Fourier Analysis of Local Fell Groups

by

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

In 1972, Baggett [9] showed that a separable locally compact group G is compact if and only if its dual space $\widehat{G}$ is discrete. Curiously however, there are non-discrete groups whose duals are compact, and such a group was identified in the same paper. In a similar vein, one can define the *Fell group*, a semidirect product of the units of the p-adic integers $\mathbb{O}_p^*$ acting via multiplication on the p-adic numbers $\mathbb{Q}_p$, which Baggett shows is a noncompact group whose dual is not countable. This Fell group forms a basis of the novel work presented in this thesis.

In Chapters 3 and 4, we look at the more general setting of the p-adic integers and numbers, known respectively as discrete valuation rings (DVRs) and local fields. We compile many known results about these objects, in order to generalise the theory of the Fell group to what we call the "local Fell groups". While this is primarily background material from a variety of sources, there is additional work required to extend these results so that the theory is coherent and complete. We also briefly study finite-dimensional vector spaces over local fields.

In Chapter 5, we analyse the Fourier and Fourier-Stieltjes algebras of these local Fell groups, which are of the form $A \rtimes K$ for A abelian and K compact. These local Fell groups fall into a particular class of groups induced by actions for which the stabilisers are 'minimal', and we call such groups *cheap groups*. For groups of this form, we show that $B(G) = B_\infty(G) \oplus A(K) \circ q_K$, where $B_\infty(G)$ is the Fourier space generated by purely infinite representations. We also show that in group with countable open orbits (such as the local Fell groups) this simplifies further to $B(G) = A(G) \oplus A(K) \circ q_K$. In an attempt to generalise this to higher-dimensional analogues, for which the above does not hold true, we examine the structure of $B_\infty(G)$. In particular, we obtain a result for dimension two in terms of the projective space, and we show that this is in some sense the 'best' decomposition that can be made.

Finally in Chapter 6, we study the amenability of the central Fourier algebra $ZA(G) = A(G) \cap ZL^1(G)$ for $G = \mathbb{O}_p \rtimes \mathbb{O}_p^*$ and its local field equivalents. We show that $ZA(G)$ contains as a quotient the Fourier algebra of a hypergroup, which is induced by action of $\mathbb{O}_p^* \curvearrowright \mathbb{O}_p$. In general, if H is a hypergroup induced by an action $K \curvearrowright A$, then there is a corresponding dual hypergroup $\widehat{H}$ by the dual action. When this is the case, we show that these satisfy $A(H) \cong L^1(\widehat{H})$, mimicking the classical result for groups. We also show that if $\widehat{H}$ has orbits which 'grow sufficiently large', then via a result of Alaghmandan [2], the algebra $L^1(\widehat{H})$ is not amenable. In particular, this shows that $ZA(G)$ is also not amenable, reaffirming a conjecture of Alaghmandan and Spronk [4].

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Oh, and I guess my brother Filip is cool too.

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List of Symbols

$\oplus_1$	ℓ^1 direct sum	23
$\otimes_2$	Hilbertian (ℓ^2) tensor product	131
$\hat{\otimes}$	projective tensor product	29
$\lvert\cdot\rvert$	absolute value of a field	55
$\lVert\cdot\rVert$	norm of a Banach space	9
$\lVert\cdot\rVert$	norm of a local space	64
$[\,\cdot\,,\,\cdot\,]$	(non-degenerate) symmetric bilinear form on a local space	68
$\langle\,\cdot\mid\cdot\,\rangle$	inner product of a Hilbert space	9
$\langle\,\cdot\,,\,\cdot\,\rangle$	dual pairing between a Banach space and its dual	9
$\langle\,\cdot\,,\,\cdot\,\rangle$	dual pairing between an abelian group and its Pontryagin dual	12
$G \curvearrowright X$	group action of G on a top. space X	31
$G \curvearrowright H$	group action of G on a group H	33
$G \curvearrowright \widehat{A}$	dual action of G on the Pontryagin dual of A	36
$H \rtimes G$	semidirect product of $G \curvearrowright H$	34
$\eta_1 \vee \eta_2$	join or supremum of two homomorphisms	77
$\mu \ll \nu$	absolute continuity of measures	79
$\mu \approx \nu$	equivalence of measures	79
$\mu \perp \nu$	measures μ, ν are singular	79
$\mu \wedge \nu$	meet or infimum of two measures	83
$\pi \preccurlyeq_u \sigma$	unitary containment of representations	14
$\pi \approx \sigma$	unitary equivalence of representations	14
$\pi \preccurlyeq_q \sigma$	quasi-containment of representations	14
$\pi \approx_q \sigma$	quasi-equivalence of representations	14
$\pi \sim_h \pi'$	the irreducible representations π and π' of $\mathscr{V} \rtimes \mathscr{U}$ share a hyperplane in their kernel	101

$\mathbf{1}$	trivial character of a group	15
$\mathbb{1}_X$	indicator function on a set X	9
$\overline{A}$	closure of a set A in a top. space X	
$A(G)$	Fourier algebra of G	18
$A^*(G)$	spine of $B(G)$	77
$A_\eta(G)$	Fourier algebra associated to $\eta : G \to H$	77
$A_\mu(G)$	shorthand for $A_{\pi_\mu}(G)$	82
$A_\pi(G)$	functions $f \in A(G)$ which are π-radial	116
$A_\pi(G)$	Fourier space associated to π	21
$\mathscr{A}^\sharp$	unitisation of a Banach algebra $\mathscr{A}$	133
ad_x	inner derivation	28
$AM(\mathscr{A})$	amenability constant of a Banach algebra $\mathscr{A}$	29
$AM_{op}(\mathscr{A})$	operator amenability constant of a contractive Banach algebra $\mathscr{A}$	31
$AP(G)$	almost-periodic functions of G	24
$\mathrm{Aut}(G)$	automorphisms of a locally compact group G	33
$B(G)$	Fourier-Stieltjes algebra of G	18
$B_0(G)$	Rajchman algebra of G	78
$B_\infty(G)$	purely infinite component of G	26
$\mathscr{B}(\mathscr{H})$	bounded linear operators on $\mathscr{H}$	9
$\mathrm{B}(x; r)$	open ball of x of radius r	9
$\mathrm{B}[x; r]$	closed ball of x of radius r	9
$\mathscr{B}_r$	ball of radius r centred at the origin in a local space	64
$\mathbb{C}$	complex numbers	
$C^*(G)$	C*-algebra of G	16
$C_r^*(G)$	reduced C*-algebra of G	20
$\mathscr{C}(X)$	compact sets of a top. space X	112
$C(X)$	continuous functions from X to $\mathbb{C}$	9
$C(X; \mathscr{H})$	continuous functions from X to a Hilbert space $\mathscr{H}$	131
$C_0(X)$	continuous functions vanishing at infinity	9
$C_b(X)$	bounded continuous functions	9
$C_c(X)$	continuous functions with compact support	9
$C_q(X; \mathscr{H})$	continuous functions from X to $\mathscr{H}$ where $q(\mathrm{supp}\, f)$ is compact	84
$\mathrm{Cl}_G(x)$	conjugacy class of $x \in G$	35
$\mathscr{C}_n$	level sets of the ord function	45
Δ	diagonal operator on a Banach algebra	29

Δ	modular function on a group	8
$\delta_G(\alpha)$, $\delta(\alpha)$	determinant of $\alpha \in \mathrm{Aut}(G)$	36
d_π	dimension of the representation π	14
δ_x	point mass at x	112
Φ	fixed non-trivial character of a local field	62
$\widehat{f}$	the Fourier transform of f	12
$\phi^{\pi}_{\xi,\zeta}$	matrix coefficient function of $\xi, \zeta \in \mathscr{H}_\pi$	18
$\phi_{f,g}$	matrix coefficient function of $f, g \in L^2(G)$ with respect to the left regular representation	18
$\mathbb{F}(x)$	Laurent series over $\mathbb{F}$ in one variable	52
$\mathbb{F}[[x]]$	formal power series over $\mathbb{F}$ in one variable	51
$\mathbb{F}_p$	finite field of order p	
$\mathrm{Frac}(\mathscr{R})$	field of fractions of a local ring $\mathscr{R}$	47
${}_x f$	left translation of f by x	9
G	locally compact group	8
$\widehat{G}$	irreducible representations of G	15
$\widehat{G}$	Pontryagin dual of G	11
$\widehat{G}_f$	finite-dim. irreducible representations of G	15
$\widehat{G}_\infty$	infinite-dim. irreducible representations of G	15
$\widehat{G}_N$	irreducible representations with kernel containing N	99
$\widehat{G}_{N,\infty}$	infinite-dim. irreducible representations with kernel containing N	99
G_x	stabiliser subgroup of x	32
$\Gamma_{\mathscr{A}}$	Gelfand spectrum of a Banach algebra $\mathscr{A}$	133
G^{ap}	almost-periodic compactification of G	24
$\mathrm{GL}_n(\mathscr{R})$	invertible matrices whose entries are in $\mathscr{R}$	98
$H^\perp$	annihilator of H	12
$\mathscr{H}$	Hilbert space	9
η^G_{ap}, η_{ap}	canonical homomorphism from G to G^{ap}	24
$\mathbf{H}(\mathscr{V})$	all hyperplanes of a local space	74
$\mathscr{H}_\pi$	Hilbert space associated to the representation π	14
$\mathrm{Homeo}(X)$	homeomorphisms of a topological space X	31
H_x	hyperplane orthogonal to $x \in \mathscr{V}$	74
$\mathrm{Ind}^G_H(\sigma)$	induced representation of σ from H to G	85
$\mathscr{K}$	local field	56
κ	residue field	50
$\lvert\mathscr{K}^\times\rvert$	value group of a local field $\mathscr{K}$	56

λ	left regular representation	15
$L^1(G)$	the group algebra	9
$L^1_\pi(G)$	functions $f \in L^1(G)$ which are π-radial	116
$L^2(X;\mathscr{H})$	the L^2-space from X to $\mathscr{H}$	131
$L^p(X,\mu)$, $L^p(X)$	the L^p-space of (X,μ)	9
$\mathscr{M}$	maximal ideal of a local ring	39
$\mathbf{m}$	Haar measure of a group	8
$M(G)$	group measure algebra	9
$M_n(\mathscr{R})$	matrices whose entries are in $\mathscr{R}$	69
$M(X)$	Banach space of complex-valued measures	112
$\mathbb{N}$	natural numbers (including 0)	9
ν	discrete valuation	48
$\mathbb{N}_{\mathscr{K}}$	inclusion of the natural numbers inside $\mathscr{K}$	57
$\mathscr{O}_G(H)$	orbit space of $G \curvearrowright H$	31
$\mathscr{O}(\mathscr{K})$	ring of integers of a field $\mathscr{K}$	48
$\mathbb{O}_p$	p-adic integers	53
$\operatorname{ord}(x)$	order function of a local ring	40
$\widehat{\pi}$	the adjoint of a homomorphism π between two abelian groups	60
$\mathbf{P}(\mathscr{V})$	projective space	72
$\operatorname{Prob}(X)$	probability measures	112
$\mathbb{Q}$	rational numbers	
$\mathbb{Q}_p$	p-adic numbers	53
$\mathscr{R}$	discrete valuation ring (DVR)	48
$\mathscr{R}$	local ring	39
$\mathbb{R}$	real numbers	
$\mathscr{R}^*$	units of a ring $\mathscr{R}$	40
$\mathbb{R}_{\geq 0}$	positive reals (including 0)	
$\mathbb{R}_{>0}$	strictly positive reals	
$\operatorname{Rep}(G)$	(equivalence classes of) representations of G	18
$\mathscr{S}_r$	sphere of radius r centred at the origin in a local space	65
$\operatorname{supp} f$	support of a function f	
$\operatorname{supp} \mu$	support of a measure μ	112
$\mathbb{T}$	unit circle in $\mathbb{C}$	
$\mathscr{U}$	unit circle of a local field	72
$U(\mathscr{H})$	unitary operators on $\mathscr{H}$	9
$\mathscr{V}$	local space	68
$\mathscr{V}^\times$	non-zero vectors of a local space	72

Chapter 1

Introduction

> " Crazy? I was crazy once. They put me in a room. A rubber room. A rubber room with rats. A rubber room with rubber rats. Rats? I hate rats. They make me crazy. Crazy? I was crazy once..."
>
> –*Anonymous*

Ever since since its inception, the Fourier transform has become an incredibly important mainstay of modern mathematics across many disciplines. It was originally developed by Joseph Fourier in the early 1800s, whose key insight was that certain periodic functions on $\mathbb{R}$ could be decomposed as an (infinite) sum of trigonometric functions. Even non-periodic functions can be decomposed this way though one needs to do this in a 'continuous' manner. This leads naturally to the idea of the *Fourier transform* on $\mathbb{R}$, given by $f \mapsto \widehat{f}$ where

$$\widehat{f}(s) := \int_{\mathbb{R}} f(t)e^{-2\pi ist}\,\mathrm{d}t \tag{1.1}$$

for $s \in \mathbb{R}$. Among the many various facets of its study, there is a generalisation of this transform to abelian locally compact groups. The key property of these groups which makes this generalisation work is the existence of the *Pontryagin dual.*

Given an abelian locally compact group G, its Pontryagin dual $\widehat{G}$ is defined as the group of continuous homomorphisms from G to the torus $\mathbb{T}$. Pontryagin showed that this is a genuine a dual pairing in the sense that the dual of $\widehat{G}$ is canonically isomorphic to G itself, and this result came to be known as *Pontryagin duality*. As such, it is common to denote $\varphi(x) = \langle \varphi, x \rangle$ for $x \in G$ and $\varphi \in \widehat{G}$. We briefly look at properties of the Pontryagin dual in Section 2.1.3.

One key feature of this dual in which we are particularly interested is that it unlocks the *Fourier transform* on any arbitrary abelian locally compact group G. Given an $f \in L^1(G)$, we define $\widehat{f} \in C_0(\widehat{G})$ by

$$\widehat{f}(\varphi) := \int_G f(x)\overline{\langle \varphi, x \rangle}\,\mathrm{d}x \tag{1.2}$$

for $\varphi \in \widehat{G}$, much akin to (1.1). This transform is also widely studied, though we will focus more on its image, rather than the transform itself. We are also interested in the extension of this map which sends the complex-valued measures $M(G)$ to $C(\widehat{G})$, known as the *Fourier-Stieltjes transform*. The images of the Fourier and Fourier-Stieltjes transforms are given the names *Fourier algebra* and *Fourier Stieltjes algebra* of $\widehat{G}$, and are denoted $A(\widehat{G})$ and $B(\widehat{G})$ respectively. These were originally introduced by Eymard [19]; we provide a brief introduction of these algebras in Section 2.3.

Of note is that they may be defined even for nonabelian groups, with the aid of representation theory. In particular, we define *coefficient functions* of a representation π by

$$u(x) = \phi^{\pi}_{\xi,\eta}(x) := \langle \pi(x)\xi \,|\, \eta \rangle \tag{1.3}$$

for $\xi, \eta \in \mathscr{H}_\pi$, and we define a special case for the left regular representation λ by $\phi_{f,g} = \phi^{\lambda}_{f,g}$ for $f, g \in L^2(G)$. We use these to define the Fourier algebra by

$$A(G) := \{\phi_{f,g} : f, g \in L^2(G)\} \tag{1.4}$$

and the Fourier-Stieltjes algebra by

$$B(G) := \{\phi^{\pi}_{\xi,\eta} : \pi \in \mathrm{Rep}(G),\ \xi, \eta \in \mathscr{H}_\pi\} \tag{1.5}$$

where $\mathrm{Rep}(G)$ is the collection of all (unitary equivalence classes of) representations of G. When we equip these spaces with appropriate norms, they both become Banach algebras under pointwise multiplication. In fact, $A(G)$ will always be a closed ideal of $B(G)$. In the abelian setting, we have that Fourier transform is an isomorphism of $L^1(G)$ with $A(\widehat{G})$, and likewise with the Fourier-Stieltjes transform. It is then reasonable to expect that these ought to behave much like the L^1 and measure algebras, perhaps even in the nonabelian setting.

This argument holds weight; for instance we have that $A(G) = B(G)$ precisely when G is a compact group. Noting that compactness is an appropriate dual notion to discreteness, this mimics the fact that $L^1(G) = M(G)$ holds precisely when G is discrete. In general, we consider $A(G)$ to be a 'nice' and 'small' algebra to work with, whereas $B(G)$ is 'large' and 'intractable' (this is much akin to the relationship between $L^1(G)$ and $M(G)$). For instance, we have that the spectrum of $A(G)$ is always G itself, contrasting the situation for $B(G)$ in which there is currently no known closed formula describing its spectrum.

The ultimate goal of this thesis is to better understand these algebras of $A(G)$ and $B(G)$. We will focus our attention to a particular class of groups, heavily inspired by the work of Baggett [9] on the so-called *Fell group*. Thanks to the Peter-Weyl Theorem, it is well known that the *dual space* $\widehat{G}$ (the collection of irreducible representations) of a compact group is always discrete. In this paper of Baggett, it is shown that the converse is also true, so that a group with a discrete dual is necessarily compact. However, one must be cautious, for if we replace the roles of compactness and discreteness, this statement is no longer true.

The aforementioned Fell group is defined as the semidirect product of the p-adic integer units $\mathbb{O}_p^*$ acting on the p-adic numbers $\mathbb{Q}_p$ by scalar multiplication. The Fell group has several interesting properties, but most notably is that it is a noncompact group whose dual is countable, something which could never occur in the abelian setting. Many groups we look at will be semidirect products, and we provide some background notes on such groups in Section 2.5.

This is not where the similarity ends however; we also model the groups underlying the semidirect products by those which are similar to the p-adics. In particular, the p-adic integers form a *discrete valuation ring* or *DVR* for short. This ring is precisely the right structure which generalises the Fell group; we make this statement concrete in Theorem 3.21. We develop the (well-known) theory of these DVRs in Chapter 3.

These rings are part of a larger structure known as a *local field*. These objects are defined in various (equivalent) forms, though one relatively natural way is to say that a local field is a non-discrete locally compact field. There is a surprising amount of structure given the simplicity of the above definition; for instance, one may show that such fields possess an *absolute value* $|\cdot|$, and that they are self-dual in the sense that they are isomorphic to their Pontryagin duals. There is even a full classification of such fields, and include examples such as: the p-adic numbers $\mathbb{Q}_p$, as well as the usual reals $\mathbb{R}$ and the complex numbers $\mathbb{C}$. These results are covered in Section 4.1, and are primarily known results.

The subsequent section, Section 4.2, covers finite-dimensional vector spaces over local fields, which we call *local spaces*. We build up some basic theory surrounding these objects, so that we may generalise the results that apply to the Fell group to higher dimensions. As alluded to earlier, these results primarily focus on the structure of the Fourier and Fourier-Stieltjes algebras. See for instance Theorem 5.60, where we provide a complete description of $B(\mathscr{V} \rtimes \mathscr{U})$, where $\mathscr{V}$ is a non-Archimedean local space of dimension 2, and $\mathscr{U}$ is the 'unit circle' of scalars, and acts on $\mathscr{V}$ via scalar multiplication. Although the material in Section 4.2 bears striking resemblance to many results already known in finite dimensions over $\mathbb{R}$ or $\mathbb{C}$, some care is needed to fully generalise these to local spaces.

As suggested by the numbering, the new results on the Fourier-Stieltjes algebras of local spaces is primarily worked through in Chapter 5. In particular, we focus on the group $G = \mathscr{V} \rtimes \mathscr{U}$ that we mentioned prior. So to analyse $B(G)$, we need to understand the representation theory of G, and in particular, the dual space $\widehat{G}$. In order to do this, we employ a technique due to Mackey known as the *Mackey machine* [41], which gives a complete characterisation of the dual space $\widehat{G}$ for certain well-behaved semidirect products. Fortunately, the semidirect product is very well-behaved, and has a property we term *cheapness*, which we define in Definition 5.36, but let us restate it as follows.

Definition 1.1. Let G be a (second-countable) locally compact group of the form $A \rtimes K$ where A is an abelian locally compact group and K is compact. We say that G is **cheap** if the stabilisers K_φ of the dual action contain only the identity for all non-trivial $\varphi \in \widehat{A}$.

In other words, if $k \cdot \varphi = \varphi$ for $\varphi \in \widehat{A}$ and $k \in K$, then either $\varphi = \mathbf{1}$ or $k = 1$. Many of the groups we study will have the property that the action $K \curvearrowright A$ is isomorphic to the dual action $K \curvearrowright \widehat{A}$, so we need only check this property on A itself. See Proposition 4.37. When $G = A \rtimes K$ is a cheap group, the Mackey machine lends itself rather nicely to give a description of the dual space.

Theorem 1.2 (The Mackey machine for cheap groups)**.** *Let $G = A \rtimes K$ be cheap, and $q : G \to K$ be the canonical quotient map. If $\pi \in \widehat{G}$, then exactly one of two cases must occur.*

- *Either $\pi = \rho \circ q$ for some $\rho \in \widehat{K}$, in which case π is always finite-dimensional. In particular $A \subseteq \ker \pi$.*
- *Otherwise, $\pi = \mathrm{Ind}_A^G(\varphi)$ for some non-trivial $\varphi \in \widehat{A}$, in which case π is infinite-dimensional if and only if K is infinite. Moreover, $\mathrm{Ind}_A^G(\varphi)$ and $\mathrm{Ind}_A^G(\psi)$ will be equivalent representations precisely when φ and ψ are in the same orbit.*

We also can use this result to describe the topology on $\widehat{G}$ (see for instance Proposition 5.39 and Fig. 5.1), and to then obtain decompositions of the Fourier-Stieltjes algebra $B(G)$. This decomposition is given in Theorem 5.43.

In the special case where the non-trivial orbits of $K \curvearrowright A$ are open and countable in number, then we can refine the previous decomposition to obtain Theorem 5.45, which we state below. Note that we say a group is *infinitely cheap* if it is cheap and K is an infinite compact group.

Theorem 1.3. *Let $G = A \rtimes K$ be infinitely cheap. If every non-zero orbit is open, and the orbit space $\mathcal{O}_K(A)$ is countable, then*

$$B(G) = A(G) \oplus A(K) \circ q$$

where $q : G \to K$ is the canonical quotient map. In particular $B(G) = A^*(G)$.

Here $A^*(G)$ is known as the *spine* of $B(G)$, and groups which satisfy $B(G) = A^*(G)$ are termed *spinal*. The spine of $B(G)$ was introduced by Ilie and Spronk [27]. In essence, this spine contains the 'nice' part of $B(G)$, in the sense that it is precisely the (closed linear span of) all subalgebras of $B(G)$ of the form $A(H) \circ \eta_H$ where $\eta_H : G \to H$ is any continuous homomorphism from G to another locally compact group H. Naturally any compact group is spinal, but as Baggett (implicitly) [9] and Runde and Spronk (explicitly) [51] show, there are noncompact groups, primarily of the form $\mathbb{Q}_p^n \rtimes \mathrm{GL}_n(\mathbb{O}_p)$. We note that aside from the local field equivalents, this is essentially the complete list of known noncompact spinal groups.

In an attempt to find more such groups, we study now the group $G = \mathscr{V} \rtimes \mathscr{U}$ where $\mathscr{V}$ is a non-Archimedean local space. This is a generalisation of the p-adic motion group $\mathbb{Q}_p^n \rtimes \mathbb{O}_p^*$. We study these in Section 5.4, and as mentioned previously, we show that in the case of $\dim \mathscr{V} = 2$ we can make the following decomposition

$$B(G) = B_0(G) \oplus \left[\bigoplus_{H \in \mathbf{H}(\mathscr{V})} A_{q_H}(G) \right] \oplus A(\mathscr{U}) \circ q \tag{1.6}$$

where $B_0(G) = B(G) \cap C_0(G)$, $\mathbf{H}(\mathscr{V})$ is the collection of all lines in $\mathscr{V}$ and $q_H : G \to G/H$ is the corresponding quotient map, and $q : G \to \mathscr{U}$ is also the canonical quotient map. We then also show that $B_0(G) \cap A^*(G) = A(G)$, so that we also compute the spine

$$A^*(G) = A(G) \oplus \left[\bigoplus_{H \in \mathbf{H}(\mathscr{V})} A_{q_H}(G) \right] \oplus A(\mathscr{U}) \circ q \tag{1.7}$$

in a similar manner. However, in Section 5.6, due to the totally disconnected nature of $\mathscr{V}$, we observe that there is an abundance of coefficient functions in $B_0(G)$. In particular, we use this to show that $B_0(G) \supsetneq A(G)$, and so this G is unfortunately not spinal. However, we do also show that the product of any two coefficient functions inside $B_0(G)$ do land inside $A(G)$, indicating that perhaps this space is not as large as one may fear.

In Chapter 6, we return to the study of compact DVRs. However, before we describe the role of these DVRs, let us first mention the amenability of Banach algebras, as was first studied by Johnson in [30]. His renowned result shows that a locally compact group G is amenable if and only if its group algebra $L^1(G)$ is amenable, although amenability is defined differently for Banach algebras (such as $L^1(G)$) compared to the definition for groups. One may ask if the amenability of G is detectable by other algebras built from the group; for instance, we have the following result on the Fourier algebra due to Leptin [38] and Ruan [48].

Theorem 1.4. *Let G be a locally compact group. The following are equivalent.*

(i) *G is amenable.*

(ii) *$A(G)$ has a bounded approximate identity.*

(iii) *$A(G)$ is operator amenable.*

Conversely, another natural question asks if the amenability of $A(G)$ is easily detectable in G. This has been answered by Forrest and Runde [21].

Theorem 1.5. *Let G be a locally compact group. The following are equivalent.*

(i) *G is virtually abelian[1].*

(ii) *$A(G)$ is amenable.*

It was conjectured by Azimifard, Samei and Spronk [7], that in the case where G is compact, the above theorem holds for the algebra $ZL^1(G)$, the centre of $L^1(G)$. It was shown by Alaghmandan and Crann [3] that $ZL^1(G)$ is always amenable for a compact virtually abelian group G, though the converse remains an open question. We note that their method involves a construction known as *hypergroups*, and we shall see these again shortly. However, in a similar vein, Alaghmandan and Spronk show that the *central Fourier algebra* $ZA(G) = A(G) \cap ZL^1(G)$ is also amenable whenever G is compact virtually abelian, and again the converse remains open.

Consider now a semidirect product of the form $G \rtimes K$, where G is abelian and both G and K are compact. We can define the central Fourier algebra as

$$ZA(G) := \{u \in A(G) : u(a^{-1}xa) = u(x) \text{ for all } a, x \in G\} \tag{1.8}$$

[1] A virtually abelian group is one with an abelian subgroup of finite index.

which naturally leads us to a slightly more general *K-central Fourier algebra*

$$Z_K A(G) := \{u \in A(G) : u(k \cdot x) = u(x) \text{ for all } k \in K,\ x \in G\} \tag{1.9}$$

where $K \curvearrowright G$. We show in Proposition 6.6 that $Z_K A(G)$ is in fact a quotient of $ZA(G \rtimes K)$, and may prove to be an easier object to study. Indeed, we have that $Z_K A(G)$ is in fact the Fourier algebra of a particular hypergroup known as a *spherical hypergroup*. These were introduced by Muruganandam [43], and in our case, these are induced by the orbits of the action $K \curvearrowright G$. Let us call such a hypergroup H.

When G is abelian, the action has a corresponding dual action $K \curvearrowright \widehat{G}$, and to this we may associate a 'dual hypergroup[2]' which we denote $\widehat{H}$. We show in this case that $A(H) = L^1(\widehat{H})$, mimicking the classical result for groups. Since H is compact, we have that $\widehat{H}$ is discrete and so it is often easier to compute $L^1(\widehat{H})$. When these hypergroups satisfy an 'amenability-like' condition, it was shown by Alaghmandan [2, Theorem 5.1] that if their dual hypergroups have unbounded orbit size, then $L^1(\widehat{H})$ is not amenable as a Banach algebra. This would then imply that the corresponding $ZA(G \rtimes K)$ would also not be amenable.

We explicitly compute $\widehat{H}$ for certain classes of hypergroups which are induced by actions on a profinite group. For instance, a compact DVR $\mathscr{R}$ is always a profinite group, and the action $\mathscr{R}^* \curvearrowright \mathscr{R}$ generates one of these aforementioned hypergroups. Since these have unbounded orbit size, then it follows that $ZA(\mathscr{R} \rtimes \mathscr{R}^*)$ is not amenable, which reaffirms the conjecture of Alaghmandan and Spronk. This result as stated below, can be found in Theorem 6.29.

Theorem 1.6. *Let $\mathscr{R}$ be a compact DVR. If $G = \mathscr{R}^d \rtimes \mathrm{GL}_d(\mathscr{R})$ for $d > 0$, then $ZA(G)$ is not amenable.*

[2] Not to be confused with the Pontryagin dual of a hypergroup.

Chapter 2

Preliminaries

" ℓ^∞ and beyond! "

–T. Bray

2.1 Locally Compact Groups

2.1.1 Terminology & Notation

Throughout this thesis, G will denote a locally compact group, and unless otherwise stated, this will implicitly mean a locally compact *Hausdorff* group, as is standard with most literature on the subject. It is outside of the scope of this thesis to revisit all the fundamentals of locally compact groups, so we shall assume the reader is well-versed in this theory. There are many good resources on the theory of locally compact groups, for instance, the books of Folland [20], Kaniuth and Lau [34], or Runde [49], provide suitable introductions. For convenience, we begin with a few definitions and results that we explicitly need, but this is by no means an exhaustive list.

Let us start by introducing some notation. Given a locally compact group G, we shall denote the group operation by concatenation, and the identity by e, though we may occasionally opt to use 1 or e_G. In the case where G is abelian, we instead adopt the usual notations of $+$ and 0. We also denote the *Haar measure* by $\mathbf{m}$, the *modular function* by Δ, and the corresponding integral by $\int_G \ldots \,\mathrm{d}x$. The notation $C(G)$ will denote the algebra of complex-valued continuous functions, with subalgebras of bounded/vanishing at infinity/compactly supported (continuous) functions being denoted by $C_b(G) \,/\, C_0(G) \,/\, C_c(G)$

respectively. We let $L^p(G)$ denote the usual L^p spaces, where in particular the *group algebra* is $L^1(G)$ equipped with the convolution

$$[f * g](x) := \int_G f(y)g(y^{-1}x)\,\mathrm{d}y \tag{2.1}$$

for $f, g \in L^1(G)$. This is an ideal inside of the complex-valued (finite) measures $M(G)$, where the embedding is given by $\mu_f(A) := \int_A f(x)\,\mathrm{d}x$ for measurable $A \subseteq G$.For an arbitrary function f, we shall let ${}_xf$ denote the *left translation* of f by $x \in G$: where ${}_xf(y) := f(x^{-1}y)$ for all $y \in G$.

In general, we shall restrict our focus to second-countable groups. These are generally more tractable and so give nicer theories. Moreover, the groups we study (which are primarily derived from local fields) will be second-countable, so these results will be applicable. So, unless otherwise stated, whenever we refer to a locally compact group, *we shall implicitly assume the group is second-countable.* Of note, the assumption of second-countability implies that G is always *metrisable* (see Theorem 2.6). For these and other metric spaces, we shall denote the open and closed balls by $\mathrm{B}(x; \varepsilon)$ and $\mathrm{B}[x; \varepsilon]$ respectively.

Also, since we are in the category of locally compact groups, whenever we say that a mapping is a *homomorphism*, we shall implicitly mean that it is a *continuous homomorphism*. We do not study non-continuous homomorphisms in this thesis.

We shall also need to borrow technology from the study of operator algebras. So given a Hilbert space, which we typically denote $\mathscr{H}$ (or some variation thereof), we let $\langle \cdot \mid \cdot \rangle_{\mathscr{H}}$ denote the inner product on $\mathscr{H}$, and $\|\cdot\|_{\mathscr{H}}$ the norm of $\mathscr{H}$. Naturally, we drop the subscripts if $\mathscr{H}$ is unambiguous. We let $\mathscr{B}(\mathscr{H})$ denote the bounded linear operators on $\mathscr{H}$, with $U(\mathscr{H})$ denoting the space of unitaries. More generally, if X is a Banach space, we let X^* denote the Banach space dual, and $\langle \cdot\,, \cdot \rangle$ denote the dual pairing.

Finally, we also use many standard notations that are common throughout mathematics, though we explicitly state here that the natural numbers $\mathbb{N}$ contain the number 0. We also adopt the somewhat unconventional notation $\mathbb{1}_F$ for the indicator function on a set $F \subseteq X$.

2.1.2 Preliminary results

We now present a few well-known facts about topological and metric spaces that shall prove useful to us. First let us recall the following definition of a *proper* map.

Definition 2.1. Let X, Y be topological spaces. If $\eta : X \to Y$ is a function such that preimages of compact sets are compact, then we say that η is **proper**.

When these maps are continuous, they provide a closure property when the codomain is a locally compact space. We state this below.

Proposition 2.2. *Let X, Y be topological spaces, with Y being locally compact. If $\eta : X \to Y$ is a continuous proper map with dense range, then η is surjective.*

Proof. Let $y \in Y$, and choose a compact neighbourhood $y \in U \subseteq \overline{U}$, so that $\eta^{-1}(U) \subseteq \eta^{-1}(\overline{U})$. Since η has dense range, we may find a net $(x_\alpha)_\alpha$ in X such that $\eta(x_\alpha) \to y$. Now, eventually this net must land in $\eta^{-1}(U) \subseteq \eta^{-1}(\overline{U})$, the latter of which is compact since η is proper. Therefore, $(x_\alpha)_\alpha$ must have a cluster point at some $x \in X$, and by continuity of η, it follows that $\eta(x) = y$. $\square$

It follows from this that a locally compact subgroup will always be closed in its ambient containing group[3].

Now let (X, d) be a metric space. If $d : X \times X \to \mathbb{R}$ is a proper map, it is not hard to see that every closed and bounded set must be compact. This gives the following definition.

Definition 2.3. Let X be a metric space. We say that X is **proper** if every closed and bounded set is compact.

These spaces are very well behaved topological spaces. If X is a proper metric space, then X is locally compact, for if $x \in X$, then $x \in \mathrm{B}(x; 1) \subseteq \mathrm{B}[x; 1]$, and so $\mathrm{B}[x; 1]$ is a compact neighbourhood of x. Moreover, by the σ-compactness of $\mathbb{R}$, X itself must also be σ-compact. Recall in the metric space setting that σ-compactness implies second-countability, which in turn is equivalent to separability[4]. Let us coalesce these results into the following statement.

Proposition 2.4. *Let X be a metric space. If X is proper, then X is locally compact, second-countable, separable and σ-compact.*

Now, let us apply these results in the group setting; recall the definition of metrisability for groups.

[3] Let $H \subseteq G$, and let $\eta : H \to \overline{H}$ be the inclusion map. This result then gives that $H = \eta(H) = \overline{H}$.

[4] See for instance Willard's book [62, Theorem 16.11].

Definition 2.5. Let G be a topological group. We say that a metric d is **left-invariant** if $d(xy, xz) = d(y, z)$ for all $x, y, z \in G$. Furthermore, we say that G is **metrisable** if there exists a left-invariant metric d on G whose topology agrees with the given topology on G.

There is a well-known result due to Birkhoff [12] and Kakutani [32] characterising the metrisability of a group to its topological structure.

Theorem 2.6 (Birkhoff-Kakutani Theorem)**.** *Let G be a topological group. Then G is metrisable if and only if G is Hausdorff and first-countable.*

This is also proven in more modern language in [25, Theorem 8.3]. In particular, this result implies that all locally compact *second-countable* groups are metrisable. Since we are making the broad assumption of second-countability, this implies that all groups we work with will be metrisable as well. For now, we shall emphasise the metrisability of these groups, though we shall later drop this practice.

These metrisable groups have a nice completion property when they are locally compact.

Proposition 2.7. *Let G be a locally compact metrisable group. Then G is complete with respect to any left-invariant metric.*

Proof. Let (x_n) be a Cauchy sequence in G. Let K be a compact neighbourhood of the identity, and choose $\varepsilon > 0$ so that $\mathrm{B}(e; \varepsilon) \subseteq K$. Let $N \in \mathbb{N}$ be sufficiently large so that $x_n \in \mathrm{B}(x_N; \varepsilon)$ for all $n > N$. Now by left invariance of the metric, we have that $\mathrm{B}(x_N; \varepsilon) \subseteq x_N K$. Since the tail of this sequence is contained in a compact set $x_N K$, it must genuinely converge to some point $x \in G$. Hence G is complete. □

2.1.3 Pontryagin Duality

When G is an abelian locally compact group, we obtain a corresponding dual group $\widehat{G}$ known as the *Pontryagin dual* of G. We again assume the familiarity with this subject, however we shall briefly present a few definitions and results which will prove to be useful. Our main reference for this section will be Chapter 4 of Folland's book [20].

We define the **Pontryagin dual** to be $\widehat{G}$, the space of (continuous) homomorphisms from G to the torus $\mathbb{T}$. We equip $\widehat{G}$ with pointwise multiplication and the compact-open topology, so that it is a locally compact group. Pontryagin duality states that the homomorphisms from $\widehat{G}$ to $\mathbb{T}$ are precisely the point evaluations, so that $\widehat{\widehat{G}} = G$. Therefore, for $x \in G$ and $\varphi \in \widehat{G}$, we shall occasionally write the evaluation $\varphi(x)$ as either $\langle x, \varphi \rangle$ or $\langle \varphi, x \rangle$, to signify this duality (it also frequently simplifies certain notations).

The **Fourier** transform of G maps $L^1(G) \to C_0(\widehat{G})$ and is written as $f \mapsto \widehat{f}$ where

$$\widehat{f}(\varphi) := \int_G f(x)\overline{\langle \varphi, x \rangle}\, \mathrm{d}x \tag{2.2}$$

for $f \in L^1(G)$. This may be generalised to the *Fourier-Stieltjes* (which maps $M(G) \to C_b(\widehat{G})$) and *Plancherel* ($L^2(G) \to L^2(\widehat{G})$) transforms, which are defined in a similar manner.

When H is a closed subgroup of G, there is a corresponding subgroup in $\widehat{G}$ which is a dual analogue of H. This subgroup is known as the *annihilator* of H.

Definition 2.8. Let H be a closed subgroup of an abelian locally compact group G. We define the **annihilator** $H^\perp$ of H to be

$$H^\perp := \{\varphi \in \widehat{G} : H \subseteq \ker \varphi\}$$

which is a closed subgroup of $\widehat{G}$.

We list of a few key properties of the annihilator without proof. Most of these results may be found in Folland's book [20, Chapter 4].

Proposition 2.9. *Let H, K be closed subgroups of an abelian locally compact group G. Then:*

(i) $(H^\perp)^\perp = H$,

(ii) *if $H \subseteq K$, then $K^\perp \subseteq H^\perp$,*

(iii) $H^\perp = \widehat{G/H}$ *and* $\widehat{H} = \widehat{G}/H^\perp$,

(iv) *H is compact if and only if $H^\perp$ is open, and*

(v) *the Haar measure on $\widehat{G}$ can be normalised so that if H is compact and open, then* $\mathbf{m}_G(H) = 1/\mathbf{m}_{\widehat{G}}(H^\perp)$.

We leave the proof for the inquisitive reader. Next we consider abelian locally compact groups which have chains of ascending or descending subgroups; in particular, we are interested in how these chains behave with respect to the annihilator. This will be particularly useful when we investigate profinite groups in Section 6.3.

Proposition 2.10. *Let G be an abelian locally compact group. If G contains an ascending chain of closed subgroups $H_1 \subseteq H_2 \subseteq \dots$ with $H = \overline{\bigcup_{n\in\mathbb{N}} H_n}$, then*

$$H^\perp = \bigcap_{n\in\mathbb{N}} {H_n}^\perp$$

Proof. The containment $H^\perp \subseteq \bigcap_{n\in\mathbb{N}} {H_n}^\perp$ is trivial. On the other hand, suppose that that $\varphi \in \bigcap_{n\in\mathbb{N}} {H_n}^\perp$. Take any $x \in H$ and choose $x_n \in H_n$ so that the sequence $(x_n)_n$ converges to x. Since $\varphi \in {H_n}^\perp$, it follows that $\varphi(x_n) = 1$, and so by continuity $\varphi(x) = 1$. Thus $\varphi \in H^\perp$. □

Naturally, we may leverage this to obtain the corresponding result for descending chains.

Corollary 2.11. *If G contains a descending chain of closed subgroups $H_1 \supseteq H_2 \supseteq \cdots$ with $H = \bigcap_{n\in\mathbb{N}} H_n$, then*

$$H^\perp = \overline{\bigcup_{n\in\mathbb{N}} {H_n}^\perp}$$

Proof. This follows by setting $H'_n = {H_n}^\perp$ and using the previous proposition. □

One special case that we shall need is when the descending chain intersects to the trivial group. In particular, if all these subgroups are compact, then the closure on the union is superfluous.

Corollary 2.12. *Let G be an abelian locally compact group. If G contains a descending chain of compact subgroups H_n such that $\bigcap_{n\in\mathbb{N}} H_n = \{0\}$, then*

$$\bigcup_{n\in\mathbb{N}} {H_n}^\perp = \widehat{G}$$

Proof. By Corollary 2.11, it is sufficient to show that $\bigcup_{n\in\mathbb{N}} {H_n}^\perp$ is closed. Since H_n is compact, then each ${H_n}^\perp$ is open, hence their union is an open subgroup, and so must be closed. □

2.2 Representations of Locally Compact Groups

In this section, we present some fundamentals of the representation theory for locally compact groups. While we assume the reader is acquainted with this theory, we shall restate several basic definitions and results for posterity. So for a locally compact group G, a **unitary representation** of G is a SOT-continuous homomorphism $\pi : G \to U(\mathscr{H}_\pi)$, where $U(\mathscr{H}_\pi)$ is the group unitaries of some Hilbert space $\mathscr{H}_\pi$. Since we only consider representations which map to unitary operators, when we refer to a representation, we shall always mean a *unitary* representation.

Given a unitary representation π, then as mentioned above, we shall denote its corresponding Hilbert space by $\mathscr{H}_\pi$. We shall also let $\|\cdot\|_\pi$ and $\langle \cdot \mid \cdot \rangle_\pi$ denote the norm and inner product on $\mathscr{H}_\pi$, though if there is no ambiguity, we shall drop the subscripts.

Naturally, we say the **dimension** of π is the dimension of $\mathscr{H}_\pi$ and we denote it d_π. A **subrepresentation** of π is the restriction of π to $\mathscr{M}$, a closed π-*invariant* subspace (that is $\pi(G)\mathscr{M} \subseteq \mathscr{M}$). A representation is **finite** if it is finite-dimensional, and is **purely infinite** if it contains no finite-dimensional subrepresentations.

We say a representation π is **cyclic** if there exists a $\xi \in \mathscr{H}_\pi$ such that the span of $\pi(G)\xi$ is dense in $\mathscr{H}_\pi$. A Zorn's lemma argument shows that any representation may be decomposed as a direct sum of cyclic representations. Henceforth, throughout this thesis, we shall impose without loss of generality that *any given representation* π *is cyclic.*

> *Remark* 2.13. As previously mentioned, we also make an implicit assumption that G is second-countable. As a result, it can be easily seen that any (cyclic) representation must be *separable*, that is $\mathscr{H}_\pi$ must be separable. Hence, unless otherwise stated, we also make this assumption going forward.

