j^T$ , we get

$$\begin{aligned} (C^{(-)} \circ A) \circ (J \epsilon_1 \epsilon_j^T) &= ((C^{(-)} \circ A) \epsilon_j \circ \mathbf{1}) \epsilon_j^T \\ &= (C^{(-)} \circ A) \epsilon_j \epsilon_j^T. \end{aligned}$$

The right-hand side evaluated at  $E_{1,j}$  equals  $(M_{j,1})^{-1} M \epsilon_1 \epsilon_j^T$ . Therefore the  $ij$ -entry of  $C^{(-)} \circ A$  equals  $(M_{j,1})^{-1} M_{i,1}$ . So if  $D$  is the diagonal matrix with  $D_{i,i} = M_{i,1}$  then

$$C^{(-)} \circ A = DJD^{-1}.$$

□

**Corollary 4.7.2** *If both  $(A, B)$  and  $(C, B)$  are invertible one-sided Jones pairs, then  $A = DC D^{-1}$  for some invertible diagonal matrix  $D$ .*

*Proof.* By Lemma 4.2.3, we have

$$\Delta_{A^{(-)}} X_A \Delta_A = X_B \Delta_{B^T} X_{B^{-1}} = \Delta_{C^{(-)}} X_C \Delta_C$$

which gives

$$X_A \Delta_{A \circ C^{(-)}} = \Delta_{A \circ C^{(-)}} X_C.$$

Applying Lemma 4.7.1 with  $M = A \circ C^{(-)}$ , we get  $C^{(-)} \circ A = DJD^{-1}$  for some invertible diagonal matrix  $D$ . Now

$$A = C \circ (DJD^{-1}) = D(C \circ J)D^{-1} = DC D^{-1}.$$

□

Combining this with Lemma 4.2.2 (c), we see that  $(A, B)$  and  $(C, B)$  are invertible one-sided Jones pairs if and only if  $A = DC D^{-1}$  for some invertible diagonal matrix  $D$ .

By Lemma 4.2.2 (a), if  $(A, B)$  is an invertible one-sided Jones pair, then so is  $(A^T, B)$ . Therefore there exists an invertible diagonal matrix  $D$  that satisfies  $A = D A^T D^{-1}$ . Since the diagonal entries of  $D$  are non-zero complex numbers, there exists diagonal matrix  $D_1$  satisfying  $D_1^2 = D$ . Then the matrix

$$D_1^{-1} A D_1 = D_1 A^T D_1^{-1} = (D_1^{-1} A D_1)^T$$

is symmetric and  $(D_1^{-1} A D_1, B)$  is odd gauge equivalent to  $(A, B)$ . So we get the following result, which generalizes Proposition 7 (ii) from Jaeger [13].

**Lemma 4.7.3** *Let  $(A, B)$  be an invertible one-sided Jones pair. Then there exists a symmetric matrix  $A'$  such that the invertible one-sided Jones pair  $(A', B)$  is odd gauge equivalent to  $(A, B)$ .* □

## 4.8 Even Gauge Equivalence

We say that the invertible one-sided Jones pairs  $(A, B)$  and  $(A, C)$  are even gauge equivalent if  $C = BP$ , for some permutation matrix  $P$ . Now we extend Jaeger's result on even gauge equivalence to invertible one-sided Jones pairs.

**Lemma 4.8.1** *Let  $F, G$ , and  $M$  be invertible matrices. If  $\Delta_F X_M = X_M \Delta_G$  then  $GF^{-1}$  is a permutation matrix.*

*Proof.* Multiplying both sides by  $\Delta_J = X_I$ , we get

$$\Delta_F X_M \Delta_J = X_M \Delta_G X_I$$

with the Exchange Lemma.

$$\Delta_M X_F \Delta_J = X_M \Delta_I X_G$$

and

$$\Delta_M = X_M \Delta_I X_{GF^{-1}}.$$

Evaluating both sides at  $E_{ij}$  yields

$$M_{i,j} e_i e_j^T = M(Ie_j \circ GF^{-1} e_i) e_j^T.$$

Then for all  $i = 1, \dots, n$ ,

$$M_{i,j} e_i = (GF^{-1})_{j,i} M e_j.$$

But  $M$  is invertible. So for each  $j$ , there exists a unique  $r$  such that  $M e_j$  is a scalar multiple of  $e_r$ . That is,  $(GF^{-1})_{j,r}$  is the only non-zero entry in the  $j$ -th row of  $GF^{-1}$ . Since  $M_{r,j} = (GF^{-1})_{j,r} M_{r,j} \neq 0$ , we conclude that  $(GF^{-1})_{j,r} = 1$ . Therefore  $GF^{-1}$  is a permutation matrix.  $\square$

**Corollary 4.8.2** *If  $(A, B)$  and  $(A, C)$  are invertible one-sided Jones pairs, then  $C = BP$  for some permutation matrix  $P$ .*

*Proof.* By Lemma 4.2.3, we have

$$X_B \Delta_{B^T} X_{B^{-1}} = \Delta_{A(-)} X_A \Delta_A = X_C \Delta_{C^T} X_{C^{-1}}$$

which gives

$$\Delta_{C^T} X_{C^{-1}B} = X_{C^{-1}B} \Delta_{B^T}.$$

Applying Lemma 4.8.1 with  $F = C^T$ ,  $G = B^T$  and  $M = C^{-1}B$ , there exists a permutation matrix  $P$  such that  $B^T C^{-T} = P$  which leads to  $C = BP$ .  $\square$

Together with Lemma 4.2.2 (d),  $(A, B)$  and  $(A, C)$  are invertible one-sided Jones pairs if and only if  $C = BP$  for some permutation matrix  $P$ .

If  $(A, B)$  is an invertible Jones pair, then both  $(A, B)$  and  $(A, B^T)$  are invertible one-sided Jones pairs. So there exists permutation matrix  $P$  such that  $B^T = BP$ . Now

$$B = (B^T)^T = (BP)^T = P^T B^T = P^T BP$$

and hence  $B$  and  $P$  commute. We focus on the case where  $P$  has order  $2r - 1$ . Let  $Q = P^r$ . Note that  $Q^T = P^{r-1}$ . Then we get

$$(BQ)^T = Q^T B^T = P^{r-1} BP = BQ.$$

Therefore  $(A, B)$  is even-gauge equivalent to  $(A, BQ)$ , where  $BQ$  is symmetric. This is a proof of Proposition 10 (ii) in [13]. In summary, if  $(A, B)$  is an invertible one-sided Jones pair with  $P = B^{-1}B^T$  having odd order, then  $(A, B)$  is gauge equivalent to some invertible one-sided Jones pair whose matrices are symmetric.

# Chapter 5

## Spin Models

In Section 6 of [12], Jaeger proposed to study the properties of association schemes that contain spin models. We use this chapter to survey some classical results in this area. We first examine spin models from the point of view of Jones pairs. In Section 5.2, we present the derivation of the modular invariance equation given by Jaeger, Matsumoto, and Nomura in [14]. In Section 5.3, we provide a new and shorter proof of Curtin and Nomura's result, which states that the Nomura algebra of a spin model is strongly hyper-self-dual [7]. Section 5.4 contains a shorter proof of Jaeger's characterization of two-class association schemes that contain spin models. This was the first connection between spin models and association schemes discovered [10]. In the last section, we give a new proof using Jones pairs of Jaeger and Nomura's result on the symmetric and non-symmetric Hadamard spin models [15].

## 5.1 The Jones Pair $(d^{-1}W, W^{(-)})$

As mentioned in Section 1.2, Kawagoe, Munemasa and Watatani defined a spin model with loop variable  $d = \pm\sqrt{n}$  to be an  $n \times n$  matrix  $W$  that satisfies

(I) There exists some non-zero scalar  $a$  such that

$$W \circ I = aI, \quad \text{and} \quad WJ = W^T J = da^{-1}J.$$

(II)  $W$  is a type-II matrix.

(III) For all  $i, j, k = 1, \dots, n$ ,

$$\sum_{x=1}^n \frac{W_{k,x} W_{x,i}}{W_{j,x}} = d \frac{W_{k,i}}{W_{j,i} W_{j,k}}. \quad (5.1)$$

An interesting example is the Higman-Sims model discovered by Jaeger [10]. The Higman-Sims graph is a strongly regular graph with parameters  $(100, 22, 0, 6)$ , defined from the unique  $3-(22, 6, 1)$  design. Let  $A_1$  and  $A_2$  be adjacency matrices of the Higman-Sims graph and its complement respectively. If  $t$  satisfies  $t^2 + t^{-2} = -3$ , then

$$W = (5t - 3)I + tA_1 + t^{-1}A_2$$

is a spin model. The Nomura algebra  $\mathcal{N}_W$  equals the span of  $\{I, A_1, A_2\}$ , see [14].

In Section 4.1, we see that if  $W$  is a spin model with loop variable  $d$  then  $(d^{-1}W, W^{(-)T})$  is a one-sided Jones pair. It turns out that  $W$  is a spin model if and only if  $(d^{-1}W, W^{(-)T})$  is an invertible Jones pair. We will prove this statement below.

**Lemma 5.1.1** *Let  $W$  be a type-II matrix. Then  $(d^{-1}W, W^{(-)T})$  is a one-sided Jones pair if and only if  $(d^{-1}W, W^{(-)})$  is also a one-sided Jones pair.*

*Proof.* We apply the Exchange Lemma to

$$X_{d^{-1}W} \Delta_{W^{(-)T}} X_{d^{-1}W} = \Delta_{W^{(-)T}} X_{d^{-1}W} \Delta_{W^{(-)T}}$$

to get

$$d^{-1}X_W \Delta_W X_{W^{(-)T}} = \Delta_W X_{W^{(-)T}} \Delta_{W^{(-)}}.$$

Since  $W^{(-)T} = nW^{-1}$ , taking the inverse of each side gives

$$dX_W \Delta_{W^{(-)}} X_{W^{-1}} = \Delta_W X_W \Delta_{W^{(-)}}.$$

which equals

$$d\Delta_{W^{(-)}} X_W \Delta_{W^{-1}} = X_W \Delta_{W^{(-)}} X_W.$$

Since every step above is reversible, the converse is also true.  $\square$

If  $W$  is a spin model then it is also a type-II matrix. So the above lemma implies that  $(d^{-1}W, W^{(-)T})$  is an invertible Jones pair.

**Lemma 5.1.2** *If  $(d^{-1}W, W^{(-)T})$  is an invertible Jones pair, then  $W$  is a spin model with loop variable  $d$ .*

*Proof.* By Lemma 4.4.1, the invertibility of  $W^{(-)T}$  implies that  $W$  is a type-II matrix and  $W \circ I = aI$  where  $a = \text{tr}(W)n^{-1}$ . Applying Theorem 4.2.1 to the

one-sided Jones pairs  $(d^{-1}W, W^{(-)T})$  and  $(d^{-1}W, W^{(-)})$  yields

$$W^{(-)}J = daJ \quad \text{and} \quad W^{(-)T}J = daJ,$$

respectively.

It follows from  $W^{-1} = n^{-1}W^{(-)T}$  that  $W^TJ = WJ = da^{-1}J$ . Lastly, using Equation (4.2) on the one-sided Jones pair  $(d^{-1}W, W^{(-)T})$  gives the type-III condition of spin models.  $\square$

**Theorem 5.1.3** *Let  $W$  be an  $n \times n$  matrix. Then  $W$  is a spin model if and only if  $(d^{-1}W, W^{(-)T})$  is an invertible Jones pair.*  $\square$

In the following, we use the theory developed in the previous chapters to reproduce two existing results about spin models.

Lemma 5.1.1 together with Theorem 4.4.3, we have a proof of a result due to Jaeger, Matsumoto and Nomura, Proposition 9 in [14].

**Theorem 5.1.4** *Suppose  $W$  is a type-II matrix. Then  $W \in \mathcal{N}_W$  if and only if  $cW$  is a spin model for some non-zero scalar  $c$ .*  $\square$

We now prove Proposition 2 in [15] due to Jaeger and Nomura.

**Theorem 5.1.5** *If  $W$  is a spin model then there exist a diagonal matrix  $D$  and a permutation matrix  $P$  such that*

$$W^{(-)T} \circ W = DJD^{-1}.$$

and

$$n^{-1}W^{(-)}W = P.$$

*Proof.* By Theorem 5.1.3, if  $W$  is a spin model with loop variable  $d$ , then

$$(d^{-1}W, W^{(-)}), (d^{-1}W^T, W^{(-)}), (d^{-1}W, W^{(-)T}), (d^{-1}W^T, W^{(-)T})$$

are invertible one-sided Jones pairs. The first equation follows from applying Corollary 4.7.2 to the first two invertible one-sided Jones pairs listed above. Applying Corollary 4.8.2 to the first and the third pairs above yields the second equality.  $\square$

Jaeger and Nomuar called the order of  $P$  the *index of the spin model*. In the same paper, they also proved that any spin model of index two has to take the following form

$$\begin{pmatrix} A & A & B & -B \\ A & A & -B & B \\ -B^T & B^T & C & C \\ B^T & -B^T & C & C \end{pmatrix}$$

where  $A$  and  $C$  are symmetric (Proposition 7 in [15]). As we will see in Section 5.5, the construction of the non-symmetric Hadamard spin models is very similar to the above form.

## 5.2 Duality and Modular Invariance Equation

Let  $P$  be the matrix of eigenvalues for a formally self-dual Bose-Mesner algebra of dimension  $m + 1$ . Let  $d = \pm\sqrt{n}$  and  $D$  be a diagonal matrix with  $D_{i,i} = t_i$  for  $i = 0, \dots, m$ . We call the equation

$$(PD)^3 = t_0 d^3 I$$

the *modular invariance equation*. We say that  $P$  satisfies the *modular invariance property* if there exists a diagonal matrix  $D$  such that the modular invariance equation holds. Bannai, Bannai and Jaeger ([3]) first discovered that this property of  $P$  is a necessary condition for a formally self-dual Bose-Mesner algebra to contain a spin model. Using this equation, they provided a method to exhaustively search for all spin models contained in a formally self-dual Bose-Mesner algebra. In Chapter 6, we use this equation to design a search for four-weight spin models.

Suppose  $W$  is a spin model. Then  $(W, W^{(-)})$  is an invertible Jones pair and by Theorem 4.5.2, we know that  $\mathcal{N}_W = \mathcal{N}_{W^T}$ . For all  $M \in \mathcal{N}_{W^T}$ , we have by Lemma (4.5.4)

$$\begin{aligned} \Theta_{W^T, W^{(-)T}}(M) &= \Theta_{W^T, W^{(-)}}(M) W^{-1} W^T \\ &= (\Theta_{W^{(-)}, W^T}(M))^T W^{-1} W^T. \end{aligned}$$

Applying Lemma (4.5.4) again, the above becomes

$$\begin{aligned} \Theta_{W^T}(M) &= (\Theta_{W^{(-)}, W}(M) W^T W^{-1})^T W^{-1} W^T \\ &= W^{-T} W (\Theta_{W^{(-)}, W}(M))^T W^{-1} W^T \\ &= W^{-T} W \Theta_W(M) W^{-1} W^T. \end{aligned}$$

Now  $\mathcal{N}_W$  is commutative and it contains  $W$ ,  $W^{-T}$ , and  $\Theta_W(M)$ . So we have  $\Theta_{W^T}(M) = \Theta_W(M)$ , for all  $M$  in  $\mathcal{N}_W$ . By Theorem 3.2.2, we have  $\Theta_W^2(M) = nM^T$ . Hence  $\mathcal{N}_W$  is formally self-dual. By Lemma 2.3.3, the duality map  $\Theta_W$  is expressed explicitly as

$$\Theta_W(M) = \frac{n}{\text{tr}(W)} W \circ ((W^T \circ M)W^{(-)}), \quad (5.2)$$

for all  $M \in \mathcal{N}_W$ .

In the following, we present Jaeger, Matsumoto and Nomura's proof that the modular invariance property is a necessary condition of  $\mathcal{N}_W$  for  $W$  to be a spin model.

**Theorem 5.2.1** *Suppose  $W$  is a spin model with loop variable  $d = \pm\sqrt{n}$  and  $\{A_0, \dots, A_m\}$  is the basis of Schur idempotents of  $\mathcal{N}_W$ . If  $W = \sum_{i=0}^m t_i A_i^T$  then the diagonal matrix  $D$  with  $D_{i,i} = t_i$  satisfies the modular invariance equation*

$$(PD)^3 = t_0 d^3 I.$$

*Proof.* Let  $\mathcal{E} = \{E_0, \dots, E_m\}$  be the basis of the principal idempotents of  $\mathcal{N}_W$  such that  $\Theta_W(E_i) = A_i$  and  $\Theta_W(A_i) = nE_i^T$ . The matrix of eigenvalues  $P$  is the matrix of  $\Theta_W$  with respect to  $\mathcal{E}$ . However,  $P$  is also the transition matrix from  $\mathcal{A}$  to  $\mathcal{E}$ . Therefore the matrix of  $\Theta_W$  with respect to  $\mathcal{A}$  is  $P^{-1}PP = P$ . Since  $\Theta_W^2(A_i) = nA_i^T$ , we have  $P^2 = nT$  where  $T$  represents the transpose map with respect to  $\mathcal{A}$ .

