Partial C*-dynamical systems and the ideal structure of partial reduced crossed products

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

Chapters 3, 4, and 5 are based on joint work with my supervisor Matthew Kennedy and Camila Sehnem.

Abstract

In this thesis, we study C*-algebras stemming from partial C*-dynamical systems. We develop equivariant injective envelopes associated to such systems, which allow us to obtain canonical connections to enveloping actions as well as results on the ideal structure of partial crossed products.

We extend the theory of equivariant injective envelopes pioneered by Hamana in the 1980s to partial C*-dynamical systems. To do so, we introduce the category of generalized unital partial actions by allowing for partial *-automorphisms acting on families of special hereditary subalgebras. Utilizing properties of injective envelopes and the notion of an injective unitization of partial C*-dynamical systems, we argue that it suffices to consider unital objects in our category. This also allows us to connect our theory to Abadie's notion of enveloping actions leading to a canonical relationship of their G-injective envelopes.

Utilizing properties of injective envelopes, we introduce novel non-triviality conditions for partial *-automorphisms inspired by global C*-dynamics. We contrast this notion with existing conditions in the literature. Lastly, we study a non-commutative generalization of stabilizer subgroups for pseudo-Glimm ideals. In particular, we show that for Glimm ideals in the G-injective envelope, these stabilizer subgroups are in fact amenable — a result which is crucial for our main theorems regarding the ideal structure of partial reduced crossed products.

Finally, our main application of the theory of G-injective envelopes is a characterization of the ideal intersection property for partial C*-dynamical systems subject to a cohomological condition as a generalization of the result for global group actions. To state this generalization, we utilize the dynamical conditions introduced previously and generalize the notion of equivariant pseudo-expectations to partial C*-dynamical systems. We also give a sufficient intrinsic condition in terms of non-commutative uniformly recurrent partial subsystems utilizing pseudo-Glimm ideals. As a consequence of our results, we obtain a full characterization of the ideal intersection property for partial actions on commutative C*-algebras in terms of freeness of the partial action on the spectrum of the G-injective envelope.

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Dedication

To the love of my life, E.K.
For bringing brightness to my life

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Chapter 1

Introduction

Dynamical systems appear in various forms in several mathematical fields. At its core, a dynamical system refers to the evolution of a mathematical object subject to some predetermined set of rules. For example, one can look at the time evolution of a particle in space given by a differential equation or at a group of symmetries acting on a geometric object.

To motivate *partial* dynamical systems, we can look at the basic example of a dynamical system given by time evolution dictated by an ordinary differential equation [Rae99]. More specifically, for a differential equation $x' = f(x)$, where x denotes an unknown time-dependent real-valued function and $f\colon \mathbb{R} \to \mathbb{R}$ describes its time evolution, we can look at its flow $\alpha\colon \mathbb{R} \times \mathbb{R} \to X$. Here, the flow α is defined as $\alpha_t(x_0) := x(t)$, where $x(t)$ is the solution to the initial value problem $x' = f(x)$, $x(0) = x_0$. Assuming that f is Lipschitz continuous on $\mathbb{R}$, each initial value problem gives rise to a unique global solution $x\colon \mathbb{R} \to \mathbb{R}$. Note that for $s, t \in \mathbb{R}$, determining $\alpha_s(\alpha_t(x_0))$ amounts to solving the initial value problem $x' = f(x)$ and $x(0) = x(t)$. The unique solution to this initial value problem is given by $x(s + t)$ and thus $\alpha_s(\alpha_t(x_0)) = \alpha_{s+t}(x_0)$, i.e., α describes a group action of $\mathbb{R}$. If f is only locally Lipschitz, we still obtain unique solutions, however, they might not be global. In this case, $\alpha_s(\alpha_t(x_0))$ might not necessarily be well-defined. If it is though, then we still observe the same relationship as above. This phenomenon captures the essence of a partial dynamical system — we cannot expect to combine arbitrary rules for all elements, but if the composition is admissible, we know how it behaves.

While the example of partial dynamics stemming from differential equations is known, partial dynamical systems were first formalized using an operator algebraic perspective, which is the focus of this thesis. For operator algebras, dynamical descrip-

tions have two main advantages: first, dynamics on C*-algebras lead to machinery for generating new examples of C*-algebras, e.g., in the form of crossed product constructions encoding both the base C*-algebra and the dynamics. Second, describing already known C*-algebras dynamically using a crossed product construction allows for new avenues for studying the C*-algebra's properties. Having more general dynamics thus leads to a potentially wider class of examples as well as more C*-algebras that can be equipped with a dynamical description. This ability to describe given C*-algebras using partial dynamics was one of the motivating reasons that led to the introduction of partial actions as a generalization of ordinary C*-dynamical systems.

In this short introduction, we will present a select history of partial actions in the context of C*-algebras, review relevant results on the ideal structure of their crossed products, and introduce relevant literature and results utilizing equivariant injective envelopes. We end with an overview of our contributions.

1.1 Related Literature

Partial Actions

A partial action $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ of a discrete group G on a C*-algebra A, also called a partial C*-dynamical system, is given by a family of ideals $\{A_g\}_{g\in G}$ and *-isomorphisms $\alpha_g \colon A_{g^{-1}} \to A_g$ satisfying certain compatibility conditions generalizing the concept of a group action on A. As described at the beginning of the introduction, we cannot expect composition of *-isomorphisms to be well-defined globally, but we do know about its behaviour on smaller domains. This concept was first introduced in [Exe94] to study the structure of C*-algebras equipped with a circle action. In particular, Exel showed that under fairly mild assumptions on the action, such a C*-algebra can be written as a crossed product by a partial action of the integers generalizing a result of Paschke [Pas81]. This description is useful, as Exel also provides a generalization of the Pimsner–Voiculescu exact sequence leading to desirable K-theoretic consequences. Furthermore, this structural result implies that any AF-algebra can be written as a partial crossed product of a commutative AF-algebra [Exe95]. In [McC95], McClanahan generalized the notion of a partial action to arbitrary discrete groups and introduced full and reduced partial crossed products. Since Exel's and McClanahan's work, the concept of partial actions has been formalized for many different settings — for an overview of developments in partial actions see the survey [Dok19]. In the realm of C*-algebras, many more known C*-algebras have been described using partial dynamics, see for example: [Exe97; EL99; ES16].

A prime example of a partial dynamical system is the restriction of a global dynamical system to a non-invariant subobject. For C*-dynamical systems, this translates to the restriction of a global C*-dynamical system to a non-invariant ideal. A natural question to ask is whether all partial actions actually arise as restrictions of global actions. This question was first studied by Abadie in [Aba03], where he answered the question affirmatively for partial actions on topological spaces. However, the corresponding global action might exhibit fairly peculiar behaviour as for example Hausdorffness is not necessarily preserved. Using this phenomenon, Abadie also showed that on the contrary not every partial action on a C*-algebra arises as a restriction. Thus, partial actions on C*-algebras are a strict generalization of ordinary group actions. By [Fer18], an enveloping action, i.e., the smallest global action such that the partial action arises as a restriction, exists for a partial C*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ on a unital C*-algebra A if and only if each ideal A_g is unital.

Partial actions can also be viewed through several other perspectives. For example, partial actions of a group correspond to actions of inverse semigroups. This connection appears in the work of Sieben in the context of C*-algebras [Sie97] and was then introduced systematically by Exel [Exe98]. Another viewpoint, at least for partial actions on topological spaces, is given by groupoids. This was first explored in [Aba04], where Abadie showed that a partial action on a second-countable locally compact Hausdorff space corresponds to a groupoid and its groupoid C*-algebra corresponds to the partial crossed product of the partial action on its induced commutative C*-algebra. More recently, [CK24] introduced a criterion of when a given étale groupoid comes from a partial action on its unit space.

Ideal Structure of Crossed Products

In order to utilize the power of a dynamical description of a C*-algebra, one is interested in identifying dynamical properties that characterize C*-algebraic properties. For example, one is interested in when a crossed product by a partial action is simple, or to what extent the ideal structure of the base C*-algebra dictates the ideal structure of the crossed product. The first result along those lines for partial C*-dynamical systems was given by McClanahan in [McC95], where he shows that for Power's groups the partial reduced crossed product is simple if and only if the base C*-algebra lacks invariant ideals. This generalizes the result of de la Harpe and Skandalis in the global setting [HS86] using similar techniques.

Besides simplicity, there are other notions regarding the ideal structure of crossed products that have been studied in the literature. For example, one can ask more gen-

erally when the ideals in the partial reduced crossed product are fully determined by invariant ideals in the base algebra, a property often referred to as the ideal separation property. A natural extension of this question is whether non-zero ideals in the crossed product always have non-zero intersection with the base algebra, which is now often referred to as the ideal intersection property. A first result along those lines for partial C*-dynamical systems was given in [ELQ02], where the authors show that for commutative C*-algebras the partial action being topologically free implies the ideal intersection property. In particular, if the partial action is additionally minimal, the reduced partial crossed product is simple. This is a generalization of a result by Kawamura and Tomiyama in the global setting [KT90]. Furthermore, Exel, Laca and Quigg give a condition that guarantees the ideal separation property in this setting. The result on topological freeness and the ideal intersection property was then generalized to the non-commutative setting by Lebedev [Leb05] giving an analogous result to work of Archbold and Spielberg in the global setting [AS94]. A full characterization of the ideal intersection property for partial actions of amenable groups on commutative C*-algebras was given in [AA17], where the authors show equivalence to topological freeness in this setting.

In addition to the conditions on the partial action mentioned above, there have been several other conditions that give some insight on the structure of partial reduced crossed products. For example, [GS14] introduces a condition guaranteeing the ideal intersection property inspired by known conditions for global C*-dynamical systems. For an overview of the plethora of non-triviality conditions for group actions, partial actions and an even more general setting, see [KM18].

Equivariant Injective Envelopes

In addition to the results in the previous section, for ordinary C*-dynamical systems there has been an influx of injective envelope techniques and tools used to study the ideal structure of ordinary reduced crossed products over the last decade. Results along these lines rely on the realization that many structural problems can be phrased in terms of the injective and for the special case of dynamical systems the G-injective envelope. After showing existence of the injective envelope of C*-algebras and operator systems [Ham79a; Ham79b], Hamana also introduced the notion of G-injective envelopes for global C*-dynamical systems in [Ham85]. Among many things, Hamana showed that injective envelopes have desirable properties. In particular, they are monotone complete and thus allow for many techniques involving projections.

The first result using G-injective envelopes to gain structural insights into C*-dyna-

mical questions was a characterization of C*-simplicity, i.e., when the reduced group C*-algebra of a discrete group is simple, by Kalantar and Kennedy [KK17]. Their key idea was that the spectrum of the G-injective envelope $I_G(\mathbb{C})$ of the trivial action of G can be identified with the Furstenberg boundary, i.e., the universal minimal strongly proximal flow of G. This realization allowed the authors to show that C*-simplicity is equivalent to topological freeness of the action on the Furstenberg boundary.

This characterization led to many interesting consequences. For example, [Kaw17] gives a full characterization of the ideal intersection property for ordinary group actions on commutative C*-algebras. Furthermore, using this equivalence [Bre+17] show that the reduced crossed product for a non-commutative C*-dynamical system for C*-simple groups is simple if and only if the base C*-algebra admits no non-trivial invariant ideals. This was then generalized to twisted C*-dynamical systems in [BK18] by giving a more general correspondence between maximal invariant ideals of the base algebra and maximal ideals of the reduced crossed product.

For general groups Kennedy and Schafhauser give a characterization of the ideal intersection property in the non-commutative setting up to a cohomological obstruction [KS19]. A more recent advance was given by Geffen and Ursu [GU23] in the case of FC-hypercentral groups, that is, groups admitting no non-trivial ICC quotient, where they show that this cohomological obstruction can be overcome for minimal actions. This was then generalized to arbitrary actions in [KKS25]. Interestingly, countable FC-hypercentral groups are exactly those groups that admit a trivial Poisson boundary for any non-degenerate probability measure [Fri+19]. This is noteworthy, as this together with the fact that results for the non-FC-hypercentral case are limited to C*-simple groups highlights the importance of group theoretic boundaries in answering these questions. Lastly, injective envelope techniques and subsequently boundary theoretic notions have also been generalized to study the ideal structure of groupoid C*-algebras [Ken+21].

1.2 Contributions

In this thesis, we develop an equivariant theory of injective envelopes for partial C*-dynamical systems and demonstrate its use with an application to the ideal intersection property. As evident by Hamana's work [Ham79a; Ham79b; Ham85], a crucial point for the existence of injective envelopes is the determination of the correct category. For example, for injective envelopes of C*-dynamical systems, it is necessary to work in the category of G-operator systems with equivariant unital completely positive (u.c.p.)

maps to guarantee the existence of G-injective envelopes. Similarly, we need to enlarge the objects in our category. To do so, we introduce the category of generalized unital partial actions. Instead of considering families of unital ideals, we consider families of well-behaved unital hereditary C*-subalgebras and corresponding *-isomorphisms following the partial action axioms. The appropriate morphisms are given by u.c.p. maps preserving the units of the subalgebras and admitting equivariance with respect to the action on the subalgebras. Our first main result gives the existence and appropriate uniqueness of the G-injective envelope of a unital partial C*-dynamical system.

Theorem A (Theorem 3.3.3)**.** *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial action. There exists a G-injective and G-essential extension $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$ of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$. Furthermore, the G-injective envelope is unique and $(I_G(A), G, I_G(A)_{g\in G}, I_G(\alpha)_{g\in G})$ is a partial C*-dynamical system.*

This G-injective envelope behaves appropriately in relation to similar objects in related categories. In particular, we show that the G-injective envelope of a global C*-dynamical system in the category of generalized unital partial actions agrees with the G-injective envelope in the category of G-operator systems introduced in [Ham85].

As our work utilizes unital partial C*-dynamical systems, it is therefore natural to study its connections to enveloping actions. We show that the G-injective envelope of a unital partial C*-dynamical system and the G-injective envelope of its enveloping action are canonically related. In particular, we show that the former arises as a restriction of the latter.

Theorem B (Theorem 3.4.2)**.** *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system and let $(\mathcal{C}^A, \zeta^A)$ denote its enveloping action. Let $(I_G(\mathcal{C}^A), I_G(\zeta^A))$ denote the G-injective envelope of the enveloping action. Then, $I_G(A) = I_G(\mathcal{C}^A)1_A$, where 1_A denotes the unit of A. In particular, $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$ is a restriction of $(I_G(\mathcal{C}^A), I_G(\zeta^A))$.*

This connection highlights an important point in the thesis: The setting of unital partial C*-dynamical systems behaves in a sense similar to global C*-dynamical systems allowing for more or less straightforward generalizations of results for global C*-dynamical systems. Passing to the non-unital setting, however, can be technically difficult and potentially requires new ideas. In this thesis, we argue that the theory of injectivity provides a valuable method to study non-unital partial C*-dynamical systems through a unital lens. The key observation here is that a partial C*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ admits a canonical unital partial C*-dynamical system $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ on the injective envelope $I(A)$ of A, which we refer to as the injective

unitization of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$. Using its properties and analogous results from the unital setting we define the notion of G-injective envelopes for not necessarily unital partial C*-dynamical systems. We establish the following equivalence.

Theorem C (Theorem 5.2.6)**.** *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system. The following are equivalent:*

(i) $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ *has the ideal intersection property;*

(ii) For all unital partial C-dynamical systems $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ with $A \subseteq B \subseteq I(A)$, $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ has the ideal intersection property;*

(iii) The injective unitization $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ has the ideal intersection property.

To establish a characterization of the ideal intersection property we generalize the notion of proper outerness of *-automorphisms to partial *-automorphisms between ideals and give a characterization of the ideal intersection property in terms of faithful equivariant pseudo-expectations. In particular, we show that each non-trivial partial *-automorphism being properly outer implies the ideal intersection property. In Example 5.3.1 we give an example of a non-properly outer partial C*-dynamical system with the ideal intersection property showing that the converse is not necessarily true. However, we identify a cohomological condition that allows us to establish an appropriate converse as seen in the next theorem, which generalizes work of [KS19].

Theorem D (Theorem 5.3.9)**.** *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system with vanishing obstruction. Then, $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has the ideal intersection property if and only if the partial C*-dynamical system $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$ is properly outer.*

We observe that this cohomological obstruction always vanishes for partial actions on commutative C*-algebras leading to a full characterization for partial actions on compact Hausdorff spaces. In this case we characterize the ideal intersection property in terms of the partial action $(\{\hat{X}_g^G\}_{g\in G}, \{\hat{\theta}_g^G\}_{g\in G})$ on the spectrum of the G-injective envelope.

Theorem E (Theorem 5.4.5)**.** *Let $(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action on a compact Hausdorff space. Then $\mathcal{C}(X) \subseteq \mathcal{C}(X) \rtimes_{\mathrm{r}} G$ has the ideal intersection property if and only if $(\{\hat{X}_g^G\}_{g\in G}, \{\hat{\theta}_g^G\}_{g\in G})$ is topologically free. If additionally, G is amenable, $\mathcal{C}(X) \subseteq \mathcal{C}(X) \rtimes_{\mathrm{r}} G$ has the ideal intersection property if and only if $(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ is topologically free.*

We apply this to characterize the ideal intersection property for the partial reduced group C*-algebra of a discrete group.

Lastly, we give a sufficient intrinsic condition for the ideal intersection property in terms of pseudo-Glimm ideals $\mathrm{Glimm}_{\mathrm{p}}(A)$ of A, which arise as intersections of proper and primal ideals in $I(A)$ with A. To do so we show that a partial C*-dynamical system induces a partial action on the space $\mathrm{Glimm}_{\mathrm{p}}(A) \times \mathrm{Sub}(G)$. The ideal intersection property is then implied by the lack of appropriate invariant closed subsets of $\mathrm{Glimm}_{\mathrm{p}}(A) \times \mathrm{Sub}(G)$ that contain an invariant subset of the form $Z = \{(I, \{e\})\}$ with $\bigcap_{(I,\{e\})\in Z} I = \{0\}$.

Theorem F (Proposition 5.3.18)**.** *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system of a countable discrete group G. If every amenable non-commutative uniformly recurrent partial subsystem contains a trivial non-commutative uniformly recurrent partial subsystem, the partial C*-dynamical system has the ideal intersection property.*

Outline

The thesis is organized as follows: in Chapter 2 we give necessary background knowledge on partial actions, their reduced crossed products, and injective envelopes of C*-algebras. We introduce the category of generalized unital partial actions, as well as G-injectivity and G-essentiality in Chapter 3. Furthermore, this section contains the proof of the existence of a G-injective envelope for unital partial C*-dynamical systems. In Section 3.4 we prove that the G-injective envelope of a unital partial C*-dynamical system arises as the restriction of the G-injective envelope of its enveloping action. Chapter 4 concerns generalizations of proper outerness and quasi-stabilizer subgroups to partial C*-dynamical systems and we compare our conditions to existing non-triviality notions for partial *-automorphisms. Finally, Chapter 5 demonstrates the utility of the techniques and theory introduced in the previous chapters by giving a characterization of the ideal intersection property subject to a cohomological obstruction, which is outlined in Section 5.3.1. First, we introduce the concept of essential unitizations in Section 5.2 showing that our results also apply to the non-unital setting. The main theorems on the characterization of the ideal intersection property can be found in Section 5.3. We end with Section 5.4 on applications, including integer partial actions and the commutative setting.

Chapter 2

Preliminaries

In this chapter we give the relevant background information on injective envelopes of C*-algebras, partial actions and their crossed products, as well as regular and essential ideals in C*-algebras. Throughout this thesis, G will denote a discrete group and countability assumptions will be specified if necessary.

2.1 Injective Envelopes

Injectives envelopes in various categories have been extensively studied by Hamana (see for example [Ham79a; Ham79b; Ham85]). We recall here key properties and definitions relevant to our work. For the present work we focus mostly on the C*-algebraic setting, however, the theory of injective envelopes for C*-algebras is strongly connected to operator systems. For more information on operator systems and the theory of unital completely positive (u.c.p) maps see [Pau02]. For a detailed account of monotone complete C*-algebras we refer to [SW15].

A unital C*-algebra A is *injective* if for any operator systems S, T and any unital completely isometric (u.c.i.) map $\kappa\colon S \to T$, and any u.c.p. map $\phi\colon S \to A$, there exists a u.c.p. map $\hat{\phi}\colon T \to A$ satisfying $\hat{\phi} \circ \kappa = \phi$, i.e., A is injective in the category of operator systems with u.c.p. maps. An *extension* of A is a pair (B, κ), where B is a unital C*-algebra and $\kappa\colon A \to B$ is an injective *-homomorphism. The extension (B, κ) of A is *injective* if B is injective, and it is *rigid* if for any u.c.p. map $\phi\colon B \to B$ with $\phi|_{\kappa(A)} = \mathrm{id}|_{\kappa(A)}$, we have $\phi = \mathrm{id}_B$. Lastly, an extension (B, κ) of A is called *essential* if given a u.c.p. map $\phi\colon B \to S$ for some operator S such that $\phi \circ \kappa$ is completely isometric, we have ϕ is completely isometric. An extension (B, κ) of A is an *injective envelope* of A, if it is an injective

and essential extension. This is equivalent to being an injective and rigid extension. By [Ham79b; Ham79a], given a unital C*-algebra A there exists an injective envelope $(I(A), \kappa)$. Furthermore, the injective envelope is unique meaning if (B, ι) is another injective and essential extension of A, there exists a unique *-isomorphism $\rho\colon I(A) \to B$ such that $\rho \circ \kappa = \iota$. For convenience we usually suppress the specific embedding κ and simply write $A \subseteq I(A)$ to denote the C*-inclusion of $\kappa(A) \subseteq I(A)$. For a non-unital C*-algebra A, we can define the injective envelope of A as the injective envelope of the minimal unitization A^1 of A given by adjoining a unit to A [Ham82b].

Injective envelopes admit many useful properties and allow for powerful techniques. For example, since the inclusion $A \subseteq I(A)$ is essential, we have for any $J \trianglelefteq I(A)$ with $J \neq \{0\}$, $J \cap A \neq \{0\}$. Additionally, injective envelopes are *monotone complete*, i.e., for any bounded increasing net $(x_\lambda)_\lambda$ of self-adjoint elements in $I(A)$, $\sup_\lambda x_\lambda$ exists in $I(A)$. Additionally, we have the following inclusion $Z(A) \subseteq Z(I(A))$. Lastly, injective envelopes behave well with respect to hereditary subalgebras and ideals. In particular, if $p \in A$ is a projection, $I(pAp) = pI(A)p$ [Ham82b, Proposition 6.3]. Hence, if $J \trianglelefteq A$ is a unital ideal, $I(J) \trianglelefteq I(A)$ with the same unit and similarly for hereditary subalgebras. Furthermore, by [Ham82b, Proposition 6.5] if $B \subseteq A$ is a not necessarily unital hereditary subalgebra, there exists a projection $p \in I(A)$ such that $I(B) = pI(A)p$. If in addition, B is an ideal in A, the projection p is central and $I(B)$ is a unital ideal in $I(A)$.

2.2 Partial Actions

We give an overview of relevant notions for partial actions on C*-algebras. For a thorough treatment of partial actions on C*-algebras and their connection to Fell bundles we refer to Exel's book [Exe17].

Definition 2.2.1. A *partial action* of a discrete group G on a C*-algebra A is given by a pair $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$, where $\{A_g\}_{g\in G}$ is a family of closed two-sided ideals $A_g \trianglelefteq A$ and $\alpha_g\colon A_{g^{-1}} \to A_g$ is a *-isomorphism for all $g \in G$ such that

(i) $A_e = A$, and $\alpha_e = \mathrm{id}$;

(ii) $\alpha_g(A_{g^{-1}} \cap A_h) = A_{gh} \cap A_g$ for all $g, h \in G$;

(iii) $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$ for all $x \in A_{h^{-1}} \cap A_{(gh)^{-1}}$ and $g, h \in G$.

We call $(A, G, \{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ a *partial C*-dynamical system*. If the group and base C*-algebra are clear, we will denote partial C*-dynamical systems by $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$

for convenience. We call a partial action $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ *unital* if $A_g = Ap_g$ for some central projection $p_g \in A$ for all $g \in G$.

Note that if $A_g = A$ for a partial action $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$, we recover the notion of an ordinary group action. We will refer to this case as a *global* group action or global C*-dynamical system denoted by (A, G, α), where $\alpha\colon G \to \mathrm{Aut}(G)$ is a group homomorphism into the group of *-automorphisms on A. A prime example of partial actions comes from global actions via restrictions to non-invariant ideals. Given a global group action α of a discrete group G on a C*-algebra A and a two-sided ideal $I \trianglelefteq A$ the *restriction* $(\{I_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ of α is given by $I_g = I \cap \alpha_g(I)$ and $\beta_g = \alpha|_{I_{g^{-1}}}$.

Motivated by the viewpoint of the global setting, we can also view a partial action as an admissible set of partial *-automorphisms. Given two ideals $B, C \trianglelefteq A$ of a C*-algebra A, we refer to a *-isomorphism $\phi\colon B \to C$ as a *partial *-automorphism*. For a C*-algebra A let $\mathrm{pAut}(G)$ denote the set of all partial *-automorphisms. Given $\phi, \psi \in \mathrm{pAut}(G)$, $\phi\psi$ is defined as $\phi \circ \psi$ on the largest possible domain. For a discrete group G and a C*-algebra A, partial actions $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ are in one-to-one correspondence with maps $\alpha\colon G \to \mathrm{pAut}(G)$ satisfying, $\alpha(e) = \mathrm{id}_A$, $\alpha(g^{-1}) = \alpha(g)^{-1}$, $\alpha(g^{-1})\alpha(g)\alpha(h) = \alpha(g^{-1})\alpha(gh)$, and $\alpha(g)\alpha(h)\alpha(h^{-1}) = \alpha(gh)\alpha(h^{-1})$ for all $g, h \in G$.

Using the duality between compact Hausdorff spaces and commutative C*-algebras, we can phrase partial actions in the language of topological partial dynamical systems. A *topological partial dynamical system* $(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ on a topological space X is given by a family of open subsets $X_g \subseteq X$ and a family of homeomorphisms $\theta_g\colon X_{g^{-1}} \to X_g$ satisfying the relations given in Definition 2.2.1. If X is a compact Hausdorff space, topological partial dynamical systems $(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ are in one-to-one correspondence with partial C*-dynamical systems $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ for $A = \mathcal{C}(X)$ unital and commutative.

A *partial *-representation* of a group G on a unital C*-algebra B is given by a map $u\colon G \to B$ such that $u_e = 1$, $u_g u_h u_{h^{-1}} = u_{gh} u_{h^{-1}}$, $u_{g^{-1}} u_g u_h = u_{g^{-1}} u_{gh}$, and $u_{g^{-1}} = u_g^*$ for all $g, h \in G$. Observe that, in particular, u_g is a partial isometry for all $g \in G$. Then, given a partial action $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ and a unital C*-algebra B, a *covariant representation* of the partial action on B is a pair (π, u), where $\pi\colon A \to B$ is a *-homomorphism and u is a partial *-representation of G on B such that

$$u_g \pi(a) u_{g^{-1}} = \pi(\alpha_g(a)) \qquad \text{for all } g \in G, a \in A.$$

Note that similarly to the global group action case, there exits a correspondence between covariant representations (π, u) of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ and *-representations of the partial crossed product $A \rtimes G$ introduced in the next section.

The next theorem establishes an important connection between a partial C*-dynamical system and a canonically associated unital partial C*-dynamical system using the injective envelope. We will use this observation throughout the thesis.

Proposition 2.2.2. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial action of G on a C*-algebra A and let $I(A)$ denote the injective envelope of A. There exists a unique partial action of G on $I(A)$, $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$. Moreover, $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ is unital.*

Proof. Let $\kappa\colon A \to I(A)$ denote the canonical embedding of A into its injective envelope and fix $g \in G$. Since $\alpha_g\colon A_{g^{-1}} \to A_g$ is u.c.p. and $I(A_g)$ is injective, the map $\kappa \circ \alpha_g$ extends to a *-isomorphism $I(\alpha)_g\colon I(A_{g^{-1}}) \to I(A_g)$ such that $I(\alpha)_g \circ \kappa = \alpha_g$. Furthermore, by [Ham82b, Theorem 6.5] $I(A_g) = I(A)p_g$ for a unique central projection p_g in $I(A)$. In particular, $I(A_g)$ is a unital ideal in $I(A)$ for all $g \in G$. Thus, $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ is a unital partial action, where $I(A)_g := I(A_g)$. □

2.3 Partial Crossed Products

As with global C*-dynamical systems, one is interested in encoding the dynamical data of a partial action in a larger C*-algebra — the crossed product. There are several constructions of partial reduced and full crossed products in the literature [McC95; Exe97; QR97]. Given a partial action $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ on a C*-algebra A McClanahan constructed the partial reduced product as a particular completion of $\{f \in \ell^1(G, A) \mid f(g) \in A_g\}$. Quigg and Raeburn took a similar approach utilizing appropriate completions of the covariance algebra focused on incorporating the notion of a coaction. Lastly, Exel's descriptions is framed in the general setting of Fell bundles over discrete groups encompassing both the global and partial group action setting. This most general approach has seen various applications over the years (see for example [KM18; ABF22] and the textbook [Exe17]). For this thesis, we will mainly use the description of Quigg and Raeburn [QR97]. We recall key elements of their construction below.

Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial action on a C*-algebra A. Given a covariant representation (π, u) of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$, we consider the closure of the *covariance algebra*

$$C^*(\pi, u) = \overline{\operatorname{span}}\{\pi(a)u_g \mid a \in A_g\},$$

where we will use $C(\pi, u)$ to refer to the not necessarily closed linear span. Using the relations of the covariant representation, we see $\pi(a)u_g\pi(b)u_h = \pi(\alpha_g(\alpha_{g^{-1}}(a)b))u_{gh}$ and

$(\pi(a)u_g)^* = \pi(\alpha_{g^{-1}}(a^*))u_{g^{-1}}$ for $g, h \in G$ and $a \in A_g, b \in A_h$. Quigg and Raeburn then construct a universal covariant representation to implement the full crossed product.

For the construction, first for $a \in A_g$ define $a\chi_g(t) = a$ for $g = t$ and 0 otherwise and consider

$$C_c((\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})) = \text{span}\{a\chi_g \mid g \in G, a \in A_g\}.$$

We can endow $C_c((\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G}))$ with a *-algebra structure inspired by the covariance algebra as follows

$$\begin{aligned} a\chi_g b\chi_h &= \alpha_g(\alpha_{g^{-1}}(a)b)\chi_{gh}, \\ (a\chi_g)^* &= \alpha_{g^{-1}}(a^*)\chi_{g^{-1}}, \end{aligned}$$

for $g, h \in G$ and $a \in A_g, b \in A_h$. Observe also that A embeds into $C_c((\{A_g\}_{g\in G}, \{\alpha)_g \}_{g\in G})$ via $a \mapsto a\chi_e$. We then define the *full partial crossed product* $A \rtimes G$ of the partial action $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ as the C*-completion of $C_c((\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G}))$.

Next, consider the unit $p_g \in A^{**}$ of A_g^{**}. Note that we can view p_g as a projection in $(A \rtimes G)^{**}$. As shown by Quigg and Raeburn, for $g \in G$ there exists $m_g \in p_g(A \rtimes G)^{**}p_g$ such that $am_g = a\chi_g$ for $a \in A_g$ and $m_g b = \alpha_g(b)\chi_g$ for $a \in A_g, b \in A_{g^{-1}}$ and subsequently $m_g a m_g^* = \alpha_g(a)$ for $a \in A_g$. In fact the inclusion $\iota\colon A \to (A \rtimes G)^{**}$ and $m\colon G \to (A \rtimes G)^{**}$ form a covariant pair of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ in $(A \rtimes G)^{**}$, which constitutes a suitable universal representation of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$.

To obtain a description of the reduced partial crossed product $A \rtimes_{\text{r}} G$, we will give a description of the corresponding covariant representation stemming from the left regular representation $\lambda\colon G \to \ell^2(G)$ of G. Recall that for $g \in G$, $\alpha_g\colon A_{g^{-1}} \to A_g$ extends to a canonical homomorphism $\overline{\alpha}_g\colon A \to M(A_g)$, where $M(A_g)$ denotes the multiplier algebra of A_g. Now let $\pi\colon A \to B(\mathcal{H})$ be a faithful non-degenerate representation of A and define the regular covariant pair (π^r, u^r) as the pair corresponding to the covariant representation $\pi^r \times u^r$ satisfying $(\pi^r \times u^r)(am_g) = \sum_{t\in G} \pi^r(\overline{\alpha}_{t^{-1}}(a)) \otimes \chi_g\lambda_t$ acting on $\mathcal{H} \otimes \ell^2(G)$. Using this covariant pair we obtain the following description of the *reduced partial crossed product* $A \rtimes_{\text{r}} G \simeq C^*(\pi^r, u^r)$. This is independent of the choice of π up to isomorphism. For convenience of notation we will denote the dense set of generators in $A \rtimes_{\text{r}} G$ as $a\lambda_g$ for $g \in G, a \in A_g$ and the corresponding covariance algebra $C(\pi^r, u^r) := C(\alpha, r)$ in analogy with the global group action case.