We say two representations π, σ are **unitarily equivalent** if there exists a unitary $U : \mathscr{H}_\pi \to \mathscr{H}_\sigma$ such that $U\pi(x) = \sigma(x)U$ for all $x \in G$. When this is the case we shall write $\pi \approx \sigma$. Similarly, if π contains a subrepresentation which is unitarily equivalent to σ, we say σ is **contained** in π and write $\sigma \preccurlyeq_u \pi$.

We also have a notion of quasi-equivalence. This is defined through amplifications: an **amplification** of π is any representation (unitarily equivalent to) $\bigoplus_\kappa \pi$ where κ is any cardinal. Alternatively, if we let $\iota : G \to U(\mathscr{H})$ be the constant representation $x \mapsto I_{\mathscr{H}}$ on any Hilbert space $\mathscr{H}$, then any representation equivalent to $\pi \otimes \iota$ is an amplification of π. We say that σ is **quasi-contained** in π, if there is an amplification of π which contains a subrepresentation equivalent to σ. When this is the case, we write $\sigma \preccurlyeq_q \pi$. If we have both $\pi \preccurlyeq_q \sigma$ and $\sigma \preccurlyeq_q \pi$, then we say that π and σ are **quasi-equivalent**, and write $\pi \approx_q \sigma$.

There are two representations which occur frequently in literature and will be extremely important. The first is the **trivial representation**, which we denote by $\mathbf{1} : G \to U(\mathbb{C})$. This constant representation maps every $x \in G$ to the identity, or if we identify $\mathscr{B}(\mathbb{C}) \cong \mathbb{C}$, then it maps $x \mapsto 1$. The other important representation is the **left regular representation**, which acts on $L^2(G)$ by $\lambda(x)f(y) := f(x^{-1}y)$ for a.e. $y \in G$. We shall present results on these representations as they are needed; a large portion of these will be near the beginning of Chapter 5. One such result is known as *Fell's absorption principle*, which we shall prove here. One may see Brown and Ozawa [15, Theorem 2.5.5] as a reference.

Proposition 2.14 (Fell's Absorption Principle)**.** *Let G be a locally compact group, and π a unitary representation on G. Then $\pi \otimes \lambda$ is quasi-equivalent to λ.*

Proof. Let ι be the constant representation on $\mathscr{H}_\pi$. One may identify $\mathscr{H}_\pi \otimes_2 L^2(G)$ with $L^2(G; \mathscr{H}_\pi)$, see Appendix A. Now let $U \in U(L^2(G; \mathscr{H}_\pi))$ be defined by

$$[Uf](x) = \pi(x)f(x)$$

which we claim is an intertwiner for $\pi \otimes \lambda$ and $\iota \otimes \lambda$. Indeed we have that

$$\begin{aligned}
[U^*(\pi \otimes \lambda)(y)Uf](x) &= \pi(x^{-1})[(\pi \otimes \lambda)(y)Uf](x) \\
&= \pi(x^{-1})\pi(y)[Uf](y^{-1}x) \\
&= \pi(x^{-1})\pi(y)\pi(y^{-1}x)f(y^{-1}x) \\
&= f(y^{-1}x) \\
&= [(\iota \otimes \lambda)(y)f](x)
\end{aligned}$$

thus showing that $\pi \otimes \lambda$ is unitarily equivalent to $\iota \otimes \lambda$. Hence $\pi \otimes \lambda \approx_q \lambda$. □

We say a representation π is **irreducible** if it has no non-trivial subrepresentations. Given a locally compact group G, we let $\widehat{G}$ denote the space of irreducible representations (modulo unitary equivalence) of G, and we call this the **dual space** of G. This space has a natural topology, which is defined through $C^*(G)$, the *C^*-algebra* of G. We shall present this definition shortly.

When G is abelian, the dual space of G coincides with the Pontryagin dual of G. Likewise, when G is compact, the Peter-Weyl theory gives us that $\widehat{G}$ is discrete and contains only the finite-dimensional representations. In general we let $\widehat{G}_{\mathfrak{f}} \subseteq \widehat{G}$ denote the finite-dimensional irreducible representations of G, and similarly we set $\widehat{G}_\infty = \widehat{G} \setminus \widehat{G}_{\mathfrak{f}}$, the collection of infinite-dimensional irreducible representations. Note that $\widehat{G}_\infty$ necessarily consists of purely infinite representations.

The Group C*-Algebra

Let G be a locally compact group, and consider the group algebra $L^1(G)$. Given a unitary representation $\pi : G \to U(\mathscr{H}_\pi)$, we may extend the action of π to $\pi_1 : L^1(G) \to U(\mathscr{H}_\pi)$ by

$$\pi_1(f) = \int_G f(x)\pi(x)\,\mathrm{d}x \tag{2.3}$$

for $f \in L^1(G)$. One may verify this is a non-degenerate $*$-representation on $L^1(G)$. Note that we interpret the above integral in the weak sense, so that

$$\langle \pi_1(f)\xi \,|\, \eta\rangle = \int_G f(x)\langle \pi(x)\xi \,|\, \eta\rangle\,\mathrm{d}x \tag{2.4}$$

for $\xi, \eta \in \mathscr{H}_\pi$. More details on this kind of integral can be found in [20, Appendix 3]. Checking that this is indeed a non-degenerate $*$-representation is fairly routine work. However, it turns out that reverse direction is also true, that is all non-degenerate $*$-representations of $L^1(G)$ arise in this manner. Indeed, the idea is that given a $*$-representation π_1 of $L^1(G)$, one may define $\pi(x)$ for $x \in G$ as the limit of $\pi({}_x\psi_\alpha)$ where ψ_α is a bounded approximate identity for $L^1(G)$. Of course, additional details need to be checked, the interested reader may find these in Folland's book [20, Section 3.2]. In general, we shall denote the $*$-representation π_1 simply by π.

Another important algebra is the *group C*-algebra*. Leveraging the above association, we define an alternative norm on $L^1(G)$ by

$$\|f\|_* := \sup_{\pi \in \widehat{G}} \|\pi(f)\| \tag{2.5}$$

which is in fact a C*-norm on $L^1(G)$. Verifying that this is a C*-norm requires some work, the details of which are shown in [20, Section 7.1].

Definition 2.15. Let G be a locally compact group. We let $C^*(G)$ be the completion of $L^1(G)$ with respect to the norm $\|\cdot\|_*$. We call this the **group C*-algebra** of G.

Since $L^1(G)$ is dense in $C^*(G)$, we can always extend representations of $L^1(G)$ up to $C^*(G)$ and vice versa. This then gives a bijective correspondence between the following:

- (unitary) representations of G,
- non-degenerate $*$-representations of $L^1(G)$, and
- non-degenerate $*$-representations of $C^*(G)$.

Our primary utilisation of the group C*-algebra is in giving us a way to define the topology on $\widehat{G}$. Given a C*-algebra $\mathscr{A}$, we let $\widehat{\mathscr{A}}$ denote the collection of irreducible $*$-representations of $\mathscr{A}$. Note that the above exposition shows that $\widehat{G} = \widehat{C^*(G)}$.

Definition 2.16. Let $\mathscr{A}$ be a C*-algebra. A closed ideal I of $\mathscr{A}$ is a **primitive ideal** of $\mathscr{A}$ if $I = \ker \pi$ for some $\pi \in \widehat{\mathscr{A}}$. The collection of all primitive ideals is called the **primitive ideal space** and is denoted $\mathrm{Prim}(\mathscr{A})$.

We equip $\mathrm{Prim}(\mathscr{A})$ with the **hull-kernel** (or the **Jacobson**) topology where for any $E \subseteq \mathrm{Prim}(\mathscr{A})$, we define the closure

$$\overline{E} := \left\{ I \in \mathrm{Prim}(\mathscr{A}) : I \supseteq \bigcap_{J \in E} J \right\} \tag{2.6}$$

for any set $E \subseteq \mathrm{Prim}(\mathscr{A})$. It is of note that this topology is frequently not 'nice'.[5] Now consider the map $\pi \mapsto \ker \pi$, which maps $\widehat{\mathscr{A}} \to \mathrm{Prim}(\mathscr{A})$. We can then induce a topology on $\mathscr{A}$ via the pullback map, and we call this the **Fell topology** on $\mathscr{A}$. Naturally, if G is a locally compact group, then the topology on the dual space $\widehat{G}$ is given by the Fell topology on $\widehat{C^*(G)}$. We shall also call this topology the *Fell topology* on $\widehat{G}$. Further details for the Fell topology may be found in [20, Section 7.1] or [35, Section 1.6].

We will briefly note an alternative method to define the Fell topology. One may define a third notion of containment of representations, known as *weak containment*. We shall omit the definition of weak containment, and instead refer the reader to [35, Definition 1.67 or Proposition 1.68]. With this definition, the topology on $\widehat{G}$ is given as follows: For any net $(\pi_\alpha)_\alpha$ in $\widehat{G}$, we have that $\pi_\alpha \to \pi \in \widehat{G}$ if and only if every subnet of π_α weakly contains in π. See for instance [35, Lemma 5.7].

2.3 Fourier & Fourier-Stieltjes Algebras

2.3.1 Definition & Properties

There are two additional algebras of G which will be central to this thesis, namely the *Fourier* and *Fourier-Stieltjes* algebras. These were introduced by Eymard in his seminal

[5]For instance, it is often not Hausdorff. Its construction mimics that of the Zariski topology from algebraic geometry, which is famously intricate.

work [19], and have since become a mainstay in abstract harmonic analysis. Given a representation π, we define a **matrix coefficient** to be any function $u \in C(G)$ of the form

$$u(x) = \phi^{\pi}_{\xi,\eta}(x) := \langle \pi(x)\xi \,|\, \eta \rangle \tag{2.7}$$

for $\xi, \eta \in \mathscr{H}_\pi$. For the special case where $\pi = \lambda$, the left regular representation, we simply write $\phi_{f,g} := \phi^{\lambda}_{f,g}$ for $f, g \in L^2(G)$. The collection of all such functions gives rise to these Fourier and Fourier-Stieltjes algebras, which we define below. For simplicity, let us denote by $\mathrm{Rep}(G)$ the collection of all representations of G (up to unitary equivalence).

Definition 2.17. Let G be a locally compact group. We define the **Fourier-Stieltjes algebra** to be

$$B(G) := \{\phi^{\pi}_{\xi,\eta} : \pi \in \mathrm{Rep}(G),\ \xi, \eta \in \mathscr{H}_\pi\}$$

where the algebra product is given pointwise multiplication, and norm

$$\|u\|_{B(G)} := \inf\{\|\xi\|\|\eta\| : \pi \in \mathrm{Rep}(G),\ \xi, \eta \in \mathscr{H}_\pi,\ u = \phi^{\pi}_{\xi,\eta}\} \tag{2.8}$$

for $u \in B(G)$. Similarly, we define the **Fourier algebra** to be

$$A(G) := \{\phi_{f,g} : f, g \in L^2(G)\}$$

which is a subalgebra of $B(G)$.

We shall primarily use Chapter 2 of Kaniuth and Lau's book [34] as a reference for this section. As one would expect, $B(G)$ and $A(G)$ are indeed Banach algebras. The following results may be found as Theorem 2.1.11 and Corollary 2.3.5 respectively in the same book of Kaniuth and Lau.

Proposition 2.18. *Let G be a locally compact group. Then $B(G)$ and $A(G)$ are unital commutative Banach algebras when equipped with pointwise multiplication and the norm in (2.8). Furthermore, $A(G)$ is a closed ideal of $B(G)$.*

The ideality of $A(G)$ follows from Fell's Absorption Principle.

Proposition 2.19. *Let G be a locally compact group. Then $A(G)$ consists of functions which vanish at infinity, and moreover it is uniformly dense in $C_0(G)$.*

We note that when $u \in A(G)$, the norm can be expressed more succinctly as

$$\|u\|_{A(G)} = \inf\{\|f\|\|g\| : f, g \in L^2(G),\ u = \phi_{f,g}\} \tag{2.9}$$

which coincides with the norm inherited from $B(G)$.

Remark 2.20. It is important to note that $A(G)$, as simply the set of coefficient functions $\phi_{f,g}$, is already a closed subalgebra. This is a surprisingly non-trivial fact! The key idea here is that $A(G)$ is the predual of the group von Neumann algebra $VN(G)$ and that this von Neumann algebra is in so-called *standard form*. We shall introduce $VN(G)$ shortly. For more details, we refer the reader to the books of Takesaki [55, 56], which develops this theory rigorously. In particular, the theory of standard forms is given in Chapter IX Section 1, whereas this specific fact for $A(G)$ is provided in Lemma 3.7 of Chapter VII.

There are a few nice duality properties of these algebras. Firstly, $B(G)$ can be seen as the natural dual of $C^*(G)$, with the pairing

$$\langle f, u \rangle = \int_G f(x)u(x)\,\mathrm{d}x \tag{2.10}$$

for $u \in B(G)$ and $f \in L^1(G)$ (where one extends this to all of $C^*(G)$ by density). On the other hand, $A(G)$ is the predual of the *group von Neumann algebra*.

Definition 2.21. Let G be a locally compact group. We define the **group von Neumann algebra** to be

$$VN(G) := \lambda(G)'' \subseteq \mathscr{B}(L^2(G))$$

where $\lambda(G)''$ is the double commutant of $\lambda(G)$ inside $\mathscr{B}(L^2(G))$.[6]

To show that this indeed the dual, we follow loosely the proof given by Arsac [5, (2.2) Theoreme].

Theorem 2.22. *Let G be a locally compact group. The dual of $A(G)$ can be isometrically identified with $VN(G)$ where for any $T \in VN(G)$ the dual pairing is given by $\langle T, \phi_{f,g}\rangle = \langle Tf \mid g\rangle$.*

Proof. It is well known that the unique predual of $\mathscr{B}(\mathscr{H})$ for any Hilbert space $\mathscr{H}$ is the set of *trace-class operators* $\mathscr{B}_1(\mathscr{H})$. These are (norm) densely spanned by linear functionals of the form

$$\varphi_{\xi,\eta}(T) := \langle T\xi \mid \eta\rangle$$

for $T \in \mathscr{B}(\mathscr{H})$ and $\xi, \eta \in \mathscr{H}$. Now let us fix $\mathscr{H} = L^2(G)$, and consider the mapping $Q : \mathscr{B}_1(L^2(G)) \to A(G)$ given by $Q(\varphi_{f,g}) = \phi_{f,g}$. With some work, one may check that this is indeed a well-defined quotient map, and so it follows that $A(G)^* \cong (\ker Q)^\perp$

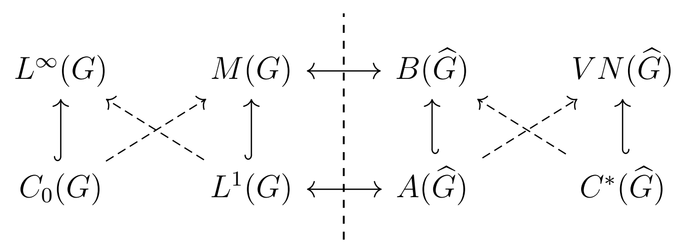
[6] Equivalently, this is the von Neumann algebra generated by $\lambda(G)$

isometrically. However, one may verify that $T \in (\ker Q)^{\perp}$ precisely when $\langle Tf \mid g \rangle = 0$ for all $f, g \in L^2(G)$ such that $\langle \lambda(x)f \mid g \rangle = 0$. In other words, $(\ker Q)^{\perp} = VN(G)$, thus completing the proof. □

When G is abelian, $B(G)$ and $A(G)$ can be identified with algebras of $\widehat{G}$. In particular, we have

$$B(G) \cong M(\widehat{G}) \qquad VN(G) \cong L^{\infty}(\widehat{G})$$
$$A(G) \cong L^1(\widehat{G}) \qquad C^*(G) \cong C_0(\widehat{G})$$

and with this one can draw a nice diagram depicting the relations between these spaces



where the dashed arrows indicate dual spaces. We note that even in the nonabelian case, each half of the diagram holds with one exception, namely that $C^*(G)$ need not embed into $VN(G)$. We do have that the **reduced group C*-algebra** $C^*_r(G) := \lambda(C^*(G))$[7] does embed into $VN(G)$; so by Hulanicki's theorem [26], this diagram will be accurate whenever G is amenable.

It is well known that G is discrete if and only if $M(G) = L^1(G)$. This works just as well for $B(G)$ and $A(G)$ in the dual property, compactness.

Proposition 2.23. *Let G be a locally compact group. Then G is compact if and only if $B(G) = A(G)$.*

Proof. If G is compact, then the constant function 1 is in $L^2(G)$. So it follows that $\phi_{1,1} = 1 \in A(G)$. Since $A(G)$ is an ideal in $B(G)$, this implies that $B(G) = A(G)$. On the other hand, suppose that $B(G) = A(G)$. Consider then the coefficient function $u \in B(G)$ on the trivial representation $\mathbf{1}$ given by $u(x) = \langle \mathbf{1}(x)1 \mid 1 \rangle = 1$. Since $A(G)$ comprises of $C_0(G)$ functions, this means that constant function 1 must vanish at infinity, and so G is compact. □

[7]Here we view λ as a *-representation of $C^*(G)$, as per the association given below Definition 2.15.

2.3.2 Fourier Spaces

We now generalise some of the ideas of $A(G)$ to similar spaces for arbitrary representations of a group.

Definition 2.24. Let G be a locally compact group, and π a representation of G. We define the **Fourier space** associated to π to be

$$A_\pi(G) := \overline{\text{Span}}\{\phi^\pi_{\xi,\eta} : \xi, \eta \in \mathscr{H}_\pi\} \subseteq B(G)$$

whose norm is inherited from $B(G)$.

Fourier spaces were first studied by Arsac [5], who provided insight into the relationship between a representation π and its associated Fourier space $A_\pi(G)$. Some of the key results are presented in Kaniuth and Lau's book [34, Section 2.8]. As is the case for Fourier algebras, the space $A_\pi(G)$ is the predual of the von Neumann algebra

$$VN_\pi(G) := \pi(G)'' \subseteq \mathscr{B}(\mathscr{H}_\pi) \tag{2.11}$$

in much the same way as in Theorem 2.22. Furthermore, we can turn $A_\pi(G)$ into a $VN_\pi(G)$-module: given $\pi(x) \in VN_\pi(G)$ and $u \in A_\pi(G)$, we set $\pi(x) \cdot u \in A_\pi(G)$ to be

$$[\pi(x) \cdot u](y) := u(yx) \tag{2.12}$$

and we extend this to all $T \in VN_\pi(G)$. Note that this module action is faithful.

In general, $A_\pi(G)$ are *not* subalgebras of $B(G)$. though they are (by construction) closed subspaces. In fact, we can say something stronger. They are precisely the closed translation-invariant subspaces of $B(G)$, and this is due to the fact that they arise from central projections in $VN_\pi(G)$. This result is proved in [34, Lemma 2.8.3].

Theorem 2.25. *Let G be a locally compact group, and π a representation of G. If σ is a subrepresentation of π, then there is a unique central projection $P \in VN_\pi(G)$ such that $A_\sigma(G) = P \cdot A_\pi(G)$. On the other hand, if V is a closed translation invariant subspace of $A_\pi(G)$, then $V = A_\sigma(G)$ for some subrepresentation σ of π.*

> *Remark* 2.26. Care must be taken, this is not an entirely trivial result. When σ is a subrepresentation of π, we may assume without loss of generality that $\mathscr{H}_\sigma$ is a closed invariant subspace of $\mathscr{H}_\pi$, and so there is a very natural projection $R_\sigma \in \mathscr{B}(\mathscr{H}_\pi)$ such that $\mathscr{H}_\sigma = R_\sigma \mathscr{H}_\pi$. Since $\mathscr{H}_\sigma$ is π-invariant, we have that
>
> $$\sigma(x) = R_\sigma \pi(x) = \pi(x) R_\sigma$$

and so it follows that $R_\sigma \in VN_\pi(G)'$. In general however, there is no reason to believe that the central projection $P_\sigma \in VN_\pi(G)$ as given by Theorem 2.25 is the same as the projection R_σ defined above. And indeed, there are many examples where these projections are distinct. In order to see this, we will need to use the upcoming results on disjoint representations.

Consider any representation σ of G. Let us write $\pi = \sigma \oplus \sigma'$, where $\sigma' \approx \sigma$. It is clear that R_σ and $R_{\sigma'}$ as defined above are disjoint. However, using the upcoming Corollary 2.31, this means that $P_\sigma = P_{\sigma'}$. In fact, one can show that in general, $R_\sigma \leq P_\sigma$, and moreover, P_σ is the minimal central projection with this property.

Another important result of these spaces is that their lattice structure under inclusion mimics the quasi-containment lattice of representations. To see this, we first need to define what it means for representations to be disjoint.

Definition 2.27. Let G be a locally compact group, with representations π and σ. We say that π and σ are **disjoint** if they do not contain any subrepresentations which are unitarily equivalent.

Disjoint representations have the rather useful property that their corresponding central projections are also mutually disjoint.

Lemma 2.28. *Let G be a locally compact group, with disjoint representations π and σ. Suppose ρ is a representation which contains both π and σ. If $P_\pi, P_\sigma \in VN_\rho(G)$ are the central projections given by Theorem 2.25, then $P_\pi P_\sigma = 0$.*

Proof. Let $P = P_\pi P_\sigma$, so that $V = P \cdot A_\rho(G)$ is a translation invariant subspace of $A_\rho(G)$. In fact, we have that $V = P_\pi P_\sigma \cdot A_\rho(G) = P_\pi \cdot A_\sigma(G)$, and so V is a translation invariant subspace of $A_\sigma(G)$ (and similarly for $A_\pi(G)$). Using Theorem 2.25, we have that $V = A_\tau(G)$ for some common subrepresentation τ of both π and ρ, but since π and σ are disjoint, it follows that $V = \{0\}$. Since $A_\rho(G)$ is a faithful module, it follows that $P = 0$. □

With this lemma, we may show that disjointedness in representations is equivalent to disjointedness in Fourier spaces. This was also originally due to Arsac; though below we follow the proof of Kaniuth and Lau [34, Lemma 2.8.7].

Proposition 2.29. *Let G be a locally compact group, with representations π and σ. Then π and σ are disjoint if and only if $A_\pi(G) \cap A_\sigma(G) = \{0\}$.*

Proof. Suppose that π and σ are not disjoint. Then there is a subrepresentation ρ of π and τ of σ such that there is a unitary U with $U \circ \rho(x) = \sigma(x) \circ U$. Given any $\xi, \eta \in \mathscr{H}_\rho$, we have that

$$\langle \rho(x)\xi \mid \eta \rangle = \langle U(\rho(x)\xi) \mid U\eta \rangle = \langle \sigma(x) U\xi) \mid U\eta \rangle$$

for all $x \in G$. Thus it follows that $\phi^\rho_{\xi,\eta} \in A_\pi(G) \cap A_\sigma(G)$.

On the other hand, suppose π and σ are disjoint, and set $\rho = \pi \oplus \sigma$. Let $P_\sigma, P_\pi \in VN_\rho(G)$ be the projections given by Theorem 2.25. By Lemma 2.28 it follows that $P_\pi P_\sigma = 0$. So now if $u \in A_\pi(G) \cap A_\sigma(G)$, then there exist $v_\pi, v_\sigma \in A_\rho(G)$ such that $u = P_\pi v_\pi = P_\sigma v_\sigma$. But then we have that

$$u = P_\pi v_\pi = (1 - P_\sigma) P_\pi v_\pi = (1 - P_\sigma) u$$

and so $u = (1 - P_\sigma) P_\sigma v_\sigma = 0$. □

One consequence of this result is that when π and σ are disjoint representations, we have that $A_{\pi\oplus\sigma}(G) = A_\pi(G) \oplus_1 A_\sigma(G)$. We note that '$\oplus_1$' denotes an ℓ^1-direct sum of these spaces in the sense that if $u_1 \in A_\pi(G)$ and $u_2 \in A_\sigma(G)$, then $\|u_1 + u_2\|_{B(G)} = \|u_1\|_{B(G)} + \|u_2\|_{B(G)}$. A proof of this fact can be found in [34, Proposition 2.8.9].

Finally, this leads us to the fact that Fourier spaces are determined uniquely by the quasi-equivalence classes of representations.

Proposition 2.30. *Let G be a locally compact group, with representations π and σ. Then $A_\pi(G) = A_\sigma(G)$ if and only if $\pi \approx_q \sigma$.*

The proof of this is given in [34, Proposition 2.8.12]. Crucially, it relies on the fact that $VN_\pi(G)$ and $VN_\sigma(G)$ are algebraically isomorphic if and only if the representations π and σ are quasi-equivalent. A proof of this fact can be found in the book of Dixmier [16, Proposition 5.3.1]. This also implies that quasi-containment characterises inclusion of these spaces.

Corollary 2.31. *Let G be a locally compact group, with representations π and σ. Then σ is quasi-contained in π if and only if $A_\sigma(G) \subseteq A_\pi(G)$.*

Proof. Combine Proposition 2.30 and Theorem 2.25 and the result follows. □

Remark 2.32. From these results, it follows that $A_\pi(G)$ is a subalgebra if and only if π is quasi-contained in $\pi\otimes\pi$. Further to the point, it shows that there is a correspondence between the quasi-equivalence classes of representations, and the Fourier spaces of G. We shall see this correspondence manifest once again when we study direct integrals and Fell groups in Chapter 5.

2.3.3 The Almost-Periodic Compactification

We now take a brief detour in order to introduce the *almost-periodic compactification*[8] G^{ap} of a locally compact group G. Roughly speaking, this compactification is important as it will allow us to describe a decomposition of $B(G)$ into the 'finite-dimensional' and 'infinite-dimensional' components.

Definition 2.33. We define the **almost-periodic compactification** of a group G to be

$$G^{ap} := \overline{\left\{(\pi(x))_{\pi\in\widehat{G}_f} : x\in G\right\}} \subseteq \prod_{\pi\in\widehat{G}_f} U(\mathscr{H}_\pi) \tag{2.13}$$

and we let $\eta_{ap}^G : G \to G^{ap}$ denote the canonical homomorphism, though we may simply write η_{ap} if the group G is unambiguous.

Note that each $\mathscr{H}_\pi$ in (2.13) is finite-dimensional, so every $U(\mathscr{H}_\pi)$ is compact. By Tychonoff's Theorem, it then follows that G^{ap} is a compact group. Note also that by construction, the homomorphism η_{ap} has dense range.

Remark 2.34. The name '*almost-periodic compactification*' arises from an equivalent definition. We say that $f\in C_b(G)$ is an **almost-periodic function** if the set $\{{}_xf : x\in G\}$ is relatively compact in $C_b(G)$, and let $AP(G)$ denote the collection of all almost-periodic functions. This is a unital commutative Banach algebra, and as such it has an associated spectrum $\Gamma_{AP(G)}$. This spectrum is the weak*-closure of the point evaluations, and has a group structure inherited from G. As it turns out, this group is precisely G^{ap}. In our situation, the definition in Definition 2.33 will prove sufficient, though the intrigued reader may find more details in the book of Kaniuth [33, Section 2.10].

Let us prove a few properties of G^{ap}. First and foremost, compact groups are fixed under the almost-periodic compactification.

[8] This is also known as the **Bohr compactification**, particularly in the abelian setting.

Proposition 2.35. *Let K be a compact group. Then $K^{ap} = K$.*

Proof. This result follows from the Peter-Weyl Theorem. Indeed, we have that $\widehat{K} = \widehat{K}_{\mathfrak{f}}$ and moreover this is a discrete space. Thus it follows that the set $\{(\pi(s))_{\pi\in\widehat{K}_{\mathfrak{f}}} : s \in K\}$ is already closed in $\prod_{\pi\in\widehat{K}} U(\mathscr{H}_\pi)$, and so K^{ap} is isomorphic to K. $\square$

Next, we have a sequence of a few related propositions. The key idea behind these is that homomorphisms between groups 'extend' to homomorphisms between their compactifications. Of course, the term 'extend' is a bit loose as G is not necessarily embedded into G^{ap}, however the principle works much the same.

Proposition 2.36. *Let G, H be locally compact groups. If $\varphi : G \to H$ is a homomorphism with dense range, then there is an induced surjective homomorphism $\varphi_{ap} : G^{ap} \to H^{ap}$ such that $\varphi_{ap} \circ \eta^G_{ap} = \eta^H_{ap} \circ \varphi$.*

Proof. For convenience, let us denote $U_G = \prod_{\pi\in\widehat{G}_{\mathfrak{f}}} U(\mathscr{H}_\pi)$ and $U_H = \prod_{\sigma\in\widehat{H}_{\mathfrak{f}}} U(\mathscr{H}_\sigma)$. Observing that $\widehat{H}_{\mathfrak{f}} \circ \varphi \subseteq \widehat{G}_{\mathfrak{f}}$, it follows that there is a natural compression map from U_G onto $\prod_{\sigma\in\widehat{H}_{\mathfrak{f}}} U(\mathscr{H}_{\sigma\circ\varphi}) = U_H$, which we denote by q. It is easy to verify that $q \circ \eta^G_{ap} = \eta^H_{ap} \circ \varphi$, which gives the commutative diagram

$$\begin{array}{ccccc} G & \overset{\eta^G_{ap}}{\rightsquigarrow} & G^{ap} & \hookrightarrow & U_G \\ {\scriptstyle\varphi}\wr\downarrow & & & & \downarrow\!\!\downarrow{\scriptstyle q} \\ H & \underset{\eta^H_{ap}}{\rightsquigarrow} & H^{ap} & \hookrightarrow & U_H \end{array}$$

where the squiggly arrows indicate homomorphisms with dense range. This in turn induces another homomorphism $\varphi_{ap} : G^{ap} \to H^{ap}$ with dense range. But since G^{ap} is a compact group, it follows that φ_{ap} is surjective. $\square$

One may drop the assumption of dense range, and the above statement would still hold, though of course without surjectivity. In any case, the combination of Propositions 2.35 and 2.36 gives the following universal property of the almost-periodic compactification.

Theorem 2.37. *Let G be a locally compact group. If K is a compact group such that $\varphi : G \to K$ a homomorphism with dense range, then φ factors through η_{ap}. That is, there exists a unique surjective homomorphism $\varphi_{ap} : G^{ap} \to K$ such that $\varphi_{ap} \circ \eta_{ap} = \varphi$.*

This theorem is simply a corollary of Propositions 2.35 and 2.36. As before, one may drop the assumption of dense range to obtain a similar statement without surjectivity. This result can be represented as the commutative diagram

$$\begin{array}{ccc} G & \overset{\eta_{ap}}{\rightsquigarrow} & G^{ap} \\ & \overset{\varphi}{\searrow} & \downarrow \varphi_{ap} \\ & & K \end{array}$$

where we use the convention of squiggly arrows as above. We can use this result to compute almost-periodic compactifications in certain situation, which we state below.

Proposition 2.38. *Let G be a locally compact group, K a compact group, and $\varphi : G \to K$ a homomorphism with dense range. If $\widehat{G}_{\mathfrak{f}} = \widehat{K} \circ \varphi$, then $G^{ap} = K$.*

Proof. Let $\varphi_{ap} : G^{ap} \to K$ be as in Theorem 2.37, and take any $\eta_{ap}(g) \in \ker \varphi_{ap}$. We have that $1 = \varphi_{ap}(\eta_{ap}(g)) = \varphi(g)$, and so $g \in \ker \varphi$. Using Definition 2.33, we can consider $\eta_{ap}(g)$ as a map on $\widehat{G}_{\mathfrak{f}}$, where $\eta_{ap}(g)(\pi) := \pi(g) \in U(\mathscr{H}_\pi)$. Then for any $\pi \in \widehat{G}_{\mathfrak{f}}$, we can find $\sigma \in \widehat{K}$ such that $\pi = \sigma \circ \varphi$, and so

$$\eta_{ap}(g)(\pi) = \pi(g) = \sigma \circ \varphi(g) = 1$$

from which it follows that $\eta_{ap}(g) = 1$. Thus φ_{ap} is injective, and by Proposition 2.35 we have that $G^{ap} = K$. □

Returning to Fourier spaces, we wish to consider the subspaces which correspond to the finite-dimensional as well as the purely infinite representations. These were first introduced by Runde and Spronk in their paper [50], and are defined as follows.

Definition 2.39. Let G be a locally compact group. We let $A_F(G)$ be the closed subspace inside $B(G)$, consisting of all coefficient functions arising from finite-dimensional representations. Similarly, we let $B_\infty(G)$ be the closed span of all coefficient functions arising from purely infinite representations, again inside $B(G)$. We call $B_\infty(G)$ the **purely infinite component** of $B(G)$.

We note that they denote $B_\infty(G)$ by $A_{PIF}(G)$ instead; the notation $B_\infty(G)$ is our own. On the other hand, they do use the notation $A_F(G)$. However, we shall rarely use this notation and instead opt to use $A(G^{ap}) \circ \eta_{ap}$, as is justified by the following result.

Proposition 2.40. *Let G be a locally compact group. Then we have that*

$$A_F(G) = AP(G) \cap B(G) = A(G^{ap}) \circ \eta_{ap}$$

This is stated as Proposition 2.1 in the same paper of Runde and Spronk, and is in part also due to Eymard [19, (2.27) corollarie 4]. Runde and Spronk also show in Theorem 2.3 that the space $B_\infty(G)$ is in fact a closed *ideal* of $B(G)$, which leads to the following decomposition.

Theorem 2.41. *Let G be a locally compact group. Then $B(G)$ admits the decomposition $B(G) = B_\infty(G) \oplus A(G^{ap}) \circ \eta_{ap}$.*

2.4 Amenability

Amenable groups were first defined by von Neumann in the 1920s, and have since become an incredibly important field of study across many disciplines of mathematics. As it is such a broad topic, we shall omit most details, and refer only to properties which are necessary. The inquiring reader may find many excellent resources on the subject, including (but not limited to) the books of Greenleaf [23], Paterson [46], or Runde [49]. Recall that we say a locally compact group G is **amenable** if there exists a left-invariant mean $\mu \in L^\infty(G)^*$. There are many characterisations of this property, and for the most part, we do not deal with amenable groups directly in this thesis. Rather we are interested in the amenability properties of *Banach algebras*.

The study of amenability in Banach algebras was started by Johnson in his celebrated paper [30]. In it, he defines *amenable* Banach algebras and justifies this terminology by connecting it to the usual amenability of groups. Let us introduce these ideas here as well; we shall primarily be following the exposition in the textbook of Runde [49].

Definition 2.42. Let $\mathscr{A}$ be a Banach algebra. We say that a Banach space E is a **left Banach $\mathscr{A}$-module** if E is a left $\mathscr{A}$-module such that there is a constant $M > 0$ with $\|a \cdot x\| \leq M\|a\|\|x\|$ for all $a \in \mathscr{A}$ and $x \in E$.

Similarly, we can define **right Banach $\mathscr{A}$-modules** and **Banach $\mathscr{A}$-bimodules**. We shall use the terminology Banach $\mathscr{A}$-module to refer to any of these three categories.

Given a left Banach $\mathscr{A}$-module E, we can turn its dual space E^* into a right Banach $\mathscr{A}$-module. Namely, given $a \in \mathscr{A}$ and $\varphi \in E^*$, we define

$$\langle \varphi \cdot a, x \rangle := \langle \varphi, a \cdot x \rangle \tag{2.14}$$

for all $x \in E$. Naturally, we also have that if E is a right Banach $\mathscr{A}$-module, then E^* is a left Banach $\mathscr{A}$-module; a similar result holds for Banach $\mathscr{A}$-bimodules. When E^* is equipped with such a module action, we say that E^* is a **dual Banach $\mathscr{A}$-module**.

An important class of mappings in the study of module actions are the *derivations* of $\mathscr{A}$. Recall the following definition, modified to fit our class of objects.

Definition 2.43. Let $\mathscr{A}$ be a Banach algebra and E a Banach $\mathscr{A}$-bimodule. We say that a bounded linear map $D : \mathscr{A} \to E$ is a **derivation** if

$$D(ab) = a \cdot D(b) + D(a) \cdot b$$

for every $a, b \in \mathscr{A}$. For a fixed $x \in E$, the map $\mathrm{ad}_x(a) := a \cdot x - x \cdot a$ is a derivation, and we call all derivations of this form **inner**.

Given this, we have the following definition of amenability in the context of Banach algebras.

Definition 2.44. Let $\mathscr{A}$ be a Banach algebra. If every derivation $D : \mathscr{A} \to E^*$ to a dual Banach $\mathscr{A}$-bimodule E^* is inner, we say that $\mathscr{A}$ is **amenable**.

This amenability notion is precisely what is needed to guarantee the amenability of G. The definition above and result below are both due to Johnson in his aforementioned paper [30].

Theorem 2.45 (Johnson's Theorem)**.** *Let G be a locally compact group. The following are equivalent.*

(i) *G is amenable.*

(ii) *$L^1(G)$ is amenable.*

With this result established, one may begin to wonder whether the amenability properties between a group G and any one of its many other Banach algebras also coincide. If not, what is the equivalent characterising condition? Let us examine this question in the context of the Fourier algebra $A(G)$. For instance, one has the following result due to Leptin [38].

Theorem 2.46 (Leptin's Theorem)**.** *Let G be a locally compact group. The following are equivalent.*

(i) *G is amenable.*

(ii) *$A(G)$ has an approximate identity bounded by 1.*

(iii) *$A(G)$ has a bounded approximate identity.*

As it turns out, amenability is a formally stronger condition than possessing a bounded approximated identity. In fact, amenability is equivalent to possessing a so-called *bounded approximate diagonal*. Let us present some notation so that we may define these bounded approximate diagonals. First, we shall let $\hat{\otimes}$ denote the *projective tensor product*[9]. Moreover, we shall imbue $\mathscr{A} \hat{\otimes} \mathscr{A}$ with the natural Banach $\mathscr{A}$-bimodule structure given by

$$a \cdot (b \otimes c) = ab \otimes c \qquad \text{and} \qquad (b \otimes c) \cdot a = b \otimes ca \tag{2.15}$$

for all $a, b, c \in \mathscr{A}$, and we shall let $\Delta : \mathscr{A} \hat{\otimes} \mathscr{A} \to \mathscr{A}$ denote the **diagonal operator** given by $b \otimes c \mapsto bc$.

Definition 2.47. Let $\mathscr{A}$ be a Banach algebra. We say that a bounded net $(m_\alpha)_\alpha$ in $\mathscr{A} \hat{\otimes} \mathscr{A}$ is a **bounded approximate diagonal** if

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0 \qquad \text{and} \qquad a\,\Delta m_\alpha \to a$$

for all $a \in \mathscr{A}$. Moreover, the smallest bound of an approximate diagonal is called the **amenability constant** of $\mathscr{A}$ and is denoted $AM(\mathscr{A})$.

Equivalently, one may define a *virtual diagonal* on $\mathscr{A}$ instead, see [49, Definition 2.2.2] for details. It was a result, also due to Johnson [29, Theorem 1.3], that the existence of a bounded approximate diagonal is equivalent to the amenability of an algebra. Let us explicitly state it here.

Theorem 2.48. *Let $\mathscr{A}$ be a Banach algebra. The following are equivalent.*

(i) *$\mathscr{A}$ is amenable.*

(ii) *$\mathscr{A}$ possesses a bounded approximate diagonal.*

(iii) *$AM(\mathscr{A}) < \infty$.*

We note that conditions (ii) and (iii) are trivially seen to be equivalent.

[9]We shall not define the projective tensor product (or other tensor products for that matter) here. Chapter 7 from the book of Effros and Ruan [17] provides a suitable introduction.

Remark 2.49. Amenability constants play a surprising role in various structural results of groups. We briefly see (a variation of) one such result in Chapter 5. There has been an effort to compute these values explicitly in certain instances, One particularly notable example is the amenability constant of $A(G)$ for finite groups G, which was computed by Johnson in [31, Theorem 4.1]. This has a surprisingly simple formula, given in terms of the dimensions of the irreducible representations. For the prying reader, this formula is given by

$$AM(A(G)) = \frac{1}{|G|} \sum_{\pi \in \widehat{G}} d_\pi^3 \tag{2.16}$$

where d_π is the dimension of π.

By Theorem 2.48, it is clear that every amenable Banach algebra possesses a bounded approximate identity. In certain classes of examples, these notions are equivalent, for instance when considering ideals of an already amenable Banach algebra (see [49, Proposition 2.2.3]). In general however, this is a stronger requirement; indeed one need only look at $L^1(G)$ for any non-amenable group G.

For Fourier algebras of compact groups, this is indeed a stronger condition. It is a result of Forrest and Runde [21, Theorem 2.3] that when G is a compact group, then $A(G)$ is amenable precisely when G has an abelian subgroup of finite index. When this is the case, we say that G is **virtually abelian**, in which case G is necessarily amenable (though the converse is not true). We examine related results in Chapter 6.

Therefore, it is not the case that amenability coincides with G and $A(G)$, where the latter is regarded as a Banach algebra. One may then ask if it is perhaps more pertinent to observe $A(G)$ not as a Banach algebra, but as a different structure (i.e. in a different category). In particular, when we view $A(G)$ as a *completely contractive Banach algebra*, it turns out that amenability in this setting is equivalent to amenability of the underlying group. The study of such spaces belongs to the field of *operator spaces*, and we refer the inquisitive reader to the book of Effros and Ruan [17], which provides a comprehensive introduction and reference to the subject. In particular, a **completely contractive Banach algebra** is a Banach algebra $\mathscr{A}$ with a compatible operator space structure, such that the multiplication map $\mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is completely contractive.

Let us provide a *very* brief explanation on how one defines amenability in this context. Roughly speaking, this is done by replacing every instance of the words *"Banach space"* with *"operator space"* and *"bounded"* with *"completely bounded"*. We call this new condition **operator amenability**. One can also define these using approximate *operator* diagonals (which are nets in the *operator* projective tensor product), as well as the **oper-**

ator amenability constant $AM_{op}(\mathscr{A})$. In much the same way as in the Banach case, the existence of these approximate operator diagonals is equivalent to operator amenability. With this new language, we have the following result due to Ruan [48, Theorem 3.6].

Theorem 2.50. *Let G be a locally compact group. The following are equivalent.*

(i) *G is amenable.*

(ii) *$A(G)$ is operator amenable.*

2.5 Group Actions

2.5.1 Actions on a Space

We now turn our attention to group actions. It is presumed the reader is familiar with the elementary definitions and results, but let us restate these and introduce the notations adopted in this thesis. Given a topological space X, we shall denote the group of homeomorphisms of X by $\mathrm{Homeo}(X)$, and imbue it with the compact-open topology.

Definition 2.51. Let G be a locally compact group, and X a topological space. We say G **acts on** the topological space X if there is a (continuous) homomorphism G to $\mathrm{Homeo}(X)$. When this is the case, we write $G \curvearrowright X$.

