

The black hole information paradox in a brane world

by

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Author's Declaration

This thesis consists of material all of which I authored or co-authored: see Statement of Contributions included in the thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

I understand that my thesis may be made electronically available to the public.

Statement of Contributions

This thesis draws on the work appearing in two of my published papers [1,2] coauthored with my supervisor, Robert C. Myers, as well as Dominik Neuenfeld, Ignacio A. Reyes, and Joshua Sandor. The introduction chapter 1 and discussion chapter 7 draws from both papers [1,2]. Chapters 2 and 3 and the accompanying appendices A and B are based on [1]. Here, I was responsible for the general analysis of intersections between Ryu-Takayanagi surfaces and the brane in section 3.2, the higher curvature analysis of section 3.3, and some of the explicit Ryu-Takayanagi surface calculations of chapter 3. Chapters 4 to 6 are based on [2]. Most of the entropy calculations here can be mapped by symmetry to special cases, examined in sections 4.4 to 4.6, for which I was responsible. I also contributed many of the calculations in the extremal case studied in chapter 5 and partly to the two-dimensional analysis covered in chapter 6.

Over the course of my PhD studies, I have also worked on other projects beyond the scope of this thesis. At the start of my degree, I completed the paper [3] on the complexity of charged states in quantum field theory, started during my time in the Perimeter Scholars International MSc program. Concurrent to [1,2], I also contributed to [4,5] which are also about the black hole information paradox, but specific to Jackiw-Teitelboim gravity. More recently, I published the single-author paper [6] on the first law of generalized entropy. To date, I have authored or co-authored nine published papers. Currently, I am working on a collaborative project on asymptotic symmetries and entanglement, whose manuscript is now in preparation.

Abstract

Recent progress in our understanding of the black hole information paradox has led to a new prescription for calculating entanglement entropy, which involves special subsystems in regions where gravity is dynamical, called quantum extremal islands. We present a simple holographic framework where the emergence of quantum extremal islands can be understood in terms of the standard Ryu-Takayanagi prescription, used for calculating entanglement entropy under the anti-de Sitter (AdS)/conformal field theory (CFT) correspondence. Our setup describes a d -dimensional boundary CFT coupled to a $(d-1)$ -dimensional defect, which are dual to a $(d+1)$ -dimensional global AdS spacetime containing a codimension-one brane. Through the Randall-Sundrum mechanism, graviton modes become localized at the brane and, in a certain parameter regime, an effective description of the brane is given by Einstein gravity on a d -dimensional AdS background coupled to two copies of the boundary CFT. Within this effective description, the standard Ryu-Takayanagi formula implies the existence of quantum extremal islands in the gravitating region, whenever Ryu-Takayanagi surfaces cross the brane. Considered with Rindler and Poincaré coordinates respectively, our setup may be viewed as a special class of non-extremal and extremal black holes on the brane, in equilibrium with non-gravitational bath systems. For non-extremal black holes in any dimension, the appearance of quantum extremal islands has the right behaviour to avoid the information paradox and we show that the calculation of the full Page curve is possible. In the case of extremal black holes in higher dimensions, we find no quantum extremal islands for a wide range of parameters. The main benefit of our setup is that it allows for a high degree of analytic control as compared to previous work in higher dimensions. In two dimensions, we find agreement with previous work at leading order; however, a finite ultraviolet cutoff introduced by the brane results in subleading corrections.

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Dedication

I dedicate this to my parents and my brother, for their love and support.

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List of Abbreviations

- AdS** anti-de Sitter v, xi, xiii, 1, 2, 6–9, 21, 23, 29, 31–33, 35–37, 41, 42, 44, 46, 47, 54, 70–74, 76, 81, 83, 85, 86, 90, 91, 93, 96, 101, 106, 110, 111, 114, 123, 124, 135, 136, 138, 141, 142, 144, 147, 150, 153, 154, 177, 180, 185
- BH** Bekenstein-Hawking xi, 1, 4–7, 9, 10, 13, 114, 119
- CFT** conformal field theory v, xi–xvii, 1, 2, 6–14, 17, 19, 21, 23, 27–29, 31–43, 46, 49, 53, 62, 67–75, 77, 80, 84–88, 90, 91, 96, 99, 101, 106, 107, 109, 114–116, 119, 122–124, 128–130, 132, 135–139, 141–146, 148–151, 153–155, 176, 177, 180
- DGP** Dvali-Gabadadze-Porrati xiii, xiv, 11, 18, 29, 31–33, 35, 37, 40–42, 45–47, 51, 57, 59, 62–65, 77–79, 93, 94, 97, 104, 116, 117, 136, 137, 139, 140, 145, 146, 149, 179, 184
- EE** entanglement entropy xi, xiii, xvi, 4–10, 13–15, 40–44, 46, 48, 49, 52, 53, 55, 57, 60, 63, 64, 66, 69, 71, 72, 75, 80, 83, 84, 89–91, 95, 99, 108, 109, 115, 116, 124, 126, 128, 136–140, 146, 148, 151, 154, 155, 176, 177, 184
- FG** Fefferman-Graham 21, 22, 24
- IR** infrared 35, 52, 86, 89–91, 110, 116, 117, 136
- JT** Jackiw-Teitelboim xii, 10–12, 27–29, 32, 39, 66, 67, 122, 125, 132, 136, 139–141, 149, 150, 154
- LHS** left hand side 9, 95

- QEI** quantum extremal island 9–11, 13, 15, 27, 39, 42, 63, 66, 72, 107, 108, 110, 137–139, 141, 142, 145–147, 150, 152, 154
- QES** quantum extremal surface xvi, 8, 9, 67–69, 77, 79, 93, 95, 109, 112–114, 117, 119, 129–133, 136
- RHS** right hand side 9, 10, 95, 119, 122
- RS** Randall-Sundrum xii, 11, 12, 14, 17, 19, 21, 29, 31, 33, 35, 36, 45, 70, 75, 135, 140, 142, 153
- RT** Ryu-Takayanagi xi–xvii, 1, 6–11, 13–15, 39–42, 44–55, 57–64, 68, 69, 72, 75–82, 84, 87, 88, 91–97, 103, 104, 107–113, 116–122, 125–129, 135–138, 140, 141, 145–151, 155, 180, 181, 183–185
- TFD** thermofield double 7, 10, 66, 68, 70, 73, 123–125, 131, 148, 149, 155
- UV** ultraviolet xvi, 8, 11, 13, 23, 35, 42, 43, 48, 49, 52, 54, 60, 69, 81, 83, 86, 89, 96, 114–116, 121, 122, 126–128, 130, 131, 133, 136, 142, 151, 181

Chapter 1

Introduction

Almost half a century ago, it was discovered that black holes behave as quantum objects, with an associated temperature, entropy, and other thermodynamic properties [8–12]. In particular, quantum black holes are not exactly black, but rather emit Hawking radiation at their prescribed temperature. Another realization of these ideas is the Bekenstein-Hawking (BH) formula, which states that the black hole entropy is a quarter of its horizon area measured in Planck units:

$$S = \frac{A}{4G_{\text{N}}} . \tag{1.1}$$

These concepts gained a wider scope in the context of the holographic anti-de Sitter (AdS)/conformal field theory (CFT) correspondence, where, as we will review, the Ryu-Takayanagi (RT) prescription [13–16] applies the same geometric expression to extremal bulk surfaces in evaluating entropy.

Understanding the quantum description of black holes remains a central question in theoretical physics. The key unresolved puzzle is the information paradox [8, 9, 17–19] — that is, the question of whether black holes destroy information as they evaporate. In recent years, some success has been made in resolving this paradox using the AdS/CFT correspondence. The goal of this thesis is to expand upon these ideas. In particular, we will set up a gravitational model on a brane which is simple enough to admit a largely analytic study of the black hole information paradox even in arbitrary dimensions.

We start here with a review of the black hole information paradox in section 1.1 and a review of recent advances in its understanding using the AdS/CFT correspondence in section 1.2. In section 1.3, we provide an overview of the doubly holographic brane world

model which will be our focus. Finally, in section 1.4, we outline the remainder of this thesis.

1.1 The black hole information paradox

As hinted above, the fate of information during black hole evaporation remains a subject of ongoing research. In his seminal work, Hawking argued that information is lost in black hole evaporation [8, 9, 17]. Below, we provide an intuitive sketch of how this occurs in semiclassical gravity. In a fully quantum theory of gravity, however, to an outside observer, the black hole looks like an ordinary, quantum mechanical system, *e.g.*, as suggested by the AdS/CFT correspondence [20, 21]. Under the tenets of quantum mechanics, the time evolution of such a system should be unitary and thus preserve quantum information. This tension is colloquially known as the black hole information paradox [18].

To understand the phenomena of Hawking radiation, for concreteness, we may consider the situation illustrated in fig. 1.1 where a black hole forms from the collapse of matter and subsequently evaporates. Near the horizon, quantum fields fluctuate, much like they do anywhere else in the spacetime. However, by its definition, the horizon separates those fluctuating modes that are doomed to fall into the black hole singularity, and those that escape as excitations to infinity. The latter constitute Hawking radiation. Due to the emission of Hawking radiation, the black hole loses mass, *i.e.*, evaporates, and will eventually disappear completely.

Information loss due to Hawking radiation is the result of entanglement between Hawking radiation and modes falling from just inside the horizon into the black hole singularity. At a very rough level, the entangled state of an excited Hawking radiation mode and its infalling partner inside the black hole can be modelled by two qubits, labelled in the following by subscripts r and b respectively. For simplicity, we may consider the Bell state

$$|\psi\rangle_{rb} = \frac{|\uparrow\rangle_r |\uparrow\rangle_b + |\downarrow\rangle_r |\downarrow\rangle_b}{\sqrt{2}}. \quad (1.2)$$

This state is clearly entangled since tracing out b leads to a state

$$\rho_r = \text{tr}_b(\rho_{rb}) = \text{tr}_b(|\psi\rangle_{rb} \langle\psi|_{rb}) = \frac{|\uparrow\rangle_r \langle\uparrow|_r + |\downarrow\rangle_r \langle\downarrow|_r}{2} \quad (1.3)$$

of r that is *mixed* — mathematically, this means the density matrix ρ_r is not a projection (in contrast to the original *pure* state $\rho_{rb} = |\psi\rangle_{rb} \langle\psi|_{rb}$). Physically, this means that

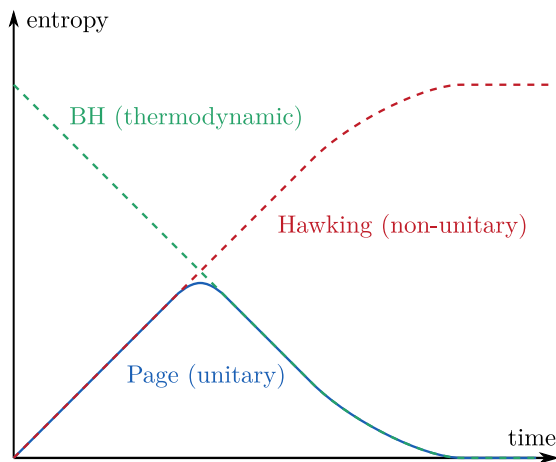


Figure 1.2: Qualitative entropy curves for black hole evaporation. The green curve shows the BH entropy (1.1) which gives the thermodynamic entropy of the black hole; the red curve shows the monotonically increasing entanglement entropy (EE) for Hawking radiation predicted by Hawking’s calculation; and the Page curve is shown in blue, giving the expected EE of Hawking radiation under a unitary evolution of the gravitational system.

information would be lost if the b qubit is discarded; in particular, the above state for r is now a classical probabilistic distribution over the spin-up and spin-down states. Indeed, in the black hole context, the infalling quanta within the black hole are inaccessible to outside observers and eventually lost altogether to the singularity. While this qubit toy example is extremely simplified, it intuitively illustrates the underlying mechanism for information loss behind Hawking’s calculation.

In reality, Hawking quanta are entangled thermally with their infalling partners, giving a tangible meaning to the temperature of a black hole, as hinted earlier in this chapter. In fact, this thermal entanglement is a general property of any Killing horizon and boost-symmetric state, including Rindler horizons in the vacuum. (In this general context, the analogue of Hawking radiation is called Unruh radiation [22–25].) So really, the entanglement between Hawking radiation and their infalling partners is a necessary ingredient for a horizon that is smooth, *i.e.*, locally vacuum-like and without “firewalls” [19].

To sharpen this discussion, let us quantify the entanglement and resulting loss of quantum information from losing a subsystem by consider entanglement entropy (EE) — to be precise, von Neumann entropy — defined for a state ρ by

$$S = -\text{tr}(\rho \log \rho) . \tag{1.4}$$

Intuitively, EE quantifies the degree to which the state ρ is mixed; in particular, note that a pure state $\rho = |\psi\rangle\langle\psi|$ has eigenvalues 1 and 0, and therefore has a vanishing EE. Further, if the state $\rho_R = \text{tr}_B \rho_{RB}$ describes a subsystem R within a pure state $\rho_{RB} = |\psi\rangle_{RB}\langle\psi|_{RB}$ of a full system $R \cup B$, the EE of ρ_R quantifies the entanglement between the R and B subsystems. (A property of EE in this case is that ρ_R and $\rho_B = \text{tr}_R \rho_{RB}$ have the same EE.)

Let us return to the situation illustrated in fig. 1.1, and take R to be the part of a Cauchy slice that is outside the black hole horizon and B to be the part inside. Of course, before the formation of the black hole, there is no horizon, so R fully covers a Cauchy slice. With access to this entire Cauchy slice, it is reasonable to suppose that the state $\rho_R(t=0) = |\psi\rangle_R\langle\psi|_R$ at this initial time $t=0$ is pure, so the EE of $\rho_R(t=0)$ is initially zero. However, after the black hole forms, observers outside its horizon lose access to the black hole interior. While R has now collected some Hawking quanta, their entangled partners within the black hole interior are part of the complementary system B . Thus, the state $\rho_R(t)$ outside becomes mixed and its EE is correspondingly nonvanishing. Moreover, as the black hole evaporates through the emission of Hawking radiation, the increase of this entropy over time gives a measure of the amount of entanglement between the radiation and the black hole. According to Hawking's original calculation, this EE increases monotonically throughout the evaporation process and becomes a nonzero constant once the black hole has disappeared, as illustrated by the red curve of fig. 1.2.

However, the above described information loss, characterized by a nonvanishing final EE for the Hawking radiation collected by R , is at odds with how a quantum system is expected to unitarily evolve. Note, in particular, that at any final time $t = t_{\text{fin}}$ after the black hole has disappeared, R again covers a full time slice. One expects, therefore, a theory of quantum gravity to describe the time evolution of R from $t=0$ to $t=t_{\text{fin}}$ in a unitary manner; that is, $\rho_R(t_{\text{fin}}) = U\rho_R(0)U^\dagger$ where U is a unitary operator. If this is true, the final state $\rho_R(t_{\text{fin}}) = U|\psi\rangle_R\langle\psi|_R U^\dagger$ must be pure and have a vanishing EE, in contrast to Hawking's calculation. Thus, under unitary evolution, after the initial increase predicted by Hawking, the EE of Hawking radiation must eventually decrease and fall back to zero. This qualitative behaviour of the radiation's entropy as a function of time, illustrated by the blue curve of fig. 1.2, is known as the Page curve [26–30] — see also [21].

The latter decreasing branch of the curve is expected to be governed by the BH entropy (1.1) proportional to the area of the black hole. This is because the BH entropy is believed to represent the thermodynamic or coarse-grained entropy of the black hole [10, 12, 31]. In other words, it counts the microstates [32] of the black hole subject to its macroscopic

properties such as mass, charges, and angular momentum¹, and therefore sets an upper bound on its EE. Since the combined system of the black hole and Hawking radiation should be pure (again, because we have started with a pure state at $t = 0$), the EEs of the Hawking radiation and black hole are equal and both bounded by the BH entropy. So, assuming the EE of Hawking radiation follows Hawking’s calculation to the furthest extent possible, it will increase until the so-called Page time when it saturates the upper bound given by the falling BH entropy, after which it will follow the latter back down to zero as the black hole completely evaporates.

While reconciling Hawking’s calculation with the idea that quantum gravity is unitary was a longstanding puzzle, recent progress has made it possible to compute the Page curve in a controlled manner [7, 34, 35], as we will now review.

1.2 Advances using the AdS/CFT correspondence

As the simplest solution to Einstein’s equations with a negative cosmological constant, an anti-de Sitter (AdS) spacetime is a spacetime with constant negative curvature. It is maximally symmetric, meaning it is homogeneous and isotropic in both space and time. In fact, the isometries $SO(2, d)$ of AdS_{d+1} (*i.e.*, $(d + 1)$ -dimensional AdS) may be recast as the conformal (*i.e.*, angle-preserving) transformations of a spacetime in one lower dimension. In physics, the matching of symmetries between different theories sometimes hints at the presence of a duality — that is, the fact that the theories are actually alternative descriptions of the same underlying physics. Originally derived in string theory, the AdS/CFT correspondence [36] indeed describes such a duality. It relates a gravitational theory in a so-called bulk spacetime that asymptotically approaches AdS near its boundary to a quantum field theory with conformal symmetry, *i.e.*, a conformal field theory (CFT), living on the AdS boundary. This correspondence, illustrated in fig. 1.3, has been dubbed ‘holographic’ due to the differing dimensions of the related theories. Since its discovery over two decades ago, AdS/CFT has proven an ideal arena in which to study the interplay of spacetime geometry and quantum information.

This is perhaps best demonstrated by the celebrated Ryu-Takayanagi (RT) formula

¹There is also a somewhat different notion of “complexity coarse-graining” that has arisen in recent discussions [33] of the black hole information paradox. In particular, it is posited that the monotonically increasing entropy in Hawking’s calculation is measuring an entropy that is coarse grained over complex operators acting on the Hawking radiation. This leaves only simple operators that fail to detect the subtle correlations required to recover the Page curve.

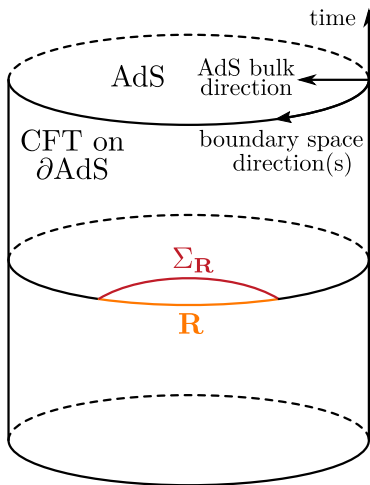


Figure 1.3: The AdS/CFT correspondence relates a gravitational theory in an asymptotically AdS spacetime (filling in the middle of the cylinder) to a CFT on the AdS boundary (the cylinder’s surface). In particular, the RT formula relates the EE of a boundary subregion $\mathbf{R}$ to the area of an extremal surface $\Sigma_{\mathbf{R}}$.

[13–16] for the EE of a boundary spatial subregion $\mathbf{R}$:

$$S_{\text{EE}}(\mathbf{R}) = \min \left\{ \text{ext}_{\Sigma_{\mathbf{R}}} \frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{N}}} \right\} . \quad (1.5)$$

Here, $\Sigma_{\mathbf{R}}$ is a bulk surface homologous to the boundary subregion $\mathbf{R}$ — that is, $\Sigma_{\mathbf{R}} \cup \mathbf{R}$ forms the boundary of some spatial subregion in the bulk — and we are instructed to take the minimal extrema among such surfaces. A simple example of a boundary subregion $\mathbf{R}$ in the CFT and the corresponding RT surface $\Sigma_{\mathbf{R}}$ in the AdS bulk are illustrated in fig. 1.3.

An interesting application of the RT formula is to (asymptotically) AdS black holes, where the BH entropy (1.1) can be found, in certain situations, to match the EE of the CFT. For example, one may consider an eternal black hole in the bulk with two asymptotic boundaries, as illustrated in fig. 1.4. Under AdS/CFT, this gravitational system is dual to a so-called thermofield double (TFD) state, which is an entangled state of two copies of the CFT (corresponding to the two asymptotic boundaries). When the RT formula is evaluated with $\mathbf{R}$ being a time slice of either asymptotic boundary, the relevant RT surface $\Sigma_{\mathbf{R}}$ coincides with the bifurcation surface of the black hole. Thus, the BH formula (1.1) is simply a special case of the RT formula (1.5).

However, as motivated by the promotion of classical laws of black hole mechanics to the semiclassical level, it is natural to collectively consider the geometric BH entropy

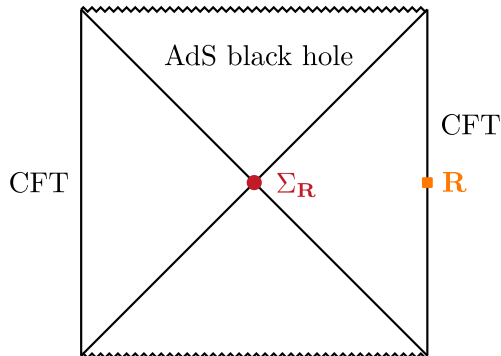


Figure 1.4: An eternal AdS black hole dual to two copies of a CFT respectively on the two asymptotic AdS boundaries. The RT formula (1.5) relates the EE of the CFT on a time slice $\mathbf{R}$ of one asymptotic boundary to the area of the extremal surface $\Sigma_{\mathbf{R}}$ coinciding, in this case, with the black hole bifurcation surface.

together with the EE of quantum fields outside the horizon. This combination, known as generalized entropy, can be shown for example to satisfy the semiclassical generalization [6, 37] of the first law of black hole mechanics [38, 39], which relates the variations of entropy, energy, charge, and angular momentum in much the same manner as the first law of thermodynamics. The generalized entropy of a black hole horizon is also non-decreasing [40–42] over time, in analogy to the second law of thermodynamics. Moreover, whereas the EE of quantum fields is by itself ultraviolet (UV) divergent, generalized entropy can be rendered finite by a renormalization of Newton’s constant [43]. These are indications that generalized entropy is the correct notion of entropy to consider for a semiclassical bulk theory of gravity.

In the context of holographic EE, consideration of semiclassical corrections in the bulk indeed leads to the natural extension of the RT prescription [44, 45]

$$S_{\text{EE}}(\mathbf{R}) = \min \left\{ \text{ext}_{\Sigma_{\mathbf{R}}} S_{\text{gen}}(\Sigma_{\mathbf{R}}) \right\} = \min \left\{ \text{ext}_{\Sigma_{\mathbf{R}}} \left(\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{N}}} + S_{\text{QFT}} \right) \right\}, \quad (1.6)$$

which now includes the EE (to be precise, the von Neumann entropy) S_{QFT} of the bulk quantum fields on the partial Cauchy surface bounded by $\Sigma_{\mathbf{R}} \cup \mathbf{R}$. The surface that extremizes the generalized entropy in the above expression is then referred to as a quantum extremal surface (QES) [45]. Again, the ‘min’ indicates that, in the situation where there is more than one extremal surface, one chooses that which yields the minimum value for $S_{\text{gen}}(\Sigma_{\mathbf{R}})$.

This quantum RT formula has produced some surprising new insights with holographic

models of black hole evaporation [7, 34, 35]. In particular, at late stages in the evaporation, the quantum term can compete with the classical BH contribution in eq. (1.6) to produce new saddle points for the QES, which could describe the late-time phase of the Page curve. Perhaps the biggest surprise is that the Page curve can be reproduced from saddle-point calculations in semi-classical gravity, *i.e.*, in a situation where the details of the black hole microstates or of the encoding of information in the Hawking radiation are still not fully revealed.

This calculation is best understood in a setting where a black hole is coupled to an auxiliary, non-gravitational reservoir — referred to as the *bath* — which captures the Hawking radiation. In this situation, it was argued that instead of using Hawking’s calculation, the true entropy of the Hawking radiation captured in a region $\mathbf{R}$ of the bath should be calculated using the so-called *island rule* [7]

$$S_{\text{EE}}(\mathbf{R}) = \min \left\{ \text{ext}_{\text{islands}} \left(\frac{A(\partial(\text{islands}))}{4G_N} + S_{\text{QFT}}(\mathbf{R} \cup \text{islands}) \right) \right\}. \quad (1.7)$$

This formula instructs us to evaluate, in the semiclassical theory, the EE S_{QFT} of fields in the bath region $\mathbf{R}$ combined with any codimension-one and possibly disconnected ‘island’ subregions in the gravitating region. The boundaries of the candidate islands also contribute a gravitational term in the form of the usual BH entropy. One extremizes the right hand side (RHS) of eq. (1.7) over all such choices, and if this yields multiple extrema, the correct choice is the minimal extremum. If this procedure yields a solution with a nontrivial island, the latter is called a quantum extremal island (QEI) — see [46] for a recent review.

In a sense, eq. (1.7) follows quite naturally from eq. (1.6), by making explicit the fact that QESs can contain disconnected pieces that form the boundaries of QEIs. The incorporation of the bath system, though, requires some explanation. As described above, it is an auxiliary spacetime region into which the bulk matter fields may propagate. Viewed as an extension of the bulk in this way, the bath region $\mathbf{R}$ therefore appears on the RHS of eq. (1.7). On the other side of the AdS/CFT duality, however, one may view the bath as a system coupled to the CFT. So, considering the bath region $\mathbf{R}$ as part of this boundary side of the holographic duality, it is natural to apply the quantum RT formula (1.6) to evaluate its EE — hence, the appearance of $\mathbf{R}$ on the left hand side (LHS) of eq. (1.7). A more careful derivation of eq. (1.7) can be obtained from an analysis of the ‘replica trick’ for evaluating holographic EE [47–49]. Further, as will become clear in the brane world model studied in this thesis, eq. (1.7) may even be viewed as the manifestation of (a mild extension to) the *classical* RT formula (1.5).

Let us now explain how eq. (1.7) allows one to recover the Page curve. For an evaporating black hole, an obvious choice for the island region that extremizes the functional on the RHS is the empty set. In the early phase of the Hawking evaporation, this is indeed the minimal extremum, and the result of eq. (1.7) is simply the semiclassical EE $S_{\text{QFT}}(\mathbf{R})$ calculated by Hawking. However, if radiation in the region $\mathbf{R}$ shares a large amount of entanglement with the quantum fields within the black hole, a nontrivial QEI can appear. In particular, this occurs for an old evaporating black hole, and in this case a QEI appears in the black hole interior, extending nearly to the horizon [35]. It turns out that after the Page time, this configuration takes over as the minimal extremum in eq. (1.7). In this phase, $S_{\text{EE}}(\mathbf{R})$ is dominated by the area term on the RHS of eq. (1.7). Hence, as the black hole evaporates, the latter shrinks to zero size and the island rule (1.7) gives a unitary Page curve, as illustrated in fig. 1.2. The discovery of islands has sparked a variety of new investigations in the contexts of both black holes [4, 5, 46–75] and cosmology [76–83].

1.3 Doubly holographic brane world model

In this thesis, we will further examine the island formula (1.7). This formula was motivated by the ‘doubly holographic’ model presented in [7], who in turn began with the two-dimensional model of [34]. The latter consists of a bath, *i.e.*, a two-dimensional CFT on a half line, and a pair of quantum mechanical systems. These quantum mechanical systems are then assumed to be holographically dual to a two-dimensional theory of gravity, called Jackiw-Teitelboim (JT) gravity (which will be described later in this thesis), on AdS_2 coupled to the same CFT as in the bath. Hence if the quantum mechanical systems begin in a TFD state, the dual description is given by a two-sided AdS_2 black hole. If the boundary of the bath is then coupled to one of the quantum systems, *i.e.*, to the asymptotic boundary of one side of the black hole, the black hole begins to evaporate as Hawking radiation leaks into the bath.

Now the insight of [7] was to examine the case where the two-dimensional CFT is itself *holographic*, and so can be replaced with a locally AdS_3 bulk. The boundary of this bulk geometry has two components: the two-dimensional asymptotic AdS_3 boundary, on which the bath lives, and a so-called Planck brane, where the JT gravity is supported. This third perspective on the system has the advantage that the generalized entropy of the gravitational theory on the Planck brane and the matter fields (described now by the two-dimensional CFT) in eq. (1.6) or (1.7) is realized completely geometrically. In particular, the EE of the CFT is computed by classical RT surfaces in the three-dimensional bulk, and the geometric BH contribution is given by the usual expression for JT gravity. Further,

calculations in this doubly holographic model produce the expected Page curve, with RT surfaces ending on the Planck brane manifesting the island rule (1.7).

One direction to generalize the above calculations is to understand the Page curve and QEIs in higher dimensions. Only limited results have been previously obtained on this front. For instance, some analyses in higher dimensions were undertaken numerically in [58], in an effective theory in flat space [59], and using a Randall-Sundrum (RS)-inspired toy model in [60]. (See also [35, 47, 61].)

In this thesis, we shall generalize the doubly holographic model described above to arbitrary dimensions. Our model will provide the first setup in which islands in black holes may be calculated largely analytically in higher dimensions, in the regime where the gravitational theory on the brane is well-approximated by Einstein gravity. This is done as follows (see figure 1.5): We consider a d -dimensional holographic CFT coupled to a codimension-one conformal defect. The gravitational dual then corresponds to an asymptotically AdS_{d+1} spacetime, containing a codimension-one brane anchored on the asymptotic boundary at the position of the defect. The gravitational backreaction of the brane warps the geometry creating localized graviton modes in its vicinity, by the mechanism originally discovered by Randall-Sundrum (RS) [84–86]. Hence at sufficiently long wavelengths, the system can then also be described by an effective theory of Einstein gravity coupled to (two copies of) the holographic CFT on the brane, all coupled to the CFT on the static boundary geometry.² To better emulate the previous model with JT gravity [7], we also consider introducing an intrinsic Einstein term to the brane action, analogous to the construction of Dvali, Gabadadze, and Porrati (DGP) [87].³

In any event, this more or less standard holographic model can be viewed from three perspectives in analogy with [7]: the *bulk perspective*, with a brane coupled to gravity in an asymptotically AdS_{d+1} space; the *boundary perspective*, with the boundary CFT coupled to a conformal defect; and the *brane perspective*, with a region where the holographic CFT couples to Einstein gravity and another region where the same CFT propagates on a fixed background geometry. Let us emphasize that while the three different perspectives were presented on a more or less equal footing, the fact that the RS gravity on the brane has a finite UV cutoff [84, 85] singles out the brane perspective as an effective low-energy description, in contrast to the boundary and bulk descriptions.⁴

²Some tuning of the parameters characterizing the brane is required to achieve this effective description. Note that the fact that the RS gravity on the brane has a finite cutoff [84, 85] makes conspicuous that this is only an effective theory.

³Without the DGP term, our construction resembles that in [47] in many respects. Our model resembles the setup in [7] even more closely if we make a $\mathbb{Z}_2$ orbifold quotient across the brane.

⁴This does not mean that the bulk description in terms of a(n infinitely thin) brane in AdS_{d+1} is UV

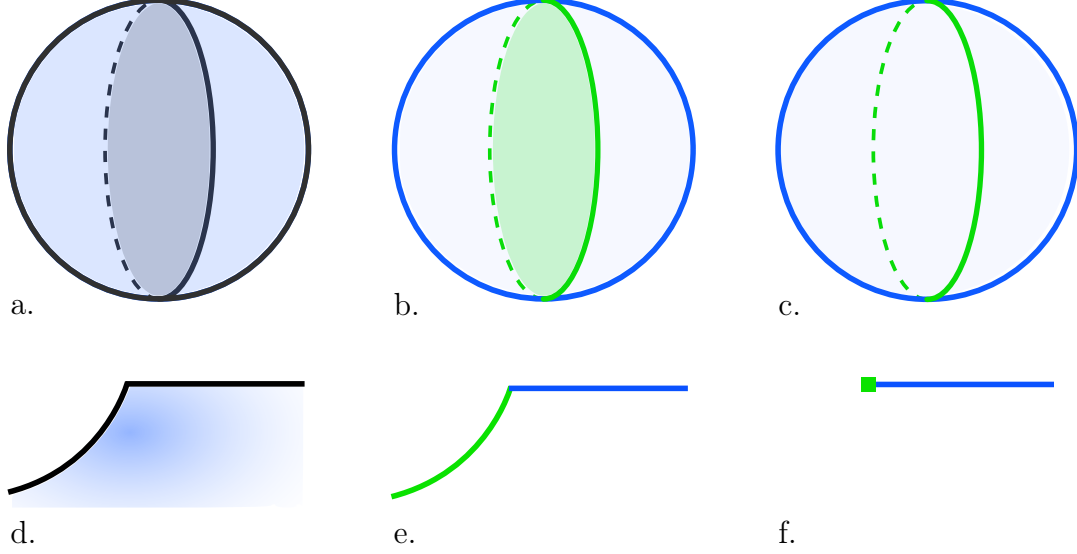


Figure 1.5: This figure shows the relation between a time-slice in our construction and the holographic setup of [7]. The top row illustrates three perspectives with which the system discussed here can be described, while the bottom row displays the analogous descriptions for the model in [7]. In the former, we are using the global conformal frame where the boundary CFT lives on $R \times S^{d-1}$ and the conformal defect appears on the equator of the $(d-1)$ -sphere — see discussion in chapter 2 and section 2.5. The comparison can be made more precise by performing a $\mathbb{Z}_2$ orbifold quotient across the bulk brane/conformal defect in the top row.

- a.** Bulk gravity perspective, with an asymptotically AdS_{d+1} space (shaded blue) which contains a co-dimension one RS brane (shaded grey).
- b.** Brane perspective, with dual CFT_d on the asymptotic boundary geometry (blue) and also extending on the AdS_d region (shaded green) where gravity is dynamical.
- c.** Boundary perspective, with the holographic CFT_d on S^{d-1} (blue) coupled to a codimension-one conformal defect (green).
- d.** AdS_3 formulation with two boundary components: the flat asymptotic boundary (straight black line) and a “Planck brane” (curved black line) with an AdS_2 geometry.
- e.** The holographic CFT extends over a region with a fixed metric (blue) and an AdS_2 region with JT gravity (green).
- f.** The microscopic description as a two-dimensional BCFT (blue) coupled to a quantum mechanical system at its boundary (green).

An advantage of working in the bulk perspective is that EEs of subregions in the bath can be computed geometrically using the usual rules of holographic EE [13–15], taking into account that the RT surfaces that can also end on the Planck brane [91, 92]. In particular, EE is realized in terms of the areas of RT surfaces, with a contribution in the bulk and another contribution on the brane. That is, we have an extension of the usual RT prescription with

$$S_{\text{EE}}(\mathbf{R}) = \min \left\{ \text{ext}_{\Sigma_{\mathbf{R}}} \left(\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} + \frac{A(\sigma_{\mathbf{R}})}{4G_{\text{brane}}} \right) \right\}, \quad (1.8)$$

where again $\Sigma_{\mathbf{R}}$ is a bulk surface homologous to the boundary subregion $\mathbf{R}$ and $\sigma_{\mathbf{R}} = \Sigma_{\mathbf{R}} \cap \text{brane}$ is a surface on the brane — this intersection of the RT surface with the brane becomes the boundary of the islands seen in the brane perspective. (See figure 1.6.) Note that the brane contribution seems natural here; we will argue for its presence by extending the derivation in [93].

In contrast to eq. (1.6), we are not considering quantum field contributions in the AdS_{d+1} bulk. Nonetheless, the equivalence between eqs. (1.7) and (1.8) can be easily understood as follows: The bulk term in eq. (1.8) describes the leading planar contributions of the EE of the boundary CFT, and so matches the second term in eq. (1.7). However, as we will show in this thesis, expanding the former near the brane also reveals a BH term that matches the induced Einstein part of the effective gravitational action on the brane. This contribution combines with the brane term in eq. (1.8) to produce the BH term appearing in eq. (1.7). In fact, as we shall demonstrate, the RT contribution also captures higher derivative contributions matching the Wald-Dong entropy [38, 39, 94, 95] of the higher curvature terms appearing in the effective gravitational action. Further, the competition between candidate QEIs, denoted by the ‘min’ in eq. (1.7) simply becomes the usual competition between different possible RT surfaces in the holographic formula (1.8), *e.g.*, see figure 1.7.

Our goal will be to study the question of QEIs for black holes in arbitrary dimensions using the purely geometric description (1.8) of the bulk gravity perspective. We emphasize the underlying *simplicity* of our holographic model. In particular, the elements of construction are more or less standard, and the entropies are evaluated with the geometric formula for holographic EE. Hence we generalize the island rule to any number of dimensions but also cast it in a framework where many of its features follow simply from the properties of the RT prescription — and in fact, can be understood analytically. For instance, the

complete. However, it is reasonable to expect that the bulk description can be completed in the UV by a more complicated configuration which can be obtained within string theory, *e.g.*, see [88–90]. In contrast, the brane theory has a fundamental cutoff.

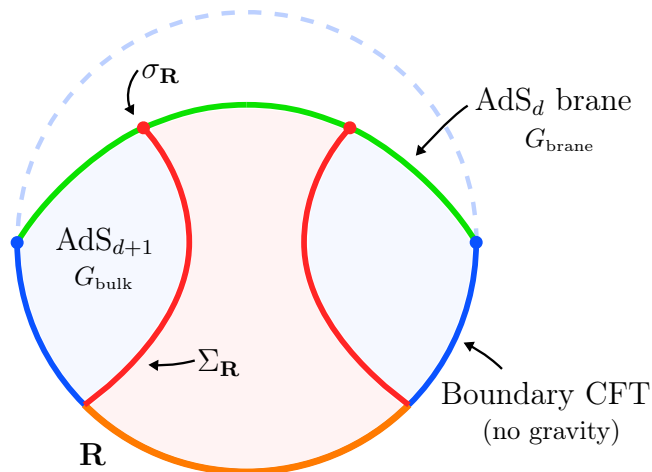


Figure 1.6: A sketch of our holographic setup illustrating the various elements appearing in eq. (1.8), which manifests the island rule in our analysis.

transition between the phase without an island and that with the island is nothing more than the usual transition between different classes of RT surfaces [96–98] — see figure 1.7. In particular, in the island phase, the RT surface crosses the brane so that a portion of the latter, *i.e.*, the island, is included in the corresponding entanglement wedge. Thus the appearance of quantum extremal islands is simply described by a well understood feature of holographic EE in a new setting. Using our model, we will moreover be able to address several issues that appeared puzzling in [7]. Our analysis, being analytic to a great extent, complements previous approaches that heavily relied on numerics [58]. In our case, the numerics required to extract quantitative results are limited to solving few ordinary differential equations.

1.4 Outline of thesis

To conclude this introduction, let us now outline the remainder of this thesis.

In chapter 2, we begin by studying a certain class of d -dimensional branes embedded in AdS_{d+1} . We show how the RS gravity induced on the brane is equivalent to the bulk description of the brane embedded in the higher dimensional geometry. Moreover, in section 2.5, we elucidate the different holographic perspectives of this system as described above, *i.e.*, we can describe the system as a d -dimensional boundary CFT coupled to a conformal defect, a d -dimensional CFT containing a region with dynamical gravity, or a

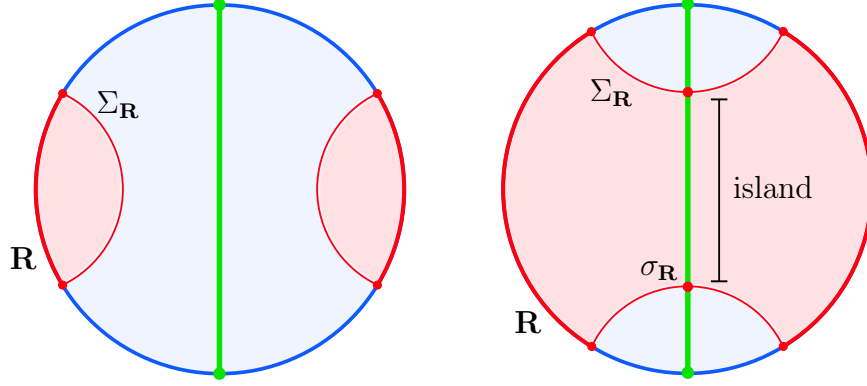


Figure 1.7: The choice of RT surfaces on a constant time slice in the presence of the brane (coloured green), showing the different ingredients involved in eq. (1.8).

$(d+1)$ -dimensional theory of gravity coupled to a codimension-one brane. Chapter 3 investigates the relation between the appearance of QEIs using eq. (1.7) and the bulk picture using eq. (1.8) with RT surfaces crossing the brane. We also present some explicit calculations illustrating the appearance of such QEIs for $d = 3$. Connected to ideas introduced in this first half of the thesis are the appendices A and B: In appendix A, we extend the arguments in [93] to support the appearance of the brane contribution to the generalized entropy in eq. (1.8). Appendix B examines a surprising class of spherical RT surfaces, which can be supported at finite size by the brane.

In this first half of the thesis, our discussion will be quite general and not necessarily linked to the physics of black holes. In fact, the explicit calculations in section 3.4 evaluate the EE of entangling regions (with components on either side of the conformal defect) in the vacuum state of the boundary system.⁵ This illustrates that QEIs are not a feature exclusive to the black hole information problem, but may play a role in more general settings where gravity and entanglement are involved.

Black holes will be the focus of the second half of this thesis, starting in chapter 4: As a warm-up, in section 4.1, we summarize the two-dimensional setup of [50], describe the connection to our model, and introduce eternal black holes. In sections 4.2 and 4.3, we construct eternal black holes on the brane in higher dimensions. As in the $d = 2$ case, these black holes are in equilibrium with the bath at finite temperature and so they do not

⁵Further, let us note that the formation of QEIs on branes in the ‘Einstein gravity regime’ require us to introduce somewhat unconventional couplings. That is, we must consider a negative Newton’s constant on the brane and/or a Gauss-Bonnet interaction in the four-dimensional bulk gravity.

evaporate. Nonetheless, there is a continuous exchange of radiation between the black hole and the bath, which has the potential to create an information paradox [50]. Hence, we use eq. (1.8) to investigate under which conditions islands appear in sections 4.4 to 4.8. We present the general analysis for the time dependence of the entropy, exploring the island and no-island phases. In section 4.9, we develop the numerics associated with some integral equations found in the previous sections and explicitly evaluate the Page curve for $d = 3$, 4 and 5. Chapter 5 examines an extremal horizon with a vanishing temperature, finding that in contrast to two dimensions [50], generally islands do not form in higher dimensions. However, this is not problematic, since at zero temperature the black hole and bath are not actually exchanging radiation and thus no information paradox arises. Details for the special case $d = 2$ appear in chapter 6. We evaluate the quantum extremal surfaces and the Page curve, and show that the brane cutoff produces subleading corrections compared to the results in [50].

Finally in chapter 7, we discuss our results, considering features of the brane world model introduced and examined in the first half of this thesis and its application to black holes and the information paradox considered in the second half. We also point towards some directions for future research.

Chapter 2

Brane Gravity

As described in the introduction in chapter 1, we are studying a holographic system where the boundary theory is a d -dimensional CFT which lives on a spherical cylinder $R \times S^{d-1}$ (where the R is the time direction). Further, this CFT is coupled to a (codimension-one) conformal defect positioned on the equator of the sphere. Hence, the defect spans the geometry $R \times S^{d-2}$ and supports a $(d-1)$ -dimensional CFT.

The bulk description of this system involves an asymptotically AdS_{d+1} spacetime with a codimension-one brane spread through the middle of the space (and extending to the position of the defect at asymptotic infinity). In this setup which we will review below, the brane has an AdS_d geometry. In particular, we consider the case in which the brane has a substantial tension and backreacts on the bulk geometry. If the brane tension is appropriately tuned, the backreaction produces Randall-Sundrum (RS) gravity supported on the brane [84, 85], as we will describe later in this chapter. That is, in the backreacted geometry, new (normalizable) modes of the $(d+1)$ -dimensional bulk graviton are localized near the brane inducing an effective theory of d -dimensional dynamical gravity on the brane.

Our overall goal in this chapter is to describe the bulk geometry produced by the backreaction of the brane, and also the gravitational theory induced on the brane.

2.1 Brane Geometry

In the bulk, we have Einstein gravity with a negative cosmological constant in $d + 1$ dimensions, *i.e.*,

$$I_{\text{bulk}} = \frac{1}{16\pi G_{\text{bulk}}} \int d^{d+1}x \sqrt{-g} \left[R(g) + \frac{d(d-1)}{L^2} \right], \quad (2.1)$$

where g_{ab} denotes the bulk metric, and we are ignoring the corresponding surface terms here [99–101]. We also introduce a codimension-one (*i.e.*, d -dimensional) brane in the bulk gravity theory. The brane action is simply given by

$$I_{\text{brane}} = -T_o \int d^d x \sqrt{-\tilde{g}}. \quad (2.2)$$

where T_o is the brane tension and $\tilde{g}_{ij}$ denotes the induced metric on the brane. We will later supplement this action with a Dvali-Gabadadze-Porrati (DGP) term [87], but for now let us consider just the tension term (2.2).

Away from the brane, the spacetime geometry locally takes the form of AdS_{d+1} with the curvature scale set by L , *i.e.*, it has constant curvature

$$R_{\mu\nu\rho\sigma} = -\frac{1}{L^2} (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}). \quad (2.3)$$

As described above, the induced geometry on the brane will be an AdS_d space, and so it is useful to consider the following metric where the AdS_{d+1} geometry is foliated by AdS_d slices

$$ds^2 = d\varpi^2 + \cosh^2(\varpi/L) g_{ij}^{\text{AdS}_d} dx^i dx^j. \quad (2.4)$$

Implicitly here, L also sets the curvature of the AdS_d metric, *e.g.*, in global coordinates,

$$g_{ij}^{\text{AdS}_d} dx^i dx^j = L^2 \left[-\cosh^2 \tilde{\varrho} dt^2 + d\tilde{\varrho}^2 + \sinh^2 \tilde{\varrho} d\Omega_{d-2}^2 \right]. \quad (2.5)$$

With the above choices, we approach the asymptotic boundary with $\varpi \rightarrow \pm\infty$, or with fixed ϖ and $\tilde{\varrho} \rightarrow \infty$. In the latter case, we arrive at the equator of the boundary S^{d-1} , where the conformal defect is located. For the following, it will be convenient to replace ϖ with a Fefferman-Graham-like coordinate [102, 103] $0 < z < \infty$, or an angular coordinate $0 < \theta < \pi$,

$$z = 2Le^{-\varpi/L}, \quad \frac{1}{\sin \theta} = \cosh(\varpi/L), \quad (2.6)$$

with which the metric (2.4) becomes

$$ds^2 = \frac{L^2}{z^2} \left[dz^2 + \left(1 + \frac{z^2}{4L^2} \right)^2 g_{ij}^{\text{AdS}_d} dx^i dx^j \right] = \frac{1}{\sin^2 \theta} \left(L^2 d\theta^2 + g_{ij}^{\text{AdS}_d} dx^i dx^j \right). \quad (2.7)$$

In these coordinates we approach the asymptotic boundary with $z \rightarrow 0$ ($\theta \rightarrow 0$) and with $z \rightarrow \infty$ ($\theta \rightarrow \pi$). Below, we will focus on the region near $z \sim 0$ ($\theta \sim 0$).

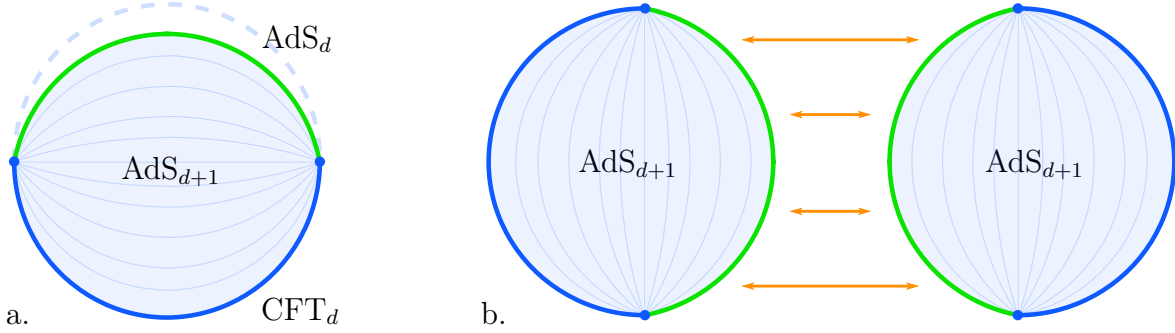


Figure 2.1: Panel (a): Our RS construction involves foliating with AdS_d slices. Then identical portions of two such AdS_{d+1} geometries are glued together along a common AdS_d slice. Panel (b): The jump in the extrinsic curvature across the interface between the two geometries is supported by a(n infinitely) thin brane. The brane is represented by a green line in the figures and the bulk AdS_{d+1} spacetime is blue with a d -dimensional CFT at the asymptotic boundary.

As described above, the brane spans an AdS_d geometry in the middle of the backreacted spacetime. Following the usual RS approach, we construct the desired solution by cutting off the AdS_{d+1} geometry at some $z = z_B$ ($\theta = \theta_B$), and then complete the space by gluing this geometry to another copy of itself — see figure 2.1. Then the Israel junction conditions (*e.g.*, see [104, 105]) fix z_B (and θ_B) by relating the discontinuity of the extrinsic curvature K_{ij} across this surface to the stress tensor introduced by the brane, *i.e.*,

$$\Delta K_{ij} - \tilde{g}_{ij} \Delta K^k_k = 8\pi G_{\text{bulk}} S_{ij} = -8\pi G_{\text{bulk}} T_o \tilde{g}_{ij}, \quad (2.8)$$

where $\Delta K_{ij} = K_{ij}^+ - K_{ij}^- = 2K_{ij}$ (given the symmetry of our construction) and $\tilde{g}_{ij}$ denotes the induced metric on the brane, located at $z = z_B$. More generally, the induced metric on an arbitrary surface of fixed z (or, equivalently, fixed θ) is given by

$$g_{ij} = \left(\frac{L}{z} + \frac{z}{4L} \right)^2 g_{ij}^{\text{AdS}_d} = \frac{1}{\sin^2 \theta} g_{ij}^{\text{AdS}_d}, \quad (2.9)$$

of which $\tilde{g}_{ij}$ is a special case with $z = z_B$ ($\theta = \theta_B$). Then, the extrinsic curvature is calculated as [105]

$$K_{ij} = \frac{1}{2} \frac{\partial g_{ij}}{\partial n} \Big|_{z=z_B} = -\frac{z}{2L} \frac{\partial g_{ij}}{\partial z} \Big|_{z=z_B} = \frac{1}{L} \frac{4L^2 - z_B^2}{4L^2 + z_B^2} \tilde{g}_{ij}, \quad (2.10)$$

where $\partial_n = -\frac{z}{L} \partial_z$ is the outward directed unit normal vector. Combining eqs. (2.8) and (2.10), we arrive at

$$\frac{4L^2 - z_B^2}{4L^2 + z_B^2} = \frac{4\pi G_{\text{bulk}} L T_o}{d-1}, \quad (2.11)$$

or alternatively in terms of the brane angle θ_B related to z_B by eq. (2.6),

$$\sin^2 \theta_B = 2\varepsilon (1 - \varepsilon/2), \quad \text{where } \varepsilon \equiv \left(1 - \frac{4\pi G_{\text{bulk}} L T_o}{d-1}\right). \quad (2.12)$$

Now if we consider $z_B \ll L$, it will ensure that the defect is well approximated by the holographic gravity theory on the brane — see the discussion in the next subsection. In this regime, we can solve eq. (2.11) in a small z_B expansion, and to leading order, we find that

$$z_B^2 \simeq z_0^2 = 2L^2 \varepsilon. \quad (2.13)$$

Hence to achieve this result, we must tune ε to be small:

$$\varepsilon \equiv 1 - \frac{4\pi G_{\text{bulk}} L T_o}{d-1} \ll 1. \quad (2.14)$$

As the notation suggests, we can think of this quantity ε as an expansion parameter in solving for the brane position from eq. (2.11). A useful check of our calculations below will come from carrying the solution to the next order, *i.e.*, $z_B^2 = z_0^2 + \delta[z_B^2]_2 + \dots$ with

$$\delta[z_B^2]_2 = \frac{(d-1)L}{4\pi G_{\text{bulk}} T_o} \varepsilon^2 = \frac{(d-1)L}{4\pi G_{\text{bulk}} T_o} \left(1 - \frac{4\pi G_{\text{bulk}} L T_o}{d-1}\right)^2. \quad (2.15)$$

To conclude, we consider the intrinsic geometry of the brane. As we noted above, the curvature scale of $g_{ij}^{\text{AdS}_d}$ is simply L , and hence given the full bulk metric (2.7), we can read off the curvature scale of the surface $z = z_B$ ($\theta = \theta_B$) as

$$\ell_B = \frac{L^2}{z_B} \left(1 + \frac{z_B^2}{4L^2}\right), \quad (2.16)$$

$$\frac{1}{\ell_B^2} = \frac{\sin^2 \theta_B}{L^2}. \quad (2.17)$$

Note that since we are considering $z_{\text{B}}/L \ll 1$, it follows that $\ell_{\text{B}}/L \gg 1$, *i.e.*, the brane is weakly curved compared to the bulk. Using eq. (2.13), we can solve for ℓ_{B} to leading order in the ε expansion to find

$$\frac{L^2}{\ell_{\text{B}}^2} \simeq 2\varepsilon = 2 \left(1 - \frac{4\pi G_{\text{bulk}} L T_o}{d-1} \right). \quad (2.18)$$

It will be useful to have the following expressions for the Ricci tensor and scalar evaluated for the brane geometry, and these are compactly written using eq. (2.16) as

$$\tilde{R}_{ij}(\tilde{g}) = -\frac{d-1}{\ell_{\text{B}}^2} \tilde{g}_{ij}, \quad \tilde{R}(\tilde{g}) = -\frac{d(d-1)}{\ell_{\text{B}}^2}. \quad (2.19)$$

2.2 Gravitational Action on the Brane

As noted above, following the usual RS scenario [84–86], new (normalizable) modes of the bulk graviton are localized near the brane in the backreacted geometry, and this induces an effective theory of dynamical gravity on the brane. The gravitational action can be determined as follows: First, one considers a Fefferman-Graham (FG) expansion near the boundary of an asymptotic AdS geometry [102, 103]. Then integrating the bulk action (including the Gibbons-Hawking-York surface term [99, 100]) over the radial direction out to some regulator surface produces a series of divergent terms, which through the FG expansion can be associated with various geometric terms involving the intrinsic curvature of the boundary metric. Usually in AdS/CFT calculations, a series of boundary counterterms are added to the action to remove these divergences, as the regulator surface is taken to infinity [101]. In the present brane world construction, the regulator surface is replaced by the brane, which remains at a finite radius, and no additional counterterms are added. Rather the ‘divergent’ terms become contributions to the gravitational action of the brane theory. Hence, drawing upon previous discussions of the boundary counterterms in AdS/CFT [101], the brane action becomes

$$I_{\text{diver}} = \frac{1}{16\pi G_{\text{bulk}}} \int d^d x \sqrt{-\tilde{g}} \left[\frac{2(d-1)}{L} + \frac{L}{(d-2)} \tilde{R} + \frac{L^3}{(d-4)(d-2)^2} \left(\tilde{R}^{ij} \tilde{R}_{ij} - \frac{d}{4(d-1)} \tilde{R}^2 \right) + \dots \right]. \quad (2.20)$$

Several comments are in order at this point: First of all, we note that the above expression is written in terms of the induced metric $\tilde{g}_{ij}$ on the brane (as in [101]) rather

than the boundary metric ${}^{(0)}g_{ij}$ that enters the FG expansion. Using the standard results, *e.g.*, [106, 107], we can relate the two with

$$\tilde{g}_{ij}(x_k) = \frac{L^2}{z_B^2} {}^{(0)}g_{ij}(x_k) + {}^{(1)}g_{ij}(x_k) + \frac{z_B^2}{L^2} {}^{(2)}g_{ij}(x_k) + \cdots, \quad (2.21)$$

where the higher order terms can be expressed in terms of the curvatures of ${}^{(0)}g_{ij}$, *e.g.*,

$${}^{(1)}g_{ij} = -\frac{L^2}{d-2} \left(R_{ij} [{}^{(0)}g] - \frac{{}^{(0)}g_{ij}}{2(d-1)} R [{}^{(0)}g] \right). \quad (2.22)$$

In other words, the two metrics are related by a Weyl scaling and a field redefinition. Further, we see a factor of $(d-2)$ appearing in the denominator of the second term, *i.e.*, the Einstein-Hilbert term, in eq. (2.20). Hence this expression only applies for $d \geq 3$ and must be reevaluated for $d = 2$, which we do in section 2.3. Similar factors, as well as a factor of $d-4$, appear in the denominator of the third term, which again indicates that this expression must be reconsidered for $d = 4$.