2.4 Ideals and the Ideal Structure of Crossed Products

In this subsection, we highlight two kinds of ideals that admit a particularly nice description in terms of their injective envelopes. Furthermore, we will give some background on the ideal intersection property.

Regular Ideals

As pointed out before, given an ideal $J \trianglelefteq A$ in a C*-algebra A, there exists a central projection $p \in Z(I(A))$ such that $I(J) = I(A)p \trianglelefteq I(A)$. While this is a somewhat concrete description, several ideals might admit the same injective envelope and it is not necessarily possible to recover the original ideal from its injective envelope. The ideals that allow for a full recovery are precisely the regular ideals.

For an ideal $J \trianglelefteq A$ define its *annihilator* as

$$J^{\perp} = \{a \in A \mid ab = ba = 0 \text{ for all } b \in J\}.$$

We call an ideal $J \trianglelefteq A$ *regular*, if $J = J^{\perp\perp}$. By [Ham82c], J is regular if and only if there exists a central open projection $p \in Z(I(A))$ such that $J = A \cap I(J) = A \cap I(A)p = A \cap Ap$. Furthermore, for a not necessarily regular $J \trianglelefteq A$ with $I(J) = I(A)p$ for a projection $p \in Z(I(A))$, we have $J^{\perp} = A \cap A(1-p)$. In particular, $J^{\perp}$ is regular.

Regular ideals have seen a recent rise in interest as quotients by regular ideals can preserve properties of C*-inclusions. This phenomenon was observed for graph C*-algebras in [Bro+22], and more generally in [Bro+24].

Essential Ideals

Let A be a C*-algebra, which is not necessarily prime. Recall that an ideal $J \trianglelefteq A$ is called *essential* if $I \cap J \neq \{0\}$ for any non-zero ideal $I \trianglelefteq A$. We note that if A is prime, any non-zero ideal is essential. As was pointed out in [KP03], essential ideals are exactly the ideals such that their injective envelope is completely isometrically isomorphic to the injective envelope of A. This essentially follows from adapting the proof of [Ham81, Theorem 7.1]. We include the required version of the result as well as its proof for completeness.

Proposition 2.4.1. *Let A be a C*-algebra and suppose $J \trianglelefteq A$. Let $p \in Z(I(A))$ be a central projection such that $I(J) = I(A)p$. Then, $p = 1$ if and only if J is essential.*

Proof. Suppose, $p \neq 1$. Without loss of generality we can assume that $p > 0$, since otherwise $J = \{0\}$ and in particular non-essential. Then $0 < 1 - p < 1$ and thus, $I(A)(1-p) \neq 0$ is an ideal such that $I(J) \cap I(A)(1-p) = \{0\}$. Furthermore, consider $I = A \cap I(A)(1-p)$. Note that $I \neq \{0\}$, since if not the map $\phi\colon A \to I(A)p\colon a \mapsto ap$ is the identity on A. However, since $I(A)p \subseteq I(A)$, this implies that the extension $\hat{\phi}\colon I(A) \to I(A)$ also satisfies $\hat{\phi}|_A = \mathrm{id}_A$. By rigidity $\hat{\phi} = \mathrm{id}_{I(A)}$. This contradicts $p < 1$. In sum, $J \cap I = \{0\}$ for $I \neq \{0\}$ and J is not essential.

Conversely, suppose that J is not essential. If $J = \{0\}$ then $I(J) = \{0\}$ and thus $p = 0$. Hence, assume $J \neq \{0\}$. Then there exists a non-zero ideal $I \trianglelefteq A$ such that $I \cap J = \{0\}$. Suppose there exists $0 \neq a \in I$ such that $a \in I(J)$. This implies that $ap = a$. However, if $(e_\lambda)_\lambda$ is an approximate identity for J, $p = \sup_\lambda e_\lambda$. Thus, there exists a λ such that $ae_\lambda \neq 0$. This is a contradiction to $I \cap J = \{0\}$. Thus, $ap = 0$ for all $a \in I$ and $p \neq 1$. □

The Ideal Intersection Property

In this thesis, we will demonstrate the use of the theory of equivariant injective envelopes for partial C*-dynamical systems by giving a characterization of the ideal intersection property subject to a cohomological obstruction.

Definition 2.4.2. Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system. We say that $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has the *ideal intersection property* if $J \cap A = \{0\}$ for an ideal $J \trianglelefteq A \rtimes_r G$ implies $J = \{0\}$.

Remark 2.4.3. The ideal intersection property can also be more generally applied to inclusions $A \subseteq B$ of C*-algebras A, B, where A, B share a unit. We say that $A \subseteq B$ has the ideal intersection property if $J \cap A = \{0\}$ for an ideal $J \trianglelefteq B$ implies $J = \{0\}$.

The ideal intersection property is useful as it allows for some insight into the ideal structure of the reduced crossed product. For example, for partial C*-dynamical systems $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ ideals in the reduced crossed product are in one-to-one correspondence with invariant ideals in A if and only if G is exact and $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ exhibits the ideal intersection property on all quotients by invariant ideals [GS14]. Furthermore, if $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has the ideal intersection property, then a unital *-homomorphism $\pi\colon A \rtimes_\mathrm{r} G \to B(\mathcal{H})$ is faithful, if and only if π is faithful on A [PZ15, Proposition 3.3].

Chapter 3

Injective Envelopes for Partial C*-dynamical Systems

The goal of this chapter is to introduce the notion of injectivity for partial actions on C*-algebras similar to injective envelopes of operator systems as well as global C*-dynamical systems [Ham79b; Ham85]. To do so we need to introduce the appropriate category of partial actions, which leads us to generalized unital partial actions. Observe that since for a partial action its canonical extension to the injective envelope as in Proposition 2.2.2 is unital, we will restrict our attention to unital objects. For the non-unital setting see Section 5.2. After showing existence and uniqueness of the injective envelope for a unital partial C*-dynamical system, we establish its connection to enveloping actions in Section 3.4.

3.1 The Category of Generalized Unital Partial Actions

Definition 3.1.1. A *generalized unital partial action* of a discrete group G on a unital C*-algebra A is given by a pair $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$, where $\{A_g\}_{g\in G}$ is a family of unital hereditary subalgebras in A and $\alpha_g\colon A_{g^{-1}} \to A_g$ is a *-isomorphism for all $g \in G$ such that

(i) $A_e = A$, and $\alpha_e = \mathrm{id}$;

(ii) $\{p_g\}_{g\in G}$ sits inside a commutative subalgebra of A, where $A_g = p_g A p_g$;

(iii) $\alpha_g(A_{g^{-1}} \cap A_h) = A_{gh} \cap A_g$ for all $g, h \in G$;

(iv) $\alpha_g(\alpha_h(x)) = \alpha_{gh}(x)$ for all $x \in A_{h^{-1}} \cap A_{(gh)^{-1}}$ and $g, h \in G$.

We call $(A, G, \{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ a *generalized unital partial C*-dynamical system*. If the group and base C*-algebra are clear, we will denote generalized unital partial C*-dynamical systems by $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ for convenience.

Note that Item (ii) says that the corresponding units form a commutative subspace lattice [Arv74]. This definition is more general than the definition of a unital partial action in Definition 2.2.1, since we allow for special families of hereditary subalgebras instead of closed two-sided ideals. If $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is a unital partial action, for any $g \in G$, $A_g = Ap_g$ for some central projection $p_g \in A$. Thus, $\{p_g\}$ forms a commutative subspace lattice and in particular, $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is a generalized unital partial action. In the case where A is commutative the two definitions coincide, since any hereditary subalgebra is an ideal in A. As with partial actions, a plethora of examples of generalized unital partial actions come from restrictions.

Definition 3.1.2. Let (A, G, α) be a global C*-dynamical system on a C*-algebra A and let $B \subseteq A$ be a unital hereditary subalgebra. We define the *restriction* of α to B as the generalized unital partial action $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ given by $B_g = B \cap \alpha_g(B)$ and $\beta_g = \alpha_g|_{B_{g^{-1}}}$.

The motivating example for generalized unital partial actions comes from covariant representations of partial actions using partial *-representations.

Example 3.1.3. Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial action and (π, u) be a covariant *-representation into $B(\mathcal{H})$. We can define a generalized unital partial action $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ on $B(\mathcal{H})$ by setting $B_g = q_g B(\mathcal{H}) q_g$, where $q_g = u_g u_g^*$ is the final projection of u_g for all $g \in G$, and $\beta_g(x) = u_g x u_g^*$ for $g \in G$ and $x \in B(\mathcal{H})$. Furthermore, by [Exe17, Proposition 9.8] we have $q_g q_h = q_h q_g$ for $g, h \in G$.

Another important example comes from partial crossed products. Recall that in the case of global C*-dynamical systems, G canonically acts on the crossed product via conjugation by unitaries. Using the framework of generalized unital partial actions we observe a similar phenomenon in the partial setting.

Proposition 3.1.4. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system. There exist partial isometries v_g and hereditary subalgebras $B_g \subseteq A \rtimes_r G$ such that $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ is a generalized unital partial action on $A \rtimes_r G$ with $\beta_g(b) = v_g b v_g^*$ for all $b \in B_{g^{-1}}$ and $v_g^* = v_{g^{-1}}$ for all $g \in G$. Furthermore, this implements the partial action on A, i.e., we have $\beta_g(a) = \alpha_g(a)$ for $a \in A \cap B_{g^{-1}}$.*

Proof. Observe that it suffices to find a suitable generalized unital partial action for the covariance algebra defined in Section 2.3

$$C(\alpha, r) = \operatorname{span}\{a\lambda_g \mid a \in A_g\},$$

where $a\lambda_g b\lambda_h = \alpha_g(\alpha_{g^{-1}}(a)b)\lambda_{gh}$ and $(a\lambda_g)^* = \alpha_{g^{-1}}(a^*)\lambda_{g^{-1}}$.

Let $A_g = Ap_g$ for central projections $p_g \in A$ for all $g \in G$. We define $B_g = \operatorname{span}\{a\lambda_s \mid a \in A_s \cap A_g \cap A_{sg}, s \in G\}$. This clearly is a subalgebra and we claim it is hereditary since $B_g = p_g\mathcal{C}(\alpha, r)p_g$. To see this note that for $a \in A_s$, $s \in G$ we have

$$p_g a\lambda_s p_g = p_g\alpha_s(\alpha_{s^{-1}}(a)p_g)\lambda_s.$$

Thus, if $a \in A_s \cap A_g \cap A_{sg}$, $p_g\alpha_s(\alpha_{s^{-1}}(a)p_g) = a$ and generally $p_g\alpha_s(\alpha_{s^{-1}}(a)p_g) \in A_s \cap A_g \cap A_{sg}$, which shows the desired. Moreover, since p_g is central in A, the units of the hereditary subalgebras in particular commute with each other. By setting $v_g = p_g\lambda_g$, we see $v_g^* = \alpha_{g^{-1}}(p_g)\lambda_{g^{-1}} = p_{g^{-1}}\lambda_{g^{-1}} = v_{g^{-1}}$ and since $v_g^* v_g v_g^* = v_g$ this is a partial isometry. Furthermore, $\beta_g \colon B_{g^{-1}} \to B_g$ given by $\beta_g(a\lambda_s) = v_g a v_g^*$ is well-defined, since

$$\beta_g(p_{g^{-1}}) = p_g\lambda_g p_{g^{-1}} p_{g^{-1}}\lambda_{g^{-1}} = \alpha_g(p_{g^{-1}})\lambda_g p_{g^{-1}}\lambda_{g^{-1}} = p_g\lambda_g p_{g^{-1}}\lambda_{g^{-1}} = p_g.$$

In general

$$\beta_g(a\lambda_s) = p_g\lambda_g a\lambda_s p_{g^{-1}}\lambda_{g^{-1}} = \alpha_g(p_g^{-1}\alpha_s(\alpha_{s^{-1}}(a)p_{g^{-1}}))\lambda_{gsg^{-1}} = \alpha_g(a)\lambda_{gsg^{-1}}$$

for $a \in A_s \cap A_{g^{-1}} \cap A_{sg^{-1}}$, so β_g is onto. Since $p_{g^{-1}}\lambda_{g^{-1}}p_g\lambda_g = p_{g^{-1}}$, the inverse is subsequently given by $\beta_{g^{-1}}(\cdot) = v_g^*(\cdot)v_g$ as desired. Lastly, for $a \in A_{g^{-1}}$ we obtain $\beta_g(a) = \alpha_g(a)$, which shows that this implements the partial action of G on A. □

Remark 3.1.5. The partial isometries v_g resemble the partial isometries m_g in the characterization of the full crossed product by [QR97], as the authors observed that these partial isometries implement the partial action.

Note that from the proof of the previous result we see that the covariance algebra is generated by A and v_g for $g \in G$.

The morphisms in the category of generalized unital partial C*-dynamical systems are given by G-equivariant u.c.p. maps that respect the hereditary subalgebras.

Definition 3.1.6. Given two generalized unital partial C*-dynamical systems $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ and $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ with $A_g = p_g A p_g$, $B_g = q_g B q_g$, a u.c.p. map $\phi \colon A \to B$ is called a *G-morphism*, if

(i) ϕ is G-*unital*, i.e., $\phi(p_g) = q_q$ for all $g \in G$;

(ii) ϕ is G-*equivariant*, i.e., for all $g \in G$, $\phi(\alpha_g(x)) = \beta_g(\phi(x))$ for $x \in A_{g^{-1}}$.

If ϕ is additionally completely isometric, we call ϕ a G-*embedding*.

In the definition of G-morphism, part (ii) in particular requires that $\phi(A_g) \subseteq B_g$ for all $g \in G$. This is automatically the case, since G-morphisms are required to be G-unital. To see this we recall the multiplicative domain of a u.c.p. map and its properties. For two C*-algebras A, B and a u.c.p. map $\phi\colon A \to B$ recall that the *multiplicative domain* Dom_ϕ *of* ϕ is defined as

$$\mathrm{Dom}_\phi = \{a \in A \mid \phi(a^*a) = \phi(a)^*\phi(a) \text{ and } \phi(aa^*) = \phi(a)\phi(a)^*\}$$

and for any $a \in \mathrm{Dom}_\phi$ we have $\phi(ab) = \phi(a)\phi(b)$ and $\phi(ba) = \phi(b)\phi(a)$ for all $b \in B$ [Pau02, Theorem 3.19]. If $\phi\colon A \to B$ is a G-unital u.c.p. map between two generalized unital partial C*-dynamical systems $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ and $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ with $A_g = p_g A p_g$ and $B_g = q_g B q_g$, then $\phi(p_g p_g^*) = \phi(p_g^* p_g) = q_g = \phi(p_g)^*\phi(p_g) = \phi(p_g)\phi(p_g)^*$ for all $g \in G$. Hence, the units belong to the multiplicative domain of a G-morphism and we obtain $\phi(A_g) = \phi(p_g)\phi(A)\phi(p_g) \subseteq B_g$.

Example 3.1.7. Consider a unital partial C*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ and let (π, u) be the unique non-degenerate covariant *-representation of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ into $B(\mathcal{H})$ corresponding to a non-degenerate *-representation $\rho\colon A \rtimes G \to B(\mathcal{H})$. Define the generalized unital partial action $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ on $B(\mathcal{H})$ as in Example 3.1.3. By [Exe17, Theorem 13.2] $q_g = u_g u_g^*$ is the orthogonal projection onto the closed linear span of $\pi(A_g)\mathcal{H}$ and thus, $\pi\colon A \to B(\mathcal{H})$ is G-unital. By construction, it is also G-equivariant and hence π is a G-morphism.

We also observe that two canonical maps associated to the partial reduced crossed product are G-morphisms in this category.

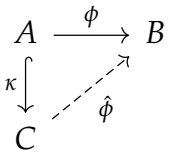
Example 3.1.8. Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system and consider the canonical generalized unital partial action on the reduced partial crossed product $A \rtimes_{\mathrm{r}} G$ as in Proposition 3.1.4. By definition the embedding $\kappa\colon A \hookrightarrow A \rtimes_{\mathrm{r}} G$ is G-equivariant and G-unital for a unital partial action $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$. Furthermore, the faithful conditional expectation $\Phi\colon A \to A \rtimes_{\mathrm{r}} G$ is a G-morphism.

Given two generalized unital partial C*-dynamical systems $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ and $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ we say that the pair $((\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G}), \kappa)$ is a G-*extension* of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$, if there exists a G-embedding $\kappa\colon A \to B$.

Example 3.1.9. Suppose $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is a generalized unital partial C*-dynamical system with $A_g = p_g A p_g$ for some projection $p_g \in A$ for all $g \in G$ and consider the corresponding partial action $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ on the injective envelope $I(A)$, which exists by similar arguments as in Proposition 2.2.2, and let $\kappa\colon A \to I(A)$ be the canonical embedding. By [Ham82b, Proposition 6.3], $I(A_g) = \kappa(p_g)I(A)\kappa(p_g)$ and thus κ is G-unital. Furthermore, as $I(\alpha)_g \circ \kappa = \alpha_g$ for all $g \in G$, it is also G-equivariant and thus a G-morphism. Since it is completely isometric, $((\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G}), \kappa)$ is a G-extension of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$.

3.2 G-Injectivity and -Essentiality

Definition 3.2.1. A generalized unital partial C*-dynamical system $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ is called *G-injective* if for any pair of generalized unital partial actions $(\{C_g\}_{g\in G}, \{\gamma_g\}_{g\in G})$, $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ with a G-embedding $\kappa\colon A \to C$ and G-morphism $\phi\colon A \to B$, there exists a G-morphism $\hat{\phi}\colon C \to B$ such that $\hat{\phi} \circ \kappa = \phi$. In other words, the following diagram



commutes.

We call a G-extension $((\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G}), \kappa)$ of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ *G-injective* if $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ is G-injective. A G-extension $((\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G}), \kappa)$ of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is called *G-essential* if given a generalized unital partial action $(\{C_g\}_{g\in G}, \{\gamma_g\}_{g\in G})$ and a G-morphism $\phi\colon B \to C$ such that $\phi \circ \kappa$ is a G-embedding, ϕ is a G-embedding.

Note that by setting $G = \{e\}$, we recover the notion of ordinary injectivity and essentiality. Furthermore, if $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is a global group action, i.e., $A_g = A$ for all $g \in G$, we obtain injectivity for C*-dynamical systems as in [Ham85].

We will show that any generalized unital partial action has some G-injective extension following techniques demonstrated in [Ham79b; Ham85]. Recall that for a C*-algebra A, $\ell^\infty(G, A)$ can be equipped with the canonical translation action τ given by $\tau_g(f)(t) = f(g^{-1}t)$ for $g, t \in G$ and $f \in \ell^\infty(G, A)$. Furthermore, [Ham85, Lemma

2.2] shows that if A is injective, $\ell^\infty(G,A)$ is G-injective in the category of C*-dynamical systems. We will define an analogue in the category of generalized unital partial C*-dynamical systems using a corner of $\ell^\infty(G,A)$. Let $(\{A_g\}_{g\in G},\{\alpha_g\}_{g\in G})$ be a generalized unital partial action with p_g denoting the unit of A_g, i.e., $A_g = p_g A p_g$. We define

$$B = \{f \in \ell^\infty(G,A) \mid f(t) \in A_{t^{-1}}\} = p[\ell^\infty(G,A)]p, \tag{3.1}$$

where $p(t) = p_{t^{-1}}$. Let $(\{B_g\}_{g\in G},\{\beta_g\}_{g\in G})$ denote the restriction of the translation action τ to B. In particular, we have

$$B_g = \{f \in \ell^\infty(G,A) \mid f(t) \in A_{t^{-1}} \cap A_{t^{-1}g}\}. \tag{3.2}$$

The unit of B_g is given by $p\tau_g(p)$. Observe that $(\{A_g\}_{g\in G},\{\alpha_g\}_{g\in G})$ can be mapped into $(\{B_g\}_{g\in G},\{\beta_g\}_{g\in G})$ via $\iota\colon A \to B$ defined by $\iota(a)(t) = \alpha_{t^{-1}}(p_t a p_t)$ for $t \in G$ and $a \in A$. The next result shows that this is in fact a G-embedding.

Proposition 3.2.2. *Let $(\{A_g\}_{g\in G},\{\alpha_g\}_{g\in G})$ be a generalized unital partial action of G on a C*-algebra A with $A_g = p_g A p_g$ for all $g \in G$ and let $B \subseteq \ell^\infty(G,A)$ be as in (3.1) and consider the restriction $(\{B_g\}_{g\in G},\{\beta_g\}_{g\in G})$ of the translation action τ to B. Then, $\iota\colon A \to B$ defined by $\iota(a)(t) = \alpha_{t^{-1}}(p_t a p_t)$ for $t \in G$ and $a \in A$ is a G-embedding.*

Proof. As α_t is a *-isomorphism, ι is completely isometric. To see that ι is a G-morphism, first observe that

$$\iota(p_g)(t) = \alpha_{t^{-1}}(p_t p_g p_t) = \alpha_{t^{-1}}(p_t p_g)\alpha_{t^{-1}}(p_t) = p_{t^{-1}} p_{t^{-1}g} p_{t^{-1}} = p_{t^{-1}} p_{t^{-1}g} = p(t)\tau_g(p)(t).$$

Here we used the fact that for unital partial actions $\alpha_g(p_{g^{-1}} p_h) = p_g p_{gh}$ for $g,h \in G$ [Exe17, Lemma 6.8]. This still holds in the case of generalized unital partial actions. Hence, the embedding preserves the units of the hereditary subalgebras, i.e., ι is G-unital. It remains to show that for $a \in A_{g^{-1}}$, $\beta_g(\iota(a)) = \iota(\alpha_g(a))$. To see this observe that for $a \in A_{g^{-1}}$ and $t \in G$ we obtain

$$\begin{aligned}\beta_g(\iota(a))(t) &= \iota(a)(g^{-1}t) = \alpha_{t^{-1}g}(p_{g^{-1}t} a p_{g^{-1}t}) = \alpha_{t^{-1}}(\alpha_g(p_{g^{-1}t} p_{g^{-1}} a p_{g^{-1}t} p_{g^{-1}}))\\ &= \alpha_{t^{-1}}(\alpha_g(p_{g^{-1}t} p_{g^{-1}})\alpha_g(a)\alpha_g(p_{g^{-1}t} p_{g^{-1}})) = \alpha_{t^{-1}}(p_t p_g \alpha_g(a) p_t p_g)\\ &= \alpha_{t^{-1}}(p_t \alpha_g(a) p_t) = \iota(\alpha_g(a))(t),\end{aligned}$$

where we again used that for generalized unital partial actions $\alpha_g(p_{g^{-1}} p_h) = p_g p_{gh}$ for $g,h \in G$. □

We now show that $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ is G-injective in the category of generalized unital partial actions if the base C*-algbera A is injective.

Proposition 3.2.3. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a generalized unital partial action of G on an injective C*-algebra A with $A_g = p_g A p_g$ for all $g \in G$, let $B \subseteq \ell^\infty(G, A)$ be as in (3.1), and let $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ denote the restriction of the translation action τ to B. The generalized unital partial C*-dynamical system $(B, G, \{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ is G-injective.*

Proof. To show that the generalized unital partial C*-dynamical system is G-injective, let $((\{D_g\}_{g\in G}, \{\delta_g\}_{g\in G}), \kappa)$ be a G-extension of $(\{C_g\}_{g\in G}, \{\gamma_g\}_{g\in G})$ and consider a G-morphism $\phi\colon C \to B$. Let q_g denote the unit of C_g and r_g be the unit of D_g for all $g \in G$. Note that $\psi\colon C \to A\colon x \mapsto \phi(x)(e)$ is a u.c.p. map. Hence, since A is injective, it extends to a u.c.p. map $\hat{\psi}\colon D \to A$ such that $\hat{\psi} \circ \kappa = \psi$. The candidate $\hat{\phi}\colon D \to B$ for the extension of ϕ is then given by $\hat{\phi}(x)(t) = \hat{\psi}(\delta_{t^{-1}}(r_t x r_t))$ for $x \in G$ and $t \in G$. To see that this is well-defined note that since ϕ is G-unital, we have $\psi(q_g) = \phi(q_g)(e) = p_g$ for all $g \in G$. Thus, since κ is also G-unital, $\hat{\psi}(r_g) = \hat{\psi}(\kappa(q_g)) = \psi(q_g) = p_g$ for all $g \in G$. In particular, $\hat{\psi}$ is G-unital, so $\hat{\psi}(D_g) \subseteq A_g$ and subsequently $\hat{\phi}(D) \subseteq B$. It remains to show $\hat{\phi}$ is G-unital, G-equivariant and $\hat{\phi} \circ \kappa = \phi$.

For G-unitality, note that for $g, t \in G$, we obtain

$$\hat{\phi}(r_g)(t) = \hat{\psi}(\delta_{t^{-1}}(r_t r_g r_t)) = \hat{\psi}(r_{t^{-1}} r_{t^{-1}g}) = p_{t^{-1}} p_{t^{-1}g} = p(t)\tau_g(p)(t),$$

where the latter describes the unit of B_g. Here we again used that for generalized unital partial actions $\delta_g(r_{g^{-1}} r_h) = r_g r_{gh}$ for $g, h \in G$.

Towards G-equivariance, let $x \in D_{g^{-1}}$ and consider

$$\begin{aligned}
\hat{\phi}(\delta_g(x))(t) &= \hat{\psi}(\delta_{t^{-1}}(r_t \delta_g(x) r_t)) = \hat{\psi}(\delta_{t^{-1}}(r_t r_g \delta_g(x) r_g r_t)) \\
&= \hat{\psi}(\delta_{t^{-1}}(\delta_g(\delta_{g^{-1}}(r_t r_g) x \delta_{g^{-1}}(r_t r_g)))) = \hat{\psi}(\delta_{t^{-1}g}(\delta_{g^{-1}}(r_t r_g) x \delta_{g^{-1}}(r_t r_g))) \\
&= \hat{\psi}(\delta_{t^{-1}g}(r_{g^{-1}t} r_{g^{-1}} x r_{g^{-1}t} r_{g^{-1}})) = \hat{\psi}(\delta_{t^{-1}g}(r_{g^{-1}t} x r_{g^{-1}t})) \\
&= \hat{\phi}(x)(g^{-1}t) = \beta_g(\hat{\phi}(x))(t).
\end{aligned}$$

Thus, $\hat{\phi}$ is G-equivariant.

To see that $\hat{\phi} \circ \kappa = \phi$, recall that since G-morphisms are G-unital, the units are in the multiplicative domain of the G-morphism. Then for $x \in C$, $t \in G$ we obtain

$$\begin{aligned}
\hat{\phi}(\kappa(x))(t) &= \hat{\psi}(\delta_{t^{-1}}(r_t \kappa(x) r_t)) = \hat{\psi}(\delta_{t^{-1}}(\kappa(q_t x q_t))) \\
&= \hat{\psi}(\kappa(\gamma_{t^{-1}}(q_t x q_t))) = \psi(\gamma_{t^{-1}}(q_t x q_t)) \\
&= \phi(\gamma_{t^{-1}}(q_t x q_t))(e) = \phi(q_t x q_t)(t) \\
&= \phi(q_t)(t)\phi(x)(t)\phi(q_t)(t) = \phi(x)(t),
\end{aligned}$$

where we used G-equivariance and G-unitality of ϕ in the last two lines. □

Note that the construction of $\hat{\phi}$ and the techniques used are similar to [ABF22], where the authors consider partial actions for von Neumann algebras. Combining Proposition 3.2.2 and Proposition 3.2.3 shows that $((\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G}), \iota)$ is a G-injective extension of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$. In particular, this gives that every generalized unital partial action embeds into a G-injective one or more precisely every generalized unital partial C*-dynamical system admits an injective G-extension. We also obtain a similar equivalent characterization of G-injectivity as in the global setting [Ham85, Remark 2.3].

Proposition 3.2.4. *A generalized unital partial C*-dynamical system $(A, G, \{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is G-injective if and only if A is injective and there exists a G-morphism $\phi\colon B \to A$ such that $\phi \circ \iota = \mathrm{id}_A$. Here, $B = \{f \in \ell^\infty(G, A) \mid f(t) \in A_{t^{-1}}\}$ and $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ is the generalized unital partial action obtained by restricting the translation action on $\ell^\infty(G, A)$. Lastly, $\iota\colon A \to B$ is defined by $\iota(a)(t) = \alpha_{t^{-1}}(p_t a p_t)$, where p_t denotes the unit of A_t.*

Proof. First, observe that Proposition 3.2.3 shows that ι is a G-embedding. Thus, if $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is G-injective, then the identity map $\mathrm{id}_A\colon A \to A$ extends to a G-morphism $\phi\colon B \to A$ such that $\phi \circ \iota = \mathrm{id}_A$. In particular, since B is injective as a corner of $\ell^\infty(G, A)$ this shows that A is injective.

Conversely, suppose that A is injective and there exists a G-morphism $\phi\colon B \to A$ such that $\phi \circ \iota = \mathrm{id}_A$. Again by Proposition 3.2.3 $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ is G-injective. Thus, if $((\{D_g\}_{g\in G}, \{\delta_g\}_{g\in G}), \kappa)$ is a G-extension of $(\{C_g\}_{g\in G}, \{\gamma_g\}_{g\in G})$ and $\psi\colon C \to A$ is a G-morphism, then $\iota \circ \psi$ extends to a G-morphism $\Psi\colon D \to B$. Setting $\hat{\psi} = \phi \circ \Psi$ finishes the proof. □

Using this alternative characterization we can give a class of examples of injective objects in the category of generalized unital partial C*-dynamical systems given by certain restrictions of global actions on injective C*-algebras.

Corollary 3.2.5. *Let (A, G, α) be a C*-dynamical system such that (A, G, α) is injective in the category of C*-dynamical systems and let $p \in A$ be a central projection. The partial C*-dynamical system $(\{C_g\}_{g\in G}, \{\gamma_g\}_{g\in G})$ obtained by restricting α to $C = Ap$ is G-injective in the category of generalized unital partial actions.*

Proof. First, since A is G-injective it is in particular injective and thus as a corner of an injective C*-algebra, C is injective. Furthermore, by [Ham85, Remark 2.3] there exists a globally G-equivariant u.c.p. map $\phi\colon \ell^\infty(G, A) \to A$ such that $\phi \circ \iota_A = \mathrm{id}_A$. In particular, ϕ is G-unital. Then, define $\psi\colon \ell^\infty(G, C) \to C$ as the restriction of ϕ to $\ell^\infty(G, C)$

composed with the compression by p. By construction this is a G-morphism and since $\phi \circ \iota_A = \mathrm{id}_A$ we have $\psi \circ \iota_C = \mathrm{id}_C$. Hence, C is G-injective. □

Note that if in the previous proposition p was G-invariant, then the resulting global C*-dynamical system would be injective in the category of global C*-dynamical systems as well. This is in line with the prime examples of partial actions arising from restrictions of global systems to not necessarily invariant ideals.

3.3 Construction of the Partial G-injective Envelope

Using the fact that $B(\mathcal{H})$ is injective and that for any partial action there exists a non-degenerate covariant representation, Proposition 3.2.3 shows that any partial action admits a G-injective extension. We will now show that in fact, we can find an extension that is minimal in the appropriate sense.

Definition 3.3.1. Let $(A, G, \{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system. We call a G-injective and G-essential G-extension a *G-injective envelope* of $(A, G, \{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$.

We will show the existence and uniqueness of the G-injective envelope of a unital partial C*-dynamical system. Instead of using G-essentiality, we consider a related property: G-rigidity. A G-extension $((\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G}), \kappa)$ of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is called *G-rigid* if whenever $\phi\colon B \to B$ is a G-morphism such that $\phi \circ \kappa = \mathrm{id}_{\kappa(A)}$, then $\phi = \mathrm{id}_B$. The following proposition shows that in the G-injective case, G-rigidity implies G-essentiality. This has been observed by Hamana in the non-equivariant as well as C*-dynamical case [Ham79b; Ham85]. We will include the proof for completeness.

Proposition 3.3.2. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a generalized unital partial action with a G-injective extension $(\{B_g\}_{g\in G}, \{\beta\}_{g\in G}, \kappa)$. If $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G}, \kappa)$ is a G-rigid extension, then it is G-essential.*

Proof. Let $(\{C_g\}_{g\in G}, \{\gamma_g\}_{g\in G})$ be a generalized unital partial action and $\phi\colon B \to C$ be a G-morphism such that $\phi \circ \kappa$ is a G-embedding. Then, since $\phi \circ \kappa$ is completely isometric, there exists a G-morphism $\psi\colon (\phi \circ \kappa)(C) \to B$ such that $\phi \circ \psi = \mathrm{id}_B$. Observe that $\phi \circ \kappa\colon A \to \phi(C)$ is a G-embedding, i.e., it gives rise to a G-extension. Thus, by G-injectivity of $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$, there exists an extension $\hat{\psi}\colon \phi(C) \to B$. By G-rigidity, we obtain $\hat{\psi} \circ \phi = \mathrm{id}_B$. Hence, ϕ is a G-embedding, which shows the desired. □

The next theorem shows the existence of a G-injective and G-essential extension for a given unital partial action $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$. This has been shown in various categories [Ham79b; Ham85; Ken+21] and we will use ideas presented in [Ken+21; Sin25].