There will frequently be multiple such homomorphisms from G to $\mathrm{Homeo}(X)$. However, when we write $G \curvearrowright X$, we shall implicitly fix a homomorphism $\pi : G \to \mathrm{Homeo}(X)$.[10] In such a case we shall write $g \cdot x := \pi(g)(x)$, removing explicit mention of the homomorphism π.

Two important properties of group actions are *orbits* and *stabilisers*.

Definition 2.52. Let G be a group action on a topological space X. For a fixed $x \in X$, we denote the **orbit** of x by

$$[x]_G := G \cdot x = \{g \cdot x : g \in G\}$$

Furthermore, if we let $\sim_G$ denote the corresponding equivalence relation, we define the **orbit space** of $G \curvearrowright X$ by

$$\mathscr{O}_G(X) := X/\sim_G$$

and we imbue this space with the final topology.

[10] Note that π need not be injective.

It will also occasionally be useful to write $[Y]_G$ for subsets $Y \subseteq X$ as the union of all orbits which meet Y. We naturally have a (continuous and surjective) projection map $q_G : X \to \mathbb{O}_G(X)$, and since this induced from a group action, this will be an open map as well. Note that this is *not* true for arbitrary projections.

Proposition 2.53. *Let $G \curvearrowright X$. Then the quotient map $q_G : X \to \mathbb{O}_G(X)$ defined by $q_G(x) = [x]_G$ is an open map.*

Proof. Let $U \subseteq X$ be open so that $[U]_G$ is open if and only if ${q_G}^{-1}([U]_G)$ is open. Thus ${q_G}^{-1}([U]_G) = G \cdot U = \bigcup_{g \in G} g \cdot U$, which is a union of open sets. Hence $[U]_G$ is open and so q_G is an open map. □

As mentioned prior, we also have stabilisers of a group actions.

Definition 2.54. Let $G \curvearrowright X$. We define the **stabiliser** of $x \in X$ as the subgroup of G which fixes x

$$G_x := \{g \in G : g \cdot x = x\}$$

Similar to the orbits, we shall occasionally write G_Y for $Y \subseteq X$ to denote the subgroup of G which fixes all of Y (so that $G_Y = \bigcap_{y \in Y} G_y$). In general, the stabiliser group G_x is not necessarily a normal subgroup of G. Nonetheless, there is a canonical map from G/G_x to $[x]_G$ which relates these concepts.

Proposition 2.55. *Let $G \curvearrowright X$, and fix $x \in X$. There is a well defined map $\varphi : G/G_x \to [x]_G$ given by $\varphi(aG_x) = a \cdot x$ for $a \in G$ which is a continuous bijection.*

Proof. First observe that for any $a, b \in G$, we have $\varphi(aG_x) = \varphi(bG_x)$ precisely when $b^{-1}a \in G_x$, so that φ is both well defined and injective. Surjectivity follows trivially. Finally, recall that the canonical quotient map $q : G \to G/G_x$ is an open map. So if we let $\psi : G \to X$ denote the (continuous) map $a \mapsto a \cdot x$, it follows that $\varphi \circ q = \psi$ and so φ is continuous as well. □

In general, this map is not a homeomorphism. For instance one may consider discrete actions on non-discrete spaces.

Example 2.56. Let $\mathbb{R}_d$ denote the reals with the discrete topology. Let $\mathbb{R}_d$ act on $\mathbb{R}$ by translation. Then, for any $x \in \mathbb{R}$, we have that $(\mathbb{R}_d)_x = \{0\}$. This means that the map φ as in Proposition 2.55 maps $\mathbb{R}_d \to \mathbb{R}$, and is clearly not open.

There are also less trivial examples.

Example 2.57. Fix any irrational point $z \in \mathbb{T}$ (so that z^n is never 1 for $n \in \mathbb{Z} \setminus \{0\}$) and define the group action $\mathbb{Z} \curvearrowright \mathbb{T}$ by $n \cdot x = z^n x$ for $x \in \mathbb{T}$. Since z is an irrational point, it follows that $\mathbb{Z}_x = \{0\}$ for any $x \in \mathbb{T}$. However one may verify that $[x]_{\mathbb{Z}}$ is always dense in $\mathbb{T}$, which is clearly not discrete.

There is one particularly important instance when this map is homeomorphic, namely when G is compact.

Proposition 2.58. *Let $K \curvearrowright X$ where K is a compact group, and fix $x \in X$. Then the map $\varphi : K/K_x \to [x]_K$ as given by Proposition 2.55 is a homeomorphism.*

Proof. This follows by the well-known result which states that if $f : X \to Y$ is a bijective continuous function from a compact space X to a Hausdorff space Y, then it is automatically a homeomorphism. □

2.5.2 The Semidirect Product

Suppose now that the locally compact group G acts on another locally compact group H. In order to align this action with the group structure of H, let us suppose that each $g \in G$ is not only a homeomorphism, but an automorphism of H as well. In other words, we have a (continuous) homomorphism from $G \to \mathrm{Aut}(H)$, where $\mathrm{Aut}(H)$ is the collection of automorphisms on H. We again denote this by $G \curvearrowright H$. Note that we also imbue $\mathrm{Aut}(H)$ with compact-open topology.

Given two actions $G_1 \curvearrowright H_1$ and $G_2 \curvearrowright H_2$, we say they are **equivalent** if there exist isomorphisms $\psi : G_1 \to G_2$ and $\pi : H_1 \to H_2$ such that $\psi(g) \cdot \pi(h) = \pi(g \cdot h)$ for all $g \in G_1$ and $h \in H_1$. Alternatively, this also occurs precisely when the diagram

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi_1} & \mathrm{Aut}\, H_1 \\ \psi \downarrow \wr & & \pi^* \downarrow \wr \\ G_2 & \xrightarrow{\varphi_2} & \mathrm{Aut}\, H_2 \end{array}$$

commutes, where we define $\pi^* : \mathrm{Aut}(H_1) \to \mathrm{Aut}(H_2)$ by $\pi^*(f) = \pi \circ f \circ \pi^{-1}$. Naturally, equivalent actions behave as one would expect, for instance they share the same orbit structures and topological properties.

We now wish to construct a group which encodes all the information in this group action.

Definition 2.59. Let H and G be locally compact groups, with $G \curvearrowright H$. The **semidirect product** $H \rtimes G$ is the group whose underlying set is $H \times G$, along with the product

$$(h, g)\ (h', g') := (h(g \cdot h'), gg') \tag{2.17}$$

for all $g, g' \in G$ and $h, h' \in H$.

> *Remark* 2.60. It is perhaps not immediately clear why one would use (2.17) to define $H \rtimes G$, nor how this 'encodes' the structure of the group action. However, recall that for any group S, there is a natural action of S on itself given by conjugations. So in order to realise the action of $G \curvearrowright H$ inside $S = H \rtimes G$ via conjugations, it would be ideal if $ghg^{-1} = g \cdot h$ for all $h \in H$ and $g \in G$. As we shall soon see, the group S as defined above does indeed implement the action in this manner.

It is a fairly standard exercise to check that this is indeed a group. In particular, one can compute the inverse to find that

$$(h, g)^{-1} = (g^{-1} \cdot h^{-1}, g^{-1}) \tag{2.18}$$

for all $h \in H$, $g \in G$. Now, it is clear that $S = H \rtimes G$ contains both H and G as subgroups with

$$H = \{(h, 1) : h \in H\}$$
$$G = \{(1, g) : g \in G\}$$

which we shall adopt as a convention. In particular, writing $h = (h, 1)$ and $g = (1, g)$, one observes that (2.17) can be expressed as $hgh'g' = h(g \cdot h')gg'$ and so it follows that

$$g \cdot h = ghg^{-1} \tag{2.19}$$

showing that the action $G \curvearrowright H$ is indeed observed via group conjugation in $H \rtimes G$. Furthermore, this shows that $S = HG$, and moreover H is normal in S with $S/H = G$. More generally, we have the following characterisation of normality inside S.

Proposition 2.61. *Let $S = H \rtimes G$. If N is a closed subgroup of H, then N is normal in S if and only if N is G-invariant and normal in H.*

Proof. If N is normal in S, then for any $h \in H$, we have $hNh^{-1} = N$, and so N is normal in H. Likewise, for any $g \in G$, we have $g \cdot N = gNg^{-1} = N$, and so N is G-invariant. The reverse direction follows similarly (noting that $S = HG$). □

Now, given an action $G \curvearrowright H$ and the corresponding semidirect product $S = H \rtimes G$, there is an implicit action of $S \curvearrowright H$ which extends $G \curvearrowright H$ simply by setting $s \cdot h = shs^{-1}$ for all $h \in H$ and $s \in S$. There is a strong correspondence between these actions, and this may be realised through the use of *conjugacy classes* and *centralisers* of a group. Recall that one defines the **conjugacy class** of $E \subseteq G$ (with respect to G) to be

$$\mathrm{Cl}_G(E) := \{gxg^{-1} : x \in E, g \in G\} \tag{2.20}$$

and the **centraliser** of E (with respect to G) to be

$$\mathrm{Z}_G(E) := \{g \in G : gxg^{-1} = x \text{ for all } x \in E\} \tag{2.21}$$

In the setting where $S = H \rtimes G$, it is then readily checked that

$$\begin{aligned} [h]_S &= \mathrm{Cl}_S(h) & \qquad S_h &= \mathrm{Z}_S(h) \\ [h]_G &= \mathrm{Cl}_G(h) & \qquad G_h &= \mathrm{Z}_G(h) \end{aligned}$$

for all $h \in H$. Naturally the corresponding equations hold for subsets as well. This leads nicely to the following proposition.

Proposition 2.62. *Let $S = H \rtimes G$. If $E \subseteq H$, then we have that $[E]_S = \mathrm{Cl}_H([E]_G)$ and $S_E = \mathrm{Z}_H(G_E)$.*

Proof. By observing that in general $\mathrm{Cl}_{HG}(\,\cdot\,) = \mathrm{Cl}_H(\mathrm{Cl}_G(\,\cdot\,))$ and $\mathrm{Z}_{HG}(\,\cdot\,) = \mathrm{Z}_H(\mathrm{Z}_G(\,\cdot\,))$, the result immediately follows. □

2.5.3 Dual Actions

The vast majority of group actions that we will be studying will be of the form $K \curvearrowright A$ where K is a compact group and A an abelian locally compact group. Let us examine each of these restrictions individually. We begin with the action of $G \curvearrowright A$, where A is abelian. As always, we let $S = A \rtimes G$, and in this case, the conjugacy classes and centralisers are always trivial in the sense that

$$\mathrm{Cl}_A(E) = E \qquad \text{and} \qquad \mathrm{Z}_A(E) = A$$

for every $E \subseteq A$. In particular, Proposition 2.62 gives

$$[a]_S = [a]_G \qquad \text{and} \qquad S_a = AG_a$$

for all $a \in A$. In particular, we have that orbits with respect to either of the actions of $G \curvearrowright A$ or $S \curvearrowright A$ are identical, and furthermore $S_x = A \rtimes G_x$. So these action are almost identical, in the sense that

$$(ag) \cdot b = a(g \cdot b)a^{-1} = g \cdot b$$

for every $ag \in S$ and $b \in A$.

As an abelian locally compact group, A also possesses a dual group $\widehat{A}$.

Definition 2.63. Let $G \curvearrowright A$ where A is an abelian locally compact group. Then the induced action $G \curvearrowright \widehat{A}$ given by

$$\langle g \cdot \varphi, a \rangle = \langle \varphi, g^{-1} \cdot a \rangle$$

for every $g \in G$, $\varphi \in \widehat{A}$ and $a \in A$ is called the **dual action** of $G \curvearrowright A$.

The inversion of g here is to ensure that this remains a *left* action. For convenience, we shall extend this notion to arbitrary representations of A as well. Given a representation π of A, we define $g \cdot \pi$ for $g \in G$ by

$$g \cdot \pi(a) = \pi(g^{-1} \cdot a) \tag{2.22}$$

for $a \in A$. Again, the inversion on g is there in order to ensure that this a left action.

2.5.4 Determinant of an Automorphism

As alluded to previously, we now consider the action of a compact group K on a locally compact group H. The most salient feature of such actions is a measure preserving property, akin to unimodularity of a group. We formalise this with the determinant map δ_G on the automorphisms $\mathrm{Aut}(G)$ of a locally compact group G. In particular, given an $\alpha \in \mathrm{Aut}(G)$, we have that $\mathbf{m} \circ \alpha$ is also Haar measure, and thus it is a scalar multiple of $\mathbf{m}$. We give this scalar a special name.

Definition 2.64. Let G be a locally compact group, and α an automorphism of G. If c is the unique constant for which $\mathbf{m} \circ \alpha = c\mathbf{m}$, then we say that c is **determinant** of α and we write $\delta_G(\alpha) = c$.

When G is unambiguous, we shall often write $\delta(\alpha) = \delta_G(\alpha)$. Naturally, if $X \subseteq G$ is any Borel set, we have that

$$\mathbf{m}(\alpha(X)) = \delta(\alpha)\mathbf{m}(X) \tag{2.23}$$

and from this it is fairly trivial to see that δ is multiplicative, in the sense that $\delta(\alpha \circ \beta) = \delta(\alpha)\delta(\beta)$ for any $\alpha, \beta \in \mathrm{Aut}(G)$. In other words, when $N \curvearrowright G$, we have that $\delta_G : N \to \mathbb{R}_{\geq 0}$ is a (multiplicative) homomorphism. Notice as well that this coincides with the definition of the modular function, namely $\Delta(x) = \delta_G(y \mapsto x^{-1}yx)$ for $x, y \in G$. As a result, this allows for easy computation of certain integrals.

Lemma 2.65. *Let G be a locally compact group, and $\alpha \in \mathrm{Aut}(G)$. If $f \in L^1(G)$, then*

$$\int_G f(\alpha(x))\,\mathrm{d}x = \delta(\alpha) \int_G f(x)\,\mathrm{d}x$$

Proof. This proof follows the usual proof for the modular function, indeed simply use (2.23) on simple functions, and use the density of these in $L^1(G)$. ☐

As a result, we can show that this determinant is continuous. We again follow a similar proof as used for the modular function, see [20, Propositions 2.24].

Proposition 2.66. *Let G be a locally compact group. Then the determinant $\delta : \mathrm{Aut}(G) \to \mathbb{R}_{>0}$ is a continuous homomorphism.*

Proof. We have already seen that δ is a homomorphism. If we can show that the map $\alpha \mapsto f \circ \alpha$ is continuous for any $f \in L^1(G)$, then Lemma 2.65 completes the proof. By density, we may fix $f \in C_c(G)$. Then it suffices to show that for any net $\alpha_\lambda \in \mathrm{Aut}(G)$ such that $\alpha_\lambda \to 1_G$ where 1_G is the trivial automorphism, then $f \circ \alpha_\lambda \to f$ in uniform norm.

So to this end, let $K = \operatorname{supp} f$ and fix $\varepsilon > 0$. Choose $U_1, \ldots, U_n$ to be a finite cover of K, each with the property that $|f(x) - f(y)| < \varepsilon$ for all $x, y \in U_i$. With a little bit of work, one may show that there then exists compact sets $K_1, \ldots K_m$ which also cover K, such that each K_j is contained in some U_{i_j}. Notice of course that $1_G(K_j) \subseteq U_{i_j}$, and so by the compact-open topology, we have that $\alpha_\lambda(K_j) \subseteq U_{i_j}$ for all sufficiently large λ. So now if $x \in K$, then $x \in K_j \subseteq U_{i_j}$ for some j, and so it follows that $\alpha_\lambda(x) \in U_{i_j}$. Hence $|f(\alpha_\lambda(x)) - f(x)| < \varepsilon$, and so we have that $f \circ \alpha_\lambda \to f$ in the uniform norm. ☐

Naturally, when K is a compact group acting on G, the determinant function is constantly one on K. Indeed this follows since $\delta(K)$ is compact, and the only compact subgroup of $\mathbb{R}_{\geq 0}$ is $\{1\}$. In general, when $\delta(K)$ is identically 1, we say that K is **special**. In particular, this gives us the following corollary.

Corollary 2.67. *Let G be a locally compact group, and K a compact (or in general special) group acting on G. If $f \in L^1(G)$, then*

$$\int_G f(k \cdot x)\,\mathrm{d}x = \int_G f(x)\,\mathrm{d}x$$

for any $k \in K$.

We can also use this to compute the modular function for semidirect products.

Proposition 2.68. *Let $S = H \rtimes K$ where K is a compact group. Then $\Delta_S(hk) = \Delta_H(h)$.*

Proof. It is known for semidirect products (see Hewitt and Ross [25, Section (15.29)]) that the modular function Δ_S takes the form

$$\Delta_S(hk) = \frac{\Delta_H(h)\Delta_K(k)}{\delta_H(k)}$$

and since $\Delta_K(K) = \delta_H(K) = 1$, it follows that $\Delta_S(hk) = \Delta_H(h)$. □

In particular, this means that if K acts on a unimodular group (for instance, an abelian group), then the semidirect product is itself unimodular.

Chapter 3

Local Rings & DVRs

" Chickens are simply the union
of hens and roosters. "
–E. Séguin

In this chapter, we briefly introduce local rings and more specifically discrete valuation rings (DVRs). This will primarily be background material, and the only new theorem that we introduce is Theorem 3.21; the rest consists of known results. We follow a variety of sources, though the books of Atiyah and Macdonald [6, Chapters 3 & 9], Serre [52, Chapter 1], or Singh [53, Chapter 16] prove to be good introductions on local rings and DVRs. Though they each provide a different perspective on the subject matter, they are all suitable reference for this chapter.

3.1 Local Rings

In this section, $\mathscr{R}$ will always denote a commutative ring (and always with identity).

Definition 3.1. Let $\mathscr{R}$ be a commutative ring. We say that $\mathscr{R}$ is a **local** ring if it has a unique non-trivial[11] maximal ideal. We denote this maximal ideal as $\mathscr{M}$.

[11]Clearly, if the only ideal is trivial, then the ring $\mathscr{R}$ is in fact a field. Non-triviality hence merely implies our ring is not a field. We note that most authors do not make this assumption.

In general, we shall let $\mathscr{R}^*$ denote the units of $\mathscr{R}$. When $\mathscr{R}$ is a local ring, we can succinctly describe $\mathscr{R}^*$ using $\mathscr{M}$. In fact, this formulation completely characterises local rings within all commutative rings.

Proposition 3.2. *Let $\mathscr{R}$ be a commutative ring and $\mathscr{M}$ an ideal of $\mathscr{R}$. Then $\mathscr{R}$ is a local ring with maximal ideal $\mathscr{M}$ if and only if $\mathscr{R}$ satisfies $\mathscr{R}^* = \mathscr{R} \setminus \mathscr{M}$.*

Proof. If $\mathscr{R}^* = \mathscr{R} \setminus \mathscr{M}$, then it is clear that $\mathscr{M}$ is the unique maximal ideal as any larger ideal must contain a unit and hence be all of $\mathscr{R}$. On the other hand, suppose $\mathscr{R}$ is a local ring. If $u \in \mathscr{R}^*$, then u cannot belong to any ideal, and hence $u \notin \mathscr{M}$. Moreover, if $r \notin \mathscr{R}^*$, then Zorn's lemma shows that r must belong to some maximal ideal, and hence $r \in \mathscr{M}$. Thus $\mathscr{R}^* = \mathscr{R} \setminus \mathscr{M}$. □

Given two ideals I and J, we define their **product** to be the ideal generated by all products xy such that $x \in I$ and $y \in J$. In other words, we set

$$I \cdot J := \langle \{xy : x \in I, y \in J\} \rangle \tag{3.1}$$

which is by construction an ideal contained in both I and J. Returning to local rings, we can extend this idea to ideal powers: for any $n \in \mathbb{N}$ we set $\mathscr{M}^n := \mathscr{M} \cdot \ldots \cdot \mathscr{M}$ with the understanding that $\mathscr{M}^0 = \mathscr{R}$. Note that we have $\mathscr{M}^0 \supseteq \mathscr{M} \supseteq \mathscr{M}^2 \supseteq \ldots$, though there is currently no reason to believe that these containments are proper. We can then use this sequence to construct a 'hierarchy' of elements in our ring.[12]

Definition 3.3. For a local ring $\mathscr{R}$ and $x \in \mathscr{R}$, we define the **order** of x to be

$$\operatorname{ord}(x) := \sup\{n \in \mathbb{N} : x \in \mathscr{M}^n\}$$

For the sake of clarity, when $x \notin \mathscr{M}$ we say x has order 0, and in the case where $x \in \bigcap_{n=0}^{\infty} \mathscr{M}^n$, we say that x has order ∞. This ordering has the following properties.

Proposition 3.4. *Let $\mathscr{R}$ be a local ring. For any $x, y \in \mathscr{R}$, and $u \in \mathscr{R}^*$, the order function satisfies the following properties:*

(O1) $\operatorname{ord}(x) \geq 0$,

(O2) $\operatorname{ord}(ux) = \operatorname{ord}(x)$,

(O3) $\operatorname{ord}(x + y) \geq \min\{\operatorname{ord}(x), \operatorname{ord}(y)\}$, *and*

[12]While the construction presented in this section is fairly well known, one may refer to [53, Section 8.2] for additional details.

(O4) $\operatorname{ord}(xy) \geq \operatorname{ord}(x) + \operatorname{ord}(y)$.

Proof. These primarily follow from the definition and basic properties of an ideal. We note that for property (O4), if $x \in \mathscr{M}^n$ and $y \in \mathscr{M}^k$, then we have that $xy \in \mathscr{M}^{n+k}$, though it of course may also be in a higher power. □

This order function allows us to define a *pseudo-metric* on the local ring $\mathscr{R}$.

Definition 3.5. Given a set X, we say $d : X \times X \to \mathbb{R}$ is a **pseudo-metric** if for all $x, y, z \in X$ it satisfies:

(i) $d(x, y) \geq 0$ with $d(x, x) = 0$,

(ii) $d(x, y) = d(y, x)$, and

(iii) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

Furthermore if d satisfies the **strong triangle inequality**:

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}$$

we say that d is a **pseudo-ultrametric**.

Notice of course, that if we impose the additional condition that if $d(x, y) = 0$ then $x = y$, we get our usual definition of a metric. In much the same way, every pseudo-(ultra)metric space has an associated topology, defined in exactly the same manner as one would define it for a metric space. In particular, a pseudo-ultrametric space has the rather nice property that the balls are always clopen sets.

Lemma 3.6. *Let d be a pseudo-ultrametric on a space X. Then the balls of (X, d) are clopen. In particular, the basis consists of clopen sets.*

Proof. Let $x \in X$, $r > 0$, and consider the open ball $\mathrm{B}(x; r)$. Take any $y \notin \mathrm{B}(x; r)$, and we claim that $\mathrm{B}(x; r) \cap \mathrm{B}(y; r) = \varnothing$. Indeed, suppose there was such a z in the intersection. It would follow that

$$d(x, y) \leq \max\{d(x, z), d(z, y)\} < r$$

which is clearly a contradiction. This means we can write

$$X \setminus \mathrm{B}(x; r) = \bigcup_{y \notin \mathrm{B}(x;r)} \mathrm{B}(y; r)$$

and hence $\mathrm{B}(x; r)$ is closed. □

Returning to a local ring $\mathscr{R}$, we can imbue $\mathscr{R}$ with a pseudo-ultrametric structure using the ord function in Definition 3.3. The following construction is partially inspired by the method presented by Singh in [53, Section 8.2].

Proposition 3.7. *Given a local ring $\mathscr{R}$, the order function induces a pseudo-ultrametric given by*

$$d_{\mathscr{M}}(x, y) = 2^{-\operatorname{ord}(x-y)}$$

Proof. Positivity and symmetry are clear. All that remains to check is the strong triangle inequality. Given $x, y, z \in \mathscr{R}$, it follows that

$$\begin{aligned} d_{\mathscr{M}}(x, z) &= 2^{-\operatorname{ord}(x-z)} \\ &= 2^{-\operatorname{ord}(x-y+y-z)} \\ &\leq 2^{-\min\{\operatorname{ord}(x-y)+\operatorname{ord}(y-z)\}} \\ &= \max\{2^{-\operatorname{ord}(x-y)}, 2^{-\operatorname{ord}(y-z)}\} \\ &= \max\{d_{\mathscr{M}}(x, y), d_{\mathscr{M}}(y, z)\} \end{aligned}$$

□

We refer to the topology generated by the pseudo-ultrametric $d_{\mathscr{M}}$ as the **$\mathscr{M}$-adic** topology on $\mathscr{R}$. Recall that this topology is generated by the basis of all (open) balls. Fortunately, it is very easy to describe all the balls of the $\mathscr{M}$-adic topology .

Lemma 3.8. *Let $\mathscr{R}$ be a local ring. If $r > 0$ and $z \in \mathscr{R}$, then $\mathrm{B}(z; r) = z + \mathscr{M}^n$ for some $n \in \mathbb{N}$.*

Proof. We start with the case where $z = 0$. Choose $n \in \mathbb{N}$ to be the (unique) integer such that so that $2^{-n} \leq r < 2^{-(n-1)}$. We claim that $\mathrm{B}(0; r) = \mathscr{M}^n$. Indeed if $x \in \mathrm{B}(0; r)$, then $2^{-\operatorname{ord}(x)} \leq r < 2^{-(n-1)}$. From this it follows that $\operatorname{ord}(x) > n - 1$, and so $x \in \mathscr{M}^n$. On the other hand if $x \notin \mathrm{B}(0; r)$, then $2^{-\operatorname{ord}(x)} > r \geq 2^{-n}$. It follows that $\operatorname{ord}(x) < n$ and so $x \notin \mathscr{M}^n$. The result for arbitrary $z \in \mathscr{R}$ follows by the observation that $d(x - z, 0) = d(x, z)$. □

Corollary 3.9. *Let $\mathscr{R}$ be a local ring. Then $z + \mathscr{M}^n$ is clopen for every $n \in \mathbb{N}$ and $z \in \mathscr{R}$.*

Note that there is a small amount of work to prove this corollary in that we strictly speaking have not shown that every $\mathscr{M}^n$ arises from some ball; however one can adapt the above proof to see that this must always be possible.

Due to Lemma 3.8, the $\mathscr{M}$-adic topology also behaves nicely under standard ring operations. To show this, we characterise convergence in the following manner.

Lemma 3.10. *Let $\mathscr{R}$ be a local ring. Then a net $(x_\alpha)_\alpha$ converges to x if and only if $(\operatorname{ord}(x_\alpha - x))_\alpha \to \infty$.*

Proof. This is a fairly routine calculation; one merely notes that $d(x, x_\alpha) = 2^{-\operatorname{ord}(x - x_\alpha)}$. □

Proposition 3.11. *Let $\mathscr{R}$ be a local ring. Then the standard ring operations (addition, negation, and multiplication) are continuous in the $\mathscr{M}$-adic topology.*

Proof. Let $(x_\alpha)_\alpha$ and $(y_\beta)_\beta$ be nets converging to $x, y \in \mathscr{R}$ respectively. Let us use Proposition 3.4 to show that convergence is preserved under ring operations. For addition, we have that

$$\operatorname{ord}((x_\alpha + y_\beta) - (x - y)) \geq \min\{\operatorname{ord}(x_\alpha - x), \operatorname{ord}(y_\beta - y)\}$$

and so $(x_\alpha + y_\beta)$ converges to $x + y$. Hence addition is continuous. In a similar fashion, it is clear that $(-x_\alpha)$ converges to $-x$.

Lastly, for multiplication, we find

$$\begin{aligned}
\operatorname{ord}(x_\alpha y_\beta - xy) &= \operatorname{ord}((x_\alpha - x)y_\beta + x(y_\beta - y)) \\
&\geq \min\{\operatorname{ord}((x_\alpha - x)y_\beta), \operatorname{ord}(x(y_\beta - y))\} \\
&= \min\{\operatorname{ord}(x_\alpha - x) + \operatorname{ord}(y_\beta), \operatorname{ord}(x) + \operatorname{ord}(y_\beta - y)\} \\
&\geq \min\{\operatorname{ord}(x_\alpha - x), \operatorname{ord}(y_\beta - y)\}
\end{aligned}$$

and so $\operatorname{ord}(x_\alpha y_\beta - xy) \to \infty$. Hence multiplication is continuous as well. □

Henceforth whenever we refer to a local ring $\mathscr{R}$, we shall implicitly assume this $\mathscr{M}$-adic topology on $\mathscr{R}$. As we have just seen, this turns $\mathscr{R}$ into a topological ring. However notice that this topology need not be Hausdorff. In fact, it is Hausdorff if and only if $d_{\mathscr{M}}$ is a genuine metric. Furthermore, from the definition of $d_{\mathscr{M}}$, it is clear that this occurs precisely when the only element of order ∞ is 0 itself. A theorem of Krull [37] gives a nice description of sufficient conditions for this to occur. First recall the following definitions.

Definition 3.12. Let $\mathscr{R}$ be a commutative ring.

- We say that $\mathscr{R}$ is **Noetherian** if every ideal is finitely generated.
- We say that $\mathscr{R}$ is an **(integral) domain** if it has no non-zero zero divisors.

In particular, if $\mathscr{R}$ is a local ring satisfying both of the above conditions, then we say that $\mathscr{R}$ is a **local Noetherian domain**.

Theorem 3.13 (Krull's Intersection Theorem)**.** *Let $\mathscr{R}$ be a local Noetherian domain. Then*

$$\bigcap_{n=0}^{\infty} \mathscr{M}^n = \{0\}$$

See Eisenbud [18, Corollary 5.4] for a proof. It follows that a local Noetherian domain is always a Hausdorff ring. Note also that the integral domain assumption implies that the chain of ideals $\mathscr{M} \supseteq \mathscr{M}^2 \supseteq \dots$ is also a strictly decreasing sequence.

Proposition 3.14. *Let $\mathscr{R}$ be a local Noetherian domain. Then the containments $\mathscr{M} \supseteq \mathscr{M}^2 \supseteq \dots$ are all proper.*

Proof. Let $n \in \mathbb{N}$ be the minimal integer such that $\mathscr{M}^n = \mathscr{M}^{n+1}$. By Krull's intersection theorem we have that $\mathscr{M}^n = \{0\}$. It follows then that $ab = 0$ for every $a \in \mathscr{M}$ and $b \in \mathscr{M}^{n-1}$. Now $\mathscr{M} \neq \{0\}$ (otherwise $\mathscr{R}$ would be a field), and $\mathscr{M}^{n-1} \neq \{0\}$ (this would contradict minimality of n), and so this implies the existence of zero divisors. This contradicts the integral domain assumption. $\square$

Recall that a topological space X is T_1 if for any two distinct points $x, y \in X$, there is some open U such that $x \in U$ and $y \notin U$. It is well known that all Hausdorff spaces are T_1. Furthermore, if the basis is clopen, the space must be totally disconnected.

Lemma 3.15. *Let X be a T_1 topological space, and suppose that X has a clopen basis. Then X is totally disconnected.*

Proof. Take any two distinct points $x, y \in X$. Since X is T_1, we can find an open set U such that $x \in U$ and $y \notin U$. Write $U = \bigcup_{i \in I} B_i$, where B_i are members of our clopen basis. We can then find some $i \in I$ so that $x \in B_i$. Hence B_i and $X \setminus B_i$ are clopen sets which separate $\{x, y\}$. So X is totally disconnected. $\square$

Corollary 3.16. *Let d be an ultrametric on a space X. Then the topology generated by d on X is totally disconnected.*

Proof. This follows from Lemmas 3.6 and 3.15. $\square$

Corollary 3.17. *Let $\mathscr{R}$ be a local Noetherian domain. Then $\mathscr{R}$ is totally disconnected.*

A key interest in this thesis will be the action $\mathscr{R}^* \curvearrowright \mathscr{R}$, or variations of it. Specifically, this action is the multiplicative action of $\mathscr{R}^*$ acting on the additive group of $\mathscr{R}$. In particular, we will be interested in the orbit structure of this action. With this in mind, it is fairly natural to define the *level sets* of the order function by

$$\mathscr{C}_n := \{x \in \mathscr{R} : \operatorname{ord}(x) = n\} \tag{3.2}$$

for $n \in \mathbb{N} \cup \{\infty\}$. Notice that for $n \in \mathbb{N}$ we have $\mathscr{C}_n = \mathscr{M}^n \setminus \mathscr{M}^{n+1}$, and otherwise $\mathscr{C}_\infty = \bigcap_{n=0}^{\infty} \mathscr{M}^n$. In particular we also have that $\mathscr{C}_0 = \mathscr{R}^*$.

Lemma 3.18. *Let $\mathscr{R}$ be a local Noetherian domain. The sets $\mathscr{C}_n$ for $n \in \mathbb{N}$ are invariant under the action of $\mathscr{R}^*$.*

Proof. This follows from (O2), where we have that the ord function is invariant under $\mathscr{R}^*$, and so $\mathscr{C}_n$ is also invariant. □

Topologically, these sets are also quite nice.

Lemma 3.19. *Let $\mathscr{R}$ be a local Noetherian domain. Then for $n \in \mathbb{N}$, the set $\mathscr{C}_n$ is clopen.*

Proof. This follows since $\mathscr{C}_n = \mathscr{M}^n \setminus \mathscr{M}^{n+1}$, and by Corollary 3.9, each $\mathscr{M}^n$ is clopen. □

Notice however this is not necessarily true for $\mathscr{C}_\infty$. Indeed, for a local Noetherian domain, $\mathscr{C}_\infty$ is the singleton $\{0\}$, and so in general will not be open.

So now there is a partition of $\mathscr{R}$ consisting of subsets invariant under the action of $\mathscr{R}^*$. So the orbits of this action must each be contained in some $\mathscr{C}_n$. In an ideal scenario, these sets would be the orbits themselves, and we would be done. In general however, this is not the case. However we can classify precisely when this is true, though we first need a lemma.

Lemma 3.20. *Let $\mathscr{R}$ be a local Noetherian domain. Then the ideal $\langle \mathscr{C}_1 \rangle$ is equal to $\mathscr{M}$.*

Proof. Since $\mathscr{R}$ is Noetherian, we can write $\mathscr{M} = \langle x_1, \cdots, x_n \rangle$ where $x_i \in \mathscr{M}$ are some generators for $\mathscr{M}$. Suppose without loss of generality that $x_1 \notin \mathscr{C}_1$ (that is, $x_1 \in \mathscr{M}^2$). This means that we can write x_1 as follows

$$x_1 = \sum_{i=1}^{n} \sum_{j=i}^{n} a_{i,j} x_i x_j$$

for some $a_{i,j} \in \mathscr{R}$. We can shift all terms with some x_1 component to the left hand side, and thus obtain

$$x_1 - \sum_{j=1}^{n} a_{1,j} x_1 x_j = \sum_{i=2}^{n} \sum_{j=i}^{n} a_{i,j} x_i x_j$$

and so

$$x_1 \left[1 - \sum_{j=1}^{n} a_{1,j} x_j \right] = \sum_{i=2}^{n} \sum_{j=i}^{n} a_{i,j} x_i x_j$$

Now, notice that each $a_{1,j} x_j \in \mathscr{M}$, while $1 \notin \mathscr{M}$. Therefore the expression $1 - \sum_{j=1}^{n} a_{1,j} x_j$ is not in $\mathscr{M}$ and is thus invertible. It follows that

$$x_1 = \left[1 - \sum_{j=1}^{n} a_{1,j} x_j \right]^{-1} \sum_{i=2}^{n} \sum_{j=i}^{n} a_{i,j} x_i x_j$$

and so $x_1 \in \langle x_2, \cdots, x_n \rangle$. Repeating this argument as necessary, we can thus express $\mathscr{M}$ using only generators within $\mathscr{C}_1$. It follows that $\langle \mathscr{C}_1 \rangle = \mathscr{M}$ □

With this lemma established, we can show that the sets $\mathscr{C}_n$ are orbits of the action $\mathscr{R}^* \curvearrowright \mathscr{R}$ precisely when $\mathscr{R}$ is a principal ideal domain. We state our result below.

Theorem 3.21. *Let $\mathscr{R}$ be a local Noetherian domain. The following are equivalent.*

(i) *The ideal $\mathscr{M}$ is principal.*

(ii) *$\mathscr{R}$ is a principal ideal domain.*

(iii) *The sets $\mathscr{C}_n$ for $n \in \mathbb{N} \cup \{\infty\}$ are (all of) the orbits of $\mathscr{R}^* \curvearrowright \mathscr{R}$.*

(iv) *$\mathscr{C}_1$ is an orbit of $\mathscr{R}^* \curvearrowright \mathscr{R}$.*

Proof. **(i) ⇒ (ii)** Let $\mathscr{M} = \langle a \rangle$ for some $a \in \mathscr{R}$. It is clear then that $\mathscr{M}^n = \langle a^n \rangle$, and that these ideals are all principal. All that remains is to check that there are no other ideals.

To this end, let I be a proper ideal, and choose $x \in I$ such that x has minimal order, say $\text{ord}(x) = n$. We claim that $I = \mathscr{M}^n$. Since the minimal order of I is n, it is clear that $I \subseteq \mathscr{M}^n$. This also means we can find some $u \in \mathscr{R}$ such that $x = ua^n$. Note that

$$n = \text{ord}(x) = \text{ord}(ua^n) \geq \text{ord}(u) + \text{ord}(a^n) = \text{ord}(u) + n$$

and so it follows that $\text{ord}(u) = 0$; in other words $u \in \mathscr{R}^*$. This means that $a^n = u^{-1}x \in I$, and thus $I = \mathscr{M}^n$.

(ii) ⇒ (iii) We have seen that the sets $\mathscr{C}_n$ form an $\mathscr{R}^*$-invariant partition, so all that remains is to show that the action of $\mathscr{R}^*$ on $\mathscr{C}_n$ is transitive.

Supposing that $\mathscr{R}$ is a principal ideal domain, we can write $\mathscr{M} = \langle a \rangle$ for some $a \in \mathscr{M}$, and in general $\mathscr{M}^n = \langle a^n \rangle$. If we take $x \in \mathscr{C}_n$, we can write $x = a^n u$ for some $u \in \mathscr{R}$. In fact, it follows that $u \in \mathscr{R}^*$, as if $u \in \mathscr{M}$, then we would have $x = a^n u \in \mathscr{M}^{n+1}$, which is a contradiction.

So let $x, y \in \mathscr{C}_n$ and write $x = a^n u$ and $y = a^n v$ for some $u, v \in \mathscr{R}^*$. Rearranging, we get $y = vu^{-1}x$, so it follows that $y \in \mathscr{R}^* x$. Hence $\mathscr{C}_n$ is an orbit.

(iii) ⇒ (iv) This is trivial.

(iv) ⇒ (i) Take any $a \in \mathscr{C}_1$, and since $\mathscr{C}_1$ is an orbit, then $\mathscr{C}_1 = \mathscr{R}^* a$. Using Lemma 3.20 it follows that

$$\mathscr{M} = \langle \mathscr{C}_1 \rangle = \langle \mathscr{R}^* a \rangle = \langle a \rangle$$ □

Any ring which satisfies (any of) the conditions of Theorem 3.21 shall be called a **local principal ideal domain** or an **LPID** for short. It will be made clear soon that this is equivalent to another notion, the so-called discrete valuation ring (DVR). Once we establish this connection, we shall phase out the term "LPID" and opt for the more standard "DVR" instead. However, in the interim, this LPID terminology will be useful as a shorthand.

3.2 Discrete Valuation Rings

Let $\mathscr{R}$ be an LPID. One can construct a field containing $\mathscr{R}$ by using the standard field of fractions construction, which we denote by $\mathrm{Frac}(\mathscr{R})$. Take any $a \in \mathscr{R}$ such that $\mathscr{M} = \langle a \rangle$, and define $\mathscr{M}^{-n} = a^{-n}\mathscr{R}$. Note that this definition is well defined since if $\langle b \rangle = \mathscr{M}$, then $a^n\mathscr{R} = \mathscr{M}^n = b^n\mathscr{R}$, and so it follows that $b^{-n}\mathscr{R} = a^{-n}\mathscr{R}$.

Proposition 3.22. *Let $\mathscr{R}$ be an LPID. Then*

$$\mathrm{Frac}(\mathscr{R}) = \bigcup_{n \in \mathbb{Z}} \mathscr{M}^n$$

Proof. Let $x \in \mathscr{K}$. Since $\mathscr{K}$ is the field of fractions of $\mathscr{R}$, there is some $a, b \in \mathscr{R}$ such that $b \neq 0$ and $x = ab^{-1}$. If we let $n = \mathrm{ord}(b)$ (which is never infinite as $b \neq 0$), then we have that $b = b_1^n$ where $b_1 \in \mathscr{M}$. It follows that $x = b_1^{-n} a \in b_1^{-n}\mathscr{R} = \mathscr{M}^{-n}$. □

In fact, this field has a special structure known as a *discrete valuation*, which is induced by the ord function on $\mathscr{R}$. Note the similarities between this definition and Proposition 3.4.

Definition 3.23. Let $\mathscr{K}$ be a field. We say that $\nu : \mathscr{K} \to \mathbb{Z} \cup \{\infty\}$ is a **discrete valuation** if for every $x, y \in \mathscr{K}$ the following hold:

(DV0) There is some $a \in \mathscr{K}$ such that $\nu(a) = 1$ (*non-triviality*),

(DV1) $\nu(x) = \infty$ if and only if $x = 0$,

(DV2) $\nu(xy) = \nu(x) + \nu(y)$, and

(DV3) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$.

When such a ν exists, we say that $\mathscr{K}$ is a **discrete valuation field**.

In particular, when $\mathscr{K} = \mathrm{Frac}(\mathscr{R})$ for an LPID $\mathscr{R}$, we define ν in an analogous way to the ord function, namely

$$\nu(x) = \min\{n \in \mathbb{Z} : x \in \mathscr{M}^n\} \tag{3.3}$$

for $x \in \mathrm{Frac}(\mathscr{R})$. From this definition it is clear that $\mathscr{R} = \mathscr{M}^0 = \{x \in \mathrm{Frac}(\mathscr{R}) : \nu(x) \geq 0\}$, and in general this ring is called the *ring of integers* of $\mathscr{K}$.

Definition 3.24. Let $\mathscr{K}$ be a discrete valuation field. We define

$$\mathscr{O}(\mathscr{K}) := \{x \in \mathscr{K} : \nu(x) \geq 0\}$$

to be the **ring of integers** of $\mathscr{K}$. We also say that the ring $\mathscr{O}(\mathscr{K})$ is a **discrete valuation ring** (**DVR**).

It is almost immediate that $\mathscr{O}(\mathscr{K})$ is indeed a ring, and furthermore one can show that $\mathrm{Frac}(\mathscr{O}(\mathscr{K})) = \mathscr{K}$ for any discrete valuation field $\mathscr{K}$. Likewise, it is an easy exercise that $\mathscr{O}(\mathrm{Frac}(\mathscr{R})) = \mathscr{R}$ for any DVR $\mathscr{R}$.

> *Remark* 3.25. One must take heed; there is an important subtlety here that is easily missed. Notice that a discrete valuation field is defined by the existence of a valuation, whereas a discrete valuation ring is not. Indeed, while one could define the notion of a '*ring with discrete valuation*', it would *not* be true that this is equivalent to a discrete valuation ring, as defined by Definition 3.24. One could argue that this is perhaps an artefact of poor naming conventions. Alas, this is the standard terminology, and it is what we shall adopt.