In any event, the gravitational action on the brane is given by combining the above expression with the brane action (2.2),

$$I_{\text{induced}} = 2 I_{\text{diver}} + I_{\text{brane}}, \quad (2.23)$$

where the factor of two in the first term accounts for integrating over the bulk geometry on both sides of the brane — see fig. 2.1. The combined result can be written as

$$\begin{aligned} I_{\text{induced}} = & \frac{1}{16\pi G_{\text{eff}}} \int d^d x \sqrt{-\tilde{g}} \left[\frac{(d-1)(d-2)}{\ell_{\text{eff}}^2} + \tilde{R}(\tilde{g}) \right] \\ & + \frac{1}{16\pi G_{\text{RS}}} \int d^d x \sqrt{-\tilde{g}} \left[\frac{L^2}{(d-4)(d-2)} \left(\tilde{R}^{ij} \tilde{R}_{ij} - \frac{d}{4(d-1)} \tilde{R}^2 \right) + \cdots \right], \end{aligned} \quad (2.24)$$

where

$$\frac{1}{G_{\text{eff}}} = \frac{1}{G_{\text{RS}}} = \frac{2L}{(d-2)G_{\text{bulk}}}, \quad \frac{1}{\ell_{\text{eff}}^2} = \frac{2}{L^2} \left(1 - \frac{4\pi G_{\text{bulk}} L T_o}{d-1} \right). \quad (2.25)$$

In the present discussion G_{eff} and G_{RS} are equal, but by adding terms to the brane action this can change. We will explain this in section 2.4. Comparing eqs. (2.12), (2.18), and (2.25), we see that ℓ_{eff} (which sets the cosmological constant term in I_{induced}) precisely matches the leading order expression for the brane curvature ℓ_B . Hence if we only consider

the first two terms in eq. (2.24), the resulting Einstein equations would reproduce the leading expression (in the ε expansion) for the curvatures in eq. (2.19). Further, it is a straightforward exercise to show that if the contribution of the curvature squared terms is also included in the gravitational equations of motion, the curvature is shifted to precisely reproduce the ε^2 term in eq. (2.19). Hence rather than using the Israel junction conditions, we could determine the position of the brane in the backreacted geometry by first solving the gravitational equations of the brane action (2.24) and then finding the appropriate surface $z = z_B$ with the corresponding curvature. More generally, the fact that these two approaches match was verified by [108],¹ which argued the bulk Einstein equations combined with the Israel junction conditions are equivalent to the brane gravity equations of motion.²

This approach of constructing an effective theory of gravity on the brane is only a good way to describe the brane physics in the limit that $\ell_{\text{eff}} \gg L$; otherwise the contributions of the higher curvature terms cannot be ignored and dealing with the effective gravity theory becomes very complicated. For example, if the curvatures are proportional to $1/\ell_{\text{eff}}^2$ at leading order, then the curvature squared term is suppressed by a factor of L^2/ℓ_{eff}^2 relative to the first two terms. Similarly the higher order curvature terms denoted by the ellipsis in eq. (2.24) are further suppressed by a further factor of L^2/ℓ_{eff}^2 for each additional curvature appearing in these terms. From eqs. (2.12) and (2.18), we can write $\frac{L^2}{\ell_{\text{eff}}^2} = 2\varepsilon$ and hence we see that the gravitational brane action and the resulting equations of motion can be organized in the same small ε expansion discussed in section 2.1.

Following the usual rules of AdS/CFT correspondence, we can give a holographic description of this system involving (two weakly interacting copies of) the boundary CFT living on the brane. However, this CFT has a finite UV cutoff because the brane resides at a finite radius in the bulk, *e.g.*, see [107, 115, 116]. The action (2.20) is then the induced gravitational action resulting from integrating out the CFT degrees of freedom. The UV cutoff is usually discussed in the context of the boundary metric $g_{ij}^{(0)}$, where the short distance cutoff would be given by $\delta \simeq z_B$. However, recall that the gravitational action (2.24) is expressed in terms of the induced metric $\tilde{g}_{ij}$ and so the conformal transformation in eq. (2.21) yields $\tilde{\delta} \simeq L$ for this description of the brane theory. Therefore the ε expansion corresponds to an expansion in powers of the short distance cutoff, *i.e.*, $\varepsilon \sim \tilde{\delta}^2/\ell_{\text{eff}}^2$.

¹See also earlier discussions, *e.g.*, [109–111].

²We note that the brane graviton acquires a small mass through interactions with the CFT residing there [86, 112, 113]. While this mass appears to be negligible in the regime of interest, *i.e.*, $L/\ell_{\text{eff}} \ll 1$ — see further discussion in section 2.5 — it has also been argued [60, 114] that this mass plays an important role in the appearance of islands.

2.3 The case of two dimensions

Recall that the curvature terms in the induced action (2.20) have coefficients with inverse powers of $(d-2)$ and so we must reconsider the calculation of this brane action for $d=2$, *i.e.*, when the bulk space is (locally) AdS_3 and the induced geometry on the brane is AdS_2 . This section sketches the necessary calculations, which are largely the same as those performed in higher dimensions, but with a few important differences.

Let us add that in contrast to the induced action, the calculations in section 2.1, where the position of the brane is determined using the Israel junction conditions, need no modifications for $d=2$. Therefore we can simply substitute $d=2$ into eqs. (2.13) and (2.15) for the brane position to find

$$z_{\text{B}}^2 \simeq 2L^2\varepsilon + \frac{L}{4\pi G_{\text{bulk}}T_o}\varepsilon^2 + \cdots, \quad \text{with} \quad \varepsilon = 1 - 4\pi G_{\text{bulk}}LT_o. \quad (2.26)$$

Of course, we must be able to reproduce the same result using the new induced gravity action.

Integration of bulk action

As discussed in section 2.2, one can determine the structure of the terms in the induced action by a careful examination of the FG expansion near the asymptotic boundary [107, 108, 117]. However, we can take the simpler route here, since in two dimensions the Riemann curvature has a single component and therefore the entire induced action can be expressed in terms of the Ricci scalar $\mathcal{R}(\tilde{g})$. Therefore, we evaluate the on-shell bulk action and match the boundary divergences to an expansion in $\mathcal{R}(\tilde{g})$. That is, we substitute the metric (2.7) into the bulk action (2.1) plus the corresponding Gibbons-Hawking-York surface term [99, 100] and integrate over the radial direction z . The result can be expressed as a boundary integral with a series of divergences as $z_{\text{B}} \rightarrow 0$,³

$$I_{\text{diver}} = \frac{L}{16\pi G_{\text{bulk}}} \int d^2x \sqrt{-g^{\text{AdS}_2}} \left[\frac{1}{z_{\text{B}}^2} + \frac{1}{L^2} \log\left(\frac{z_{\text{B}}}{L}\right) - \frac{z_{\text{B}}^2}{16L^4} + \cdots \right]. \quad (2.27)$$

³This expression also includes $\mathcal{O}(z_{\text{B}}^2)$ contributions (which are not divergent as $z_{\text{B}} \rightarrow 0$); these are necessary to match eq. (2.26) to $\mathcal{O}(\varepsilon^2)$ in the following. Further, note that we are ignoring the contributions coming from asymptotic boundaries at $z \rightarrow \infty$.

Now we rewrite the above expression in terms of the induced metric and the corresponding Ricci scalar combining eqs. (2.7), (2.16) and (2.19), which yield

$$\sqrt{-\tilde{g}} = \frac{L^2}{z_B^2} \left(1 + \frac{z_B^2}{4L^2}\right)^2 \sqrt{-g^{\text{AdS}_2}}, \quad \mathcal{R} = -2 \frac{z_B^2}{L^4} \left(1 + \frac{z_B^2}{4L^2}\right)^{-2}. \quad (2.28)$$

Using these expressions, the induced action becomes⁴

$$I_{\text{diver}} = \frac{L}{16\pi G_{\text{bulk}}} \int d^2x \sqrt{-\tilde{g}} \left[\frac{2}{L^2} - \frac{1}{2} \tilde{R} \log \left(-\frac{L^2}{2} \tilde{R} \right) + \frac{1}{2} \tilde{R} + \frac{L^2}{16} \tilde{R}^2 + \dots \right]. \quad (2.29)$$

The most striking feature of this induced action is the term proportional to $\mathcal{R} \log |\mathcal{R}|$. The appearance of this logarithm is related to the conformal anomaly [118–120], and points towards the fact that the corresponding gravitational action is nonlocal,⁵ as we discuss next. Further, since the Einstein-Hilbert term is topological in two dimensions, it turns out that this unusual action is precisely what is needed to match the dynamics of the bulk gravity described above, *i.e.*, the position of the brane in eq. (2.26).

The logarithmic contribution should correspond to that coming from the nonlocal Polyakov action [117]. Schematically, we would have

$$I_{\text{bulk}} \simeq I_{\text{Poly}} = -\frac{\alpha L}{16\pi G_{\text{bulk}}} \int d^2x \sqrt{-\tilde{g}} \tilde{R} \frac{1}{\square} \tilde{R}, \quad (2.30)$$

where we have introduced an arbitrary constant α here but it will be fixed by comparing with the divergences in the integrated action. Of course, $\frac{1}{\square} \tilde{R}$ indicates a convolution of the Ricci scalar with the scalar Green's function, but there are subtleties here in dealing with constant curvatures. The latter are ameliorated by making the action (2.30) local by introducing a auxiliary field ϕ (*e.g.*, see [117, 121]),

$$I_{\text{Poly}} = \frac{\alpha L}{8\pi G_{\text{bulk}}} \int d^2x \sqrt{-\tilde{g}} \left[-\frac{1}{2} \tilde{g}^{ij} \tilde{\nabla}_i \phi \tilde{\nabla}_j \phi + \phi \tilde{R} + \chi e^{-\phi} \right]. \quad (2.31)$$

⁴Our derivation of eq. (2.29) will miss terms involving derivatives of $\tilde{R}$ as these vanish for the constant curvature geometry of our brane. However, apart from the total derivative $\square \tilde{R}$, such terms will only appear at higher orders, *i.e.*, in the ‘ $\dots$ ’.

⁵Similar nonlocalities appear in the curvature-squared or four-derivative contributions with $d = 4$, or more generally in the interactions with $d/2$ curvatures for higher (even) d . Hence they do not play a role in higher dimensions if we work in the regime where the induced action (2.24) is well approximated by Einstein gravity coupled to a cosmological constant.

where χ is a fixed constant.⁶

The equation of motion resulting from eq. (2.31) is

$$0 = \tilde{\square}\phi + \tilde{R} - \chi e^{-\phi}, \quad (2.32)$$

which has a simple solution when $\tilde{R}$ is a constant, namely,

$$\phi = \phi_0 = \log(\chi/\tilde{R}). \quad (2.33)$$

Evaluating the Polyakov action with $\phi = \phi_0$ yields

$$I_{\text{Poly}}|_{\phi=\phi_0} = -\frac{\alpha L}{8\pi G_{\text{bulk}}} \int d^2x \sqrt{-\tilde{g}} \left[\tilde{R} \log(\tilde{R}/\chi) - \tilde{R} \right]. \quad (2.34)$$

Comparing this expression with the log term in eq. (2.29), we fix $\alpha = \frac{1}{4}$ and $\chi = -\frac{2}{\ell^2}$.

Varying the action (2.31) with respect to the metric, we find the corresponding contribution to the ‘gravitational’ equations of motion

$$T_{ij}^{\text{Poly}} = -\frac{2}{\sqrt{-g}} \frac{\delta I_{\text{Poly}}}{\delta g^{ij}} = \frac{L}{32\pi G_{\text{bulk}}} \left[\tilde{\nabla}_i \phi \tilde{\nabla}_j \phi + 2 \tilde{\nabla}_i \tilde{\nabla}_j \phi - \tilde{g}_{ij} \left(\frac{1}{2} (\tilde{\nabla} \phi)^2 + 2 \tilde{\square} \phi - \chi e^{-\phi} \right) \right], \quad (2.35)$$

where we have used $\tilde{R}_{ij} - \frac{1}{2} \tilde{g}_{ij} \tilde{R} = 0$ for $d = 2$ to eliminate the terms linear in ϕ (without any derivatives). Now substituting ϕ_0 , we find that this expression reduces to

$$T_{ij}^{\text{Poly}}|_{\phi=\phi_0} = \frac{L}{32\pi G_{\text{bulk}}} \tilde{g}_{ij} \tilde{R}, \quad (2.36)$$

which we will substitute into evaluating the equations of motion below to fix the position of the brane. As an aside, we can take the trace of the above expression to find that it reproduces the trace anomaly, *e.g.*, [123, 124]

$$\langle T^i_i \rangle = \frac{c}{24\pi} \tilde{R}, \quad (2.37)$$

⁶The last term is needed to take care of zero mode problem [121]. Examining the equation of motion (2.32), one can think of ϕ as a conformal factor relating the metric $\tilde{g}_{ij}$ to a canonical constant curvature metric $\hat{g}_{ij}$, *i.e.*, $\tilde{g}_{ij} = e^\phi \hat{g}_{ij}$ with $\hat{R}(\hat{g}) = \chi$ [121, 122]. Hence we choose χ to be negative to match the sign of $\tilde{R}$. Further, note that with the interaction $\chi e^{-\phi}$ in the action (2.31), ϕ becomes an interacting field [117].

where we recall that $c = \frac{3L}{2G_{\text{bulk}}}$ for the boundary CFT. In our case, the trace anomaly will be twice as large, since there are two copies of the CFT living on the brane.

The induced action $I_{\text{induced}} = 2 I_{\text{diver}} + I_{\text{brane}}$ can be written as

$$I_{\text{induced}} = \frac{1}{16\pi G_{\text{eff}}} \int d^2x \sqrt{-\tilde{g}} \left[\frac{2}{\ell_{\text{eff}}^2} - \tilde{R} \log \left(-\frac{L^2}{2} \tilde{R} \right) + \tilde{R} + \frac{L^2}{8} \tilde{R}^2 + \dots \right], \quad (2.38)$$

where the two effective scales are

$$\left(\frac{L}{\ell_{\text{eff}}} \right)^2 = 2(1 - 4\pi G_{\text{bulk}} L T_o), \quad G_{\text{eff}} = G_{\text{bulk}}/L. \quad (2.39)$$

Notice that while the first equality follows the same definition as in higher dimensions, the second one must be redefined for $d = 2$ (c.f. eq. (2.25)). The metric variation then yields the following equation of motion

$$0 = \frac{2}{\ell_{\text{eff}}^2} \tilde{g}_{ij} + \tilde{g}_{ij} \tilde{R} + \frac{L^2}{8} \tilde{R} (\tilde{g}_{ij} \tilde{R} - 4\tilde{R}_{ij}) + \dots, \quad (2.40)$$

where we dropped the terms involving derivatives of curvatures arising from the variation of the $\tilde{R}^2$ term. To leading order, we find $\tilde{R} \sim -2/\ell_{\text{eff}}^2 = -4\varepsilon/L^2$ in agreement with eqs. (2.12), (2.18), and (2.19). Hence, the gravitational equations of motion again fix the (leading-order) position of the brane for $d = 2$, and further it is a straightforward exercise to match to second order corrections in eq. (2.26) using the curvature-squared contributions in eq. (2.40).

Adding JT gravity

Much of the recent literature on QEIs examines models involving two-dimensional gravity, *e.g.*, [4, 7, 34, 48, 50, 53, 65], however, the gravitational theory in these models is Jackiw-Teitelboim (JT) gravity [125, 126]. One can incorporate JT gravity into the current model by dropping the usual tension term (2.2), and instead using the following brane action⁷

$$I_{\text{brane}} = I_{\text{JT}} + I_{\text{ct}}, \quad (2.41)$$

where the JT action takes the usual form,

$$I_{\text{JT}} = \frac{1}{16\pi G_{\text{brane}}} \int d^2x \sqrt{-\tilde{g}} \left[\Phi_0 \tilde{R} + \Phi \left(\tilde{R} + \frac{2}{\ell_{\text{JT}}^2} \right) \right]. \quad (2.42)$$

⁷Alternatively, one could simply add I_{JT} to the usual tension term. With this approach, an extra source term appears in eq. (6.4), but it can be eliminated by shifting the dilaton in a manner similar to eq. (2.48).

Here, as in previous actions, we have ignored the boundary terms associated with the JT action, *e.g.*, see [127], and we have introduced the dilaton Φ . Recall that Φ_0 is simply a constant and so the first term is topological but contributes to the generalized entropy. In eq. (2.41), we have also included a counterterm

$$I_{\text{ct}} = -\frac{1}{4\pi G_{\text{bulk}} L} \int d^2x \sqrt{-\tilde{g}}, \quad (2.43)$$

which is tuned to cancel the induced cosmological constant on the brane. This choice ensures that the JT gravity (2.42) couples to the boundary CFT in the expected way, *e.g.*, as in [34, 127] — see further comments below.

The full induced action now takes the form

$$I_{\text{induced}} = \frac{1}{16\pi G_{\text{eff}}} \int d^2x \sqrt{-\tilde{g}} \left[-\tilde{R} \log \left(-\frac{L^2}{2} \tilde{R} \right) + \frac{L^2}{8} \tilde{R}^2 + \dots \right] \\ + \frac{1}{16\pi G_{\text{brane}}} \int d^2x \sqrt{-\tilde{g}} \left[\tilde{\Phi}_0 \tilde{R} + \Phi \left(\tilde{R} + \frac{2}{\ell_{\text{JT}}^2} \right) \right], \quad (2.44)$$

where we have combined the two topological contributions in the second line with⁸

$$\tilde{\Phi}_0 = \Phi_0 + G_{\text{brane}}/G_{\text{eff}}. \quad (2.45)$$

Now, with the JT action (2.42), the dilaton equation of motion fixes $\tilde{R} = -2/\ell_{\text{JT}}^2$, *i.e.*, the brane geometry is locally AdS_2 everywhere with $\ell_{\text{B}} = \ell_{\text{JT}}$. Then the position z_{B} of the brane is fixed by eq. (2.16) and implicitly we assume that $\ell_{\text{JT}} \gg L$, which ensures that $z_{\text{B}} \ll L$ as in our previous discussions. The gravitational equation of motion coming from the variation of the metric becomes

$$-\nabla_i \nabla_j \Phi + \tilde{g}_{ij} \left(\nabla^2 \Phi - \frac{\Phi}{\ell_{\text{JT}}^2} \right) = 8\pi G_{\text{brane}} \tilde{T}_{ij}^{\text{CFT}} = -\frac{G_{\text{brane}}}{G_{\text{eff}}} \frac{1}{\hat{\ell}_{\text{eff}}^2} \tilde{g}_{ij}, \quad (2.46)$$

where $\hat{\ell}_{\text{eff}}$ is the effective curvature scale produced by ℓ_{JT} . That is, in the case without JT gravity, we can combine eqs. (2.11), (2.16) and (2.25) to find

$$\frac{L^2}{\ell_{\text{eff}}^2} = f\left(\frac{L^2}{\ell_{\text{B}}^2}\right) \equiv 2 \left(1 - \sqrt{1 - \frac{L^2}{\ell_{\text{B}}^2}} \right). \quad (2.47)$$

⁸In [34], $\tilde{\Phi}_0$ would also absorb a logarithmic constant $-2\log(L/z_{\text{B}})$, which would be accompanied by a shift in the prefactor in the argument of the logarithmic term in eq. (2.44), *i.e.*, $2/L^2 \rightarrow 2/z_{\text{B}}^2$.

We can understand this expression as the gravitational equation of motion coming from the two-dimensional action (2.38), where a Taylor expansion of the right-hand side for $L/\ell_{\text{B}} \ll 1$ corresponds to varying the curvature terms and subsequently substituting $\tilde{R}_{ij} = -\frac{1}{\ell_{\text{B}}^2} \tilde{g}_{ij}$, as in eq. (2.19). Now in the JT equation of motion (6.4), the effective curvature scale $\hat{\ell}_{\text{eff}}$ satisfies $L^2/\hat{\ell}_{\text{eff}}^2 = f(L^2/\ell_{\text{JT}}^2)$. We have indicated in eq. (6.4) that the left-hand side corresponds to the stress tensor of the boundary CFT which lives on the brane. In the present arrangement,⁹ this takes a particularly simple form, with $T_{ij}^{\text{CFT}} \propto \tilde{g}_{ij}$. Of course, this source term in eq. (6.4) can be easily absorbed by shifting the dilaton,

$$\tilde{\Phi} \equiv \Phi - \frac{G_{\text{brane}}}{G_{\text{eff}}} \frac{\ell_{\text{JT}}^2}{\hat{\ell}_{\text{eff}}^2}, \quad (2.48)$$

so that $\tilde{\Phi}$ satisfies the usual source-free equation studied in *e.g.*, [127].

At this point, we observe that the trace of eq. (6.4) yields on the right-hand side,

$$\langle [\tilde{T}^{\text{CFT}}]^i_i \rangle = -\frac{L}{4\pi G_{\text{bulk}}} \frac{1}{\hat{\ell}_{\text{eff}}^2} = -\frac{L}{4\pi G_{\text{bulk}}} \frac{1}{\ell_{\text{JT}}^2} \left(1 + \frac{1}{4} \frac{L^2}{\ell_{\text{JT}}^2} + \frac{1}{8} \frac{L^4}{\ell_{\text{JT}}^4} + \dots \right), \quad (2.49)$$

where in the final expression, we are Taylor expanding $f(L^2/\ell_{\text{JT}}^2)$ assuming $L^2/\ell_{\text{JT}}^2 \ll 1$, as above. Noting that $\tilde{R} = -2/\ell_{\text{JT}}^2$ and comparing to eq. (2.37),¹⁰ we see that the expected trace anomaly has received an infinite series of higher order corrections. We can interpret the latter as arising from the finite UV cutoff on the brane, recalling that $\tilde{\delta} \simeq L$ as discussed at the end of section 2.2.

2.4 Dvali-Gabadadze-Porrati Gravity on the Brane

The previous discussion of $d = 2$ motivates that it is interesting to add an intrinsic gravity term to the brane action. Here, we extend this discussion to higher dimensions, *i.e.*, extend the brane action to include an Einstein-Hilbert term. Of course, this scenario can be viewed as a version of Dvali-Gabadadze-Porrati (DGP) gravity [87] in an AdS background. Hence, it combines features of both RS and DGP gravity theories. We discuss the modifications

⁹In more interesting scenarios, *e.g.*, with evaporating black holes as in [4, 7, 34], it is more appropriate to work directly with the CFT's stress tensor, rather than replacing these degrees of freedom by an effective gravity action after integrating out the CFT.

¹⁰Recall that the central charge here is twice that appearing in eq. (2.37) because the brane supports two (weakly interacting) copies of the boundary CFT.

of the brane dynamics and the induced action below, but it also produces interesting modifications of the generalized entropy, as discussed in chapter 3 and appendices A and B.

We write the extended brane action, replacing eq. (2.2), as

$$I_{\text{brane}} = -(T_o - \Delta T) \int d^d x \sqrt{-\tilde{g}} + \frac{1}{16\pi G_{\text{brane}}} \int d^d x \sqrt{-\tilde{g}} \tilde{R}. \quad (2.50)$$

In general, for a fixed brane tension, the position of the brane will be modified with the additional Einstein-Hilbert term. Hence we have parametrized the full brane tension as $T_o - \Delta T$ and the contribution ΔT will be tuned to keep the position of the brane fixed. This choice will facilitate the comparison of the generalized entropy between different scenarios in the following.

As in section 2.1, the position of the brane can be determined using the Israel junction conditions (2.8). Hence we begin by evaluating the brane's stress tensor,

$$S_{ij} \equiv -\frac{2}{\sqrt{-\tilde{g}}} \frac{\delta I_{\text{brane}}}{\delta \tilde{g}^{ij}} = -\tilde{g}_{ij}(T_o - \Delta T) - \frac{1}{8\pi G_{\text{brane}}} \left(\tilde{R}_{ij} - \frac{1}{2} \tilde{g}_{ij} \tilde{R} \right). \quad (2.51)$$

As commented above, we choose ΔT to cancel the curvature contributions in this expression, *i.e.*, the stress tensor reduces to $S_{ij} = -T_o \tilde{g}_{ij}$. With this tuning, the Israel junction conditions in eq. (2.8) are unchanged as the analysis which follows from there. Therefore the brane position and curvature remain identical to those determined in eqs. (2.11) and (2.16). This allows use to determine the desired tuning as

$$\Delta T = \frac{(d-1)(d-2)}{16\pi G_{\text{brane}} \ell_{\text{B}}^2} \simeq \frac{(d-1)(d-2)}{8\pi G_{\text{brane}} L^2} \varepsilon. \quad (2.52)$$

We have used eqs. (2.12) and (2.18) to show that the shift in the brane tension is small in the ε expansion.

We return to the induced gravitational action on the brane that takes the same form as in eq. (2.24) but with the effective Newton's constant in eq. (2.25) replaced by

$$\frac{1}{G_{\text{eff}}} = \frac{2L}{(d-2) G_{\text{bulk}}} + \frac{1}{G_{\text{brane}}}. \quad (2.53)$$

By construction, ℓ_{eff} and the position of the brane are unchanged. Note that the gravitational couplings in the Einstein terms and in the higher curvature interactions, *i.e.*, in the first and second lines of eq. (2.24), are now distinct. That is, G_{eff} no longer equals G_{RS} .

In the following, it will be useful to define the ratio

$$\lambda_b = \frac{G_{\text{RS}}}{G_{\text{brane}}} \quad \text{with} \quad \frac{1}{G_{\text{RS}}} = \frac{2L}{(d-2) G_{\text{bulk}}}, \quad (2.54)$$

where G_{RS} is the induced Newton's constant on an RS brane appearing in eq. (2.25), while the dimensionless ratio λ_b controls the relative strength of the Newton's constants in the bulk and on the brane. With these definitions, the induced Newton's constant on the DGP brane, in eq. (2.53), can be rewritten as

$$\frac{1}{G_{\text{eff}}} = \frac{1}{G_{\text{RS}}} (1 + \lambda_b). \quad (2.55)$$

Of course, one can also consider other modifications of the brane action beyond adding the Einstein-Hilbert term in eq. (2.50) — see discussion in the next subsection and [128]. Further, we will discuss adding topological gravitational terms on the brane or in the bulk in chapters 3 and 7. In particular, we will see in section 3.4 that adding a Gauss-Bonnet term to the four-dimensional bulk gravity theory yields another tunable parameter which, for a certain parameter range, makes it possible to find quantum extremal islands in the absence of black holes.

2.5 Three perspectives: Bulk/Brane/Boundary

Our setup can be interpreted from three different ‘holographic’ perspectives, which are analogous to the three descriptions of [7], suitably generalized to arbitrary dimensions. A set of analogous descriptions for gravity on a brane in higher dimensions was discussed in the context of the Karch-Randall model [86], and in fact, these are the models discussed here with the addition of the DGP term (2.50). In this section we review each of the dual descriptions, and explore their relation.

First, consider the *bulk gravity perspective* corresponding to the geometric picture portrayed in section 2.1: we have an AdS_{d+1} bulk region where gravity is dynamical, containing a DGP brane with tension running through the middle of the spacetime — see figure 1.5a. The induced geometry on the brane is AdS_d . In the second picture, we integrate out the bulk action from the asymptotic boundary where gravity is frozen up to the brane, giving rise to RS gravity [84–86] on the brane. From the resulting *brane perspective*, the CFT_d is then supported in a region with dynamical gravity (*i.e.*, the brane) and another non-dynamical one (*i.e.*, the asymptotic boundary) — figure 1.5b. Finally, the third description makes full use of the AdS/CFT dictionary, by using holography along the brane.

This *boundary perspective* describes the system as a CFT_d coupled to a conformal defect that is located at the position where the brane intersects the asymptotic boundary — see figure 1.5c.

A holographic system was presented in [7] to describe the evaporation of two-dimensional black holes in JT gravity. This system has three descriptions analogous to those above. Of course, it also includes certain elements that we did not introduce in our model, *i.e.*, end-of-the-world branes to give a holographic description of conformal boundaries separating various components [91, 92] and performing a $\mathbb{Z}_2$ orbifold quotient across the Planck brane, *i.e.*, the brane supporting JT gravity. However, the essential ingredients are the same as above. The boundary perspective in [7] describes the system as a two-dimensional holographic conformal field theory with a boundary, at which it couples to a (one-dimensional) quantum mechanical system — figure 1.5f. With the brane perspective, the quantum mechanical system is replaced by its holographic dual, the Planck brane supporting JT gravity coupled to another copy of the two-dimensional holographic CFT — see figure 1.5e. Finally, the bulk gravity perspective replaces the holographic CFT with three-dimensional Einstein gravity in an asymptotically AdS_3 geometry. Because of the $\mathbb{Z}_2$ orbifolding, the latter effectively has two boundaries, the standard asymptotically AdS boundary and the dynamical Planck brane — see figure 1.5d.

This initial model [7] raised a number of intriguing puzzles. For example, as emphasized in [50], implicitly two different notions of the radiation degrees of freedom are being used: one being the semi-classical approximation and the other one in the purely quantum theory. Here, we will explain some details of the higher dimensional construction which allow us to provide a resolution of several of these questions in chapter 7.

2.5.1 Bulk gravity perspective

As discussed in section 2.1, the system has a bulk description in terms of gravity on an asymptotically AdS_{d+1} spacetime containing a codimension-one brane, which splits the bulk into two halves — see figure 1.5a. The brane is characterized by the tension T_o and also the DGP coupling $1/G_{\text{brane}}$, introduced in eqs. (2.2) and (2.50), respectively. We can use the Israel junction conditions (2.8) to determine the location of the brane as embedded in the higher dimensional space. The backreaction causes warping around the brane, and after a change of coordinates, tuning the brane tension can be understood as moving the brane further into a new asymptotic AdS region, as seen in eq. (2.11) or (2.13). For large brane tension, *i.e.*, with $\varepsilon \ll 1$, the spectrum of graviton fluctuations in the bulk is almost unchanged with respect to the modes in empty AdS space. However, a new set of

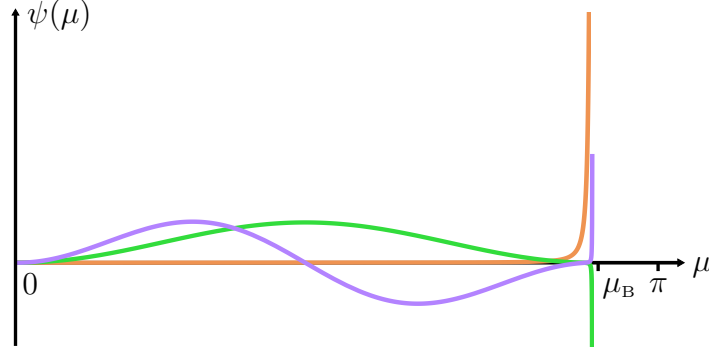


Figure 2.2: This figure illustrates the spatial profile of the first few normalized graviton modes in the presence of a large tension brane, and a $\mathbb{Z}_2$ orbifolding across the brane. We use the spatial coordinate μ , related to ϖ in eq. (2.4) by $\cot \mu = \sinh \varpi/L$. The tension is adjusted such that the location of the brane is at $\mu = \mu_B$ with $\mu_B \lesssim \pi$. As discussed in the main text, the presence of the brane creates new bulk modes (orange), which are highly localized at the brane, and which play the role of a (nearly massless) graviton on the brane. The remaining bulk modes appear as KK modes in the brane theory.

graviton states also appear localized at the brane [84, 85], as illustrated figure 2.2. These are created by the nonlinear coupling of gravity to the brane. Unlike in the RS model with a flat or de Sitter brane, the new graviton modes are not actually massless on the brane, but merely very light states whose wavefunction peaks around the brane [86, 112]. The remaining bulk graviton modes appear as a tower of Kaluza-Klein states, from the point of view of the theory on the brane, with masses of $\mathcal{O}(1/\ell_{\text{eff}})$ set by the curvature scale of the d -dimensional AdS geometry on the brane. These results have been studied in quite some detail [86, 112, 113, 129–131] for RS branes, but it is interesting to examine how the spectrum is modified by the DGP term (2.50). We will make some qualitative statements about this question below, but leave a detailed quantitative discussion and the interpretation of this mechanism from the point of view of the CFT for future work [128].

2.5.2 Brane perspective

This second perspective, discussed in section 2.2, effectively integrates out the spatial direction between the asymptotically AdS boundary and the brane to produce an effective action (2.24) for RS/DGP gravity on the brane, with the new localized graviton state playing the role of the d -dimensional graviton. Hence, we are left with a d -dimensional

theory of gravity coupled to (two copies of) the dual CFT on the brane — see figure 1.5b. As discussed in the description of the bulk perspective, amongst the new localized bulk modes, we have an almost massless graviton but also a tower of massive Kaluza-Klein states with masses of $\mathcal{O}(1/\ell_{\text{eff}})$. In section 2.2, we demonstrated the consistency between the bulk gravity perspective and the brane perspective by observing how the equations of motion of the new effective action fix the brane position in the ambient spacetime. Of course, the bulk physics is also dual to the dual CFT on the asymptotic AdS_{d+1} boundary, and so this description is completed by coupling the gravitational and CFT degrees of freedom on the brane to the CFT on the fixed boundary geometry. We refer to that latter as the *bath* CFT. Next, we discuss how different parameters in the brane perspective are related to bulk parameters.

There are four independent parameters that characterize the gravitational theory on the brane: the curvature scale ℓ_{eff} , the effective Newton’s constant G_{eff} , the central charge of the boundary CFT c_{T} , and the effective short-distance cutoff $\tilde{\delta}$. These emerge from the bulk theory through the four parameters characterizing the latter: the bulk curvature scale L , the bulk Newton’s constant G_{bulk} , the brane Newton’s constant G_{brane} and the brane tension T_o .¹¹ From eq. (2.25), we see that ℓ_{eff} is determined by a specific combination of T_o , G_{bulk} and L . Similarly, G_{eff} is determined by G_{brane} , G_{bulk} and L in eq. (2.53). The central charge of the boundary CFT is given by the standard expression $c_{\text{T}} \sim L^{d-1}/G_{\text{bulk}}$, *e.g.*, see [132].

Lastly, as discussed in section 2.2, the theory on the brane comes with a short-distance cutoff $\tilde{\delta}$ [107, 115, 116] at which the description of the brane theory in terms of (two copies of) the boundary CFT coupled to Einstein gravity breaks down. Following a standard bulk analysis, one would see that correlators of local operators (with appropriate gravitational dressings) no longer exhibit the expected CFT behaviour at short distances of order

$$\tilde{\delta}_{\text{CFT}} \sim L. \quad (2.56)$$

We denote this cutoff with the subscript ‘CFT’ to emphasize that the description of the matter degrees of freedom on the brane as a local d -dimensional CFT is failing at distances smaller than this short-distance cutoff. However, we stress that there is another scale $\tilde{\delta}_{\text{GR}}$, which is the distance at which the approximation of Einstein gravity on the brane breaks down. The simple parameter counting above shows that this cannot be an independent scale. For the brane perspective, the true cutoff $\tilde{\delta}$ where the description in terms of the dual CFT coupled to Einstein gravity fails is

$$\tilde{\delta} = \max \left\{ \tilde{\delta}_{\text{CFT}}, \tilde{\delta}_{\text{GR}} \right\}. \quad (2.57)$$

¹¹Recall that ΔT is determined by these parameters in eq. (2.52), as well as eqs. (2.11) and (2.16).

We now discuss how $\tilde{\delta}_{\text{GR}}$ is related to the other scales in the brane theory.

Recall that integrating out the bulk degrees of freedom produces a series of higher curvature terms in the effective action (2.24), and hence demanding that d -dimensional Einstein gravity provides a good approximation of the brane theory introduces constraints. The suppression of these higher curvature corrections requires that the ratio L/ℓ_{eff} be small. However, if we examine eq. (2.24) carefully and note the distinction $G_{\text{eff}} \neq G_{\text{RS}}$, then suppressing the curvature-squared terms requires that

$$\frac{1}{1 + \lambda_{\text{b}}} \frac{L^2}{\ell_{\text{eff}}^2} \ll 1, \quad (2.58)$$

using eq. (2.55). Note that for fixed bulk and boundary curvature scales, this implies a lower bound on the DGP term, such that λ_{b} cannot be arbitrarily close to -1 . For a pure RS brane with no additional DGP gravity, *i.e.*, $\lambda_{\text{b}} = 0$, we conclude that the cutoff below which we find Einstein gravity coincides with the CFT cutoff $\tilde{\delta}_{\text{GR}} \sim \tilde{\delta}_{\text{CFT}} \sim L$. More generally then, the above expression suggests that the DGP term (2.50) affects a shift producing a new short-distance cutoff for gravity,

$$\tilde{\delta}_{\text{GR}} \sim \frac{L}{\sqrt{1 + \lambda_{\text{b}}}} \sim \frac{\tilde{\delta}_{\text{CFT}}}{\sqrt{1 + \lambda_{\text{b}}}}. \quad (2.59)$$

Hence the true cutoff (2.57) depends on the sign of λ_{b} — we return to this point below. We should note that this result only applies for $d > 4$. For $d = 4$, the coefficient of the curvature-squared term is logarithmic in the cutoff, while for $d = 2$ or 3 , this interaction is not associated with a UV divergence.

While the above are UV effects, there are also infrared (IR) effects resulting from having a large number of matter degrees of freedom propagating on the brane, as explained in [133–135]. The usual regime of validity for QFT in semiclassical gravity lies at energy scales below the Planck mass, or at distance scales larger than $G_{\text{eff}}^{1/(d-2)}$. However, the boundary CFT has a large number of degrees of freedom, as indicated by the large c_{T} , and hence the semiclassical description of gravity in fact breaks down much earlier. A direct way to see this breakdown [134] is to consider the computation of the (canonically normalized) graviton two-point function. In the high energy approximation, *i.e.*, ignoring the AdS geometry, we have here:¹²

$$\langle h(p) h(-p) \rangle \sim p^{-2} [1 + c_{\text{T}} G_{\text{eff}} p^{d-2} + \dots]. \quad (2.60)$$

¹²This propagator argument can also be applied for the higher curvature terms discussed above. For example, the curvature-squared terms gives a perturbative correction: $\langle h(p) h(-p) \rangle \sim p^{-2} [1 + \frac{L^2}{1 + \lambda_{\text{b}}} p^2 + \dots]$. Hence this approach yields the same result for the cutoff in eq. (2.59).

The leading correction arises from a diagram involving the external gravitons coupling to the CFT stress tensor two-point function. We see that such corrections are only suppressed relative to the ‘tree-level’ result for momenta below a cutoff scale of order $(c_T G_{\text{eff}})^{-1/(d-2)}$. For our model, the gravitational theory of the brane can therefore only be treated semi-classically for distance scales larger than

$$\tilde{\delta}_{\text{GR}} \sim (c_T G_{\text{eff}})^{1/(d-2)} \sim \frac{L}{(1 + \lambda_b)^{1/(d-2)}} \sim \frac{\tilde{\delta}_{\text{CFT}}}{(1 + \lambda_b)^{1/(d-2)}}. \quad (2.61)$$

Again, for a pure RS brane with $\lambda_b = 0$, the cutoffs for Einstein gravity and the CFT agree, yielding $\tilde{\delta} \sim L$. However, the addition of a DGP gravity term modifies the cutoff, but in a manner distinct from eq. (2.59), produced by the higher curvature terms. Note that the above result applies for $d \geq 3$.

The distinction between these two cutoffs indicates that these are really two different physical phenomena contributing to the breakdown of Einstein gravity in the brane perspective. Note that for $\lambda_b > 0$, in both eqs. (2.59) and (2.61), the effect is to produce a shorter cutoff scale; however, the second limit (2.61) is the first to contribute (where we are assuming $d > 4$). However, this result is smaller than $\tilde{\delta}_{\text{CFT}}$ and hence from eq. (2.57), we find

$$\lambda_b > 0 \quad : \quad \tilde{\delta} \sim \tilde{\delta}_{\text{CFT}} \sim L. \quad (2.62)$$

On the other hand with $\lambda_b < 0$, the cutoff $\tilde{\delta}_{\text{GR}}$ is pushed to larger distance scales. In this case, eq. (2.59) is the first to modify the gravitational physics on the brane as we move to smaller distances. Further since this result is now larger than the CFT cutoff, in this regime, eq. (2.57) yields

$$\lambda_b < 0 \quad : \quad \tilde{\delta} \sim \tilde{\delta}_{\text{GR}} \sim \frac{L}{\sqrt{1 + \lambda_b}}. \quad (2.63)$$

Let us also note that the latter effect, *i.e.*, CFT corrections to the graviton propagator, is also responsible for the mass of the brane graviton [129]. It is interesting to note that if we take the high energy limit of the corrections to the graviton propagator, eq. (2.60), we can estimate a mass correction for low energy gravitons mode of roughly

$$\frac{c_T G_{\text{eff}}}{\ell_{\text{eff}}^d} \sim \frac{1}{(1 + \lambda_b) \ell_{\text{eff}}^2} \left(\frac{L}{\ell_{\text{eff}}} \right)^{d-2}, \quad (2.64)$$

where we have substituted the d -dimensional AdS scale as a lower bound on the momentum. The scaling with the d -dimensional cosmological constant $-\frac{1}{\ell_{\text{eff}}^2}$ agrees with predictions

in the Karch-Randall model [130, 131]. However, we caution the reader that the above argument by which we obtained the scaling is heuristic at best. Importantly, whether or not the graviton actually obtains a mass correction depends on the boundary conditions of the matter fields in AdS and can therefore not be determined by a local argument alone [129]. However, taking eq. (2.64) at face value, we also see that a negative DGP coupling increases the mass scale, and vice versa for a positive coupling. This can be confirmed explicitly from bulk calculations [128].

2.5.3 Boundary perspective

As the preceding discussion has made clear, the theory obtained by integrating out the bulk between the asymptotic boundary and the brane, has an effective description of the brane in terms of a local d -dimensional CFT coupled to Einstein gravity up to some cutoff (2.57). However, the standard rules of AdS/CFT also allow for a fully microscopic description of the system in terms of the boundary theory. This is obtained by integrating out the bulk — including the brane — and the result is given by the bath CFT on the fixed d -dimensional boundary geometry coupled to a $(d - 1)$ -dimensional conformal defect (positioned where the brane reaches the asymptotic boundary, *i.e.*, the equator of the boundary sphere) — see figure 1.5c.

The bath CFT is characterized by the central charge $c_T \sim L^{d-1}/G_{\text{bulk}}$, while the defect is characterized by its defect central charge $\tilde{c}_T \sim \ell_{\text{eff}}^{d-2}/G_{\text{eff}}$. We note that in the absence of a DGP term, increasing the brane tension increases the defect central charge $\tilde{c}_T$. Further, we note that the ratio of these two charges is given by

$$\frac{\tilde{c}_T}{c_T} \sim \left(\frac{\ell_{\text{eff}}}{L} \right)^{d-2} (1 + \lambda_b). \quad (2.65)$$

Following the standard AdS/CFT dictionary, the ratio ℓ_{eff}/L also translates to a ratio of couplings in the defect and bath CFTs,¹³

$$\tilde{\lambda}/\lambda \sim \ell_{\text{eff}}/L. \quad (2.66)$$

Since we do not have a particular string construction in mind here, λ should be thought of some positive power of the ‘t Hooft coupling of the bath CFT, while $\tilde{\lambda}$ will be some (different) positive power of the analogous coupling for the defect CFT.

¹³Remember that the AdS/CFT dictionary tells us that $G_N \sim \ell_{\text{AdS}}^{d-1}/N_{\text{dof}}$ and $\lambda_{\text{Hooft}} \sim (\ell_{\text{AdS}}/\ell_s)^d$.

Now the parameters in this boundary description must be constrained if we want to be in the regime where the brane perspective is valid. In particular, the latter requires that the brane curvature scale must be much larger than the effective cutoff, *i.e.*,

$$\ell_{\text{eff}}/\tilde{\delta} \gg 1. \quad (2.67)$$

Now as described above, the cutoff has a separate form depending on whether λ_b is positive or negative. Eq. (2.62) applies for $\lambda_b > 0$, which then yields $\ell_{\text{eff}}/L \gg 1$. Hence we must have $\tilde{\lambda}/\lambda \gg 1$ and also $\tilde{c}_T/c_T \gg 1$ since $1 + \lambda_b > 1$ in this case. Similarly for $\lambda_b < 0$, combining eqs. (2.63) and (2.67) yields $\ell_{\text{eff}}/L \gg 1/\sqrt{1 + \lambda_b}$. In this case, $1 + \lambda_b < 1$ and it is straightforward to again show that the ratios must be constrained in the same manner. Hence for either sign of λ_b , we have

$$\tilde{\lambda}/\lambda \gg 1 \quad \text{and} \quad \tilde{c}_T/c_T \gg 1. \quad (2.68)$$

The large ratio of the central charges can also be heuristically understood requiring that energy and information are only leaking very slowly from the dynamical gravity region into the bath [47]. It has been argued that this ratio also sets the Page time [47]. With the boundary perspective, this can be understood as a requirement that ensures that the degrees of freedom on the defect and the CFT only slowly mix.

Lastly, the d -dimensional graviton can be understood as a field dual to the lightest operator appearing in the boundary OPE expansion of the CFT stress energy tensor [136]. At weak coupling, one would naively assume that the lightest operator has dimension $\Delta = d$. However, due to strong coupling effects it becomes possible that a negative anomalous dimension of roughly -1 is obtained, so that the corresponding operator can act as the holographic dual to a d -dimensional graviton. The mass of the lightest state then signals that the anomalous dimension is not quite -1 , such that the dimension of the boundary operator dual to the graviton is $\Delta \geq d - 1$.

Chapter 3

Holographic entanglement entropy on the Brane

In this chapter, we shall look for QEIs using the holographic setup described in chapter 2. Of course, QEIs have recently proven especially enlightening in the context of the black hole information paradox in two-dimensional JT gravity, where the emergence of these islands has signalled a transition to a phase where entropy of the Hawking radiation decreases over time, *e.g.*, [4, 7, 34, 48, 50, 53, 65]. Some preliminary investigations of quantum extremal islands in higher dimensions also appeared in [35, 58]. In later chapters, we will use the holographic model developed here to further extend these discussions to consider black holes in arbitrary dimensions. However, in our present discussion, black holes are not involved. Rather, we are simply considering the holographic entanglement entropies for certain regions in the vacuum of the boundary CFT coupled to the conformal defect. In situations to be discussed below, we find that the corresponding RT surfaces cross the brane in the bulk and this can be interpreted in terms of the appearance of a QEI in the effective theory of gravity on the brane.

In section 3.1 we will describe the regions we are considering and the possible RT surfaces. Section 3.2 discusses the extremization procedure of the RT surface in the presence of a brane and derives the conditions an RT surface needs to obey in our setting. Further, section 3.3 shows that the leading contribution of the RT surface close to the brane can be understood as the Dong-Wald entropy on the brane, as seen from the brane perspective. In section 3.4 we show an explicit calculation in $d = 3$ and investigate the choices of parameters necessary to obtain QEIs. In particular, there, we will consider adding a DGP coupling to the bulk theory.

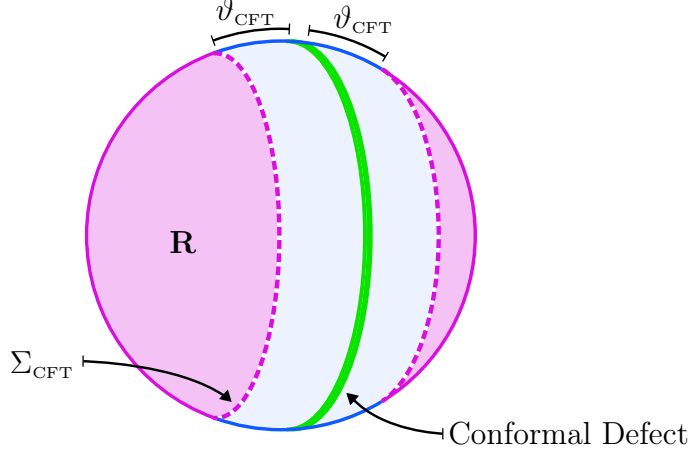


Figure 3.1: A time slice of our CFT setup. A conformal defect running along the equator separates the two halves of $\mathbf{R}$ and its corresponding entangling surface Σ_{CFT} .

3.1 Holographic setup

In the remainder of this chapter, we will focus on a specific calculation of the EE: We consider the vacuum state of our boundary CFT on $\mathbb{R}(\text{time}) \times S^{d-1}(\text{space})$ with a conformal defect running along the equator of the S^{d-1} . As described in chapter 2, the bulk spacetime has locally an AdS_{d+1} geometry and is bisected by a brane extending out to the defect position on the asymptotic boundary. Now we wish to evaluate the EE in the boundary CFT for a region $\mathbf{R}$ comprised of the union of two polar spherical caps on the S^{d-1} — see figure 3.1. We follow the usual holographic prescription to compute the EE. That is, we examine the bulk surfaces $\Sigma_{\mathbf{R}}$ which are homologous to $\mathbf{R}$ and extremize the generalized entropy functional

$$S_{\text{EE}}(\mathbf{R}) = \min \{ \text{ext } S_{\text{gen}}(\Sigma_{\mathbf{R}}) \} = \min \left\{ \text{ext } \left(\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} + \frac{A(\Sigma_{\mathbf{R}} \cap \text{brane})}{4G_{\text{brane}}} \right) \right\}. \quad (3.1)$$

Of course, the first term above corresponds to the usual Ryu-Takayanagi (RT) term [13,14] while, as discussed in appendix A, we expect the second term to arise whenever the bulk surface crosses a DGP brane.¹ Let us denote the extremal bulk surface as $\Sigma_{\mathbf{R}}$, and the intersection with the brane $\sigma_{\mathbf{R}} = \Sigma_{\mathbf{R}} \cap \text{brane}$, see figure 3.3. Importantly, if there are

¹Implicitly, eq. (3.1) assumes that the bulk and brane gravitational actions both correspond to the Einstein-Hilbert action (with a cosmological constant term), as in eqs. (2.1) and (2.50).

multiple extrema, the EE is given by choosing the extremal surface yielding the smallest value for $S_{\text{gen}}(\Sigma_{\mathbf{R}})$, as indicated above.

For the calculation described above, the candidate RT surfaces are anchored at the AdS boundary to the entangling surface $\Sigma_{\text{CFT}} = \partial\mathbf{R} = \partial\Sigma_{\mathbf{R}}$, *i.e.*, the boundaries of these surfaces are comprised of two $(d-2)$ -spheres, which are the boundaries of the two polar caps. We will find that there are two topologically distinct candidates for $\Sigma_{\mathbf{R}}$ which extremize the generalized entropy in eq. (3.1), see figure 3.4. The first consists of two disconnected disks on either side of the brane (in which case $\sigma_{\mathbf{R}} = \{\emptyset\}$). The second candidate has a cylindrical geometry which pierces the brane. Hence it is only in this latter case that the second term contributes in eq. (3.1). As noted above, the correct RT surface is chosen from these two candidates as the one which yields a smaller generalized entropy. Generally, we shall find that when the two polar caps are small, the disconnected discs are favoured, while the cylindrical surface can be the leading saddle for when the polar caps are large. We will denote the first situation as the ‘disconnected’ phase and the latter as the ‘connected’ phase. As we will describe in section 3.4, the details of the transition between these two phases also depends on other parameters in the holographic model, *e.g.*, the tension and gravitational coupling of the brane.

To understand the interpretation of these results in terms of quantum extremal islands, we turn to the ‘brane perspective’ described in section 2.5.2. This effective description gives the ‘island rule’ proposed in [7] for the EE,

$$\begin{aligned} S_{\text{EE}}(\mathbf{R}) &= \min \{ \text{ext } S_{\text{gen}}(R \cup \text{islands}) \} \\ &= \min \left\{ \text{ext} \left(S_{\text{EE}}(R \cup \text{islands}) + \frac{A(\partial(\text{islands}))}{4G_{\text{brane}}} \right) \right\}. \end{aligned} \quad (3.2)$$

As geometries $\mathbf{R} = R$, but we have used a different font on the right-hand side to emphasize the fact that the effective ‘brane perspective’ does not give the same detailed description of the CFT state on $\mathbf{R}$, as the boundary or bulk perspectives. Based on our discussion in chapter 2, one might have expected that the gravitational term in eq. (3.2) would involve G_{eff} rather than G_{brane} . This is implicit in (3.2), as we will see below. In the presence of an island, the first term $S_{\text{EE}}(R \cup \text{islands})$ receives two large contributions, coming from the asymptotic AdS boundary and the region close to the brane. It is this second term, proportional to $1/G_{\text{RS}}$, which combines with the last term in eq. (3.2) to yield the expected island contribution proportional to $1/G_{\text{eff}}$, *c.f.* eq. (2.53).

It is now straightforward to interpret the previous holographic discussion in terms of the effective theory on the brane, eq. (3.2). In the connected phase, the holographic RT surface crosses the brane and (if DGP couplings are turned on) we see an explicit

brane contribution in eq. (3.1). From the brane perspective, a QEI has formed in the gravitational region (*i.e.*, the region on the brane enclosed by $\sigma_{\mathbf{R}}$)² and the analogous gravitational term appears in the island rule (3.2). The bulk RT contribution in eq. (3.1) corresponds to $S_{\text{EE}}(R \cup \text{islands})$ in eq. (3.2). As alluded to above, this makes clear that the gravitational contribution to the island is comprised of two components: the bare contribution $\sim 1/G_{\text{brane}}$, which arises from a DGP coupling added to the brane, and the bulk contribution proportional to L/G_{bulk} , which arises from the volume of the RT surface close to the brane. To see how the latter arises in the effective theory, notice that we can split S_{EE} into UV-finite and UV-‘divergent’ contributions close to the brane, where the latter are contributions proportional to inverse powers of z_{B} . These are the analog of the UV divergent boundary contributions for the boundary CFT in the regions without gravity. As discussed in section 2.5, the brane position imposes a UV cutoff for the CFT on the brane, and hence the corresponding ‘divergent’ contributions to the EE are in fact finite and instead yield contributions which match those expected for the gravitational entropy from the induced contributions to eq. (2.24). This makes contact with the usual notion of generalized entropy as the sum of the geometric gravitational entropy and the entropy of the quantum fields [44, 45].

In the disconnected phase, the EE only involves the modes enclosed within the two polar caps and there is no contribution from the CFT in the gravitational region, *i.e.*, on the AdS_d brane. In passing, let us recall that the short wavelength modes in the vicinity of the entangling surface Σ_{CFT} produce various UV divergent boundary contributions, such as the celebrated area law term [144–146]. Of course, in both phases these contributions are regulated in the holographic calculation by introducing a cutoff surface near the asymptotic AdS boundary [16].

3.2 Ryu-Takayanagi meets the Brane

In this section, we shall introduce some technical details, which are useful to calculate the EE associated with the two polar caps in the boundary CFT. In particular, we examine the behaviour of the bulk RT surface $\Sigma_{\mathbf{R}}$ as it crosses the brane, *i.e.*, how the intersection surface $\sigma_{\mathbf{R}}$ is determined. However, we begin by specifying our EE calculation more precisely and reviewing the metrics describing the bulk spacetime.

²Let us add that from the bulk perspective, entanglement wedge reconstruction [137–143] ensures that operators within this island can be reconstructed from boundary CFT data in $\mathbf{R}$.

Let us describe the $\mathbb{R} \times S^{d-1}$ geometry on which the boundary CFT lives with,

$$ds^2 = R^2 \left[-dt^2 + d\vartheta^2 + \sin^2\vartheta d\Omega_{d-2}^2 \right], \quad (3.3)$$

where R is the radius of curvature of the $(d-1)$ -sphere. The polar angle ϑ runs over $0 \leq \vartheta \leq \pi$, and the conformal defect sits at the equator $\vartheta = \pi/2$. As illustrated in figure 3.1, we wish to evaluate the EE in the boundary CFT for a region $\mathbf{R}$ comprised of two polar caps on the S^{d-1} . More specifically, we choose the entangling surface Σ_{CFT} to be two circles placed symmetrically on either side of the defect at $\vartheta = \pi/2 \pm \vartheta_{\text{CFT}}$. Hence we are evaluating the EE between these two balls and the complementary region, which corresponds to a ‘belt’ of width $2\vartheta_{\text{CFT}}$ centred on the conformal defect.

Turning now to the bulk geometry, recall that in section 2.1, we discussed the back-ground solution in terms a metric where the AdS_{d+1} geometry was foliated by AdS_d slices. Eq. (2.7) describes the local geometry on either side of the brane located at $z = z_{\text{B}}$ with

$$ds^2 = \frac{L^2}{z^2} \left[dz^2 + L^2 \left(1 + \frac{z^2}{4L^2} \right)^2 \left(-\cosh^2\tilde{\varrho} dt^2 + d\tilde{\varrho}^2 + \sinh^2\tilde{\varrho} d\Omega_{d-2}^2 \right) \right]. \quad (3.4)$$

While these coordinates are well suited to discuss the brane geometry, we also consider ‘global’ coordinates for the AdS_{d+1} geometry

$$ds^2 = L^2 \left[-\cosh^2\varrho dt^2 + d\varrho^2 + \sinh^2\varrho (d\vartheta^2 + \sin^2\vartheta d\Omega_{d-2}^2) \right], \quad (3.5)$$

which are better adapted to discuss the boundary theory. That is, up to a Weyl rescaling, the geometry on fixed ϱ surfaces matches eq. (3.3) in the asymptotic region, and the UV regulator surface needed to properly define the holographic EE can be simply chosen as some slice $\varrho = \varrho_{\text{UV}} \gg L$.