Theorem 3.3.3. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C^*-dynamical system. There exists a G-injective and G-essential extension $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G}), \iota)$ of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$. Furthermore, the G-injective envelope is unique and $(I_G(A), G, I_G(A)_{g\in G}, I_G(\alpha)_{g\in G})$ is a unital partial C^*-dynamical system.*

Proof. We will show the existence of a G-injective and G-rigid extension of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$, which by the previous proposition suffices to show the existence of the G-injective envelope. Let (π, u) be a non-degenerate covariant representation of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ on a Hilbert space $\mathcal{H}$. Furthermore, consider the generalized unital partial action induced by this representation as in Example 3.1.3 and consider the generalized unital partial action $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ obtained by restricting the translation action on $\ell^\infty(G, B(\mathcal{H}))$ as in Proposition 3.2.3. This is G-injective, because $B(\mathcal{H})$ is injective. Since $B(\mathcal{H})$ is a von Neumann algebra, B is in particular w*-closed as a corner of $\ell^\infty(G, B(\mathcal{H}))$. As seen in Proposition 3.2.4, G-injective extensions are in correspondence with G-morphisms into B that fix a certain subalgebra. Thus, consider the set

$$S = \{\phi\colon B \to B \mid \phi \text{ is a } G\text{-morphism and } \phi \circ \iota = \mathrm{id}_{\iota(A)}\}, \tag{3.3}$$

where $\iota\colon A \to B$ denotes the canonical embedding of A into B via π. Observe that, since $\phi \circ \psi \in S$ for $\phi, \psi \in S$ and $\mathrm{id} \in S$, S is a right semigroup. Furthermore, it is convex and since B is w*-closed, S is closed with respect to the point w*-topology. In summary, S is a compact right topological semigroup. Thus, there exists a minimal left ideal $L \subseteq S$ that is closed and contains a minimal idempotent $\psi \in L$ [HS12, Corollary 2.6]. Since ψ is an idempotent, $\psi(B)$ is injective in the category of operator systems and thus, by [CE77] isomorphic to a C*-algebra via the Choi-Effros product $a \cdot b = \psi(ab)$. Furthermore, since $\psi \in S$, $\psi \circ \iota = \mathrm{id}_{\iota(A)}$ and ψ is a G-morphism. Thus, the generalized unital partial action structure on B restricts to a generalized unital partial action on $\psi(B)$ denoted by $(\{\psi(B)_g\}_{g\in G}, \{\psi(\beta)_g\}_{g\in G})$ and $((\{\psi(B)_g\}_{g\in G}, \{\psi(\beta)_g\}_{g\in G}), \iota)$ is a G-extension of A. By Proposition 3.2.4 this is a G-injective extension.

Using the convexity of S, we will show that $(\{\psi(B)_g\}_{g\in G}, \{\psi(\beta)_g\}_{g\in G})$ is in fact a G-rigid extension. Observe that since L is minimal and $S\psi \subseteq L$ we must have $L = S\psi$. Now, since S is convex, L is convex. Also, due to the fact that ψ is idempotent, the set $\{\phi \in S \mid \phi \circ \psi = \psi\}$ is non-empty and a closed left ideal. Thus, by minimality $\{\phi \in S \mid \phi \circ \psi = \psi\} = L$. Now, due to minimality $L = S\psi'$ for any $\psi' \in L$ and since L

is convex, the map $\phi \mapsto \phi \circ \psi'$ is a continuous affine map for arbitrary $\psi' \in S$. Hence, it admits a fixed point and we can repeat the above argument to show that L is a left-zero semigroup, i.e., $\phi \circ \psi = \phi$ for all $\phi, \psi \in L$. Now for G-rigidity, let $\eta \colon \psi(B) \to \psi(B)$ be a G-morphism such that $\eta \circ \iota = \mathrm{id}_{\iota(A)}$. To see that $\eta = \mathrm{id}_{\psi(B)}$, let $\kappa \colon \psi(B) \to B$ denote the inclusion map. Then, $\kappa \circ \eta \circ \psi \colon B \to B$ and $\kappa \circ \eta \circ \psi \circ \iota = \mathrm{id}_{\iota(A)}$, so $\kappa \circ \eta \circ \psi \in S$. Moreover, since ψ is idempotent and $L = S\psi$, $\kappa \circ \eta \circ \psi \in L$. Using the fact that L is left-zero, we must have $\psi \circ \kappa \circ \eta \circ \psi = \kappa \circ \eta \circ \psi$. Since this gives that for all $x \in B$, $\eta(\psi(x)) = (\kappa \circ \eta \circ \psi)(x) = \psi(x)$, we obtain $\eta = \mathrm{id}_{\psi(B)}$. Thus, $(\{\psi(B)_g\}_{g\in G}, \{\psi(\beta)_g\}_{g\in G})$ is a G-rigid extension and since it is G-injective this implies it is G-essential by Proposition 3.3.2.

To see uniqueness, we will again use G-rigidity. Suppose there exists another G-injective and G-essential extension $((\{C_g\}_{g\in G}, \{\gamma_g\}_{g\in G}), \rho)$ of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$. Using G-injectivity and G-essentiality we can extend ρ to a G-embedding $\hat{\rho} \colon \psi(B) \to C$. Similarly, we obtain a G-embedding $\hat{\iota} \colon C \to \psi(B)$. Furthermore, $\hat{\rho} \circ \hat{\iota}|_A = \mathrm{id}_A$ and thus by G-rigidity $\hat{\rho} \circ \hat{\iota} = \mathrm{id}$. Hence, we can write $((\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G}), \iota)$ for the G-injective envelope.

Lastly, we will show that $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha_g\}_{g\in G})$ is a unital partial action in the usual sense, i.e., $I_G(A)_g \trianglelefteq I_G(A)$ for all $g \in G$. First note that since A_g is an ideal, the unit $p_g \in A_g$ is central and it suffices to show that the unit for $I_G(A)_g$ is central in $I_G(A)$. Observe that since ι is G-unital, the unit of $I_G(A)_g$ will be of the form $\iota(p_g)$. Hence, it suffices to show that $Z(A) \subseteq Z(I_G(A))$, where here we identify A with $\iota(A) \subseteq I_G(A)$. To see this, note that if $u \in Z(A)$ is a unitary, the map defined by $\phi \colon A \to I_G(A) \colon x \mapsto uxu^*$ is G-equivariant and G-unital, since for any $g \in G$ and $x \in A_{g^{-1}}$ we have

$$\phi(I_G(\alpha_g)(x)) = u\alpha_g(x)u^* = I_G(\alpha_g)(x) = I_G(\alpha_g)(uxu^*)$$

using $u \in Z(A)$. Thus, it extends to a G-morphism $\hat{\phi} \colon I_G(A) \to I_G(A)$ by G-injectivity. By G-rigidity, this extension will be unique. However, again using the fact that $u \in Z(A)$, we see that $\hat{\phi}|_A = \mathrm{id}_A$, which shows that $\hat{\phi} = \mathrm{id}_{I_G(A)}$ by G-rigidity. Hence, $u \in Z(I_G(A))$. Since $Z(A)$ is spanned by unitaries, it follows that $Z(A) \subseteq Z(I_G(A))$, which finishes the proof. □

Since global C*-dynamical systems on unital C*-algebras are in particular unital partial C*-dynamical systems, given such a global C*-dynamical system $(A, G) = (\{A\}_{g\in G}, \{\alpha_g\}_{g\in G})$, we can look at the injective envelopes of (A, G) in both categories, the injective envelope in the category of global C*-dynamical systems $(I_G^{\mathrm{glob}}(A), \kappa)$ and $(I_G(A), \iota)$ as in Theorem 3.3.3. Fortunately, the two coincide. To see this, first observe

that if $A_g = A$ for all $g \in G$, the hereditary subalgebra $B \subseteq \ell^\infty(G, B(\mathcal{H}))$ in the proof of Theorem 3.3.3 equals $\ell^\infty(G, B(\mathcal{H}))$, since $p(t) = 1$ in the construction of B. Furthermore, the elements in S are ordinarily G-equivariant u.c.p. maps, i.e., morphisms in the category of global C*-dynamical systems. In summary, $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$ is a global C*-dynamical system and by construction it is a G-injective and G-essential extension of (A, G), i.e., $I_G(A) = I_G^{\text{glob}}(A)$.

We summarize the relations between the injective envelope and the G-injective envelope given a unital partial C*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$. First, note, by Proposition 3.2.4, $I_G(A)$ is in particular injective. Thus, there exists a u.c.p. map $\phi\colon I(A) \to I_G(A)$ extending the embedding of A into $I_G(A)$. By essentiality of $I(A)$, this is completely isometric. However, we can even arrange for a G-embedding. To see this note that again using injectivity of $I_G(A)$ and the fact that $I(A)$ sits inside $I_G(A)$ completely isometrically, we obtain a u.c.p. map $\psi\colon I_G(A) \to I(A)$ such that $\psi|_{I(A)} = \mathrm{id}_{I(A)}$. Now, recall that the embedding $\kappa\colon A \to I(A)$ is a G-morphism, so $\iota\colon A \to I_G(A)$ extends to a G-morphism $\hat{\iota}\colon I(A) \to I_G(A)$. In fact, this is completely isometric, since $\psi \circ \hat{\iota} = \mathrm{id}_{I(A)}$ by rigidity of $I(A)$. Hence, we obtain the following inclusions as unital partial C*-dynamical systems

$$A \subseteq I(A) \subseteq I_G(A).$$

Also note that since $\hat{\iota}\colon I(A) \to I_G(A)$ is a G-embedding, it is in particular G-unital and we obtain for all $g \in G$, $I(A)_g = I(A)p_g$ and $I_G(A)_g = I_G(A)p_g$ where $A_g = Ap_g$. Furthermore, in the last paragraph of the proof of Theorem 3.3.3 we saw the relation between the centres of A and $I_G(A)$. Using the chain of inclusions above as well as a [Ham82c] we obtain

$$Z(A) \subseteq Z(I(A)) \subseteq Z(I_G(A)),$$

where for the last inclusion $I_G(I(A)) = I_G(A)$ was used. This follows simply since $I_G(A)$ is in particular a G-injective extension for $I(A)$ and thus we just need to verify it is G-rigid. But clearly any G-morphism $\phi\colon I_G(A) \to I_G(A)$ with $\phi|_{I(A)} = \mathrm{id}$ satisfies $\phi|_A = \mathrm{id}$ and thus since $I_G(A)$ is G-rigid for A, $\phi = \mathrm{id}$.

We will end this section with a quick observation on the inclusion of a partial C*-dynamical system into its G-injective envelope. This is due to Hamana in the setting of global C*-dynamical systems and the generalization of the proof is rather straightforward.

Proposition 3.3.4. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system and consider the inclusion $\kappa_G\colon A \hookrightarrow I_G(A)$ into its G-injective envelope $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g$*

$\}_{g\in G}$). *The inclusion is normal, i.e., if* $(x_\lambda)_\lambda$ *is an increasing net in* A *with* $\sup_\lambda x_\lambda = x \in A$, *then* $\sup_\lambda \kappa_G(x_\lambda) = \kappa_G(x)$.

Proof. Note that since $p\ell^\infty(G, I(A))p$ is G-injective by Proposition 3.2.3, it contains a copy of $I_G(A)$. Furthermore, by [Ham81, Lemma 3.6], the inclusion $\kappa\colon A \to I(A)$ is normal. Lastly, observe that the image $\iota(I(A))$ of the inclusion $\iota\colon I(A) \to p\ell^\infty(G, I(A))p$ with $\iota(a)(t) = I(\alpha)_{t^{-1}}(ap_t)$ for all $a \in I(A)$, $t \in G$ is closed under taking suprema. Hence, $\iota(A) \subseteq \iota(I(A)) \subseteq I_G(A) \subseteq p\ell^\infty(G, I(A))p$ and the inclusion κ_G is normal. □

Moving forward we will often identify A with its image in $I(A)$ and $I_G(A)$ respectively, and omit a specific mention of the corresponding embedding.

3.4 Enveloping Actions

Restrictions of global C*-dynamical systems give rise to a rich class of examples of partial C*-dynamical systems. It is therefore natural to ask when this process can be reversed, i.e., when does a given partial C*-dynamical arise as a restriction of a global C*-dynamical system. This problem goes back to [Aba03], where the author answered it fully in the case of topological partial dynamical systems. In particular, Abadie showed that every topological partial dynamical system arises as a restriction of a global topological dynamical system. However, one might lose Hausdorffness of the underlying space in the process. In this section, specifically Theorem 3.4.2, we establish a natural relationship between G-injective envelopes of unital partial C*-dynamical systems and of their enveloping actions, defined below.

Definition 3.4.1. Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system. We call $(\mathcal{C}^A, \zeta^A)$ an *enveloping action* for $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ if $(\mathcal{C}^A, \zeta^A)$ is a global C*-dynamical system, $A \trianglelefteq \mathcal{C}^A$, α arises as the restriction of ζ^A to A, and $\operatorname{span}\{\zeta^A_t(x) \mid x \in A, t \in G\}$ is dense in $\mathcal{C}^A$.

One can think of an enveloping action as a minimal global action such that the original partial C*-dynamical system arises as a restriction. In fact if an enveloping action exists, the enveloping action is unique up to isomorphism as shown in [Aba03, Theorem 2.1]. Regarding existence, in [Fer18, Corollary 4.8] it is shown that a partial action on a unital C*-algebra arises as a restriction if and only if it is a unital partial action.

A common construction of an enveloping action given a unital partial C*-dynamical system is the following (see for example [Exe17, Section 6]): Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be

a unital partial C*-dynamical system with $A_t = Ap_t$ for some projection $p_t \in Z(A)$. As in Proposition 3.2.2 we consider $\iota(A) \subseteq \ell^\infty(G, A)$, where $\iota(a)(t) = \alpha_{t^{-1}}(p_t a)$ for all $a \in A$ and $t \in G$. Then, ι is G-equivariant, where we consider the canonical translation action τ on $\ell^\infty(G, A)$. Setting $\mathcal{C}^A = \overline{\operatorname{span}}\{\tau_g(\iota(a)) \mid g \in G, a \in A\}$ we see that $\mathcal{C}^A$ is invariant under the translation action, thus choosing $\zeta^A = \tau|_{\mathcal{C}^A}$ gives rise to a global C*-dynamical system. By definition, $\iota(A) \subseteq \mathcal{C}^A$ and since ι is a G-embedding, the restriction of ζ^A is α. Lastly, $\iota(A) \trianglelefteq \mathcal{C}^A$ by the proof of [Exe17, Theorem 6.13].

Given a unital partial action $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$, as A is a unital ideal in $\mathcal{C}^A$, i.e., $A = \mathcal{C}^A 1_A$, we have $I(A) = I(\mathcal{C}^A)1_A$ by [Ham82b, Propostion 6.3]. Furthermore, note that the C*-dynamical system $(\mathcal{C}^A, \zeta^A)$ extends uniquely to a C*-dynamical system $(I(\mathcal{C}^A), I(\zeta^A))$ and the restriction to $I(\mathcal{C}^A)1_A$ is a partial action extending α. Thus, since the extension to the injective envelope is unique, the partial C*-dynamical system $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ is a restriction of $(I(\mathcal{C}^A), I(\zeta^A))$. The next theorem shows that the G-injective envelopes exhibit a similar relationship.

Theorem 3.4.2. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system with $A_g = Ap_g$ for projections $p_g \in Z(A)$ for all $g \in G$ and let $(\mathcal{C}^A, \zeta^A)$ denote its enveloping action. Let $(I_G(\mathcal{C}^A), I_G(\zeta^A))$ denote the G-injective envelope of the enveloping action. Then, $I_G(A) = I_G(\mathcal{C}^A)1_A$, where 1_A denotes the unit of A. In particular, $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$ is a restriction of $(I_G(\mathcal{C}^A), I_G(\zeta^A))$.*

Proof. Let β denote the partial action given by the restriction of $I_G(\zeta^A)$ to $I_G(\mathcal{C}^A)1_A$ and define $B := I_G(\mathcal{C}^A)1_A$. First, observe that since $A \trianglelefteq \mathcal{C}^A$ and $\mathcal{C}^A \subseteq I_G(\mathcal{C}^A)$ completely isometrically, $A \subseteq B$ completely isometrically. We claim this embedding is also G-equivariant and G-unital. For the former note that since α is a restriction of ζ^A and $I_G(\zeta^A)$ is an extension of ζ^A, the embedding is G-equivariant. Furthermore, recall that for $g \in G$, the unit of B_g is given by $p_g I_G(\zeta^A)_g(p_g) = 1_A \zeta^A_g(1_A)$. By definition of restriction we have $A_g = A \cap \zeta^A_g(A)$ and thus $A_g = A \cap \zeta^A_g(A) \subseteq B \cap I_G(\zeta^A)_g(B)$. Hence, $p_g \leq 1_A \zeta^A_g(1_A)$. Conversely, since A is an ideal in $\mathcal{C}^A$, $1_A \zeta^A_g(1_A) \in A$ and also, since $1_A \zeta^A_g(1_A) = \zeta^A_g(\zeta^A_{g^{-1}}(1_A)1_A)$, $1_A \zeta^A_g(1_A) \in \zeta^A_g(A) \cap A = A_g$, we obtain $1_A \zeta^A_g(A) \leq p_g$. Thus, $p_g = 1_A \zeta^A_g(1_A)$ and in summary this is a G-embedding. Additionally, since $I_G(\mathcal{C}^A)$ is G-injective and B is a restriction, by Corollary 3.2.5 $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ is G-injective, so it is a G-injective extension of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$. To show the desired result, it thus suffices to show that this extension is also G-rigid.

Towards G-rigidity, let $\phi\colon B \to B$ be a G-morphism with $\phi|_A = \mathrm{id}_A$. Consider the extension $\tilde{\phi}\colon \ell^\infty(G, B) \to \ell^\infty(G, B)$ of ϕ given by $\tilde{\phi}(f)(t) = \phi(f(t))$ for $f \in \ell^\infty(G, B)$

and $t \in G$. Using the canonical G-embedding $\kappa\colon B \to \ell^\infty(G,B)$ via $\kappa(b)(t) = \beta_{t^{-1}}(bp_t)$ for $b \in B$ and $t \in G$, we see that $\tilde{\phi}|_{\kappa(A)} = \mathrm{id}_{\kappa(A)}$, since for $a \in A$ we have

$$\tilde{\phi}(\kappa(a))(t) = \phi(\kappa(a)(t)) = \phi(\beta_{t^{-1}}(ap_t)) = \beta_{t^{-1}}(ap_t) = \kappa(a)(t).$$

In particular, we obtain $\kappa(A) \subseteq \ell^\infty(G,B)$. Hence, by the uniqueness of the enveloping action $\mathcal{C}^A \subseteq \ell^\infty(G,B)$ via taking the closure of the appropriate orbits. Furthermore, since $\ell^\infty(G,B)$ is G-injective by [Ham85, Lemma 2.2], we can find a completely isometric copy $I_G(\mathcal{C}^A) \subseteq \ell^\infty(G,B)$. Lastly, using G-injectivity of $I_G(\mathcal{C}^A)$, there exists a G-morphism $\psi\colon \ell^\infty(G,B) \to I_G(\mathcal{C}^A)$ such that $\psi|_{\mathcal{C}^A} = \mathrm{id}_{\mathcal{C}^A}$. Notice that for $g \in G$ and $a \in A \subseteq B$ by definition of $\tilde{\phi}$ we have

$$\tilde{\phi}(\tau_g(\kappa(a)))(t) = \phi(\kappa(a)(g^{-1}t)) = \tilde{\phi}(\kappa(a))(g^{-1}t) = (\tau_g \circ \tilde{\phi})(\kappa(a))(t),$$

where τ_g denotes the translation action on $\ell^\infty(G,B)$. Hence, $\tilde{\phi}$ is G-equivariant on the orbits of A. In particular, $\tilde{\phi}|_{\mathcal{C}^A}$ is G-equivariant. Using the fact that $\ell^\infty(G,B)$ is G-injective, there exists a map $\Phi\colon I_G(\mathcal{C}^A) \to \ell^\infty(G,B)$ such that $\Phi|_{\mathcal{C}^A} = \mathrm{id}_{\mathcal{C}^A}$. The composition $\psi \circ \Phi\colon I_G(\mathcal{C}^A) \to I_G(\mathcal{C}^A)$ satisfies $\psi \circ \Phi|_{\mathcal{C}^A} = \mathrm{id}_{\mathcal{C}^A}$ and thus by G-rigidity of $I_G(\mathcal{C}^A)$, $\psi \circ \Phi = \mathrm{id}$. In particular, $\psi \circ \Phi(\kappa(b)) = \kappa(b)$ and thus $\phi = \mathrm{id}$. Hence, $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ is a G-rigid and G-injective extension of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$, which shows the desired. □

The enveloping action is characterized by orbits being dense. We observe a similar relationship for the injective and G-injective envelopes of the enveloping action in the next lemma.

Lemma 3.4.3. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial action with 1_A denoting the unit of A and let $(\mathcal{C}^A, \zeta^A)$ denote its enveloping action. Suppose $c \in I(\mathcal{C}^A)$ and $cI(\zeta^A)_g(1_A) = 0$ for all $g \in G$, then $c = 0$. Similarly, let $d \in I_G(\mathcal{C}^A)$ with $dI_G(\zeta^A)_g(1_A) = 0$ for all $g \in G$, then $d = 0$.*

Proof. First, observe that $cI(\zeta^A)_g(1_A) = 0$ for all $g \in G$, if and only if $I(\zeta^A)_g(c)1_A = 0$ for all $g \in G$. Now consider

$$S := \{c \in I(\mathcal{C}^A) \mid cI(\zeta^A)_g(1_A) = 0 \text{ for all } g \in G\} \subseteq I(A).$$

This is an ideal, since for $b \in I(\mathcal{C}^A)$ and $c \in S$ we have $0 = I(\zeta^A)_g(b)I(\zeta^A)_g(c)1_A = I(\zeta^A)_g(bc)1_A$ and thus $bcI(\zeta^A)_g(1_A) = 0$. Hence, $bc \in S$. Using that 1_A is central, we also obtain $cb \in S$. However, $S \cap A = \{0\}$ and thus $S = \{0\}$, since the extension $A \subseteq I(A)$ is essential.

Second, using the same arguments as above, consider the ideal

$$S_G := \{d \in I_G(\mathcal{C}^A) \mid dI_G(\zeta^A)_g(1_A) = 0 \text{ for all } g \in G\} \trianglelefteq I_G(A).$$

In fact, S_G (and similarly S) is G-invariant. To see this, let $d \in S_G, h \in G$ and observe $I_G(\zeta^A)_h(d)I_G(\zeta^A)_g(1_A) = I_G(\zeta^A)_h(dI_G(\zeta^A)_{h^{-1}g}(1_A)) = 0$ for all $g \in G$, showing $I_G(\zeta^A)_h(d) \in S_G$. Again $S_G \cap A = \{0\}$, so by G-essentiality $S_G = \{0\}$. □

The relation between $I_G(A)$ and $I_G(\mathcal{C}^A)$ can be helpful when trying to use tools established for global C*-dynamical systems for unital partial C*-dynamical systems.

Proposition 3.4.4. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ and $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ be unital partial C*-dynamical systems. If $A \subseteq B \subseteq I_G(A)$, then, $A \rtimes_r G \subseteq B \rtimes_r G \subseteq I(A \rtimes_r G)$.*

Proof. Suppose $A \subseteq B \subseteq I_G(A)$. Note that it suffices to show that $I_G(A) \rtimes_r G \subseteq I(A \rtimes_r G)$ as then $A \rtimes_r G \subseteq B \rtimes_r G \subseteq I_G(A) \rtimes_r G \subseteq I(A \rtimes_r G)$. Let $(\mathcal{C}^A, \zeta^A)$ denote the enveloping action of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$. Now by Theorem 3.4.2 $I_G(A) = I_G(\mathcal{C}^A)1_A$ and thus $I_G(A) \rtimes_r G = 1_A(I_G(\mathcal{C}^A) \rtimes_r G)1_A$. Using the corresponding result for global C*-dynamical systems [Ham85, Theorem 3.4] we obtain $1_A(I_G(\mathcal{C}^A) \rtimes_r G)1_A \subseteq 1_A I(\mathcal{C}^A \rtimes_r G)1_A = I(1_A(\mathcal{C}^A \rtimes_r G)1_A) = I(A \rtimes_r G)$. In sum, $I_G(A) \rtimes_r G \subseteq I(A \rtimes_r G)$. □

Corollary 3.4.5. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system. The reduced crossed product $A \rtimes_r G$ is prime if and only if $I(A) \rtimes_r G$ (or $I_G(A) \rtimes_r G$ respectively) is prime.*

Proof. This follows from the fact that $A \rtimes_r G$ is prime if and only if $I(A \rtimes_r G)$ is an AW*-factor [Ham81, Theorem 6.3 and 7.1]. □

We will end with a characterization of the ideal intersection property in terms of the enveloping action. As established by Abadie in [Aba03, Theorem 3.3], for a partial C*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ and its enveloping action $(\mathcal{C}^A, \zeta^A)$, their reduced crossed products $A \rtimes_r G$ and $\mathcal{C}^A \rtimes_r G$ are Morita equivalent. Hence, the next result follows quite readily and we suspect this is already well-known in the literature. We include the argument for completeness. Note that similar to the partial setting, we say that a global C*-dynamical system (B, β) has the ideal intersection property if for all $\{0\} \neq J \trianglelefteq B \rtimes_r G$ we have $J \cap B \neq \{0\}$.

Proposition 3.4.6. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial action and let $(\mathcal{C}^A, \zeta^A)$ denote its enveloping action. The partial C*-dynamical system has the ideal intersection property if and only if $(\mathcal{C}^A, \zeta^A)$ does.*

Proof. First, suppose $(\mathcal{C}^A, \zeta^A)$ has the ideal intersection property and consider $I \trianglelefteq A \rtimes_{\mathrm{r}} G$ with $I \cap A = \{0\}$. Now, consider $J = \overline{(\mathcal{C}^A \rtimes_{\mathrm{r}} G) I (\mathcal{C}^A \rtimes_{\mathrm{r}} G)}$. By definition, $J \trianglelefteq \mathcal{C}^A \rtimes_{\mathrm{r}} G$ and we claim $J \cap \mathcal{C}^A = \{0\}$. If the latter were not the case, there exists $0 \neq a \in J \cap \mathcal{C}^A$. As the orbits of A are dense in $\mathcal{C}^A$, this implies that there exists $g \in G$ such that $\alpha_g(a) 1_A \neq 0$. However, this leads to a contradiction, as $\alpha_g(a) 1_A \in J \cap A = I \cap A$. The latter equality follows from the fact that $A \rtimes_{\mathrm{r}} G$ is isomorphic to a full hereditary (unital) subalgebra of $\mathcal{C}^A \rtimes_{\mathrm{r}} G$ [Exe17, Theorem 28.8]. Since $(\mathcal{C}^A, \zeta^A)$ has the ideal intersection property, $J = \{0\}$, giving $I = \{0\}$. Thus, $(\{A_g\}_{g \in G}, \{\alpha_g\}_{g \in G})$ has the ideal intersection property.

Conversely, suppose $(\mathcal{C}^A, \zeta^A)$ does not have the ideal intersection property and take $J \trianglelefteq \mathcal{C}^A \rtimes_{\mathrm{r}} G$ satisfying $J \cap \mathcal{C}^A = \{0\}$. Note it suffices to show that $J \cap A \rtimes_{\mathrm{r}} G \neq \{0\}$, since then $(\{A_g\}_{g \in G}, \{\alpha_g\}_{g \in G})$ does not have the ideal intersection property. However, we have $(A \rtimes_{\mathrm{r}} G)(\mathcal{C}^A \rtimes_{\mathrm{r}} G) J (\mathcal{C}^A \rtimes_{\mathrm{r}} G)(A \rtimes_{\mathrm{r}} G) \subseteq J \cap (A \rtimes_{\mathrm{r}} G)$ using the fact that J is an ideal and $A \rtimes_{\mathrm{r}} G \subseteq \mathcal{C}^A \rtimes_{\mathrm{r}} G$ is hereditary. Now, using fullness, the ideal generated by $A \rtimes_{\mathrm{r}} G$ coincides with $\mathcal{C}^A \rtimes_{\mathrm{r}} G$. Thus,

$$J \subseteq \langle (A \rtimes_{\mathrm{r}} G)(\mathcal{C}^A \rtimes_{\mathrm{r}} G) J (\mathcal{C}^A \rtimes_{\mathrm{r}} G)(A \rtimes_{\mathrm{r}} G) \rangle \subseteq \langle J \cap (A \rtimes_{\mathrm{r}} G) \rangle$$

showing $J \cap (A \rtimes_{\mathrm{r}} G) \neq \{0\}$, which finishes the proof. □

Remark 3.4.7. Note that if $(\{A_g\}_{g \in G}, \{\alpha_g\}_{g \in G})$ is not necessarily unital, by Proposition 2.2.2, the induced action on the injective envelope $(\{I(A)_g\}_{g \in G}, \{I(\alpha)_g\}_{g \in G})$ always is. Thus, together with Theorem 5.2.6 the previous proposition shows that a (not necessarily unital) partial C*-dynamical system $(\{A_g\}_{g \in G}, \{\alpha_g\}_{g \in G})$ has the ideal intersection property if and only if $(\mathcal{C}^{I(A)}, \zeta^{I(A)})$ does.

Chapter 4

Partial Noncommutative Dynamics

4.1 Dynamical Conditions for Partial C*-dynamical Systems via Injective Envelopes

We introduce analogous notions of inner and properly outer automorphisms of C*-algebras in the setting of partial *-automorphisms. This approach follows ideas presented in [Kal69; Ham85].

4.1.1 Quasi-inner Partial *-Automorphisms

Definition 4.1.1. Let A be a C*-algebra and let B, C be ideals in A. Moreover, suppose $\alpha\colon B \to C$ is a *-isomorphism. Then, α is *inner* if there exists a partial isometry $w \in A$ such that $\alpha(x) = wxw^*$ for all $x \in B$. We call α *quasi-inner* if there exists $w \in I(A)$ such that $\alpha(x) = wxw^*$ for all $x \in B$.

Note that a similar notion for arbitrary C*-subalgebras was introduced recently in [EGL24]. The key difference is that instead of requiring $w \in I(A)$, the authors assume $w \in A^{**}$. Furthermore, the authors also require the inverse to be of the form $\alpha^{-1}(x) = w^*xw$. In the case of Definition 4.1.1 this assumption is not necessary, as we are working with ideals, which the next proposition shows.

Proposition 4.1.2. *Let A be a C*-algebra and let B, C be ideals in A. Suppose the *-isomorphism $\alpha\colon B \to C$ is quasi-inner, then α^{-1} is also quasi-inner and there exists a partial isometry $w \in I(A)$ such that $\alpha(x) = wxw^*$ for all $x \in B$ and $\alpha^{-1}(x) = w^*xw$ for $x \in C$.*

Proof. Let $w \in I(A)$ be a partial isometry such that $\alpha(x) = wxw^*$ for all $x \in B$ and let $p, q \in I(A)$ be central projections such that $I(B) = I(A)p$ and $I(C) = I(A)q$. Observe that we can extend α to a unique *-isomorphism $I(\alpha)\colon I(B) \to I(C)$ using injectivity and rigidity. Now note that by [Ham82b, Lemma 6.5] we can find a positive and increasing approximate identity $(u_\lambda)_\lambda$ for B, such that $p = \sup_\lambda u_\lambda$. But then, $w(\sup_\lambda u_\lambda)w^* = \sup_\lambda wu_\lambda w^*$ and in particular, $wpw^* \in I(C)$, since the latter is monotone complete. Hence, we see $wI(B)w^* \subseteq I(C)$. Since $I(\alpha)^{-1} \circ \mathrm{Ad}(w)\colon I(B) \to I(B)$ satisfies $I(\alpha)^{-1} \circ \mathrm{Ad}(w)|_B = \mathrm{id}_B$, $I(\alpha)^{-1} \circ \mathrm{Ad}(w) = \mathrm{id}_{I(B)}$ by rigidity. Hence, the extension is given by $I(\alpha)(x) = \mathrm{Ad}(w)(x) = wxw^*$. Furthermore, since α is a *-isomorphism, $\alpha(u_\lambda)_\lambda$ is a positive and increasing approximate identity for C and we also obtain $\sup_\lambda \alpha(u_\lambda) = q$. In particular, we see that $I(\alpha)(p) = q$ and thus, the extension is a unital *-isomorphism.