The following is an example that illustrates the previous remark.

Example 3.26. Let $\mathbb{F}_p$ be a finite field, and set $\mathscr{R} = \mathbb{F}_p[[X, Y]]$ to be the ring of *formal power series* in X and Y (see Section 3.3.1). In particular, $\mathscr{R}$ is a local Noetherian domain with maximal ideal $\mathscr{M} = \langle X, Y \rangle$ Define $\nu(x) = \operatorname{ord}(x)$ as given by Definition 3.3. It is clear by Proposition 3.4 that ν satisfies (DV0) to (DV3), and so is a '*ring with discrete valuation*'. Notice though that taking $XY^{-1} \in \mathscr{K} = \operatorname{Frac}(\mathscr{R})$, one finds that

$$\nu(XY^{-1}) = \nu(X) - \nu(Y) = 0$$

and so $XY^{-1} \in \mathscr{O}(\operatorname{Frac}(\mathscr{R}))$. Clearly however $XY^{-1} \notin \mathscr{R}$, and thus $\mathscr{R} \neq \mathscr{O}(\operatorname{Frac}(\mathscr{R}))$.

In fact, as the next proposition shows, the issue with the above example arises from the non-principality of the ideal $\mathscr{M}$.

Theorem 3.27. *Let $\mathscr{R}$ be a commutative ring. Then $\mathscr{R}$ is a DVR if and only if it is an LPID.*

Proof. Suppose $\mathscr{R}$ is a DVR, contained in a field $\mathscr{K}$ with discrete valuation ν. Define $\mathscr{M} = \{x \in \mathscr{R} : \nu(x) \geq 1\}$, which is an ideal by the properties of ν. Now, if $u \in \mathscr{R} \setminus \mathscr{M}$, then $\nu(u^{-1}) = -\nu(u) = 0$, and so $u^{-1} \in \mathscr{R}$. The converse direction holds similarly, and so $\mathscr{R}^* = \mathscr{R} \setminus \mathscr{M}$, which implies that $\mathscr{R}$ is local. Now, fix any $a \in \mathscr{R}$ such that $\nu(a) = 1$. Then if $x \in \mathscr{M}$, we have that $\nu(xa^{-1}) = \nu(x) - 1 \geq 0$, and so there is some $y \in \mathscr{R}$ such that $x = ya$. Hence $\langle a \rangle = \mathscr{M}$, and so $\mathscr{R}$ is an LPID.

For the reverse direction, let $\mathscr{R}$ be an LPID and define the map $\nu(xy^{-1}) = \operatorname{ord}(x) - \operatorname{ord}(y)$ for $xy^{-1} \in \operatorname{Frac}(\mathscr{R})$. We claim ν is a discrete valuation. Indeed, note that for any $a \in \mathscr{K}$, we have that $\nu(a)$ is the smallest integer n such that $a \in \mathscr{M}^n$. From this, one may use Proposition 3.4 to show that ν is a discrete valuation, and we leave this as an exercise to the reader. □

> *Remark* 3.28. As alluded to earlier, we shall henceforth drop the term "LPID" and instead opt for the standard "DVR" terminology. There are in fact more equivalent conditions which turn $\mathscr{R}$ into a DVR, beyond those stated in Theorems 3.21 and 3.27. For instance, an integrally closed local Noetherian domain with Krull dimension one is also a DVR. We refer the curious reader to [6, Proposition 9.2] for more conditions, though this is list is by no means exhaustive.

We now wish for our DVR $\mathscr{R}$ to be a locally compact space. However, there is no reason to believe that this should always hold; and indeed, there exist DVRs $\mathscr{R}$ which are not locally compact, as we shall soon see. As it turns out, in the setting of DVRs, local compactness is equivalent to compactness. Moreover, there is an additional algebraic condition on $\mathscr{R}$ which also gives compactness. Namely it asks that the **residue field** κ of $\mathscr{R}$, defined as

$$\kappa := \mathscr{R}/\mathscr{M} \tag{3.4}$$

is a *finite* field. This is in part due to the fact that that the residue field characterises not just $\mathscr{R}/\mathscr{M}$ but the entire nested quotient field structure $\mathscr{M}^n/\mathscr{M}^{n+1}$ in $\mathscr{R}$.

Proposition 3.29. *Let $\mathscr{R}$ be a DVR. For any $n \in \mathbb{N}$, we have that κ is field isomorphic to $\mathscr{M}^n/\mathscr{M}^{n+1}$.*

Proof. Let $\mathscr{M} = \langle a \rangle$. Consider the ring homomorphism $\pi : \mathscr{R}/\mathscr{M} \to \mathscr{M}^n/\mathscr{M}^{n+1}$ given by $\pi(x + \mathscr{M}) = a^n x + \mathscr{M}^{n+1}$. This is a well-defined ring homomorphism, and one can perform a routine check to verify. On the other hand, if $y + \mathscr{M}^{n+1} \in \mathscr{M}^n/\mathscr{M}^{n+1}$, then we can write $y = a^n x$ for some $x \in \mathscr{R}$. Then it follows that the map $y + \mathscr{M}^{n+1} \mapsto x + \mathscr{M}$ is the inverse map of π. □

In the case where this quotient is finite, we can calculate the size of $\mathscr{R}/\mathscr{M}^n$ explicitly.

Corollary 3.30. *Let $\mathscr{R}$ be a DVR with a finite residue field, say $q = |\kappa|$. Then for $n \in \mathbb{N}$ we have $|\mathscr{R}/\mathscr{M}^n| = q^n$.*

We are now ready to prove this next result.

Theorem 3.31. *Let $\mathscr{R}$ be a DVR. The following are equivalent.*

(i) *$\mathscr{R}$ is compact.*

(ii) *$\mathscr{R}$ is locally compact.*

(iii) *$\mathscr{R}$ is complete and κ is a finite field.*

Proof. **(i) ⇒ (ii)** This is trivial.

(ii) ⇒ (iii) Consider $\mathscr{R}$ as a locally compact group under addition, so that the metric $d_{\mathscr{M}}$ is left-invariant. It follows by Proposition 2.7 that $\mathscr{R}$ is complete. Furthermore, take some open U and compact K such that $0 \in U \subseteq K$. In particular, since $\{\mathscr{M}^n\}_{n\in\mathbb{N}}$ forms a neighbourhood base of 0, there must be some $n \in \mathbb{N}$ such that $\mathscr{M}^n \subseteq U \subseteq K$. Now form an open cover of K by

$$K \subseteq \bigcup_{x\in\mathscr{R}} (x + \mathscr{M}^{n+1})$$

Since K is compact, there exists a finite subcover of K and hence of $\mathscr{M}^n$. Thus $|\mathscr{M}^n/\mathscr{M}^{n+1}| = |\kappa|$ is finite.

(iii) ⇒ (i) For any $\varepsilon > 0$, choose $n \in \mathbb{N}$ so that $2^{-n} < \varepsilon$. We can then cover $\mathscr{R}$ with the cosets of $\mathscr{M}^n$, which by Corollary 3.30 is a finite cover. Moreover, for a given $x + \mathscr{M}^n \in \mathscr{R}/\mathscr{M}^n$, we can write

$$x + \mathscr{M}^n = \mathrm{B}[x; 2^{-n}]$$

so each coset has diameter at most 2^{-n}. Thus $\mathscr{R}$ is totally bounded (and complete) so is therefore compact. □

In general, when we have a DVR $\mathscr{R}$, then there are natural quotient maps from $\mathscr{R}/\mathscr{M}^n \to \mathscr{R}/\mathscr{M}^k$ for $k > n$. One can then show that

$$\mathscr{R} = \varprojlim \mathscr{R}/\mathscr{M}^n \tag{3.5}$$

where '$\varprojlim$' indicates a *projective limit* (or *indirect limit*) of rings. We shall not define projective limits here, though the investigative reader may refer to either of the books by Atiyah and Macdonald [6, Chapter 10] or Ramakrishnan and Valenza [47, Section 1.3].

In the case where $\mathscr{R}$ is a *compact* DVR, we know by Corollary 3.30 that the quotients $\mathscr{R}/\mathscr{M}^n$ are finite. In particular, combined with (3.5), this then means that $\mathscr{R}$ is a *profinite* ring. We shall discuss this in more detail in Section 6.3. In particular, the profiniteness of $\mathscr{R}$ can be seen as a result of van Dantzig's Theorem, since $\mathscr{R}$ is a compact, totally disconnected group, though (3.5) provides an explicit construction.

3.3 Examples

It would be remiss to present this theory without mentioning a few examples. There are two classical examples of compact DVRs (and their corresponding fields) that are often presented when one talks about these objects. In keeping with tradition, we shall present a brief description of both here.

3.3.1 Formal Power Series

Given a field $\mathbb{F}$, we define the **formal power series** of $\mathbb{F}$ to be

$$\mathbb{F}[[x]] := \left\{ \sum_{n=0}^{\infty} a_n x^n : a_n \in \mathbb{F} \right\} \tag{3.6}$$

where the summation is a 'formal summation'. In other words, we define addition and multiplication on $\mathbb{F}[[x]]$ in the same manner as we do for normal polynomials. One may check that the ideal $\mathscr{M} = \langle x \rangle$ is in fact the unique maximal ideal of $\mathbb{F}[[x]]$. Using our previous terminology, this shows by Theorem 3.21 that $\mathbb{F}[[x]]$ is an LPID. This induces a metric d on $\mathbb{F}[[x]]$ where $d(a, b)$ is the degree of the *smallest* non-zero coefficient of $a - b$.

Of course, by Theorem 3.27, $\mathbb{F}[[x]]$ is also a DVR. Observe now that $\mathbb{F}[[x]]/\mathscr{M}$ is isomorphic to $\mathbb{F}$, so by Theorem 3.31, we have that $\mathbb{F}[[x]]$ is a compact DVR precisely when $\mathbb{F}$ is a finite field. Furthermore, the characteristic of $\mathbb{F}[[x]]$ is always the characteristic of $\mathbb{F}$, so when $\mathbb{F}$ is finite, this will necessarily be non-zero. We also have a more general form for the quotients, namely that $\mathbb{F}[[x]]/\mathscr{M}^n = P_n(\mathbb{F})[x]$ where $P_n(\mathbb{F})[x]$ is the ring of polynomials of degree at most n. It follows by (3.5) that $\mathbb{F}[[x]]$ can be expressed as the projective limit

$$\mathbb{F}[[x]] = \varprojlim P_n(\mathbb{F})[x] \tag{3.7}$$

though this is perhaps clear from the construction.

When we consider the field of fractions of $\mathbb{F}[[x]]$, this gives the **Laurent series** of $\mathbb{F}$. In particular the Laurent series is given by

$$\mathbb{F}(x) := \left\{ \sum_{n=-N}^{\infty} a_n x^n : a_n \in \mathbb{F}, N \in \mathbb{N} \right\} \tag{3.8}$$

where this summation is the formal summation as before. It is fairly routine to see that this indeed the field of fractions of $\mathbb{F}[[x]]$, and is an example of a *local field* which we study further in the following chapter.

3.3.2 The p-adic Numbers

Of the two examples presented here, the p-adics are arguably more ubiquitous. There are several equivalent ways to define them, each with its own utility. In order to do this, let us explicitly set the notation $\mathbb{Z}_n$ to be the ring of integers mod n (that is $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$).

One way to concisely construct $\mathbb{O}_p$ is as an inverse limit. Given a prime p and positive integers $n > m$, we can map $\mathbb{Z}_{p^n}$ onto $\mathbb{Z}_{p^m}$ in the canonical way: simply take the integer and evaluate it mod p^m. This will be a well-defined quotient map, so let us denote it $q_{n,m}$. This then allows us to define the inverse limit

$$\mathbb{O}_p := \varprojlim \mathbb{Z}_{p^n} = \{(a_0, a_1, \cdots) \in \prod_{n \in \mathbb{N}} \mathbb{Z}_{p^n} : q_{j,i}(a_j) = a_i \text{ for all } j \geq i\} \tag{3.9}$$

and we call the ring $\mathbb{O}_p$ the ***p*-adic integers**. This mimics the construction given in (3.5), and shows that our residue field is $\mathbb{Z}_p$. This ring has characteristic 0, and contains $p\mathbb{O}_p$ as the unique maximal ideal, where we identify $p \in \mathbb{O}_p$ as the element $p = (0, p, p, p, \ldots)$.

As is the case for integers, one way to represent an element $a \in \mathbb{O}_p$, is via a string of p-igits[13]. However, the crucial difference between $\mathbb{Z}$ and $\mathbb{O}_p$ is that these strings of p-igits may have infinite length. More precisely we can write

$$a = \sum_{n=0}^{\infty} a_n p^n \qquad \text{where } a_n \in \{0, \ldots, p-1\} \tag{3.10}$$

where the values a_n represent the p-igits of a. In other words, the *p-igital representation* of a is $a = \ldots a_3 a_2 a_1 a_0$. It follows that a is an integer if and only if $a_n = 0$ for all sufficiently large n. With this p-igital representation, one may perform the usual operations such as addition and multiplication using the standard algorithm one would use for integers.

If we now consider the field of fractions of $\mathbb{O}_p$, we arrive at a similar construction as in the formal power series case. We denote the field of fractions as $\mathbb{Q}_p$ and we call it the ***p*-adic numbers**. Given a $b \in \mathbb{Q}_p$, we can always write

$$b = \sum_{n=-N}^{\infty} b_n p^n \qquad \text{where } b_n \in \{0, \ldots, p-1\} \text{ and } N \in \mathbb{N} \tag{3.11}$$

where addition and multiplication are define in the expected manner.

An alternative method to construct $\mathbb{O}_p$ is to apply a different metric to $\mathbb{Z}$. We define $\|x\|_p := p^{-n}$ where n is the largest integer such that $p^n \mid x$. When we complete with respect to this metric, this also results in the p-adic integers $\mathbb{O}_p$. Similarly, if one applies a similar norm to $\mathbb{Q}$, then its completion will give rise to $\mathbb{Q}_p$. We note that due to Ostrowski's Theorem (see Theorem 4.11), these are in essence the only such norms one can assign to $\mathbb{Q}$ other than the trivial norms and the usual norm $\|q\| = |q|$.

[13] That is, digits in base p.

Chapter 4

Local Fields & Spaces

> " I'm speaking from a scientific perspective, mermaids don't produce milk, they lay eggs. "
>
> *–A. Lafrance*

In the previous chapter, we saw that the field of fractions played an important role in defining and working with (compact) DVRs. In particular, these were fields with a discrete valuation ν, which was in essence an additive homomorphism. Taking an exponent, we arrive at a multiplicative homomorphism which we call an "*absolute value*". Locally compact fields with such a map are known as *local fields*, and serve as a generalisation of discrete valuation fields. In fact, it is a major theorem that all non-discrete locally compact fields are in actuality local fields. This leads to a nice classification result, as well as some nice duality properties. In particular, the Pontryagin dual of any local field is isomorphic to itself.

We also introduce the concept of a *local space*, which is a finite-dimensional vector space over a local field. Naturally, a local space $\mathscr{V}$ will also have nice duality properties, and furthermore the duality can be implemented via a symmetric bilinear form. Eventually this will show that the characters on $\mathscr{V}$ must have large kernels. This will be particularly useful when we examine the Fourier-Stieltjes and other related algebras of semidirect products of these spaces in Chapter 5.

In general, the content of this chapter is largely known material, particularly Section 4.1. However, our goal is to extract this content from various sources, and present in a coherent manner that will suit our needs. Section 4.2 on local spaces does introduce some new terminology, though it very closely mirrors existing results for finite-dimensional vector spaces over $\mathbb{R}$ or $\mathbb{C}$. Again, the purpose is to compile this material into a single source.

4.1 Local Fields

4.1.1 Definition

As we saw in the previous chapter, a useful construction when working with a DVR $\mathscr{R}$ is its field of fractions $\mathscr{K} = \text{Frac}(\mathscr{R})$, which is always equipped with a discrete valuation. By definition, this discrete valuation is an additive homomorphism from $\mathscr{K}$ to $\mathbb{Z}$. We may modify this to a multiplicative homomorphism from $\mathscr{K}^\times$ to $\mathbb{R}_{>0}$ (where $\mathscr{K}^\times$ is the collection of non-zero elements of $\mathscr{K}$). Such a homomorphism is called an *absolute value*; we define this precisely as follows.

Definition 4.1. Let $\mathscr{K}$ be a field. We say $|\cdot| : \mathscr{K} \to \mathbb{R}_{\geq 0}$ is an **absolute value** if for every $x, y \in \mathscr{K}$:

(AV0) $|\cdot|$ is non-trivial (admits values other than 0 or 1),

(AV1) $|x| = 0$ if and only if $x = 0$,

(AV2) $|xy| = |x||y|$, and

(AV3) $|x + y| \leq |x| + |y|$.

Furthermore if $|\cdot|$ satisfies the stronger condition

(AV3') $|x + y| \leq \max\{|x|, |y|\}$

then we say it is a **non-Archimedean** absolute value. An absolute value which does not satisfy this condition is called **Archimedean**.

Naturally, if $\mathscr{K} = \text{Frac}(\mathscr{R})$ for a DVR $\mathscr{R}$, then one can always construct an absolute value on $\mathscr{K}$ via

$$|x| := e^{-\nu(x)} \tag{4.1}$$

which we leave as an exercise to verify. Notice that we can choose any base in the exponent, which leads us to the following notion of equivalence.

Definition 4.2. Let $\mathscr{K}$ be a field. We say two absolute values $|\cdot|_1$ and $|\cdot|_2$ on $\mathscr{K}$ are **equivalent** if there exists some constant $r \in \mathbb{R}_{>0}$ such that $|\cdot|_1 = |\cdot|_2^r$.

Note that non-triviality and (AV2) together imply that $|\cdot|$ is unbounded. One can induce a metric on $\mathscr{K}$ by $d(x,y) = |x-y|$, similar to the construction given by Definition 3.5. In particular, by Proposition 2.7, if $\mathscr{K}$ is locally compact then this metric is complete. Let us distinguish such fields.

Definition 4.3. Let $\mathscr{K}$ be a field. We say that $\mathscr{K}$ is a **local field** if $\mathscr{K}$ has an absolute value and is locally compact with respect to the metric induced by the absolute value.

> *Remark* 4.4. If we return to the case where $\mathscr{K} = \mathrm{Frac}(\mathscr{R})$ for a DVR $\mathscr{R}$, then we see that by Theorem 3.31, $\mathscr{K}$ will be a local field precisely when $\mathscr{R}$ is a *compact* DVR. In particular, such fields will always be locally compact and non-discrete. This follows from Theorem 3.31 since $\mathscr{R}$ is compact and infinite, it cannot be discrete.

Not only are local fields complete metric spaces, but additionally they are also proper metric spaces, as defined in Definition 2.3.

Proposition 4.5. *Let $\mathscr{K}$ be a local field. Then $\mathscr{K}$ is proper.*

Proof. For convenience, let us denote the closed ball $\mathrm{B}[0;r]$ simply as B_r. Let K be a compact neighbourhood of 0. Since K contains an open set, there must be some $\varepsilon > 0$ so that $B_\varepsilon \subseteq K$, which implies B_ε is compact. Now take any $r > 0$, and choose $x \in \mathscr{K}$ such that $|x| \geq r/\varepsilon$. By (AV2), it follows that $B_r \subseteq B_{|x|\varepsilon} = xB_\varepsilon$, which is clearly compact. Finally, for any $y \in \mathscr{K}$, we have that $\mathrm{B}[y;r] = y + \mathrm{B}[0;r]$, and so $\mathscr{K}$ is proper. □

By Proposition 2.4, one obtains several other topological properties for free.

Corollary 4.6. *Let $\mathscr{K}$ be a local field. Then $\mathscr{K}$ is σ-compact, separable, and second-countable.*

4.1.2 Non-Archimedean Fields

As we saw in (4.1), a compact DVR $\mathscr{R}$ always gives rise to a local field $\mathscr{K} = \mathrm{Frac}(\mathscr{R})$. In fact, it follows from (DV3) that any such absolute value must be non-Archimedean. However, this construction is reversible: any non-Archimedean local field gives rise to a compact DVR via $\mathscr{R} = \{x \in \mathscr{K} : |x| \leq 1\}$. We shall prove this shortly, though we need the following definition first.

Definition 4.7. Let $\mathscr{K}$ be a local field. We define

$$|\mathscr{K}^\times| := \{|x| : x \in \mathscr{K}^\times\}$$

to be the **value group** of $\mathscr{K}$.

Note that by (AV2) and properness of $\mathcal{K}$, $|\mathcal{K}^\times|$ will always be a closed multiplicative subgroup of $\mathbb{R}_{>0}$. This means that it will either be $\mathbb{R}_{>0}$ itself, or of the form $\{\alpha^n : n \in \mathbb{Z}\} \cong \mathbb{Z}$ for some $\alpha > 1$.

We will also need to use the notation $\mathbb{N}_{\mathcal{K}}$ to denote the inclusion of the natural numbers inside of $\mathcal{K}$. In other words, if we define $n_{\mathcal{K}} := 1_{\mathcal{K}} + \ldots + 1_{\mathcal{K}}$, then $\mathbb{N}_{\mathcal{K}} = \{n_{\mathcal{K}} : n \in \mathbb{N}\}$.

Theorem 4.8. *Let $\mathcal{K}$ be a local field. The following are equivalent.*

(i) *$\mathcal{K}$ is non-Archimedean.*

(ii) *$\mathcal{K} = \mathrm{Frac}(\mathcal{R})$ where $\mathcal{R}$ is a compact DVR.*

(iii) *The value group $|\mathcal{K}^\times|$ is discrete.*

(iv) *The set $\{|x| : x \in \mathbb{N}_{\mathcal{K}}\}$ is bounded.*

Proof. **(i) ⇒ (ii)** Note that since $\mathcal{K}$ is non-Archimedean, the metric generated by $|\cdot|$ will in fact be an ultrametric. By Corollary 3.16, the topology on $\mathcal{K}$ is totally disconnected, hence the value group must be of the form $|\mathcal{K}^\times| = \{\alpha^n : n \in \mathbb{Z}\}$ for some $\alpha \in \mathbb{R}_{>0}$ with $\alpha > 1$. Set $\nu(x) = -\log_\alpha(x)$. It is a standard exercise to verify that ν turns $\mathcal{K}$ into a discrete valuation field, and so it arises from some DVR $\mathcal{R}$. Furthermore, since by definition

$$\mathcal{R} = \{x \in \mathcal{K} : \nu(x) \geq 0\} = \{x \in \mathcal{K} : |x| \leq 1\}$$

then it is indeed a compact DVR.

(ii) ⇒ (iii) This is clear since $|\mathcal{K}^\times| = \exp(-\nu(\mathcal{K}^\times)) \subseteq \exp(\mathbb{Z})$.

(iii) ⇒ (iv) This argument is borrowed from an online post by Jonathan Lubin [40]. Suppose that $|\mathbb{N}_{\mathcal{K}}|$ is unbounded. Then there is some sequence $a_n \in \mathbb{N}_{\mathcal{K}}$ such that $|a_n| \geq n$ and that $|a_n| > |a_n - 1|$.[14] We can use these inequalities to see that

$$1 - \frac{1}{n} \leq 1 - \left|\frac{1}{a_n}\right| \leq \left|1 - \frac{1}{a_n}\right| = \left|\frac{1 - a_n}{a_n}\right| < 1$$

It follows that the sequence $1 - \frac{1}{a_n}$ never has absolute value 1, but converges to something with absolute value 1. Hence the value group $|\mathcal{K}^\times|$ must be non-discrete.

[14] For example, the sequence $a_n = \min\{k_{\mathcal{K}} : k \in \mathbb{N}, |k_{\mathcal{K}}| \geq n\}$ satisfies this property.

(iv) ⇒ (i) This argument follows an idea as presented by Ramakrishnan and Valenza [47, Proposition 4-28]. Suppose that the non-Archimedean property does not hold; there is some $x, y \in \mathscr{K}$ such that $|x+y| > |x|, |y|$. Without loss of generality, let $|x| \leq |y|$, and set $z = xy^{-1}$. Then it follows that $|z+1| > 1 \geq |z|$. Now, we use an analogue of the binomial theorem to observe that

$$|z+1|^n \leq \sum_{k=0}^{n} \left| \binom{n}{k}_{\mathscr{K}} \right| |z|^k \leq \sum_{k=0}^{n} \left| \binom{n}{k}_{\mathscr{K}} \right|$$

where $\binom{n}{k}_{\mathscr{K}}$ is the copy of $\binom{n}{k}$ inside $\mathscr{K}$. Now, if $|\mathbb{N}_{\mathscr{K}}| \leq M$ for some $M > 0$, then it would follow that

$$\frac{|z+1|^n}{n+1} \leq \sum_{k=0}^{n} \frac{M}{n+1} = M$$

for all $n \in \mathbb{N}$. But since $|z+1| > 1$, the left hand side grows without bound, so $|\mathbb{N}_{\mathscr{K}}|$ must also be unbounded. □

4.1.3 Characterisation & Classification of Local Fields

Perhaps one of the most surprising results is that local fields characterise all the 'interesting' locally compact fields. Indeed, given a non-discrete locally compact field, its Haar measure can be used to explicitly construct an absolute value. Let us state this as a theorem.

Theorem 4.9. *Let $\mathscr{K}$ be a topological field. Then $\mathscr{K}$ is a local field if and only if it is locally compact and non-discrete.*

The forward direction of this proof is fairly straightforward, whereas the reverse direction is rather non-trivial. We shall present a brief outline of the proof, but we omit many technical details and instead refer the reader to either of Weil [61, Chapter 1] or Ramakrishnan and Valenza [47, Chapter 4] for the full proof.

So let $\mathscr{K}$ be a locally compact non-discrete field. Recall the definition of the *determinant* $\delta(\alpha)$ of an automorphism α, as defined in Definition 2.64. We can extend this to multiplication by associating to $a \in \mathscr{K}$ the automorphism[15] $x \mapsto ax$. That is, we define $\delta : \mathscr{K} \to \mathbb{R}_{\geq 0}$ by $\delta(a) = \delta_G(x \mapsto ax)$. Of course, the zero map is not an automorphism, so we explicitly set $\delta(0) = 0$.

[15] Note that this is a *group* automorphism, not a field automorphism.

Proposition 4.10. *Let $\mathscr{K}$ be a locally compact, non-discrete field. Then $\delta : \mathscr{K} \to \mathbb{R}_{\geq 0}$ as defined above is continuous.*

Proof. See [47, Proposition 4-1]. □

This determinant function can be used to define an absolute value. Namely, for any $r > 0$, we define

$$|a|_r := \delta(a)^r \tag{4.2}$$

which for all $a, b \in \mathscr{K}$ satisfy the following properties.

(i) $|ab|_r = |a|_r |b|_r$. This follows by the multiplicative property of δ.

(ii) $|a|_r = 0$ if and only if $a = 0$; indeed this is immediate from the definition.

(iii) $|\cdot|_r$ is non-trivial. This follows from continuity of δ.

This gives us a collection of 'near absolute values' which satisfy (AV0) to (AV2). However, (AV3) is considerably more difficult. To fix this issue, we first note that by [47, Theorem 4-9], there is some positive constant $M \geq 1$ such that

$$\delta(a+b) \leq M \sup\{\delta(a), \delta(b)\} \tag{4.3}$$

for every $a, b \in \mathscr{K}$. Writing this in terms of $|\cdot|_r$, we find that

$$|a+b|_r \leq M^r \sup\{|a|_r, |b|_r\} \tag{4.4}$$

From this we choose r sufficiently large so that $M^r \geq 2$, and then by [47, Lemma 4-27], we have that $|a+b|_r \leq |a|_r + |b|_r$. Hence this r (and moreover any larger r also proves adequate) will provide a genuine absolute value on $\mathscr{K}$. This completes the proof of Theorem 4.9.

A particularly important result that follows from this is the classification of all local fields. This is intimately related to Ostrowski's Theorem[16], which gives a classification of all possible absolute values on the field $\mathbb{Q}$.

Theorem 4.11 (Ostrowski's Theorem)**.** *Let $|\cdot|$ be an absolute value on $\mathbb{Q}$. Then $|\cdot|$ is equivalent to either the standard absolute value (sometimes denoted $|\cdot|_\infty$) or to any of the p-adic absolute values $|\cdot|_p$.*

This allows us to classify local fields (up to isomorphism) into three categories.

[16] As the name may suggest this is due to a paper of Ostrowski from 1916 [45]. One may refer to the book of Ramakrishnan and Valenza for a more modern proof [47, Theorem 4.30].

- *Archimedean local fields.*
  The only instances of these are $\mathbb{R}$ and $\mathbb{C}$, and as such, will always have characteristic zero.
- *Non-Archimedean fields with characteristic zero.*
  These fields must be a finite algebraic extension of some p-adic number field $\mathbb{Q}_p$ (see Section 3.3.2).
- *Non-Archimedean fields with non-zero characteristic.*
  The fields must be of the form $\mathbb{F}_{p^k}(x)$ — the Laurent series of the finite field $\mathbb{F}_{p^k}$ (see Section 3.3.1).

This classification is well known in the study of local fields, and one proof (among many) is presented in [47, Theorem 4.12].

> *Remark* 4.12. Our focus in this thesis will primarily be on the latter two cases, which by Theorem 4.8 will always arise from some compact DVR. However, unless stated otherwise, we shall strive to prove results for all local fields, including the Archimedean fields.
>
> The Archimedean fields are rather special. Indeed there are only two such fields – $\mathbb{R}$ and $\mathbb{C}$ – and they are two of the most ubiquitous objects in mathematics. That these are the only examples can in fact be seen as a result of the *Gelfand-Mazur theorem.* This theorem is usually stated as: "any complex Banach algebra where every non-zero element is invertible must be isomorphic to $\mathbb{C}$". More generally this also works for real Banach algebras, where any such algebra must now be either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ (the quaternions). All that remains to show is that any Archimedean local field will be a real Banach algebra, so that it must be either $\mathbb{R}$ or $\mathbb{C}$. We of course exclude $\mathbb{H}$ as it is not a field.

4.1.4 Self-Duality

Our next goal is to establish the self-duality of local fields. This shall be realised by premultiplying characters in $\widehat{\mathcal{K}}$ by any non-zero element in $\mathcal{K}$. This can be shown with the aid of *adjoint* homomorphisms. Recall that in this thesis, homomorphisms between topological groups are by definition continuous.

Definition 4.13. Given a homomorphism $\pi : G \to H$ between two abelian locally compact groups, we define the **adjoint** of π to be $\widehat{\pi} : \widehat{H} \to \widehat{G}$ where

$$\langle \widehat{\pi}(\varphi), x \rangle = \langle \varphi, \pi(x) \rangle$$

for all $x \in G$ and $\varphi \in \widehat{H}$.

It is a standard exercise to verify that this is indeed a homomorphism. Furthermore, by Pontryagin duality, we have that $\widehat{\widehat{\pi}} = \pi$. We can also relate the kernels and and ranges of homomorphisms and their adjoints in the following manner.

Lemma 4.14. *Let $\pi : G \to H$ be a homomorphism between two abelian locally compact groups. Then* $\ker \pi = \widehat{\pi}(\widehat{H})^{\perp}$

Proof. This follows almost directly from the definition of the adjoint. Indeed $x \in \ker \pi$ precisely when for all $y \in H$

$$1 = \langle \pi(x), y \rangle = \langle x, \widehat{\pi}(y) \rangle$$

and so $x \in \widehat{\pi}(\widehat{H})^{\perp}$. □

Proposition 4.15. *Let $\pi : G \to H$ be a homomorphism between two abelian locally compact groups. Then π is injective if and only if $\widehat{\pi}$ has dense range.*

Proof. Recall that for any closed subgroup $N \subseteq G$, we have that $\widehat{N} = \widehat{G}/N^{\perp}$. [17] Applying Lemma 4.14, we find that $\widehat{\ker \pi} = \widehat{G}/\overline{\widehat{\pi}(\widehat{H})}$. Hence $\ker \pi$ is trivial if and only if $\widehat{\pi}(\widehat{H})$ is dense in $\widehat{G}$. □

In the special case where π maps $G \to \widehat{G}$, then by Pontryagin duality, the adjoint map $\widehat{\pi}$ also maps $G \to \widehat{G}$. Furthermore, if this adjoint is equal to π, that is

$$\langle \pi(x), y \rangle = \langle x, \pi(y) \rangle \tag{4.5}$$

for all $x, y \in G$, then we say that π is **self-adjoint**. When this is the case, this homomorphism becomes a prime candidate for exhibiting self-duality. Recall that we say a map is **relatively open** if it is open in the subspace topology of its range.

Corollary 4.16. *Let $\pi : G \to \widehat{G}$ be a self-adjoint homomorphism. If π is relatively open and injective, then π is surjective, and hence G is self-dual.*

Proof. By Proposition 4.15, we get that π has dense range. However, since π is open, then $\pi(G)$ is a locally compact subgroup of $\widehat{G}$, and so must also be closed. Hence π is surjective, and G self-dual. □

[17] For a proof, see for instance Folland [20, Theorem 4.39].

Let us now apply this to local fields. First, let $\mathscr{K}$ be a (possibly discrete) locally compact field, and fix a distinguished non-trivial character Φ of $\mathscr{K}$ (as an additive group). Let $\tau_\Phi : \mathscr{K} \to \widehat{\mathscr{K}}$ be the map by

$$\langle \tau_\Phi(a), x \rangle = \Phi(ax) \tag{4.6}$$

for all $a \in \mathscr{K}$. It is easily verified that $x \mapsto \Phi(ax)$ is indeed a character of $\mathscr{K}$.

Proposition 4.17. *Let $\mathscr{K}$ be a locally compact field. Then τ_Φ is a continuous, injective, and self-adjoint group homomorphism.*

Proof. First, let $a, b, x \in \mathscr{K}$, and fix $x_0 \in \mathscr{K}$ such that $\Phi(x_0) \neq 1$. We then have

$$\langle \tau_\Phi(a+b), x \rangle = \Phi((a+b)x) = \Phi(ax)\Phi(bx) = \langle \tau_\Phi(a) + \tau_\Phi(b), x \rangle$$

so that τ_Φ is a homomorphism. Next, if $a \neq 0$, then it follows that $\langle \tau_\Phi(a), a^{-1}x_0 \rangle = \Phi(x_0) \neq 1$, and so $\tau_\Phi(a)$ is not the trivial character. Thus $\ker \tau_\Phi = 0$ and hence τ_Φ is injective. Continuity of τ_Φ follows by continuity of Φ. Finally, we have that $\langle \tau_\Phi(a), x \rangle = \Phi(ax) = \langle \tau_\Phi(x), a \rangle$, and so τ_Φ is self-adjoint. □

Notice that the above proposition holds for any locally compact field, including discrete fields. To show self-duality, the only additional property we need to show is that the map τ_Φ is relatively open. However, almost all self-dual locally compact fields are necessarily local fields.[18] So in order to show that our local field $\mathscr{K}$ is self-dual, we will need to exploit the existence of an absolute value. The following is a proof of this fact which follows closely the proof that is presented in Tate's thesis [58, Lemma 2.2.1 (5)].

Proposition 4.18. *Let $\mathscr{K}$ be a local field. Then the map τ_Φ as defined in (4.6) is a relatively open map.*

Proof. Suppose that $(a_\alpha)_\alpha$ is a net such that $\tau_\Phi(a_\alpha)$ converges to the trivial character. Fix any $x_0 \in \mathscr{K}$ such that $\Phi(x_0) \neq 1$. Take any (large) $M > 0$ and consider the compact ball $B_M := \mathrm{B}[0; M] \subseteq \mathscr{K}$. Since $\widehat{\mathscr{K}}$ is equipped with the compact-open topology, and since $\Phi(x_0) \neq 1$, we have that if α is sufficiently large, then $|\langle \tau_\Phi(a_\alpha), x \rangle - 1| < |\Phi(x_0) - 1|$ for all $x \in B_M$. When this is the case, it follows that $x_0 \notin a_\alpha B_M$. In other words, $|x_0| > |a_\alpha| M$, and since this holds for every M, it follows that $|a_\alpha|$ must eventually be small, and hence $a_\alpha \to 0$. Thus τ_Φ is relatively open. □

[18] Recall that the only non-local locally compact fields are discrete fields. However, a discrete group is self-dual if and only if it is finite. So excluding the finite fields, all self-dual fields are necessarily local.

Corollary 4.19. *Let $\mathscr{K}$ be a local field. Then $\mathscr{K}$ is self-dual.*

Proof. This follows from Corollary 4.16 and Propositions 4.17 and 4.18. □

While this is an incredibly useful result, it is often a boon to avoid directly identifying $\mathscr{K}$ and $\widehat{\mathscr{K}}$, and we strive to do this when reasonable. To compensate, let us briefly provide a more concrete description of the characters of $\mathscr{K}$. When $\mathscr{K} = \mathbb{R}$, the characters are well known, and take the form

$$\varphi_s(x) = e^{2\pi i x s} \tag{4.7}$$

for some fixed $s \in \mathbb{R}$. This mapping $s \to \varphi_s$ is one of these aforementioned self-adjoint homomorphisms exhibiting self-duality. When $\mathscr{K} = \mathbb{C}$, then this is group isomorphic to $\mathbb{R}^2$, so the self-duality of $\mathbb{C}$ is exhibited in a similar manner.

For non-Archimedean fields $\mathscr{K}$, recall that $\mathscr{M}$ is the unique maximal ideal inside its ring of integers $\mathscr{R}$. Then the annihilator chain property of Corollary 2.12 gives the following observation.

Corollary 4.20. *Let $\mathscr{K}$ be a non-Archimedean local field. Then for any non-trivial $\varphi \in \widehat{\mathscr{K}}$, there is some minimal $n \in \mathbb{Z}$ such that $\mathscr{M}^n \subseteq \ker \varphi$.*

Proof. Since $\bigcap \mathscr{M}^n = \{0\}$, this follows by Corollary 2.12. Minimality holds by the non-triviality of φ. □

We shall call the integer n with this property the **order** of φ, and denote it by $\mathrm{ord}(\varphi)$ (noting the similarity to order functions we have seen prior). Indeed, this satisfies many of the usual properties of the order map, though it is exhibited in $\widehat{\mathscr{K}}$. In general, when referring to a fixed non-trivial character Φ, we shall assume without loss of generality that Φ has order 0.

One particular use for this description is in computing the dual space of compact DVRs. Recall that for a compact DVR $\mathscr{R}$ is the ring of integers of a non-Archimedean field $\mathscr{K}$. Since in particular this is a subgroup of $\mathscr{K}$, we may identify $\widehat{\mathscr{R}}$ with $\widehat{\mathscr{K}}/\mathscr{R}^\perp$. It is easy to check that $\mathscr{R}^\perp = \{\varphi \in \widehat{\mathscr{K}} : \mathrm{ord}(\varphi) \leq 0\}$. In particular, when we write $\widehat{\mathscr{K}} = \tau_\Phi(\mathscr{K})$, we also observe that $\mathscr{R}^\perp = \tau_\Phi(\mathscr{R})$, and hence, $\widehat{\mathscr{R}} = \mathscr{K}/\mathscr{R}$. Moreover, since $\mathscr{R}$ is compact, it follows that $\widehat{\mathscr{R}}$ is discrete.

If we now define $\mathscr{N}_n := (\mathscr{M}^n)^\perp$, we have that $\mathscr{N}_n = (\widehat{\mathscr{R}/\mathscr{M}^n})$. However, we know by Corollary 3.30 that $\mathscr{R}/\mathscr{M}^n$ is finite, and so it follows that $\mathscr{N}_n = \mathscr{R}/\mathscr{M}^n$. In particular, $|\mathscr{N}_n| = q^n$ where $q = |\kappa|$ and κ is the residue field.

Furthermore, by Krull's Intersection Theorem, it is clear that the collection $\{\mathscr{N}_n\}_{n\in\mathbb{N}}$ forms an increasing chain whose union is all of $\widehat{\mathscr{R}}$. Thus for any $\varphi \in \widehat{\mathscr{R}}$, we can find a minimal integer n such that $\varphi \in \mathscr{N}_n$. We shall call its negation the **order** of φ and we write $\operatorname{ord}(\varphi) = -n$. Note that if we identify $\widehat{\mathscr{R}}$ with $\mathscr{K}/\mathscr{R}$, and we write $\varphi = \psi + \mathscr{R}$, then $\operatorname{ord}(\varphi) = \operatorname{ord}(\psi)$, and so these notions of order coincide.

4.2 Local Spaces

4.2.1 Vector Spaces over Local Fields

We now turn our focus to vector spaces over local fields $\mathscr{K}$. Given such a field, recall that we say that $\mathscr{V}$ is a **topological vector space** (**TVS**) over $\mathscr{K}$ if $\mathscr{V}$ is equipped with a topology which makes the standard vector operations (vector addition, scalar multiplication) continuous. It shall be assumed that any such topological vector space is Hausdorff.

Given any vector space $\mathscr{V}$ over $\mathscr{K}$, we may attempt to equip it with a *norm*. While normed spaces are typically defined over $\mathbb{R}$ and $\mathbb{C}$, one may borrow the definition and use it for local fields more generally. We state the definition here for posterity, though its formulation should hardly be surprising.

Definition 4.21. Let $\mathscr{V}$ be a vector space over a local field $\mathscr{K}$. We say that $\|\cdot\| : \mathscr{V} \to \mathbb{R}_{\geq 0}$ is a **norm** if it satisfies:

(N1) $\|x\| = 0$ if and only if $x = 0$,

(N2) $\|cx\| = |c|\|x\|$, and

(N3) $\|x + y\| \leq \|x\| + \|y\|$.

for all $x, y \in \mathscr{V}$ and $c \in \mathscr{K}$.

Naturally, we can thus induce a topology on $\mathscr{V}$ via the metric $d(x, y) = \|x - y\|$, and this topology is indeed a TVS. The procedure for this should be a standard exercise by now. We say a TVS $\mathscr{V}$ is a **normed space** or that it is **normed** by $\|\cdot\|$ if the topology generated by a norm $\|\cdot\|$ coincides with the pre-existing topology on $\mathscr{V}$.

Definition 4.22. Let $\mathscr{V}$ be normed by $\|\cdot\|$. For $r > 0$ we define the **ball** of radius r to be the *closed* ball

$$\mathscr{B}_r := \{x \in \mathscr{V} : \|x\| \leq r\}$$

and similarly we define the **sphere** of radius r to be

$$\mathscr{S}_r := \{x \in \mathscr{V} : \|x\| = r\}$$

We also define the **line** through y to be

$$\mathscr{K}y := \{ay : a \in \mathscr{K}\}$$

for a fixed non-zero $y \in \mathscr{V}$

It is clear that if we consider $\mathscr{K}$ as a one-dimensional TVS over itself, then $\mathscr{K}$ is in fact normed by $|\cdot|$. In higher dimensions where $\mathscr{V} = \mathscr{K}^d$, we generalise this norm to

$$\|(x_1, \ldots, x_n)\|_\infty = \max_{i=1,\ldots,d} |x_i| \tag{4.8}$$

which one may verify is indeed a norm. While there are many similar norms that one could define (for instance any of the ℓ^p-norms), this ∞-norm has a special property which we call *exactness*. Recall that we let $|\mathscr{K}| \subseteq \mathbb{R}_{\geq 0}$ denote the image of $\mathscr{K}$ under the absolute value map $|\cdot|$.