However, while we refer to eq. (3.5) as ‘global’ coordinates, they do not cover the entire back-reacted bulk solution depicted in figure 2.1. Rather we use the coordinates in eq. (3.5) to cover two patches on either side of the brane and near the asymptotic AdS_{d+1} boundary.³ Comparing eqs. (3.4) and (3.5), it is straightforward to identify the transformation between the two coordinate systems as

$$\tanh \tilde{\varrho} = \tanh \varrho \sin \vartheta, \quad \frac{z}{L} = -2 \sinh \varrho \cos \vartheta \pm 2 \sqrt{\sinh^2 \varrho \cos^2 \vartheta + 1}. \quad (3.6)$$

With the $+$ ($-$) sign, the brane at $z = z_{\text{B}} \ll L$ resides near the boundary hemisphere with $0 \leq \vartheta \leq \pi/2$ ($\pi/2 \leq \vartheta \leq \pi$) and $\varrho \rightarrow \infty$. Therefore letting ϑ run from 0 to π on the

³Of course, the same applies for the previous coordinates in eq. (3.4).

boundary with the defect at $\vartheta = \pi/2$, we choose the $- (+)$ sign to cover the patch covering the asymptotic boundary hemisphere $0 \leq \vartheta \leq \pi/2$ ($\pi/2 \leq \vartheta \leq \pi$).

Using the AdS foliation (3.4), the position of the brane was specified by $z = z_B$. In terms of the global coordinates (3.5), the brane position can be specified with

$$\sinh^2 \varrho \cos^2 \vartheta = \frac{L^2}{z_B^2} \left(1 - \frac{z_B^2}{4L^2} \right)^2. \quad (3.7)$$

The specific sign of $\cos \vartheta$ depends on whether one considers the coordinate patch above or below the brane — see comments below eq. (3.6). Further, we reach the asymptotic boundary on the brane by taking $\tilde{\varrho} \rightarrow \infty$, which in the global coordinates then corresponds to $\varrho \rightarrow \infty$ and $\vartheta \rightarrow \pi/2$. Hence, we see that the brane intersects the asymptotic boundary at the position of the conformal defect, as expected.

To examine the behaviour of the bulk RT surface $\Sigma_{\mathbf{R}}$ where it crosses the brane, it is useful to consider the problem of extremal surfaces using the metric (3.4). Because the bulk geometry is static, the RT surfaces will be confined to a constant time slice in the bulk. The entangling surfaces in the boundary are spherically symmetric and so we only need to consider bulk surfaces with the same rotational symmetry on the S^{d-2} , that is, we parametrize the surfaces as $\tilde{\varrho} = \tilde{\varrho}(z)$ and the bulk contribution to the holographic EE is then given by

$$S_{\text{bulk}} = 2 \frac{L^{d-1} \Omega_{d-2}}{4 G_{\text{bulk}}} \int \frac{dz}{z} \left[\frac{L}{z} \left(1 + \frac{z^2}{4L^2} \right) \sinh \tilde{\varrho} \right]^{d-2} \sqrt{1 + L^2 \left(1 + \frac{z^2}{4L^2} \right)^2 \left(\frac{d\tilde{\varrho}}{dz} \right)^2} \quad (3.8)$$

where Ω_{d-2} is the area of a unit $(d-2)$ -sphere.⁴ An overall factor of 2 is included here because we assume that the profile $\tilde{\varrho}(z)$ will be reflection symmetric about the brane, and hence S_{RT} receives the same contribution from both sides.

Treating eq. (3.8) as an action, we would derive an ‘Euler-Lagrange’ equation for the profile whose solution corresponds to an extremal surface in the bulk, *i.e.*, away from the brane.⁵ However, this equation is second order and so the solutions are parameterized by two integration constants. One of these constants is fixed by the angle ϑ_{CFT} on the asymptotic boundary (*i.e.*, the position of the entangling surface Σ_{CFT} in the boundary theory), and the other, by the radius $\tilde{\varrho}_B$ at which the RT surface intersects the brane, *i.e.*, $\tilde{\varrho}(z = z_B) = \tilde{\varrho}_B$. We are thus left with the question of fixing the boundary condition at the brane.

⁴Recall that the area of a unit n -sphere is given by $\Omega_n = 2\pi^{\frac{n+1}{2}}/\Gamma(\frac{n+1}{2})$.

⁵This equation is rather involved and the details are not important here.

There are two contributions that come into play at the brane. The first is the DGP contribution in eq. (3.1),

$$S_{\text{brane}} = \frac{L^{d-2} \Omega_{d-2}}{4 G_{\text{brane}}} \left[\frac{L}{z_B} \left(1 + \frac{z_B^2}{4 L^2} \right) \sinh \tilde{\varrho}_B \right]^{d-2}. \quad (3.9)$$

The second is a boundary term that comes from integrating by parts in the variation of the RT functional (3.8). Combining these, one arrives at the following expression⁶

$$L \frac{d\tilde{\varrho}}{dz} \Big|_{z=z_B} = \frac{1}{G_{\text{brane}}} \frac{z_B/L}{1 + \frac{z_B^2}{4 L^2}} \left[\left(1 + \frac{z_B^2}{4 L^2} \right)^2 \left(\frac{2L}{G_{\text{bulk}}} \tanh \tilde{\varrho}_B \right)^2 - \left(\frac{z_B/L}{G_{\text{brane}}} \right)^2 \right]^{-\frac{1}{2}}. \quad (3.10)$$

Hence, scanning through the family of RT surfaces parametrized by $\tilde{\varrho}_B$, the solution which satisfies the above boundary condition is the one that properly extremizes the full entropy functional in eq. (3.1). One observation is that without the DGP term, *i.e.*, $1/G_{\text{brane}} = 0$, the boundary condition simplifies to $L d\tilde{\varrho}/dz|_{z=z_B} = 0$. That is, the RT surface intersects the brane at a right angle. Turning on the gravitational action on the brane (with a positive coupling) produces $L d\tilde{\varrho}/dz|_{z=z_B} > 0$, which arises from pushing $\tilde{\varrho}_B$ to a smaller value. The decrease in $\tilde{\varrho}_B$ is natural here because the DGP contribution in eq. (3.9) adds an additional penalty for large areas on the brane and the effect is to shrink the area of $\sigma_{\mathbf{R}}$.⁷ This is illustrated in the left panel of figure 3.2.

We observe that the above analysis has a simple interpretation in terms of the island rule (3.2). Recall that extremizing the RT functional (3.8) leads to a family of bulk solutions that are parametrized by $\tilde{\varrho}_B$, their radius on the brane. Evaluating $S_{\text{bulk}} + S_{\text{brane}}$ for these different solutions is equivalent to evaluating the S_{gen} in eq. (3.2) with different candidates for the island geometry. The final step of extremizing with respect to variations of $\tilde{\varrho}_B$ then matches the extremization in the island rule and identifies the quantum extremal surface $\sigma_{\mathbf{R}}$ on the brane.

Next, we provide a more general geometric discussion of the boundary conditions. The orthogonality between the RT surface and the RS brane is a special feature of the reflection symmetry of our setup. For a better geometric understanding of the boundary

⁶Implicitly, we assume that we are considering the RT surfaces with a cylindrical topology, *i.e.*, in the connected phase. Examining these boundary terms carefully, one also finds that they are eliminated with $\tilde{\varrho}_B = 0$. This solution points towards the existence of the second phase of disconnected surfaces, which do not intersect the brane.

⁷If $1/G_{\text{brane}} < 0$ as we consider in section 3.4, then the DGP entropy (3.9) facilitates a larger area for $\sigma_{\mathbf{R}}$ and so we find that $\tilde{\varrho}_B$ increases.

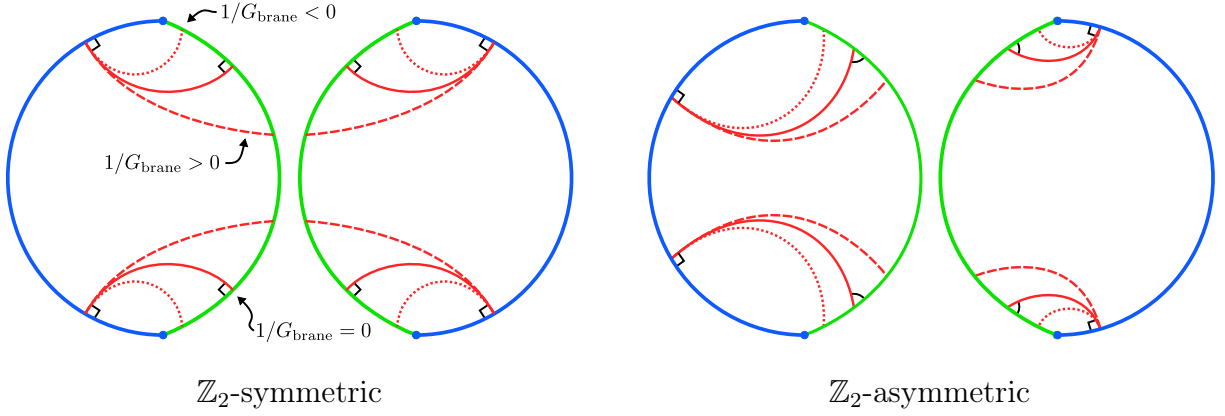


Figure 3.2: Families of extremal surfaces anchored at fixed positions on the asymptotic AdS boundary. The true RT surfaces are the members of these families which extremize area in the bulk, or equivalently, generalized entropy in the brane perspective. The RT surfaces in the case of zero, positive, and negative $1/G_{\text{brane}}$ are respectively shown in solid, dashed, and dotted red. In the absence of a DGP Einstein-Hilbert action ($1/G_{\text{brane}} = 0$), the RT surfaces pass ‘straight’ through the brane. The left (right) panel shows the computation of EE for a region in the boundary CFT that is $\mathbb{Z}_2$ -symmetric (-asymmetric) about the defect.

conditions, let us examine eq. (3.1) in more detail. Consider a $(d-1)$ -dimensional surface $\Sigma_{\mathbf{R}}$ parametrized by intrinsic coordinates ξ^α , embedded in the $(d+1)$ -dimensional bulk spacetime with coordinates X^μ and metric $g_{\mu\nu}$. Hence we describe the embedding of this surface in the bulk spacetime as $X^\mu = X^\mu(\xi^\alpha)$, and the induced metric then becomes

$$h_{\alpha\beta} = g_{\mu\nu} \frac{\partial X^\mu}{\partial \xi^\alpha} \frac{\partial X^\nu}{\partial \xi^\beta}. \quad (3.11)$$

Hence the bulk contribution in eq. (3.1) becomes

$$S_{\text{bulk}} = \frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} = \frac{1}{4G_{\text{bulk}}} \int_{\Sigma_{\mathbf{R}}} d^{d-1}\xi \sqrt{h}. \quad (3.12)$$

Next to evaluate the brane contribution in eq. (3.1), we introduce $(d-2)$ coordinates y^a to parameterize the intersection of $\Sigma_{\mathbf{R}}$ and the brane. The induced metric on this intersection surface then becomes

$$\tilde{h}_{ab} = h_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial y^a} \frac{\partial \xi^\beta}{\partial y^b}, \quad (3.13)$$

and the corresponding contribution to the generalized entropy is

$$S_{\text{brane}} = \frac{1}{4G_{\text{brane}}} \int d^{d-2}y \sqrt{\tilde{h}}. \quad (3.14)$$

Now following the prescription in eq. (3.1), we wish to extremize the sum of the two quantities above. So we begin with the variation of S_{bulk} , which yields

$$\begin{aligned} \delta S_{\text{bulk}} = & \frac{1}{4G_{\text{bulk}}} \left[\int_{\Sigma_{\mathbf{R}} \cap \text{brane}} d^{d-2}y \sqrt{\tilde{h}} g_{\mu\nu} (\partial_{n_{\mathbf{R}}} X^\mu + \partial_{n_{\mathbf{L}}} X^\mu) \delta X^\nu \right. \\ & \left. + \int_{\Sigma_{\mathbf{R}}} d^{d-1}\xi \sqrt{h} [\text{e.o.m.}]_\nu \delta X^\nu \right]. \end{aligned} \quad (3.15)$$

Here we assume that the equations of motion along the bulk of $\Sigma_{\mathbf{R}}$ can be satisfied and so the second term above vanishes. However, one must integrate by parts to arrive at these equations and so we are left with a boundary term where $\Sigma_{\mathbf{R}}$ crosses the brane.⁸ Here we are assuming that the extremal surface is not necessarily smooth at the brane and so $n_{\mathbf{R}}^\alpha$ and $n_{\mathbf{L}}^\alpha$ are unit normals to the intersection surface directed along the extremal surface approaching the brane from either side.

In the absence of the DGP term (2.50), there is no brane contribution (3.14) and then the vanishing of the boundary term in eq. (3.15) dictates $n_{\mathbf{R}}^\alpha + n_{\mathbf{L}}^\alpha = 0$.⁹ That is, with $1/G_{\text{brane}} = 0$, the boundary condition is that the RT surface should pass smoothly through the brane — this is illustrated by the solid red RT surfaces in figure 3.2. In the reflection symmetric setup considered above, this can only be accomplished if the RT surface is orthogonal to the brane, *i.e.*, both $n_{\mathbf{R}}^\alpha$ and $n_{\mathbf{L}}^\alpha$ are orthogonal to the brane.

Of course, with a DGP brane, we must also consider the variation of S_{brane} in eq. (3.14), which yields

$$\delta S_{\text{brane}} = \frac{1}{4G_{\text{brane}}} \int d^{d-2}y \sqrt{\tilde{h}} \tilde{\mathcal{K}}_i \frac{\partial x^i}{\partial X^\nu} \delta X^\nu, \quad (3.16)$$

⁸Dirichlet boundary conditions remove the analogous boundary contributions at the asymptotic AdS boundary.

⁹Actually, the requirement is $\tilde{g}_{i\alpha}(n_{\mathbf{R}}^\alpha + n_{\mathbf{L}}^\alpha) = 0$, *i.e.*, the projection into the brane of the sum of the two normals vanishes — see the discussion after eq. (3.17). However, the vanishing of the full vector sum follows from this restriction.

where $\tilde{\mathcal{K}}_i$ denotes the trace of the extrinsic curvature of the intersection surface on the brane, as viewed from the brane geometry (with the d coordinates x^i).¹⁰ Requiring the sum of eqs. (3.15) and (3.16) to vanish then yields the boundary condition

$$0 = \tilde{g}_j{}^\nu \left(g_{\mu\nu} (\partial_{n_R} X^\mu + \partial_{n_L} X^\mu) + \frac{G_{\text{bulk}}}{G_{\text{brane}}} \tilde{\mathcal{K}}_i \partial_\nu x^i \right). \quad (3.17)$$

Here, we think of the induced metric on the brane as the bulk tensor $\tilde{g}_{\mu\nu} = g_{\mu\nu} - N_\mu N_\nu$, where N_μ is the unit normal orthogonal to the brane. Then, the initial factor $\tilde{g}_j{}^\nu$ above projects the vector expression in the brackets on to the brane. This projection is required because δX^ν in eqs. (3.15) and (3.16) is restricted to be parallel to the brane.¹¹ Hence the brane contribution (3.14) leads to a discontinuity in the first derivative of the RT surface at the brane, as was implicitly found in eq. (3.10) above.

3.3 Wald-Dong entropy

As alluded to above, one of the striking features of EE for subregions in quantum field theory is that the result is dominated by short wavelength modes in the vicinity of the entangling surface and the EE is UV divergent. Of course, the leading contribution is the famous area law term [144–146] and in higher dimensions, there are subleading UV divergences which are also determined by the geometry of the entangling surface (as well as the dimensionful couplings of the underlying theory). In the holographic context, these divergences arise because the RT surface in the bulk extends out to the asymptotic boundary and hence the unregulated area is infinite [13, 14, 16]. In the context of brane world gravity, like the construction in the previous sections with $z_B \ll L$, one expects large UV contributions when the RT surface crosses the brane. However, in this instance, the corresponding UV cutoff remains finite and set by the position of the brane, as discussed above. We show below that the corresponding UV contributions to the holographic EE can be interpreted as the Wald-Dong entropy [38, 39, 94, 95]¹² of the induced gravity on the brane [115, 116].

¹⁰In deriving eq. (3.16), we used that $\tilde{\mathcal{K}}_i$ gives the expansion of the area element $\sqrt{\tilde{h}}$ under the map produced by geodesics shooting out normal to the intersection surface, $\text{RT} \cap \text{brane}$.

¹¹In writing eq. (3.15), we have assumed that the same domain for the coordinates ξ^a mapped to the portion of the RT surface on either side of the brane under both $X^\mu(y^a)$ and $X^\mu(y^a) + \delta X^\mu(y^a)$. Said another way, S_{bulk} has the same integration limits in y^a both before and after the variation.

¹²Our calculations will include the subleading contributions arising from the curvature-squared terms in eq. (2.24). Because the corresponding quantum extremal surfaces have nonvanishing extrinsic curvature, we will need the full expression for the gravitational entropy derived by Dong [95].

Of course, the leading UV contributions studied here do not probe the full bulk profile of the RT surface, and we leave the full calculation of holographic EE to section 3.4.

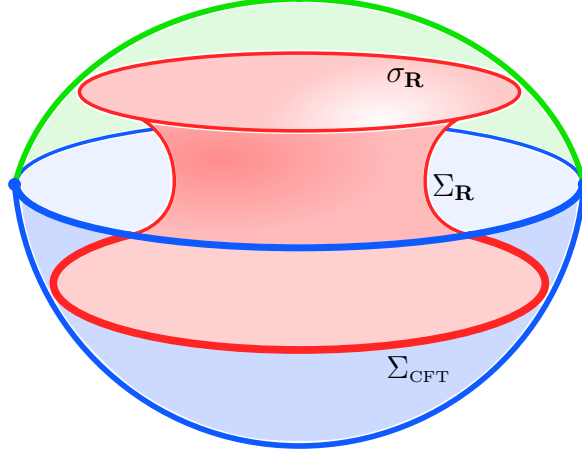


Figure 3.3: A time slice of AdS_{d+1} space. The entangling surface Σ_{CFT} lies on the CFT boundary and the RT surface Σ_{R} intersects the brane at σ_{R} .

To evaluate the leading contributions to the holographic entropy where the RT surface crosses the brane, first recall the bulk metric (2.7) with bulk coordinates $X^\mu = (z, x^i)$ and the brane positioned at $z = z_{\text{B}}$. Now, for the RT area functional (3.12), we choose the coordinates on the RT surface as $\xi^\alpha = (z, y^a)$ where z is the same radial coordinate as in the bulk and y^a are the $d-2$ spatial coordinates describing the profile of the RT surface in slices of constant z (and time). Now following [147], we can construct a Fefferman-Graham expansion for the transverse profile $x^i(\xi)$ of the RT surface for small z (*i.e.*, in the vicinity of the brane) to find¹³

$$x^i(z, y^a) = x^{(0)i}(y^a) + \frac{z^2}{L^2} x^{(1)i}(y^a) + \frac{z^4}{L^4} x^{(2)i}(y^a) + \dots \quad (3.18)$$

In principle, the functions $x^{(n)i}(y^a)$ are determined recursively through extremization of the RT area functional (3.12), however, we will simply quote the next-to-leading result found

¹³We will assume that the RT surface is $\mathbb{Z}_2$ symmetric across the brane. However, in principle, there are two independent profiles on either side of the brane, *i.e.*, $x_{\text{R}}^i(z, y^a)$ and $x_{\text{L}}^i(z, y^a)$. Of course, the profiles agree where they meet on the brane, $x_{\text{R}}^i(z = z_{\text{B}}, y^a) = x_{\text{L}}^i(z = z_{\text{B}}, y^a)$ and satisfy the boundary condition (3.17). At this point, let us also recall that the profile in the time direction is trivial here, *i.e.*, $x^t(z, y^a) = x^{(0)t}(y^a) = \text{constant}$.

in [147]:

$$x^{(1)i} = -\frac{L^2 \mathcal{K}^{(0)i}}{2(d-2)} = -\frac{L^4 \tilde{\mathcal{K}}^i}{2(d-2)z_B^2} + \mathcal{O}\left(\frac{z_B^2}{L}\right), \quad (3.19)$$

where $\mathcal{K}^{(0)i}$ is the trace of the extrinsic curvature of the surface $x^{(0)i}(y^a)$ (at $z = 0$) with the boundary metric, $g_{ij}^{(0)} = g_{ij}^{\text{AdS}_d}$ as given in eq. (2.5). As the latter is an unphysical surface in the present context, we introduced the second expression with $\tilde{\mathcal{K}}^i$, the trace of extrinsic curvature of intersection surface $\sigma_{\mathbf{R}}$ on the brane, *i.e.*, $x^i(z = z_B, y^a)$, evaluated with induced metric $\tilde{g}_{ij}$. This expression follows using the relation (2.21) between the boundary metric and the induced metric on the brane,¹⁴ and the relation (3.18) between $x^{(0)i}(y^a)$ and $x^i(z_B, y^a)$. Note that the leading term on the right-hand side of eq. (3.19) scales as z_B^0 since $\tilde{\mathcal{K}}^i \sim \mathcal{K}^{(0)i} z_B^2 / L^2$ and by this counting, the first correction is $\mathcal{O}(z_B^2/L)$.

Using eq. (3.11), we now evaluate the non-vanishing components of the induced metric on the RT surface. First, the h_{zz} component is given by

$$h_{zz} = \frac{L^2}{z^2} \left[1 + \frac{z^2}{L^2} h_{zz}^{(1)} + \mathcal{O}\left(\frac{z^4}{L^4}\right) \right], \quad (3.20)$$

$$\text{where} \quad h_{zz}^{(1)} = \frac{4}{L^2} g_{ij}^{\text{AdS}_d} x^{(1)i} x^{(1)j} = \frac{L^4}{(d-2)^2 z_B^2} \tilde{\mathcal{K}}_i \tilde{\mathcal{K}}^i + \mathcal{O}\left(\frac{z_B^2}{L^2}\right).$$

In the final expression and throughout the following, the indices on $\tilde{\mathcal{K}}^i$ are contracted using the induced metric $\tilde{g}_{ij}$. The remaining nonvanishing components are

$$h_{ab} = \frac{L^2}{z^2} \left(1 + \frac{z^2}{4L^2} \right)^2 \mathfrak{h}_{ab}, \quad \text{with} \quad \mathfrak{h}_{ab} \equiv g_{ij}^{\text{AdS}_d} \frac{\partial x^i}{\partial y^a} \frac{\partial x^j}{\partial y^b} = \mathfrak{h}_{ab}^{(0)} + \frac{z^2}{L^2} \mathfrak{h}_{ab}^{(1)} + \mathcal{O}\left(\frac{z^4}{L^4}\right). \quad (3.21)$$

The leading term in $\mathfrak{h}_{ab}$ is simply given by

$$\mathfrak{h}_{ab}^{(0)} = g_{ij}^{\text{AdS}_d} \frac{\partial x^{(0)i}}{\partial y^a} \frac{\partial x^{(0)j}}{\partial y^b}, \quad (3.22)$$

and while the individual components $\mathfrak{h}_{ab}^{(1)}$ will not be needed, we will use the next-to-leading order expansion of the measure

$$\sqrt{\mathfrak{h}} = \sqrt{\mathfrak{h}^{(0)}} \left\{ 1 - \frac{L^2 z^2}{2(d-2)z_B^2} \tilde{\mathcal{K}}_i \tilde{\mathcal{K}}^i \left[1 + \mathcal{O}\left(\frac{z_B^2}{L^2}\right) \right] + \mathcal{O}\left(\frac{z^4}{L^4}\right) \right\}. \quad (3.23)$$

¹⁴Note that in contrast to [147], the indices i on the extrinsic curvatures in eq. (3.19) are coordinate indices, rather than orthonormal frame indices. This introduces an extra factor of L/z_B in the leading term on the right-hand side of eq. (3.19). We also note that the sign of our extrinsic curvatures differs from that in [147], *i.e.*, the extrinsic curvature of a sphere embedded in flat space is positive here.

The latter is obtained by interpreting $\tilde{\mathcal{K}}^i \sim L^2 \tilde{\mathcal{K}}^i / z_B^2$ as giving an expansion of the area element $\sqrt{\mathfrak{h}}$.

Combining these expressions, the area functional (3.12) for the RT surface $\Sigma_{\mathbf{R}}$ in the vicinity of the brane becomes

$$\begin{aligned} \frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} &\simeq \frac{1}{2G_{\text{bulk}}} \int_{z_B} dz \left\{ \left(\frac{L}{z} \right)^{d-1} \left(1 + \frac{z^2}{4L^2} \right)^{d-2} \right. \\ &\quad \times \left. \int d^{d-2}y \sqrt{\mathfrak{h}} \left[1 + \frac{z^2}{2L^2} h_{zz}^{(1)} + \mathcal{O}\left(\frac{z^4}{L^4}\right) \right] \right\} \\ &= \frac{L^{d-1}}{2G_{\text{bulk}} z_B^{d-2}} \int d^{d-2}y \sqrt{\mathfrak{h}^{(0)}} \left[\frac{1}{d-2} + \frac{d-2}{4(d-4)} \left(\frac{z_B}{L} \right)^2 \right. \\ &\quad \left. - \frac{d-3}{2(d-2)^2(d-4)} L^2 \tilde{\mathcal{K}}_i \tilde{\mathcal{K}}^i + \mathcal{O}\left(\frac{z_B^4}{L^4}\right) \right] \end{aligned} \quad (3.24)$$

where an overall factor of 2 was included to account for the contributions coming from both sides of the brane.¹⁵ Next, we evaluate the area of the intersection surface $\sigma_{\mathbf{R}} = \Sigma_{\mathbf{R}} \cap \text{brane}$ using the metric induced on this surface, *i.e.*, $\tilde{h}_{ab} = h_{ab}|_{z=z_B}$ where h_{ab} appears in eq. (3.21):

$$\begin{aligned} A(\sigma_{\mathbf{R}}) &= \int_{\sigma_{\mathbf{R}}} d^{d-2}y \sqrt{\tilde{h}} \\ &= \left(\frac{L}{z_B} \right)^{d-2} \int_{\sigma_{\mathbf{R}}} d^{d-2}y \sqrt{\mathfrak{h}^{(0)}} \left[1 + \frac{d-2}{4} \left(\frac{z_B}{L} \right)^2 - \frac{L^2 \tilde{\mathcal{K}}_i \tilde{\mathcal{K}}^i}{2(d-2)} + \mathcal{O}\left(\frac{z_B^4}{L^4}\right) \right]. \end{aligned} \quad (3.25)$$

Hence we may rewrite the result in eq. (3.24) as

$$\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} \simeq \frac{L A(\sigma_{\mathbf{R}})}{2(d-2)G_{\text{bulk}}} + \frac{L}{4(d-4)G_{\text{bulk}}} \int_{\sigma_{\mathbf{R}}} d^{d-2}y \sqrt{\tilde{h}} \left[\frac{z_B^2}{L^2} - \frac{L^2 \tilde{\mathcal{K}}_i \tilde{\mathcal{K}}^i}{(d-2)^2} \right] + \mathcal{O}\left(\frac{L^{d-6}}{z_B^{d-6}}\right). \quad (3.26)$$

Of course, if the brane action also includes a DGP contribution (2.50), one would add the corresponding Bekenstein-Hawking term, as in eq. (3.1), to produce

$$\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} + \frac{A(\sigma_{\mathbf{R}})}{4G_{\text{brane}}} \simeq \frac{A(\sigma_{\mathbf{R}})}{4G_{\text{eff}}} + \frac{L}{4(d-4)G_{\text{bulk}}} \int_{\sigma_{\mathbf{R}}} d^{d-2}y \sqrt{\tilde{h}} \left[\frac{z_B^2}{L^2} - \frac{L^2 \tilde{\mathcal{K}}_i \tilde{\mathcal{K}}^i}{(d-2)^2} \right] + \mathcal{O}\left(\frac{L^{d-6}}{z_B^{d-6}}\right), \quad (3.27)$$

¹⁵Recall that we are assuming that the RT surface is symmetric under reflection across the brane.

where the two leading contributions proportional to $A(\sigma_{\mathbf{R}})$ were combined using eq. (2.53).

It is clear that the first term in eq. (3.27) corresponds to the Bekenstein-Hawking entropy of the surface $\sigma_{\mathbf{R}}$ for the gravity action (2.24) induced on the brane. We now show that leading corrections in eq. (3.27) match the contributions to the Wald-Dong entropy [95] coming from the curvature-squared terms. That is, given the gravity action (2.24), the corresponding Wald-Dong entropy is given by

$$S_{\text{WD}} = \frac{A(\sigma_{\mathbf{R}})}{4G_{\text{eff}}} + \frac{L^3}{4(d-2)^2(d-4)G_{\text{bulk}}} \int_{\sigma_{\mathbf{R}}} d^{d-2}y \sqrt{\tilde{h}} \left(2\tilde{R}_{ij}n^{im}n^j{}_m - \frac{d}{d-1}\tilde{R} - \tilde{\mathcal{K}}_i\tilde{\mathcal{K}}^i \right), \quad (3.28)$$

where n_m^i are two unit normals to the entangling surface $\sigma_{\mathbf{R}}$ embedded in the d -dimensional brane geometry, and as in chapter 2, $\tilde{R}_{ij}$ and $\tilde{R}$ are the Ricci tensor and scalar curvatures, respectively, evaluated with $\tilde{g}_{ij}$. Comparing eqs. (3.27) and (3.28), we immediately see that the coefficients precisely match for the term proportional to $\tilde{\mathcal{K}}_i\tilde{\mathcal{K}}^i$. Then using eqs. (2.16) and (2.19), we can evaluate the remaining two curvature terms in eq. (3.28),

$$\begin{aligned} \frac{L^3}{4(d-2)^2(d-4)G_{\text{bulk}}} \int d^{d-2}y \sqrt{\tilde{h}} \left(2\tilde{R}_{ij}n^{im}n^j{}_m - \frac{d}{d-1}\tilde{R} \right) \\ = \frac{z_{\text{B}}^2}{4(d-4)G_{\text{bulk}}L} \int d^{d-2}y \sqrt{\tilde{h}}, \end{aligned} \quad (3.29)$$

which matches the $\mathcal{O}(z_{\text{B}}^2/L^2)$ term in eq. (3.27). Hence, as expected [115,116], in the regime $z_{\text{B}} \ll L$, one finds that the leading contributions to the holographic EE (3.1) where the RT surface crosses the brane reproduce the Wald-Dong entropy of the intersection surface derived for the gravity action (2.24).

To close this section, we briefly remark on the case of $d = 2$, which is somewhat special in that the intersection between the RT surface and the brane is a point. Consequently, the leading UV contribution to entropy is not a standard area term, but rather a logarithmic term. Integrating the RT area (in this case, length) across gives

$$S \simeq \frac{L}{2G_{\text{bulk}}} \log \left(\frac{\ell_{\text{IR}}}{z_{\text{B}}} \right). \quad (3.30)$$

where an IR length scale ℓ_{IR} must appear to make the argument of the logarithm dimensionless.¹⁶

¹⁶As in eq. (3.24), a factor of two has been included to account for both sides of the brane.

Following [148], we can find (the leading contribution to) the gravitational entropy for the brane theory evaluating the Wald entropy formula [38] to the Polyakov-Liouville action (2.31), and then substituting the on-shell solution (2.33) for the scalar ϕ ,¹⁷

$$S = \frac{L}{4G_{\text{bulk}}} \phi_0 = -\frac{L}{4G_{\text{bulk}}} \log \left(-\frac{L^2 \tilde{R}}{2} \right). \quad (3.31)$$

Now substituting $\tilde{R} \simeq -2z_{\text{B}}^2/L^4$ reproduces the leading singular behaviour in the holographic result (3.30). The same answer can be obtained by evaluating the Wald-Dong entropy formula [38, 95] directly on the induced gravity action (2.38). Hence, once again in this special case, the holographic EE (3.1) reproduces the Wald-Dong entropy for the corresponding gravity action on the brane.

3.4 Explicit Calculations

In this section, we explicitly evaluate the holographic EE and examine the transition between the two classes of RT surfaces. While we set up the calculations for general $d > 2$, our explicit results are given for $d = 3$ in which case the bulk spacetime locally has the geometry of AdS_4 . We add some comments about $d = 2$, and the addition of Jackiw-Teitelboim gravity (2.42) on the brane, in the discussion of chapter 7.

3.4.1 Setting up the calculation for general dimension

In section 3.2, we reviewed two different coordinate systems in AdS_{d+1} . The AdS_d foliation (3.4) was well suited to discuss the brane geometry, while the global coordinates are adapted to discuss the background geometry of the boundary CFT. However, our explicit calculations of the holographic EE are best performed in a new ‘cylindrical’ coordinate system. In particular, following [149], we introduce cylindrical coordinates P, ς where ς specifies the position along the axis of the cylinder while P measure the distance from the axis. These are related to the global coordinates in eq. (3.5) by

$$\cosh \varrho = \sqrt{P^2 + 1} \cosh \varsigma, \quad (3.32)$$

$$\tan \vartheta = \frac{P}{\sqrt{1 + P^2}} \frac{1}{\sinh \varsigma}, \quad (3.33)$$

¹⁷Note that the action (2.31) is multiplied by a factor of two for the full induced brane action, *i.e.*, $I_{\text{induced}} = 2 I_{\text{diver}} + I_{\text{brane}}$.

while the rest of the spherical angles remain unchanged. With this transformation, the metric becomes

$$ds^2 = L^2 \left[-(P^2 + 1) \cosh^2 \varsigma dt^2 + \frac{dP^2}{1 + P^2} + (1 + P^2) d\varsigma^2 + P^2 d\Omega_{d-2}^2 \right]. \quad (3.34)$$

The range of these coordinates is $P \in (0, \infty)$ and $\varsigma \in (-\infty, \infty)$. The conformal boundary is reached with $P \rightarrow \infty$ (or $\varsigma \rightarrow \pm\infty$ with fixed P). The upper ($0 \leq \vartheta \leq \pi/2$) and lower ($\pi/2 \leq \vartheta \leq \pi$) hemispheres are mapped to the upper ($\varsigma \geq 0$) and lower ($\varsigma \leq 0$) halves of the cylindrical system. The conformal defect is positioned at $\varsigma = 0$. As noted above, the RT surfaces will be restricted to a constant time surface and hence the convenience of the cylindrical coordinates becomes evident, *i.e.*, ς becomes an extra Killing coordinate in the corresponding spatial geometry.

A few more technical details are needed for our calculations: in cylindrical coordinates (3.34), the boundary entangling surface corresponds to the two circles $\varsigma = \pm\varsigma_{\text{CFT}}$, where

$$\sinh \varsigma_{\text{CFT}} = \tan \vartheta_{\text{CFT}}, \quad (3.35)$$

seen in the limit $P \rightarrow \infty$ of the second line in eq. (3.33). Using the AdS foliation of eq. (3.4), the position of the brane was $z = z_{\text{B}}$. Using eq. (3.7), the brane position can be specified in cylindrical coordinates (3.34) according to

$$(1 + P^2) \sinh^2 \varsigma = \frac{L^2}{z_{\text{B}}^2} \left(1 - \frac{z_{\text{B}}^2}{4L^2} \right)^2. \quad (3.36)$$

Recall that the brane intersects the asymptotic boundary at the position of the conformal defect, *i.e.*, at $\vartheta = \pi/2$ with $\varrho \rightarrow \infty$, which corresponds to $\varsigma = 0$ with $P \rightarrow \infty$ in cylindrical coordinates. Further recall that RT surface areas are UV divergent since they extend to the asymptotic boundaries. Hence we introduced a UV regulator surface at $\varrho = \varrho_{\text{UV}}$, which in cylindrical coordinates becomes

$$(P^2 + \tanh^2 \varsigma) \cosh^2 \varsigma = \sinh^2 \varrho_{\text{UV}}. \quad (3.37)$$

We will be mainly interested in comparing the areas of different surfaces for fixed ς_{CFT} , as discussed above. Since the UV divergent terms only depend of the geometry of the entangling surface, they will cancel in the difference of the two areas. Hence, we can then safely take the UV cutoff to infinity.

As noted, the RT surfaces all lie in a fixed time slice and thus we only need consider configurations with cylindrical symmetry (*i.e.*, rotational symmetry on the S^{d-2}). Hence

it is convenient to use the cylindrical coordinates (3.34) and parametrize the profile of the bulk surfaces as $\varsigma = \varsigma(P)$. The bulk contribution to the holographic EE is given by

$$S_{\text{bulk}} = \frac{L^{d-1} \Omega_{d-2}}{2 G_{\text{bulk}}} \int dP P^{d-2} \sqrt{\frac{1}{1+P^2} + (1+P^2) \varsigma'^2} \quad (3.38)$$

where again Ω_{d-2} is the area of the unit $(d-2)$ -sphere — see footnote 4. As in eq. (3.8), an overall factor of 2 is included here to account for the reflection symmetry of the profile $\varsigma(P)$ about the brane. Since this expression does not contain an explicit ς dependence, it is straightforward to derive

$$\varsigma'(P) = \pm \frac{1}{1+P^2} \sqrt{\frac{P_0^{2(d-2)} (1+P_0^2)}{P^{2(d-2)} (1+P^2) - P_0^{2(d-2)} (1+P_0^2)}} \quad (3.39)$$

where the two branches correspond to two identical surfaces related by a reflection with respect to $\varsigma = 0$. P_0 corresponds to the turning point, where the surface makes its closest approach to the symmetry axis.

We now discuss the disconnected phase described in section 3.1. It corresponds to the ‘trivial’ solution with $P_0 = 0$. We find $\varsigma(P) = \pm \varsigma_{\text{CFT}}$, which in cylindrical coordinates looks simply as a pair of disks anchored at the boundary entangling surface. Substituting $\varsigma' = 0$ into eq. (3.38), the area of the two discs can be integrated up to some cutoff radius P_{UV} , and the corresponding holographic EE is

$$S_{\text{disc}} = \frac{L^{d-1} \Omega_{d-2}}{2(d-1) G_{\text{bulk}}} P_{\text{UV}}^{d-1} {}_2F_1 \left[\frac{1}{2}, \frac{d-1}{2}, \frac{d+1}{2}, -P_{\text{UV}}^2 \right]. \quad (3.40)$$

In this case, the entanglement wedge corresponds to two identical disconnected pieces contained between each component of the RT surface and the asymptotic boundary, *i.e.*, the regions $\varsigma \geq +\varsigma_{\text{CFT}}$ and $\varsigma \leq -\varsigma_{\text{CFT}}$, as sketched in the upper panel of figure 3.4.

The connected phase corresponds to $P_0 > 0$, which leads to a cylindrical RT surface. Integrating eq. (3.39) yields a family of bulk surfaces, which are symmetric about the brane and which are anchored on the asymptotic boundary at $\varsigma = \pm \varsigma_{\text{CFT}}$. Recalling the discussion below eq. (3.8), we observe that in this configuration, P_0 is the second integration constant which must be tuned in order to satisfy the appropriate boundary condition (3.10) at the brane, see the lower panel of figure 3.4.

Before we calculate the entropy in the most general setting, let us consider the case of a zero-tension brane with $1/G_{\text{brane}} = 0$, *i.e.*, empty AdS_{d+1} . In this case, the brane

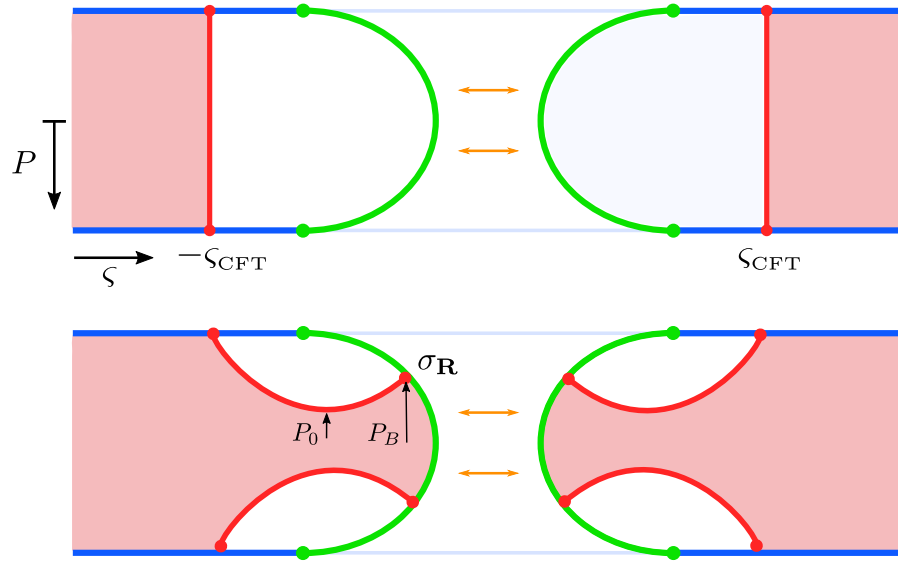


Figure 3.4: Sketch of fixed time slices of our symmetric setup, showing the two possible configurations. The shaded red region corresponds to the entanglement wedge. The connected solution contains an island on the brane, where gravity is dynamical.

is positioned at $z_B = 2L$ or simply, $\varsigma = 0$. Now, the ‘plus’ branch of eq. (3.39) can be integrated to produce a profile extending from $P = P_{UV}$ at $\varsigma = +\varsigma_{\text{CFT}}$ to the maximal depth $P = P_0$ at some $\varsigma = \varsigma_0(\varsigma_{\text{CFT}}, P_0) < \varsigma_{\text{CFT}}$. Since eq. (3.10) indicates that the RT surface must intersect the brane orthogonally, we must tune P_0 (with fixed ς_{CFT}) such that $\varsigma_0 = 0$, *i.e.*, the RT surface reaches its maximal depth at the brane position. Now, substituting eq. (3.39) into eq. (3.38), the holographic EE (for empty AdS_{d+1}) becomes

$$S_{\text{conn}}(T_o = 0) = \frac{L^{d-1} \Omega_{d-2}}{2 G_{\text{bulk}}} \int_{P_0}^{P_{UV}} dP \frac{P^{2(d-2)}}{\sqrt{P^{2(d-2)}(1+P^2) - P_0^{2(d-2)}(1+P_0^2)}}. \quad (3.41)$$

In the general case, this exercise is slightly more complicated for the case of interest with a finite-tension DGP brane at some $z = z_B \ll L$, and the geometry of the corresponding RT surface is illustrated in the lower panel of figure 3.4. The RT surface is again symmetric about the brane and so as above, we focus on the portion starting at $\varsigma = +\varsigma_{\text{CFT}}$ at the asymptotic boundary (*i.e.*, at $P = P_{UV}$). As before, the ‘plus’ branch of eq. (3.39) produces a surface reaching its maximal depth $P = P_0$ at some $\varsigma = \varsigma_0(\varsigma_{\text{CFT}}, P_0) < \varsigma_{\text{CFT}}$.¹⁸ Now one continues from this point using the ‘minus’ branch of eq. (3.39), which then meets the brane as some $P = P_B(\varsigma_{\text{CFT}}, P_0)$ and $\varsigma = \varsigma_B(\varsigma_{\text{CFT}}, P_0)$.¹⁹ One would again tune P_0 (for fixed ς_{CFT}) to ensure the appropriate boundary condition (3.10) is satisfied at the brane. The bulk contribution to the holographic EE then becomes

$$S_{\text{conn}}(T_o > 0) = \frac{L^{d-1} \Omega_{d-2}}{2 G_{\text{bulk}}} \left[\int_{P_0}^{P_{UV}} dP \frac{P^{2(d-2)}}{\sqrt{P^{2(d-2)}(1+P^2) - P_0^{2(d-2)}(1+P_0^2)}} + \int_{P_0}^{P_B} dP \frac{P^{2(d-2)}}{\sqrt{P^{2(d-2)}(1+P^2) - P_0^{2(d-2)}(1+P_0^2)}} \right]. \quad (3.42)$$

Of course, if there is no gravitational term on the brane (*e.g.*, as in eq. (2.50)), then this expression yields the entire generalized entropy (1.6) for the connected phase. Now rather than explicitly examining the brane boundary condition (3.10) in cylindrical coordinates, we will simply evaluate the generalized entropy and find the minimum numerically in the following. Hence to proceed further we will have to choose a specific value for the boundary dimension d .

¹⁸In fact, $\varsigma_0(\varsigma_{\text{CFT}}, P_0)$ is precisely the same function introduced above, since the turning point of the RT surfaces are completely independent of the brane properties.

¹⁹Of course, P_B and ς_B are related as in eq. (3.36).

3.4.2 Explicit results for $d = 3$

In this section, we consider the above discussion for $d = 3$, in which case the boundary geometry becomes $\mathbb{R} \times S^2$, the bulk spacetime is locally AdS_4 , and the branes have an AdS_3 geometry. We will also consider supplementing the four-dimensional bulk action (2.1) with a Gauss-Bonnet term,

$$I_{\text{top}} = \frac{\lambda_{\text{GB}}}{16\pi^2} \int d^4x \sqrt{-g} [R_{abcd}R^{abcd} - 4 R_{ab}R^{ab} + R^2] . \quad (3.43)$$

Note that we have ignored the necessary boundary terms which ensure that this interaction is proportional to the Euler density, *e.g.*, see [150]. Although this curvature-squared term does not effect the bulk equations of motion, it will contribute to the generalized entropy [95, 151]²⁰

$$S_{\text{JM}} = \frac{\lambda_{\text{GB}}}{4\pi} \int_{\Sigma_{\mathbf{R}}} d^2x \sqrt{h} \mathbb{R} + \frac{\lambda_{\text{GB}}}{2\pi} \int_{\partial\Sigma_{\mathbf{R}}} dx \sqrt{h} \mathcal{K}_g , \quad (3.44)$$

where $\mathbb{R}$ denotes the Ricci scalar for the intrinsic geometry on the RT surface $\Sigma_{\mathbf{R}}$. Similarly, $\mathcal{K}_g$ denotes the geodesic curvature of the boundary $\partial\Sigma_{\mathbf{R}}$. Of course, eq. (3.44) gives a topological contribution proportional to the Euler character of the two-dimensional extremal surfaces²¹ and so their geometry remains unaffected by this term. However, in the following, this additional contribution will provide an extra parameter which allows us to adjust the transition between the connected and disconnected phases.

For $d = 3$, some analytic expressions for the extremal surfaces can be obtained [149]. For example, integrating eq. (3.39) yields the following profile for the extremal surface in empty AdS_4 [149]

$$\begin{aligned} \varsigma_{\pm}(P; P_0, \varsigma_0) &= \varsigma_0 \pm \frac{P_0}{\sqrt{(1 + P_0^2)(1 + 2P_0^2)}} \\ &\times \left[(1 + P_0^2) F\left(\text{Arcos}\frac{P_0}{P}, \sqrt{\frac{1 + P_0^2}{1 + 2P_0^2}}\right) - P_0^2 \Pi\left(\text{Arccos}\frac{P_0}{P}, \frac{1}{1 + P_0^2}, \sqrt{\frac{1 + P_0^2}{1 + 2P_0^2}}\right) \right] \end{aligned} \quad (3.45)$$

where F and Π correspond to incomplete elliptic integrals of the first and third kind, respectively.²² Again, the $\pm$ branches correspond to the two portions of the surface, sym-

²⁰One may worry that the topological nature of I_{top} undercuts the usual derivations of the generalized entropy. However, individually the three terms in eq. (3.43) are dynamical and one can apply the results of [95] for each separately and then take the sum of the corresponding contributions to the holographic entropy, which one finds matches the result in eq. (3.44).

²¹The normalization is chosen so that for an RT surface with two-sphere topology, $S_{\text{JM}} = 2\lambda_{\text{GB}}$.

²²Our notation for the elliptic integrals matches that in [152], section 8.1.

metric with respect to $\varsigma_0 = 0$. Of course, we need to know where this surface is anchored at the boundary. Hence we define

$$\begin{aligned}\varsigma_\infty &\equiv \varsigma_+(P \rightarrow \infty; P_0, \varsigma_0) - \varsigma_0 \\ &= \frac{P_0 \left[(1 + P_0^2) K\left(\sqrt{\frac{1+P_0^2}{1+2P_0^2}}\right) - P_0^2 \Pi\left(\frac{1}{1+P_0^2}, \sqrt{\frac{1+P_0^2}{1+2P_0^2}}\right) \right]}{\sqrt{(1 + P_0^2)(1 + 2P_0^2)}}\end{aligned}\tag{3.46}$$

and the surface reaches the asymptotic boundary at $\varsigma_\pm(P \rightarrow \infty) = \varsigma_0 \pm \varsigma_\infty$. Hence the two components of the entangling surface in the boundary theory are separated by $2\varsigma_\infty$, in the cylindrical coordinates.

Figure 3.5 plots ς_∞ as a function of P_0 . The maximum is obtained at $P_0 = P_0^{\text{crit}} \approx 0.51633$ with $\varsigma_\infty = \varsigma_\infty^{\text{crit}} \approx 0.5011$. An interesting observation in [149] was that, for $P_0 < P_0^{\text{crit}}$, there exist *two* values of P_0 with the same ς_∞ . That is, if the two components of the entangling surface are sufficiently ‘close’ on the boundary sphere, there actually exist *two* extremal RT surfaces that connect them in the bulk. However, one branch (with the smaller value of P_0) is always subdominant, and therefore will be of little interest in our analysis. On the other hand, if the separation of the two entangling spheres is larger than the critical value $2\varsigma_\infty^{\text{max}}$ (in cylindrical coordinates), there is no connected extremal surface that joins them.

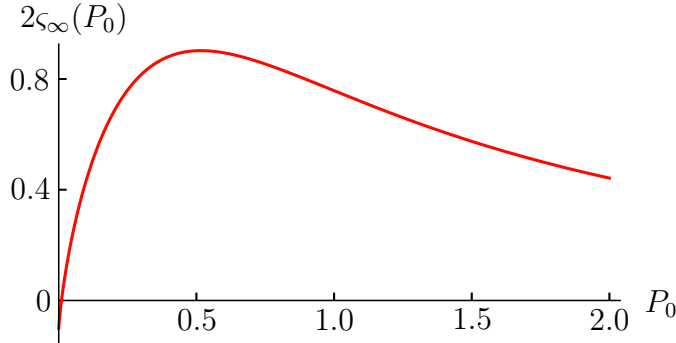


Figure 3.5: Plot of the ‘height’ of the RT surface in cylindrical coordinates, as a function of the turning point P_0 characterizing the surface. For $\varsigma_\infty < \varsigma_\infty^{\text{crit}}$, there are two minimal surfaces anchored at the same regions; otherwise there exists none.

Let us now describe the solutions corresponding to different values of the tension and DGP term:

a) $T_o = 0$; $1/G_{\text{brane}} = 0$: First we consider the holographic EE in empty AdS_4 as a lead-in to the case with a brane. As emphasized above, the area of these surfaces is divergent, and so one introduces a UV regulator surface, integrating of the area from P_0 to some $P_{\text{UV}} \gg 1$ [149]. For the disconnected solution (*i.e.*, a pair of disks), eq. (3.40) with $d = 3$ gives

$$S_{\text{disc}}(P_{\text{UV}}) = \frac{\pi L^2}{G_{\text{bulk}}} \left(\sqrt{1 + P_{\text{UV}}^2} - 1 \right) + 2\lambda_{\text{GB}} \quad (3.47)$$

$$= \frac{A(S_{P_{\text{UV}}}^1)}{4G_{\text{eff}}} - \frac{\pi L^2}{G_{\text{bulk}}} + 2\lambda_{\text{GB}} + \mathcal{O}(P_{\text{UV}}^{-1}). \quad (3.48)$$

where

$$\frac{A(S_P^1)}{4G_{\text{eff}}} = \frac{\pi L^2}{G_{\text{bulk}}} P, \quad (3.49)$$

is the length of S_P^1 , a circle with radius P , and we used eq. (2.25) to write $\frac{1}{G_{\text{eff}}} = \frac{2L}{G_{\text{bulk}}}$. We have included in eq. (3.47) the topological contribution in eq. (3.44). On the other hand, for the connected surfaces the area formula (3.41) yields

$$\begin{aligned} S_{\text{conn}}(P_{\text{UV}}, P_0) &= \frac{\pi L^2}{G_{\text{bulk}}} \frac{P_0^2}{\sqrt{1 + 2P_0^2}} \Pi \left(\text{Arccos} \frac{P_0}{P_{\text{UV}}}, 1, \sqrt{\frac{1 + P_0^2}{1 + 2P_0^2}} \right) \\ &= \frac{A(S_{P_{\text{UV}}}^1)}{4G_{\text{eff}}} + \frac{\pi L^2}{G_{\text{bulk}}} \left[-\sqrt{1 + 2P_0^2} E \left(\sqrt{\frac{1 + P_0^2}{1 + 2P_0^2}} \right) + \frac{P_0^2}{\sqrt{1 + 2P_0^2}} K \left(\sqrt{\frac{1 + P_0^2}{1 + 2P_0^2}} \right) \right] \\ &\quad + \mathcal{O}(P_{\text{UV}}^{-1}), \end{aligned} \quad (3.51)$$

where E is the elliptic integral of the second kind. We emphasize that this result only applies for vanishing T_o and vanishing $1/G_{\text{brane}}$, *i.e.*, for the AdS_4 vacuum. Note that the Euler character of the cylindrical RT surface is zero and hence there is no contribution proportional to λ_{GB} . As expected, the divergence in the $P_{\text{UV}} \rightarrow \infty$ limit matches for the areas of the connected and disconnected surfaces. Hence we can safely take the limit when considering the difference

$$\Delta S(P_0) = \lim_{P_{\text{UV}} \rightarrow \infty} (S_{\text{conn}}(P_{\text{UV}}, P_0) - S_{\text{disc}}(P_{\text{UV}})), \quad (3.52)$$

given by the difference in $\mathcal{O}((P_0/P_{\text{UV}})^0)$ terms in eq. (3.51) and eq. (3.48). A plot of ΔS is shown in figure 3.6. When $\Delta S > 0$, the disconnected RT surface is the dominant saddle,

while for $\Delta S < 0$, the connected solution dominates. Notice that with a larger (positive) topological coupling λ_{GB} , the entropy in eq. (3.47) increases while eq. (3.50) is unaffected, and hence the range of the disconnected phase is decreased in figure 3.6.

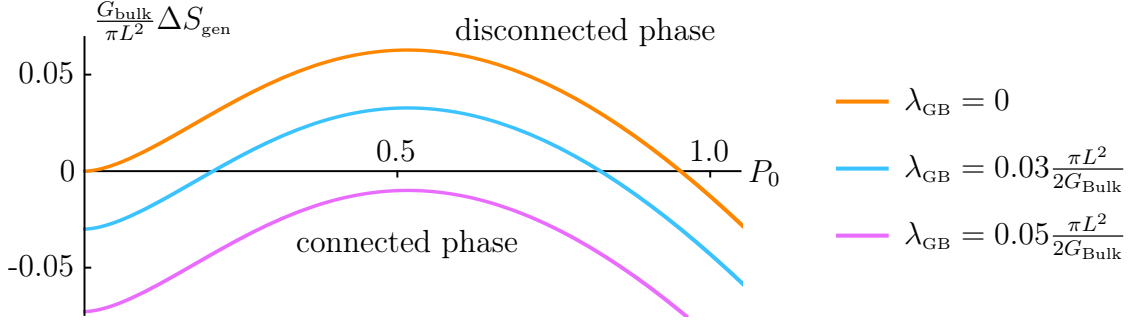


Figure 3.6: Renormalized entropy from eq. (3.52). The connected (disconnected) surface dominates when $\Delta S < 0$ ($\Delta S > 0$). When λ_{GB} becomes very large, $\lambda_{\text{GB}} \sim c_T$, the connected solution becomes favoured.

b) $T_o \neq 0$; $1/G_{\text{brane}} = 0$: The next step is to introduce the brane, however, we do not include a gravitational term in the brane action yet, *i.e.*, $1/G_{\text{brane}} = 0$. In this case, we saw in eq. (3.42) that there is an additional contribution as the RT surface extends from the maximal depth P_0 back out to meet the brane at P_B . Both contributions in eq. (3.42) take the same form except for the limits of integration, hence the $d = 3$ result in eq. (3.50) is replaced by

$$S_{\text{conn}}(P_{\text{UV}}, P_0) = \frac{\pi L^2}{G_{\text{bulk}}} \frac{P_0^2}{\sqrt{1 + 2P_0^2}} \left[\Pi \left(\text{Arccos} \frac{P_0}{P_{\text{UV}}}, 1, \sqrt{\frac{1 + P_0^2}{1 + 2P_0^2}} \right) + \Pi \left(\text{Arccos} \frac{P_0}{P_B}, 1, \sqrt{\frac{1 + P_0^2}{1 + 2P_0^2}} \right) \right]. \quad (3.53)$$

Of course, the entropy for the disconnected phase remains the same as in eq. (3.47) and we can consider the difference of the generalized entropy evaluated on the connected and disconnected extremal surfaces, as in eq. (3.52). Just as we saw a leading divergent contribution in eq. (3.50) for $P_{\text{UV}} \rightarrow \infty$, we expect that eq. (3.53) will contain an analogous large contribution for $P_B \gg P_0$. However, this term will not be cancelled in ΔS . In fact,

in this regime, we can expand the difference as

$$\begin{aligned} \Delta S(P_0) &= \frac{A(\sigma_{\mathbf{R}})}{4G_{\text{eff}}} + \frac{\pi L^2}{G_{\text{bulk}}} \left[1 - 2\sqrt{1 + 2P_0^2} E\left(\sqrt{\frac{1 + P_0^2}{1 + 2P_0^2}}\right) + \frac{2P_0^2}{\sqrt{1 + 2P_0^2}} K\left(\sqrt{\frac{1 + P_0^2}{1 + 2P_0^2}}\right) \right] \\ &\quad - 2\lambda_{\text{GB}} + \mathcal{O}(P_{\text{B}}^{-1}). \end{aligned} \tag{3.54}$$

Here, the intersection $\sigma_{\mathbf{R}}$ of the RT surface and the brane is a circle of radius P_{B} with area $A(\sigma_{\mathbf{R}}) = 2\pi L P_{\text{B}}$ given by eq. (3.34). The fact that the leading term can be expressed as the gravitational entropy for the induced gravity action (2.24) on the brane is in perfect agreement with our discussion in section 3.3. As we will see below, the finite terms will play a role once we turn on the DGP term, allowing for the appearance of a different island on the brane.