Next, note that we can without loss of generality assume that $ww^* = q$. To see this, observe that the partial isometry wp also implements the isomorphism and since the extension is unital, $I(\alpha)(p) = pww^* = q$. Using the fact that $I(C)$ is an ideal and w is a partial isometry, we obtain $qw = ww^*w = w \in I(C)$. Also, since $I(B), I(C)$ are ideals and $I(\alpha)$ is a *-isomorphism, $I(C) = I(\alpha)(I(B)) \subseteq I(B)$. Thus the map $\beta\colon I(C) \to I(B)\colon y \mapsto w^*yw$ is well-defined. Moreover, since $ww^* = q$, $I(\alpha)(\beta(y)) = y$ for all $y \in I(C)$. □

Remark 4.1.3. In the proof of Proposition 4.1.2 it is also shown that, if $\alpha\colon B \to C$ is inner, then the canonical extension $I(\alpha)\colon I(B) \to I(C)$ is also inner, since by rigidity $I(\alpha)(x) = wxw^*$ for $x \in I(B)$. Furthermore, this shows that $\alpha\colon B \to C$ is quasi-inner, if and only if its extension $I(\alpha)\colon I(B) \to I(C)$ is inner. This is in line with the corresponding result for automorphisms [Ham85, Theorem 7.4].

From the proof of the previous proposition we immediately obtain the following corollary, which shows that a partial *-automorphism being inner imposes a lot of structure.

Corollary 4.1.4. *Let A be a C*-algebra, let B, C be ideals in A and suppose the *-isomorphism $\alpha\colon B \to C$ is quasi-inner. Furthermore, let $I(B) = I(A)p$ and $I(C) = I(A)q$ for central projections $p, q \in I(A)$. Then, $p = q$ and $I(B) = I(C)$. In particular, the extension $I(\alpha)\colon I(B) \to I(C)$ is a *-automorphism.*

Proof. Let $w \in I(A)$ be a partial isometry such that $\alpha(x) = wxw^*$ for all $x \in B$. As we have seen in the proof of Proposition 4.1.2, the extension $I(\alpha)\colon I(B) \to I(C)\colon x \mapsto wxw^*$ is a unital *-isomorphism. Since p, q are central, this gives $q = I(\alpha)(p) = ww^*p \leq p$ as w is a partial isometry. Applying the same logic to the inverse $p = I(\alpha)^*(q) = w^*wq \leq q$, showing $p = q$. In particular, $I(B) = I(C)$. □

In the previous result, if B, C are already unital to begin with, this actually gives $B = C$. We can even go a little bit further, recalling the role of regular ideals and annihilators in injective envelopes.

Corollary 4.1.5. *Let A be a C*-algebra and let B, C be ideals in A. If there exists a quasi-inner *-isomorphism $\alpha\colon B \to C$, then $B^\perp = C^\perp$. In particular, if B and C are regular ideals, $B = C$.*

Proof. By Corollary 4.1.4 as α is quasi-inner, we have $I(B) = I(C) = I(A)p$ for some central projection $p \in Z(I(A))$. By [Ham82c, Lemma 1.1], $B^\perp = A(1-p) \cap A = C^\perp$. Now, this immediately gives $B^{\perp\perp} = C^{\perp\perp}$ and thus assuming B and C are regular, we obtain $C = B$. □

The next definition gives an analogue of pointwise inner and inner global group actions on C*-algebras, which will be crucial for understanding the obstruction in Section 5.3.1.

Definition 4.1.6. Let $(A, G, \{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system. We call the partial C*-dynamical system *pointwise (quasi)-inner* if α_g is (quasi)-inner for all $g \in G$. The partial C*-dynamical system is *inner* if there exists a partial *-representation $u\colon G \to A$ such that $\alpha_g = \mathrm{Ad}(u_g)$ for $g \in G$.

4.1.2 Properly Outer Partial *-Automorphisms

Since we are interested in an analogue of free actions for non-commutative partial actions, we are interested in partial *-automorphisms that do not impose as much structure as (quasi-)inner partial *-automorphisms. Hence, we intend to identify a maximal subalgebra on which the extension of a partial *-automorphism acts inner. This is possible in the injective case, since the injective envelope is a monotone complete C*-algebra. The proof is hence similar to the one of Kallman [Kal69] in the von Neumann algebra setting. We include it for completeness.

Proposition 4.1.7. *Let A be a C*-algebra and let $\alpha\colon B \to C$ be a partial *-automorphism between ideals $B, C \trianglelefteq A$. Then, there exists a maximal $I(\alpha)$-invariant projection $r \in Z(I(A))$ such that the extension $I(\alpha)\colon I(B) \to I(C)$ acts inner restricted to $I(B)r$.*

Proof. Let $I(B) = I(A)p$ and $I(C) = I(A)q$ for projections $p, q \in Z(I(A))$ and consider the set

$$S = \{s \in Z(I(A)) \mid s = s^2 = s^*,\ s \leq p,\ \text{and } I(\alpha)|_{I(B)s} \text{ is inner}\}.$$

We claim that $r = \bigvee_{s\in S} s$ satisfies the desired. To see this, note that if we could express $r = \sum_\lambda r_\lambda$, where r_λ are mutually orthogonal central projections such that $I(\alpha)$ acts inner on $I(A)r_\lambda$, then $I(\alpha)$ would be inner on the direct sum. Hence, it suffices to show $r = \sum_\lambda r_\lambda$ for an appropriate orthogonal family of projections. To show the existence of such a family, consider

$$T = \{\{r_\lambda\}_\lambda \mid \{r_\lambda\}_\lambda \text{ family of orthogonal central projections such that } r_\lambda \leq s \text{ for some } s \in S\}.$$

Then, $T \neq \varnothing$, since $\{0\} \in T$ and the union is an upper bound for any chain in T. Thus, by Zorn's lemma there exists a maximal family of orthogonal projections $\{r_\lambda\}_\lambda$ such that $r_\lambda \leq s$ for some $s \in S$. We claim $r = \sum_\lambda r_\lambda$. Since if not, then $r - \sum_\lambda r_\lambda \neq 0$ and $(r - \sum_\lambda r_\lambda)s \neq 0$ for some $s \in S$. However, $(r - \sum_\lambda r_\lambda)s$ is orthogonal to all r_λ and $(r - \sum_\lambda r_\lambda)s \leq s$, which contradicts the maximality of the family. Now, observe that since by definition $r_\lambda \leq s$ for some $s \in S$, $I(\alpha)$ is inner on $I(A)r$. This concludes the proof. □

Remark 4.1.8. Note that even though the proof of Proposition 4.1.2 is not directly applicable, if $\alpha\colon B \to C$ and $I(\alpha)$ are as above and $r \in Z(I(A))$ is a projection with $r \leq p$ such that $I(\alpha)|_{I(A)r}$ acts inner, we still have $I(\alpha)(r) = r$. To see this, let $w \in I(A)$ be a partial isometry such that $I(\alpha)(xr) = wxrw^*$ for all $x \in I(B)$. Furthermore, note that since $I(\alpha)$ is surjective, $I(\alpha)(I(A)r)$ is an ideal in $I(C)$ and in particular unital with unit $I(\alpha)(r)$. We would like to apply the same arguments as in Proposition 4.1.2 to this restriction. However, to overcome the fact that the image is only an ideal in $I(C)$ and not $I(A)$, we need to define the modified partial isometry $\tilde{w} = qw$. We have $I(\alpha)(xr) = qwxrw^* = \tilde{w}xr\tilde{w}^*$, and so the same arguments as in Proposition 4.1.2 and Corollary 4.1.4 show that $I(\alpha)(r) = r$. In particular, $I(\alpha)$ acts as a *-automorphism on the ideal $I(B)r$.

The next definition identifies partial *-automorphisms that do not admit an inner part.

Definition 4.1.9. Let A be a C*-algebra and let B, C be ideals in A and $\alpha\colon B \to C$ a *-isomorphism. Denote by $I(\alpha)\colon I(B) \to I(C)$ its extension to the injective envelope, where $I(B) = I(A)p$, and $I(C) = I(A)q$. We say that α is *properly outer* if for any projection $r \in Z(I(A))$ with $r \leq p$ the restriction $I(\alpha)|_{I(B)r}$ is not inner.

Remark 4.1.10. By Proposition 4.1.7 $\alpha\colon B \to C$ being properly outer is equivalent to the vanishing of the maximal projection $r \in Z(I(A))$ such that the restriction is inner.

Corollary 4.1.11. *Let A be a unital C*-algebra and let B, C be ideals in A. Furthermore, suppose $\alpha\colon B \to C$ is a *-isomorphism and let $I(\alpha)\colon I(B) \to I(C)$ be its extension to the injective envelope. Then there exists a projection $p \in I(A)$ such that $I(\alpha)$ acts properly outer on $I(B)p$ and inner on $I(B)(1-p)$.*

Another very useful characterization of a *-isomorphism being as far away from being inner as possible was introduced for von Neumann algebras in [Kal69] as the notion of freely acting. To motivate this notion first note that if a partial *-automorphism $\alpha\colon B \to C$ between unital ideals on a C*-algebra A is inner, then $\alpha(x)w = wx$ for all $x \in B$. This holds, since there exists a partial isometry $w \in A$ with $\alpha(x) = wxw^*$ and $ww^* = p = w^*w$, where p denotes the unit of $B = C$. In fact, in the injective setting we obtain an equivalent characterization of an inner partial *-automorphism using such an intertwining relation as seen in Proposition 4.1.12.

First, we establish some notation in case A is injective. Given $y \in A$, we denote by $z(y)$ its central support projection and by $\ell(y), r(y)$ its left and right support projection respectively. If $\ell(y) = r(y)$, we denote this by $s(y)$. Since A is injective, it is in particular monotone complete, and thus, these always exist. Also recall that two projections $p, q \in A$ are *Murray-von Neumann equivalent*, denoted $p \simeq q$, if there exists a partial isometry $w \in A$ such that $p = w^*w$ and $q = ww^*$. Similarly, we say q is *sub-equivalent* to p and write $q \preceq p$ if there exists a partial isometry such that w such that $w^*w = p$ and $ww^* \leq q$. The proof of the next proposition uses some ideas of [Kal69] and [CK72].

Proposition 4.1.12. *Let A be an injective C*-algebra and let B, C be unital ideals in A with $B = Ap$ and $C = Aq$. Furthermore, suppose that A is injective and let $\alpha\colon B \to C$ be a *-isomorphism. Then, α is inner if and only if there exists some $y \in A$ such that $z(y) \simeq q$ and $\alpha(x)y = yx$ for all $x \in B$.*

Proof. The forward direction of this is clear, since if $\alpha\colon B \to C$ is inner, we obtain the desired intertwining condition by setting $y = w$.

For the other direction, suppose there exists $y \in A$ such that $z(y) \simeq q$ and $\alpha(x)y = yx$. Note that since $qy = \alpha(p)y = yp$ we can assume $y \in B \cap C$ by instead looking at yp. Then, $\alpha(x)y = yx$, implies $y^*\alpha(x^*) = x^*y^*$ and thus $yy^*\alpha(x^*) = yx^*y^* = \alpha(x^*)yy^*$. Since α is surjective, this shows $yy^* \in Z(C)$. Similarly, $y^*yx^* = y^*\alpha(x^*)y = x^*y^*y$ and $y^*y \in Z(B)$. Now using this we obtain

$$(yy^*)^2 = y(y^*y)y^* = y^*y(yy^*) = y^*yy^*y = (y^*y)^2.$$

Hence, $y^*y = yy^*$ and $yy^* \in Z(B)$ as well. Now, consider the polar decomposition of $y = w|y|$ for some partial isometry $w \in C$. Since $z(y) \simeq q$, we have $ww^* = q$. Moreover,

since $yy^* \in Z(B) \cap Z(C)$, $|y| \in Z(B) \cap Z(C)$. We obtain

$$\alpha(x)w|y| = \alpha(x)y = yx = w|y|x = wx|y|.$$

Since $s(|y|) = s(yy^*) = z(yy^*) = z(y) \simeq q$, this gives $\alpha(x)w = wx$ for all $x \in B$ and hence $\alpha(x) = wxw^*$ as desired. □

Note that if $\alpha\colon B \to C$ is inner on Br for a central projection $r \leq q$, then the condition $z(y) \simeq p$ in the previous proposition can be replaced with $z(y) \simeq r$. Hence, the following definition naturally covers the lack of an inner part.

Definition 4.1.13. Let A be a C*-algebra and let B, C be ideals in A. Furthermore, suppose $\alpha\colon B \to C$ is a partial *-automorphism. We say that α is *freely acting* if $\alpha(x)y = yx$ for all $x \in B$ for some $y \in B$ implies that $y = 0$.

Remark 4.1.14. The intertwining condition in the definition of a freely acting partial *-automorphism is a condition on the intersection of B and C. To illustrate this, assume that A is injective, B, C are unital and that α is unital. Observe that $\alpha(x)y = yx$, especially implies $\alpha(p)y = y = y$. Since α is unital, this gives $qy = y$ and $y \in B \cap C$. In particular, if $B \cap C = \{0\}$, then α is freely acting. This is in stark contrast to $B = C$ if α is inner.

In the case of A being injective and the ideals being unital (i.e., for extensions of partial *-automorphisms to the injective envelope), proper outerness is equivalent to the partial *-automorphism acting freely as the next proposition shows. The proof is again similar to the one of Kallman in the von Neumann algebra setting [Kal69].

Proposition 4.1.15. *Let A be an injective C*-algebra and let B, C be unital ideals in A with $B = Ap$ and $C = Aq$. Furthermore, suppose that $\alpha\colon B \to C$ is a unital *-isomorphism. Then, α is properly outer if and only if it acts freely.*

Proof. Suppose that α is not properly outer. By definition there exists a central projection $r \in A$ with $r \leq p$ such that α acts inner on Br. By the discussion after Proposition 4.1.12, there exists $y \in A$ such that $z(y) \simeq r$ and $\alpha(x)y = yx$ for all $x \in B$, i.e., α is not freely acting.

Conversely, suppose that α is not freely acting. Then, there exists $y \in B \cap C \setminus \{0\}$ such that $\alpha(x)y = yx$ for all $x \in B$. However, by Proposition 4.1.12 this implies that α acts inner on $Bz(y)$ and thus, α is not properly outer. □

Similar to the global C*-dynamical system setting, we can define proper outerness for partial C*-dynamical systems.

Definition 4.1.16. Let $(A, G, \{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system. We call the partial C*-dynamical system *properly outer* if each α_g for $g \neq e$ is properly outer.

By definition of proper outerness for partial C*-dynamical systems, we immediately obtain the following characterization analogously to the global C*-dynamical system case.

Corollary 4.1.17. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system. Then, $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is properly outer if and only if its extension $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ is properly outer.*

By Corollary 4.1.11 given a partial C*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$, each partial *-automorphism $I(\alpha)_g$ on the induced system on the injective envelope $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ decomposes into an inner and properly outer part. Similarly, since the G-injective envelope $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha_g\}_{g\in G})$ is in particular injective, the partial *-automorphisms $I_G(\alpha)_g$ also decompose for the same reason. The next theorem gives a relation between these decompositions. The proof is similar to the global C*-dynamical system setting as proven in [KS19, Theorem 2.14]. Beforehand, we prove a useful lemma, which gives a dynamical relation between the inner parts of the action in the injective setting.

Lemma 4.1.18. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial action on an injective C*-algebra with $A_g = Ap_g$ for central projections $p_g \in A$ for $g \in G$. Furthermore, for $g \in G$, let $q_{g^{-1}} \leq p_{g^{-1}}$ denote the largest central projection such that α_g acts inner on $Aq_{g^{-1}} \subseteq A_{g^{-1}}$ and assume $w_g \in Aq_{g^{-1}}$ is a partial isometry such that $\alpha_g(x) = w_g x w_g^*$ for $x \in Aq_{g^{-1}}$. Then, for $h \in G$, $\alpha_h(p_{h^{-1}}w_g)$ is a partial isometry such that $\alpha_{hgh^{-1}}(x) = \alpha_h(w_g p_{h^{-1}})x\alpha_h(w_g p_{h^{-1}})$ for all $x \in A_h \cap A_{hg^{-1}} \cap Aq_{g^{-1}}$. Moreover, $\alpha_h(q_g p_{h^{-1}}) = q_{hgh^{-1}}p_h$.*

Proof. Let $g \in G$ be fixed. Observe that since α_g acts inner on $Aq_{g^{-1}}$, we have $w_g b = w_g b q_{g^{-1}} = \alpha_g(bq_{g^{-1}})w_g$. In particular, for $h \in G$ if $b \in \alpha_h(Aq_{g^{-1}}p_{h^{-1}})$, $w_g\alpha_{h^{-1}}(b) = \alpha_g(q_{g^{-1}}\alpha_{h^{-1}}(b))w_g = \alpha_{gh^{-1}}(\alpha_h(q_{g^{-1}}p_{h^{-1}})b)w_g = \alpha_{gh^{-1}}(b)w_g$. Since $\alpha_{h^{-1}}(b) \in A_{h^{-1}} \cap A_{h^{-1}g}$, applying α_h leads to

$$\alpha_h(w_g p_{h^{-1}})b = \alpha_{hgh^{-1}}(b)\alpha_h(w_g p_{h^{-1}}).$$

Note that if $A_{h^{-1}} \cap Aq_{g^{-1}} \neq \{0\}$, then $\alpha_h(w_g p_{h^{-1}})^*\alpha(w_g p_{h^{-1}}) = \alpha_h(q_{g^{-1}}p_{h^{-1}}) \neq 0$. So $\alpha_h(w_g p_{h^{-1}})$ is a partial isometry such that $\alpha_{hgh^{-1}}$ is inner on $A\alpha_{h^{-1}}(q_{g^{-1}}p_{h^{-1}})$. Hence we see that $\alpha_h(q_{g^{-1}}p_{h^{-1}}) \leq q_{hg^{-1}h^{-1}}$. This gives $\alpha_h(q_{g^{-1}}p_{h^{-1}}) \leq q_{hg^{-1}h^{-1}}p_h$ and applying $\alpha_{h^{-1}}$

as well as the same reasoning as before leads to $q_{g^{-1}}p_{h^{-1}} \leq \alpha_{h^{-1}}(q_{hg^{-1}h^{-1}}p_h) \leq q_{g^{-1}}p_{h^{-1}}$. In sum, $\alpha_{h^{-1}}(q_{hg^{-1}h^{-1}}p_h) = q_{g^{-1}}p_{h^{-1}}$. □

In particular, the previous lemma shows that we can associate a pointwise inner partial action to a unital partial action on an injective C*-algebra. This will be important for the characterization of the ideal intersection property and thus will be highlighted in the definition below.

Definition 4.1.19. Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system on a C*-algebra A and for all $g \in G$ let $q_{g^{-1}} \in Z(I(A))$ denote the largest projection such that $I(\alpha)_g$ is inner. Furthermore, for all $g \in G$ suppose $w_g \in I(A)q_{g^{-1}}$ is a partial isometry implementing $I(\alpha)_g$ on $I(A)q_{g^{-1}}$. The *associated pointwise inner* unital partial C*-dynamical system is given by the partial C*-dynamical system $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ where $B_g = I(A)q_g$ and $\beta_g = \mathrm{Ad}(w_g) = I(\alpha)_g|_{I(A)q_{g^{-1}}}$.

Note that since $\beta_g = \mathrm{Ad}(w_g) = I(\alpha)_g|_{I(A)q_{g^{-1}}}$, the definition of the associated pointwise inner action does not depend on the choice of partial isometries. Besides identifying this associated pointwise inner unital partial C*-dynamical system, Lemma 4.1.18 allows us to relate the inner parts of automorphisms on $I_G(A)$ to the inner parts on $I(A)$.

Theorem 4.1.20. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial action and consider its induced action $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ as well as the action on its G-injective envelope $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$. Let $q_{g^{-1}} \in Z(I(A))$ denote the largest projection such that $I(\alpha)_g$ acts inner on $I(A)q_{g^{-1}}$ and $r_{g^{-1}} \in Z(I_G(A))$ denote the largest projection such that $I_G(\alpha)_g$ acts inner on $I_G(A)r_{g^{-1}}$. Then, $r_g \leq q_g$.*

Proof. Let v_g, w_g denote the partial isometries implementing $I(\alpha)_g$ and $I_G(\alpha)_g$ on $I(A)q_{g^{-1}}$ and $I_G(A)r_{g^{-1}}$ respectively and let p_g denote the unit of A_g. Furthermore, we can assume that $v_g v_g^* = q_g = v_g^* v_g = q_{g^{-1}}$ and $w_g w_g^* = r_g = w_g^* w_g = r_{g^{-1}}$. We will show that $w_g(1 - q_{g^{-1}}) = 0$, since then $r_{g^{-1}} = w_g^* w_g = w_g^* w_g q_{g^{-1}} = r_{g^{-1}} q_{g^{-1}}$.

To see this note that since $I(A)$ is injective and $I(A) \subseteq I_G(A)$, there exists a u.c.p. map $\phi\colon I_G(A) \to I(A)$ with $\phi|_{I(A)} = \mathrm{id}_{I(A)}$. In particular, since $I(A)_g = I(A)p_g$ and $I_G(A)_g = I_G(A)p_g$, ϕ is G-unital. Now define

$$\psi\colon I_G(A) \to p\ell^\infty(G, I(A))p\colon a \mapsto [t \mapsto \phi(I_G(\alpha)_{t^{-1}}(p_t a))],$$

where $p(t) = p_{t^{-1}}$ and we view $B = p\ell^\infty(G, I(A))p$ as the partial C*-dynamical system $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ described in Eq. (3.1). Since ϕ is G-unital, $\phi(I_G(A)_g) \subseteq B_g$ and ψ is G-unital. Furthermore, by similar arguments as in Proposition 3.2.2, $\beta_g \circ \psi(a) = \psi \circ I_G(\alpha)_g(a)$ for all $a \in I_G(A)_{g^{-1}}$. Thus, ψ is a G-morphism. By construction ψ is a complete isometry on $I(A)$ and since $I_G(I(A)) = I_G(A)$, ψ itself is completely isometric by G-essentiality. Hence, it suffices to show $\psi(w_g(1 - q_{g^{-1}})) = 0$. Using that $I_G(\alpha)_{t^{-1}}(p_t w_g)$ implements $I_G(\alpha)_{tgt^{-1}}$ for all $t \in G$ from Lemma 4.1.18 as well as the fact that $I(A)$ is contained in the multiplicative domain of ϕ, observe for $t \in G$ and $a \in I(A)_t \cap I(A)_{tg^{-1}} \cap I(A)q_{g^{-1}}$

$$\begin{aligned} a\psi(w_g)(t) &= \phi(aI_G(\alpha)_{t^{-1}}(p_t w_g)) \\ &= \phi(I_G(\alpha)_{t^{-1}}(p_t w_g) I_G(\alpha)_{tgt^{-1}}(a)) \\ &= \psi(w_g)(t) I(\alpha)_{tgt^{-1}}(a), \end{aligned}$$

where in the last equality we used that $I_G(\alpha)$ extends $I(\alpha)$. This shows that $\psi(w_g)(t)$ implements $I(\alpha)_{tgt^{-1}}$ and in particular, since the embedding of $I(A)$ into $p\ell^\infty(G, I(A))p$ satisfies $\iota(q_g)(t) = I(\alpha)_t(p_{t^{-1}}q_g) = q_{tgt^{-1}}p_t$, this shows $\psi(w_g)(t) \in I(A)\iota(q_g)(t)$. But then again using multiplicative domain arguments and the fact that $I_G(\alpha)$ extends $I(\alpha)$:

$$\begin{aligned} 0 &= \psi(w_g)(t)(1 - I(\alpha)_{t^{-1}}(p_t q_g)) = \psi(w_g)(t)(p_{t^{-1}} - I(\alpha)_{t^{-1}}(p_t q_g)) \\ &= \phi(I_G(\alpha)_{t^{-1}}(p_t w_g(1 - p_t q_g)) = \psi(w_g(1 - q_g))(t) \end{aligned}$$

for all $t \in G$. Since ψ is completely isometric, this concludes the proof. □

We immediately obtain the following.

Corollary 4.1.21. *Suppose the partial C*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is properly outer. Then, $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$ is properly outer as well.*

We end this section by highlighting the connection between proper outerness for partial actions and proper outerness for the enveloping action. This result is in line with [KM18, Proposition 6.8], where the authors show the equivalence of Kishimoto's condition, which corresponds to proper outerness for global C*-dynamical systems, for Hilbert bimodules and their dense Morita coverings.

Proposition 4.1.22. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system and let $(\mathcal{C}^A, \zeta^A)$ denote its enveloping action. The partial C*-dynamical system is properly outer if and only if $(\mathcal{C}^A, \zeta^A)$ is. Furthermore, the G-injective partial C*-dynamical system $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$ is properly outer, if and only if $(I_G(\mathcal{C}^A), I_G(\zeta^A))$ is.*

Proof. First, suppose $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is not properly outer, then there exists $g \in G$ and a central non-zero projection $q_{g^{-1}} \in I(A)$ such that $I(\alpha)_g(xq_{g^{-1}}) = u_g x u_g^*$ for some partial isometry $u_g \in I(A)_g$ with $u_g u_g^* = q_{g^{-1}}$. Since $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ is a restriction of $(I(\mathcal{C}^A), I(\zeta^A))$ this implies in particular, that $I(\zeta^A)_g(xq_{g^{-1}}) = u_g x u_g^*$ for all $x \in I(A)$. Hence, $I(\zeta^A)_g$ is not properly outer giving that $(I(\mathcal{C}^A), I(\zeta^A))$ is not properly outer. The argument can be adapted verbatim to show the desired result for $(I_G(\mathcal{C}^A), I_G(\zeta^A))$.

Conversely, suppose $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is properly outer and let $r_g \in I(\mathcal{C}^A)$ be the maximal central projection such that $I(\zeta^A)_g(xr_g)$ is inner. Then, since $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is properly outer, $r_g p_s = 0$ for all $s \in G \setminus \{e\}$. If $0 \neq r_g$, by Lemma 3.4.3 there exists $s \in G$ such that $I(\zeta^A)_s(r_g)1_A \neq 0$. But then by [KS19, Theorem 2.14] $r_{sgs^{-1}}1_A \neq 0$. However, then there exists $t \in G$ such that

$$\begin{aligned} 0 \neq I(\zeta^A)_t(r_{sgs^{-1}}1_A)1_A &= I(\zeta^A)_t(r_{sgs^{-1}}I(\zeta^A)_{t^{-1}}(1_A)1_A)1_A \\ &= I(\zeta^A)_t(r_{sgs^{-1}}p_{t^{-1}})1_A = I(\alpha)_t(r_{sgs^{-1}}p_{t^{-1}}), \end{aligned}$$

where we used that $I(\zeta^A)_{t^{-1}}(1_A)1_A$ denotes the unit of $I(A)_t$. This is a contradiction. Hence, $r_g = 0$ for all $g \neq e$ and $(\mathcal{C}^A, \zeta^A)$ is properly outer. The same argument using $(I_G(\mathcal{C}^A), I_G(\zeta^A))$ gives that, if $(\{I_G(A)_{g_g}\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$ is properly outer, so is $(I_G(\mathcal{C}^A), I_G(\zeta^A))$. □

4.1.3 Connection to Other Non-Triviality Conditions

Over the years, there have been many definitions for non-triviality conditions for (partial) *-automorphisms as non-commutative analogues of freeness. For a thorough overview for ordinary *-automorphisms see [OP82]. A generalization of many of these notions to partial actions and even Fell bundels is given in [KM18].

In this thesis we introduce a generalization of proper outerness based on a result by Hamana. More specifically, [Ham85] gave an equivalent characterization of a non-triviality condition by Kishimoto in the group setting in terms of the injective envelope. To study the connection of our proposed condition to other generalizations in the partial setting we first prove a characterization of proper outerness using particular u.c.p. maps first introduced by Pitts [Pit17].

Definition 4.1.23. Let $A \subseteq B$ be an inclusion of unital C*-algebras. A *pseudo-expectation* for the inclusion $A \subseteq B$ is a u.c.p. map $\phi\colon B \to I(A)$ such that $\phi|_A = \mathrm{id}_A$.

Remark 4.1.24. As implied by the name, pseudo-expectations can be viewed as a generalization of conditional expectations, i.e., a u.c.p. map $E\colon B \to A$ such that $E|_A = \mathrm{id}_A$. Note that this coincides with the notion of a conditional expectation as an A-bimodule u.c.p. map, as $E|_A = \mathrm{id}_A$ implies that A is in the multiplicative domain of E. By enlarging the codomain, conditional expectations can easily be seen as pseudo-expectations. The key benefit of allowing for an injective codomain however, is the fact that pseudo-expectations always exist.

For the case of partial C*-dynamical systems, there always exists a canonical pseudo-expectation stemming from the conditional expectation $E\colon A \rtimes_{\mathrm{r}} G \to A$. In the next result, we show that proper outerness of a C*-dynamical system is equivalent to uniqueness of pseudo-expectation, i.e., the only pseudo-expectation is the canonical one. This is a generalization of the analogous result for global C*-dynamical systems by [Zar19] and the proof follows similarly. We provide the details for convenience. First, we observe a quick lemma.

Lemma 4.1.25. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system. There is a one-to-one correspondence between pseudo-expectations $\phi\colon A \rtimes_{\mathrm{r}} G \to I(A)$ and conditional expectations $\Phi\colon I(A) \rtimes_{\mathrm{r}} G \to I(A)$.*

Proof. Clearly, if $\Phi\colon I(A) \rtimes_{\mathrm{r}} G \to I(A)$ is a conditional expectation, its restriction to $A \rtimes_{\mathrm{r}} G$ gives rise to a pseudo-expectation. Conversely, if $\phi\colon A \rtimes_{\mathrm{r}} G \to I(A)$ is a pseudo-expectation, then by injectivity this extends to a u.c.p. map $\Phi\colon I(A) \rtimes_{\mathrm{r}} G \to I(A)$ with $\Phi|_A = \mathrm{id}_A$. It remains to show that $\Phi|_{I(A)} = \mathrm{id}_{I(A)}$. This follows since $I(A)$ is in particular rigid. This also shows that the extension is unique. □

Using this lemma, we can now prove the equivalence of proper outerness and admitting a unique pseudo-expectation utilizing the injective envelope.

Proposition 4.1.26. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system. The partial C*-dynamical system is properly outer if and only if it admits a unique pseudo-expectation.*

Proof. First, observe that using the one-to-one correspondence given in Lemma 4.1.25 and Corollary 4.1.17, it suffices to show that $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ admits a unique pseudo-expectation if and only if $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ is properly outer. Let p_g denote the unit of $I(A)_g$ for $g \in G$ and consider the partial isometries v_g corresponding to the left-regular covariant pair. In particular, $v_g v_g^* = p_g$.

Now, suppose $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ is properly outer, and consider $\phi\colon I(A) \rtimes_{\mathrm{r}} G \to I(A)$ with $\phi|_{I(A)} = \mathrm{id}_{I(A)}$. For $g \in G, a \in I(A)$, we have $v_g a v_g^* = I(\alpha)_g(a p_{g^{-1}})$

and thus $v_g a p_{g^{-1}} = I(\alpha)_g(a p_{g^{-1}}) p_g v_g$. Applying ϕ and using that $I(A) \subseteq \mathrm{Dom}_\phi$, we see $\phi(v_g) a p_{g^{-1}} = I(\alpha)_g(a p_{g^{-1}})\phi(v_g)$. But, since $I(\alpha)_g$ is freely acting for $g \neq e$ by assumption, $0 = p_g \phi(v_g) = \phi(v_g)$. In particular, ϕ is the canonical conditional expectation for $I(A) \rtimes_{\mathrm{r}} G$ showing the desired.