Suppose  $W = \sum_{i=0}^m t_i A_i^T$ . Since  $(d^{-1}W, W^{(-)})$  is a one-sided Jones pair, we have  $W^{(-)} = \Theta_W(d^{-1}W) = d \sum_{i=0}^m t_i E_i$ . Now  $D$  is the matrix representing the map  $M \rightarrow W^T \circ M$  with respect to  $\mathcal{A}$ . Similarly  $TD T$  represents the map  $M \rightarrow W \circ M$

with respect to  $\mathcal{A}$ . Moreover, the matrix  $dP^{-1}DP$  represents the map  $M \rightarrow MW^{(-)}$  with respect to  $\mathcal{A}$ . So Equation (5.2) holds if and only if

$$P = t_0^{-1}(TDT)(dP^{-1}DP)D.$$

Since  $P^2 = nT$ , we have  $TP^{-1} = P^{-1}T = n^{-1}P$  and

$$\begin{aligned} I &= t_0^{-1}d(P^{-1}T)D(TP^{-1})DPD \\ &= t_0^{-1}d^{-3}PDPDPD. \end{aligned}$$

Hence  $(PD)^3 = t_0d^3I$ . □

### 5.3 Strongly Hyper-Self-Duality

In this section, we extend a result due to Curtin and Nomura, (Theorem 5.5 in [7]).

Suppose  $\mathcal{A} = \{A_0, \dots, A_d\}$  is an association scheme with its Bose-Mesner algebra denoted by  $\mathcal{B}$ . The Terwilliger algebra of  $\mathcal{B}$  can be defined as

$$\mathcal{T}_{\mathcal{B}} = \{\Delta_M, X_M : M \in \mathcal{B}\}.$$

Now consider the subspace  $S_p$  of  $\mathbf{M}_n(\mathbb{C})$  spanned by  $\{E_{i,p} : i = 1, \dots, n\}$ . This space is isomorphic to  $\mathbb{C}^n$ . For each endomorphism  $Y$  of  $\mathbf{M}_n(\mathbb{C})$ , we use  $(Y)_p$  to denote  $Y$  restricted to  $S_p$ . The Terwilliger algebra of  $\mathcal{B}$  with respect to  $p$  is defined as

$$\mathcal{T}_{\mathcal{B},p} = \{(\Delta_M)_p, (X_M)_p : M \in \mathcal{B}\}.$$

A *hyper-duality* of  $\mathcal{T}_{\mathcal{B},p}$  is an automorphism  $\Psi_p$  that swaps the sets

$$\{\Delta_M : M \in \mathcal{N}_W\} \quad \text{and} \quad \{X_M : M \in \mathcal{N}_W\}.$$

and satisfies  $\Psi_p^2((X_M)_p) = (X_M)_p^T$  for all  $M \in \mathcal{B}$ . Furthermore,  $\mathcal{T}_{\mathcal{B},p}$  is *strongly hyper-self-dual* if there exists a hyper-duality that can be expressed as a conjugation of some invertible element of  $\mathcal{T}_{\mathcal{B},p}$ . That is, there exists some  $Y \in \mathcal{T}_{\mathcal{B},p}$  such that  $\Psi_p(Z) = Y^{-1}ZY$ , for all  $Z \in \mathcal{T}_{\mathcal{B},p}$ .

Theorem 5.5 of [7] states that if  $W$  is a spin model then  $\mathcal{T}_{\Lambda_W,p}$  is strongly hyper-self-dual for all  $p = 1, \dots, n$ . In the following, we extend this result to  $\mathcal{T}_{\Lambda_W}$  by showing that there exists  $\Lambda \in \mathcal{T}_{\Lambda_W}$  such that the map  $\Psi : Z \mapsto \Lambda^{-1}Z\Lambda$  interchanges the sets  $\{\Delta_M : M \in \mathcal{B}\}$  and  $\{X_M : M \in \mathcal{B}\}$ , and it satisfies  $\Psi^2(X_M) = X_M^T$  for all  $M \in \mathcal{N}_W$ .

Suppose  $W$  is a spin model, so  $\Theta_W(W) = dW^{(-)}$ . By Lemma 3.2.6,

$$\Theta_W(W^T) = \Theta_W(W)^T = dW^{(-)T}.$$

Hence we have

$$X_{W^T} \Delta_{W^{(-)}} X_W = \Delta_{W^{(-)}} X_W \Delta_{dW^{(-)T}}.$$

We use  $\Lambda$  to denote this operator.

**Lemma 5.3.1** *Let  $W$  be a spin model.*

a. *If  $R_1 \in \mathcal{N}_{W,W^{(-)}}$  and  $S_1 = \Theta_{W,W^{(-)}}(R_1)$  then*

$$\Lambda^{-1} X_{R_1} \Lambda = \Delta_{S_1}.$$

b. If  $R_2 \in \mathcal{N}_{W^{(-)}, W}$  and  $S_2 = \Theta_{W^{(-)}, W}(R_2)$  then

$$\Lambda^{-1} \Delta_{S_2} \Lambda = X_{R_2}.$$

*Proof.* First we prove (a). Using  $\Lambda = d \Delta_{W^{(-)}} X_W \Delta_{W^{(-)T}}$ , we get

$$\Lambda^{-1} X_{R_1} \Lambda = \Delta_{W^T} X_{W^{-1}} \Delta_W (X_{R_1} \Delta_{W^{(-)}} X_W) \Delta_{W^{(-)T}},$$

since  $\Theta_W(R_1) = S_1$ , applying Theorem 2.2.1 gives

$$\begin{aligned} \Lambda^{-1} X_{R_1} \Lambda &= \Delta_{W^T} X_{W^{-1}} \Delta_W (\Delta_{W^{(-)}} X_W \Delta_{S_1}) \Delta_{W^{(-)T}} \\ &= \Delta_{W^T} \Delta_{S_1} \Delta_{W^{(-)T}} \\ &= \Delta_{S_1}. \end{aligned}$$

So part (a) holds.

Now we prove (b). Using  $\Lambda = X_{W^T} \Delta_{W^{(-)}} X_W$ , we get

$$\Lambda^{-1} \Delta_{S_2} \Lambda = X_{W^{-1}} \Delta_W X_{W^{-T}} \Delta_{S_2} X_{W^T} \Delta_{W^{(-)}} X_W.$$

Since  $W^{-T} = n^{-1} W^{(-)}$ ,

$$\begin{aligned} \Lambda^{-1} \Delta_{S_2} \Lambda &= \frac{1}{n} X_{W^{-1}} (\Delta_W X_{W^{(-)}} \Delta_{S_2}) X_{W^T} \Delta_{W^{(-)}} X_W \\ &= \frac{1}{n} X_{W^{-1}} (X_{R_2} \Delta_W X_{W^{(-)}}) X_{W^T} \Delta_{W^{(-)}} X_W \\ &= X_{W^{-1} R_2 W}. \end{aligned}$$

Since both  $R_2$  and  $W$  belong to  $\mathcal{N}_W$ , they commute and  $W^{-1}R_2W = R_2$ . Therefore

$$\Lambda^{-1}\Delta_{S_2}\Lambda = X_{R_2}.$$

□

Define an isomorphism  $\Psi$  of  $\mathcal{T}_{\mathcal{N}_W}$  as

$$\Psi(Y) = \Lambda^{-1}Y\Lambda.$$

for all  $Y \in \mathcal{T}_{\mathcal{N}_W}$ . Note that  $\Psi$  is expressed as a conjugation of  $\Lambda$  in  $\mathcal{T}_{\mathcal{N}_W}$ . Now we show that  $\Psi$  acts as a hyper-duality for  $\mathcal{T}_{\mathcal{N}_W}$ .

**Theorem 5.3.2** *If  $W$  is a spin model, then the map  $\Psi$  defined above interchanges the sets*

$$\{\Delta_M : M \in \mathcal{N}_W\} \quad \text{and} \quad \{X_M : M \in \mathcal{N}_W\}.$$

Moreover, it satisfies  $\Psi^2(X_M) = X_M^T$ , for all  $M \in \mathcal{N}_W$ .

*Proof.* Since  $\mathcal{N}_W = \mathcal{N}_{W^T}$ , Lemma 5.3.1 says that  $\Psi$  interchanges the sets

$$\{\Delta_M : M \in \mathcal{N}_W\} \quad \text{and} \quad \{X_M : M \in \mathcal{N}_W\}.$$

Now consider  $\Psi^2(X_R)$ . It follows from Lemma 5.3.1 that

$$\Psi^2(X_R) = \Psi(\Delta_{\Theta_{W, W(-)}(R)}) = X_{R'}.$$

for some  $R'$  satisfying  $\Theta_{W, W(-)}(R) = \Theta_{W(-), W}(R')$ . But  $\Theta_{W(-), W}(R') = \Theta_{W, W(-)}(R'^T)$

by Lemma 3.8.2. Since  $\Theta_{W,W}(\cdot)$  is an isomorphism, we have  $R = R'^T$  and

$$\Psi^2(X_R) = X_{R^T} = X_R^T.$$

□

## 5.4 Spin Models in Two-class Association Schemes

We give a new proof of one direction of Jaeger's result about the triply-regularity of two-class association schemes that contain a spin model [10]. His result was the first indication that spin models have strong combinatorial properties.

Suppose  $\mathcal{A} = \{A_0, \dots, A_d\}$  is an association scheme. It is *triply-regular* if for all  $i, j, k, r, s, t \in \{0, 1, \dots, d\}$ , the cardinality of the set

$$\{w : (A_i)_{w,x} = (A_j)_{w,y} = (A_k)_{w,z} = 1\}$$

depends only on  $(i, j, k, r, s, t)$ , where  $(A_r)_{x,y} = (A_s)_{x,z} = (A_t)_{y,z} = 1$ .

The following Lemma is a direct translation of Lemma 4 in Munemasa's notes [19] into the language of endomorphisms.

**Lemma 5.4.1** *If for all  $i, j, k = 0, \dots, d$ ,*

$$X_{A_i} \Delta_{A_j} X_{A_k} \in \text{span}(\Delta_{A_r} X_{A_s} \Delta_{A_t} : r, s, t = 0, 1, \dots, d)$$

*then  $\mathcal{A}$  is triply-regular.*

*Proof.* For each  $i, j, k$ , the operator  $X_{A_i^T} \Delta_{A_j} X_{A_k}$  lies in the span of

$$\{\Delta_{A_r} X_{A_s} \Delta_{A_t^T} : r, s, t = 0, 1, \dots, d\} = \{\Delta_{A_r} X_{A_s} \Delta_{A_t^T} : r, s, t = 0, 1, \dots, d\}.$$

So there exists scalars  $\kappa(ijk|rst)$  such that

$$X_{A_i^T} \Delta_{A_j} X_{A_k} = \sum_{r,s,t=0}^d \kappa(ijk|rst) \Delta_{A_r} X_{A_s} \Delta_{A_t^T}.$$

Consider the left-hand side,

$$\begin{aligned} X_{A_i^T} \Delta_{A_j} X_{A_k}(E_{zy}) &= A_i^T(A_j e_y \circ A_k e_z) e_y^T \\ &= \sum_x (A_i^T(A_j e_y \circ A_k e_z))_x e_x e_y^T \\ &= \sum_x \left( \sum_w (A_i)_{w,x} (A_j)_{w,y} (A_k)_{w,z} \right) e_x e_y^T \\ &= \sum_x |\{w : (A_i)_{w,x} = (A_j)_{w,y} = (A_k)_{w,z} = 1\}| E_{xy}. \end{aligned}$$

Consider

$$\begin{aligned} \Delta_{A_r} X_{A_s} \Delta_{A_t^T}(E_{zy}) &= (A_t)_{y,z} (A_r e_y \circ A_s e_z) e_y^T \\ &= \sum_x (A_r)_{x,y} (A_s)_{x,z} (A_t)_{y,z} E_{xy}. \end{aligned}$$

So we get

$$|\{w : (A_i)_{w,x} = (A_j)_{w,y} = (A_k)_{w,z} = 1\}| = \sum_{r,s,t=0}^d \kappa(ijk|rst) (A_r)_{x,y} (A_s)_{x,z} (A_t)_{y,z}$$

and  $\mathcal{A}$  is triply-regular.  $\square$

**Theorem 5.4.2** *Suppose  $\mathcal{A} = \{I, A_1, A_2\}$  is a two-class association scheme with Bose-Mesner algebra  $\mathcal{B}$ . If there exists a spin model  $W = t_0I + t_1A_1 + t_2A_2$  with  $t_1 \neq t_2$ , then  $\mathcal{A}$  is triply-regular.*

*Proof.* It is well known that all two-class association schemes are symmetric, so  $W = W^T$ . Let  $A_0 = I$  and

$$S = \text{span}(\Delta_{A_r} X_{A_s} \Delta_{A_t} : r, s, t = 0, 1, 2).$$

Since  $\{A_0, A_1, A_2\}$  is a basis for  $\mathcal{B}$ ,

$$S = \text{span}(\Delta_{F'} X_{G'} \Delta_{H'} : F', G', H' \in \mathcal{B}).$$

If we can show that  $X_F \Delta_G X_H \in S$  for all  $F, G, H$  in  $\mathcal{B}$ , then  $\mathcal{A}$  is triply-regular by Lemma 5.4.1.

Now the two sets  $\{I, W, J\}$  and  $\{I, W^{(-)}, J\}$  are bases for  $\mathcal{B}$ . So it is sufficient to show that for all  $F, H \in \{I, W, J\}$  and  $G \in \{I, W^{(-)}, J\}$

$$X_F \Delta_G X_H \in S.$$

When  $H = I$ ,  $F \in \{I, W, J\}$  and  $G \in \{I, W^{(-)}, J\}$ , we get

$$X_F \Delta_G X_I = X_F \Delta_G = \Delta_J X_F \Delta_G \in S.$$

Secondly, when  $H = J$ ,  $F \in \{I, W, J\}$  and  $G \in \{I, W^{(-)}, J\}$ , we can apply the

Exchange Lemma to

$$X_F \Delta_J X_G = X_{FG} = \Delta_J X_{FG} \Delta_J.$$

and get

$$X_F \Delta_G X_J = \Delta_{FG} X_J \Delta_J$$

which belongs to  $S$  because  $FG \in \mathcal{B}$ .

Thirdly, when  $H = W$ , we enumerate the cases where  $G = J, I$  or  $W^{(-)}$ . When  $H = W$  and  $G = J$ , we obtain for all  $F \in \{I, W, J\}$  that

$$X_F \Delta_J X_W = X_{FW} = \Delta_J X_{FW} \Delta_J \in S.$$

When  $H = I, G = I$  and  $F \in \{I, W, J\}$ , we apply the Exchange Lemma to

$$X_F \Delta_W X_I = X_F \Delta_W = \Delta_J X_F \Delta_W,$$

and get

$$X_F \Delta_I X_W = \Delta_F X_J \Delta_W \in S.$$

When  $H = W, G = W^{(-)}$  and  $F = W$ , we get

$$X_W \Delta_{W^{(-)}} X_W = d \Delta_{W^{(-)}} X_W \Delta_{W^{(-)}} \in S.$$

because  $(d^{-1}W, W^{(-)})$  is a one-sided Jones pair. When  $H = W, G = W^{(-)}$  and

$F = I$ .  $\Theta_W(I) = J$  gives

$$X_I \Delta_{W^{(-)}} X_W = \Delta_{W^{(-)}} X_W \Delta_J \in S.$$

Lastly when  $H = W$ ,  $G = W^{(-)}$  and  $F = J$ ,

$$\Theta_W(J) = \Theta_{W^T}(\Theta_W(I)) = nI$$

yields

$$X_J \Delta_{W^{(-)}} X_W = n \Delta_{W^{(-)}} X_W \Delta_I \in S.$$

By Lemma 5.4.1, we conclude that  $\mathcal{A}$  is triply-regular.  $\square$

Suppose  $\mathcal{A} = \{I, A_1, A_2\}$  is a triply-regular two-class association scheme. Let  $G$  be the strongly regular graph whose adjacency matrix is  $A_1$ . The fact that  $\mathcal{A}$  is triply-regular implies that the neighborhoods of any vertex in  $G$  and its complements  $\overline{G}$  induce strongly regular graphs, see [11]. In this case, we say that both  $G$  and  $\overline{G}$  are *locally strongly regular*. Now we have proved one direction of the following result due to Jaeger. For the proof of the converse, please see [10].

**Theorem 5.4.3** *If  $G$  is a strongly regular graph and  $A_1$  is its adjacency matrix, then there exist  $t_0, t_1, t_2$  with  $t_1 \neq t_2$  such that  $W = t_0 I + t_1 A_1 + t_2 (J - A_1 - I)$  is a spin model if and only if both  $G$  and  $\overline{G}$  are locally strongly regular.*  $\square$

## 5.5 Symmetric and Non-Symmetric Hadamard Spin Models

In [15], Jaeger and Nomura constructed two  $4n \times 4n$  spin models from an  $n \times n$  Hadamard matrix. They are called the symmetric and non-symmetric Hadamard spin model. They are one of the three known infinite families of spin models that do not result from tensor product. The other two families are the Potts model and the spin models that come from finite Abelian groups, see Section 3.7. We present a new proof of Jaeger and Nomura's result here.