From the above expansion, we see that there is a strong penalty for having a large $\sigma_{\mathbf{R}}$ in the connected phase. From the brane perspective, the gravitational entropy results in a large penalty against forming an island on the brane. In fact, generally we expect that $\Delta S > 0$ in this regime and hence the disconnected solution provides the dominant saddle point. However, if we tune the topological coupling λ_{GB} to be large²³ (and positive), this contribution can compensate for the leading gravitational entropy term, at least for $\sigma_{\mathbf{R}}$ up to a certain size.

On the other hand, we must note that P_{B} is not an independent parameter. Rather it is implicitly determined by ς_{CFT} and the brane tension T_o , as well as the value of P_0 that minimizes the area functional in eq. (3.54). P_{B} can be determined in the following way (see figure 3.4). One begins by solving for ς_0 using $\varsigma_0 + \varsigma_{\infty}(P_0) = \varsigma_{\text{CFT}}$ where $\varsigma_{\infty}(P_0)$ is given in eq. (3.46). Then one finds ‘sample’ values of $P_{\text{B}}, \varsigma_{\text{B}}$ where the extremal surface meets the brane by combining eqs. (3.36) and (3.45) and simultaneously solving

$$\begin{aligned} (1 + P_{\text{B}}^2) \sinh^2 \varsigma_{\text{B}} &= \frac{L^2}{z_{\text{B}}^2} \left(1 - \frac{z_{\text{B}}^2}{4L^2} \right)^2, \\ \varsigma_{-}(P_{\text{B}}; P_0, \varsigma_0) &= \varsigma_{\text{B}}. \end{aligned} \tag{3.55}$$

This yields P_{B} as a function of $P_0, \varsigma_{\text{CFT}}$ and T_o , and substituting P_{B} into eq. (3.53) gives the area of the associated extremal surface. Below, we perform this calculation numerically. However, we have not yet considered the boundary conditions (3.10) in this analysis.

²³We note that this requires $\lambda_{\text{GB}} \sim L^2/G_{\text{bulk}} \sim c_{\text{T}}$, the central charge of the boundary CFT — see further discussion in chapter 7.

Rather than explicitly examining the latter, we simply evaluate the area (or rather the difference ΔS) over the range of possible P_0 (with fixed $\varsigma_{\text{CFT}}, T_o$), as shown in figure 3.7a. The correct RT surfaces are then identified as the minima in these plots. Further, the examples in the figure illustrate that without the topological contribution, $\Delta S > 0$ for all minima and so the disconnected phase dominates, as generally expected. That is, no QEs form on the brane in this case. However, as shown in figure 3.7b, we see that with a sufficiently large topological coupling λ_{GB} one can achieve $\Delta S < 0$, where a first order transition leads to the formation of an island.

Although the above recipe is valid for arbitrary brane tensions, in the limit of very large tension we can approximate the solution analytically. Since, as stated above, the leading contribution to the entropy (3.49) scales as $A(\sigma_{\mathbf{R}}) \sim P_{\text{B}}$, the RT surface corresponds to that which has the minimal value of P_{B} . Moreover, since the function $\varsigma_{\text{B}}(P)$ defining embedding of the brane in (3.55) is monotonically decreasing with P , the surface must maximize $\varsigma_{\infty}(P_0)$, which is achieved for $P_0 = P_0^{\text{crit}}$, by definition (see discussion around figure 3.5). This can be readily checked in figure 3.7a, where the curves attain a minimum around $\arctan(P_0^{\text{crit}}) \approx 0.47$, with a small correction due to the finite terms in (3.54), which becomes smaller and smaller as we increase the tension. We shall refer to this solution with $P_0 \approx P_0^{\text{crit}}$ as the *small island*, in order to distinguish it from a second island appearing below which corresponds to a circle with a larger radius.

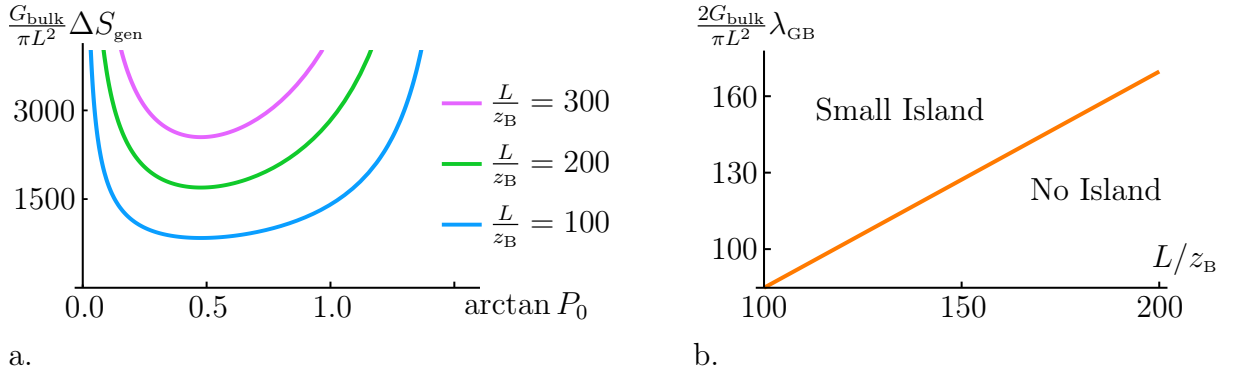


Figure 3.7: Panel a. illustrates the renormalized area from eq. (3.54) of connected RT surfaces, anchored at $\varsigma_{\text{CFT}} = 0$, with $\lambda_{\text{GB}} = 0$. Panel b. is a plot of the critical value of λ_{GB} such that $\min(\Delta S) < 0$.

c) $T_o \neq 0$; $1/G_{\text{brane}} \neq 0$: Finally, we examine the holographic EE in the presence of a DGP brane. The only difference in this analysis is the additional contribution coming at

the intersection of the RT surface with the brane in eq. (3.1). In the present setting, this means that we add the following,

$$S_{\text{brane}} = \frac{A(\sigma_{\mathbf{R}})}{4G_{\text{brane}}} = \frac{\pi L}{2G_{\text{brane}}} P_{\text{B}} , \quad (3.56)$$

to the bulk contribution in eq. (3.53). In fact, the expansion of ΔS for $P_{\text{B}} \gg P_0$ takes precisely the same form as in eq. (3.54). The only difference is that the induced Newton's constant on the brane is now given by eq. (2.53), *i.e.*, $\frac{1}{G_{\text{eff}}} = \frac{2L}{G_{\text{bulk}}} + \frac{1}{G_{\text{brane}}}$.

Generally, we might think of $1/G_{\text{brane}}$ as a positive quantity, and so the DGP contribution (3.56) would simply increase the penalty for having a large $\sigma_{\mathbf{R}}$ in the connected phase, and enhance the dominance of the disconnected phase. However, there is no a priori reason why we should not also consider a negative gravitational coupling on the brane,²⁴ in which case the DGP term serves as another mechanism to reduce the penalty for forming an island on the brane. It is this scenario that we will examine further here — as well as in appendix B.

It will prove convenient to work with the ratio λ_{b} introduced in eq. (2.54). Let us recall what parameters are in play. The tension of the brane is controlled by z_{B} , which we keep small but finite. The dimensionless ratio between the bulk and brane gravitational constants is controlled by λ_{b} . As discussed above, interesting things happen when $\lambda_{\text{b}} < 0$, which is when $G_{\text{brane}} < 0$ while $G_{\text{bulk}} > 0$.

Using the same approach described above, we can explore the transition between the connected and disconnected phases numerically. In figure 3.8a, we plot ΔS as function of P_0 for a fixed $\varsigma_{\text{CFT}} = 0.095$, $L/z_{\text{B}} = 100$ and $\lambda_{\text{GB}} = 0$, for different values of λ_{b} . These plots are analogous to those presented in figure 3.8a where $\lambda_{\text{b}} = 0$ (but L/z_{B} is varied). Again, these plots are made in lieu of a detailed examination of the boundary conditions where the RT surfaces meet the brane, rather the correct boundary conditions (3.10) will be achieved where P_0 is tuned to produced an minimum in these plots. For small λ_{b} the curves show a single minimum but $\Delta S > 0$, indicating that the disconnected solution dominates in this case. As λ_{b} becomes more negative, the curves are pulled down and eventually ΔS enters the negative region so that the connected solution becomes the dominant saddle point. This behaviour is as expected but we note that λ_{b} is very close to -1 in this regime, which according to eq. (2.54) means there is almost a complete cancellation between the induced

²⁴For example, integrating out quantum fields on the brane could produce either a positive or negative shift in Newton's constant. In particular, it can be negative for gauge fields or non-minimally coupled scalar fields, as discussed in the context of EE in [153, 154] — see further discussion in chapter 7 and appendix B.

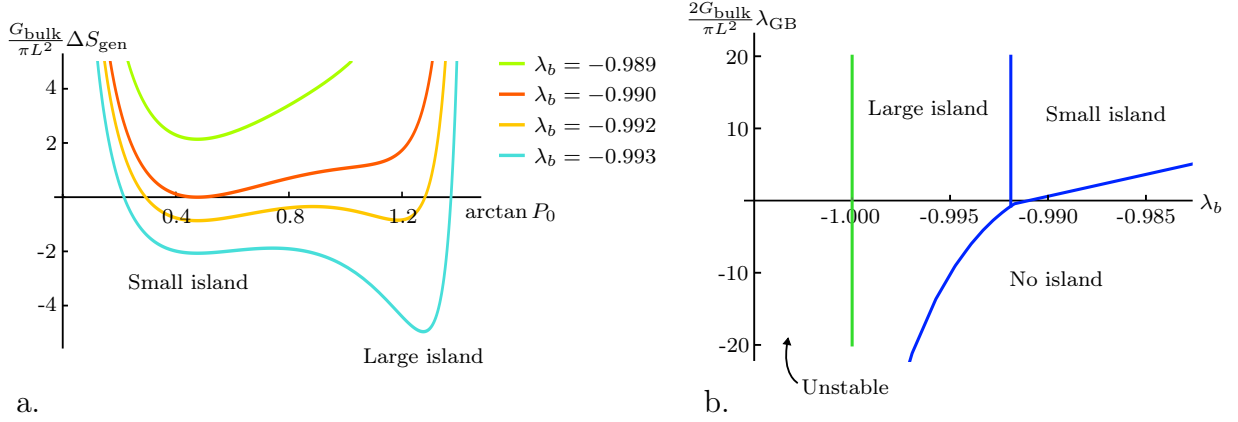


Figure 3.8: Panel a.: Generalized (renormalized) area from eq. (3.54) as function of P_0 , for different values of the DGP coupling λ_b . Notice the appearance of a ‘large’ island when λ_b approaches -1 , due to the partial cancellation of the induced and DGP area terms. Panel b.: Phase diagram: the blue lines correspond to first order phase transitions, while the green one at $\lambda_b = -1$ bounds the region where gravity becomes unstable due to $G_{\text{eff}} < 0$ (see eq. (2.55)). Both plots are done for fixed $L/z_B = 100$.

gravitation coupling $1/G_{\text{RS}}$ and the DGP term $1/G_{\text{brane}}$. Of course, this near cancellation is alleviated by turning on the topological coupling λ_{GB} , as shown in figure 3.8b.

Another interesting feature shown in figure 3.8a is the appearance of a second minimum in the curves. This second solution occurs at a larger value of P_0 and also of P_B , and corresponds to a larger circle $\sigma_{\mathbf{R}}$ on the brane, and therefore we refer to it as a *large island*. The existence of this second island is due to the finite terms in (3.54). Indeed, these terms are essentially what is plotted in figure 3.6, and they are unbounded from below for large P_0 . Therefore, when λ_b becomes sufficiently negative as to produce a significant cancellation between the induced and DGP gravitational entropies, there is a new competition, now between $A(\sigma_{\mathbf{R}})/4G_{\text{eff}}$ and the finite terms, producing the large island. As $\lambda_b \rightarrow -1$, the minimum rolls down to infinity ($P \rightarrow \infty, \Delta S_{\text{gen}} \rightarrow -\infty$), indicating an instability at this point, which we explore further in appendix B.

Figure 3.8b summarizes the phase diagram of the system, for a fixed value of the tension $L/z_B = 100$, as we vary both the DGP coupling λ_b and the topological coupling λ_{GB} . The lines between no/small/large islands correspond to first order phase transitions, while the blue line at λ_b indicates the region where the theory becomes unstable.

Chapter 4

The brane world framework and black holes

Having introduced our brane world model and performed some preliminary entropy calculations, our goal in the second half of this thesis is to apply our model to study black holes, in particular, the information paradox. As we will see, our brane world model will allow us to extend the analysis of QEIs to arbitrary dimensions. As a starting point, in section 4.1, we will briefly review the two-dimensional case studied in [50] and reframe their discussion in terms of our model. This will provide some intuition to set us up for the study of higher dimensional black holes, which we overview in section 4.2 and carry out explicitly in following sections.

4.1 The two-dimensional case: a review

Ref. [50] interprets the two Rindler patches of AdS_2 as exteriors of an eternal non-zero temperature black hole and subsequently considers coupling each exterior to a flat half-space, constituting a bath region. A matter CFT_2 theory spans both the bath and AdS_2 regions and JT gravity is placed on the AdS_2 region. Invoking $\text{AdS}_2/\text{CFT}_1$, this setup is alternatively described by the TFD state of a BCFT living on two half-lines (the bath regions) coupled to quantum mechanics (dual to the AdS_2 spacetime) on the boundaries of the half-lines. The authors then compute the EE of a region consisting of intervals on both sides of the TFD including the defect and with endpoints in the bath regions. From the AdS_2 perspective, this entropy is obtained using eq. (1.7), allowing for the possibility

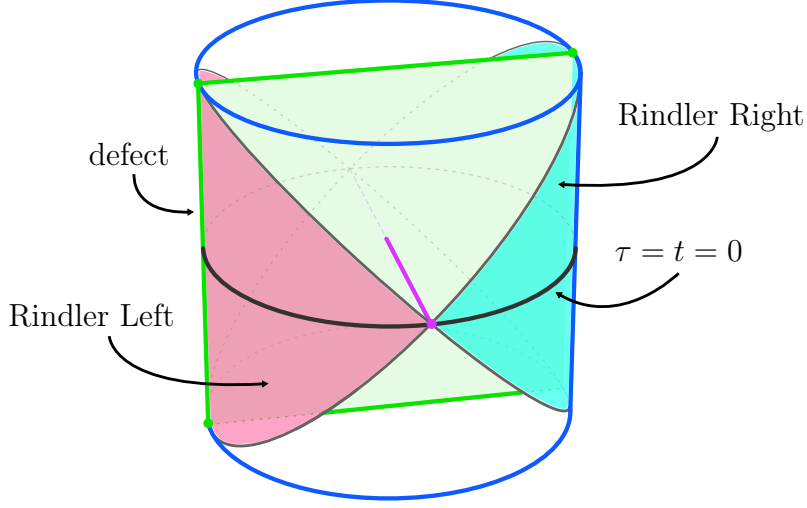


Figure 4.1: Our eternal black hole coupled to the CFT bath, as seen from the *bulk perspective*.

of islands in the AdS_2 spacetime. In particular, this gives rise to a competition between a no-island phase and an island phase, with the former dominating at early times and the latter at late times. In the island phase, QESs appear in the AdS_2 spacetime just outside the horizon, marking the boundaries of an island, stretching through the AdS_2 wormhole, which now belongs to the entanglement wedge of the bath complements to the intervals.

Let us return to our brane world to see how our setup mimics that of [50] described above. The key extension, as described in chapter 2, is to view the matter CFT_2 itself as holographically dual to a locally AdS_3 bulk. Through this higher dimensional bulk runs a brane, lying along an AdS_2 slice (see fig. 4.1) and supporting an intrinsic theory of JT gravity, as described in section 2.3. We may reproduce the AdS_2 black hole on the brane by taking Rindler-AdS coordinates in the AdS_3 bulk — this equips the AdS_3 bulk with a horizon and ‘left’ and ‘right’ exterior regions. The resulting picture is that of a Hartle-Hawking state prepared by the Euclidean path integral drawn in fig. 4.2. The Rindler AdS_3 coordinates also induce a horizon on the brane. In fact, the geometry of the brane is itself Rindler-AdS,

$$ds^2 = \ell_{\text{B}}^2 \left[-(\rho^2 - 1)d\tau^2 + \frac{d\tilde{\rho}^2}{\rho^2 - 1} \right], \quad (4.1)$$

supporting a dilaton profile $\Phi \propto \rho$. In the brane perspective, we then have a matter CFT_2 spanning the left and right asymptotic boundary regions — the baths — and the Rindler- AdS_2 brane supporting a theory of JT gravity. Illustrated in figure 4.3, this is essentially

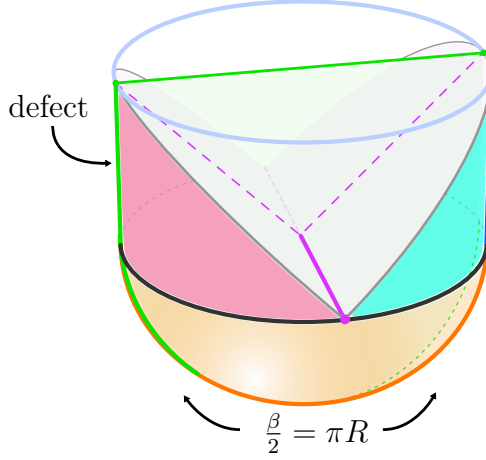


Figure 4.2: The Euclidean path integral (orange region) prepares the Hawking-Hartle state. The black hole temperature $T = 1/(2\pi R)$ is derived in section 4.2.

the same setup as in [50], up to a $\mathbb{Z}_2$ -quotient across the brane. We may alternatively take the boundary perspective, wherein the bulk AdS_3 plus brane theory is dual to a CFT_2 plus defect theory. More precisely, the Euclidean path integral preparing the Hartle-Hawking bulk is equated to a thermal path integral preparing a TFD state of two copies of a CFT_2 with a defect running through its middle. We are thus led to the boundary picture drawn in figure 4.4. Taking a $\mathbb{Z}_2$ quotient across the defect, this, of course, is the alternative description of the setup in [50] as a thermal BCFT coupled to quantum mechanics.

With our setup in place, we can then consider subregions of the boundary CFT and use the RT formula (1.8) to compute the corresponding entanglement entropies. Analogous to [50], we may choose ‘belt’ subregions consisting of intervals symmetric about the defect; equivalently (and for better alignment with the notation of this thesis), we consider the entanglement of the bath regions $\mathbf{R}$ complementary to the belt. The details of the resulting entropy calculation in two dimensions are provided in chapter 6. The upshot is that we find a competition between a no-island phase and an island phase, as sketched in fig. 1.7, with the former dominating at early times and the latter past a Page time. Notice that these phases are analogous to the no-island and island phases of [50], with now the QESs demarked by the intersection between our bulk RT surface and the brane. Namely, it is clear from the bulk picture shown in the right panel of fig. 1.7 that the island region between these intersection points belongs to the entanglement wedge of the bath region $\mathbf{R}$.

In chapter 6, we also explicitly demonstrate that our bulk RT calculation using eq. (1.8) precisely reproduces the results of [50], in the limit where the brane approaches the would-

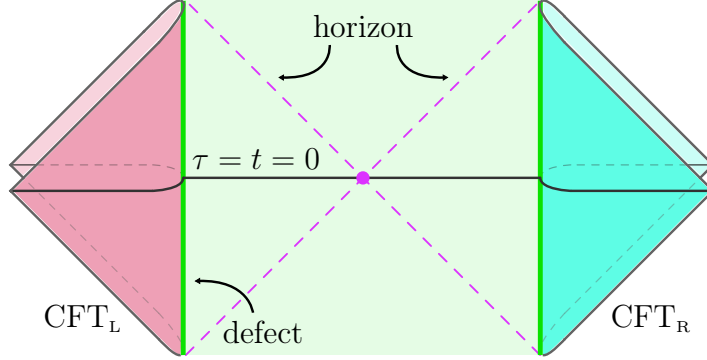


Figure 4.3: Our eternal black hole coupled to the CFT bath, as seen from the effective *brane perspective*. Each point in the Penrose diagram represents a hyperbolic space H_{d-2} . For $d = 2$ this is simply a point.

be AdS_3 boundary by slicing through the bulk at a small brane angle θ_B (that is, the high-tension limit of higher dimensions). For early times, we find that the EE grows linearly in the no-island phase¹ as $4\pi ct/(3\beta)$ (see eq. (6.30)), whereas for late times it is dominated by the island and given by a constant, $\frac{1}{2G_{\text{brane}}}(\tilde{\Phi}_0 + \Phi_r)$ (see eq. (6.43)). Thus, as in [50], the appearance of an island caps off the entropy growth at the expected coarse-grained entropy of two copies of the black hole on the brane, rescuing the system from a potential information paradox (the resulting Page curve is shown in fig. 6.2). While we find perfect agreement with [50] at leading order in θ_B , we also find corrections to these results due to the brane imposing a UV cutoff at finite θ_B . The result is $O(\theta_B^2)$ corrections which, for instance, push the QES further from the horizon, lower the entropy of the island phase, and lead to a hastened Page transition. (Note that, in the no-island phase, no such corrections appear as the bulk RT surface does not intersect the brane.)

It would be straightforward to use our setup to perform the zero-temperature analysis also covered in [50] for $d = 2$. Here one would instead take Poincaré coordinates which would equip the AdS_3 bulk and AdS_2 brane with an extremal horizon. We then expect EE of large regions in the bath to require the inclusion of islands on the gravitating brane. In particular, intervals stretching from some location in the bath out to infinity require the inclusion of an island localized around the horizon. (This is to be contrasted with our findings in $d \geq 3$, where islands are lacking in the extremal case at small brane angle θ_B .)

¹Recall our setup is related to that of [50] by a $\mathbb{Z}_2$ -orbifold, hence factors of 2 must be accounted for when comparing results.

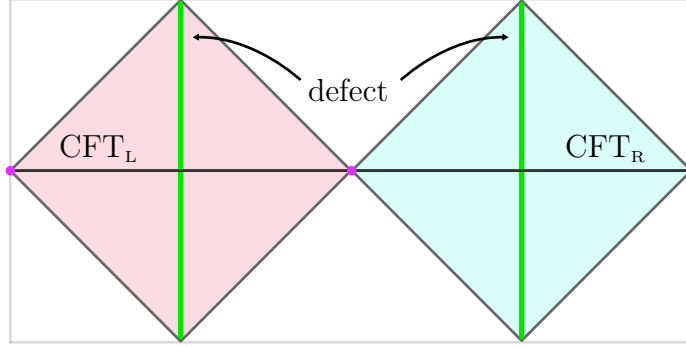


Figure 4.4: Conformal defects along a CFT bath in the *boundary perspective*.

The benefit of our RS setup is that it allows great flexibility in generalizing the construction of [50] to higher dimensions. Indeed, it is straightforward to re-interpret figures 4.1, 4.3, and 4.4 with a suppressed hyperbolic H_{d-2} direction. In the following sections, we shall apply our setup to extend the results mentioned here to higher dimensions.

4.2 Higher dimensions: an overview

In this and following sections, we discuss how islands arise in the presence of certain topological, non-extremal black holes in higher-dimensional brane-world models. Topological black holes are characterized as having nontrivial horizon topology, and we will be interested in the case of neutral black holes with a hyperbolic horizon [155, 156]. The general metric is given by

$$ds^2 = -f(r) \frac{L^2}{R^2} dt^2 + \frac{dr^2}{f(r)} + r^2 dH_{d-1}^2, \quad (4.2)$$

with the blackening factor

$$f(r) = \frac{r^2}{L^2} - 1 - \frac{\omega^{d-2}}{r^{d-2}}. \quad (4.3)$$

Here, L denotes the AdS curvature scale and dH_{d-1}^2 denotes the line element on a $(d-1)$ -dimensional hyperbolic plane with unit curvature. This bulk geometry (4.2) is dual to a TFD state for two copies of the boundary CFT [157], where each resides on a spatial geometry H_{d-1} . After an appropriate Weyl rescaling, the boundary metric for each CFT reads

$$ds_{\text{CFT}}^2 = -dt^2 + R^2 dH_{d-1}^2, \quad (4.4)$$

and hence the scale R (introduced in eq. (4.2)) corresponds to the curvature scale of the spatial geometry. The full boundary geometry is then two copies of $\mathbb{R} \times H_{d-1}$, where the $\mathbb{R}$ corresponds to the time direction in each of the CFTs.

Turning back to eq. (4.2), the relation between the position of the horizon r_h , the black hole mass M , and the ‘mass’ parameter ω is [101, 158, 159]

$$\omega^{d-2} = r_h^{d-2} \left(\frac{r_h^2}{L^2} - 1 \right) = \frac{16\pi G_N}{(d-1) \text{vol}_{H_{d-1}}} \frac{R}{L} M. \quad (4.5)$$

Here and in the following, we use $\text{vol}_{H_{d-1}}$ to denote the dimensionless volume of the spatial boundary geometry, *i.e.*, the volume measured by the metric dH_{d-1}^2 . Of course, this volume is infinite and we must introduce an infrared regulator — see below.

In the following, we will consider the special case of a topological black hole with vanishing mass $M = \omega = 0$. Note that despite the fact that $\omega = 0$, we still find a horizon at $r_h = L$ from eq. (4.3). In fact, the bulk geometry corresponds to the AdS vacuum (as expected for $M = 0$), but we are describing this geometry with the AdS-Rindler coordinates where the metric resembles that of black hole [160]. In this case, it is straightforward to evaluate the entropy and the temperature of the black hole

$$S = \frac{\text{vol}_{H_{d-1}} L^{d-1}}{4G_N}, \quad T = \frac{1}{2\pi R}. \quad (4.6)$$

In terms of the dual CFT, we are considering a pure state (*i.e.*, the vacuum) in the conformal frame where the boundary geometry corresponds to $\mathbb{R} \times S^{d-1}$. However, with an appropriate conformal transformation, we produce the TFD state on two copies of $\mathbb{R} \times H_{d-1}$ with temperature $T = 1/(2\pi R)$ [160]. The entropy in eq. (4.6) corresponds to the EE between the two copies of the CFT — and alternatively, can be interpreted as the EE between two halves of the sphere in the original conformal frame. From the point of view of the CFT, masslessness of the black hole corresponds to a fine tuning of the temperature to $T = \frac{1}{2\pi R}$.

Following the brane world construction outlined in the previous section, we locate a codimension-one defect at the centre of each CFT. By the holographic dictionary, this corresponds to a brane which cuts through the bulk and orthogonally intersects the horizon — see figure 4.1. Since, with $\omega = 0$, the bulk geometry is just the AdS vacuum, our previous discussion of the brane geometry in section 2.1 is still applicable. Hence, the brane position in the bulk is determined precisely as described above in terms of the brane tension T_o . In fact, this bulk geometry provides a higher dimensional generalization of the construction

discussed in section 4.1, and we will see that the brane inherits a black hole metric with temperature $T = 1/(2\pi R)$, from the AdS-Rindler coordinates in the bulk.

Our aim will be to use eq. (1.8) to investigate the appearance of QEIs, from the brane perspective, where (two copies of) the boundary CFT are supported in this black hole geometry on the brane. Further, we will compute the EE associated with symmetric regions $\mathbf{R}$ on each side of the defect as a function of time — see figure 1.7. The regions $\mathbf{R}$ of interest consist of those points on a CFT time slice that are further than a distance $\chi = \chi_\Sigma$ away from the defect.² The EE is evaluated using the holographic prescription of the bulk perspective and as described in the introduction, the corresponding RT surfaces can be in one of two phases. Either they connect through the horizon, which we will call the no-island phase, or they connect through the brane, which we will call the island phase. The reason for those names is apparent from the d -dimensional effective gravity on the brane, *i.e.*, the region bounded by the intersection of the RT surface and the brane is a QEI, which now contributes to the entropy of $\mathbf{R}$. This also implies that from the $(d + 1)$ -dimensional bulk perspective, the appearance of islands is simply explained as a standard phase transition of an RT surface. We will see in the following sections that at early times, the RT surfaces start out in the no-island phase, *i.e.*, connects through the horizon. As is well known [161], the volume of the corresponding surfaces grows linearly with time. At some point its volume will have grown so large, that the RT surface in the island phase has smaller area and gives the correct EE.

The calculation of the time-dependence of the area of RT surfaces will proceed in two steps: In sections 4.4 to 4.6, we will derive expressions for the area of three special cases of extremal surfaces. The first one will be RT surfaces in the island phase anchored at Rindler time $\tau_\Sigma = 0$. The second and third special cases will be RT surfaces $\Sigma_{\mathbf{R}}$ in the no-island phase that either end on entangling surfaces $\partial\mathbf{R}$ at $\chi = \pm\chi_\Sigma$ and $\tau_\Sigma = 0$, or end on entangling surfaces located at the defect ($\chi_\Sigma = 0$) and arbitrary τ_Σ . While these special cases naively might seem not to contain enough information to completely reconstruct the time-evolution of the EE, we will argue in section 4.7 that the time-evolution of any symmetric RT surface in the no-island phase can always be reduced to one of those three cases.

We remind the reader that as described in section 4.1, we are considering eternal black holes that do not evaporate. Nonetheless, from the effective brane point of view, the black hole on the brane and the fields on the asymptotic boundary are in contact, and can

²The coordinate χ is introduced in eq. (4.8) below. Of course, since the global state that we are considering is pure, we could equivalently discuss the EE of the belt regions $-\chi_\Sigma < \chi < \chi_\Sigma$ in both CFTs, including the conformal defects.

therefore continuously exchange radiation. If islands are not accounted for appropriately, this leads to information loss [7]. In section 4.8 we will argue, using results obtained below, that also in higher dimensions the presence of islands makes the entanglement dynamics of the joint system of black holes and radiation compatible with unitarity.

4.3 Geometry on the brane

To set the stage for the following calculations, we will start by discussing the bulk and brane geometry. As noted above, the bulk metric is described by AdS-Rindler coordinates

$$ds^2 = L^2 \left(-(r^2 - 1)d\tau^2 + \frac{dr^2}{r^2 - 1} + r^2 dH_{d-1}^2 \right), \quad (4.7)$$

which is obtained from eq. (4.2) by taking the massless limit $\omega, M \rightarrow 0$ and rescaling the coordinates $t \rightarrow R\tau$ and $r \rightarrow Lr$, such that the coordinates in eq. (4.7) are dimensionless. Although the underlying geometry is simply the AdS vacuum, the metric (4.7) resembles a black hole metric with horizons at $r = \pm 1$ and an apparent singularity at $r = 0$. We can also extend the spacetime at a fixed time-slice through the bifurcation surface and arrive at a second Rindler wedge. The bulk spacetime thus has two asymptotic regions, located at $r \rightarrow \infty$, each of which hosts one copy of the boundary CFT on the $\mathbb{R} \times H_{d-1}$ geometry. As noted above (in terms of the dimensionful coordinates), the corresponding TFD state has a (dimensionful) temperature $T = 1/(2\pi R)$, which is tuned in relation to the curvature scale R of the hyperbolic geometries (4.4). Lastly, note that since the Rindler wedges are simply a reparametrization of pure AdS, it is clear that the singularity at $r = 0$ is only a coordinate singularity.³ In fact, we can extend the coordinates smoothly through the interior to negative r where we can exit the region behind the (inner) horizon at $r = -1$ and enter a new set of Rindler wedges.

For each CFT, we introduce a codimension-one conformal defect (with zero extrinsic curvature) at the centre of the hyperbolic spatial geometry. It is convenient to choose slicing coordinates for the hyperbolic boundaries, such that

$$dH_{d-1}^2 = d\chi^2 + \cosh^2 \chi dH_{d-2}^2. \quad (4.8)$$

In these coordinates, the location of the conformal defect is $\chi = 0$.

From the bulk perspective, the CFT defects are dual to a co-dimension one brane, which spans a slice of constant extrinsic curvature of the bulk spacetime and intersects the

³This is in contrast to the general metric (4.2) where $r \rightarrow 0$ does yields a curvature singularity.

asymptotic boundary at the location of the CFT defect. In order to describe its trajectory, it is convenient to write the bulk metric in terms of the slicing coordinates in eq. (2.7). The brane is located at constant $\theta = \theta_{\text{B}}$, which is determined by the tension T_o through eq. (2.12) with

$$\ell_{\text{B}} = \frac{L}{\sin \theta_{\text{B}}} . \quad (4.9)$$

The trajectory of a hypersurface of constant θ_{B} in the bulk spacetime is then given by

$$r^2 \sinh^2 \chi = \cot^2 \theta_{\text{B}} = \left(\frac{\ell_{\text{B}}}{L} \right)^2 - 1 . \quad (4.10)$$

As noted in section 2.1, this means that a brane with positive tension (*i.e.*, $T_o \geq 0$) creates additional geometry by its backreaction. Of course, the backreaction of a negative-tension brane would remove geometry. However, let us add that there is no (nearly) massless graviton induced on a negative-tension brane⁴ and therefore we will only consider positive tensions in the following, *i.e.*, $0 \leq \theta_{\text{B}} \leq \frac{\pi}{2}$.

For such a (positive-tension) brane, the bulk geometry to one side of the brane can be described by eq. (4.7), with $r \sinh \chi \leq \cot \theta_{\text{B}}$, while the geometry to the other side of the brane is given by the same metric with $r \sinh \chi \geq -\cot \theta_{\text{B}}$. We can therefore treat either side of the brane as an AdS-Rindler geometry which is cut off by the brane.

Using eq. (4.10), we can determine the induced metric on the brane. After a short calculation, one finds

$$ds^2 = \ell_{\text{B}}^2 \left(-(\rho^2 - 1) d\tau^2 + \frac{d\rho^2}{\rho^2 - 1} + \rho^2 dH_{d-2}^2 \right) , \quad (4.11)$$

where we have changed the radial coordinate with

$$\ell_{\text{B}}^2 (\rho^2 - 1) = L^2 (r^2 - 1) . \quad (4.12)$$

This brane metric again takes the form of an AdS-Rindler metric, c.f. eq. (4.7). Further, this demonstrates that the Rindler horizon in the bulk (at $r = 1$) induces a Rindler horizon on the brane (at $\rho = 1$), as one would expect from the bulk perspective.⁵ From the boundary perspective, this behaviour is readily explained by the fact that the conformal

⁴We thank Raman Sundrum for explaining this point to us.

⁵However, it is interesting to note that $r = 0$ corresponds to $\rho = \cos \theta_{\text{B}} = 1 - (L/\ell_{\text{B}})^2$, and hence one cannot reach $\rho = 0$ in the r -coordinate system (unless $\theta_{\text{B}} = \pi/2$).

defect is in thermal equilibrium with the surrounding CFT. In the effective RS description of the brane perspective, this behaviour arises because the region of dynamical gravity is coupled to the bath CFT along an accelerated trajectory, so that the temperature felt by the accelerated boundary agrees with the temperature of the CFT, *e.g.*, see [162–164]. As already mentioned, this setup generalizes the two-dimensional framework presented in [50] to higher dimensions.

All calculations below will be done for the case of positive tension branes. However, when it comes to interpretation, we will be particularly interested in the case where $1 \gg \theta_{\text{B}} \simeq \frac{L}{\ell_{\text{B}}}$, for which the brane theory is well described as Einstein gravity coupled to two copies of the boundary CFT (with a high cutoff). The reason is that in this limit, we can interpret the intersection of the brane and the RT surface as bounding an island in this effective gravitational theory.

4.4 Island phase at $\tau_{\Sigma} = 0$

We will start our analysis by calculating the area of the RT surface for an entangling surface lying in the $\tau = \tau_{\Sigma} = 0$ plane and crossing the Planck brane. In other words, the RT surface is in the connected phase — see figure 4.5. We are interested in the EE of $\mathbf{R}$ comprised of the combined regions $\chi > \chi_{\Sigma}$ and $\chi < -\chi_{\Sigma}$ in both the left and right CFTs. Hence the entangling surfaces of interest have two components (in each CFT) sitting at a constant distance away from the defect at $\chi = \pm\chi_{\Sigma}$. We note that the induced metric on the latter surfaces is proportional to $\cosh^{d-2}\chi_{\Sigma}$.

In two dimensions, the analysis of the RT surfaces is simplified because the metric (4.7) has a shift symmetry $\chi \rightarrow \chi + \text{const}$, but the latter is absent in higher dimensions. However, we can find a similar simplification by going to a different coordinate system defined via [149]

$$(1 + \zeta^2) = r^2 \cosh^2 \chi, \quad \tan \xi = \frac{r}{\sqrt{r^2 - 1}} \sinh \chi, \quad (4.13)$$

such that the horizon is located at $\xi = \pm\frac{\pi}{2}$. By time-translation invariance, we know that the RT surface lies on a constant Rindler time slice and hence we consider the metric on the $\tau = 0$ slice in the new coordinates,⁶ which reads

$$ds_{\text{E}}^2 = L^2 \left(\frac{d\zeta^2}{1 + \zeta^2} + \zeta^2 d\xi^2 + (1 + \zeta^2) dH_{d-2}^2 \right). \quad (4.14)$$

⁶Note that the full metric takes the form $ds^2 = L^2 \zeta^2 \cos^2 \xi d\tau^2 + ds_{\text{E}}^2$, and hence the shift symmetry does not extend to the full spacetime metric.

Hence the geometry of this spatial slice (or any constant τ slice) is invariant under $\xi \rightarrow \xi + \text{const}$, which will simplify the following.

Making the ansatz $\zeta = \zeta(\xi)$ for the profile of the RT surface, the induced metric on these surfaces takes the form

$$ds_{\text{ind}}^2 = L^2 \left[\left(\left(\frac{\partial \zeta}{\partial \xi} \right)^2 + \zeta^2 (1 + \zeta^2) \right) \frac{d\xi^2}{(1 + \zeta^2)} + (1 + \zeta^2) dH_{d-2}^2 \right], \quad (4.15)$$

with metric determinant

$$\det(\gamma) = L^{2(d-1)} (1 + \zeta^2)^{d-3} \left(\left(\frac{\partial \zeta}{\partial \xi} \right)^2 + \zeta^2 (1 + \zeta^2) \right). \quad (4.16)$$

To obtain the correct RT surface, we now need to extremize the area functional

$$A(\Sigma_{\mathbf{R}}) = \int_{\Sigma_{\mathbf{R}}} \sqrt{\det(\gamma)}, \quad (4.17)$$

subject to the correct boundary conditions. Even in higher dimensions, the extremization of the RT surface is simple, as it can be cast as an effectively two-dimensional problem with metric

$$ds_{2D}^2 = L^{2(d-1)} \text{vol}_{H_{d-2}} (1 + \zeta^2)^{d-2} \left(\frac{d\zeta^2}{1 + \zeta^2} + \zeta^2 d\xi^2 \right). \quad (4.18)$$

Note that the area functional does not explicitly depend on ξ . Rather, ξ plays the role of an angular coordinate and its associated Hamiltonian is conserved. This allows us to turn the second order equation which determines extremal surfaces into a first order expression,

$$\frac{d\zeta}{d\xi} = \pm \sqrt{\zeta^2 (1 + \zeta^2) \left(\frac{\zeta^2 (1 + \zeta^2)^{d-2}}{\zeta_*^2 (1 + \zeta_*^2)^{d-2}} - 1 \right)}, \quad (4.19)$$

where we have introduced ζ_* which is the turn-around point for ζ as a function of ξ — see figure 4.5. The sign depends on whether ζ is going towards (+) or away (−) from the boundary as ξ increases. In the latter case, where the RT surface does not turn around before it intersects the brane we have to think of ζ_* as a coordinate of vacuum AdS extended past the brane, as shown in figure 4.6. More generally, the sign starts out negative and generally flips after $\zeta = \zeta_*$ has been reached.

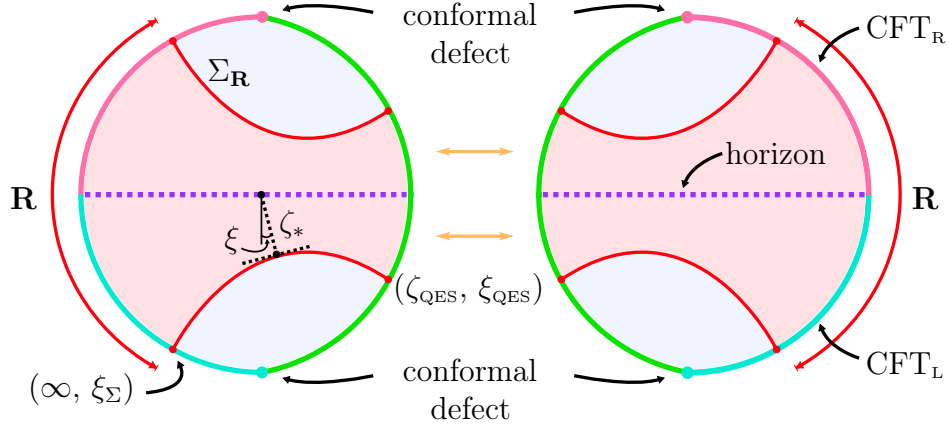


Figure 4.5: This figure shows the RT surface and various quantities defined in the text for the RT surface in the connected phase. The entangling region $\mathbf{R}$ in the boundary is composed of the two regions $|\chi| > \chi_\Sigma$ (where $\tan \xi_\Sigma = \sinh \chi_\Sigma$) in both the left and right CFTs. Note that the right (left) CFT occupies the region on the asymptotic boundary marked in pink (aqua). The conformal defects (*i.e.*, $\chi = 0$ or $\xi = 0$ and π) are positioned where the brane (green) reaches these boundary regions.

The area functional for the RT surfaces satisfying eq. (4.19) then becomes

$$A(\Sigma_{\mathbf{R}}) = 4L^{d-1} \text{vol}_{H_{d-2}} \left(\int_{\zeta_*}^{\infty} \pm \int_{\zeta_{\text{QES}}}^{\zeta_*} \right) d\zeta \frac{\zeta(1+\zeta^2)^{d-\frac{5}{2}}}{\sqrt{\zeta^2(1+\zeta^2)^{d-2} - \zeta_*^2(1+\zeta_*^2)^{d-2}}}, \quad (4.20)$$

where here and below, we use the subscript QES to mark coordinates of the intersection between RT surface and brane, which corresponds to a quantum extremal surface in the brane theory. The upper limit of integration indicated as ∞ must be regulated, since the area of the RT surface is infinite. The sign here is the same sign as in eq. (4.19). We have also included a factor of four, since there is one RT surface to each side of the defect and considering both CFTs, we need to multiply the result by another factor of two.

Eq. (4.19) yields a family of RT surfaces (parameterized by ζ_*) that are locally extremal in the bulk away from the brane. However, fully extremizing the area functional (4.17) requires that we also extremize over the possible locations where these candidate surfaces intersect the brane. That is, we consider the extremization condition of the RT surfaces' area (plus possibly the area of the QES, should there be extra DGP gravity) with respect to the position of the intersection $\sigma_{\mathbf{R}}$,

$$0 = \frac{\partial}{\partial \rho_{\text{QES}}} \left(\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} + \frac{A(\sigma_{\mathbf{R}})}{4G_{\text{brane}}} \right), \quad (4.21)$$

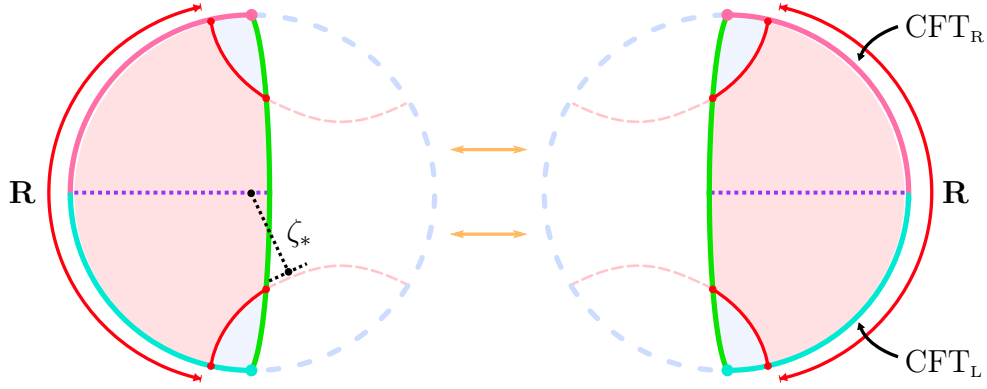


Figure 4.6: This figure shows how RT surfaces can intersect the brane before reaching the turnaround point ζ_* , with relatively small brane tension T_o , *i.e.*, $\theta_B \sim O(1)$, and positive DGP coupling.

where the two contributions reflect the two contributions in eq. (1.8). Here, ρ_{QES} denotes the location of $\sigma_{\mathbf{R}}$ in coordinates along the brane in eq. (4.11).

As described in section 3.2, this extremization leads to a boundary condition restricting the angle at which the RT surface meets the brane. Normally, this would be a difficult problem in higher dimensions. However, here we are leveraging the hyperbolic symmetry along the transverse directions, which reduces the present case to a two-dimensional problem. That is, we need only extremize a one-dimensional profile $\zeta(\xi)$ of the RT surface in the effective two-dimensional geometry given by eq. (4.18). Assuming that we consider an extremal bulk surface that is anchored at the asymptotic boundary, the variation of the surface's area with respect to its intersection point with the brane is given by

$$\delta_{\sigma_{\mathbf{R}}} A(\Sigma_{\mathbf{R}}) = h_{ij} T^i X^j|_{\text{end-point}}, \quad (4.22)$$

where h_{ij} is the two-dimensional metric (4.18) and T^i is a normalized (w.r.t. h_{ij}) tangent vector to the RT surface, which can be obtained from eq. (4.10). The vector X^i determines the variation along the brane.

In the absence of a DGP gravity term in the action, this variation must vanish for X^j along the brane; hence we have a boundary condition that sets the RT surface perpendicular to the brane. More generally, we must balance the above variation against the variation of the entropy contribution intrinsic to the brane, as can be seen from eq. (4.21).

The first contribution to eq. (4.21) is then calculated using eq. (4.22) and yields

$$\partial_\rho A(\Sigma_{\mathbf{R}}) = 4L^{d-1} \text{vol}_{H_{d-2}} \frac{\zeta_*(1 + \zeta_*^2)^{\frac{d-2}{2}}}{\zeta^2 \sin \theta_B} \left(\sqrt{\frac{\zeta^2 + 1}{\tan^2 \theta_B \zeta^2 - 1}} \pm \sqrt{\frac{\zeta^2(1 + \zeta^2)^{d-2}}{\zeta_*^2(1 + \zeta_*^2)^{d-2}} - 1} \right), \quad (4.23)$$

which is evaluated at $\zeta = \zeta_{\text{QES}}$. Here we have used the brane angle θ_B defined in eq. (4.9).

If the brane DGP coupling is turned on, the variation of the area also obtains a contribution from the second term in eq. (4.21),

$$\partial_\rho A(\sigma_{\mathbf{R}}) = 2L^{d-2} \text{vol}_{H_{d-2}} \frac{\zeta_*(1 + \zeta_*^2)^{\frac{d-2}{2}}}{\zeta^2 \sqrt{\zeta^2 \sin^2 \theta_B - \cos^2 \theta_B}} (d-2)(\zeta^2 + 1)^{\frac{d-2}{2}}. \quad (4.24)$$

Substituting eqs. (4.23) and (4.24) into eq. (4.21), we obtain the following relation between the QES position ζ_{QES} and the deepest point ζ_* reached by the RT surface:

$$\begin{aligned} \zeta_*(1 + \zeta_*^2)^{\frac{d-2}{2}} &= (\zeta_{\text{QES}}^2 + 1)^{\frac{d-3}{2}} \sqrt{\zeta_{\text{QES}}^2 \sin^2 \theta_B - \cos^2 \theta_B} \\ &\times \left[\lambda_b \cos(\theta_B) \sqrt{1 + \zeta_{\text{QES}}^2} + \sqrt{1 + \zeta_{\text{QES}}^2 - \lambda_b^2 (\zeta_{\text{QES}}^2 \sin^2 \theta_B - \cos^2 \theta_B)} \right], \end{aligned} \quad (4.25)$$

where λ_b was defined in eq. (2.54).

A final relation associating ζ_{QES} and the belt width ξ_Σ comes from integrating eq. (4.19) from the boundary to the brane,

$$\xi_{\text{QES}} = \xi_\Sigma + \int_{\zeta_*}^{\infty} d\zeta \left| \frac{d\zeta}{d\xi} \right|^{-1} \pm \int_{\zeta_*}^{\zeta_{\text{QES}}} d\zeta \left| \frac{d\zeta}{d\xi} \right|^{-1}. \quad (4.26)$$

After using eq. (4.25), this can then be rewritten as a relation between the location of the entangling surface ξ_Σ and the QES ζ_{QES} only, if we further use eq. (4.10) together with eq. (4.13) to find the brane trajectory in ζ, ξ coordinates and determine the relationship between ξ and ζ on the brane

$$\zeta^2 \sin^2 \xi = \cot^2 \theta_B. \quad (4.27)$$

In section 4.9, we will use eqs. (4.20), (4.25) and (4.26) to produce the late-time part of the Page curve for a topological black hole coupled to a bath in higher dimensions.

4.5 No-island phase for $\tau_\Sigma = 0$

We can use the result of the previous subsection to obtain a solution for the no-island phase. The first order equation (4.19) (where we choose the minus sign) again determines the shape of extremal surface. By symmetry, we know that ζ_* must lie on the bifurcate horizon and is thus determined by solving

$$\int_{\infty}^{\zeta_*} \left| \frac{d\zeta}{d\xi} \right|^{-1} d\zeta = -\frac{\pi}{2} - \xi_\Sigma. \quad (4.28)$$

Here we have implicitly chosen to perform the calculation in the asymptotic CFT which sits at negative ξ , *i.e.*, to a particular side of the brane. By symmetry the calculation on the other side of the brane yields the same result. The total area of the two RT surfaces which connect both CFTs through the horizon is then given by

$$A(\Sigma_{\mathbf{R}}) = 4L^{d-1} \text{vol}_{H_{d-2}} \int_{\zeta_*}^{\infty} d\zeta \frac{\zeta(1+\zeta^2)^{d-\frac{5}{2}}}{\sqrt{\zeta^2(1+\zeta^2)^{d-2} - \zeta_*^2(1+\zeta_*^2)^{d-2}}}, \quad (4.29)$$

with ζ_* given by eq. (4.28). In the case of small brane angle θ_b this phase always dominates at early times. The reason is that the RT surface in the competing phase, *i.e.*, the phase where the RT surface crosses the brane, has to travel a large distance to the brane before it can return to the asymptotic boundary across the brane. This additional distance can be made arbitrarily small by choosing a small enough brane angle. We will furthermore see in section 4.7 how the time evolution of an RT surface at early times can be mapped to this case.

4.6 No-island phase for $\chi_\Sigma = 0$

Lastly, we will consider the case of a zero-width belt, *i.e.*, the case where the location of the entangling surface is taken towards the defect, so that the RT surface falls straight through the bulk along a constant value of the boundary slicing coordinate $\chi = \chi_\Sigma$, c.f. figure 4.7. Note that this setup is essentially the same as considered in [161], which studied EE of identical half-spaces in the two sides of a time-evolved TFD.

Due to symmetry, the trajectory of the RT surface is determined by its radial coordinate r as a function of time τ . However, it is convenient to introduce Eddington-Finkelstein

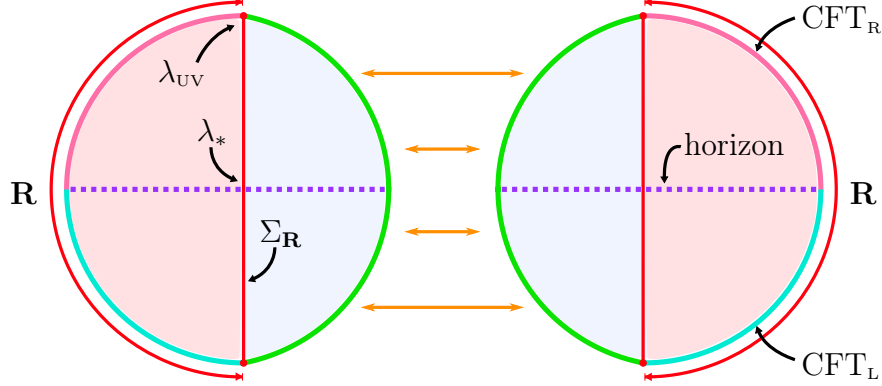


Figure 4.7: The RT surface of an entangling surface located at the defect in the no-island phase.

coordinates to avoid the coordinate singularity at $r = 1$. Hence, describing ingoing null rays, we have

$$v = \tau + r_{\text{tor}}(r) \quad \text{where} \quad r_{\text{tor}}(r) = \frac{1}{2} \log \left(\frac{|r-1|}{r+1} \right), \quad (4.30)$$

where $r_{\text{tor}}(r)$ denotes the usual tortoise coordinate.⁷ Note that with the above definitions, $r_{\text{tor}}(r \rightarrow \infty) \rightarrow 0$ and hence $v = \tau$ at the asymptotic AdS boundary. Then the metric becomes

$$ds^2 = L^2 \left(-(r^2 - 1) dv^2 + 2 dv dr + r^2 dH_{d-1}^2 \right). \quad (4.31)$$

Now the extremal surface will fall from the asymptotic boundary, through the exterior, across the Rindler horizon, reaching a minimal radius at r_* , within the interior. Then the surface will continue emerging into the second exterior region. Due to reflection symmetry, we need only track the trajectory of the RT surface until it reaches r_* . Using eq. (4.31), the area functional can be written as

$$A(\Sigma_{\mathbf{R}}) = 4 \text{vol}_{H_{d-2}} L^{d-1} \int_{\lambda_*}^{\lambda_{\text{UV}}} d\lambda r^{d-2} \sqrt{-(r^2 - 1)\dot{v}^2 + 2\dot{v}\dot{r}}, \quad (4.32)$$

where λ is a radial coordinate intrinsic to the surface, which increases along the surface moving from the left asymptotic AdS boundary to the right boundary. The limits of integration here correspond to λ_* , the value at the minimal radius r_* , and λ_{UV} , the value at the UV cutoff near the right boundary — see figure 4.7. We have also included a factor

⁷We extend our definition of $r_{\text{tor}}(r)$ across the horizon using the standard prescription given in [159].

of 4 to account for the fact that we only integrate from the $\mathbb{Z}_2$ symmetric point λ_* out to the right boundary, and the fact that there are two such RT surfaces, one on either side of the brane. Of course, we have also integrated out the directions along the belt, *i.e.*, along the H^{d-2} . Now, we fix the reparametrization symmetry of the area functional with the following convenient gauge choice

$$\sqrt{-(r^2 - 1)\dot{v}^2 + 2\dot{v}\dot{r}} = r^{d-2}. \quad (4.33)$$

The integrand in eq. (4.32) is independent of v and so we have a conserved ‘ v -momentum’

$$P_v = \frac{\partial \mathcal{L}}{\partial \dot{v}} = \frac{r^{d-2}(\dot{r} - (r^2 - 1)\dot{v})}{\sqrt{-(r^2 - 1)\dot{v}^2 + 2\dot{v}\dot{r}}} = \dot{r} - (r^2 - 1)\dot{v}, \quad (4.34)$$

where the second expression results from substituting in the gauge choice (4.33). Using eqs. (4.33) and (4.34) to solve for $\dot{r}$ and $\dot{v}$, we find

$$\begin{aligned} \dot{r}[P_v, r] &= \sqrt{(r^2 - 1)r^{2(d-2)} + P_v^2}, \\ \dot{v}[P_v, r] &= \frac{\dot{r} - P_v}{r^2 - 1} = \frac{1}{r^2 - 1} \left(-P_v + \sqrt{(r^2 - 1)r^{2(d-2)} + P_v^2} \right). \end{aligned} \quad (4.35)$$

Note that we have implicitly chosen a positive sign for $\dot{r}$ indicating that r is increasing as we move along the surface out towards the asymptotic boundary.