Definition 4.23. Let $\mathscr{V}$ be a TVS normed by $\|\cdot\|$. We say that that the norm $\|\cdot\|$ is **exact** if $\|\mathscr{V}\| = |\mathscr{K}|$.

This definition is particularly useful for totally disconnected (non-Archimedean) fields; when $\mathscr{K}$ is Archimedean, then it is clear that every norm is exact. It is clear that the ∞-norm defined in Eq. (4.8) is exact. Such norms have a special property: any non-empty sphere $\mathscr{S}_r$ and any line $\mathscr{K}x$ *always* intersect. As a result of this intersection property, all exact norms will be relatively open.

Proposition 4.24. *Let $\mathscr{V}$ be normed by an exact norm $\|\cdot\|$. Then the map $x \mapsto \|x\|$ is relatively open.*

Proof. When the field $\mathscr{K}$ is Archimedean, it is a well-known fact that the norm is an open map. Otherwise, if $\mathscr{K}$ is non-Archimedean, then $|\mathscr{K}^\times|$ is discrete by Theorem 4.8, and so $\|\mathscr{V}\|$ is also discrete. Thus $x \mapsto \|x\|$ is relatively open. □

> *Remark* 4.25. Exactness does not necessarily hold for other norms: for instance, the standard ℓ^1-norm of $\mathbb{Q}_p^2$. Consider the line through the point $x = (1, p)$. If there were some $a \in \mathscr{K}$ such that $ax \in \mathscr{S}_1$, then
>
> $$1 = \|ax\| = |a|\|x\| = |a|(1 + p^{-1})$$
>
> and so $|a| = (1 + p^{-1})^{-1}$. However, this is not of the form p^n for some $n \in \mathbb{Z}$, and so

the line $\mathscr{K}x$ does not intersect the unit sphere.

Compounding this issue, this norm is not relatively open either. Take the open ball $U = \mathrm{B}((1,1);2^{-1})$, and note that if $x \in U$, then $x = (1+a, 1+b)$ where $|a|+|b| \leq 2^{-1}$. Since both a and b have absolute value less than 1, then $|1+a| = |1+b| = 1$. Hence $\|x\| = 2$, and so $\|U\| = \{2\}$. However, we have $\|\mathscr{V}\| = \{2^n + 2^m : n, m \in \mathbb{Z}\}$, so clearly the subset $\{2\}$ is not open. This shows that $\|\cdot\|_1$ on $(\mathbb{Q}_2)^2$ is not (relatively) open.

Another important property of the ∞-norm is that the metric it induces is proper. In general, when a norm induces a proper metric, we shall naturally say that the norm is **proper** as well.

Lemma 4.26. *The norm $\|\cdot\|_\infty$ on $\mathscr{K}^d$ is proper.*

Proof. Recall that $\mathscr{K}$ is proper and so $B_r := \{a \in \mathscr{K} : |a| \leq r\}$ is compact for $r > 0$. Since $\mathscr{B}_r = (B_r)^d$, this ball must also be compact as well. Finally, for arbitrary $x \in \mathscr{K}^d$, compactness of $\mathrm{B}[x;r]$ follows by continuity of vector addition. □

As is well known in the real and complex cases, any two norms on a finite-dimensional vector space are equivalent, that is, they generate the same topology. In addition, the following statement true: any finite-dimensional TVS is normed, and is therefore unique up to dimension. This seems to be a well-known result originally due to Weil [61, Chapter II, §1, Proposition 1], however we shall follow a proof given by an online post of Tao's [57]. We have modified this proof to work for local fields as well.

Proposition 4.27. *Let $\mathscr{V}$ be a finite-dimensional TVS with basis $\{e_1, \ldots, e_d\}$ over a local field $\mathscr{K}$. Define $f : \mathscr{K}^d \to \mathscr{V}$ by $f(x_1, \ldots, x_d) = \sum_{i=1}^d x_i e_i$ where $\mathscr{K}^d$ is equipped with the ∞-norm. Then f is a homeomorphism.*

Proof. It is readily checked that f is a (linear) bijection. Furthermore, f is also continuous. Indeed, suppose that $x_\alpha \to 0$ in $\mathscr{K}^d$ where $x_\alpha = (x_{1,\alpha}, \ldots, x_{d,\alpha})$. Since $\max_{i=1,\ldots,d}|x_{i,\alpha}| = \|x_\alpha\|_\infty \to 0$, it follows that $x_{i,\alpha} \to 0$ for each i, and so

$$f(x_\alpha) = \sum_{i=1}^{d} x_{i,\alpha} e_i \to 0$$

by continuity of vector operations.

All that remains now is to show that f is open. First we show that there is a 'bounded' neighbourhood of 0 in $\mathscr{V}$, in the sense that its preimage is bounded. To this end, consider the unit sphere $\mathcal{S}_1$ in $\mathcal{K}^d$. Note that $\mathcal{S}_1$ is compact, so that $f(\mathcal{S}_1)$ is compact as well. Let U be the complement of $f(\mathcal{S}_1)$, which will be an open neighbourhood of 0. By joint continuity of scalar multiplication, there is some $\varepsilon > 0$ and open neighbourhood V of 0 such that $B_\varepsilon V \subseteq U$, where $B_\varepsilon = \{a \in \mathcal{K} : |a| < \varepsilon\}$.

Let $x = (x_1, \ldots, x_d) \in \mathcal{K}^d$, and suppose that $f(x) \in V$. It follows then that for any $a \in B_\varepsilon \setminus \{0\}$ we have $f(ax) \in U$. Since U does not contain the unit sphere, this means that $\|ax\| \neq 1$, and in particular $\|x\|_\infty \neq |a|^{-1}$. This holds for every $a \in B_\varepsilon \setminus \{0\}$, and since $\|\cdot\|_\infty$ is exact, it follows that $\|x\|_\infty < \varepsilon^{-1}$. By definition, we then have that each $x_i \in B_{\varepsilon^{-1}}$.

Thus we have obtained V, a neighbourhood of 0 with bound $M = \varepsilon^{-1}$. Let us confirm that its existence does indeed turn f into an open map. Take $U = \mathrm{B}(0; r)$ for some radius $r > 0$. It then follows that if $v \in \frac{r}{M}V$, then $\|f^{-1}(v)\|_\infty < r$, and so $f(v) \in f(U)$. That is, $f(U)$ contains an open neighbourhood around 0, and by continuity of translation, this shows that f is an open map. □

The previous result shows that finite-dimensional spaces are very well behaved. For one, they are always isomorphic to a copy of $\mathcal{K}^d$, equipped with the ∞-norm. However, even when they reside in an ambient TVS, they are still closed subspaces.

Corollary 4.28. *Let $\mathcal{V}$ be a TVS over a local field $\mathcal{K}$, with a finite-dimensional subspace W. Then W is closed in $\mathcal{V}$.*

Proof. Take any $x \in \mathcal{V} \setminus W$, and suppose that x_α is a net in W such that $x_\alpha \to x$. However, if we consider the finite-dimensional space $S = \mathrm{Span}(W \cup \{x\})$, then since W is closed in S (by Proposition 4.27), it is not possible that $x_\alpha \to x$. Hence x lies in the exterior of W, and so W is closed. □

This leads us to the following theorem, which gives a nice characterisation for locally compact topological vector spaces. This result also borrows ideas from the aforementioned post by Tao [57].

Theorem 4.29. *Let $\mathcal{V}$ a TVS over a local field $\mathcal{K}$. The following are equivalent.*

(i) *$\mathcal{V}$ is locally compact.*

(ii) *$\mathcal{V}$ is finite-dimensional.*

(iii) *$\mathcal{V}$ is normed by an exact proper norm $\|\cdot\|$.*

Proof. **(i) ⇒ (ii)** Let K be a compact neighbourhood of the origin. Fixing any $\lambda \in \mathscr{K}$ such that $0 < |\lambda| < 1$, there is a finite set S with $K \subseteq S + |\lambda|K$. Let W be the finite-dimensional subspace spanned by S so that $K \subseteq W + |\lambda|K$. If we 'iterate' this inclusion, we find

$$K \subseteq W + |\lambda|(W + |\lambda|K) = W + |\lambda|^2 K$$

and in general, $K \subseteq W + |\lambda|^n K$ for all $n \in \mathbb{N}$.

Now let U be any neighbourhood of the origin. Since $\lambda^n \to 0$, it must be that λ^n is eventually inside U. Thus by joint continuity of multiplication at zero, $\lambda^n K \subseteq U$ for sufficiently large n. So $K \subseteq W + U$ for every neighbourhood U. Since W is finite-dimensional and hence closed by Corollary 4.28, it thus follows that $K \subseteq W$. Now K is a compact neighbourhood of the origin, so for any $x \in \mathscr{V}$, we can find some $\alpha \in \mathscr{K}$ such that $\alpha x \in K \subseteq W$. Thus $W = \mathscr{V}$.

(ii) ⇒ (iii) By Proposition 4.27, there exists a linear homeomorphism $f : \mathscr{K}^d \to \mathscr{V}$. Then we equip $\mathscr{V}$ with the exact proper norm $\|x\|_{\mathscr{V}} := \|f^{-1}(x)\|_\infty$.

(iii) ⇒ (i) By Proposition 2.4, we have that any proper metric space is immediately locally compact. □

This result leads to the following definition.

Definition 4.30. Given a Hausdorff topological vector space $\mathscr{V}$ over a local field $\mathscr{K}$, we say that $\mathscr{V}$ is a **local space** (or a **local $\mathscr{K}$-space**) if $\mathscr{V}$ is finite-dimensional.

We note that this terminology is non-standard. If it is not explicitly stated, we shall assume that the associated local field for a local space $\mathscr{V}$ is $\mathscr{K}$. By Theorem 4.29 we may assume that all local spaces are locally compact and come equipped with an exact proper norm $\|\cdot\|$.

Proposition 4.31. *Let $\mathscr{V}$ be a local space. Then $\mathscr{V}$ is separable and second-countable.*

Proof. Since $\mathscr{V}$ is a proper metric space the result follows by Proposition 2.4. □

4.2.2 Local Space Duality

Recall that a map $[\,\cdot\,,\,\cdot\,] : \mathscr{V} \times \mathscr{V} \to \mathscr{K}$ is called a **symmetric bilinear form** (or simply a **symmetric form**) if it is $\mathscr{K}$-linear in each argument, and it is symmetric. It will also be assumed that $[\,\cdot\,,\,\cdot\,]$ is **non-degenerate**; that is, for every non-zero $x \in \mathscr{V}$, there is some $y \in \mathscr{V}$ such that $[x, y] \neq 0$. We may always consider $[\cdot, \cdot]$ as a $\mathscr{K}$-bilinear map from $\mathscr{V} \times \mathscr{V} \to \mathscr{K}$, and so it must be separately continuous. A symmetric form always exists; one may be constructed by mimicking the standard dot product definition.

Proposition 4.32. *Let $\mathscr{V}$ be a local space. Then there exists a non-degenerate symmetric form $[\cdot, \cdot]$ on $\mathscr{V}$.*

Proof. Let $\{b_1, \ldots, b_d\}$ be a basis for $\mathscr{V}$. For $x = \sum_{i=1}^{d} x_i b_i$ and $y = \sum_{i=1}^{d} y_i b_i$ in $\mathscr{V}$, define

$$[x, y] := \sum_{i=1}^{d} x_i y_i$$

It is readily verified that $[\cdot, \cdot]$ is symmetric and bilinear. Furthermore, consider a non-zero $x \in \mathscr{V}$ as above, and suppose that $x_j \neq 0$. Then

$$[x, b_j] = \sum_{i=1}^{d} x_i \delta_{i,j} = x_j \neq 0$$

thus showing non-degeneracy. □

Henceforth, given a local space $\mathscr{V}$, we shall implicitly imbue it with some non-degenerate symmetric form $[\cdot, \cdot]$ (such as the one given above). It can be shown that any symmetric form can be written as $[x, y] = [x]_B^T A [y]_B$ for some matrix $A \in M_n(\mathscr{K})$, where $[x]_B, [y]_B$ denote the basis coordinates of $x, y \in \mathscr{V}$. Moreover, we have that A is invertible if and only if $[\cdot, \cdot]$ is non-degenerate. This can be shown in the same manner as one would for $\mathbb{R}$ or $\mathbb{C}$.

Recall now that we let Φ be a fixed non-trivial character of $\mathscr{K}$. Since $[\cdot, \cdot]$ maps $\mathscr{V} \times \mathscr{V}$ to $\mathscr{K}$, we may then leverage Φ to generate characters of $\mathscr{V}$. In fact, the structure of these characters is precisely another copy of $\mathscr{V}$.

Theorem 4.33. *Let $\mathscr{V}$ be a local space. A non-degenerate symmetric bilinear form $[\cdot, \cdot]$ witnesses an isomorphism $\mathscr{V} \to \widehat{\mathscr{V}}$ given by $x \mapsto \hat{x}$ where*

$$\langle \hat{x}, y \rangle = \Phi([x, y]) \tag{4.9}$$

for all $x, y \in \mathscr{V}$. In particular, it follows that $\mathscr{V}$ is self-dual.

Proof. For convenience, let us denote the mapping $x \mapsto \hat{x}$ by π. It is a routine check that $\hat{x} \in \widehat{\mathscr{V}}$ for any $x \in \mathscr{V}$, that the mapping π is a homomorphism. Continuity of π is also fairly straightforward to see, since both Φ and $[\,\cdot\,,\,\cdot\,]$ are continuous. To show that π is relatively open, one may adapt the proof as in Proposition 4.18; we leave this as an exercise to the reader.

To show injectivity, fix an $\alpha \in \mathscr{K}$ for which $\Phi(\alpha) \neq 1$. Then take any $x \in \mathscr{V}$ such that $x \neq 0$, and choose $y \in \mathscr{V}$ such that $[x, y] = \beta \neq 0$. It follows that

$$\langle \hat{x}, \alpha\beta^{-1}y\rangle = \Phi([x, \alpha\beta^{-1}y]) = \Phi(\alpha\beta^{-1}[x, y]) = \Phi(\alpha) \neq 1$$

and so $\hat{x}$ is not trivial. Hence the kernel of π is zero and thus π is injective. Lastly, it can be seen that π is self dual, since $[\,\cdot\,,\,\cdot\,]$ is symmetric. Thus by Corollary 4.16, π is an isomorphism of $\mathscr{V} \to \widehat{\mathscr{V}}$. □

It should be noted that converse of Theorem 4.33 is not true: not all isomorphisms $\mathscr{V} \to \widehat{\mathscr{V}}$ can be associated to a symmetric form. For instance, if one precomposes the dual pairing with a non-trivial field automorphism, the result will not necessarily be $\mathscr{K}$-linear. However, dual pairings which arise from a symmetric forms are somewhat preferable, as they give a nicer description of structures such as the (pre)annihilators of a subspace.

The key to this is that annihilation in the Pontryagin sense is identical to orthogonality of this symmetric form. For any $x, y \in \mathscr{V}$, we say that x and y are **orthogonal** with respect to $[\,\cdot\,,\,\cdot\,]$ if $[x, y] = 0$. Furthermore, if $W \subseteq \mathscr{V}$ is a subspace we define the **orthogonal complement** $W^\perp$ of W by

$$W^\perp := \{x \in \mathscr{V} : [x, y] = 0 \text{ for all } y \in \mathscr{V}\} \tag{4.10}$$

As one may expect, these complements have the usual properties.

Proposition 4.34. *Let $\mathscr{V}$ be a local space, and W a subspace of $\mathscr{V}$. Then the following hold:*

(i) *$W^\perp$ is a subspace of $\mathscr{V}$,*

(ii) $\dim W + \dim W^\perp = \dim \mathscr{V}$, *and*

(iii) $(W^\perp)^\perp = W$.

Proof. **(i)** This is a standard exercise.

(ii) Let $\{b_1, \ldots, b_n\}$ be a basis for W, and define

$$f(x) = \sum_{i=1}^{n} [x, b_i] b_i$$

Note that $y \in W^\perp$ precisely when $[y, b_i] = 0$ for every i, so $\ker f = W^\perp$. Furthermore, since the symmetric form is non-degenerate, it is readily shown that $\operatorname{Ran} f = W$. The result then follows by rank-nullity.

(iii) It is clear that $W \subseteq (W^\perp)^\perp$. By (ii), it follows that $\dim W = \dim (W^\perp)^\perp$, and so equality holds. □

As alluded to earlier, they coincide with the Pontryagin notion of orthogonality (and this justifies the notation used). To help us distinguish the different notions of orthogonality, we will let $W^\perp$ denote the orthogonal subspace as defined in (4.10), whereas $\hat{W}^\perp$ shall denote the annihilator of $\hat{W}$ (the image of W under the isomorphism given by Theorem 4.33).

Proposition 4.35. *Let $\mathscr{V}$ be a local space, and W a subspace of $\mathscr{V}$. Then $W^\perp = \hat{W}^\perp$.*

Proof. If $x \in W^\perp$, then for every $y \in W$, we have that

$$\langle \hat{y}, x \rangle = \Phi([y, x]) = \Phi(0) = 1$$

and so $x \in \hat{W}^\perp$. On the other hand, if $x \in \hat{W}^\perp$, take $y \in W$ and set $\alpha = [y, x]$. Suppose that $\alpha \neq 0$. Then for any $\beta \in \mathscr{K}$, we would have that

$$\Phi(\beta) = \Phi(\beta\alpha^{-1}[y, x]) = \langle \widehat{\beta\alpha^{-1}y}, x \rangle = 1$$

with the final equality holding since W is a subspace. This would imply that Φ is trivial, which is a contradiction. Hence $\alpha = 0$, and so $x \in W^\perp$. □

4.2.3 Actions on Local Spaces & the Projective Space

In the prior chapter, we expressed interest in the action $\mathscr{R}^* \curvearrowright \mathscr{R}$ where $\mathscr{R}$ is a compact DVR and the action was implemented via multiplication. We also want examine variants of this action to local fields $\mathscr{K}$ and local spaces $\mathscr{V}$. For instance, we can extend $\mathscr{R}^* \curvearrowright \mathscr{R}$ to $\mathscr{R}^* \curvearrowright \mathscr{K}$ for a non-Archimedean local field $\mathscr{K}$. It is readily verified that the orbits of this action are of the form $\mathscr{M}^n \setminus \mathscr{M}^{n+1}$ for $n \in \mathbb{Z}$, which precisely correspond to the (discrete) level sets of $|\mathscr{K}|$. We can generalise this to any local field.

Definition 4.36. Let $\mathscr{K}$ be a local field. We define the **unit circle** of $\mathscr{K}$ to be

$$\mathscr{U} := \{x \in \mathscr{K} : |x| = 1\}$$

as a subgroup of $\mathscr{K}^\times$.

In this case, the orbits of the action $\mathscr{U} \curvearrowright \mathscr{K}$ are indexed precisely by the level sets of $|\mathscr{K}|$. A particularly nice property of this action, as opposed to $\mathscr{K}^\times \curvearrowright \mathscr{K}$, is that the group $\mathscr{U}$ is compact. As mentioned in Section 2.5, we are primarily interested in compact actions, as the representation structures of the semidirect products are considerably easier to compute.

We also wish to extend this action to local spaces $\mathscr{V}$. Perhaps the most natural way to realise this is via the action $\mathscr{U} \curvearrowright \mathscr{V}$, where this action is implemented via scalar multiplication. We will eventually need to utilise the dual action of this group, however, we are fortunate in that $\mathscr{U} \curvearrowright \mathscr{V}$ is isomorphic to its dual.

Proposition 4.37. *Let $\mathscr{V}$ be a local space. The action $\mathscr{U} \curvearrowright \mathscr{V}$ via multiplication is isomorphic to its dual action $\mathscr{U} \curvearrowright \widehat{\mathscr{V}}$.*

Proof. Let $u \in \mathscr{U}$ and $x \in \mathscr{V}$. For $y \in \mathscr{V}$ we have

$$\widehat{u \cdot x}(y) = \Phi([ux, y]) = \Phi([x, uy]) = \hat{x}(u \cdot y) = u^{-1} \cdot \hat{x}(y)$$

and so $\widehat{u \cdot x} = u^{-1} \cdot \hat{x}$. Note that $\mathscr{U}$ is abelian, so that the inversion map is a genuine isomorphism. □

Another important aspect of this action is its orbit structure $\mathscr{O}_{\mathscr{U}}(\mathscr{V})$. A key insight in determining this structure is that $\mathscr{U} \curvearrowright \mathscr{V}$ will fix all lines in $\mathscr{V}$. We call the space of all lines in $\mathscr{V}$ the *projective space* of $\mathscr{V}$. We note however that the origin is problematic as it intersects all lines, so let us set $\mathscr{V}^\times := \mathscr{V} \setminus \{0\}$.

Definition 4.38. Let $\mathscr{V}$ be a local space. Given $x, y \in \mathscr{V}^\times$, we write $x \sim_\ell y$ if there is some $\lambda \in \mathscr{K}^\times$ such that $x = \lambda y$. We define the **projective space** of $\mathscr{V}$ to be the quotient space $\mathbf{P}(\mathscr{V}) := \mathscr{V}^\times / \sim_\ell$.

Given an element $x \in \mathscr{V}^\times$, we will let $[x]_\ell$ denote its image in $\mathbf{P}(\mathscr{V})$, and we occasionally identify this with the line $\mathscr{K}x \subseteq \mathscr{V}$. Moreover, by Proposition 2.53, the quotient map $x \mapsto [x]_\ell$ must be open. As a result, we obtain a particularly useful feature in that the projective space is compact.

Proposition 4.39. *If $\mathscr{V}$ is a local space, then $\mathbf{P}(\mathscr{V})$ is compact.*

Proof. Let $\|\cdot\|$ be an exact proper norm. In particular, the unit sphere $\mathcal{S}_1$ is compact, and by exactness, must intersect every line. It follows that $[\mathcal{S}_1]_\ell = \mathbf{P}(\mathscr{V})$, and so $\mathbf{P}(\mathscr{V})$ must be compact. □

We can then use the projective space to describe precisely the orbit space $\mathbb{O}_{\mathcal{U}}(\mathscr{V}^\times)$. This will correspond to a projective space component which determines the line containing the orbit, and another component which further divides the line.

Proposition 4.40. *Let $\mathscr{V}$ be a local space over $\mathscr{K}$, and let $\mathcal{U}$ act on $\mathscr{V}$ by scalar multiplication. Then $\mathbb{O}_{\mathcal{U}}(\mathscr{V}^\times) \cong |\mathscr{K}^\times| \times \mathbf{P}(\mathscr{V})$.*

Proof. Define the map $f : \mathscr{V}^\times \to |\mathscr{K}^\times| \times \mathbf{P}(\mathscr{V})$ by

$$f(v) = (\|v\|, [x]_\ell)$$

where $\|\cdot\|$ is an exact proper norm on $\mathscr{V}$. We already know that both $\|\cdot\|$ and $[\,\cdot\,]_\ell$ are bicontinuous maps, so f must be as well. It is also easy to verify that f is surjective as well.

Furthermore if $f(v) = f(w)$, then since $[v]_\ell = [w]_\ell$ there is some $\lambda \in \mathscr{K}^\times$ such that $v = \lambda w$. But since $\|v\| = \|w\|$ then $|\lambda| = 1$, so $v \in [w]_{\mathcal{U}}$. Conversely, the orbits of $\mathcal{U} \curvearrowright \mathscr{V}^\times$ map to singletons, and so the mapping of f induces a homeomorphism between $\mathbb{O}_{\mathcal{U}}(\mathscr{V}^\times)$ and $|\mathscr{K}^\times| \times \mathbf{P}(\mathscr{V})$. □

We note that we explicitly exclude the origin in the above proof, as it makes the result cleaner. One could extend this result to show that $\mathbb{O}_{\mathcal{U}}(\mathscr{V})$ will be homeomorphic to $|\mathscr{K}| \times \mathbf{P}(\mathscr{V})/{\sim}$, where $\sim$ identifies all elements of the form $(0, [x]_\ell)$. However, the above result for $\mathscr{V}^\times$ proves sufficient, so we leave this extension as an exercise to the reader.

Of course, if $\mathscr{V}$ has dimension 1, then $\mathbf{P}(\mathscr{V})$ is trivial, and so $\mathbb{O}_{\mathcal{U}}(\mathscr{V}^\times)$ is homeomorphic to $|\mathscr{K}^\times|$, as we have already seen. If we now consider the case where $\mathscr{V}$ is dimension 2, we have a fairly nice description of the structure of $\mathbf{P}(\mathscr{V})$.

Proposition 4.41. *Let $\mathscr{V}$ be a local space over $\mathscr{K}$. If $\dim \mathscr{V} = 2$, then $\mathbf{P}(\mathscr{V})$ is homeomorphic to $\mathscr{K}^\circ$, the one-point compactification of $\mathscr{K}$.*

Proof. Let us assume without loss of generality that $\mathscr{V} = \mathscr{K}^2$. Consider the set $D = \{(x, y) \in \mathscr{V} : x = 1\}$. It is easy to verify that the quotient map is a homeomorphism from D onto its image $[D]_\ell$ in $\mathbf{P}(\mathscr{V})$. So in particular we have that $[D]_\ell \cong D$ and these are clearly homeomorphic to $\mathscr{K}$ (consider the mapping $(x, y) \mapsto y$).

Now, observe that $\mathbf{P}(\mathscr{V}) \setminus [D]_\ell$ contains precisely one point, namely the line $[(0, 1)]_\ell$. Since $\mathbf{P}(\mathscr{V})$ is compact, we have by Corollary B.7 that the one-point compactification $[D]_\ell^\circ$ is homeomorphic to $\mathbf{P}(\mathscr{V})$. Thus $\mathscr{K}^\circ \cong D^\circ \cong \mathbf{P}(\mathscr{V})$. □

This gives us a concrete description of the orbit space of $\mathscr{U} \curvearrowright \mathscr{V}^\times$ in dimension 2.

Corollary 4.42. *Let $\mathscr{V}$ be a local space over $\mathscr{K}$, and let $\mathscr{U}$ act on $\mathscr{V}$ by scalar multiplication. If $\dim \mathscr{V} = 2$, then $\mathfrak{S}_{\mathscr{U}}(\mathscr{V}^\times) \cong |\mathscr{K}^\times| \times \mathscr{K}^\circ$.*

The symmetric form we defined earlier gives a duality pairing between the line of a space $\mathscr{V}$ and its *hyperplanes*. We define these as follows.

Definition 4.43. Let $\mathscr{V}$ be a local space. We say that a subspace $H \subseteq \mathscr{V}$ is a **hyperplane** of $\mathscr{V}$ if $\dim H = \dim \mathscr{V} - 1$. Furthermore, we let $\mathbf{H}(\mathscr{V})$ denote the collection of all hyperplanes of $\mathscr{V}$.

There is a natural bijective correspondence between the lines and the hyperplanes of a projective space of $\mathscr{V}$. Given an $x \in \mathscr{V}^\times$, we set

$$H_x := (\mathscr{K}x)^\perp = \{y \in \mathscr{V} : [x, y] = 0\} \tag{4.11}$$

which is easily verified to be a hyperplane. It is clear by Proposition 4.34 that this also implements the aforementioned bijective correspondence between $\mathbf{H}(\mathscr{V})$ and $\mathbf{P}(\mathscr{V})$ via $[x]_\ell \mapsto H_x$. The hyperplanes will play an important role when studying the irreducible representation structure of $\mathscr{V} \rtimes \mathscr{U}$ in Section 5.5. We can see this even in the character structure of $\mathscr{V}$, since by Theorem 4.33, any character must takes the form $\Phi([x, \cdot])$ for some $x \in \mathscr{V}$. In particular, every character will always annihilate a hyperplane. For now, let us finish with a (fairly intuitive) result which characterises dimension 2 local spaces.

Proposition 4.44. *Let $\mathscr{V}$ be a local space of dimension d. The intersection of two distinct hyperplanes $H, H' \in \mathbf{H}(\mathscr{V})$ has dimension $d - 2$. In particular, we have that $H \cap H' = \{0\}$ precisely when $d = 2$.*

Proof. Choose $x, y \in \mathscr{V}$ such that $H = (\mathscr{K}x)^\perp$ and $H' = (\mathscr{K}y)^\perp$. It is easily verified that the map $\pi : \mathscr{V} \to \mathscr{K}^2$ given by $\pi(z) = ([x, z], [y, z])$ is linear and has kernel $H \cap H'$. Moreover, since $H \neq H'$, we have that π is surjective. By rank-nullity, it follows that $\dim(H \cap H') = d - 2$. □

Chapter 5

The Spine of Local Fell Groups

" To have good posture, you must always be thinking about your posture. "

–Z. Zhang

In this chapter we explore the structure of the Fourier-Stieltjes algebras of a certain class of groups, which we call *cheap groups*. If we let G denote such a group, then G is the semidirect product of a compact group K acting on an abelian locally compact group A in an 'almost-free' manner. Due to this, the dual space $\widehat{G}$ has an accessible description, both in terms of the algebraic as well as the topological structure. We then apply this to "*local Fell*" groups in order to determine the nature of their irreducible representations. In these constructions, we see close connections with questions relating to the structure of the Fourier and Fourier-Stieltjes algebras, as well as the so-called *Rajchman algebra* $B_0(G)$, and the *spine of the Fourier-Stieltjes algebra* $A^*(G)$, the latter which was introduced by Ilie and Spronk in [27]. We examine these connections and discuss their implications.

The first section of this chapter details the background to the motivation behind these cheap groups, and defines some key concepts. Sections 5.2 and 5.3 examine two methods of obtaining new representations from old: the direct integral and the inducing construction respectively. We provide a brief overview of each as well as some of the key results that we shall need. Later, in Section 5.3 specifically, we briefly examine the *Mackey machine*, a tool that can be used to compute the representation theory of certain semidirect products. We then define *cheap groups* in Section 5.4 and examine them through the lens of the Mackey machine in order to determine their representation structure. Furthermore, we provide a description of the various subalgebras of $B(G)$ that were mentioned beforehand. This is then applied to local Fell groups in Section 5.5, where we outline the structure of the Fourier-Stieltjes algebra and its spine. Finally, in Section 5.6, we show that higher-dimensional analogues of the local Fell group do not exhibit *spinality*.

5.1 The Spine of the Fourier-Stieltjes Algebra

Recall the following result on Fourier and Fourier-Stieltjes algebras, which can be found in some form in almost every standard text on the subject.

Theorem 5.1. *Let G be an abelian locally compact group. Then $B(G) \cong M(\widehat{G})$ and $A(G) \cong L^1(\widehat{G})$.*

This can be seen as these algebras are the images of the Fourier-Stieltjes and Fourier transforms respectively. This means, at least in the abelian group setting, that the Fourier-Stieltjes algebra $B(G)$ has the structure of a measure algebra, whereas the Fourier algebra $A(G)$ has the structure of an L^1 algebra. In some sense, this is true more generally. For instance, we have that the spectrum of $A(G)$ is G (see [34, Theorem 2.3.8]), mimicking the statement that the spectrum of $L^1(G)$ is $\widehat{G}$ in the abelian case. Meanwhile, a general description for the spectrum of $B(G)$ has not been found. Another way in which we see the parallels between these algebras is that $B(G) = A(G)$ precisely when G is compact. This again mimics the well-known result which states that $M(G) = L^1(G)$ precisely when G is discrete; this is often considered to be a dual notion of compactness. In this way, we often consider $B(G)$ to be 'large' and 'intractable', whereas $A(G)$ is more 'accessible'.

With this as motivation, perhaps a natural question to ask is the following: when is the Fourier-Stieltjes algebra $B(G)$ 'small' in some sense? One instance of this was answered by Runde and Spronk in [50, 51], where they show that a locally compact group G is compact precisely when $AM_{op}(B(G)) < 5$ [50, Theorem 3.2]. It was initially conjectured that the above statement would hold when 5 was replaced with ∞, however, in their second paper [51], they show that $AM_{op}(B(\mathbb{Q}_p \rtimes \mathbb{O}_p{}^*)) = 5$. This group is called the **Fell group** and is noncompact, showing that the bound of 5 is in fact sharp. This result involves computing the Fourier-Stieltjes algebra of $G = \mathbb{Q}_p \rtimes \mathbb{O}_p{}^*$ explicitly, and we state it below.

Theorem 5.2. *Let $G = \mathbb{Q}_p \rtimes \mathbb{O}_p{}^*$ be the Fell group. Then*

$$B(G) = A(G) \oplus A(\mathbb{O}_p{}^*) \circ q$$

where $q : G \to \mathbb{O}_p{}^*$ is the canonical quotient map.

This was first shown explicitly by Runde and Spronk [51, Proposition 2.1], though it follows from earlier work by Baggett in [9, Theorem 4.6] and Walter in [60]. The proof of this uses the Mackey machine, which as alluded to earlier we shall describe in Section 5.3. The keen-eyed reader will also note that the Fell group is defined in terms of the local field $\mathbb{Q}_p$, where the unit circle acts on the field itself. Indeed, we shall prove this result in the more general setting $G = \mathcal{K} \rtimes \mathcal{U}$ where $\mathcal{K}$ is a non-Archimedean local field and $\mathcal{U}$ its unit circle. We call such a group a **local Fell group**. Alas, the proof of this more general result shall have to wait until we have presented some Mackey machinery — see Proposition 5.47.

Notice though, that while $B(G) \neq A(G)$ (since G is not compact), we do see that $B(G)$ is very 'small' in some sense. In particular, it is the direct sum of two Fourier algebras, one of which is passed through a homomorphism. We wish to characterise this notion of smallness, and we shall do so by construction of the *spine*. This definition, introduced by Ilie and Spronk [27], is as follows.

Definition 5.3. Let G be a locally compact group. The **spine** of the Fourier-Stieltjes Algebra of G is a closed subalgebra of $B(G)$ defined by

$$A^*(G) := \overline{\text{Span}}\{u : u \in A(H) \circ \eta, (H, \eta) \in \text{Hom}(G, \cdot)\}$$

where $\text{Hom}(G, \cdot)$ denotes the collection of all pairs (H, η) such that H is a locally compact group and $\eta : G \to H$ is a homomorphism. For convenience, we define $A_\eta(G) := A(H) \circ \eta$.

> *Remark* 5.4. We note that in [27], this algebra is actually defined in terms of the so-called *locally precompact topologies*. These are the topologies induced by the given homomorphisms. So for a given continuous homomorphism $\eta : G \to H$, we define a topology τ on G by $\tau = \{\eta^{-1}(U) : U \subseteq H \text{ is open}\}$. This then induces a subspace $A_\tau(G)$ of $B(G_\tau)$ by considering the Fourier algebra of the group G_τ equipped with this new topology. However, the definition we present here is readily seen to be equivalent, and simplifies some discussions of this algebra.

Given two homomorphisms from G, there is a natural way to define their *join* that is compatible with the spine structure.

Definition 5.5. Let G, H_1, H_2 be locally compact groups, with homomorphisms $\eta_1 : G \to H_1$ and $\eta_2 : G \to H_2$. We define a homomorphism $\eta_1 \vee \eta_2$ by

$$\eta_1 \vee \eta_2(x) := (\eta_1(x), \eta_2(x)) \in H_1 \times H_2$$

for $x \in G$. We shall set the codomain of $\eta_1 \vee \eta_2$ to be the closure of its range inside $H_1 \times H_2$.

A particularly useful aspect of this definition is that it captures the multiplicative structure of $A^*(G)$. This result is due to [27, Proposition 3.1], and states that if η_1 and η_2 are homomorphisms, then $A_{\eta_1}(G)A_{\eta_2}(G) \subseteq A_{\eta_1 \vee \eta_2}(G)$. In a similar vein, we have that automorphisms of G preserve the Fourier algebra.

Proposition 5.6. *Let G be a locally compact group. If α is a (bicontinuous) automorphism of G, then $A_\alpha(G) = A(G)$.*

The proof of this result follows readily from the definition of the Fourier algebra and Lemma 2.65.

A natural question to ask is whether the Fourier-Stieltjes algebra can be small enough so that $B(G) = A^*(G)$. When this holds we shall say that G is **spinal**. Of course, all compact groups are spinal, as is the Fell group G. In [50], Runde and Spronk also computed $B(G)$ for higher dimensions, namely for groups of the form $G_{p,n} = \mathbb{Q}_p^n \rtimes \mathrm{GL}_n(\mathbb{O}_p)$. It was found that $B(G_{p,n})$ had a similar form as the Fell group, showing that these groups were spinal as well. Aside from the local field variations of these groups, this is the complete list of known noncompact spinal groups.

A key aspect of this proof is that the left regular representation λ is **completely reducible**: it can be written as a direct sum of irreducible representations. When this is the case, we say that G is an **AR-group**. If all representations can be decomposed in this manner, we say that G is an **AU-group**. A characterisation of the latter in the separable case is as follows, originally proved by Taylor [59, Theorem 4.5].

Theorem 5.7. *Let G be a separable locally compact group. Then G is an AU-group if and only if $\widehat{G}$ is countable.*

Such a characterisation has not been found for AR-groups, though there is a sufficient condition due to Baggett and Taylor [11, Theorem 2.1] in terms of the *Rajchman algebra*. Here, the **Rajchman algebra** of G is defined to be the subalgebra of $B(G)$ consisting of all functions which vanish at infinity, and is denoted by $B_0(G)$.

Theorem 5.8. *Let G be a separable locally compact group. If $B_0(G) = A(G)$, then G is an AR-group.*

It was originally suspected that the converse of this statement is true, but several counterexamples have been constructed which exhibit its failure. A connected counterexample was originally given by Baggett and Taylor [10], and a further unimodular example was provided by Knudby [36, Theorem 8.17].

5.2 Direct Integrals

In this section we shall very briefly introduce direct integrals and state some key results. In essence, the direct integral is a continuous analogue of the direct sum, much the same way an integral is a continuous analogue of summation. We shall present a brief overview of the machinery at play, but we shall provide little in the way of details, as they are rather technical and are not the focus of this thesis. Instead, we refer the keen reader to two sources which do provide a more thorough description of these objects: see either Dixmier's book [16, Chapter 8 & Appendices A69-98] or Folland's book [20, Section 7.4].

Without further delay, let us present a summary of the key ideas. Let Z be a Borel space. Whenever we refer to a measure μ on Z, unless otherwise noted, *we shall implicitly assume that μ is positive and σ-finite.* Let us pair to each $z \in Z$ a Hilbert space $\mathscr{H}_z$, from which one may define the **direct integral**

$$\mathscr{H} = \int_Z^{\oplus} \mathscr{H}_z \,\mathrm{d}\mu(z) \tag{5.1}$$

which will be a Hilbert space in its own right. For each $\xi \in \mathscr{H}$, one may associate a map $z \mapsto \xi_z \in \mathscr{H}_z$, though this map need not be unique. Rather, this association $z \mapsto \xi_z$ is determined by μ-a.e. equivalence. Furthermore, the map $z \mapsto \langle \xi_z \,|\, \zeta_z \rangle_z$ will be measurable for each $\xi, \zeta \in \mathscr{H}$ (where $\langle \cdot \,|\, \cdot \rangle_z$ denotes the inner product on $\mathscr{H}_z$). With this being the case, we have that the inner product on $\mathscr{H}$ is

$$\langle \xi \,|\, \zeta \rangle = \int_Z \langle \xi_z \,|\, \zeta_z \rangle_z \,\mathrm{d}\mu(z) \tag{5.2}$$

for $\xi, \zeta \in \mathscr{H}$. Note that $\mathscr{H}$ consists precisely of the elements ξ for which $\|\xi\|^2 = \langle \xi \,|\, \xi \rangle < \infty$. This construction is only unique up to equivalence classes of μ; recall that we say two positive measures μ and ν are **equivalent** if they are absolutely continuous with respect to one another. When this is the case, we write $\mu \approx \nu$, and otherwise we denote absolute continuity by $\mu \ll \nu$. We also let $\mu \perp \nu$ denote that the measures μ and ν are **singular**; that is there exists a measurable set $A \subseteq Z$ such that $\mu(A) = 0$ and $\nu(Z \setminus A) = 0$.

When $\mu \approx \nu$, the spaces $\int_Z^{\oplus} \mathscr{H}_z \,\mathrm{d}\mu(z)$ and $\int_Z^{\oplus} \mathscr{H}_z \,\mathrm{d}\nu(z)$ will be isomorphic. Let us present a few examples, to illustrate these ideas.

Example 5.9. If μ is the counting measure (or any discrete measure), then the direct integral $\int_Z^{\oplus} \mathscr{H}_z \,\mathrm{d}\mu(z)$ is isomorphic to the actual direct sum $\bigoplus_{z \in Z} \mathscr{H}_z$. An obvious corollary to this is that if δ_a is the point mass at some fixed point $a \in Z$, then $\int_Z^{\oplus} \mathscr{H}_z \,\mathrm{d}\delta_a(z) = \mathscr{H}_a$.

Example 5.10. If the field of Hilbert spaces is constant, that is $\mathscr{H}_z = \mathscr{H}$ for every $z \in Z$, then $\int_Z^{\oplus} \mathscr{H}_z \,\mathrm{d}\mu(z)$ is isomorphic to $L^2(Z, \mu; \mathscr{H})$.

We may also build operators on $\mathscr{H}$ in a similar manner: given operators $T_z \in \mathscr{B}(\mathscr{H}_z)$ such that $\operatorname{ess\,sup}_{z\in Z}\|T_z\| < \infty$, we may define

$$T = \int_Z^{\oplus} T_z \,\mathrm{d}\mu(z) \qquad \text{where} \qquad (T\xi)_z := T_z \xi_z \tag{5.3}$$

for all $\xi \in \mathscr{H}$ and μ-a.e. $z \in Z$. Note that for any two operators on $\mathscr{H}$, we have $T = S$ precisely when $T_z = S_z$ for μ-a.e. $z \in Z$.

As a result, we may also take direct integrals of group representations. Given a locally compact group G, let $\pi_z : G \to U(\mathscr{H}_z)$ be representations such that the map $z \mapsto \pi_z(x)$ is measurable for every $x \in G$ — we call such a map a **measurable field of representations**. Then we define a representation $\pi := \int_Z^{\oplus} \pi_z \,\mathrm{d}\mu(z)$ on $\mathscr{H} = \int_Z^{\oplus} \mathscr{H}_z \,\mathrm{d}\mu(z)$ given by

$$\pi(x) := \int_Z^{\oplus} \pi_z(x) \,\mathrm{d}\mu(x) \tag{5.4}$$

which one may verify is indeed a unitary representation. Moreover, we have that $(\pi(x)\xi)_z = \pi_z(x)\xi_z$ for every $x \in G$, $\xi \in \mathscr{H}$, and μ-a.e. $z \in Z$. We shall often say that the map $z \mapsto \pi_z$ (along with the measure μ) is a **disintegration** or a **decomposition** of π. When the space Z and the measurable field $z \mapsto \pi_z$ is fixed, we shall adopt a shorthand for writing direct integrals. We write $\pi_\mu := \int_Z^{\oplus} \pi_z \,\mathrm{d}\mu(z)$ for any positive measure μ on Z.