**Lemma 5.5.1** *Let  $A, B, C \in M_n(\mathbb{C})$  and let  $W$  be the following  $4n \times 4n$  matrix with  $\epsilon = \pm 1$*

$$W = \begin{pmatrix} A & A & B & -B \\ A & A & -B & B \\ \epsilon B^T & -\epsilon B^T & C & C \\ -\epsilon B^T & \epsilon B^T & C & C \end{pmatrix}.$$

*Then  $W$  is a spin model with loop variable  $2d$  with  $d = \pm\sqrt{n}$  if and only if the following conditions hold:*

- a.  $B$  is a type II matrix;*
- b.  $A$  and  $C$  are symmetric spin models with loop variable  $d$ ;*
- c.  $X_C \Delta_{B(-)^T} X_{B^T} = d \Delta_{B(-)^T} X_{B^T} \Delta_{A(-)}$*
- d.  $X_C \Delta_{B(-)^T} X_{B(-)^T} = \epsilon d \Delta_{B^T} X_{B^T} \Delta_{A(-)}$*

*Proof.* The  $4n \times 4n$  matrix  $W$  is a spin model if and only if  $(d^{-1}W, W^{(-)})$  is an invertible one-sided Jones pairs. By construction,  $W$  is type II if and only if  $A, B$

and  $C$  are type II. The type-III condition is equivalent to

$$W^* W^* e_h \circ W^{(-)} e_k = \frac{2d}{W_{h,k}} W^* e_h \circ W^{(-)} e_k, \quad \text{for } h, k = 1, \dots, 4n.$$

The eigenvector constructed from column  $i$  and  $j$  of  $W^*$  with  $i, j = 1, \dots, n$  is

$$\begin{pmatrix} Ae_i \circ A^{(-)} e_j \\ Ae_i \circ A^{(-)} e_j \\ B^T e_i \circ B^{(-)T} e_j \\ B^T e_i \circ B^{(-)T} e_j \end{pmatrix},$$

and the corresponding eigenvalue of  $W^*$  is  $2d(A_{i,j})^{-1}$ . Similarly, for  $i, j = 1, \dots, n$ , the following lists all other eigenvectors for  $W^*$ :

$$\begin{pmatrix} Ae_i \circ A^{(-)} e_j \\ Ae_i \circ A^{(-)} e_j \\ -B^T e_i \circ B^{(-)T} e_j \\ -B^T e_i \circ B^{(-)T} e_j \end{pmatrix}, \begin{pmatrix} Be_i \circ B^{(-)} e_j \\ Be_i \circ B^{(-)} e_j \\ Ce_i \circ C^{(-)} e_j \\ Ce_i \circ C^{(-)} e_j \end{pmatrix}, \begin{pmatrix} -Be_i \circ B^{(-)} e_j \\ -Be_i \circ B^{(-)} e_j \\ Ce_i \circ C^{(-)} e_j \\ Ce_i \circ C^{(-)} e_j \end{pmatrix}, \\ \begin{pmatrix} Ae_i \circ B^{(-)} e_j \\ -Ae_i \circ Be_j \\ \epsilon B^T e_i \circ C^{(-)} e_j \\ -\epsilon B^T e_i \circ C^{(-)} e_j \end{pmatrix}, \begin{pmatrix} Ae_i \circ B^{(-)} e_j \\ -Ae_i \circ Be_j \\ -\epsilon B^T e_i \circ C^{(-)} e_j \\ \epsilon B^T e_i \circ C^{(-)} e_j \end{pmatrix}, \begin{pmatrix} Be_i \circ A^{(-)} e_j \\ -Be_i \circ A^{(-)} e_j \\ \epsilon Ce_i \circ B^{(-)T} e_j \\ -\epsilon Ce_i \circ B^{(-)T} e_j \end{pmatrix}, \begin{pmatrix} Be_i \circ A^{(-)} e_j \\ -Be_i \circ A^{(-)} e_j \\ -\epsilon Ce_i \circ B^{(-)T} e_j \\ \epsilon Ce_i \circ B^{(-)T} e_j \end{pmatrix}.$$

with eigenvalues

$$\frac{2d}{A_{i,j}}, \frac{2d}{C_{i,j}}, \frac{2d}{C_{i,j}}, \frac{2d}{B_{i,j}}, -\frac{2d}{B_{i,j}}, \frac{2\epsilon d}{B_{j,i}}, -\frac{2\epsilon d}{B_{j,i}},$$

respectively.

Multiplying the first eigenvector by  $W$  gives

$$\begin{pmatrix} A & A & B & -B \\ A & A & -B & B \\ \epsilon B^T & -\epsilon B^T & C & C \\ -\epsilon B^T & \epsilon B^T & C & C \end{pmatrix} \begin{pmatrix} A\epsilon_i \circ A^{(-)}\epsilon_j \\ A\epsilon_i \circ A^{(-)}\epsilon_j \\ B^T\epsilon_i \circ B^{(-)T}\epsilon_j \\ B^T\epsilon_i \circ B^{(-)T}\epsilon_j \end{pmatrix} = 2 \begin{pmatrix} A (A\epsilon_i \circ A^{(-)}\epsilon_j) \\ A (A\epsilon_i \circ A^{(-)}\epsilon_j) \\ C (B^T\epsilon_i \circ B^{(-)T}\epsilon_j) \\ C (B^T\epsilon_i \circ B^{(-)T}\epsilon_j) \end{pmatrix} \\ = \frac{2d}{A_{i,j}} \begin{pmatrix} A\epsilon_i \circ A^{(-)}\epsilon_j \\ A\epsilon_i \circ A^{(-)}\epsilon_j \\ B^T\epsilon_i \circ B^{(-)T}\epsilon_j \\ B^T\epsilon_i \circ B^{(-)T}\epsilon_j \end{pmatrix}$$

if and only if

$$X_A \Delta_{A^{(-)}} X_A = d \Delta_{A^{(-)}} X_A \Delta_{A^{(-)}}$$

and

$$X_C \Delta_{B^{(-)T}} X_{B^T} = d \Delta_{B^{(-)T}} X_{B^T} \Delta_{A^{(-)}}.$$

Repeating the computation on other eigenvectors, the type-III condition on  $W$  is equivalent to the following set of relations

$$X_A \Delta_{A(-)} X_A = d \Delta_{A(-)} X_A \Delta_{A(-)} \quad (5.3)$$

$$X_C \Delta_{B(-)T} X_{B^T} = d \Delta_{B(-)T} X_{B^T} \Delta_{A(-)} \quad (5.4)$$

$$X_A \Delta_{B(-)} X_B = d \Delta_{B(-)} X_B \Delta_{C(-)} \quad (5.5)$$

$$X_C \Delta_{C(-)} X_C = d \Delta_{C(-)} X_C \Delta_{C(-)} \quad (5.6)$$

$$X_B \Delta_{C(-)} X_{B^T} = \epsilon d \Delta_{B(-)} X_A \Delta_{B(-)} \quad (5.7)$$

$$X_{B^T} \Delta_{B(-)} X_A = d \Delta_{C(-)} X_{B^T} \Delta_{B(-)} \quad (5.8)$$

$$X_B \Delta_{B(-)T} X_C = d \Delta_{A(-)} X_B \Delta_{B(-)T} \quad (5.9)$$

$$X_{B^T} \Delta_{A(-)} X_B = \epsilon d \Delta_{B(-)T} X_C \Delta_{B(-)T}. \quad (5.10)$$

Now (5.3) and (5.6) hold if and only if  $A$  and  $C$  are spin models with loop variable  $d$ . If we take the transpose of each side of (5.8) and compare it with (5.5), then we see that these two equations hold if and only if  $A$  is symmetric. Similarly, (5.9) and (5.4) hold simultaneously if and only if  $C$  is symmetric.

It remains to show the equivalence of (5.4) and (5.5), and the equivalence of (5.7), (5.10) and (d).

We can rewrite (5.5) as

$$X_B \Delta_{C(-)} X_{B^{-1}} = \frac{1}{d} \Delta_B X_A \Delta_{B(-)},$$

by taking the inverse of each side, we get

$$X_B \Delta_C X_{B^{-1}} = d \Delta_B X_{A^{-1}} \Delta_{B(-)}.$$

Taking the transpose of each side gives

$$X_{B^{-T}} \Delta_C X_{B^T} = d \Delta_{B(-)} X_{A^{-T}} \Delta_B.$$

Now  $nM^{-T} = M^{(-)}$  for any type II matrix  $M$ . After applying the Exchange Lemma, we get

$$X_{B^{-T}} \Delta_{B^T} X_C = d \Delta_{A(-)} X_{B^{-T}} \Delta_{B^T}$$

and whence (c), which is identical to (5.4), holds.

By taking the inverse of each side of (5.7), we have

$$X_{B^{-T}} \Delta_C X_{B^{-1}} = \frac{\epsilon}{d} \Delta_B X_{A^{-1}} \Delta_B.$$

Now apply the exchange lemma to get

$$X_{B^{-T}} \Delta_{B^{-1}} X_C = \frac{\epsilon}{d} \Delta_{A^{-1}} X_B \Delta_{B^T}.$$

By taking the transpose of each side and using the symmetry of  $A$  and  $C$ , we have

$$X_C \Delta_{B(-)^T} X_{B(-)^T} = \epsilon d \Delta_{B^T} X_{B^T} \Delta_{A(-)}$$

which equals to (d) and can be easily rewritten as (5.10). □

The following construction gives the symmetric Hadamard spin models described in [15] and [22] when  $\epsilon = 1$ . When  $\epsilon = -1$ , it gives the non-symmetric Hadamard spin models. It is an easy consequence of Lemma 5.5.1.

**Corollary 5.5.2** *Let  $H$  be an  $n \times n$  Hadamard matrix. Let  $A = -u^3 I + u^{-1}(J - I)$ ,*

with  $-u^2 - u^{-2} = d$ , be the Potts model. For  $\epsilon = \pm 1$  and  $\omega$  such that  $\omega^4 = \epsilon$ , the  $4n \times 4n$  matrix

$$W = \begin{pmatrix} A & A & \omega H & -\omega H \\ A & A & -\omega H & \omega H \\ \epsilon \omega H^T & -\epsilon \omega H^T & A & A \\ -\epsilon \omega H^T & \epsilon \omega H^T & A & A \end{pmatrix}$$

is a spin model.

*Proof.* We use the construction in Lemma 5.5.1 with  $B = \omega H$  and  $C = A$ . The matrix  $B$  is type II.  $\mathcal{N}_{B^T} = \mathcal{N}_{H^T}$  is a Bose Mesner algebra and therefore contains any linear combinations of  $I$  and  $J$ . Since

$$\begin{aligned} \Theta_{B^T}(A) &= \Theta_{H^T}(-u^3 I + u^{-1}(J - I)) \\ &= -u^3 J + u^{-1} n I - u^{-1} J \\ &= d(-u^{-3} I + u(J - I)) \\ &= dA^{(-)}. \end{aligned}$$

Condition (c) holds. Moreover, since  $B^T = \omega^2 B^{(-)T}$ , Condition (d) is equivalent to Condition (c). As a result, Lemma 5.5.1 implies that  $W$  is a spin model.  $\square$

# Chapter 6

## Association Schemes

We present our main results in this thesis. In Sections 6.1 and 6.2, we construct an  $2n \times 2n$  type-II matrix  $W$  and a pair of  $4n \times 4n$  symmetric spin models,  $V$  and  $V'$ , from each  $n \times n$  invertible Jones pair  $(A, B)$ . We exhibit the intricate relations among the Nomura algebras of these matrices in Sections 6.4 to 6.6. In Section 6.3, we design a strategy that allows us to find invertible Jones pairs, or equivalently four-weight spin models, up to odd-gauge equivalence.

The constructions of the type-II matrix and the spin models provide three new Bose-Mesner algebras attached to a four-weight spin model. In particular, we get a formally dual pair of Bose-Mesner algebras from  $W$  and a formally self-dual Bose-Mesner algebra from  $V$ . So our constructions extend the existing theory of Bose-Mesner algebras associated with four-weight spin models, which only concerns  $\mathcal{N}_A$ . In addition, these algebras form an interesting web of relations. So we do not have just four Bose-Mesner algebras, we have a structured set of four Bose-Mesner algebras.

Our construction of the pair of symmetric spin models generalizes Nomura's in [22]. It places  $A$  and  $B^{(-)}$  as submatrices of  $V$  and  $V'$ . Since both spin models and

four-weight spin models are invertible Jones pairs, they become submatrices of a pair of symmetric spin models four times their sizes. Hence if we can enumerate all symmetric spin models, then we will have found all spin models and all four-weight spin models. This observation leads us to the only known strategy of finding four-weight spin models described in Section 6.3, and answers Bannai's request [1] for such a method.

## 6.1 A Dual Pair of Association Schemes

Suppose  $A$  and  $B$  are  $n \times n$  matrices and  $(A, B)$  is an invertible Jones pair. We will use these matrices to define an  $2n \times 2n$  type-II matrix  $W$ . Consequently, we get two new Nomura algebras,  $\mathcal{N}_W$  and  $\mathcal{N}_{W\tau}$ , associated to an invertible Jones pair, or equivalently a four-weight spin model. In this section, we show that the dimension of  $\mathcal{N}_W$  is twice that of  $\mathcal{N}_A$ . We also exhibit the basis of Schur idempotents of  $\mathcal{N}_W$  and the basis of principal idempotents of  $\mathcal{N}_{W\tau}$ . Understanding these algebras allows us to see their connections to the other two Bose-Mesner algebras associated with the same invertible Jones pair, in Sections 6.4 to 6.7.

Godsil constructed the  $2n \times 2n$  matrix  $W$  mentioned above and he hypothesized correctly about the dimension of  $\mathcal{N}_W$ . The author proved his conjecture and we present this proof here. (Subsequently Godsil found a shorter proof, but this assumed  $A$  is symmetric, and gives less information.)

Let  $(A, B)$  be an invertible Jones pair. Recall that by Theorem 4.5.2,

$$\mathcal{N}_A = \mathcal{N}_{A\tau} = \mathcal{N}_B = \mathcal{N}_{B\tau}.$$

Moreover, Theorem 3.8.5 tells us that  $\mathcal{N}_{A,B}$ ,  $\mathcal{N}'_{A,B}$  and  $\mathcal{N}_A$  have the same dimension.

As discussed in Section 4.7, there exists an invertible diagonal matrix  $C$  such that  $A^T = C^{-2}AC^2$ . We define an  $2n \times 2n$  matrix  $W$  by

$$W = \begin{pmatrix} A^T & -A^T \\ B^{(-)T}C & B^{(-)T}C \end{pmatrix}. \quad (6.1)$$

It is easy to check that  $W$  is type II. In the following, we show that

$$\dim \mathcal{N}_W = 2 \dim \mathcal{N}_A.$$

and we find bases for  $\mathcal{N}_W$  and  $\mathcal{N}_{W^T}$ .

**Lemma 6.1.1** *We have*

$$\begin{pmatrix} J_n & 0 \\ 0 & J_n \end{pmatrix} \in \mathcal{N}_W, \quad \text{and} \quad \begin{pmatrix} I_n & I_n \\ I_n & I_n \end{pmatrix} \in \mathcal{N}_{W^T}.$$

*Proof.* The eigenvectors for the matrices in  $\mathcal{N}_W$  are

$$\begin{pmatrix} \pm A^T e_i \circ A^{(-)T} e_j \\ B^{(-)T} C e_i \circ B^T C^{-1} e_j \end{pmatrix} = \begin{pmatrix} \pm A^T e_i \circ A^{(-)T} e_j \\ \frac{C_{i,i}}{C_{j,j}} B^{(-)T} e_i \circ B^T e_j \end{pmatrix}.$$

Applying Lemma 3.2.1 to the type-II matrices  $A^T$  and  $B^{(-)T}C$ , we get

$$\Theta_{A^T}(J) = \Theta_{B^{(-)T}C}(J) = nI.$$

So for  $i, j = 1, \dots, n$ .

$$\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} \pm A^T \epsilon_i \circ A^{(-)T} \epsilon_j \\ B^{(-)T} C \epsilon_i \circ B^T C^{-1} \epsilon_j \end{pmatrix} = \delta_{ij} n \begin{pmatrix} \pm A^T \epsilon_i \circ A^{(-)T} \epsilon_j \\ B^{(-)T} C \epsilon_i \circ B^T C^{-1} \epsilon_j \end{pmatrix}.$$

As a result,

$$\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \in \mathcal{N}_W.$$

and its image under  $\Theta_W$  is

$$n \begin{pmatrix} I & I \\ I & I \end{pmatrix}.$$

□

Now the Schur idempotents of  $\mathcal{N}_W$  that sum to

$$\begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$$

have the form

$$\begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}.$$

and the rest have the form

$$\begin{pmatrix} 0 & C_1 \\ C_2 & 0 \end{pmatrix}.$$

We examine the two types of Schur idempotents below.

The following two lemmas analyze the Schur idempotents with zero diagonal blocks.