An intuitive picture of the dynamics of the extremal surfaces is given by recasting the $\dot{r}$ equation above as a Hamiltonian constraint,

$$\dot{r}^2 + U(r) = P_v^2, \quad (4.36)$$

where the effective potential is given by

$$U(r) = -(r^2 - 1)r^{2(d-2)}. \quad (4.37)$$

In this framework, P_v^2 plays the role of the conserved energy and the minimum radius r_* corresponds to the turning point where $\dot{r} = 0$, *i.e.*,

$$(1 - r_*^2)r_*^{2(d-2)} = P_v^2. \quad (4.38)$$

The area (4.32) of the extremal surface becomes

$$A(\Sigma_{\mathbf{R}}) = 4 \text{vol}_{H_{d-2}} L^{d-1} \int_{r_*}^{r_{\text{UV}}} dr \frac{r^{2(d-2)}}{\sqrt{(r^2 - 1)r^{2(d-2)} + P_v^2}}, \quad (4.39)$$

using eqs. (4.33) and (4.35). Note that r_{UV} denotes the position of the UV cutoff surface near the asymptotic AdS boundary.

With eq. (4.38), the extremal surface can be specified by the integration constant P_v or the boundary condition r_* . However, we want to examine the time evolution of the EE and so we must determine a relation between these constants and the boundary time. In particular, using eq. (4.35), we can integrate out to the right boundary to determine

$$v_{\text{bound}} - v_* = \int_{r_*}^{r_{\text{UV}}} dr \frac{\dot{v}}{\dot{r}} = \int_{r_*}^{r_{\text{UV}}} dr \frac{1}{r^2 - 1} \left[1 - \frac{P_v}{\sqrt{(r^2 - 1) r^{2(d-2)} + P_v^2}} \right], \quad (4.40)$$

where v_* denotes the value of the Eddington-Finkelstein time at the turning point. However, because of the $\mathbb{Z}_2$ symmetry of the extremal surface, we know that the turning point lies on the surface $t = 0$, and so we may use eq. (4.30) to write

$$v_* = r_{\text{tor}}(r_*) = \frac{1}{2} \log \left(\frac{1 - r_*}{1 + r_*} \right). \quad (4.41)$$

Further, we know that $v_{\text{bound}} = \tau[P_v]$ and hence we find

$$\tau[P_v] = \frac{1}{2} \log \left(\frac{1 - r_*}{1 + r_*} \right) + \int_{r_*}^{r_{\text{UV}}} dr \frac{1}{r^2 - 1} \left[1 - \frac{P_v}{\sqrt{(r^2 - 1) r^{2(d-2)} + P_v^2}} \right]. \quad (4.42)$$

Note that the integrand is nonsingular in the vicinity of the horizon, *i.e.*, near $r = 1$.

The time derivative of the area (4.32) admits a very simple form

$$\frac{dA(\Sigma_{\mathbf{R}})}{d\tau_{\Sigma}} = 4 \text{vol}_{H_{d-2}} L^{d-1} P_v = 4 \text{vol}_{H_{d-2}} L^{d-1} r_*^{d-2} \sqrt{1 - r_*^2}, \quad (4.43)$$

where τ is the boundary time parameter.⁸ Further, we also observe that the critical radius where $\partial_r U = 0$ is given by

$$r_c^2 = \frac{d - 2}{d - 1}. \quad (4.44)$$

⁸A quick derivation of this result follows by considering a small variation of the surface profile in eq. (4.32). The bulk contributions naturally vanishes by the equations of motion determining the extremal surface. However, deriving the latter requires an integration by parts which produces boundary terms. These are usually eliminated by fixing the boundary conditions at infinity. In the above result, we instead allow for a small variation in the boundary time.

At late times, the turning point is very close to this critical radius, *i.e.*, the critical surface lies near the surface $r = r_c$ for a long time, and so we can replace $r_* \rightarrow r_c$ into eq. (4.43). Hence we expect the growth of the area is fixed at late times, *i.e.*,

$$\frac{dA(\Sigma_{\mathbf{R}})}{d\tau_{\Sigma}} = 4 \text{vol}_{H_{d-2}} L^{d-1} \frac{(d-2)^{(d-2)/2}}{(d-1)^{(d-1)/2}}. \quad (4.45)$$

As we will see momentarily, the late time behaviour of the entropy of any subregion bounded by constant χ in the no-island phase is determined by a zero-belt width calculation. Thus, as in the two-dimensional case studied in [61] (as well as the higher dimensional case [58]), the entropy corresponding to the no-island phase grows without bound.

4.7 Time-evolution for general $\chi_{\Sigma}, \tau_{\Sigma} \neq 0$

Given the region $\mathbf{R}$ of interest,⁹ we can ask how the RT surface changes under time evolution. If we are in the island phase, the RT surface is completely contained inside the Rindler patch so that time translations are a symmetry and the entropy is a constant. On the other hand, in the no-island phase, the RT surface connects to both bath CFTs. Forward time evolution of both sides is not a symmetry and the area of the RT surface changes.

Obtaining RT surfaces in the no-island phase that are anchored on symmetric entangling surfaces of arbitrary width and at arbitrary times in higher dimensions is generally difficult. However, as we will now show, our choice of entangling surfaces with the hyperbolic symmetry of H_{d-2} allows us to map the RT surface at any $(\chi_{\Sigma}, \tau_{\Sigma})$ either to some RT surface in the $\tau = 0$ slice, *i.e.*, with $(\chi'_{\Sigma}, \tau'_{\Sigma} = 0)$ or to the case where the entangling surface is at $\chi = 0$, *i.e.*, with $(\chi'_{\Sigma} = 0, \tau'_{\Sigma})$. In particular, this means that the solutions obtained in the last two subsections are sufficient to discuss the full time evolution of the symmetric entangling surfaces of interest.

The strategy we will employ in this chapter is the following. We will perform a coordinate change from Rindler space to a particular Poincaré coordinate system defined below. In the new coordinates, the entangling surfaces are straight lines. By exploiting the boost symmetry of the Poincaré patch and mapping back to Rindler space, the task of calculating EE of a subregion with χ_{Σ} at time τ_{Σ} can be reduced to one of the cases discussed in sections 4.5 and 4.6.

⁹Recall that $\mathbf{R}$ consists of all points more than a distance χ_{Σ} away from the defect in both CFTs.

To understand the required coordinate changes it is convenient to embed AdS_{d+1} into $\mathbb{R}^{d,2}$, *i.e.*, we are looking for a parameterization of (parts of) the hyperboloid defined via

$$-T_1^2 - T_2^2 + X_1^2 + \dots + X_d^2 = -L^2. \quad (4.46)$$

Our original two Rindler patches correspond to the parameterization

$$\begin{aligned} T_1 &= \pm L\sqrt{r^2 - 1} \sinh \tau, & T_2 &= Lr \cosh \chi \cosh \eta, \\ X_1 &= \pm L\sqrt{r^2 - 1} \cosh \tau, & X_2 &= \pm Lr \sinh \chi, \\ X_i &= Lr \cosh \chi \sinh \eta \mu_i & \text{with } i &= 3, 4, \dots, d, \end{aligned} \quad (4.47)$$

where μ_i denotes further angular coordinates, *e.g.*, $\mu_3 = \cos \phi_1$, $\mu_4 = \sin \phi_1 \cos \phi_2$, $\dots$, which, together with η parametrize the H_{d-2} slice of the metric (4.8). The AdS boundary is located at $r \rightarrow \infty$, and each sign corresponds to one of the two Rindler wedges. On a fixed r slice, we can reach the boundary by taking $\chi \rightarrow \pm\infty$ or $\eta \rightarrow \pm\infty$. For any constant Rindler time (*i.e.*, fixed τ), the bifurcation surface reached with $r \rightarrow 1$. The defect in the CFT is located at $\chi = 0 = X_2$. The entangling surfaces are defined to be at $\chi = \pm\chi_\Sigma$ in both CFTs.

We will now consider a particular Poincaré coordinate system, which covers both Rindler wedges and is defined in terms of embedding coordinates as

$$\begin{aligned} T_1 &= L \frac{\tilde{t}}{\tilde{z}}, & X_1 &= L \frac{\tilde{x}_1}{\tilde{z}}, & X_2 &= L \frac{\tilde{x}_2}{\tilde{z}}, & \dots \\ X_d &= \frac{\tilde{z}^2 + \tilde{x}^2 - \tilde{t}^2 - L^2}{2\tilde{z}}, & T_2 &= \frac{\tilde{z}^2 + \tilde{x}^2 - \tilde{t}^2 + L^2}{2\tilde{z}}, \end{aligned} \quad (4.48)$$

where $\tilde{x}^2 = \tilde{x}_1^2 + \tilde{x}_2^2 + \dots + \tilde{x}_{d-1}^2$. In these coordinates, the bifurcation surface intersects the boundary ($\tilde{z} \rightarrow 0$) at $\tilde{x}_1 = \tilde{t} = 0$,¹⁰ while the defects are located at $\tilde{x}_2 = 0$. The two CFTs are mapped to the regions $\tilde{x}_1 > 0$ and $\tilde{x}_1 < 0$, respectively. We will denote the CFT at $\tilde{x}_1 > 0$ as the right CFT, and the one at $\tilde{x}_1 < 0$ as the left CFT. Comparing eqs. (4.47) and (4.48) in the boundary limit, it is easy to see that the entangling surfaces in the right CFT get mapped to

$$\tilde{x}_2 = \pm \frac{\sinh \chi_\Sigma}{\cosh \tau_\Sigma} \cdot \tilde{x}_1, \quad \tilde{t} = \tanh \tau_\Sigma \cdot \tilde{x}_1. \quad (4.49)$$

This shows the convenient property of the new Poincaré coordinates: entangling surfaces lie along rays (*i.e.*, straight lines) in the positive half-space with $\tilde{x}_1 > 0$, whose slope

¹⁰The full Rindler horizons reach the boundary along $\tilde{x}_1^2 - \tilde{t}^2 = 0$.

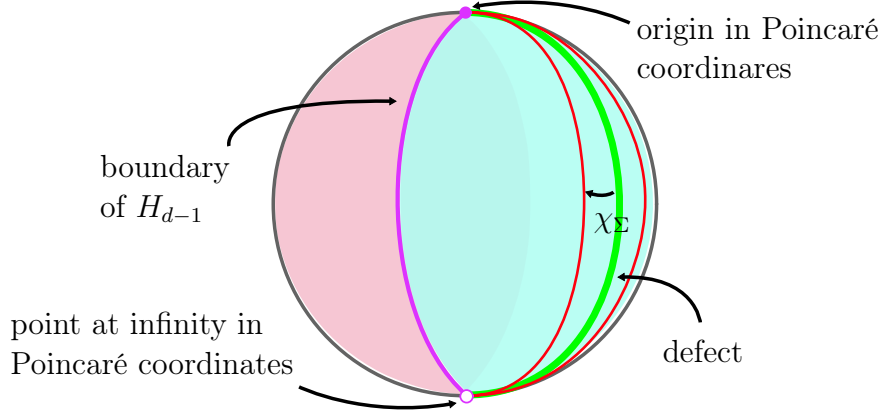


Figure 4.8: A time-slice of our setup. The spatial boundary S^{d-1} (in global coordinates) is split into two hyperbolic discs H_{d-1} , shown in pink and aqua, which are glued together at infinity. At the same location, the bifurcate horizon intersects the boundary. The CFT on either disc is dual to a Rindler wedge in the bulk. The defect (green) is a great circle on the global boundary. As indicated in the figure, the Poincaré coordinates introduced in this section cover the full sphere, with the point at infinity appearing on the south pole of the sphere. Entangling surfaces are the semi-circles shown in red.

depends on the spatial location χ_Σ and the Rindler time τ_Σ at which the entangling surfaces are defined. Further, flipping the sign of $\tilde{x}_1$ to $-\tilde{x}_1$ in the above expressions yields the entangling surfaces in the left CFT. The relation between the Rindler coordinate given in eq. (4.47) and the new Poincaré coordinates of eq. (4.49) is illustrated in figure 4.8.

We now need to choose cutoffs in order to regulate the area integrals of the RT surfaces. First, we need to regulate the UV divergence in the entanglement entropy by introducing a maximum radius in both AdS-Rindler patches $r_{UV} \gg 1$. This translates to a $\tilde{z}$ -dependent cutoff in the new coordinates,

$$\tilde{z}^2 > \tilde{z}_{\min}^2 = \frac{\tilde{x}_1^2 - \tilde{t}^2}{r_{UV}^2 - 1} \sim \frac{\tilde{x}_1^2 - \tilde{t}^2}{r_{UV}^2}, \quad (4.50)$$

where in the last step, we used that $r_{UV} \gg 1$.

Second, we need an IR cutoff which we impose in the transverse directions along the entangling surface. Since the solution is independent of shifts in all directions along the brane, the transverse directions should just contribute an overall volume factor. We choose

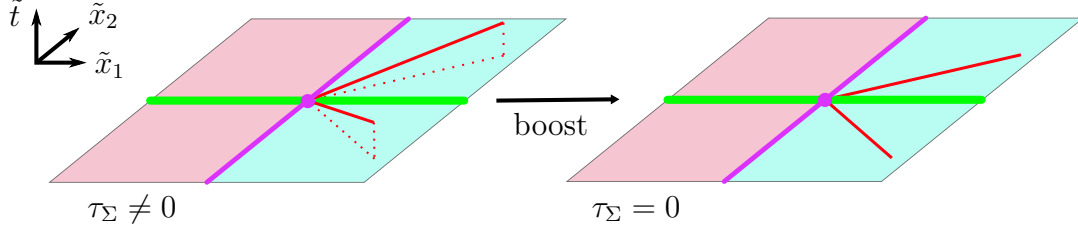


Figure 4.9: The left panel shows two components of the entangling surface (red) at non-zero Rindler time τ_Σ in the right CFT in the Poincaré coordinates (4.48). In the island phase, these two rays in the boundary geometry are connected by an RT surface in the bulk. We can perform a boost in $\tilde{x}_1$ direction to map this set of entangling surfaces to the $t = 0$ slice, which also corresponds to $\tau = 0$ slice of the hyperbolic boundary geometry. The boost is a symmetry of the defect (green).

$\eta_{\max} = \frac{\ell_{\text{IR}}}{R} \gg 1$, which translates to

$$\frac{\tilde{z}^2 - \tilde{t}^2 + \tilde{x}^2 + L^2}{\sqrt{\tilde{z}^2 - \tilde{t}^2 + \tilde{x}_1^2 + \tilde{x}_2^2}} < 2L \cosh \frac{\ell_{\text{IR}}}{R}. \quad (4.51)$$

4.7.1 Island phase

As a warm-up exercise, we will show that the entropy on the island phase is in fact invariant under time evolution. This is obviously true, since the RT surface is completely contained within one Rindler wedge and τ is a Killing coordinate for the corresponding metric (4.11). Hence the corresponding time evolution of a single Rindler wedge is an isometry of that wedge. In this case, we are looking for an extremal surface that ends on the boundary at the location defined by eq. (4.49) for either $\tilde{x}_1 > 0$ or $\tilde{x}_1 < 0$, depending on which Rindler wedge we are interested in. Here, we choose $\tilde{x}_1 > 0$. We can express the problem in a boosted coordinate systems

$$\tilde{t}' = \gamma(\tilde{t} - \beta\tilde{x}_1), \quad \tilde{x}_1' = \gamma(\tilde{x}_1 - \beta\tilde{t}), \quad (4.52)$$

with boost parameter $\beta = \tanh \tau_\Sigma$. This is depicted in figure 4.9. This boost leaves the cutoffs given in eqs. (4.50) and (4.51) invariant, and changes the equation for the entangling surface to

$$\tilde{x}_2' = \pm \tilde{x}_1' \sinh \chi_\Sigma, \quad \tilde{t}' = 0. \quad (4.53)$$

This is precisely the entangling surface of the same region at $\tau = \tau'_\Sigma = 0$ with the appropriate cutoffs. We may thus conclude that entropy of the region $\mathbf{R}$ remains constant in the

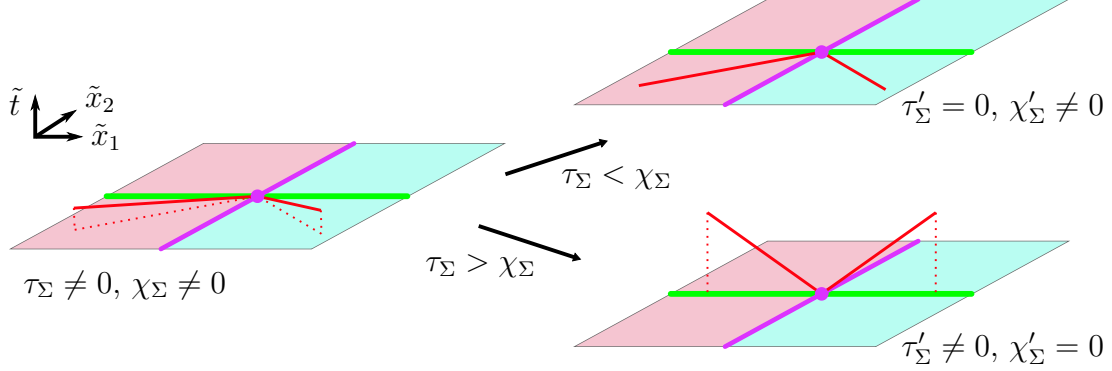


Figure 4.10: The left panel shows two components of the entangling surface (red) at non-zero Rindler time τ_Σ in the right CFT in the Poincaré coordinates (4.48). These two rays are located in different CFTs so that in the no-island phase, they are joined by an RT surface in the bulk, which passes through the Rindler horizon. In this case, we can now boost in $\tilde{x}_2$ direction to map these two rays to $\tau'_\Sigma = 0$ when $\tau_\Sigma < \chi_\Sigma$ or to $\chi'_\Sigma = 0$ when $\tau_\Sigma > \chi_\Sigma$.

island or connected phase, as anticipated. Again because we have a pure state globally, we can see that the entropy of the complementary region, *i.e.*, the two belts centred on the conformal defects in each of the two CFTs, is independent of τ_Σ in this connected phase.

4.7.2 No-island phase

For the no-island phase, we focus on the case in which the RT surface connects entangling surfaces in the CFTs dual to different Rindler patches. The entangling surfaces are located at

$$\tilde{t} = -\frac{\sinh \tau_\Sigma}{\sinh \chi_\Sigma} \tilde{x}_2, \quad \tilde{x}_1 = \pm \frac{\cosh \tau_\Sigma}{\sinh \chi_\Sigma} \tilde{x}_2, \quad (4.54)$$

where we have chosen to focus on $\tilde{x}_2 < 0$, *i.e.*, to the region on one side of the defect. Similarly to the island phase, we want to go to a new coordinate system in which the calculation becomes simpler. Now, however, we have to distinguish two cases.

Case 1: If $\tau_\Sigma < \chi_\Sigma$,¹¹ we can boost this problem in $\tilde{x}_2$ direction with boost parameter $\beta = -\frac{\sinh \tau_\Sigma}{\sinh \chi_\Sigma}$. This is depicted in the upper panel of figure 4.10. The new entangling surfaces are then located at

$$\tilde{t}' = 0, \quad \tilde{x}'_1 = \pm \tilde{x}'_2 \sqrt{\frac{\cosh^2 \tau_\Sigma}{\sinh^2 \chi_\Sigma - \sinh^2 \tau_\Sigma}}, \quad (4.55)$$

where $\tilde{x}'_2 < 0$. Expressing the result in Rindler coordinates, we are dealing with the case of an entangling surface in the $\tau = \tau'_\Sigma = 0$ plane. The new location of the entangling surface χ'_Σ is given by

$$\cosh \chi'_\Sigma = \frac{\cosh \chi_\Sigma}{\cosh \tau_\Sigma}. \quad (4.56)$$

Note that as $\cosh \tau_\Sigma \rightarrow \cosh \chi_\Sigma$ (and so as $|\frac{\sinh \tau_\Sigma}{\sinh \chi_\Sigma}| \rightarrow 1$), the new entangling surface gets closer and closer to the defect, *i.e.*, $\chi'_\Sigma \rightarrow 0$.

Importantly, the cutoffs are not boost invariant in this case. The IR cutoff given in eq. (4.51) remains unchanged, but the UV cutoff in eq. (4.50) changes along the trajectory of the entangling surface to

$$r'_{\text{UV}} = r_{\text{UV}} \cosh \tau_\Sigma. \quad (4.57)$$

We should caution the reader that we arrived at eq. (4.57) by substituting the trajectory of the entangling surface into the boosted cutoff. This means that eq. (4.57) is only correct for a small cutoff. Luckily, the corrections to the new cutoff only change the EE at order $\mathcal{O}(1/r_{\text{UV}})$.

In conclusion, we found that if $\tau_\Sigma < \chi_\Sigma$, the entanglement entropy of the region $|\chi| > \chi_\Sigma$ at time $\tau = \tau_\Sigma$ is the same as that of a region $|\chi| > \chi'_\Sigma$ given in eq. (4.56) at time $\tau = \tau'_\Sigma = 0$ calculated with a different cutoff, given by eq. (4.57).

Case 2: The other case, $\tau_\Sigma > \chi_\Sigma$, is shown in the lower panel of figure 4.10. Now we can boost in the $\tilde{x}_2$ direction again, but using $\tilde{\beta} = -\frac{\sinh \chi_\Sigma}{\sinh \tau_\Sigma}$. The new entangling surfaces are located at

$$\tilde{x}'_2 = 0, \quad \tilde{x}'_1 = \pm \tilde{t}' \sqrt{\frac{\cosh^2 \tau_\Sigma}{\sinh^2 \tau_\Sigma - \sinh^2 \chi_\Sigma}}. \quad (4.58)$$

¹¹We are assuming that both τ_Σ and χ_Σ are positive (or zero). Let us also note here that $\tau_\Sigma = \chi_\Sigma$ is a special case, where the entangling surfaces lie in the null plane $\tilde{t} = -\tilde{x}_2$. Our approach of boosting in the $\tilde{x}_2$ direction fails in this case, but the results for the time evolution are smooth across this point.

While this does not reduce to a surface lying in the $\tau' = 0$ plane, in Rindler coordinates it reduces to an entangling surface for a belt width $\chi'_\Sigma = 0$ and

$$\cosh \tau'_\Sigma = \frac{\cosh \tau_\Sigma}{\cosh \chi_\Sigma}. \quad (4.59)$$

Again, the IR cutoff in eq. (4.51) is unchanged, however, the UV cutoff changes to¹²

$$r'_{\text{UV}} = r_{\text{UV}} \cosh \chi_\Sigma. \quad (4.60)$$

Let us note that the cutoff location still is continuous. In the previous case, the new cutoff was the old cutoff multiplied by $\cosh^2 \tau_\Sigma$. The latter was reliable as long as $\tau_\Sigma < \chi_\Sigma$. However, we see here that once $\tau_\Sigma > \chi_\Sigma$, the cutoff is no longer time-dependent.

4.8 The information paradox

Now the preceding results can be combined to give a qualitative description of the time evolution of the EE. Following the discussion in section 4.1 for two dimensions, at time $\tau = 0$, we have a standard thermofield double state of the two CFTs on hyperbolic spatial geometries, including the conformal defects at $\chi = 0$. If we restrict the observations to either the left or right side, the reduced state is a thermal one and in particular, the bath CFT is in thermal equilibrium with the corresponding conformal defect, with temperature $T = 1/(2\pi R)$.

Using the brane perspective and an appropriate choice of parameters,¹³ we can describe the conformal defects as replaced by (two copies of) the boundary CFT coupled to Einstein gravity on an AdS_d region. For the configuration described above, this yields a topological black hole solution shown in eq. (4.11). We emphasize that the latter really describes an AdS_d geometry in AdS-Rindler coordinates, and hence the thermal equilibrium between this ‘black hole’ and the finite temperature CFT on the asymptotic boundary can be understood as arising because the two systems are coupled along an accelerated trajectory in the region of dynamical gravity. While the black hole is in equilibrium with the bath CFT, under time evolution, the two systems are constantly exchanging thermal quanta.

¹²Note that, like above, we have substituted the trajectory of the entangling surface into the boosted expression. Thus, this equation is only strictly correct in the $r_{\text{UV}} \rightarrow \infty$ limit, but the corrections are subleading to the finite part of the EE.

¹³Recall that we obtain a good approximation to (semiclassical) Einstein gravity on the brane if we choose $\frac{L}{\ell_B} \ll 1$ and λ_b not too close to -1 — see chapter 2.

The immediate effect of this process after $\tau = 0$ is to increase the entanglement between one side of the black hole, *i.e.*, one of the AdS-Rindler wedges on the brane, and its respective bath CFT.

A standard measure for the entanglement between both AdS-Rindler wedges and their respective baths is given by the EE of the complement of two belt subregions centred around the conformal defects in the boundary as discussed above. In section 4.7, we saw that by a judicious change of coordinates (and cutoff), the calculation of the EE of these regions can be mapped at late times (*i.e.*, $\tau_\Sigma \geq \chi_\Sigma$) to the case of a zero-width belt. Further, in section 4.6, we found that the entanglement entropy grows linearly in time, as shown in eq. (4.45).¹⁴

As in the two-dimensional case [50, 165], this linear growth of entropy would lead to an information paradox for our eternal black holes, if it was valid for all times. The reason is that the entanglement entropy must be bound from above by the defect entropy, since the defects need to purify the bath system. In the case of interest, the theory is well approximated by weakly coupled Einstein gravity. This allows us to view the quantum fields on the gravitational background as giving a small correction to the entropy and thus, the defect entropy is well-approximated by two times the black hole entropy.¹⁵

The appearance of an island in the effective gravity theory from the brane perspective is simply related to a phase transition of the RT surfaces in the bulk description of our system. The RT surface changes from the no-island phase, in which it connects both CFTs through the horizon, to the island phase, in which it connects both sides of the defect in a single Rindler wedge. The fact that there will always be an extremal surface crossing the brane is easy to see: Before we invoke the extremization condition at the brane, there is an infinite family of candidate RT surfaces, which start in the bath and meet at the brane. To get the correct RT surface, we need only extremize the area by varying the position of the surface where they meet the brane. Subregion duality and the homology constraint guarantee that there will be one extremal surface for every belt configuration (although the boundary of the island might sit at the horizon or at the CFT defect).

¹⁴Implicitly, to apply eq. (4.45), we must also show $\partial_{\tau'_\Sigma} \simeq \partial_{\tau_\Sigma}$. The latter follows at late times from eq. (4.59), which yields

$$\frac{\partial}{\partial \tau'_\Sigma} = \left(1 - \frac{\sinh^2 \chi_\Sigma}{\sinh^2 \tau_\Sigma}\right)^{1/2} \frac{\partial}{\partial \tau_\Sigma}. \quad (4.61)$$

Alternatively, the same result also follows by simply observing that eq. (4.59) implies that at late times: $\tau' = \tau - \log(\cosh \chi_\Sigma) + O(e^{-2\tau})$. Let us add that this linear growth is analogous to that found for planar black holes in [161].

¹⁵The black hole entropy is proportional to the horizon area of the black hole, which in our case is infinite. Hence to be precise, we must consider an IR regulated entropy, as discussed with eq. (4.51).

In order to establish unitarity of the Page curve, we still need to argue that the island appears before the black hole fails to purify the bath region $\mathbf{R}$ under consideration. In the case of interest here, we have that $\frac{\ell_B}{L} \gg 1$. In this approximation, it follows from eq. (4.25) that

$$\zeta_{\text{QES}} = \zeta_{\text{hor}} \left(1 + \frac{\zeta_*^2 (1 + \zeta_*^2)^{d-2}}{2\zeta_{\text{hor}}^{(2d-4)} (1 + \lambda_b)^2} + \dots \right). \quad (4.62)$$

In deriving this equation, we have used that the location of the horizon on the brane is at $\zeta_{\text{hor}} \sim \frac{\ell_B}{L} \gg 1$ and that ζ_* cannot scale with $\frac{\ell_B}{L}$ at leading order. The reason is that ζ_* is bounded from above by a function of the belt width. We can see that the location of the new quantum extremal surface will always be close to the horizon — see also section 4.9 for numerical plots. The leading order contribution to the generalized entropy is given by the area of the horizon which gives the black hole entropy. While a more involved analysis is needed to demonstrate that the appearance of the island saves unitarity, this shows that the island mechanism has the right qualitative behaviour to unitarize the Page curve.

4.9 Numerical results

In the previous sections, we found a phase transition between the no-island and island phases that has the right qualitative properties to yield a Page curve consistent with unitarity. The calculations involved differential equations that have no known closed form solution. However, the reader might have realized that all of these equations were ordinary differential equations and are thus easily solved numerically. In this section, we will first present numerical solutions to the equations for the RT surface in the island phase, and then use the arguments of the previous sections to obtain the Page curve for massless, topological black holes in equilibrium with a bath.

4.9.1 General behaviour of the islands

As discussed previously, by choosing entangling surfaces with the hyperbolic symmetry of H_{d-2} , the problem of finding the corresponding RT surfaces reduces to a two-dimensional problem. Choosing the convenient coordinates in eq. (4.13), we can express the profile of the RT surface as $\zeta(\xi)$. We start here by discussing examples of extremal surfaces in the island phase for different choices of parameters. Instead of working with ζ as a radial

coordinate, we conformally compactify the geometry and use the coordinate

$$\kappa = \arctan(\zeta), \quad (4.63)$$

which maps time slices of AdS to a finite region. In order to calculate the profile of the RT surface, we fix the location of the entangling surface χ_Σ at the boundary. Applying the large r limit of eq. (4.13), we relate this to ξ_Σ , the location of the entangling surface in ζ, ξ coordinates. We can then use eqs. (4.25) and (4.26) to determine ζ_* and ζ_{QES} numerically as a function of ξ_Σ . The shape of the RT surface is obtained by integrating eq. (4.19) from the boundary.

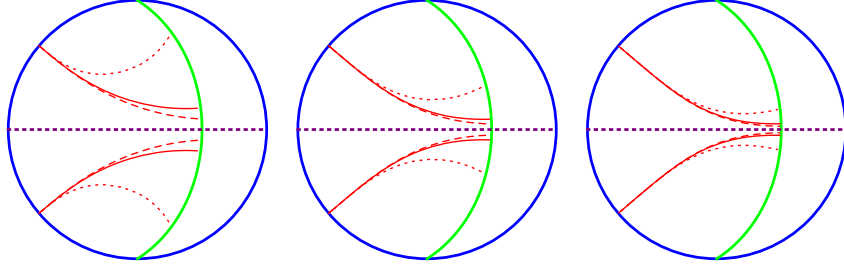
Figure 4.11 shows a few examples of RT surfaces in the connected phase for $d = 3, 4$ and 5 , *i.e.*, in four, five and six bulk dimensions, respectively. Here, we only show the geometry on one side of the brane. The other side is determined by a reflection across the brane. Since the RT surfaces do not cross the horizon, the configuration is independent of the choice of Rindler time τ .

Figure 4.11a shows RT surfaces with fixed χ_Σ for different values of the dimension and selected values of the DGP coupling λ_b . We can see that positive DGP coupling pushes the point of intersection between brane and RT surface towards the horizon, *i.e.*, it reduces the area of the island's boundary. Similarly, negative DGP coupling causes the island to become bigger. This behaviour is readily explained through eq. (2.55) which shows that by increasing (decreasing) the value of λ_b , the gravitational coupling in the brane theory, *i.e.*, the effective Newton's constant, becomes smaller (bigger). In turn, the coefficient of the Bekenstein-Hawking contribution is bigger (smaller) in the island rule (1.7) and therefore creating an island of fixed size becomes harder (easier).

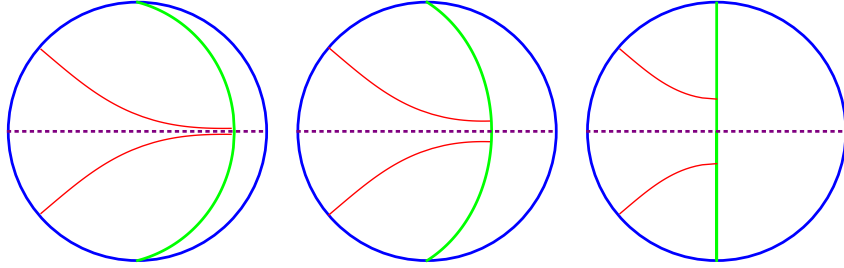
Figure 4.11b shows how the RT surface in the island phase behaves as we vary the brane angle given by $\sin \theta_B = L/\ell_B$ (or equivalently the brane tension — see eqs. (2.12) and (2.18)). Recall that Einstein gravity is a good approximation when θ_B is small. As we depart from the limit of small brane angle, the island grows.

Finally, figure 4.11c shows that the size of the island varies with χ_Σ , the location of the entangling surface in the bath. Moreover, as we will discuss momentarily, we see that an island phase for the RT surface seems to exist for all values of the belt width, although of course it will generally not dominate at early times.

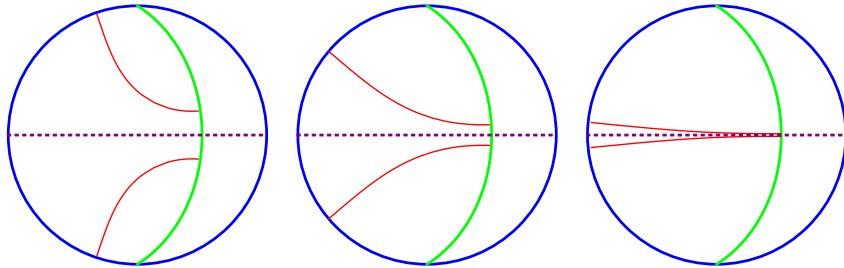
We can get an even better idea of the qualitative features of the islands in higher dimensions by plotting the turning point ζ_* and the QES position $(\zeta_{\text{QES}}, \xi_{\text{QES}})$ as a function of the brane angle θ_B for different dimensions — see figure 4.12. A general feature is that in the $\theta_B \rightarrow 0$ limit, the QES always approaches the horizon on the brane at $\xi = \pi/2$, as



(a) RT surfaces for the island phase in (left to right) $d = 3, 4, 5$. The DGP coupling λ_b is chosen to be $1/0/-0.9$ for the dashed/solid/dotted curves. The brane angle is $\theta_B = \frac{\pi}{4}$ and the location of the entangling surface is $\chi_\Sigma = 1$.



(b) RT surfaces in $d = 4$ with $\chi_\Sigma = 1$ and brane angle of (left to right) $\theta_B = \frac{1}{8}\pi, \frac{1}{4}\pi, \frac{1}{2}\pi$. The DGP coupling is set to zero.



(c) RT surfaces in $d = 4$ with brane angle $\theta_B = \frac{\pi}{4}$ and (left to right) $\chi_\Sigma = \frac{1}{3}, 1, 3$. The DGP coupling is set to zero.

Figure 4.11: RT surfaces in the island phase in higher dimensions. We only show one side of the brane. The asymptotic boundary of the spacetime is shown in blue, the Planck brane in green and the RT surfaces in red. The radial coordinate is κ defined in eq. (4.63). On each side of the horizon (dashed purple line) the angular coordinate ξ runs between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

discussed around eq. (4.62). In terms of ξ_{QES} and the distance from the horizon on the brane, ρ_{QES} , we have

$$\xi_{\text{QES}} = \frac{\pi}{2} - \frac{\zeta_*(1 + \zeta_*^2)^{\frac{d-2}{2}}}{1 + \lambda_b} \theta_B^{d-2} + O(\theta_B^d) \quad (4.64)$$

$$\rho_{\text{QES}} = 1 + \frac{\zeta_*^2(1 + \zeta_*^2)^{d-2}}{2(1 + \lambda_b)^2} \theta_B^{2(d-2)} + O(\theta_B^{2(d-1)}), \quad (4.65)$$

where the first terms on the RHSs give the location of the horizon. Granted ζ_* tends towards a finite value as $\theta_B \rightarrow 0$, the above formulas tell us that the QES tends towards the horizon on the brane. Applying eq. (4.64) to eq. (4.26) and noting from eq. (4.19) that

$$\frac{d\zeta}{d\xi} \sim \mp \frac{\zeta^{d+1}}{\zeta_*(1 + \zeta_*^2)^{\frac{d-2}{2}}} \quad \text{for } \zeta \gg 1, \quad (4.66)$$

we find that ζ_* at small θ_B is determined by the equation

$$\frac{\pi}{2} - \frac{\zeta_*(1 + \zeta_*^2)^{\frac{d-2}{2}}}{1 + \lambda_b} \theta_B^{d-2} + O(\theta_B^d) = \xi_{\text{QES}} = \xi_\Sigma + 2 \int_{\zeta_*}^{\infty} d\zeta \left| \frac{d\zeta}{d\xi} \right|^{-1} + O(\theta_B^d), \quad (4.67)$$

with $d\zeta/d\xi$ given by eq. (4.19). At leading order in θ_B , the second term on the LHS can be ignored and the above equation is just the statement that the RT surface should stretch from the belt boundary to approximately the bifurcation surface on the brane.

4.9.2 The Page curve in $d > 2$

As discussed in section 4.2, the benefit of our model is that calculating the entropy of (the complement of) the belt-shaped subregions centred on the conformal defects reduces to calculating areas in an effectively two-dimensional geometry. Further we produced explicit formulas for the areas of a number of special RT surfaces, which — as shown in section 4.7 — are sufficient to calculate the full time evolution of the RT surfaces and thus of the EE.

Figure 4.13a shows the Page curves for d -dimensional topological black holes, coupled to a bath on a hyperbolic background, for the cases $d = 3, 4, 5$. More precisely, we consider the entropy of the region defined by $\chi_\Sigma = 1$, which is given by

$$4G_{\text{bulk}} S(\tau) = \min \left(\left[A(\Sigma_{\mathbf{R}}) + \frac{2L\lambda_b}{(d-2)} A(\sigma_{\mathbf{R}}) \right]_{\text{isl.}}, [A(\Sigma_{\mathbf{R}})]_{\text{isl.}} \right). \quad (4.68)$$

Here $A(\Sigma)$ are the regulated areas of the RT surfaces, and the subscript indicates whether we consider the extremal surface in the island or no-island phase. Since eq. (4.68) is a cutoff dependent quantity, it is convenient to subtract off $[A(\Sigma_{\mathbf{R}})]_{\text{is}, \tau=0}$. That is, we subtract off the value of the entropy at $\tau = 0$, at which point the minimal RT surfaces in the no-island phase, to define

$$\Delta S(\tau) = S(\tau) - S(\tau = 0). \quad (4.69)$$

Even though the UV divergences have been removed, eq. (4.68) would still be infinite, as a result of the infinite extent of the entangling surface. Hence the plots in figure 4.13 show the change in the entropy density,

$$\Delta s = \frac{\Delta S}{\text{vol}_{H_{d-2}} L^{d-2}}, \quad (4.70)$$

with respect to the entropy at $\tau = 0$.¹⁶ The kinks in the plots of figure 4.13 indicate the time at which the island phase of the RT surface begins to dominate. The corresponding time is, of course, the natural analog of the Page time for eternal black holes coupled to a bath at finite temperature. The slope of the (linearly) rising portion of the Page curve has been determined in section 4.6 and is given by

$$4G_N \Delta s / \tau \sim 4 \frac{(d-2)^{(d-2)/2}}{(d-1)^{(d-1)/2}}. \quad (4.71)$$

Moreover, recall that τ is a dimensionless time such that the temperature of the hyperbolic black hole is $\frac{1}{2\pi}$ (cf. the discussion in section 4.3). The dimensionful time t is related to τ by

$$t = \tau R = \frac{\tau}{2\pi T}, \quad (4.72)$$

where R is the curvature scale for the spatial sections in the bath CFT, as defined in eq. (4.4), and the bath CFT is taken at temperature $T = \frac{1}{2\pi R}$.

The calculation of the RT surfaces is performed as follows: the area in the island phase is computed by substituting eqs. (4.25) and (4.27) into eq. (4.26) and numerically solving for ζ_{QES} . The result is then used together with eq. (4.25) to numerically integrate the area in eq. (4.20). There are three different regimes for the calculation in the *no-island phase*. At early times, $\tau_{\Sigma} \leq \chi_{\Sigma}$, the calculation of the entropy of the subregion with

¹⁶Note that we are actually plotting $4G_{\text{bulk}} \Delta s$, which is a dimensionless quantity. For the horizontal axes, also recall that the AdS-Rindler time τ is also dimensionless — see further comments below.

boundaries at $\pm\chi_\Sigma$ can be translated to the calculation of the entropy of a belt with boundary $\chi'_\Sigma = \pm \text{arccosh}\left(\frac{\cosh\chi_\Sigma}{\cosh\tau_\Sigma}\right)$ in the $\tau = 0$ time-slice, as explained in section 4.7. As also explained in the same section, we need to choose a different cutoff on r in this case. However, working in ζ, ξ coordinates, it turns out that the cutoff on ζ does not change. At intermediate times, $\tau_\Sigma \gtrsim \chi_\Sigma$, the entropy can be computed by calculating the area of an RT surface for a zero-belt-width entangling surface at a time given in eq. (4.59). Accidentally, the relation between r and ζ works out in such a way that the cutoff of r agrees with the cutoff on ζ in the previous calculation. As τ_Σ becomes larger, the numerics become less reliable. However, for moderately sized belt widths we are already well into the regime in which the area of the RT surface grows linearly in time. Therefore, we use a linear fit to extrapolate the last few numeric data points to late times, $\tau_\Sigma \gg \chi_\Sigma$. We verified that the resulting slope agrees with the analytic result given in eq. (4.71).

In figure 4.13b, we show how the Page curve and Page time change as we vary the brane angle. As we see, increasing θ_B decreases the Page time, or in other words decreases the number of microstates available to the black hole on the brane. This can also be understood from the CFT point of view where the defect entropy is given in terms of an RT surface in the island phase [91,92]. As the brane angle approaches zero, the Page time diverges. The reason is that in this limit the area of the island diverges. The absence of islands in this limit was already noted in [60]. The divergence as $\theta_B \rightarrow 0$ goes like θ_B^{2-d} , and in the small-angle approximation we find that

$$\tau_P \sim \frac{(d-1)^{\frac{d-1}{2}}}{(d-2)^{\frac{d}{2}}} \frac{1}{\theta_B^{d-2}}. \quad (4.73)$$

For example, the numerical coefficient which multiplies θ_B^{2-d} can be estimated from the above formula to be 1.30 for $d = 4$. A fit to the numerical data plotted in figure 4.13b agrees with this value.

Figure 4.13c shows the dependence of the Page curve and Page time on the DGP coupling. As we decrease the DGP coupling (*i.e.*, increase G_{eff}) the Page time goes to zero. The linearity can be easily explained by recalling that in the small θ_B regime we are interested in the island sits close to the horizon and thus has a fixed location for varying values of λ_b . The Page transition occurs whenever the area of the RT surface in the no-island phase exceeds the area of the RT surface in the island phase. Since the area in the no-island phase approximately grows linearly with time and the area in the island phase depends approximately linearly on λ_b , c.f. eq. (4.68), we obtain a linear relationship between the Page time τ_P and λ_b . Based on this argument, we can estimate the slope of

the graph to be

$$\tau_P/\lambda_b \sim \frac{(d-1)^{\frac{d-1}{2}}}{(d-2)^{\frac{d}{2}}} \frac{1}{\theta_B^{d-2}}, \quad (4.74)$$

which for the parameters in 4.13c (*i.e.*, $\theta_B = 0.1$ and $d = 4$) evaluates to $\tau_P \sim 130 \lambda_b$ and agrees with the fitted value of the slope.

The Page curve and Page time only depends very weakly on the belt size. In fact, the only significant effect can be seen at very early times of the evaporation. Figure 4.14a shows that for wide belts, the entanglement between the belts and baths starts growing convexly (*i.e.*, $\partial^2 \Delta s / \partial \tau_\Sigma^2 > 0$), then enters a period of concave growth (*i.e.*, $\partial^2 \Delta s / \partial \tau_\Sigma^2 < 0$) before entering the linear regime.

Generally, we can separate the time-dependence of the Page curve into four different regimes. At times of the order of the thermal scale β (~ 0.16 in figure 4.14a) the entanglement growth increases until it enters a phase of fast growth between $\tau_\Sigma \sim O(\beta)$ and $\tau_\Sigma \sim O(\chi_\Sigma)$. This fast growth depends on the belt size. At time $\tau_\Sigma \sim O(\chi_\Sigma)$ a universal, linear behaviour takes over, which is independent of the belt width. The entanglement keeps growing until at the Page time τ_P it saturates and stays constant.

In the following we will explain the region of fast growth and its transition into the region of universal linear growth. To understand the behaviour of the Page curve, first consider a few characteristics of our belt geometries. As can be seen from the metric in eq. (4.8), points on any of our entangling surfaces are a fixed distance χ_Σ from the surface at $\chi = 0$, where the defect is located, *i.e.*, where the bath is coupled to the black hole.¹⁷ However, the extrinsic curvature of the entangling surfaces which we consider depends on this distance. Similarly, the entangling surfaces with larger χ_Σ have a larger regulated volume.

In [166], it was proposed that the growth of entropy $S[\Sigma]$ for an arbitrary entangling surface Σ is bound by

$$\frac{1}{R} \frac{dS[\Sigma]}{d\tau} = \frac{dS[\Sigma]}{dt} \leq s_{\text{th}} v_{\text{ent}} A(\Sigma), \quad (4.75)$$

where $A(\Sigma)$ is the area of the entangling surface Σ , as measured by the boundary metric in eq. (4.8). The thermal entropy density s_{th} and the entanglement velocity v_{ent} are region independent constants. The entropy density is given by the black hole entropy (*i.e.*, $\frac{1}{4G_{\text{bulk}}}$

¹⁷The proper distance would be $R\chi_\Sigma$ in the boundary metric (4.4).

times horizon area) divided by the CFT volume of the spatial slices (again, measured by the metric (4.8)):

$$s_{\text{th}} = \frac{1}{4G_{\text{bulk}}} \frac{L^{d-1}}{R^{d-1}}. \quad (4.76)$$

In [166] which primarily considers flat space, v_{ent} is defined such that eq. (4.75) is saturated at times just above the thermal scale for sufficiently straight entangling surfaces – this definition is well-defined in the sense that v_{ent} turns out to be independent of the shape of the entangling surface, provided it is sufficiently straight [167, 168]. In hyperbolic space, v_{ent} can be similarly defined by demanding that the straight surface $\chi = 0$ saturates eq. (4.75) — we shall justify this choice further below — specifically,

$$v_{\text{ent}} = \frac{(d-2)^{\frac{d-2}{2}}}{(d-1)^{\frac{d-1}{2}}}, \quad (4.77)$$

obtained by comparison of eq. (4.75) with the zero-width belt result in eq. (4.45).

It is clear that (4.75) cannot be tight at late times for belts of finite width. The reason is that the area factor on the right hand side $A[\chi > 0]$ is exponentially large compared to $A[\chi = 0]$, while, as can be seen from figure 4.14a, all belts share the same rate of entanglement growth at late times. To more tightly bound the late time behaviour of finite width belts, we will therefore need to combine eq. (4.75) with the monotonicity of mutual information. It will turn out that the optimal bound obtained in this way for finite-width belts uses eq. (4.75), but always evaluated for the $\chi = 0$ surface Σ at late times; thus we will find that the $\chi = 0$ surface acts as a bottleneck for entanglement growth even for finite width belts.

To see why the surface at $\chi_{\Sigma} = 0$ acts as a bottleneck, let us formulate the more refined combined bound now, following closely [166]. To this end, it will be less helpful to consider the EE of the bath intervals $\mathbf{R}$; instead we will consider their complement $\bar{\mathbf{R}}$, *i.e.*, belts surrounding the defects, whose entropy is the same as $\mathbf{R}$ since the state of both Rindler patches is pure. Considering $\bar{\mathbf{R}}$ instead of $\mathbf{R}$ is equivalent to looking at the Page curve of the black hole instead of that of the radiation. It is useful to rewrite the entropy displayed in the Page curve as

$$\Delta S = I[\bar{\mathbf{R}}_L : \bar{\mathbf{R}}_R](0) - I[\bar{\mathbf{R}}_L : \bar{\mathbf{R}}_R](\tau), \quad (4.78)$$

where $I[\bar{\mathbf{R}}_L : \bar{\mathbf{R}}_R](\tau) = S(\bar{\mathbf{R}}_L) + S(\bar{\mathbf{R}}_R) - S(\bar{\mathbf{R}}_L \cup \bar{\mathbf{R}}_R)$ is the mutual information between the regions $\mathbf{R}$ in the left (L) and right (R) CFT at time τ .

Similar to [166], we now assume that information is only transported with the butterfly velocity v_{but} or less.¹⁸ For the hyperbolic geometries considered here and the temperature $T = \frac{1}{2\pi R}$, this velocity is given by [170, 171]

$$v_{\text{but}} = \frac{1}{d-1}. \quad (4.79)$$

This implies that a belt region $\bar{\mathbf{R}}'$ at time τ' can be considered a subsystem of the original belt $\bar{\mathbf{R}}$ at τ if $\frac{\chi_{\Sigma} - \chi'_{\Sigma}}{v_{\text{but}}} \geq |\tau - \tau'|$. We can then use monotonicity of mutual information

$$I[\bar{\mathbf{R}}_L : \bar{\mathbf{R}}_R](\tau) \geq I[\bar{\mathbf{R}}'_L : \bar{\mathbf{R}}'_R](\tau') = S[\bar{\mathbf{R}}'_L](\tau') + S[\bar{\mathbf{R}}'_R](\tau') - S[\bar{\mathbf{R}}'_L \cup \bar{\mathbf{R}}'_R](\tau'). \quad (4.80)$$

In our setup, we have that the one-sided entropies are time-independent, $S[\bar{\mathbf{R}}'_{R/L}](\tau') = S[\bar{\mathbf{R}}'_{R/L}](0)$. Using eq. (4.75) we can then bound $S[\bar{\mathbf{R}}'_L \cup \bar{\mathbf{R}}'_R](\tau')$ from above

$$S[\bar{\mathbf{R}}'_L \cup \bar{\mathbf{R}}'_R](\tau') \leq R s_{\text{th}} v_{\text{ent}} A(\partial \bar{\mathbf{R}}') \tau' + S[\bar{\mathbf{R}}'_L \cup \bar{\mathbf{R}}'_R](0). \quad (4.81)$$

Collecting everything, we find a bound on the Page curve of the black hole,

$$\Delta S[\bar{\mathbf{R}}_L \cup \bar{\mathbf{R}}_R] \leq R s_{\text{th}} v_{\text{ent}} A(\partial \bar{\mathbf{R}}') \tau' + I[\bar{\mathbf{R}}_L : \bar{\mathbf{R}}_R](0) - I[\bar{\mathbf{R}}'_L : \bar{\mathbf{R}}'_R](0). \quad (4.82)$$

To find a tightest bound this has to be minimized over all choices of χ'_{Σ} , see below. For any fixed χ'_{Σ} it is sufficient to focus on the case where $\tau' < \tau$, which will always give the smaller bound. The mutual information terms appearing on the right hand side are evaluated on the initial time slice and can be obtained numerically by using the results of section 4.5.

From eq. (4.82), it is now easy to see why the entanglement growth becomes universal at late times. Note that eq. (4.82) is in fact a family of inequalities, parametrized by a choice of regions $\bar{\mathbf{R}}'$. The time τ' is chosen such that $\bar{\mathbf{R}}'$ at τ' is just barely a subsystem of $\bar{\mathbf{R}}$ at τ , in the sense described below eq. (4.79). For times before τ' we assume that the mutual information of subregions $\bar{\mathbf{R}}'$ is allowed to decrease as fast as possible, while still compatible with eq. (4.75). Since the regions $\bar{\mathbf{R}}'$ at time τ' are subregions of $\bar{\mathbf{R}}$ at time τ , their mutual information bounds the mutual information of regions $\bar{\mathbf{R}}$. We can find a tight bound on the Page curve by minimizing over all choices of $\bar{\mathbf{R}}'$, or in other words, by minimizing over all χ'_{Σ} with $\tau' = \tau - \frac{\chi_{\Sigma} - \chi'_{\Sigma}}{v_{\text{but}}}$. It turns out that, for sufficiently large τ , the tightest bound is obtained for $\chi'_{\Sigma} = 0$, yielding the prescription stated below eq. (4.77). We thus see from the first term on the right hand side of eq. (4.82) that this surface acts

¹⁸The butterfly velocity is defined as the spread of the region in which the commutator of an operator $O_1(t)$ with $O_2(t)$ is bigger than 1 [169].

as a bottle neck for information transfer and thus controls the late time growth of entropy. Matching this behaviour to the late time rate of growth of the exact Page curve provides further justification, *a posteriori*, for the choice of the entanglement velocity stated in eq. (4.77).

The bounds found in this way are presented in figure 4.14b. We see that a fast growth at early times is allowed by the bounds, before the linear growth phase is entered. Further, as can be seen from the figure, these bounds are fairly loose. It would be interesting to understand how to make them tighter. Note that the blue curve in figure 4.14b behaves qualitatively different than the other curves. The reason is that the early convex onset of the curve is controlled by the thermal scale and thus lasts for roughly $\Delta\tau \sim O(\beta)$, independent of the belt width. The rescaling in figure 4.14b magnifies the early time behaviour of belts with $\chi_\Sigma < 1$ while it reduces the early time behaviour of belts of width $\chi_\Sigma > 1$. Thus, while all other curves show the linear entanglement spreading for time scales $\tau \sim O(\chi_\Sigma) > O(\beta)$, the behaviour of the blue curve is dominated by entanglement spreading through thermalization, since the belt width is of order of the thermalization scale. The quadratic¹⁹ growth at times below the thermal scale is reminiscent of the ‘pre-local-equilibration growth’ described in [167, 168].

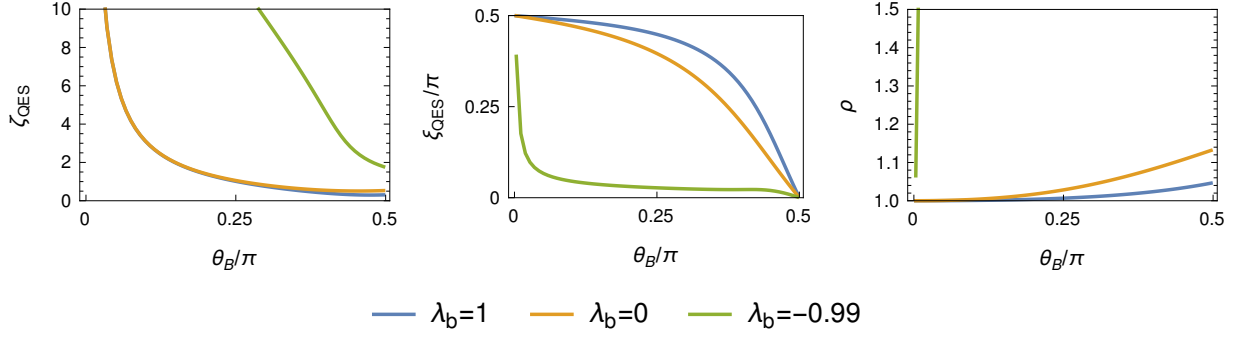
Let us end with a few observations regarding the structure of entanglement spreading in our system. First, we note that the entanglement velocity (4.77) for Rindler spacetime with hyperbolic spatial slices differs from the analogous velocity $\sqrt{d}(d-2)^{\frac{1}{2}-\frac{1}{d}}/[2(d-1)]^{1-\frac{1}{d}}$ in flat space [161] dual to AdS planar black holes. Furthermore, for $d > 3$, the entangling velocity for a CFT on hyperbolic space exceeds the butterfly velocity, eq. (4.79). Typically, whenever $v_{\text{ent}} > v_{\text{but}}$, one might worry about contradictions to entanglement monotonicity laws [166, 172], which apply above the thermal scale. However, no immediate contradictions appear in the present case, as we now explain.

For concreteness, let us interpret eq. (4.45) as describing the entanglement growth in hyperbolic space without defects, specifically, computing the entropy for a region consisting of half-spaces $\chi > 0$ on either side of the TFD.²⁰ This growth saturates eq. (4.75) with $v_{\text{ent}} > v_{\text{but}}$ in $d > 3$ so one might worry that entanglement is spreading faster than the speed

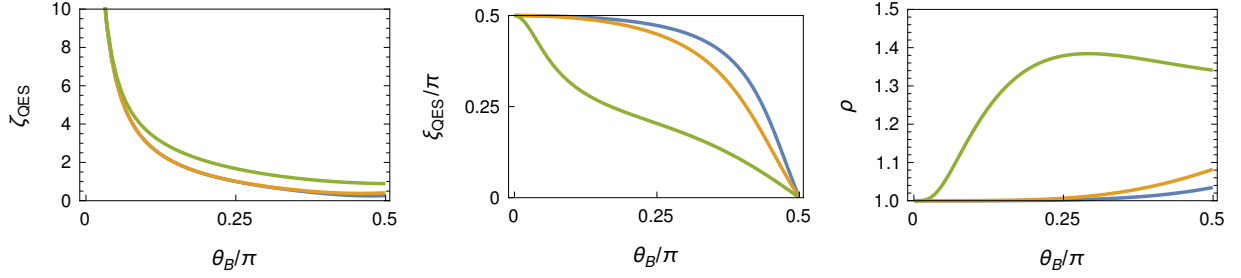
¹⁹Note that time-reflection symmetry demands that the Page curve have an early time expansion containing only even powers of τ . For the zero-width, it is easily verified, at least numerically, from eq. (4.42) that $\sqrt{1-r_*} \sim \tau$ so that the growth is indeed quadratic by eq. (4.43). For finite-width belts, plugging eqs. (4.56) and (4.57) into eq. (4.13) shows that early time evolution is equivalent to holding the cutoff at fixed ζ and shifting the ξ of the entangling surface by $\sim \tau^2$, again leading to quadratic entanglement growth.