Conversely, suppose $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ is not properly outer. Then, there exists $g \neq e$ such that $I(\alpha)_g$ is not freely acting, i.e., there exists a $y \in I(A)_g \cap I(A)_{g^{-1}} \setminus \{0\}$ such that $I(\alpha)_g(x)y = yx$ for all $x \in I(A)_{g^{-1}}$. Now, let $E\colon I(A) \rtimes_{\mathrm{r}} G \to I(A)$ denote the canonical conditional expectation and define $\phi(x) = E(x v_{g^{-1}})y$. In particular, $\phi(v_g) = y \neq 0$. Observe that as E is a $I(A)$-bimodule map, we have $\phi(ax) = a\phi(x)$ for $a, x \in I(A)$. Furthermore, using that $p_{g^{-1}}$ is central in $I(A)$, we see

$$\begin{aligned}\phi(xa) &= E(xa v_{g^{-1}})y = E(x p_{g^{-1}} a v_{g^{-1}})y = E(x v_g^* v_g a v_{g^{-1}})y \\ &= E(x v_{g^{-1}}) I(\alpha)_g(a p_{g^{-1}})y = E(x v_{g^{-1}}) y a p_{g^{-1}} = E(x v_{g^{-1}}) p_{g^{-1}} y a = \phi(x)a,\end{aligned}$$

showing that ϕ is in fact also an $I(A)$-bimodule map. Now, since E is u.c.p., ϕ is completely bounded and so using a result of Wittstock [Wit81], we can write ϕ as a linear combination of four completely positive $I(A)$-bimodule maps. As $\phi(v_g) \neq 0$, one of them must be non-zero for v_g. By a slight abuse of notation, denote this completely positive $I(A)$-bimodule map again by ϕ. By [Zar19, Lemma 3.4] there exists a u.c.p. $I(A)$-bimodule map $\tilde{\phi}\colon I(A) \rtimes_{\mathrm{r}} G \to I(A)$ such that $\phi(x) = \phi(1)\tilde{\phi}(x)$. But then $\tilde{\phi}$ is a u.c.p. $I(A)$-bimodular map with $\tilde{\phi}(v_g) \neq 0$ and $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ does not admit a unique pseudo-expectation. □

In the remainder of this subsection, we use the previous result to establish a connection with aperiodicity, which Kwaśniewski and Meyer identify as a suitable notion to study dynamical phenomena [KM18; KM22].

Definition 4.1.27. A partial C*-dynamical system is called *aperiodic* if for every $g \neq e$, every $a \in A_g$, and every hereditary C*-subalgebra $B \subseteq A$, we have

$$\inf\{\, \|x(av_g)x\| \mid x \in B, x \geq 0, \|x\| = 1\} = 0.$$

As shown by [GS14, Theorem 3.12] aperiodicity guarantees the ideal intersection property using techniques similar to [ELQ02], where they show that topologically free partial C*-dynamical systems on abelian C*-algebras have the ideal intersection property. Giordano and Sierakowski's result was generalized to Fell bundles in [KM18] using a globalization approach. Later on, the authors connected aperiodicity to pseudo-expectations in [KM22]. The next result summarizes its connection to proper outerness, using their result for pseudo-expectations.

Proposition 4.1.28. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system. If $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is aperiodic, then it is properly outer. Furthermore, if A contains an essential ideal that is separable or Type I and G is countable, then the converse is also true.*

Proof. The first point follows, since by [KM22, Theorem 3.6] aperiodicity implies uniqueness of pseudo-expectation. As this is equivalent to proper outerness by Proposition 4.1.26, this gives the desired. The additional assertion follows from [KM22, Theorem 7.2], as they show that in this setting aperiodicity is actually equivalent to uniqueness of pseudo-expectation. □

The previous result gives some first insight on the question whether proper outerness and aperiodicity are equivalent. In the case of ordinary C*-dynamical systems, the two notions coincide. Furthermore, Kwaśniewski and Meyer conjecture that aperiodicity for more general inclusions into crossed products for inverse semigroup actions is in fact also equivalent to uniqueness of pseudo-expectation [KM22, Remark 7.5], which would give the equivalence of the two conditions.

4.2 Quasistabilizer Subgroups

For global commutative C*-dynamical systems the point-stabilizers have shown to be crucial in the study of the ideal intersection property (see for example [Kaw17]). A non-commutative analogue has been introduced in [KS19]. In this section we will give a generalization to partial C*-dynamical systems. To do so we review the notions of Glimm and pseudo-Glimm ideals given in [KS19]. Their proofs involving the G-injective envelope only make use of the inclusions $A \subseteq I(A) \subseteq I_G(A)$ and $Z(A) \subseteq Z(I(A)) \subseteq Z(I_G(A))$ as well as the fact that $I_G(A)$ is ordinarily injective. Thus, the arguments translate verbatim to our setting.

Let A be a unital C*-algebra. We call the spectrum of $Z(A)$ the Glimm space of A, denoted $\mathrm{Glimm}(A)$. An ideal $I \trianglelefteq A$ is called a *Glimm ideal* if there exists an $x \in \mathrm{Glimm}(A)$ such that $I = AK_x$, where $K_x \trianglelefteq Z(A)$ is the maximal ideal given by $K_x := \{f \in Z(A) \mid f(x) = 0\}$. We will often refer to I itself as $x \in \mathrm{Glimm}(A)$. Since $Z(A) \hookrightarrow Z(I(A)) \hookrightarrow Z(I_G(A))$, we obtain surjections $\mathrm{Glimm}(I_G(A)) \twoheadrightarrow \mathrm{Glimm}(I(A)) \twoheadrightarrow \mathrm{Glimm}(A)$. Recall that an ideal $I \trianglelefteq A$ is called *primal*, if for any $n \geq 2$ and $J_1, \ldots, J_n \trianglelefteq A$ with $J_1 J_2 \cdots J_n = \{0\}$, there exists some $1 \leq i \leq n$ such that $J_i \subseteq I$. Also recall that an ideal $\{0\} \neq J \trianglelefteq A$ is *proper* if $J \neq A$. We call an ideal $I \trianglelefteq A$ a *pseudo-Glimm ideal* if there exists a proper primal ideal $J \trianglelefteq I(A)$ with $J \cap A = I$. The space of pseudo-Glimm ideals of A is denoted by $\mathrm{Glimm}_{\mathrm{p}}(A)$. Now consider the space $\mathrm{Id}(A)$ of ideals of

A equipped with the *Fell topology*, where a net $(I_\lambda)_\lambda$ in $\mathrm{Id}(A)$ converges to $I \in \mathrm{Id}(A)$ if the corresponding quotient norms converge, i.e., $\lim_\lambda \|a + I_\lambda\| = \|a + I\|$ for all $a \in A$. Then, $\mathrm{Glimm_p}(A) \subseteq \mathrm{Id}(A)$ is compact. By [KS19, Proposition 3.8] any Glimm ideal of $I(A)$ and $I_G(A)$ is proper and primal and thus gives rise to a pseudo-Glimm ideal. Specifically, for $x \in \mathrm{Glimm}(I(A))$ and $y \in \mathrm{Glimm}(I_G(A))$, $x \cap A, y \cap A \in \mathrm{Glimm_p}(A)$. Furthermore, if A is already injective, any Glimm ideal is a pseudo-Glimm ideal. For general A, we still obtain that any $z \in \mathrm{Glimm}(A)$ is contained in a pseudo-Glimm ideal and every pseudo-Glimm ideal contains a Glimm ideal.

Definition 4.2.1. Given a partial C*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$, let $q_{g^{-1}} \in Z(I(A))$ denote the largest $I(\alpha)_g$-invariant projection such that $I(\alpha)_g$ acts inner on $I(A)q_{g^{-1}}$ for $g \in G$ and let $I \in \mathrm{Glimm_p}(A)$ be a pseudo-Glimm ideal. We define the *quasistabilizer subgroup* $G_I \subseteq G$ of I as

$$G_I = \{g \in G \,|\, 1 - q_{g^{-1}} \in J \text{ for every proper primal ideal } J \trianglelefteq I(A) \text{ with } I = J \cap A\}.$$

Remark 4.2.2. Note that for a unital C*-algebra A, if $J \trianglelefteq A$ is proper and primal, then for any central projection $p \in Z(A)$ either $p \in J$ or $1 - p \in J$. Since J is proper, it clearly cannot contain both. Additionally, by primality and since $Ap, A(1-p) \trianglelefteq A$ with $Ap \cdot A(1-p) = \{0\}$, J must contain one of them.

The next result shows that the terminology is warranted.

Proposition 4.2.3. *Given a partial C*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ and a pseudo-Glimm ideal $I \in \mathrm{Glimm_p}(A)$, the quasistabilizer subgroup G_I is a subgroup of G.*

Proof. For $g \in G$ let $q_{g^{-1}}$ denote the maximal projection in $Z(I(A))$ such that $I(\alpha)_g$ acts inner. Furthermore, let $w_g \in I(A)q_{g^{-1}}$ be a partial isometry such that $I(\alpha)_g(x) = w_g x w_g^*$ for $x \in Aq_{g^{-1}}$. Clearly, $e \in G_I$ and by Proposition 4.1.2, the inverse of $I(\alpha)_g|_{I(A)q_{g^{-1}}}$ is also inner and $I(A)q_{g^{-1}} = I(A)q_g$. In particular, we have $q_{g^{-1}} = q_g$. Thus, if $s \in G_I$, then $s^{-1} \in G_I$. Now let $s, t \in G$. To show $st \in G_I$ it suffices to show $q_{s^{-1}}q_{t^{-1}} \leq q_{(st)^{-1}}$, since then

$$1 - q_{(st)^{-1}} \leq 1 - q_{s^{-1}}q_{t^{-1}} = -(1 - q_{s^{-1}})(1 - q_{t^{-1}}) + (1 - q_{s^{-1}}) + (1 - q_{t^{-1}}) =: x.$$

However, since $(1 - q_{s^{-1}}), (1 - q_{t^{-1}}) \in J$ for all $J \trianglelefteq I(A)$ proper and primal with $I = J \cap A$, $x \in J$. Ideals are in particular hereditary, so this gives $1 - q_{(st)^{-1}} \in J$, which shows $st \in G_I$. To see the claim, observe that for $b \in I(A)q_{s^{-1}}q_{t^{-1}} \subseteq I(A)_{s^{-1}} \cap I(A)_{t^{-1}}$, $I(\alpha)_t(b) = I(\alpha)_t(bq_{s^{-1}}q_{t^{-1}}) = I(\alpha)_t(b)q_{s^{-1}}$, since $q_{s^{-1}}$ is central and $I(\alpha)_t$ is inner on

$I(A)q_{t^{-1}}$. Thus, $I(\alpha)_t(b) \in I(A)q_{s^{-1}}$ and $I(\alpha)_s(I(\alpha)_t(b)) = w_s w_t b (w_s w_t)^*$. Hence, it suffices to show that $I(\alpha)_s(I(\alpha)_t(b)) = I(\alpha)_{st}(b)$ to see that $I(\alpha)_{st}$ is inner on $I(A)q_{s^{-1}}q_{t^{-1}}$. This holds because $I(\alpha)_t(b) \in I(A)_t \cap I(A)_{s^{-1}}$ and thus $b \in I(A)_{t^{-1}} \cap I(A)_{t^{-1}s^{-1}}$. In sum, $I(\alpha)_{st}$ is inner on $I(A)q_{s^{-1}}q_{t^{-1}}$ and $q_{s^{-1}}q_{t^{-1}} \leq q_{(st)^{-1}}$. □

The next two results are analogous to the case of global C*-dynamical systems [KS19, Lemma 4.3]. We include the arguments for convenience.

Lemma 4.2.4. *Suppose $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is a unital partial C*-dynamical system on an injective C*-algebra A. Let $q_{g^{-1}} \in Z(A)$ denote the largest projection such that α_g is inner on $Aq_{g^{-1}}$ and let $J \trianglelefteq A$ be proper and primal. If $J = AK_y$ for $y \in \mathrm{Glimm}(A)$, we obtain the following description for the quasistabilizer subgroup*

$$G_y := G_{AK_y} = \{s \in G \,|\, y \in \mathrm{supp}(q_{s^{-1}})\}.$$

Proof. First, by Remark 4.2.2, J contains either $q_{g^{-1}}$ or $1 - q_{g^{-1}}$ as J is proper and primal. Furthermore, by [AS90, Lemma 2.2] there exists a unique $y \in \mathrm{Glimm}(A)$ such that $AK_y \subseteq J$. Since $y \in \mathrm{Glimm}(A)$ is in particular a pseudo-Glimm ideal and contained in any proper and primal ideal J with $J \cap A = AK_y$, $1 - q_{g^{-1}} \in J$ if and only if $1 - q_{g^{-1}} \in AK_y$, which is equivalent to $y \in \mathrm{supp}(q_{g^{-1}})$. □

Recall that for a global group action on a compact Hausdorff space (G, X, θ), θ_g is free if and only if $g \notin G_x$ for all $x \in X$, where $G_x = \{h \in G \,|\, \theta_h(x) = x\}$ denotes the ordinary point stabilizer. The next result can be seen as a non-commutative analogue of this for partial C*-dynamical systems.

Proposition 4.2.5. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system and fix $g \in G$. The partial *-automorphism α_g is properly outer, if and only if $g \notin G_I$ for any $I \in \mathrm{Glimm_p}(A)$.*

Proof. Take $I \in \mathrm{Glimm_p}(A)$ and let $J \trianglelefteq I(A)$ be a proper and primal ideal such that $I = J \cap A$. By definition, if α_g is properly outer, the maximal projection $q_{q^{-1}} \in Z(I(A))$ such that $I(\alpha)_g$ is inner vanishes. Since J is proper, $1 - q_{g^{-1}} = 1 \notin J$ and thus $g \notin G_I$.

Conversely, assume $g \notin G_I$ for all $I \in \mathrm{Glimm_p}(A)$ and let $q_{g^{-1}} \in Z(I(A))$ denote the maximal projection such that $I(\alpha)_g$ is inner. In particular, since Glimm ideals in $I(A)$ give rise to pseudo-Glimm ideals, for any $y \in \mathrm{Glimm}(I(A))$ there exists a proper primal ideal $J \trianglelefteq I(A)$ with $I(A)K_y \cap A = J \cap A$ such that $1 - q_{g^{-1}} \notin J$. By Remark 4.2.2, $q_{g^{-1}} \in J$. In summary, $I(A)q_{g^{-1}} \cap A \subseteq J \cap A = I(A)K_y \cap A$ for all $y \in \mathrm{Glimm}(A)$. However, by the definition of K_y, we see $\cap_{y\in \mathrm{Glimm}(I(A))} I(A)K_y = \{0\}$ and thus $I(A)q_{g^{-1}} \cap A = \{0\}$.

However, $I(I(A)q_{g^{-1}} \cap A) = I(A)q_{g^{-1}} = \{0\}$, which shows $q_{g^{-1}} = 0$, i.e., α_g is properly outer. □

Theorem 4.2.6. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C^*-dynamical system with $A_g = Ap_g$. For $z \in \mathrm{Glimm}(I_G(A))$ the quasistabilizer subgroup $G_z := G_{I_G(A)K_z}$ is amenable.*

Proof. Since G is discrete, $\ell^\infty(G_z) \subseteq \ell^\infty(G)$, so it suffices to construct a G_z-invariant state on $\ell^\infty(G)$. Fix $g \in G_z$ and let $r_{g^{-1}} \in Z(I_G(A))$ denote the largest projection such that $I_G(\alpha)_g$ is inner on $I_G(A)r_{g^{-1}}$. By Lemma 4.2.4, $r_{g^{-1}} \in \mathrm{supp}(z)$ or equivalently $1 - r_{g^{-1}} \in I_G(A)K_z$. Now consider the quotient map $\pi\colon I_G(A) \to I_G(A)/(I_G(A)K_z)$. Since $1 - r_{g^{-1}} \in I_G(A)K_z$ we obtain for $x \in I_G(A)$

$$I_G(\alpha)_g(\pi(xp_{g^{-1}})) = I_G(\alpha)_g(\pi(xr_{g^{-1}} + x(1 - r_{g^{-1}}))) = I_G(\alpha)_g(\pi(xr_{g^{-1}})), \tag{4.1}$$

where $I_G(\alpha)_g$ denotes the induced action on the quotient. In particular, $I_G(\alpha)_g$ acts inner on the quotient. Now let $\rho\colon I_G(A) \to B(\mathcal{H}_z)$ be a representation of the G-injective envelope with $\ker(\rho) = I_G(A)K_z$. Then, G_z acts inner on $\rho(I_G(A))$. For a unit vector $\xi \in B(\mathcal{H})$, define a state σ on $I_G(A)$ via $\sigma(x) = \langle \rho(x)\xi, \xi\rangle$ for $x \in I_G(A)$.

Using G-injectivity of $I_G(A)$, by Proposition 3.2.4 there exists a G-morphism $\phi\colon B \to I_G(A)$ such that $\phi|_{I_G(A)} = \mathrm{id}_{I_G(A)}$, where $B = p\ell^\infty(G, I_G(A))$ with $p(t) = p_{t^{-1}}$. Here the partial action on B is given by the restriction of the translation action. Consider the state $v\colon \ell^\infty(G) \to \mathbb{C}\colon f \mapsto \sigma(\phi(fp))$, where we view $\ell^\infty(G) \subseteq \ell^\infty(G, I_G(A))$. Observe that since $\phi|_{I_G(A)} = \mathrm{id}_{I_G(A)}$, $I_G(A)$ belongs to the multiplicative domain of ϕ, and thus $b\phi(pf) = \phi(bpf) = \phi(pfb) = \phi(pf)b$ for all $f \in \ell^\infty(G)$ and $b \in I_G(A)$. In particular, $\phi(p\ell^\infty(G)) \subseteq Z(I_G(A))$. Now observe for $g \in G_z$ that since $r_{g^{-1}} \leq p_{g^{-1}}$ and $1 - r_{g^{-1}} \in I_G(A)K_z$, we also have $1 - p_{g^{-1}} \in I_G(A)K_z$. Thus, by similar arguments as in Eq. (4.1), $\sigma(p_{g^{-1}}\phi(pf)) = \sigma(\phi(pf))$ and similarly for p_g, since G_z is a group. Using the fact that inner actions act trivially on the centre as well as G-equivariance and -unitality of ϕ, we obtain

$$\begin{aligned} v(\tau_g(f)) &= \sigma[p_g\phi(p\tau_g(f))] = \sigma[\phi(p\tau_g(p))\phi(p\tau_g(f))] \\ &= \sigma[\phi(p\tau_g(pf))] = \sigma[\phi(\tau_g(\tau_{g^{-1}}(p)pf))] \\ &= \sigma[\alpha_g(\phi(\tau_{g^{-1}}(p)pf))] = \sigma[\phi(\tau_{g^{-1}}(p)pf)] \\ &= \sigma[p_{g^{-1}}\phi(pf)] = v(f) \end{aligned}$$

for $g \in G_z$ and $f \in \ell^\infty(G)$. Hence, v is a G_z-invariant state on $\ell^\infty(G)$ and G_z is amenable. □

Chapter 5

The Ideal Intersection Property

5.1 Equivariant Pseudo-Expectations

As demonstrated in Section 4.1.3, dynamical phenomena can be characterized using an appropriate generalization of conditional expectations. However, the pseudo-expectations introduced in Definition 4.1.23 do not necessarily take advantage of all the available structure of the inclusion $A \subseteq A \rtimes_{\mathrm{r}} G$. In particular, if $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is a unital partial C*-dynamical system, the reduced crossed product $A \rtimes_{\mathrm{r}} G$ can be equipped with a canonical generalized unital partial action, which has not been taken into account so far. Thus, in this section we generalize the concept of pseudo-expectations in an equivariant manner for the inclusion $A \subseteq A \rtimes_{\mathrm{r}} G$. The next definition generalizes the notion of an equivariant conditional expectation for the inclusion of generalized unital partial C*-dynamical systems $A \subseteq A \rtimes_{\mathrm{r}} G$. In the case of global C*-dynamical systems this was introduced in [KS19].

Definition 5.1.1. Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system. A u.c.p. map $\phi\colon A \rtimes_r G \to I_G(A)$ is called *equivariant pseudo-expectation* if ϕ is a G-morphism and $\phi|_A = \mathrm{id}_A$.

Remark 5.1.2. As mentioned above the previous definition can be interpreted as an equivariant version of pseudo-expectations introduced by Pitts [Pit17] as defined in Definition 4.1.23. Since potentially $I(A) \subsetneq I_G(A)$ and equivariance is not guaranteed, an equivariant pseudo-expectation does not necessarily restrict to a pseudo-expectation whereas a pseudo-expectation does not always lift to an equivariant one. Hence, these two notions are different.

Observe that analogous to Lemma 4.1.25 equivariant pseudo-expectations arise from equivariant conditional expectations for the inclusion $I_G(A) \subseteq I_G(A) \rtimes G$. We record the result in the next proposition. The proof is similar to the non-equivariant setting, and we are using that $A \rtimes_{\mathrm{r}} G$ embeds equivariantly into $I_G(A) \rtimes_{\mathrm{r}} G$ similar to the global C*-dynamical system case.

Proposition 5.1.3. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system. There is a one-to-one correspondence between equivariant pseudo-expectations $\phi\colon A \rtimes_{\mathrm{r}} G \to I_G(A)$ and equivariant conditional expectations $\Phi\colon I_G(A) \rtimes_{\mathrm{r}} G \to I_G(A)$.*

Also note that any equivariant conditional expectation $E\colon A \rtimes_{\mathrm{r}} G \to A$ gives rise to an equivariant pseudo-expectation by enlarging the codomain. Hence, the set of equivariant pseudo-expectations is not empty, since the canonical conditional expectation always exists. We call the resulting equivariant pseudo-expectation the *canonical equivariant pseudo-expectation* and will also denote it by E. If $\psi\colon A \rtimes_{\mathrm{r}} I_G(A) \to I_G(A)$ is an equivariant pseudo-expectation with $\psi \neq E$, i.e., if there exists $g \in G \setminus \{e\}$ such that $\psi(\lambda_g) \neq 0$, we say ψ is *non-canonical*.

In the works of [PZ15; Zar25] and [KS19] the connection between pseudo-expectations and the ideal intersection property has been explored in the non-equivariant and in the equivariant setting for global C*-dynamical systems respectively. The remainder of this sections is devoted to showing similar results in the case of unital partial C*-dynamical systems. The next result shows that equivariant pseudo-expectations actually give rise to ideals that potentially violate the ideal intersection property.

Proposition 5.1.4. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system and let $\phi\colon A \rtimes_{\mathrm{r}} G \to I_G(A)$ be an equivariant pseudo-expectation. Define*

$$J_\phi := \{x \in A \rtimes_{\mathrm{r}} G \mid \phi(x^*x) = 0\}.$$

This is an ideal in $A \rtimes_{\mathrm{r}} G$ with $J_\phi \cap A = \{0\}$. We call J_ϕ the faithful kernel *of ϕ.*

Proof. Note that clearly J_ϕ is a left ideal, since $\phi(x^*x) = 0$ implies in particular for all $y \in A \rtimes_{\mathrm{r}} G$, $\phi(x^*y^*xy) \leq \|y\|^2\phi(x^*x) = 0$ and so $yx \in J_\phi$. To see that it is also a right ideal, it suffices to check for elements that generate the covariance algebra. First, observe that $\phi|_A = \mathrm{id}_A$ implies that A is in the multiplicative domain of ϕ. Thus, for any $a \in A$, $x \in J_\phi$ we have

$$\phi(a^*x^*xa) = \phi(a^*)\phi(x^*x)\phi(a) = 0. \tag{5.1}$$

For $g \in G$ consider, $v_g = p_g \lambda_g$ as in the proof of Proposition 3.1.4. We obtain

$$\begin{aligned}\phi((xv_{g^{-1}})^* xv_{g^{-1}}) = \phi(v_g x^* x v_{g^{-1}}) = \phi(v_g p_{g^{-1}} x^* x p_{g^{-1}} v_{g^{-1}}) \\ = I_G(\alpha)_g(\phi(p_{g^{-1}} x^* x p_{g^{-1}})) = 0\end{aligned}$$

for any $g \in G$, $x \in J_\phi$. Here, we used that ϕ is equivariant as well as the previous argument (5.1). Thus, J_ϕ is an ideal in $A \rtimes_{\mathrm{r}} G$. Lastly, $J_\phi \cap A = \{0\}$ follows easily from the observation that if $x \in A$, $\phi(x^* x) = x^* x$ by definition of equivariant pseudo-expectation. □

The previous result suggests that the ideal intersection property is linked to faithfulness of equivariant pseudo-expectations. This is indeed the case as given in the next theorem.

Theorem 5.1.5. *Given a unital partial C*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$, $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has the ideal intersection property if and only if every equivariant pseudo-expectation faithful.*

Proof. If there exists a non-faithful equivariant pseudo-expectation, then by Proposition 5.1.4 the faithful kernel of the equivariant pseudo-expectation gives rise to an ideal that violates the ideal intersection property. Conversely, suppose that all equivariant pseudo-expectations are faithful and let $J \trianglelefteq A \rtimes_{\mathrm{r}} G$ be such that $J \cap A = \{0\}$. Then, the quotient map $\pi\colon A \rtimes_{\mathrm{r}} G \to (A \rtimes_{\mathrm{r}} G)/J$ is faithful when restricted to A. Hence, we see $\pi|_A$ is a G-embedding. By G-injectivity there exists a G-morphism ϕ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\mathrm{id}} & I_G(A) \\ {\scriptstyle \pi|_A}\downarrow & \nearrow_{\phi} & \\ (A \rtimes_{\mathrm{r}} G)/J & & \end{array}$$

But then, $\phi \circ \pi\colon A \rtimes_{\mathrm{r}} G \to I_G(A)$ is a G-morphism with $\phi \circ \pi|_A = \mathrm{id}$. Thus, $\phi \circ \pi$ is an equivariant pseudo-expectation with $J \subseteq J_{\phi\circ\pi} = \{0\}$ by assumption. □

If the partial C*-dynamical system is properly outer, the set of equivariant pseudo-expectations reduces to a singleton similar to the non-equivariant setting.

Proposition 5.1.6. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system and suppose that $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$ is properly outer. Then, the partial C*-dynamical system admits a unique equivariant pseudo-expectation.*

Proof. Let $\phi\colon A \rtimes_{\mathrm{r}} G \to I_G(A)$ be an equivariant pseudo-expectation and consider the corresponding equivariant conditional expectation $\hat{\phi}\colon I_G(A) \rtimes_{\mathrm{r}} G \to I_G(A)$. It suffices to show that $\hat{\phi}(v_g) = 0$ if $g \in G \setminus \{e\}$. Recall that since $\hat{\phi}$ is a conditional expectation, $I_G(A)$ belongs to the multiplicative domain of $\hat{\phi}$. Now, for $x \in I_G(A)_{g^{-1}}$ we obtain

$$\hat{\phi}(v_g)x = \phi(v_g x) = \phi(v_g x v_{g^{-1}} v_g) = \hat{\phi}(I_G(\alpha)_g(x)v_g) = I_G(\alpha_g(x))\hat{\phi}(v_g)$$

using the fact that $v_{g^{-1}}v_g$ is the unit of $I_G(A)_{g^{-1}}$. Since $\hat{\phi}$ is a G-morphism, $\hat{\phi}(v_g) \in I_G(A)_g$. Since $I_G(\alpha)_g$ is properly outer, by Proposition 4.1.15 it is freely acting. It follows that $\hat{\phi}(v_g) = 0$. Thus, ϕ is the canonical equivariant pseudo-expectation. □

Remark 5.1.7. Note that we have seen in Proposition 4.1.26 that proper outerness is equivalent to having a unique pseudo-expectation. However, this is not true in the equivariant setting, not even for C*-dynamical systems, as [KKS25, Example 10.1] shows.

Since the canonical equivariant pseudo-expectation is in particular faithful, Theorem 5.1.5 together with Proposition 5.1.6 give the first sufficient condition for the ideal intersection property in relation to proper outerness. Similar results for other non-triviality conditions on the morphisms can be found for example in [ELQ02; Leb05; GS14; KM18].

Corollary 5.1.8. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system such that $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$ is properly outer. Then, $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has the ideal intersection property.*

Lastly, we will use equivariant pseudo-expectations to establish an equivalent characterization of the ideal intersection property known in the global setting [Bry22, Theorem 3.2].

Theorem 5.1.9. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system and suppose $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ is another unital partial C*-dynamical system with $A \subseteq B \subseteq I_G(A)$. Then, $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has the ideal intersection property if and only if $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ does.*

Proof. First, observe that since $A \subseteq B \subseteq I_G(A)$, $I_G(A) = I_G(B)$ and thus, equivariant pseudo-expectations for $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ are in one-to-one correspondence with equivariant pseudo-expectations on $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ by Proposition 5.1.3. Furthermore, by Proposition 3.4.4, $A \rtimes_{\mathrm{r}} G \subseteq B \rtimes_{\mathrm{r}} G \subseteq I(A \rtimes_{\mathrm{r}} G)$. In particular, $A \rtimes_{\mathrm{r}} G \subseteq B \rtimes_{\mathrm{r}} G$ has the ideal intersection property.

Now, suppose $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ does not have the ideal intersection property. Then, by Theorem 5.1.5, there exists a non-faithful equivariant pseudo-expectation $\phi\colon B \rtimes_{\mathrm{r}} G \to I_G(A)$. Let $J = \ker\phi$ and observe that $\phi|_{A\rtimes_{\mathrm{r}}G}$ is an equivariant pseudo-expectation for $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$. But $\ker\phi|_{A\rtimes_{\mathrm{r}}G} = \ker\phi \cap A \rtimes_{\mathrm{r}} G \neq \{0\}$ as $A \rtimes_{\mathrm{r}} G \subseteq B \rtimes_{\mathrm{r}} G$ has the ideal intersection property, which shows the desired.

Conversely, suppose $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ does not have the ideal intersection property and let $\phi\colon A \rtimes_{\mathrm{r}} G \to I_G(A)$ be a non-faithful equivariant pseudo-expectation. Using injectivity and the fact that $A \subseteq B \subseteq I_G(A)$ we can extend it to an equivariant pseudo-expectation $\hat{\phi}\colon B \rtimes_{\mathrm{r}} G \to I_G(B)$. In particular, $\ker\phi \subseteq \ker\hat{\phi}$, and $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ does not have the ideal intersection property. □

5.2 The Non-unital Setting

Observe that by Proposition 2.2.2 if $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is a not necessarily unital partial C*-dynamical system, the corresponding partial action on $I(A)$ will always be unital. Thus, one can think of the canonical partial action $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ as an *injective unitization* of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$. With this viewpoint, we can define the notion of G-injective envelope for non-unital partial C*-dynamical systems utilizing the fact that $I_G(I(A)) = I_G(A)$ in the unital setting as shown at the end of Section 3.3. This is similar to the definition of injective envelopes for non-unital C*-algebras as given in [Ham82b, Section 6].

Definition 5.2.1. Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a not necessarily unital partial C*-dynamical system and let $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ denote the associated unital partial C*-dynamical system on the injective envelope of A. We define the *G-injective envelope* of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ as the G-injective envelope of $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ in the category of generalized unital partial C*-dynamical systems, i.e., $I_G(A) := I_G(I(A))$.

Remark 5.2.2. In a similar spirit to the previous definition, for a not necessarily unital partial C*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$, we can define an equivariant pseudo-expectation as a G-morphism $\phi\colon I(A) \rtimes_{\mathrm{r}} G \to I_G(A)$ such that $\phi|_{I(A)} = \mathrm{id}_{I(A)}$. Using the one-to-one correspondence in the unital setting, this definition is justified.

A key observation in studying the ideal intersection property for global C*-dynamical systems using injective envelope techniques is that $A \subseteq A \rtimes_{\mathrm{r}} G$ having the ideal intersection property is characterized by the ideal intersection property for $I(A) \subseteq I(A) \rtimes_{\mathrm{r}} G$ as demonstrated by [GU23; KS19; Kaw17]. We have already observed this for

unital partial C*-dynamical systems in Theorem 5.1.9. In the remainder of this section we argue that this philosophy in fact also allows us to consider the non-unital setting to begin with; culminating in an analogous result for not necessarily unital partial C*-dynamical systems. While the unital setting allowed us to leverage known techniques from global C*-dynamical systems such as equivariant pseudo-expectations leading to a concise proof, the non-unital analogue offers a new challenge as many of the previous techniques do not apply. We will overcome this using a series of lemmas and propositions of varying technical difficulty. First, we will show that for a not necessarily unital partial C*-dynamical system the ideal intersection property for $A \subseteq A \rtimes_{\mathrm{r}} G$ implies the ideal intersection property for its injective unitization.