**Lemma 6.1.2** *If  $\begin{pmatrix} 0 & H_1 \\ H_2^T & 0 \end{pmatrix} \in \mathcal{N}_W$ , then  $H_1, H_2 \in \mathcal{N}'_{A,B}$ .*

*Proof.* Since

$$\begin{pmatrix} A^T \epsilon_i \circ A^{(-)T} \epsilon_j \\ B^{(-)T} C \epsilon_i \circ B^T C^{-1} \epsilon_j \end{pmatrix}$$

is an eigenvector, there exists some matrix  $S$  such that

$$\begin{aligned} H_1 (B^{(-)T} C \epsilon_i \circ B^T C^{-1} \epsilon_j) &= S_{i,j} (A^T \epsilon_i \circ A^{(-)T} \epsilon_j), \\ H_2^T (A^T \epsilon_i \circ A^{(-)T} \epsilon_j) &= S_{i,j} (B^{(-)T} C \epsilon_i \circ B^T C^{-1} \epsilon_j). \end{aligned}$$

This is equivalent to

$$X_{H_1} \Delta_{B^T C^{-1}} X_{B^{(-)T} C} = \Delta_{A^{(-)T}} X_{A^T} \Delta_S, \quad (6.2)$$

$$X_{H_2^T} \Delta_{A^{(-)T}} X_{A^T} = \Delta_{B^T C^{-1}} X_{B^{(-)T} C} \Delta_S. \quad (6.3)$$

Since  $B^{(-)T} = n B^{-1}$ , Equation (6.2) can be rewritten as

$$\Delta_{A^T} X_{H_1} \Delta_{B^T C^{-1}} = X_{A^T} \Delta_S X_{n^{-1} C^{-1} B},$$

and applying the Exchange Lemma, we get

$$\Delta_{H_1} X_{A^T} \Delta_{C^{-1} B} = \frac{1}{n} X_{A^T} \Delta_{C^{-1} B} X_S.$$

Recall that  $C^{-1}$  is a diagonal matrix, so  $\Delta_{C^{-1}B}$  equals  $\Delta_B X_{C^{-1}}$  and the above equation becomes

$$\Delta_{H_1} X_{A^T} \Delta_B X_{C^{-1}} = \frac{1}{n} X_{A^T} \Delta_B X_{C^{-1}} X_S.$$

Taking the transpose of each side gives

$$X_{C^{-1}} \Delta_B X_A \Delta_{H_1} = \frac{1}{n} X_{S^T} X_{C^{-1}} \Delta_B X_A.$$

Therefore

$$\frac{1}{n} X_{CS^T C^{-1}} \Delta_B X_A = \Delta_B X_A \Delta_{H_1}.$$

and  $H_1 = n^{-1} \Theta_{A,B}(CS^T C^{-1}) \in \mathcal{N}'_{A,B}$ .

Equation (6.3) is equivalent to

$$\Delta_{B(-)^TC} X_{H_1^T} \Delta_{A(-)^T} = X_{B(-)^TC} \Delta_S X_{(A^T)^{-1}}.$$

Now replacing  $(A^T)^{-1}$  by  $n^{-1} A^{(-)}$  and applying the Exchange Lemma gives.

$$\Delta_{H_1^T} X_{B(-)^TC} \Delta_{A(-)} = X_{B(-)^TC} \Delta_{n^{-1} A^{(-)}} X_S.$$

Taking the transpose of each side yields

$$\Delta_{A(-)} X_{CB(-)} \Delta_{H_1^T} = \frac{1}{n} X_{S^T} \Delta_{A(-)} X_{CB(-)}.$$

Again,  $C$  is a diagonal matrix, so  $\Delta_{A(-)}X_C = X_C\Delta_{A(-)}$  and

$$X_C\Delta_{A(-)}X_{B(-)}\Delta_{H_2^T} = \frac{1}{n}X_{S^T}X_C\Delta_{A(-)}X_{B(-)}.$$

Consequently,

$$\frac{1}{n}X_{C^{-1}S^TC}\Delta_{A(-)}X_{B(-)} = \Delta_{A(-)}X_{B(-)}\Delta_{H_2^T},$$

and

$$H_2^T = \frac{1}{n}\Theta_{B(-),A(-)}(C^{-1}S^TC).$$

This is the same as

$$\begin{aligned} H_2 &= \frac{1}{n}\Theta_{A(-),B(-)}(C^{-1}S^TC) \\ &= \frac{1}{n}\Theta_{A,B}(CSC^{-1}). \end{aligned}$$

by Lemma 3.8.2. So  $H_2 \in \mathcal{N}'_{A,B}$ . □

Now if  $F \in \mathcal{N}_{A,B}$ , then by Theorem 3.8.5 (a), there exists a matrix  $H \in \mathcal{N}_A$  such that  $F = H \circ A$ . Taking the transpose of each side, we have

$$\begin{aligned} F^T &= H^T \circ A^T \\ &= H^T \circ (C^{-2}AC^2) \\ &= C^{-2}(H^T \circ A)C^2. \end{aligned}$$

So  $C^2F^TC^{-2} = H^T \circ A$ . Since  $H^T \in \mathcal{N}_A$ , we know that  $C^2F^TC^{-2} \in \mathcal{N}_{A,B}$  by Theorem 3.8.5 (a). If  $\mathcal{N}_{A,B}$  has dimension  $r$ , the following lemma gives  $r$  Schur

idempotents of  $\mathcal{N}_W$ .

**Lemma 6.1.3** *Let  $\{F_0, \dots, F_{r-1}\}$  be the basis of the principal idempotents of  $\mathcal{N}_{A,B}$ . Then for  $k = 0, \dots, r-1$ , the matrix*

$$\widehat{F}_k = \begin{pmatrix} 0 & \Theta_{A,B}(F_k) \\ \Theta_{A,B}(C^2 F_k^T C^{-2})^T & 0 \end{pmatrix} \quad (6.4)$$

is a Schur idempotent of  $\mathcal{N}_W$ , and

$$\frac{1}{2} \begin{pmatrix} C F_k^T C^{-1} & -C F_k^T C^{-1} \\ -C F_k^T C^{-1} & C F_k^T C^{-1} \end{pmatrix}$$

is a principal idempotent of  $\mathcal{N}_{W,T}$ .

*Proof.* Let  $S = C F_k^T C^{-1}$ . It follows from the proof of the previous lemma that the following two equations hold:

$$X_{\Theta_{A,B}(F_k)} \Delta_{B^T C^{-1}} X_{B(-)^T C} = \Delta_{A(-)^T} X_{A^T} \Delta_{nS}. \quad (6.5)$$

$$X_{\Theta_{A,B}(C^2 F_k^T C^{-2})^T} \Delta_{A(-)^T} X_{A^T} = \Delta_{B^T C^{-1}} X_{B(-)^T C} \Delta_{nS}. \quad (6.6)$$

We conclude that  $\widehat{F}_k \in \mathcal{N}_W$ .

By Equations (6.5) and (6.6), we have

$$\widehat{F}_k \begin{pmatrix} A^T e_i \circ A(-)^T e_j \\ B(-)^T C e_i \circ B^T C^{-1} e_j \end{pmatrix} = n(C F_k^T C^{-1})_{i,j} \begin{pmatrix} A^T e_i \circ A(-)^T e_j \\ B(-)^T C e_i \circ B^T C^{-1} e_j \end{pmatrix}.$$

Thus the image of  $\widehat{F}_k$  under  $\Theta_W$  is

$$n \begin{pmatrix} CF_k^T C^{-1} & -CF_k^T C^{-1} \\ -CF_k^T C^{-1} & CF_k^T C^{-1} \end{pmatrix} \in \mathcal{N}_{W\tau}.$$

Consider

$$\widehat{F}_k \circ \widehat{F}_l = \begin{pmatrix} 0 & \Theta_{A,B}(F_k) \circ \Theta_{A,B}(F_l) \\ (\Theta_{A,B}(C^2 F_k^T C^{-2}) \circ \Theta_{A,B}(C^2 F_l^T C^{-2}))^T & 0 \end{pmatrix}.$$

for any  $l, k = 0, \dots, r-1$ . By Lemma 2.1.2, we have

$$\widehat{F}_k \circ \widehat{F}_l = \begin{pmatrix} 0 & \Theta_{A,B}(F_k F_l) \\ \Theta_{A,B}((C^2 F_k^T C^{-2})(C^2 F_l^T C^{-2}))^T & 0 \end{pmatrix}.$$

Since  $F_k$  and  $F_l$  are the principal idempotents of  $\mathcal{N}_{A,B}$ , we have  $F_k F_l = \delta_{kl} F_k$  and  $(C^2 F_k^T C^{-2})(C^2 F_l^T C^{-2}) = \delta_{kl} C^2 F_k^T C^{-2}$ . So  $\widehat{F}_k \circ \widehat{F}_l = \delta_{kl} \widehat{F}_k$ , that is,  $\widehat{F}_0, \dots, \widehat{F}_{r-1}$  are the Schur idempotents of  $\mathcal{N}_W$ . Moreover, for any Schur idempotent  $M$  in  $\mathcal{N}_W$ , the matrix  $\frac{1}{2n} \Theta_W(M)$  is a principal idempotent of  $\mathcal{N}_{W\tau}$ . So the result follows.  $\square$

We conclude from Lemmas 6.1.2 and 6.1.3 that the matrices in (6.4) form  $r$  Schur idempotents which span the subspace of  $\mathcal{N}_W$  consisting matrices with  $n \times n$  zero diagonal blocks. Further, we see in the next corollary that  $A$  and  $B$  are encoded in  $\mathcal{N}_W$  and  $\mathcal{N}_{W\tau}$ .

**Corollary 6.1.4** *We have*

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \in \mathcal{N}_W.$$

$$\begin{pmatrix} C^{-1}AC & -C^{-1}AC \\ -C^{-1}AC & C^{-1}AC \end{pmatrix} \in \mathcal{N}_{W\tau}.$$

*Proof.* Since  $A \in \mathcal{N}_{A,B}$  and  $A = C^2 A^T C^{-2}$ , we know that  $\Theta_{A,B}(A) = B$  and  $\Theta_{A,B}(C^2 A^T C^{-2}) = \Theta_{A,B}(A) = B$ . Hence  $B \in \mathcal{N}'_{A,B}$  and it follows from Lemma 6.1.3 that

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \in \mathcal{N}_W.$$

Moreover its image under  $\Theta_W$  equals

$$n \begin{pmatrix} C A^T C^{-1} & -C A^T C^{-1} \\ -C A^T C^{-1} & C A^T C^{-1} \end{pmatrix} = n \begin{pmatrix} C^{-1}AC & -C^{-1}AC \\ -C^{-1}AC & C^{-1}AC \end{pmatrix}$$

belongs to  $\mathcal{N}_{W\tau}$ . □

**Lemma 6.1.5** *All matrices in  $\mathcal{N}_W$  with  $n \times n$  zero off-diagonal blocks have the form*

$$\begin{pmatrix} \Theta_A(M) & 0 \\ 0 & \Theta_B(M)^T \end{pmatrix},$$

for some  $M$  in  $\mathcal{N}_A$ .

*Proof.* Suppose

$$\begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \in \mathcal{N}_W.$$

By examining the eigenvectors  $W\epsilon_i \circ W^{(-)}\epsilon_j$ , we conclude that  $C_1 \in \mathcal{N}_{A^T}$  and  $C_2 \in \mathcal{N}_{B^{(-)T}C}$ . By Corollary 6.1.4,  $\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \in \mathcal{N}_W$ . Since  $\mathcal{N}_W$  is commutative, we have

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix},$$

which leads to  $BC_2 = C_1B$  and  $C_2 = B^{-1}C_1B$ . Since  $C_1 \in \mathcal{N}_{A^T}$ , there exists  $M \in \mathcal{N}_A$  such that  $\Theta_A(M) = C_1$ . By Theorem 4.5.2, we have

$$C_2 = B^{-1}\Theta_A(M)B = \Theta_B(M)^T.$$

□

Since  $A$  and  $B$  are type II, applying Theorem 3.8.5 (a) to  $A \in \mathcal{N}_{A,B}$  give  $A \circ \mathcal{N}_A = \mathcal{N}_{A,B}$ . Therefore the dimensions of  $\mathcal{N}_{A,B}$  and  $\mathcal{N}_A$  are both  $r$ . We see below that the Schur idempotents of  $\mathcal{N}_W$  with zero diagonal blocks can be expressed in terms of matrices in  $\mathcal{N}_A$ .

**Lemma 6.1.6** *Let  $\{E_0, \dots, E_{r-1}\}$  be the basis of the principal idempotents of  $\mathcal{N}_A$ .*

Then for  $k = 0, \dots, r-1$ , the matrix

$$\tilde{E}_k = \begin{pmatrix} \Theta_A(E_k) & 0 \\ 0 & \Theta_B(E_k)^T \end{pmatrix}$$

is a Schur idempotent of  $\mathcal{N}_W$ , and

$$\frac{1}{2} \begin{pmatrix} E_k^T & E_k^T \\ E_k^T & E_k^T \end{pmatrix}$$

is a principal idempotent of  $\mathcal{N}_{W^T}$ .

*Proof.* Since  $\Theta_{A^T}(\Theta_A(E_k)) = nE_k^T$  and

$$\Theta_{B(-)^T}(\Theta_B(E_k)^T) = \Theta_{B(-)^T}(\Theta_{B(-)}(E_k)) = nE_k^T.$$

we have

$$\begin{pmatrix} \Theta_A(E_k) & 0 \\ 0 & \Theta_B(E_k)^T \end{pmatrix} \begin{pmatrix} \pm A^T e_i \circ A^{(-)^T} e_j \\ \frac{C_{i,i}}{C_{j,j}} B^{(-)^T} e_i \circ B^T e_j \end{pmatrix} = n(E_k)_{j,i} \begin{pmatrix} \pm A^T e_i \circ A^{(-)^T} e_j \\ \frac{C_{i,i}}{C_{j,j}} B^{(-)^T} e_i \circ B^T e_j \end{pmatrix}.$$

Hence  $\tilde{E}_k \in \mathcal{N}_W$  and

$$\Theta_W(\tilde{E}_k) = n \begin{pmatrix} E_k^T & E_k^T \\ E_k^T & E_k^T \end{pmatrix}.$$

Consider

$$\tilde{E}_k \circ \tilde{E}_l = \begin{pmatrix} \Theta_A(E_k) \circ \Theta_A(E_l) & 0 \\ 0 & (\Theta_B(E_k) \circ \Theta_B(E_l))^T \end{pmatrix}.$$

for any  $k, l = 0, \dots, r-1$ . By Lemma 2.1.2, we have

$$\tilde{E}_k \circ \tilde{E}_l = \begin{pmatrix} \Theta_A(E_k E_l) & 0 \\ 0 & \Theta_B(E_k E_l)^T \end{pmatrix}.$$

Since  $E_k$  and  $E_l$  are the principal idempotents of  $\mathcal{N}_A$ , we have  $E_k E_l = \delta_{kl} E_k$ . So  $\tilde{E}_k \circ \tilde{E}_l = \delta_{kl} \tilde{E}_k$ . That is,  $\tilde{E}_0, \dots, \tilde{E}_{r-1}$  are the Schur idempotents of  $\mathcal{N}_W$ . Moreover, for any Schur idempotent  $M$  in  $\mathcal{N}_W$ , the matrix  $\frac{1}{2n} \Theta_W(M)$  is a principal idempotent of  $\mathcal{N}_{W^T}$ . So the result follows.  $\square$

Combining the Lemmas 6.1.3 and 6.1.6, we find the basis of Schur idempotents for  $\mathcal{N}_W$  and the basis of principal idempotents for  $\mathcal{N}_{W^T}$ .

**Theorem 6.1.7** *Suppose  $\dim(\mathcal{N}_A) = \dim(\mathcal{N}_{A,B}) = r$ . Let  $\{E_0, \dots, E_{r-1}\}$  be the basis of the principal idempotents of  $\mathcal{N}_A$ . Let  $\{F_0, \dots, F_{r-1}\}$  be the basis of the principal idempotents of  $\mathcal{N}_{A,B}$ . Then the set*

$$\left\{ \begin{pmatrix} \Theta_A(E_i) & 0 \\ 0 & \Theta_B(E_i)^T \end{pmatrix}, \begin{pmatrix} 0 & \Theta_{A,B}(F_j) \\ \Theta_{A,B}(C^2 F_j^T C^{-2})^T & 0 \end{pmatrix} : i, j = 0, \dots, r-1 \right\}$$

*is the basis of Schur idempotents for  $\mathcal{N}_W$ . Further, the set*

$$\left\{ \frac{1}{2} \begin{pmatrix} E_i^T & E_i^T \\ E_i^T & E_i^T \end{pmatrix}, \frac{1}{2} \begin{pmatrix} C F_j^T C^{-1} & -C F_j^T C^{-1} \\ -C F_j^T C^{-1} & C F_j^T C^{-1} \end{pmatrix} : i, j = 0, \dots, r-1 \right\}$$

is the basis of principal idempotents for  $\mathcal{N}_{W^T}$ . Hence

$$\dim \mathcal{N}_W = \dim \mathcal{N}_{W^T} = 2 \dim \mathcal{N}_A.$$

□

In general  $\mathcal{N}_W$  is not equal to  $\mathcal{N}_{W^T}$ . We now examine the situation where these two algebras coincide.