When G is sufficiently nice, all representations of G can be expressed as a direct integral of its irreducible representations.

Definition 5.11. Let G be a locally compact group. We say G is **type I** if the Fell topology on $\widehat{G}$ is T_0.

> *Remark* 5.12. The term "type I" is derived from the theory of von Neumann algebras. Recall that for a representation π of G, we define the von Neumann algebra $VN_\pi(G)$ to be generated by the image of π in $\mathscr{B}(\mathscr{H}_\pi)$. We say that π is a **primary** or **factor** representation if $VN_\pi(G)$ is a factor von Neumann algebra. Then, G is type I if and only $VN_\pi(G)$ is a type I von Neumann algebra for every factor representation π of G. This was originally shown in a paper of Glimm [22], in particular, as a result of the equivalence of (a2) and (a6) in Theorem 1.

As alluded to earlier, a key feature of type I groups is that every representation is disintegrable into irreducible representations. This theorem is stated below, though we shall omit its proof and instead refer the reader to [16, Theorem 8.4.2] and [20, Theorem 7.40].

Theorem 5.13. *Let G be a (second-countable) type I group, and π a representation on a (separable) Hilbert space. Then there exists a unique positive measure μ on $\widehat{G}$ such that*

$$\pi \approx \int_{\widehat{G}}^{\oplus} \rho \, \mathrm{d}\mu(\rho)$$

up to unitary equivalence.

One particularly important disintegration is that of the left regular representation λ. If G is a type I group, then there is a positive measure μ on $\widehat{G}$ such that

$$\lambda \approx \int_{\widehat{G}}^{\oplus} \pi \, \mathrm{d}\mu(\pi) \tag{5.5}$$

and if G is furthermore unimodular, then such a measure satisfies particularly nice properties. Namely, we get that

$$\int_G |f(x)|^2 \, \mathrm{d}x = \int_{\widehat{G}} \mathrm{tr}[\widehat{f}(\pi)\widehat{f}(\pi)^*] \, \mathrm{d}\mu(\pi) \tag{5.6}$$

for every $f \in L^1(G) \cap L^2(G)$, where $\widehat{f}$ is the *Fourier transform* of f, given in the usual way:

$$\widehat{f}(\pi) := \int_G f(x)\pi(x^{-1}) \, \mathrm{d}x \tag{5.7}$$

See Folland [20, 7.44] for a comprehensive version of this statement.

Definition 5.14. Let G be a (second-countable) unimodular type I group. When μ is the unique measure on $\widehat{G}$ as given by (5.6), we say that μ is the **Plancherel measure** of G.

The term *Plancherel measure* naturally comes from the Plancherel theorem of standard Fourier analysis. Indeed, when G is abelian, the Plancherel measure coincides the usual Haar measure on $\widehat{G}$. There is also a special case for compact groups: one finds via the Peter-Weyl theory that the Plancherel measure on G is the discrete measure with weights d_π for $\pi \in \widehat{G}$.

Whenever there is no ambiguity in the choice of the measurable field of representations $z \mapsto \pi_z$, we shall make the following shorthand. Given the representation $\pi_\mu = \int_Z^\oplus \pi_z \, \mathrm{d}\mu(z)$ for some positive measure μ, we shall denote

$$A_\mu(G) := A_{\pi_\mu}(G) \tag{5.8}$$

for short. One particular instance where this will occur is when the underlying space Z is $\widehat{G}$ or some subspace thereof, in which case there is an obvious canonical choice for the field of representations. In other instances, the field of representations will be implied.

Theorem 5.15. *Let G be a locally compact group, and Z a Borel space with an associated measurable field of representations $z \mapsto \pi_z$. If $u \in A_\mu(G)$ for some positive measure μ, then*

$$u = \int_{\widehat{G}} u_\pi \, \mathrm{d}\mu(\pi)$$

where each $u_\pi \in A_\pi(G)$.

This result is due to Arsac [5, Proposition 3.43]. Recall that $A_\mu(G)$ is a predual of a von Neumann algebra, so we may understand the integral as given above in the weak sense. Additionally, these Fourier spaces behave nicely with respect to absolute continuity of measures.

Theorem 5.16. *Let G be a locally compact group. Suppose Z is a Borel space with an associated measurable field of representations $z \mapsto \pi_z$, and let μ, ν be positive measures on Z. The following are equivalent.*

(i) $\mu \ll \nu$.

(ii) $\pi_\mu \preccurlyeq_q \pi_\nu$.

(iii) $A_\mu(G) \subseteq A_\nu(G)$.

Proof. The equivalence between (i) and (ii) follows from a result present in Dixmier's book [16, Proposition 8.4.5] and the equivalence of (ii) and (iii) follows as a result of Corollary 2.31. □

Corollary 5.17. *Let G be a locally compact group, and Z a Borel space with an associated measurable field of representations $z \mapsto \pi_z$. If μ is a positive measure such that $\mu(\{z\}) > 0$ for some $z \in Z$, then π_z is quasi-contained in π_μ.*

Proof. By Theorem 5.16, since $\delta_z \ll \mu$ it therefore follows that $\pi_z = \pi_{\delta_z} \preccurlyeq_q \pi_\mu$. □

These results suggest that the positive measures, representations, and Fourier spaces have a lattice structure, and that furthermore they are order isomorphic. The lattice operations on Fourier spaces are clear, the infimum is given simply by intersection. For measures, we define the meet in the following manner.

Definition 5.18. Let μ, ν be positive measures on a measurable space Z. We define the **meet** or **infimum** of μ and ν to be the measure

$$\mu \wedge \nu(A) := \inf\{(\mu - \nu)(E) + \nu(A) : E \subseteq A \text{ measurable}\} \tag{5.9}$$

One may quickly verify that this is a well defined measure, and that it is equivalent to the following definition:

$$\mu \wedge \nu(A) = \inf\{\mu(E) + \nu(A \setminus E) : E \subseteq A \text{ measurable}\} \tag{5.10}$$

Lemma 5.19. *Let μ, ν be positive measures on a measurable space Z. Then $\mu \wedge \nu$ is a genuine infimum of μ and ν with respect to absolute continuity.*

Proof. Suppose that $A \subseteq Z$ is measurable and $\mu(A) = 0$. It then follows that

$$\begin{aligned} \mu \wedge \nu(A) &= \inf\{\mu(E) + \nu(A \setminus E) : E \subseteq A \text{ measurable}\} \\ &= \inf\{\nu(A \setminus E) : E \subseteq A \text{ measurable}\} \\ &= 0 \end{aligned}$$

and so $\mu \wedge \nu \ll \mu$. Likewise $\mu \wedge \nu \ll \nu$.

On the other hand, suppose that η is any measure such that $\eta \ll \mu, \nu$. Let $A \subseteq Z$ be measurable so that $\mu \wedge \nu(A) = 0$. Then there is some $E_n \subseteq A$ such that $(\mu - \nu)(E_n) + \nu(A) < \frac{1}{n}$. We can use the Hahn-Jordan decomposition to write $\mu - \nu = \lambda_+ - \lambda_-$ where $\lambda_\pm$ are positive measures with $\lambda_+ \perp \lambda_-$.

With this decomposition, there must be infinitely many n such that $\lambda_-(E_n) > 0$; if not, then the quantity $(\mu - \nu)(E_n) + \nu(A)$ would eventually be bounded below by $\nu(A)$, and eventually would never be small. So, passing to a subsequence if necessary, we may then assume without loss of generality that $\lambda_+(E_n) = 0$. In this case, we have that $-\lambda_-(E_n) + \nu(A) < \frac{1}{n}$. Taking $E = \bigcup_{n=1}^\infty E_n$, we see that $-\lambda_-(E) + \nu(A) \leq 0$ and $\lambda_+(E) = 0$. Thus $(\mu - \nu)(E) + \nu(A) = 0$, and rewriting we find $\mu(E) + \nu(A \setminus E) = 0$, and so by absolute continuity we have

$$\eta(A) = \eta(E) + \eta(A \setminus E) = 0$$

from which it follows that $\eta \ll \mu \wedge \nu$. $\square$

This result, in conjunction with Theorem 5.16, tells us that the semilattice structures on measures of Z, representations of G, and Fourier spaces of G must be isomorphic. Let us clarify this statement briefly. Given two Fourier spaces $A_\pi(G)$ and $A_\sigma(G)$, we define their lattice meet to simply be their intersection. One may verify that $A_\pi(G) \cap A_\sigma(G)$ is a translation invariant subspace of both $A_\pi(G)$ and $A_\sigma(G)$, and so by Theorem 2.25, there is a subrepresentation ρ of both π and σ such that $A_\rho(G) = A_\pi(G) \cap A_\sigma(G)$. This ρ is also how we may define $\pi \wedge \sigma$, though we shall not explicitly use such a representation.

It is clear that if μ and ν are singular measures, then $\mu \wedge \nu = 0$. For convenience, it will be useful to associate to the 0 measure the '0 representation' $\pi_0(x) = 0$. Naturally $A_0(G) = \{0\}$. This observation yields the following result.

Proposition 5.20. *Let G be a locally compact group, and and Z be a Borel space with an associated measurable field of representations $z \mapsto \pi_z$. If μ, ν are positive measures on Z, then they are mutually singular if and only if $A_\mu(G) \cap A_\nu(G) = \{0\}$.*

Proof. Since the semilattice structures on measures and Fourier spaces are isomorphic, it must follow that $A_\mu(G) \cap A_\nu(G) = A_{\mu\wedge\nu}(G)$. Now if μ and ν are singular, then $\mu \wedge \nu = 0$, and so $A_\mu(G) \cap A_\nu(G) = A_0(G) = \{0\}$. The reverse direction holds similarly. □

5.3 Induced Representations

5.3.1 Construction

In this section, we shall focus on the *induced representation* construction — a method of taking representations on a subgroup and 'lifting' them to representations on the larger group. We shall follow a simplified construction for unimodular groups that is presented by Folland [20, Section 6.1], though this is described in the general case later in the same section.

So, let G be a *unimodular* locally compact group, N a closed subgroup, and $q : G \to G/N$ its canonical quotient map. Take any unitary representation σ of N acting on $\mathscr{H}_\sigma$. We define $C_q(G; \mathscr{H}_\sigma)$ to be the collection of continuous functions $f : G \to \mathscr{H}_\sigma$ for which $q(\operatorname{supp} f)$ is compact. From this we construct

$$\mathscr{F}_0 = \{f \in C_q(G; \mathscr{H}_\sigma) : f(xy) = \sigma(y)^* f(x) \text{ for } x \in G, y \in N\} \tag{5.11}$$

which shall soon serve as a dense subspace of our Hilbert space. Notice that for $x \in G$ and $y \in N$, we have

$$\langle f(xy) \,|\, g(xy) \rangle_\sigma = \langle \sigma(y)^* f(x) \,|\, \sigma(y)^* g(x) \rangle_\sigma = \langle f(x) \,|\, g(x) \rangle_\sigma \tag{5.12}$$

for $f, g \in \mathscr{F}_0$, and so this quantity is invariant on cosets of N. Thus if we take any quasi-invariant measure μ on G/N [19], we can imbue this space with the inner product

$$\langle f \,|\, g \rangle_\pi := \int_{G/N} \langle f(y) \,|\, g(y) \rangle_\sigma \, \mathrm{d}\mu(yN) \tag{5.13}$$

which is readily verified to be an inner product on $\mathscr{F}_0$. Note that it is in the construction of this inner product that we need unimodularity, for otherwise the existence of μ is not guaranteed. We then take the completion of $\mathscr{F}_0$ with respect to this inner product to obtain a Hilbert space $\mathscr{F}$.

We are now ready to define the induced representation. Let L_x be the left translation operator on $\mathscr{F}_0$ so that $L_x f = {}_x f$. Since μ is G-invariant, we have ${L_x}^* = L_{x^{-1}}$ and so $L_x \in U(\mathscr{F}_0)$. We then extend L_x to a unitary operator $\pi(x)$ on $\mathscr{F}$. One may verify that π is indeed strong operator continuous, and hence a (unitary) representation.

Definition 5.21. Given G, N, σ and π as above, we say that π is the **induced representation** of σ. We shall write the induced representation π with the notation $\mathrm{Ind}_N^G(\sigma)$.

First and foremost, this is a well-defined operation in the class of unitary representations.

Proposition 5.22. *Let G be a unimodular group, N a closed subgroup, and let σ_1, σ_2 be equivalent representations on N. Then* $\mathrm{Ind}_N^G(\sigma_1) \approx \mathrm{Ind}_N^G(\sigma_2)$.

This is well known, see for instance [20, Proposition 6.9]. Note that the converse of this statement is in general not true: two distinct representations may induce to the same representation. The next results are concerned with inducing from further subgroups and quotients of the subgroup N, the former of which is frequently known as *induction in stages*. We offload the details of the proof to other resources.

Proposition 5.23 (Induction in stages)**.** *Let G be a unimodular group, and let $L \subseteq N \subseteq G$ be a series of nested closed subgroups. If σ is a representation of L, then*

$$\mathrm{Ind}_N^G(\mathrm{Ind}_L^N(\sigma)) \approx \mathrm{Ind}_L^G(\sigma)$$

[19] Such a measure always exists due to a result sometimes known as *Weil's formula*, see [20, Theorem 2.49]

Proof. See [20, Theorem 6.14] or [35, Section 2.7]. □

Proposition 5.24. *Let G be a unimodular group, N be a closed subgroup of G, and L a closed normal subgroup of G contained in N. Let $q : G \to G/L$ be the canonical quotient map. Then for any representation of π of N/L we have that*

$$\mathrm{Ind}_N^G(\pi \circ q) \approx \mathrm{Ind}_{N/L}^{G/L}(\pi) \circ q$$

Proof. See [35, Proposition 2.38]. □

We now wish to provide concrete examples of the inducing construction, particularly of the trivial and left regular representations.

Lemma 5.25. *Let G be a unimodular group and $\mathbf{1}$ the trivial representation on the trivial subgroup $E = \{e\}$. Then $\mathrm{Ind}_E^G(\mathbf{1})$ is unitarily equivalent to the left regular representation λ.*

Proof. By definition, the inner product space $\mathscr{F}_0$ will simply be $C_c(G)$. It follows immediately then that $\mathrm{Ind}_E^G(\mathbf{1})$ acts on $L^2(G)$ by left translations. □

There is a more general result which states that the induced representation of the trivial representation on any closed subgroup N will be the quasi-regular representation on the cosets of N. Again, this more general form shall not be necessary. However, we do have the following result.

Proposition 5.26. *Let G be a unimodular group, N a closed subgroup, and λ_N the left regular representation on N. Then $\mathrm{Ind}_N^G(\lambda_N)$ is unitarily equivalent to λ_G, the left regular representation on G.*

Proof. By Lemma 5.25, we can write $\lambda_N \approx \mathrm{Ind}_E^N(\mathbf{1})$ where $E = \{e\}$ is the trivial group. It then follows by induction in stages (Proposition 5.23) that

$$\mathrm{Ind}_N^G(\lambda_N) \approx \mathrm{Ind}_N^G(\mathrm{Ind}_E^N(\mathbf{1})) \approx \mathrm{Ind}_E^G(\mathbf{1}) \approx \lambda_G$$ □

5.3.2 Induced Representations of Semidirect Products

We now move to a particular class of examples which will be of interest to us. Recall that when we have a group H acting on a now *abelian* locally compact group A, we can define the semidirect product $G = A \rtimes H$. In particular, we shall be looking at the induced construction of representations on A up to G. Fortunately, this construction is rather well behaved.

While the proof presented here is of the author's own work, one may find the statement of this result in [35] under "*Realization III for Semidirect Products*" in Section 2.4. Before we proceed with the theorem, recall the following notations.

- We define $C_q(G; \mathscr{H}_\sigma)$ to be the collection of continuous functions $f \in C(G; \mathscr{H}_\sigma)$ for which $q(\operatorname{supp} f)$ is compact, where $q : G \to A$ is the quotient map.
- For a representation σ of A, we define $l \cdot \sigma(a) = \sigma(l^{-1} \cdot a)$ for $a \in A$ and $l \in H$.

Theorem 5.27. *Let $G = A \rtimes H$ be unimodular and take σ a representation of A. Then the induced representation $\operatorname{Ind}_A^G(\sigma)$ is unitarily equivalent to the representation $\pi : G \to U(L^2(H; \mathscr{H}_\sigma))$ given by*

$$[\pi(ah)f](l) = l \cdot \sigma(a) f(h^{-1}l) \tag{5.14}$$

for $ah \in G$, $f \in L^2(H; \mathscr{H}_\sigma)$, and almost every $l \in H$.

Proof. Let $q : G \to H$ be the canonical quotient map, and recall that

$$\mathscr{F}_0 = \{f \in C_q(G; \mathscr{H}_\sigma) : f(ahb) = \sigma(b)^* f(ah), \text{ for } ah \in G, b \in A\}$$

Observe then that for $f \in \mathscr{F}_0$,

$$f(ah) = f(hh^{-1}ah) = f(h\,(h^{-1} \cdot a)) = h \cdot \sigma(a)^* f(h)$$

for all $ah \in G$. Armed with this knowledge, consider now the mapping $U : C_c(H; \mathscr{H}_\sigma) \to \mathscr{F}_0$ given by $[Ug](ah) = h \cdot \sigma(a)^* g(h)$ for $g \in C_c(H; \mathscr{H}_\sigma)$. As we shall show, this is the inverse of the restriction map $V : \mathscr{F}_0 \to C_c(H; \mathscr{H}_\sigma)$.

Firstly, it is clear that $VU = I$. On the other hand, for $f \in \mathscr{F}_0$, we have

$$[UVf](ah) = h \cdot \sigma(a)^* [Vf](h) = h \cdot \sigma(a)^* f(h) = f(ah)$$

and thus $V = U^{-1}$. Moreover, for any $f \in \mathscr{F}_0$, it follows that

$$\langle Vf \mid Vg \rangle_{L^2(H; \mathscr{H}_\sigma)} = \int_H \langle f(h) \mid g(h) \rangle_\sigma \, \mathrm{d}h = \langle f \mid g \rangle_{\mathscr{F}_0}$$

showing that the maps U and V are invertible and inner-product preserving. We may then extend U to a unitary $\tilde{U} : L^2(H; \mathscr{H}_\sigma) \to \mathscr{F}$, and V to $\tilde{V}$ in a similar manner. Let us now fix π as given in (5.14). Then for $f \in \mathscr{F}_0$ and $ah, bl \in G$ we have

$$\begin{aligned}
[U\pi(ah)Vf](bl) &= l \cdot \sigma(b)^*[\pi(ah)Vf](l) \\
&= l \cdot \sigma(b)^* l \cdot \sigma(a)[Vf](h^{-1}l) \\
&= l \cdot \sigma(a^{-1}b)^* f(h^{-1}l) \\
&= f(h^{-1} \cdot (a^{-1}b)h^{-1}l) \\
&= f((ah)^{-1}bl) \\
&= [\mathrm{Ind}_A^G(\sigma)(ah)f](bl)
\end{aligned}$$

and so $U\pi(ah)V = \mathrm{Ind}_A^G(\sigma)(ah)$. By extending to their completions, we see these representations are equivalent. □

It should be clear that in this situation, the vast majority of induced representations will be infinite-dimensional. The natural exception of course being when H is finite.

Corollary 5.28. *Let $G = A \rtimes H$ be unimodular and $\varphi \in \widehat{A}$. Then the representation $\pi = \mathrm{Ind}_A^G(\varphi)$ is finite if and only if H is finite.*

Proof. This follows since π acts on $L^2(H)$, which is finite-dimensional precisely when H is finite. □

Another result of Theorem 5.27 is that the kernel is fairly stable under the inducing construction.

Proposition 5.29. *Let $G = A \rtimes H$ be unimodular, let σ be a representation of A, and set $\pi = \mathrm{Ind}_A^G(\sigma)$. Then $\ker \pi$ is the largest H-invariant subgroup of $\ker \sigma$. In particular, if $\ker \sigma$ is H-invariant, then $\ker \pi = \ker \sigma$.*

Proof. For convenience, let us denote by M the largest H-invariant subgroup of $\ker \sigma$. If $a \in M$, then for $f \in L^2(H; \mathscr{H}_\sigma)$ we have

$$[\pi(a)f](l) = l \cdot \sigma(a)f(l) = \sigma(l^{-1} \cdot a)f(l) = f(l)$$

where the last equality holds since $l^{-1} \cdot a \in M \subseteq \ker \sigma$. Thus $M \subseteq \ker \pi$.

On the other hand, take $ah \in \ker \pi$. Then $f(l) = [\pi(ah)f](l) = l \cdot \sigma(a)f(h^{-1}l)$ for all $f \in L^2(H; \mathscr{H}_\pi)$. Noting that $l \cdot \sigma(a)$ acts on $f(l)$, we may choose an f with an appropriately small support to conclude that $h = e$. In other words, $\ker \pi \subseteq A$. Since $\ker \pi$ is normal, it must be H-invariant, so all that remains is to show that $\ker \pi \subseteq \ker \sigma$.

So choose $a \in \ker \pi$ so that $f(l) = l \cdot \sigma(a) f(l)$. In a similar manner, we may again choose appropriate $f \in L^2(H; \mathscr{H}_\sigma)$, in order to conclude that $l \cdot \sigma(a)$ must act trivially on $L^2(H; \mathscr{H}_\sigma)$ for almost every $l \in H$. Hence $l^{-1} \cdot a \in \ker \sigma$, again for almost every $l \in H$. As a result, one may show that there must exist a sequence $l_n \in H$ such that $l_n \to 1$ and ${l_n}^{-1} \cdot a \in \ker \sigma$. Furthermore, since $\ker \sigma$ is closed, it follows that $a \in \ker \sigma$. $\square$

There is a more general version of the above result which can be found in [35, Theorem 2.45]. Another nice property of induced representations is that they behave well with direct summations. Namely, we have that if π_λ for $\lambda \in \Lambda$ is a family of representations of a closed subgroup H, then

$$\operatorname{Ind}_H^G\Big(\bigoplus_{\lambda \in \Lambda} \pi_\lambda\Big) \approx \bigoplus_{\lambda \in \Lambda} \operatorname{Ind}_H^G(\pi_\lambda) \tag{5.15}$$

so that in other words, the operations of direct summation and inducing commute. This works for any (unimodular) group G, and this is shown in [35, Proposition 2.42]. One may ask if this works in a generic setting where we replace direct sums with direct integrals. Fortunately this holds for at least the specific circumstances we require.

Theorem 5.30. *Let $G = A \rtimes H$ be unimodular. Let μ be a positive measure on a Borel space Z, with a measurable field of representations $z \mapsto \varphi_z \in \widehat{A}$. Then it follows that*

$$\operatorname{Ind}_A^G\Big(\int_Z^\oplus \varphi_z \,\mathrm{d}\mu(z)\Big) \approx \int_Z^\oplus \operatorname{Ind}_A^G(\varphi_z) \,\mathrm{d}\mu(z)$$

Proof. For convenience, let ρ be the direct integral $\int_Z^\oplus \varphi_z \,\mathrm{d}\mu(z)$, so that it acts on the space $L^2(Z)$ by $[\rho(a)f](z) = \varphi_z(a) f(z)$ for all $a \in A$ and $f \in L^2(Z)$. By Theorem 5.27, we have that $\pi := \operatorname{Ind}_A^G(\rho)$ acts on the space $L^2(H; L^2(Z))$, which by Corollary A.2 is isomorphic to $L^2(H \times Z)$. This action is given by

$$[\pi(ah)g](l, z) = l \cdot \rho(a) g(h^{-1}l, z) = l \cdot \varphi_z(a) g(h^{-1}l, z)$$

for $g \in L^2(H \times Z)$.

On the other hand, again by Theorem 5.27, each representation $\sigma_z := \operatorname{Ind}_A^G(\varphi_z)$ acts by

$$[\sigma_z(ah)f](l) = l \cdot \varphi_z(a) f(h^{-1}l)$$

for every $f \in L^2(H)$. Now, set π' to be $\int_Z^{\oplus} \sigma_z \, \mathrm{d}\mu(z)$, from which we see that π' acts on the space $L^2(Z; L^2(H))$ by 'pointwise' application, in the sense that $[\pi'(ah)g](z) = \sigma_z(ah)f'(z)$ for $g \in L^2(Z; L^2(H))$. In particular, we get that

$$[[\pi'(ah)g](z)](l) = [\sigma_z(ah)g(z)](l) = l \cdot \varphi_z(a)g(z)(h^{-1}l)$$

and if we again use Corollary A.2 to associate $L^2(Z; L^2(H))$ to the space $L^2(Z \times H)$, then

$$[\pi'(ah)g](z, l) = l \cdot \varphi_z(a)g(z, h^{-1}l)$$

for $g \in L^2(Z \times H)$. From this it is clear that π' is unitarily equivalent to π. □

Question 5.31. Let G be a locally compact group G and H a closed subgroup. Suppose that μ is a positive measure on a Borel space Z, with a measurable field of representations $z \mapsto \pi_z$ on H. Does it then follow that

$$\mathrm{Ind}_H^G\Big(\int_Z^{\oplus} \pi_z \, \mathrm{d}\mu(z)\Big) \approx \int_Z^{\oplus} \mathrm{Ind}_H^G(\pi_z) \, \mathrm{d}\mu(z)$$

and if not, under which circumstances does the above equation hold?

It is almost surely the case that the above question is true to some extent, but as to which conditions are necessary (if any) is not immediately clear. Alas, we must press forward. It follows from the previous result that if π is any representation on G whose disintegration consists only of induced characters from A, then it can be written $\pi = \mathrm{Ind}_A^G(\sigma)$ for some representation σ of A, where the disintegration of σ into characters will be mimic the disintegration of π.

5.3.3 The Mackey Machine

Next, we move to the question of irreducibility. This is answered by the **Mackey machine**, which gives a full description of the irreducible representation structure of semidirect products via the inducing construction. We present the standard theory here, as described in [20, Section 6.6], and in Section 5.4 we apply this theory to our own class of examples. Let us proceed with the following definition.

Definition 5.32. Let H be a locally compact group acting on an abelian locally compact group A. We say that H acts **regularly** on A if all of the following properties hold.

(R1) The orbit space $\mathcal{O}_H(A)$ is **countably separated**: that is there exists a countable family $\{E_i\}$ of Borel sets in A such that $[a]_H = \bigcap\{E_i : [a]_H \subseteq E_i\}$ for all $a \in A$.

(R2) For every $a \in A$, the natural map from H/H_a to $[a]_H$ given by $xH_a \mapsto xa$ is a homeomorphism.

Recall we always assume that $G = A \rtimes H$ is second-countable, in which case these conditions are equivalent and we need only check one. This is a result due to Mackey [42, Theorem 5.2].

Proposition 5.33. *Let $G = A \rtimes H$, where G is second-countable. Then the properties (R1) and (R2) are equivalent.*

When the *dual action* $G \curvearrowright \widehat{A}$ is regular, the representation theory of the semidirect product $G = A \rtimes H$ can be computed using the Mackey machine, a method which unsurprisingly is due to Mackey [41]. This major result for semidirect products is as follows.

Theorem 5.34 (The Mackey machine)**.** *Let $G = A \rtimes H$, and suppose that G acts regularly on $\widehat{A}$. If $\varphi \in \widehat{A}$ and σ is an irreducible representation of H_φ, then $\mathrm{Ind}_{G_\varphi}^{G}(\varphi\sigma)$ is an irreducible representation of G, where $(\varphi\sigma)(ah) := \langle \varphi, a \rangle \sigma(h)$. Furthermore, all irreducible representations of G are equivalent to one of this form. In particular, the representations $\mathrm{Ind}_{G_\varphi}^{G}(\varphi\sigma)$ and $\mathrm{Ind}_{G_{\varphi'}}^{G}(\varphi'\sigma')$ will be equivalent precisely when there exits some $h \in H$ such that $\varphi' = h \cdot \varphi$ and $\sigma' \approx h \cdot \sigma$.*

This is a very deep and non-trivial theorem, and as such, we omit its proof. The captivated reader may find a proof in the book of Folland [20, Theorem 6.42].

5.4 Cheap Groups

We shall now impose that the group H acting on A is compact; as such we shall denote it by K. This assumption of compactness immediately implies nice properties. For one, such groups satisfy the regularity assumption, and hence may be used in the Mackey machine.

Lemma 5.35. *Let $G = A \rtimes K$ where A is an abelian locally compact group and K is compact. Then $K \curvearrowright \widehat{A}$ is regular.*

Proof. By Proposition 2.58, (R2) holds, thus by Proposition 5.33 the result follows. □

Moreover, these groups are also unimodular (see Proposition 2.68), so any considerations for the inducing construction as described in the previous section shall require no modification.

Let us now introduce a particular class of examples where the Mackey machine gives an incredibly simple yet powerful description of the representation structure of G. We call such groups *cheap*. As we shall soon see, the Fell group, and the more general versions of it, are all indeed cheap groups.

Definition 5.36. Let G be a (second-countable) locally compact group of the form $A \rtimes K$ where A is an abelian locally compact group and K is compact. We say that G is **cheap**[20] if the stabilisers K_φ of the dual action contain only the identity for all non-trivial $\varphi \in \widehat{A}$.

In other words, if there is $\varphi \in \widehat{A}$ and $k \in K$ such that $k \cdot \varphi = \varphi$, then either $\varphi = \mathbf{1}$ or $k = e$. With only this assumption, we get the following strengthening of the Mackey machine.

Theorem 5.37 (The Mackey machine for cheap groups)**.** *Let $G = A \rtimes K$ be cheap, and $q : G \to K$ be the canonical quotient map. If $\pi \in \widehat{G}$, then exactly one of two cases must occur.*

- *Either $\pi = \rho \circ q$ for some $\rho \in \widehat{K}$, in which case π is always finite-dimensional. In particular $A \subseteq \ker \pi$.*
- *Otherwise, $\pi = \mathrm{Ind}_A^G(\varphi)$ for some non-trivial $\varphi \in \widehat{A}$, in which case π is infinite-dimensional if and only if K is infinite. Moreover, $\mathrm{Ind}_A^G(\varphi)$ and $\mathrm{Ind}_A^G(\psi)$ will be equivalent representations precisely when φ and ψ are in the same orbit.*

Proof. By the Mackey machine, if a representation arises from the trivial orbit in $\widehat{A}$, then we have that $K_{\mathbf{1}} = K$, and so $\pi(ak) = \mathbf{1}(a)\rho(k)$, where $\mathbf{1}$ is the trivial character on A and $\rho \in \widehat{K}$. Thus we may write $\pi = \rho \circ q$, and so clearly π is finite-dimensional.

On the other hand, suppose π arises from any other orbit and choose some φ in this orbit. Since $K_\varphi = \{1\}$, it follows that $G_\varphi = A$, and thus $\pi = \mathrm{Ind}_A^G(\varphi)$ for some non-trivial $\varphi \in \widehat{A}$. Then by Corollary 5.28, π is infinite precisely when K infinite as well. □

[20] This terminology originates from the following observation. Recall that a group action on $K \curvearrowright X$ on a topological space is *free* if whenever $k \cdot x = x$, then $k = e$. While dual actions of cheap groups are never free, since $k \cdot \mathbf{1} = \mathbf{1}$ for any $k \in K$, they are almost free in the sense that $K \curvearrowright \widehat{A} \setminus \{\mathbf{1}\}$ is free. As such, it is only natural that a group action which is 'almost-free' should be called 'cheap'.

This result can be stated more colloquially. Namely, the irreducible representations of G come in two varieties: either they arise from K and are always finite-dimensional, or they are induced from A in which case they are finite-dimensional if and only if K is finite.

While we now have an evident description of the irreducible characters on G, we wish to also describe the topology on $\widehat{G}$. The key idea behind this is that the operations of inducing and restricting representations are continuous in a reasonable sense (see [35, Theorems 5.37 & 5.39]). In particular, the topological structure on $\widehat{G}$ will have two 'components'. The representations induced from characters will have the orbit space structure $\mathcal{O}_K(\widehat{A})$, though the trivial character will be replaced with a copy of $\widehat{K}$ — indeed the trivial character induces to the left quasi-regular representation, which by Peter-Weyl is a sum of all irreducible representations. In particular, this means that an open set $U \subseteq \widehat{G}$ containing some element of $\widehat{K}$ *must* also contain some (punctured) open neighbourhood of the identity in $\mathcal{O}_K(\widehat{A})$. A diagram illustrating this structure has been provided in Fig. 5.1.

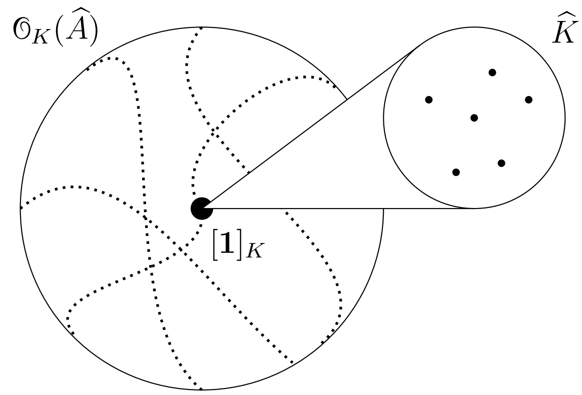


Figure 5.1: The dual space of a cheap group. The dashed lines indicate the partitioning of $\widehat{A}$ into its orbit structure $\mathcal{O}_K(\widehat{A})$. Note also that $\widehat{K}$ is discrete.

Let us provide a formal description of this space. The general theory was originally laid out by Baggett [8], though we opt for a presentation of this work given by Kaniuth and Taylor [35, Theorem 5.58]. They provide a general framework for describing the structure of the dual space of a more general semidirect product, but we shall apply this to the context of cheap groups. In order to do this, let us introduce a helper function $\iota : \widehat{G} \to \mathcal{O}_K(\widehat{A})$ by mapping

$$\iota(\mathrm{Ind}_A^G(\varphi)) = [\varphi]_K \qquad \text{and} \qquad \iota(\rho \circ q) = [\mathbf{1}]_K \tag{5.16}$$

for $\varphi \in \widehat{A}$ and $\rho \in \widehat{K}$. We restate Kaniuth and Taylor's Theorem 5.58 for cheap groups below.

Proposition 5.38. *Let $G = A \rtimes K$ be a cheap group. Suppose that $\pi \in \widehat{G}$, and let $(\pi_\alpha)_\alpha$ be a net in $\widehat{G}$. Define ι as in (5.16). Then $\pi_\alpha \to \pi$ if and only if:*

- *$\iota(\pi_\alpha) \to \iota(\pi)$, and*
- *if $\pi = \rho \circ q$, and $\pi_\beta = \rho_\beta \circ q$ is a subnet of π_α, then ρ_β is eventually equal to ρ.*

We can use this result to give a concrete description of the topology on $\widehat{G}$.

Proposition 5.39. *Let $G = A \rtimes K$ be cheap. Then $\widehat{G}$ is homeomorphic to $[\widehat{K} \times \mathbb{O}_K(\widehat{A})]/\sim$ where $(\rho_1, [\varphi_1]_K) \sim (\rho_2, [\varphi_2]_K)$ if and only if:*

- *$[\varphi_1]_K = [\varphi_2]_K$, and*
- *if $[\varphi_1]_K = [\mathbf{1}]_K$ (in other words $\varphi_1 = \mathbf{1}$), then $\rho_1 = \rho_2$*

for every $(\rho_1, [\varphi_1]_K), (\rho_2, [\varphi_2]_K) \in \widehat{K} \times \mathbb{O}_K(\widehat{A})$.

Proof. First, let us define the map $\kappa : \widehat{G} \to \widehat{K}$ as a sort of 'complement' to ι where $\kappa(\mathrm{Ind}_A^G(\varphi)) = \mathbf{1}_K$ and $\kappa(\rho \circ q) = \rho$. Combining these, we define the map on to the quotient as $\eta : \widehat{G} \to [\widehat{K} \times \mathbb{O}_K(\widehat{A})]/\sim$ by $\eta(\pi) = [(\kappa(\pi), \iota(\pi))]_\sim$. As a result of Theorem 5.37, it is readily checked that this is a bijection, and we leave this for the reader to verify.

Bicontinuity follows from Proposition 5.38. Indeed if $\pi_\alpha \to \pi \in \widehat{G}$, then we have that $\iota(\pi_\alpha) \to \iota(\pi)$. Moreover, if π is of the form $\mathrm{Ind}_A^G(\varphi)$, then $\iota(\pi_\alpha)$ is eventually not $[\mathbf{1}]_K$, and so it follows that $\kappa(\pi_\alpha) = \mathbf{1}_K = \kappa(\pi)$. On the other hand, if π is of the form $\rho \circ q$, then by the second condition of Proposition 5.38, we have that any subnet of the form $\pi_\beta = \rho_\beta \circ q$ must eventually satisfy $\rho_\beta = \rho = \kappa(\pi)$. This tells us that if α is sufficiently large, then either $\pi_\alpha = \rho_\alpha \circ q$ so that $\kappa(\pi_\alpha) = \kappa(\pi)$, or $\kappa(\pi_\alpha) = \mathbf{1}_K$, in which case we have that

$$(\kappa(\pi_\alpha), \iota(\pi_\alpha)) \sim (\kappa(\pi), \iota(\pi_\alpha)) \to (\kappa(\pi), \iota(\pi))$$

and so it follows that η is continuous. A similar analysis shows that η^{-1} is also continuous, and so η must be a homeomorphism. □

Recall that a locally compact group G is *type I* if and only if $\widehat{G}$ is a T_0 space. We shall show that cheap groups in fact have that $\widehat{G}$ is T_1.

Proposition 5.40. *Let $G = A \rtimes K$ be cheap. Then $\widehat{G}$ is T_1 and so G is type I.*

Proof. Firstly, recall that the quotient of a T_1 space is T_1 if and only if the equivalence classes are closed. Now $\widehat{A}$ is clearly Hausdorff, so the orbits $[a]_K$ (which are compact since K is compact) are closed. Hence $\mathcal{O}_K(\widehat{A})$ is T_1, and therefore so is $\widehat{K} \times \mathcal{O}_K(\widehat{A})$. All that remains now is to show that the equivalence classes of $\sim$ as defined in Proposition 5.39 are closed.

By observation, the equivalence classes of $\sim$ have one of two forms. Either they are singletons $\{(\rho, [\mathbf{1}]_K)\}$ for $\rho \in \widehat{K}$, in which case they are clearly closed. Otherwise, they are of the form $(\widehat{K}, [\varphi]_K)$ for $[\varphi]_K \in \mathcal{O}_K(\widehat{A})$ with $\varphi \neq \mathbf{1}$. In this case as well, it is evident that these are closed sets. Thus $\widehat{G}$ is T_1, and therefore the group G is type I. □

Recall in Theorem 5.37, we saw that the dimension of the representations of the form $\mathrm{Ind}_A^G(\varphi)$ depended on whether the group K was finite or not.

Definition 5.41. Let $G = A \rtimes K$ be cheap. If K is a finite group, we shall say that G is **finitely cheap**. Otherwise, if K is an (uncountably) infinite compact group, we shall say that G is **infinitely cheap**.

When G is finitely cheap, every irreducible representation is finite-dimensional, and so $\widehat{G} = \widehat{G}_\ell$. Otherwise if G is an infinitely cheap group, then by Theorem 5.37 and Proposition 5.39 we have $\widehat{G}_\ell = \widehat{K}$ and $\widehat{G}_\infty = \mathcal{O}_K(\widehat{A} \setminus \{\mathbf{1}\})$. In particular, since the finite representations have simple structure, we can describe the almost-periodic compactification of the group easily. Indeed it will simply be the group K. We remind the reader that the definition (and subsequently properties) of this compactification may be found in Definition 2.33.

Proposition 5.42. *Let $G = A \rtimes K$ be infinitely cheap. Then $G^{ap} = K$.*

Proof. From Theorem 5.37, it is clear that $\widehat{G}_\ell = \widehat{K} \circ q$. So by Proposition 2.38, we get that $G^{ap} = K$. □

We can then apply this result to Theorem 2.41, to obtain a description of the Fourier-Stieltjes algebra for cheap groups. Recall in Definition 2.39 that we define $B_\infty(G)$ to be the subalgebra of $B(G)$ whose matrix coefficients arise from *purely infinite* representations.

Theorem 5.43. *Let $G = A \rtimes K$ be infinitely cheap. Then*

$$B(G) = B_\infty(G) \oplus A(K) \circ q$$

where $q : G \to K$ is the canonical quotient map. Furthermore, if $u \in B_\infty(G)$, then there is some representation σ on A such that $u(x) = \langle \mathrm{Ind}_A^G(\sigma)(x) f \mid g\rangle$ where $f, g \in L^2(K; \mathcal{H}_\sigma)$.

Proof. The first part follows from Theorem 2.41 and Proposition 5.42. When $u \in B_\infty(G)$, then there is some purely infinite representation π such that $u(x) = \langle \pi(x)\xi \,|\, \eta \rangle$ for some $\xi, \eta \in \mathscr{H}_\pi$. Now since G is type I (Proposition 5.40), we may use Theorem 5.13 to write $\pi \approx \pi_\mu$ for some positive measure μ on $\widehat{G}$. However, as π is purely infinite, we have by Corollary 5.17 that $\mu(\{\tau\}) = 0$ for any $\tau \in \widehat{G}_\ell$. Since $\widehat{G}_\ell$ is countable, it follows that $\mu(\widehat{G}_\ell) = 0$. Hence by Theorem 5.30 we have

$$\pi \approx \int_{\widehat{G}_\infty}^{\oplus} \rho \, \mathrm{d}\mu(\rho) \approx \int_{\widehat{G}_\infty}^{\oplus} \mathrm{Ind}_A^G(\varphi_\rho) \, \mathrm{d}\mu(\rho) \approx \mathrm{Ind}_A^G\Big(\int_{\widehat{G}_\infty}^{\oplus} \varphi_\rho \, \mathrm{d}\mu(\rho)\Big)$$

where φ_ρ is given by $\rho = \mathrm{Ind}_A^G(\varphi_\rho)$ for $\rho \in \widehat{G}_\infty$. If we let $\sigma = \int_{\widehat{G}_\infty}^{\oplus} \varphi_\rho \, \mathrm{d}\mu(\rho)$, then it follows that $\pi = \mathrm{Ind}_A^G(\sigma)$ as required. □

Finally, we can in fact 'compress' $B_\infty(G)$ down to $A(G)$ when the orbits of $K \curvearrowright \widehat{A}$ are open and countable in number. We begin with a lemma.