²⁰To be precise, we should multiply eq. (4.45) by $\frac{1}{2} \cdot \frac{1}{4G_{\text{bulk}}}$ with the factor of 1/2 due to our focus on just two copies of the entangling surface $\chi = 0$ — one on either side of the TFD.

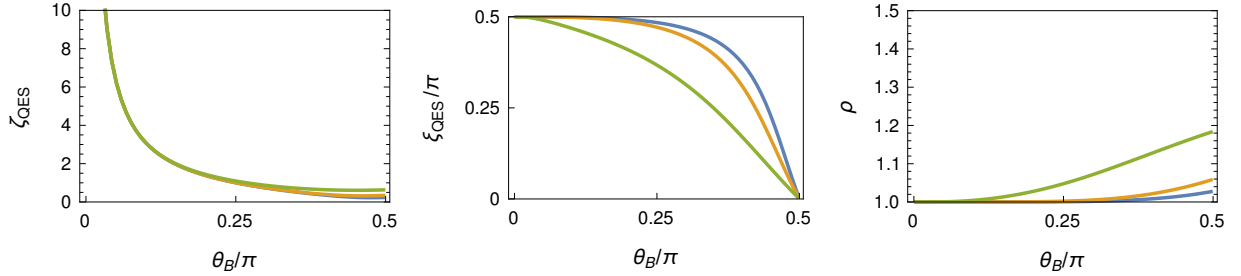
v_{but} permitted by operator commutator growth [169]. Specifically, by applying an analysis similar to the one reviewed around eq. (4.79) to thermal relative entropies, [166, 172] argue that, for regions and times above the thermal scale, entanglement growth must be bounded by the thermal entropy density s_{th} times the volume between the entangling surface and a tsunami wavefront propagating with speed v_{but} away from the entangling surface (in either direction). Said differently, the rate dS/dt of entanglement growth is bounded by $s_{\text{th}}v_{\text{but}}$ times the area of the tsunami wavefront — this is essentially eq. (4.75) with v_{but} replacing v_{ent} and the tsunami wavefront replacing the entangling surface. In flat space, the tsunami wavefront can be typically chosen to propagate in a direction away from the entangling surface such that it shrinks or does not grow in time (*e.g.*, propagating inward from a spherical entangling surface). Thus, for the flat space equivalent of eq. (4.75) to be saturated, one must require $v_{\text{ent}} < v_{\text{but}}$. In hyperbolic space however, it is possible for the tsunami wavefront to grow in both directions away from the entangling surface. Indeed, this is precisely what happens for the hyperbolic half-space which has an entangling surface $\chi = 0$ of minimal area; within a few thermal times, the tsunami wavefront propagating in either direction grows to an area exponentially large compared to the entangling surface. We thus see that, though the hyperbolic half-space saturates eq. (4.75) with $v_{\text{ent}} > v_{\text{but}}$, this does not contradict the bound on entanglement spreading due to the butterfly velocity.



(a) The dependence of RT surface parameters on the brane angle θ_B for $d=3$.

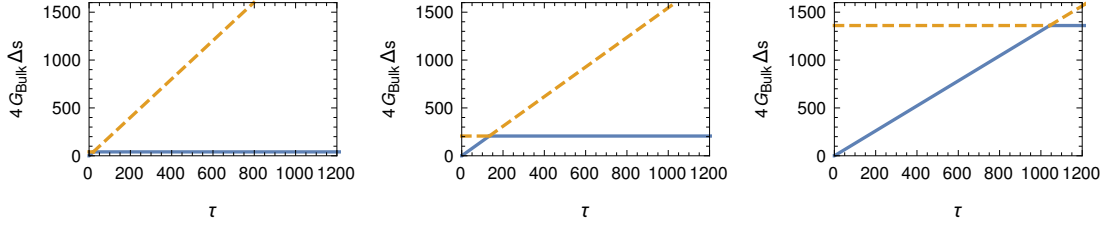


(b) The dependence of RT surface parameters on the brane angle θ_B for $d=4$.

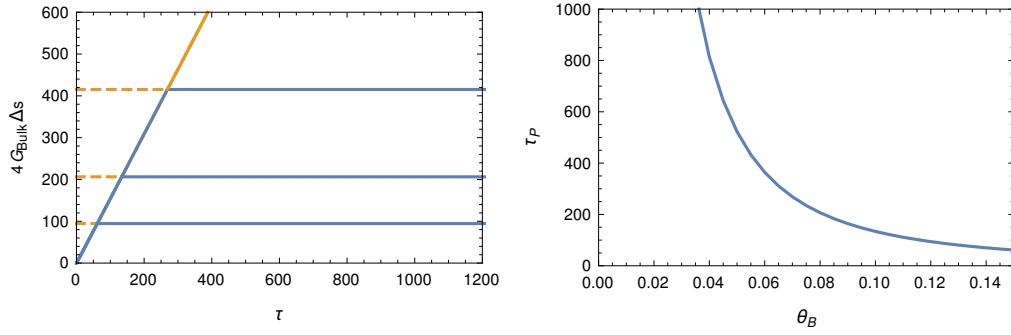


(c) The dependence of RT surface parameters on the brane angle θ_B for $d=5$.

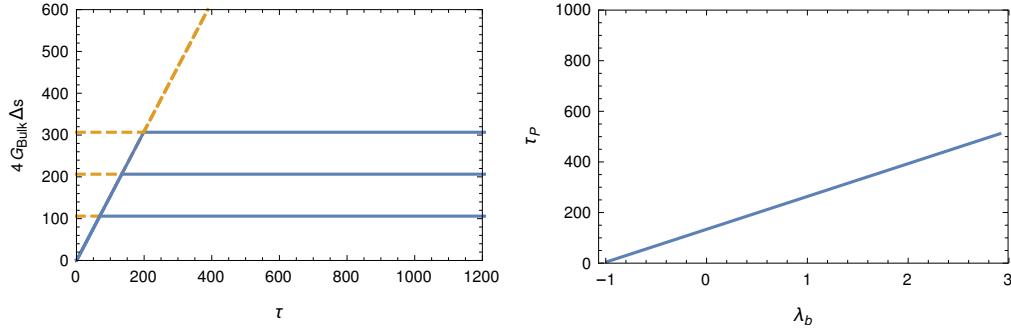
Figure 4.12: The dependence of the RT surface and the quantum extremal surface on the brane angle θ_B for $d=3, 4$ and 5 . The location of the entangling surface is chosen to be $\chi_\Sigma = 1$.



(a) The Page curve for dimensions $d = 3, 4, 5$ (left to right). The entangling surface is located at $\chi_\Sigma = 1$ and the DGP coupling is set to zero. The brane angle is chosen as $\theta_B = 0.1$.

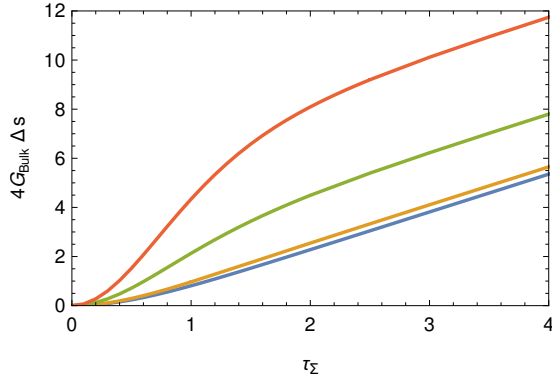


(b) Left: The Page curve for selected brane angles $\theta_B = 0.07, 0.10, 0.15$ (top to bottom). Right: The Page time τ_P as a function of the brane angle θ_B . The constant parameters are set to $\lambda_b = 0$, $\chi_\Sigma = 1$, and $d = 4$.

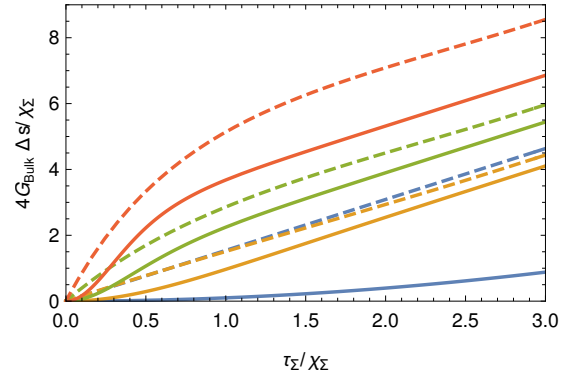


(c) Left: The Page curve for selected values of the DGP coupling $\lambda_b = 0.5, 0, -0.5$ (top to bottom). Right: The Page time τ_P as a function of the DGP coupling λ_b . The constant parameters are set to $\theta_B = 0.1$, $\chi_\Sigma = 1$, and $d = 4$.

Figure 4.13: The Page curve in various dimensions. The solid blue line indicates the physical Page curve. The dashed orange lines correspond to entropies calculated by non-minimal extremal surfaces. At early times, the RT surface in the no-island phase is the minimal surface. After some time, the minimal surface transitions to the RT surface in the island phase.



(a) This figure shows the onset of the Page curve for different values of the location of the entangling surface $\chi_\Sigma = 0.1, 1, 2, 2.5$ (bottom to top) in four dimensions.



(b) The same plot, but with axes rescaled by χ_Σ . The solid lines are numerical results, while the dashed lines are the bounds explained in the main text.

Figure 4.14: The initial behaviour of the Page curve in four dimensions (left) and a rescaled version of the same plot with bounds (dashed) on the onset (right).

Chapter 5

Extremal horizon in equilibrium with $T = 0$ bath

Here we turn our attention to extremal black holes. In particular, we consider the same bulk geometry described in chapter 2, *i.e.*, a backreacting codimension-one brane extending across the spacetime which locally has the geometry of AdS_{d+1} . However, we replace the AdS-Rindler coordinates introduced in eq. (4.7) with Poincaré coordinates,

$$ds^2 = \frac{L^2}{z^2} (dz^2 - dt^2 + dx_1^2 + \cdots + dx_{d-1}^2) . \quad (5.1)$$

Of course, the coordinate singularity at $z \rightarrow \infty$ corresponds to an extremal $T = 0$ horizon. Figure 5.1 illustrates the Poincaré patch in our bulk geometry.

For the most part, we will be interested in limit of large tension (*i.e.*, $\ell_B \gg L$), for which the brane theory can be described as Einstein gravity coupled to two copies of the boundary CFT. As we describe in a moment, the brane geometry naturally inherits a Poincaré metric from the bulk geometry. Hence the brane supports an extremal black hole which is equilibrium with the $T = 0$ bath CFT on the asymptotic AdS boundary. We note that with Poincaré coordinates, we are examining the system in a new conformal frame where the bath CFT is living on flat d -dimensional Minkowski space,

$$ds_{\text{CFT}}^2 = -dt^2 + dx_1^2 + \cdots + dx_{d-1}^2 . \quad (5.2)$$

This brane perspective is illustrated in figure 5.2a.

Of course, we may also have the boundary perspective where the d -dimensional CFT in Minkowski space is coupled to a codimension-one conformal defect. For simplicity, we

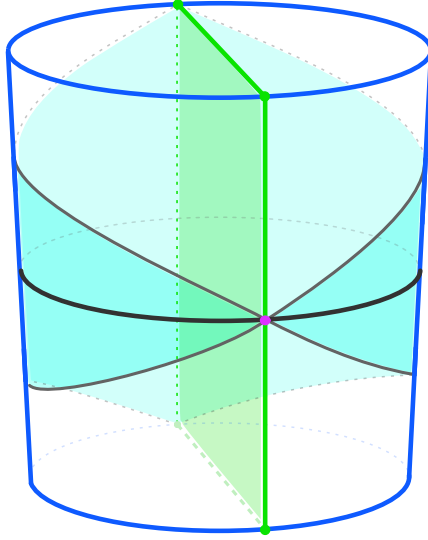


Figure 5.1: The Poincaré patch models a zero temperature extremal black hole. The brane intersects the CFT Poincaré patch at the origin and infinity.

insert the latter at $x_1 = 0$ for the metric in eq. (5.2) and so the induced geometry on the defect is also flat, *i.e.*, $(d-1)$ -dimensional Minkowski space. The Penrose diagram for this perspective is shown in figure 5.2b. Note that in contrast to the finite temperature TFD state (entangling two copies of the bath CFT) in section 4.2, here for the $T = 0$ scenario, we only have a single copy of the bath CFT, *e.g.*, compare the above to figures 4.3 and 4.4. Of course, at $T = 0$, we are simply studying the vacuum state of the defect CFT in flat space.¹

We may recall from [50] that for the extremal case in $d = 2$, one always finds islands for the analogous belt regions. This result is a consequence of two features that hold for $d = 2$: firstly, there always exists a bulk RT surface intersecting the brane to produce an island; secondly, the alternative no-island RT candidate surface has an additional IR divergence² and this surface is therefore subdominant. However, neither of these statements hold in $d \geq 3$. Indeed, we will find in higher dimensions that QEIs do not appear in the large tension limit. Nonetheless, no information paradox arises since extremal black holes do not radiate, *i.e.*, the black hole and the bath are not exchanging radiation. This contrasts

¹Of course, this is a pure state, as is manifest in bulk since the Poincaré time slices constitute complete Cauchy slices.

²Coming from integrating the length of the surface down to the extremal horizon.

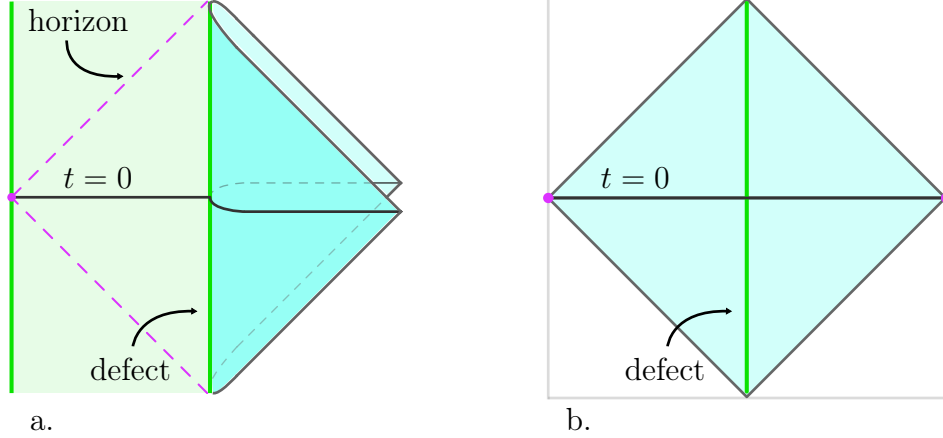


Figure 5.2: The brane and boundary perspectives of the extremal black hole setup. (As visible in fig. 5.1 but suppressed here, the brane also intersects the non-gravitational boundary of the AdS_{d+1} spacetime at the left edge of panel (a) here.)

with the non-extremal case in section 4.2, where the information paradox for the eternal black hole in the effective d -dimensional gravity theory arises because of the continuous exchange of quanta between the black hole and the bath. Of course, the paradox is avoided by the appearance of QEIs.

The remainder of this section is organized as follows. We shall begin by first explicitly constructing the bulk and brane metrics to be used in the extremal case and by introducing the EE calculation which we wish to consider. Then, in subsections 5.1 and 5.2, we carry out this calculation using RT surfaces corresponding to island and no-island phases, respectively. Finally, we collect these results in subsection 5.3 to determine when each phase dominates.

The Poincaré coordinates (5.1) cover a wedge of the AdS_{d+1} vacuum geometry. However, in the present geometry with a backreacting brane, a portion of two such wedges would appear on either side of the brane — see figure 5.1. If we consider the coordinate transformation

$$z = y \sin \theta, \quad x_1 = y \cos \theta, \quad (5.3)$$

the metric (5.1) is transformed to the form given in eq. (2.7), where the AdS_d slices each inherit a Poincaré metric. As described in chapter 2, the brane spans one such slice at a fixed $\theta = \theta_{\text{B}}$ determined by the brane tension T_o according to eq. (2.12). The induced

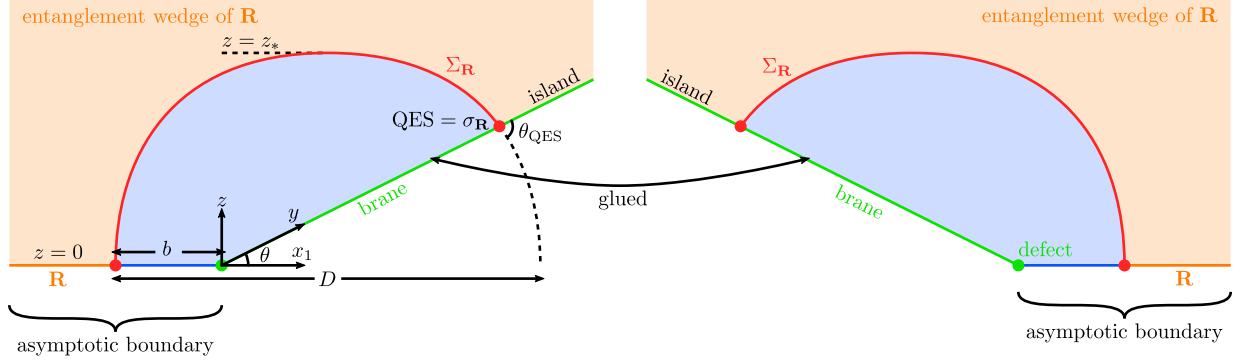


Figure 5.3: The bulk dual to a d -dimensional Minkowski CFT with a defect (green dot) along a line $x_1 = 0$. The CFT lives on the asymptotic boundary of a Poincaré AdS_{d+1} spacetime with a brane (green line) running through it. We consider the EE of the complement $\mathbf{R} = (-\infty, -b] \cup [b, \infty)$ of a belt geometry in the CFT. As considered in section 5.1, one candidate RT surface $\Sigma_{\mathbf{R}}$, shown in red, intersects the brane at a QES $\sigma_{\mathbf{R}}$, forming an island on the brane belonging to the entanglement wedge of $\mathbf{R}$. Various quantities defined in section 5.1 are marked in this figure.

metric on the brane then becomes

$$ds_{\text{AdS}_d}^2 = \frac{L^2}{y^2 \sin^2 \theta_B} (dy^2 - dt^2 + dx_2^2 + \cdots + dx_{d-1}^2) , \quad (5.4)$$

and we may then read off the curvature scale of the brane as $\ell_B = L/\sin \theta_B$, as expected from eq. (2.18). Here, y is interpreted as the radial Poincaré coordinate running along the brane, and the Poincaré horizon on the brane, located at $y \rightarrow \infty$, is inherited from the bulk. As usual, we wish to work in the regime $L^2/\ell_B^2 \ll 1$, or alternatively $\theta_B \ll 1$.

Following the brane perspective described above (and in chapter 2 and section 2.5), eq. (5.4) is interpreted as an extremal black hole solution of the gravity theory induced brane at $\theta = \theta_B$ and the CFT of the flat asymptotic boundary at $z = 0$ becomes the zero temperature bath. This then provides a direct extension of the extremal scenario in [50] to d dimensions. The question which interests us here is then whether the entanglement wedge of certain subregions in the bath includes islands residing on the brane.

Specifically, we consider the EE calculation for a boundary region $\mathbf{R}$ that is the complement of a “belt” geometry centred on the defect at $x_1 = 0$, *i.e.*, the boundary subregion $\mathbf{R} = (-\infty, -b] \cup [b, \infty)$. According to the RT formula we should consider codimension-two surfaces $\mathbf{V}$ sharing the same boundary $\partial \mathbf{V} = \partial \mathbf{R} \equiv \Sigma_{\text{CFT}}$. To determine RT surface candidates among these surfaces, we must search for surfaces that extremize their area. As

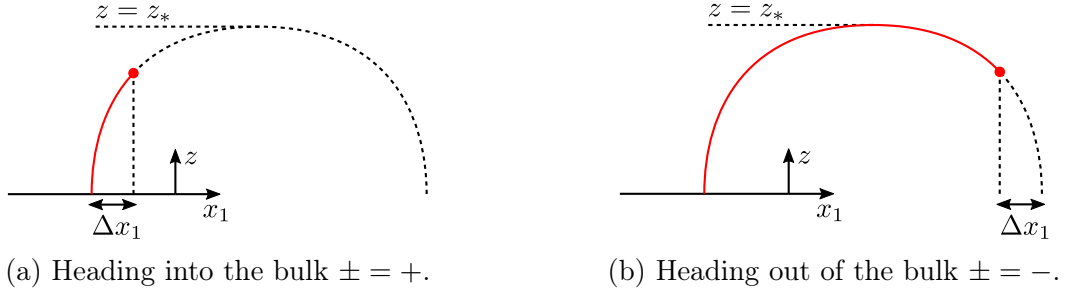


Figure 5.4: Definitions for the choice of $\pm$ in eq. (5.6) and for the corresponding Δx_1 (> 0) from eq. (5.7) on the two branches of the RT surface.

we discussed in the introduction, there are generally two sets of surfaces that achieve this extremization; the RT prescription then instructs us to choose the one with the smallest area. The first class of surfaces are those that intersect the brane, forming a QEI on the brane, which belongs to the entanglement wedge of $\mathbf{R}$ — see figure 5.3. We will say that this RT surface is in the *island phase*. The second set of surfaces fall trivially into the bulk and do not produce islands on the brane, *i.e.*, these surfaces are in the *no-island phase*.

5.1 Island phase

As a starting point, let us review the calculation for RT surfaces of belt geometries in pure AdS [13]. That is, we are considering the complement of $\mathbf{R}$, but the RT calculations for this region and for its complement, $\overline{\mathbf{R}} = [-b, b]$, are equivalent. Integrating out the $x_2, \dots, x_{d-1}$ directions in which the brane is constant, the area functional of a codimension-2 surface $\mathbf{V}$ becomes

$$A(\mathbf{V}) = L^{d-1} \text{vol}_{d-2}^\perp \int_{\mathbf{V}} dx_1 \frac{\sqrt{1 + \left(\frac{dz}{dx_1}\right)^2}}{z^{d-1}}, \quad (5.5)$$

where $\text{vol}_{d-2}^\perp$ is the volume of transverse directions $\{x_2, \dots, x_{d-1}\}$.³

The RT surface $\Sigma_{\mathbf{R}}$ is obtained by extremizing the area functional (5.5) with respect to the profile $z(x_1)$. This functional, viewed as a Lagrangian, contains no explicit dependence

³Note that in contrast to $\text{vol}_{H_{d-2}}$ introduced in section 4.2, $\text{vol}_{d-2}^\perp$ has the dimensions of length^{d-2} and so is essentially given by ℓ_{IR}^{d-2} where ℓ_{IR} is an IR cutoff in the $x_2, \dots, x_{d-1}$ directions.

on x_1 and hence the corresponding Hamiltonian is a constant along $\Sigma_{\mathbf{R}}$, allowing us to deduce

$$\frac{dz}{dx_1} = \pm \frac{\sqrt{z_*^{2(d-1)} - z^{2(d-1)}}}{z^{d-1}} \quad (5.6)$$

for some constant z_* . Further, the sign $\pm$ above is determined by whether we are on the portion of the RT surface heading into the bulk (+) or heading out of the bulk (−) with increasing x_1 — see figures 5.3 and 5.4.⁴ From eq. (5.6), we see that $dz/dx_1 = 0$ at $z = z_*$ and therefore z_* is the maximal z -value attained by $\Sigma_{\mathbf{R}}$. We can integrate eq. (5.6) to obtain the trajectory of the RT surface:

$$\Delta x_1 = \frac{z^d}{d z_*^{d-1}} {}_2F_1 \left[\frac{1}{2}, \frac{d}{2(d-1)}; \frac{d}{2(d-1)} + 1; \left(\frac{z}{z_*} \right)^{2(d-1)} \right] \quad (5.7)$$

Here $\Delta x_1 > 0$ is the absolute x_1 -separation between a point on the RT trajectory and the initial (final) endpoint on the asymptotic boundary, on the portion of the RT surface heading into (out of) the bulk — see figure 5.4. If we evaluate this expression at $z = z_*$, we obtain half of the width of the boundary strip (in the x_1 direction) defined by the RT surface. Denoting this width as D , which we emphasize is in the empty AdS vacuum (see figure 5.3), we have

$$\frac{D}{2} = \frac{\sqrt{\pi} \Gamma \left[\frac{d}{2(d-1)} \right]}{\Gamma \left[\frac{1}{2(d-1)} \right]} z_*. \quad (5.8)$$

Now returning to the geometry with the backreacting brane, each half of the RT surface $\Sigma_{\mathbf{R}}$ on either side of the brane will follow the trajectory given in eq. (5.7) for pure AdS prior to meeting the brane. We have placed the defect at $x_1 = 0$ and the RT surface begins on the asymptotic boundary at $x_1 = -b$. Further, if we were to extend the RT surface past the brane, it would hit the asymptotic boundary again at $x_1 = -b + D$. In terms of eq. (5.7), x_1 along the trajectory is then given by

$$x_1 = -b + \begin{cases} \Delta x_1 & \text{when heading into bulk (towards } z = z_*) \\ D - \Delta x_1 & \text{when heading out of bulk (away from } z = z_*) \end{cases}. \quad (5.9)$$

⁴As noted previously, if we restrict our attention to positive tension T_o , we will have $0 < \theta_B < \pi/2$. In this case, the RT surface must be increasing in x_1 as one heads away from the boundary $(z, x_1) = (0, -b)$, in order for the RT surface to meet the brane.

In general, as illustrated in figure 5.3, $\frac{D}{2} \neq b$, rather, the relation between D (or z_*) and b must be determined by demanding that the choice of the intersection $\sigma_{\mathbf{R}}$ of the RT surface with the brane should extremize the RT surface's area (plus the area of the QES, when brane action includes an extra DGP term). As described in section 3.2, this extremization leads to a boundary condition restricting the angle at which the RT surface meets the brane. Again, we may reduce this to a two-dimensional problem where we view the RT surface as a geodesic in an effective two-dimensional geometry

$$ds_{2D}^2 = L^{2(d-1)} (\text{vol}_{d-2}^\perp)^2 \frac{dz^2 + dx_1^2}{z^{2(d-1)}}, \quad (5.10)$$

and the area becomes the length of the geodesic in this geometry.

As before, we may use eq. (4.22) to determine the variation of the RT surface area under perturbations of $\sigma_{\mathbf{R}}$, the QES on the brane. Here, h_{ij} is given by eq. (5.10), the deviation vector X^j is chosen to be ∂_y , and the tangent T^i determined from eq. (5.6), with both X^j and T^i normalized with respect to h_{ij} . Hence, upon perturbing the intersection of the RT surface with the brane, the RT area varies as

$$\frac{\partial A(\Sigma_{\mathbf{R}})}{\partial y_{\text{QES}}} = \frac{2L^{d-1} \text{vol}_{d-2}^\perp}{z_{\text{QES}}^{d-1}} \cos \theta_{\text{QES}} = 2L^{d-1} \text{vol}_{d-2}^\perp \left(\frac{\cos \theta_{\text{B}}}{z_*^{d-1}} \pm \sqrt{\frac{1}{z_{\text{QES}}^{2(d-1)}} - \frac{1}{z_*^{2(d-1)}}} \sin \theta_{\text{B}} \right), \quad (5.11)$$

where θ_{QES} is the angle between the RT surface and the brane, y_{QES} is the y coordinate of $\sigma_{\mathbf{R}}$ — see figure 5.3 — and the $\pm$ sign is the same one as introduced in eq. (5.6) and illustrated in figure 5.4. An extra factor of 2 is included to account for the two components of the RT surface on either side of the brane. From eq. (5.4), we read off the area of $\sigma_{\mathbf{R}}$:

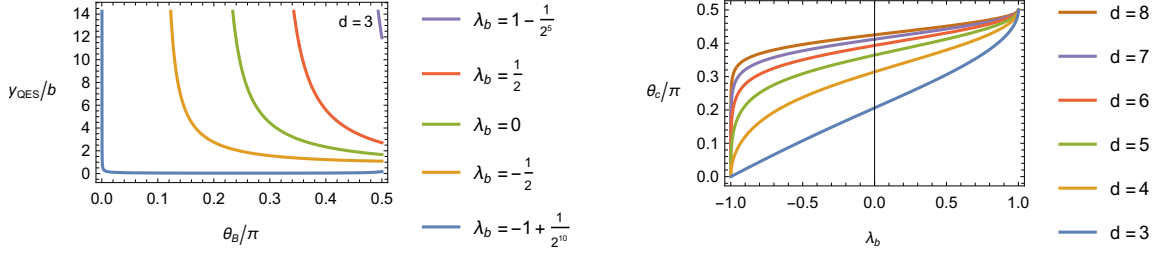
$$A(\sigma_{\mathbf{R}}) = \text{vol}_{d-2}^\perp \left(\frac{L}{y_{\text{QES}} \sin \theta_{\text{B}}} \right)^{d-2}, \quad \frac{\partial A(\sigma_{\mathbf{R}})}{\partial y_{\text{QES}}} = - \frac{(d-2) \text{vol}_{d-2}^\perp L^{d-2} \sin \theta_{\text{B}}}{z_{\text{QES}}^{d-1}}. \quad (5.12)$$

The extremality condition

$$0 = \frac{\partial}{\partial y_{\text{QES}}} \left(\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} + \frac{A(\sigma_{\mathbf{R}})}{4G_{\text{brane}}} \right) \quad (5.13)$$

is satisfied if

$$\cos \theta_{\text{QES}} = \lambda_{\text{b}} \sin \theta_{\text{B}} \quad \Longleftrightarrow \quad z_{\text{QES}} = z_* \left[\sin \theta_{\text{B}} \left(\lambda_{\text{b}} \cos \theta_{\text{B}} + \sqrt{1 - \lambda_{\text{b}}^2 \sin^2 \theta_{\text{B}}} \right) \right]^{\frac{1}{d-1}}, \quad (5.14)$$



(a) Position of $\sigma_{\mathbf{R}}$ as a function of brane angle (b) Critical brane angle as a function of the ratio of G_{bulk} to G_{brane} .

Figure 5.5: Plots of the position of $\sigma_{\mathbf{R}}$, the intersection of the RT surface with the brane, and the critical brane angle at which this surface runs off to $y_{\text{QES}} \rightarrow +\infty$.

where λ_b is defined in eq. (2.54). The relationship between z_* and b may then be determined by substituting $(x_1, z) = (z_{\text{QES}} \cot \theta_B, z_{\text{QES}})$ into eq. (5.9), and using eqs. (5.8), (5.7) and (5.14) to find

$$b = \pm \Delta x_1 + \frac{1 \mp 1}{2} D - z_{\text{QES}} \cot \theta_B = \mathcal{F}(d, \lambda_b, \theta_B) z_* \quad (5.15)$$

$$\begin{aligned} \mathcal{F}(d, \lambda_b, \theta_B) \equiv & \pm \frac{z_{\text{QES}}^d}{d z_*^d} {}_2F_1 \left[\frac{1}{2}, \frac{d}{2(d-1)}; \frac{d}{2(d-1)} + 1; \left(\frac{z_{\text{QES}}}{z_*} \right)^{2(d-1)} \right] \\ & + (1 \mp 1) \frac{\sqrt{\pi} \Gamma \left[\frac{d}{2(d-1)} \right]}{\Gamma \left[\frac{1}{2(d-1)} \right]} - \frac{z_{\text{QES}}}{z_*} \cot \theta_B \end{aligned} \quad (5.16)$$

where the top (bottom) signs chosen above if the RT surface intersects the brane to the left (right) of the extremal point $z = z_*$. We have noted in the second equality of eq. (5.15) that all terms of the previous expression are linear in z_* ; in particular, note in eq. (5.16) that the ratio z_{QES}/z_* is determined by eq. (5.14). In figure 5.5a, we have plotted the position of the intersection $\sigma_{\mathbf{R}}$ between the RT surface and the brane as a function of the brane angle θ_B for various λ_b and $d = 3$. In section 5.3, we shall discuss the fact that, for θ_B below some critical angle θ_c , the extremal surfaces discussed here fail to exist. That is, y_{QES} , the position of the QES on the brane, runs off to infinity as $\theta_B \rightarrow \theta_c$ from above.

Having determined the profile of the RT surfaces, we may proceed to evaluate their corresponding entropies using the RT formula (1.8) — keeping in mind that we have not shown that these surfaces minimize the entropy functional yet. Inserting eqs. (5.6) and (5.12) into the generalized entropy functional, we find that the entropy of the belt geometry

$\overline{\mathbf{R}}$ and hence of the complementary bath region $\mathbf{R}$ is given by

$$\begin{aligned}
& \frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} + \frac{A(\sigma_{\mathbf{R}})}{4G_{\text{brane}}} \\
&= \frac{L^{d-2}}{4G_{\text{brane}}} \frac{\text{vol}_{d-2}^{\perp}}{z_{\text{QES}}^{d-2}} + \frac{L^{d-1}}{4G_{\text{bulk}}} \left\{ \frac{(1 \mp 1)\sqrt{\pi}\Gamma\left[\frac{2-d}{2(d-1)}\right]}{(d-1)\Gamma\left[\frac{1}{2(d-1)}\right]} \frac{\text{vol}_{d-2}^{\perp}}{z_*^{d-2}} \right. \\
&\quad \left. + \frac{2}{d-2} \left[\frac{\text{vol}_{d-2}^{\perp}}{\delta^{d-2}} \mp \frac{\text{vol}_{d-2}^{\perp}}{z_{\text{QES}}^{d-2}} {}_2F_1\left(\frac{1}{2}, \frac{d}{2(d-1)} - 1; \frac{d}{2(d-1)}; \left(\frac{z_*}{z_{\text{QES}}}\right)^{2(d-1)}\right) \right] \right\}
\end{aligned} \tag{5.17}$$

where $z = \delta$ defines the UV cutoff surface near the asymptotic AdS boundary, and z_{QES} and z_* are linearly related to b by eqs. (5.14) and (5.16). For $z_{\text{QES}} \ll z_*$, the hypergeometric function becomes $1 + O[(z_{\text{QES}}/z_*)^{2(d-1)}]$, giving

$$\begin{aligned}
\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} + \frac{A(\sigma_{\mathbf{R}})}{4G_{\text{brane}}} &= \frac{L^{d-1}}{4G_{\text{bulk}}} \left\{ \frac{2}{d-2} \frac{\text{vol}_{d-2}^{\perp}}{\delta^{d-2}} - \frac{2}{d-2} \left(\frac{2\sqrt{\pi}\Gamma\left[\frac{d}{2(d-1)}\right]}{(d-2)\Gamma\left[\frac{1}{2(d-1)}\right]} \right)^{d-1} \frac{\text{vol}_{d-2}^{\perp}}{D^{d-2}} \right\} \\
&\quad + \text{vol}_{d-2}^{\perp} \left(\frac{L}{z_{\text{QES}}} \right)^{d-2} \left\{ \frac{1}{4G_{\text{eff}}} + O\left[\frac{1}{G_{\text{bulk}}} \left(\frac{z_{\text{QES}}}{z_*} \right)^{2(d-1)} \right] \right\},
\end{aligned} \tag{5.18}$$

where we have used eq. (5.8) to replace z_* with D in the first line. Note from eq. (5.14) that $z_{\text{QES}}/z_* \sim [(\lambda_b + 1)\theta_b]^{1/(d-1)}$ so the correction is indeed smaller than the other terms shown here in high tension limit.

Using the brane perspective, let us examine the various contribution to the generalized entropy on the right-hand side of eq. (5.18). Beginning with the leading term of the second line in eq. (5.18), we find that it corresponds to the BH entropy of the QES, *i.e.*, $\frac{1}{4G_{\text{eff}}}$ times the area of $\sigma_{\mathbf{R}}$. It is interesting to note that there are no higher curvature corrections to the generalized entropy of the QES as might have been expected from the Wald-Dong entropy formula.⁵ Turning to the first term in the first line of eq. (5.18), we have the

⁵One can argue that all of the higher curvature corrections to the Wald-Dong entropy must cancel against one another as follows: In the present case, these terms would arise from integrating out the boundary CFT on the gravitating brane and so should be conformally invariant, *e.g.*, see [173]. However, by a simple Weyl transformation, the brane metric becomes flat and further both the intrinsic and the extrinsic curvatures of $\sigma_{\mathbf{R}}$ vanish. Hence in this flat conformal frame, the higher curvature corrections to the Wald-Dong entropy individually vanish. Hence while these curvatures do not vanish in the original conformal frame, the higher curvature entropy corrections must all cancel against one another.

area law divergence associated with the two components of the entangling surface Σ_{CFT} at $x_1 = \pm b$. This leaves us with the second term in the first line. Upon closer examination, this can be recognized as the finite contribution to the EE for a belt of width D , up to an additional factor of 2, *e.g.*, see [13, 14]. Further, we note that both contributions on the first line of eq. (5.17) contain a prefactor proportional to $L^{d-1}/G_{\text{bulk}} \sim c_T$, which measures the number of degrees of freedom in the boundary CFT, *e.g.*, [132].

We can see that these results correspond approximately to the expected entropy from the brane perspective as follows: We begin by considering the contribution from the CFT to one side of the conformal defect, say $x_1 < 0$. Imagine we begin with a single copy of the CFT in flat space (5.2), and evaluate the entropy of a belt of width D with entangling surfaces at $x_1 = -b$ and $x_1 = D - b$. For this geometry, the holographic EE becomes [13, 14]

$$S_{\text{EE}} = \frac{L^{d-1}}{4G_{\text{bulk}}} \left\{ \frac{1}{d-2} \frac{\text{vol}_{d-2}^\perp}{\delta^{d-2}} + \frac{1}{d-2} \frac{\text{vol}_{d-2}^\perp}{\delta^{d-2}} - \frac{1}{d-2} \left(\frac{2\sqrt{\pi} \Gamma\left[\frac{d}{2(d-1)}\right]}{(d-2) \Gamma\left[\frac{1}{2(d-1)}\right]} \right)^{d-1} \frac{\text{vol}_{d-2}^\perp}{D^{d-2}} \right\}, \quad (5.19)$$

where we have separated the area law contributions of the two components of the entangling surface. Now from the brane perspective in our system, the bath CFT resides in flat space for $x_1 < 0$ but the corresponding copy of the CFT resides on the AdS_d geometry of the brane for $x_1 > 0$. However the latter can be produced by making a local Weyl transformation in the positive x_1 domain:

$$ds^2 = \frac{\delta^2}{x_1^2 \sin^2 \theta_B} ds_{\text{CFT}}^2 = \frac{\delta^2}{x_1^2 \sin^2 \theta_B} (-dt^2 + dx_1^2 + \cdots + dx_{d-1}^2). \quad (5.20)$$

Note that this geometry is not the induced metric (5.4) but rather we are considering the standard conformal frame where one strips off the factor of $(L/\delta)^2$ from the bulk metric.⁶ Now the net effect of this Weyl transformation on the EE (5.19) is to modify the cutoff appearing in the area law contribution for the surface at $x_1 = D - b$, *i.e.*, $\delta \rightarrow (D - b) \sin \theta_B \simeq z_{\text{QES}}$, where the latter assumes that $\theta_B \ll 1$. Hence the entropy (5.19) becomes

$$S'_{\text{EE}} \simeq \frac{L^{d-1}}{4G_{\text{bulk}}} \left\{ \frac{1}{d-2} \frac{\text{vol}_{d-2}^\perp}{\delta^{d-2}} - \frac{1}{d-2} \left(\frac{2\sqrt{\pi} \Gamma\left[\frac{d}{2(d-1)}\right]}{(d-2) \Gamma\left[\frac{1}{2(d-1)}\right]} \right)^{d-1} \frac{\text{vol}_{d-2}^\perp}{D^{d-2}} \right\} + \frac{1}{4} \frac{L}{(d-2)G_{\text{bulk}}} \left(\frac{L}{z_{\text{QES}}} \right)^{d-2} \frac{\text{vol}_{d-2}^\perp}{\delta^{d-2}}. \quad (5.21)$$

⁶Further, we are only performing the Weyl transformation (5.20) for $x_1 > \delta / \tan \theta_B$, which corresponds to the intersection of the brane with the UV cutoff surface $z = \delta$.

Now using eq. (2.54), the term on the second line can be recognized as the contribution of one of the boundary CFTs to the Bekenstein-Hawking entropy of the quantum extremal surface on the brane. Hence combining the above contribution (5.21) with that from the other copy of the boundary CFT (which extends to the bath for $x_1 > 0$) and the DGP contribution to the Bekenstein-Hawking entropy, we precisely recover the leading contributions in eq. (5.18). Hence this simple CFT argument allows us to match the leading contributions in the holographic result with the expected EE.

5.2 No-island phase

Above, we studied the set of candidate RT surfaces that intersect the brane. In fact (for $\theta_B < \pi/2$), there exists another set of simple extremal surfaces that must also be considered under the RT prescription (1.8). These surfaces are constant x_1 planes anchored on the entangling surface Σ_{CFT} on the asymptotic boundary and fall straight into the bulk. By reflection symmetry about $x_1 = \pm b$, these planes trivially extremize the area functional, which becomes

$$A(\mathbf{V}) = 2L^{d-1} \text{vol}_{d-2}^\perp \int_{\mathbf{V}} \frac{dz}{z^{d-1}}. \quad (5.22)$$

A factor of 2 has been included above to account for the two planes at $x_1 = \pm b$.⁷ Unlike the surfaces considered in section 5.1, these planes do not intersect the brane and thus no islands are formed on the brane. The entropy in this no-island phase is easily obtained from evaluating the area functional (5.22), which then yields

$$\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} = \frac{L^{d-1}}{2(d-2)G_{\text{bulk}}} \frac{\text{vol}_{d-2}^\perp}{\delta^{d-2}}, \quad (5.23)$$

where δ is again the UV cutoff in the boundary CFT.

5.3 Islands at $T = 0$ for $d > 2$

Altogether, we have two candidate RT surfaces: the extremal surfaces described in section 5.1, which intersect the brane to form a quantum extremal island, and the extremal planes described in section 5.2 corresponding to the no-island phase. To determine which

⁷Further, let us note that for the special case $d = 2$, the integral produces an IR divergence at $z \rightarrow \infty$. However, there is no such IR divergence for $d \geq 3$.

is the correct RT surface, we must first study the parameter space for which each surface exists. When both types of surfaces exist simultaneously, the correct RT surface is given by the one that has the smallest generalized entropy, as in eq. (1.8). Below, we first observe that on a brane at angle $\theta_B < \pi/2$, an island can only possibly exist when $-1 < \lambda_b < 1$; more specifically, for this range of the DGP parameter λ_b , there is a critical angle $\theta_c < \pi/2$ that gives the minimum θ_B which supports the island phase — recall that this critical angle was plotted in figure 5.5b. For $\theta_B > \theta_c$, the island phase exists and is dominant. At $\theta_B = \theta_c$ the entropies computed by the island and no-island RT surfaces equalize, leading to a transition to the no-island phase below θ_c . As we shall find that θ_c scales as $(1 + \lambda_b)^{\frac{1}{d-2}}$ at its smallest, this precludes the possibility of islands in the regime where the brane is well-described by QFT on semiclassical gravity — see section 2.5. This differs from the $d = 2$ case, where the island phase always exists; furthermore, while the no-island RT surface in $d > 3$ has an IR-finite area, the analogous surfaces in $d = 2$ produce an IR divergence and thus are never dominant.

Let us begin our analysis by constraining the parameter space in which each type of RT candidate surface exists. It is easy to see that the extremal planes of the no-island phase exist if and only if $\theta_B \leq \pi/2$.⁸ It is slightly more involved to determine when the extremal surfaces in the island phase exist. For a start, the first equality of eq. (5.14) indicates that for $\theta_B < \pi/2$, sensible extremal surfaces intersecting the brane can only possibly exist when $-1 < \lambda_b < 1$.⁹ From figure 5.5a, we see that this is the range of λ_b for which there exists some $\theta_B < \pi/2$ such that the DGP gradient has not overpowered the bulk term of eq. (5.13) to push the QES to the asymptotic boundary $y = 0$ or to the horizon $y = \infty$.

To be more precise, we must consider properties of the $\mathcal{F}$ function introduced in eq. (5.16). For $-1 < \lambda_b < 1$, some (numerically deduced) facts about $\mathcal{F}(d, \lambda_b, \theta_B)$ are

⁸Of course, this was our regime of interest, as this was the regime where a (nearly) massless graviton is induced on the brane.

⁹Specifically, this can be seen as follows: Let us take the extreme case of $\lambda_b = 1$ ($\lambda_b = -1$). Then eq. (5.14) indicates that $\theta_{\text{QES}} = \theta_B - \pi/2$ ($\theta_{\text{QES}} = \theta_B + \pi/2$). For $\lambda_b = 1$, this implies that when $\theta_B > \pi/2$, the RT surface falls straight into the bulk until it hits the brane, *i.e.*, $z_* = \infty$ — see figure 5.3. Now as $\theta_B \rightarrow \pi/2$ from above, the QES runs off towards the horizon and consequently no QES exists for $\theta_B < \pi/2$. For $\lambda_b = -1$, one can argue that for $\theta_B < \pi/2$, the QES is stuck to the defect, *i.e.*, $z_{\text{QES}} = \infty$. As increasing (decreasing) λ_b beyond 1 (-1) means the DGP entropy contribution exerts a greater force pushing the QES towards the horizon (the defect), it follows that no QES exists for $\theta_B < \pi/2$ when $\lambda_b > 1$ ($\lambda_b < -1$). In these parameter ranges, the naive ‘solutions’ obtained from eq. (5.14) are unphysical, *i.e.*, have the RT surface anchored in the unphysical region behind the brane.

that it is decreasing in λ_b and increasing in θ_B . Moreover,

$$\mathcal{F}(d, \lambda_b, \theta_B \text{ close to } 0) = -(1 + \lambda_b)^{\frac{1}{d-1}} \theta_B^{-\frac{d-2}{d-1}} [1 + O(\theta_B^2)] + \frac{D}{z_*} \quad (5.24)$$

$$\mathcal{F}(d, \lambda_b, \theta_B \text{ close to } \pi) = (1 - \lambda_b)^{\frac{1}{d-1}} (\pi - \theta_B)^{-\frac{d-2}{d-1}} \{1 + O[(\pi - \theta_B)^2]\}. \quad (5.25)$$

Since the former diverges negatively while the latter diverges positively, it follows that there exists a critical angle θ_c for which $\mathcal{F}(d, \lambda_b, \theta_c) = 0$. For $-1 < \lambda_b < 1$, we have $0 < \theta_c < \pi/2$ with $\theta_c \rightarrow 0, \pi/2$ as $\lambda_b \rightarrow -1, 1$, respectively.¹⁰ The physical significance of θ_c can be seen from the second equality of eq. (5.15): for θ_B above θ_c , there exist extremal surfaces that intersect the brane; as $\theta_B \rightarrow \theta_c$ from above, $z_*, z_{\text{QES}}, y_{\text{QES}}$ run off to $+\infty$ as $\sim (\theta_B - \theta_c)^{-1}$; finally, for $\theta_B < \theta_c$, no extremal surfaces exist which intersect the brane. In Figure 5.5b, we plot the critical angle θ_c as a function of λ_b for various d .

Before continuing, let us briefly note a number of peculiarities which arise when $|\lambda_b| > 1$. First, for $\lambda_b > 1$, there exists a range of $\theta_B \gtrsim \pi/2$ for which no extremal surfaces of any kind exists, *i.e.*, the RT prescription fails completely. This may indicate that there is no CFT plus defect theory that can be dual to a bulk with this range of parameters – of course, the brane has a negative tension in this regime and so there is no effective gravitational theory on the brane. Second, recall that as $\lambda_b \rightarrow -1$ from above, the coefficient of the Einstein-Hilbert action vanishes, leading to a breakdown of semiclassical Einstein gravity, as mentioned in section 2.5. Further taking $\lambda_b < -1$ then corresponds leads to an unphysical ghost-like gravity action in the brane theory. At any rate, from here on, we shall restrict our focus to $-1 < \lambda_b < 1$.

Now we have two competing possible RT surfaces: for $\theta_B \leq \pi/2$, extremal planes anchored on the entangling surfaces to either side of the brane, which correspond to a no-island phase; and, for $\theta_B > \theta_c$, extremal surfaces that intersect the brane, corresponding to an island phase. As both types of surfaces exist for $\theta_c < \theta_B < \pi/2$, the RT formula instructs us to choose the surface with the smallest area in this parameter space. Thus, we consider the area difference:

$$\left[\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} + \frac{A(\sigma_{\mathbf{R}})}{4G_{\text{brane}}} \right]_{\text{isl.}} - \left[\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} \right]_{\text{isl.}} = - \frac{L^{d-1} \text{vol}_{d-2}^\perp}{2(d-2)G_{\text{bulk}} z_*^{d-2}} \mathcal{F}(d, \lambda_b, \theta_B) \quad (5.26)$$

¹⁰In particular then, no islands form with $\lambda_b > 1$ in the regime of interest with $\theta_B \leq \pi/2$.

where we have used eqs. (5.16), (5.17), (5.23), and the hypergeometric function identity¹¹

$$\begin{aligned} {}_2F_1 \left[\frac{1}{2}, \frac{d}{2(d-1)} - 1; \frac{d}{2(d-1)}; w \right] \\ = \sqrt{1-w} + \left(\frac{w}{d} \right) {}_2F_1 \left[\frac{1}{2}, \frac{d}{2(d-1)}; \frac{d}{2(d-1)} + 1; w \right]. \end{aligned} \quad (5.27)$$

From eq. (5.26), we see that whenever the island- and no-island-type surfaces coexist, the island-type surface always gives a lower area and is thus the surface picked out by the RT formula. Moreover, we see that entropy transitions continuously between the island and no-island phases at the critical angle θ_c where $\mathcal{F}(d, \lambda_b, \theta_c) = 0$. Altogether, we find that, for $\theta_B < \theta_c$, we are in the no-island phase where the RT surface is given by planes falling straight into the bulk, and, for $\theta_B > \theta_c$, we transition to an island phase where the RT surface is given by extremal surfaces which intersect the brane and form an island.

To gain intuition for the critical angle θ_c from the brane perspective, we note from eq. (5.18) that eq. (5.26) can be approximated as

$$\begin{aligned} & \left[\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} + \frac{A(\sigma_{\mathbf{R}})}{4G_{\text{brane}}} \right]_{\text{isl.}} - \left[\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} \right]_{\text{isl.}} \\ &= - \frac{L^{d-1}}{4G_{\text{bulk}}} \frac{4\sqrt{\pi} \Gamma \left[\frac{d}{2(d-1)} \right]}{(d-2) \Gamma \left[\frac{1}{2(d-1)} \right]} \frac{\text{vol}_{d-2}^\perp}{z_*^{d-2}} \\ &+ \text{vol}_{d-2}^\perp \left(\frac{L}{z_{\text{QES}}} \right)^{d-2} \left\{ \frac{1}{4G_{\text{eff}}} + O \left[\frac{1}{G_{\text{bulk}}} \left(\frac{z_{\text{QES}}}{z_*} \right)^{2(d-1)} \right] \right\} \end{aligned} \quad (5.28)$$

in the small θ_B limit. Building upon the discussion given below eq. (5.18), we interpret the RHS as giving a change in generalized entropy due to the introduction of the island in the effective theory of the asymptotic boundary and brane. Namely, comparing with the island rule (1.7), the first term on the RHS of eq. (5.28) gives the change in S_{QFT} due to the introduction of the island, and the second term gives BH entropy of the QES. Hence, for $\theta_B > \theta_c$, the island phase is favoured as the introduction of the island reduces generalized entropy. For $\theta_B < \theta_c$, the QES ceases to exist and only the no-island phase is possible.

We briefly comment that, unlike for the CFT region considered in chapter 3, the addition of topological terms to the bulk gravity theory does not change the favourability between the island and no-island phases of the belt geometry. This is because such a modification can only effect a topological contribution to the Wald-Dong entropy formula and,

¹¹This can be proven using eq. (15.1.8) and (15.2.25) of [174].

for the belt geometry, the RT surfaces in both phases have vanishing Euler characteristic. Namely, the RT surface of the island phase has the topology of an infinite strip while the RT surface of the no-island phase consists of two half-planes. Thus, the topological contribution would not favour one phase over the other.

In closing, we note that, unlike the $d = 2$ case [50], we have found that in the small θ_{B} limit, where an effective theory of gravity plus quantum matter emerges on the brane, islands typically do not exist for extremal black holes in $d \geq 3$. To be more precise, eq. (5.24) and figure 5.5b suggest that $\theta_{\text{c}}^{d-2} \sim 1 + \lambda_{\text{b}}$. It is still possible to stay in the island phase by tuning $1 + \lambda_{\text{b}}$ to scale as $\sim \theta_{\text{B}}^{d-2}$. However, from eq. (2.55), we see that this limit $\lambda_{\text{b}} \rightarrow -1^+$ corresponds to $G_{\text{eff}} \rightarrow +\infty$, leading to a breakdown of the semiclassical description of the effective brane theory (as described in section 2.5). We remark that, unlike for non-extremal black holes discussed in section 4.2, there is no immediate information paradox that arises as a result of the lack of islands in the extremal case here.

Chapter 6

Two dimensions revisited with black holes

In this chapter, we study the black hole information paradox in the specialized case of $d = 2$ which, as previously described in section 2.3, requires a slightly different treatment. We begin by introducing some field redefinitions relative to section 2.3, to facilitate more direct comparisons with the results of [50]. Next, we review the bulk AdS_3 and brane AdS_2 geometries. Finally, we study extremal surfaces serving as candidate RT surfaces to determine the entropy in the two phases, with and without an island, leading to the Page curve. At leading order in an expansion in terms of small brane angles, *i.e.*, $\theta_B \rightarrow 0$, our results precisely agree with those of [50]. However, we can also retain the subleading terms, which produce corrections due to the finite UV cutoff on the brane.

6.1 Field redefinitions

A goal of this chapter is to compare our results for the quantum extremal surfaces and the Page curve to those derived in [50]. To facilitate this comparison, we make the following field redefinitions

$$\phi_0 = \frac{\Phi_0}{4G_{\text{brane}}} , \qquad \phi = \phi_0 + \frac{\Phi}{4G_{\text{brane}}} , \qquad (6.1)$$

$$\tilde{\phi}_0 = \phi_0 - \frac{1}{2G_{\text{eff}}} \log \left(\frac{L}{\ell_{\text{JT}}} \right) , \qquad \tilde{\phi} = \phi - \frac{1}{2G_{\text{eff}}} \log \left(\frac{L}{\ell_{\text{JT}}} \right) , \qquad (6.2)$$

giving the bare and renormalized values of the dilaton — we shall clarify the meaning of this renormalization shortly. In terms of the latter, induced action (2.44) now reads

$$I_{\text{induced}} = \frac{1}{16\pi G_{\text{eff}}} \int d^2x \sqrt{-\tilde{g}} \left[-\tilde{R} \log \left(-\frac{\ell_{\text{JT}}^2}{2} \tilde{R} \right) + \tilde{R} + \frac{L^2}{8} \tilde{R}^2 + \dots \right] \\ + \frac{1}{4\pi} \int d^2x \sqrt{-\tilde{g}} \left[\tilde{\phi} \tilde{R} + \frac{2}{\ell_{\text{JT}}^2} (\tilde{\phi} - \tilde{\phi}_0) \right]. \quad (6.3)$$

Here, the first line eq. (6.3) may be interpreted as the renormalized effective action produced by integrating out the brane CFT, and the second line contains the renormalized JT action, which can be compared to eq. (2) in [50]. Here, ‘renormalized’ means that we have absorbed the logarithmic UV divergence that would otherwise appear in the induced action¹ as $L/\ell_{\text{B}} \rightarrow 0$ into the JT action, which was achieved by the renormalization of $\phi_0 \rightarrow \tilde{\phi}_0$ in eq. (6.2).

As before, the dilaton $\tilde{\phi}$ acts as a Lagrange multiplier which fixes the brane geometry to be locally AdS_2 with radius of curvature $\ell_{\text{B}} = \ell_{\text{JT}}$. The equation of motion for the induced metric $\tilde{g}_{ij}$, on the other hand, yields the dilaton equation of motion

$$-\nabla_i \nabla_j \tilde{\phi} + \tilde{g}_{ij} \left(\nabla^2 \tilde{\phi} - \frac{\tilde{\phi} - \tilde{\phi}_0}{\ell_{\text{JT}}^2} \right) = 2\pi \tilde{T}_{ij}^{\text{CFT}} = -\frac{\tilde{g}_{ij}}{4L^2 G_{\text{eff}}} f \left(\frac{L^2}{\ell_{\text{JT}}^2} \right). \quad (6.4)$$

In the final expression, we evaluated the renormalized CFT stress tensor $\tilde{T}_{ij}^{\text{CFT}}$ using the function f defined in eq. (2.47). The standard discussions of JT gravity (*e.g.*, [34, 127]) refer to the source-free dilaton equation, *i.e.*, the RHS vanishes, but this is easily accommodated by a further shift²

$$\hat{\phi}_0 = \tilde{\phi}_0 + \frac{\ell_{\text{JT}}^2}{4L^2 G_{\text{eff}}} f \left(\frac{L^2}{\ell_{\text{JT}}^2} \right). \quad (6.5)$$

6.2 Bulk and brane geometries

Let us now review the geometry for our current setup. Due to the simplicity of AdS_3 , we will find it convenient to describe RT surfaces using global coordinates, even though we

¹Recall that we also removed the power law divergence corresponding to the induced cosmological constant term by introducing the counterterm (2.43).