**Lemma 6.1.8** *If  $\mathcal{N}_W = \mathcal{N}_{W^T}$ , then there exists non-zero scalar  $\alpha$  such that  $\alpha B$  is a spin model.*

*Proof.* By Corollary 6.1.4, we have

$$\hat{B} = \begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \in \mathcal{N}_{W^T}.$$

Now

$$W^T = \begin{pmatrix} A & CB^{(-)} \\ -A & CB^{(-)} \end{pmatrix},$$

and

$$\begin{pmatrix} CB^{(-)}e_i \circ C^{-1}Be_j \\ CB^{(-)}e_i \circ C^{-1}Be_j \end{pmatrix} = \begin{pmatrix} B^{(-)}e_i \circ Be_j \\ B^{(-)}e_i \circ Be_j \end{pmatrix}$$

is an eigenvector of  $\hat{B}$ . That is,  $B(B^{(-)}e_i \circ Be_j) = \beta(B^{(-)}e_i \circ Be_j)$ , for some  $\beta \in \mathbb{C}$ . This is equivalent to  $B$  belongs to  $\mathcal{N}_{B^{(-)}}$ , which is identical to  $\mathcal{N}_B$ . By Theorem 5.1.4, there exists non-zero scalar  $\alpha$  such that  $\alpha B$  is a spin model. □

This lemma tells us that if  $(A, B)$  is not gauge equivalent to  $(\alpha^{-1}d^{-1}B^{(-)}, \alpha B)$  for  $\alpha B$  a spin model, then we do have a formally dual pair of Bose-Mesner algebras.

## 6.2 Nomura's Extension

In this section, we build two  $4n \times 4n$  symmetric spin models,  $V$  and  $V'$ , from an invertible Jones pair. We show that they share the same formally self-dual Nomura algebra. We obtain an explicit form of the matrices belonging to this algebra.

In Section 5 of [22], Nomura constructed an  $4n \times 4n$  symmetric spin model from a four-weight spin model whose matrices are symmetric. Our construction is a generalization of Nomura's, which is motivated by the fact that Jones pairs are equivalent to four-weight spin models. Moreover, our construction delivers the fourth Bose-Mesner algebra associated with a four-weight spin model. It becomes evident in Section 6.3 that this Bose-Mesner algebra is the ticket to our strategy of finding four-weight spin models.

By Lemma 4.7.3 and the discussion prior to it, given any invertible Jones pair  $(A, B)$ , we can construct an invertible Jones pair  $(A', B)$  with  $A'$  symmetric which is odd-gauge equivalent to  $(A, B)$ . From this new invertible Jones pair  $(A', B)$ , we construct the  $4n \times 4n$  symmetric spin models  $V$  and  $V'$ . So in the following results, we can focus on only the invertible Jones pairs with their first matrix symmetric.

Suppose  $(A, B)$  is an invertible one-sided Jones pair with  $A$  symmetric. We

define

$$V = \begin{pmatrix} dA & -dA & B^{(-)} & B^{(-)} \\ -dA & dA & B^{(-)} & B^{(-)} \\ B^{(-)T} & B^{(-)T} & dA & -dA \\ B^{(-)T} & B^{(-)T} & -dA & dA \end{pmatrix}.$$

It follows from the construction of  $V$  together with  $A$  and  $B$  being type II that  $V$  is a type-II matrix. For easier reading, we separate the columns and the rows into four groups 1, 2, 3, and 4. For instance, the (1, 2)-block of  $V$  equals  $-dA$  while its (2, 4)-block equals  $B^{(-)}$ .

For  $\alpha, \beta \in \{1, 2, 3, 4\}$  and  $i, j = 1, \dots, n$ , we use  $\mathbf{Y}_{i,j}^{\alpha,\beta}$  to denote the eigenvectors we get from  $V$ . For example,

$$\mathbf{Y}_{i,j}^{3,1} = V e_{2n+1} \circ V^{(-)} e_j.$$

In the following, We show that  $V$  is a spin model. Our plan is to find the explicit block structure of the matrices in  $\mathcal{N}_V$ . After that, we will prove that  $V \in \mathcal{N}_V$  and  $\Theta_V(V) = (2d)^{-1} V^{(-)}$ , which imply that  $V$  is a spin model with loop variable  $2d$ .

In the following, we use  $J_k$  and  $I_k$  denote the  $k \times k$  matrix of all ones and the  $k \times k$  identity matrix, respectively.

**Lemma 6.2.1** *Let  $(A, B)$  be an invertible Jones pair with  $A$  symmetric. We have*

$$(I_2 \otimes J_{2n}) = \begin{pmatrix} J_n & J_n & 0 & 0 \\ J_n & J_n & 0 & 0 \\ 0 & 0 & J_n & J_n \\ 0 & 0 & J_n & J_n \end{pmatrix} \in \mathcal{N}_V.$$

and

$$\begin{pmatrix} I_n & I_n & 0 & 0 \\ I_n & I_n & 0 & 0 \\ 0 & 0 & I_n & I_n \\ 0 & 0 & I_n & I_n \end{pmatrix} \in \mathcal{N}_{V^T}.$$

*Proof.* Consider

$$\mathbf{Y}_{i,j}^{1,1} = \begin{pmatrix} Ae_i \circ A^{(-)}e_j \\ Ae_i \circ A^{(-)}e_j \\ B^{(-)T}e_i \circ B^Te_j \\ B^{(-)T}e_i \circ B^Te_j \end{pmatrix}.$$

Since  $\Theta_A(J_n) = \Theta_{B^{(-)T}}(J_n) = nI_n$ , we have  $(I_2 \oplus J_{2n}) \mathbf{Y}_{i,j}^{1,1} = 2n\delta_{ij} \mathbf{Y}_{i,j}^{1,1}$  and the  $(1, 1)$ -block of  $\Theta_V(I_2 \oplus J_{2n})$  equals  $2nI_n$ . Similarly, when  $\alpha, \beta \in \{1, 2\}$ , we have

$$(I_2 \oplus J_{2n}) \mathbf{Y}_{i,j}^{\alpha,\beta} = 2n\delta_{ij} \mathbf{Y}_{i,j}^{\alpha,\beta}.$$

for all  $i, j = 1, \dots, n$ , and the same holds for  $\alpha, \beta \in \{3, 4\}$ .

Consider

$$\mathbf{Y}_{i,j}^{1,3} = \begin{pmatrix} d Ae_i \circ Be_j \\ -d Ae_i \circ Be_j \\ d^{-1} B^{(-)T}e_i \circ A^{(-)}e_j \\ -d^{-1} B^{(-)T}e_i \circ A^{(-)}e_j \end{pmatrix}.$$

We have  $(I_2 \oplus J_{2n}) \mathbf{Y}_{i,j}^{1,3} = 0$  and so the  $(1, 3)$ -block of  $\Theta_V(I_2 \oplus J_{2n})$  is the  $n \times n$  zero matrix. The same holds for  $\mathbf{Y}_{i,j}^{1,4}$ ,  $\mathbf{Y}_{i,j}^{2,3}$ ,  $\mathbf{Y}_{i,j}^{2,4}$ ,  $\mathbf{Y}_{i,j}^{3,1}$ ,  $\mathbf{Y}_{i,j}^{3,2}$ ,  $\mathbf{Y}_{i,j}^{4,1}$  and  $\mathbf{Y}_{i,j}^{4,2}$ . Hence

$I_2 \oplus J_{2n} \in \mathcal{N}_V$  and

$$\Theta_V(I_2 \oplus J_{2n}) = 2n \begin{pmatrix} I_n & I_n & 0 & 0 \\ I_n & I_n & 0 & 0 \\ 0 & 0 & I_n & I_n \\ 0 & 0 & I_n & I_n \end{pmatrix}$$

and the lemma holds.  $\square$

One consequence is that any Schur idempotent of  $\mathcal{N}_V$  has one of the following two forms:

$$\begin{pmatrix} M_1 & N_1 & 0 & 0 \\ P_1 & Q_1 & 0 & 0 \\ 0 & 0 & M_2 & N_2 \\ 0 & 0 & P_2 & Q_2 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & M_1 & N_1 \\ 0 & 0 & P_1 & Q_1 \\ M_2 & N_2 & 0 & 0 \\ P_2 & Q_2 & 0 & 0 \end{pmatrix}.$$

We need the next two lemmas to anatomize the Schur idempotents with zero off-diagonal blocks.

**Lemma 6.2.2** *Let  $(A, B)$  be invertible Jones pair with  $A$  symmetric. If  $M \in \mathcal{N}_A$ , then*

$$\begin{aligned} \Theta_A(M) &= \Theta_{B(-)\tau}(B^{-1}MB), \\ \Theta_{B(-)}(M) &= \Theta_A(B^{-1}MB). \end{aligned}$$

*Proof.* Let  $S = \Theta_A(M)$ . Then

$$X_M \Delta_{A(-)} X_A = \Delta_{A(-)} X_A \Delta_S.$$

Multiplying each sides by  $\Delta_A$  gives

$$X_M \Delta_{A(-)} X_A \Delta_A = \Delta_{A(-)} X_A \Delta_S \Delta_A.$$

By Lemma 4.2.3, we have  $\Delta_{A(-)} X_A \Delta_A = X_B \Delta_{B^T} X_{B^{-1}}$ . Replacing  $\Delta_{A(-)} X_A \Delta_A$  by  $X_B \Delta_{B^T} X_{B^{-1}}$  on each side yields

$$X_M X_B \Delta_{B^T} X_{B^{-1}} = X_B \Delta_{B^T} X_{B^{-1}} \Delta_S.$$

The matrix  $B$  is type II, therefore  $B^{-1} = n^{-1} B^{(-)T}$  and the above is equivalent to

$$X_{B^{-1}MB} \Delta_{B^T} X_{B^{(-)T}} = \Delta_{B^T} X_{B^{(-)T}} \Delta_S.$$

So the first equation of the lemma follows.

Let  $S' = \Theta_{B^{(-)}}(M)$  and

$$X_M \Delta_B X_{B^{(-)}} = \Delta_B X_{B^{(-)}} \Delta_{S'}.$$

Multiplying both sides by  $\Delta_A$ ,

$$X_M \Delta_B X_{B^{(-)}} \Delta_A = \Delta_B X_{B^{(-)}} \Delta_{S'} \Delta_A.$$

By Corollary 2.3.2 (d) on  $\Theta_{A,B^T}(A) = B^T$ , we get  $\Delta_B X_{B^{(-)}} \Delta_A = X_B \Delta_{A^{-1}} X_{A^T}$  and

$$X_M X_B \Delta_{A^{-1}} X_{A^T} = X_B \Delta_{A^{-1}} X_{A^T} \Delta_{S'}.$$

Since  $A$  is symmetric, we have  $nA^{-1} = A^{(-)T} = A^{(-)}$  and

$$X_{B^{-1}MB} \Delta_{A^{(-)}} X_A = \Delta_{A^{(-)}} X_A \Delta_{S'}.$$

So the second equation of the lemma holds.  $\square$

**Lemma 6.2.3** *Let  $(A, B)$  be an invertible Jones pair with  $A$  symmetric. If  $M$  lies in  $\mathcal{N}_{A,B}$  and  $N$  lies in  $\mathcal{N}_{A,B^T}$ , then the following are equivalent:*

- a.  $\Theta_{A,B}(M) = \Theta_{B^{(-)T}, A^{(-)}}(N).$
- b.  $\Theta_A(M \circ A) = \Theta_{B^T}(N \circ A)^T.$
- c.  $\Theta_{B^{(-)}, A^{(-)}}(M) = \Theta_{A,B^T}(N).$

*Proof.* First note that by Theorem 3.8.5 (a), we have  $\mathcal{N}_{A,B^T} = \mathcal{N}_{A,B} = A \circ \mathcal{N}_A$ . Since  $A$  is symmetric,  $\mathcal{N}_{A,B}$  is closed under transpose. Therefore  $M^T, N^T$  also belong to  $\mathcal{N}_{A,B} = \mathcal{N}_{A,B^T}$ .

Now the right-hand side of (a) equals  $\Theta_{A^{(-)}, B^{(-)T}}(N)^T$ . After applying Lemma 3.8.2, it becomes  $\Theta_{A,B^T}(N^T)^T$ . Multiplying each side of (a) by  $n^{-1}B^T$  gives

$$\frac{1}{n} \Theta_{A,B}(M) B^T = \frac{1}{n} \Theta_{A,B^T}(N^T)^T B^T,$$

which is equivalent to

$$\frac{1}{n} \Theta_{A,B}(M) \Theta_{A,B}(A)^T = \frac{1}{n} (\Theta_{A,B^T}(A)^T \Theta_{A,B^T}(N^T))^T.$$

Applying the first part of Theorem 3.8.4 to  $(A, B)$  and its second part to  $(A, B^T)$ ,

we have

$$\Theta_A(M \circ A) = \Theta_{B^T}(A \circ N)^T.$$

So (a) is equivalent to (b).

By Theorem 3.8.4, (b) can be rewritten as

$$\frac{1}{n} \Theta_{A,B}(A) \Theta_{A,B}(M^T)^T = \frac{1}{n} (\Theta_{A,B^T}(N)^T \Theta_{A,B^T}(A))^T,$$

which is equal to

$$B \Theta_{A,B}(M^T)^T = B \Theta_{A,B^T}(N).$$

So

$$\Theta_{B,A}(M^T) = \Theta_{A,B^T}(N).$$

By Lemma 3.8.2,

$$\Theta_{B(-),A(-)}(M) = \Theta_{A,B^T}(N).$$

Hence we have shown the equivalence of (b) and (c).  $\square$

Now we are ready to determine the structure of the matrices in  $\mathcal{N}_V$  with zero off-diagonal blocks.

**Lemma 6.2.4** *Let  $(A, B)$  be an invertible Jones pair with  $A$  symmetric. The set*

of matrices

$$\begin{pmatrix} F+R & F-R & 0 & 0 \\ F-R & F+R & 0 & 0 \\ 0 & 0 & B^{-1}FB+R_1 & B^{-1}FB-R_1 \\ 0 & 0 & B^{-1}FB-R_1 & B^{-1}FB+R_1 \end{pmatrix}$$

satisfying

$$F \in \mathcal{N}_A.$$

$$R \in \mathcal{N}_{A,B}.$$

$$\Theta_{B^T}(A \circ R_1)^T = \Theta_A(A \circ R)$$

equals the subspace of  $\mathcal{N}_V$  consisting matrices with  $2n \times 2n$  zero off-diagonal blocks.

*Proof.* Suppose

$$Z = \begin{pmatrix} M_1 & N_1 & 0 & 0 \\ P_1 & Q_1 & 0 & 0 \\ 0 & 0 & M_2 & N_2 \\ 0 & 0 & P_2 & Q_2 \end{pmatrix} \in \mathcal{N}_V.$$

Since

$$\mathbf{Y}_{i,j}^{1,1} = \mathbf{Y}_{i,j}^{2,2} = \begin{pmatrix} Ae_i \circ A^{(-)}e_j \\ Ae_i \circ A^{(-)}e_j \\ B^{(-)T}e_i \circ B^Te_j \\ B^{(-)T}e_i \circ B^Te_j \end{pmatrix}.$$

$Z$  having  $\mathbf{Y}_{i,j}^{1,1}$  and  $\mathbf{Y}_{i,j}^{2,2}$  as eigenvectors for all  $i, j = 1, \dots, n$  implies

$$\Theta_A(M_1 + N_1) = \Theta_A(P_1 + Q_1) = \Theta_{B^{(-)T}}(M_2 + N_2) = \Theta_{B^{(-)T}}(P_2 + Q_2). \quad (6.7)$$

From the first and the third equalities, we know that  $M_1 + N_1 = P_1 + Q_1$  and  $M_2 + N_2 = P_2 + Q_2$ .

By the first equation of Lemma 6.2.2, the second equality in (6.7) holds if and only if  $M_2 + N_2 = B^{-1}(M_1 + N_1)B$ , which is true if we let  $F = \frac{1}{2}(M_1 + N_1)$ . Therefore both the (1,1)- and the (2,2)-blocks of  $\Theta_V(Z)$  equal  $\Theta_A(M_1 + N_1)$ . It is an easy consequence that for  $i, j = 1, \dots, n$ ,

$$\mathbf{Y}_{i,j}^{1,2} = \mathbf{Y}_{i,j}^{2,1} = \begin{pmatrix} -Ae_i \circ A^{(-)}e_j \\ -Ae_i \circ A^{(-)}e_j \\ B^{(-)T}e_i \circ B^Te_j \\ B^{(-)T}e_i \circ B^Te_j \end{pmatrix}$$

are also eigenvectors of  $Z$ , and the (1,2)- and the (2,1)-blocks of  $\Theta_V(Z)$  also equal  $\Theta_A(M_1 + N_1)$ .