Lemma 5.44. *Let $G = A \rtimes K$ be cheap. If $\varphi \in \widehat{A}$ is contained in an orbit of non-zero measure, then $\mathrm{Ind}_A^G(\varphi)$ is quasi-contained in the left representation λ.*

Proof. We can write the left regular representation of A by $\lambda_A = \int_{\widehat{A}} \psi \, \mathrm{d}\mu(\psi)$ where μ is the Plancherel measure on $\widehat{A}$. Since A is abelian, μ is also the Haar measure on $\widehat{A}$. By Proposition 5.26 and Theorem 5.30 we have that

$$\lambda_G \approx \mathrm{Ind}_A^G(\lambda_A) \approx \int_{\widehat{A}}^{\oplus} \mathrm{Ind}_A^G(\psi) \, \mathrm{d}\mu(\psi)$$

Now, the Mackey machine gives us that the map $\psi \mapsto \mathrm{Ind}_A^G(\psi)$ is constant on orbits in $\widehat{A}$. So we can define the pushforward measure ν on the orbit space $\mathcal{O}_K(\widehat{A})$ by $\nu = \mu \circ q^{-1}$ where q is the canonical quotient map. It follows that $\lambda_G \approx \int_{\widehat{A}}^{\oplus} \mathrm{Ind}_A^G(\psi) \, \mathrm{d}\nu([\psi]_K)$, and in particular, since the orbit of φ has non-zero measure, then $\mathrm{Ind}_A^G(\varphi)$ is quasi-contained in λ_G. □

Theorem 5.45. *Let $G = A \rtimes K$ be infinitely cheap. If every non-zero orbit in $\widehat{A}$ has positive measure, and the orbit space $\mathcal{O}_K(A)$ is countable, then*

$$B(G) = A(G) \oplus A(K) \circ q$$

where $q : G \to K$ is the canonical quotient map. In particular, G is spinal.

Proof. By Theorem 5.43, all that remains is to check that $B_\infty(G) = A(G)$. To this end, let $u \in B_\infty(G)$ so that $u(x) = \langle \pi(x)\xi \,|\, \eta\rangle$ for some infinite-dimensional representation π and $\xi, \eta \in \mathscr{H}_\pi$. Since the orbit space is countable, then G is an AU-group by [59, Theorem 4.5]. Hence we may write $\pi = \bigoplus \alpha_\varphi \pi_\varphi$ where $\pi_\varphi := \mathrm{Ind}_A^G(\varphi)$ and each α_φ is the corresponding multiplicity constant. Moreover, π_φ is quasi-contained in λ by Lemma 5.44, and so $A_{\pi_\varphi}(G) \subseteq A(G)$ follows from Corollary 2.31. Thus $u \in A(G)$. □

5.5 Local Fell Groups

5.5.1 Structure of the Fourier-Stieltjes Algebra

We shall now explore an application of the prior theory, by computing the Fourier-Stieltjes algebra of $G = \mathscr{V} \rtimes \mathscr{U}$. For this section, we will let $\mathscr{K}$ be a *non-Archimedean* local field. Recall that we have a local space $\mathscr{V}$ over $\mathscr{K}$, we define $\mathscr{U}$ to be the unit circle of $\mathscr{K}$, which is the collection of all $x \in \mathscr{K}$ with $|x| = 1$. We then have an action of $\mathscr{U} \curvearrowright \mathscr{V}$ via scalar multiplication, note in particular this action is isomorphic to its dual by Proposition 4.37. This shows that the group $G = \mathscr{V} \rtimes \mathscr{U}$ is (infinitely) cheap.

Lemma 5.46. *Let $\mathscr{V}$ be a local $\mathscr{K}$-space. Then $G = \mathscr{V} \rtimes \mathscr{U}$ is infinitely cheap.*

Proof. Cheapness of the action is easy to verify: if $v \in \mathscr{V}$ and $a \in \mathscr{U}$, then the statement $av = v$ clearly implies that either $a = 1$ or $v = 0$. It is also easy to see that $\mathscr{U}$ is compact and infinite. Finally, G is second-countable by Proposition 4.31. □

Recall that we say G is a *local Fell group* if it is of the form $G = \mathscr{K} \rtimes \mathscr{U}$ for some non-Archimedean local field $\mathscr{K}$. When this is the case, we can directly compute the Fourier-Stieltjes algebra of G.

Proposition 5.47. *Let $G = \mathscr{K} \rtimes \mathscr{U}$ be a local Fell group. Then G is spinal, and in particular $B(G) = A(G) \oplus A(\mathscr{U}) \circ q$.*

Proof. By Lemma 5.46, G is infinitely cheap. All that remains to be checked is that the orbit structure is countable and consists of open sets. Each non-trivial orbit of $\mathscr{U} \curvearrowright \mathscr{K}$ can be uniquely identified to some real number in the value group $|\mathscr{K}^\times|$. However, the value group is a discrete subgroup of $\mathbb{R}_{>0}$ (Theorem 4.8), so it follows that there are countably many non-trivial orbits, all of which are open. □

With this, we have a small class of groups which satisfy the conditions of Theorem 5.45, and are therefore necessarily spinal. As of the time of writing, this is an exhaustive list of groups known to satisfy these properties. It is suspected there are other examples, so we ask the following question.

Question 5.48. Do there exist groups other than local Fell groups for which the conditions of Theorem 5.45 hold?

There do exist a few additional noncompact spinal groups, due to Runde and Spronk [51, Proposition 2.1], in particular groups of the form $\mathbb{Q}_p \rtimes \mathrm{GL}_n(\mathbb{O}_p)$. It is likely that the equivalent non-Archimedean local field groups also satisfy this property, though this has yet to be checked.

Many results we have presented in this section thus far have partially mimicked results from the same paper of Runde and Spronk [51]. However, we can use the theory of cheap groups to understand the structure of $B(G)$ for groups of the form $G = \mathscr{V} \rtimes \mathscr{U}$. As mentioned previously, we understand the case where $\dim \mathscr{V} = 1$, so let us henceforth assume that $\dim \mathscr{V} \geq 2$. By Theorem 5.43, the Fourier-Stieltjes algebra of G can be decomposed as $B(G) = B_\infty(G) \oplus A(\mathscr{U})$, so let us examine the structure of $B_\infty(G)$.

We begin with the irreducible representations: by Theorem 5.37 we know that such representations can be written as $\pi_\varphi := \mathrm{Ind}_{\mathscr{V}}^{G}(\varphi)$ where $\varphi \in \widehat{\mathscr{V}}$. Recall from Definition 4.43 that a *hyperplane* is a $(d-1)$-dimensional subspace of $\mathscr{V}$.

Proposition 5.49. *Let $G = \mathscr{V} \rtimes \mathscr{U}$ with $\pi_\varphi \in \widehat{G}_\infty$ as defined above. Then there is some hyperplane H such that $H \subseteq \ker \pi_\varphi$.*

Proof. By Theorem 4.33, there exists some $x \in \mathscr{V}$ such that $\varphi = \Phi([x, \cdot])$. From this it follows that $H_x \subseteq \ker \varphi$, where $H_x := (\mathscr{K}x)^\perp$. Since H_x is a subspace, and hence $\mathscr{U}$-invariant, then by Proposition 5.29 it follows that $H_x \subseteq \ker \pi_\varphi$. ☐

This result will be key in our analysis of these representations. It tells us that in essence, all these irreducible representations essentially 'live on' only a single dimension of $\mathscr{V}$. This means they will behave a lot like the irreducible representations of the local Fell group, of which we know have a good understanding. Let us formalise this in a more general setting first.

Given an arbitrary locally compact group G, and N a closed normal subgroup, we define the space $\widehat{G}_N$ to be

$$\widehat{G}_N := \{\pi \in \widehat{G} : N \subseteq \ker \pi\} \tag{5.17}$$

and a similar space for infinite representations

$$\widehat{G}_{N,\infty} := \widehat{G}_N \cap \widehat{G}_\infty \tag{5.18}$$

from which it is readily verified that $\widehat{G}_N = \widehat{G/N} \circ q$ and $\widehat{G}_{N,\infty} = \widehat{G/N}_\infty \circ q$ where $q : G \to G/N$ is the quotient map. Now, in the case of $G = \mathscr{V} \rtimes \mathscr{U}$ and $H \in \mathbf{H}(\mathscr{V})$, it is a standard exercise to verify that $G/H = \mathscr{K} \rtimes \mathscr{U}$, which is a local Fell group. In particular, $\mathscr{K} \rtimes \mathscr{U}$ is an AR-group (recall that an *AR-group* is one for which the left regular representation can be written as a direct sum of irreducible representations, see the paragraph prior to Theorem 5.7), and this gives the following result. Recall also that for a homomorphism $q : G \to H$, we define in Definition 5.3 that $A_q(G) = A(H) \circ q$.

Proposition 5.50. *Let $G = \mathscr{V} \rtimes \mathscr{U}$. If $H \in \mathbf{H}(\mathscr{V})$ and $q_H : G \to G/H$ is the canonical quotient map, then*

$$\bigoplus_{\pi \in \widehat{G}_{H,\infty}} A_\pi(G) = A_{q_H}(G)$$

and in particular, $A_\pi(G) \subseteq A_{q_H}(G)$.

Proof. For any $\pi \in \widehat{(G/H)}$ it is easy to verify that $A_{\pi \circ q_H}(G) = A_\pi(G/H) \circ q_H$. Since $G/H = \mathscr{K} \rtimes \mathscr{U}$ which is an AR-group, one may write

$$\bigoplus_{\pi \in \widehat{G}_{H,\infty}} A_\pi(G) = \bigoplus_{\pi \in \widehat{(G/H)}_\infty} A_\pi(G/H) \circ q_H = A(G/H) \circ q_H$$

which is by definition $A_{q_H}(G)$. □

If we now sum over all hyperplanes of $\mathscr{V}$, this will give us a description of the purely infinite representations which are direct sums of irreducible representations.

Proposition 5.51. *Let $G = \mathscr{V} \rtimes \mathscr{U}$. Then*

$$\bigoplus_{\pi \in \widehat{G}_\infty} A_\pi(G) = \bigoplus_{H \in \mathbf{H}(\mathscr{V})} A_{q_H}(G)$$

Proof. Firstly, we get that

$$\bigoplus_{H \in \mathbf{H}(\mathscr{V})} A_{q_H}(G) = \bigoplus_{H \in \mathbf{H}(\mathscr{V})} \bigoplus_{\pi \in \widehat{G}_{H,\infty}} A_\pi(G)$$

by summing over every $H \in \mathbf{H}(\mathscr{V})$ in (5.50). Now, if $\pi \in \widehat{G}_\infty$, then by Proposition 5.49 there is always some $H \in \mathbf{H}(\mathscr{V})$ such that $\pi \in \widehat{G}_{H,\infty}$. Clearly such H must be unique, and so it follows that

$$\bigoplus_{H \in \mathbf{H}(\mathscr{V})} A_{q_H}(G) = \bigoplus_{\pi \in \widehat{G}_\infty} A_\pi(G) \qquad \square$$

We can then incorporate the finite-dimensional representations as well.

Corollary 5.52. *Let $G = \mathscr{V} \rtimes \mathscr{U}$. If π is a direct sum of irreducible representations, then*

$$A_\pi(G) \subseteq \left[\bigoplus_{H \in \mathbf{H}(\mathscr{V})} A_{q_H}(G) \right] \oplus A(\mathscr{U}) \circ q$$

5.5.2 Continuous Measures

Now recall that if π is an arbitrary representation, then since G is type I, we may write

$$\pi = \pi_\mu = \int_{\widehat{G}}^{\oplus} \rho \, \mathrm{d}\mu(\rho) \tag{5.19}$$

where μ is a positive measure on $\widehat{G}$. By Corollary 5.52, if μ is discrete, then we have that

$$A_\mu(G) \subseteq \left[\bigoplus_{H \in \mathbf{H}(\mathscr{V})} A_{q_H}(G) \right] \oplus A(\mathscr{U}) \circ q \subseteq A^*(G) \tag{5.20}$$

We now ask the question: what if μ is a non-discrete measure? In particular we will be interested in measures with no atomic component.

Definition 5.53. Let (μ, X) be a measure space. We say that μ is **continuous** or **atomless** if $\mu(\{x\}) = 0$ for every $x \in X$.

It should be clear that given an arbitrary measure μ on $\widehat{G}$, we may always decompose it into a discrete and continuous component. Since the discrete component is already understood, let us assume that μ is continuous. First, let us note that $\widehat{G}_\ell = \widehat{K}$ is discrete, and so it follows that $\mu(\widehat{G}_\ell) = 0$. So we need not concern ourselves with this component of $\widehat{G}$. If we take $u \in A_\mu(G) = A_{\pi_\mu}(G)$, then by Theorem 5.15 we can write

$$u = \int_{\widehat{G}_\infty} u_\pi \, \mathrm{d}\mu(\pi) \tag{5.21}$$

where each $u_\pi \in A_\pi(G)$. Now consider π_ν defined in a similar manner for any arbitrary measure ν on $\widehat{G}_\infty$. Then if $v \in A_\nu(G)$, it follows by Fubini's theorem that

$$uv = \int_{\widehat{G}_\infty \times \widehat{G}_\infty} u_\pi v_{\pi'} \,\mathrm{d}(\mu \times \nu)(\pi, \pi') \tag{5.22}$$

This now raises the question, for fixed $u_\pi \in A_\pi(G)$ and $v_{\pi'} \in A_{\pi'}(G)$ where $\pi, \pi' \in \widehat{G}_\infty$, what can we say about $u_\pi v_{\pi'}$? By Proposition 5.50, we have that if $H \in \mathbf{H}(\mathscr{V})$ such that $H \subseteq \ker \pi$ and $q_H : G \to G/H$ is the quotient map, then $u_\pi \in A_{q_H}(G)$. Similarly $v_{\pi'} \in A_{q_{H'}}(G)$. So it follows by [27, Proposition 3.1] that we have that $u_\pi v_{\pi'} \in A_{q_H}(G) A_{q_{H'}}(G) \subseteq A_{q_H \vee q_{H'}}(G)$, where we recall that we define the join operation in Definition 5.5. This leads us to the following useful lemma.

Lemma 5.54. *For $i \in 1, 2$, let W_i be a subspace of a local space $\mathscr{V}$, with $q_i : \mathscr{V} \to \mathscr{V}/W_i$ the corresponding quotient map. Then $q_1 \vee q_2$ is a linear map of rank $\dim \mathscr{V} - \dim(W_1 \cap W_2)$.*

Proof. The maps q_1 and q_2 are linear maps. So we can write $q_1 \vee q_2 = (q_1 \oplus q_2) \circ \alpha$ where $\alpha : \mathscr{V} \to \mathscr{V} \oplus \mathscr{V}$ is the (linear) amplification map $x \mapsto (x, x)$. The kernel of this map is $W_1 \cap W_2$, so the result follows by rank-nullity. □

Now let us suppose that $\dim \mathscr{V} = 2$. It follows by Proposition 4.44 that this is exactly the dimensions for which we can have hyperplanes $H, H' \in \mathbf{H}(\mathscr{V})$ such that $H \cap H' = \{0\}$. In fact, this works for any pair of distinct hyperplanes in dimension 2. When this is the case, the rank of the map $q_H \vee q_{H'}$ is simply $\dim \mathscr{V}$, and so this map must be an *automorphism* of $\mathscr{V}$ (recall that local spaces over $\mathscr{K}$ are unique up to dimension). In particular, it follows by Proposition 5.6 that $A_{q_H \vee q_{H'}}(G) = A(G)$. Let us state this as the following lemma.

Lemma 5.55. *Let $\mathscr{V}$ be a local space of dimension 2. If $H, H' \in \mathbf{H}(\mathscr{V})$ are distinct, then $A_{q_H \vee q_{H'}}(G) = A(G)$.*

The reason we are interested in such a result, is that yields the pleasing property that $u_\pi v_{\pi'} \in A(G)$ for $u_\pi, v_{\pi'}$ defined as in (5.22). Since the above lemma only works for distinct hyperplanes, let us then consider the collection of unruly pairs (π, π') whose kernels do share a hyperplane. Let us define an equivalence relation on $\widehat{G}_\infty$ by

$$\sim_h = \{(\pi, \pi') \in \widehat{G}_\infty \times \widehat{G}_\infty : \exists H \in \mathbf{H}(\mathscr{V}), H \subseteq \ker \pi \cap \ker \pi'\} \tag{5.23}$$

so that in other words, $\pi \sim_h \pi'$ if and only if the unique hyperplanes H, H' which satisfy $H \subseteq \ker \pi$ and $H' \subseteq \ker \pi'$ are in fact one and the same. We define this collection of pairs to be the **saturated diagonal** $\Delta_h \subseteq \widehat{G}_\infty \times \widehat{G}_\infty$ given by

$$\Delta_h := \{(\pi, \pi') \in \widehat{G}_\infty \times \widehat{G}_\infty : \pi \sim_h \pi'\} \tag{5.24}$$

Our next few results aim to show that this set is in fact $(\mu \times \nu)$-null. We start by showing that the usual diagonal on an arbitrary product measure space is null when one of the constituent measures is continuous.

Lemma 5.56. *Let μ, ν be σ-finite measures on a measurable space X, with μ being continuous. Let $\Delta = \{(x, x) : x \in X\}$ be the diagonal in $X \times X$. Then $\mu \times \nu(\Delta) = \nu \times \mu(\Delta) = 0$.*

Proof. This follows by Fubini's Theorem. Indeed, we have that

$$\mu \times \nu(\Delta) = \int_{X \times X} \mathbb{1}_\Delta(x, y) \, \mathrm{d}(\mu \times \nu)(x, y) = \int_X \mu(\{x\}) \, \mathrm{d}\nu(x) = 0 \qquad \square$$

We can use this result to show more generally that saturated diagonals will be $\mu \times \nu$-null whenever the equivalence classes are μ-null (or ν-null).

Proposition 5.57. *Let μ and ν be measures on a measurable space X. Furthermore, let $\sim$ be an equivalence relation on X such that the equivalence classes are μ-null. Then the saturated diagonal $\Delta_\sim = \{(x, y) \in X \times X : x \sim y\}$ is $\mu \times \nu$-null (and $\nu \times \mu$-null).*

Proof. Let $q : X \to X/\sim$ be the canonical quotient map, and let $\mu' = \mu \circ q^{-1}$ and $\nu' = \nu \circ q^{-1}$ be the pushforward measures. Then μ' is continuous, since for any singleton set its preimage inside X will be μ-null by assumption. Now, let $\Delta = \{([x]_\sim, [x]_\sim) : x \in X\}$ be the diagonal inside $(X/\sim) \times (X/\sim)$. Since μ' is continuous, by Lemma 5.56, this diagonal will be $\mu' \times \nu'$-null. Observe that $(q^{-1} \times q^{-1})(\Delta) = \Delta_\sim$, and so it follows that

$$\mu \times \nu(\Delta_\sim) = \mu \times \nu((q^{-1}) \times q^{-1})(\Delta)) = \mu' \times \nu'(\Delta) = 0 \qquad \square$$

Let us apply this to the saturated diagonal Δ_h we defined in (5.24). Of course, this will be $\mu \times \nu$-null whenever μ is continuous, however we can say more. These were the pairs for which Lemma 5.55 fails, and since it is $\mu \times \nu$-null, then we can say that $u_\pi v_{\pi'} \in A(G)$ for almost every pair (π, π'). In particular, the integral defined in (5.22) will genuinely be in $A(G)$ as well.

Proposition 5.58. *Let $G = \mathscr{V} \rtimes \mathscr{U}$ with $\dim \mathscr{V} = 2$. Let μ, ν be positive σ-finite measures on $Z = \widehat{G}_\infty$, and suppose further that μ is continuous. Then for any $u \in A_\mu(G)$ and $v \in A_\nu(G)$, we have $uv \in A(G)$.*

Proof. It is clear that equivalence classes of $\sim_h$ are given by $\widehat{G}_{H,\infty}$. Since G/H is a local Fell group, then these are countable in size, and furthermore, this implies that $\mu(\widehat{G}_{H,\infty}) = 0$

since μ is continuous. So by Proposition 5.57, it follows that $\mu \times \nu(\Delta_h) = 0$. Thus if we write

$$u = \int_{\widehat{G}_\infty} u_\pi \,\mathrm{d}\mu(\pi) \qquad \text{and} \qquad v = \int_{\widehat{G}_\infty} v_\pi \,\mathrm{d}\nu(\pi)$$

where $u_\pi, v_\pi \in \pi$, then it follows that

$$\begin{aligned} uv &= \int_{\widehat{G}_\infty \times \widehat{G}_\infty} u_\pi v_{\pi'} \,\mathrm{d}(\mu \times \nu)(\pi, \pi') \\ &= \int_{(\widehat{G}_\infty \times \widehat{G}_\infty) \setminus \Delta_h} u_\pi v_{\pi'} \,\mathrm{d}(\mu \times \nu)(\pi, \pi') \end{aligned}$$

which for $(\pi, \pi') \notin \Delta_h$, we have that $u_\pi v_{\pi'} \in A(G)$ by Lemma 5.55. Recalling that the above integral is defined in the weak sense, it follows that $uv \in A(G)$. □

Corollary 5.59. *Let $G = \mathscr{V} \rtimes \mathscr{U}$ with $\dim \mathscr{V} = 2$. If μ is a positive continuous σ-finite measure on Z, then $A_\mu(G) \subseteq B_0(G)$.*

Proof. Let us apply Proposition 5.58. Take any $u \in A_\mu(G)$, and choose $\nu = \mu$ and $v = \overline{u}$. It follows that $uv = |u|^2 \in A(G)$ and hence $u \in C_0(G)$. □

For any σ-finite measure on Z, we may always decompose it into the sum of a continuous and discrete measure. This gives the following result.

Theorem 5.60. *Let $G = \mathscr{V} \rtimes \mathscr{U}$ with $\dim \mathscr{V} = 2$. Then*

$$B(G) = B_0(G) \oplus \left[\bigoplus_{H \in \mathbf{H}(\mathscr{V})} A_{q_H}(G) \right] \oplus A(\mathscr{U}) \circ q$$

Moreover, if $u, v \in B_0(G)$, then $uv \in A(G)$.

Proof. Let π be a representation of G. Then by Theorem 5.13, there is a positive measure μ on $\widehat{G}$ such that $\pi \approx \pi_\mu = \int_{\widehat{G}}^{\oplus} \rho \,\mathrm{d}\mu(\rho)$. We can write $\mu = \mu_c + \mu_d$ where μ_c is a continuous measure, and μ_d is discrete. Now, by Corollary 5.52, we know that

$$A_{\mu_d}(G) \subseteq \left[\bigoplus_{H \in \mathbf{H}(\mathscr{V})} A_{q_H}(G) \right] \oplus A(\mathscr{U}) \circ q$$

Similarly, by Corollary 5.59, we have $A_{\mu_c}(G) \subseteq B_0(G)$. Thus since $\pi \approx \pi_{\mu_c} \oplus \pi_{\mu_d}$, it follows that

$$A_\pi(G) \subseteq B_0(G) \oplus \left[\bigoplus_{H \in \mathbf{H}(\mathscr{V})} A_{q_H}(G) \right] \oplus A(\mathscr{U}) \circ q$$

as claimed. □

One may wish to obtain a tighter decomposition of $B(G)$. Indeed, if we could write $B_0(G)$ as a Fourier algebra of some sort, then $G = \mathscr{V} \rtimes \mathscr{U}$ would be a spinal group. In general, the only Fourier algebra inside $B_0(G)$ is $A(G)$ itself, though as it turns out, $B_0(G)$ is strictly larger than $A(G)$ for our particular G. We shall see a proof of the latter result in the following section, but let us show the former result first.

Theorem 5.61. *Let G be a locally compact group. Then $B_0(G) \cap A^*(G) = A(G)$.*

Proof. First, let $\eta : G \to H$ be any homomorphism with dense range such that $A_\eta(G) := A(H) \circ \eta \subseteq B_0(G)$. If $L \subseteq H$ is a compact set, then $\eta^{-1}(L)$ must also be compact. To see this, suppose to the contrary that $(x_\alpha)_\alpha$ is a sequence in $\eta^{-1}(L)$ such that $x_\alpha \to \infty$. As L is compact, we may find a $u \in A(H)$ such that $u(h) > \varepsilon$ for any $h \in L$. By assumption we have $u \circ \eta \in B_0(G)$, however the previous statement shows that $u \circ \eta(x_\alpha) \not\to 0$. This contradicts compactness, see Proposition B.11.

Thus η is a proper map, and so by Proposition 2.2 it is also surjective and hence must be open by the open mapping theorem for groups[21]. It follows that $H = G/K$, where $K = \ker \eta$ is compact. If we let $q : G \to G/K$ denote the canonical quotient map, then

$$A_\eta(G) = A_q(G)$$

So now all that remains is to show that $A_q(G) \subseteq A(G)$. Indeed let us take $u(xK) := \langle \lambda_K(xK) f \,|\, g \rangle \in A(G/K)$ for some $f, g \in L^2(G/K)$. Then if we define $f' \in L^2(G)$ by $f'(x) = f(xK)$ and g' similarly, we see that $u \circ q(x) = \langle \lambda(x) f' \,|\, g' \rangle$, and so $A_q(G) = A(G/K) \circ q \subseteq A(G)$. □

This result does allow us to leverage Theorem 5.60 in order to obtain a complete description of the spine of $B(G)$.

[21]See [25, Theorem 5.29] for a statement of this variation of the open mapping theorem. Do note that we require that G be σ-compact, which in our case holds since we are assuming that the locally compact groups we work with are second-countable.

Corollary 5.62. *Let* $G = \mathscr{V} \rtimes \mathscr{U}$ *with* $\dim \mathscr{V} = 2$. *Then*

$$A^*(G) = A(G) \oplus \left[\bigoplus_{H \in \mathbf{H}(\mathscr{V})} A_{q_H}(G) \right] \oplus A(\mathscr{U}) \circ q$$

5.6 Failure of Spinality

We shall now exhibit failure of spinality of the groups $\mathscr{V} \rtimes \mathscr{U}$ for $\dim \mathscr{V} \geq 2$. While a large portion of the material in this section is well known, the results we present here has been manually worked out for lack of finding a contiguous source on this material. We demonstrate the failure of spinality by constructing a continuous measure on Z which is singular with respect to the Plancherel measure. The key idea underlying this is that $\widehat{G}_\infty$ contains a Cantor space, which is teeming with mutually singular continuous measures. Recall that we say a space X is a **Cantor space** if it is a perfect compact totally disconnected metric space. A well-known result of classical analysis is that all such spaces are isomorphic, and as such, we shall often refer to X as *the* Cantor space.

For convenience, we shall let X be the specific Cantor space $\{0,1\}^{\mathbb{Z}}$, given the usual product topology. Elements of this space will typically be denoted by $x = (x_n)$, with the understanding that this is a $\mathbb{Z}$-sequence of zeroes and ones. Let us now construct the aforementioned measures on X. Given some $p \in (0,1)$, let μ_p be the probability measure on $\{0,1\}$ where

$$\mu_p := p\delta_1 + (1-p)\delta_0 \tag{5.25}$$

where δ_0, δ_1 are the point masses at 0 and 1 respectively. Now define $\nu_p := \prod_{\mathbb{Z}} \mu_p$ to be the probability measure on all of X. These measures are sometimes called the **Bernoulli measures**, and they are clearly continuous, since if $p \geq 1/2$, then we have

$$\nu_p(x) = \prod_{n \in \mathbb{Z}} (p\delta_1(x_n) + (1-p)\delta_0(x_n)) \leq \prod_{n \in \mathbb{Z}} p = 0$$

for any $x \in X$. A similar argument holds for $p \leq 1/2$.

Given an arbitrary element $x \in X$, we will define the **partial average** $R_{x,k}$ of x by

$$R_{x,k} := \frac{1}{k} \sum_{i=1}^{k} x_i \tag{5.26}$$

for any $k \in \mathbb{N}$. For fixed $x \in X$, we are interested in the long-term behaviour of these partial averages, so we shall define the limit

$$L_x := \lim_{k\to\infty} R_{x,k} \tag{5.27}$$

provided the limit exists.

We now wish to show that $L_x = p$ for ν_p-almost every $x \in X$. A direct proof of this can be rather non-trivial, so we shall borrow some technology from dynamical systems. As such we shall be rather terse with descriptions, as they may be found in greater detail in most textbooks on dynamical systems and ergodicity. For any probability measure μ of X, we say that a **μ-preserving transformation** is a measurable map $T : X \to X$ such that $\mu(T^{-1}(E)) = \mu(E)$ for all measurable $E \subseteq X$. We say that T is **ergodic** with respect to μ if for any measurable set E such that $T^{-1}(E) = E$, then $\mu(E)$ is either 0 or 1. There are many ergodic theorems relating to ergodic transformations, but the one that shall be of use to us is the *Birkhoff Ergodic Theorem*: we present a simplified version of the statement below.

Theorem 5.63 (Birkhoff Ergodic Theorem)**.** *Let T be an ergodic μ-preserving transformation on a finite measure space (X, μ). Then, for any $f \in L^1(X)$ we have*

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} f(T^{(i)}(x)) = \int_X f(x)\, \mathrm{d}\mu(x)$$

where $T^{(i)}$ denotes i many compositions of T with itself.

One may see [14, Theorem 4.5.5 & Corollary 4.5.7] for a proof of this result. In the case of the Cantor space, it is a well-known result that the **Bernoulli left shift** T, defined by $(Tx)_n = x_{n+1}$ is a ν_p-ergodic transformation[22]. We can use this to obtain the following result.

Lemma 5.64. *Let $r \in (0, 1)$, and set $A_r = \{x \in X : L_x = r\}$. Then $\nu_p(A_r) = 1$ if and only if $r = p$ (and is zero otherwise).*

[22]Indeed, one may check that T is *(strongly) mixing*, meaning that

$$\lim_{n\to\infty} \nu_p(T^{-n}(A) \cap B) = \nu_p(A)\nu_p(B)$$

for any measurable $A, B \subseteq X$. By [14, Proposition 4.3.3], this property implies ergodicity.

Proof. Let T be the Bernoulli left shift on X, which we have seen is ergodic. So by Theorem 5.63, we have that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} f(T^{(i)}(x)) = \int_X f(x)\,\mathrm{d}\nu_p(x)$$

for any $f \in L^1(X, \nu_p)$, and ν_p-almost every $x \in X$. Choosing f to be

$$f(x) = \begin{cases} 1 & \text{if } x_0 = 1 \\ 0 & \text{if } x_0 = 0 \end{cases}$$

we see that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=1}^{k} x_i = \int_X f(x)\,\mathrm{d}\nu_p(x) = p$$

for ν_p-almost every $x \in X$. Thus $\nu_p(A_r) = 1$ precisely when $r = p$. □

From this, it is clear that for any pair $p, q \in (0,1)$ with $p \neq q$, the measures ν_p and ν_q are singular. This will allow us to always construct singular measures, and we prove the general statement below.

Proposition 5.65. *If μ is any measure*[23] *on the Cantor space X, then there exists a (non-trivial) continuous measure ν on X such that $\mu \perp \nu$.*

Proof. Firstly, suppose that $\nu_p \ll \mu$ for every $p \in (0,1)$. By Lemma 5.64, this means that $\mu(A_p) \neq 0$ for every $p \in (0,1)$. However, the collection of sets $\{A_p\}_{p\in(0,1)}$ are pairwise disjoint. Since this collection is uncountable, this would imply that μ is not σ-finite, a contradiction.

So now, choose any $p \in (0,1)$ such that $\nu_p \not\ll \mu$. Then by Lebesgue's decomposition theorem there exist measures η_0, η_1 such that $\nu_p = \eta_0 + \eta_1$ with $\eta_0 \ll \mu$ and $\eta_1 \perp \mu$. Since ν_p is continuous, it must follows that both η_0 and η_1 are also continuous. Moreover, by choice of p, it follows that $\eta_1 \neq 0$, and so choosing $\nu = \eta_1$ is sufficient. □

Question 5.66. What conditions need to be imposed on a measure space (X, μ) to guarantee the existence of a continuous singular measure ν?

Lemma 5.67. *Let $G = \mathscr{V} \rtimes \mathscr{U}$, as in prior sections, and suppose that $\dim \mathscr{V} \geq 2$. Then the space $\widehat{G}$ contains a closed subspace which is a Cantor space.*

[23]Recall that we assume all measures to be *σ-finite*.

Proof. Recall that $\widehat{G}_\infty$ is isomorphic to $\mathcal{O}_\mathcal{U}(\mathcal{V}^\times)$, which by Corollary 4.42 is isomorphic to $|\mathcal{K}^\times| \times \mathcal{K}^\circ$, where $\mathcal{K}^\circ$ is the one-point compactification of $\mathcal{K}$. Since $\mathcal{K}$ is a non-Archimedean local field, then $\mathcal{K}^\circ$ will contain the ring of integers $\mathcal{R} = \mathcal{O}(\mathcal{K})$, which is a Cantor space. $\square$

Proposition 5.68. *Let $G = \mathcal{V} \rtimes \mathcal{U}$. If $\dim \mathcal{V} = 2$, then $B_0(G) \neq A(G)$, and in particular, G is not spinal.*

Proof. Let μ be the Plancherel measure on $\widehat{G}$. Since μ is σ-finite by Proposition 5.65 and Lemma 5.67, there exists a (non-trivial) continuous measure ν on $\widehat{G}$ such that $\mu \perp \nu$. Since ν is continuous, we have by Corollary 5.59 that $A_\nu(G) \subseteq B_0(G)$. But by Proposition 5.20 we also have that $A(G) \cap A_\nu(G) = \{0\}$, and so it follows that $A(G) \neq B_0(G)$. $\square$

Chapter 6

Algebra Amenability in Profinite Groups

" You cannot delete empty space. "

–J. Bal

6.1 Central Fourier Algebra

In Section 2.4, we described various conditions which are equivalent to the amenability of G, especially conditions of various algebras on G. Most notably, we saw that G is amenable if and only if $L^1(G)$ is amenable. Naturally, one may ask the converse question: what if these various algebras are amenable, what can we say about the structure of G? Of course this question is already answered for $L^1(G)$, and we have a result, due to Forrest and Runde [21, Theorem 2.3], for $A(G)$. Recall that we say a group G is **virtually abelian** if it has an abelian subgroup of finite index.

Theorem 6.1. *Let G be a locally compact group. Then $A(G)$ is amenable if and only if G is virtually abelian.*

Another algebra of interest is $ZL^1(G)$, the centre of $L^1(G)$. The following conjecture was brought forward by Azimifard, Samei, and Spronk [7], with the reverse direction being proved by Alaghmandan and Crann [3, Corollary 4.5].

Conjecture 6.2. *Let G be a compact group. Then $ZL^1(G)$ is amenable if and only if G is virtually abelian.*

The forward direction, as of the time of writing, remains open. Notice here that G is now compact, and we shall henceforth be working in this setting. The reason we ask for compactness is that such groups have that $ZL^1(G)$ is comprised of elements $f \in ZL^1(G)$ with the property that $f(yx) = f(xy)$ for a.e. $x, y \in G$. In particular, this means that f is constant on the conjugacy classes of G, and so such elements are densely spanned by the characters of G. However, some work has been done on the noncompact case in [7, Section 2].

Let us now present another algebra introduced by Alaghmandan and Spronk [4]. Its definition, and known results parallel that of $ZL^1(G)$.

Definition 6.3. Let G be a locally compact group. We define

$$ZA(G) := \{u \in A(G) : u(a^{-1}xa) = u(x) \text{ for all } a, x \in G\}$$

to be the **central Fourier algebra** of G (or sometimes the G**-centre** of $A(G)$).

The name is perhaps initially a little misleading, for $A(G)$ is always abelian, and so this object is unrelated to the algebraic centre of $A(G)$. Rather, the terminology is inspired by the fact that $ZA(G) = A(G) \cap ZL^1(G)$. In a paper of Alaghmandan and Spronk [4], they show that if G is virtually abelian (and compact), then $ZA(G)$ is amenable. Naturally, they also conjecture the converse statement holds, though this again remains open.

As before, $ZA(G)$ contains precisely all $u \in A(G)$ which are constant on conjugacy classes, that is they are invariant under inner conjugations. Let us extend this to more general groups of automorphisms.

Definition 6.4. Let G be a locally compact group, and let K be a compact group such that $K \curvearrowright G$. Then

$$Z_K A(G) := \{u \in A(G) : u(k \cdot x) = u(x) \text{ for all } k \in K,\, x \in G\}$$

is called the K**-centre** of $A(G)$.

There is an easy way to determine if a given $u \in A(G)$ belongs to $Z_K A(G)$.

Lemma 6.5. *Let G be a locally compact group, and let K be a compact group such that $K \curvearrowright G$. We have that $u \in Z_K A(G)$ if and only if $u = \int_K v \circ k \, \mathrm{d}k$ for some $v \in A(G)$.*

Proof. The forward direction is trivial, simply take $v = u$. For the reverse direction, let $u = \int_K v \circ k \, \mathrm{d}k$ for some $v \in A(G)$. It is clear that $v \circ k \in A(G)$ for any $k \in K$, and so it follows that $u \in A(G)$. Moreover we have that $u \circ l = \int_K v \circ (kl) \, \mathrm{d}k = \int_K v \circ k \, \mathrm{d}k = u$, and so $u \in Z_K A(G)$. □

Naturally, if we identify G with the inner automorphisms of G, we have that $Z_G A(G) = ZA(G)$, so this is a generalisation of the previous definition. When G is abelian, we have that the K-centre of $A(G)$ is closely related to the central Fourier algebra of its semidirect product.

Proposition 6.6. *Let G be an abelian locally compact group, and K a compact group such that $K \curvearrowright G$. Then $Z_K A(G)$ is a quotient algebra of $ZA(G \rtimes K)$.*

Proof. We know that that the restriction map $u \mapsto u\big|_G$ is a quotient map from $A(G \rtimes K)$ to $A(G)$. So all that remains to verify is that it also maps $ZA(G \rtimes K)$ to $Z_K A(G)$. Indeed, for $u \in ZA(G \rtimes K)$ we have that $u\big|_G(k \cdot x) = u(kxk^{-1}) = u(x)$, and so $u\big|_G \in Z_K A(G)$.

Let us also verify surjectivity. Take any $u \in Z_K A(G)$ and write $u = \int_K v \circ k \, \mathrm{d}k$ for some $v \in A(G)$. Then we can choose $v' \in A(G \rtimes K)$ such that $v'\big|_G = v$, see for instance [34, Theorem 2.6.4]. Now let $u' = \int_K v' \circ k \, \mathrm{d}k$, from which it is clear that $u'\big|_G = u$. It is immediate that u' is invariant by conjugations in K, and furthermore, since G is abelian, it is trivially invariant under conjugations in G. Hence we have that $u' \in ZA(G \rtimes K)$, so that this restriction map is surjective, and thus a quotient map. □

Since amenability is passed through quotients, this means that one way we can exhibit failure of amenability of $ZA(G \rtimes K)$ is by considering the smaller algebra $Z_K A(G)$. Indeed, this will be our plan. In particular, this $Z_K A(G)$ is actually the Fourier algebra of the *hypergroup* generated by the action $K \curvearrowright G$. Let us introduce the terminology and make this precise.

6.2 Hypergroups

6.2.1 Definition

Hypergroups are a generalisation of groups which were originally introduced by Jewett [28] under the name "*convolution spaces*" or "*convos*" for short. The core idea behind these objects is that when we multiply two elements, our result is allowed to be 'spread out' among different elements. For example, the result of some product may be supported on two elements whose weights sum to 1. In general this may be any arbitrary probability measure, and we capture this notion using measure spaces. For most basic definitions and properties, we shall refer to the book of Bloom and Heyer [13], though we shall of course note when reference to other material is made.

Before we begin, let us state some basic definitions and properties. Most of these should be familiar to the reader, but we present them here for completeness. When H is a locally compact Hausdorff space, we adopt the following conventions.

- We let $M(H)$ be the Banach space of complex-valued (finite) measures on H.
- We let $\operatorname{Prob}(H) \subseteq M(H)$ be the space of probability measures.
- For $x \in H$ we let δ_x denote the point measure at x.

Recall that for $\mu \in M(H)$, we define

$$\operatorname{supp} \mu := \bigcap \{F \subseteq H : F \text{ is closed}, \mu(F) \neq 0\} \tag{6.1}$$

to be the **support** of μ. Another space we will need is the collection of compact subsets of H, which we shall denote by $\mathscr{C}(H)$. This collection is given a topology known as the **Michael topology**, where the open sets are generated by collections of the form

$$\mathscr{C}_{U,V} := \{C \in \mathscr{C}(H) : C \subseteq U \text{ and } C \cap V \neq \varnothing\} \tag{6.2}$$

where $U, V \subseteq H$ are open.

With these notations in place, let us present the definition of a hypergroup.

Definition 6.7. Let H be a locally compact space. Suppose there exists an associative binary operation $* : M(H) \times M(H) \to M(H)$ satisfying the following properties.

(H1) For $x, y \in H$, we have $\delta_x * \delta_y \in \operatorname{Prob}(H)$ and $\operatorname{supp}(\delta_x * \delta_y)$ is compact.

(H2) The map $(x, y) \mapsto \delta_x * \delta_y$ from $H \times H$ to $\operatorname{Prob}(H)$ is continuous.

(H3) The map $(x, y) \mapsto \operatorname{supp}(\delta_x * \delta_y)$ from $H \times H$ to $\mathscr{C}(H)$ is continuous.

(H4) There exists an element $e \in H$ known as the *identity* such that $\delta_x * \delta_e = \delta_e * \delta_x = \delta_x$ for all $x \in H$.

(H5) There exists a map $x \mapsto x^*$ from H onto itself such that:

- $x \mapsto x^*$ is involutive, so $(x^*)^* = x$ for all $x \in H$,
- $x \mapsto x^*$ is a homeomorphism, and

- $(\delta_x * \delta_y)^* = \delta_{y^*} * \delta_{x^*}$ for all $x, y \in H$.

  Note that we have extended this map to $M(H)$ by $\mu^*(A) = \mu(A^*)$ for every measurable $A \subseteq H$.

(H6) For $x, y \in H$, we have $e \in \operatorname{supp}(\delta_x * \delta_y)$ if and only if $x = y^*$.

When all of these properties are satisfied, we say that $(H, *)$ is a **hypergroup**.

For convenience, we shall often let x stand in place of the point measure δ_x. In particular we shall write $x * y = \delta_x * \delta_y$, and in this case, we shall call the map $* : H \times H \to M(H)$ the **hyperproduct** of H. Of course, if clarifications are necessary, we shall write δ_x explicitly. For subsets $A, B \subseteq H$, we define $A * B$ as a subset of H given by

$$A * B := \bigcup_{x \in A, y \in B} \operatorname{supp}(x * y) \tag{6.3}$$

When the hyperproduct maps into H, we recover the usual group axioms.

Proposition 6.8. *Let H be a hypergroup. If for every $x, y \in H$ there is some $z \in H$ such that $x * y = z$, then H is a locally compact group.*

Proof. See [28, Proposition 4.1]. □

In general, hypergroups behave much like classical groups, though often to a somewhat limited capacity. For instance, many classes of hypergroups possess the notion of a **Haar measure**: a left-invariant[24] positive measure $\mathbf{m}$ on H. Classes of hypergroups which possess a Haar measure include discrete, compact, and commutative hypergroups; see [28, Theorems 7.1A & 7.2A]. However, the question of whether all hypergroups possess a Haar measure remains open. In the discrete case, there is an explicit formula to compute the Haar measure of H.