²Note that implementing this shift in the action (6.3) introduces a new cosmological constant term. Hence an alternative approach would be to introduce a general brane tension T_o in eq. (2.43) and then tune the latter to absorb both the corresponding (power law) UV divergence in the induced action and the RHS of the dilaton equation (6.4).

will be considering Rindler time evolution, as in the main text. In global coordinates, we may write the bulk AdS₃ metric as

$$ds^2 = \frac{L^2}{\cos^2 \tilde{r}} \left[-d\tilde{\tau}^2 + d\tilde{r}^2 + \sin^2 \tilde{r} d\varphi^2 \right] \quad (6.6)$$

where $\tilde{\tau} \in \mathbb{R}$, $\tilde{r} \in [0, \pi/2]$ and $\varphi \in [-\pi, \pi]$.

In the AdS-Rindler coordinates, the AdS₃ geometry becomes

$$ds^2 = L^2 \left(-(r^2 - 1) d\tau^2 + \frac{dr^2}{r^2 - 1} + r^2 d\chi^2 \right), \quad (6.7)$$

which is just the special case of eq. (4.7) for $d=2$. Here, $\tau, \chi \in (-\infty, \infty)$ and one exterior region is given by $r > 1$. As described in section 4.1, the AdS-Rindler coordinates are useful for the description of vacuum AdS as a topological black hole, such that the boundary CFT is in a TFD state. The inverse temperature with respect to time τ is 2π , giving the periodicity of $i\tau$ necessary for a smooth Euclidean continuation — we shall also define a dimensionful time and temperature shortly. Indeed, these coordinates describe a horizon at $r = 1$. Note that in $d = 2$, the boundary geometry is flat, *i.e.*, it is simply two copies of $\mathbb{R}^2$. The AdS-Rindler coordinates (τ, r, χ) are related to the global coordinates $(\tilde{\tau}, \tilde{r}, \varphi)$ in eq. (6.6) by

$$\tanh \tau = \frac{\sin \tilde{\tau}}{\cos \varphi \sin \tilde{r}}, \quad \tanh \chi = \frac{\sin \varphi \sin \tilde{r}}{\cos \tilde{\tau}}, \quad r^2 = \frac{\cos^2 \tilde{\tau} - \sin^2 \varphi \sin^2 \tilde{r}}{\cos^2 \tilde{r}}. \quad (6.8)$$

As described above in section 2.3, extremizing the brane action in eq. (2.44) with respect to Φ (or eq. (6.3) with respect to $\tilde{\phi}$) fixes the intrinsic brane geometry to be AdS₂ with radius of curvature $\ell_B = \ell_{\text{JT}}$. This becomes the $\theta = \theta_B$ slice of the AdS₃ metric written as in eq. (2.7), where θ_B is determined by

$$\sin \theta_B = \frac{L}{\ell_{\text{JT}}}, \quad (6.9)$$

as in eq. (4.9). We write the induced metric on the brane as

$$\ell_{\text{JT}}^2 ds_{\text{AdS}_2}^2 = \ell_{\text{JT}}^2 \left(-(\rho^2 - 1) d\tau^2 + \frac{d\rho^2}{\rho^2 - 1} \right) = -\frac{4\pi^2 \ell_{\text{JT}}^2}{\beta^2} \frac{dy^+ dy^-}{\sinh^2 \left(\frac{\pi(y^+ - y^-)}{\beta} \right)}. \quad (6.10)$$

The first line element with (τ, ρ) is simply the special case of AdS-Rindler coordinates given in eq. (4.11) with $d = 2$. The light-cone coordinates (y^+, y^-) in the second line element are

those used by [50], whose results we wish to compare against. The relationship between (τ, ρ) and (y^+, y^-) is given by

$$\tau = \frac{\pi(y^+ + y^-)}{\beta} = \frac{2\pi t}{\beta}, \quad \rho = \coth \left[\frac{\pi(y^+ - y^-)}{\beta} \right]. \quad (6.11)$$

Given that the TFD has temperature $\frac{1}{2\pi}$ with respect to dimensionless time τ , we have introduced the dimensionful time $t = \frac{y^+ + y^-}{2}$ where the temperature becomes $T = 1/\beta$.³

On the brane, eq. (6.4) is easily solved for the dilaton profile in terms of ρ or $y^\pm$:

$$\tilde{\phi} = \hat{\phi}_0 + \frac{2\pi\phi_r}{\beta} \rho = \hat{\phi}_0 + \frac{2\pi\phi_r}{\beta} \coth \left[\frac{\pi(y^+ - y^-)}{\beta} \right], \quad (6.12)$$

where ϕ_r is a constant introduced in [50] (see eq. (18) and discussion below (2) there).

In the AdS-Rindler metric given in eq. (6.7), we introduce a surface of large constant $r = r_{\text{UV}}$ which will serve as the UV cutoff surface. Then following [50], we take the induced metric on this surface as the background metric for the bath CFT, *i.e.*,

$$ds_{\text{CFT}}^2 = L^2 r_{\text{UV}}^2 (-d\tau^2 + d\chi^2), \quad (6.13)$$

with the conformal defect at $\chi = 0$. Now the light-cone coordinates $y^\pm$ can be extended to describe the geometry of the AdS₃ bulk, and in particular the bath region on the asymptotic boundary near $\theta = \pi$ as well as the brane geometry given in eq. (6.10) at $\theta = \theta_{\text{B}}$, by taking an AdS₃ metric in the form eq. (2.7). Indeed, on the asymptotic boundary, with metric given in eq. (6.13), $y^\pm$ are related to (τ, χ) with⁴

$$y^\pm = \frac{\beta(\tau \mp \chi)}{2\pi}, \quad ds_{\text{CFT}}^2 = - \left(\frac{2\pi L r_{\text{UV}}}{\beta} \right)^2 dy^+ dy^-. \quad (6.14)$$

As in higher dimensions, we are interested in computing the EE of a boundary region $\mathbf{R}$ comprised of all of the points with $|\chi| \geq \chi_\Sigma$ in the two baths (associated with the two

³This is the same time coordinate introduced below eq. (4.7), though the relation $\beta = 2\pi R$ loses its meaning as there is no spatial curvature in $d = 2$.

⁴We should note that the geometry in [50] can be seen as a $\mathbb{Z}_2$ orbifold of our setup (see section 4.1). Hence they would only consider $\chi < 0$ of the flat boundary geometry in eq. (6.13). Therefore, the extension of the null coordinates that we are discussing here has to be considered separately for each side of the conformal defect. As a technical point, let us add that in [50], the sign of the spatial coordinate on the brane is reversed so that $y^+ - y^- > 0$ describes the asymptotic boundary while $y^+ - y^- < 0$ describes the brane. Here, $y^+ - y^-$ is always positive and $\theta = \pi, \theta_{\text{B}}$ correspond respectively to the bath and brane.

copies of the CFT entangled in the TFD state). That is, this region is the complement of two intervals (‘belts’) centred on the conformal defects in the two boundaries (which corresponds to the intersection of the brane with the asymptotic boundary — see figure 6.1). Focusing on a single Rindler wedge and on one side of the brane, the entangling surface is located at a fixed $\chi = -\chi_\Sigma < 0$, which we define as

$$\frac{y^+ - y^-}{2} = b > 0 \quad \text{with} \quad b = \frac{\beta}{2\pi} \chi_\Sigma, \quad (6.15)$$

for all Rindler times τ . Similar assignments apply for the patches covering the other portions of the boundary.

Finally, we note that going to the asymptotic boundary (with $\tilde{r} \rightarrow \pi/2$ and $r \rightarrow \infty$), eq. (6.8) yields the relation of the global and Rindler coordinates on the boundary:

$$\tan \varphi = \frac{\sinh \chi}{\cosh \tau}, \quad \tan \tilde{\tau} = \frac{\sinh \tau}{\cosh \chi}, \quad (6.16)$$

which allow us to simplify some calculations below. It will be useful to denote the (time-dependent) global coordinate angle of the entangling surface at $\chi = -\chi_\Sigma$ as φ_Σ .

6.3 Entropies: Island and no-island phases

Now we turn to the problem of computing entropies using the RT formula in the background of the hyperbolic AdS_3 black hole coupled to the AdS_2 brane with JT gravity. Specifically, we wish to compute the entropy of the region $\mathbf{R}$ complementary to belts centred on the defects, as described at the end of subsection 6.2. In the island and no-island phases the RT formula equates the entropy to:

$$\left[\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} + \phi_{\text{QES}} \right]_{\text{isl.}}, \quad \left[\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} \right]_{\text{isl.}}. \quad (6.17)$$

The RT variational problem instructs us to consider extremal co-dimension two surfaces $\Sigma_{\mathbf{R}}$ in the bulk, which in AdS_3 are simply geodesics. Although we are primarily concerned with evolution in Rindler time, the boundaries of the entangling surface are simply four points; these can always be simultaneously placed on a surface of constant global time. This property, not present in higher dimensions, allows us to simplify the analysis by using global coordinates⁵ as seen below.

⁵The fact that the endpoints reside at constant global time, together with the conservation of the charge associated with the global time Killing vector (obtained by dotting with the RT tangent) implies that the RT surfaces themselves must reside on constant global time slices.

Now just as in higher dimensions, the minimization procedure yields two competing phases. At early times, the minimal surfaces cross the Rindler horizon avoiding the brane and the entropy is given purely by the bulk length of the RT surface, as in the second of the expressions (6.17). This length stretches with Rindler time and leads to a growing entropy. At late times the RT surfaces go across the brane instead, leading to an island where the contribution of the dilaton becomes important, as shown in the first of the expressions (6.17). As in the rest of the paper, we restrict to the regime of small brane angle θ_B .

We begin by considering geodesics and their lengths in global coordinates. As is well known, a convenient way to parametrize the RT surfaces on constant global time $\tilde{\tau}$ is by using two anchoring points φ_1, φ_2 , where geodesics are given by

$$\sin(\tilde{r}) \cos\left(\varphi - \frac{\varphi_1 + \varphi_2}{2}\right) = \cos\left(\frac{\varphi_2 - \varphi_1}{2}\right). \quad (6.18)$$

such that the curves hit the boundary $\tilde{r} \rightarrow \pi/2$ at φ_1 and φ_2 . The area (length in $d = 2$) of an RT surface with this trajectory is given by

$$A = L \sum_{i \in \{1,2\}} \tanh^{-1} \left[\csc\left(\frac{\Delta\varphi}{2}\right) \sqrt{-\cos\left(\frac{\Delta\varphi}{2} + \tilde{r}_i\right) \cos\left(\frac{\Delta\varphi}{2} - \tilde{r}_i\right)} \right] \quad (6.19)$$

$$= L \log \left[\frac{4 \sin^2(\Delta\varphi/2)}{\epsilon_1 \epsilon_2} \right] - \frac{L}{12} \left(1 + 3 \cot^2 \frac{\Delta\varphi}{2} \right) (\epsilon_1^2 + \epsilon_2^2) + O(\epsilon_1^4) + O(\epsilon_2^4), \quad (6.20)$$

where

$$\Delta\varphi = |\varphi_1 - \varphi_2|, \quad \epsilon_i = \frac{\pi}{2} - \tilde{r}_i \quad (i \in \{1, 2\}) \quad (6.21)$$

are respectively the opening angle of the RT surface and the UV cutoffs (in the global radial coordinate) at which the area integral is terminated, see figure 6.1.

The leading order term in eq. (6.20) corresponds to the standard EE formula of an interval on the circle [14, 175] (but allowing now for two different UV cutoffs). We have also included the next-to-leading order terms as these will be important for computing corrections to entropy formulas on the brane.

Now as usual, one must appropriately regularize the areas of the RT surfaces. As explained above, we place the cutoff surface at a large holographic radius $r = r_{UV}$ in the Rindler radial coordinate. In terms of global coordinates, this describes the surface

$$\sin^2(\tilde{\tau}) = (\sin \tilde{r} \cos \varphi)^2 - (r_{UV}^2 - 1) \cos^2(\tilde{r}). \quad (6.22)$$

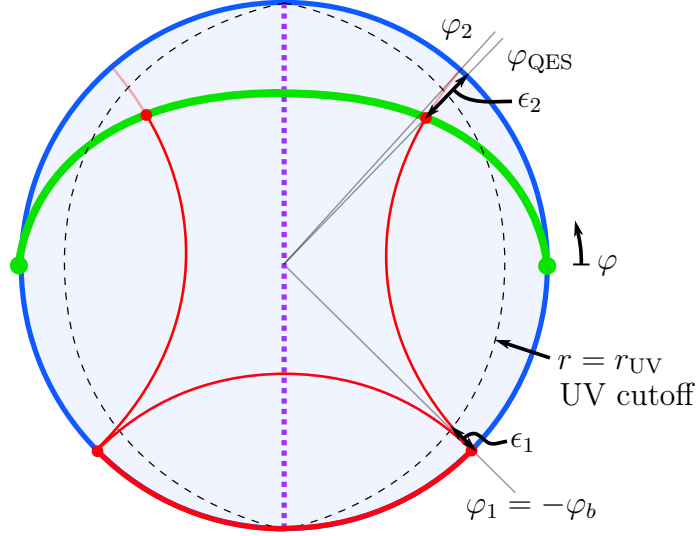


Figure 6.1: A slice of constant global time in AdS_3 , showing the two phases of the generalized entropy. The two cutoffs $\epsilon_{1,2}$ involved in the computation are associated with the UV cutoff at the asymptotic boundary and the brane, respectively. The global coordinate angles φ_1, φ_2 relate to the RT surface opening angle, while φ_{QES} is the angle at which the RT surface intersects the brane and corresponds to the boundary of the island. Recall the geometry is cut at the brane and continued by gluing it to another copy.

Expanding to leading order in r_{UV} , one finds that the UV cutoff is associated with a length in eq. (6.21) given by

$$\epsilon_1 = \frac{1}{r_{\text{UV}}} \sqrt{\frac{2}{\cosh(2\tau) + \cosh(2\chi)}} + O(r_{\text{UV}}^{-3}) . \quad (6.23)$$

where we have used eq. (6.16). Here and below, we shall use ϵ_1 to denote the cutoff at the end-point of the RT surface at the asymptotic boundary; ϵ_2 , on the other hand, will either be a cutoff at the asymptotic boundary or due to the brane, depending on whether we are in the no-island or island phase. Note that although the entropies diverge with the regulator r_{UV} , these contributions will cancel once we consider the difference between the island and no island phases, as seen below.

Equipped with this, we can now compute the generalized entropy in the two phases and reproduce the Page curve found in [50].

No-island phase. We begin with the no-island phase. Here once again due to the simplicity of AdS_3 , the minimal surfaces lie on constant global time slices. The RT surface consists of two pieces, one connected piece on either side of the brane with trajectory given by eq. (6.18) where $\varphi_1 = -\varphi_\Sigma$ and $\varphi_2 = -\pi + \varphi_\Sigma$ (recall the definition of φ_Σ below eq. (6.16)). The total RT length is given by doubling eq. (6.20) (due to the two pieces) with both cutoffs ϵ_1, ϵ_2 given by eq. (6.23). Substituting this into eq. (6.20) with $\Delta\varphi = \pi - 2\varphi_\Sigma$ and using eq. (6.16), the associated EE in the no-island phase is

$$\begin{aligned} \left[\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} \right]_{\text{island}} &= \frac{L}{G_{\text{bulk}}} \log(2r_{\text{UV}} \cosh \tau) \\ &= \frac{2c}{3} \log \left[\frac{\beta}{\pi\delta} \cosh \left(\frac{2\pi t}{\beta} \right) \right], \end{aligned} \quad (6.24)$$

where we used the Brown-Henneaux central charge

$$c = \frac{3L}{2G_{\text{bulk}}}. \quad (6.25)$$

In the second line of eq. (6.24), we have expressed the answer in terms of the dimensionful time t , as in eq. (6.11) (see also below eq. (4.7)) and the short-distance cutoff in the boundary CFT

$$\delta = \frac{\beta}{2\pi r_{\text{UV}}} \quad (6.26)$$

in the $y^\pm$ coordinates on the boundary⁶. Eq. (6.24) matches the entropy from eq. (29) of [50], accounting for the fact that here the central charge is doubled since we include the regions on both sides of the brane⁷. For times much larger than the thermal scale,

$$\left[\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} \right]_{\text{island}} = \frac{2c}{3} \left[\log \left(\frac{\beta}{2\pi\delta} \right) + \frac{2\pi t}{\beta} \right] + O\left(ce^{-4\pi t/\beta}\right), \quad (6.30)$$

which corresponds to the linear growth predicted by Hawking.

⁶To be precise, eq. (6.24) computes the entropy of $\mathbf{R}$ in a CFT with metric $-dy^+dy^-$ and short distance cutoff δ — here, δ is both the proper distance cutoff and the cutoff in $y^\pm$. We may equivalently take the CFT metric to be the induced metric $-\left(\frac{L}{\delta}\right)^2 dy^+dy^-$, in eq. (6.14), with coordinate cutoff δ in $y^\pm$, corresponding to a proper distance cutoff L as measured by the induced metric.

⁷There is a typo in eq. (29) of [50]: inside the logarithm, it should be β/π rather than π/β . The UV cutoff δ is also hidden. The full answer is obtained by applying the conformal transformation

$$w^\pm = \tanh \left(\frac{\pi y_R^\pm}{\beta} \right) = -\coth \left(\frac{\pi y_L^\pm}{\beta} \right) \quad (6.27)$$

Island phase. Let us next consider the island phase. Since translations in Rindler time are an isometry, we can use this symmetry to bring the problem to the $\tilde{\tau} = 0 = \tau$ slice. Notice that this is also a symmetry of the dilaton profile as is clear from eq. (6.12).

We will leave point 1 anchored on the cutoff surface near the asymptotic boundary at global coordinate $\varphi_1 = -\varphi_\Sigma$, as in the no-island phase. But, the RT surface will now intersect the brane at its other endpoint. Here it is important to distinguish between two different angles appearing in the island calculation — see figure 6.1. First, φ_2 (together with φ_1) characterize the trajectory of the RT surface, as in eq. (6.18), such that the trajectory, when maximally extended (even behind the brane), reaches the asymptotic boundary at φ_1 and φ_2 . The opening angle $\Delta\varphi$ appearing in eq. (6.20) is defined in terms of φ_1 and φ_2 as per eq. (6.21). Second, there is the global angular coordinate $\varphi = \varphi_{\text{QES}}$ of the QES where the RT surface intersects the brane. In the limit of vanishing brane angle $\theta_B \rightarrow 0$, $\varphi_{\text{QES}} \rightarrow \varphi_2$ but, at finite θ_B , $\varphi_{\text{QES}} \neq \varphi_2$.

While ϵ_1 is still given by eq. (6.23), the regulator ϵ_2 is now provided by the brane position and is given by

$$\epsilon_2 = \tan^{-1} [\tan(\theta_B) \sin(\varphi_{\text{QES}})] \quad (6.31)$$

$$= \theta_B \sin(\varphi_{\text{QES}}) + \frac{\theta_B^3}{3} \sin(\varphi_{\text{QES}}) \cos^2(\varphi_{\text{QES}}) + O(\theta_B^5), \quad (6.32)$$

which we use below perturbatively in the regime of $\theta_B \ll 1$. From eq. (6.20), the area of the RT surface (including the pieces to either side of the brane and to either side of the

(mapping the vacuum to a TFD) to the entropy formula

$$S[-dw^+ dw^-, \delta] = \frac{c}{6} \log \left[-\frac{(w_R^+ - w_L^+)(w_R^- - w_L^-)}{\delta^2} \right] \quad (6.28)$$

$$\rightarrow S[-dy^+ dy^-, \delta] = S[-dw^+ dw^-, \delta] - \frac{c}{12} \log(\partial_{y_R^+} w_R^+ \partial_{y_R^-} w_R^- \partial_{y_L^+} w_L^+ \partial_{y_L^-} w_L^-), \quad (6.29)$$

where $y_R^\pm = t \pm b$ and $y_L^\pm = t \mp b$ are the entangling surfaces on the R and L sides respectively. We have used the notation $S[ds^2, \delta]$ to denote entropy in a CFT living in ds^2 with proper distance cutoff δ as measured by ds^2 . Eq. (6.29) gives the length of the piece of the RT surface to one side of the brane; eq. (6.24) is then exactly double eq. (6.29).

horizon) is given in terms of φ_Σ and φ_{QES} by

$$\begin{aligned} \left[\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} \right]_{\text{isl.}} &= \frac{L}{G_{\text{bulk}}} \log \left[\frac{4}{\epsilon_1 \theta_{\text{B}}} \frac{\sin^2 \left(\frac{\varphi_\Sigma + \varphi_{\text{QES}}}{2} \right)}{\sin(\varphi_{\text{QES}})} \right] \\ &+ \frac{L\theta_{\text{B}}^2}{G_{\text{bulk}}} \left[-\frac{1}{3} + \frac{\sin^2 \varphi_{\text{QES}}}{4 \sin^2 \left(\frac{\varphi_\Sigma + \varphi_{\text{QES}}}{2} \right)} \right] + O \left(\frac{L\theta_{\text{B}}^4}{G_{\text{bulk}}} \right). \end{aligned} \quad (6.33)$$

We can also write this in terms of the $y^\pm$ coordinates of [50], reviewed around eqs. (6.10) and (6.14) (see also footnote 4). Placing the belt boundary at $\theta = \pi$, $\frac{y^+ - y^-}{2} = b$ and the QES at $\theta = \theta_{\text{B}}$, $\frac{y^+ - y^-}{2} = a$ (matching the a and b of [50]), we find

$$\begin{aligned} \left[\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} \right]_{\text{isl.}} &= \frac{L}{G_{\text{bulk}}} \log \left\{ \frac{4r_{\text{UV}}}{\theta_{\text{B}}} \frac{\sinh^2 \left[\frac{\pi(a+b)}{\beta} \right]}{\sinh \frac{2\pi a}{\beta}} \right\} + \frac{L\theta_{\text{B}}^2}{12G_{\text{bulk}}} \left\{ \frac{3 \sinh^2 \left[\frac{\pi(a-b)}{\beta} \right]}{\sinh^2 \left[\frac{\pi(a+b)}{\beta} \right]} - 1 \right\} + O \left(\frac{L\theta_{\text{B}}^4}{G_{\text{bulk}}} \right) \end{aligned} \quad (6.34)$$

$$= \frac{2c}{3} \log \left\{ \frac{2\beta\ell_{\text{JT}}}{\pi\delta\tilde{\delta}} \frac{\sinh^2 \left[\frac{\pi(a+b)}{\beta} \right]}{\sinh \frac{2\pi a}{\beta}} \right\} - \frac{c\tilde{\delta}^2}{6\ell_{\text{JT}}^2} \frac{\sinh \left(\frac{2\pi a}{\beta} \right) \sinh \left(\frac{2\pi b}{\beta} \right)}{\sinh^2 \left[\frac{\pi(a+b)}{\beta} \right]} + O \left(\frac{c\tilde{\delta}^4}{\ell_{\text{JT}}^4} \right), \quad (6.35)$$

where, in the second line, we have written the answer in terms of the CFT central charge c and cutoff δ (in $y^\pm$) in the bath, given in eqs. (6.25) and (6.26); we have also used the proper distance UV cutoff $\tilde{\delta} = L$ on the brane (hinted at earlier below eq. (6.4)) with the induced metric given in eq. (6.10). Using eq. (6.9) (as well as $\ell_{\text{B}} = \ell_{\text{JT}}$), we can write

$$\theta_{\text{B}} = \frac{\tilde{\delta}}{\ell_{\text{JT}}} \left(1 + \frac{\tilde{\delta}^2}{6\ell_{\text{JT}}^2} + O \left(\frac{\tilde{\delta}^4}{\ell_{\text{JT}}^4} \right) \right). \quad (6.36)$$

Eq. (6.35) is to be interpreted as the von Neumann entropy of the effective CFT spanning the asymptotic boundary and the brane. The first term of eq. (6.35) precisely recovers the

expected CFT result⁸, while the higher orders in $\tilde{\delta}/\ell_{\text{JT}}$ may be interpreted as corrections due to the finite UV cutoff on the brane. Curiously, the leading order correction in eq. (6.35) vanishes for the case of a zero-width belt $b = 0$, *i.e.*, when $\mathbf{R}$ completely contains the baths. We may add eq. (6.35) to the bare dilaton profile ϕ , given by eqs. (6.2) and (6.12), evaluated at the QES, to obtain the generalized entropy

$$\begin{aligned} \left[\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} + \phi_{\text{QES}} \right]_{\text{isl.}} = & 2\hat{\phi}_0 + \frac{4\pi\phi_r}{\beta} \coth\left(\frac{2\pi a}{\beta}\right) + \frac{2c}{3} \log \left\{ \frac{2\beta \sinh^2 \left[\frac{\pi(a+b)}{\beta} \right]}{\pi\delta \sinh \frac{2\pi a}{\beta}} \right\} \\ & - \frac{c\tilde{\delta}^2}{6\ell_{\text{JT}}^2} \frac{\sinh\left(\frac{2\pi a}{\beta}\right) \sinh\left(\frac{2\pi b}{\beta}\right)}{\sinh^2 \left[\frac{\pi(a+b)}{\beta} \right]} + O\left(\frac{c\tilde{\delta}^4}{\ell_{\text{JT}}^4}\right), \end{aligned} \quad (6.39)$$

where we have included dilaton contributions from the QES points on both the left and the right of the TFD. Recall that $\hat{\phi}_0$ conveniently absorbs the part of eq. (6.35) which becomes logarithmically divergent on the brane as we take the UV limit $\tilde{\delta}/\ell_{\text{JT}} \rightarrow 0$ — see eqs. (6.2) and (6.5). This is unsurprising given that the renormalized entropy is derivable from the renormalized matter effective action, and that the renormalization of $\phi_0 \rightarrow \tilde{\phi}_0 \sim \hat{\phi}_0$ is precisely designed to eliminate the UV divergence of the matter effective action on the brane. The first line of eq. (6.39) matches exactly⁹ eq. (19) of [50], accounting for the doubling and quadrupling of the dilaton and von Neumann entropies here (since eq. (19) of [50] considers only one side of the TFD and they work with an end of the world brane with bulk spacetime only to one side). The terms of higher order in $\tilde{\delta}/\ell_{\text{JT}}$ are the corrections due to the UV cutoff, inherited from the von Neumann entropy in eq. (6.35).

⁸To see this, we may apply the transformation between w and y_R written in eq. (6.27) to

$$\begin{aligned} S \left[\left\{ \begin{array}{ll} -\frac{L^2 dw^+ dw^-}{\delta^2} & \text{bath} \\ -\frac{4\ell_{\text{JT}}^2 dw^+ dw^-}{(w^+ - w^-)^2} & \text{brane} \end{array} \right\}, L \right] = & \frac{c}{6} \log \left[\frac{2\ell_{\text{JT}}}{L(w_{\text{QES}}^+ - w_{\text{QES}}^-)} \frac{-(w_{\Sigma}^+ - w_{\text{QES}}^+)(w_{\Sigma}^- - w_{\text{QES}}^-)}{\delta} \right] \\ \rightarrow S \left[\left\{ \begin{array}{ll} -\frac{L^2 dy^+ dy^-}{\delta^2} & \text{bath} \\ -\frac{4\ell_{\text{JT}}^2 dw^+ dw^-}{(w^+ - w^-)^2} & \text{brane} \end{array} \right\}, L \right] = & S \left[\left\{ \begin{array}{ll} -\frac{L^2 dw^+ dw^-}{\delta^2} & \text{bath} \\ -\frac{4\ell_{\text{JT}}^2 dw^+ dw^-}{(w^+ - w^-)^2} & \text{brane} \end{array} \right\}, L \right] - \frac{c}{12} \log(\partial_{y_{\Sigma}^+} w_{\Sigma}^+ \partial_{y_{\Sigma}^-} w_{\Sigma}^-), \end{aligned} \quad (6.37)$$

where we have used the notation $S[\bullet, \bullet]$ introduced in footnote 7, and $y_{\Sigma}^{\pm} = t \pm b$ and $y_{\text{QES}}^{\pm} = t \mp a$ correspond to the entangling surface and the QES respectively. (In this footnote, we have swapped the sign of $y^+ - y^-$ on the AdS_2 brane relative to the main text, so that here $y^+ - y^- > 0$ and $y^+ - y^- < 0$ correspond respectively to the bath and brane.) Then, the first term of eq. (6.35) is precisely four times eq. (6.38).

⁹In fact, the match between the first line of eq. (6.39) and (19) in [50] is exact even after keeping all terms collected in their “constant”. This can be checked by keeping all constant terms in the von Neumann entropy calculation, described in eq. (6.38), as well as the topological dilaton contribution $\hat{\phi}_0$.

To find the location $\frac{y^+ - y^-}{2} = a$ of the QES, the RT prescription instructs us to extremize the generalized entropy given in eq. (6.39). Symmetry has already allowed us to restrict the QES to the same slice of Rindler time $\tau \propto t = \frac{y^+ + y^-}{2}$ as the anchoring point on the asymptotic boundary. It thus remains only to extremize eq. (6.39) in the spatial direction. Setting the derivative of eq. (6.39) in $\frac{y^+ - y^-}{2} = a$ to zero, we obtain the extremization condition:

$$\frac{6\pi\phi_r}{c\beta} = \frac{\sinh\left(\frac{2\pi a}{\beta}\right) \sinh\left[\frac{\pi(a-b)}{\beta}\right]}{\sinh\left[\frac{\pi(a+b)}{\beta}\right]} \left\{ 1 + \frac{\tilde{\delta}^2}{4\ell_{\text{JT}}^2} \frac{\sinh\left(\frac{2\pi a}{\beta}\right) \sinh\left(\frac{2\pi b}{\beta}\right)}{\sinh^2\left[\frac{\pi(a+b)}{\beta}\right]} \right\} + O\left(\frac{\tilde{\delta}^4}{\ell_{\text{JT}}^4}\right). \quad (6.40)$$

At leading order in $\tilde{\delta}/\ell_{\text{JT}}$, this matches eq. (20) in [50] accounting for the fact that we have two copies of the CFT versus a single copy of JT gravity. This equation can be solved for the QES position a in terms of the belt width b numerically or analytically with an additional expansion in $\frac{\phi_r}{c\beta} \gg 1$:

$$a = b + \frac{\beta}{2\pi} \left[\log\left(\frac{12\pi\phi_r}{c\beta}\right) - \frac{\tilde{\delta}^2}{4\ell_{\text{JT}}^2} \left(1 - e^{-\frac{4\pi b}{\beta}}\right) + O\left(\frac{\tilde{\delta}^4}{\ell_{\text{JT}}^4}\right) + O\left(\frac{c\beta}{\phi_r}\right) \right], \quad (6.41)$$

matching eq. (21) in [50] at leading order in $\tilde{\delta}/\ell_{\text{JT}}$, again accounting for the doubling of the CFT. We see that the leading order correction due to finite $\tilde{\delta}/\ell_{\text{JT}}$ is to push the QES further from the bifurcation point at $\frac{y^+ - y^-}{2} = +\infty$.

Having found the location of the QES, we may re-evaluate the generalized entropy of the island phase by substituting eq. (6.41) into eq. (6.39), obtaining

$$\begin{aligned} \left[\frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} + \phi_{\text{QES}} \right]_{\text{isl.}} &= 2 \left(\hat{\phi}_0 + \frac{2\pi\phi_r}{\beta} \right) + \frac{2c}{3} \log\left(\frac{\beta}{\pi\delta}\right) + \frac{4\pi cb}{3\beta} - \frac{c\tilde{\delta}^2}{6\ell_{\text{JT}}^2} \left(1 - e^{-\frac{4\pi b}{\beta}}\right) \\ &\quad - \frac{c^2\beta}{18\pi\phi_r} e^{-\frac{4\pi b}{\beta}} + c \left[O\left(\tilde{\delta}^4/\ell_{\text{JT}}^4\right) + O\left(c^2\beta^2/\phi_r^2\right) \right]. \end{aligned} \quad (6.42)$$

(We have also dropped terms of order $\frac{c^2\beta\tilde{\delta}^2}{\phi_r\ell_{\text{JT}}^2}$ as these are inherently smaller than either the $c\tilde{\delta}^4/\ell_{\text{JT}}^4$ or $c^3\beta^2/\phi_r^2$ corrections.) The first line simply evaluates the generalized entropy, given in eq. (6.39), at the bifurcation surface, *i.e.*, taking $a \rightarrow +\infty$. In particular, we recognize the first term as giving the Bekenstein-Hawking result for the coarse-grained entropy of two black holes

$$2S_{\text{BH}} = 2 \left(\hat{\phi}_0 + \frac{2\pi\phi_r}{\beta} \right). \quad (6.43)$$

This classical contribution dominates eq. (6.42) in the limit $S_{\text{BH}} \gg c$ and corresponds to eq. (30) in [50]. The other terms on the first line of eq. (6.42) evaluate the von Neumann entropy, given in eq. (6.35), after re-absorbing the UV divergence on the brane into $\hat{\phi}_0$. Specifically, the second term gives the UV contribution from the entangling surface on the asymptotic boundary (also appearing in the no-island phase in eq. (6.24)), and the third and fourth terms give finite contributions to the renormalized entropy including a $\tilde{\delta}^2/\ell_{\text{JT}}^2$ correction. Moving to the second line in eq. (6.42), we have a correction due to the displacement of the QES location a from the bifurcation point. Here, the dilaton and von Neumann components of generalized entropy both receive contributions at order $\frac{\phi_r}{\beta} \cdot \frac{c^2\beta^2}{\phi_r^2} \sim \frac{c^2\beta}{\phi_r}$. Note that there are no dilaton corrections at orders $\frac{\phi_r}{\beta} \cdot \frac{c\beta}{\phi_r}$ and $\frac{\phi_r}{\beta} \cdot \frac{c\beta\tilde{\delta}^2}{\phi_r\ell_{\text{JT}}^2}$ because the bifurcation point extremizes the dilaton profile¹⁰. The order $\tilde{\delta}^2/\ell_{\text{JT}}^2$ correction in the QES location given in eq. (6.41) is not visible at the order shown in eq. (6.42).

6.4 Page curve

Collecting together the results of the previous section, we have two phases. At early times, we have the no-island phase, with generalized entropy given by eq. (6.24). Over time, this entropy grows at a rate proportional to the temperature $1/\beta$ and the number c of matter degrees of freedom participating in Hawking radiation, as emphasized in eq. (6.30). This growth, however is capped off by an island phase, where quantum extremal surfaces on the brane just outside the black hole horizon surround an island, containing a portion of the black hole interior, now belonging to the entanglement wedge of the bath. In this latter phase, generalized entropy is given by the constant value written in eq. (6.42) which is dominated by double the Bekenstein-Hawking black hole entropy, as given in eq. (6.43). Viewing eq. (6.43) as the coarse-grained entropy for the two sides of the black hole, this is precisely the expected maximal entropy of the system.

To find the Page time $\tau_P = 2\pi t_P/\beta$ marking the transition between the two phases, we

¹⁰It is helpful to consider the coordinate $\tilde{\rho} = \sqrt{\rho^2 - 1}$, in terms of which eq. (6.12) reads $\tilde{\phi} = \hat{\phi}_0 + \frac{2\pi\phi_r}{\beta} \sqrt{1 + \tilde{\rho}^2}$ and the brane metric $\ell_{\text{JT}}^2 ds_{\text{AdS}_2}^2 = \ell_{\text{JT}}^2 \left(-\tilde{\rho}^2 d\tau^2 + \frac{d\tilde{\rho}^2}{\tilde{\rho}^2 + 1} \right)$, near the horizon $\tilde{\rho} = 0$, resembles the standard flat metric $-\tilde{\rho}^2 d\tau^2 + d\tilde{\rho}^2$ in polar coordinates with $\tilde{\rho}$ the usual radial coordinate. The dilaton and the von Neumann entropy in eq. (6.39) should then have an expansion in terms of non-negative integer powers of $\tilde{\rho}_{\text{QES}}$. Eq. (6.41) gives the first corrections to $\tilde{\rho}_{\text{QES}} = 0$ at orders $\frac{c\beta}{\phi_r}$ and $\frac{c\beta\tilde{\delta}^2}{\phi_r\ell_{\text{JT}}^2}$, leading to the corrections mentioned in the main text.

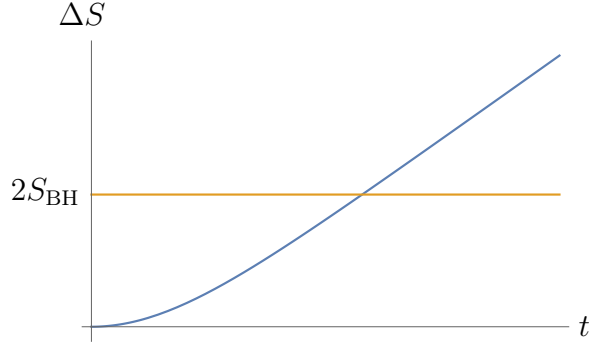


Figure 6.2: Page curve for the equilibration of our topological black hole in $d = 2$. We plot the entropy $\Delta S = S(t) - S(0)$ of the subregion on the CFT which is associated to the radiation, where we subtract the value of the entropy at $t = 0$.

equate the corresponding generalized entropies given in eqs. (6.30) and (6.42):

$$\begin{aligned} \tau_P = \frac{2\pi t_P}{\beta} &= \frac{3}{c} \left(\hat{\phi}_0 + \frac{2\pi\phi_r}{\beta} \right) + \log(2) + \frac{2\pi b}{\beta} - \frac{\tilde{\delta}^2}{4\ell_{\text{JT}}^2} \left(1 - e^{-\frac{4\pi b}{\beta}} \right) - \frac{c\beta e^{-\frac{4\pi b}{\beta}}}{12\pi\phi_r} \\ &\quad + O\left(\tilde{\delta}^4/\ell_{\text{JT}}^4\right) + O\left(c^2\beta^2/\phi_r^2\right). \end{aligned} \quad (6.44)$$

Overall, we recover a Page curve, with entropy growing linearly in a no-island phase up to the Page time, and saturating to a constant maximal value in an island phase after the Page time. In figure 6.2, we plot the Page curve after subtracting off the initial entropy (which includes the UV divergences from the asymptotic boundary).

Chapter 7

Discussion

In this thesis, we have described a holographic framework where quantum extremal surfaces and the island rule (1.7) can be examined in higher dimensions, *i.e.*, for gravity theories in $d \geq 2$. In particular, the background is simple enough that the construction given in chapter 2 is straightforward and purely analytic, in contrast to the numerical approach of [58]. In section 2.5, we were also able to describe the system from three different perspectives, analogous to the three descriptions of the two-dimensional system examined in [7]. In particular, we have the boundary perspective, where the system is described as a d -dimensional CFT coupled to a $(d - 1)$ -dimensional conformal defect; the bulk gravity perspective, where $(d + 1)$ -dimensional gravity with a negative cosmological constant is coupled to a codimension-one brane; and the brane perspective, where the boundary CFT is coupled to an AdS_d region which supports Einstein gravity and two copies of the same CFT, which are weakly coupled to each other. As we emphasized, this last perspective is an effective theory, as is made clear by the cut-off arising in this RS brane world scenario. As discussed and examined in some detail in chapter 3, this effective gravity theory lends itself to the appearance of quantum extremal islands in the brane perspective, although these have a conventional interpretation from the bulk gravity perspective, in terms of RT surfaces which cross the brane for certain of choices of the entangling geometry on the boundary.

We have considered the vacuum state of the system with respect to global time, which simplifies the bulk geometry to be pure AdS. However, as discussed in chapter 4, by viewing this setup in AdS-Rindler coordinates, the global vacuum can be re-interpreted as in terms of a massless hyperbolic black hole. This induces a similar description of the brane geometry as a black hole of one lower dimension. The ‘two’ asymptotic boundaries then play the role of bath CFTs in equilibrium with the black hole on the brane at a finite temperature

$T = \frac{1}{2\pi R}$. Similarly, as explained in chapter 5, viewing our setup in Poincaré coordinates, we have an extremal horizon in the bulk and on the brane. The latter was coupled to a (single) $T = 0$ bath CFT on the asymptotic boundary.

While islands have been numerically studied previously in [58], our approach provides a relatively simple setting in which analytic calculations are possible. In particular, the doubly holographic nature of our model reduces the entropy calculations involving islands in the presence of massless hyperbolic, or extremal black holes of arbitrary dimension to holographic EE calculations in (locally) pure AdS in one dimension higher. From the d -dimensional brane perspective, when computing the entropy of a boundary region $\mathbf{R}$ in the island phase, a quantum extremal surface $\sigma_{\mathbf{R}}$ marks the boundary of an island on the brane; this island belongs to the entanglement wedge of the bath region $\mathbf{R}$. From the bulk perspective, the RT surface of $\mathbf{R}$ runs into the bulk from its anchoring surface $\Sigma_{\text{CFT}} = \partial\mathbf{R}$ and intersects the brane at $\sigma_{\mathbf{R}}$. As noted in [58], the entanglement wedge of $\sigma_{\mathbf{R}}$ stretches through the bulk and is manifestly connected to the island on the brane in this higher-dimensional picture, despite the apparent disconnection in the effective d -dimensional theory. To determine the RT surface in an island phase, we must not only extremize the area functional locally within the bulk, but also extremize with respect to the intersection of the RT surface and the brane. Since the deep bulk (IR) and near-brane (UV) contributions (further modified by DGP contributions) to the RT area, respectively, can be interpreted as renormalized von Neumann and gravitational Wald-Dong entropies, this bulk calculation is equivalent to the island prescription of extremizing generalized entropy over candidate quantum extremal surfaces.

The most striking difference between our holographic construction and the two-dimensional model of [50] is that, as detailed in chapter 6, JT gravity does not appear automatically but has to be added by hand to the brane theory for $d = 2$, in analogy to the DGP terms in higher dimensions. However, this may be contrasted with the induced gravity on the branes in higher dimensions, where adding a DGP term provides finer control over the model, but is not strictly necessary for interpreting the brane perspective as an effective CFT coupled to gravity. Having added JT gravity as a DGP term, we showed in chapter 6 that applying the RT formula in the AdS_3 bulk and including the DGP entropy, as in the $d = 2$ analogue of eq. (1.8), correctly reproduces the results of [50] at leading order in an expansion in terms of small brane angles, *i.e.*, $\theta_{\text{B}} \ll 1$. A finite θ_{B} imposes a finite UV cutoff in the effective brane theory, as shown in eq. (6.36), and therefore subleading corrections to entropy formulas appear in the island phase — see eq. (6.42). Of course, with a finite UV cutoff, we would not, for instance, expect the holographic entropy to precisely satisfy the CFT transformation rules of the entanglement entropy used by [50] in deriving their results. These corrections have the effect of pushing the QES slightly further from

the horizon, lowering the entropy of the island phase, and shifting the Page transition to an earlier time.

Below in sections 7.1 and 7.2, we provide further discussion of the results presented in this thesis and summarized above. Finally, we end in section 7.3 by suggesting future directions in which to extend our work.

7.1 Comments on the brane world model

Setting black holes to the side for now, let us begin our more detailed discussion by considering various features of the brane world model, introduced and examined in the first half of this thesis.

7.1.1 Unconventional features

The analysis presented in the first half of this thesis is somewhat unusual in that we are finding QEIs but there are no black holes, no horizons and no Hawking radiation involved. Rather we simply considered the EE of various entangling regions in the vacuum state of the boundary system. However, to favour the formation of these quantum extremal islands, and at the same time have the brane in the ‘Einstein gravity regime,’ *i.e.*, $L/\ell_{\text{eff}} \ll 1$, we had to introduce somewhat unconventional couplings. That is, we considered a negative Newton’s constant on the brane $\lambda_b < 0$ and nonzero Gauss-Bonnet coupling λ_{GB} for a four-dimensional bulk. Both of these choices were enhancing the connected RT surfaces over the disconnected RT surfaces in calculating the holographic EE. Of course, an interesting question is the interpretation of these ‘exotic’ bulk couplings in terms of data describing the boundary CFT (and the conformal defect). While we do not have a precise interpretation, some qualitative results can be stated below. Before continuing, let us emphasize that these unconventional features introduced to favour QEIs are unnecessary in the discussion of black holes in the second half of this thesis, where islands always arise at late times for (non-extremal) black holes, even with conventional parameters.

As observed in section 2.5, using standard holographic techniques, one finds that the gravitational coupling in the DGP brane action (2.50) affects the spectrum of defect operators in the boundary theory [128]. Now let us reiterate that there is no a priori reason not to consider $\lambda_b < 0$. For example, integrating out quantum fields on the brane could produce either a positive or negative shift of Newton’s constant. In particular, the shift can be negative for gauge fields or non-minimally coupled scalar fields, as was discussed

in the context of EE in [153, 154] — see also discussion in appendix B. However, this scenario is not the one we are describing here. In particular, additional brane fields such as these would make significant contributions to the EE which are not accounted for in our calculations. Hence, implicitly, we simply assume that the gravitational coupling $1/G_{\text{brane}}$ (either positive or negative) is induced by some unknown UV physics.

Introducing the Gauss-Bonnet term (3.43) does not modify the gravitational dynamics in the four-dimensional bulk, considered in section 3.4, and hence the correlators of the stress tensor are not modified in the dual three-dimensional boundary theory.¹ However, the topological coupling λ_{GB} affects the entanglement structure of the boundary CFT states. To see this, consider calculating the entanglement entropy holographically for two nearby regions in the boundary. The phase transition between connected and disconnected phases of the RT surfaces is sensitive to a Gauss-Bonnet term. For positive λ_{GB} , the transition from the disconnected to connected phase takes place earlier (and vice versa for negative λ_{GB}). This means that with $\lambda_{\text{GB}} > 0$, the mutual information between these two regions remains of order c_{T} for larger separations, *e.g.*, [96]. Note, however, that choosing positive λ_{GB} favours higher genus surfaces. A concern with this choice might be if higher genus extremal surfaces exist, they may produce unusual results. Finally, we note that the topological coupling appears directly in the expressions for the holographic EE, *e.g.*, see eq. (3.47). Therefore to have an appreciable effect, we must choose this coupling to be of the order of the central charge of the boundary theory, *i.e.*, $\lambda_{\text{GB}} \sim L^2/G_{\text{bulk}} \sim c_{\text{T}}$.

Let us add that in section 3.4, we focused on the example of $d = 3$ with a four-dimensional bulk. In this case, the natural topological term to add to the bulk gravity is the Gauss-Bonnet term (3.43). Of course, the scenario extends straightforwardly to any $d = 2n - 1$ for which there is a corresponding topological term which can be added to the bulk gravity action, *i.e.*, the Euler character for $2n$ -dimensional manifolds, *e.g.*, see [151]. Similarly, for even boundary dimensions ($d = 2n$), the analogous topological terms could be added to the brane action, where they would not modify the dynamics of gravity on the brane but they would modify the gravitational entropy associated with the boundary of the QEIs.

In light of these unconventional features, a natural question therefore is whether we find QEIs in our analysis with both $\lambda_{\text{b}} = 0 = \lambda_{\text{GB}}$. The answer is affirmative; however, one must reduce the tension of the brane to reduce its backreaction and the extent of the additional geometry in the vicinity of the brane's location. As a result, the connected RT surfaces will have a smaller (bulk) area contribution as they cross the brane. However, in this case, the curvature of the AdS geometry on the brane is also smaller, and hence the effective

¹Of course, such modifications arise for holographic constructions in higher dimensions [132].

description of the brane theory in terms of Einstein gravity breaks down. That is, with $\ell_{\text{eff}} \sim L$, the contributions of the higher curvature corrections in the induced action (2.24) are no longer suppressed relative to the Einstein term and these new interactions play an important role in the dynamics of gravity in the brane perspective. Furthermore, the cutoff of the corresponding CFT on the brane will be much lower. Alternatively, one could think about computing the EE in settings beyond the vacuum state. Indeed, reinterpreting the global vacuum as eternal black holes in equilibrium with external heat baths, we have already seen in chapter 4 how QEIs appear without additional Gauss-Bonnet or DGP couplings.

Let us conclude here by comparing our approach, as described in chapter 2, with that of [60], which appeared while our work in [1] was being completed. The latter examines essentially the same model (with no DGP term) but concentrates on a very different regime. The authors of [60] focused on the formation of islands for the case of a tensionless brane, where the brane gravity becomes very nonstandard, as explained above. Further, in the limit where the graviton becomes massless, *i.e.*, $\ell_{\text{eff}} \rightarrow \infty$, they observe that no islands form [60]. On the other hand, the present work focuses the regime of large brane tension, where the theory on the brane can be well approximated by Einstein gravity (*i.e.*, the graviton mass and higher curvature interactions are negligible). We moreover show that by allowing either a topological term or a negative G_{brane} , islands can appear even in the absence of horizons.

7.1.2 Resolving Puzzles

Our construction clarifies certain conceptual puzzles that arose in early discussions of QEIs in a holographic framework, *e.g.*, for the two-dimensional gravity models introduced in [7] and studied in [4, 50]. For example in these models the Planck brane, which supports the JT gravity theory, appears as part of the boundary of the three-dimensional bulk spacetime. Hence one might have wondered if the brane degrees of freedom (including the JT gravity) are a part of the boundary theory or part of the bulk theory. In our construction, the Planck brane is in the middle of the spacetime geometry and so this question does not arise — these degrees of freedom belong to the bulk. An important corollary of this observation is that when a QEI appears on the brane, *e.g.*, see the lower panel in figure 3.4, we are able to recover information about the island with data from the boundary CFT in the corresponding boundary subregion, by applying standard entanglement wedge reconstruction [137–143]. Of course, the latter would not apply if the brane degrees of freedom were a part of the boundary theory.

Further, our construction circumvents the question of whether RT surfaces are allowed to end on the Planck brane. Rather in our paper, the extremal surfaces just pass through the bulk and only end on the asymptotic boundary as usual. It is simply that in certain situations, the RT surfaces will pass through the brane, which of course, corresponds to the formation of a quantum extremal island.

Another ‘novel’ feature of the two-dimensional JT gravity model of [7] was that the holographic EE included an extra boundary term, *i.e.*, the gravitational entropy of the JT model, where the RT surface terminated on the Planck brane. That is, the holographic entanglement entropy was given by extremizing the sum of the bulk area of the RT surface and this additional boundary term. An analogous gravitational entropy term on the brane arises in our construction with a DGP brane — see eq. (3.1). In fact, our derivation in appendix A suggests that if the brane supports intrinsic gravitational interactions then the corresponding Wald-Dong entropy on the brane is part of the holographic EE formula, as shown in eq. (A.7). Hence this general result agrees with the boundary term introduced in the two-dimensional JT gravity models, mentioned above. A shortcoming of the derivation in appendix A is that the geometric configuration involved a high degree of symmetry, which precluded finding the expected extrinsic curvature terms [95]. Therefore it would be interesting to extend our construction there to more general configurations along the lines of [176, 177].

We want to emphasize the above discussion is distinct from the finding in section 3.2 that the leading contribution to the holographic EE where the RT surface crosses the brane matches the Wald-Dong entropy of the induced gravitational action on the brane (2.24).² For example, the leading contribution is $A(\sigma_{\mathbf{R}})/4G_{\text{eff}}$, where $\sigma_{\mathbf{R}}$ is the cross-section of the RT surface on the brane. As shown in eq. (3.27), the DGP term is one important contribution to this result, but the bulk area of the RT surface in the vicinity of the brane is also necessary. Of course, we still find the leading contributions reproduce the gravitational entropy of the induced gravity theory on the brane even without the DGP term, *i.e.*, with $1/G_{\text{brane}} = 0$. This must be closely related to the fact that the bulk Einstein equations combined with the Israel junction conditions are equivalent to the gravity equations of motion on the brane in the RS scenario [108].

In passing, we note here that $d = 2$ is distinguished in the above discussion. In this case, the leading contribution corresponds to the Wald-Dong entropy for the Polyakov-Liouville action (2.31) and takes the form given in eq. (3.31). However, since it only depends on the curvature scalar which is constant across the AdS_2 geometry of the brane, this contribution

²Recall that this analysis was general enough to see the extrinsic curvature contributions coming from the higher curvature interactions in eq. (2.24).

takes the same value no matter where the RT surface crosses the brane. This contrasts with the higher dimensional result $A(\sigma_{\mathbf{R}})/4G_{\text{eff}}$, which rapidly grows as the position of $\sigma_{\mathbf{R}}$ moves to larger radii on the brane. That is, there is an enormous penalty against forming large QEIs for $d \geq 3$. In contrast, no such penalty arises for $d = 2$ facilitating the formation of islands, as discussed in detail in [47]. Of course, if one adds JT gravity (2.42) to the two-dimensional brane action, as in eq. (2.41), then the gravitational entropy on the brane includes $(\Phi_0 + \Phi(x))/4G_{\text{brane}}$, which will favour smaller quantum extremal islands because the dilaton profile grows with the radius on the brane [127].

Of course, we can modify our higher dimensional construction to make it more analogous to the two-dimensional model introduced in [7] by taking a $\mathbb{Z}_2$ orbifold quotient across the brane. With this orbifold, the brane appears as the edge of the bulk geometry but clearly the association with the bulk degrees of freedom has not changed. The brane now only supports a single copy of the boundary CFT and there are factors of $1/2$ appearing in various expressions, *e.g.*, we make the following replacement in eq. (2.25): $1/G_{\text{eff}} = L/((d-2)G_{\text{bulk}})$. Similarly, the RT surfaces will now end on the orbifolded brane while satisfying the boundary condition,

$$0 = \tilde{g}_j{}^\nu \left(g_{\mu\nu} \partial_n X^\mu + \frac{G_{\text{bulk}}}{G_{\text{brane}}} \tilde{\mathcal{K}}_i \partial_\nu x^i \right), \quad (7.1)$$

which replaces eq. (3.17). Further, the conformal defect becomes a conformal boundary in the orbifolded theory, *i.e.*, the spatial geometry on which the CFT lives is now a $(d-1)$ -dimensional hemisphere with the conformal boundary being the S^{d-2} at the edge of the hemisphere.

Other questions that may have arisen from the early discussions of quantum extremal islands which focused on JT gravity might include the importance of having a low spacetime dimension, *i.e.*, $d = 2$, or of the JT model itself. The early work of [35] considered black hole evaporation with Einstein gravity in higher dimensions, and the holographic model of [7] was extended to a holographic framework with $d = 4$ in [58] using numerical calculations. Hence our paper reinforces these results by describing QEIs in a new setting, in particular, in higher dimensions and with Einstein gravity. Our construction is also simple enough that further investigations of the role of QEIs in higher dimensions are straightforward. Let us add that JT gravity can be seen as the gravitational dual of the so-called SYK model [178–181]. This duality involves an ensemble average over the couplings in the boundary quantum mechanics and so one may expect that this averaging plays a role in the appearance of QEIs. However, it seems that this is not the case as our construction relies on the standard holographic rules of the AdS/CFT correspondence where there is no such averaging of the couplings in the boundary theory.

One other perplexing issue with the island rule (3.2) is the appearance of the entanglement of the CFT degrees of freedom in the region $\mathbf{R}$ on both sides of the equation [7]. As explained in [50], we should distinguish the “full quantum description” of, *e.g.*, the Hawking radiation in the presence of black holes on the left-hand side from the “semiclassical description” which includes the outgoing radiation and purifying partners on the QEI on the right-hand side. Our holographic construction makes clear that the description of quantum states with islands in the brane picture is on a different footing than that solely in terms of the boundary theory. In particular, referring to the three perspectives discussed in section 2.5, it is clear that the boundary perspective (with the boundary CFT coupled to a conformal defect) gives a complete description of quantum state. By the standard rules of the AdS/CFT correspondence, the bulk perspective (where Einstein gravity with a negative cosmological constant is coupled to a codimension-one brane) gives an equivalent description.³ However, the brane perspective has a different character. In particular, the description in terms of a CFT coupled to the dynamical AdS_d region is only an effective one. Indeed, as emphasized in section 2.5, the RS gravity is only valid down to the short distance cutoff $\tilde{\delta} \sim L$, *i.e.*, see eqs. (2.62) and (2.63). Beyond this cutoff, gravity is no longer localized to the brane and the additional ‘Kaluza-Klein’ modes of the graviton are strongly coupled to the brane and their contribution cannot be ignored.