Since  $M_2 + N_2 = B^{-1}(M_1 + N_1)B$ , the second equation of Lemma 6.2.2 with  $M = M_1 + N_1$  implies the following

$$\begin{aligned} X_{(M_1+N_1)} \Delta_B X_{B^{(-)}} &= \Delta_B X_{B^{(-)}} \Delta_{S'}, \\ X_{(M_2+N_2)} \Delta_{A^{(-)}} X_A &= \Delta_{A^{(-)}} X_A \Delta_{S'}. \end{aligned}$$

where  $S' = \Theta_{B(-)}(M_1 + N_1) = \Theta_A(B^{-1}(M_1 + N_1)B)$ . Hence for all  $i, j = 1, \dots, n$ ,

$$Y_{i,j}^{3,3} = Y_{i,j}^{4,4} = \begin{pmatrix} B^{(-)}e_i \circ Be_j \\ B^{(-)}e_i \circ Be_j \\ Ae_i \circ A^{(-)}e_j \\ Ae_i \circ A^{(-)}e_j \end{pmatrix}$$

and

$$Y_{i,j}^{3,4} = Y_{i,j}^{4,3} = \begin{pmatrix} B^{(-)}e_i \circ Be_j \\ B^{(-)}e_i \circ Be_j \\ -Ae_i \circ A^{(-)}e_j \\ -Ae_i \circ A^{(-)}e_j \end{pmatrix}$$

are eigenvectors of  $Z$ . We see that both the (3.3)- and the (4.4)-blocks of  $\Theta_V(Z)$  are

$$\Theta_{B(-)}(M_1 + N_1) = \Theta_B(M_1 + N_1)^T = B^{-1}\Theta_A(M_1 + N_1)B.$$

with the last equality implied by Theorem 4.5.2.

Consider

$$Y_{i,j}^{1,3} = -Y_{i,j}^{2,4} = \begin{pmatrix} d Ae_i \circ Be_j \\ -d Ae_i \circ Be_j \\ d^{-1} B^{(-)T}e_i \circ A^{(-)}e_j \\ -d^{-1} B^{(-)T}e_i \circ A^{(-)}e_j \end{pmatrix}.$$

and

$$\mathbf{Y}_{i,j}^{1,4} = -\mathbf{Y}_{i,j}^{2,3} = \begin{pmatrix} d \cdot A e_i \circ B e_j \\ -d \cdot A e_i \circ B e_j \\ -d^{-1} B^{(-)T} e_i \circ A^{(-)} e_j \\ d^{-1} B^{(-)T} e_i \circ A^{(-)} e_j \end{pmatrix}.$$

They are eigenvectors of  $Z$  if and only if the following hold:

$$M_1 - N_1 = -P_1 + Q_1 \in \mathcal{N}_{A,B}.$$

$$M_2 - N_2 = -P_2 + Q_2 \in \mathcal{N}_{A,B^T} = \mathcal{N}_{A,B}.$$

$$\Theta_{A,B}(M_1 - N_1) = \Theta_{B^{(-)T}, A^{(-)}}(M_2 - N_2).$$

If we let  $R = \frac{1}{2}(M_1 - N_1)$  and  $R_1 = \frac{1}{2}(M_2 - N_2)$  then by Lemma 6.2.3, the third equation above is equivalent to

$$\Theta_A(A \circ R) = \Theta_{B^T}(A \circ R_1)^T. \quad (6.8)$$

So the (1, 3)-, (2, 4)-, (1, 4)- and (2, 3)-blocks of  $\Theta_V(Z)$  equals  $\Theta_{A,B}(M_1 - N_1) = 2\Theta_{A,B}(R)$ .

Now, consider

$$\mathbf{Y}_{i,j}^{3,1} = -\mathbf{Y}_{i,j}^{4,2} = \begin{pmatrix} d^{-1} B^{(-)} e_i \circ A^{(-)} e_j \\ -d^{-1} B^{(-)} e_i \circ A^{(-)} e_j \\ d \cdot A e_i \circ B^T e_j \\ -d \cdot A e_i \circ B^T e_j \end{pmatrix}$$

and

$$\mathbf{Y}_{i,j}^{4,1} = -\mathbf{Y}_{i,j}^{3,2} = \begin{pmatrix} d^{-1} B^{(-)}e_i \circ A^{(-)}e_j \\ -d^{-1} B^{(-)}e_i \circ A^{(-)}e_j \\ -d A e_i \circ B^T e_j \\ d A e_i \circ B^T e_j \end{pmatrix}.$$

They are eigenvectors of  $Z$  if and only if

$$\Theta_{B^{(-)}, A^{(-)}}(M_1 - N_1) = \Theta_{A, B^T}(M_2 - N_2).$$

This equation holds because it is equivalent to Equation (6.8) by Lemma 6.2.3. So these vectors are indeed eigenvectors of  $Z$  and the (3.1)-, (4.2)-, (4.1)- and (3.2)-blocks of  $\Theta_V(Z)$  equal

$$\Theta_{B^{(-)}, A^{(-)}}(M_1 - N_1) = \Theta_{A^{(-)}, B^{(-)}}(2R)^T = 2\Theta_{A, B}(R^T)^T.$$

If we let  $F = \frac{1}{2}(M_1 + N_1)$ ,  $R = \frac{1}{2}(M_1 - N_1)$  and  $R_1 = \frac{1}{2}(M_2 - N_2)$ , then the result follows and

$$\Theta_V(Z) = 2 \begin{pmatrix} \Theta_A(F) & \Theta_A(F) & \Theta_{A, B}(R) & \Theta_{A, B}(R) \\ \Theta_A(F) & \Theta_A(F) & \Theta_{A, B}(R) & \Theta_{A, B}(R) \\ \Theta_{A, B}(R^T)^T & \Theta_{A, B}(R^T)^T & B^{-1}\Theta_A(F)B & B^{-1}\Theta_A(F)B \\ \Theta_{A, B}(R^T)^T & \Theta_{A, B}(R^T)^T & B^{-1}\Theta_A(F)B & B^{-1}\Theta_A(F)B \end{pmatrix}.$$

□

**Corollary 6.2.5** *Let  $(A, B)$  be an invertible Jones pair with  $A$  symmetric. Suppose*

$F \in \mathcal{N}_A$  and  $R \in \mathcal{N}_{A,B}$  then

$$\begin{pmatrix} \Theta_A(F) & \Theta_A(F) & \Theta_{A,B}(R) & \Theta_{A,B}(R) \\ \Theta_A(F) & \Theta_A(F) & \Theta_{A,B}(R) & \Theta_{A,B}(R) \\ \Theta_{A,B}(R^T)^T & \Theta_{A,B}(R^T)^T & B^{-1}\Theta_A(F)B & B^{-1}\Theta_A(F)B \\ \Theta_{A,B}(R^T)^T & \Theta_{A,B}(R^T)^T & B^{-1}\Theta_A(F)B & B^{-1}\Theta_A(F)B \end{pmatrix} \in \mathcal{N}_{V^T}.$$

□

If  $\mathcal{N}_A$  has dimension  $r$ , then Lemma 6.2.4 says that the subspace of  $\mathcal{N}_V$  spanned by the matrices with zero off-diagonal blocks has dimension  $2r$ . Unfortunately, we are not able to determine the dimension of the subspace of  $\mathcal{N}_V$  spanned by the matrices with zero diagonal blocks. Lemma 6.2.8 describes the structure of this subspace. We need the following two lemmas to prove Lemma 6.2.8.

**Lemma 6.2.6** *Let  $A$  and  $B$  be  $n \times n$  type-II matrices. Assume further that  $A$  is symmetric. If  $M \in \mathcal{N}'_{A,B}$  then  $\Theta_{A,B}(R^T) = nM$  if and only if*

$$X_M \Delta_{B^T} X_{B^{(-)T}} = \Delta_{A^{(-)}} X_A \Delta_R.$$

*Proof.* Since  $B^{(-)T} = nB^{-1}$ , the equality above becomes

$$n\Delta_A X_M \Delta_{B^T} = X_A \Delta_R X_B.$$

Applying the Exchange Lemma yields

$$n\Delta_M X_A \Delta_B = X_A \Delta_B X_R.$$

Taking the transpose of each side, we get

$$n\Delta_B X_A \Delta_M = X_{R^T} \Delta_B X_A$$

and  $\Theta_{A,B}(R^T) = nM$ . □

**Lemma 6.2.7** *Let  $A$  be a symmetric matrix. If  $(A, B)$  is an invertible one-sided Jones pair and  $R \in \mathcal{N}_{A,B}$ , then*

$$\begin{aligned} \Theta_{A,B}(R)^T &= B^{-1} \Theta_{A,B}(R^T) B^T \\ &= B^T \Theta_{A,B}(R^T) B^{-1}. \end{aligned}$$

*Proof.* Since  $A$  is symmetric,  $\mathcal{N}_{A,B} = A \circ \mathcal{N}_A$  is closed under transpose and  $R^T$  belongs to  $\mathcal{N}_{A,B}$ . Applying Theorem 3.8.4, we get

$$\Theta_A(R^T \circ A) = \frac{1}{n} \Theta_{A,B}(R^T) B^T$$

and

$$\Theta_A(A \circ R^T) = \frac{1}{n} B \Theta_{A,B}(R)^T.$$

Hence the first equality follows.

Similarly, applying the same theorem, we get

$$\Theta_A(R^T \circ A^{-1}) = \frac{1}{n} \Theta_{A,B}(R^T) B^{(-)T}$$

and

$$\Theta_A(A^{-1} \circ R^T) = \frac{1}{n} B^{(-)} \Theta_{A,B}(R)^T.$$

So the second equality holds.  $\square$

Now we are ready to examine the matrices of  $\mathcal{N}_V$  with zero diagonal blocks.

**Lemma 6.2.8** *Suppose  $A$  is a symmetric matrix and  $(A, B)$  is an invertible Jones pair. Let  $\mathcal{S}$  be the set of matrices*

$$\begin{pmatrix} 0 & 0 & G+H & G-H \\ 0 & 0 & G-H & G+H \\ B^{-1}GB^T + H_1 & B^{-1}GB^T - H_1 & 0 & 0 \\ B^{-1}GB^T - H_1 & B^{-1}GB^T + H_1 & 0 & 0 \end{pmatrix}$$

satisfying

$$G \in \mathcal{N}'_{A,B}$$

and there exist  $n \times n$  matrices  $S$  and  $S_1$  such that

$$\begin{aligned} X_{A^{-1}} \Delta_{B^{(-)}} X_H \Delta_{A^{(-)}} X_{B^{-1}} &= \Delta_S = X_B \Delta_A X_{H_1} \Delta_B X_A \\ X_{B^T} \Delta_A X_H \Delta_{B^T} X_A &= \Delta_{S_1} = X_{A^{-1}} \Delta_{B^{(-)T}} X_{H_1} \Delta_{A^{(-)T}} X_{B^{-T}}. \end{aligned}$$

Then  $\mathcal{S}$  equals the subspace of  $\mathcal{N}_V$  consisting matrices with  $2n \times 2n$  zero diagonal blocks.

*Proof.* Suppose

$$Z = \begin{pmatrix} 0 & 0 & M_1 & N_1 \\ 0 & 0 & P_1 & Q_1 \\ M_2 & N_2 & 0 & 0 \\ P_2 & Q_2 & 0 & 0 \end{pmatrix}.$$

There exists an  $n \times n$  matrix  $R$  satisfying the following two equations if and only if  $\mathbf{Y}_{i,j}^{1,1}$ ,  $\mathbf{Y}_{i,j}^{2,2}$ ,  $\mathbf{Y}_{i,j}^{1,2}$  and  $\mathbf{Y}_{i,j}^{2,1}$  are eigenvectors of  $Z$ .

$$X_{P_1+Q_1} \Delta_{B^T} X_{B(-)^T} = X_{M_1+N_1} \Delta_{B^T} X_{B(-)^T} = \Delta_{A(-)} X_A \Delta_R. \quad (6.9)$$

$$X_{P_2+Q_2} \Delta_{A(-)} X_A = X_{M_2+N_2} \Delta_{A(-)} X_A = \Delta_{B^T} X_{B(-)^T} \Delta_R. \quad (6.10)$$

By Lemma 6.2.6, Equation (6.9) implies  $P_1 + Q_1 = M_1 + N_1 = n^{-1} \Theta_{A,B}(R^T)$ .

Applying Corollary 2.3.2 (d), Equation (6.10) is equivalent to

$$\Theta_{A,B}(R) = n(M_2 + N_2)^T = n(P_2 + Q_2)^T.$$

By Lemma 6.2.7, we conclude that

$$\begin{aligned} M_2 + N_2 &= B^{-1}(M_1 + N_1)B^T \\ &= B^T(M_1 + N_1)B^{-1}. \end{aligned}$$

Note that the (1, 1)- and the (2, 2)-blocks of  $\Theta_V(Z)$  equal to  $R$ , while its (1, 2)- and (2, 1)-blocks equal  $-R$ .

We know that  $M_1 + N_1 = \Theta_{A,B}(n^{-1}R^T)$  lies in  $\mathcal{N}'_{A,B}$  and  $M_2 + N_2 = B^{-1}(M_1 + N_1)B^T$ . Now by Corollary 4.5.3,  $M_2 + N_2$  also belongs to  $\mathcal{N}'_{A,B^T}$ . Let  $R_1$  be the

matrix such that  $M_2 + N_2 = \Theta_{A,B^T}(n^{-1}R_1^T)$ . By Lemma 6.2.6, this equation is equivalent to

$$X_{M_2+N_2} \Delta_B X_{B(-)} = \Delta_{A(-)} X_A \Delta_{R_1}. \quad (6.11)$$

Applying the second equation of Lemma 6.2.7 to  $(A, B^T)$ , we get

$$\begin{aligned} \Theta_{A,B^T}(R_1)^T &= B \Theta_{A,B^T}(R_1^T) B^{-T} \\ &= n B (M_2 + N_2) B^{-T} \\ &= n (M_1 + N_1). \end{aligned}$$

Applying Corollary 2.3.2 (d),  $\Theta_{A,B^T}(R_1) = n(M_1 + N_1)^T$  is equivalent to

$$X_{M_1+N_1} \Delta_{A(-)} X_A = \Delta_B X_{B(-)} \Delta_{R_1}. \quad (6.12)$$

Equations (6.11) and (6.12) imply that  $\mathbf{Y}_{i,j}^{3,3}$ ,  $\mathbf{Y}_{i,j}^{4,4}$ ,  $\mathbf{Y}_{i,j}^{3,4}$  and  $\mathbf{Y}_{i,j}^{4,3}$  are eigenvectors of  $Z$ . Moreover, the  $(3,3)$ - and the  $(4,4)$ - blocks of  $\Theta_V(Z)$  equal  $R_1$ , while its  $(3,4)$ - and  $(4,3)$ -blocks equal  $-R_1$ .

Now  $\mathbf{Y}_{i,j}^{1,3}$ ,  $\mathbf{Y}_{i,j}^{2,4}$ ,  $\mathbf{Y}_{i,j}^{1,4}$  and  $\mathbf{Y}_{i,j}^{2,3}$  are eigenvectors of  $Z$  if and only if there exists an  $n \times n$  matrix  $S$  such that

$$\begin{aligned} d^{-1} X_{M_1-N_1} \Delta_{A(-)} X_{B(-)^T} &= d \Delta_B X_A \Delta_S \\ d X_{M_2-N_2} \Delta_B X_A &= d^{-1} \Delta_{A(-)} X_{B(-)^T} \Delta_S \end{aligned}$$

which is equivalent to the first equation in the lemma if we let  $H = M_1 - N_1$  and  $H_1 = M_2 - N_2$ . The  $(1,3)$ - and the  $(2,4)$  blocks of  $\Theta_Z(V)$  are  $S$ , while its  $(1,4)$ - and  $(2,3)$ -blocks are  $-S$ .

Similarly, there exists an  $n \times n$  matrix  $S_1$  such that

$$\begin{aligned} dX_{M_1-N_1} \Delta_{B^T} X_A &= d^{-1} \Delta_{A(-)} X_{B(-)} \Delta_{S_1} \\ d^{-1} X_{M_2-N_2} \Delta_{A(-)} X_{B(-)} &= d \Delta_{B^T} X_A \Delta_{S_1} \end{aligned}$$

if and only if  $Y_{i,j}^{3,1}$ ,  $Y_{i,j}^{4,2}$ ,  $Y_{i,j}^{3,2}$  and  $Y_{i,j}^{4,1}$  are eigenvectors of  $Z$ . The above equations are equivalent to the second equation in the lemma. Hence the (3.1)- and the (4.2) blocks of  $\Theta_Z(V)$  are  $S_1$ , while its (3.2)- and (4.1)-blocks are  $-S_1$ .

The lemma follows after we let  $G = M_1 + N_1$ ,  $H = M_1 - N_1$  and  $H_1 = M_2 - N_2$ . In addition, it is worth noting that

$$\Theta_V(Z) = \begin{pmatrix} R & -R & S & -S \\ -R & R & -S & S \\ S_1 & -S_1 & R_1 & -R_1 \\ -S_1 & S_1 & -R_1 & R_1 \end{pmatrix}.$$

□

The lemma above describes the matrices in  $\mathcal{N}_V$  with zero diagonal blocks. The dimension of this subspace of  $\mathcal{N}_V$  is greater than or equal to the dimension of  $\mathcal{N}'_{A,B}$ . The matrix  $H_1$  is determined by  $H$ . However, we do not understand the conditions on  $H$  listed in the lemma. We only know that  $\mathcal{N}_V$  has dimension at least three times the dimension of  $\mathcal{N}_A$ .