Proposition 6.9. *Let H be a discrete hypergroup. Then H possesses a (discrete) Haar measure $\mathbf{m}$, with*

$$\mathbf{m}(\{x\}) = \frac{1}{(x^* * x)(\{e\})}$$

for every $x \in H$.

[24] Here, left-invariant means $x * \mathbf{m} = \mathbf{m}$ for all $x \in H$. One does need to be careful in this definition as $\mathbf{m}$ is $[0, \infty]$-valued, so it is also assumed that $x * \mathbf{m}$ is well defined for all $x \in H$.

Remark 6.10. Notice here that unlike the group case, there is a lack of homogeneity in the structure of H. Indeed, the Haar measure is frequently not constant on the singletons of H. This sort of structure is common to many hypergroups.

Let us present a few more examples which are derived from groups and frequently occur in other mathematical literature.

Example 6.11. Let G be a compact group, with $\widehat{G}$ the (classes of) irreducible representations. By Peter-Weyl, we know that any representation of G can be decomposed into a direct sum of irreducible representations. In particular, if we take π, σ, then $\pi \otimes \sigma$ is also a direct sum of irreducible representations.

With this as motivation, let us write $\chi_\pi := (1/d_\pi)\,\mathrm{tr}(\pi)$, the normalised character of $\pi \in \widehat{G}$ (where d_π is the dimension of π). Let $H = \{\chi_\pi : \pi \in \widehat{G}\}$ with the discrete topology. For $\pi, \sigma, \tau \in \widehat{G}$, let $\alpha^\pi_{\sigma,\tau}$ denote the multiplicity of π in $\sigma \otimes \tau$. Then it follows that

$$\chi_\sigma \chi_\tau = \sum_{\pi \in \widehat{G}} \alpha^\pi_{\sigma,\tau} \chi_\pi \tag{6.4}$$

and we extend this to convolution on $M(H)$. Naturally, some work is required to verify that this is indeed a hypergroup. Notably, we have that the inverse of $\pi \in \widehat{G}$ is $\bar{\pi}$, the contragredient of π, and moreover, the *Haar measure* of H is a discrete measure with $\mathbf{m}(\{\pi\}) = d_\pi$ (see Proposition 6.9).

Example 6.12. Let G be a locally compact group and K a (non-normal) compact subgroup of G. We shall denote by $G/\!\!/K$ the collection of *double cosets* of K in G, so that $G/\!\!/K := \{KxK : x \in G\}$. This will be our underlying space H, and we equip it with the quotient topology. Then H is a hypergroup with

$$KxK * KyK = \int_K \delta_{HxkyH}\,\mathrm{d}k \tag{6.5}$$

where $x, y \in G$ and $\mathrm{d}k$ is the normalised Haar measure on K. We quickly see that $e_H = KeK = K$ and $(KxK)^* = Kx^{-1}K$.

The example which will be of importance to us will be the orbit space of a compact action. Given a locally compact group G and a compact group K acting on G, let us set H to be orbit space $\mathbb{O}_K(G)$. This space is naturally equipped with the quotient topology, and along with a hyperproduct that we shall introduce shortly, this turns H into a hypergroup. This hypergroup is known as a **spherical hypergroup**, and arises from a more general construction. Spherical hypergroups were introduced by Muruganandam [44], and are constructed using a map known as a **spherical projector**. Such a projector is defined in Definition 2.1 of the same paper, in terms of the conditions $\mathbf{SH}_1$, $\mathbf{SH}_2$, and $\mathbf{SH}_3$. In essence, spherical projectors are maps $\pi : C_c(G) \to C_c(G)$ with nice 'averaging' properties. We note that they may be extended in the obvious way to maps $\pi_0 : C_0(G) \to C_0(G)$ and $\pi^1 : L^1(G) \to L^1(G)$ with a corresponding adjoint map $\pi_0^* : M(G) \to M(G)$. When we wish to be explicit, we use these notations of π_0, π^1 and π_0^*, though frequently we shall refer to all these simply as π.

Muruganandam shows in Example 2.3 of the same paper that the double cosets $G/\!/K$ are also a spherical hypergroup. In much the same way, we shall observe that the orbit space $H = \mathcal{O}_K(G)$ is also a spherical hypergroup. Indeed, the spherical projector $\pi : C_c(G) \to C_c(G)$ that induces H is given by

$$\pi(f)(x) := \int_K f(k \cdot x)\,\mathrm{d}k \tag{6.6}$$

for $f \in C_c(G)$ and $x \in G$. We let $H = \{\pi(\delta_x) : x \in G\}$ and we equip $M(H)$ with the usual convolution inherited from G. To check that this is indeed a hypergroup, it suffices to show that π is a spherical projector.

Proposition 6.13. *Let G be a locally compact group and K a compact group acting on G. Then π as defined in (6.6) is a spherical projector.*

Proof. The proofs of $\mathbf{SH}_1$ and $\mathbf{SH}_2$ are fairly straightforward algebraic computations, so we leave these for the reader to compute. For $\mathbf{SH}_3$, it is clear that $\operatorname{supp} \pi(\delta_x) = [x]_K$ from which it follows that π satisfies points (i) and (ii) of $\mathbf{SH}_3$. Lastly, one may check that the Michael topology on orbits of K coincides with the usual quotient topology, and so the map $x \mapsto [x]_K$ is continuous. □

For discrete groups, we can model these hypergroups quite nicely.

Proposition 6.14. *Let G be a discrete group and K a compact group acting on G. If we let π be the spherical projector, then $\pi(\delta_x) \in \ell^1(G)$ with*

$$\pi(\delta_x) = \frac{1}{|K \cdot x|} \mathbb{1}_{K \cdot x}$$

and moreover the hyperproduct on H is given by the convolutions of these as functions in $\ell^1(G)$.

Proof. Firstly, we have that

$$\langle \pi(\delta_x), f \rangle = \langle \delta_x, \pi(f) \rangle = \pi(f)(x) = \int_K f(k \cdot x) \, \mathrm{d}k$$

for $f \in C_0(G)$ and $x \in G$. Since $[x]_K$ is finite, it follows by the orbit stabiliser theorem (Proposition 2.55) that K/K_x is finite and we can write

$$\begin{aligned}
\int_K f(k \cdot x) \, \mathrm{d}k &= \frac{1}{|K/K_x|} \sum_{lK_x \in K/K_x} \int_{K_x} f(lk \cdot x) \, \mathrm{d}k \\
&= \frac{1}{|K/K_x|} \sum_{lK_x \in K/K_x} f(x \cdot l) \\
&= \frac{1}{|K \cdot x|} \sum_{y \in K \cdot x} f(y) \\
&= \sum_{y \in G} \frac{1}{|K \cdot x|} \mathbb{1}_{K \cdot x}(y) f(y)
\end{aligned}$$

thus showing that

$$\pi(\delta_x) = \frac{1}{|K \cdot x|} \mathbb{1}_{K \cdot x}$$

□

Let us introduce some additional terminology from this paper of Muruganandam.

Definition 6.15. Let G be a locally compact group and π a spherical projector. We say that a function $f \in C_c(G)$ is π**-radial** if $\pi(f) = f$, or equivalently, if f is constant on the orbits of π.

Naturally, we extend this definition to π_0, π^1 and π_0^* as well. When the modular function on H is π-radial, we say that the projector π is **ultraspherical**. Since the groups we are dealing with are abelian and hence unimodular, then by [44, Proposition 2.14 (ii)] we have that π is ultraspherical. We also define the algebras

$$\begin{aligned}
L^1_\pi(G) &:= \{f \in L^1(G) : f \text{ is } \pi^1\text{-radial}\} \\
A_\pi(G) &:= \{u \in A(G) : u \text{ is } \pi_0\text{-radial}\}
\end{aligned}$$

consisting of π-radial functions. Muruganandam shows in Theorem 3.4 that when π is ultraspherical, then $A_\pi(G) = A(H)$. Here, the Fourier algebra of a hypergroup is defined in much the same way as it is for classical groups. In order to avoid repetition, we shall not define this Fourier algebra or other related spaces here, but instead refer the reader to any of the papers [2, 3, 43].

Remark 6.16. Care must be taken. While we can always define $A(H)$ of a hypergroup H in the usual manner, it is not the case that this is always an algebra under pointwise multiplication, let alone a Banach algebra. When this is the case, we call such a hypergroup a **regular Fourier hypergroup**. It is shown by Muruganandam [44, Theorem 3.13] that all ultraspherical hypergroups are indeed regular Fourier hypergroups. Thus we need not concern ourselves with this issue here.

One may quickly observe that a similar result holds for $L^1_\pi(G)$.

Proposition 6.17. *Let G be a locally compact group and π a spherical projector. Then $L^1_\pi(G) = L^1(H)$.*

Proof. Given an $f \in L^1(H)$, we have by [44, Proposition 2.14] that

$$\|f\|_{L^1(H)} = \int_H f(\dot{x})\,\mathrm{d}\dot{x} = \int_G f(x)\,\mathrm{d}x = \|f\|_{L^1(G)}$$

and so it follows that $L^1(H) \subseteq L^1(G)$. Moreover, $f \in L^1(H)$ precisely when f is constant on the orbits of π, or in other words, f is π-radial. □

Now let us consider the case where a compact group K acts on an *abelian* group G. Since both $K \curvearrowright G$ and $K \curvearrowright \widehat{G}$ are both group actions, they have associated hypergroups which we shall denote by H and $\widehat{H}$[25] respectively.

Theorem 6.18. *Let K be a compact group acting on an abelian locally compact group G. Let π be the associated spherical projector and H the corresponding hypergroup of $K \curvearrowright G$. Likewise $\hat{\pi}$ and $\widehat{H}$ for $K \curvearrowright \widehat{G}$. Then*

$$L^1(H) = L^1_\pi(G) = A_{\hat{\pi}}(\widehat{G}) = A(\widehat{H})$$

Proof. We have already seen that $L^1(H) = L^1_\pi(G)$ by Proposition 6.17. Since $\widehat{H}$ is abelian, it follows that $\hat{\pi}$ ultraspherical, and so $A_{\hat{\pi}}(\widehat{G}) = A(\widehat{H})$ by [44, Theorem 3.4]. For $f \in L^1(G)$,

[25] Despite the suggestive notation, this does not indicate the Pontryagin dual of the hypergroup. Even though this is a reasonable candidate for the dual of H (if such a dual exists), it is not clear if this is truly the dual in the Pontryagin sense. Since we do not concern ourselves with duals of hypergroups, this notation shall cause no confusion.

we have that

$$\begin{aligned}\hat{\pi}(\widehat{f})(\varphi) &= \int_K \widehat{f}(k \cdot \varphi)\,\mathrm{d}k \\ &= \int_K \int_G f(x)\,\overline{\langle k \cdot \varphi, x\rangle}\,\mathrm{d}x\,\mathrm{d}k \\ &= \int_K \int_G f(x)\,\overline{\langle \varphi, k^{-1} \cdot x\rangle}\,\mathrm{d}x\,\mathrm{d}k\end{aligned}$$

Since K is compact, it is a *special* action, and so we can apply Corollary 2.67. Hence we can write

$$\begin{aligned}\hat{\pi}(\widehat{f})(\varphi) &= \int_K \int_G f(k \cdot x)\,\overline{\langle \varphi, x\rangle}\,\mathrm{d}x\,\mathrm{d}k \\ &= \int_G \pi(f)(x)\,\overline{\langle \varphi, x\rangle}\,\mathrm{d}x \\ &= \widehat{\pi(f)}(\varphi)\end{aligned}$$

and from this, it is clear that f is π-radial if and only if $\widehat{f}$ is $\hat{\pi}$-radial. Thus we have $L^1_\pi(G) = A_{\hat{\pi}}(\widehat{G})$, and so the result follows. □

6.2.2 Amenability

The notion of amenability on hypergroups is somewhat more tortured than in the group setting. Naturally we say a hypergroup H is **amenable** if there is a $\mu \in L^\infty(H)^*$ which is left-invariant. In the group setting, there are a number of other similar or equivalent conditions, most of which translate nicely into the language of hypergroups. However, many of the equivalences between these conditions that one obtains in the group setting break down for hypergroups. As such, their amenability theory can at times be significantly more delicate.

Early work on the study of amenable hypergroups was done by Skantharajah [54], who introduced the (P_p) Reiter conditions for hypergroups; these are defined in an analogous manner to the group setting. In Theorem 4.1 of their paper, Skantharajah shows that (P_1) is equivalent to amenability of H, and also shows in Theorem 4.3 that (P_2) is stronger than (P_1). However, unlike the classical case, (P_1) need not imply (P_2), see Lemma 4.5. For us, condition (P_2) will be of particular interest.

Other conditions were introduced by Alaghmandan [1, 2], and in particular, we shall focus on the *1-Leptin condition* (L_1). Recall that we define the hyperproduct of sets in (6.3).

Definition 6.19. A hypergroup H satisfies the **1-Leptin** condition if for every compact $K \subseteq H$ and $\varepsilon > 0$, there exists a measurable $V \subseteq H$ such that $0 < \mathbf{m}(V) < \infty$ with $\mathbf{m}(K * V)/\mathbf{m}(V) < 1 + \varepsilon$.

It is shown by [2, Proposition 4.1 and Theorem 4.4] that (L_1) implies (P_2) for discrete Fourier hypergroups[26]. When a hypergroup is (P_2), we have a necessary condition for the amenability of $L^1(H)$. This condition is given as Theorem 5.1 in the same paper of Alaghmandan.

Theorem 6.20. *Let H be a discrete commutative hypergroup which satisfies (P_2). If $L^1(H)$ is amenable, then there is some $M \geq 1$ such that $\{x \in H : \mathbf{m}(x) \leq M\}$ is infinite.*

6.3 Actions on Profinite Groups

Recall that we say a locally compact group is **profinite** if it is the projective (inverse) limit of finite groups. Equivalently, it is a well-known result that a group is profinite if and only if it is a totally disconnected compact group. Consider the example of compact DVR $\mathscr{R}$. As we have seen in Chapter 3, such a ring will have a collection of ideals $\mathscr{M}^n$, and moreover, these ideals form a local basis for the topology of $\mathscr{R}$. Let us give a name to this property.

Definition 6.21. Let G be a locally compact group. We say G has the **small invariant neighbourhood** property (or **[SIN]** for short) if there exists a local basis of the identity consisting of compact normal subgroups.

As it turns out, all profinite groups are [SIN] groups. This result is in essence the statement of *van Dantzig's Theorem*, a proof of which may be found in [25, Theorem 7.7].

Theorem 6.22 (van Dantzig's Theorem)**.** *Let G be a locally compact totally disconnected group. Then G contains a neighbourhood basis of compact subgroups. Moreover, if G itself is compact, then G is [SIN].*

In addition, G is second-countable precisely when there are at most countably many open subgroups; see [63, Proposition 4.1.3]. So if G is compact second-countable, then there is a sequence of normal open subgroups $G = G_0 \supsetneq G_1 \supsetneq \ldots$ such that $\bigcap_{n \in \mathbb{N}} G_n = \{e\}$. Henceforth we shall let G_n denote the subgroups in such a sequence. We can quantify the rate at which these subgroups shrink.

[26] In this same paper, Alaghmandan presents a nice diagram summarising some known implications of amenability conditions of discrete Fourier hypergroups. It may be found below Example 4.5.

Lemma 6.23. *Let G be a second-countable profinite group. If we set $q_n = \mathbf{m}(G_n)^{-1}$, then $\lim_{n\to\infty} (q_{n+1} - q_n) = \infty$.*

Proof. Since G_{n+1} is a strict subgroup of G_n, then $\mathbf{m}(G_n) \geq 2\mathbf{m}(G_{n+1})$. Rearranging, we find that $q_{n+1} - q_n \geq q_n$. Since $\bigcap_{n\in\mathbb{N}} G_n = \{e\}$, we have that $\mathbf{m}(G_n) \to 0$ and so the result follows. □

When G is abelian, this result has a more direct interpretation. If we label $F_n := {G_n}^{\perp}$, then we may use Corollary 2.11 in order to see that this forms an increasing chain of (finite) subgroups $\{\mathbf{1}\} = F_0 \subsetneq F_1 \subsetneq \dots$ whose union is all of $\widehat{G}$. Noting that $|F_n| = \mathbf{m}(G_n)^{-1} = q_n$, it follows by Lemma 6.23 that the (finite) sets $F_{n+1} \setminus F_n$ grow without bound. Let us then consider a special class of compact actions on G, which behave nicely with respect to these chains of groups.

Definition 6.24. Let G be a (second-countable) profinite abelian group, with a decreasing sequence of compact open subgroups G_n such that $\bigcap_{n\in\mathbb{N}} G_n = \{0\}$. If K is a compact group which acts on G, we say that the action $K \curvearrowright G$ is **layered by** G_n if the orbits of the *dual group* are precisely the sets $B_n := F_{n+1} \setminus F_n$ and $B_0 := \{\mathbf{1}\}$. If such a layering exists, we shall say that $K \curvearrowright G$ is a **layered action**.

> *Remark* 6.25. One may ask why we require the orbits match the dual group and not the group itself. This shall be made apparent soon, and later in this section we discuss the alternative of asking that the group itself be "layered". See in particular Conjecture 6.30.

When $K \curvearrowright G$ is layered by G_n, let H denote the hypergroup of the action $K \curvearrowright G$, and $\widehat{H}$ the hypergroup of the corresponding dual action. We shall let b_n be the elements of the dual hypergroup $\widehat{H}$, whose associated orbit is $B_n = F_n \setminus F_{n-1}$ where $F_n = {G_n}^{\perp}$. In particular, the hypergroup elements are given by

$$b_n = \frac{1}{|B_n|}\mathbb{1}_{B_n} = \frac{1}{q_n - q_{n-1}}\left(\mathbb{1}_{F_n} - \mathbb{1}_{F_{n-1}}\right) \tag{6.7}$$

which follows from Proposition 6.14. This is a useful representation as the product of two elements b_n, b_m is precisely their convolution as L^1 functions on $\widehat{G}$. Thus we compute the algebraic structure of $\widehat{H}$ as follows.

Proposition 6.26. *Let G be a (second-countable) profinite abelian group, and K a layered action on G, and assume the notation as given above. For $n > m \geq 0$ we have*

$$b_n * b_m = b_n \qquad \text{and} \qquad b_n * b_n = \frac{1}{q_n - q_{n-1}}\left[\sum_{i=0}^{n}(q_i - q_{i-1})b_i - q_{n-1}b_n\right]$$

where we set $q_{-1} := 0$ for convenience.

Proof. Let us make the conventions $g_n = \mathbb{1}_{G_n}$, $f_n = \mathbb{1}_{F_n}$ and $\alpha_n = q_n/q_{n-1}$. One may easily verify that $\widehat{f_n} = q_n g_n$. By considering $b_n \in L^1(\widehat{G})$, we have that

$$\widehat{b_n} = \left[\frac{f_n - f_{n-1}}{q_n - q_{n-1}}\right]^{\frown} = \frac{q_n g_n - q_{n-1} g_{n-1}}{q_n - q_{n-1}} = \frac{\alpha_n g_n - g_{n-1}}{\alpha_n - 1}$$

Now for $n > m$, we have the absorption property $g_n \cdot g_m = g_n$ since $G_n \cdot G_m = G_n$. Thus we compute

$$\begin{aligned}
\widehat{b_n} \cdot \widehat{b_m} &= \left(\frac{\alpha_n g_n - g_{n-1}}{\alpha_n - 1}\right)\left(\frac{\alpha_m g_m - g_{m-1}}{\alpha_m - 1}\right) \\
&= \frac{1}{(\alpha_n - 1)(\alpha_m - 1)} \left(\alpha_n \alpha_m g_n - \alpha_n g_n - \alpha_m g_{n-1} + g_{n-1}\right) \\
&= \frac{1}{(\alpha_n - 1)(\alpha_m - 1)} \left(\alpha_m - 1\right)\left(\alpha_n g_n - g_{n-1}\right) \\
&= \frac{\alpha_n g_n - g_{n-1}}{\alpha_n - 1}
\end{aligned}$$

and so

$$b_n * b_m = [\widehat{b_n} \cdot \widehat{b_m}]^{\frown} = b_n$$

Similarly, we compute

$$\begin{aligned}
\widehat{b_n} \cdot \widehat{b_n} &= \frac{1}{(\alpha_n - 1)^2} \left(\alpha_n g_n - g_{n-1}\right)^2 \\
&= \frac{1}{(\alpha_n - 1)^2} \left((\alpha_n - 1)\alpha_n g_n - \alpha_n g_n + g_{n-1}\right) \\
&= \frac{\alpha_n g_n}{\alpha_n - 1} - \frac{\alpha_n g_n - g_{n-1}}{(\alpha_n - 1)^2}
\end{aligned}$$

and so

$$\begin{aligned}
b_n * b_n &= \frac{f_n}{q_{n-1}(\alpha_n - 1)} - \frac{b_n}{\alpha_n - 1} \\
&= \frac{1}{q_n - q_{n-1}} \left[f_n - q_{n-1} b_n\right]
\end{aligned}$$

Since b_n is a weighted difference of f_n and f_{n+1}, we can reverse this to obtain that $f_n = \sum_{i=0}^{n}(q_i - q_{i-1})b_i$. Thus we prove our claim that

$$b_n * b_n = \frac{1}{q_n - q_{n-1}} \left[\sum_{i=0}^{n} (q_i - q_{i-1}) b_i - q_{n-1} b_n \right] \qquad \square$$

This immediately shows that such a hypergroup is (P_2).

Corollary 6.27. *Let G and K be as in Proposition 6.26, and let $\widehat{H}$ be the corresponding dual hypergroup. Then $\widehat{H}$ satisfies (L_1), and hence (P_2).*

Proof. Let $F \subseteq \widehat{H}$ be any finite subset of $\widehat{H}$, and choose $N > \max\{n \in \mathbb{N} : b_n \in F\}$. Then $b_N * F = b_N$, and so $|\{b_N\} * F|/|\{b_N\}| = 1$. $\square$

Theorem 6.28. *Let G be a (second-countable) profinite abelian group, and suppose that $K \curvearrowright G$ is a layered action. Then $Z_K A(G)$ is not amenable. In particular $ZA(G \rtimes K)$ is also not amenable.*

Proof. From Proposition 6.26 we get that $\mathbf{1} \in \operatorname{supp}(b_n * b_n)$, so that the inverse of b_n in $\widehat{H}$ is itself. Moreover by Proposition 6.9, the Haar measure for $\widehat{H}$ is

$$\mathbf{m}(b_n) = \frac{1}{(b_n * b_n)(\mathbf{1})} = q_{n+1} - q_n$$

which by Lemma 6.23 grows without bound. Thus it follows that for any $M > 0$, the equation $\mathbf{m}(b_n) \geq M$ holds for all but finitely many n. Since $\widehat{H}$ is (P_2), then by [2, Theorem 5.1] we have that $L^1(\widehat{H}) = A(H) = Z_K A(G)$ is not amenable. $\square$

It is likely that this method may be used for a broader class of hypergroups, with perhaps a more flexible structure. Nonetheless, this covers several examples, and perhaps one of the more prominent examples is the general linear action on d copies of a compact DVR.

Theorem 6.29. *Let $\mathscr{R}$ be a compact DVR. If $G = \mathscr{R}^d \rtimes \mathrm{GL}_d(\mathscr{R})$ for $d > 0$, then $ZA(G)$ is not amenable.*

Proof. Firstly it is clear that $G = \mathscr{R}^d$ is a second-countable profinite abelian group.[27] Consider then the action of $K = \mathrm{GL}_d(\mathscr{R})$ on G. Recall that we let $\mathscr{M}$ be the maximal ideal in $\mathscr{R}$. Borrowing this notation, we define $\mathscr{M}_d^n := \{(x_1, \ldots, x_d) \in \mathscr{R}^d : x_i \in \mathscr{M}^n\}$, so that these form the nested subgroups G_n. If we let $\mathscr{K}$ be the local field associated to $\mathscr{R}$, then the dual of $\mathscr{R}^d$ is (isomorphic to) $\mathscr{K}^d/\mathscr{R}^d$. Moreover, one can verify that the annihilators of $\mathscr{M}_d^n$ are of the form $\mathscr{M}_d^{-n}/\mathscr{R}^d$. Now, the dual action will act by row premultiplication (so $A \in \mathrm{GL}_d(\mathscr{R})$ acts on $x \in \mathscr{K}^d/\mathscr{R}^d$ by $x^T A$). Clearly this action is isomorphic to the usual action of K on $\mathscr{K}^d/\mathscr{R}^d$. So it suffices to show that the non-trivial orbits of $K \curvearrowright \mathscr{K}^d$ are of the form $\mathscr{M}_d^m \setminus \mathscr{M}_d^{m-1}$ for $m \in \mathbb{Z}$.

It is not difficult to check that the sets $\mathscr{M}_d^m$ are invariant under the action of K, we leave this as an exercise to the reader. The difficulty is to show that the their differences are in fact orbits. Without loss of generality, let us show that $\mathscr{M}_d^0 \setminus \mathscr{M}_d^1 = \mathscr{R}^d \setminus \mathscr{M}_d$ is an orbit. Let us set $e_i \in \mathscr{R}^d$ to be the usual basis element with 1 at the i-th position and 0 everywhere else. These are all in the same orbit since $K = \mathrm{GL}_d(\mathscr{R})$ contains the permutation matrices. Let us now take $x = (x_1, \ldots, x_d) \in \mathscr{R}^d \setminus \mathscr{M}_d$. Since K contains the permutation matrices, let us assume without loss of generality that $x_1 \in \mathscr{R} \setminus \mathscr{M}$. Now take $A \in K$ to be the matrix

$$A = \begin{bmatrix} x_1 & 0 & 0 & \cdots & 0 \\ x_2 & 1 & 0 & \cdots & 0 \\ x_3 & 0 & 1 & & 0 \\ \vdots & \vdots & & \ddots & \vdots \\ x_d & 0 & 0 & \cdots & 1 \end{bmatrix}$$

where A is invertible since $x_1 \in \mathscr{R}^*$. Clearly $Ae_1 = x$, and so x is in the orbit of e_1. By transitivity, $\mathscr{R} \setminus \mathscr{M}$ is an orbit, and $K \curvearrowright G$ is a layered action. The result then follows by Theorem 6.28. □

This reaffirms the conjecture of Alaghmandan and Spronk, as $G = \mathscr{R}^d \rtimes \mathrm{GL}_d(\mathscr{R})$ is not virtually abelian. With this previous example, some readers may find it an attractive prospect that if the sets $G_n \setminus G_{n+1}$ are orbits, then $K \curvearrowright G$ is a layered action. However, the proof of this is not so straightforward, there are surprising complications which arise. We shall present some progress towards the proof of this result, though the final step currently remains unsolved (and is a rather interesting problem in its own right).

[27] As a word of caution, we will be overloading the superscript notation. Occasionally it will mean algebraic products of ideals, such as in $\mathscr{M}^n$, but it may also mean a Cartesian product, as in $\mathscr{R}^d$. Regardless, there will be sufficient context provided to determine which interpretation is in use, so no ambiguity should arise.

Conjecture 6.30. *Let G be a second-countable profinite abelian group, and let K be a compact group acting on G. If the orbits of $K \curvearrowright G$ are precisely the trivial orbit and those of the form $G_n \setminus G_{n+1}$, then $K \curvearrowright G$ is a layered action.*

Partial proof. Without loss of generality, it suffices to show that $F_1 \setminus \{\mathbf{1}\}$ is an orbit (where we adopt the notation $F_n = {G_n}^{\perp}$ as before). To this end, consider the action of K on the quotient G/G_1. By assumption this has two orbits: the trivial orbit and everything else. Let us say that such actions are **transitive**.

Notice now that G/G_0 is finite. So by transitivity, every (non-trivial) element must have the same order, and hence it is of the form $\mathbb{F}_p^n$ where $\mathbb{F}_p$ is the finite field of order p. We also have that $F_1 = \widehat{G/G_0} = \widehat{\mathbb{F}_p^n} = \mathbb{F}_p^n$. Now if one could show that dual action of $K \curvearrowright F_1$ is also transitive, then this would complete the proof of this result. $\square$

So the question remains, is this dual action also transitive? In general, it is not too difficult to see that the dual action of $G \curvearrowright \mathbb{F}_p^n$ is in fact given by $G^T \curvearrowright \mathbb{F}_p^n$ where $G^T := \{A^T : A \in G\}$.

Conjecture 6.31. *Let G be a transitive subgroup of $\mathrm{GL}_n(\mathbb{F}_p)$ acting on $\mathbb{F}_p^n$. Then the transpose group G^T is also transitive on $\mathbb{F}_p^n$.*

This question is surprisingly difficult to answer. There is a classification of all transitive subgroups of $\mathrm{GL}_n(\mathbb{F}_p)$ due to Hering [24]: in particular there are four infinite classes of transitive subgroups, as well as a handful of sporadic examples. This is summarised quite succinctly in Appendix 1 of Liebeck's paper [39].

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APPENDICES

Appendix A

Hilbert-Valued Functions

" Figerters. "
–*G. Wong*

When we write $C(X)$, or $L^2(X, \mu)$, we have that the underlying functions (or equivalence classes thereof) are complex-valued. We can modify these definitions so that functions take value on some Hilbert space $\mathscr{H}$, and we denote these by $C(X; \mathscr{H})$ and $L^2(X, \mu; \mathscr{H})$ respectively. Of course, when $\mathscr{H} = \mathbb{C}$, these will coincide with their usual definitions. The construction of $C(X; \mathscr{H})$ is straightforward, and needs no extra work, as is the case for its various subspaces such as $C_c(X; \mathscr{H})$. However, some care is needed when defining $L^2(X, \mu; \mathscr{H})$.

Given a measure space (X, μ) and a Hilbert space $\mathscr{H}$, we define $L^2(X, \mu; \mathscr{H})$ in the following manner. First, let us set $\mathscr{F}$ to be the collection of measurable functions $f : X \to \mathscr{H}$ where the quantity $\int_X \|f(x)\|^2_{\mathscr{H}} \, \mathrm{d}\mu(x)$ is finite. Then we define the degenerate inner product on $\mathscr{F}$ by

$$\langle f \mid g \rangle_{\mathscr{F}} := \int_X \langle f(x) \mid g(x) \rangle \, \mathrm{d}\mu(x) \tag{A.1}$$

for $f, g \in \mathscr{F}$. Following the standard method, one can quotient by the norm zero elements, then complete this quotient space, and this will create a Hilbert space. We then define $L^2(X, \mu; \mathscr{H})$ to be this resultant Hilbert space. When it is clear which measure we are referring to, we shall omit it and simply write $L^2(X; \mathscr{H})$.

We can identify $L^2(X, \mu; \mathscr{H})$ with the Hilbert space $L^2(X, \mu) \otimes_2 \mathscr{H}$, where $\otimes_2$ denotes the usual Hilbert space tensor product.

Proposition A.1. *Let (X, μ) be a measure space, and $\mathscr{H}$ a Hilbert space. Then*

$$L^2(X;\mathscr{H}) \cong L^2(X) \otimes_2 \mathscr{H}$$

Proof. Let us denote by $\mathcal{S}(X)$ the collection of simple functions on X, and similarly $\mathcal{S}(X;\mathscr{H})$ for $\mathscr{H}$-valued simple functions. It is well known that the Hilbert space completion of $\mathcal{S}(X)$ gives $L^2(X)$, and in a similar manner, the completion of $\mathcal{S}(X;\mathscr{H})$ with respect to the inner product in (A.1) gives $L^2(X;\mathscr{H})$.

So now, let $U : \mathcal{S}(X) \otimes_2 \mathscr{H} \to \mathcal{S}(X;\mathscr{H})$ be defined by

$$U[\mathbb{1}_A \otimes \xi] = \mathbb{1}_A(\cdot)\,\xi$$

for any measurable $A \subseteq X$ and $\xi \in \mathscr{H}$. We claim U is surjective and inner product preserving. For surjectivity, note that if $f \in \mathcal{S}(X)$, then there are measurable $A_i \subseteq X$ and $\xi \in \mathscr{H}$ such that $f(x) = \sum_{i=1}^n \mathbb{1}_{A_i}(x)\xi_i$, from which it follows that $U[\sum_{i=1}^n \mathbb{1}_{A_i} \otimes \xi_i] = f$.

Now take any measurable $A, B \subseteq X$ and $\xi, \eta \in \mathscr{H}$. Then

$$\begin{aligned}
\langle U[\mathbb{1}_A \otimes \xi] \,|\, U[\mathbb{1}_B \otimes \eta]\rangle &= \langle \mathbb{1}_A(\cdot)\,\xi \,|\, \mathbb{1}_B(\cdot)\,\eta\rangle \\
&= \int_X \langle \mathbb{1}_A(x)\,\xi \,|\, \mathbb{1}_B(x)\,\eta\rangle_{\mathscr{H}}\,\mathrm{d}\mu(x) \\
&= \int_X \mathbb{1}_A(x)\mathbb{1}_B(x)\,\mathrm{d}\mu(x)\ \langle \xi \,|\, \eta\rangle_{\mathscr{H}} \\
&= \langle \mathbb{1}_A \,|\, \mathbb{1}_B\rangle_{L^2(X)}\ \langle \xi \,|\, \eta\rangle_{\mathscr{H}} \\
&= \langle \mathbb{1}_A \otimes \xi \,|\, \mathbb{1}_B \otimes \eta\rangle
\end{aligned}$$

and so U is inner product preserving. Thus we are able to extend U to a unitary map $\tilde{U} : L^2(X) \otimes_2 \mathscr{H} \to L^2(X;\mathscr{H})$, and so these spaces are isomorphic as Hilbert spaces. □

As a result, we can use the associativity of tensor products to show that nesting these Hilbert-valued L^2 spaces is the same as working on L^2 of the product measure space.

Corollary A.2. *Let (X, μ) and (Y, ν) be measure spaces, and let $\mathscr{H}$ be a Hilbert space. Then*

$$L^2(X; L^2(Y;\mathscr{H})) \cong L^2(X \times Y;\mathscr{H})$$

Proof. One need only observe that

$$\begin{aligned}
L^2(X; L^2(Y;\mathscr{H})) &\cong L^2(X) \otimes_2 L^2(Y;\mathscr{H}) \\
&\cong L^2(X) \otimes_2 L^2(Y) \otimes_2 \mathscr{H} \\
&\cong L^2(X \times Y) \otimes_2 \mathscr{H} \\
&\cong L^2(X \times Y;\mathscr{H})
\end{aligned}$$

□

Appendix B

One-Point Compactification

"$\bullet$"

–J. Zhu

Recall that when X is a locally compact Hausdorff space, we have that $C_0(X)$ is a commutative Banach algebra whose spectrum is X itself. Naturally, this will be a unital algebra precisely when X is compact. So if X is a noncompact space, we may embed X into a compact space by embedding $C_0(X)$ into a unitary algebra. Of course, the simplest way to do this is to take the unitisation of $C_0(X)$. For a general commutative Banach algebra $\mathscr{A}$, we shall let $\mathscr{A}^\sharp = \mathscr{A} \oplus \mathbb{C}$ denote its *unitisation*, and in particular, we let $\mathbb{1}$ denote the unit inside $\mathscr{A}^\sharp$. This leads us to the following definition.

Definition B.1. Let X be a locally compact Hausdorff space. We define the **one-point compactification** (or the **Alexandroff compactification**) X° of X to be the space $\Gamma_{C_0(X)^\sharp}$, the Gelfand spectrum of $C_0(X)^\sharp$.

Since $C_0(X)^\sharp$ is a unital algebra by construction, it follows that $X^\circ = \Gamma_{C_0(X)^\sharp}$ will always be a compact Hausdorff space, and so this is a genuine compactification of the space X. Contrast this with the larger space $C_b(X)$, whose spectrum will be the *Stone-Čech compactification* βX, which one may consider to be a 'larger' compactification in a certain sense.

> *Remark* B.2. Some authors will only define the unitisation only for non-unital algebras. In our case, we always set $\mathscr{A}^\sharp := \mathscr{A} \oplus \mathbb{C}$, so that even unital algebras will 'grow' in size. This in turn affects how the one-point compactification X° of a locally compact

space X is defined. In particular, it implies that K° will not be isomorphic to K, even for compact K: instead it will contain one additional isolated point. Strictly speaking, this also means that this is not a true compactification, but if we accept this shortcoming, we will have a simpler working theory.

Let us give a more concrete description of X°.

Proposition B.3. *Let X be a locally compact Hausdorff space. Then X canonically embeds into X°, in such a way so that $X^\circ \setminus X$ contains a single point.*

Proof. We do this by giving a concrete description of the multiplicative linear functionals of $C_0(X)^\sharp$. To this end, if we let $x \in X$, we can construct $\phi_x \in X^\circ$ by $\phi_x(f + \alpha\mathbb{1}) = f(x)+\alpha$. We leave it as an exercise for the reader to verify that ϕ_x is indeed a multiplicative functional. It should be clear as well, that this is a genuine embedding of X into X°, and so we shall identify $X \subseteq X^\circ$.

Now, take $\phi_0 \in X^\circ \setminus X$. If we restrict ϕ_0 to act on the $\mathbb{C}$ component of $C_0(X) \oplus \mathbb{C}$, we see that either $\phi_0(\alpha\mathbb{1}) = \alpha$ or $\phi_0(\alpha\mathbb{1}) = 0$ for $\alpha \in \mathbb{C}$. Suppose the latter, and take any $f, g \in C_0(X)$ and $\alpha \in \mathbb{C}$. Then

$$\phi_0(fg + \alpha g) = \phi_0(f + \alpha\mathbb{1})\phi_0(g) = \phi_0(f)\phi_0(g) = \phi_0(fg)$$

and so $\phi_0(\alpha g) = 0$. So in other words, ϕ_0 is identically zero, and by definition is not in the spectrum of $C_0(X)^\sharp$. Thus we must have $\phi_0(\alpha\mathbb{1}) = \alpha$. In a similar manner, we must also have that $\phi_0\big|_{C_0(X)} \in \Gamma_{C_0(X)}$ or that $\phi_0\big|_{C_0(X)} = 0$. In the former case, we recover an element of X as identified above. Hence only the latter case can hold, and so we have uniquely specified ϕ_0. Thus $X^\circ \setminus X = \{\phi_0\}$ where $\phi_0(f + \alpha\mathbb{1}) = \alpha$. $\square$

Henceforth we shall set $\mathring{\infty} := \phi_0$ where ϕ_0 is as in the proof above, so that as a set, X° is the disjoint union of X and $\{\mathring{\infty}\}$. This gives a natural isomorphism between $C(X^\circ)$ and $C_0(X)^\sharp$. In fact, it is easy to see that this is implemented via the mapping $g \mapsto g\big|_X + g(\mathring{\infty})\mathbb{1}$ for $g \in C(X^\circ)$. Following this mapping, we see that $C_0(X)$ is embedded into $C(X^\circ)$ via the map $f \mapsto \mathring{f}$ where $\mathring{f}(x) = f(x)$ for $x \in X$ and $\mathring{f}(\mathring{\infty}) = 0$. We summarise this in the following lemma.

Lemma B.4. *Let X be a locally compact space, and $f \in C(X)$. Then $f \in C_0(X)$ if and only if $\mathring{f}(\mathring{\infty}) = 0$.*

This leads naturally to the notion of convergence at infinity. Recall the usual definition.

Definition B.5. Let X be a locally compact Hausdorff space. We say that a net $(x_\alpha)_\alpha$ in X **converges to infinity** and write $x_\alpha \to \infty$ if x_α eventually leaves every compact subset of X.

Notice the above definition makes no explicit mention of the point $\mathring{\infty}$ nor of X°. However, this above notion of convergence is equivalent to the usual convergence inside X°, as the following result shows.

Lemma B.6. *Let K be a compact Hausdorff space, and set $X = K \setminus \{a\}$ for some fixed $a \in K$. If (x_α) is a net in X, then $x_\alpha \to \infty$ in X if and only if $x_\alpha \to a$ in K.*

Proof. A quick observation reveals that $U \subseteq K$ is a neighbourhood of a if and only if $K \setminus U$ is a compact set inside X. So if $x_\alpha \to a$, then x_α is eventually inside U, and therefore must eventually leave $K \setminus U$. Thus $x_\alpha \to \infty$, and the converse direction is identical. □

Since we may set $K = X^\circ$ and $a = \mathring{\infty}$, then we shall henceforth treat the statements of "$x_\alpha \to \infty$" and "$x_\alpha \to \mathring{\infty}$" as one and the same. Moreover, this gives an easy way to verify if a given compact space is the one-point compactification of another space.

Corollary B.7. *Let K be a compact Hausdorff space. If $X = K \setminus \{a\}$, then $K \cong X^\circ$.*

Proof. Let $\varphi : X^\circ \to K$ be defined by $\varphi(x) = x$ for $x \in X$ and $\varphi(\mathring{\infty}) = a$. It is clear that φ is bijective. Furthermore, since $\varphi\big|_X$ is the identity, we need only check continuity at ∞. However this follows from Lemma B.6. □

> *Remark* B.8. The one-point compactification is occasionally defined in a different manner. More commonly, X° is constructed as the set $X \cup \{\mathring{\infty}\}$, where the open sets take the form of either U for any open set $U \subseteq X$, or of $\{\mathring{\infty}\} \cup X \setminus K$ for any compact $K \subseteq X$. However, as the previous result shows, these constructions are equivalent.

Combining this with the identification of $C_0(X)^\sharp$ to $C(X^\circ)$, we obtain a characterisation of convergence at infinity inside any locally compact space. Recall that for $f \in C_0(X)$ we define the extension $\mathring{f} \in C(X^\circ)$ by setting $\mathring{f}(\mathring{\infty}) = 0$.

Proposition B.9. *Let X be a locally compact Hausdorff space. For $f \in C(X)$, we have that $f \in C_0(X)$ if and only if $f(x_\alpha) \to 0$ for every net $x_\alpha \to \infty$ in X.*

Proof. If $f \in C_0(X)$, then $\mathring{f} \in C(X^\circ)$. Clearly if $x_\alpha \to \infty$, then

$$f(x_\alpha) = \mathring{f}(x_\alpha) \to \mathring{f}(\mathring{\infty}) = 0$$

and so the forward direction follows. On the other hand, if $f(x_\alpha) \to 0$ for every net $x_\alpha \to \infty$, then $\mathring{f}$ will be a genuinely continuous function on X°. However, since $\mathring{f}$ vanishes at $\mathring{\infty}$, then by Lemma B.4 we have $f \in C_0(X)$. □

This naturally holds for sequential spaces as well.

Corollary B.10. *Let X be a sequential space. If $f \in C(X)$, then $f \in C_0(X)$ if and only if $f(x_n) \to 0$ for all sequences $x_n \to \infty$.*

Lastly, we present the following characterisation of compactness.

Proposition B.11. *Let X be a locally compact Hausdorff space. The following are equivalent.*

(i) *X is compact.*

(ii) *The point $\mathring{\infty}$ is isolated in X°.*

(iii) *There is no net $(x_\alpha)_\alpha \in X$ such that $x_\alpha \to \infty$.*

The proof of this result follows from the observation that X is compact precisely when the singleton $\{\mathring{\infty}\}$ is open inside X°. The rest of the proof follows trivially.