Further, this brane perspective also provides an effective description of the coupling to the defect CFT. That is, it only accounts for the couplings localized at the defect, which dominate at low energies, but ignores the subtle nonlocal couplings, which could be seen as coming through the bulk AdS geometry in the dual description. Of course, the QEIs in the effective description of the brane perspective are a clear example of this. These islands are a remnant of replica wormholes in the limit $n \rightarrow 1$ [47–49]. However, in the replica trick construction of the corresponding Rényi entropies in the bath CFT, one can ask why the gravity on the different branes in the replica copies should connect with one another. However, these effective gravity theories are UV completed by a single theory of gravity in the bulk and so it is natural to consider geometries connecting the branes, *i.e.*, replica wormholes if the effective theory. Hence the connection of the brane and boundary through the bulk provides a simple explanation of these wormholes. Given the simplicity of our construction, it may provide a useful framework in which to further understand the nonlocal couplings which implicitly provide subtle correlations between the island degrees

³In this paper, we modelled the CFT defect with a simple brane in the bulk. This bottom-up approach is neither sufficient, nor completely correct. For example, in the case of $\mathcal{N} = 4$ SYM theory on S^4 , the presence of an interface breaks at least half of the supersymmetry generators and the R symmetry. In a complete description, this will result in a deformation of the bulk S^5 . For top-down models, see [88–90, 182–187].



Figure 7.1: Left: In the brane perspective, the bath CFT on the asymptotic boundary (blue) is connected to two copies of the effective CFT on the brane (green) but the resulting geometry is not a manifold. Right: For excitations below the effective CFT cutoff the system behaves as if it consists of two systems on a manifold which are weakly coupled in the gravitational region (green).

of freedom and those in the bath CFT [128].

As a final note here, we observe that the finite cutoff for the CFT on the brane has noticeable effects even for $d = 2$, *e.g.*, see eq. (2.49). In contrast, the early discussions of *e.g.*, [4, 7, 34, 48, 50, 65] assumed that one could use standard formulae for conformal transformations in the $d = 2$ CFT in the gravitational region (*i.e.*, on the brane). It would be interesting to understand if the cutoff modifies any of this analysis in a significant way.

7.1.3 Coupling across the defect and brane

The geometry of the setup presented in this paper might look unconventional. As seen from the brane perspective, we have the bath CFT on the asymptotic boundary with geometry $S^{d-1} \times \mathbb{R}$, and two copies of the same CFT on the brane with an AdS_d geometry. These two geometries are joined by introducing a cutoff surface (with topology $S^{d-2} \times \mathbb{R}$) near the asymptotic boundary of the AdS_d geometry and gluing it to the equator of the $S^{d-1} \times \mathbb{R}$ geometry. In particular, the resulting geometry is not a manifold in the vicinity of the gluing region — see the left panel of figure 7.1. Of course, we can obtain a manifold by taking the $\mathbb{Z}_2$ quotient which identifies the two halves of the bath CFT, such that the theory is again defined on a manifold with topology $S^{d-1} \times \mathbb{R}$. However, we will ignore this simplification here. Rather, we want to comment on the theory before taking the $\mathbb{Z}_2$ quotient.

First, we note that constructions where multiple CFTs are joined at a common defect are not rare. For example they appear in the study of boundary and interface CFTs (*e.g.*, see [185]), and sometimes seem to be required to remove anomalies [188].

Second, we would like to argue that in the regime where the defect theory can be

described by two copies of the boundary CFT coupled to Einstein gravity, we can approximately think of the full theory as two copies of the orbifolded theory (each living on a manifold), which are weakly coupled in the gravitational region — see the right panel of figure 7.1. This is particularly easy to see from the bulk perspective. For brevity we restrict ourselves to the discussion of graviton modes, but a similar story applies to all bulk fields.

Let us begin by recalling that for $\varepsilon \ll 1$, the spectrum of graviton fluctuations in the bulk is almost unchanged with respect to the modes in (two copies of) empty AdS space. Hence much of the corresponding physics should be very similar that of two copies of the AdS_{d+1} , or to two copies of the dual CFT_d on the boundaries of two independent AdS_{d+1} geometries. Of course, one exception to the preceding is that, upon gluing the two AdS_{d+1} geometries together, a new set of very light graviton states appear localized in the vicinity of the brane [84–86, 112], as discussed in section 2.5. For simplicity, we refer to the latter as the brane graviton modes, while we refer to the former as the standard normalizable modes.⁴

On a fixed time slice, as shown in the right panel of figure 2.1, the standard normalizable modes will describe stress energy excitations in the dual CFT on both the left and right halves of the asymptotic boundary. If we assume an approximate extrapolate dictionary [189] for the brane theory as well, these normalizable modes will also describe analogous excitations for the effective CFT on the brane. However, there will be two sets of such excitations: those described by bulk excitations⁵ with support primarily in the right copy of the AdS_{d+1} geometry, and those described by the analogous excitations primarily in the left AdS_{d+1} geometry. Hence, the stress tensor on the brane can be decomposed into two pieces that correspond to subsectors of the brane theory, each of which is determined by bulk excitations that essentially live on one side of the brane or the other. If these subsectors were truly superselection sectors (*e.g.*, as one might imagine arises in the limit $\varepsilon \rightarrow 0$), our brane theory would contain two independent copies of the boundary CFT and each of these copies would only interact with the bath CFT on the corresponding half of the asymptotic boundary. That is, each of these systems would live on an independent manifold with topology $S^{d-1} \times \mathbb{R}$.

However, this is not strictly correct and the two copies of the CFT on the brane are

⁴These bulk modes are $\mathbb{Z}_2$ graded under reflection across the Planck brane, and the even modes survive the $\mathbb{Z}_2$ orbifold discussed above include the brane graviton states as well as half of the standard normalizable modes. However, this organization of the modes is not useful for the following discussion.

⁵We stress here that the localized excitations considered here do not correspond to individual energy eigenmodes, which were implicit in the previous paragraph. Rather they will consist of linear combinations of such eigenmodes evaluated on the fixed time slice being examined here. Of course, having superpositions of energy eigenmodes is what produces the complicated time evolution described below.

weakly coupled with $\varepsilon \ll 1$ but finite. In particular, localized stress energy excitations of the form considered above will not remain localized with time evolution. Rather they will eventually spread across the entire asymptotic boundary after a sufficiently long time. For example, an excitation localized on the right asymptotic boundary will evolve to eventually produce excitations of the stress tensors on the left asymptotic boundary and on the brane as well. From the boundary perspective, excitations moving onto the brane correspond to excitations that are absorbed by the conformal defect (and remain there for a long time).

The spreading of the localized excitations can be seen to arise through two physical effects: First, the bulk excitations can tunnel between the two AdS_{d+1} regions shown in figure 2.1. Recall that (the radial part of) the linearized bulk equation of motion can be reduced to a Schrödinger equation with a double-well potential, where the height of the barrier is determined by the brane tension [86]. With $\varepsilon \ll 1$ but finite, the barrier height while large remains finite and there will be a finite probability for a bulk excitation on one side of the Planck brane to tunnel to the other. A second independent coupling comes because the stress tensors of the two copies of the CFT couple to the same gravity theory on the brane. From the bulk perspective, the nonlinear Einstein equation produces interactions between the brane graviton modes with excitations on either side of the brane. Hence, bulk excitations on one side can leak to the other side by scattering process involving the brane gravitons. However, we note that both effects become smaller as the brane tension approaches its critical value, *i.e.*, as ε approaches zero. Thus, to a good approximation, the brane theory can be treated as two copies of the boundary CFT, which only interact weakly.

7.1.4 Ryu-Takayanagi Bubbles and Wormholes

In appendix B, we consider a surprising class of RT surfaces with the topology of a sphere, *i.e.*, S^{d-1} in the $(d+1)$ -dimensional bulk. The appearance of these extremal ‘bubbles’ is quite unusual as they are homologous to the entire boundary. Hence the standard RT prescription would assign an entropy to the ground state of the dual boundary system. Further, presence of a ‘zero mode’ that allows the bubbles to be translated along the brane makes their interpretation even more puzzling. An essential feature for the appearance of the RT bubbles was that the gravitational coupling in the DGP term (2.50) was negative, *i.e.*, $\lambda_b < 0$. We also noted that the bubbles do not appear to be macroscopic objects in the brane theory. Rather, as shown in eq. (B.9), their size is always of order of the effective cutoff $\tilde{\delta}$.

Despite the unusual features of these RT bubbles, the discussion in appendix B highlights a general feature of the QEIs in a simple way. In particular, as discussed below

eq. (B.5), there are two competing terms contributing to the generalized entropy of these surfaces: the bulk area, which describes the entropy of the CFT fields on the brane enclosed by the bubble, and the area of the boundary where they intersect the brane, which appears in the gravitational entropy of the DGP term. The bulk contribution naturally acts to contract the bubble but with $\lambda_b < 0$, the brane contribution acts to expand the bubble. As described in the appendix, there is an equilibrium radius where these two effects balance one another. Of course, with $\lambda_b > 0$, the brane contribution also acts to contract the boundary of the bubble and so no closed extremal surfaces appear, as expected.

As noted above, a similar competition is a general feature in the formation of QEIs. However, in this case as discussed in section 3.2, the bulk and brane contributions combine to produce a Bekenstein-Hawking term $A(\sigma_{\mathbf{R}})/4G_{\text{eff}}$ on the boundary of the island. This contribution, of course, imposes a large penalty to the formation of a large island and acts to contract the boundary towards a smaller (*i.e.*, vanishing) radius. For an island to appear, this contraction must be balanced by an expanding contribution. From the bulk perspective, this is simply coming from the remaining⁶ bulk area contribution of the RT surface, which we can ascribe to the quantum EE of the CFT state from the brane perspective. The point to be noted here is that for this to provide an expansion the RT surface must be anchored far from the island, *i.e.*, in the asymptotic (nongravitational) region associated with the boundary CFT. While perhaps self-evident, this discussion highlights the nonlocal nature of the physics producing the QEIs.

Let us add that the QEIs discussed here (as well as the RT bubbles) are remnants of replica wormholes in the limit $n \rightarrow 1$. This follows from the fact that we are simply studying holographic EE with RT surfaces in a new bulk background, *i.e.*, with a back-reacted brane. Hence, the analysis of [176]⁷ introduces a smooth n -fold covering geometry for the corresponding Rényi entropies with positive integer indices. These covering geometries produce smooth wormhole geometries on the brane analogous to those discussed in [48, 65] for two dimensions.

Now assuming replica symmetry, one can then take a $\mathbb{Z}_n$ orbifold quotient which leaves a single copy of the boundary geometry but the bulk solution now contains a codimension-two cosmic brane with tension $T_n = (n-1)/(4G_{\text{bulk}} n)$. In the presence of a DGP brane, we expect that there is an additional contribution where the two branes intersect, *i.e.*, the intersection surface carries an intrinsic tension $\hat{T}_n = (n-1)/(4G_{\text{brane}} n)$. In this setting, our discussion above for the formation of QEIs extends to the Rényi entropies in a relatively straightforward way. In particular, we expect that an area contribution associated with

⁶We combined part of the bulk area into the Bekenstein-Hawking term above.

⁷Following [142, 177], the same applies for general time dependent situations.

the boundary of the island now carries an effective tension $\tilde{T}_n = (n - 1)/(4G_{\text{eff}} n)$, which combines the intrinsic tension of this intersection surface and the contribution of the cosmic brane in the vicinity of the Planck brane. The contraction created by this term must be balance by the expansion provided by the remaining cosmic brane contributions. However, to provide an expansion the cosmic brane must be anchored by a twist operator in the asymptotic (nongravitational) boundary. Again, this highlights the nonlocal nature of the physics that implicitly supports the replica wormholes.

Of course, these dynamical considerations are emergent in the topological models considered in [48, 77]. Hence it would be interesting to understand the implications of this dynamics to extend the new discussions of baby universes and ensembles to higher dimensions. Some recent work describing baby universes and the Page curve in arbitrary dimensions can be found in [190, 191].

7.2 Black holes and the information paradox

Let us now turn to our analysis of black holes and the information paradox considered in the second half of this thesis using our brane world model.

7.2.1 Non-extremal black holes in higher dimensions

As previously mentioned, in section 4.2, we considered AdS-Rindler coordinates in the bulk, providing a description of the pure AdS spacetime as a two-sided massless non-extremal black hole. A similar black hole geometry is induced on the brane, coupled to and in equilibrium with bath regions on the asymptotic boundary in both Rindler wedges. We considered the entropy of bath regions $\mathbf{R}$ complementary to belts centred around the defects in the two Rindler wedges. This setup, from the perspective of the effective theory on the brane and asymptotic baths, is analogous to the two-dimensional setup at finite temperature considered in [50].

We find, in particular, that the information paradox for eternal black holes and its resolution studied in [50] makes an expected re-appearance in higher dimensions, as reviewed in section 4.8. Again, this information paradox is resolved by the appearance of a QEI when a second quantum extremal surface minimizes the generalized entropy in the island rule (1.7). Our holographic construction translates this competition between quantum extremal surfaces to the usual competition between different possible RT surfaces in the holographic

formula (1.8). In particular, at late times, the minimal RT entropy is provided by a second extremal surface with components that cross the brane, as illustrated in figure 1.7. From the brane perspective, the intersection of this RT surface with the brane becomes the quantum extremal surfaces bounding the island in the black hole background. The island belongs to the entanglement wedge of the bath region $\mathbf{R}$. Without the appearance of islands, the entropy of bath subregions would grow ad infinitum. With the islands however, the ever-growing entropy of the no-island phase is eventually capped off by the constant finite entropy of this island phase at late times. Further, our higher-dimensional discussion provides a simple explanation for the saturation of entropy: the connected pieces of the RT surface in the island phase are isolated to individual Rindler wedges and are thus invariant under time translation (*i.e.*, forward boosts in both wedges).

Recall that the global state is pure, *i.e.*, from the boundary perspective, it is a TFD state of two copies of the boundary CFT plus conformal defect. Hence the entropy of $\mathbf{R}$ is identical to that of its complement $\overline{\mathbf{R}}$. This gives a useful alternative view of the evolution of the entropy. The region $\overline{\mathbf{R}}$ consists of a belt region centred on the conformal defect in the two bath regions. Hence from this point of view, we are considering the EE of two isolated boundary regions A and B on either side of the corresponding eternal black hole in the bulk. This is essentially the same system studied in [161], except that here the spatial sections of the bath geometry are hyperbolic in the present case. As in [161], the entropy grows at early times but then quickly thermalizes. In this case, the growth of the entropy stops, because it is bounded by subadditivity, *i.e.*, $S(A \cup B) \leq S(A) + S(B)$. In fact, for the holographic system, the late time entropy saturates this inequality, which erases the mutual information between two boundary subregions. The primary difference between the framework studied in [161] and our setup is the addition of a backreacting brane that creates extra spacetime geometry for the RT surfaces to traverse in this late-time island phase and so delays the onset of this phase where the entropy is saturated. From the boundary perspective, this longer thermalization time relative to [161] can be understood as a consequence of the large number of degrees of freedom introduced by the conformal defect.

Further, as in [50], we find that the island extends outside the event horizon, *i.e.*, the quantum extremal surfaces appear outside of the horizon. If we focus on the entropy of $\overline{\mathbf{R}}$ as above, this feature again has a simple explanation in our holographic setup, in terms of entanglement wedge nesting. Recall in the island phase, the individual components of the RT surface yield the entropy of the individual belt regions on the boundary of either Rindler wedge. Since these belts are subregions of the full hyperbolic slice on which the corresponding CFT resides, the RT surface must remain within the corresponding Rindler wedge. That is, the bifurcation surface of the Rindler horizon in the bulk is the RT

surface corresponding to either of the copies of the CFT in the TFD state [160], and the Rindler wedge is the corresponding entanglement wedge. Hence, by entanglement wedge nesting [139,192], the RT surface and entanglement wedge for any subregion of H_{d-1} on the boundary must lie within the corresponding Rindler wedge. Finally it was straightforward to see from eq. (4.12) that the horizon on the brane is precisely the intersection of the Rindler horizon in the bulk with the brane. Hence the quantum extremal surface on the brane, *i.e.*, the intersection of RT surface with the brane, must lie outside of the black hole horizon. This also means that if we consider regions $\mathbf{R}$ far away from the defect, the RT surface will pass close to the horizon. Thus, analogously to the situation discussed in [50], information about the horizon seems to be contained in the entanglement of CFT regions of the bath which are furthest from the black hole.

7.2.2 Extremal black holes in higher dimensions

In chapter 5, by taking a Poincaré patch of the bulk, we considered an extremal black hole on the brane coupled to a (single) bath CFT in a flat background. As in [50], we calculated the entanglement entropy for a bath region $\mathbf{R}$ which corresponded to points greater than some distance b from the conformal defect. In the case of extremal black holes, we did not find a transition as the system was time evolved, but instead found that the appearance of an island is linked to the choice of brane angle θ_B (or brane tension) and the DGP coupling.

Due to the scale invariance of Poincaré coordinates, it is clear that as we push the entangling surface out in the bath region, *i.e.*, increase b , we proportionately reduce the size of the island. Again, this behaviour reproduces the intuition suggested in [50] that the region near the extremal horizon deep in the gravitating region (our brane) is contained within the far-away portion of the bath. Actually, our higher-dimensional picture shows that these regions are not far from each other at all — they are both close to the spatial infinity of the Poincaré patch which corresponds to a single point in the global frame. In the other extreme $b \rightarrow 0$, we find that regions of the brane arbitrarily close to the asymptotic boundary can be recovered by portions of the bath sufficiently close to the defect. This is in contrast to the two dimensional JT model, where a maximum island size exists.

Interestingly, a further qualitative deviation from the two-dimensional case is seen at small brane angles θ . Recall that, in the two-dimensional JT model, the island phase is always dominant for belt geometries in the extremal case [50]. In contrast, we have found in $d > 3$ that islands cease to exist for θ_B below some critical $\theta_c > 0$. As θ_B approaches θ_c from above, the quantum extremal surface of the island phase runs off to infinity (*i.e.*, towards the extremal horizon). For $\theta_B < \theta_c$, no quantum extremal surface exists on the

brane and the bulk RT surface is simply given by two planes on either side of the brane running straight into the bulk. Since the area of these latter surfaces is IR finite in $d > 2$, their candidacy for RT surfaces must be considered even when the alternative island-phase surfaces exist. In fact, we find that θ_c is precisely the angle at which the entropies of the no-island-type and island-type surfaces match — above this angle, the island-type surfaces remain favourable as RT surfaces. The relevance of small θ_B (and in particular $\theta_B < \theta_c$) is that in this limit, the effective theory on the brane is described by Einstein gravity with small higher curvature corrections, which is the most interesting parameter regime. While the lack of islands for $\theta_B < \theta_c$ is strikingly different from the two-dimensional case, we remark that, in the extremal case, islands are not required from an information-theoretic standpoint and their absence should perhaps not be terribly surprising. This is to be contrasted with the non-extremal case, where islands are necessary, at all brane angles, to tame the otherwise unbounded growth of black hole entropy at late times and avoid the information paradox.

Of course, an interesting question may be to examine how varying the geometry of the entangling surface affects the appearance of QEIs at $T = 0$. For example, rather than belt geometries, one might consider spherical regions bisected by the conformal defect.

7.2.3 Not an ensemble

In order to derive the island formula, a crucial ingredient was the appearance of wormholes in the replica trick. In the two-dimensional models involving JT gravity studied so far [48, 65], the existence of wormholes follows from the fact that JT gravity is defined by averaging over an ensemble of Hamiltonians. For example, JT gravity emerges as the low energy effective description of the SYK model [178–181], or has a definition in terms of a matrix model [193].

On the contrary, our construction relies only on the standard holographic rules of the AdS/CFT correspondence where there is no such averaging of the couplings in the boundary theory. This is in line with the general expectations for higher dimensional holography. This lack of averaging characterizes the UV-complete description of the system, *i.e.*, the boundary perspective. Nonetheless, quantum extremal islands appear in the effective description of the brane perspective and once again one likes to understand them as remnants of replica wormholes in the limit $n \rightarrow 1$. One might then wonder why — despite the absence of ensemble averaging — replica wormholes should appear and connect the gravitating region in different copies of replica trick calculations.

In fact, this is not a problem: since the different effective gravity theories in the brane

picture are UV completed by a single theory of gravity in the bulk as discussed above, it is natural to consider geometries connecting the branes, *i.e.*, replica wormholes in the effective theory. In fact considering Rényi entropy calculations in the boundary theory, one sees that the corresponding bulk geometry induces connections between the different copies of the brane theories, *i.e.*, replica wormholes on the brane. This becomes particularly clear in our setup where the brane lives in the bulk and does not serve as a boundary of spacetime. We emphasize that here this discussion implicitly relies on the standard derivation of the RT prescription for holographic EE [176, 177] in the bulk perspective, where again we assume that there is no ensemble averaging.⁸

Following the logic of [77], one might be tempted to turn the logic around and, given the appearance of wormholes in the brane description of our model, conclude that there is some form of ensemble averaging in the dual boundary theory. However, this line of argument implicitly assumes a precise equivalence between the boundary theory and the ‘bulk’ gravity theory (containing wormholes). We stress that this equivalence does not hold in our construction. Rather the gravitational theory on the brane is an effective theory and so the arguments of [77] do not extend to this situation. Instead, in our situation replica wormholes appear, but wormholes connecting independent instances of the boundary theory do not play a role. For example, this implies that higher powers of the partition function of the boundary CFT with a conformal defect will still factorize.

Nonetheless, this issue is certainly worth further examination since in two dimensions, replica wormholes have now been shown to play an important role in a variety of situations, *e.g.*, calculations of Rényi entropies [48, 75], the spectral form factor [193, 194], correlation functions [157, 195], and overlap of black hole microstate wavefunctions [48, 75]. Apart from Rényi entropies, it is not clear how to reproduce these effects in our construction, or in higher dimensions more generally. Furthermore, it was suggested in [48, 196] that in non-averaged theories wormholes might appear as a result of some diagonal approximation. To obtain a full quantum gravitational answer, additional off-diagonal terms need to be added. Given that we have a system, where wormholes appear in an approximate formulation, while at the same time having some control over a UV complete description, one might hope that studying our system will give an idea of how this suggestion might be realized.

⁸Ref. [81] formulates a point of view where integrating out the bath CFT generates an averaging over couplings in the theory of the conformal defect.

7.3 Future directions

Having produced a setup in which QEIs can be studied with relative ease, some possible avenues of further investigation were already suggested above. We shall conclude this thesis by mentioning below a number of other possible extensions to our work to be explored in the future.

7.3.1 Questions about entanglement wedges

A few directions for future research may be inspired by information-theoretic questions similar to those raised in [50]. There, the authors investigated whether a protocol can be implemented to retrieve information from the island. In particular, the entanglement wedge of the complete left system plus an interval of the right bath contains an island that naively appears causally disconnected from the left and the right bath interval. However, by acting with operators in the left and right baths, it was argued that sufficient negative null energy can be generated to pull information from this region into the left exterior, to be picked up by the left defect and bath. One could try to reproduce this protocol in our higher-dimensional setup using insertions of operators on the left and right asymptotic boundaries. The negative null energy produced would then shift the bulk horizon and hence the induced horizon on the brane.

Recall that above, we described how in the present discussion the appearance of the quantum extremal surfaces outside of the horizon was a simple result of the nesting of entanglement wedges from the bulk perspective. However, another question raised by [50] is whether this protrusion of islands outside the horizon violates causality. In particular, the portion of the island of the baths outside the horizon appears to be causally connected to the defects. Naively, this appears to allow communication between the baths and defects even if the coupling between these systems is severed. The resolution of this paradox comes from noting that a splitting quench between the defect and bath systems would inevitably create a positive energy shock causing an outward shift of the horizon. It was argued in [50], using a JT version of the quantum focusing conjecture [197, 198], that this shift would have the final event horizon swallow the island, preventing post-quench communication between the bath and defect. It would be interesting to re-create this problem in our setup to probe the quantum focusing conjecture in higher dimensions. From the bulk perspective, a splitting quench would be implemented by a bulk end-of-the-world brane anchored asymptotically on the splitting surface [199]. In $d = 2$, the splitting surface on the asymptotic boundary can be obtained by a conformal transformation from a full plane; in the bulk, the end-of-the-world brane can similarly be obtained by a diffeomorphism from a planar brane in

pure AdS. In $d > 3$, however, the calculations will become more complicated, *e.g.*, the end-of-the-world brane will, in general, backreact on the geometry such that the bulk is no longer locally pure AdS.

Returning to the issue of extracting information from the island, entanglement wedge reconstruction [53, 137–143] allows us to recover information about the island with data from the boundary CFT in the corresponding boundary subregion. One interesting question would be to evaluate the expectation value of various CFT operators in the island, *e.g.*, reconstructing $\langle T_{ij} \rangle$ in the vicinity of the horizon.⁹ The latter is particularly interesting because while the appearance of quantum extremal islands pointed out a new resolution of the information paradox, this does not directly address the issue of firewalls [19, 200]. Here asking if the black hole horizon develops a firewall in the late time phase of the Page curve can be addressed by evaluating $\langle T_{ij} \rangle$ on the horizon. While a direct boundary reconstruction of the latter remains to be done, we are confident that no singularities arise in our framework. The reason is that in the bulk, the system is in the vacuum state and we are simply examining this state from a Rindler frame of reference. Hence in fact, we expect that $\langle T_{ij} \rangle = 0$ on the horizon and throughout the black hole solution on the brane.¹⁰

7.3.2 Other geometries

Other choices of brane curvature As described in chapter 2, we choose the brane tension to produce a negative cosmological constant in the gravity theory on the brane, in accord with eqs. (2.24) and (2.25). As a result, the d -dimensional geometry on the brane is AdS space. However, it is straightforward to consider the case where the brane tension takes its critical value, such that $1/\ell_{\text{eff}}^2 = 0$, as is usually done in the RS scenario [84, 85]. In this case, the analogous brane geometry is simply flat space, and the brane is easily embedded in the bulk AdS_{d+1} geometry on a slice of constant radius (or constant z) in standard Poincaré coordinates. An interesting feature of this embedding is that the brane reaches the asymptotic AdS_{d+1} boundary along the null boundaries of the flat space geometry (as well as a timelike and spacelike infinity) *e.g.*, see [182]. Hence we can naturally investigate quantum extremal surfaces and the island formula in flat space using the usual expressions

⁹We thank Ahmed Almheiri for raising this question.

¹⁰The vanishing of the stress tensor on the brane is an essential feature of our construction as the AdS_d brane geometry must be a solution of the corresponding gravitational equations. That is, the CFT on the brane cannot provide a source in these equations (at least to leading order for large c_T) otherwise the geometry would deviate from AdS space. Recall that while the brane CFT is in its vacuum state, the bath CFT is coupled to the brane along an accelerating trajectory — see discussion under eq. (4.12). This acceleration allows the bath CFT to achieve equilibrium at a finite temperature.

for holographic entanglement entropy in this construction as long as we consider regions on null infinity. Notably this matches the approach pursued in [49], but contrasts with studies of *e.g.*, [51] which considered spacelike regions. It would, of course, be interesting to use this framework to study QEIs in the context of asymptotically flat brane world black holes, *e.g.*, as described in [201, 202]. We should note however that there are undoubtedly subtleties with the proposed construction, *e.g.*, as the brane completely cuts out the asymptotic AdS_{d+1} boundary (except for a single point) on constant time slices.

Of course, one can also consider the case where the brane tension is chosen such that $1/\ell_{\text{eff}}^2 < 0$. That is, the brane gravity theory would have a positive cosmological constant and the corresponding brane geometry becomes de Sitter space. In this case, one constructs a foliation of the bulk AdS_{d+1} geometry in terms of d -dimensional de Sitter slices and the brane can be embedded along the slice with the appropriate curvature, *e.g.*, see [182]. In this case, the brane reaches the asymptotic AdS_{d+1} boundary on the future and past timelike infinities of the de Sitter geometry. Hence, this construction provides a framework to use holographic entanglement entropy for investigating the island formula in de Sitter space as long as we consider regions on the timelike future of the latter geometry. Let us add that this would be similar to the work of [80, 203], which studies related questions in the context of JT gravity with a positive cosmological constant [204]. The de Sitter evolution of the Hartle-Hawking vacuum prepares a two-dimensional CFT state on the circle and the EE of various regions in the latter state are investigated, revealing new islands in the de Sitter geometry. Further discussions of islands in de Sitter spacetime can be found in [82, 83, 205].

Massive black holes In this thesis, for the sake of simplicity, we have chosen to work with a bulk that is pure AdS, *i.e.*, the temperature was tuned to $T = \frac{1}{2\pi R}$. The Rindler horizon in this geometry consequently corresponds to a massless hyperbolic black hole. An obvious extension would then be to consider massive black holes. Again, calculations will be made difficult by the fact that the brane and bulk equations of motion must be solved simultaneously with the former back-reacting on the latter. In particular, the equilibrium configuration will now involve excitations of the CFT on the brane, *i.e.*, the effective Einstein equations on the brane will be sourced by the stress tensor of the boundary CFT residing there. Recently, islands for massive black holes in the doubly holographic model have been studied by [206] for $d = 2$; however, as described in section 2.3, this is a rather special case where Einstein gravity on the brane is topological and one may instead consider adding intrinsic JT gravity.

7.3.3 Other information-theoretic objects and considerations

Entanglement wedge cross-sections Recent work [207, 208] has drawn attention to the entanglement wedge cross-section, *i.e.*, for disconnected boundary regions, the codimension-two surfaces in the bulk which have minimal area and which split the entanglement wedge in two. In particular, there are a number of proposals relating these holographic surfaces to various entanglement measures: entanglement of purification [207, 208], odd EE [209–211], entanglement negativity [212, 213], and (perhaps the most promising) reflected entropy [214].

Turning to our model and examining figure 3.4, we see that there are two such minimal surfaces in the connected phase, for which a quantum extremal island appears on the brane. These surfaces are simply disks of radius $P = P_0$ on either side of the brane, with area

$$A = \frac{2 L^{d-1} \Omega_{d-2}}{d-1} P_0^{d-1} {}_2F_1 \left[\frac{1}{2}, \frac{d-1}{2}, \frac{d+1}{2}, -P_0^2 \right], \quad (7.2)$$

as can be seen from eq. (3.40). The fact that both disks have the same area results from the fact that the corresponding boundary regions are symmetric on either of the conformal defect — see figure 3.1. Of course, if one of the two caps comprising the boundary regions was smaller, the minimal area disk closer to this cap would provide the global minimum and hence become the entanglement wedge cross-section. It would be interesting to understand if the second minimal disk also plays an interesting role in characterizing the entanglement of the boundary state. In this vein, let us add that there are also two additional extremal disks which divide the entanglement wedge in two but their area is actually a local maximum. These disks again lie on either side of the brane but end on $\sigma_{\mathbf{R}}$, the intersection of the RT surface with the brane. Again, it is natural to wonder if these surfaces have an interpretation in terms of the boundary entanglement.

Tensor networks and complexity Yet another direction to take would be to consider our setup from the perspective of tensor networks and error correction codes [141, 215–217]. For instance, as noted in [161], the MERA-like tensor network constructing the time-evolved CFT TFD state on the asymptotic boundary shares a similar geometry to codimension-one bulk spatial slices stretching through the bulk wormhole. One might then be motivated, as in [216], to view these spatial slices as supporting tensor networks implementing quantum error correction codes between the bulk and boundary. It would be interesting to see what such a network would tell us about the effective theory (see *e.g.*, [218, 219]) on the brane and how information on the brane is ultimately encoded in the asymptotic CFT and defect theory. On a related note, one might also study the

complexity of these brane configurations, for example, using the higher-dimensional bulk to probe holographic complexity conjectures [220–223], *e.g.*, see [224].

Entanglement growth Lastly, in order to explain the fast growth of entanglement at early times for large regions, in section 4.9.2 we computed bounds on entanglement growth in hyperbolic space. While they display the expected qualitative behaviour, they are not particularly tight. Instead, the difference between bounds and numerical data becomes bigger as χ_Σ grows. It would be interesting to improve these bounds.

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APPENDICES

Appendix A

Generalized Entropy on the Brane

In sections 2.3 and 2.4, we introduced intrinsic gravitational terms to the brane action. Following [7],¹ we assumed that these terms contribute to the generalized entropy, *e.g.*, see eq. (1.8) or (3.1). In this appendix, we present an extended version of an argument in [93], which will support this assumption and our formula for generalized entropy.

As in the main text, we begin with a d -dimensional holographic CFT on $R \times S^{d-1}$ with a conformal defect on the equator of the sphere, sweeping out $R \times S^{d-2}$. On a fixed time-slice, we choose an entangling surface Σ_{CFT} which divides the sphere into two equal halves along a maximal S^{d-2} which lies orthogonal to the conformal defect. Now we wish to determine the EE between the two halves of the system, as sketched in figure A.1. Recall that with the geometric approach [225], we must evaluate the partition function on a (Euclidean) background geometry with an infinitesimal conical defect. In order to construct a symmetric geometry where introducing such a defect is well-defined, we perform a Wick rotation on the boundary time (*i.e.*, $t_{\text{E}} = it$) and then conformally transform the Euclidean background metric to a round S^d with the conformal defect lying on a maximal S^{d-1} on this background. Now Σ_{CFT} remains a maximal S^{d-2} which runs orthogonal to the defect and pierces the latter on a S^{d-3} . With this construction, there is a rotational symmetry in the two dimensions orthogonal to Σ_{CFT} . To evaluate the corresponding entanglement entropy, we construct $\mathcal{M}_{1-\epsilon}$, the ‘ n -fold cover’ with $n = 1 - \epsilon$, by introducing an infinitesimal conical defect at Σ_{CFT} . The EE is then given by

$$S = \lim_{\epsilon \rightarrow 0} \left(\frac{\partial}{\partial \epsilon} + 1 \right) \log Z_{1-\epsilon}, \quad (\text{A.1})$$

¹See also [4, 34, 48, 50, 65].

where $Z_{1-\epsilon}$ is the partition function of the holographic CFT on the covering space $\mathcal{M}_{1-\epsilon}$. Of course, the latter has a dual description in terms of the bulk gravity, and using the usual saddle point approximation, eq. (A.1) becomes [93]

$$S = -\lim_{\epsilon \rightarrow 0} \left(\frac{\partial}{\partial \epsilon} + 1 \right) I_{E,1-\epsilon}, \quad (\text{A.2})$$

where $I_{E,1-\epsilon}$ is the Euclidean bulk action evaluated on the appropriate dual solution.

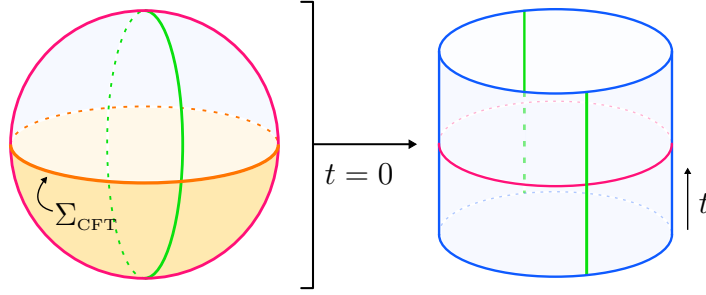


Figure A.1: A time slice of our d -dimensional CFT setup with entangling surface Σ_{CFT} and an equatorial conformal defect (the green line). In the right panel, one dimension is suppressed relative to the left panel.

Setting $n = 1$ for a moment, the bulk dual of $\mathcal{M}_1$ is simply the Euclidean version of the geometry constructed in section 2.1, which we denote $\widetilde{\mathcal{M}}_1$. Recall the boundary geometry is S^d and the conformal defect runs around a maximal S^{d-1} . In the bulk, the geometry is locally EAdS_{d+1} everywhere away from the brane, and the brane has a EAdS_d geometry which extends out to the conformal defect at the asymptotic boundary and with the curvature scale given by eq. (2.16) — see figure A.2. Now the entangling surface Σ_{CFT} on the asymptotic AdS boundary is the boundary of an extremal surface $\Sigma_{\mathbf{R}}$ in the bulk, which runs straight across the bulk solution and has a EAdS_{d-1} geometry with curvature scale L . This surface pierces the brane at a right angle and the intersection, another extremal surface $\sigma_{\mathbf{R}}$, has the geometry of a EAdS_{d-2} with curvature scale ℓ_{B} — see figure A.2. Now because of the symmetry of this configuration, the rotational symmetry about the entangling surface in the boundary extends to a rotational symmetry about $\Sigma_{\mathbf{R}}$ in the bulk. Hence we can calculate the EE $\widetilde{\mathcal{M}}_{1-\epsilon}$ with the same geometric approach as we applied in the boundary. That is, we construct $\widetilde{\mathcal{M}}_{1-\epsilon}$, the n -fold cover (with $n = 1 - \epsilon$) of the bulk solution with a infinitesimal conical defect at $\Sigma_{\mathbf{R}}$ and by extension, at $\sigma_{\mathbf{R}}$ on the brane.

That is, the angle around $\Sigma_{\mathbf{R}}$ runs through a range $2\pi(1-\epsilon)$. Now [226,227] developed a description of such conical defects in which the singular geometry is replaced by a ‘regulator’

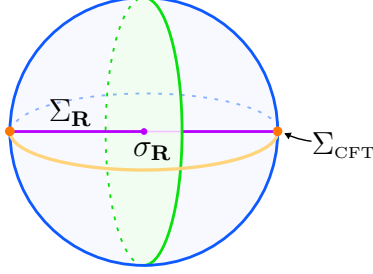


Figure A.2: A cross-section of the Euclidean geometry $\widetilde{\mathcal{M}}_1$. The orange semicircle and its complement along a time slice represent the orange shaded region of figure A.1 and its complement. The rotation that keeps Σ_{CFT} fixed represents euclidean time. An infinitesimal conical defect $\Sigma_{\mathbf{R}}$ runs through the bulk and intersects the brane at $\sigma_{\mathbf{R}}$.

geometry where the region around the conical singularity is smoothed out. Applying their key result, we can write the bulk Riemann tensor as a “smooth” contribution away from $\Sigma_{\mathbf{R}}$, the conical defect, and a singular order ϵ contribution at $\Sigma_{\mathbf{R}}$,²

$${}^{(\epsilon)}R^ab_{cd} = R^ab_{cd} + 2\pi\epsilon\epsilon^{ab}\epsilon_{cd}\delta_{\Sigma_{\mathbf{R}}}, \quad (\text{A.3})$$

where ϵ_{ab} is the Euclidean volume form in the two-dimensional transverse space to $\Sigma_{\mathbf{R}}$, and R^ab_{cd} is the “smooth” curvature piece. The $\delta_{\Sigma_{\mathbf{R}}}$ is a two-dimensional delta function defined in [93]. The conical singularity intersects the brane at $\sigma_{\mathbf{R}}$ and so we have a similar decomposition for the Riemann tensor on the brane,

$${}^{(\epsilon)}\tilde{R}^{ij}_{kl} = \tilde{R}^{ij}_{kl} + 2\pi\epsilon\tilde{\epsilon}^{ij}\tilde{\epsilon}_{kl}\delta_{\sigma_{\mathbf{R}}}. \quad (\text{A.4})$$

Now recall that our aim is to evaluate the Euclidean action in eq. (A.2). This action is the sum of the Euclidean versions³ of the bulk and brane actions in eqs. (2.1) and (2.50) (or perhaps eq. (2.42) for $d = 2$), as well as the associated boundary terms. Equipped with eqs. (A.3) and (A.4), it can be shown that in the limit of small ϵ that the Euclidean action can be expanded as

$$I_{E,1-\epsilon} = (1-\epsilon)I_{E,1} + \int_{\text{bulk}} d^{d+1}x\sqrt{g}2\pi\epsilon\epsilon^{ab}\epsilon_{cd}\delta_{\Sigma_{\mathbf{R}}}\frac{\partial\mathcal{L}_{\text{E,bulk}}}{\partial R^ab_{cd}} \quad (\text{A.5})$$

$$+ \int_{\text{brane}} d^dx\sqrt{\tilde{g}}2\pi\epsilon\tilde{\epsilon}^{ij}\tilde{\epsilon}_{kl}\delta_{\sigma_{\mathbf{R}}}\frac{\partial\mathcal{L}_{\text{E,brane}}}{\partial\tilde{R}^{ij}_{kl}} + \mathcal{O}(\epsilon^2). \quad (\text{A.6})$$

²This order ϵ contribution is universal, whereas the details of the regulator come into play at order ϵ^2 and higher.

³Note that the difference in signs in going between Minkowski and Euclidean signatures [93].

Noting the symmetry of our configuration, *i.e.*, the curvatures are constant everywhere along the surfaces $\Sigma_{\mathbf{R}}$ and $\sigma_{\mathbf{R}}$, we then find the entropy in eq. (A.2) is given by

$$S = -2\pi \frac{\partial \mathcal{L}_{\text{E,bulk}}}{\partial R^{ab}_{cd}} \varepsilon^{ab} \varepsilon_{cd} \int_{\Sigma_{\mathbf{R}}} d^{d-1}x \sqrt{h} - 2\pi \frac{\partial \mathcal{L}_{\text{E,brane}}}{\partial \tilde{R}^{ij}_{kl}} \tilde{\varepsilon}^{ij} \tilde{\varepsilon}_{kl} \int_{\sigma_{\mathbf{R}}} d^{d-2}x \sqrt{h'}, \quad (\text{A.7})$$

where h and h' are the induced metrics along the $\Sigma_{\mathbf{R}}$ and $\sigma_{\mathbf{R}}$, respectively. Hence we see that there is a contribution of the Wald entropy from both the bulk action and the brane action. Further, let us note that various signs appear upon analytically continuing back to Lorentzian spacetime, *i.e.*, in the Lagrangian and the transverse volume form [93].

For the case where the Einstein-Hilbert action appears both in the bulk and on the brane, as in eqs. (2.1) and (2.50), we find the formula for the generalized entropy (A.7) becomes

$$S = \frac{A(\Sigma_{\mathbf{R}})}{4G_{\text{bulk}}} + \frac{A(\sigma_{\mathbf{R}})}{4G_{\text{brane}}}, \quad (\text{A.8})$$

as given in equation (3.1). The present derivation only applies to special symmetric configuration, as in [93]. The symmetry of this configuration precludes finding any extrinsic curvature terms in eq. (A.7), as would be expected for the Dong entropy [95]. We note however that no such terms would correct eq. (A.8) for the generalized entropy coming from the Einstein-Hilbert term. It would, of course, be interesting to extend our derivation to more general configurations involving bulk DGP branes, along the lines of [176] or [177].

Appendix B

Ryu-Takayanagi Bubbles

In this appendix, we consider a simple but surprising class of RT surfaces. In particular, we show below that there are closed extremal surfaces with the topology of a sphere, *i.e.*, S^{d-1} in the locally AdS_{d+1} bulk geometry. In empty AdS space, one could consider such spherical surfaces, but their area would be extremized when they collapse to zero size. In the present case, we will show that in certain situations, the spherical RT surfaces can be supported at finite size by the brane. To illustrate the situation, we continue with the special case of $d = 3$ as in section 3.4, and afterwards comment on the situation with general d .

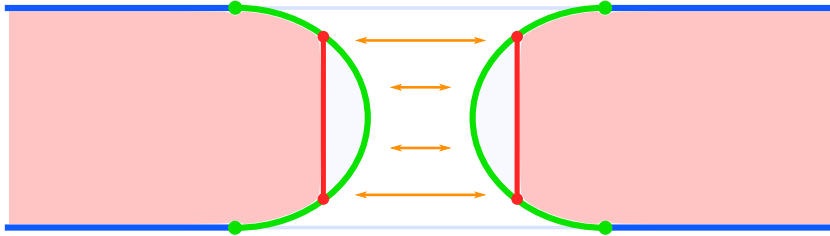


Figure B.1: An RT ‘bubble’ on the brane: even for the vacuum, when $G_{\text{brane}} < 0$ the competing bulk and brane area terms can lead to a stable extremal surface, which is homologous to the entire time slice for the boundary CFT. The entanglement wedge then corresponds to the shaded red region. Since the two sides of the brane are glued together, the RT surface has the topology of S^{d-1} .

Consider the geometry illustrated in figure B.1. On either side of the brane, we have a disk satisfying $\varsigma = \text{constant}$, *i.e.*, satisfying eq. (3.39) with $P_0 = 0$. Hence locally these

surfaces extremize the entropy functional (3.38) in the bulk. However, rather than extending out to the asymptotic boundary, as shown in the figure, the two disks intersect the brane and meet at some radius P_B . Hence this RT surface has the topology of a sphere and we use the nomenclature ‘bubble’ to describe these surfaces. For $d = 3$, the generalized entropy (3.1) of this bubble is

$$S_{\text{gen}} = \frac{\pi L^2}{G_{\text{bulk}}} \left(\sqrt{1 + P_B^2} - 1 + \lambda_b P_B \right) + 2 \lambda_{\text{GB}} \quad (\text{B.1})$$

with λ_b defined in eq. (2.54). We have also included the topological term introduced in eq. (3.44). Of course, since these surfaces never reach the asymptotic boundary, this quantity is finite, *i.e.*, there are no UV divergences in eq. (B.1).

Extremizing eq. (B.1) with respect to the radius of the bubble, we find

$$\partial_{P_B} S_{\text{gen}} = 0 \quad \implies \quad \frac{P_B}{\sqrt{1 + P_B^2}} = -\lambda_b = -\frac{G_{\text{bulk}}}{2L G_{\text{brane}}}. \quad (\text{B.2})$$

Now recall that we will always have $G_{\text{bulk}} > 0$, but considered the possibility of G_{brane} becoming negative in section 3.4. Let us first consider the case $\lambda_b \geq 0$, which implies $1/G_{\text{brane}} \geq 0$. In this case, we cannot satisfy eq. (B.2), since both the bulk and brane contributions to the generalized entropy (B.1) are positive and monotonically increasing functions of P_B . Therefore the minimum lies at $P_B = 0$, *i.e.*, where the bubble collapses to zero size — see figure (B.2).

Of course, the more interesting scenario is when λ_b , and hence $1/G_{\text{brane}}$, are negative. Then eq. (B.2) has the solution

$$P_{B,0} = -\frac{\lambda_b}{\sqrt{1 - \lambda_b^2}}, \quad (\text{B.3})$$

for which the generalized entropy (B.1) becomes

$$S_{\text{gen}} = \frac{\pi L^2}{G_{\text{bulk}}} \left(\sqrt{1 - \lambda_b^2} - 1 \right) + 2 \lambda_{\text{GB}}. \quad (\text{B.4})$$

We note that these expressions are only sensible for $-1 < \lambda_b < 0$. In fact, for $\lambda_b < -1$, there is no minimum for the generalized entropy (B.1), *i.e.*, there is no solution for eq. (B.2), and rather P_B runs off to infinity — see figure (B.2). This is, perhaps, not so surprising since we can see from eq. (2.55) that this regime is pathological, with the graviton localized on the brane becoming a ghost.

Therefore we only consider the regime $-1 < \lambda_b < 0$ where eqs. (B.3) and (B.4) apply. As illustrated in figure (B.2), eq. (B.3) is indeed the global minimum of the generalized entropy (B.1). We might note that the sum of the bulk and brane terms in eq. (B.4) is negative. That is, the combined contributions of the two area terms in eq. (3.1) is in fact *negative!* Hence we only get a sensible (*i.e.*, positive) result for the generalized entropy (B.1) with the inclusion of the topological term (3.44) and with λ_{GB} sufficiently positive, which was also favoured in section 3.4.

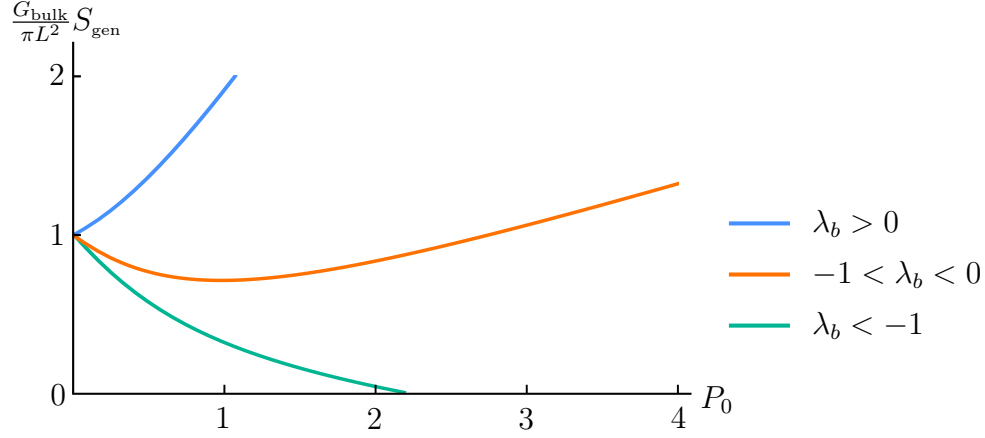


Figure B.2: The generalized area (B.1) for a bubble as a function of its radius. For $\lambda_b > 0$, the area is minimal for vanishing size, whereas for $-1 < \lambda_b < 0$ it has a finite size. For $\lambda_b < -1$, there is no global minimum, signalling an instability of the system. Further note that as P_0 approaches zero, $S_{\text{gen}} \rightarrow \pi L^2 / G_{\text{bulk}}$ since we have set $\lambda_{\text{GB}} = \pi L^2 / (2G_{\text{bulk}})$.

These calculations are easily extended to higher dimensions, where eq. (B.1) is replaced by

$$S_{\text{gen}} = \frac{L^{d-1} \Omega_{d-2}}{2(d-1)G_{\text{bulk}}} P_{\text{B}}^{d-1} {}_2F_1 \left[\frac{1}{2}, \frac{d-1}{2}, \frac{d+1}{2}, -P_{\text{B}}^2 \right] + \frac{L^{d-2} \Omega_{d-2}}{4G_{\text{brane}}} P_{\text{B}}^{d-2}. \quad (\text{B.5})$$

We have not included contributions from any topological gravity terms in this expression for general d — see further comments below. To produce a qualitative understanding of this expression, we note that

$${}_2F_1 \left[\frac{1}{2}, \frac{d-1}{2}, \frac{d+1}{2}, -P_{\text{B}}^2 \right] \simeq \begin{cases} 1 & \text{if } P_{\text{B}} \ll 1, \\ \frac{d-1}{d-2} \frac{1}{P_{\text{B}}} & \text{if } P_{\text{B}} \gg 1. \end{cases} \quad (\text{B.6})$$

Now, we observe that for large P_{B} , the leading contribution in eq. (B.5) takes the expected

form

$$S_{\text{gen}} \simeq \frac{A(\sigma_{\mathbf{R}})}{4G_{\text{eff}}} + \dots \quad \text{where} \quad \frac{A(\sigma_{\mathbf{R}})}{4G_{\text{eff}}} = \frac{L^{d-1} \Omega_{d-2}}{2(d-2)G_{\text{bulk}}} (1 + \lambda_{\text{b}}) P_{\text{B}}^{d-2}, \quad (\text{B.7})$$

again using eq. (2.54). Hence, there is a large penalty for having the RT surface meet the brane at a large radius P_{B} , which will tend to push the intersection $\sigma_{\mathbf{R}}$ to smaller radii. However, for small P_{B} , the bulk contribution to S_{gen} grows like the volume, *i.e.*, it is proportional to P_{B}^{d-1} . Hence in this regime, the brane contribution dominates since it is proportional to $\lambda_{\text{b}} P_{\text{B}}^{d-2}$, and for $\lambda_{\text{b}} < 0$, this term will favour larger values of P_{B} . Hence for the interesting case of $\lambda_{\text{b}} < 0$, we can expect that the generalized entropy for general d is extremized at some finite value of P_{B} of order $-\lambda_{\text{b}}$, just as we found for $d = 3$. Of course, the denominator in eq. (B.3) is also important for λ_{b} close to -1 , but this cannot be seen with this simple qualitative analysis. Now, in fact, the extremality condition can in fact be solved exactly for any d . One finds

$$\partial_{P_{\text{B}}} S_{\text{gen}} = \frac{L^{d-1} \Omega_{d-2}}{2G_{\text{bulk}}} P_{\text{B}}^{d-3} \left(\frac{P_{\text{B}}}{\sqrt{P_{\text{B}}^2 + 1}} + \lambda_{\text{b}} \right) = 0. \quad (\text{B.8})$$

Of course, for $\lambda_{\text{b}} \geq 0$, the only solution is $P_{\text{B}} = 0$, *i.e.*, the bubble collapses to zero size, as expected. However, for $\lambda_{\text{b}} < 0$, the minimum is given by $P_{\text{B}} = P_{\text{B},0}$, precisely the same critical radius as in eq. (B.3). Substituting this critical radius into the generalized entropy (B.5) does not yield any simplifications, however the result is easily evaluated numerically as a function of λ_{b} . Of course, the generalized entropy (B.5) is negative at this minimum and so one would really need to add a topological term to the gravitational theory, either in the bulk or on the brane, to produce a sensible entropy, as we did for the $d = 3$ example.

Wormholes and Cutoffs

The appearance of these extremal bubbles is quite unusual, of course. Since they are homologous to the entire boundary, this suggests that the ground state of the dual boundary system has an entropy by the standard RT prescription. The bulk construction makes clear that it is the conformal defect which introduces this large degeneracy of ground states.¹

¹As we see in figure B.2, entropy associated with the zero-size bubble is nonvanishing and higher than that of the stable finite-size bubble due to the topological contribution. However, we note that it may be that the correct RT prescription is to choose ‘empty surface’ in this case, giving zero entropy.

We should note, however, that the evaluation of this ground state entropy presented above is incomplete. In particular, there is a ‘zero mode’ associated with these bubbles that allows them to be translated along the brane. Recall that while the empty AdS_{d+1} geometry has an $SO(2, d)$ isometry (reflecting the conformal symmetry group of the boundary CFT), the backreacted brane geometry preserves an $SO(2, d - 1)$ subgroup of these symmetries. Now our construction places the centre of the bubbles at $P = 0$, however, by acting with these symmetries, we can position the centre anywhere on the brane. Further recall that one arrives at the RT prescription by evaluating (a particular limit of) a saddlepoint in the gravitational path integral [176]. Hence we have discovered that there is a zero mode associated with the saddlepoints connected to the bubbles. Hence the integral over this zero mode would add a contribution to the entropy proportional to the logarithm of the (regulated) brane volume. It is interesting to speculate that this contribution may lift the negative value for $S_{\text{gen}}(P_{\text{B},0})$ to some positive entropy.

An essential feature required for the appearance of these bubbles was that the gravitational coupling associated with the DGP term (2.50) was negative, *i.e.*, $1/G_{\text{brane}} < 0$. While this may seem unusual, let us note that integrating out quantum fields on the brane can produce either a positive or negative shift in Newton’s constant. In particular, the shift is found to be negative for a $U(1)$ gauge field when $d < 8$ [153, 154]. With the connection between the renormalization of Newton’s constant and the area law contribution in EE [43, 225], this negative renormalization generates a puzzle which, however, was finally resolved in terms of edge modes in [228, 229]. There is a similar negative renormalization for non-minimally coupled scalars [153], for which the resolution of the associated puzzle appears in [44]. However, we should add that if we imagine $1/G_{\text{brane}} < 0$ is induced by additional quantum fields on the brane, then our EE calculations are incomplete as they do not fully include the contributions of these extra fields. Hence our perspective here is to simply view the DGP term as a counterterm as would appear in the usual quantization of gravity on the brane, and in this context, the sign of $1/G_{\text{brane}}$ is not proscribed but rather is chosen as needed to produce the ‘observed’ value of $1/G_{\text{eff}}$.

Another remark in this vein is that the bubble solutions appear as soon as $1/G_{\text{brane}}$ is negative, *i.e.*, these solutions (B.3) exist for very small values of λ_{b} as long as $\lambda_{\text{b}} < 0$. However, it is important to recall that the short distance cutoff is given by eq. (2.63) in this regime. Hence combining eqs. (3.34) and (B.3), the areal size of the bubbles becomes

$$LP_{\text{B},0} = \frac{|\lambda_{\text{b}}| L}{\sqrt{1 - \lambda_{\text{b}}^2}} \simeq \frac{|\lambda_{\text{b}}|}{\sqrt{1 + |\lambda_{\text{b}}|}} \tilde{\delta}. \quad (\text{B.9})$$

where we have substituted eq. (2.59) in the second expression. This expression approaches the maximum size $\tilde{\delta}/\sqrt{2}$ as $\lambda_{\text{b}} \rightarrow -1$. That is, the radius of bubbles is always smaller than

the cutoff scale $\tilde{\delta}$ on the brane! Therefore, these solutions are not reliable in the regime where Einstein gravity gives a good description of the brane. On the other hand, our calculations in this appendix involved evaluating RT surfaces in the bulk, *i.e.*, they only depended on bulk perspective. Further, for $|\lambda_b| \gtrsim 1/\sqrt{2}$, the corresponding RT surfaces grow much larger than the bulk AdS scale, and so would be seen as valid solutions. However, one may ask if there are physical constraints which will not allow us to realize theories with λ_b which are that negative and so prevent us from considering scenarios where these bubbles have a macroscopic size.

We close here with two final remarks: These bubbles are a remnant of replica wormholes in the limit $n \rightarrow 1$ [48, 49]. In the discussion of chapter 7, we explore if there are any lessons that they may hold for the new discussions of baby universes and ensembles [77]. Another comment is that the bubble surfaces produce an interesting entanglement wedge, which extends to a band covering a finite time interval on the boundary. Of course, this is reminiscent of the holographic construction of differential entropy [230–234], which can be used to evaluate the area of closed surfaces in the bulk. It would be interesting to examine these connections further.