Proposition 5.2.3. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system. If $A \subseteq A \rtimes_{\mathrm{r}} G$ has the ideal intersection property, so does $I(A) \subseteq I(A) \rtimes_{\mathrm{r}} G$.*

Proof. Suppose $A \subseteq A \rtimes_{\mathrm{r}} G$ has the ideal intersection property and take $J \trianglelefteq I(A) \rtimes_{\mathrm{r}} G$ with $J \cap I(A) = \{0\}$. Then, $J \cap A = \{0\}$ and thus by the ideal intersection property for $A \subseteq A \rtimes_{\mathrm{r}} G$, we obtain $J \cap A \rtimes_{\mathrm{r}} G = \{0\}$. We claim this implies $J = \{0\}$. To see this, note that since $J \cap A \rtimes_{\mathrm{r}} G = \{0\}$, there exists an embedding $\phi\colon A \rtimes_{\mathrm{r}} G \to I(A) \rtimes_{\mathrm{r}} G/J$. Furthermore, consider the conditional expectation $E\colon A \rtimes_{\mathrm{r}} G \to A$. Viewing $A \subseteq I(A)$ and using the fact that ϕ is an embedding, we can extend the conditional expectation to $\hat{E}\colon I(A) \rtimes_{\mathrm{r}} G/J \to I(A)$, such that $\hat{E} \circ \phi = E$. Now since $J \cap I(A) = \{0\}$, $\hat{E}|_{I(A)} = \mathrm{id}_{I(A)}$. In particular, $I(A)$ belongs to the multiplicative domain of $\hat{E}$ and thus $\hat{E}(\lambda_g) = p_g \hat{E}(\lambda_g)$ for all $g \in G$, where p_g denotes the unit of $I(A)_g$. Then, if $(u_\mu)_\mu$ is an approximate identity for A_g, $\sup_\mu u_\mu = p_g$ and we obtain

$$\hat{E}(\lambda_g)^* \hat{E}(\lambda_g) = \hat{E}(\lambda_g)^* (\sup_\mu u_\mu) \hat{E}(\lambda_g) = \sup_\mu \hat{E}(\lambda_g)^* u_\mu \hat{E}(\lambda_g) = \sup_\mu \hat{E}(\lambda_g)^* E(u_\mu \lambda_g).$$

As $E(u_\mu \lambda_g)$ is 0 if $g \neq e$, this shows $\hat{E}(\lambda_g) = 0$ for all $g \neq e$. Thus, $\hat{E}$ is the conditional expectation from $I(A) \rtimes_{\mathrm{r}} G$ to $I(A)$ and $J = \{0\}$. □

Remark 5.2.4. In the proof of the previous proposition we observe that we are actually showing that $A \rtimes_{\mathrm{r}} G \subseteq I(A) \rtimes_{\mathrm{r}} G$ has the ideal intersection property.

For the other implication, instead of looking at the injective unitization, we will focus on a potentially smaller unital partial C*-dynamical system that is closely related to $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ and $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$. In particular, we say that for an inclusion of partial C*-dynamical systems $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G}) \subseteq (\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ the C*-algebra B is *generated by the orbits* of A, if $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ arises as the restriction

of $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ and $B = \mathrm{C}^*(\{\beta_g(Ap_{g^{-1}}) \mid g \in G\})$, where p_g denotes the unit of B_g. We first observe the following.

Lemma 5.2.5. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G}) \subseteq (\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ be an inclusion of partial C^*-dynamical systems, where $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ is a unital partial C^*-dynamical system with $B_g = Bp_g$ for projections $p_g \in Z(B)$ such that B is generated by the orbits of A. Furthermore, consider a *-homomorphism $\pi\colon A \rtimes_{\mathrm{r}} G \to B(\mathcal{H})$ with a u.c.p. extension $\Pi\colon (B \rtimes_{\mathrm{r}} G)^{**} \to B(\mathcal{H})$ and suppose that $C^*(\{q_g v_g | g \in G\})$ is in the multiplicative domain of Π, where q_g denotes the unit of A_g^{**} and v_g denote the partial isometries corresponding to the left-regular covariant pair for $B \rtimes_{\mathrm{r}} G$. If $\Pi(p_g) = \Pi(q_g)$ for all $g \in G$, then Π is multiplicative on $B \rtimes_{\mathrm{r}} G$.*

Proof. To see that Π is multiplicative on $B \rtimes_{\mathrm{r}} G$, it suffices to show it is multiplicative on B and v_g as they generate the covariance algebra, which is dense in the reduced partial crossed product. Note that by assumption Π is multiplicative on $\mathrm{C}^*(\{q_g v_g | g \in G\})$. Now, observe that $q_g v_g (q_g v_g)^* = q_g p_g = q_g$ as $A_g \subseteq B_g$ by assumption and thus $q_g \leq p_g$ and $q_g \in \mathrm{Dom}_\Pi$. As $\Pi(p_g) = \Pi(q_g)$ by assumption, this shows $p_g \in \mathrm{Dom}_\Pi$. To see that v_g is also in the multiplicative domain, first observe that as Π extends π, in particular $A \subseteq \mathrm{Dom}_\Pi$. Now, since B is generated by orbits of A, this gives that $B \subseteq \mathrm{Dom}_\Pi$. Thus, we obtain

$$\Pi(v_g) = \Pi(p_g v_g) = \Pi(p_g)\Pi(v_g) = \Pi(q_g)\Pi(v_g) = \Pi(q_g v_g),$$

which finishes the proof. □

Using the results given above together with the result for G-essential extensions in the unital setting (see Theorem 5.1.9), we obtain the following non-unital analogue.

Theorem 5.2.6. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C^*-dynamical system. The following are equivalent:*

(i) *$(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has the ideal intersection property;*

(ii) *For all unital partial C^*-dynamical systems $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ with $A \subseteq B \subseteq I(A)$, $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ has the ideal intersection property;*

(iii) *The injective unitization $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ has the ideal intersection property.*

Proof. We will show $(i) \implies (iii) \implies (ii) \implies (i)$. The first implication is exactly Proposition 5.2.3. To see that *(iii)* implies *(ii)*, observe that if $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ is a partial C^*-dynamical system with $A \subseteq B \subseteq I(A)$, $I(B) = I(A)$. Hence, if $(\{I(A)_g\}_{g\in G},$

$\{I(\alpha)_g\}_{g\in G})$ has the ideal intersection property, $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ does so as well by Theorem 5.1.9. It remains to show the last implication.

Lastly, to see that *(ii)* implies *(i)*, we will show the contrapositive. Suppose $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ does not have the ideal intersection property and let $\{0\} \neq J \trianglelefteq A \rtimes_r G$ be an ideal with $J \cap A = \{0\}$. We will construct a unital partial C*-dynamical system $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ with $A \subseteq B \subseteq I(A)$ that also fails the ideal intersection property. To do so, let p_g again denote the units of the injective unitization $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ and consider the C*-algebra $B = C^*(\{I(\alpha)_g(Ap_{g^{-1}}) \mid g \in G)\})$ generated by A, the family of units $\{p_g\}_{g\in G}$ and the corresponding orbits. We can now define a partial C*-dynamical system $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ as the restriction of $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ to B (see [Exe17, p. 13]).

We claim $B_g = Bp_g$. To see this first observe $B_g = I(A)_g \cap B \cap I(\alpha)_g(I(A)_{g^{-1}} \cap B) = Bp_g \cap I(\alpha)_g(Bp_{g^{-1}})$. Next note that for $g, h \in G$, $I(\alpha)_g(ap_{g^{-1}})p_h = I(\alpha)_g(ap_{g^{-1}}p_{g^{-1}h}) \in B$ for all $a \in A$. This immediately gives

$$\begin{aligned} I(\alpha)_g(I(\alpha)_h(ap_{h^{-1}})p_{g^{-1}}) &= I(\alpha)_g(I(\alpha)_h(ap_{h^{-1}})p_hp_{g^{-1}})p_g \\ &= I(\alpha)_g(I(\alpha)_h(ap_{h^{-1}})I(\alpha)_h(p_{h^{-1}}p_{h^{-1}g^{-1}}))p_g \\ &= I(\alpha)_{gh}(ap_{h^{-1}}p_{(gh)^{-1}})p_g \in Bp_g \end{aligned}$$

for all $g, h \in G, a \in A$. As B is generated by linear combinations of products of this form, we see $I(\alpha)_g(Bp_{g^{-1}}) \subseteq Bp_g$. Applying the same reasoning and $I(\alpha)_{g^{-1}}$ shows $Bp_{g^{-1}} \subseteq I(\alpha)_{g^{-1}}(Bp_g) \subseteq Bp_{g^{-1}}$, giving $B_g = Bp_g$ as desired.

We will now construct an ideal $\tilde{J} \trianglelefteq B \rtimes_r G$ such that $\tilde{J} \cap B = \{0\}$ and $J \subseteq \tilde{J} \neq \{0\}$, i.e., $\tilde{J}$ fails the ideal intersection property. To do so, we construct an appropriate *-homomorphism Ψ from $B \rtimes_r G$ to another C*-algebra. First consider the quotient map $\pi\colon A \rtimes_r G \to A \rtimes_r G/J$. As $J \cap A = \{0\}$, this is faithful on A. Representing $A \rtimes_r G/J$ as bounded operators on a Hilbert space, $A \rtimes_r G/J \subseteq B(\mathcal{H})$, let $\hat{\pi}\colon B \rtimes_r G \to B(\mathcal{H})$ be a c.p. extension of π. Since $A \subseteq B \subseteq I(A)$, $A \subseteq B$ is essential, and thus $\hat{\pi}|_B$ is completely isometric. Next, consider $\Pi := \hat{\pi}^{**}$ the unique normal extension $\Pi\colon (B \rtimes_r G)^{**} \to B(\mathcal{H})$. We note that $\Pi_{(A\rtimes_r G)^{**}} = \pi^{**}$ and thus $(A \rtimes_r G)^{**} \subseteq \mathrm{Dom}_\Pi$. As in Lemma 5.2.5, let v_g denote the partial isometries coming from the left-regular covariant pair for $B \rtimes_r G$ and let $q_g \in A_g^{**} \subseteq (B \rtimes_r G)^{**}$ denote the unit of A_g^{**} for $g \in G$. Then, we claim $q_gv_g \in (A \rtimes_r G)^{**}$. To see this, let $(u_\mu)_\mu$ be an approximate identity for A_g. By [Exe17, Proposition 21.3 and 21.9], $u_\mu v_g \in A \rtimes_r G$. Since q_gv_g is the weak*-limit of $u_\mu v_g$, we see $q_gv_g \in (A \rtimes_r G)^{**}$ and in particular $q_gv_g \in \mathrm{Dom}_\Pi$. Furthermore, q_gv_g implements the partial action on A, as $q_gv_g a(q_gv_g)^* = v_gav_g^* = \alpha_g(a)$ for $a \in A_{g^{-1}}$.

Fixing a copy of $I(A)$ in $B(\mathcal{H})$ there exists a u.c.p. map $\rho\colon B(\mathcal{H}) \to I(A)$ such that $\rho \circ \hat{\pi} = \mathrm{id}_B$. Now, consider the C*-algebra $C := \mathrm{C}^*(\{\hat{\pi}(B \rtimes_{\mathrm{r}} G) \cup \mathrm{span}\{\Pi(B_q)\Pi(q_g v_g) | g \in G\}\}) \subseteq \Pi((B \rtimes_{\mathrm{r}} G)^{**})$, where $B_q := \mathrm{C}^*(\{q_t v_t a (q_t v_t)^* \mid a \in A, t \in G\})$, and define the ideal

$$I := \{d \in C \mid \rho(c^* d^* d c) = 0 \text{ for all } c \in C\}.$$

Let $\psi\colon C \to C/I$ denote the quotient map. It suffices to show that $\Psi := \psi \circ \hat{\pi}\colon B \rtimes_{\mathrm{r}} G \to C/I$ is multiplicative, since then $\tilde{J} := \ker(\Psi)$ is an ideal in $B \rtimes_{\mathrm{r}} G$ with $\tilde{J} \cap B = \{0\}$. The latter follows from the fact $\rho \circ \pi = \mathrm{id}_A$ and thus if $\rho(c^* d^* d c) = 0$ for all $c \in C$ for some $d \in \pi(A)$, $d = 0$, i.e., $I \cap \pi(A) = \{0\}$. In particular, Ψ is faithful on A, and by essentiality of the inclusion it is faithful on B. Furthermore, as $\hat{\pi}$ extends π, we also have $J \subseteq \ker \Psi$ and in particular $\ker \Psi \neq \{0\}$. By Lemma 5.2.5 to show that Ψ is multiplicative, it is enough to show $\psi \circ \Pi(p_g) = \psi \circ \Pi(q_g)$ for all $g \in G$. We will show $r_g := \Pi(p_g) - \Pi(q_g) \in I$ via induction. To do so, we need the following technical results.

Claim 1: *For $a \in A$, $\rho(\pi^{**}(v_g q_g a (q_g v_g)^*)) = I(\alpha)_g(a p_{g^{-1}})$ for $g \in G$. In particular, $\pi^{**}(v_h^* q_h \tilde{b} q_h v_h)$ is in the multiplicative domain of ρ for $\tilde{b} \in B_q$ and $\rho(\pi^{**}(q_g)) = p_g$ for all $g \in G$.*

To see the first assertion assume $a = c^* c$ for some $c \in A$ and let $(u_\mu)_\mu \subseteq A_{g^{-1}}$ be an increasing approximate identity for $A_{g^{-1}}$. We see

$$\begin{aligned}\hat{\pi}(I(\alpha)_g(c^* c p_{g^{-1}})) = \hat{\pi}(I(\alpha)_g(c^* p_{g^{-1}} c)) &\geq \hat{\pi}(I(\alpha)_g(c^* u_\mu c)) \\ &= \pi(\alpha_g(c^* u_\mu c)) = \pi^{**}(q_g v_g c^* u_\mu c (q_g v_g)^*)\end{aligned}$$

for all μ as $q_g v_g$ implements the action on $A_{g^{-1}}$. In particular, as π^{**} is normal, $\rho \circ \hat{\pi} = \mathrm{id}_B$ and $q_{g^{-1}}$ is central in A^{**}, we obtain

$$I(\alpha)_g(c^* c p_{g^{-1}}) \geq \rho(\sup_\mu \pi^{**}(q_g v_g c^* u_\mu c (q_g v_g)^*)) = \rho(\pi^{**}(q_g v_g c^* c (q_g v_g)^*)).$$

But, then

$$I(\alpha)_g(c^* c p_{g^{-1}}) \geq \rho(\pi^{**}(q_g v_g c^* c (q_g v_g)^*)) \geq \rho(\pi(\alpha_g(c^* u_\mu c))) = \alpha_g(c^* u_\mu c).$$

Using similar reasoning as before, this gives

$$I(\alpha)_g(c^* c p_{g^{-1}}) \geq \rho(\pi^{**}(q_g v_g c^* c (q_g v_g)^*)) \geq \sup_\mu I(\alpha)_g(c^* u_\mu c) = I(\alpha)_g(c^* c p_{g^{-1}}),$$

and since A is spanned by its positive elements, this shows the desired.

Regarding the multiplicative domain, first observe that as B_q is generated by elements of the form $v_h^* q_h a q_h v_h$ and since π^{**} is multiplicative, it suffices to check that $y = \pi^{**}(v_h^* q_h a q_h v_h)$ is in the multiplicative domain of ρ. However, this can be seen quickly since $\rho(yy^*) = \rho(\pi^{**}(v_h^* q_h a a^* q_h v_h)) = I(\alpha)_{h^{-1}}(aa^* p_h) = \rho(y)\rho(y)^*$ and similarly $\rho(y^*y) = \rho(y)^*\rho(y)$.

Lastly, to see that $\rho(\pi^{**}(q_g)) = p_g$ for all $g \in G$, if A is unital this follows immediately from the first part of the claim as $\rho(\pi^{**}(q_g)) = \rho(\pi^{**}(v_g q_g (q_g v_g)^*)) = I(\alpha)_g(p_{g^{-1}}) = p_g$. Otherwise consider an increasing approximate identity $(u_\mu)_\mu$ for A. Using the same reasoning as above we observe

$$\begin{aligned}\rho(\pi^{**}(q_g)) = \rho(\pi^{**}(v_g q_g (q_g v_g)^*)) &= \rho(\sup_\mu \pi^{**}(v_g q_g u_\mu (q_g v_g)^*)) \\ &\geq \sup_\mu I(\alpha)_g(u_\mu p_{g^{-1}}) = I(\alpha)_g(p_{g^{-1}}) = p_g.\end{aligned}$$

However, as $q_g \leq p_g$ we have $\Pi(q_g) \leq \Pi(p_g)$ giving $\rho(\pi^{**}(q_g)) \leq p_g$. $\diamond$

<u>*Claim 2:*</u> *Let* $b = I(\alpha)_{t_1}(a_1 p_{t_1^{-1}}) \cdots I(\alpha)_{t_n}(a_n p_{t_n^{-1}}) \in B$ *and consider* $b_q = q_{t_1} v_{t_1} a_1 (q_{t_1} v_{t_1})^* \cdots q_{t_n} v_{t_n} a_n (q_{t_n} v_{t_n})^* \in B_q$ *for* $a_1, \ldots, a_n \in A$ *and* $t_1, \ldots, t_n \in G$. *For all* $d \in C$ *and* $g \in G$, $\rho(\hat{\pi}(bv_g)^* d^* d \hat{\pi}(bv_g)) = 0$ *if and only if* $\rho(\pi^{**}((b_q q_g v_g)^* d^* d \pi^{**}(b_g q_g v_g)) = 0$.

To see this let $x_g := \Pi(bv_g) - \Pi(b_q q_g v_g) = \hat{\pi}(bv_g) - \pi^{**}(b_q q_g v_g)$. We will first show that $\rho(x_g^* x_g) = 0$. We have

$$\begin{aligned}\rho(x_g^* x_g) &= \rho(\hat{\pi}(bv_g)^* \hat{\pi}(bv_g)) - \rho(\hat{\pi}(bv_g)^* \pi^{**}(b_q q_g v_g)) \\ &\quad - \rho(\pi^{**}(b_q q_g v_g)^* \hat{\pi}(bv_g)) + \rho(\pi^{**}((b_q q_g v_g)^* b_q q_g v_g)) \\ &\leq \rho(\hat{\pi}((bv_g)^* bv_g)) - \rho(\hat{\pi}((bv_g)^* b_q q_g v_g)) \\ &\quad - \rho(\hat{\pi}((b_q q_g v_g)^* bv_g)) + \rho(\pi^{**}(v_{g^{-1}} q_g b_q^* b_q q_g v_g)) \\ &= 2I(\alpha)_{g^{-1}}(b^* b p_g) - \rho(\hat{\pi}((bv_g)^* b_q q_g v_g)) - \rho(\hat{\pi}((b_q q_g v_g)^* bv_g)),\end{aligned}$$

where the first inequality follows from the Cauchy-Schwarz inequality and the fact that $b_q q_g v_g \in \mathrm{Dom}_\Pi$. The second equality follows since v_g implements the action on B and $\rho \circ \hat{\pi}|_B = \mathrm{id}_B$, thus $\rho(\hat{\pi}((bv_g)^* bv_g)) = \rho(\hat{\pi}(I(\alpha)_{g^{-1}}(b^* b p_g))) = I(\alpha)_{g^{-1}}(b^* b p_g)$. Lastly, by applying *Claim 1* to products, we see $\rho(\pi^{**}(v_{g^{-1}} q_g b_q^* b_q q_g v_g)) = I(\alpha)_{g^{-1}}(b^* b p_g)$. Furthermore, recall that $v_g v_g^* = p_g$ for all $g \in G$. Combining this with the fact that $q_g \leq p_g$

and thus $q_g p_g = q_g$ we obtain

$$\begin{aligned}\rho(\hat{\pi}((bv_g)^* b_q q_g v_g)) &= \rho(\hat{\pi}((bv_g)^* p_g q_g b_q q_g v_g)) = \rho(\hat{\pi}(v_g^* b^* v_g v_g^* q_g b_q q_g v_g)) \\ &= \rho(\hat{\pi}(v_g^* b^* v_g)\pi^{**}(v_g^* q_g b_q q_g v_g)) = I(\alpha)_{g^{-1}}(b^* p_g)\rho(\pi^{**}(v_g^* q_g b_q q_g v_g)) \\ &= I(\alpha)_{g^{-1}}(b^* p_g) I(\alpha)_{g^{-1}}(b p_g) = I(\alpha)_{g^{-1}}(b^* b p_g),\end{aligned}$$

where in the fourth equality we used that $\hat{\pi}(B)$ is in the multiplicative domain of ρ. This is true since $\rho \circ \hat{\pi}|_B = \mathrm{id}_B$ and thus for $b \in B$

$$b^* b = \rho(\hat{\pi}(b))^* \rho(\hat{\pi}(b)) \leq \rho(\hat{\pi}(b)^* \hat{\pi}(b)) \leq \rho(\hat{\pi}(b^* b)) = b^* b,$$

using again Cauchy-Schwarz. By the same reasoning we see $\rho(\hat{\pi}((b_q q_g v_g)^* b v_g)) = I(\alpha)_{g^{-1}}(b^* b p_g)$. In sum,

$$0 \leq \rho(x_g^* x_g) \leq 2I(\alpha)_{g^{-1}}(b^* b p_g) - \rho(\hat{\pi}((bv_g)^* b_q q_g v_g)) - \rho(\hat{\pi}((b_q q_g v_g)^* b v_g)) = 0,$$

showing the desired. This immediately shows $\rho(x_g^* d^* d x_g) = 0$ as $\rho(x_g^* d^* d x_g) \leq \|d\|^2 \rho(x_g^* x_g)$. Expanding gives

$$\begin{aligned}\rho(x_g^* d^* d x_g) = \rho(\hat{\pi}(bv_g)^* d^* d\hat{\pi}(b_q q_g v_g)) - \rho(\hat{\pi}(bv_g)^* d^* d\pi^{**}(b_q q_g v_g)) \\ - \rho(\pi^{**}(b_q q_g v_g)^* d^* d\hat{\pi}(bv_g)) + \rho(\pi^{**}(b_q q_g v_g)^* d^* d\pi^{**}(b_q q_g v_g)) = 0.\end{aligned} \tag{5.2}$$

Now, if $\rho(\hat{\pi}(bv_g)^* d^* d\hat{\pi}(b_q q_g v_g)) = 0$, then since $\|\rho(a^* b)\| \leq \|\rho(a^* a)\|\|\rho(b^* b)\|$, we obtain $\rho(\hat{\pi}(bv_g)^* d^* d\pi^{**}(b_q q_g v_g)) = 0 = \rho(\pi^{**}(b_q q_g v_g)^* d^* d\hat{\pi}(bv_g))$ by setting $a^* = \hat{\pi}(bv_g)^* d^*$, $b = d\pi^{**}(b_q q_g v_g)$. By (5.2) we must have $\rho(\pi^{**}(b_q q_g v_g)^* d^* d\pi^{**}(b_q q_g v_g)) = 0$. The other direction follows using the exact same reasoning, finishing the proof of *Claim 2*. ◇

With this, we can now proceed to show $r_g = \Pi(p_g - q_g) = \hat{\pi}(p_g) - \pi^{**}(q_g) \in I$, by induction. To do so, recall that by definition of I, $r_g \in I$ if and only if $\rho(c^* r_g^* r_g c) = 0$ for all $c \in C$, where $C = \mathrm{C}^*(\hat{\pi}(B \rtimes_{\mathrm{r}} G) \cup \mathrm{span}\{\Pi(B_q q_g v_g) \mid g \in G\})$. In particular, it suffices to show $\rho(C_n^* r_g^* r_g C_n) = 0$ for all C_n, where C_n acts as a placeholder for possible products of length $n \in \mathbb{N}$, i.e.,

$$C_n = \prod_{k=1}^{n} x_k \qquad \text{for } x_k = \hat{\pi}(b_k v_{h_k}) \text{ or } x_k = \pi^{**}(b_{k,q} q_{h_k} v_{h_k}), \text{where } b_k \in B, b_{k,q} \in B_q. \tag{5.3}$$

Here, $b_{k,q}$ is of the same form as in the statement of *Claim 2*. Note that since $(A \rtimes_{\mathrm{r}} G)^{**} \subseteq \mathrm{Dom}_\Pi$, we can assume that the factors either alternate or $x_k x_{k+1}$ are both of the form

$\hat{\pi}(b_k v_{h_k}), \hat{\pi}(b_{k+1} v_{h_{k+1}})$ for some $b_k, b_{k+1} \in B$, i.e., we assume the product is appropriately simplified.

For the proof by induction, consider the base case $C_1 = \hat{\pi}(b v_g)$ or $C_1 = \pi^{**}(b_q q_g v_g)$ for $b \in B$, $b_q \in B_q$ and $g \in G$. By *Claim 2* we can restrict to the latter setting. Using Cauchy-Schwarz, $p_g q_g = q_g$, and the fact that $b_q q_g v_g \subseteq \mathrm{Dom}_\Pi$ we observe

$$\begin{aligned}\rho(\pi^{**}(b_q q_g v_g)^* r_g^* r_g \pi^{**}(b_q q_g v_g)) = \rho(\Pi(p_g b_q q_g v_g)^* \Pi(p_g b_q q_g v_g)) \\ - \rho(\Pi(p_g b_q q_g v_g)^* \Pi(b_q q_g v_g)) \\ - \rho(\Pi(b_q q_g v_g)^* \Pi(p_g b_q q_g v_g)) \\ + \rho(\pi^{**}((b_q q_g v_g)^* b_q q_g v_g)) = 0\end{aligned}$$

giving the base case. Now, assume $\rho(C_n^* r_g^* r_g C_n) = 0$ for all C_n as in (5.3) for a fixed $n \in \mathbb{N}$. We will show $\rho(C_{n+1}^* r_g^* r_g C_{n+1}) = 0$. Again by *Claim 2*, it suffices to look at $C_{n+1} = C_n \pi^{**}(b_{n+1,q} q_{h_{n+1}} v_{h_{n+1}})$. Furthermore, appealing to the multiplicative domain, we can assume that $x_n = \hat{\pi}(b_n v_{h_n})$ for some $b_n \in B$. Now, again appealing to $\pi^{**}(b_{n+1,q} q_{h_{n+1}} v_{h_{n+1}})$ being in the multiplicative domain of $\hat{\pi}$, observe

$$\begin{aligned}C_{n+1} &= C_{n-1} x_n x_{n+1} = C_{n_1} \Pi(b_n v_{h_n} b_{n+1,q} q_{h_{n+1}} v_{h_{n+1}}) \\ &= C_{n-1} \Pi(b_n v_{h_n} v_{h_{n+1}} v_{h_{n+1}}^* q_{h_{n+1}} b_{n+1,q} q_{h_{n+1}} v_{h_{n+1}}) \\ &= C_{n-1} \hat{\pi}(b_n p_{h_n} v_{h_n h_{n+1}}) \pi^{**}(v_{h_{n+1}}^* q_{h_{n+1}} b_{n+1,q} q_{h_{n+1}} v_{h_{n+1}}) \\ &= C_n \pi^{**}(v_{h_{n+1}}^* q_{h_{n+1}} b_{n+1,q} q_{h_{n+1}} v_{h_{n+1}}),\end{aligned} \tag{5.4}$$

where in the second-last equality we used

$$v_{h_n} v_{h_{n+1}} = I(\alpha)_{h_n}(p_{h_n^{-1}} p_{h_{n+1}}) v_{h_n h_{n+1}} = p_{h_n} p_{h_n h_{n+1}} v_{h_n h_{n+1}}.$$

However, using that $\pi^{**}(v_h^* q_h \tilde{b} q_h v_h)$ is in the multiplicative domain of ρ combined with (5.4) immediately shows

$$\begin{aligned}\rho(C_{n+1}^* d^* d C_{n+1}) = \rho\Big(\pi^{**}(v_{h_{n+1}}^* q_{h_{n+1}} b_{n+1,q} q_{h_{n+1}} v_{h_{n+1}})^* C_n^* r_g^* \\ r_g C_n \pi^{**}(v_{h_{n+1}}^* q_{h_{n+1}} b_{n+1,q} q_{h_{n+1}} v_{h_{n+1}})\Big) \\ = I(\alpha)_{h_{n+1}^{-1}}(b_{n+1}^* p_{h_{n+1}}) \rho(C_n^* r_g^* r_g C_n) I(\alpha)_{h_{n+1}^{-1}}(b_{n+1} p_{h_{n+1}}) = 0\end{aligned}$$

by hypothesis. In sum, $r_g \in I$ for all $g \in G$ and by Lemma 5.2.5 this gives that $\tilde{J} = \ker(\psi \circ \hat{\pi})$ is an ideal in $B \rtimes_{\mathrm{r}} G$ violating the ideal intersection property. □

As $I(A)$ is also characterized as the maximal essential extension of a (unital) C*-algebra A, the unital partial C*-dynamical systems described in part *(ii)* in the theorem above can be thought of as *essential unitizations* of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$. Using the previous characterization together with Theorem 5.1.9 we obtain an analogous result for non-unital partial C*-dynamical systems, demonstrating the use of these essential unitizations for partial C*-dynamical systems.

Corollary 5.2.7. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a not necessarily unital partial C*-dynamical system and suppose $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ is a (not necessarily unital) partial C*-dynamical system with $A \subseteq B \subseteq I_G(A)$. Then, $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has the ideal intersection property if and only if $(\{B_g\}_{g\in G}, \{\beta_g\}_{g\in G})$ does.*

As mentioned at the beginning of this section and evident in the proof of Theorem 5.2.6, the transition from $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ to $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ and back can be challenging in the non-unital setting as it often requires new techniques. For example, properties characterized by equivariant maps on the crossed product do not have an immediate analogue meaning proofs such as the one of Theorem 5.1.9 do not immediately apply. However, we have seen in Section 4.1.3 that characterizations using non-equivariant maps can translate quite readily to the partial setting as we can exploit unitality of the injective unitization without worrying too much about pulling the result back to $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$. This method also works for showing the following stronger result than the ideal intersection property. The next definition is due to [PZ15] for more general C*-inclusions.

Definition 5.2.8. Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system. We say that $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is *hereditarily essential*, if for all $A \subseteq B \subseteq A \rtimes_{\mathrm{r}} G$, $A \subseteq B$ has the ideal intersection property.

Using the machinery of pseudo-expectations, we can fairly quickly show that $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is hereditarily essential if and only if its injective unitization $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ is. To do so, we will show that proper outerness of the partial C*-dynamical system is in fact equivalent to the partial C*-dynamical system being hereditarily essential. In [PZ15, Theorem 3.5] the authors show that for more general C*-dynamical inclusions being hereditarily essential is equivalent to each pseudo-expectation being faithful. In [Zar25] it was shown that this is not generally equivalent to uniqueness of pseudo-expectation, however, for twisted global C*-dynamical systems uniqueness of pseudo-expectation is equivalent to faithfulness of pseudo-expectations. Zarikian's proof generalizes to the (non-twisted) partial C*-dynamical systems setting, which we include below.

Proposition 5.2.9. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system. The system admits a unique pseudo-expectation if and only if each pseudo-expectation is faithful.*

Proof. As $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ always admits the pseudo-expectation stemming from the canonical conditional expectation, which is in particular faithful, one direction of this result is clear. For the converse, suppose that every pseudo-expectation for $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is faithful, but that there exists more than one. In particular, by Proposition 5.1.6 the partial C*-dynamical system is not properly outer. Thus, there exists $g \neq e$ and an $I(\alpha)_g$-invariant non-zero projection $q \in Z(I(A))$ such that $I(\alpha)_g|_{I(A)q}$ acts inner. Furthermore, by Corollary 4.1.4, $I(\alpha)_g|_{I(A)q}$ is in fact a *-automorphism. In particular, defining $H = \langle g \rangle$ we see that $I(\alpha)_h$ is defined on $I(A)q$ for all $h \in H$. Now, as $I(A)q$ is injective, by [Zar25, Lemma 2.1.1], there exists a unitary representation $u\colon H \to I(A)q$ such that $I(\alpha)_h|_{I(A)q} = \mathrm{Ad}(u_h)$. Then, consider the compression map $\pi\colon I(A) \to I(A)q$ and notice that (π, u) is a covariant pair for the partial C*-dynamical system $(\{I(A)_h\}_{h\in H}, \{I(\alpha)_h\}_{h\in H})$. As H is amenable, let $\rho\colon I(A) \rtimes_{\mathrm{r}} H \to I(A)q$ denote the corresponding *-homomorphism such that $\rho(av_h) = \pi(a)u_h = aqu_h = au_h$ for all $a \in I(A)_h$ and $h \in H$. Next, consider the canonical conditional expectation $E\colon I(A) \rtimes_{\mathrm{r}} H \to I(A)$ and define $\phi\colon I(A) \rtimes_{\mathrm{r}} H \to I(A)$ via $\phi(x) = \rho(x) + E(x)(1-q)$ for $x \in I(A) \rtimes_{\mathrm{r}} H$. By definition, we have $\phi(a) = \mathrm{id}_{I(A)}$ for $a \in I(A)$. As $I(A) \rtimes_{\mathrm{r}} H \subseteq I(A) \rtimes_{\mathrm{r}} G$ and using the injectivity of $I(A)$, we can extend ϕ to $\hat{\phi}\colon I(A) \rtimes_{\mathrm{r}} G \to I(A)$. In particular, $\hat{\phi}|_{I(A)} = \mathrm{id}_{I(A)}$ and thus the restriction $\hat{\phi}|_{A\rtimes_{\mathrm{r}}G}$ is a pseudo-expectation. Hence, by assumption $\hat{\phi}|_{A\rtimes_{\mathrm{r}}G}$ is faithful and subsequently $\phi|_{A\rtimes_{\mathrm{r}}H}$ is faithful. Note also that as by Remark 5.2.4 this implies that $\hat{\phi}$ and thus ϕ are faithful.