**Theorem 6.2.9** *Suppose  $(A, B)$  is an invertible Jones pair with  $A$  symmetric. Let*

$S$  be the set of matrices of the following form

$$\begin{pmatrix} F + R & F - R & G + H & G - H \\ F - R & F + R & G - H & G + H \\ B^{-1}GB^T + H_1 & B^{-1}GB^T - H_1 & B^{-1}FB + R_1 & B^{-1}FB - R_1 \\ B^{-1}GB^T - H_1 & B^{-1}GB^T + H_1 & B^{-1}FB - R_1 & B^{-1}FB + R_1 \end{pmatrix}$$

where  $F \in \mathcal{N}_A$ ,  $R \in \mathcal{N}_{A,B}$ ,  $G \in \mathcal{N}'_{A,B}$ ,  $H$  and  $H_1$  satisfy the conditions in Lemma 6.2.8 and  $R_1$  such that

$$\Theta_{BT}(A \circ R_1)^T = \Theta_A(A \circ R).$$

Then the Nomura algebra  $\mathcal{N}_V$  equals  $S$ . □

**Theorem 6.2.10** *Let  $A$  and  $B$  be type-II matrices. If  $A$  is symmetric, then the  $4n \times 4n$  matrix*

$$V = \begin{pmatrix} dA & -dA & B^{(-)} & B^{(-)} \\ -dA & dA & B^{(-)} & B^{(-)} \\ B^{(-)T} & B^{(-)T} & dA & -dA \\ B^{(-)T} & B^{(-)T} & -dA & dA \end{pmatrix}$$

*is a symmetric spin model with loop variable  $2d$  if and only if  $(A, B)$  is an invertible Jones pair.*

*Proof.* Suppose  $(A, B)$  is an invertible Jones pair. If we let  $F = H = H_1 = 0$ ,  $R_1 = R = dA$  and  $G = B^{(-)}$  in Theorem 6.2.9, then we have  $V \in \mathcal{N}_V$ .

Let

$$\check{A} = \begin{pmatrix} dA & -dA & 0 & 0 \\ -dA & dA & 0 & 0 \\ 0 & 0 & dA & -dA \\ 0 & 0 & -dA & dA \end{pmatrix}$$

and

$$\check{B} = \begin{pmatrix} 0 & 0 & B^{(-)} & B^{(-)} \\ 0 & 0 & B^{(-)} & B^{(-)} \\ B^{(-)T} & B^{(-)T} & 0 & 0 \\ B^{(-)T} & B^{(-)T} & 0 & 0 \end{pmatrix}.$$

From the proof of Lemma 6.2.4, we know that

$$\Theta_V(\check{A}) = 2d \begin{pmatrix} 0 & 0 & B & B \\ 0 & 0 & B & B \\ B^T & B^T & 0 & 0 \\ B^T & B^T & 0 & 0 \end{pmatrix}.$$

Moreover, from the proof of Lemma 6.2.8, we have

$$\Theta_V(\check{B}) = 2d \begin{pmatrix} d^{-1}A^{(-)} & -d^{-1}A^{(-)} & 0 & 0 \\ -d^{-1}A^{(-)} & d^{-1}A^{(-)} & 0 & 0 \\ 0 & 0 & d^{-1}A^{(-)} & -d^{-1}A^{(-)} \\ 0 & 0 & -d^{-1}A^{(-)} & d^{-1}A^{(-)} \end{pmatrix}.$$

So we have  $\Theta_V(V) = 2dV^{(-)}$  and  $(V, 2dV^{(-)})$  is therefore an invertible one-sided

Jones pair. It follows from Lemmas 5.1.1 and 5.1.2 that  $V$  is in fact a spin model with loop variable  $2d$ .

Conversely, if  $(V, 2dV^{(-)})$  is an invertible Jones pair, then

$$V Y_{i,j}^{1,3} = 2dV^{(-)} Y_{i,j}^{1,3}$$

implies  $A(Ae_i \circ Be_j) = B_{i,j}(Ae_i \circ Be_j)$ . So  $(A, B)$  is an invertible one-sided Jones pair. Furthermore,

$$V Y_{i,j}^{3,1} = 2dV^{(-)} Y_{i,j}^{3,1}$$

implies  $A(Ae_i \circ B^T e_j) = B_{j,i}(Ae_i \circ B^T e_j)$ . So  $(A, B^T)$  is also an invertible one-sided Jones pair. Hence  $(A, B)$  is an invertible Jones pair.  $\square$

**Theorem 6.2.11** *Let  $(A, B)$  be an invertible Jones pair with  $A$  symmetric. Then the matrix*

$$V' = \begin{pmatrix} dA & -dA & -B^{(-)} & -B^{(-)} \\ -dA & dA & -B^{(-)} & -B^{(-)} \\ -B^{(-)T} & -B^{(-)T} & dA & -dA \\ -B^{(-)T} & -B^{(-)T} & -dA & dA \end{pmatrix}$$

*is a symmetric spin model with loop variable  $-2d$ . Moreover,  $\Theta_{V'} = \Theta_V$ .*

*Proof.* Let

$$D = \begin{pmatrix} I_{2n} & 0 \\ 0 & -I_{2n} \end{pmatrix}$$

Note that  $V' = DV'D$ . By Lemma 2.4.1, we conclude that  $\mathcal{N}_{V'} = \mathcal{N}_V$ . By setting  $F = H = H_1 = 0$ ,  $R_1 = R = dA$  and  $G = -B^{(-)}$  in Theorem 6.2.9, we have  $V' \in \mathcal{N}_{V'}$ . Moreover, by the proof of the previous theorem, we have

$$\begin{aligned} \Theta_V(V') &= 2d \begin{pmatrix} 0 & 0 & B & B \\ 0 & 0 & B & B \\ B^T & B^T & 0 & 0 \\ B^T & B^T & 0 & 0 \end{pmatrix} - 2d \begin{pmatrix} d^{-1}A^{(-)} & -d^{-1}A^{(-)} & 0 & 0 \\ -d^{-1}A^{(-)} & d^{-1}A^{(-)} & 0 & 0 \\ 0 & 0 & d^{-1}A^{(-)} & -d^{-1}A^{(-)} \\ 0 & 0 & -d^{-1}A^{(-)} & d^{-1}A^{(-)} \end{pmatrix} \\ &= -2dV'^{(-)}. \end{aligned}$$

Since  $D^2 = I_{4n}$ ,

$$\begin{aligned} V'e_h \circ (V')^{(-)}e_k &= (DV'D)e_h \circ DV'^{(-)}De_k \\ &= (VD)e_h \circ V'^{(-)}De_k \\ &= \pm(V)e_h \circ V'^{(-)}e_k, \end{aligned}$$

for all  $h, k = 1, \dots, 4n$ . So for any  $M \in \mathcal{N}_V$ , we have  $\Theta_{V'}(M)_{h,k} = \Theta_V(M)_{h,k}$ . As a result, the two duality maps  $\Theta_V$  and  $\Theta_{V'}$  are identical. Further,  $V'$  is a spin model because

$$\Theta_{V'}(V') = \Theta_V(V') = -2d(V')^{(-)}.$$

□

### 6.3 The Modular Invariance Equation

In [1], Bannai asked for a strategy to find four-weight spin models. In this section, we answer her question. In [3], Bannai, Bannai and Jaeger used the modular invariance equation to design a method that exhaustively searches for spin models. Their design relies on the fact that a spin model always belongs to some formally self-dual Bose-Mesner algebra. Since this is usually not the case for the matrices in a four-weight spin model, their method does not apply directly.

Given an  $n \times n$  invertible Jones pair  $(A, B)$ , there exist two  $4n \times 4n$  spin models,  $V$  and  $V'$ , with  $A$  and  $B^{(-)}$  as their submatrices. So we can apply Bannai, Bannai and Jaeger's method to formally self-dual Bose-Mesner algebras on  $4n$  elements, hoping to find a symmetric spin model with the same structure as  $V$ . From  $V$ , we retrieve  $A$  and  $B$ . The algorithm we outline at the end of this section is the only known strategy to find four-weight spin models.

Suppose that  $(A, B)$  is an invertible Jones pair and that  $\mathcal{N}_V$  is the Nomura algebra constructed from this pair. We want to see how we can recover  $(A, B)$  from this algebra. Let  $\mathcal{A} = \{A_0, \dots, A_m\}$  be the basis of Schur idempotents of  $\mathcal{N}_V$ . Let  $\mathcal{E} = \{E_0, \dots, E_m\}$  be the basis of principal idempotents of  $\mathcal{N}_V$  such that  $\Theta_V(E_i) = A_i$  and  $\Theta_V(A_i) = nE_i^T$ . Let  $P$  be the matrix of eigenvalues with respect to this ordering. So if  $T$  is the matrix representing the transpose map with respect to  $\mathcal{A}$ , then  $P^2 = nT$ . Note that the ordering of the matrices in  $\mathcal{E}$  and the matrix of eigenvalues depend on the duality map  $\Theta_V$ .

From Lemma 6.2.1, we can always define two sets of indices  $\mathcal{I}, \mathcal{J} \subset \{0, \dots, m\}$

satisfying

$$\sum_{i \in \mathcal{I}} A_i^T = \begin{pmatrix} J_n & J_n & 0 & 0 \\ J_n & J_n & 0 & 0 \\ 0 & 0 & J_n & J_n \\ 0 & 0 & J_n & J_n \end{pmatrix}, \quad \sum_{j \in \mathcal{J}} A_j^T = \begin{pmatrix} 0 & 0 & J_n & J_n \\ 0 & 0 & J_n & J_n \\ J_n & J_n & 0 & 0 \\ J_n & J_n & 0 & 0 \end{pmatrix}. \quad (6.13)$$

So  $\mathcal{I} \cup \mathcal{J} = \{0, \dots, m\}$ .

Suppose  $V = \sum_{k=0}^m t_k A_k^T$ . Then  $V' = \sum_{i \in \mathcal{I}} t_i A_i^T - \sum_{j \in \mathcal{J}} t_j A_j^T$ . Define two  $(m+1) \times (m+1)$  diagonal matrices  $D_{\mathcal{I}}$  and  $D_{\mathcal{J}}$  as

$$(D_{\mathcal{I}})_{i,i} = \begin{cases} t_i & \text{if } i \in \mathcal{I}, \\ 0 & \text{if } i \in \mathcal{J} \end{cases}$$

and

$$(D_{\mathcal{J}})_{j,j} = \begin{cases} t_j & \text{if } j \in \mathcal{J}, \\ 0 & \text{if } j \in \mathcal{I}. \end{cases}$$

By Theorem 5.2.1, since  $V$  is a spin model with loop variable  $2d$ , we get

$$(P(D_{\mathcal{I}} + D_{\mathcal{J}}))^3 = 8d^3 t_0 I_{m+1}.$$

Similarly, since  $V'$  is a spin model with loop variable  $-2d$  and  $\Theta_{V'} = \Theta_V$ , we have

$$(P(D_{\mathcal{I}} - D_{\mathcal{J}}))^3 = -8d^3 t_0 I_{m+1}.$$

Define  $D_{\mathcal{I}}^{-1}$  to be the diagonal matrix with its  $ii$ -entry equals to  $t_i^{-1}$  if  $i \in \mathcal{I}$ .

and zero otherwise. Define  $D_{\mathcal{J}}$  similarly. Then the above two equations can be rewritten as

$$\begin{aligned}(D_{\mathcal{I}} + D_{\mathcal{J}})P(D_{\mathcal{I}} + D_{\mathcal{J}}) &= 8d^3t_0 P^{-1}(D_{\mathcal{I}}^- + D_{\mathcal{J}}^-)P^{-1}, \\ (D_{\mathcal{I}} - D_{\mathcal{J}})P(D_{\mathcal{I}} - D_{\mathcal{J}}) &= -8d^3t_0 P^{-1}(D_{\mathcal{I}}^- - D_{\mathcal{J}}^-)P^{-1},\end{aligned}$$

which are equivalent to

$$D_{\mathcal{I}}PD_{\mathcal{I}} + D_{\mathcal{J}}PD_{\mathcal{J}} = 8d^3t_0 P^{-1}D_{\mathcal{J}}^-P. \quad (6.14)$$

$$D_{\mathcal{I}}PD_{\mathcal{J}} + D_{\mathcal{J}}PD_{\mathcal{I}} = 8d^3t_0 P^{-1}D_{\mathcal{I}}^-P. \quad (6.15)$$

So Equations (6.14) and (6.15) are necessary conditions on  $\mathcal{N}_V$  for the existence of the invertible Jones pair  $(A, B)$  where  $A$  is symmetric. By Lemma 6.2.1, the matrix  $I_2 \otimes J_{2n} \in \mathcal{N}_V$ . We say that  $\mathcal{N}_V$  is the Bose-Mesner algebra of an *imprimitive* association scheme. Note that it is possible for a formally self-dual Bose-Mesner to have more than one duality map, each of which has a corresponding matrix of eigenvalues.

Here is the strategy for constructing invertible Jones pairs:

- a. Given a formally self-dual Bose-Mesner algebra on  $4n$  elements that contains  $I_2 \otimes J_{2n}$ , define  $\mathcal{I}$  and  $\mathcal{J}$  as in Equation (6.13).
- b. Enumerate its duality maps.
- c. For each duality map, form the matrix of eigenvalues  $P$ .
- d. Solve Equations (6.14) and (6.15) for the  $t_i$ 's.
- e. For each solution, compute the matrix  $V = \sum_{i \in \mathcal{I}} t_i A_i^T + \sum_{j \in \mathcal{J}} t_j A_j^T$ .

f. Check if it has the same structure as the matrix described in Theorem 6.2.10.

If so, retrieve  $A$  and  $B$ .

g. Verify if  $(A, B)$  is an invertible Jones pair.

If the formally self-dual Bose-Mesner algebra we start with is not the  $\mathcal{A}_V$  constructed from some invertible Jones pair  $(A, B)$ , then we have three possible outcomes at each iteration:

**Step (d):** We have no solution at this step.

**Step (f):** We have a solution at Step (d). But the  $4n \times 4n$  matrix  $V$  constructed does not have the desired structure.

**Step (g):** The  $4n \times 4n$  matrix  $V$  has the desired structure, but the pair  $(A, B)$  retrieved is not an invertible Jones pair.

Note that any invertible Jones pair  $(A, B)$  with  $A$  symmetric could be found by this method. It follows from Lemma 4.7.3 that this method does provide all invertible Jones pairs up to odd-gauge equivalence. Although this method is admittedly quite inefficient, it is all we have.

## 6.4 Quotients

Up to this point, we have four Bose-Mesner algebras constructed from an invertible Jones pair  $(A, B)$  where  $A$  is symmetric. In this section and the next one, we examine the relations among these algebras. We summarize them in a picture at the end of the next section. They are interesting because the constructions in Sections 6.1 and 6.2 are done independently. In particular, we have not been

able to spot any connections or common features between these two constructions.

However their resulting Bose-Mesner algebras are closely related to each other.

When  $A$  is symmetric, we have  $A^T = I^{-2}AI^2$ . So we let  $C = I$  and define

$$W = \begin{pmatrix} A^T & -A^T \\ B^{(-)T} & B^{(-)T} \end{pmatrix}.$$

Suppose  $\mathcal{B}$  is the Bose-Mesner algebra of an association scheme on  $n$  elements. Let  $\pi = (C_1, \dots, C_r)$  be a partition of  $\{1, \dots, n\}$ . Define the characteristic matrix  $S$  of  $\pi$  to be the  $n \times r$  matrix with

$$S_{i,k} = \begin{cases} 1 & \text{if } i \in C_k, \\ 0 & \text{otherwise.} \end{cases}$$

We say  $\pi$  is *equitable relative to  $\mathcal{B}$*  if and only if for all  $M \in \mathcal{B}$ , there exists matrix  $B_M$  satisfying

$$MS = SB_M.$$

We call the  $r \times r$  matrix  $B_M$  the *quotient of  $M$  with respect to  $\pi$* . We name the set  $\{B_M : M \in \mathcal{B}\}$  the *quotient of  $\mathcal{B}$  with respect to  $\pi$* .