We will now conclude the proof in showing that $I(A)q = \{0\}$ which is a contradiction to $I(\alpha)_g$ being not properly outer. Observe that $I(A)q$ by construction is invariant under $I(\alpha)_h$ for $h \in H$ and thus, we obtain the following inclusion $I(A)q \rtimes_{\mathrm{r}} H \subseteq I(A) \rtimes_{\mathrm{r}} H$. Furthermore, by construction $\phi(x) = \rho(x) + E(x)(1-q) = \rho(x)$ for $x \in I(A)q \rtimes_{\mathrm{r}} H$ and thus ρ is faithful on $I(A)q \rtimes_{\mathrm{r}} H$. In particular, the restriction $\tilde{\rho}\colon I(A)q \rtimes_{\mathrm{r}} H \to I(A)q$ is faithful and thus using injectivity of $I(A)q$, we can extend the conditional expectation $\Phi\colon I(A)q \rtimes_{\mathrm{r}} H \to I(A)q$ to a u.c.p. map $\psi\colon I(A)q \to I(A)q$ such that $\psi \circ \rho = \Phi$. Hence, we see for $a \in I(A)q$, $\psi(a) = \psi(aq) = \psi(\pi(a)) = \Phi(a)$ and thus $\psi = \Phi$. But then for $h \in H \setminus \{e\}$, $qu_h = \pi(v_h) = \Phi(v_h) = 0$, which shows that $q = 0$ — a contradiction. This shows that $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is properly outer and thus, there exists a unique pseudo-expectation, which shows the desired. □

We end this subsection with a corollary of Proposition 5.2.9 giving the desired characterization of when a partial C*-dynamical system is hereditarily essential.

Corollary 5.2.10. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system. The following are equivalent:*

(i) $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ *is hereditarily essential;*

(ii) $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ *is hereditarily essential;*

(iii) $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ *is properly outer.*

Proof. By [PZ15, Theorem 3.5], $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is hereditarily essential if and only if each pseudo-expectation is faithful. Combining Proposition 4.1.26 and Proposition 5.2.9, we see that this occurs if and only if the partial C*-dynamical system is properly outer. As this is equivalent to the proper outerness of the injective unitization $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$, this concludes the proof. □

5.3 Characterizations of the Ideal Intersection Property

5.3.1 Cohomological Obstruction

As observed in [KS19, Example 5.6] even in the case of global C*-dynamical systems, proper outerness is not equivalent to the ideal intersection property, which is in line with Corollary 5.2.10 as there we show that proper outerness is equivalent to the stronger condition of the partial C*-dynamical system being hereditarily essential. However, Kennedy and Schafhauser identify a cohomological obstruction, the vanishing of which allows for a full characterization of when proper outerness on the G-injective envelope is in fact equivalent to the ideal intersection property. In the case of partial C*-dynamical systems we observe a similar obstruction.

While global C*-dynamical systems are in particular examples of partial C*-dynamical systems, the finite-dimensional example given by Kennedy and Schafhauser was very instructional and influential for overcoming this obstruction in follow-up work [GU23; KKS25]. Thus, in the next example we give a truly partial analogue of [KS19, Example 5.6] showcasing the cohomological obstruction in a very elementary setting.

Example 5.3.1. Consider $G = \mathbb{Z}_3 \times K_4$, where $K_4 = \{e, u, v, uv\}$ denotes the Klein four-group, and $A = M_2 \oplus M_2$. Let $A_1 := M_2 \oplus 0$ and $A_2 := 0 \oplus M_2$ denote the non-trivial

ideals in A. Furthermore, define the flip $\sigma\colon A_2 \to A_1\colon 0 \oplus a \mapsto a \oplus 0$ and

$$W_u = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad W_v = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad W_{uv} = W_u W_v = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Note that since $W_u W_v = -W_u W_v$, we have $\mathrm{Ad}(W_u) \circ \mathrm{Ad}(W_v) = \mathrm{Ad}(W_v) \circ \mathrm{Ad}(W_u) = \mathrm{Ad}(W_{uv})$. Hence, we define the following partial action of G on A via

$$\begin{aligned} \alpha_{(0,e)} &= \mathrm{id}_A, & \alpha_{(0,u)} &= \mathrm{Ad}(W_u \oplus W_u), \\ \alpha_{(0,v)} &= \mathrm{Ad}(W_v \oplus W_v), & \alpha_{(0,uv)} &= \mathrm{Ad}(W_{uv} \oplus W_{uv}), \end{aligned}$$

and

$$\begin{aligned} &\alpha_{(1,h)}\colon A_2 \to A_1\colon 0 \oplus a \mapsto (\mathrm{Ad}(W_h) \circ \sigma)(0 \oplus a), \text{ for all } h \in K_4, \\ &\alpha_{(2,h)}\colon A_1 \to A_2\colon a \oplus 0 \mapsto (\mathrm{Ad}(W_h) \circ \sigma^{-1})(a \oplus 0), \text{ for all } h \in K_4, \end{aligned}$$

where W_e denotes the identity matrix. Since $\alpha_{(0,u)}, \alpha_{(0,v)}, \alpha_{(0,uv)}$ are inner, this partial C^*-dynamical system is not properly outer. However, it has the ideal intersection property, since $A \rtimes_{\mathrm{r}} G$ is simple.

Proof of Simplicity: By definition of the covariance algebra, we see that $A \rtimes_{\mathrm{r}} G$ is 64-dimensional. We will show that $A \rtimes_{\mathrm{r}} G \simeq M_8$ by showing that the centre of the crossed product is trivial. Suppose $x = \sum_{g \in G} a_g \lambda_g \in Z(A \rtimes_{\mathrm{r}} G)$, then for all $b \in A$ we must have

$$bx = \sum_{g \in G} b a_g \lambda_g = \sum_{g \in G} \alpha_g(\alpha_{g^{-1}}(a_g) b) \lambda_g = xb.$$

Thus, $b a_g = \alpha_g(\alpha_{g^{-1}}(a_g) b)$ for all $g \in G$ and $b \in A$. If $g = (0, h)$ for $h \in K_4$, this gives $b a_g (W_h \oplus W_h) = a_g (W_h \oplus W_h) b$ for all $b \in A$, so $a_g (W_h \oplus W_h) \in \mathbb{C} \oplus \mathbb{C}$. For $g = (1, h)$ with $h \in K_4$, $a_g \in A_1$ and we obtain

$$b a_g = b_1 a_g = \mathrm{Ad}(W_h)((\mathrm{Ad}(W_h) \circ \sigma^{-1})(a_g) b_2) = a_g W_h b_2 W_h,$$

where $b = b_1 \oplus b_2$. Since this holds in particular for all $b \in A$, setting $b_1 = b_2$ shows that $a_g W_h \in \mathbb{C}$. However, setting $b_1 = -b_2$, gives that $a_g = 0$. Similarly, one can see that $a_g = 0$ for $g = (2, h)$ with $h \in K_4$. Secondly, for all $t \in G$, we must have

$$\lambda_t x = \sum_{g \in K_4} \alpha_t(p_{t^{-1}} a_{(0,g)}) \lambda_{t(0,g)} = \sum_{g \in K_4} \alpha_{(0,g)}(\alpha_{(0,g)}(a_{(0,g)}) p_t) \lambda_{t(0,g)},$$

where p_t denotes as usual the unit of A_t. Now if $t = (0,h)$ for $h \in K_4$, then this gives $(W_h \oplus W_h)a_{(0,g)} = a_{(0,g)}(W_h \oplus W_h)$. Hence, since W_e, W_u, W_v, W_{uv} span M_2, this shows that $a_{(0,g)} \in \mathbb{C} \oplus \mathbb{C}$. Together with $a_{(0,g)}W_g \in \mathbb{C} \oplus \mathbb{C}$ from above, this shows $a_{(0,g)} = 0$ for $g \neq e$. Lastly, it remains to show that $a_{(0,e)} \in \mathbb{C}$. To see this consider $t = (1,e)$. Then $\alpha_t(p_{t^{-1}}a_e) = \alpha_t(0 \oplus a_{e,2}) = a_{e,2} \oplus 0 = a_{e,1} \oplus 0 = a_e p_t$, showing $a_{e,1} = a_{e,2}$. In summary, $a_e \in \mathbb{C}$ and $Z(A \rtimes_{\mathrm{r}} G) \simeq \mathbb{C}$, which shows $A \rtimes_{\mathrm{r}} G \simeq M_8$. □

Recall that given a partial action $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$, we can look at its associated pointwise inner partial C*-dynamical system $(\{B_g\}_{g\in G}, \{\mathrm{Ad}(w_g)\}_{g\in G})$ (see Definition 4.1.19), where $B_g = I(A)q_g$ with $q_g \in Z(I(A))$ being the largest projection such that $I(\alpha)_{g^{-1}}$ is inner on $I(A)q_g$ and $w_g \in I(A)q_{g^{-1}}$ is a partial isometry implementing the action. As mentioned, this definition does not depend on the choice of partial isometry. In particular, since by Lemma 4.1.18 we have that $I(\alpha)_g(q_g p_{h^{-1}}) = q_{hgh^{-1}}p_h$, we obtain by a similar calculation as in the proof of the lemma

$$\begin{aligned} I(\alpha)_g(I(\alpha)_{h^{-1}}(bp_h)q_{g^{-1}})w_g &= I(\alpha)_g(I(\alpha)_{h^{-1}}(bp_h)I(\alpha)_{h^{-1}}(q_{hg^{-1}h^{-1}}p_h))w_g \\ &= I(\alpha)_{gh^{-1}}(bq_{hg^{-1}h^{-1}}p_h)w_g \\ &= w_g I(\alpha)_{h^{-1}}(bp_h) \end{aligned}$$

for $b \in I(A)$. Hence, applying $I(\alpha)_h$ shows that $I(\alpha)_h(w_g p_{h^{-1}})$ is a partial isometry implementing $I(\alpha)_{hgh^{-1}}$ on $I(A)q_{hgh^{-1}} \cap I(A)_h$. However, it is not clear that the family of partial isometries can be chosen such that $I(\alpha)_h(w_g p_{h^{-1}}) = w_{hgh^{-1}}p_h$. The next definition highlights an obstruction in characterizing the ideal intersection property in terms of proper outerness.

Definition 5.3.2. Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system. If there exists a partial *-representation $u\colon G \to I(A)$ such that the associated pointwise inner partial C*-dynamical system for $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is given by $(\{I(A)u_g u_g^*\}_{g\in G}, \{\mathrm{Ad}(u_g)\}_{g\in G})$ with $I(\alpha)_h(u_g p_{h^{-1}}) = u_{hgh^{-1}}p_h$ for $I(A)_h = I(A)p_h, h \in G$, we say that $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has *vanishing obstruction*.

Remark 5.3.3. The definition of vanishing obstruction precisely states that the associated pointwise inner partial C*-dynamical system is in fact inner *and* $I(\alpha)$ acts on u_g by conjugation. By definition, if the associated pointwise inner partial C*-dynamical system $(\{I(A)u_g u_g^*\}_{g\in G}, \{\mathrm{Ad}(u_g)\}_{g\in G})$ to the unital partial C*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is given by a partial *-representation $u\colon G \to I(A)$, we have $I(\alpha)_h(u_g q_{h^{-1}}) = q_{h^{-1}}u_h u_g u_h^* = u_h u_{gh^{-1}} u_h u_h^* = u_h u_h^* u_{hgh^{-1}} = u_{hgh^{-1}}q_h$. However, this is not necessarily equivalent to $I(\alpha)_h(u_g p_{h^{-1}}) = u_{hgh^{-1}}p_h$, since $I(\alpha)_h$ does not necessarily act inner on $A_h \cap Aq_{g^{-1}}$.

Example 5.3.4. Suppose $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is a partial C*-dynamical system on a commutative C*-algebra A, then $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has vanishing obstruction. To see this note that since A is commutative (subsequently so is $I(A)$), the only inner automorphism is the trivial automorphism. Hence, by Proposition 4.1.7, there exist maximal projections $q_{g^{-1}} \in Z(I(A))$ such that $I(\alpha)_g$ is trivial on $I(A)q_{g^{-1}}$. In particular, the associated pointwise inner partial C*-dynamical system is given by $(\{I(A)q_g\}_{g\in G}, \{\mathrm{Ad}(q_g)\}_{g\in G})$, i.e., the partial *-automorphisms act via compression by q_g. Since $q_g q_h \leq q_{gh}$ for all $g, h \in G$ as seen in the proof of Proposition 4.2.3, the map $q\colon G \to I(A)\colon g \mapsto q_g$ is a partial *-representation by [Exe17, Proposition 9.6]. Furthermore, by Lemma 4.1.18 $I(\alpha)_g(q_h p_{g^{-1}}) = q_{ghg^{-1}} p_g$ and thus, $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has vanishing obstruction.

Example 5.3.5. The partial C*-dynamical system in Example 5.3.1 does not have vanishing obstruction. To see this first observe, since $A_1 \cap A_2 = \{0\}$, $\alpha_{(1,h)}, \alpha_{(2,h)}$ are properly outer for $h \in K_4$. Also by definition $\alpha_{(0,h)}$, $h \in K_4$ are inner. So the corresponding pointwise inner partial C*-dynamical system is given by K_4 acting on A via unitary adjugation using W_u, W_v, W_{uv}. However, this system is not inner since W_u, W_v anticommute.

In the remainder of this section, we show that this can be in fact thought of as a cohomological obstruction that is the same as in the global setting. To do so we recall the theory of cohomology for unital partial actions introduced in [DK15].

Definition 5.3.6. Suppose $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is a unital partial C*-dynamical system with $A_g = Ap_g$ for a projection $p_g \in Z(A)$ and all $g \in G$. A function $f\colon G \to A$ is called a *partial 1-cochain* if $f(t)$ is invertible in A_t for all $t \in G$. Similarly we call $f\colon G \times G \to A$ a *partial 2-cochain* if $f(s,t)$ is invertible in $A_s \cap A_{st}$. We say that a partial 2-cochain $\sigma\colon G \times G \to A$ is a *partial 2-cocycle* if

$$\alpha_r(p_{r^{-1}}\sigma(s,t))\sigma(r,st) = \sigma(rs,t)\sigma(r,s)$$

for all $r, s, t \in G$. Lastly, σ as above is called a *partial 2-coboundary* if there exists a partial 1-cochain $f\colon G \to A$ such that

$$\sigma(s,t) = \alpha_s(p_{s^{-1}}f(t))f(st)^{-1}f(s)$$

for $s, t \in G$.

The partial isometries implementing the inner parts of the action give rise to a partial 2-cocycle as the next result shows.

Proposition 5.3.7. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C^*-dynamical system on an injective C^*-algebra A and let $(\{Aw_gw_g^*\}_{g\in G}, \{\mathrm{Ad}(w_g)\}_{g\in G})$ denote the corresponding pointwise inner system. The map $\sigma\colon G\times G\to A\colon (s,t)\mapsto w_sw_tw_{st}^*$ is a partial 2-cocycle. Furthermore, $\sigma(s,t)\in Z(A)$. If the map $\{w_g\}$ can be chosen to be a partial *-representation, $\sigma(s,t)$ is a partial 2-coboundary.*

Proof. Consider the associated pointwise inner partial C^*-dynamical system $(\{Aw_gw_g^*\}_{g\in G}, \{\mathrm{Ad}(w_g)\}_{g\in G})$ and let $q_g = w_gw_g^*$. By Corollary 4.1.4, we have $q_g = q_{g^{-1}}$. First, observe for $s,t\in G$, $\sigma(s,t)\in Aq_s\cap Aq_{st}$. To see that it is invertible note that $\alpha_{s^{-1}}(\sigma(s,t)) = w_s^*\sigma(s,t)w_s = q_sw_tw_{st}^*w_s$. This is invertible in $Aq_s\cap Aq_t$, since $q_sw_tw_{st}^*w_s(w_tw_{st}^*w_s)^* = q_tq_s$, which shows the desired. Second, note that for $a\in A$, since $\sigma(s,t)a\in Aq_s\cap Aq_t\cap Aq_{st}$, we have

$$\begin{aligned}\sigma(s,t)a &= \sigma(s,t)aq_sq_tq_{st} = w_sw_tw_{st}^*aw_{st}w_t^*w_s^*\sigma(s,t)\\ &= \alpha_s(\alpha_t(\alpha_{st}^{-1}(aq_sq_tq_{st})))\sigma(s,t) = a\sigma(s,t).\end{aligned}$$

Hence, σ is a $Z(A)$-valued partial 2-cochain.

To see that σ is in fact a $Z(A)$-valued partial 2-cocycle, take $r,s,t\in G$ and notice

$$\alpha_r(q_{r^{-1}}\sigma(s,t))\sigma(r,st) = w_rw_sw_tw_{st}^*w_r^*w_rw_{st}w_{rst}^* = q_{st}w_rw_sw_tw_{rst}^*,$$

whereas

$$\sigma(rs,t)\sigma(r,s) = \sigma(r,s)\sigma(rs,t) = q_{rs}w_rw_sw_tw_{rst}^*$$

using the fact that $\sigma(r,s)\in Z(A)$. Now, recall as seen in the proof of Proposition 4.2.3 $q_gq_h\leq q_{gh}$ for all $g,h\in G$. In particular, $q_gq_h\in Aq_{gh}$ and thus, $\alpha_r(q_{r^{-1}}\sigma(s,t))\sigma(r,st) = \sigma(rs,t)\sigma(r,s)$.

Lastly, suppose that $\{w_g\}_{g\in G}$ is in fact given by a partial *-representation and consider the partial 1-cochain $f\colon G\to A\colon g\mapsto q_g$. Then,

$$\alpha_s(q_{s^{-1}}f(t))f(st)^{-1}f(s) = w_sq_sq_tw_s^*q_{st}q_s = q_sq_t,$$

for all $s,t\in G$, where we again used $q_sq_t\in Aq_{st}$. Now since $\{w_g\}$ is a partial *-representation, we have $w_{st}q_tq_s = w_sw_tw_t^*w_tq_s = w_sw_t$ for all $s,t\in G$. But this immediately gives

$$\sigma(s,t) = w_sw_tw_{st}^*q_sq_t = w_sw_t(w_sw_t)^* = q_sq_t,$$

showing that σ is a partial 2-coboundary. □

5.3.2 Proper Outerness

In this section, we will show that under vanishing obstruction, Corollary 5.1.8 can actually be reversed. To do so, we will build a non-faithful equivariant pseudo-expectation in case the partial C*-dynamical system is not properly outer but has vanishing obstruction. The next lemma gives the building block for this equivariant pseudo-expectation. The proof is an adaption of [KS19, Lemma 9.1] to the partial setting.

Lemma 5.3.8. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system with vanishing obstruction and let $u\colon G \to I(A)$ be a partial *-representation for the associated pointwise inner partial C*-dynamical system as in Definition 5.3.2. Furthermore, let $r_g \in Z(I_G(A))$ denote the largest projection such that $I_G(\alpha)_g$ is inner on $I_G(A)r_g$ for $g \in G$. There exists an equivariant conditional expectation $\phi\colon I_G(A) \rtimes_{\mathrm{r}} G \to I_G(A)$ satisfying $\phi(a\lambda_g) = au_g r_g$ for $g \in G$.*

Proof. For $g \in G$ let $A_g = Ap_g$, where $p_g \in Z(A)$ is a projection. We will build ϕ using Glimm ideals $z \in \mathrm{Glimm}(I_G(A))$. As $u\colon G \to I(A)$ is a partial representation of $I(A)$ and not $I_G(A)$, define the compression map $\pi\colon I(A) \to I_G(A)^{**}(1 - P_z^{**})$, where P_z^{**} denotes the open projection in $I_G(A)^{**}$ such that $z^{**} = I_G(A)^{**}P_z^{**}$ and $(I_G(A)/z)^{**} \cong I_G(A)^{**}(1 - P_z^{**})$. Recall that for $z \in \mathrm{Glimm}(I_G(A))$ the quasistabilizer subgroup $G_z \leq G$ is amenable by Theorem 4.2.6, thus we obtain a covariant pair for the partial C*-dynamical system $(I(A), G_z, \{I(A)_g\}_{g\in G_z}, \{I(\alpha)_g\}_{g\in G_z})$ given by $(\pi_z, \pi_z \circ u)$. Consider the conditional expectation $E_{G_z}\colon I(A) \rtimes_{\mathrm{r}} G \to I(A) \rtimes G_z$ given by $E_{G_z}(a\lambda_t) = a\lambda_t$ for $t \in G_z$ and $a \in I(A)_t$ and $E_{G_z}(a\lambda_t) = 0$ otherwise. Let $\rho_z\colon I(A) \rtimes G_z \to I_G(A)^{**}(1 - P_z^{**})$ denote the partial *-representation corresponding to the covariant pair $(\pi_z, \pi_z \circ u)$. Then, recalling $G_z = \{s \in G \,|\, z \in \mathrm{supp}(r_s)\}$, we see that $\rho_z(a\lambda_t) = au_t(1 - P_z^{**})$ and subsequently

$$\rho_z \circ E_{G_z}(a\lambda_t) = \begin{cases} au_t(1 - P_z^{**}) & \text{for } t \in G_z, \\ 0 & \text{otherwise,} \end{cases}$$

for $a \in I(A)$, $t \in G$. Using the fact that the projections $1 - P_z^{**}$ are mutually orthogonal for $z \in \mathrm{Glimm}((I_G(A))$ the u.c.p. map $\Phi_1 = \bigoplus_{z\in\mathrm{Glimm}(I_G(A))} \rho_z \circ E_{G_z}$ is well-defined. Furthermore, the intersection over all Glimm ideals $z \in \mathrm{Glimm}(I_G(A))$ is trivial, thus we obtain an embedding $I_G(A) \hookrightarrow \bigoplus_{z\in\mathrm{Glimm}(I_G(A))} I_G(A)(1 - P_z^{**})$ and there exists a u.c.p. map $\Phi_2\colon \bigoplus_{z\in\mathrm{Glimm}(I_G(A))} I_G(A)(1 - P_z^{**}) \to I_G(A)$ such that the composition $\phi := \Phi_2 \circ \Phi_1\colon I(A) \rtimes_{\mathrm{r}} G \to I_G(A)$ satisfies $\phi(a\lambda_t) = au_t r_t$ for $a \in I(A)$ and $t \in G$.

To see that this is a G-morphism first observe that it is G-unital by definition. Second, note that it is enough to check equivariance on the covariance algebra and let $a\lambda_s \in B_g$ for some $g \in G$, then

$$\phi(\beta_g(a\lambda_s)) = \phi(I(\alpha)_g(a)\lambda_{gsg^{-1}}) = I(\alpha)_g(a)u_{gsg^{-1}}r_{gsg^{-1}} = I(\alpha)_g(ap_{g^{-1}}u_s r_s),$$

where in the last equality we used that $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has vanishing obstruction. □

Theorem 5.3.9. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system with vanishing obstruction. Then $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has the ideal intersection property if and only if the partial C*-dynamical system $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$ is properly outer.*

Proof. Without loss of generality, assume $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is unital, since otherwise we can work with its injective unitization. Suppose $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has vanishing obstruction and let $u\colon G \to I(A)$ be a partial *-representation for the associated pointwise inner partial C*-dynamical system. If $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$ is not properly outer, then there exits $g \in G$ and a non-zero central projection $r_{g^{-1}} \in I_G(A)$ such that $I_G(\alpha)_g$ is inner on $I_G(A)r_{g^{-1}}$. By Lemma 5.3.8 there exists an equivariant pseudo-expectation $\phi\colon I_G(A) \rtimes_{\mathrm{r}} G \to I_G(A)$ such that $\phi(a\lambda_g) = au_g r_g$. Note that $u_g \neq 0$ as $0 \neq r_g \leq q_g = u_g u_g^*$ by Theorem 4.1.20. But then considering $x = r_g(u_g - \lambda_g)$ we obtain

$$\begin{aligned}\phi(x^*x) &= r_g\phi(u_g u_g^* - u_g^*\lambda_g - \lambda_g^* u_g + \lambda_g\lambda_g^*)\\ &= 2r_g - u_g^*\phi(\lambda_g) - \phi(\lambda_g^*)u_g\\ &= 2r_g - 2u_g^* u_g r_g = 0,\end{aligned}$$

where we used that $I_G(A)$ is in the multiplicative domain of ϕ. Hence, ϕ is not faithful and by Theorem 5.1.5, $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ does not have the ideal intersection property.

Conversely, suppose that $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$ is properly outer, then by Proposition 5.1.6 the only equivariant pseudo-expectation is the canonical one. Since it is in particular faithful, Theorem 5.1.5 shows the desired. □

Notice that it was necessary to work with $I_G(A)$ in Lemma 5.3.8 to obtain amenable quasi-stabilizer subgroups. If in turn, G is amenable the stabilizer for any $I \in \mathrm{Glimm}_{\mathrm{p}}(I(A))$ will be amenable and thus, we can restrict our attention to $I(A)$ instead. We obtain the following result in this case.

Corollary 5.3.10. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system with vanishing obstruction and suppose that G is amenable. Then, $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has the ideal intersection property if and only if the partial C*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is properly outer.*

Proof. By Corollary 5.1.8 together with Corollary 4.1.21 if $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is properly outer, then it has the ideal intersection property. Conversely, if $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has vanishing obstruction and is not properly outer, one can construct an equivariant conditional expectation $\phi\colon I(A) \rtimes_{\mathrm{r}} G \to I(A)$ similar to the proof of Lemma 5.3.8 with $\phi(a\lambda_g) = au_g q_g$, where q_g denotes the largest projection in $I(A)$ such that $I(\alpha)_g$ acts inner, by working with $\mathrm{Glimm}(I(A))$ instead. A similar argument to the previous proof then shows the desired. □

5.3.3 Non-commutative Uniformly Recurrent Partial Subsystems

We will now establish an intrinsic condition guaranteeing the ideal intersection property for unital partial actions on C*-algebras by countable discrete groups generalizing the notion of uniformly recurrent subgroups introduced in [GW15]. This characterization is in line with similar results given in [Kaw17; KS19; Ken20]. First, we show that given a unital partial C*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$, the compact space of pseudo-Glimm ideals $\mathrm{Glimm}_{\mathrm{p}}(A)$ can be equipped with an induced partial action.

Proposition 5.3.11. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system with $A_g = Ap_g$ for some central projection $p_g \in Z(A)$. Defining $X_g = \{I \in \mathrm{Glimm}_{\mathrm{p}}(A) \mid 1 - p_g \in I\}$ and*

$$\theta_g\colon X_{g^{-1}} \to X_g\colon I \mapsto \alpha_g(Ip_{g^{-1}}) + A(1 - p_g),$$

for $g \in G$ gives rise to a topological partial dynamical system $(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$.

Proof. First, we will show that the action is well-defined. To see this, observe that for $g \in G$, X_g is in fact a clopen subset of X, since if $(I_\lambda)_\lambda$ is a net with $1 - p_g \in I_\lambda$ for all λ and $\lim_\lambda I_\lambda = I$ for some $I \in \mathrm{Glimm}_{\mathrm{p}}(A)$, then $\|(1 - p_g) + I_\lambda\| = 0$ for all λ. Thus $\lim_\lambda \|(1 - p_g)\| = \|(1 - p_g) + I\| = 0$ and subsequently $1 - p_g \in I$, showing that X_g is closed. Similarly, suppose that $1 - p_g \notin I$, then for all proper and primal ideals $J \in I(A)$ with $J \cap A = I$, $1 - p_g \notin J$. But since J is proper and primal $p_g \in J$ and subsequently $p_g \in I$. A similar argument as before then shows that the complement is closed and thus, X_g is clopen.

Next, to see that $\theta_g(I)$ for $I \in X_{g^{-1}}$ is well-defined, note that since $I \in \mathrm{Glimm_p}(A)$, there exists a proper and primal ideal $J \trianglelefteq I(A)$ with $J \cap A = I$. Note that $I(\alpha)_g(Jp_{g^{-1}}) + I(A)(1-p_g)$ is also proper and primal. The former follows from the fact that $Jp_{g^{-1}}$ is proper and to see the latter, consider $K_1, \dots, K_n \trianglelefteq I(A)$ with $K_1 \cdots K_n = \{0\}$. If $K_i \subseteq I(A)(1-p_g)$ for some i , then $K_i \subseteq J$, so without loss of generality suppose $K_i p_g \neq \{0\}$ for all $i = 1, \dots, n$. Then $\{0\} = I(\alpha)_{g^{-1}}(K_1 p_g) \cdots I(\alpha)_{g^{-1}}(K_n p_g)$ and since J is primal and $I(\alpha)_{g^{-1}}(K_i p_g) \subseteq I(A)p_{g^{-1}}$ for all i, there exists an index i with $I(\alpha)_{g^{-1}}(K_i p_g) \subseteq Jp_{g^{-1}}$ or equivalently $K_i \subseteq I(\alpha)_g(Jp_{g^{-1}})$, which shows that $I(\alpha)_g(Jp_{g^{-1}}) + I(A)(1-p_g)$ is primal. Note also that $[I(\alpha)_g(Jp_{g^{-1}}) + I(A)(1-p_g)] \cap A = I(\alpha)_g(Jp_{g^{-1}} \cap A) + A(1-p_g)$ and $I(\alpha)_g(Jp_{g^{-1}}) \cap A = I(\alpha)_g((J \cap A)p_{g^{-1}}) = \alpha_g(Ip_{g^{-1}})$. In sum, $\theta_g(I) \in \mathrm{Glimm_p}(A)$.

Second, to see that the partial action axioms are satisfied, clearly $X_e = \mathrm{Glimm_p}(A)$ and $\theta_e = \mathrm{id}_X$. Now, let $g, h \in G$ and consider $I \in X_{g^{-1}} \cap X_h$. As $1 - p_h \in I$, $\alpha_g(p_{g^{-1}}(1-p_h)) = p_g(1-p_{gh}) \in \theta_g(I)$. Additionally, $(1-p_g)(1-p_{gh}) \in A(1-p_g) \subseteq \theta_g(I)$ and thus, $1 - p_{gh} \in \theta_g(I)$. Therefore, $\theta_g(I) \in X_{gh}$ and the second axiom is satisfied. For the third axiom, take $I \in X_{h^{-1}} \cap X_{(gh)^{-1}}$ and consider

$$\begin{aligned}
\theta_g(\theta_h(I)) &= \alpha_g(\alpha_h(Ip_{h^{-1}})p_{g^{-1}}) + \alpha_g(A(1-p_h)p_{g^{-1}}) + A(1-p_g) \\
&= \alpha_{gh}(Ip_{h^{-1}}p_{(gh)^{-1}}) + Ap_{gh}(1-p_g) + A(1-p_{gh}), \\
\theta_{gh}(I) &= \alpha_{gh}(Ip_{(gh)^{-1}}) + A(1-p_{gh}).
\end{aligned}$$

Note that clearly $\alpha_{gh}(Ip_{h^{-1}}p_{(gh)^{-1}}) \subseteq \alpha_{gh}(Ip_{(gh)^{-1}})$ and observe $Ap_{gh}(1-p_g) \subseteq \alpha_{gh}(A(1-p_{h^{-1}})p_{(gh)^{-1}})$, since $\alpha_{gh}(Ap_{h^{-1}}p_{(gh)^{-1}}) = A_g \cap A_{gh}$. Thus, $\theta_g(\theta_h(I)) \subseteq \theta_{gh}(I)$. For the other inclusion, notice that $\alpha_{gh}(Ip_{(gh)^{-1}})(1-p_g) \subseteq Ap_{gh}(1-p_g)$ and also

$$\alpha_{gh}(Ip_{(gh)^{-1}})p_g = \alpha_{gh}(Ip_{(gh)^{-1}})p_g p_{gh} = \alpha_{gh}(Ip_{(gh)^{-1}}p_{h^{-1}}),$$

where we again used $\alpha_{gh}(p_{h^{-1}}p_{(gh)^{-1}}) = p_g p_{gh}$. In sum, $\theta_{gh}(I) \subseteq \theta_g(\theta_h(I))$, which shows the third partial action axiom. □

Using this induced action we can now define a partial action on the space $\mathrm{Glimm_p}(A) \times \mathrm{Sub}(G)$. Here, $\mathrm{Sub}(G)$ denotes the set of subgroups equipped with the Chabauty topology, which can be identified as the relative topology coming from the product topology viewing $\mathrm{Sub}(G)$ as a subset of $\{0,1\}^G$. In particular, a net $(H_\lambda)_\lambda$ in $\mathrm{Sub}(G)$ converges to $H < G$, if for all $h \in H$, there exists μ such that for all $\lambda \geq \mu$, $h \in H_\lambda$ and for every subnet $(H_\nu)_\nu$, $\bigcap_\nu H_\nu \subseteq H$. We then obtain a compact global dynamical

system on $\mathrm{Sub}(G)$ via conjugation. Furthermore, the set of amenable subgroups of G, $\mathrm{Sub}_{\mathrm{a}}(G) \subseteq \mathrm{Sub}(G)$ is closed and invariant in $\mathrm{Sub}(G)$ and so we also obtain a compact subsystem $(\mathrm{Sub}_{\mathrm{a}}(G), G) \subseteq (\mathrm{Sub}(G), G)$. As $\mathrm{Glimm}_{\mathrm{p}}(A)$ is compact in the Fell topology, $\mathrm{Glimm}_{\mathrm{p}}(A) \times \mathrm{Sub}(G)$ is compact when equipped with the product topology. Combining the induced action in Proposition 5.3.11 with the conjugation action leads to the following action.