**Theorem 6.4.1** *For  $i = 1, \dots, n$ , let  $C_i = \{i, n+i\}$  and let  $\pi = (C_1, \dots, C_n)$ . If  $(A, B)$  is an invertible Jones pair with  $A$  symmetric, then  $\mathcal{N}_A$  is the quotient of  $\mathcal{N}_{WT}$  with respect to  $\pi$ .*

*Proof.* It follows from Theorem 6.1.7 that any matrix in  $\mathcal{N}_{W\tau}$  can be expressed as

$$\begin{aligned} M &= \begin{pmatrix} F & F \\ F & F \end{pmatrix} + \begin{pmatrix} R^T & -R^T \\ -R^T & R^T \end{pmatrix} \\ &= \begin{pmatrix} F + R^T & F - R^T \\ F - R^T & F + R^T \end{pmatrix} \end{aligned}$$

for some  $F \in \mathcal{N}_A$  and  $R \in \mathcal{N}_{A,B}$ . The characteristic matrix of  $\pi$  is

$$S = \begin{pmatrix} I_n \\ I_n \end{pmatrix}.$$

and

$$MS = \begin{pmatrix} 2F \\ 2F \end{pmatrix} = S(2F).$$

So the quotient of  $\mathcal{N}_{W\tau}$  with respect to  $\pi$  equals  $\mathcal{N}_A$ . □

Although we are assuming  $A$  is symmetric throughout this section, the above proof does not require this condition.

**Theorem 6.4.2** *For  $i = 1, \dots, n, 2n+1, \dots, 3n$ , let  $C_i = \{i, n+i\}$  and let  $\pi = (C_1, \dots, C_n, C_{2n+1}, \dots, C_{3n})$ . If  $(A, B)$  is an invertible Jones pair with  $A$  symmetric, then  $\mathcal{N}_W$  is the quotient of  $\mathcal{N}_V$  with respect to  $\pi$ .*

*Proof.* The characteristic matrix of  $\pi$  is

$$S = \begin{pmatrix} I_n & 0 \\ I_n & 0 \\ 0 & I_n \\ 0 & I_n \end{pmatrix}.$$

Let  $N$  be a matrix in  $\mathcal{N}_V$ . By Theorem 6.2.9, we have

$$NS = 2 \begin{pmatrix} F & G \\ F & G \\ B^{-1}GB^T & B^{-1}FB \\ B^{-1}GB^T & B^{-1}FB \end{pmatrix} = S(2 \begin{pmatrix} F & G \\ B^{-1}GB^T & B^{-1}FB \end{pmatrix}),$$

for some  $F \in \mathcal{N}_A$  and  $G \in \mathcal{N}'_{A,B}$ .

Now  $F$  belongs to  $\mathcal{N}_A$  which equals  $\mathcal{N}_{A^T}$ , so there exists matrix  $M$  in  $\mathcal{N}_A$  such that  $F = \Theta_A(M)$ . By Theorem 4.5.2,  $B^{-1}FB = \Theta_B(M)^T$ . Since  $G \in \mathcal{N}'_{A,B}$ , there exists  $M_1 \in \mathcal{N}_{A,B}$  such that  $G = \Theta_{A,B}(M_1)$ . By Lemma 6.2.7, we know  $B^{-1}GB^T = \Theta_{A,B}(M_1^T)^T$ .

So the quotient of  $\mathcal{N}_V$  with respect to  $\pi$  is

$$\left\{ \begin{pmatrix} \Theta_A(M) & \Theta_{A,B}(M_1) \\ \Theta_{A,B}(M_1^T)^T & \Theta_B(M)^T \end{pmatrix} : M \in \mathcal{N}_A, M_1 \in \mathcal{N}_{A,B} \right\}.$$

which equals  $\mathcal{N}_W$  according to Theorem 6.1.7. □

## 6.5 Induced Schemes

We now complete the picture of the relations of the association schemes obtained from an invertible Jones pair.

Suppose  $\mathcal{A} = \{A_0, \dots, A_d\}$  is an association scheme on  $n$  elements. Let  $Y$  be a non-empty subset of  $\{1, \dots, n\}$ . For any  $n \times n$  matrix  $M$ , we use  $M_Y$  to denote the  $|Y| \times |Y|$  matrix obtained from the rows and the columns of  $M$  indexed by the elements in  $Y$ . We define  $\mathcal{A}_Y = \{(A_0)_Y, \dots, (A_d)_Y\}$ . If  $\mathcal{A}_Y$  is also an association scheme, we say it is an *induced scheme* of  $\mathcal{A}$ .

**Theorem 6.5.1** *If  $(A, B)$  is an invertible Jones pair, then  $\mathcal{N}_A$  is an induced scheme of  $\mathcal{N}_W$ .*

*Proof.* Let  $Y = \{1, \dots, n\}$ . The result follows immediately from Theorem 6.1.7.

□

**Theorem 6.5.2** *Let  $A$  be an  $n \times n$  symmetric matrix. If  $(A, B)$  is an invertible Jones pair, then  $\mathcal{N}_{W\tau}$  is an induced scheme of  $\mathcal{N}_V$ .*

*Proof.* Let  $Y = \{1, \dots, 2n\}$ . Define  $(\mathcal{N}_V)_Y$  to be the set  $\{M_Y : M \in \mathcal{N}_V\}$ . By Theorem 6.2.10, we have

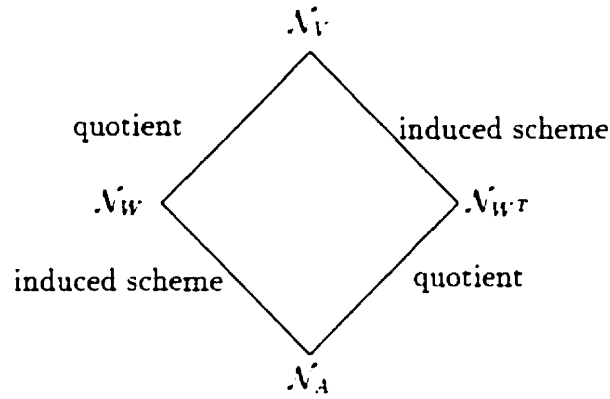
$$(\mathcal{N}_V)_Y = \left\{ \begin{pmatrix} F + R & F - R \\ F - R & F + R \end{pmatrix} : F \in \mathcal{N}_A, R \in \mathcal{N}_{A,B} \right\}.$$

By Theorem 6.1.7,  $\mathcal{N}_{W\tau}$  is spanned by

$$\begin{pmatrix} F & F \\ F & F \end{pmatrix}, \quad \begin{pmatrix} R^T & -R^T \\ -R^T & R^T \end{pmatrix},$$

for all  $F \in \mathcal{N}_A$  and  $R \in \mathcal{N}_{A,B}$ . Since  $A$  is symmetric, both  $\mathcal{N}_A$  and  $\mathcal{N}_{A,B}$  are closed under the transpose. Therefore  $R^T \in \mathcal{N}_{A,B}$  and  $\mathcal{N}_{W^T} = (\mathcal{N}_V)^T$ .  $\square$

The following diagram summarizes the relations among the four Bose-Mesner algebras obtained from an invertible Jones pair  $(A, B)$  where  $A$  is symmetric.



## 6.6 Subschemes

Now we give a stronger version of the two theorems in the previous section.

Godsil noticed that if we let  $U = \{2n+1, \dots, 4n\}$ , then the set of  $2n \times 2n$  matrices of  $\mathcal{N}_V$  induced by  $U$  also equals  $\mathcal{N}_{W^T}$ . From this observation, he spotted a subscheme of  $\mathcal{N}_V$  which is a union of two copies of  $\mathcal{N}_{W^T}$ . This fact is stronger than Theorem 6.5.2. It turns out that similar situation happens to  $\mathcal{N}_W$ , which we describe below.

Suppose  $\mathcal{B}$  is the Bose-Mesner algebra of some association scheme  $\mathcal{A}$ . If  $\mathcal{B}' \subset \mathcal{B}$  is also a Bose-Mesner algebra, we call its corresponding association scheme the *subscheme* of  $\mathcal{A}$ .

**Lemma 6.6.1** *Let  $U = \{2n + 1, \dots, 4n\}$ . Then the set*

$$S = \{M_U : M \in \mathcal{N}_V\}$$

*equals  $\mathcal{N}_{W^T}$ .*

*Proof.* From Lemma 6.2.4, we see that  $S$  is the span of

$$\left\{ \begin{pmatrix} B^{-1}FB & B^{-1}FB \\ B^{-1}FB & B^{-1}FB \end{pmatrix}, \begin{pmatrix} R_1 & -R_1 \\ -R_1 & R_1 \end{pmatrix} : F \in \mathcal{N}_A, A \circ R_1 \in \mathcal{N}_{B^T} \right\}.$$

Since  $\mathcal{N}_A = \mathcal{N}_{A^T} = \mathcal{N}_{B^T} = \mathcal{N}_B$ , we have  $B^{-1}FB \in \mathcal{N}_A$  by Theorem 3.5.5 (d). Moreover, part (a) of the same theorem implies that  $R_1 \in \mathcal{N}_{A,B}$ . Consequently, by Theorem 6.1.7, we conclude that  $S$  and  $\mathcal{N}_{W^T}$  are equal.  $\square$

Now we define  $\widehat{S}$  to be the set of  $4n \times 4n$  matrices having the form

$$\begin{pmatrix} F+R & F-R & 0 & 0 \\ F-R & F+R & 0 & 0 \\ 0 & 0 & B^{-1}FB+R_1 & B^{-1}FB-R_1 \\ 0 & 0 & B^{-1}FB-R_1 & B^{-1}FB+R_1 \end{pmatrix},$$

where  $F \in \mathcal{N}_A$ ,  $R \in \mathcal{N}_{A,B}$  and  $\Theta_{B^T}(A \circ R_1)^T = \Theta_A(A \circ R)$ . We also define

$$\widehat{J} = \begin{pmatrix} 0 & 0 & J_n & J_n \\ 0 & 0 & J_n & J_n \\ J_n & J_n & 0 & 0 \\ J_n & J_n & 0 & 0 \end{pmatrix}.$$

**Theorem 6.6.2** *The space  $\mathcal{B}'$  spanned by the matrices in  $\widehat{S} \cup \{\widehat{J}\}$  is a Bose-Mesner*

algebra contained in  $\mathcal{N}_V$ .

*Proof.* It is straightforward to show that this space is a commutative algebra containing  $I_{4n}$  and  $J_{4n}$ , and it is also closed under the Schur product and transpose.

□

Note that if  $\mathcal{N}_A$  has dimension  $r$ , then  $\mathcal{B}'$  has dimension  $2r + 1$ .

Similarly, if we let  $U = \{n + 1, \dots, 2n\}$ , then  $\mathcal{N}_A$  equals the set of matrices  $\{M_U : M \in \mathcal{N}_W\}$ . Moreover, if  $E_0, \dots, E_{r-1}$  is the basis of principal idempotents of  $\mathcal{N}_A$ , then the following set is a subscheme of  $\mathcal{N}_W$  with  $r$  classes:

$$\left\{ \begin{pmatrix} \Theta_A(E_i) & 0 \\ 0 & \Theta_B(E_i)^T \end{pmatrix} : i = 0, \dots, r-1 \right\} \cup \left\{ \begin{pmatrix} 0 & J_n \\ J_n & 0 \end{pmatrix} \right\}.$$

## 6.7 Dimension Two

In this section, we look at the simplest kind of invertible Jones pairs.

We define the *dimension* of a Jones pair  $(A, B)$  to be the dimension of  $\mathcal{N}_{A,B}$ , and we define the *degree* of  $(A, B)$  to be the number of distinct entries of  $B$ . If  $B$  is invertible, then the degree of  $(A, B)$  is at least two. Moreover, the number of distinct entries of  $B$  equals the number of eigenvalues of  $A$ , and  $A \in \mathcal{N}_{A,B}$ . Therefore the dimension of  $(A, B)$  is greater than or equal to its degree. In the following, we consider invertible Jones pairs of dimension two.

**Lemma 6.7.1** *If  $(A, B)$  is an  $n \times n$  invertible Jones pair of dimension two, then  $B = a(J - N) + bN$ , where  $N$  is the incidence matrix of some symmetric design.*

*Proof.* Let  $B = a(J - N) + bN$  for some 01-matrix  $N$  and some non-zero scalars  $a$

and  $b$ . By Corollary 6.1.4.

$$\begin{pmatrix} 0 & B \\ B^T & 0 \end{pmatrix} \in \mathcal{N}_W.$$

which implies

$$\hat{N} = \begin{pmatrix} 0 & N \\ N^T & 0 \end{pmatrix} \in \mathcal{N}_W.$$

Since  $\mathcal{N}_W$  is closed under matrix multiplication, we have

$$\hat{N}^2 = \begin{pmatrix} N N^T & 0 \\ 0 & N^T N \end{pmatrix} \in \mathcal{N}_W.$$

By Theorem 6.1.7, we know that  $N N^T$  belongs to  $\mathcal{N}_A$ . Since  $\mathcal{N}_A = \dim \mathcal{N}_{A,B}$  have dimension two,  $\mathcal{N}_A$  equals to the span of  $\{I, J\}$ . Therefore we can write

$$N N^T = \lambda(J - I) + kI,$$

for some integers  $\lambda$  and  $k$ . As a result,  $N$  is the incidence matrix of a symmetric  $(n, k, \lambda)$ -design. □

This lemma is a weaker version of Bannai and Sawano's result [4]. They showed that  $N$  is the incidence matrix of a symmetric design whose derived design with respect to any block is a quasi-symmetric design.

Later, Godsil showed that if  $A$  is an  $n \times n$  symmetric matrix and  $(A, B)$  is an invertible Jones pair of dimension two, then  $A$  comes from a regular two-graph. A *regular two-graph* is a symmetric matrix  $M$  that satisfies the following conditions:

- (a) all diagonal entries of  $M$  are zero and all off-diagonal entries of  $M$  are equal to  $\pm 1$ , and
- (b)  $M$  has quadratic minimal polynomial.

Now we give Godsil's argument.

**Lemma 6.7.2** *If  $A$  is an  $n \times n$  symmetric matrix and  $(A, B)$  is an invertible Jones pair of dimension two, then  $A = cI + dM$ , where  $M$  is a regular two-graph.*

*Proof.* By Theorem 3.8.5 (a), we have  $A \circ A \in \mathcal{N}_A$ . Since  $\mathcal{N}_A$  equals to the span of  $\{I, J\}$ , there exist non-zero scalars  $a$  and  $b$  such that  $A \circ A = aI + b(J - I)$ . Hence we can write

$$A = cI + dM,$$

for some symmetric matrix  $M$  such that  $M \circ I = 0$  and  $M \circ M = J - I$ . So the off-diagonal entries of  $M$  equal to  $\pm 1$ .

Moreover, since  $\mathcal{N}_{A,B}$  has dimension two, the minimal polynomial of  $A$  is quadratic. As a result,  $M$  also has quadratic minimal polynomial, and so it is a regular two-graph.  $\square$

When  $(A, B)$  is an  $n \times n$  invertible Jones pair of dimension two, then  $B$  comes from a symmetric design on  $n$  points. From such design, we can construct a bipartite distance regular graph on  $2n$  vertices with diameter three. Theorem 6.1.7 tells us that  $\mathcal{N}_W$  is the Bose-Mesner algebra of this graph. Furthermore, if  $A$  is symmetric, then  $A$  comes from a regular two-graph  $M$ . By the same theorem, we see that  $\mathcal{N}_{W,r}$  is the Bose-Mesner algebra of the association scheme associated to  $M$ . For more information on two-graphs, please see Seidel's survey in [25].

## 6.8 Unfinished Business

We believe Jones pairs are the natural way to view problems on spin models and four-weight spin models. We hope to extend their existing theory using Jones pairs. In a narrower scope, we feel that our results have great potential for extensions. Now we propose several research problems.

Firstly, from Theorem 6.2.9, we know that the dimension of  $\mathcal{N}_V$  is bounded between  $3r$  and  $3r + n$  where  $r$  is the dimension of  $\mathcal{N}_A$ . We conjecture that

$$\dim \mathcal{N}_V = 4r.$$

If we can determine the dimension of  $\mathcal{N}_V$ , then we can limit our search for four-weight spin models to the formally self-dual Bose-Mesner algebras having the right dimension. In order to prove this, we need to better understand the conditions in Lemma 6.2.8 and the interactions among  $\mathcal{N}_A$ ,  $\mathcal{N}_{A,B}$  and  $\mathcal{N}'_{A,B}$ .

Secondly, given an invertible Jones pair  $(A, B)$ , we get a link invariant. But we also get a link invariant from the  $4n \times 4n$  symmetric spin models constructed from  $(A, B)$ . It is natural to ask for the relations between these two link invariants.

Further, we want to investigate more thoroughly the two constructions described in this chapter. In particular, we would like to understand why the resulting Nomura algebras relate to each other in such interesting fashion. In addition, we want to examine other known constructions for possible extensions. Nomura's construction of the symmetric and non-symmetric Hadamard spin models, described in Section 5.5, is a good candidate.

In Section 6.7, we have studied invertible Jones pairs of dimension two. The next open case is the invertible Jones pairs of dimension three. On the other hand, Jaeger characterized the three-dimensional Bose-Mesner algebras that contain spin

models. The characterization of symmetric Bose-Mesner algebras of dimension four that contain spin models still remains open. This case corresponds to a special class of invertible Jones pairs of dimension four, so any results on these Jones pairs may help solving this open case.

Lastly, we have designed Jones pairs in an attempt to answer Jones' question in [17]. However we have not been able to answer his question yet. One obstacle is that there are very few Jones pairs where the matrices are not type II. Here we ask for new examples of non-invertible Jones pairs that are not tensor products of existing Jones pairs.

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