Definition 5.3.12. Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system with $A_g = Ap_g$ for some central projection $p_g \in Z(A)$ and let $Y = \mathrm{Glimm}_{\mathrm{p}}(A) \times \mathrm{Sub}(G)$. For $g \in G$, define $Y_{g^{-1}} = \{(I, H_I) \mid 1 - p_{g^{-1}} \in I\}$, and consider

$$v_g \colon Y_{g^{-1}} \to Y_g \colon (I, H_I) \mapsto (\alpha_g(Ip_{g^{-1}}) + A(1 - p_g), gH_Ig^{-1}).$$

We call the resulting partial dynamical system $(\{Y_g\}_{g\in G}, \{v_g\}_{g\in G})$ the *partial subgroup action induced by* $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$.

Remark 5.3.13. If $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is in fact a global C*-dynamical system, i.e., $p_g = 1$ for all $g \in G$, then the action in Definition 5.3.12 coincides with the one defined in [KS19].

We are now ready to introduce the notion of non-commutative uniformly recurrent partial subsystem for a unital partial C*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$. We will say that a family of ideals $(I_\lambda)_\lambda$ in A *covers* A, if $\bigcap_\lambda I_\lambda = \{0\}$.

Definition 5.3.14. Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system and consider the corresponding partial subgroup action $(\{Y_g\}_{g\in G}, \{v_g\}_{g\in G})$ on $\mathrm{Glimm}_{\mathrm{p}}(A) \times \mathrm{Sub}(G)$ as given in Definition 5.3.12. We call an invariant closed subset $Z \subseteq Y$ a *non-commutative uniformly recurrent partial subsystem (URpS)* if the set of ideals $\{I \mid (I, H_I) \in Z\}$ covers A and $H_I < G_I$ for all $(I, H) \in Z$. Furthermore, if $\{H_I \mid (I, H_I) \in Z\} \subseteq \mathrm{Sub}_{\mathrm{a}}(G)$, we say the non-commutative URpS Z is *amenable*. Lastly, Z is called *trivial*, if $\{H_I \mid (I, H_I) \in Z\} = \{\{e\}\}$.

Remark 5.3.15. The definition of non-commutative URpS is well-defined, since $gH_Ig^{-1} < G_{\theta_g(I)}$ for $I \in Y_{g^{-1}}$, if $H_I < G_I$, where $(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ denotes the partial action on $\mathrm{Glimm}_{\mathrm{p}}(A)$ as in Proposition 5.3.11. To see this it suffices to show for $s \in G_I$, $gsg^{-1} \in G_{\theta_g(I)}$. Suppose $s \in G_I$, and let $J \trianglelefteq A$ be a proper and primal ideal with $J \cap A = \theta_g(I)$. By the same arguments as above $I(\alpha)_{g^{-1}}(Jp_g) + I(A)(1 - p_{g^{-1}})$ is a proper and primal ideal with $[I(\alpha)_{g^{-1}}(Jp_g) + I(A)(1 - p_{g^{-1}})] \cap A = I$. Since $s \in G_I$, $1 - q_s \in I(\alpha)_{g^{-1}}(Jp_g) + I(A)(1 - p_{g^{-1}})$. In particular, by Lemma 4.1.18 $I(\alpha)_g(p_{g^{-1}} -$

$q_s p_{g^{-1}}) = p_g - q_{gsg^{-1}} p_g = (1 - q_{gsg^{-1}}) p_g \in J p_g$. Since $(1 - q_{gsg^{-1}})(1 - p_g) \in J$ as $1 - p_g \in J$, this shows $1 - q_{gsg^{-1}} \in J$, which shows the desired. Applying the same argument with $\theta_{g^{-1}}$ shows that $g^{-1} G_{\theta_g(I)} g \subseteq G_I$ and we obtain $G_{\theta_g(I)} = g G_I g^{-1}$.

Remark 5.3.16. If $Z \subseteq Y$ is a non-commutative URpS, by definition it is in particular invariant under the corresponding partial subgroup action $(\{Y_g\}_{g\in G}, \{v_g\}_{g\in G})$ on $\mathrm{Glimm}_{\mathrm{p}}(A) \times \mathrm{Sub}(G)$. Hence, Z itself corresponds to a topological partial dynamical system $(\{Z_g\}_{g\in G}, \{v_g\}_{g\in G})$ with $Z_g = Y_g \cap Z$ for all $g \in G$.

To see the connection between the ideal intersection property and amenable non-commutative URpS, the next lemma gives a potential starting point to construct an amenable non-commutative URpS for countable discrete groups. This is [KS19, Proposition 4.9] for partial C*-dynamical systems. By Proposition 3.3.4 the inclusion $\iota\colon A \hookrightarrow I_G(A)$ is normal and thus the proof is similar and will be omitted. Observe that the proof of Proposition 3.3.4 also works in the non-unital setting.

Lemma 5.3.17. *Let G be a countable discrete group and suppose $(A, G, \{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is a partial C*-dynamical system with G-injective envelope $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$. There exists a dense subset $X \subseteq \mathrm{Glimm}(I_G(A))$ such that the stabilizer map sending x to its quasi-stabilizer subgroup G_x in $I_G(A)$ is continuous when restricted to X. Furthermore, X is a G_δ set and $G_x \leq G_{x\cap A}$ for all $x \in X$. Lastly, there exists an $x \in X$ with $G_x \neq \{e\}$, if $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$ is not properly outer.*

We can now prove that the lack of amenable non-commutative uniformly recurrent partial subsystems that do not contain a trivial non-commutative URpS guarantees the ideal intersection property.

Proposition 5.3.18. *Let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a unital partial C*-dynamical system of a countable discrete group G. If every amenable non-commutative URpS contains a trivial non-commutative URpS, the partial C*-dynamical system has the ideal intersection property.*

Proof. We will prove the contrapositive. If $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ does not have the ideal intersection property, the partial action $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$ is not properly outer by Corollary 5.1.8. By Lemma 5.3.17 there exists a dense G_δ-subset $X \subseteq \mathrm{Glimm}(I_G(A))$ such that the map $x \mapsto G_x$ is continuous on X with $\{x \in X \mid G_x \neq \{e\}\} \neq \varnothing$. We will construct a non-commutative URpS Z from a closed invariant subset in $\mathrm{Glimm}(I_G(A)) \times \mathrm{Sub}(G)$.

First, observe that we can restrict the partial subgroup action induced by $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$ to $\mathrm{Glimm}(I_G(A)) \times \mathrm{Sub}(G)$. If $1 - p_{g^{-1}} \in I_G(A) K_x$ for $x \in$

$\mathrm{Glimm}(I_G(A))$, $x \in \mathrm{supp}(p_{g^{-1}})$. In particular, $I_G(A)(1-p_{g^{-1}}) \in I_G(A)K_x$. Additionally, observe

$$\begin{aligned} I_G(\alpha)_g(I_G(A)K_x p_{g^{-1}}) + I_G(A)(1-p_g) &= I_G(A)_g I_G(\alpha)_g(K_x p_{g^{-1}}) + I_G(A)(1-p_g) \\ &= I_G(A)_g K_{\overline{I_G(\alpha)}_g(x)} + I_G(A)(1-p_g), \end{aligned} \tag{5.5}$$

where $\overline{I_G(\alpha)}_g(\cdot)$ denotes the induced action on the spectrum of $Z(I_G(A))$. In sum, $I_G(\alpha)_g(I_G(A)K_x p_{g^{-1}}) + I_G(A)(1-p_g)$ is again a Glimm ideal corresponding to $\overline{I_G(\alpha)}_g(x)$.

Now, as shown in Remark 5.3.15 for $x \in X$, $gG_xg^{-1} = G_{\theta_g(x)}$, where θ is the partial action defined in Proposition 5.3.11. Moreover, observe that if $x \in \mathrm{Glimm}(I_G(A))$ is a point of continuity for the stabilizer map and $I_G(\alpha)_g(x)$ is well-defined, then $I_G(\alpha)_g(x)$ is also a point of continuity. Similarly, if $G_x \leq G_{x\cap A}$ for $x \in X$, $G_{I_G(\alpha)_g(x)} \leq G_{I_G(\alpha)_g(x)\cap A}$. Thus, if necessary we can replace X with the orbit of X and the set $\{(x, G_x) \,|\, x \in X\}$ is invariant under the partial subgroup action induced by $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$. Let $\tilde{Z}$ denote the closure of $\{(x, G_x) \,|\, x \in X\}$, which is again invariant and clearly $\tilde{Z} \subseteq \mathrm{Glimm}(I_G(A)) \times \mathrm{Sub}(G)$. By compactness of $\mathrm{Glimm}(I_G(A)) \times \mathrm{Sub}(G)$, $\tilde{Z}$ is compact and since X is dense, $\{z \,|\, (z, H) \in \tilde{Z}\} = \mathrm{Glimm}(I_G(A)))$. Recall that $z \cap A \in \mathrm{Glimm}_{\mathrm{p}}(A)$ for $z \in I_G(A)$, thus we can consider

$$Z = \{(z \cap A, H) \,|\, (z, H) \in \tilde{Z}\} \subseteq \mathrm{Glimm}_{\mathrm{p}}(A) \times \mathrm{Sub}(G).$$

This is invariant as $\tilde{Z}$ is. Furthermore, since the intersection map is continuous in the Fell topology [KS19, Lemma 3.5], Z is closed by compactness of $\tilde{Z}$.

To see that Z is an amenable non-commutative URpS, observe that for $(z, H) \in \tilde{Z}$, $H < G_z$. To see this, let $(z_\lambda, G_{z_\lambda})$ be a net converging to (z, H). By definition of the Chabauty topology, for $h \in H$ there exists μ such that for $\lambda \geq \mu$, $h \in G_{z_\lambda}$. By definition of quasistabilizer groups, this implies $z_\lambda \in \mathrm{supp}(r_h)$, where $r_h \in Z(I_G(A))$ denotes the largest invariant projection such that $I_G(\alpha)_h$ is inner. Since $\mathrm{supp}(r_h)$ is clopen, $z \in \mathrm{supp}(r_h)$ and thus $h \in G_z$. Now, $H < G_z < G_{z\cap A}$ and since G_z is amenable by Theorem 4.2.6, so is H. Lastly, note that $\{A \cap z \,|\, (z, H) \in Z\} = \{z \cap A \,|\, z \in \mathrm{Glimm}(I_G(A)))\}$. As $\bigcap_{z\in\mathrm{Glimm}(I_G(A))} z = \{0\}$, this shows that Z is an amenable non-commutative URpS.

Now, suppose that Z does contain a trivial non-commutative URpS Z', i.e.,

$$Z' = \{(z \cap A, \{e\}), \,|\, z \in X'\}$$

for some invariant closed subset $X' \subseteq \mathrm{Glimm}(I_G(A))$ with $\bigcap_{x\in X'} x \cap A = \{0\}$. Since $J = \bigcap_{x\in X'} x$ is a closed invariant ideal and by assumption $J \cap A = \{0\}$, $J = \{0\}$ as

$I_G(A)$ is G-essential. This implies $X' = \mathrm{Glimm}(I_G(A))$. In particular, $\{(z, \{e\}), z \in X\} \subseteq \tilde{Z}$. However, this leads to a contradiction. To see this, recall that X consists of points of continuity for the stabilizer map $z \mapsto G_z$, and thus if $(z_\lambda, G_{z_\lambda})$ is a net in $\{(x, G_x) \mid x \in X\}$ converging to $(z, H) \in \tilde{Z}$, then $H = G_z$. In other words, for all $z \in X$, $\{H \mid (z, H) \in \tilde{Z}\} = \{G_z\}$. Together with the above, this implies $G_x = \{e\}$ for all $x \in X$, which is a contradiction to the original choice of X. Hence, Z does not contain a trivial non-commutative URpS. □

5.4 Applications

5.4.1 Commutative Partial C*-dynamical Systems

As seen in Example 5.3.4 for partial actions on commutative C*-algebras the obstruction given in Definition 5.3.2 vanishes and we obtain a full characterization of when the reduced crossed product has the ideal intersection property. In this section, we will give an equivalent characterization of proper outerness in terms of the underlying topological partial dynamical system leading to a full characterization when the corresponding reduced crossed product has the ideal intersection property. Similar to the case of global C*-dynamical systems, proper outerness is a non-commutative generalization of topological freeness.

Definition 5.4.1. Let $(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action on a compact Hausdorff space X. For $g \in G$, define the fixed point set of θ_g as $F_g = \{x \in X_{g^{-1}} | \theta_g(x) = x\}$. We call the partial action *free*, if $F_g = \varnothing$ for all $g \in G \setminus \{e\}$. Similarly, we say the partial action is *topologically free*, if $(F_g)^\circ = \varnothing$ for all $g \in G \setminus \{e\}$.

Note that if $(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ is a topological partial dynamical system on a compact Hausdorff space X, then the injective envelope $I(\mathcal{C}(X))$ of the corresponding partial C*-dynamical system on $\mathcal{C}(X)$ is again commutative, i.e., $I(\mathcal{C}(X)) \cong \mathcal{C}(\hat{X})$. The corresponding spectrum is projective in the category of compact Hausdorff spaces. By Gleason's Theorem $\hat{X}$ is extremally disconnected [Gle58, Theorem 2.5]. We will denote the induced partial action of $(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ on the projective cover $\hat{X}$ by $(\{\hat{X}_g\}_{g\in G}, \{\hat{\theta}_g\}_{g\in G})$.

The next lemma is well-known in the case of global group actions on compact Hausdorff spaces. It relies on Frolík's Theorem [Fro71, Theorem 3.1] as stated and adapted in [Pit17, Proposition 2.11].

Lemma 5.4.2. *Let $(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action on an extremally disconnected compact Hausdorff space. Then, $(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ is topologically free if and only if it is free.*

Proof. By [Pit17, Proposition 2.11] the set of fixed points F_g of θ_g is clopen. Thus, $F_g = (F_g)^\circ$, which shows the desired. □

The next proposition shows that proper outerness is a non-commutative analogue of topological freeness.

Proposition 5.4.3. *Let $(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action on a compact Hausdorff space. Then $(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ is topologically free if and only if the corresponding partial C^*-dynamical system is properly outer.*

Proof. If $(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ is not topologically free, there exists $g \in G$ and a non-empty open set $U \subseteq X$ such that $\theta_g(x) = x$ for all $x \in U$. In particular, for $K = \{f \in \mathcal{C}(X) \mid f(x) = 0 \text{ for } x \in X \setminus U\}$, $f \circ \theta_{g^{-1}} = f$ for all $f \in K$. In particular, we can see the induced partial *-automorphism is quasi-inner by picking $u_g = q_g$ for $I(K) = I(\mathcal{C}(X))q_g$. In sum, the partial action on $\mathcal{C}(X)$ is not properly outer.

Conversely, assume the partial action $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ on a unital commutative C*-algebra A is not properly outer, i.e., there exists $p \in I(A)$ such that $\alpha_g(f) = f$ for all $f \in I(A)p$. However, then $\mathcal{C}(U) \simeq I(A)p \cap A$ corresponds to a non-empty open set U so that the corresponding induced homeomorphism θ_g on the spectrum does have non-empty interior. □

There exist several other equivalent formulations of topological freeness for partial dynamical systems similar to the case of global dynamical systems giving multiple candidates for non-commutative analogues. For a summary and a proof of these equivalences we refer to [GS14, Proposition A.7].

Combining Proposition 5.4.3 and Lemma 5.4.2 we obtain the partial analogue of a known result for group actions, see for example [Pit17, Proposition 2.17].

Corollary 5.4.4. *Let $(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action on a compact Hausdorff space and consider its induced action on its projective cover $(\{\hat{X}_g\}_{g\in G}, \{\hat{\theta}_g\}_{g\in G})$. Then $(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ is topologically free if and only if $(\{\hat{X}_g\}_{g\in G}, \{\hat{\theta}_g\}_{g\in G})$ is free.*

Instead of looking at $I(\mathcal{C}(X))$, we can also look at $I_G(\mathcal{C}(X))$. By construction $B = \{f \in \ell^\infty(G, A) \mid f(t) \in A_{t^{-1}}\}$ as in (3.1) is commutative if A is. Thus, as $B = \{f \in \ell^\infty(G, \mathcal{C}(\hat{X})) \mid f(t) \in \mathcal{C}(\hat{X})_{t^{-1}}\}$ is G-injective by Proposition 3.2.3, we obtain

a u.c.p. idempotent map $\phi\colon B \to I_G(\mathcal{C}(X))$. Equipping $I_G(\mathcal{C}(X))$ with the Choi-Effros product, we see that $I_G(\mathcal{C}(X))$ is commutative by commutativity of B. In particular, $I_G(\mathcal{C}(X)) \simeq \mathcal{C}(\hat{X}^G)$ and we obtain a corresponding topological partial dynamical system $(\{\hat{X}_g^G\}_{g\in G}, \{\hat{\theta}_g^G\}_{g\in G})$.

Proposition 5.4.3 together with Theorem 5.3.9 immediately gives the next result. This is a generalization of [ELQ02, Theorem 2.6]. We recover the known result for amenable groups, which can be obtained as a consequence of [AA17, Theorem 4.5].

Theorem 5.4.5. *Let $(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action on a compact Hausdorff space. Then $\mathcal{C}(X) \subseteq \mathcal{C}(X) \rtimes_{\mathrm{r}} G$ has the ideal intersection property if and only if $(\{\hat{X}_g^G\}_{g\in G}, \{\hat{\theta}_g^G\}_{g\in G})$ is topologically free. If additionally, G is amenable, $\mathcal{C}(X) \subseteq \mathcal{C}(X) \rtimes_{\mathrm{r}} G$ has the ideal intersection property if and only if $(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ is topologically free.*

Now, recall that given a discrete group G, the partial group C*-algebra $\mathrm{C}_p^*(G)$ is defined as the universal crossed product of the partial Bernoulli shift $(\{\Omega_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ [Exe98; ELQ02]. More precisely, define $\Omega = \{\omega \in \{0,1\}^G \mid \omega(e) = 1\} \subseteq \{0,1\}^G$ equipped with the relative topology from $\{0,1\}^G$. This is a compact Hausdorff space. To define the partial action, consider $\Omega_{g^{-1}} = \{w \in \{0,1\}^G \mid \omega(e) = \omega(g^{-1}) = 1\}$ and define $\theta_g(\omega)(t) = \omega(gt)$ for $\omega \in X_{g^{-1}}$ and $t \in G$, i.e., $(\{\Omega_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ is the restriction of the Bernoulli shift on $\{0,1\}^G$ to Ω. Then, $\mathcal{C}(\Omega) \rtimes G \simeq \mathrm{C}_p^*(G)$. By [Exe98, Theorem 6.5] partial group representations on $\mathcal{H}$ are in one-to-one correspondence between C*-algebra representations of $\mathrm{C}_p^*(G)$ on $\mathcal{H}$. The reduced partial group C*-algebra is analogously defined as $\mathrm{C}_{p,r}^*(G) = \mathcal{C}(\Omega) \rtimes_{\mathrm{r}} G$. In [ELQ02, Proposition 5.1], the authors show that the partial Bernoulli shift is topologically free if and only if G is infinite. Furthermore, they assert that in this setting $\mathcal{C}(\Omega) \subseteq \mathrm{C}_{p,r}^*(G)$ has the ideal intersection property. Using Exel, Laca and Quigg's result together with Theorem 5.4.5 and the fact that finite groups are amenable we obtain the following.

Proposition 5.4.6. *For a discrete group G, $\mathcal{C}(\Omega) \subseteq \mathcal{C}(\Omega) \rtimes_{\mathrm{r}} G = \mathrm{C}_{p,r}^*(G)$ has the ideal intersection property if and only if G is infinite.*

Note that as pointed out in [ELQ02, Remark 5.3], $(\{\Omega_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ is never minimal as $\{G\} \subseteq \Omega$ is closed and invariant, i.e., $\mathrm{C}_{p,r}^*(G)$ is not simple.

Lastly, we will end with the analogue of Proposition 5.3.18, an intrinsic condition for the ideal intersection property for partial actions on compact Hausdorff spaces. If A is commutative, by [KS19, Proposition 3.12], $\mathrm{Glimm}(A) = \mathrm{Glimm}_{\mathrm{p}}(A)$. In particular, each (pseudo-)Glimm ideal corresponds to a point in the spectrum of A. Now, recall that

for an arbitrary unital partial C*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ by the proof of Proposition 5.3.18, the partial subgroup action $(\{Y_g\}_{g\in G}, \{v_g\}_{g\in G})$ on $\mathrm{Glimm_p}(A) \times \mathrm{Sub}(G)$ induced by $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ respects Glimm ideals. Furthermore, by the same arguments as in (5.5) we see that if $A = \mathcal{C}(X)$, the corresponding action on $\bar{Y} = X \times \mathrm{Sub}(G)$ is given as follows:

$$\overline{Y}_g = X_g \times \mathrm{Sub}(G) \text{ for } g \in G,$$
$$\overline{v}_g((x,H)) = (\alpha_g(x), gHg^{-1}) \text{ for } (x,H) \in \overline{Y}_{g^{-1}}.$$

Observe also that the covering condition can be easily translated to Glimm ideals. In particular, for Glimm ideals K_x, $\bigcap_{x\in E} K_x = \{0\}$, if and only if $E = X$. This together with Proposition 5.3.18 immediately gives an intrinsic condition implying the ideal intersection property in the commutative setting in terms of uniformly recurrent partial subsystems, which is in line with the characterization in the global C*-dynamical system setting [Kaw17; Ken20].

Theorem 5.4.7. *Let $(\{X_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a topological partial C*-dynamical system of a countable discrete group G with $X_g \subseteq X$ clopen for $g \in G$ on a compact Hausdorff space X. Then, $\mathcal{C}(X) \subseteq \mathcal{C}(X) \times G$ has the ideal intersection property if for any closed and invariant subset $Z \subseteq X \times \mathrm{Sub}(G)$ with $\{H \mid (x,H) \in Z\} \subseteq \mathrm{Sub_a}(G)$ and $\{x \mid (x,H) \in Z\} = X$, we have $X \times \{e\} \subseteq Z$.*

5.4.2 Partial Actions by Integers and Cyclic Groups

Partial actions of the integers on C*-algebras induced by a single partial *-automorphism played an important role in Exel's influential work on partial actions in [Exe94]. Given a C*-algebra A, ideals $B, C \trianglelefteq A$ and a partial *-automorphism $\phi\colon B \to C$, for $n \in \mathbb{Z}$ we define

$$A_0 = A,$$
$$A_{n+1} = \{a \in C \mid \phi^{-1}(a) \in A_n\}, \text{ for } n \geq 0,$$
$$A_{n-1} = \{a \in B \mid \phi(a) \in A_n\}, \text{ for } n \leq 0.$$

By [Exe94, Proposition 3.2] these are ideals and thus we can define a partial action $(\{A_n\}_{n\in\mathbb{Z}}, \{\phi_n\}_{n\in\mathbb{Z}})$ of $\mathbb{Z}$ on A by setting $\phi_n = \phi|_{A_n}$. Observe that in particular $A_1 = C$ and $A_{-1} = B$.

Using our notion of proper outerness, we obtain a sufficient condition for the ideal intersection property in this setting. The next result is a strengthening of Remark 4.1.14.

Proposition 5.4.8. *Let A be a C*-algebra and consider a partial *-automorphism $\phi\colon C \to B$ between ideals $B, C \trianglelefteq A$. Let $(\{A_n\}_{n\in\mathbb{Z}}, \{\phi_n\}_{n\in\mathbb{Z}})$ be the corresponding integer partial action on A induced by ϕ. Then, ϕ is properly outer if $\bigcap_{n\in\mathbb{Z}} A_n^{\perp\perp} = \{0\}$. In particular, in this case $(\{A_n\}_{n\in\mathbb{Z}}, \{\phi_n\}_{n\in\mathbb{Z}})$ has the ideal intersection property.*

Proof. Suppose $\phi\colon B \to C$ is not properly outer, then there exists an invariant projection $r \in Z(I(A))$ such that $I(\phi)\colon I(B) \to I(C)$ is inner on $I(B)r \neq \{0\}$. In particular, $I(\phi)(I(B)r) = I(B)r$ by Corollary 4.1.4 and $\phi|_{I(B)r}$ is a *-automorphism. But then, $\phi(B \cap I(B)r) = B \cap I(B)r$ and thus $\{0\} \neq I(B)r \cap B \subseteq \bigcap_{n\in\mathbb{Z}}(I(A_n) \cap A_n) = \bigcap_{n\in\mathbb{Z}} A_n^{\perp\perp}$, which shows the desired. By Corollary 5.1.8 $(\{A_n\}_{n\in\mathbb{Z}}, \{\phi_n\}_{n\in\mathbb{Z}})$ has the ideal intersection property. □

For group actions, Olesen and Pedersen gave a characterization of the ideal intersection property using several equivalent non-triviality conditions for integer partial actions on separable C*-algebras [OP82, Theorem 10.4]. This was then generalized by Kwaśniewski and Meyer to Fell bundles over $\mathbb{Z}$ or $\mathbb{Z}/p\mathbb{Z}$ for p square-free under the additional assumption that the unit fibre contains an essential ideal [KM18]. Using injective envelope techniques, Geffen and Ursu remove this additional assumption for global C*-dynamical systems [GU23]. The next result shows that we can often lift results from the group action setting to partial C*-dynamical systems as a straightforward application of the injective unitization of a partial C*-dynamical system and the connection with its enveloping action.

Proposition 5.4.9. *Consider $G = \mathbb{Z}$ or $G = \mathbb{Z}/p\mathbb{Z}$ for p square-free. The partial C*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has the ideal intersection property if and only if it is properly outer.*

Proof. By [GU23, Corollary 9.6] the statement holds if $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is a global C*-dynamical system. However, $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is properly outer if and only if $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ is properly outer by Corollary 4.1.17 and Proposition 3.4.6 shows that the ideal intersection property for $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ is equivalent to the ideal intersection property for $(\mathcal{C}^{I(A)}, \zeta^{I(A)})$. Now by Proposition 4.1.22 $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ is properly outer if and only if $(\mathcal{C}^{I(A)}, \zeta^{I(A)})$ is properly outer. Hence, we can reduce the case of partial C*-dynamical systems to the case of global C*-dynamical systems in this setting, which shows the desired. □

5.4.3 Prime C*-algebras and FC-hypercentral Groups

As discussed in Section 5.2 given a partial C*-dynamical system $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ one can view the induced partial action on the injective envelope $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ as the injective unitization of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$. By Theorem 5.2.6 it suffices to characterize the ideal intersection property for the unital partial C*-dynamical system $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$. Now observe that by Proposition 2.2.2 the ideals are of the form $I(A)_g = I(A_g) = I(A)p_g$, where $p_g \in Z(I(A))$ is a projection. Thus, if the centre of the injective envelope is trivial, i.e., $Z(I(A)) = \{0,1\}$, then for any $\{0\} \neq A_g \trianglelefteq A$, we have $I(A)_g = I(A)$. This section investigates when the injective unitization is in fact a global C*-dynamical system. It is worth pointing out that in this setting $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ is not an enveloping action of $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ as in Definition 3.4.1, since A is not an ideal in $I(A)$.

In [Ham81, Theorem 7.1], it was shown that the regular monotone completion of a unital C*-algebra A is an AW*-factor if and only if A is prime. Using [Ham81, Lemma 3.7, Theorem 6.3] this gives that the injective envelope $I(A)$ is an AW*-factor, i.e., has trivial centre, if and only if A is prime. This was also pointed out in [GU23]. This immediately leads to our next result.

Theorem 5.4.10. *Let A be a prime C*-algebra and let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system with $A_g \neq \{0\}$ for all $g \in G$. Then, the induced partial action on $I(A)$ is a global action and the G-injective envelope $(\{I_G(A)_g\}_{g\in G}, \{I_G(\alpha)_g\}_{g\in G})$ coincides with the ordinary G-injective envelope of $(I(A), I(\alpha))$.*

Proof. Consider the induced partial action $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$. Since A is prime, $Z(I(A)) = \{0,1\}$. Thus, $I(A)_g = I(A)$ for all $g \in G$, as $A_g \neq \{0\}$. In particular, $I(\alpha)_g \colon I(A)_{g^{-1}} \to I(A)_g$ is a *-automorphism of $I(A)$ for all $g \in G$ and the action is indeed a global group action $(I(A), I(\alpha))$. By construction of the G-injective envelope in Theorem 3.3.3, we see that $I_G(A) = I_G(I(A))$. Since the action on $(I(A), I(\alpha))$ is global, the canonical action on $I_G(I(A))$ is a global group action as well. As seen in the discussion after Theorem 3.3.3, the latter coincides with the G-injective envelope of $(I(A), I(\alpha))$ in the category of global C*-dynamical systems. □

The previous theorem shows that for any partial action on a prime C*-algebra, the G-injective envelope will form a global C*-dynamical system as long as none of the ideals are trivial. This is not the only possible scenario where the induced action on the injective envelope (and thus the action on the G-injective envelope) is a global group action. The key ingredient in Theorem 5.4.10 was that the injective envelope of any

ideal was equal to the injective envelope of A. Hence, it suffices to characterize the ideals $J \trianglelefteq A$ such that $I(J) \simeq I(A)$. By Proposition 2.4.1 these are exactly the essential ideals. This together with the proof of Theorem 5.4.10 immediately gives the following.

Corollary 5.4.11. *Let A be a C*-algebra and consider a partial action $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ such that $A_g \trianglelefteq A$ is essential for all $g \in G$. Then, the G-injective envelope is a global C*-dynamical system and coincides with the G-injective envelope of the induced action on $(\{I(A)_g\}_{g\in G}, \{I(\alpha)_g\}_{g\in G})$ in the category of global C*-dynamical systems.*

In the setting of Corollary 5.4.11, we can immediately use known results from global C*-dynamical systems for partial C*-dynamical systems on essential ideals. In particular, we obtain a full characterization of the ideal intersection property for actions of FC-hypercentral groups on such systems using results of [KKS25]. Given a discrete group G, denote by $\mathrm{FC} = \{g \in G \,|\, |\mathrm{Cl}(g)| < \infty\}$ the *FC-center* of G, where $\mathrm{Cl}(g) = \{hgh^{-1} \,|\, h \in G\}$ is the conjugacy class of $g \in G$. This is a normal subgroup of G and thus we can look at the quotient group $G/\mathrm{FC}(G)$. Iterating this process we obtain a chain of groups F_α for each ordinal α such that $F_\alpha/F_{\alpha-1} \simeq FC(G/\mathrm{FC}(F_{\alpha-1}))$, where we set $F_0 = \{e\}$. For a limit ordinal β, we define the *FC-hypercenter* as $\mathrm{FCH}(G) = \bigcup_{\alpha<\beta} F_\alpha$. We say that G is *FC-hypercentral* if $\mathrm{FCH}(G) = G$. For more information on these groups see for example [Rob72, Chapter 4.3]. FC-hypercentral groups can be seen as being in a sense opposite to groups with the infinite conjugacy class (ICC) property, meaning each non-trivial group element has infinite conjugacy class. More precisely, as proven in [BO16, Corollary 2.6], G is FC-hypercentral if and only if it has no non-trivial ICC quotients. Utilizing Corollary 5.4.11 we immediately obtain the following characterization from [KKS25, Theorem 9.3].

Theorem 5.4.12. *Let G be an FC-hypercentral group and let $(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ be a partial C*-dynamical system such that $A_g \trianglelefteq A$ is essential for all $g \in G$. The following are equivalent:*

(i) *$(\{A_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ has the ideal intersection property;*

(ii) *The ordinary C*-dynamical system $(I_G(A), I_G(\alpha))$ has a unique equivariant pseudo-expectation;*

(iii) *Suppose there exists an invariant regular ideal $K \trianglelefteq I(A)$, a subgroup $H \leq G$, and an H-invariant regular ideal $J \trianglelefteq K$ such that $K \simeq \ell^\infty(G/H)\overline{\otimes}J$. Here $\overline{\otimes}$ denotes the monotone complete tensor product introduced by Hamana [Ham82b; Ham82a]. Let (J, H, β) be the C*-dynamical system obtained by restricting α. If there exists an $r \in H \setminus \{e\}$ and a unitary $u_r \in J$ that is invariant for the centralizer subgroup $C_H(r) = \{h \in H \,|\, hrh^{-1} = r\}$ such that $\beta_r = \mathrm{Ad}(u_r)$, the H-conjugacy class of r, $\{hrh^{-1} \,|\, h \in H\}$, must be infinite.*

(iv) $Z(I(A))^G = Z(I(I(A) \rtimes_{\mathrm{r}} G))$, *where* $Z(I(A))^G$ *denotes the fixed point* C^**-subalgebra, i.e.,* $Z(I(A))^G = \{a \in Z(I(A)) \mid I(\alpha)_g(a) = a \text{ for all } g \in G\}$.

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